The support function of conves sets in locally context spaces

We expose a theorem due to Lars Hörmander from the article *Sur la fonction d'appui des ensembles convexes dans un espaces localementt convexe* [Arkiv för mat. Vol 3: 1954]. As pointed out by Hörmander in his cited article, Theorem 2 from § 2 below is related to earlier studies by Fenchel in the article *On conjugate convex functions* Canadian Journ. of math. Vol 1 p. 73-77) where Legendre transforms are studied in infinite dimensional topological vector spaces. The novelty in Theorem 2 is the generality and we remark that various separation theorems in text-books dealing with notions of convexity are easy consequences of Hörmander's result. In § 1 we collect preliminary facts about locally convex vector spaces over the real numbers which are used in § 2. The material in §1 has independent interest and teaches the beginner basic facts about locally convex vector spaces which give a starting point for further study in functional analysis.

Topological vector spaces

Throughout E denotes a vector space over the real numbers.

0.1 Convex sets and their ρ-functions. A convex set U in E which contains the origin is said to be absorbing if there for each vector $x \in E$ exists some real $s > 0$ such that $s \cdot x \in U$. The vector is fully absorbed by U if we have the inclusion

$$\mathbf{R}^+ \cdot x \subset U$$

The function ρ_U. If x is a vector which is not fully absorbed we put

$$\mu(x) = \max\{s : sx \in U\} \quad \& \quad \rho_U(x) = \frac{1}{\mu(x)}$$

If x is fully absorbed we put $\mu(x) = +\infty$ so that $\rho_U(x) = 0$. Notice that

$$x \in U \implies \mu(x) \geq 1 \implies \rho_U(x) \leq 1$$

Exercise. Show that the convexity of U entails that ρ_U satisfies the triangle inequality

$$\rho_U(x_1 + x_2) \leq \rho_U(x_1) + \rho_U(x_2) \tag{0.1.1}$$

for all pairs of vectors in E, and that ρ_U is positively homogeneous, i.e.

$$\rho_U(sx) = s\rho_U(x) \quad : s > 0 \tag{0.1.2}$$

Conversely,, let $\rho \colon E \to \mathbf{R}^+$ satisfy (0.1.1) and (0.1.2). Put

$$U = \{\rho \leq 1\}$$

and show that $\rho_U = \rho$.

A ρ-map as above is called a subadditive and positively homogeneous function on E. So every absorbing convex set U gives such a map ρ_U. If ρ is given we get the convex and absorbing sets

$$U_* = \{\rho < 1\} \quad \& \quad U^* = \{\rho \leq 1\}$$

The reader can check that $\rho_{U^*} = \rho_{U_*}$. Moreover, for every convex set U such that $\rho_U = \rho$ one has

$$U_* \subset U \subset U^*$$

One refers to U_* as the minimal absorbing convex set of ρ, and U^* is the maximal associated convex set. So $U \mapsto \rho_U$ is surjective from the family of absorbing convex sets but not injective. The failure is exiled via the two associated minimal and maximal convex sets for a given ρ.

The Hahn-Banach theorem.

Let ρ be subadditive and positively homogeneous. An $\mathbf{R}$-linear map λ from E to the 1-dimensional real line is majorised by ρ if

$$(*) \qquad \lambda(x) \leq \rho(x)$$

hold for every vector x. Let E_0 be a subspace of E and $\lambda_0 \colon E_0 \to \mathbf{R}$ a linear map such that (*) hold for vectors in E_0. Then there exists a linear map $\lambda \colon E \to \mathbf{R}$ which extends λ_0 and is majorised by ρ.

Exercise. Prove the Hahn-Banach Theorem using the following hint. Zorn's Lemma gives a maximal subspace E^* which contains E_0 such that λ_0 can be extend to a linear map λ^* on E^* which is majored by ρ. There remains to show that $E^* = E$. Assume the contrary and pick a non-zero vector $\xi \in E \setminus E^*$. For every real number α we get an extension of λ^* to a linear functional on $E^* + \mathbf{R}\xi$ by

$$\Lambda(x + s\xi) = \lambda^*(x) + s\alpha$$

when $x \in E^*$ and s is a real number. Since ρ is positively homogeneous we see that it majorises Λ if and only if

$$\Lambda(x + \xi) \leq \rho(x + \xi) \,\&\, \Lambda(x - \xi) \leq \rho(x - \xi)$$

hold for all $x \in E^*$. It means that

$$\alpha \leq \rho(x + \xi) - \lambda^*(x) \,\&\, \alpha \geq \lambda^*(x) - \rho(x - \xi)$$

