6. Interpolation and solutions to the $\bar{\partial}$-equation.

Contents.

I. Carleson's Interpolation Theorem

Introduction. Let $U = \mathfrak{Im}(z) > 0$ be the upper half-plane. Denote by $\mathbf{c}_*$ the family of sequences of complex numbers $\{c_\nu\}$ where every $|c_\nu| \leq 1$. A sequence $z_\bullet = \{z_\nu\}$ in U has a finite interpolation norm if there exists a constant K such that for every sequence $\{c_\nu\}$ in $\mathbf{c}_*$ gives an analytic function $f(z)$ in U such that

$$(*) \qquad f(z_\nu) = c_\nu \quad : \quad \nu = 1, 2, \ldots \quad \text{and} \quad |f|_U \leq K$$

The smallest constant K for which (*) holds is denoted by $\mathfrak{int}(z_\bullet)$.

0.1 Theorem. *A sequence $z_\bullet$ has a finite interpolating norm if and only if*

$$(1) \qquad \min_\nu \prod_{k \neq \nu} \left|\frac{z_\nu - z_k}{z_\nu - \bar{z}_k}\right| > 0$$

Moreover, if $\delta(z_\bullet)$ denotes the minimum above then

$$(2) \qquad \mathfrak{int}(z_\bullet) \leq \frac{4A}{\delta(z_\bullet)} \cdot \text{Log}\, \frac{1}{\delta(z_\bullet)}\,.$$

where A is an absolute constant.

Remark. That the condition (1) is necessary is easily verified. See Exercise XX below. The proof of sufficieny is more involved. In [Ca] the proof is carried out in the unit disc D where (1) in Theorem 0.1 means that a sequence $\{z_\nu\}$ in D should satisfy

$$(3) \qquad \min_\nu \prod_{k \neq \nu} \frac{|z_\nu - z_k|}{|1 - \bar{z}_k \cdot z_\nu|} > 0$$

Carleson's result will be proved in the upper half-plane where certain constructions become a bit more transparent compared to the unit disc. Let $\{z_\nu\}$ be a sequence in U where (1) holds in Theorem 0.1. Since a family of analytic functions in U with a uniform upper bound for the maximum norm is a normal in Montel's sense, it is sufficient to prove the requested interpolation by bounded functions for every finite subsequence of $\{z_\nu\}$. The Nevanlinna-Pick theorem assigns to each finite sequence $\{z_\nu\}$ and every sequence $\{c_\nu\}$ a unique interpolating analytic function $F(z)$ with smallest maximum norm. So Carleson's result gives a uniform bound in the Nevannlinna-Pick interpolation expressed via the numbers $\delta(z_\bullet)$.

0.1 Carleson measures.

For every $h > 0$ we denote by $\mathfrak{square}(h)$ the family of squares of the form

$$\square = \{(x, y) \quad : \quad x_0 - h/2 < x < x_0 + h/2 \quad : 0 < y < h\} \quad : x_0 \in \mathbf{R}$$

0.2. Definition. *A non-negative measure μ in U is called a Carleson measure if there exists a constant Ksuch that*

$$\mu(\square) \leq K \cdot h \quad : \quad \square \in \mathfrak{square}(h) \quad : 0 < h < \infty$$

The least constant K is denoted by $\mathfrak{car}(\mu)$ and called the Carleson norm of μ.

An essential step during the proof Theorem 0.1 is the following inequality:

0.3 Theorem. *Let $\{z_\nu\}$ be a sequence with a positive δ-number. Then*

$$\mathfrak{car}\Big(\sum_{\nu=1}^{\nu=\infty} \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu}\Big) \leq 2 \cdot \text{Log}\,\frac{1}{\delta(z_\bullet)}$$

where $\{\delta_{z_\nu}\}$ denote Dirac measures.

Use of duality. Theorem 0.3 gives the interpolation theorem via a duality argument where the Hardy space $H^1(\mathbf{R})$ appears. Namely, to each $h \in H^1(\mathbf{R})$ we associate the maximal function h^* and the following inequality is proved in § 2:

0.4 Theorem. *Let μ be a Carleson measure in the upper half-plane. Then*

$$\int_U |h(z)| \cdot d\mu(z) \leq \mathfrak{car}(\mu) \cdot ||h^*||_1 \quad : \ h \in H^1(\mathbf{R})$$

Armed with Theorems 0.3-0.4 we derive Theorem 0.1 in § 3.

1. Proof of Theorem 0.3

First we establish an inequality which is attributed to L. Hörmander.

1.1 Lemma *Let $z_1, \ldots, z_N$ be a finite sequence in U and put $\delta = \delta(z_\bullet)$. Then*

$$(*) \qquad \sum_{\nu \neq k} \mathfrak{Im}(z_k) \cdot \frac{\mathfrak{Im}(z_\nu)}{|z_k - \bar{z}_\nu|^2} \leq \frac{1}{2} \cdot \log \frac{1}{\delta} \quad : \quad 1 \leq k \leq N$$

Proof. The left hand side as well as the δ-norm of the z-sequence are unchanged if we translate all points to $z_\nu + a$ where a is a real number. Similarly, the δ-norm and the left hand side in (*) are unchanged when the sequence is replaced by $\{A \cdot z_\nu\}$ for some $A > 0$. To prove (*) for a fixed k which we may therefore take $k = N$ and $z_N = i$. Put $z_\nu = a_\nu + ib_\nu$ when $1 \leq \nu \leq N-1$. Then we must show

$$\text{(i)} \qquad \sum_{\nu=1}^{\nu=N-1} \frac{b_\nu}{(1+b_\nu)^2 + a_\nu^2} \leq \frac{1}{2} \cdot \log \frac{1}{\delta}$$

