Mathematics by Arne Beurling.

Let X be a complex manifold of some dimension $n \geq 2$. To each $0 \leq k \leq n-1$ we denote by $\mathcal{V}_k$ the family of analytic sets of pure dimension k, i.e, the irreducible components are all k-dimensional. If $V \in \mathcal{V}_k$ and $p = n - k$ there exists the temperate local cohomology sheaf $\mathcal{H}^p_{[V]}(\mathcal{O}_X)$ and we recall, from basic $\mathcal{D}$-module theory that it is a regular holonomic left $\mathcal{D}_X$-module. Moreover, it can be identfied with the sheaf CH_V whose sections are Collef-Herrera currents of bi-degree $(0,p)$ with support on V. Indeed, this can be expressed via the Dolbeault isomrphism and the (non-trivial !) direct sum decomposition

$$\ker_{\bar\partial}(\mathcal{C}_V^{0,p}) = \bar\partial(\mathcal{C}_V^{0,p-1}) \oplus \mathrm{CH}_V$$

Let us now consider some $\phi \in \mathcal{O}(X)$ which yields the holomorphic 1-form $\partial(\phi)$. If $\gamma \in \mathrm{CH}_V$ we obtain the current $\partial\phi \wedge \gamma$ of bi-degree $(1,p)$ acting on test-form $\psi^{n-1,k}$ by

$$\partial\phi \wedge \gamma(\psi^{n-1,k}) = \gamma(\partial\phi \wedge \psi^{n-1,k})$$

Let $\mathcal{J}(\gamma)$ be the ideal in $\mathcal{O}_X$ whose sections anihilate γ. From the above it is clear that the annihilating ideal of $\partial\phi \wedge \gamma$ is given by

$$(\mathcal{J}(\gamma); \partial\phi) = \{g : \ \mathrm{Jac}(\phi) \cdot g \in \mathcal{J}(\gamma)\}$$

where $\mathrm{Jac}(\phi)$ is the dieal in $\mathcal{O}_X$ generated by the first order derivatives of ϕ.

A special case. Let T be a hypersurface in X given as the zero set $\phi^{-1}(0)$ where ϕ has no multiple factors. Let $f_1, \ldots, f_{p-1}$ be holomprphic functions which form a complete intersection with respect to T, i.e. the analytic set

$$V = T \bigcap f_\nu^{-1}(0)$$

has codimension p. In CH_V we find the current γ defined by the Coleff-Herrera product of the principal residue currents given by $\phi.f_1, \ldots, f_{p-1}$. A wellknown result due to Passare in [xx] asserts that

$$\mathcal{J}(\gamma) = (\phi.f_1, \ldots, f_{p-1})$$

Next, we get the ideal from (x) which in general is strictly larger than $\mathcal{J}(\gamma)$. The question arises if this ideal can be described in some way. To begin with we recall that up to a constant the cureent of bi-degree $(1,1)$ defined by $\partial\phi \wedge \bar\partial(\phi^{-1})$ is equal to Lelong's integration current $\square_T$. Moreover, the f-functions can be restricted to T and one constructs the current

$$\rho = \square_T \bigwedge_{\nu=1}^{p-1} \bar\partial(f_\nu^{-1})$$

Up to a constant one has the equality

$$\rho = \partial\phi \wedge \gamma$$

Mathematics by Arne Beurling.

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Introduction.

We expose some results by Beurling which not only offer striking theorems but have the merit that the proofs are both highly original and instructive. We have not tried to include material from his deepest work which appear in joint articles with Lars Ahlfors, P. Deny and Paul Malliavin. For an account about his collected work we refer to [Beurling: Collected Works 1-2] whose foreword contains a very informative text written by Lars Ahlfors and Lennart Carleson. A feature in Beurling's work is the mixture of geometric constructions and "hard calculus". An example is the notion of extremal distances which he constructed already in 1929 and applied in many different situations, foremost to estimate harmonic measures.

Beurling's ph.d-students.

From 1937 until 1955 as professor at Uppsala University, Beurling advised to many students who later became eminent mathematicians, the most prominent was Lennart Carleson who presented his thesis in 1950 at the age of 22 years. In addition to students at Uppsala university, he inspired and gave crucial ideas to visiting mathematicans like Helson, Hewitt and Wermer. In 1955 he emigrated to United States where he held a position at the Advanced Study in Princeton until his decease in 1989. A topic which is not treated in these notes is devoted to inverse spectral analysis of differential equations. Göran Borg was one of Beurling's ph.d-students at Uppsala University. His dissertation was presented in May 1945 where many crucial ideas, apart from posing the problem had been given with great generosity by Beurling. Let us recall that inverse spectral theory after became a quite extensive subject. See chapter xx in Volume 2 from the text-book by Dunford and Schwartz for an account. Another example of Beurling's active role as advisor stems from his first ph.d student Karl Dagerholm who presented his thesis at Uppsala in 1938. In § xx we expose a very elegant theorem which goes as follows: One seeks a sequence of real numbers $x_1, x_2, \ldots$ such that the additive series

$$\sum_{p=1}^{\infty} \frac{x_p}{p}$$

converges and

$$\sum_{p \neq q}^{\infty} \frac{x_p}{p-q} = 0 \quad : \; q = 1, 2, \ldots$$

The result is that this system up to a multiple with a constant has a unique solution. Dagerholm's proof relies upon specific solutions to certain Wiener-Hopf equations which had been suggested to him by Beurling.

Early study years. Like Torsten Carleman, Beurling studied at Uppsala University, where Carleman entered in 1911 and Beurling in 1924. During their years as students, the professors at Uppsala were Erik Holmgren and Anders Wiman.

The major contents in Beurling's Ph.D-thesis was already carried out in 1929, but not presented until November 4 1933 with the title *Études sur un problème de majoration.* At that time Carleman had served during five years as director at Institute Mittag-Leffler. In the period 1933-1937 Beurling attended Carleman's lectures at the Mittag-Leffler institute and got considerable inspiration from this. One can mention Carleman's lectures about the generalised Fourier transform in 1935 which contained refined versions of Ikehara's theorem and some remarkable results about analytic extensions across linear boundaries. A few years later Beurling proved Tauberian theorems with remainder terms. This work was never published but submitted in his application for the chair at Uppsala University when Beurling replaced Homgren after his retirement in 1937. Crucial idea from this manuscript were used and extended by his former ph.d-student Sonja Lyttkens whose articles contain quite impressive Tauberian theorems with remaineder terms. Preserved letters between Carleman and Beurling show that they esteemed each others work. For example, in his thesis Beurling inserted the following comments upon Carleman's proof from 1933 of the Denjoy conjecture with a sharp bound $2n$ for entire functions of order n: *Plus récemmment M. Carleman en perfectionnant sa méthode initiale est arrivé au resultat* $\leq 2n$ *d'une manière fort élegante.* His own proof from 1929 has like that by Carleman, the merit that it can be adapted to other cases than asymptotic values and relies upon the following remarkable result from the thesis:

Theorem *Let* $f(z)$ *be a bounded analytic function in the half space* $\mathfrak{Re}\, z > 0$*. Set*

$$\mu(r,f) = \min_{-\pi/2<\theta<\pi/2} |f(re^{i\theta})|$$

Then f *converges uniformly to zero in every sector* $-\pi/2+\delta \leq \theta \leq \pi/2-\delta$ *under the condition that*

$$\lim_{r\to\infty} \mu(r,f) = 0$$

Contents.

Introduction.

Familarity with measure theory and basic results in functional analysis is assumed. Relevant background appears in my notes devoted to analytic function theory of one complex variable. The reader will recognize that most of the results in these notes are concerned with inequalities which in the present author's opinion is the core of mathematical analysis. To make these notes reasonably self-contained we have included an Appendix which reviews facts about Hardy spaces, and § A:2 treats convolution algebras on the real line where major contributions are due to Wiener and Beurling. Let us start with a comment about § 0 in the Special Section devoted to the heat equation. In the analytic case, diffusion equations are solved by the usual heat kernel. But this simple device does not apply for solutions which increase considerably faster that than e^{x^2}. So for parabolic equations where the initial value functions increase fast there arise subtle problems about uniqueness. This was put forward in work by Gevrey, Hadamard and Holmgren around 1900. A conclusive theorem was proved by Holmgren in 1924 which is essentially sharp for the 1-dimensional heat-equation. *So whether one likes it or not*, it is necessary to perform constructions which involve rather tedious verifications of inequalities to obtain non-analytic solutions of the heat-equation where Holmgren's uniqueness result applies. See § 0 for further details.

A first example. To illustrate the flavour of these notes we announce a result where the reader is supposed to pursue the details. Let $u(z)$ be a subharmonic function in the open unit disc D. Assume that it has continous boundary values on the unit circle T with the exception of the point 1 and $u(e^{i\theta}) \leq 0$ for each $0 < \theta < 2\pi$. In addition we assume that there exists some $0 < \alpha < 1$ such that

$$u(z) \leq |1-z|^{-\alpha}$$

for every $z \in D$. Then $u(z) \leq 0$ for all $z \in D$. To prove this we take a small $\epsilon > 0$ and put $D_\epsilon = D \cap \{|z-1| \geq \epsilon\}$. On the boundary ∂D_ϵ we define the function h^* to be zero on $T \cap \{|z-1| \geq \epsilon\}$ while h^* is $\epsilon^{-\alpha}$ on the circular arc $\mathcal{C}_\epsilon$ of ∂D_ϵ which stays in D. Then $h^* \geq u$ on ∂D_ϵ. So if h is its harmonic extension to D_ϵ one has $u \leq h$ in this domain. For each $z \in D_\epsilon$ the principle of harmonic majorisation entails that

$$u(z) \leq \epsilon^{-\alpha} \cdot \mathfrak{m}_{D_\epsilon}(z_0; \mathcal{C}_\epsilon)$$

Now one uses an upper bound for the harmonic measure. More precisely, with $r < 1$ kept fixed there exists a constant $C(r)$ such that

$$\mathfrak{m}_{D_\epsilon}(z_0; \mathcal{C}_\epsilon) \leq C(r) \cdot \epsilon$$

hold when $|z_0| \leq r$ and $\epsilon \leq (1-r)/2$. Since $\alpha < 1$, a passage to the limit gives $u(z) \leq 0$ in every disc $|z| \leq r$ which proves the claim above.

Some inequalities. To illustrate material in the subsequent sections we state some theorems by Beurling.

0.1 A min-max inequality for polynomials.

Let $P(z)$ be a polynomial of some degree $n \geq 2$ where $P(0) = 1$. To each pair $0 < a < b < 1$ we set

$$\omega_P(a,b) = \max_{a \leq r \leq b} \min_{\theta} |P(re^{i\theta})|$$

Thus, while the radius r varies between a and b we seek some r such that the minimum modulus of P on the circle $|z| = r$ is as large as possible. In his thesis Beurling found a lower bound hold for these ω-functions.

0.1.1.Theorem. *The following hold for all pairs $0 < a < b$ and each polynomial P as above with degree n:*

$$(*) \qquad \omega_P(a,b) \geq \Big(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\Big)^n$$

In § XX we expose the proof which relies upon a study of logarithmic potentials and remark that (*) becomes sharp when n increases.

0.2 The Riemann-Schwarz inequality.

Consider a Riemann surface X whose universal covering space is the open unit disc equipped with a hyperbolic metric σ. It means that the metric in a chart with local coordinate z is defined by $e^{u(z)} \cdot |dz|$ where $u(z)$ is a subharmonic function. If γ is an arbitrary rectifiable curve in X σ-length is denoted by $\ell(\gamma)$. To each pair of points p, q in X we denote by $\mathcal{C}(p,q)$ the family of curves in x with end-points at p and q.

0.2.1 Theorem. *For every pair γ_1, γ_2 in $\mathcal{C}(p,q)$ and each point $\xi \in \gamma_1$, there exist two other curves curves α, β with $\alpha \in \mathcal{C}(p,q)$, while the end-points of β are ξ and some point on γ_2 such that*

$$(*) \qquad \ell(\alpha)^2 + \ell(\beta)^2 \leq \frac{1}{2}\big(\ell(\gamma_1)^2 + \ell(\gamma_2)^2\big)$$

This implies the following: Let

$$d(p,q) = \min_{\gamma \in \mathcal{C}(p,q)} \ell(\gamma)$$

Then, if ℓ_1, ℓ_2 is pair in this family and $\epsilon > 0$ are such that such that

$$\ell(\gamma_\nu)^2 \leq d(p,q)^2 + \epsilon^2$$

It follows that the distance between the two courves is majorised by $\sqrt{2\epsilon}$. In § xx we give a proof due to Beurling. Using the uniformisation theorem for Riemann surfaces, Beurling reduced the proof to a local result where D is the open unit disc in which case the following crucial result holds:

0.2.2 Proposition. *To each $0 < \theta < \pi/2$ we denote by C_θ the unique circle which contains $e^{i\theta}$ and $e^{-i\theta}$ and intersects the unit circle at right angles at these points, and γ_θ is the circular subarc of C_θ contained in D. Then the following inequality holds for every subharmonic function u in D:*

$$\Big(\int_{-1}^{1} e^{u(x)}\,dx\Big)^2 + \Big(\int_{\gamma_\theta} e^{u(z)}\,|dz|\Big)^2 \leq \frac{1}{4}\cdot\Big(\int_0^{2\pi} e^{u(e^{i\phi})}\,d\phi\Big)^2$$

and equality holds only for special rhombic configurations as explained in § xx.

0.3 Duality on Hardy spaces.

Tbe appendix devoted to Hardy spaces contains more than is needed as background for the special sections. But it has been included because the materail fits with the spirit of these notes. Here is a theorem from this appendix due to Fefferrman and Stein. Denote by $H_0^1(T)$ the class of functions in the Hardy space on the unit circle whose analytic extensions to the disc D vanish at the origin. Let F be a function in $L^1(T)$ and denote by H_F its harmonic extension to D. For each $0 < r < 1$ we get a linear functional on $H_0^1(T)$ defined by

$$L_r(g) = \int_0^{2\pi} H_F(re^{i\theta})\cdot\mathfrak{Re}(g)(re^{i\theta})\,d\theta$$

These functionals depend on F and we set

$$L^*(F) = \max_{rg}\,|L_r(g)|$$

with the maximum taken over all pairs $0 < r < 1$ and $H_0^1(T)$-functions g for which

$$\int_0^{2\pi} |\mathfrak{Re}(g)(re^{i\theta})|\,d\theta = 1$$

Recall also that $L^1(T)$ constains the subspace BMO(T) of functions with a bounded mean oscillation in the sense of John and Nirenberg

0.3.1 Theorem. *There exists an absolute constant C such that*

$$|L_F^*| \leq C\cdot||F||_{\text{BMO}} \tag{1}$$

for all $F \in BMO(T)$, and conversely if $L^*(F) < \infty$ for some $F \in L^1(T)$, then F has bounded mean oscillation and

$$||F||_{\text{BMO}} \leq L^*(F) \tag{2}$$

The inequality (1) is proved in a rather straightforward way while (2) requires considerably more work based upon studies of Carleson measures.

A. Beurling's work in analytic function theory.

Riemann's mapping theorem. Before Theorem A.1 is announced announced we recall the connection between the conformal mappings and the equilibrium potential on plane curves. Let Γ be a closed Jordan curve of class C^1. One seeks a positive density function μ on Γ with total mass one such that the logarithmic potential

$$(*) \qquad \int_\Gamma \log \frac{1}{|z-\zeta|} \cdot \mu(\zeta) \cdot |d\zeta|$$

is constant as z varies in Γ, where $|d\zeta|$ is the arc-length measure. The existence of μ is expected since μ corresponds to an equilibrium density for an electric field. It turns out that μ is found via the conformal mapping function f from the exterior domain bordered by Γ onto the exterior of the unit disc. More precisely one has

$$(**) \qquad \mu(\zeta) = \frac{1}{|f'(\zeta)|}$$

A proof of this classic result appears in Chapter VI from my notes in analytic function theory. Beurling introduced a general set-up for conformal mappings as follows: Denote by $\mathcal{C}$ the family of analytic functions f in the open unit disc D such that f yields a conformal mapping onto some simply connected domain and $f'(0)$ is real and positive. For each positive and bounded continuous function Φ defined in the whole complex w-plane we get a subclass of $\mathcal{C}$ as follows:

Definition. *The family of all $f \in \mathcal{C}$ such that*

$$(*) \qquad \lim_{r\to 1} \max_{0\leq\theta\leq 2\pi} \big|\, |f'(re^{i\theta})| - \Phi(f(re^{i\theta}))\,\big| = 0$$

is denoted by $\mathcal{C}_\Phi$.

Thus, when $f \in \mathcal{C}_\Phi$ the difference of the absolute values of $|f'(z)|$ and $\Phi(f(z))$ tends uniformly to zero as $|z| \to 1$. Let M be the upper bound of Φ. The maximum principle applied to the complex derivative $f'(z)$ gives

$$|f'(z)| \leq M \quad : \quad z \in D$$

Hence $f(z)$ is a continuous function in the open disc D whose Lipschitz norm is uniformly bounded by M. This implies that f extends to a continuous function in the closed disc, i.e. f belongs to the disc algebra $A(D)$.

A.1 Theorem. *If $\log \frac{1}{\Phi(w)}$ is subharmonic, it follows that $\mathcal{C}_\Phi$ contains a unique function f^*.*

A.2. Invariant subspaces of $H^2(T)$. The Hilbert space $L^2(T)$ of square integrable functions on the unit circle T contains the closed subspace $H^2(T)$ whose elements are boundary values of analytic functions in D. Each $f \in H^2(T)$ is expanded as

$$\sum_{n=0}^{\infty} a_n \cdot e^{in\theta}$$

and Parseval's theorem gives the equality

$$\sum_{n=0}^{\infty} |a_n|^2 = \frac{1}{2\pi} \cdot \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$$

Moreover, in D we get the analytic function $f(z) = \sum a_n z^n$ whose radial limits

$$\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})$$

exist almost everywhere. In fact, this follows from the Brothers Riesz theorem established in 1916, and the inclusion $H^2(T) \subset H^1(T)$. Let us now consider some $f \in H^2(T)$ whose analytic extension $f(z)$ is zero-free in D. Then $\log f(z)$ exists as an analytic function in D and the Brothers Riesz theorem gives an absolutely continuous measure on T via radial boundary values of $\log |f(z|$. Put

$$(*) \qquad \mu = \log |f(e^{i\theta})| \quad : \; g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot d\mu(\theta)$$

So g_μ is the Herglotz extension of the Riesz measure μ. Following Beurling one says that f is of outer type if

$$f(z) = e^{g_\mu(z)}$$

A.2.1 Beurling's closure theorem. *Let $f \in H^2(T)$ be of outer type . Then the family $\{e^{in\theta} f(e^{i\theta}) : n = 0, 1, 2, \ldots\}$ generates a dense subspace of $H^2(T)$.*

We prove this in § xx where we also expose applications of this closure theorem due to Helson and Szegö.

A.3 Analytic extensions across a boundary. Consider the open square

$$\square = \{z = x + iy|, : \; -1 < x, y < 1\}$$

The x-interval divides $\square$ into the upper part $\square^* = \square \cap \{0 < y < 1\}$. and the lower part $\square_*$ where $-1 < y < 0$. Consider a pair $f \in \mathcal{O}(\square^*)$ and $g \in \mathcal{O}(\square_*)$. One seeks condition in order that they are analytic extensions of each other, i.e. when there exists $\phi \in \mathcal{O}(\square)$ whose restriction to $\square^*$ is f, and $\phi|\square_* = g$. The following sufficiency results was established by Carleman in lectures at Mittag Leffler in 1935:

A.3.1 Theorem. *The analytic extension exists if*

$$\lim_{\epsilon \to 0} \int_{-1}^{1} |f(x + i\epsilon) - g(x - i\epsilon)| \, dx = 0$$

Remark. Notice that from the start no growth conditions are imposed on the functions f or g as z approaches the real x-interval. The proof by Carleman uses the subharmonic property of the radius of convergence while one takes Taylor series of f and g in discs which stay in the half-squares. Inspired by this, Beurling established further results in the article [Acta: xxx]. Here is one example:

A.3.2 An extension of Runge's theorem. Consider the open set $\Omega = \square^* \cup \square_*$ and let $\omega(y)$ be a continuus function on the closed square $\overline{\square}$ which is > 0 for $y \neq 0$ while $\omega(0) = 0$. Moreover, $y \to \omega(y)$ decreases as y tends to zero through positive or negative values. Denote by $C_\omega(\square)$ the space of complex-valued continuous functions

$f(z)$ in Ω such that the product $\omega(y)f(z)$ extends to a continuous function on $\overline{\Box}$ which is zero on the real interval $[-1,1]$. It becomes a Banach space under the norm:

$$||f|| = \max_{x+iy\in\Box} \omega(y)|f(x+iy)|$$

Define

$$A_\omega(\Box) = \{f \in C_\omega(\Box) :\ f|\Box \in \mathcal{O}(\Box)\}$$

We have also the larger subspace $A^*_\omega(\Box)$ of functions f in $C_\omega(\Box)$ whose restrictions to Ω is analytic. It is clear that $A^*_\omega(\Box)$ is a closed subspace of $C_\omega(\Box)$ and hence the closure of $A_\omega(\Box)$ is contained in $A^*_\omega(\Box)$. It turns out that the equality depends upon the ω-function.

A.3.3 Theorem. *The equality* $\overline{A_\omega(\Box)} = A^*_\omega(\Box)$ *holds if and only if*

(*) $$\int_{-b}^{b} \log\log\frac{1}{\omega(y)}\,dy = -\infty$$

We remark that a crucial point in Beurling's proof employs a result by Carleman from 1920 which asserts that if $g(z)$ is an entire function of exponential type and in addition bounded on the real x-line, then

$$\int_1^\infty \log^+\frac{1}{|g(x)|}\,dx < \infty$$

A.4 Gabriel's problem and interpolation.

The interpolation problem below was posed and solved in the joint article [N-P] by F. Nevanlinna and Pick. To each n-tuple of distinct points $E = \{\alpha_1,\dots,\alpha_n\}$ in the open unit disc D and some n-tuple $w_1,\dots,w_n$ of complex numbers we put:

(*) $$\rho(E;w(\cdot)) = \min_{f\in\mathcal{O}(D)} |f|_D \quad : \quad f(\alpha_\nu) = w_\nu \quad : 1 \le \nu \le n$$

Thus we seek to interpolate preassigned values on E with an analytic function $f(z)$ whose maximum norm is minimal. Denote by $\mathfrak{B}_{n-1}$ the family of finite Blaschke products:

(**) $$f(z) = e^{i\theta}\cdot\prod_{\nu=1}^{\nu=n-1}\frac{z-z_\nu}{1-\bar{z}_\nu\cdot z}$$

where $0 \le \theta \le 2\pi$ and $(z_1,\dots,z_{n-1})$ is some $(n-1)$-tuple of points in D which are not necessarily distinct.

