Special chapter: ODE-equations.

On the real x-line the space of distributions is denoted by $\mathfrak{Db}(\mathbf{R})$. Notice that we do not insist that the distriobutions are tempered, i.e. $\mathcal{S}^*$ appears as a proper subspace. As a first example we take the first order differential operator

$$\nabla = x \cdot \frac{\partial}{\partial x}$$

When $x \neq 0$ the equation $\nabla(f) = 0$ has solutions given by constant functions. To pass beyond $x = 0$ we take the Heaviside densities H^+ and H_-, where $H^+(x) = 1$ when $x > 0$ and zero if $x < 0$, while $H_- = 1 - H^+$. It turns out that these two linearly independent distributions on the x-line generate the vector space of all distribution solutions to the equation $\nabla(\mu) = 0$. See § xx below for details. On the other hand, if $f(x)$ is a C^1-function, i.e. continuously differentiable which satisfies $\nabla(f) = 0$, then it is clear that f must be a constant. So the set of distribution solutions is more extensive. Next, let s be a complex number which is not an integer. If $\mathfrak{s} > -1$ then x^s is integrable on intervals $(0, a)$ with $a > 0$ and there exists a distribution denoted by x_+^s acting as a linear functional on test-functions $\phi(x)$ by

$$x_+^s(\phi) = \int_0^\infty x^s \cdot \phi(x)\,dx$$

On the open interval $(0, +\infty)$ we notice that

$$(\nabla - s)(x^s) = 0 \quad : x > 0$$

It we apply $\nabla - s$ to the distribution x_+^s the construction of distribution derivatives means that $(\nabla - s)(x_+^s)$ acts on test-functions ϕ by

$$\text{(i)} \qquad \phi \mapsto \int_0^\infty x^s \cdot (-\partial(x\phi) - s\phi)\,dx$$

When $\mathfrak{Re}\, s > -1$ a partial integration shows that (i) is zero. Hence $(\nabla - s)(x_+^s) = 0$. It turns out that there exist more distributions μ such that $(\nabla - s)(\mu) = 0$. Namely, there exists the boundary value distribution $\mu = (x + i0)^s$ which also satisfies the equation $(\nabla - s)(\mu) = 0$. Here we recall that μ is defined on test-functions $\phi(x)$ by

$$\text{(ii)} \qquad \mu(\phi) = \lim_{\epsilon \to 0} \int (x + i\epsilon)^s \cdot \phi(x)\,dx$$

Hence we have found two linearly independent distribution solutions to the equation $\nabla - s(\mu) = 0$. It turns out that they give a basis for the null solutions which gives the dimension formula:

$$\dim_{\mathbf{C}} (\mathrm{Ker}_{\nabla - s}(\mathfrak{Db})) = 2$$

A special example. Take $Q = \nabla + 1$. Here two boundary value distributions $(x + i0)^{-1}$ and $(x - i0)^{-1}$ are null solutions. Let us also recall that the difference

$$(x - i0)^{-1} - (x + i0)^{-1} = \pi i \cdot \delta_0$$

So the Dirac measure at $x = 0$ is also a null solution which of course could have been verified directly. By the general result in § xx the space of null solutions is 2-dimensional so above we have found a basis.

In § xx we consider general differential operators with polynomial coefficients

$$P(x, \partial) = p_m(x)\partial^m + \ldots + p_0(x)$$

Under the assumption that the real zeros of the leading polynomial $p_m(x)$ are simple and consists of some k-tuple $\{a_1 < \ldots a_k\}$ we show in § xx that the P-kernel on $\mathfrak{Db}$ has dimension $k + m$.

0.1 A first order ODE-equation. Let p and q be a pair of polynomials and set

$$Q = q(x) \cdot \partial - p(x)$$

Assume that q is a monic polynomial of some degree $k \geq 2$ whose zeros are real and simple and arranged in strictly increasing order $\{a_1 < a_2 < \ldots < a_k\}$. The polynomial p is such that $p(a_\nu) \neq 0$ for every ν and in general it has complex coefficients and no condition is imposed upon its degree. Now we seek distributions μ on the x-line such that $Q(\mu) = 0$. One such solution is found as the boundary value of the analytic function defined in the upper half-plane by

$$f(z) = e^{\int_i^z \frac{p(\zeta)}{q(\zeta)}\, d\zeta}$$

Tos see this we notice that when $\mathfrak{Im} z > 0$ it is evident that the complex derivative

$$\text{(i)} \qquad \frac{\partial f}{\partial z} = \frac{p(z)}{q(z)}$$

Since the passage to boundary value distributions commute with derivations it follows that the boundary value distribution $f(x + i0)$ is a null solution to Q. Less obvious is that each simple and real zero a_ν of q yields a null solution μ_ν supported by the half-line $[a_\nu. + \infty)$.This is a consequence of general results in § xx. Let us remark that without using boundary values of analytic functions it is not easy to discover all this.