To prove (i) we first notice that

$$\text{(iii)} \qquad \frac{|i - \bar{z}_\nu|^2}{|i - z_\nu|^2} = \frac{(1+b_\nu)^2 + a_\nu^2}{(1-b_\nu)^2 + a_\nu^2}$$

By inverting the $\delta = \delta(z_\bullet)$ we also have:

$$\text{(iii)} \qquad \prod_{\nu=1}^{\nu=N-1} \frac{(1+b_\nu)^2 + a_\nu^2}{(1-b_\nu)^2 + a_\nu^2} \leq \delta^{-2} \implies \sum_{\nu=1}^{\nu=N-1} \log\Big[\frac{(1+b_\nu)^2 + a_\nu^2}{(1-b_\nu)^2 + a_\nu^2}\Big] \leq 2 \cdot \log \frac{1}{\delta}$$

Next, for each ν we have the integral formula

$$\log \frac{(1+b_\nu)^2 + a_\nu^2}{(1-b_\nu)^2 + a_\nu^2} = \int_{-b_\nu}^{b_\nu} \frac{2(1+s)}{(1+s)^2 + a_\nu^2}\, ds$$

Next, for every pair $(, b)$ of real numbers with $b > 0$ the reader may verify that

$$\text{(iv)} \qquad \frac{b}{(1+b)^2 + a^2} \leq \frac{1}{2} \cdot \int_b^b \frac{1+s}{(1+s)^2 + a^2}\, ds$$

Now (iii-iv) and a summation over ν gives (i).

Final part of the proof of Theorem 0.3

If $z_\bullet \in \mathcal{S}(\delta)$ and a is a real number the sequence $z_\bullet + a = \{z_\nu + a\}$ also belongs to $\mathcal{S}(\delta)$. Since Theorem 0.3 asserts an a priori estimate we may assume that $\square$ is centered at $x = 0$, i.e.

$$\square = \{(x,y) : \ -h/2 < x < h/2 \text{ and } 0 < y < h\}$$

There remains to estimate

(i) $$\sum_{z_\nu \in \square} \mathfrak{Im}\, z_\nu$$

Set

$$y^* = \max\{\mathfrak{Im}(z_\nu) : \ z_\nu \in \square\}$$

Let k give the equality $y^* = \mathfrak{Im}(z_k)$. With $z_k = x_k + iy^*$ and $z_\nu = x_\nu + iy_\nu \in \square$ we have

$$|z_k - \bar{z}_\nu|^2 = (x_k - x_\nu)^2 + (y^* - y_\nu)^2 \le h^2 + (y^*)^2 \implies$$

$$\frac{\mathfrak{Im}(z_k)}{|z_k - \bar{z}_\nu|^2} \ge \frac{y^*}{h^2 + (y^*)^2} \quad : \nu \ne k$$

Next, notice that

$$y^* \ge h/2 \implies \frac{y^*}{h^2 + (y^*)^2} \ge \frac{1}{5h}\,.$$

Lemma 1.1. applied with $\nu = k$ gives therefore

(ii) $$\sum_{z_\nu \in \square} \mathfrak{Im}(z_\nu) \le y^* + \frac{5h}{2} \cdot \mathrm{Log}\,\frac{1}{\delta} \le h \cdot \big(1 + \frac{5}{2} \cdot \mathrm{Log}\,\frac{1}{\delta}\big)$$

So if $y^* \ge h/2$ we are done. Suppose now that $y^* < h/2$ and regard the cubes:

$$\square_1 = \{-h/2 < x < 0 \,\text{and } 0 < y < h/2\} \quad \square_2 = \{0 < x < h/2 \text{ and } 0 < y < h/2\}$$

We want to estimate

$$S_1 + S_2 = \sum_{z_\nu \in \square_1} \mathfrak{Im}(z_\nu) + \sum_{z_\nu \in \square_2} \mathfrak{Im}(z_\nu)$$

We have also two sequences:

$$\{z_\nu : \ z_\nu \in \square_1\} \quad \text{and} \quad \{z_\nu : \ z_\nu \in \square_2\}$$

Since all factors defining the δ-norm are ≤ 1 these two smaller sequences both belong to $\mathcal{S}(\delta)$. Suppose that:

$$y_1^* = \max_{z_\nu \in \square_1} \mathfrak{Im}(z_\nu) \ge \frac{h}{4}$$

When this holds we obtain exactly as above:

$$S_1 \le \frac{h}{2} \cdot 2 \cdot \mathrm{Log}\,\frac{1}{\delta}$$

If $y_1^* < \frac{h}{4}$ we continue to split the cube $\square_1$. In a similar fashion we treat the sequence which stays in $\square_2$. After a finite number of steps we get the required inequality in Theorem 0.3.

2. Proof of Theorem 0.4

Let $h \in H^1(\mathbf{R})$ and recall that its maximal function is defined by

(i) $$h^*(t) = \max|h(x + iy)| \quad : \quad |x - t| < y$$

To each $\lambda > 0$ we consider the open subset on the real line defined by $\{h^* > \lambda\}$. It is some union of disjoint intervals $\{(a_j, b_j)\}$ and (i) gives the set-theoretic inclusion:

(ii) $$\{|h(x + iy)| > \lambda\} \subset \cup\, T_j \quad :$$

where T_j is the triangle side standing on the interval (a_j, b_j) as explained in XXX. (Hardy space). In particular we have the inclusion:

(iii) $$T_j \subset \square(a_j, b_j) = \{x + iy \ : |x - \frac{a_j + b_j}{2}| < b_j - a_j \ : \ 0 < y < b_j - a_j\}$$

See figure XXX. So if μ is a positive measure in U we obtain:

(iv) $$\mu(\{|h| > \lambda\}) \leq \sum \mu(T_j) \leq \sum \mu(\square(a_j, b_j))$$

If μ is a Carleson measure the right hand side is estimated by

(v) $$\mathfrak{car}(\mu) \cdot \sum (b_j - a_j) = \mathfrak{car}(\mu) \cdot \mathfrak{m}(\{h^* > \lambda\})$$

where $\mathfrak{m}$ refers to the 1-dimensional Lebesgue measure. Here (v) holds for every $\lambda > 0$ and the general inequality for distribution functions from XXX gives:

$$\int_U |h| \cdot d\mu \leq \mathfrak{car}(\mu) \cdot ||h^*||_1$$

This finishes the proof of Theorem 0.4.