The existence of α follows if

$$\text{(i)} \qquad \rho(x_1 + \xi) - \lambda^*(x_1) \geq \lambda^*(x_2) - \rho(x_2 - \xi)$$

hold for all pairs x_1, x_2 in E^*. Now (i) means that

$$\text{(ii)} \qquad \rho(x_1 + \xi) + \rho(x_2 - \xi) \geq \lambda^*(x_2) + \lambda^*(x_1) = \lambda^*(x_1 + x_2)$$

Finally, (ii) holds since $\lambda^*(x_1 + x_2) \leq \rho^*(x_1 + x_2)$ and because ρ is subadditive we have

$$\rho(x_1 + x_2) \leq \rho(x_1 + \xi) + \rho(x_2 - \xi)$$

Pseudo-norms.

Denote by $\mathcal{C}_E$ the family of absorbing convex sets U which in addition are symmetric, i.e.

$$x \in U \implies -x \in U$$

The symmetry entails that $\rho_U(-x) = \rho_U(x)$ and in general

$$\text{(i)} \qquad \rho_U(sx) = |s| \cdot \rho_U(x)$$

hold for every real s. If $\rho \colon E \to \mathbf{R}^+$ is a sub additive and (i) holds we say that it is a pseudo-norm. The Hahn-Banach theorem for pseudo-norms asserts that if ρ is a given pseudo-norm and λ a linear map on a subspace E_0 for which

$$|\lambda(x)| \leq \rho(x) \quad : \; x \in E_0$$

then it can be extended to a linear map Λ for which

$$|\Lambda(x)| \leq \rho(x) \quad : \; x \in E$$

The proof of this symmetric version of the Hahn-Banach theorem is left as an exercise to the reader.

1. Locally convex topologies.

Denote by $\mathcal{C}_E$ the family of symmetric and absorbing convex sets U. Let $\mathfrak{U} = \{U_\alpha\}$ be a family in $\mathcal{C}_E$ such that

$$\bigcap \mathcal{L}(U_\alpha) = \{0\} \tag{1.1}$$

Now there exists a topology on E where a base for open neighborhoods of the origin consists of sets:

$$\cap \{\rho_{U_{\alpha_i}}(x) < \epsilon\}$$

where $\epsilon > 0$ and $\{\alpha_1, \ldots, \alpha_k\}$ is a finite set of indices from the $\mathfrak{U}$-family. If x_0 is a vector in E, then a basis for its open neighborhoods are given by sets of the for $x_0 + U$ where U is a set from (1). In general, a subset Ω in E is open if there to each $x_0 \in \Omega$ exists some U from (1.2) such that $x_0 + U \subset \Omega$. This gives a topology and (1.1) entails that it is a Hausdorff topology. The sets in (1.2) are convex and therefore one refers to a locally convex topology on E.

Remark. The locally convex topology above depends upon the chosen family $\mathfrak{U}$. It is unchanged if we enlarge the family to consist of all finite intersection of its sets. When this has been done we notice that if $U_1, \ldots, U_n$ is a finite family in $\mathfrak{U}$ then the norm defined by $U = U_1 \cap \ldots \cap U_n$ is stronger than the individual ρ_{U_i}-norms. Hence a fundamental system of neighborhoods consists of single ρ-balls:

$$\{\rho_U < \epsilon\} \quad : U \in \mathfrak{U}$$

1.2. The dual space E^*. Let E be equipped with a locally convex $\mathfrak{U}$-topology where $\mathfrak{U}$ has been enlarged so that the balls above give a basis for neighborhoods of the origin. A linear functional ϕ on E is $\mathfrak{U}$-continuous if there exists some $U \in \mathfrak{U}$ and a constant C such that

$$|\phi(x)| \leq C \cdot \rho_U(x)$$

The family of such ϕ-maps give vectors in a vector space denoted by E^* and called the dual space of E.

1.3 The weak topology on E. It is by definition the coarsest topology for which the funcleons

$$x \mapsto \phi(x)$$

become continuous functions on E for every fixed $\phi \in E^*$. A fundamental system of open neighborhood of the origin in the weak topology consist of sets

$$\cap \{|\phi_k(x)| < \epsilon\}$$

where $\epsilon > 0$ and $\{\phi_k\}$ is a finite family in E^*. It is clear that every weakly open set in E is open with respect to the given locally convex topology.