0.2 The equation $\nabla^2(\mu) = 0$. The Fuchsian operator is defined by $\nabla = x\partial$. It turns out that the space of distributions μ satisfying $\nabla^2(\mu)$ is a 4-dimensional vector space. One solution is the Heavisde function H_+ defined by the density 1 if $x > 0$ and zero if $x \leq 0$. Here

$$\partial(H_+)(g) = -\int_0^\infty g'(x)\, dx = g(0) \quad : g \in C_0^\infty(\mathbf{R})$$

This means that the distribution derivative $\partial(H_+) = \delta_0$ and since $x \cdot \delta_0 = 0$ we have $\nabla(H_+) = 0$. Next, on $\{x > 0\}$ we see that the density $\log x$ satisfies $\nabla^2(\log x) = 0$. It is tempting to extend the locally integrable function $\log x$ on the positive half-line to $\mathbf{R}$ by setting the value zero if $x \leq 0$. Denote the resulting distribution by $\log_+ x$. Now

$$\nabla(\log_+ x)(g) = -\int_0^\infty -\partial(xg) \cdot \log x\, dx = \int_0^\infty xg \cdot \frac{1}{x}\, dx = \int_0^\infty g dx$$

Hence $\nabla(\log_+ x) = H_+$ Since $\nabla(H_+) = 0$ we get $\nabla^2(\log_+ x) = 0$. Hence we have found two linearly independent null solutions given by the pair $(H_+, \log_+ x)$ which are supported by $x \geq 0$. In addition we find two other null solutions. The first is the constant density 1, The second is the boundary v alue distribution $\log(x + i0)$. By the gneral resu,t in § xx the space of null solutions is 4-dimensional so above we have found a basis for these.

0.3. Higher order Fuchsian equations. Let $m \geq 2$ and consider an operator of the form

$$Q = \nabla^m + q_{m-1}(x)\nabla^{m-1} + \ldots + q_1(x)\nabla + q_0(x)$$

where $\{q_\nu(x)\}$ are polynomials. With $\{c_\nu = q_\nu(0)\}$ we associate the polynomial

$$Q^*(s) = s^m + c_{m-1}s^{m-1} + \ldots + c_1 s + c_0$$

Under the assumption that $Q^*(k) \neq 0$ for all non-negative integers the solution space $\mathcal{S} = \{\mu \colon Q(\mu) = 0\}$ has dimension $2m$ and a basis is found as follows: In the upper half-plane the Picard-Fuchs theory about holomorphic differential equations entails that there exists an m-tuple of linearly independent analytic functions $\{\phi_\nu(z)\}$ which solve $Q(z, \partial_z)(\phi_\nu) = 0$. Similarly one finds an m-tuple $\{\psi_\nu\}$ of linearly independent analytic functions in the lower half-plane. The boundary value distributions $\{\phi_\nu(x+i0)\}$ and $\{\psi_\nu(x-i0)\}$ belong to $\mathcal{S}$ and are linearly indepedent. For if $\sum c_\nu\phi_\nu(x+i0)+\sum d_\nu\psi_\nu(x-i0)=0$ where at least some c_ν or d_ν is $\neq 0$ then

(i) $$\phi_*(x+i0)=\psi_*(x-i0)=0$$

where $\phi_*=\sum c_\nu\phi(x+i0)\neq 0$ and $\psi_*=-\sum d_\nu\psi(x-i0)\neq 0$. Now (i) cannot hold. The reason is that the assumption about $Q^*(s)$ entails that the equation $Q(z,\partial_z)(f)=0$ has no holomorphic solutions at $z=0$. This fact stems from local $\mathcal{D}$-module theory and is exposed in § xx. Now the reflection principle for analytic functions entails that the analytic wave front sets of the distributions ϕ_* and ψ_* both are non-empty. On the other hand the material in § xx shows that these non-empty wave fronts have opposed directions and hence the equality (i) cannot hold. This proves that $\mathcal{S}$ is at least $2m$-dimensional and by the general results in § xx we have equality. So above we have constrcuted a basis for the null solutions.

§ 1. Fundamental solutions to ODE-equations with constant coefficients

We consider differential operators with constant coefficients acting on the real x-line. To simplify the passage to Fourier transform we introduce the first order operator
$$D=\frac{1}{i}\cdot\frac{d}{dx}$$
If $P(\xi)$ is a polynomial of the ξ-variable and μ is a tempered distributionon the x-line this gives the equality:

(*) $$\widehat{P(D)\mu}(\xi)=P(\xi)\cdot\widehat{\mu}(\xi)$$

By a tempered fundamental solution to $P(D)$ we mean a distribution $\mu\in\mathcal{S}^*$ such that
$$P(D)\mu=\delta_0$$
where δ_0 is the Dirac measure at $x=0$. Since the Fourier transform of δ_0 is the identity on the ξ-line the Fourier transform of a fundamental solution satisfies
$$P(\xi)\cdot\widehat{\mu}(\xi)=1$$
When $P(\xi)$ has no zero on the real ξ-line there exists a fundamental solution given as the inverse Fourier transform of the smooth density $P(\xi)^{-1}$. If $P(\xi)$ has some real zeros we can write
$$P(\xi)=Q(\xi)\cdot R(\xi)$$
where R has real zeros and the zeros of Q are all non-real. The factorisation is unique when we choose constants so that $Q(\xi)$ is a monic polynomial. The case $\deg Q=0$ is not excluded, i.e. this holds when all zeros of $P(\xi)$ are real. But in general one has a mixed case where $n=\deg P$ and $1\leq\deg Q\leq n-1$.