3. Proof of Theorem 0.1.

The Banach space $H^1(\mathbf{R})$ contains a dense subspace of functions $h(z)$ with polynomial decay at infinite, i.e. functions for which

$$|h(z)| \leq C_N \cdot (1 + |z|)^{-N} \quad : \text{hold for some constant } C_N \quad : N = 1, 2, \ldots$$

This is used below to ensure that various integrals are defined where it suffices to to use "nice" functions while an *a priori* inequality is established. Consider a finite sequence $z_1, \ldots, z_N$ in U and a finite sequence $c_1, \ldots, c_N$ in $\mathbf{c}_*$. Newton's interpolation gives a unique polynomial $P(z)$ of degree $N - 1$ such that:

(i) $$P(z_k) = c_k \quad : \quad 1 \leq k \leq N$$

Let $B(z)$ be the Blascke product associated to the z-sequence:

(ii) $$B(z) = \prod_{\nu=1}^{\nu=N} \frac{z - z_\nu}{z - \bar{z}_\nu}$$

Let $h \in H^1(\mathbf{R})$ have the polynomial decay $\geq N + 2$. Residue calculus gives

(iii) $$\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx = \sum_{k=1}^{k=N} \frac{c_k}{B'(z_k)} \cdot h(z_k)$$

If k is fixed we have

(iv) $$\frac{1}{B'(z_k)} = \prod_{\nu \neq k} \frac{z_k - \bar{z}_\nu}{z_k - z_\nu} \cdot 2 \cdot \mathfrak{Im}(z_k)$$

It follows that

(v) $$\Big|\frac{1}{B'(z_k)}\Big| \leq \frac{2}{\delta(z_\bullet)} \cdot \mathfrak{Im}(z_k)$$

Since $\{c_\nu\} \in \mathbf{c}_*$ we see that (v) and the triangle inequality applied to (iii) give:

(*) $$\Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx\,\Big| \leq \frac{2}{\delta(z_\bullet)} \cdot \sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k)$$

Now Theorem 0.4 gives the inequality

(5) $$\sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k) \leq \mathfrak{car}(\sum \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu} \cdot ||h^*||_1$$

3.1 Use of duality.

Let us put

$$C_\delta = \frac{2}{\delta(z_\bullet)} \cdot 2 \cdot \log \frac{1}{\delta(z_\bullet)}$$

Then (5) and Theorem 0.3 together with (*) give

$$\Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx \,\Big| \leq C_\delta \cdot ||h^*||_1$$

Next, the result in (Hardy Chapter) gives an absolute constant A such that

$$||h^*||_1 \leq A \cdot ||h||_1$$

Hence the densely defined linear functional

$$h \mapsto \int_{-\infty}^{\infty} \frac{P(x)}{B(x)} \cdot h(x) \cdot dx$$

has norm $\leq C_\delta \cdot A$. The Duality Theorem from XXX implies that if $\epsilon > 0$, then there exists some $G(z) \in \mathcal{O}(U)$ such that the maximum norm

$$\Big|\frac{P(x)}{B(x)} - G(x)\Big|_U < A \cdot C_\delta + \epsilon \tag{6}$$

Since $B(x)$ is a Blaschke product we have $|B(x)| = 1$ almost everywhere and hence (6) gives:

$$\big|P(x) - B(x) \cdot G(x)\,\big| < A \cdot C_\delta + \epsilon$$

Now $f(z) = P(z) - B(z)G(z)$ is analytic in U and since $B(z_\nu) = 0$ for every ν we have

$$f(z_\nu) = P(z_\nu) = c_\nu$$

So the bounded analytic function $f(z)$ interpolates and since $\epsilon > 0$ can be arbitrary small and $c_1, \ldots, c_N$ was an arbitrary sequence in $\mathbf{c}_*$ we conclude that the interpolation norm of the finite sequence $z_1, \ldots, z_N$ is at most $A \cdot C_\delta$. Since this uniform bound holds for all N we get

$$\mathfrak{int}(z_\bullet) \leq A \cdot C_\delta$$

which finishes the proof of Theorem 0.1.

Exercise. Prove that (1) in the Interpolation Theorem is necessary.

II. Wolff's Theorem.

Introduction. The Pompieu formula solves the inhomogeneous $\bar{\partial}$-equation in the unit disc D. So if $h(z)$ is a C^∞-function defined in some open neighborhood of the closed disc there exists a C^∞-function v such that

$$(*) \qquad \bar{\partial}(v)(z) = h(z) \quad : \quad z \in D$$

We seek conditions in order that (*) has a solution v whose maximum norm over D is controlled by extra properties of h. Such conditions were imposed in [Wolff]. To every C^∞-function h on $\bar{D}$ we set

$$(**) \qquad \mu_h(z) = \log \frac{1}{|z|} \cdot |\partial(h)(z)| \quad : \quad \nu_h(z) = \text{Log}\, \frac{1}{|z|} \cdot |h(z)|^2$$

Before we announce Theorem 0.4 we recall the following.