1.3 The weak-star topology on E^*. This is the locally convex topology on the vector space E^* where a base for open neighborhoods of the zero-vectors consist of sets defined as finite intersections of sets defined by

$$\{\phi : \ -\delta < \phi(x) < \delta\} \quad : \quad x \in X \quad \& \quad \delta > 0$$

1.4 The separation theorem. To each pair $\phi \in E^*$ and a real number a one assigns the set

$$H = \{x \in X : \ \phi(x) \leq a\}$$

Notice that $a < 0$ can occur in which case H does not contain the origin.

1.5 Theorem. *Each closed convex set K in E is the intersection of closed half-spaces.*

Proof. Assume first that K contains the origin and consider a vector $x_0 \in E \setminus K$. Since K is closed we find a pseudo-norm ρ_U with U in the defining family $\mathfrak{U}$ such that

$$(\{x_0\} + \{\rho_U < \epsilon\}) \cap K = \emptyset$$

Put

$$V = K + \{\rho_U < \epsilon\}$$

This yields an open a convex set in E and we construct ρ_V. If $s > 0$ and $x_0 \in sV$ we have $k \in K$ and a vector ξ with $\rho(\xi) < \epsilon$ such that

$$x_0 = sk + s\xi \implies x_0 + \{\rho_U < s\epsilon\} \in sK$$

Since K is convex and contains the origin this implies that $s \geq 1$. Hence

$$\rho_V(x_0) \geq 1$$

Now we apply the Hahn-Banch Therem to the absorbing convex set V and find a linear functional λ such that

$$\lambda(x_0) = \rho_V(x_0) \geq 1$$

and at the the same time the range

$$\text{(i)} \qquad \lambda(x) \leq \rho_V(x) \leq 1 \quad : \quad x \in V$$

Here λ belongs to E^* and is not identically zero and therefore its restriction to the open ball $\{\rho_U < \epsilon\}$ cannot vanish identically. Choose

$$\xi \in \{\rho < \epsilon\} \quad \& \quad \lambda(\xi) > 0$$

Now $k + \xi \in V$ hold for every $k \in K$ and (i) gives

$$\lambda(k) + \lambda(\xi) \leq 1 \implies \lambda(k) \leq 1 - \lambda(\xi)$$

So the half-space

$$H = \{x \colon \lambda(x) \leq 1 - \lambda(\xi)\}$$

contains K while x_0 is outside since $\lambda(x_0) \geq 1$.

Remark. The half-spaces in Theorem 1.5 are closed in the weak topology. Hence every a closed convex set in the original topology is also closed in the weak topology.

1.6 Normed spaces. A pseudo-norm ρ on a vector space E is called a norm of

$$x \neq 0 \implies \rho(x) > 0$$

This gives the ρ-topology on E where the open balls $\{\rho(x) < \epsilon\}$ is a fundamental system of open neighborhoods of the origin. One often uses the notation

$$||x|| = \rho(x)$$

and refer to E as a normed space.

2. Support functions of convex sets.

Let E be a locally convex space. Vectors in E are denoted by x, while y denote vectors in E^*. To each closed and convex subset K of E we define a function $\mathcal{H}_K$ on E^* by:

$$\mathcal{H}_K(y) = \sup_{x \in K} y(x)$$

Notice that $\mathcal{H}_K$ take values in $(-\infty, +\infty]$, i.e. it may be $+\infty$ for some vectors $y \in E^*$. For example, let $K = \mathbf{R}^+ x_0$ be a half-line. Then $\mathcal{H}_K(y) = +\infty$ when $y(x_0) > 0$ and otherwise zero. It is clear that

(i) $$\mathcal{H}_K(sy) = s\mathcal{H}_K(y)$$

hold when s is a positive real number, i.e $\mathcal{H}_K$ is positively homogeneous.

2.0 Exercise. Show that the convexity of K entails that

(ii) $$\mathcal{H}_K(y_1 + y_2) \leq \mathcal{H}_K(y_1) + \mathcal{H}_K(y_2)$$

for each pair of vectors in E^*. Show also that if K and K_1 is a pair of closed convex sets such that $\mathcal{H}_K = \mathcal{H}_{K_1}$ then $K = K_1$.