A.4.1. Theorem *For each pair E and $w(\cdot)$ as above there exists a unique function $f_* \in \mathfrak{B}_{n-1}$ such that $\rho(E;w(\cdot))\cdot f_*(z)$ solves the interpolation (*).*

The interpolation bound. For each finite set E in D with n points we set

$$\lambda(E) = \max_w \rho(E;w(\cdot))$$

with the maximum taken over all n-tuples w in D. In the article [1986] Beurling considered a mini-max problem which goes as follows: For each n-tuple $E = \{\alpha_\nu\}$ of distinct points in the unit disc we set

$$\tau(E) = \min_{\phi \in \mathcal{B}_{n-1}} \max_{\alpha \in E} |\phi(\alpha)|$$

A.4.2 Theorem. *Up to a multiple with $e^{i\theta}$ the mini-max problem has a unique solution $\phi_E(z)$. Moreover, one has the equality $\tau(E) = \lambda(E)^{-1}$ and*

$$|\phi_E(\alpha)| = \tau(E) \quad : \alpha \in E$$

Remark. For a given set E the zeros of ϕ_E are in general not distinct. Counted with eventual multiplicities the zero-set is denoted by E^* and called the associated set of E for the mini-max problem. To prove this theorem, Beurling considered the following Hilbert space problem: Given the set E we have the Blaschke product

$$(**) \qquad B_E(z) = \prod_{\nu=1}^{\nu=n} \frac{z - \alpha_\nu}{1 - \bar{\alpha}_\nu \cdot z}$$

In the unit disc we introduce the positive discrete measure

$$\mu = \sum_{\nu=1}^{\nu=n} \frac{1}{|B'_E(\alpha_\nu)|} \cdot \delta_{\alpha_\nu}$$

Now one has the extremal problem

$$\text{(i)} \qquad \max_F \int_D |F(z)|^2 \, d\mu(z)$$

with the maximum taken over analytic functions F for which

$$\text{(ii)} \qquad \int_0^{2\pi} |F(e^{i\theta})|^2 \, d\theta = 1$$

Dividing with Blaschke products which do not affect the L^2-norms in (ii), it follows that the extremal function F^* is zero-free and unique when we require that $F^*(0)$ is real and positive. Beurling proved that F^* satisfies the folloeing two eqautions

$$\int_D \frac{1 - |z|^2}{|e^{i\theta} - z -|^2} \cdot [F(z)|^2| \, d\mu(z) = ||\mu|| \cdot |F(e^{i\theta})|^2 \quad : 0 \leq \theta \leq 2\pi$$

$$\int_D \frac{F(z)}{1 - \bar{\zeta} \cdot z} \, d\mu(z) = ||\mu|| \cdot |F(\zeta)|^2 \quad : |\zeta| > 1/r$$

where $r = \{\max \{\alpha_\nu\}$.

From the above one has found two criteria for extremal solutions. As expressed by Beurling it is this *multiple information that creates its usefulness.* The results above were established in 1959, while the final part of the cited article contains results which Beurling carried out in 1985, where the major aim was to investigate the behaviour of infinite Blaschke products. So in [Beurling 1987] appears one of his last contributions to mathematics. The notion of Blascke products of the so called second kind appear in the cited article, and Beurling proved that their zero sets conicide with the family of discrete subsets of D for which the bounded interpolation by analytic functions is solvable. Let us recall that Carleson in a work from 1958 found necessary and sufficient conditions for discrete sets to have the interpolation property expressed via properties of th hyperboic distances between the points in the interpolating sequence. The analysis in [1987] gives a new perspective upon this result. Let us also remark that the final part of Beurling's cited article contains a quite exciting analysis where future research remains to be done.

A.5 A uniqueness theorem for Riemann's ζ-function.

Let $\mathcal{F}$ be the family of all non-decreasing increasing sequence of positive numbers $\lambda_1 \leq \lambda_2 \leq \ldots$ for which there exists some $\delta > 0$ such that

(i) $$\lambda_n \geq \delta \cdot n$$

hold for every n where δ can depend on the given sequence. A result due to Hadamard asserts that the product

(ii) $$f(z) = \prod (1 + \frac{z^2}{\lambda_n^2})$$

is an entire function in the class $\mathcal{E}$, i.e. there are constants C and A such that $|f(z)| \leq C \cdot e^{A|z|}$ for all z. Next, we construct the Dirichlet series

(iii) $$\Lambda(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

By (i) this gives an analytic function in the half-plane $\mathfrak{Re}\, s > 1$ and we recall that $\Lambda(s)$ is obtained from $f(z)$ by the formula

(iv) $$\Lambda(s) = s \cdot \sin \frac{\pi s}{2} \cdot \frac{1}{\pi} \cdot \int_0^{\infty} \log f(x) \cdot \frac{dx}{x^{s+1}}$$

From this one gets:

A.5.1 Theorem. *Every Λ -function obtained from a seqence in $\mathcal{F}$ extends to a meromorphic function in the complex s-plane.*

Example. Riemann proved that the associated f-function to the Dirchlet series defining $\zeta(s)$ is equal to

$$f(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$$

Thus, this choice of f gives $\zeta(s)$ in the left hand side from (iv).

An extremal property of $\zeta(s)$. In a lecture at Harvard University in 1949, Beurling proved that Riemann's zeta-function has a distinguished position in a class of functions defined by Dirichlet series. For each positive number k we denote by $\mathcal{C}_k$ the class of series $\Lambda(s)$ from (ii) with the properties:

(a) $$\Lambda(s) - \frac{1}{s-1} \quad \text{is entire}$$

(b) $$\Lambda(-2n) = 0 \quad \text{for all positive integers}$$

(c) $\max_{|s|=r} |\Lambda(s)| \leq C \cdot \frac{\Gamma(r)}{(2\pi k)^r}$ hold for a constant C and all $r \geq 2$

Notice that the class $\mathcal{C}_k$ becomes more restrictive as k increases. Using Riemann's original work one verifies that $\zeta(s)$ satisfies (a-b). By a more elaborate analysis Beurling proved that the zeta-function belongs to $\mathcal{D}_1$, i.e. (c) above hold with $k = 1$ for a certain constant C. Moreover one has

A.5.2 Theorem. *If $k > 1$ the class $\mathcal{C}_k$ is empty. Moreover, when $1/2 < k \leq 1$, then $\mathcal{C}_k$ only consists of constants times the zeta-function.*

Remark. Beurling's result illustrates the special role of the ζ-function. The proof requires several steps.

§ B Harmonic analysis

Beurling's work devoted to harmonic analysis is extensive. Here we are content to decribe a few of his results which will be presented with detailed proofs in the subsequent sections.

B.1 A result about weak limits. The theorem below was by Beurling in seminars at Uppsala university in 1942. On the real x-line we have the space of continuous functions $f(x)$ whose Fourier transforms are Riesz measures with a finite total variation, i.e.

$$f(x) = \frac{1}{2\pi} \int e^{ix\xi} d\gamma(\xi)$$

where γ is a complex-valued Riesz measure of finite total mass:

$$\int |d\gamma(\xi)| < \infty$$

It is clear that every such f is a bounded and even uniformly continuous function on the real x-line. Its closure taken in the Banach space $C^0(\mathbf{R})$ equipped with the maximum norm is denoted by $\mathcal{A}$. Next, a sequence $\{\mu_n\}$ of Riesz measures on the x-line converges weakly in Schauder's sense if there is a constant M such that the total variations $||\mu_n|| \leq M$, and the Fourier transforms converge pointwise to zero:

$$\lim_{n \to \infty} \int e^{ix\xi} \, d\mu(x) = 0$$

for each real number ξ. It is easily verified that when $f \in \mathcal{B}$ and $\{\mu_n\}$ converge to zero in Schauder's sense then

$$\text{(*)} \qquad \lim_{n \to \infty} \int f(x) \, d\mu_n(x) = 0$$

Beurling proved that the converse is true, i.e. if $f(x)$ is a uniformly continuous and bounded function for which (*) holds whenever $\mu_n \to 0$ in the sense of Schauder, then $f \in \mathcal{B}$. Let us remark that in contrast to "soft results" in functional analysis derived via Baire's theorem, the proof to be exposed in § xx relies on a quite involved variational problems whose mere construction is not evident from the start.

An L^1-inequality for inverse Fourier transforms. Let $g(\xi)$ be a function defined on $\xi \geq 0$ where the inverse Fourier transform of $\xi g(\xi)$ is integrable, i.e. the function defined on the x-axis by

$$(*) \qquad f(x) = \int_0^\infty e^{ix\xi} \cdot \xi g(\xi)\, d|xi$$

belongs to $L^1(\mathbf{R})$. In §§ we show:

B.2 Theorem. *When (*) holds it follows that g is integrable and one has the inequality*

$$\int_0^\infty |g(\xi)|\, d\xi \leq \frac{1}{2} \int_{-\infty}^\infty |f(x)|\, dx$$

Functions with spectral gap. A fore-runner to distribution theory appears in a result from a seminar by Beurling at at Uppsala University in March 1942.

B.3 Theorem. *Let $f(x)$ be a bounded and continuous function on the real x-line such that $\widehat{f}(\xi)$ is zero on $\{-1 \leq \xi \leq 1\}$ and*

$$f(x+h) - f(x) \leq h$$

hold for all $h > 0$ and every x. Then its maximum norm is at most π.

§ C. Extremal distances and harmonic measure.

Beurling's thesis *Études sur un problème de majoration* form Uppsala in 1933 contains many proofs which are utmost intstructive and transparent. So [ibid] constitutes a veritable classic in analytic function theory. Here follow some results from [ibid].

C.1 The Milloux problem.

Here one studies the minimum modulus of analytic functions and tries to derive upper bounds for the maximum modulus. The following result is proved in [ibid: p. xxx]:

C.1.1 Theorem. *Let f be an analytic function $f(z)$ in the unit disc where $|f(z)| < 1$ for all z and suppose there exist some $\delta > 0$ such that*

$$\min_\theta |f(re^{i\theta})| \leq \delta \quad \forall\ 0 \leq r < 1$$

Then it follows that

$$(*) \qquad \max_\theta |f(re^{i\theta})| \leq \delta^{\frac{2}{\pi} \cdot \arcsin \frac{1-r}{1+r}} \quad : \ 0 < r < 1$$

Remark. Prior to this, M.E Schmidt had proved (*) under the constraint that there exists a Jordan arc from the origin to the unit circle along which $|f|$ is $\leq \delta$ in which case one profits upon a suitable conformal mapping which in a simllar context had been used by Koebe. To prove (*) in the general case Beurling introduced totally new ideas.

C.2 Minimum modulus of analytic functions.

When $f(z)$ is analytic in a disc $\{|z| < R\}$ we set

$$m_f(r) = \max_{|z|=r} |f(z)| \quad : \ \mu_f(r) = \min_{|z|=r} |f(z) \quad : 0 < r < R$$

The following is proved in [ibid: page 94]:

C.2.1 Theorem. *Let $0 < r_1 < r_2 < R$ and $\alpha > 0$ satisfies $0 < \alpha \leq m_f(r_1)$. Then*

$$(**) \qquad \int_{r_1}^{r_2} \chi(\{\mu_f \leq \alpha\} \cdot \log r \, dr \leq \log 4 + 2 \int_{r_1}^{r_2} \log \log \frac{m_f(r)}{\alpha} \, dr$$

where $\chi(\{\mu_f \leq \alpha\})$ is the characteristic function of the set where $\mu_f(r) \leq \alpha$.

Remark. In the article [xxx] from 1933 Nevanlinna found another proof of (*). Using Nevanlinna's constructions , Beurling demonstrated that the inequality (**) gets sharp when α tends to zero which is the interesting part of (**) above.

C.3 Upper bounds for harmonic measures

Let D be a simply connected domain and $\gamma \subset \partial D$ a subarc. If $z \in D$ we denote by $r(z; \partial D)$, resp. $r(z, \gamma)$ the euclidian distance from z to the boundary and to γ

respectively. With these notations Theorem III : page 55 in [ibid] asserts that

$$(***) \qquad \mathfrak{m}(D,\gamma;z) \leq \frac{4}{\pi}\cdot \arctan \sqrt{\frac{r(z,\partial D)}{r(z,\gamma)}}$$

where the left hand side is the harmonic measure at z with respect to γ. Examples show that this upper bound is sharp.

C.4 Regular points for the Dirichlet problem.

Sections in [ibid:page 63-69] are devoted to Dirichlet's problem. Let Ω be an open subset of $\mathbf{C}$. If f is a real-valued and continuous function on $\partial\Omega$ it can be extended to some $F \in C^0(\bar{\partial}\Omega)$. Now Ω can be exhausted by an increasing sequence of subdomains $\{\Omega_n\}$ for which the Dirichlet problem is solvable. Such exhaustions were considered by Lebesgue and explicit constructions were given by de Vallé Poussin in 1910. To each n we find the harmonic function H_n in Ω_n which solves the Dirichlet problem with boundary values given by the restriction of F to $\partial\Omega_n$. Norbert Wiener proved that the sequence $\{H_n\}$ converges to a unique harmonic function W_f in Ω which is independent of the chosen exhaustion. One refers to W_f as Wiener's generalised solution. A boundary point z_0 is called regular if

$$\lim_{z\to z_0} W_f(z) = f(z_0) \quad : \forall\, f \in C^0(\partial\Omega)$$

A criterion for a boundary point to be regular was established by Boulignad in 1923. Namely, a boundary point z_0 is regular if and only if there exists a positive harmonic function V in Ω such that

$$\lim_{z\to z_0} V(z) = 0$$

This condition is rather implicit so one seeks geometric properties to decide if a boundary point is regular or not. In [ibid] Beurling established a sufficient regularity condition which goes as follows: Let $z_0 \in \partial\Omega$ and for a given $R > 0$ we consider the circular projection of the closed complement of Ω onto the real interval $0 < r < R$ defined by:

$$E_\Omega(0,R) = \{0 < r < R :\ \exists\, z \in \{|z - z_0| = r\} \cap \mathbf{C} \setminus \Omega\}$$

Following Beurling one says that z_0 is logarithmically dense (Point frontière de condensation logarithmique) if the integral

$$(1) \qquad \int_{E_\Omega(0,R)} \log \frac{1}{r}\, dr = +\infty$$

Beurling proved that (1) entails that then z_0 is a regular boundary point. His proof relies upon an inequality which has independent interest. Here is the situation considered in [ibid: page 64-66]: Let Ω be an open set - not necessarily connected and $R > 0$ where Ω is general contains points of absolute value $\geq R$. Consider a harmonic function U in Ω with the following properties:

$$\text{(i)} \qquad \limsup_{z\to z_*} U(z) \leq 0 \quad : \ z_* \in \partial\Omega \cap \{|z| \leq R\}$$

$$\text{(ii)} \qquad U(z) \leq M \quad : \ z \in \{|z| = R\} \cap D$$

C.4.1 Theorem. *When (i-ii) hold one has the inequality below for every* $0 < r < R$

$$\max_z U(z) \leq 2M \cdot e^{-\frac{K}{2}}$$

where the maximum in the left hand side is taken over $\Omega \cap \{r < |z| < R\}$ *and*

$$K = \int_{E_\Omega(r,R)} \log r \, dr$$

D. Extremal length.

Let us first cite Beurling where he gives the following attribute to original work by Poincaré: *Rappelons que dans la théorie des fonctions analytiques, on a introduit des élements géometriques non euclidiennes, invariants par rapports a certaines transformation, et cela surtout pour simplifier la thérie dont il s'agit.* In his lecture at the Scandinavian Congress in Copenhagen 1946, Beurling describes in more detail the usefulness of extremal lengths.

In geometric function theory one often tries to characterize or determine a certain mapping or quantity by an extremal property. The method goes back to Riemann who introduced variational methods in function theory in the form of Dirichlet's principle. When you want to characterize a function by an extremal property , then the class of competing functions is very important. The wider you can make this class the more you can say about the extremal function and the easier it becomes to find good majorants or minorants, as the case may be.

In his thesis Beurling defined an extremal distance between pairs of points z_0, z_1 in simply conected domains Ω with a finite area. The construction goes as follows: First the interior distance is defined by

$$\rho(z_0, z_1; \Omega) = \inf_\gamma \int_\gamma |dz|$$

where the infimimum is taken over rectifiable Jordan arcs which stay in Ω and join the two points. Set

$$\lambda_*(z_0, z_1; \Omega) = \sqrt{\frac{\pi}{\text{Area}(\Omega)}}| \cdot \rho(z_0, z_1, \Omega)$$

One refers to λ_* as the reduced distance. It is not a conformal invariant and to overcome this default Beurling considered the family of *all triples* (Ω^*, z_0^*, z_1^*) which are conformally equivalent to the given triple, i.e. there exists a conformal mapping $f \colon \Omega \to \Omega^*$ such that $f(z_\nu) = z_\nu^*$. The *extremal distance* is defined by

$$\lambda(z_0, z_1; \Omega) = \sup \, \lambda_*(z_0^*, z_1^*; \Omega^*)$$

with the supremum taken over all equivalent triples. By this construction λ yields a conformal invariant. Next, consider a triple (z_0, z_1, Ω) is the Green's function $G(z_0, z_1; \Omega)$. Recall that G is a symmetric function of the pair z_0, z_1 and keeping z_1 fixed

$$z \mapsto G(z, z_1; \Omega) - \log \frac{1}{|z - z_1|}$$

is a harmonic function Ω which is zero on $\partial\Omega$. In [ibid: Théorème 1: page 29] the following fundamental result is proved:

D.1 Theorem. *For each triple* $(z_0, z_1; \Omega)$ *one has the equality*

$$e^{-2G} + e^{-\lambda^2} = 1 \tag{*}$$

About the proof. By conformal invariance it suffices to prove the equality for a triple $(0, a, D)$ where D is the unit disc where the pair of points is the origin and a real point $0 < a < 1$. In this case

$$G = \log \frac{1}{a}$$

Hence the theorem amounts to prove the equality

$$e^{-\lambda^2} = 1 - a^2$$

or equivalently that

$$\lambda^2 = \log \frac{1}{1 - a^2} \tag{*}$$

We will show (*) in § XX.

E. General extremal metrics.

Ten years later Beurling realised the need for a more extensive class of extremal distances which can be applied for domains which are not simply connected. The constructions below appear in the section *Extremal Distance and estimates for Harmonic Mesure* [Collected work. Vol 1. page 361-385].

The numbers $\lambda(E, K; \Omega)(z_0)$. Let Ω be a connected domain in the complex z-plane whose boundary is a finite union of closed Jordan curves $\{\Gamma_\nu\}$. Consider a pair of sets E, K where $E \subset \Gamma_\nu$ for one of the Jordan curves in $\partial\Omega$, while K is a compact subset of Ω. To a point $z_0 \in \Omega \setminus K$ we introduce the family $\mathcal{J}(E, K; z_0)$ of rectifiable Jordan arcs γ with the following properties: Apart from its end-points γ stays in $\Omega \setminus K$ and passes through z_0. Moreover, the end-points of γ divides the Jordan curve Γ_ν into a pair of closed intervals ω_1 where E is contained in one of these ω-intervals. Notice that neither K or E are assumed to be connected. Next, let $\mathcal{A}$ be the family of positive and continuous functions ρ in Ω for which the squared area integral

$$\iint_\Omega \rho^2(x, y)\, dxdy = 1$$

Put $\mathcal{J} = \mathcal{J}(E, K; z_0)$ and for each ρ-function above we set

$$L(\rho) = \inf_{\gamma \in \mathcal{J}} \int \rho(z)\, |dz| \tag{*}$$

The extremal distance between E and z_0 taken in $\Omega \setminus K$ is defined by

$$\lambda(E, K; \Omega)(z_0) = \sup_{\rho \in \mathcal{A}} L(\rho) \tag{**}$$

These λ-numbers are conformal invariants. More precisely, let (E^*, Ω^*, K^*) be another triple and $f \colon \Omega \setminus K \to \Omega^* \setminus K^*$ is a conformal mapping which extends continuously to the boundary and gives a homeomorphism between $\partial\Omega$ and $\partial\Omega^*$ where $E^* = f(E)$. Then one has the equality

$$\lambda(E, K; \Omega)(z_0) = \lambda(E^*, K^*; \Omega^*)(f(z_0))$$

Next, for each triple $(E; \Omega, K)$ as above and every point $z_0 \in \Omega \setminus K$ there exists the harmonic measure $\mathfrak{m}(E; \Omega, K)(z_0)$. With these notstions Beurling proved the following:

E.1 Theorem. *For every triple $E; \Omega, K)$ and each $z_0 \in \Omega \backslash K$ where $\lambda(E; \Omega, K)(z_0) \geq 2$ one has the inequality*

$$\mathfrak{m}(E; \Omega, K)(z_0) \leq 3\pi \cdot e^{-\pi \cdot \lambda(E; \Omega, K)(z_0)}$$

This result is often used to estimate harmonic measures. The point is that in the right hand side we can use ρ-functions which need not maximize the L-functional in (**) and since $L(\rho) \leq \lambda(E; \Omega, K)(z_0)$ one has in particular

$$\mathfrak{m}(E; \Omega, K)(z_0) \leq 3\pi \cdot e^{-\pi \cdot L(\rho)}$$

E.2 The simply connected case. Here Ω is a Jordan domain and $K = \emptyset$. Consider a pair of closed subsets E and F in $\partial\Omega$ which both are a finite unions of closed subintervals. Denote by $\mathcal{J}(E, F)$ the family of Jordan arcs γ with one end-point in E and the other in F while the interior stays in Ω. Set

$$\lambda(E; F : \Omega) = \max_{\rho} \min_{\gamma \in \mathcal{J}} \int_{\gamma} \rho \cdot d|z| \quad : \rho \in \mathcal{A}$$

This yields a conformal invariant which can be determined under the extra condition that the sets E and F are *separated* which means that there exists a pair of points p, q in $\partial\Omega \setminus E \cup F$ which divide $\partial\Omega$ in two intervals where E belongs to one and F to the other. Under this condition one has the equality

$$\lambda(E; F : \Omega) = \mathcal{NS}(E; F : \Omega) \tag{1}$$

Here $\mathcal{NS}(E; F : \Omega)$ is the Neumann-Schwarz number associated to the triple $(E, F; \Omega)$ which is found as follows: Let u be the harmonic function in Ω with boundary values $u = 1$ on E and zero on F while the normal derivative $\frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega \setminus E \cup F$. If v is the harmonic conjugate then v is increases on E and decreases on F and when v is normalised so that the range $v(E)$ is an interval $[0, h]$ it follows that $f = u + iv$ is a conformal mapping from Ω onto a rectangular slit domain:

$$\{0 < x < 1\} \times \{0 < y < h\} \cup S_\nu$$

where $\{S_\nu\}$ are horizontal intervals directed into the rectangle and with end-points on one of the vertical sides $\{x = 0\}$ or $\{y = 0\}$. See § XX for a figure. The Neumann-Schwarz number is given by

$$\mathcal{NS}(E, F; \Omega) = \frac{1}{\sqrt{h}}$$

In § xx we prove that it is a conformal invariant which reduces the proof of (1) to the case when Ω is a rectangle as above with some removed spikes while E and F are the opposed vertical lines. If ρ^* is the constant function $\frac{1}{\sqrt{h}}$ the fact that horisontal straight lines minimize arc-lengths when we move from E to F imply that

$$L(\rho^*) = \frac{1}{\sqrt{h}} \tag{2}$$

On the other hand, let $\rho \in \mathcal{A}$ be non-constant. Now

$$0 < \iint_\Omega (\rho - \frac{1}{\sqrt{h}})^2 dxdy = 2 - \frac{2}{\sqrt{h}} \iint_\Omega \rho \, dxdy \implies \iint_\Omega \rho \, dxdy < \sqrt{h}$$

It follows that

$$\min_{0 \le y \le h} \int_0^1 \rho(x,y)\, dx < \frac{1}{\sqrt{h}}$$

In the left hand side appear competing γ-curves and hence

$$L(\rho) < \frac{1}{\sqrt{h}}$$

This proves that ρ^* maximizes the L-function and the equality (2) entails (1).