0.1 Carleson measures in D**.** Consider the family of sector domains defined for all pairs $0 < h < 1$ and $0 \leq \theta \leq 2\pi$ by:

$$S_h(\theta) = \{\, z = r \cdot e^{i\phi} \ : 1 - h < r < 1 : |e^{i\phi} - e^{i\theta}| \leq \frac{h}{2} \,\}$$

0.2 Definition. *A non-negative measure μ in D is called a Carleson measure if there exists a constant K such that*

$$\iint_{S_h(\theta)} d\mu \leq K \cdot h \quad : \quad 0 < h > 1 \quad : \quad 0 \leq \theta < 2\pi$$

The least constant K is denoted by $\mathfrak{car}(\mu)$ and called the Carleson norm of μ.

0.3 Remark. Using a conformal mapping between D and the upper half-plane one shows exactly as in § xx that there exists an absolute constant A such that the following holds for every pair of a Carleson measure μ in D and a function $f(z)$ in the Hardy space $H^1(T)$

$$\iint_D \big|f(z)\big| \cdot d\mu(z) \leq A \cdot \mathfrak{car}(\mu) \cdot ||f||_1$$

0.4 Theorem. *Let A be as in (0.3). For every C^∞-function $h \in C^\infty(\bar{D})$ the equation (*) has a C^∞-solution v_* where*

$$\max_\theta |v_*(e^{i\theta})| \leq 2 \cdot A \cdot \mathfrak{car}(\mu_h) + 2 \cdot \sqrt{A \cdot \mathfrak{car}(\nu_h)}$$

For the proof we need an integral formula due to Jensen.

0.5 The Fourier-Jensen formula. *Let $F(z)$ be an analytic function in D with a simple zero at $z = 0$ and otherwise it is $\neq 0$. Then one has the equality:*

$$(*) \qquad \iint_D \text{Log}\, \frac{1}{|z|} \cdot \frac{|F'(z)|^2}{|F(z)|} \cdot dxdy = \int_0^{2\pi} |F(e^{i\theta})| \cdot d\theta$$

To prove (*) we set $F(z) = z \cdot G(z)$ where the hypothesis means that G is zero-free so we can construct a square root function and write

$$\text{(i)} \qquad F(z) = z \cdot \Psi^2(z) \quad : \ \Psi \in \mathcal{O}(D)$$

This implies that

$$\frac{|F'(z)|^2}{|F(z)|} = \frac{|\Psi(z) + 2z \cdot \Psi'(z)|^2}{|z|}$$

Hence the left hand side in (*) becomes:

$$\text{(ii)} \qquad \iint_D \log \frac{1}{|z|} \cdot \big|\Psi(z) + 2z \cdot \Psi'(z)\big|^2 \cdot \frac{1}{|z|} \cdot dxdy$$

To evaluate this integral we consider the series expansion $\Psi(z)=\sum a_n z^n$. In polar coordinates the double integral becomes

(iii) $$\int_0^1\int_0^{2\pi}\log\frac{1}{r}\cdot\bigl|\sum(2n+1)\cdot a_n\cdot r^n\cdot e^{in\theta}\bigr|^2\cdot drd\theta$$

Exercise. Show that (iii) is equal to

$$2\pi\cdot\sum|a_n|^2=\int_0^{2\pi}|\Psi(e^{i\theta})|^2\cdot d\theta=\int_0^{2\pi}|F(e^{i\theta})|\cdot d\theta$$

which gives the requested equality (*).

1. Proof of Theorem 0.4

The Pompieu formula gives a solution v to the $\bar\partial$-equation

(i) $$\bar\partial(v)=h$$

We get new solutions to (i) by $v_*=v-G$ when $G(z)$ are analytic functions in D. So in order to minimize the maximum norm of a solution to (i) we seek a bounded analytic function G_* such that

(ii) $$|v-G_*|_D=\min_G\ |v-G|_D\quad : G\in H^\infty(T)$$

Let m_* be the minimum value in (ii). To estimate m_* we use the duality between $H^\infty(T)$ and $H_0^1(T)$ where $H_0^1(T)$ is the space of functions $F(z)$ in the Hardy space $H^1(T)$ for which $F(0)=0$. Denote by $S_*^(T)$ the set of functions $F\in H_0^1(T)$ such that

(1) $$\int_0^{2\pi}\bigl|F(e^{i\theta})\bigr|\cdot d\theta=1$$

The duality result from (XX) gives:

(2) $$m_*=\max_F\Bigl|\int_0^{2\pi}v(e^{i\theta})\cdot F(e^{i\theta})\cdot d\theta\Bigr|\quad : F\in S_*(T)$$

Since $F(0)=0$ Green's formula shows that (2) becomes

(3) $$\iint_D\text{Log}\,\frac{1}{|z|}\cdot\Delta(vF)\cdot dxdy$$

Since $\Delta=\partial\bar\partial$ and v solves (i) while $\bar\partial(F)=0$, we get

(4) $$\Delta(vF)=4\cdot\partial(hF)=4\cdot F\cdot\partial(h)+4\cdot h\cdot F'$$

Hence we have proved the following

1. Lemma. *One has the equality*

$$m_*=\max_F\Bigl|\iint_D\log\frac{1}{|z|}\cdot\bigl[F\cdot\partial(h)+h\cdot F'\bigr]\cdot dxdy\Bigr|\quad :\quad F\in S_*^1(T)$$

To profit upon this expression for m_* we use the Jensen-Nevanlinna factorisation and reduce the estimate to the case when $F(z)$ has a simple zero at $z=0$ while it is $\neq 0$ in the punctured disc $D\setminus\{0\}$. Thus, consider some F in $S_*(T)$. Since $F(0)=0$ we there exists the Jensen-Nevanlinna factorisation:

(i) $$F(z)=z\cdot B(z)\cdot G(z)$$

where $B(z)$ is a Blaschke product and G has no zeros in D. Moreover, since $|B|=1$ holds almost everywhere on T it follows that G belong to $S_*(T)$. Set:

(ii) $$F_1(z) = \frac{z}{2}\big(B(z)-1\big)G(z) \quad \text{and} \quad F_2(z) = \frac{z}{2}\big(B(z)+1\big)G(z)$$

It follows that $F = F_1 + F_2$ and since the maximum norms of $B(z)-1$ and $B(z)+1$ are at most 2 we have

(iii) $$||F_\nu||_1 \leq 1 \quad : \quad \nu = 1,2$$

From (ii) we see that F_1 and F_2 both have a simple zero at the origin and are otherwise $\neq 0$ in the punctured disc. Hence we can apply the Fourier-Jensen formula from (0.4) which gives

(iv) $$[\iint_D \operatorname{Log}\frac{1}{|z|}\cdot\frac{|F_\nu'(z)|^2}{|F_\nu(z)|}\,dxdy = \int_0^{2\pi} |F_\nu(e^{i\theta})|\,d\theta \leq 1 \quad : \nu = 1,2$$

Final part of the proof. For each $\nu = 1,2$ we set

(1) $$V(F_\nu) = \iint_D \log\frac{1}{|z|}\cdot|\partial(h)|\cdot|F_1(z)|\,dxdy + \iint_D \log\frac{1}{|z|}\cdot|h(z)|\cdot|F_1'(z)|\,dxdy$$

By the triangle inequality the right hand side in Lemma 1 is $\leq V(F_1)+V(F_2)$. Let us for example estimate $V(F_1)$. By the inequality (0.3) the first integral in (1) is estimated by

(2) $$A\cdot\mathfrak{car}(\operatorname{Log}\frac{1}{|z|}\cdot|\partial(h)|)\cdot||F_1||_1$$

Since $||F_1||_1 \leq 1$ the definition of μ_h means that (2) is majorised by

(*) $$A\cdot\mathfrak{car}(\mu_h)$$

To estimate the second integral in (1) we insert $\sqrt{|F_1|}$ as a factor and Cauchy-Schwartz inequality majorises the second integral by the square root of

(3) $$[\iint_D \log\frac{1}{|z|}\cdot\frac{|F_1'(z)|^2}{|F_1(z)|}\,dxdy]\cdot[\iint_D \log\frac{1}{|z|}\cdot|h(z)|^2\cdot|F_1(z)|\,dxdy$$

In this product the first factor is given by (iv) and is therefore $\leq ||F_1||_1 \leq 1$. Finally, the definition of the Carleson norm entails that the last factor is majorised by $A\cdot\mathfrak{car}(\nu_h)$. Taking the square root together with (*) above we have proved that

(4) $$V(F_1) \leq +A\cdot\mathfrak{car}(\mu_h) + \sqrt{A\cdot\mathfrak{car}(\nu_h)}$$

The same holds for F_2 and thanks to the factor 2 the requested inequality in Wolff's theorem follows.

III. A class of Carleson measures

Let $f(z)$ be a bounded analytic function in D and associate the non-negative measure in D by:

$$\mu_f = |f'(z)|^2\cdot\operatorname{Log}\frac{1}{|z|}$$

3.1 Theorem. *There exists an absolute constant A_* such that*

$$\sqrt{\mathfrak{car}(\mu_f)} \leq A_*\cdot|f|_D \quad : \quad f\in H^\infty(D)$$

Proof. To prove the a prioor inequality it suffices to treat the case when $|f|_D = 1$ and by a rotation it suffices to study integrals over sectors of the form

$$S_h = \{re^{i\theta} :\ 1-h/4 < r < 1 : \pi - h/4 < \theta < \pi + h/4\}$$

where $0 < h < 1/2$. Consider the conformal mapping from $\mathfrak{Im}\,\zeta > 0$ to D defined by

$$z = \frac{\zeta - i}{\zeta + i}$$

With $\zeta = \xi + i\eta$ it is easily seen that $G^{-1}(S_h)$ is contained in the rectangle

$$\square = \{-h/2 < \xi < h/2\} \times \{0 < \eta < h/2\}$$

Next , the expansion $\log(1-s) = -s + O(s^2)$ for small $s > 0$ gives a constant C which is independent of h such that

$$\log \frac{1}{|G(\xi,\eta)|} \leq C \cdot \eta \quad : \xi + i\eta \in \square$$

With $g(\zeta) = f(\frac{\zeta - i}{\zeta + i})$ we conclude that

$$\text{(i)} \qquad \frac{1}{h} \cdot \iint_{S_h} |f'(z)|^2 \cdot \log \frac{1}{|z|}\, dxdy \leq \frac{C}{h} \cdot \int_\square \eta \cdot |g'(\zeta)|^2 \, d\xi d\eta$$

Set $g = u + iv$ which gives $|g'(|^2 = u_\xi^2 + u_\eta^2$. The right hand side in (i) becomes:

$$\text{(ii)} \qquad \frac{1}{h} \int_{\square_h} \eta \cdot (u_\xi^2 + u_\eta^2) \cdot d\xi d\eta$$

Replace $\square_h$ by the larger semi-disc

$$D_h = \{\zeta = |xi + i\eta \quad : ||zeta| < h \quad : y > 0\}$$

Since $4h^2 - |\zeta|^2 \geq 3h^2$ when $\zeta \in D_h$ we get a larger contribution by integrating over the larger semi-disc D_{2h} and it suffices to find an absolute constant A such that

$$(*) \qquad \int_{D_{2h}} \eta(4h^2 - |\zeta|^2) \cdot (u_\xi^2 + u_\eta^2)\, d\xi d\eta \leq A \cdot h^3$$