1. The case $\deg Q=0$. When all zeros of $P(\xi)$ real there exists the boundary value distribution on the ξ-line defined by

(1.1) $$\gamma=\frac{1}{P(\xi-i0)}$$

By the general results from § XX its inverse Fourier transform is supported by the half-line $\{x\geq 0\}$. Let μ_+ denote this distribution. Then
$$\widehat{P(D)\mu_+}=P\xi)\cdot\gamma=1$$
and hence μ_+ is a fundamental solution.

2. The mixed case. If $P = Q \cdot R$ where $1 \leq \deg Q \leq n-1$ we proceed as follows. First one has a bijective map on the space of tempeted distributions on the ξ-line defined by

$$\gamma \mapsto Q(\xi)^{-1} \cdot \gamma$$

Fourier's inversion fomrula gives a bijective linear operator T_Q on the space of tempered distribtutions on the x-line such that

$$\widehat{T_Q(\mu)} = Q(\xi)^{-1} \cdot \widehat{\mu}$$

So if μ is a temepred distribution we get

$$P(D)(T_Q(\mu)) = R(D)(\mu) \tag{2.1}$$

The zeros of $R(\xi)$ are real which gives the fundamental solution ν_+ to $R(D)$ and now

$$\mu = T_Q(\nu_+) \tag{2.2}$$

yields a fundamental solution to $P(D)$. In this way we have constructed a fundamental solution in a canonical fashion. In contrast to the real case where $\deg Q = 0$ the distribution μ above is in general not supported by the half-line $\{x \geq 0\}$. We give examples in § XX below.

3. The determination of μ_+. Consider the case when $\deg Q = 0$ so that the fundamental solution μ_+ is the inverse Fourier transform of (xx) above. Let us for the moment assume that the real zeros of $P(\xi)$ are all simple and given by an n-tuple $\{\alpha_k\}$. Define the distribution ρ on the real x-line by the density

$$\rho(x) = \sum \frac{1}{P'(\alpha_k)} \cdot e^{i\alpha_k x} \quad : x \geq 0$$

while $\rho(x) = 0$ when $x < 0$. It is clear that the distribution $P(D)\rho$ vanishes when $x \neq 0$, i.e. supported by the singleton set $\{x = 0\}$. Newton's formula from § xx gives

$$\sum_{k=1}^{n} \frac{1}{P'(\alpha_k)} \cdot \alpha_k^m = 0 \quad : \quad 0 \leq m \leq n-2$$

This entails that the derivatives up to order $n-2$ of ρ vanish at $x = 0$. Using this we show that ρ up to a constant gives a fundamental solution to $P(D)$. For consider a test-function $f(x)$ and let $P^*(D)$ be the adjoint of $P(D)$. The vanishing of the derivatives of ρ at $x = 0$ above gives after partial integration

$$\int \rho(x) \cdot P^*(D)(f)(x)\,dx = (-1)^{xx} \cdot \rho^{(n-1)}(0) \cdot f(0)$$

4. Conclusion. The fundamental solution μ_+ supported by $x \geq 0$ is given by the density

$$\mu_+(x) = \frac{n}{xx} \cdot \rho(x) = \sum \frac{1}{P'(\alpha_k)} \cdot e^{i\alpha_k x}$$

5. Example. Consider $P(D) = D^2 - 1$ so that $P(\xi) = \xi^2 - 1$. Here 1 and -1 are the simple zeros and (xx) gives

$$\mu_+(x) = XXX \cdot \sum \frac{1}{-2} \cdot e^{-ix} + \frac{1}{2} \cdot e^{ix} = -\sin x \quad : \ x \geq 0$$

6. An example in the mixed case. Let $P(D) = (D^2 + 1)(D - a)$ where a is some real number $\neq 0$. So here $Q(\xi) = \xi^2 + 1$ and the fundamental solution from § 2 becomes

$$\mu = T_Q(\nu_+) \tag{6.1}$$

where ν_+ is the inverse Fourier transform derived from the linear polynomial. $R(\xi) = \xi - a$. This gives

$$\nu_+(x) = -e^{iax} \quad : x \geq 0 \tag{6.2}$$

7. The expression of μ. By the above μ is the convolution of ν_+ and the continuous density

$$\phi(x) = \frac{1}{2\pi} \int \frac{e^{ix\xi}}{1+\xi^2}\, d\xi$$

We leave it to the reader to verify that

$$\phi(x) = \frac{1}{2} \cdot e^{-|x|}$$

Hence

$$\mu(x) = -\frac{1}{2} \cdot \int_0^\infty e^{-[x-y|} \cdot e^{-aiy}\, dy$$

The reader is invited to analyze this function using a computer to plot this function with different choice of a.

2.0. ODE-equations on the real line

To grasp the notion of distributions it is natural to start with a study of distribution solutions to ordinary differential operators which leads to more systematic results as compared to studies before distribution theory was established. An example is the confluent hypergeometric function which arises as a solution to a differential operator of the form

$$P = x\partial^2 + (\gamma - x)\partial - a$$

where γ is a non-zero complex number while a is arbitrary. In the classic literature one solves this equation via the Laplace method which involves a rather cumbersome use of residue calculus. More information about the operator P arises when one determines its kernel on the space of distributions on the real x-line. The result in Theorem 0.0.1 below shows that this P-kernel is a 3-dimensional subspace of $\mathfrak{Db}$. Moreover, there exists a fundamental solution supported by the half-line $\{x \geq 0\}$.