E.3 Another inequality. Let Ω be a bounded and connected domain whose boundary consists of a finite family of disjoint closed Jordan curves $\{\Gamma_\nu\}$. Consider a pair (z_0, γ) where $z_0 \in \Omega$ and γ is a Jordan arc starting at z_0 and with an end-point p on one boundary curve, say Γ_1. Next, let K be a compact subset of $\Gamma_1 \setminus \{p\}$. For each point $z \in \gamma$ we get the harmonic measure $\mathfrak{m}_K(z)$ which is the value at z of the harmonic function in Ω whose boundary values are one on K and zero in $\partial\Omega \setminus K$. Set

$$h^*(K, \gamma; z_0) = \max_{z \in \gamma} \mathfrak{m}_K(z)$$

The interpretation is that one seeks a point $z \in \gamma$ where the probability for the Browninan motion which starts at z and escapes at some point in K before it has reached points in $\partial\Omega \setminus K$, is as large as possible. An upper bound of h^* is found using the notion of extremal length. Namely, let $\lambda(K; \gamma)$ be the extremal length for the family of curves in Ω with one end-point in K and the other on γ. Then Beurling proved that

$$h^*(K, \gamma; z_0) \le 5 \cdot e^{-\pi \cdot \lambda(K;\gamma)} \tag{*}$$

Remark. In random walks under a Brownian motion the inequality above shows that if K is a large obstacle and γ small, then the h^*-function can be majorised if some lower bound for the extremal length can be proved. This is of interest in configurations with "narrow channels" when one seeks paths out to the boundary and illustrates a typical application of extremal lengths.

F. On quasianalytic series.

Let us finish with a result which illustrates Beurling's vigour in "hard analysis". The theorem below was presented in a lecture at the Scandinvian congress in Helsinki 1938. A closed subset E of $\mathbf{C}$ has positive perimeter if there exists a positive constant c such that

$$\sum \ell(\gamma_\nu) \ge c$$

hold for every denumerable famly of closed and rectifiable Jordan curves $\{\gamma_\nu\}$ which surround E, i.e. E is contained in the union of the associated open Jordan domains. Let $\{\alpha_k\}$ be a sequences of distinct complex numbers and $\{A_k\}$ another sequence sequence such that

(i) $$\sum |A_k| < \infty$$

Set

(*) $$F(z) = \sum \frac{A_k}{z - \alpha_k}$$

To each k we put $r_k = |A_1| + \ldots |A_k|$ and define $\mathcal{E}_f$ by

$$\mathcal{E}_f = \{z \in \mathbf{C} \quad : \sum_{k=2}^{\infty} \frac{A_k}{\sqrt{r_{k-1}}} \cdot \frac{1}{|z - \alpha_k|} < \infty\}$$

It is easily seen that (i) entails that the complement of $\mathcal{E}_f$ is a null set in $\mathbf{C}$. One says that (*) is quasi-analytic if $F(z)$ cannot vanish on a closed subset E of $\mathcal{E}_F$ with positive perimeter. The following was proved in [ibid];

F.1 Theorem. *The series (*) is quasi-analytic in the sense above if*

$$\liminf_{n \to \infty} r_n^{\frac{1}{n}} < 1$$

In § XX we expose some steps in the proof based upon some remarkable inequalities for rational functions.

§ 0. Holmgren's uniqueness theorem and the heat equation.

Given some $A > 0$ we put $\Omega = \{-\infty < x < +\infty\} \times \{0 < y < A\}$, and consider a solution $u(x, y)$ to the heat equation

$$\text{(i)} \qquad \partial_x^2(u) = \partial_y(u)$$

In addition we assume that u extends to $\{y = 0\}$ and the question arises if the function $x \mapsto u(x, 0)$ determines the solution in Ω. If u satisfies the growth condition

$$\text{(ii)} \qquad |u(x, y) \leq e^{kx^2}$$

for some constant k, then it is determined by its values on $\{y = 0\}$ and found via the usual heat kernel applied to $u_0(x) = u(x, 0)$. Without any imposed growth condition uniqeuness can fail. Such examples were known by Hadamard at an early stage around 1900. In the article *Sur les solutions quasianalytiques de léquation de la chaleur*[Arkiv för matematik och fysik, Vol 18: 1926], Holmgren proved that the uniqueness holds if there exists a constant k such that

$$(*) \qquad |u(x, y) \leq e^{kx^2 \cdot \log(|x|+1)}$$

That this result is essentially sharp was demonstrated in work by Täcklund. For example, if $\alpha > 1$ there exist solutions whixh vanish identically on a line $\{y = 0\}$ while

$$|u(x, y) \leq e^{kx^2 \cdot (\log(|x|+1))^\alpha}$$

About Holmgren's proof. To establish the uniquenss when (*) holds, the cited article starts with constructions if suitable Greens functions which are used to estimate higher order derivaties of u. More precisely, under the condition that $x \mapsto u(x, 0) = 0$ and (*) hold, it is prived in [ibid] that there exists a constant C such that

$$\frac{\partial^n u}{\partial y^n}(0, y)| \leq C^n \cdot (n \log(n + e))^n$$

hold for every $y > 0$ and every positive integer n, In addition the hypothesis that $x \mapsto u(x, 0) = 0$ and the heat equation entail that $y \mapsto u(0, y)$ is flat at $y = 0$, i.e.

$$|\frac{\partial^n u}{\partial y^n}(0, 0) = 0$$

hold for every n. At this stage Holmgren concludes that $y \mapsto u(0, y)$ is the zero function since (xx) means that it satisfies the Carleman-Denjoy inequality which means that we have a quasi-analytic function of y and henbce (xx) implies that $u(0, y) = 0$ for all y.

Solutions by integral kernels. The question arises if there exists a substitute for the heat kernel when the relaxed growth condition (*) holds. For a subclass of solutions satisfying (*) we shall construct such a kernel function. Denote by $\mathcal{L}$ the class of solutions u to the heat equations for which there exists a function $\rho(r)$ which tends to zero as $r \to +\infty$ and

$$(**) \qquad |u(x, y)| \leq e^{\rho(|x|)x^2 \cdot \log|x|}$$

holds in Ω.

Main theorem. *Each $u \in \mathcal{L}$ is determined by $u_0(x)$. Moreover there exists an explicit kernel function which gives an integral representation of $u(x,y)$ for $0 < y < A$.*

Following Beurling we shall contruct such kernel which cn be applied to solutions in the class $\mathcal{L}$.

1. Preliminary constructions. Set

$$\gamma(z) = (\log(z+e))^z \quad : \quad f(z) = \frac{\Gamma(z)}{\gamma(z+1/2)}$$

where $\Gamma(z)$ is the usual gamma-function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} \cdot t^{z-1}\, dt$$

Some properties of $\gamma(z)$. Let $z = 1/2 + iy$ and put $a = e + 1/2$. Now

$$\log \gamma(a+iy) = (a + iy \log\log(a+iy)$$

When $y > 0$ we have

$$\log(a+iy) = \log\,|a+iy| + i \arg(a+iy)$$

Hence the right hand side in (i) becomes

$$(a+iy)\big(\log\log\,|a+iy| + i\arg(1 + \frac{a+iy}{\log\,|a+iy|})\big)$$

Set

$$\rho(y) = \arg(1 + \frac{a+iy}{\log\,|a+iy|})$$

This entails that the real part in (i) becmoes

$$a \cdot \log\log\,|a+iy| - y \cdot \rho(y)$$

Hence

$$|\gamma(a+iy)| = (\log\,|a+iy|)^a \cdot e^{-y\rho(y)}$$

Notice that

$$\lim_{y \to +\infty} \rho(y) = \frac{\pi}{2}$$

So

$$\frac{1}{\gamma(1/2+iy)|} \simeq \frac{1}{\log\,|a+iy|)^a} \cdot e^{\pi y/2}$$

when y is positive and large. At the same time the functional equation for the Γ-function gives

$$|\Gamma(1/2+iy)|^2 = \frac{\pi}{\sin(\pi(1/2+iy)}$$

This entails that

$$|\Gamma(1/2+iy)| \simeq 2\pi \cdot e^{-\pi y/2}$$

when $y >> 0$. So the factor $e^{piy/2}$ which appears in (xx) is compensated by the negative exponential for the Γ-function. We can improve this because $\rho(y)$ stys below $\pi/2$ for large y. In fact, when $y >> 0$ we notice that

$$\rho(y) \simeq \arctan \frac{y}{\log y} = \pi/2 - \int_{\frac{y}{\log y}} \frac{dt}{1+t^2} \simeq \pi/2 - \frac{\log y}{y}$$

Hence

$$e^{y\rho(y)} \simeq e^{\pi y/2} \cdot y^{-1}$$

From the above the reader can check that there exists a constant C such that

$$\frac{|\Gamma(1/2+iy)|}{|\gamma(a+iy)|} \leq C \cdot \frac{1}{y \cdot \log\, |a+iy|)^a} \quad : \ y \geq 1$$

Here $a = e + 1/2 > 1$ which entials that with f as above one has an absolutely convergent integral

$$\int_0^\infty |f(1/2+iy)|\, dy$$

In the same way the reader can check that we get an absolutely convergent integral when $-\infty < y \leq 0$.

1.3 The function $W(z)$. To begin with one applies Stirling's formula and find that if $-\pi/2 < \theta < \pi/2$ then

(1.3.1) $$\log\, f(re^{i\theta})| = \quad \text{Check (23) on page 422}$$

Using this and estimates as in (xx) the reader should verify that there exists an analytic function in the half-plane $\mathfrak{Re}\, z > 0$ defined by

(1.3.2) $$W(z) = \frac{1}{2\pi i} \cdot \int_{\mathfrak{Re}\zeta = a} f(\zeta)\, z^{-\zeta} \quad : a > 0$$

where the integral in (1.3.2) is independent of a.

1.4 Exercise. Show that there exists positive numbers $\alpha > 0$ and $x_0 > 0$ such that

(1.4.1) $$x > x_0 \implies W(x) \leq e^{-\alpha x \log x}$$

The hint is to change the contour in (1.3.2) with respoect to a in a suitable way as z is real and positive.

2. The function $K(t)$. It is defined for real $t > 0$ by

(2.1) $$K(t) = \frac{1}{2\pi i} \int_{\mathfrak{Re}\, z = a} t^{-z} \gamma(z)\, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(a+iy) \cdot t^{-a-iy}\, dy$$

where $a > 0$ and the reader should verify that (2.1) is independent of a.

2.2 Exercise. Apply the Fourier-Mellin inversion formula to show that

(2.2.1) $$\gamma(z) = \int_0^\infty t^{z-1}\, K(t)\, dt$$

and shoiw via rules for convolutions under the Fourier transform that

$$\int_0^\infty W(\frac{x}{t})\cdot\frac{K(t)}{\sqrt{t}}\,dt = e^{-x} \tag{2.2.2}$$

(See page 422 for details of the proof.)

2.3 Exercise. Consider the ordinary heat kernel

$$U(x,y) = e^{-x^2/y} \quad : y > 0$$

and show the equality

$$U(x,y) = \int_0^\infty W(\frac{x^2}{4yt})\cdot\frac{K(t)}{\sqrt{t}}\,dt \tag{2.3.1}$$

3. The function $v(x,y)$. When $y > 0$ we set

$$V(x,y) = \frac{W(\frac{x^2}{4y})}{\sqrt{y}}$$

The previous estimates show that if $u(x,y)$ belongs to $\mathcal{L}$ then the there exists a function $v(x,y)$ defined for $y > 0$ by the convolution integral

$$v(x,y) = \frac{1}{2\sqrt{\pi}}\cdot\int_{-\infty}^\infty V(x-\xi,y)\cdot u(\xi,0)\,d\xi \tag{3.1}$$

4. Holmgren's integral formula. With v determined as above we set

$$u^*(x,y) = \int_0^\infty v(x,yt)\cdot K(t)\,dt \tag{5-1}$$

where the previous estimates entail that the integral conveges absolutely for each $0 < y < A$.

4.1 Exercise. Show via dominated convergence under the integrals above that

$$\lim_{y\to 0} u^*(x,y) = u(x,0)$$

hold for every x. Moreover, use (2.3.1) to show that $u^*(x,y)$ satisfies the heat equation in Ω.

Finally, by Holmgren's cited uniquenss theorem, it follows that the given solution u in the class $\mathcal{L}$ is equal to u^* and hence (5.1) gives the requested integral formula in the main theorem.

§ 1. Radial limit of functions with finite Dirichlet integral

The article *Ensembles exceptionnels* [Acta math. 1940] is devoted to the study of functions $f(\theta)$ on the unit circle T whose harmonic extensions H_f to D have a finite Dirichlet integral. A real-valued functions $f(\theta)$ on the unit circle T has a Fourier series:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos n\theta + \sum_{n=1}^{\infty} b_n \cdot \sin n\theta$$

We say that f belongs to the class $\mathcal{D}$ if

$$\sum_{n=1}^{\infty} n(a_n^2 + b_n^2) < \infty \tag{*}$$

The sum in (*) is denoted by $D(f)$ and called the squared Dirichlet norm. Denote by $\mathcal{E}_f$ the set of all θ where the partial sums of the Fourier series of f does not converge.

0.1 Theorem. *For each $f \in \mathcal{D}$ the outer capacity of $\mathcal{E}_f$ is zero.*

Remark. Recall from XXX that if $E \subset T$ then its outer capacity is defined by

$$\text{Cap}^*(E) = \inf_{E \subset U} \text{Cap}(U)$$

with the infimum taken over open neighborhoods of E.

The proof of Theorem 0.1 relies upon Theorem 0.2 and 0.3 below. First, given some f in the class $\mathcal{D}$ with constant term $a_0 = 0$ we obtain the harmonic function in the open disc defined by

$$f(r,\theta) = \sum_{n=1}^{\infty} r^n(a_n \cdot \cos n\theta + b_n \cdot \sin n\theta)$$

The partial derivative with respect to r becomes:

$$f'_r(r,\theta) = \sum_{n=1}^{\infty} n \cdot r^{n-1}(a_n \cdot \cos n\theta + b_n \cdot \sin n\theta) \tag{1}$$

Define the function F in D by

$$F(r,\theta) = \int_0^r |f'_s(s,\theta)| \cdot ds \tag{2}$$

Thus, for each θ we integrate the absolute value of (1) along a ray from the origin. Now $r \mapsto F(r,\theta)$ is non-decreasing for every fixed θ. Hence there exists a limit

$$\lim_{r \to 1} F(r,\theta) = F^*(\theta) \tag{3}$$

whcih can be finite or $+\infty$. It is clear that if (3) is finite then there exists the radial limit

$$\lim_{r \to 1} f(r,\theta) = f^*(\theta) \tag{4}$$

Next, recall from the result in [Series] that when the radial limit (4) exists, then Fourier's partial sums converge to $f^*(\theta)$ which entails that the following inclusion holds for every $\rho > 0$:

$$\mathcal{E}_f \subset \{F^*(\theta) > \rho\} \tag{5}$$

We conclude that Theorem 0.1 follows if

$$\lim_{\rho\to+\infty} \text{Cap}\{F^* > \rho\} = 0 \tag{6}$$

Here (6) follows from the following more precise result:

0.2 Theorem. *Let $f \in \mathcal{D}$ where $a_0 = 0$ and $D(f) = 1$. Then*

$$\text{Cap}(\{F^* > \rho\}) \leq e^{-\rho^2} \quad \text{hold for every } \ \rho > 0$$

The essential step to get Theorem 0.2 relies upon the following inequality:

0.3 Theorem. *For each $f \in \mathcal{D}$ with $a_0 = 0$ one has $F^* \in \mathcal{D}$ and*

$$D(F^*) \leq D(f)$$

Theorem 0.3 is proved in § 1 and after we deduce deduce Theorem 0.2 in § 2. Before we proceed to § 1 we need some preliminary results.

0.4 On logarithmic potentials.

Let μ be a probability measure on T and put

$$U_\mu(z) = \int \log \frac{1}{|z-\zeta|} \cdot d\mu(\zeta)$$

This is a harmonic function in $\{|z| < 1\}$ and passing to its radial limits as $r \to 1$ the energy integral is defined by:

$$J(\mu) = \lim_{r\to 1} \int U_\mu(r,\theta) \cdot d\mu(\theta) = \int U_\mu(\theta) \cdot d\mu(\theta) \tag{*}$$

One says that μ has finite energy when (*) is finite. To check when this holds we use polar coordinates in D and the series expansion:

$$U_\mu(r,\theta) = \sum \frac{r^n}{n}(h_n \cos n\theta + k_n \sin n\theta)$$

where $\{h_n\}$ and $\{k_n\}$ are real numbers which will be determined in (2) below. Then $J(\mu)$ is the limit of the following expression as $r \to 1$:

$$\int U_\mu(r,\phi) \cdot d\mu(\phi) = \iint \log \frac{1}{|1 - re^{i(\phi-\theta)}|} d\mu(\phi) \cdot d\mu(\theta) \tag{1}$$

To compute the right hand side we expand the complex log-function:

$$\log \frac{1}{1 - re^{i(\phi-\theta)}} = \sum_{n=1}^{\infty} \frac{r^n}{n} \cdot e^{in(\phi-\theta)}$$

Taking real parts it follows that (1) is equal to

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cdot \cos n(\phi-\theta) \cdot d\mu(\phi) \cdot d\mu(\theta)$$

Now we use the trigonometric formula

$$\cos\, n(\phi-\theta)=\cos n\phi\cdot\cos n\theta+\sin n\phi\cdot\sin n\theta$$

Put

$$h_n=\int\cos\, n\theta\cdot d\mu(\theta)\quad\text{and}\quad k_n=\int\sin\, n\theta\cdot d\mu(\theta) \tag{2}$$

From the above it follows that

$$J(\mu)=\sum\frac{1}{n}(h_n^2+k_n^2) \tag{3}$$

Next, let $g(\theta)\in\mathcal{D}$ with Fourier coefficients $\{a_n\}$ and $\{b_n\}$ where $a_0=0$. Then we have

$$\int g\cdot d\mu=\sum a_n\cdot h_n+b_n\cdot k_n$$

and Cauchy-Schwarz inequality gives:

$$[\int g\cdot d\mu]^2\leq S(g)\cdot J(\mu) \tag{4}$$

From the above we obtain the following:

0.5 Theorem. *For each probability measure μ with finite energy and every function $g(\theta)\in\mathcal{D}$ which is lower semi-continuous one has the inequality*

$$\big[\int g(\theta)\cdot d\mu(\theta)\big]^2\leq S(g)\cdot J(\mu)$$

Remark. Above the lower semi-continuity is imposed in order to ensure that the Borel integral of g with respect to μ is defined.

1. Proof of Theorem 0.3

To begin with one has

1.1 Lemma. *The function F is subharmonic in D.*

For each fixed $0<\alpha<1$ we define the function ϕ_α in D by

$$\phi_\alpha(x,y)=\frac{\partial}{\partial\alpha}f(\alpha x,\alpha y)=x\cdot f'_x(\alpha x,\alpha y)+y\cdot f'_y(\alpha x,\alpha y)$$

Notice that the function $f_\alpha(x,y)=f(\alpha x,\alpha y)$ is harmonic and (1) means that

$$\phi_\alpha=(x\partial_x+y\partial_y)(f_\alpha)$$

where $\mathfrak{e}=x\partial_x+y\partial_y$ is the Euler field. As explained in XX this first order operator satisfies the identity

$$\Delta\circ\mathfrak{e}=\Delta+\mathfrak{e}\cdot\Delta$$

in the ring of differential operators and hence ϕ_α is harmonic. Next, the absolute value of a harmonic function is subharmonic so $\{|\phi_\alpha|\}$ yield subharmonic functions and a change of variables gives:

$$F = \int_0^1 |\phi_\alpha| \cdot d\alpha$$

This shows that F is a Riemann integral of subharmonic functions which in compact subsets of D is uniformly approximated by finite sums

$$\frac{1}{N}\sum_{k=1}^{k=N} |\phi_{k/N}|$$

Lemma 1.1 follows since a convex sum of subharmonic functions again is subharmonic.

An inequality. The function $F(r,\theta)$ is continuous and its derivative with respect to r exists and equals $|f'_r(r,\theta)|$. But the partial derivative $\partial F/\partial\theta$ may have jump discontinuities along rays where the derivative f'_r has a zero. However, this cannot occur too often so when $0 < r < 1$ is fixed there exists the integral

$$I(r) = \int_0^{2\pi} (\frac{\partial F}{\partial \theta}(r,\theta))^2 \cdot d\theta$$

We have proved that F is subharmonic and by its construction the partial derivative $\partial F/\partial r$ is non-negative. The result in Chapter V:B:xxx gives

1.2 Lemma. *The inequality below holds for each* $0 < r < 1$*:*

$$(*) \qquad I(r) \leq r^2 \cdot \int_0^{2\pi} \big(\frac{\partial F}{\partial r}(r,\theta)\big)^2 \cdot d\theta$$

1.3 Dirichlet integrals. Let $f \in \mathcal{D}$ with $a_0 = 0$ and construct the Dirichlet integral

$$\text{Dir}(f) = \frac{1}{\pi} \cdot \iiint_D [(f'_x)^2 + (f'_y)^2] \cdot dxdy$$

Then one has the equality:

$$(*) \qquad \text{Dir}(f) = D(f)$$

To see this we identify $f(r,\theta)$ with the real part of the analytic function

$$G(z) = \sum (a_n - i \cdot b_n) \cdot z^n$$

The Cauchy-Riemann equations give

$$\text{Dir}(f) = \frac{1}{\pi} \cdot \iiint_D |G'(z)|^2 \cdot dxdy$$

Now the reader can verify that the double integral above is equal to $D(f)$. Notice that (*) identifies $\mathcal{D}$ with the space of real-valued functions on T whose harmonic extensions to D have a finite Dirichlet integral.