To get A in (*) we use the equality $\Delta(u^2) = 2(u_x^2 + u_y^2)$ and since the function $y(4h^2 - |z|^2)$ is zero on the boundary of D_{2h}, Green's formula shows that the integral in (*) is equal to

$$\int_{D_{2h}} u^2 \cdot \Delta(y(4h^2 - |\zeta|^2)\, d\xi d\eta - \int_{\partial D_{2h}} u^2 \cdot \partial_{\mathbf{n}}(\eta(4h^2 - |\zeta|^2)\, ds$$

Notice that $\Delta(y(4h^2 - |z|^2) = -8y < 0$ in D_{2h} and an easy computation gives

$$-\int_{\partial D_{2h}} u^2 \cdot \partial_{\mathbf{n}}(\eta(4h^2 - ||zeta|^2) \cdot ds =$$

$$\int_{-2h}^{2h} u^2(\xi, 0) \cdot (4h^2 - \xi^2)\, d\xi + \int_0^\pi u^2(2he^{i\theta}) \cdot \sin\theta \cdot \left[-4h^2 + 3 \cdot (2h)^2\right] \cdot h \cdot d\theta$$

Since the maximum norem of u is ≤ 1 we conclude that (*) is majorised by

$$\int_{-2h}^{2h} (4h^2 - \xi^2)\, d\xi + \int_0^\pi \sin\theta \cdot \left[-4h^2 + 3 \cdot (2h)^2\right] \cdot h\, d\theta$$

At this stage the reader can evaluate the requested constant A which estimates(*) by $A \cdot h^3$.

IV. Berndtsson's $\bar{\partial}$-estimate

Denote by $\mathcal{B}$ the family of bounded open sets Ω in $\mathbf{C}$ for which there exists a real-valued C^2-function $\rho(z)$ defined in some neighborhood of $\bar{\Omega}$ such that the following conditions hold:

$$\text{(1)} \qquad \Omega = \{\rho(z) < 0\}$$

$$\text{(2)} \qquad \Delta(\rho)(z) > 0 \;:\; z \in \Omega \quad \text{and} \quad \nabla(\rho)(z) \neq 0 \; z \in \partial\Omega$$

4.0 Defintion. *To each pair $\Omega \in \mathcal{B}$ and a bounded and subharmonic function $\phi(z)$ in Ω a C^∞-function f in Ω is of the Berndtsson class if*

(*) $$|f(z)| \leq -\rho(z) \cdot \Delta\, \phi(z) \quad : \quad z \in \Omega$$

The family of these functions is denoted by $\mathfrak{Bernt}(\Omega, \phi)$.

4.1 Theorem. *To each $f \in \mathfrak{Bernt}(\Omega, \phi)$ the equation*

(*) $$\bar{\partial}(u) = f$$

has a solution $u(z)$ which satisfies

(**) $$\max_{z \in \partial\Omega} \frac{\big|u(z)| \cdot e^{-\phi(z)/2}\big|}{\big|\nabla\rho(z)\big|} \leq \max_{z \in \Omega} \frac{\big|f(z)| \cdot e^{-\phi(z)/2}\big|}{\big| -\rho(z) \cdot \Delta\, \phi(z) + \nabla\rho(z)\big|}$$

Remark. A special case occurs when $\Omega = D$ and $\rho(z) = |z|^2 - 1$. Then (*) means that a function f in the Berndtsson class satisfies

$$|f(z)| \leq (1 - |z|^2) \cdot \Delta(\phi)(z)$$

So here $|f|$ decays as $|z| \to 1$ and when $\Delta(\phi)$ is bounded this inequality estimates the Carleson norm of f. The reader s invited to analyze the specific case when

$$\phi(z) = \log\, 1 + |z|^2 \implies \Delta(\phi) = \frac{4}{(1 + x^2 + y^2)^2} \geq 1 \quad : z \in D$$

Theorem 4.1 is proved via an extremal problem in a Hilbert space which goes as follows: Given the ϕ-function one has the Hilbert space of functions g in D which are square integrable with respect to $e^{-\phi}$, i.e.

(1) $$\iint_D |g(z)|^2 \cdot e^{-\phi(z)} \cdot dxdy < \infty$$

In Berndtsson's article it is proved that there exists the unique extremal solution u to the equation $\bar{\partial}(u) = f$ whose norm in $L^2(e^{-\phi})$ is minimal among all functions ψ in D satisfying $\bar{\partial}\psi = f$ and this extremal solution u satisfies the inequality (**) in Theorem 4.1.

V. Hörmander's L^2-estimate

Let Ω be an open set in $\mathbf{C}$. If ϕ is a real-valued continuous and non-negative function we get the Hilbert space $\mathcal{H}_\phi$ whose elements are Lebesgue measurable functions f in Ω such that

$$\int_\Omega |f|^2 \cdot e^{-\phi}\, dxdy < \infty$$

The square root yields norm and is denoted by $||f||_\phi$. Let ψ be another continuous and non-negative function which gives the Hilbert space $\mathcal{H}_\psi$ where the norm of an element g is denoted by $||g||_\psi$. We shall consider the $\bar\partial$-operator which sends a function $f \in \mathcal{H}_\phi$ to $\bar\partial(f)$ and want to solve the equation

$$\bar\partial(f) = w \quad : w \in H_\psi \tag{5.0}$$

where one seeks a constant C such that (5.0) has a solution f with

$$||f||_\phi \leq C \cdot ||w||_\psi$$

To attain this we impose the following:

5.1 Hörmander's condition. The pair ϕ, ψ satisfies the Hörmander condition if there exists some positive constant c_0 such that the following pointwise inequality holds in Ω:

$$\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y \geq 2 \cdot c_0^2 \cdot e^{\psi(z)-\phi(z)} \tag{*}$$

where we have put $|\nabla(\psi)|^2 = \psi_x^2 + \psi_y^2$.