Let us now regard a differential operator with polynomial coefficients

$$(*) \qquad P(x, \partial) = q_m(x) \cdot \partial^m + q_{m-1}(x)\partial^{m-1} + \ldots + q_0(x)$$

where $m \geq 1$ and $q_0(x), \ldots, q_m(x)$ are polynomials which in general have complex coefficients. Let $\mathfrak{Db}$ be the space of distributions on the real x-line. A first question is to determine the P-kernel, i.e. one seeks all distributions μ such that $P(\mu) = 0$. Following material from the thesis by Ismael (xxx - University of xxx) we expose some general facts about null solutions to general operators as above.

The local Fuchsian condition. We shall restrict the study to operators P which are *locally Fuchsian* at every real zero of the leading polynomial $q_m(x)$. This means the following: Let a be a real zero of $q_m(x)$ with some multiplicity $e \geq 1$ so that $q_m(x) = q(x)(x-a)^e$ where the polynomial q is $\neq 0$ at a. Then we can write

$$P(x, \partial) = q(x) \cdot [(x-a)^e\partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_0(x)]$$

where $\{r_\nu = \frac{p_\nu}{q}\}$ are rational functions with no pole at a and therefore define analytic functions in a neighbrohood of a. Hence

$$P_*(x, \partial) = (x-a)^e\partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_0(x)$$

can be identified with a germ of a differential operator with coefficients in the local ring $\mathcal{O}(a)$ of germs of analytic functions at a. The ring $\mathcal{D}$ of such germs of differential operators is studied in § x where we define the subfamily of Fuchsian operators. For example, if $a = 0$ then a Fuchsian operator in $\mathcal{D}$ can be expressed as

$$\rho(x) \cdot [\nabla^m + g_{m-1}(x)\nabla^{m-1} + \ldots + g_0(x)]$$

where $g_{m-1}, \ldots, g_0$ belong to $\mathcal{O}$ and $\nabla = x\partial$, while ρ in general is a meromorphic function, i.e. it may have a pole of some order at $x = 0$. A local study of null solutions to fuchsian operators in $\mathcal{D}$ is carried out in § xx. From this one can derive the following conclusive result:

0.1 Theorem *Let $P(x, \partial)$ in (*) above be locally Fuchsian at the real zeros of p_m. Then $Ker_P(\mathfrak{Db})$ is a complex vector space of dimension $m + e_1 + \ldots + e_k$, where $\{e_\nu\}$ are the multiplicities at the real zeros of p_m. Moreover, for each real zero of p_m there exists a fundamental solution μ supported by $\{x \geq a\}$ such that $P(\mu) = \delta_a$.*

Remark. In addition to this the following supplement to Theorem 0.1 hold. For each real zero a of $p_m(x)$ with some multiplicity e there exists a distinguished e-dimensional subspace V_a of $\mathrm{Ker}_P(\mathfrak{Db})$ which consists of distributions μ supported by the closed half-line $[a, +\infty]$ whose analytic wave front sets satisfy the following: First, they contain the whole fiber above a and the remaining part of the analytic wave front set is either empty or a union of half-lines above some of the real zeros of p_m which are $> a$. Moreover, one has a direct sum decomposition

$$(**) \qquad \mathrm{Ker}_P(\mathfrak{Db}) = \mathcal{F}_+ \oplus V_{a_\nu}$$

where the last direct sum is taken over the real zeros of p_m, and $\mathcal{F}_+$ is an m-dimensional subspace of $\mathfrak{Db}$ with a basis given by an m-tuple of boundary value distributions $\{\phi_k(x+i0)\}$. Here $\{\phi_k(z)\}$ are analytic functions in a strip domain $U = \{-\infty < x < +\infty\} \times \{0 < y < A\}$ with $A > 0$ chosen so that the complex polynomial $p_m(z)$ is zero-free in this domain and each $\phi_k(z)$ satisfies the homogeneous equation $P(z, \partial)(\phi) = 0$ in U.

Example. Consider the first order differential operator

$$P = x\partial + 1$$

Outside $x = 0$ the density x^{-1} is a solution. Now the Euler distribution x_+^{-1} is suppored by $[0, +\infty)$. The 1-dimensional $\mathcal{F}_+$-space in (**) is generated by the boundary value distribution $(x + i0)^{-1}$ and in V_0 we find the Dirac measure δ_0 which together with $(x + i0)^{-1}$ is a basis for the null solutions. Next, an easy computation gives

$$P(x_+^{-1}) = \delta_0$$

and hence the Euler distribution x_+^{-1} yields a fundamental solution.