1.4 Exercise. Show that the Dirichlet integral of a function g of class C^2 in D also is given by the double integral

$$\text{(i)} \qquad \frac{1}{\pi}\cdot\int_0^1\int_0^{2\pi}\left[r^2\cdot\left(\frac{\partial g}{\partial r}\right)^2+\frac{1}{r^2}\cdot\left(\frac{\partial g}{\partial\theta}\right)^2\right]\cdot r\cdot d\theta dr$$

Show also that if g is harmonic then

$$\text{(ii)} \qquad \text{Dir}(g)=\frac{2}{\pi}\cdot\int_0^1\int_0^{2\pi}\left(\frac{\partial g}{\partial r}\right)^2\cdot r\cdot d\theta dr$$

1.5 Final part of the proof of Theorem 0.3

Apply (i) in 1.4 with $g=F$ where the inequality in Lemma 1.2 and an integration with respect to r give

$$\text{(1)} \qquad \text{Dir}(F)\leq\frac{2}{\pi}\cdot\int_0^1\int_0^{2\pi}\left(\frac{\partial F}{\partial r}\right)^2\cdot r\cdot d\theta dr$$

Next, the construction of F gives the equality

$$\left(\frac{\partial F}{\partial r}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2$$

in the whole disc D. Then (1) and the equality (ii) applied to the harmonic function f give:

$$\text{(2)} \qquad \text{Dir}(F)\leq\text{Dir}(f)=D(f)$$

where the last equality used (*) in 1.3. Next, construct the harmonic extension of the boundary function $F^*(\theta)$ which we denote by H_F. Here we have the equations

$$\text{(3)} \qquad D(F^*)=D(H_F)$$

Next, recall that the Dirchlet integral is minimized when we take a harmonic extension which entails that

$$\text{(4)} \qquad \text{Dir}(H_F)\leq\text{Dir}(F)$$

Hence (2-4) give the requested inequality

$$D(F^*)\leq D(f)$$

2. Proof of Theorem 0.2

Let $\rho>0$ and apply Theorem 0.5 to the function $g=F^*$ and the equilibrium distribution μ assigned to the set $E=\{F^*>\rho\}$. This gives

$$\text{(1)} \qquad \rho^2\leq\left[\int F^*\cdot d\mu\right]^2\leq S(F^*)\cdot J(\mu)$$

Now $D(F^*)\leq D(f)=1$ holds by Theorem 0.3 and hence we have:

$$\text{(2)} \qquad \rho^2\leq J(\mu)$$

Next, recall from § XX that $J(\mu)$ is the the constant value $\gamma(E)$ of the potential function U_μ restricted to E. Hence (1) gives

$$\text{(3)} \qquad e^{-\gamma(E)}\leq e^{-\rho^2}$$

By definition the left hand side is the capacity of E which proves Theorem 0.2.

3. An application

Let Ω be a simply connected domain which contains the origin in the complex ζ-plane and $\partial\Omega$ contains a relatively open set given by an interval ℓ situated on the line $\mathfrak{Re}\,\zeta = \rho$ for some $\rho > 0$. Consider the harmonic measure $\mathfrak{m}_0^\Omega(\ell)$. In other words, the value at the origin of the harmonic function in Ω which is 1 on ℓ and zero on $\partial\Omega \setminus \ell$. We shall find an upper bound for (*) from the introduction in the family of simply connected domains which contain the origin and ℓ and have area π. To attain this we consider the conformal map ϕ from the unit disc onto Ω with $\phi(0) = 0$. The invariance of harmonic measures gives:

$$\mathfrak{m}_0^\Omega(\ell) = \mathfrak{m}_0^D(\alpha)$$

where α is the interval on T such that $\phi(\alpha) = \ell$. For an interval on the unit circle one has the equality

$$\text{Cap}(\alpha) = \sin\,\alpha/4$$

At the same time $\mathfrak{m}_0^D(\alpha) = \frac{\alpha}{2\pi}$ which entails that

$$\mathfrak{m}_0^\Omega(\ell) = \frac{2}{\pi}\,\arcsin\,\text{Cap}(\alpha) \tag{1}$$

There remains to estimate last term above. Put $u = \mathfrak{Re}\,\phi$. The inclusion $\ell \subset \mathfrak{Re}\zeta = \rho$ means that $u = \rho$ on ℓ. So when ϕ is considered in the class $\mathcal{S}$ we have the inclusion

$$\alpha \subset \{|\phi| > \rho - \epsilon\}$$

for each $\epsilon > 0$. Next, since the area of $\phi(D) = \pi$ we have $S(u) = 1$ and Theorem 0.2 gives

$$\text{Cap}(\alpha) \leq e^{-\rho^2}$$

Hence we have proved the general inequality

$$\mathfrak{m}_0^\Omega(\ell) \leq \frac{2}{\pi} \cdot \arcsin\,e^{-\rho^2} \tag{**}$$

Remark. There exists a special simply connected domain Ω for which equality holds in (**). See [Frostman: p. 39] : Potential theory.

4. Sets of uniqueness for analytic functions in the unit disc.

When E is a closed subset of the unit circle T we get the function $\phi_E(t)$ which to every $t > 0$ is the linear measure of the set of points on T whose distance to E is $\leq t$. Denote by $\mathcal{E}$ the family of closed subsets in T such that

$$\int_0^1 \frac{\phi_E(t)}{t}\,dt < \infty \tag{*}$$

An equivalent condtion is that

$$\sum \ell_\nu \cdot \log \ell_\nu < \infty \tag{**}$$

where $\{\ell_\nu\}$ are the length of the open intecvals in the complementary set $T \setminus E$. The proof of this equivalence is easy and left as an exercise. See also my notes in measure theory for a proof. Next, for each positive integer m we denote by $A^m(D)$ the family of m-times continuously differentiable functions on T which extend to analytic functions in the unit disc. The following result is due to Carleson and proved in the article article [Acta-1952].

4.1 Theorem. *If E belongs to $\mathcal{E}$. there exists for every positive integer m a function $f \in A^m(D)$ such that $f = 0$ on E while f is not identically zero.*

That condition (*) is crucial for the existence of smooth functions in the disc algebra which are zero on E can be seen from a uniqueness result due to Beurling which goes as follows:

4.2 Theorem. *Let E be a closed subset in T outside the family $\mathcal{E}$. Then each $f \in A(D)$ which is Hölder continuous of some order $\alpha > 0$ which is zero on E must vanish identically.*

Proofs of the results above are given in § XXX. Next we shall consider closed sets E with a positive logarithmic capacity which means that there exists a probability measure μ on E whose energy intergal

$$\iint \log \frac{1}{|e^{i\theta} - e^{i\phi}|} \, d\mu(\theta) \cdot d\mu(\phi) < \infty$$

Denote by $\mathcal{D}$ the class of analytic functions f in D with a finite Dirichlet integral, i.e.

$$\operatorname{Dir}(f) = \iint_D |f'(z)|^2 \, dxdy < \infty$$

In § XX from Special Tpoics we proved a result due to Beurling which asserts that if $f \in \mathcal{D}$ then the radial limits

$$\lim_{r \to 1} f(re^{i\theta}) \tag{1}$$

exists for all θ outside a set whose logartihmic capacity is zero.

4.3 The class $\mathcal{D}_E$. Let E be a closed subset of T with positive logarithmic capacity and belongd to $\mathcal{E}$. If $f \in \mathcal{D}$ Beurling's result in Theorem 0.1 gives a set $\mathcal{N}_f$ of outer logarthmic capactiy zero such that the radial limits exist in $T \setminus \mathcal{N}_f$. Now $\mathcal{D}_E$ is the family of functions $f \in \mathcal{D}$ for which the set of non-zero Beurling limits in $E \setminus \mathcal{N}_f$ has outer logarithmic capacaity zero. Notice that Theorem 4.1 implies that $\mathcal{D}_E$ contains functions which are not identically zero. Denote by $\mathcal{D}_E^*$ the functions f in $\mathcal{D}_E$ which are normalised so that $f(0) = 1$.

4.3.1 Theorem. *Let E be as above. Then the variational problem*

$$\min_{f \in \mathcal{D}_E^*} \operatorname{Dir}(f)$$

has a unique solution. Moreover, the extremal function f_E extends to a continuous function in $\bar{D} \setminus E$ without zeros and the complex derivative $f'(z)$ extends to an analytic function in $\mathbf{C} \setminus E$.

Remark. This is the major result in Carleson's article [ibid]. That the variational problem has a unique solution f_E whose Dirichlet intergal is > 0 is fairly easy to establish. The remaining parts of the proof involve several steps. First Theorem 0.1 is used to show that f_E extends continuously to $\bar{D} \setminus E$ and the boundary value function $f(e^{i\theta})$ is locally Lipschitz continuous on the open set $T \setminus E$. Using this regularity the next step is to show that f_E has no zeros on $T \setminus E$. To prove this one uses the extremal property which entails that if $\tau(z)$ is an arbitrary function in $A^2(D)$ which is zero at the origin, then

$$t \mapsto D(f \cdot e^{t\tau})$$

achieves its minimum when t varies over real numbers. The final step in proof is to show that the previous facts imply that the derivative f' extends to be analytic in $\mathbf{C} \setminus E$. We refer to § xx where material from [ibd] is exposed.

§ 2. An automorphism on product measures

Introduction. The results is expose material from the article [Beurling]. Before the measure theoretic study starts we insert comments from [Beurling] about the significance of the main theorem in 0.§§ below.

Schrödinger equations. The article *Théorie relativiste de l'electron et l'interprétation de la mécanique quantique* was published 1932. Here Schrödinger raised a new and unorthodox question concerning Brownian motions leading to new mathematical problems of considerable interest. More precisely, consider a Brownian motion which takes place in a bounded region Ω of some euclidian space $\mathbf{R}^d$ for some $d \geq 2$. At time $t = 0$ the densities of particles under observation is given by some non-negative function $f_0(x)$ defined on Ω. Classically the density at a later time $t > 0$ is equal to a function $x \mapsto u(x,t)$ where $u(x,t)$ solves the heat equation

$$\frac{\partial u}{\partial t} = \Delta(u)$$

with boundary conditions

$$(1) \qquad u(x,0) = f_0(x) \quad \text{and} \quad \frac{\partial u}{\partial \mathbf{n}}(x,t) = 0 \quad \text{when} \quad x \in \partial\Omega \quad \text{and} \quad t > 0$$

Schrödinger took into the account the reality of quantum physics which means that in an actual experiment the observed density of particles at a time $t_1 > 0$ does not coincide with $u(x,t_1)$. He posed the problem to find the most probable development during the time interval $[0, t_1)$ which leads to the state at time t_1. He concluded that the the requested density function which substitutes the heat-solution $u(x,t)$ should belong to a non-linear class of functions formed by products

$$(*) \qquad w(x,t) = u_0(x,t) \cdot u_1(x,t)$$

where u_0 is a solution to (1) while $u_1(x,t)$ is a solution to an adjoint equation

$$(2) \qquad \frac{\partial u_1}{\partial t} = -\Delta(u) \quad : \quad \frac{\partial u_1}{\partial \mathbf{n}}(x,t) = 0 \quad \text{on} \quad \partial\Omega$$

defined when $t < t_1$. This leads to a new type of Cauchy problems where one asks if there exists a w-function given by (*) satisfying

$$w(x,0) = f_0(x) \quad : \quad w(x,t_1) = f_1(x)$$

where f_0, f_1 are non-negative functions such that

$$\int_\Omega f_0 \cdot dx = \int_\Omega f_1 \cdot dx$$

The solvability of this non-linear boundary value problem was left open by Schrödinger and the search for solutions has remained as an active field in mathematical physics. When Ω is a bounded set and has a smooth boundary one can use the Poisson-Greens function for the classical equation (*) and rewrite Schrödinger's equation to a system of non-linear integral equations. The interested reader should consult the talk by I.N. Bernstein a the IMU-congress at Zürich 1932 for a first account about mathematical solutions to Schrödinger equations. Examples occur already on the product of two copies of the real line where Schrödinger's equations lead to certain non-linear equation for measures which goes as follows: Consider the Gaussian density function

$$g(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

Next, consider the family $\mathcal{S}_g^*$ of all non-negative product measures $\gamma_1 \times \gamma_2$ for which

$$\text{(i)} \qquad \iint g(x_1 - x_2) \cdot d\gamma_1(x_1) \cdot d\gamma_2(x_2) = 1$$

The product measure gives another product measure

$$\mathcal{T}_g(\gamma_1 \times \gamma_2) = \mu_1 \times \mu_2$$

where

$$\mu_1(E_1) \cdot \mu_2(E_2) = \iint_{E_1 \times E_2} g(x_1 - x_2) \cdot d\gamma_1(x_1) \cdot d\gamma_2(x_2)$$

hold for all pairs of bounded Borel sets. Notice that $\mu_1 \times \mu_2$ becomes a probability measure since (i) above holds. With these notations one has

0.1 Theorem. *For every product measure $\mu_1 \times \mu_2$ which in addition is a probability measure there exists a unique $\gamma_1 \times \gamma_2$ in S_g^* such that*

$$\mathcal{T}_g(\gamma_1 \times \gamma_2) = \mu_1 \times \mu_2$$

In [Beurling] a more general result is established where the g-function can be replaced by an arbitrary non-negative and bounded function $k(x_1, x_2)$ such that

$$\iint_{\mathbf{R}^2} \log k \cdot dx_1 dx_2 > -\infty$$

1. The $\mathcal{T}$-operator and product measures

Let $n \geq 2$ and consider an n-tuple of sample spaces $\{X_\nu = (\Omega_\nu, \mathcal{B}_\nu)\}$. We get the product space

$$Y = \prod X_\nu$$

whose sample space is the set-theoretic product $\prod \Omega_\nu$ and Boolean σ-algebra $\mathcal{B}$ generated by $\{\mathcal{B}_\nu\}$.

0.1 Product measures. Let $\{\gamma_\nu\}$ be an n-tuple of signed measures on $X_1, \ldots, X_n$ where each γ_ν has a finite total variation. There exists a unique measure γ^* on Y such that

$$\gamma^*(E_1 \times \ldots \times E_n) = \prod \gamma_\nu(E_\nu)$$

hold for every n-tuple of $\{\mathcal{B}_\nu\}$-measurable sets. We refer to γ^* as the product measure. It is uniquely determined because $\mathcal{B}$ is generated by product sets $E_1 \times \ldots \times E_n)$ with each $E_\nu \in \mathcal{B}_\nu$. When no confusion is possible we put

$$\gamma^* = \prod \gamma_\nu$$

The family of all such product measures is denoted by $\text{prod}(\mathcal{M}_B)$.

0.2 Remark. The set of product measures is a proper non-linear subset of the space $\mathcal{M}_B$ of all signed measures on Y. This is already seen when $n = 2$ with two discrete sample spaces, i.e. X_1 and X_2 consists of N points for some integer N. A Every $N \times n$-matrix with non-negative elements $\{a_{jk}\}$ give a probability measure μ on $X_1 \times X_2$ when the double sum $\sum\sum a_{jk} = 1$ The condition that μ is a product measure is tha there exist N-tuples $\{\alpha_j$ and $\{\beta_k\}$ such that $\sum \alpha_\nu = \sum \beta_k = 1$ and $a_{jk} = \alpha_j \cdot \beta_k$.

0.3 The space $\mathcal{A}$. We have the linear space of functions on Y whose elements are of the form

(i) $$a = g_1^* + \ldots + g_n^*$$

where $\{g_\nu\}$ are functions on the separate product factors $\{X_\nu\}$. It is clear that a pait of product measures γ and μ on Y are equal if and only if

$$\int_Y a \cdot d\gamma = \int_Y a \cdot d\mu$$

hold for every $a \in \mathcal{A}$.

0.4 The measure $e^a \cdot \gamma^*$ Let $a = \sum g_\nu^*$ be as above. Then we get the exponential function

$$e^a = \prod e^{g_\nu^*}$$

If $\gamma^* = \prod \gamma_\nu$ is some product measure we get a new product measure defined by

$$e^a \cdot \gamma_* = \prod e^{g_\nu} \cdot \gamma_\nu$$

0.5 The $\mathcal{T}$-operators. To every bounded $\mathcal{B}$-measurable function k we shall construct a map $\mathcal{T}_k$ from the space of product measures into itself. First, let $1 \leq \nu \leq n$ be given and g_ν is some $\mathcal{B}_\nu$-measurable function. Then there exists the function g_ν^* on the product space Y defined by

$$g_\nu^*(x_1, \ldots, x_n) = g_\nu(x_\nu)$$

Let us now consider a product measure γ. If $1 \leq \nu \leq n$ we find a unique measure on X_ν denoted by $(k \cdot \gamma)_\nu$ such that

$$\int_Y g_\nu^* \cdot k \cdot d\gamma = \int_{X_\nu} g_\nu \cdot d(k \cdot \gamma)_\nu$$

hold for every bounded $\mathcal{B}_\nu$-measurable function g_ν on X_ν. Now we get the product measure

(*) $$\mathcal{T}_k(\gamma) = \prod (k\gamma)_\nu$$

Remark. In the the case when

$$k(x_1, \ldots, x_n) = g_1^* \cdots g_n^*$$

we see that

$$\mathcal{T}_k(\gamma) = \prod g_\nu \cdot \gamma\nu$$

Exercise. Consider the case $n=2$ where X_1 and X_2 both consist of two points, say a_1, a_2 and b_1, b_2 respectively. A measure $\gamma \in S_1^*$ is given by $\gamma_1 \times \gamma_2$ and we can identify this product measure by a 2×2-matrix

$$XXX$$

where $\alpha_i \cdot \beta_\nu$ is the mass of γ at the point (a_i, b_ν). Next, let k be a positive function on the product space which means that we assign four positive numbers

$$k_{i,\nu} = k(a_i, b_\nu)$$

Find the measure $\mathcal{T}_k(\gamma)$ and express it as above by a 2×2-matrix.

Now we are prepared to announce the main result in this section. Consider a positive $\mathcal{B}$-measurable function k such that k and k^{-1} both are bounded functions. Denote by $\mathcal{S}_k^*$ the family of non-negative product measures γ on Y such that

$$\int_Y k \cdot d\gamma = 1$$

We have also the set $\mathcal{S}_1^*$ of product measures μ which are non-negative and have total mass one, i.e.

$$\int_Y d\mu = 1$$

It is easily seen that $\mathcal{T}_k$ yields an injective map from S_k^* into S_1^*. It turns out that the map also is surjective, i.e. the following hold:

Main Theorem. *$\mathcal{T}_k$ yields a homeomorphism between S_k^* and S_1^*.*

0.6 Remark. Above we refer to the norm topology on the space of measure, i.e. if γ_1 and γ_2 are two measures on Y then the norm $||\gamma_1 - \gamma_2||$ is the total variation of the signed measure $\gamma_1 - \gamma_2$. The reader may verify that S_k^* and S_1^* both appear as closed subsets in the normed space of all signed measures on Y. Recall also from XX that the space of measures on Y is complete under this norm. In particular, let $\{\mu_\nu\}$ be a Cauchy sequence with respect to the norm where each $\mu_\nu \in \mathcal{S}_1^*$. Then there exists a strong limit μ^* where μ^* again belongs to $\mathcal{S}_1^*$ and

$$||\mu_\nu - \mu^*|| \to 0$$

This completeness property will be used in the subsequent proof. We shall also need some inequalities which are announced below.

0.7 Some useful inequalities. Let γ_1 and γ_2 be a pair of product measures such that

$$|\int_Y g_\nu^* \cdot d\gamma_1 - \int_Y g_\nu^* \cdot d\gamma_2| \leq \epsilon \quad : \quad 1 \leq \nu \leq n$$

hold for some $\epsilon > 0$ and every function g_ν on X_ν with maximum norm ≤ 1. Then the norm

$$\text{(i)} \qquad ||\gamma_1 - \gamma_2|| \leq n \cdot \epsilon$$

The proof of (i) is left to the reader where the hint is to make repeated use of Fubini's theorem. More generally, let k be a bounded measurable function on Y and γ, μ is a pair of product measures. Denote by $\mathcal{A}_*$ the set of $\mathcal{A}$-functions a with maximum norm ≤ 1. Then there exists a constant C which only depends on k and n such that

$$(*) \qquad ||\mathcal{T}_k(\mu) - \gamma|| \leq \max_{a \in A_*} \Big| \int_Y a(kd\mu - d\gamma) \Big|$$

Again we leave the proof as an exercise.

0.8 A variational problem. Since we already have observed that $\mathcal{T}_k$ is injective there remains to prove surjectivity. For this we shall study a a variational problem which we begin to describe before the proof is finished in 0.§§ X below. We are given the function k on Y where both k and k^{-1} are bounded and the space $\mathcal{A}$ was defined in 0.3. For every pair $\gamma \in \mathcal{S}_1^*$ and $a \in \mathcal{A}$ we set

$$W(a, \gamma) = \int_Y (e^a k - a) \cdot d\gamma \quad \text{and} \quad W_*(\gamma) = \min_{a \in \mathcal{A}} W(a, \gamma)$$

0.9 Proposition. *Let $\{a_\nu\}$ be a sequence in $\mathcal{A}$ such that*

$$\lim W(a_\nu, \gamma) = W_*(\gamma)$$

Then the sequence $\{e^{a_\nu} \cdot \gamma\}$ converges to a measure $\mu \in S_1^*$ such that $\mathcal{T}_k(\gamma) = \mu$.

Before we enter the proof we insert a preliminary result which will be used later on.

0.10. Lemma. *Let $\epsilon > 0$ and $a \in \mathcal{A}$ be such that $W(a) \leq W_*(\gamma) + \epsilon$. Then it follows that*

$$\int e^a \cdot k \cdot \gamma \leq \frac{1+\epsilon}{1 - e^{-1}}$$

Proof. For every real number s the function $a - s$ again belongs to $\mathcal{A}$ and by the hypothesis $W(a - s) \geq W(a) - \epsilon$. This entails that

$$\int e^a k \cdot d\gamma \leq \int_Y e^{a-s} \cdot kd\gamma + s \int k \cdot d\gamma + \epsilon \implies$$

$$\int (1 - e^{-s}) \cdot e^a \cdot kd\gamma \leq s + \epsilon$$

Lemma 0.10 follows if we take $s = 1$.