5.2 Theorem. *If (ϕ, ψ) satisfies (*) for some $c_0 > 0$ then the equation $\bar\partial(f) = w$ has a solution for every $w \in \mathcal{H}_\psi$ where*

$$||f||_\phi \leq \frac{1}{c_0} \cdot ||w||_\psi$$

Proof. Since $C_0^\infty(\Omega)$ is a dense subspace of $\mathcal{H}_\phi$ the linear operator $T\colon f \mapsto \bar\partial(f)$ from $\mathcal{H}_\phi$ to $\mathcal{H}_\psi$ is densely defined. Let w be in the domain of definition for the adjoint operator T^*. If $f \in C_0^\infty(\Omega)$. Stokes theorem gives

$$\langle T(f), w\rangle = \int \bar\partial(f) \cdot \bar w \cdot e^{-\psi}\, dxdy = -\int f \cdot \big[\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)\big] \cdot e^{-\psi}\, dxdy \tag{i}$$

Since ψ is real-valued, $\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)$ is the complex conjugate of $\partial(w) - w \cdot \partial(\psi)$ and (i) gives

$$T^*(w) = -\big[\partial(w) - w \cdot \partial(\psi)\big] \cdot e^{\phi-\psi} \tag{ii}$$

Taking the squared L^2-norm in $\mathcal{H}_\phi$ we obtain

$$||T^*(w)||_\phi^2 = \int |\partial(w) - w \cdot \partial(\psi)|^2 \cdot e^{\phi-2\psi} =$$

$$\int \big(|\partial(w)|^2 + |w|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi-2\psi} - 2 \cdot \mathfrak{Re}\Big(\int \partial(w) \cdot \bar w \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}\Big) \tag{iii}$$

By partial integration the last integral in (iii) is equal

$$2 \cdot \mathfrak{Re}\Big(\int w \cdot [\partial(\bar w) \cdot \bar\partial(\psi) + \bar w \cdot \partial\bar\partial(\psi) - 2\bar w \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar w \cdot \bar\partial(\psi) \cdot \partial(\phi)] \cdot e^{\phi-2\psi}\Big) \tag{iv}$$

Next, the Cauchy-Schwarz inequality gives

$$\Big|2 \cdot \int w \cdot \partial(\bar w) \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}\Big| \leq \int \big(|\partial(w)|^2 + |w|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi-2\psi} \tag{v}$$

Together (iii-v) give

$$||T^*(w)||_\phi^2 \geq 2 \cdot \mathfrak{Re} \int |w|^2 \cdot \big[\partial\bar\partial(\psi) - 2 \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar\partial(\psi) \cdot \partial(\phi)\big] \cdot e^{\phi-2\psi} =$$

$$2 \cdot \mathfrak{Re} \int |w|^2 \cdot \frac{1}{4}\big[\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y\big] \cdot e^{\phi-2\psi} \tag{vi}$$

where the last equality follows since ϕ and ψ are real-valued. When (*) holds in (5.1) it follows that

(vi) $$||T^*(w)||_\phi^2 \geq c_0^2 \cdot \int |w|^2 \cdot e^{\psi-\phi} \cdot e^{\phi-2\psi} = c_0^2 \cdot ||w||_\psi^2$$

This lower bound implies that the norm of T is bounded by c_0 and Theorem 5.2 follows.

5.3 Remark. We refer to the article [Kiselman] by C. Kiselman for some specific applications of L^2-estimates in $\mathbf{C}$ applied to carriers of Borel transforms. The full strength of L^2-estimate appears in dimension $n \geq 2$ where one works with *plurisubharmonic functions* and impose the condition that Ω is a strictly pesudo-convex set in $\mathbf{C}^n$ and solve inhomogeneous $\bar{\partial}$-equations for differential forms of bi-degree (p,q). In addition to Hörmander's original article [Hörmander] we refer to his text-book [Hörmander] and Chapter XX in [Hömander XX-Vol 2] where bounds for $\bar{\partial}$-equations are established with certain relaxed assumptions which are used to settle the fundamental principle for over-determined systems of PDE-equations in the smooth case. Working on complex manifolds Hörmander's L^2-estimates give very powerful tools. For a wealth of results of this nature we refer to the notes by Demailly in [Dem].

The case $n = 2$

The special case below may help the reader to pursue details from page xx-xx in Hörmander's text book (xx edition). See also Hörmander's article [Acta: 1962]. Let D^2 be the 2-dimensional polydisc in $\mathbf{C}^2$ with coordinates $z = (z_1, z_2)$ where $\bar{\partial}_1$ and $\bar{\partial}_2$ are pairwise commuting operators. Let $\phi(z)$ be a real-valued function in D^2 which is at least of class C^2. We get the Hilbert space $\mathcal{H}$ of locally square integraböe functions with finite norm:

$$||a||_\phi = \sqrt{\int |a(z)|^2 \cdot e^{-\phi(z)}\, d\lambda(z)}$$

where $d\lambda(z)$ is the 4-dimensional Lebesgue meaure and the integral is taken over D^2. We have the densely defined linear operator T from $\mathcal{H}$ into $\mathcal{H} \oplus \mathcal{H}$ defined by

$$T(a) = \bar{\partial}_1(a) \oplus \bar{\partial}_2(a)$$

A. Exercise. Let T^* be the adjoint of T which sends a pair $(f,g) \in \mathcal{H} \oplus \mathcal{H}$ to $\mathcal{H}$. Show that

(A) $$T^*(f,g) = -(\partial_1(f) - f \cdot \partial_1(\phi) + \partial_1(g) - g \cdot \partial_1(\phi)]$$