0.0.2 Tempered solutions. The P-kernel in Theorem 0.0.1 need not consust of tempered distributions. The reason is that we have not imposed the condition that P is locally Fuchsian at infinity. So if $\mathcal{S}^*$ denotes the space of tempered distributions, then $\mathrm{Ker}_P(\mathcal{S}^*)$ can have strictly smaller dimension than $m + k$ and the determination of the tempered solution space leads to a more involved analysis. Already the case $P = \partial - 1$ illustrates the situation. Here the P-kernel on $\mathfrak{Db}$ is the 1-dimensional space given by the exponential density e^x which is not tempered so the P-kernel on $\mathcal{S}^*$ is reduced to zero. During the search for tempered fundamental solutions to P supported by half-lines $\{x \geq a\}$ one can use a result due to Poincaré under the extra assumption that $\deg p_k \leq \deg p_m$ hold for every $0 \leq k < m$. For in this case there are series expansions when x is large and positive:

$$\frac{p_k(x)}{p_m(x)} = c_k + \sum_{\nu=1}^{\infty} c_{k\nu} x^{-\nu} \quad : 0 \leq k \leq m-1$$

The leading coeficients $c_0, \ldots, c_{m-1}$ give a monic polynomial

$$\phi(\alpha) = \alpha^m + c_{m-1}\alpha^{m-1} + \ldots c_0$$

Let us also chose $A > 0$ so large that the leading polynomial p_m has no real zeros on $[A, +\infty]$. This gives an m-dimensional space of null solutions where a basis consists of real-analytic densities $u_1(x), \ldots, u_m(x))$ on this interval.

0.0.3 Poincaré's theorem. *Suppose that ϕ has simple simple zeros $\alpha_1, \ldots, \alpha_m$. Then, with A as above one can arrange the u-basis so that*

$$u_k(x) = e^{\alpha_k x} \cdot g_k(x)$$

and there exists a non-negative integer w and a constant C such that

$$|g_k(x) \leq C \cdot (1+x)^w : 1 \leq k \leq m$$

hold for all $x \geq A$.

So for indices k such that $\mathfrak{Re}\, \alpha_k \leq 0$, it follows that $u_k(x)$ has tempered growth as $x \to +\infty$. In particular Poincaré's result entails that if the real parts are all ≤ 0, then the fundamental solutions from Theorem 0.0.1 are all tempered.

0.0.4 Example. Consider the operator

$$P = x\partial^2 - x\partial - B$$

where $B > 0$. In this case

$$\phi(\alpha) = \alpha^2 - \alpha = \alpha(\alpha - 1)$$

so one of the u-solutions above increse exponentially while the other has tempered growth as $x \to +\infty$. It is easily seen that the there exists an entire solution

$$\text{(i)} \qquad f(x) = x + c_2x^2 + \ldots$$

such that $P(f) = 0$, whose coefficients are found by the recursive formulas

$$k(k-1)c_k = (k - 1 + B(c_{k-1} \quad : k \geq 2$$

Hence $\{c_k\}$ are positive and it is clear that f has exponential growth as $x \to +\infty$. In addition we have a solution on $x > 0$ of the form

$$g(x) = f(x) \cdot \log x + a(x)$$

In §§ we explain that $P(g_+) = a \cdot \delta_0$ hold for a non-zero constant while $P(f_+) = 0$. Next, let et u_1 be the tempered solution and u_2 the non-tempered solution in Poincaré's theorem on the half-line $x > 0$. There are constants c_1, c_2 such that

$$f(x) = c_1u_1(x) + c_2u_2(x)$$

Here $c_2 \neq 0$ because f increases exponentially on $(0, +\infty)$. At the same time

$$g(x) = d_1u_1(x) + d_2u_2(x)$$

hold for some constants d_1, d_2. Set

$$\gamma = g_+ - \frac{d_2}{c_2} \cdot f_+$$

From the above γ has tempered growth as $x \to +\infty$ and $P(\gamma) = a \cdot \delta_0$ with $a \neq 0$. Hence $\mu = a^{-1} \cdot \gamma$ yields a tempered fundamental solution supported by $\{x \geq 0\}$.

In § xx we give further examples of tempered fundamental solutions.

0.5 Another example. Here we take

$$\text{(0.3.1)} \qquad P = \nabla^2 + q(x)$$

where $q(x)$ is a polynomial such that $q(0) = -1$ and $q'(0) = 0$. For example, if $q(x) = x^2 - 1$ we encounter a wellknown Bessel operator. It is easily seen that there exists a unique entire solution $f(x)$ which satisfies $P(f) = 0$ with a series expansion

$$f(x) = x + c_3x^3 + \ldots$$

Moreover, one verifies easily that there exists another entire function $g(x)$ with $g(0) = 0$ such that the multi-valued function

$$\text{(i)} \qquad \phi(z) = f(z) \cdot \log z + g(z)$$

satisfies $P(\phi) = 0$. Theorem 0.0.1 predics that the P-kernel on $\mathfrak{Db}$ is 4-dimensional. To begin with f restricts to a real analytic densitiy on the x-line and gives a null solution. A second solution is obtained by the boundary value distribution

$$\gamma = \phi(x + i0) = f(x) \cdot \log(x + i0) + g(x)$$

Together they give a basis in the 2-dimensional space $\mathcal{F}_+$ from (*) in the remark after Theorem 0.0.1. There remains to find two linearly independent distributions in V_0 since the leading polynomial of P has a double zero at $x = 0$. To attain such a pair we first consider the boundary value distribution