Proof of Proposition 0.9 Keeping γ fixed we set $W(a) = W(a, \gamma)$. Let $0 < \epsilon < 1$ and consider a pair a, b in $\mathcal{A}$ such that $W(a)$ and $W(b)$ both are $\leq W_*(\gamma) + \epsilon$. Since $\frac{1}{2}(a + b)$ belongs to $\mathcal{A}$ we get

$$\text{(i)} \qquad 2 \cdot W(\frac{1}{2}(a+b)) \geq 2 \cdot W_*(\gamma) \geq W(a) + W(b) - 2\epsilon$$

Notice that

$$\text{(ii)} \qquad W(a) + W(b) - 2 \cdot W(\frac{1}{2}(a+b)) = \int_Y [e^a + e^b - 2 \cdot e^{\frac{1}{2}(a+b))}] \cdot kd\gamma$$

Next, we have the algebraic identity

$$e^a + e^b - 2 \cdot e^{\frac{1}{2}(a+b))}] = (e^{a/2} - e^{b/2})^2$$

It follows from (i-ii) that

$$\text{(iii)} \qquad \int_Y (e^{a/2} - e^{b/2})^2 \cdot k \cdot d\gamma \leq 2\epsilon$$

Next, the identity $|e^a - e^b)| = (e^{a/2} + e^{b/2}) \cdot |e^{a/2} - e^{b/2}|$ and the Cauchy-Schwarz inequality give:

$$\text{(iv)} \qquad \Big[\int_Y |e^a - e^b| \cdot k \cdot d\gamma\Big]^2 \leq 2\epsilon \cdot \int_Y (e^{a/2} + e^{b/2}) \cdot k \cdot d\gamma$$

By Lemma 0.6 the last factor is bounded by a fixed constant and hence (iv) gives a constant C such that

$$\text{(v)} \qquad \int_Y |e^a - e^b| \cdot k \cdot d\gamma \leq C \cdot \sqrt{\epsilon}$$

Next, let k_* be the minimum value taken by k on Y which by assumption is positive since k^{-1} is bounded. Replacing C by C/k_* where we get

$$\text{(vi)} \qquad \int_Y |e^a - e^b| \cdot d\gamma \leq C \cdot \sqrt{\epsilon}$$

Now (v) applies to pairs in the sequence $\{a_\nu\}$ and shows that $\{e^a \cdot d\gamma\}$ is a Cauchy sequence with respect to the norm of measures on Y. So from Remark 0.6 there exists a non-negative measure μ such that

$$\text{(vii)} \qquad \lim_{\nu \to \infty} ||e^{a_\nu} \cdot \gamma - \mu|| = 0$$

The equality $\mathcal{T}_k(\mu) = \gamma$. Consider the a-functions in the minimizing sequence. If $\rho \in \mathcal{A}$ is arbitrary we have

$$W(a_\nu + \rho) \geq W(a_\nu) - \epsilon_\nu$$

where $\epsilon_\nu \to 0$. This gives

$$(1) \qquad \int_Y [ke^{a_\nu}(1 - e^\rho) + \rho] \cdot d\gamma \leq \epsilon_\nu$$

When the maximum norm $|\rho|_Y \leq 1$ we can write

$$(2) \qquad e^\rho = 1 + \rho + \rho_1 \quad \text{where} \quad 0 \leq \rho_1 \leq \rho^2$$

Then we see that (1) gives

$$(3) \qquad \int_Y (\rho - ke^{a_\nu} \cdot \rho) \cdot d\gamma \leq \epsilon_\nu + \int \rho_1 \cdot \gamma \leq \epsilon + ||\rho||_Y^2$$

where the last inequality follows since γ is a probability measure and the inequality in (2) above. The same inequality holds with ρ replaced by $-\rho$ which entails that

$$\Big|\int_Y (ke^{a_\nu} - 1) \cdot \rho \cdot d\gamma\Big| \leq \epsilon_\nu + ||\rho||_Y^2$$

Notice that Lemma 0.10 entails that the sequence of functions $\{ke^{a_\nu}\}$ are uniformly bounded. Now we apply the inequality (*) from 0.7 while we use ρ-functions in $\mathcal{A}$

of norm $\leq \sqrt{\epsilon_\nu}$. It follows that there exists a constant C which is independent of ν such that the following inequality for the total variation:

$$||\mathcal{T}_k(e^{a_\nu} \cdot \gamma) - \gamma|| \leq C \cdot n \cdot \frac{1}{\sqrt{\epsilon}} \cdot (\epsilon_\nu + \epsilon_\nu) = 2 \cdot Cn \cdot \sqrt{\epsilon_\nu}$$

Passing to the limit it follows from (vii) that we have the equality

$$\mathcal{T}_k(\mu) = \gamma$$

Since $\gamma \in S_1^*$ was arbitrary we have proved that the $\mathcal{T}_k$ yields a surjective map from S_k^* to S_1^* which finishes the proof of the Main Theorem.

0.11 The singular case.

We restrict to the case $n = 2$ where $k(x_1, x_2)$ is a bounded and strictly positive continuous function on $Y = X_1 \times X_2$. Let $\gamma \in S_1^*$ satisfy:

(1) $$\int_Y \log k \cdot d\gamma > -\infty$$

Under this integrability condition the following hold:

2. Theorem. *There exists a unique non-negative product measure μ on Y such that $\mathcal{T}_k(\mu) = \gamma$.*

Remark. In general the measure μ need not have finite mass but the proof shows that k belongs to $L^1(\mu)$, i.e.

$$\int_Y k \cdot d\mu < \infty$$

As pointed out by Beurling Theorem 0.12 can be applied to the case $X_1 = X_2 = \mathbf{R}$ both are copies of the real line and

$$k(x_1, x_2) = g(x_1 - x_2)$$

where g is the density of a Gaussian distribution which after a normalisation of the variance is taken to be

$$\frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2}$$

So the integrability condition for μ becomes

$$\iint (x_1 - x_2)^2 \cdot d\mu(x_1, x_2) < \infty$$

A proof of Theorem 0.12 is given on page 218-220 in [loc.cit] and relies upon similar but technically more involved methods as in the Main Theorem. Concerning higher dimensional cases, i.e. singular versions of the Main Theorem when $n \geq 3$, Beurling gives the following comments at the end of [ibid] where the citation below has changed numbering of the theorems as compared to [ibid]:

The proof of the Main Theorem relies heavily on the condition that $k \geq a$ for some $a > 0$. If this lower bound condition is dropped the individual equation $\mathcal{K}(\gamma) = \mu$ may still be meaningful, but serious complications will arise concerning the global *uniqueness if* $n \geq 3$ *and the proof of Theorem 0.12 for the case* $n \geq 3$ *cannot be duplicated.*

§ 3. Beurlings criterion for the Riemann hypothesis.

Let $\rho(x)$ be the 1-periodic function on the positive real x-line where $\rho(x) = x$ if $0 < x < 1$. So if $\{x\}$ is the integral part of x then

$$\rho(x) = x - \{x\} \quad : \; x > 0$$

To each $0 < \theta < 1$ we get the function

$$\rho_\theta(x) = \rho(\theta/x)$$

whose restriction to $(0, 1)$ gives a non-negative function with jumps at points in the discrete set $\{x = \frac{\theta}{k} \colon\; k = 1, 2, \ldots\}$. Denote by $\mathcal{D}$ the linear space of functions on $(0, 1)$ of the form

$$f(x) = \sum c_\nu \cdot \rho_{\theta_\nu}(x)$$

where $0 < \theta_1 < \ldots < \theta_N < 1$ is a finite set and $\{c_\nu\}$ are complex numbers such that

$$\sum c_\nu \cdot \theta_\nu = 0$$

1.1 Theorem. *The Riemann hypothesis is valid if and only if the identity function belongs to the closure of $\mathcal{D}$ in $L^2(0, 1)$.*

The proof uses the following formula:

1.2 Proposition. *For each $0 < \theta < 1$ one has the equality*

$$(*) \qquad \int_0^1 \rho(\theta/x) x^{s-1} \cdot dx = \frac{\theta}{s-1} - \frac{\theta^s \cdot \zeta(s)}{s} \quad \text{when} \quad \mathfrak{Re}\, s > 1$$

Proof. The variable substitutions $x \to \theta/u$ identifies the left hand side with

$$\text{(i)} \qquad \theta^s \cdot \int_\theta^\infty \rho(u) \cdot u^{-s-1} \cdot du$$

To evaluate (i) we consider the integral

$$\text{(ii)} \qquad \int_1^\infty \rho(u) \cdot u^{-s-1}\, du = \sum_{n=1}^\infty \int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du$$

where the last equation used the periodicity of ρ. Integration by parts gives for each $n \geq 1$:

$$\int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du = -\frac{1}{s}(n+1)^{-s} + \frac{1}{s} \int_0^1 \frac{du}{(n+u)^s}$$

A summation over n implies that (ii) is equal to:

$$\text{(iii)} \qquad -\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s} \int_1^\infty u^{-s} \cdot du = -\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s(s-1)} = -\frac{\zeta(s)}{s} + \frac{1}{s-1}$$

Next we have

$$\text{(iv)} \qquad \theta^s \cdot \int_\theta^1 \rho(u) \cdot u^{-s-1}\, du = \frac{\theta}{s-1} - \frac{\theta^s}{s-1}$$

Together (i-iv) give (*).

1.3 The sufficiency part

Suppose that the identity function belongs to the L^2-closure of $\mathcal{D}$. So if $\epsilon > 0$ there exists $f \in \mathcal{D}$ expressed by (*) above such that the L^2-norm of $1 + f$ is $< \epsilon$. Since $\sum c_\nu \cdot \theta_\nu = 0$ Proposition 1.2 gives:

$$\text{(i)} \quad \int_0^1 (1 + f(x)) \cdot x^{s-1} \cdot dx = \frac{1}{s} + \sum c_\nu \int_0^1 \rho(\theta_\nu/x)\, dx = \frac{1}{s} - \sum c_\nu \cdot \theta_\nu^s \cdot \frac{\zeta(s)}{s}$$

With $s = \sigma + it$ and $\sigma > 1/2$ we have x^{s-1} in L^2 and the Cauchy-Schwarz inequality gives:

$$\text{(ii)} \qquad \Big| \int_0^1 (1 + f(x)) \cdot x^{s-1} \cdot dx \Big| \leq ||f||_2 \cdot \sqrt{\int_0^1 x^{2\sigma-2} \cdot dx} = ||f||_2 \cdot \frac{1}{\sqrt{2\sigma - 1}}$$

Hence we obtain

$$\Big|\frac{1}{s} - \frac{\zeta(s)}{s} \cdot \sum c_\nu \cdot \theta_\nu^s\Big| \leq \frac{\epsilon}{\sqrt{2\sigma - 1}} \quad : \quad \sigma > 1/2$$

If $\zeta(s_*) = 0$ for some $s_* = \sigma_* + it_*$ with $\sigma_* > 1/2$, the left hand side is reduced to $\frac{1}{|s_*|}$. Since we can find f as above for every $\epsilon > 0$ it would follow that

$$\frac{1}{|s_*|} \leq \epsilon \cdot \frac{\epsilon}{\sqrt{2\sigma_* - 1}} \quad \text{for every} \quad \epsilon > 0$$

But this is impossible so if 1 belongs to the L^2-closure of $\mathcal{D}$ then the Riemann-Hypothesis is true.

1.4 Proof of necessity.

We shall need a closure theorem while the proof of necessity is postponed until § 2. In general, let $f(x)$ be a complex-valued function in $L^2(0,1)$ which is not identically close to $x = 0$, i.e.

$$(*) \qquad \int_0^\delta |f(x)| \cdot dx > 0 \quad : \forall\, \delta > 0$$

For each $0 < a < 1$ we set

$$f_a(x) = f(ax)$$

We restrict each f_a to $(0,1)$ and let $\mathcal{C}_f$ be the linear space of functions on $(0,1)$ generated by $\{f_a : 0 < a < 1\}$. Thus, a function in $\mathcal{C}_f$ is expressed as a finite sum

$$\sum c_k \cdot f_{a_k}(x)$$

where $\{c_k\}$ are complex numbers and $0 < a_1 < \ldots < a_n < 1$ some finite tuple in $(0,1)$. For every $1 < p < 2$ we can consider $\mathcal{C}_f$ as a subspace of $L^p(0,1)$ whose closure taken in the L^p-norm is denoted by $\mathcal{C}_f(p)$.

The function $F(s)$. With f as above we set

$$(1.4.1) \qquad F(s) = \int_0^1 f(x) \cdot x^{s-1} \cdot ds$$

Then F is analytic in the half-plane $\mathfrak{Re}(s) > 1/2$ for if $\sigma = \mathfrak{Re}(s) > 1/2$ the Cauchy-Schwarz inequality gives

$$(1.4.2) \qquad |F(\sigma + it)| \leq \sqrt{\int_0^1 |f(x)|^2 \cdot dx} \cdot \sqrt{\int_0^1 |x|^{2\sigma-2} \cdot dx} = ||f||_2 \cdot \sqrt{\frac{1}{2\sigma - 1}}$$

1.4.3 Theorem. *If there exists some $1 < p < 2$ such that $\mathcal{C}_f(p)$ is a proper subspace of $L^p[0,1]$, then $F(s)$ extends to a meromorphic function in the whole complex s-plane whose poles are confined to the open half-plane $\mathfrak{Re}(s) < 1/2$. Moreover, for every pole λ the function $x^{-\lambda}$ belongs to $\mathcal{C}_f(p)$.*

Proof. Set $\frac{1}{q} = 1 - \frac{1}{p}$ and recall that $L^q(0,1)$ is the dual of $L^p(0,1)$. The assumption that $\mathcal{C}_f(p) \neq L^p(0,1)$ gives a non-zero $k(x) \in L^q(0,1)$ such that

$$(1) \qquad \int_0^1 k(x) f(ax) \cdot dx = 0 \quad : \quad 0 < a < 1$$

To the k-function we associate the transform

$$(2) \qquad K(s) = \int_0^1 k(x) \cdot x^{-s} \cdot dx$$

Hölder's inequality implies that $K(s)$ is analytic in the half-plane $\mathfrak{Re} s < \frac{1}{p}$. Define a function $g(\xi)$ for every real $\xi > 1$ by

$$g(\xi) = \int_0^1 k(x) \cdot f(\xi x) \cdot dx$$

Hölder's inequality gives

$$|g(\xi)| \leq \Big[\int |k(x)|^q \cdot dx\Big]^{\frac{1}{q}} \cdot \Big[\int_0^{1/\xi} |f(\xi x)|^p \cdot dx\Big]^{\frac{1}{p}}$$

If $\xi > 1$ a variable substitution shows that the last factor is equal to $||f||_p \cdot |\xi|^{-1/p}$. Hence

$$(3) \qquad |g(\xi) \leq ||k||_q \cdot ||f||_p \cdot \xi^{-1/p} \quad : \quad \xi > 1$$

Next, put

$$(4) \qquad G(s) = \int_1^\infty g(\xi) \cdot \xi^{s-1} \cdot d\xi$$

From (3) it follows that $G(s)$ is analytic in the half-space $\mathfrak{Re}\, s < 1/p$. Consider the strip domain:

$$\square = 1/2 < \mathfrak{Re}\, s < 1/p$$

Variable substitutions of double integrals show that the following holds in $\square$:

$$(5) \qquad G(s) = F(s) \cdot K(s)$$

where $F(s)$ is defined as in (1.1.1).

Conclusion. Since $G(s)$ and $K(s)$ are meromorphic in $\mathfrak{Re}\, s < 1/p$ and $F(s)$ from the start is holomorphic in $\mathfrak{Re}\, s > 1/2$, it follows that F extends to a meromorphic function in the whole s-plane. The inequality (1.1.2) shows that no poles appear during the meromorphic continuation across $\mathfrak{Re}\, s = 1/2$. Hence F either is an entire function or else it has a non-empty set of poles where each pole λ has real part $< 1/2$.

Existence of at least one pole. We shall prove that F has at least one pole. To achieve this we argue by contradiction and suppose that F is an entire function. Fix some real number $1/2 < \alpha < 1/p$. The construction of F shows that its restriction to the half-space $\mathfrak{Re}\, s \geq \alpha$ is bounded. Next, in the half-space $\mathfrak{Re}\, s \leq \alpha$ we have

$$F = \frac{G}{K}$$

where G and K both are bounded analytic functions. Moreover their constructions imply that

$$\lim_{\sigma \to -\infty} G(\sigma + it) = 0 \qquad \lim_{\sigma \to \infty} K(\sigma + it) = 0$$

A wellknown result due to F. and R. Nevanlinna gives some constant $M > 0$ and a real number c such that

$$F(\sigma + it)| \leq M \cdot e^{c(\sigma - \alpha)} \quad \text{holds when} \quad \sigma \leq \alpha \tag{6}$$

If $c \geq 0$ the entire function F is bounded and hence reduced to a constant. But then it is easily seen from (1.1.1) that f is identically zero. Next, if $c < 0$ we put $a = e^c$ so that $0 < a < 1$ and define the function:

$$F_1(s) = \int_0^1 f(ax)x^{s-1} \cdot ds = a^s \bigl(F(s) - \int_a^1 f(x)x^{s-1} \cdot ds\bigr) \tag{7}$$

From this expression and the hypotheis that F is entire it is easily seen that $F_1(s)$ is a bounded entire function and hence constant which entails that the function $x \mapsto f(ax)$ is identically zero on $(0,1)$. This means that f vanishes on the interval $[0, a)$ which was excluded from by (*) above. Hence F cannot be an entire function.

The case at pole. Suppose that F as a pole at some λ with real part $< 1/2$. Since G is analytic in $\mathfrak{Re}(s) < 1/2$ the equality (5) implies that λ is a zero of K and observe that the pole of F at λ is *independent* of the chosen L^q-function k which is $\perp$ to $\mathcal{C}_f(p)$. Hence the following implication holds:

$$k \perp \mathcal{C}_f(p) \implies K(\lambda) = \int_0^1 k(x)x^{-\lambda} \cdot dx = 0$$

The Hahn-Banach theorem entails that the L^p-function $x^{-\lambda}$ belongs to $\mathcal{C}_f(p)$ which proves the last claim in Theorem 1.2.

§ 2. The necessity in Theorem 1.1.

We shall use a family of linear operators $\{T_a\}$ defined as follows: If $0 < a < 1$ and $g(x)$ is a function on $(0,1)$ we set

$$T_a(g)(x) = g(x/a) \quad : \quad 0 < x < a$$

while $T_a(g) = 0$ when $x \geq a$.

Assume that the identity function 1 is outside the L^2-closure of $\mathcal{D}$. Hence its orthogonal complement in the Hilbert space $L^2(0,1)$ is $\neq 0$ and we find a non-zero $g \in L^2(0,1)$ such that

$$\int_0^1 f(x) \cdot g(x) \cdot dx = 0 \quad : \quad f \in \mathcal{D} \tag{2.1}$$

Since $\mathcal{D}$ is invariant under the T-operators it follows that if $0 < a < 1$ then we also have

$$0 = \int_0^a f(x/a) \cdot g(x) \cdot dx = a \cdot \int_0^1 f(x) \cdot g(ax) \cdot dx \tag{2.2}$$

Now we show that the g-function satisfies the condition (*) above Theorem 1.1. For suppose that $g = 0$ on some interval $(0,a)$ with $a > 0$. Choose b where

$$a < b < \min(1, 2a)$$

Now $\mathcal{D}$ contains the function $f(x) = b\rho(x/a) - a\rho(x/b)$. The reader may verify that $f(x) = 0$ for $x > b$ and is equal to a on the interval (a,b). Hence (2.1) applied to f gives

$$\int_a^b g(x) \cdot dx = 0$$

This means that the primitive function

$$G(x) = \int_0^x g(u) \cdot du$$

has a vanishing derivative on the interval (a,b). The derivative is also zero on $(0,a)$ where $g = 0$. We conclude that $G = 0$ on the interval $(0,b)$ so the L^2-function g is almost everywhere a constant on this interval which in addition is zero since $g = 0$ on $(0,a)$. Hence $g = 0$ on the interval $(0,b)$. Repating this with a replaced by b it follows that g also is zero on the interval

$$0 < x < \min(1, 2b) = \min(1, 4a)$$

After a finite number of steps $2^m a \geq 1$ and hence g would be identically zero on $(0,1)$ which is not the case.

Now Theorem 1.4.3 applies to g and gives some λ_* with $\mathfrak{Re}\, \lambda_* < 1/2$ such that one has the inclusion below for every $p < 2$:

$$x^{-\lambda_*} \in \mathcal{C}_g(p) \tag{2.3}$$

Next, for each $\theta > 0$ we there exists the $\mathcal{D}$-function

$$x \mapsto \rho(1/x) - \frac{1}{\theta} \cdot \rho(\theta/x)$$

Since (2.1) holds for all $0 < a < 1$ it follows that

$$\int_0^1 [\rho(1/x) - \frac{1}{\theta} \cdot \rho(\theta/x)] \cdot x^{-\lambda_*} \cdot dx = 0 \tag{2.4}$$

Put $s_* = 1 - \lambda_*$. The formula in Proposition 1.2 shows that the vanishing in (2.4) gives

$$\frac{\theta^{s_*} - 1}{s_*} \cdot \zeta(s_*) = 0$$

This hold for every $0 < \theta < 1$ and choosing θ so that $\theta^{s_*} - 1 \neq 0$ it follows that $\zeta(s_*) = 0$. Here $\mathfrak{Re}(s_*) = 1 - \mathfrak{Re}(\lambda_*) > 1/2$ and hence the Riemann hypothesis is violated which proves the necessity in Theorem 1.1.

§ 4. The Laplace operator and the Helmholtz equation

Introduction. Let D be a domain in the (x,y)-plane where one has a stationary irrational flow of an ideal fluid described a stream function $v(z)$ which is harmonic in D. The gradient vector $\nabla(v)$ is the velocity vector of the flow and assumed to be everywhere $\neq 0$. In general v is only defined in a subdomain D_* of D. and the part of ∂D_*. On each component of the full boundary ∂D_* we assume that v is constant on every connected component which means that these curves are streamlines of the flow. In 1867 Helmoltz posed the problem to find a pair (v, D_*) such that the length $|\nabla(v)|$ is constant on $\partial D_* \cap D$. More generally one can impose the condition that the function

$$p \mapsto |\nabla(v)(p)|$$

agrees with a given positive and continuous function Φ on $\partial D_* \cap D$. This leads to a free boundary value problem which goes as follows:

Let D^* be the exterior disc $\{|z| > 1\}$ in $\mathbf{C}$ and $\Phi(z)$ is a given continuous and positive function in D^*. Denote by $\mathcal{J}^*$ the family of closed Jordan curves γ in D^* which together with $\{|z| = 1\}$ borders an annulus. For every such γ there exists the unique harmonic function v in the annulus bordered by $\{|z| = 1\}$ and γ, where $v = 1$ on $\{|z| = 1\}$ and $v = 0$ on γ. One seek γ so that

$$(*) \qquad |\nabla(v)(p)| = \Phi(p) \quad : p \in \gamma$$

This problem is studied in a series of articles by Beurling. Here we expose results from *On free-boundary problems for the Laplace operator* [Princeton: Seminars analytic functions 1957]. To solve (*) Beurling proceeds as follows. Let us say that γ is of type $\mathcal{B}(Q)$ if

$$(1) \qquad \limsup_{z \to p} |\nabla(v)(z)| \leq \Phi(p) \quad : p \in \gamma$$

We consider also the class $\mathcal{A}(Q)$ where

$$(2) \qquad \liminf_{z \to p} |\nabla(v)(z)| \geq \Phi(p) \quad : p \in \gamma$$

If a curve γ belongs to the intersection $A(Q) \cap B(Q)$ then (*) holds. There remains to find conditions on Q in order that this intersection is non-empty and the uniqueness amounts to show that this intersection is reduced to a single curve in $\mathcal{J}^*$. Following [ibid] we shall establish some sufficiency results for existence as well as uniqueness. First we impose a growth condition on Q when one approaches the unit circle:

$$(3) \qquad \lim_{|z| \to 1} \left[\log \frac{1}{|z|}\right]^{-1} \cdot Q(z) = 0$$

Remark. As explained in § XX this condition is invariant under conformal mappings between our chosen annulus and others. From now on (3) is assumed. A first major result in [ibid] is:

Theorem. *If $\mathcal{B}(Q) \neq \emptyset$ then the free boundary value problem has at least one solution.*

Next, we seek conditions in order that $B(Q) \neq \emptyset$. To each $a < 1$ we denote by $B_a(Q)$ the family of curves γ such that

(i) $$\limsup_{z \to p} |\nabla(v)(z)| \leq a \cdot \Phi(p) \quad : p \in \gamma$$

Set

$$\mathcal{B}_*(Q) = \bigcup_{a<1} B_a(Q)$$

Theorem. *Suppose there exists a pair of curves $\gamma \in \mathcal{A}(Q)$ and $\gamma_0 \in \mathcal{B}(Q)$ such that γ stays in the annulus bordered by $\{|z| = 1\}$ and γ_0. Then (*) has a solution γ_* where γ_* is between γ and γ_0.*

Uniqueness.