B. Exercise. Put

$$\delta_1(f) = \partial_1(f) - f \cdot \partial_1(\phi) \quad : \quad \delta_2(g) = \partial_2(g) - g \cdot \partial_2(\phi)$$

Use (A) to show that

(B.1) $$||T^*(f,g)||^2 = ||\delta_1(f)||^2 + ||\delta_2(g)||^2 + 2 \cdot \mathfrak{Re} \int \delta_1(f) \cdot \overline{\delta_2(g)} \cdot e^{-\phi}\, d\lambda$$

Next, use Stokes theorem to show that

(B.2) $$\int \delta_1(f) \cdot \overline{\delta_2(g)} \cdot e^{-\phi}\, d\lambda = -\int f \cdot \overline{\partial_1(\delta_2(g))} \cdot e^{-\phi}\, d\lambda$$

C. Exercise. Put

(C.0) $$H(z) = \frac{\partial^2 \phi}{\partial z_1 \bar{\partial} z_2}$$

and by multiplication one identifies H with a zero-order differential operator. Show the following equality in the ring of differential operators in $\mathbf{C}^2$:

(C.1) $$\partial_1 \circ \delta_2 = \delta_2 \circ \partial_1 - H$$

Conclude that (B.2) becomes

(C.2) $$\int f\cdot \bar{g}\cdot \bar{H}\cdot e^{-\phi}\,d\lambda - \int f\cdot \overline{\delta_2\partial_1((g))}\cdot e^{-\phi}\,d\lambda$$

D. The case $\bar{\partial}_1(g)=\bar{\partial}_2(f)$. Use the above to show that this equality gives

(D.1) $$-2\cdot \mathfrak{Re}\int \bar{f}\cdot \delta_2(\bar{\partial}_2((f))\cdot e^{-\phi}\,d\lambda = -2\cdot \mathfrak{Re}\int |\partial_2(f)|^2\cdot e^{-\phi}\,d\lambda$$

E. Conclusion. Show that (A), (B.1-2) and (D.1) and the equality $\bar{\partial}_1(g)=\bar{\partial}_1(f)$ give:

(E.1) $$||T^*(f,g)||^2 = ||\delta_1(f)||^2 + ||\delta_2(g)||^2 + 2\cdot \mathfrak{Re}\int f\cdot \bar{g}\cdot H(z)\cdot e^{-\phi}\,d\lambda + ||\partial_2(f)||^2$$

Above the last term is always ≥ 0. To ensure that there exists a constant c_0 such that

(E.2) $$||f||^2 + ||g||^2 \leq c_0^2\cdot ||T^*(f,g)||^2$$

one must impose suitable conditions upon ϕ. Above the mixed derivative which defines the H-function appears while norms $||\delta_1(f)||^2$ and $||\delta_2(g)||^2$ can be estimated as in the case $n=1$ applied with $\phi=\psi$. The reader is invited to contemplate upon conditions on ϕ which give a constant c_0 in (E.2) and at this stage Hörmander's work can be consulted for examples and further dertails. Notice that above we treated a special case since we used the same weight function ϕ and not a pair as in the case $n=1$.

VI. The Corona Theorem.

Introduction. In the unit disc D we have the Banach algebra $H^\infty(D)$ of bounded analytic functions. Let $\mathfrak{M}_\infty(D)$ denote its maximal ideal space. Via point evaluations in D we get a map

$$i\colon D \mapsto \mathfrak{M}_\infty(D)$$

The Corona problem asked if $i(D)$ is dense in $\mathfrak{M}_\infty(D)$. The affirmative answer to this question was found by Carleson. It is easily seen that the density of the i-image is equivalent to the following result which was proved in [Carleson]:

6.1 Theorem. *The ideal generated by a finite family $f_1, \ldots, f_n$ in $H^\infty(D)$ is equal to $H^\infty(D)$ if and only if there exists $\delta > 0$ such that*

$$|f_1(z)| + \ldots + |f_n(z)| \geq \delta \quad \text{hold for all } \ z \in D$$

Remark. Just as in the Interpolation Theorem an essential ingredient of the proof relies upon Carleson measures. An alternative to Carleson's original proof was given by Wolff and relies upon his result for the inhomogeneous $\bar{\partial}$-equation from XX. The deduction from Wolff's Theorem in XX to the solution of the Corona problem is exposed a several places. See for example Chapter XX in [Narasimhan] and the article [Gamelin] by T. Gamelin. So we refrain from giving further details. Let us remark that one may consider related problems where the boundedness of the f-function is relaxed. For example, using L^2-estimates with weight functions one can solve a problem where $f_1, \ldots, f_n$ are analytic functions in D with moderate growth, i.e there is an integer m and a constant A such that

$$(1 - |z|)^m \cdot |f_\nu(z)| \leq K$$

holds in D for every ν. Assume also that there is an integer m_* and $\delta > 0$ such that

$$|f_1(z)| + \ldots + |f_n(z)| \geq \delta \cdot (1 - |z|)^{m_*} \quad \text{hold for all } \ z \in D$$

Then one can show that there exists an n-tuple $g_1, \ldots g_n$ of analytic functions with moderate growth such that

$$g_1 \cdot f_1 + \ldots + g_n \cdot f_n = 1 \tag{1}$$

holds in D.

Question. Find the smallest possible number $\rho = \rho(m, m_*)$ such that (1) has a solution where the g-functions satisfy

$$|g_\nu(z(| \leq C \cdot (1 - |z|)^{-\rho}$$

for some constant C. It appears that the best constant ρ is not known. However, upper bounds for ρ can be established using L^2-estimates for the $\bar{\partial}$-equation. But there remains to investigate how sharp such bounds are.