$$\gamma_* = f(x) \cdot \log(x - i0) + g(x)$$

which also is a null solution. Here the multi-valuedness of the complex log-function entails that

$$\gamma - \gamma_* = 2\pi i \cdot f(x) \cdot H_-(x)$$

where $H_-(x)$ is the Heaviside distribution supported by the negative half-line. Then

$$\gamma^* = \gamma - \gamma_* - 2\pi i \cdot f(x)$$

is a null solution supportec by the half-line $x \geq 0$ and hence belongs to the 2-dimensional space V_0. A second null solution in V_0 is given by the Dirac measure δ_0. To see that δ_0. is a null solution for P we recall that in the non-commutative ring of differential operators one has the equality $\nabla = \partial x \circ x - 1$. Since $x \cdot \delta_0 = 0$ we get the distribution equation

$$\nabla(\delta_0) = -\delta_0 \implies \nabla^2(\delta_0) = \delta_0$$

Since $q(0) = -1$ is assumed in (0.3.1) it follows that $P(\delta_0) = 0$. Hence we have found four linearly independent null solutions $f_+ . f_-, \gamma^*, \delta_0$ in accordance with Theolrem 0.0.1.

The fundamental solution. A fundamental solution μ supported by $x \geq 0$ is found as follows: From (i) we have the real-analytic density $\phi(x)$ on the open half-line $\{x > 0\}$ which gives the distribution ϕ_+ supported by $\{x \geq 0\}$ defined by

$$\phi_+ = f(x) \cdot (\log x) \cdot H_+ g(x) \cdot H_+$$

In § xx we shall explain that

$$\nabla(\log x \cdot H_+) = \delta_0$$

and from this deduce that

$$P(\phi_+) = -\delta_0$$

Hence $\mu = -\phi_+$ gives the requested fundamental solution.

0.2 PDE-equations with constant coefficients.

The study of PDE-equations with contant coefficients in $\mathbf{R}^n$ for arbitrary $n \geq 2$ is a rich subject. The interested reader may consult Chapter xx in [Hörmander:Vol 2] for an extensive study of PDE-equations with constant coefficients. Here we shall give a construction from Hörmander's article [xxx] which illustrates how analytic function theory can be used with PDE-theory. Fourier's inversion formula for an arbitrary $n \geq 1$ asserts the following: Let $f(x) = f(x_1, \ldots, x_n)$ be a C^∞-function which is rapidly decreasing as $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ tends to $+\infty$. Then

$$(*) \qquad f(x) = \frac{1}{(2\pi)^n} \cdot \int e^{i\langle x,\xi\rangle} \cdot \widehat{f}(\xi)\, d\xi \quad \text{where} \quad \widehat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} \cdot f(x)\, dx$$

The inversion inversion formula (*) entails that the Fourier transform of the partial derivative $\frac{\partial f}{\partial x_j}(x)$ is equal to $i\xi_j \cdot \widehat{f}(\xi)$. In PDE-theory one introduces the first order differential operators

$$D_j = -i \cdot \frac{\partial}{\partial x_j} \quad : 1 \leq j \leq n$$

When $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index we get the higher order differential operator

$$D^\alpha = D_1^{\alpha_1} \cdots D_1^{\alpha_n}$$

We can take polynomials of these and get differential operatators with constant coefficients

$$P(D) = \sum c_\alpha \cdot D^\alpha$$

Fourier's inversion formula gives

$$(**) \qquad P(D)(f)(x) = \frac{1}{2\pi^n} \cdot \int e^{i\langle x,\xi\rangle} \cdot P(\xi) \cdot \widehat{f}(\xi)\, d\xi$$

Thus, applying a differential operator with constant coefficients to f corresponds to the product of its Fourier transform with the polynomial $P(\xi)$ and (**) can be used to construct solutions of the homogeneous equation $P(D)(f) = 0$. Following [Hörmander] we construct distributions μ such that $P(D)(\mu) = 0$ for a suitable class of PDE-operators. Let $\phi_1(s), \ldots, \phi_n(s)$ be some n-tuple of analytic functions of the complex variable s which extend to continuous functions on the boundary of the domain

$$\Omega = \{\mathfrak{Im}(s) < 0\} \cap \{|s| > M\}$$

where M is some positive number. Assume that the ϕ-functions satisfy the growth conditions

$$\text{(i)} \qquad |\phi_k(s)| \leq C|s|^a$$

for a constant C and some $0 < a < 1$. If $a < \rho < 1$ there exists the analytic function in Ω defined by

$$\psi(s) = e^{-(is)^\rho}$$

As explained in § xx one has the estimate

$$|\psi(s)| \leq e^{-\cos\frac{\pi\rho}{2}\cdot|s|^\rho} \quad : \mathfrak{Im}\, s \leq 0$$

The inequality $a < \rho$ and (i) entail that the functions

$$s \mapsto e^{a_1\cdot\phi_1(s)+\cdots+a_n\phi_n(s)} \cdot \psi(s)$$

decrease like $e^{-\cos\frac{\pi\rho}{2}\cdot|s|^\rho}$ in Ω.