For each rectifiable and closed Jordan curve γ in the exterior disc with winding number one we set

$$\ell_\gamma(Q) = \int_\gamma Q \cdot |ds_\gamma|$$

where ds_γ is the arc-length measure. Set

(1) $$\ell_*(Q) = \inf_\gamma \ell_\gamma(Q)$$

Definition. *The circle $\{|z| = 1\}$ is said to be convex with respect to Q if the infimum in (1) can be achieved by a sequence of Jordan curves γ which tend to the circle.*

Thus, convexity means that for each $\delta > 0$ when we take the infimum over γ curves which stay in the narrow annulus $1 < |z| < 1 + \delta$ for arbitrary small $\delta > 0$, then this infimum is $\geq \ell_*(Q)$.

Theorem. *If $\{|z| = 1\}$ is Q-convex and* $\log Q$ *is subharmonic there cannot exist more than one solution to (*).*

About the proofs.

They rely upon similar methods as in Beurling's conformal mapping theorem to be presented in § III.B. See also chapter VI: vol 2 [Collected work] for further studies related to the free boundary value problem.

5. Beurling's conformal mapping theorem.

Introduction. Let D be the open unit disc $|z| < 1$. Denote by $\mathcal{C}$ the family of conformal maps $w = f(z)$ which map D onto some simply connected domain Ω_f which contains the origin and satisfy:

$$f(0) = 0 \quad \text{and} \quad f'(0) \text{ is real and positive}\,.$$

Riemann's mapping theorem asserts that for every simply connected subset Ω of $\mathbf{C}$ which is not equal to $\mathbf{C}$ there exists a unique $f \in \mathcal{C}$ such that $\Omega_f = \Omega$. We are going to construct a subfamily of $\mathcal{C}$. Consider a positive and bounded continuous function a function Φ defined in the whole complex w-plane.

0.1 Definition. *The set of all $f \in \mathcal{C}$ such that*

$$(*) \qquad \lim_{r \to 1} \max_{0 \leq \theta \leq 2\pi} \left[|f'(re^{i\theta})| - \Phi(f(re^{i\theta})) \right] = 0$$

is denoted by $\mathcal{C}_\Phi$.

Remark. Thus, when $f \in \mathcal{C}_\Phi$ then the difference of the absolute value $|f'(z)|$ and $\Phi(f(z))$ tends uniformly to zero as $|z| \to 1$. Let M be the upper bound of Φ. The maximum principle applied to the complex derivative $f'(z)$ gives

$$|f'(z)| \leq M \quad : \quad z \in D$$

Hence $f(z)$ is a continuous function in the open disc D whose Lipschitz norm is uniformly bounded by M. This implies that f extends to a continuous function in the closed disc, i.e. f belongs to the disc algebra $A(D)$.

1. Theorem. *Assume that Log $\frac{1}{\Phi(w)}$ is subharmonic. Then $\mathcal{C}_\Phi$ contains a unique function f^*.*

Remark. When $\Phi(w) = \Phi(|w|)$ is a radial function and $\Phi(\rho) = \rho$ holds for some $\rho > 0$ then the function $f(z) = \rho \cdot z$ belongs to $\mathcal{C}_\Phi$. So for a radial Φ-function where different ρ-numbers exist uniqueness fails for the family $\mathcal{C}_\Phi$. The reader may verify that a radial function Φ for which $\Phi(\rho) = \rho$ has more than one solutions cannot satisfy the condition in Theorem 1. Next, let us give examples of Φ-functions which satisfy the condition in Theorem 1. Consider an arbitrary real-valued and non-negative L^1-function $\rho(t,s)$ which has compact support. Set

$$\Phi(w) = \exp\Big[\int \log \frac{1}{|w - t - is[} \cdot \rho(t,s) \cdot dtds\,\Big]$$

Here $\log \frac{1}{\Phi}$ is subharmonic and Theorem 1 gives a unique simply connected domain Ω such that the normalised conformal mapping function $f \colon D \to \Omega$ satisfies

$$|f'(e^{i\theta})| = \Phi(f(e^{i\theta})) \quad : \quad 0 \leq \theta \leq 2\pi\,.$$

The proof of Theorem 1 requires several steps where notble point is that we also estblish some existence results under the sole assumption that Φ is continuous and positive.

1.1 The family $\mathcal{A}_\Phi$. Let Φbe continuous and positive. Denote by $\mathcal{A}_\Phi$ the subfamily of all $f(z) \in \mathcal{C}$ such that

$$\limsup_{|z|\to 1} \; |f'(z)| - \Phi(f(z)) \leq 0 \tag{1}$$

Remark. By the definition of limes superior this means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f'(z)| \leq \Phi(f(z)) + \epsilon \quad : \text{for all } 1 - \delta < |z| < 1 \,.$$

The maximal region $\Omega^*(\Phi)$. With Φ given we get a bounded open subset in the complex plane as follows:

$$\Omega^*(\Phi) = \cup f(D) \quad : \quad \text{union taken over all } f \in \mathcal{A}_\Phi \tag{1.2}$$

1.3 Theorem. *The maximal region $\Omega^*(\Phi)$ is simply connected. Moreover, the unique normalised conformal mapping for which $f^*(D) = \Omega^*(\Phi)$ belongs to $\mathcal{C}_\Phi$.*

1.4 The family $\mathcal{B}_\Phi$. It consists of all $f \in \mathcal{C}$ such that

$$\liminf_{|z|\to 1} \; |f'(z)| - \Phi(f(z)) \geq 0$$

To this family we assign minimal region

$$\Omega_*(\Phi) = \cap f(D) \quad \text{The intersection taken over all } f \in \mathcal{B}_\Phi \tag{1.5}$$

1.6 Theorem. *The set $\Omega_*(\Phi)$ is simply connected and the unique $f_* \in \mathcal{C}$ for which $f_*(D) = \Omega_*(\Phi)$ belongs to $\mathcal{C}_\Phi$.*

Remark. The constructions of the maximal and the minimal region give

$$\Omega_*(\Phi) \subset \Omega^*(\Phi) \tag{1}$$

This inclusion is in general strict as seen by the example when Φ is radial. But when $\operatorname{Log} \frac{1}{|\Phi|}$ is subharmonic the uniqueness in Theorem 1 asserts that one has the equality $\Omega_*(\Phi) = \Omega^*(\Phi)$.

Before we enter the proofs of Theorem 1.4 and 1.6 we show the uniqueness part in Theorem 1.

A. Proof of Theorem 1.

Let Φ be as in Theorem 1. Admitting Theorem 1.4 and 1.6 we get the two simply connected domains $\Omega^*(\Phi)$ and $\Omega_*(\Phi)$. Keeping Φ fixed we set $\Omega^* = \Omega^*(\Phi)$ and $\Omega_* = \Omega_*(\Phi)$. Let $f \colon D \to \Omega_*$ and $g \colon D \to \Omega^*$ be the corresponding conformal mappings. Since $\Omega_* \subset \Omega^*$ Riemann's mapping theorem gives an inequality for the first order derivative at $z = 0$:

$$f'(0) \leq g'(0) \tag{i}$$

Set

$$\Phi(w) = e^{U(w)}$$

where $U(w)$ by assumption is super-harmonic. Solving the Dirichlet problem with respect to the domain Ω^* we get the harmonic function U^* in Ω^* where

$$U^*(w) = U(w) \quad w \in \partial\Omega^* .$$

Similarly we find the harmonic function U_* in Ω_* such that

$$(*) \qquad U_*(w) = U(w) \quad w \in \partial\Omega_* .$$

Next, since $g \in \mathcal{C}_\Phi$ we have the equality

$$\text{(ii)} \qquad \log|g'(z)| = U(g(z)) \quad |z| = 1$$

Now $\log|g'(z)|$ and $U^*(g(z))$ are harmonic in D and since g is normalised so that $g'(0)$ is real and positive it follows from (ii) that:

$$\log g'(0) = U^*(0)$$

In a similar way we find that

$$\log f'(0) = U_*(0)$$

Next, U is super-harmonic in Ω^* since and $\partial\Omega_*$ is a closed subset of $\bar{\Omega}^*$ one has:

$$U(w) \geq U^*(w) \quad w \in \partial\Omega_*$$

Then (*) entails that $U_* \geq U^*$ holds in Ω_*. In particular

$$\log f'(0) = U_*(0) \geq U^*(0) = \log g'(0)$$

Together with (i) we conclude that $f'(0) = g'(0)$and by the uniqueness in Riemann's mapping theorem it follos that $\Omega_* = \Omega^*$ which gives the uniqueness part in Theorem 1.

B. Proof of Theorem 2.

The first step in the proof is to construct a certain "union map" defined by a finite family $f_1, \ldots, f_n$ of functions $\mathcal{A}_\Phi$. Set

$$(*) \qquad S_\nu = f_\nu(D) \quad \text{and } S_* = \cup\, S_\nu$$

So above S_* is a union of Jordan domains which in general can intersect each other in a rather arbitrary fashion.

B.1 Definition. *The extended union denoted by $EU(S_*)$ is defined as follows: A point w belongs to the extended union if there exists some closed Jordan curve $\gamma \subset S_*$ and the Jordan domain bordered by γ contains w.*

Exercise. Verify that the extended union is simply connected.

B.2 Lemma *Let f_* be the unique normalised conformal map from D onto the extended union above. Then $f_* \in \mathcal{A}_\Phi$.*

Proof. First we reduce the proof to the case when all the functions $f_1, \ldots, f_n$ extend to be analytic in a neighborhood of the closed disc $\bar{D}$. In fact, with $r < 1$ we set $f_\nu^r(z) = f_\nu(rz)$ and get the image domains $S_\nu[r] = f_\nu^r(D) = f_\nu(D_r)$. Put $S_*[r) = \cup S_\nu[r]$ and construct its extended union which we denote by $S_{**}[r]$. Next, let $\epsilon > 0$ and consider the new function $\Psi(w) = \Phi(w)+\epsilon$. Let $f_*[r]$ be the conformal map from D onto $S_{**}[r]$. If Lemma B.2 has been proved for the n-tuple $\{f_\nu^r\}$ it follows by continuity that $f_*[r]$ belongs $\mathcal{A}_\Psi$ if r is sufficently close to one. Passing to the limit we see that $f_* = \lim_{r\to 1} f_*[r]$ and we get $f_* \in \mathcal{A}_\Psi$. Since $\epsilon > 0$ is arbitrary we get $f_* \in \mathcal{A}_\Phi$ as required.

After this preliminary reduction we consider the case when each f-function extends analytically to a neighborhood of the closed disc $|z| \leq 1$. Then each S_ν is a closed real analytic Jordan curve and the boundary of S_* is a finite union of real analytic arcs and some corner points. In particular we find the outer boundary which is a piecewise analytic and closed Jordan curve Γ and the extended union is the Jordan domain bordered by Γ. It is also clear that Γ is the union of some connected arcs $\gamma_1, \ldots, \gamma_N$ and a finite set of corner points and for each $1 \leq k \leq N$ there exists $1 \leq \nu(k) \leq n$ such that

$$\gamma_k \subset \partial S_{\nu(k)}$$

Denote by $\{F_\nu = f_\nu^{-1}\}$ and $F = f_*^{-1}$ the inverse functions and put:

$$G = \operatorname{Log} \frac{1}{|F|} \quad : G_\nu = \operatorname{Log} \frac{1}{|F_\nu|} \quad : 1 \leq \nu \leq n\,.$$

With $1 \leq \nu \leq n$ kept fixed we notice that G_ν and G are super-harmonic functions in S_ν and the difference

$$H = G - G_\nu$$

is superharmonic in S_ν. Next, consider a point $p \in \partial S_\nu$. Then $|F_\nu(p)| = 1$ and hence $G_\nu(p) = 0$. At the same time p belongs to ∂S_* or the interior of S_* so $|F(p)| \leq 1$ and hence $G(p) \geq 0$. This shows that $H \geq 0$ on ∂S_ν and by the minimum principle for harmonic functions we obtain:

(i) $$H(q) \geq 0 \quad \text{for all } q \in S_\nu$$

Let us then consider some boundary arc γ_k where $\gamma \subset \partial S_\nu$, i.e. here $\nu = \nu(k)$. Now $H = 0$ on γ_k and since (i) holds it follows that the *outer normal derivative*:

(ii) $$\frac{\partial H}{\partial n}(p) \leq 0 \quad p \in \gamma_k$$

Since $|F| = |F_\nu| = 1$ holds on γ_k and the gradient of H is parallell to the normal we also get:

$$\frac{\partial G}{\partial n}(p) = -|F'(w)| \quad \text{and} \quad \frac{\partial G_\nu}{\partial n}(p) = -|F_\nu'(w)| \quad : w \in \gamma_k$$

Hence (ii) above gives

(iii) $$|F'(w)| \geq |F_\nu'(w)| \quad \text{when} \quad w \in \gamma_k$$

Next, since $f_\nu \in \mathcal{A}_\Phi$ we have

(iv) $$|f'_\nu(F_\nu(w))| \leq \Phi(w)$$

and since F_ν is the inverse of f_ν we get

$$1 = f'_\nu(F_\nu(w)) \cdot F'_\nu(w)$$

Hence (iv) entails

(v) $$|F'_\nu(w)| \geq \frac{1}{\Phi(w)}$$

We conclude from (iii) that

(vi) $$|F'(w)| \geq \frac{1}{\Phi(w)} \quad : \; w \in \gamma_k$$

This holds for all the sub-arcs $\gamma_1, \ldots, \gamma_n$ and hence we have proved the inequality

(*) $$|F'(w)| \geq \frac{1}{\Phi(w)} \quad \text{for all} \quad w \in \Gamma$$

except at a finite number of corner points. To settle the situation at corner points we notice that Poisson's formula applied to the harmonic function $\log|f'_*(z)|$ in the unit disc gives

(vii) $$\log|f'_*(z)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} \cdot \log|f'(e^{i\theta})| \cdot d\theta \, .$$

Next, since F is the inverse of f_* we have

$$|f'_*(z)| \cdot |F'(f(z))| = 1 \quad \text{for all } |z| = 1 \, .$$

Hence (vi) gives

$$|f'(z)| \leq \Phi(f_*(z)) \quad \text{for all } |z| = 1 \, .$$

With $\Phi = e^U$ we therefore get

$$\log|f'_*(z)| \leq U(f(z)) \quad \text{for all } |z| = 1 \, .$$

From the Poisson integral (vii) it follows that

$$\log|f'_*(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} \cdot U(f_*(e^{i\theta})) \cdot d\theta \, . \quad z \in D$$

A passage to the limit. I addition to the obvious equi-continuity the passage to the limit requires some care which is exposed in [Beurling: Lemma 1, page 122]. Passing to the limit as $|z| \to 1$ the continuity of Φ implies that f_* belongs to $\mathcal{A}_\Phi$ which proves Lemma B.2.

B.3 The construction of Ω^*

By the uniform bound for Lipschitz norms the family $\mathcal{A}_\phi$ is equi-continuous. We can therefore find a denumerable dense subset $\{h_\nu\}$. It means that to every $f \in \mathcal{A}_\Phi$ and every $\epsilon > 0$ there exists some h_ν such that the maximum norm $|f - h_\nu|_D < \epsilon$. From this it is clear that

$$\text{(i)} \qquad \Omega^* = \cup\, h_\nu(D)$$

Next, to every $n \geq 2$ we have the n-tuple $h_1, \ldots, h_n$ Lemma B. 2 gives the function f_n where $f_n(D)$ is the extended union of $\{h_\nu(D)\}$. In particular

$$h_\nu(D) \subset f_n(D) \quad : \ 1 \leq \nu \leq n$$

The image domains $\{f_n(D)\}$ increase with n. and (i) gives

$$\text{(ii)} \qquad \Omega^* = \cup\, f_n(D)$$

Next, $\{f_n\}$ is a normal family of analytic functions and since their image domains increase there exists the limit function f^* and it belongs to $\mathcal{C}$ by the general results from Chapter VI. Morover, (ii) entails that

$$f^*(D) = \Omega^*$$

There remains to prove that

$$(*) \qquad f^* \in \mathcal{C}_\Phi$$

To establish (*) we shall need a relation between Φ and the maximal domain $\Omega^*(\Phi)$.

B.4 Proposition. *Let Ψ be a positive continuous function which is equal to Φ outside $\Omega^*(\Phi)$ while its restriction to $\Omega^*(\Phi)$ is arbitrary. Then one has the equality*

$$\Omega^*(\Phi) = \Omega^*(\Psi)$$

Proof. The assumption gives

$$\text{(i)} \qquad \Psi(w) = \Phi(w) \quad \text{for all } w \in \partial\Omega^*(\Phi)$$

It follows that $f^* \in \mathcal{A}_\Psi$. Hence the equality $f^*(D) = \Omega^*(\Phi)$ and the construction of the maximal domain $\Omega^*(\Psi)$ give the inclusion

$$\text{(ii)} \qquad \Omega^*(\Phi) \subset \Omega^*(\Psi)$$

Next, let $h^* \colon D \to \Omega^*(\Psi)$ be the conformal mapping function associated to Ψ. The equality (i) and the construction of the maximal region $\Omega^*(\Phi)$ gives $h^* \in \mathcal{A}_\phi$ and then

$$\text{(iii)} \qquad \Omega^*(\Psi) = h^*(D) \subset \Omega^*(\Phi)$$

Hence (ii-iii) give the requested equality in Proposition B.4.

B.5 A special choice of Ψ

.

Keeping Φ fixed we put $\Omega^*(\Phi) = \Omega^*$ to simplify the notations. We have the U-function such that

(B.5.i) $$\Phi(w) = e^{U(w)}$$

Here $U(w)$ is a continuous function on $\partial\Omega^*$ and solving the Dirichlet problem we obtain the function $U_*(w)$ where $U_* = U$ outside Ω^*, and in Ω^* the function U_* is the harmonic extension of the boundary function U restricted to $\partial\Omega^*$. Set

$$\Psi(w) = e^{U_*(w)}$$

Proposition B.4 gives

(B.5.ii) $$f^* \in \mathcal{A}_\Psi$$

Next, consider the function in D defined by:

$$V(z) = \log\left|\frac{df^*(z)}{dz}\right| - U_*(f(z))$$

From the above $V(z)$ is either identically zero in D or everywhere < 0. It is also clear that if $V = 0$ then $f^* \in \mathcal{C}_\Phi$ as required. So there remains only to prove:

B.6 Lemma. *The function $V(z)$ is identically zero in D.*

Proof. Assume the contrary. So now

(i) $$\left|\frac{df^*(z)}{dz}\right| < e^{U_*(z)} \quad \text{for all } z \in D\,.$$

Let $F(w)$ be the inverse of f so that :

$$F'(f^*(z)) \cdot \frac{df^*(z)}{dz} = 1 \qquad z \in D$$

Then (i) gives:

(ii) $$\big|F'(w)\big| > e^{-U_*(w)} \quad w \in \Omega^*$$

Let $V(w)$ be the harmonic conjugate of $U_*(w)$ normalised so that $V(0) = 0$ and put

$$H(w) = \int_0^w e^{-U_*(\zeta)+iV(\zeta)} \cdot d\zeta\,.$$

Then (ii) gives

(iii) $$\big|H'(w)\big| < \big|F'(w)\big| \quad w \in \Omega^*$$

Since $H(w((0)) = F(w(0)) = 0$ it follows that

(iv) $$\inf_{w\in\Omega^*} \frac{|H(w)|}{|F(w)|} = r_0 < 1$$

Now $|F(w)| < 1$ in Ω^* and $|F(w)| \to 1$ as w approaches $\partial\Omega^*$. Hence (iv) entails the domain

(v) $$R_0 = \{w \in \Omega^* \quad : \ |H(w)| < r_0\}$$

has at least one boundary point w_* which also belongs to $\partial\Omega^*$. Next, the function $H(w)$ is analytic in R_0 and its derivative is everywhere $\neq 0$ while $|H(w)| = 1$ on

∂R_0. It follows that H gives a conformal map from R_0 onto the disc $|z| < r_0$. Let $h(z)$ be the inverse of this conformal mapping. Now we get the analytic function in D defined by

$$g(z) = h(r_0 z)$$

Next, let $|z| = 1$ and put $w = h(r_0 z)$. Then

$$\text{(v)}\quad |g'(z)| = r_0 \cdot |h'(r_0 z)| = r_0 \cdot \frac{1}{|H'(g(z))|} = r_0 \cdot e^{U_*(g(z))} = r_0 \cdot \Psi(g(z)) < \Psi(g(z))$$

At the same time we have a common boundary point

$$w_* \in \partial\Omega^* \cap \partial g(D)$$

Since the g-function extends to a continuous function on $|z| \leq 1$ there exists a point $e^{i\theta}$ such that

$$g(e^{i\theta}) = w_*$$

Now we use that $r_0 < 1$ above. The continuity of Ψ gives the existence of $\epsilon > 0$ such that for any complex number a which belongs to the disc $|a-1| < \epsilon$, it follows that the function

$$z \mapsto a \cdot g(z)$$

belongs to $\mathcal{A}_\Psi$. Finally, by Proposition B.4 the maximal region for the Ψ-function is equal to Ω^* and we conclude that

$$ag(e^{i\theta}) = aw_* \in \Omega^* \quad |a-1| < \epsilon$$

This would mean that w_* is an *interior point* of Ω^* which contradicts that $w_* \in \partial\Omega_*$ and Lemma B.6 is proved.