2.1 Upper semi-continuity. For each fixed vector $x \in E$ the function

$$y \mapsto y(x)$$

is weak-star continuous on E^*. Since the supremum function attached to an arbitrary family of weak-star continuoes functions is upper semi-continuous, it follows that $\mathcal{H}_K$ is upper semi-continuous.

2.3 The class $\mathcal{S}(E)$. It consists of all all upper semi-continuous functions G on E^* with values in $(-\infty, +\infty]$ which satisfy (i) and (ii). The next result was proved by Hörmander in the article *Sur la fonction d'appui des ensembles convexes dans un espaces localementt convexe* [Arkiv för mat. Vol 3: 1954].

2.4 Theorem. *Each $G \in \mathcal{S}(E)$ is of the form $\mathcal{H}_K$ for a unique closed convex subset K in E.*

Proof Put $F = E \oplus \mathbf{R}$ which is a new vector space where the 1-dimensional real line is added. It dual space $F^* = E^* \oplus \mathbf{R}$. We are given $G \in \mathcal{S}(E)$ and put

(i) $$G_* = \{(y, \eta) \in E^* \oplus \mathbf{R} \quad : \; G(y) \leq \eta\}$$

here G_* is a convex cone in F^* and the semi-continuous hypothesis on G implies that G_* is closed with respect to the weak-star toplogy on F^*. Next, in F we define the set

(ii) $$G_{**} = \{(x, t) \in E \oplus \mathbf{R}^+ : \; y(x) \leq \eta t : \; (y, \eta) \in G_*\}$$

This gives a set $\widehat{C}$ in F^* which consists of vectors (y, η) such that

$$\max_{(x,t) \in G_{**}} y(x) - \eta t \leq 0$$

It is clear that $G_* \subset \widehat{C}$. Now we prove the equality

(*) $$G_* = \widehat{C}$$

To get (*) we use that the two sets in (*) are weak-star closed. Hence a strict inequality gives a separating vector $(x_*, t_*) \in F$, i.e. there exists $(y_*, \eta_*) \in \widehat{C}$ and a real number α such that

(iii) $$y_*(x_*) - \eta_* t_* > \alpha \quad \text{and} \quad (y, \eta) \in G_* \implies y(x_*) - \eta t_* \leq \alpha$$

Since G_* contains $(0, 0$ we have $\alpha \leq 0$. and since it also is a cone the last implication gives $(x_*, t_*) \in G_{**}$. Now the construction of $\widehat{C}$ contradicts the strict inequality in the left hand side of (iii). Hence there cannot exist a separating vector and (*) follows.

Next, in E we consider the convex set

$$K = \{x : (x, 1) \in G_{**}\}$$

Using (*) the reader can check that

$$\mathcal{H}_K(y) = G(y)$$

for all $y \in E^*$ which proves that G has the requested form. The uniqueness of K follows from Exercise B.4.0.

2.5 The case of normed spaces. If X is a normed vector space Theorem 2.4 leads to a certain isomorphism of two families. Denote by $\mathcal{K}$ the family of all convex subsets of E which are closed with respect to the norm topology. A topology on $\mathcal{K}$ is defined when we for each $K_0 \in \mathcal{K}$ and $\epsilon > 0$ declare an open neighborhood

$$U_\epsilon(K_0) = \{K \in \mathcal{K} :\ \mathrm{dist}(K, K_0) < \epsilon\}$$

where the norm defines the distance between K and K_0 in the usual way. Denote by $\mathfrak{H}$ the family of all functions G on E^* which satisfy (*) in 5.B.1 and are continuous with respect to the norm topology on E^*. A subset M of $\mathfrak{H}$ is equi-continuous if there to each $\epsilon > 0$ exists $\delta > 0$ such that

$$||y_2 - y_1|| < \delta \implies ||G(y_2) - G(y_1)|| < \epsilon$$

for every $G \in M$ and all pairs y_1, y_2 in E^*. The topology on $\mathfrak{H}$ is defined by uniform convergence on equi-continuous subsets.

2.5.1 Theorem. *If E is a normed vector space the set-theoretic bijective map $K \to \mathcal{H}_K$ is a homeomorphism when $\mathcal{K}$ and $\mathfrak{H}$ are equipped with the described topologies.*

Exercise. Deduce this result from Theorem 2.4. If necessary, consult Hörmander's cited article.