Exercise. Verify that the complex line integrals below converge absolutely for every s-polynomial $Q(s)$ and every n-tuple of real numbers $x_1, \ldots, x_n$:

$$\text{(ii)} \qquad \frac{1}{(2\pi)^n} \cdot \int_{\partial\Omega} e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_ns} \cdot Q(s) \cdot \psi(s)\, ds$$

and show that when x varies in $\mathbf{R}^n$ this gives a C^∞-function $f(x)$. If $1 \leq j \leq n-1$ one has for example

$$\frac{\partial f}{\partial x_j} = \frac{1}{(2\pi)^n} \cdot \int_{\partial\Omega} \phi_j(s) \cdot e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_ns} \cdot Q(s) \cdot \psi(s)\, ds$$

Less obvious is that the C^∞-function $f(x)$ is supported by the half-space $\{x_n \geq 0\}$. To prove it one uses the analyticity of the integrand as a function of s which enable us to shift the contour of integration so that (ii) is unchanged while we integrate on a horisontal line $\mathfrak{Im}\, s = -N$ for every $N > M$. With $s = u - iN$ we have

$$|e^{ix_n \cdot s}| = e^{N \cdot x_n}$$

If $x_n < 0$ this term tends to zero as $N \to +\infty$ and from this the resder should confirm that the C^∞-function $f(x)$ is identically zero in $\{x_n < 0\}$.

Suppose now that we are given a PDE-operator $P(D)$ and the ϕ-functions are chosen so that

$$s \mapsto P(\phi_1(s) \ldots \phi_{n-1}(s), \phi_n(s) + s) = 0 \quad : s \in \Omega$$

Then it is clear that

$$P(D)(e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_ns}) = 0$$

hold for all $x \in \mathbf{R}^n$ and $s \in \Omega$. Hence $P(D)(f) = 0$ where f is a C^∞-function supported by the half-space $\{x_n \geq 0\}$. In § xx we will show that the construction of solutions as above is not so special for PDE-operators P such that the hyperplane $\{x_n = 0\}$ is non-characteristic.

0.1 The distributions x_+^s

If s is a complex number where $\mathfrak{Re}\, s > -1$ the function defined by x^s for $x > 0$ and zero on the half-line $x \leq 0$ is locally integrable and defines a distribution denoted by x_+^s acting on test-functions g by

$$x_+^s(g) = \int_0^\infty x^s \cdot g(x)\, dx$$

The distribution-valued function $s \mapsto x_+^s$ is analytic in $\mathfrak{Re}\, s > -1$. Indeed if $x < 0$ we have $\frac{d}{ds}(x^s = \log x \cdot x^s$ which entails that the complex derivative of x_+^s is the distribution defined by

$$g \mapsto \int_0^\infty \log x \cdot x^s \cdot g(x)\, dx$$

It turns out that x_+^s extends to a meromorphic distribution-valued function in the whole s-plane. To prove this we perform a partial integration which gives

$$x_+^{s+1}(g') = \int_0^\infty x^{s+1} \cdot g(x)\, dx = -(s+1) \cdot \int_0^\infty x^s \cdot g(x)\, dx \tag{0.0.1}$$

By the construction of distribution derivatives this means that

$$\frac{d}{dx}(x_+^s + 1) = (s+1) \cdot x_+^s$$

Euler's functional equation. Set $\partial = \frac{d}{dx}$. We can iterate (0.0.1) which for every positive integer m gives

$$(s+1) \cdots (s+m) x_+^s = \partial^m (x_+^{s+m}) \tag{0.0.2}$$

We refer to (0.0.2) as Euler's functional equation. It entails that the distribution-valued function x_+^s extends to a meromorphic function with at most simple poles at negative integers. Let us investigate the situation close to a negative integer. With $s = -m + t$ and t small one has

$$t(t-1) \cdot (t - m + 1) x_+^{-m+t} = \partial^m (x_+^t)$$

When $x > 0$ one has the expansion

$$x^t = 1 + t \log x + \frac{t^2}{2} \cdot (\log x)^2 + \frac{t^3}{3!} \cdot (\log x)^3 + \ldots$$

From this we obtain a series expansion

$$x_+^{-m+t} = t^{-1} \cdot \rho_m + \gamma_0 + t\gamma_1 + \ldots$$

where ρ_m and $\{\gamma_\nu\}$ are distributions. In particular the reader may verify that

$$(-1)^{m-1}(m-1)! \cdot \rho_m = \partial^m (H_+)$$

Let us then consider the constant term γ_0. The linear t-term in the expansion of $t(t-1) \cdot (t - m + 1) x_+^{-m+t}$ becomes

$$(-1)^{m-1}(m-1)! \cdot \gamma_0 + \frac{m(m-1)}{2} \cdot \rho_m$$

If $x > 0$ we notice that

$$\partial^m (\log x) = (-1)^{m-1} \cdot (m-1)! \cdot x^{-m}$$

From the above γ_0 restricts to the density x^{-m} when $x > 0$. At the same time γ_0 is a distribution defined on the whole x–line supported by $\{x \geq 0\}$. We set

$$x_+^{-m} = \gamma_0 \tag{*}$$

and refer to this as Euler's extension of the density x^{-m} which from the start is defined on $\{x > 0\}$. So in (*) we have found distributions for every positive integer m.