C. Proof of Theorem 3.

First we have a companion to Lemma B.2. Namely, let $g_1, \ldots, g_n$ be a finite set in $\mathcal{B}_\Phi$. Set $S_\nu = g_\nu(D)$. Following [Beur: page 123] we give

C.1 Definition. *The reduced intersection of the family $\{S_\nu\}$ is defined as the set of these points w which can be joined with the origin by a Jordan arc γ contained in the intersection $\cap S_\nu$. The resulting domain is denoted by $RI\{S_\nu\}$.*

C.2 Proposition. *The domain $RI\{S_\nu\}$ is simply connected and if $g \in \mathcal{C}$ is the normalised conformal mapping onto this domain, then $g \in \mathcal{B}_\Phi$.*

The proof of this result can be carried out in a similar way as in the proof of Lemma B.2 so we leave out the details. Next, starting from a dense sequence $\{g_\nu\}$ in $\mathcal{B}_\Phi$ we find for each n the function $f_n \in \mathcal{B}_\Phi$ where

$$f_n(D) = RI\{S_\nu\} \quad : \; S_\nu = g_\nu(D) : \; 1 \leq \nu \leq n\,.$$

Here the simply connected domains $\{f_n(D)\}$ decrease and there exists the limit function $f_* \in \mathcal{C}$ where

$$f_*(D) = \Omega_*$$

There remains to prove

C.3 Proposition. *One has $f_* \in \mathcal{C}_\Phi$.*

Remark. Proposition C.3 requires a quite involved proof as comared to the case of maximal regions. The details are given in [Beur: page 127-130]. Let us just sketch the strategy in the proof. Put

$$m = \inf_{g \in \mathcal{B}_\Phi} g'(0)$$

Next, f_* belongs to $\mathcal{B}_\Phi$ because we have the trivial inclusion $\mathcal{C}_\Phi \subset \mathcal{B}_\Phi$ and using this Beurling proved that

$$m \geq \min_{w \in \Omega^*} \Phi(w)$$

Next, starting with Proposition C.2 above, Beurling introduces a normal family and proves that

$$f_*'(0) = m$$

Thus, f_* is a solution to an extremal problem which Beurling used to establish the inclusion $f_* \in \mathcal{C}_\Phi$. Let us remark that this part of the proof relies upon some some very interesting set-theoretic constructions where the family of regions of the *Schoenfliess' type* are introduced in [Beur: page 121]. So the whole proof involves topological investigations of independent interest.

§ 6. The Riemann-Schwarz inequality.

By the uniformisation theorem for Riemann surfaces it suffices to prove the result in §§ when $X = D$ and u is a continuous and subharmonic function in D where $\lambda(z) = e^{u(z)}$ which yields a metric. Consider a pair of points a, b in D and a pair of rectifiable Jordan arcs γ_1, γ_2 in $\mathcal{C}(a,b)$. Suppose for the moment that the intersection of the γ-curves only contains the end-points a and b. Their union gives a closed Jordan curve Γ which borders a Jordan domain Ω. The inequality (*) in Theorem XX amounts to show that for every point $p \in \gamma_1$, there exists a Jordan arc β in Ω which joins p with some $q \in \gamma_2$ and a Jordan arc α in Ω which joins a and b such that

$$(1) \quad \Big(\int_\alpha \lambda(z)\,|dz|\Big)^2 + \Big(\int_\beta \lambda(z)\,|dz|\Big)^2 \leq \frac{1}{2}\cdot\Big[\Big(\int_{\gamma_1} \lambda(z)\,|dz|\Big)^2 + \Big(\int_{\gamma_2} \lambda(z)\,|dz|\Big)^2\Big]$$

To prove (1) we employ a conformal mapping $\psi \colon D \to \Omega$ where D is another unit disc with the complex coordinate w. The Koebe-Lindelöf theorem entails that ψ extends to a homeomorphism from D onto the closed Jordan domain $\bar{\Omega}$ where the inverse images of γ_1 and γ_2 is a pair of closed intervals on the unit circle $|w| = 1$ which instersect at two points. Now $\log \lambda \circ \psi$ is subharmonic in D and if γ is a rectifiable arc in Ω one has the equation:

$$\int_\gamma \lambda(z)\cdot|dz| = \int_{\psi(\gamma)} \lambda\circ\psi(w)\cdot\frac{|dw|}{|\psi'(w)|}$$

Set

$$\lambda^*(w) = \frac{1}{|\psi'(w)|}\cdot\lambda\circ\psi(w)$$

Then

$$\log\lambda^* = -\log|\psi'(w)| + \log\lambda\circ\psi$$

Here $\log|\psi'(w)|$ is the real part of the analytic function $\log\psi'(w)$ in D and hence harmonic which entails that $\log\lambda^*$ is subharmonic. The proof of (1) is therefore reduced to the case when Ω is replaced by the unit w-disc and λ by λ^*. If J_1 and J_2 are intervals on $\{|w| = 1\}$ whose union is the whole circle, then the Cauchy-Schwarz inequality gives

$$\Big(\int_{J_1} \lambda^*(w)\,|dw|\Big)^2 + \Big(\int_{J_2} \lambda^*(w)\,|dw|\Big)^2 \leq \frac{1}{2}\cdot\Big(\int_{|w|=1} \lambda^*(w)\,|dw|\Big)^2$$

Passin to the w-disc with λ^* as above there remains to find a pair of curves α^* and β^* in the w-disc where α^* joins the two points $\{e^{i\theta_\nu}\}$ on the unit circle where the boundary arcs $\psi^{-1}(\gamma_1)$ and $\psi^{-1}(\gamma_2)$ intersect, while the β^*-curve joins some point $p \in \psi^{-1}(\gamma_1)$ with a point $q \in \psi^{-1}(\gamma_2)$.

The choice of α^* and β^*. Let α^* be the circular arc with end-points at $e^{i\theta_1}$ and $e^{i\theta_2}$ which intersects $\{|w| = 1\}$ at right angles. Next, given a point $p \in \psi^{-1}(\gamma_1)$ there exists the unique circular arc β^* which intersects both α^* and $\{|w| = 1\}$ at right angles. See figure § XX. Togehter with the inequality (xx) there remains to prove that α^* snd β^* can be chosen so that

$$(2) \qquad \big(\int_{\alpha^*} \lambda^*(w)\cdot |dw|\,\big)^2 + \big(\int_{\beta^*} \lambda^*(w)\cdot |dw|\,\big)^2 \leq \frac{1}{4}\cdot\big(\int_{|w|=1\}} \lambda^*(w)\,|dw|\,\big)^2$$

To prove (2) we will use a symmetrisation of λ^*. Namely, let T_1 be a Möbius transformation on the unit disc which is a reflection of α^*, i.e. it restricts to the identity map on α^* and the composed map T_1^2 is the identity in D. Similarly we find the reflection T_2 of β^*. One easily verifies that $T_2\circ T_1 = T_1 \circ T_1$. Let $S_0(w) = w$ be the identity while

$$S_1 = T_1 \quad \& \quad S_2 = T_2 \quad \& \quad S_3 = T_2 \circ T_1$$

Set

$$(3) \qquad \lambda^{**}(w) = \frac{1}{4}\cdot \sum_{\nu=0}^{\nu=3} \lambda(S_\nu(w))\cdot |\frac{S_\nu(w)}{dw}|$$

Notice that T_2 maps α^* into itself, and similarly T_1 maps β^* into itself. If follows that the left hand side in (2) is unchanged when λ^* is replaced by λ^{**}, and the right hand side is unchanged since the S-transformations map $\{|w| = 1\}$ onto itself. Hence it suffices to prove (2) when λ^* from the start is invariant with respect to the S-transformations above. In this situation we solve the Dirichlet problem using the boundary value function $\log \lambda^*$ on $\{|w|\}$ so that in the open disc

$$\log \lambda^* = g + H$$

where H is harmonic in D while g is subharmonic and zero on $\{|w|\}$. The maximum principle entails that $g \leq 0$ in D which gives

$$(4) \qquad \lambda^* = e^u \cdot e^H \leq e^H$$

Since $g = 0$ on $\{|w| = 1\}$ the right hand side in (2) is does not change while the left hand side is majorised when λ^* is replaced by e^H. Moroever, the reader csn check that the S-invariance of λ^* implies that H also is S-invariant. Next, we find an analytic function $g(w)$ in D such that

$$e^{H(w)} = |g'(w)|$$

where the map $w \mapsto g(w)$ sends α^* to a real interval $[-A, A]$ and β^* to an imaginary interval $[-iB, iB]$. Hence (4) implies that the left hand side in (2) is majorized by

$$(5) \qquad \big(\int_\alpha |g'(w)|\,|dw|)^2 + \big(\int_\beta |g'(w)|\,|dw|\,\big)^2 = 4A^2 + 4B^2$$

Next, since $\lambda^* = |g'|$ holds on $\{|w|\}$ the right hand side in (2) becomes

$$(6) \qquad \frac{1}{4}\cdot\big(\int_{|w|=1} |g'(w)|\,|dw|\,\big)^2$$

Finally, the S-symmetry of g entails that $\int_{|w|=1} |g'(w)|\,|dw|$ is 4 times the integral taken along a subarc of T which joins consequtive points where α^* and β^* intersect. Each such integral is the euclidian length of the image curve under g which by the above joins the real point A with iB. So its euclidian length is $\geq \sqrt{A^2 + B^2}$, i.e.

since the shortest distance between a pair of points is a straight line, together with Pythagoras' theorem. Hence (6) majorizes

$$\frac{1}{4} \cdot \left(4 \cdot \sqrt{A^2 + B^2}\right)^2 = 4(A^2 + B^2) \tag{7}$$

Then (5) and (7) give the requested inequality (2).

§ 7. A uniqueness result for the ζ-function

In § 0.5 from the introduction we defined a class $\mathcal{D}_k$ of Dirichlet series for every $k > 0$ and announced Beurling's result in Theorem § 0.5.1. Recall that each Dirichlet series $\Lambda(s)$ in the family $\mathcal{D}_k$ gives an even and entire function of exponential type defined by an Hadamard product:

$$f(z) = \prod (1 + \frac{z^2}{\lambda_n^2})$$

Using Phragmén-Lindelöf inequalities together with properties of the Γ-function and Mellin's inversion formula, we shall prove that for each $\epsilon > 0$ there exists a constant C_ϵ such that the following hold for each real $x > 0$:

(*) $$\big|f(x) - ax^p \cdot e^{\pi x}\big| \leq C_\epsilon \cdot e^{\pi(1-2k+\epsilon)\cdot x}$$

where

$$a = e^{2\Lambda(0)} \quad \text{and} \quad p = 2\Lambda'(0)$$

When $k > 1/2$ we can choose ϵ small so that $1 - k + \epsilon = -\delta$ for some $\delta > 0$ which means that $f(x) - ax^p \cdot e^{\pi x}$ has exponential decay as $x \to +\infty$. From this we shall deduce that $f(z)$ is of a special form in 1 7.1 below and after deduce Theorem § 0.5.1 via an inversion formula for Dirichlet series.

7.1 A uniqueness result in $\mathcal{E}$

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Let a and δ be positive real numbers and p some real number. Consider an even entire function $f(z)$ of exponential type for which there exists a constant C such that

(*) $$\big|f(x) - ax^p \cdot e^{\pi x}\big| \leq Ce^{-\delta x} \quad : x \geq 1$$

7.1.1 Theorem. *When (*) amd $f(0) = 1$ it follows that f is one of the following two functions:*

$$f_1(z) = \frac{e^{\pi z} + e^{-\pi z}}{2} \quad : \quad f_2(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$$

Proof. In the half-space $\mathfrak{Re}(z) > 0$ we have the analytic function

(i) $$h(z) = f(z) - az^p \cdot e^{\pi z}$$

where the branch of z^p is taken so that $x^p > 0$ when $z = x$ is real and > 0. Consider the domain

$$\Omega = \{z = x + iy \quad y > 0 \quad \text{and} \quad x > 1\}$$

Since $f \in \mathcal{E}$ there is a constant A such that $e^{-A|z|} \cdot f(z)$ is bounded which gives constants C and B such that

(ii) $$|h(1 + iy)| \leq C \cdot e^{By}$$

for all $y > 0$. At the same time (*) gives

(iii) $$|h(x)| \leq C \cdot e^{-\delta x}$$

The Phragmén-Lindelöf theorem applied to the quarter planer Ω therefore gives a constant C such that

$$\text{(iv)} \qquad |h(x+iy)| \leq Ce^{Ay-\delta x} \quad \text{for all} \quad x+iy \in \Omega$$

In exactly the same way one proves (iv) with y replaced by $-y$ in the quarter-plane where $x > 0$ and $y < 0$. Let us then consider the strip domain

$$S = \{x + iy \quad |y| \leq 1 \quad \text{and} \quad x > 1\}$$

Then we see that there is a constant such that

$$\text{(v)} \qquad h(x+iy)| \leq C \cdot e^{-\delta x} \quad : \quad x+iy \in S$$

If $n \geq 1$ we consider the complex derivative $h^{(n)}$ and when $x > 2$, Cauchy's inequality and (*) give a constant C_n such that

$$\text{(vi)} \qquad |h^{(n)}(x)| \leq C_n \cdot e^{-\delta x} \quad : x \geq 2$$

Next, consider the second order differential operator

$$L = x^2 \partial_x^2 - 2p \cdot x \partial_x - \pi^2 x^2 + p(p+1)$$

The functions $x^p e^{\pi x}$ and $x^p e^{-\pi x}$ are solutions to the homogeneous equation $L = 0$ when $x > 0$ and hence

$$L(f) = L(h)$$

holds on $x > 0$. Now L also yields the holomorphic differential operator where ∂_x is replaced by ∂_z and here $g = L(f)$ is en entire function exponential type. Now (*) above Theorem 7.1.1 and the estimates (vi) for $n = 0, 1, 2$ give a constant C such that

$$\text{(vii)} \qquad |g(x)| \leq C(1 + x^2) \cdot e^{-\delta |x|} \quad x > 0$$

Moreover, since f is even it follows that g is so and hence (vii) hold for all real x. Then the $\mathcal{E}$-function g is identically zero by a wellknown and easily verified result. Hence f satisfies the differential equation

$$\text{(viii)} \qquad L(f) = 0$$

The uniquenss for solutions of the in ODE-equation (viii) gives contants c_1, c_2 such that

$$f(x) = c_1 x^p e^{\pi x} + c_2 x^p e^{-\pi x} \quad x > 0$$

Since f is an even entire function it is clear that this entails that p must be an integer and if we moreover assume that $f(0) = 1$ then the reader may verify that we have $p = 0$ or $p = -1$ which yield corresponding f-functions

$$f_1(z) = \frac{e^{\pi z} + e^{-\pi z}}{2} \quad : \quad f_2(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$$

B. Dirichlet series and their transforms.

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be a non-decreasing sequence of positive real numbers in the family $\mathcal{F}$ from § xx. Then the Dirichlet series

$$\text{(1)} \qquad \Lambda(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

is analytic in the half-space $\mathfrak{Re}\, s > 1$, and we also have the entire function

$$f(z) = \prod \big(1 + \frac{z^2}{\lambda_n^2}\big) \tag{2}$$

which is of exponential type.

B.1 Inversion formula. *One has the equation*

$$\int_0^\infty \log f(x) \cdot \frac{dx}{x^{s+1}} = \frac{\pi \cdot \Lambda(s)}{s \sin \frac{\pi s}{2}} \quad : \quad \mathfrak{Re}\, s > 1 \tag{B.1.}$$

Proof. When $0 < \mathfrak{Re}\, s < 2$ and $a > 0$ is real the reader may verify the equality:

$$\int_0^\infty \log\big(1 + \frac{x^2}{a^2}\big) \cdot \frac{dx}{x^{s+1}} = \frac{1}{a^s} \cdot \frac{\pi}{\cdot \sin \frac{\pi s}{2}} \tag{i}$$

Apply (i) with $a = \lambda_n$ and then (B.1) follows after a summation over n.

B.2 Meromorphic extensions. The inversion formula (B.1) entails that Λ extends to a meromorphic function in the complex s-plane. To see this we notice that if $x > 0$ then the logartihmic derivative

$$\frac{f'(x)}{f(x)} = \sum \frac{2x}{\lambda_n^2 + x^2} \tag{i}$$

Next, a partial integration gives

$$(s+1) \cdot \int_0^\infty \log\, f(x) \cdot \frac{dx}{x^{s+1}} = \int_0^\infty \frac{f'(x)}{f(x)} \cdot \frac{dx}{x^s} \tag{ii}$$

Since $\sum \lambda_n^{-2}$ is convergent, the right hand side is analytic in the half-space $\mathfrak{Re}\, s > 0$ and by further integrations by parts the reader may verify that it extends to a meromorphic function in the s-plane. Together with the inversion for ua we conclude that $\Lambda(s)$ extends to a meromorphic function in the s-plane.

The case when $\Lambda \in \mathcal{D}_k$

.

Suppose this holds for some $k > 1/2$. In particular $\Lambda(-2n) = 0$ for every positive integer and then the right hand side in (B.1) is a meromorphic function whose poles are confined to $s = 0$ and $s = 1$. Denote this function with $\Phi(s)$. Now we shall estimate certain L^1-integrals.

B.3 Proposition. *There exists a constant C such that*

$$\int_{-\infty}^\infty |\Phi(-\sigma + it)| \cdot dt \leq C \cdot \frac{\sigma^3 \cdot \Gamma(\sigma)}{(2\pi k)^\sigma} \quad : \quad \sigma \geq 2$$

Proof. Consider the function

$$\psi_*(s) = (2\pi k)^s \cdot \Gamma(2 - s)$$

The series expression for $\Lambda(s)$ gives a constant C such that

(i) $$\Lambda(3/2+it)|\leq C \quad : \quad -\infty<t<+\infty$$

It follows that

(ii) $$\Big|\frac{\Phi(3/2+it)}{\psi_*(3/2+it)}\Big|\leq \frac{C\pi}{|3/2+it|}\cdot\frac{1}{2\pi k)^{3/2}}\cdot\frac{1}{\sin\left(\pi(3/4+it/2)\cdot\Gamma(1/2-it)\right.}$$

The complex sine-function increases along this vertical line, i.e. there is a constant $c>0$ such that

(iii) $$|\sin\left(\pi(3/4+it/2)\right|\geq c\cdot e^{\pi|t|/2}$$

At the same time, a classic result about the Γ-function gives the lower bound

(iv) $$|\Gamma(1/2-it)|\geq\sqrt{\pi}\cdot e^{-\pi|t|/2}$$

From (iii-iv) we conclude that the function $\frac{\Phi}{\psi_*}$ is bounded on the line $\mathfrak{Re}(s)=3/2$.

Sublemma 1. The function $\frac{\Phi}{\psi_*}$ *is a bounded function in the domain*

$$\Omega=\{\mathfrak{Re}(s)<3/2\}\cap\{|s|>2\}$$

Proof. Follows easily via the Phragmén-Lindelöf theorem and the bound above on $\mathfrak{Re}(s)=3/2$.

Next, Sublemma 1 gives a constant C such that

(v) $$|\Phi(s)|\leq C\cdot\Big|(2\pi k)^s\cdot\Gamma(2-s)\Big| \quad : \quad s\in\Omega$$

We shall need another classic inequality for the Γ-function which asserts that there exists a constant C such that

(vi) $$\int_{-\infty}^{\infty}\big|\,\Gamma(\sigma+2+it)\,\big|\cdot dt\leq C\cdot\sigma^3\cdot\Gamma(\sigma) \quad : \quad \sigma\geq 2$$

The verification of (vi) is left to the reader. Together (v) and (vi) give the inequality in Proposition B.3.

B.4 Mellin's inversion formula.

The integral inequality in Proposition B.3 enable us to apply the Fourier-Mellin inversion formula via (**) from XX. This gives

$$\log f(x)=\frac{1}{2\pi i}\cdot\int_{c-i\infty}^{c+i\infty}\Phi(s)\cdot x^s\cdot ds \quad : \quad 1<c<2$$

Using Proposition B.3 we can shift the contour the left and perform integrals over lines $\mathfrak{Re}\,s=-c$ where $c>0$. During such a shift we pass the poles of Φ which appear at $s=0$ and $s=1$. Using condition (a) from (0.2) in the introduction the reader can deduce the integral formula:

$$(*)\ \log f(x)-\pi x-2\phi(0)\cdot\log x-2\phi'(0)=\frac{1}{2\pi i}\cdot\int_{-c-i\infty}^{-c+i\infty}\Phi(s)\cdot x^s\cdot ds\quad\text{for all}\quad c>0$$

B.4.1 A clever estimate. To profit upon (*) we adapt the c-values when x are real and large. With $x\geq 2$ we take $c=x$ and notice that

$$|x^{(-x+it)}|=x^{-x}$$

Proposition B.3 and the triangle inequality show that the absolute value of the right hand side integral in (*) is majorized by

$$(**)\qquad 2\pi\cdot x^{-x}\cdot C\cdot\frac{x^3\cdot\Gamma(x)}{(2\pi k)^x}\quad:\quad x\geq 2$$

B.4.2 Exercise. Recall that $\Gamma(N)=N!$ for positive integers. Use this and Stirling's formula to conclude that for every $\epsilon>0$ there is a constant C_ϵ such that (**) is majorized by

$$C_\epsilon\cdot e^{-2\pi(k-\epsilon)x}$$

B.4.3 Consequences. With $x\geq 2$, $p=2\phi(0)$ and $a=2\phi'(0)$ we obtain from above:

$$\Big|\log f(x)-\pi x-p\cdot\log x-a\Big|\leq C_\epsilon\cdot e^{-2\pi(k-\epsilon)x}$$

Now (B.4.3) gives a constant C^*_ϵ such that

$$\text{(B.4.4)}\qquad |f(x)-e^a\cdot x^p\cdot e^{\pi x}|\leq C^*_\epsilon\cdot e^{\pi(1-2k+2\epsilon)\cdot x}\quad:\quad x\geq 2$$

Since $k>1/2$ and ϵ can be arbitrary small we conclude the foillowing

B.4.5 Theorem. *When $\Lambda\in\mathcal{D}_k$ for some $k>1/2$ there exists $\delta>0$ and constants a,p such that*

$$|f(x)-e^a\cdot x^p\cdot e^{\pi x}|\leq C^*_\epsilon\cdot e^{-\delta\cdot x}\quad:\quad x\geq 2$$

Final part of the proof. Together with Theorem 7.1.1 the reader can check that this gives Beurling's Theorem form the introduction.

*

Part II. Some results about subharmonic functions.