0.1.2 Further formulas. With $s=-1+z$ where z is a small non-zero complex number one has

(i) $$z\cdot x_+^{-1+z}=\partial(x_+^z)$$

Next, if $x>0$ we have

$$x^z=e^{z\log x}=1+\sum_{k=1}^\infty\frac{(\log x)^k}{k!}\cdot z^k$$

Introducing the Heaviside distribution H_+ which is 1 on $x\geq 0$ and zero on $x<0$ this means that

(ii) $$\partial(x_+^z)=\partial(H_+)+\sum_{k=1}^\infty\partial(\frac{(\log x)^k}{k!}\cdot H_+(x))z^k$$

From this we get a Laurent expansion of x_+^s at $s=-1$. The crucial point is that the distribution derivative

(iii) $$\partial(H_+)=\delta_0$$

where δ_0 is the Dirac distribution at $x=0$. It follows that

(iv) $$x_+^{-1+z}=z^{-1}\cdot\delta_0+\sum_{k=1}^\infty\partial(\frac{(\log x)^k}{k!}\cdot H_+(x))z^{k-1}$$

In particular the constant term becomes

(v) $$\partial(\log x\cdot H_+(x))$$

To find this distribution we take a test-function g and a partial integration gives

$$-\int_0^\infty(\log x\cdot g'(x)\,dx=\int_0^1\frac{g(x)-g(0)}{x}\,dx+\int_1^\infty\frac{g(x)}{x}\,dx$$

From this we conclude that the distribution x_+^{-1} is defined on test-functions by the formula:

$$x_+^{-1}(g)=\frac{(-1)^{m-1}}{(m-1)!}\cdot\int_0^1\frac{g(x)-g(0)}{x}\,dx+\int_1^\infty\frac{g(x)}{x}\,dx$$

0.1.3 Exercise. For each test-function g and integer $m\geq 2$ we set

$$T_{m-1}(g)(x)=g(0)+g'(0)x\ldots\frac{g^{(m-1)}(0)}{(m-1)!}\cdot x^{m-1}$$

Show from the above via partial integrations that

$$x_+^{-m}(g)=\frac{(-1)^{m-1}}{(m-1)!}\cdot\int_0^1\frac{g(x)-T_{m-1}(g)(x)}{x^m}\,dx+\int_1^\infty\frac{g(x)}{x^m}\,dx$$

0.1.4 The distributions $(x+i0)^\lambda$ and $(x-i0)^\lambda$. In the upper half plane there exists the single valued branch of $\log z$ whose argument stays in $(0,\pi)$ and for every complex number λ we have

$$z^\lambda=e^{\lambda\cdot\log z}$$

In § 3 we shall learn how to contruct boundary value distributions of analytic functions defined in strip domains above or below the real x-line. In particular there exists the distribution $(x+i0)^\lambda$ defined on test-functions $g(x)$ by the limit formula

$$\lim_{\epsilon\to 0}\int(x+i\epsilon)^\lambda\cdot g(x)\,dx$$

Notice that this limit exists for all complex λ, i.e even when the real part becomes very negative. In the same way we have the single valued branch of $\log z$ in the lower half-plane whose argument stays in $(-\pi, 0)$ and construct the distribution $(x - i0)^\lambda$ defined by

$$(x - i0)^\lambda(g) = \lim_{\epsilon \to 0} \int (x - i\epsilon)^\lambda \cdot g(x)\, dx$$

Since $\lambda \mapsto e^{\lambda \cdot \log z}$ are entire in λ, we get two entire distribution valued functions by $(x - i0)^\lambda$ and $(x - i0)^\lambda$. Regarding the choice of branches for the log-functions we see that

$$(x - i0)^\lambda = e^{-2\pi i \lambda} \cdot (x + i0)^\lambda \quad : x < 0$$

At the same time

$$(x + i0)^\lambda = (x - i0)^\lambda = x^\lambda \quad : x > 0$$

From this we see that the distribution

$$(x + i0)^\lambda - e^{2\pi i \lambda} \cdot (x - i0)^\lambda$$

is supported by $x \geq 0$ and expressed by the density $(1 - e^{2\pi i \lambda}) \cdot x^\lambda$. The conclusion is that one has the equation

$$\mu_\lambda = \frac{(x + i0)^\lambda - e^{2\pi i \lambda} \cdot (x - i0)^\lambda}{1 - e^{2\pi i \lambda}}$$

Remark. The equation (xx) is more involved compared to the previous description of the meromorphic μ-function found via Euler's functional equation. But (xx) has the merit that the denominator is an entire distribution valued function and when one passes to Fourier transforms it turns out that (xx) is quite useful.

Principal value integrals. If $g(x)$ is a test-function there exists a limit

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{g(x)}{x}\, dx$$

This yields a distribution denoted by $\mathrm{VP}(x^{-1})$. Outside $\{x = 0\}$ it is given bt the density x^{-1} where it agrees with $(x + i0)^{-1}$ and hence the difference

$$\mu = \mathrm{VP}(x^{-1}) - (x + i0)^{\cdot -1}$$

is supported by $\{x = 0\}$.

Exercise. Notoice that

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{1}{x + i\epsilon}\, dx = \log(1 + i\epsilon) - \log(-1 + i\epsilon) = -\pi i$$

and use this to show that

$$\mu = -\pi i \cdot \delta_0$$