Introduction. Subharmonic functions in the complex plane are used in many situations. Perron's solution to the Dirichlet problem is an example and passing to several complex variables one introduces pluri-subharmonic functions whose basis properties rely upon subharmonic functions of a single complex variable. In general, if $u(z)$ is subharmoic it is represented by the Riesz formula over discs. For example, let $u(z)$ be subharmonic in $D = \{|s| < 1\}$ and assume that $u(0) = 0$ and that $u(z) < 1$ for every $z \in D$. Then one has the non-negative masure $\mu = \Delta(u)$ and there exists a harmonic function $H(z)$ in D such that the function

$$H(z) = u(z) - \frac{1}{\pi}\int_D \log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right| \cdot d\mu(z)$$

is harmonic. Since $u(0) = 0$ it follows that $H(0) = 0$ and the maximum principle entails that $H(z) \leq 1$ for all $z \in D$. It follows that $1 - H$ is a non-negative harmonic function and therefore given by the Poisson extension of a non-negative Resz measure on T. Hence $H(z)$ is the Poisson extension of a Riesz measure on T with a finite total variation. This representation formula gives a local control on a subharmonic function $u(z)$ which in general is not continuous and may attain the value $-\infty$ at certain points. However, u cannot be "too negative too often". This follows by an a priori inequality due to Rado which asserts that for every $r < 1$ there exists a constant C_r such that for every subharmonic function u as above one has the inequality

$$\iint_{|z|\leq r} e^{-u(z)}\, dxdy \leq C_r$$

The proof relies upon Harnack's inequality and is given in § xx below. We shall establish scattered results about subharmonic functions in the separate sections below.

A. On zero sets of subharmonic functions.

Let Ω in $\mathbf{C}$ be a bounded open set and denote by $\mathrm{SH}_0(\Omega)$ the set of subharmonic functions in Ω whose Laplacian is a Riesz measure supported by a compact null set. Every such function v is locally a logarithmic potential of $\Delta(v)$ plus a harmonic function and can therefore be taken to be upper semi-continuous. Moreover, the distribution derivatives $\partial v/\partial x$ and $\partial v/\partial y$ belong to $L^1_{\mathrm{loc}}(\Omega)$. Before Theorem A.1 is announced we introduce a geometric construction. If U is an open subset of Ω we construct its forward star-domain as follows: To each $\zeta \in U$ we find the largest $s(\zeta) > 0$ such that the line segment

$$\ell_\zeta(U) = \{\zeta + x : \; 0 \leq x < s(\zeta)\} \subset \Omega$$

The forward star domain is defined by

$$(*) \qquad \mathfrak{s}(U) = \bigcup_{\zeta \in U} \ell_\zeta(U)$$

A.1 Theorem. *Let $V \in SH_0(\Omega)$ and put $K = Supp(\Delta(V))$. Suppose that $V = 0$ in an open subset U of $\Omega \setminus K$ and furthermore*

$$(*) \qquad \partial V/\partial x(z) < 0, \quad \text{holds in} \quad \Omega \setminus (K \cup U)\,.$$

Then $V = 0$ in $\mathfrak{s}(U)$.

Remark. This result was proved by Bergqvist and Rullgård in the article [Be-Ru] under the assumption that the range of V is a finite set. The crucial step to attain the more general result above is to regard the complex log-function from (2) in the subsequent proof and employ similar regularisations as in [Be-Ru].

Proof. It is clear that it suffices to show the following: Let $z_0 \in U$ and consider a horisontal line segment

$$\ell = \{z = z_0 + s :\ 0 \leq s \leq s_0\}$$

which is contained in Ω. Then, if $0 < \delta < \text{dist}(\ell, \partial\Omega)$ and the open disc $D_\delta(z_0)$ of radius δ centered at z_0 is contained in U, it follows that V vanishes in the open set

$$(1) \qquad \{z :\ \text{dist}(z, \ell) < \delta\}$$

Notice that (1) is a relatively compact subset of Ω. Consider the complex derivative

$$\partial V/\partial z = \frac{1}{2}(\partial V/\partial x - i\partial V/\partial y)$$

This yields a complex-valued and locally integrable function and since $V = 0$ in $D_\delta(z_0)$ it is clear that $V = 0$ in the open set from (1) if we prove that $\partial V/\partial z = 0$ holds almost everywhere in (1) To prove this we take some $\epsilon > 0$ and put

$$(2) \qquad \Psi_\epsilon(z) = \log\left(\partial V/\partial z - \epsilon\right)$$

where the single-valued branch of the complex Log-function is chosen so that

$$(3) \qquad \pi/2 < \mathfrak{Im}\,\Psi_\epsilon < 3\pi/2$$

Hence we can write

$$(4) \qquad \Psi_\epsilon(z) = \text{Log}|\epsilon - \partial V/\partial z| + i\tau(z) \quad : \quad \pi/2 < \tau(z) < 3\pi/2$$

A regularisation. Choose a non-negative test-function ϕ with compact support in $|z| \leq \delta$ while $\phi(z) > 0$ if $|z| < \delta$ and $\iint \phi(z)dxdy = 1$. We construct the convolution $\sigma * \Psi_\epsilon$ which is defined in the subset of Ω whose points have distance $> \delta$ to $\partial\Omega$. Rules for first order derivations of a convolution give

$$\bar\partial/\bar\partial\bar z\big(\phi * \Psi_\epsilon\big) = \frac{\phi * \bar\partial\partial(V)}{\partial V/\partial z) - \epsilon} = \frac{1}{4} \cdot \frac{1}{\partial V/\partial z - \epsilon} \cdot \phi * \Delta(V)$$

Taking the real part we get

$$(5) \qquad \mathfrak{Re}(\bar\partial/\bar\partial\bar z(\phi * \Psi_\epsilon)) = \frac{\partial V/\partial x - \epsilon}{4|\epsilon - \partial V/\partial z|^2} \cdot \phi * \Delta(V)$$

To simplify notations we set

$$(6) \qquad \sigma(z) = \text{Log}|\epsilon - \partial V/\partial z|$$

The definition of the $\bar{\partial}$-derivative and the decomposition $\Psi_\epsilon = \sigma + i \cdot \tau$ together with the inequality (5) give

$$\partial_x(\phi * \sigma) \leq \partial_y(\phi * \tau) \tag{7}$$

In the right hand side we use the partial y-derivative on ϕ, i.e. we use the general formula:

$$\partial_y(\phi * \tau) = \partial_y(\phi) * \tau$$

Since $\pi/2 \leq \tau \leq 3\pi/2$ the absolute value of this function is majorized by

$$M = \frac{3\pi}{2} \cdot ||\partial_y(\phi)||_1 \tag{8}$$

where $||\partial_y(\phi)||_1$ denotes the L^1-norm. Next, consider the function $s \mapsto \phi * \sigma(z_0 + s)$ where $0 \leq s \leq s_0$ whose s-derivative becomes $\partial_x(\phi * \sigma)(z + s)$. Hence (7-8) give:

$$\frac{d}{ds}(\phi * \sigma(z + s)) \leq M$$

$$\implies \phi * \sigma(z_0 + s_0) \leq \phi * \sigma(z_0) + M \cdot s_0 \tag{9}$$

From now on $\epsilon < 1$ so that $\log \epsilon < 0$. Since $V = 0$ in $D_\delta(z_0)$ we also have $\partial V/\partial z = 0$ in this disc and conclude that

$$\phi * \sigma(z_0) = \log \epsilon \tag{10}$$

Next, we have

$$\sigma = \log \epsilon + \log |1 - \frac{\partial V/\partial z}{\epsilon}|$$

Put

$$f_\epsilon = \log |1 - \frac{\partial V/\partial z}{\epsilon}| \tag{11}$$

Then (9-10) give the inequality

$$\phi * f_\epsilon(z_0 + s_0) \leq M \cdot s_0 \tag{12}$$

So from the construction of a convolution this means that

$$\iint_{|\zeta| \leq \delta} f_\epsilon(z_0 + s_0 + \zeta) \cdot \phi(\zeta) \cdot d\xi d\eta \leq M \cdot s_0 \tag{13}$$

Next, we can write

$$f_\epsilon(z) = \frac{1}{2} \cdot \log\big((1 - \frac{\partial V/\partial x}{\epsilon})^2 + (\frac{\partial V/\partial y}{\epsilon})^2\big) \tag{14}$$

Since $\partial V/\partial x \leq 0$ holds almost everywhere we have $f \geq 0$ almost everywhere and at each point z where $\partial V/\partial z \neq 0$ we have

$$\lim_{\epsilon \to 0} f_\epsilon(z) = +\infty \tag{15}$$

Since (13 holds for each $\epsilon > 0$ and $\phi > 0$ in the open disc $|\zeta| < \delta$ we conclude that $\partial V/\partial z = 0$ must hold almost everywhere in the disc $D_\delta(z_0+s_0)$. Here we considered the largest s-value along the horisontal line ℓ. Of course, we get a similar conclusion for each $0 < s < s_0$ and hence $\partial V/\partial z = 0$ holds almost everywhere in the open set

(1). At the same time $V = 0$ in $D_\delta(z_0)$ and we conclude that $V = 0$ in the open set from (1) as requested.

B. A subharmonic majorization.

The result below appears in Carleman's book on quasi-analytic functions. Let $\{0 < \beta_1 < \beta_2 < \ldots\}$ be a sequence of positive real numbers. Given $a > 0$ we construct harmonic functions in the right half-plane $\Omega = \mathfrak{Re}\, z > a$ as follows: Set $q_\nu = a + i\beta_\nu$ and To each $z \in \Omega$ we get the triangle with corner points at $z, eq_\nu, eq_{\nu+1}$ where e is Neper's constant. Denote the angle at z by $\theta_\nu(z)$. Notice that $0 < \theta_\nu(z) < \pi$. As explained in § XX $\theta_\nu(z)$ is a harmonic function with the boundary value π on $\mathfrak{Re}\, z = a$ when $\beta_\nu < y < \beta_{\nu+1}$ while the boundary value is zero outside the closed y-interval $[\beta\nu, \beta_{\nu+1}]$. If we instead consider the points $\{q^*_\nu = a - ie\beta_\nu\}$ we get similar harmonic angle functions $\{\theta^*_\nu\}$ when we regard the angle at z formed by the triangle with corner points at $z - eq_\nu, -eq_{\nu+1}$.

Exercise. Show by euclidian geometry that if $b < 0$ is real and positive then

$$\sin\,\theta_\nu(a+b) = \frac{eb(\beta_{\nu+1} - \beta_\nu)}{\sqrt{(\beta_\nu^2 + b^2)(\beta_{\nu+1}^2 + b^2)}} \tag{1}$$

Use also that $\beta_\nu < \beta_{\nu+1}$ and show that (1) gives

$$\theta_\nu(a+b) > \frac{eb\beta_1^2 \cdot (\beta_{\nu+1} - \beta_\nu)}{\beta_{\nu+1} \cdot \beta_\nu \cdot (e^2\beta_1^2 + b^2)} = C(b, \beta_1) \cdot (\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) \quad : \quad C(b, \beta_1) = \frac{eb\beta_1^2}{e^2\beta_1^2 + b^2}$$

In addition to these θ_ν-functions we get the angle functions $\{\theta^*_n\}$ where we for each $n \geq 2$ consider the harmonic extension to the half-plane whose boundary values are π on $y > \beta_n$ and zero when $y < \beta_n$. This harmonic function is denoted by $\theta^*_n(z)$ and here $\theta^*_n(z)$ and by a figure the reader can verify that

$$\sin\,\theta^*_n(a+b) = \frac{b}{\beta_n^2 + b^2)} \implies \theta^*_n(a+b) > \frac{eb\beta_1^2}{e^2\beta_1^2 + b^2} \cdot \frac{1}{\beta_n} \tag{2}$$

7.1 A class of harmonic functions. Given the β-sequence above we also consider a sequence of positive real numbers $\{\lambda_\nu\}$. To each $n \geq 2$ we get the harmonic function $u_n(x, y)$ in Ω defined by

$$u_n(x, y) = \frac{1}{\pi} \cdot \sum_{\nu=1}^{\nu=n-1} \lambda_\nu \cdot (\theta_\nu + \theta^*_\nu) + \lambda_n \cdot (\theta_n + \theta^*_n) \tag{3}$$

If $z = a + b$ is real with $b > 0$ the inequalities in (1-2) give

$$u_n(a+b) \geq \frac{C(b, \beta_1)}{\pi} \cdot [\, \sum_{\nu=1}^{\nu=n-1} \lambda_\nu(\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) + \frac{\lambda_n}{\beta_n}\,] \tag{4}$$

From the above we get:

7.2 Proposition. *Let $\{\lambda_\nu\}$ and $\{\beta_\nu\}$ be such that*

$$(*)\qquad \lim_{n\to\infty}\Big[\sum_{\nu=1}^{\nu=n-1}\lambda_\nu\Big(\frac{1}{\beta_\nu}-\frac{1}{\beta_{\nu+1}}\Big)=+\infty$$

Then the sequence $\{u_n(a+b)\}$ increases to $+\infty$ for every $b>0$.

Remark. If the λ-sequence increases a partial summation gives

$$\sum_{\nu=1}^{\nu=n}\frac{\lambda_\nu-\lambda_{\nu-1}}{\beta_\nu}=\sum_{\nu=1}^{\nu=n-1}\lambda_\nu\Big(\frac{1}{\beta_\nu}-\frac{1}{\beta_{\nu+1}}\Big)$$

where we have put $\lambda_0=0$. Hence (*) is equivalent to the divergence of the positive series

$$(**)\qquad \sum_{\nu=1}^{\infty}\frac{\lambda_\nu-\lambda_{\nu-1}}{\beta_\nu}$$

7.3 An application. Let $\{\beta_\nu\}$ be a strictly increasing sequence and $\{\lambda_\nu\}$ a sequence of positive real numbers. Consider an analytic function $\Phi(z)$ defined in the half-plane $\mathfrak{Re}\,z>a$ with continuous boundary values on $\mathfrak{Re}\,z=a$ which satisfies the inequalities

$$(*)\qquad |\Phi(z)|\leq\Big(\frac{\beta_\nu}{|z|}\Big)^{\lambda_\nu}\quad:\quad \nu=1,2,\ldots$$

7.4 Exercise. Denote by $u_*(z)$ the harmonic function in the half-plane whose boundary values are one on $-\beta_1<y<\beta_1$ and otherwise zero. Show that (*) gives the following inequality on $\mathfrak{Re}\,z=a$ for every $n\geq 0$ and $-\infty<y<+\infty$

$$(7.4.1)\quad \log|\Phi(a+iy)|+u_n(a+iy)\leq\log K\cdot u_*(a+iy)\quad:\quad K=\max_{-\beta_1\leq y\leq\beta_1}|\Phi(a+iy)|$$

Conclusion. Since u_n and u_* are harmonic functions while $\log|\Phi|$ is subharmonic, the principle of harmonic majorization implies that (7.4.1) holds in Ω. In particular, for every real $b>0$ we have

$$(7.4.2)\qquad \log|\Phi(a+ib)|+u_n(a+ib)\leq\log K\cdot u_*(a+ib)$$

When Φ is not identically zero we can fix some $b>0$ where $\Phi(a+ib)\neq 0$ and (7.4.2) entails that the sequence $\{u_n(a+ib)\}$ is bounded. Hence Proposition 7.2 gives

7.5 Theorem. *Assume there exists a non-zero analytic function $\Phi(z)$ in the half-plane $\mathfrak{Re}\,z>a$ such that (*) holds in (7.3) that $\{\lambda_\nu\}$ is increasing. Then*

$$\sum_{\nu=1}^{\infty}\frac{\lambda_\nu-\lambda_{\nu-1}}{\beta_\nu}<\infty$$

Remark. We can rephrase the result above and get a vanishing principle. Namely, if the positive series in Theorem 7.5 is divergent an analytic function $\Phi(z)$ in the half-plane satisfying (7.3) must be identically zero.

7.6 Example. Let $\{c_n\}$ be a sequence of positive real numbers. Suppose that $\Phi(z)$ is analytic in the half-space and satisfies

$$|\Phi(z)| \leq \frac{c_n}{|z|^n} \quad : \quad n = 1, 2, \ldots$$

Then Φ must vanish identically if the series

$$\sum_{n=1}^{\infty} \frac{1}{c_n^{\frac{1}{n}}} = +\infty$$

C. Subharmonic minorant of a given function.

Introduction. Let Ω be a bounded open and connected subset of $\mathbf{C}$ and F a non-negative and upper semicontinuous function in Ω which may take values $+\infty$ at some points. Denote by $\mathcal{F}_*$ the class of subharmonic functions u in Ω such that $u(x) \leq F(x)$ hold for every $x \in \Omega$. Set

$$\mathcal{S}_F(x) = \max_{u \in \mathcal{F}_*} u(x)$$

We refer to $\mathcal{S}_F$ as Sjöberg's maximal function associated to F. Next, recall that every subharmonic function u is upper semicontinuous and for every compact subset K of Ω there is a constant C such that $u(x) \leq C$ on K. It turns out that this is the sole obstruction in order that $\mathcal{S}_F$ itself subharmonic. More precisely the following result was proved by Sjöberg in [1938; Scand. congress Helsinki]

1. Theorem. *If $\mathcal{S}_F$ is bounded on every compact subset of Ω then it is subharmonic and gives therefore the largest subharmonic function in the class $\mathcal{F}_*$.*

Remark. In addition to [Sjöberg[we refer to Domar's article *Subharmonic minorants of a given function* [Arkiv 1954] where Sjöberg's resuts is extended in a wider context and a similar result is proved for subharmonic functions in $\mathbf{R}^k$ when $k \geq 3$. One expects that if $F(x)$ enjoys some finiteness condition then Sjöberg's condition holds so that $\mathcal{S}_F$ gives the largest subharminic minorant to F. The following sufficency result was proved by Beurling in [Beurling xx]:

Theorem. *The function $\mathcal{S}_F$ is subharmonic if there exists $\epsilon > 0$ such that*

$$(*) \qquad \iint_\Omega \big[\log^+ F(x,y)\,\big]^{1+\epsilon} < \infty$$

Proof. Assume (*) and with ϵ kept fixed the value of the finite integral is denoted by $J(F)$. We shall prove that if K is a compact subset of Ω then there exists an integer n such that

$$(1) \qquad \max_{x \in K} u(x) \leq e^n$$

hold for every $u \in \mathcal{F}_*$. If this holds then Sjöberg's result entails that $\mathcal{S}_F$ is subharmonic. To prove (1) we fix a positive integer λ and a positive constant C such that

$$(2) \qquad \frac{e}{\pi C^2} + e^{-\lambda} \leq 1$$

Let us then take some $u \in \mathcal{F}_*$ and to each integer ν we set

$$(3) \qquad U_\nu = \{e^\nu \leq u < e^{\nu+1}\}$$

Then $\{U_\nu\}$ is a family of disjoint sets whose union is Ω and to each ν we denote by ℓ_ν the area of U_ν. Next, suppose that for some integer $n > \lambda$ there exists a point $z_n \in \Omega$ such that

$$(4) \qquad u(x_n) \geq e^n \quad \text{and} \quad \operatorname{dist}(x,\partial\Omega) > C \cdot \sqrt{\ell_{n-\lambda} + \cdot + \ell_n}$$

Consider the disc $D_n^* = \{|z - z_n| \leq C \cdot \sqrt{\ell_{n-\lambda} + \cdot + \ell_n}\}$.

Sublemma.The inequality (4) entails that

$$\max_{z \in D_n^*} u(z) \geq e^{n+1} \tag{5}$$

Proof. We argue by a contradiction. Set

$$\rho = C \cdot \sqrt{\ell_{n-\lambda} + \cdot + \ell_n}$$

(5) fails the upper semi-continuity of u gives some $\rho_* > \rho$ such that the disc $\Delta = \{|z - z_n| \leq \rho^*$ is contained in Ω and $u \leq e^{n+1}$ holds in Δ. The mean-value inequality gives

$$e^n \leq u(z_n) \leq \frac{1}{\pi \rho_*^2} \iint_\Delta u(x,y)\, dxdy \tag{6}$$

Now $U_\nu \cap \Delta = \emptyset$ if $\nu > n$ and since the U-sets are disjoint the right hand side in (6) is majorised by

$$\frac{1}{\pi \rho_*^2} \cdot e^{n+1}(\ell_n + \cdots + \ell_{n-\lambda}) + e^{n-\lambda} = e^n \cdot \frac{\rho^2}{\rho_*^2} \cdot [e \cdot \frac{\ell_n + \cdots + \ell_{n-\lambda}}{\pi} + e^{-\lambda}] \tag{7}$$

The last factor is $\frac{e}{\pi C^2} + e^{-\lambda}$ which is ≤ 1 by (2) above. Hence (6-7) would give

$$e^n \leq e^n \cdot \frac{\rho^2}{\rho_*^2}$$

Thus is a contradiction since $\rho_* > \rho$ and hence (4) $\implies$ (5) holds.

Proof continued. Next, given $z_n \in \Omega$ where $u(z_n) \geq e^n$ we set

$$\xi(m) = C \cdot \sum_{\nu=n}^{\nu=m} \sqrt{\ell_{\nu-\lambda} + \ldots + \ell_\nu} \quad : \forall\, m > n$$

Repeated use of (4) $\implies$ (5) shows that when $m > n$ and the disc $\{|z - z_n| \leq \xi(m)\}$ stays in Ω then it contains a point z_m where $u(z_m) \geq e^m$. Since u is bounded over compact subsets of Ω we must have

$$\operatorname{dist}(z_n, \partial\Omega) \leq C \cdot \sum_{\nu=n}^{\infty} \sqrt{\ell_{\nu-\lambda} + \ldots + \ell_\nu} \tag{*}$$

Let us majorize the right hand side in (*). Since $\sqrt{a_1 + \ldots + a_k} \leq \sqrt{a_1} + \ldots + \sqrt{a_k}$ hold for all tuples of positive numbers the right hand side in (*) is majorized by

$$C \cdot \sum_{\nu=n}^{\infty} [\sqrt{\lambda_{\nu-\lambda} + \ldots + \lambda_\nu}] \leq C \cdot (\lambda + 1) \sum_{\nu=n-\lambda}^{\infty} \sqrt{\lambda_\nu}$$

To estimate the sum of the square roots we pick the positive ϵ in the Theorem and write

$$\sum_{\nu=n-\lambda}^{\infty} \sqrt{\lambda_\nu} = \sum_{\nu=n-\lambda}^{\infty} \nu^{-1/2-\epsilon/2} \cdot \sqrt{\lambda_\nu} \cdot \nu^{1/2+\epsilon/2} \leq \sqrt{\sum_{\nu=n-\lambda}^{\infty} \nu^{-1-\epsilon}} \cdot \sqrt{\sum_{\nu=n-\lambda} \lambda_\nu \cdot \nu^{1+\epsilon}}$$

where the Cauchy-Schwarz inequality was used. Since $\epsilon > 0$ the first factor is a function $n \mapsto \tau(n)$ given as a square root of tail sums of a convergent series and hence $\tau(n) \to 0$ when $n \to +\infty$. Now (*) is majorised by

$$(**) \qquad C \cdot \tau(n) \cdot \sqrt{\sum_{\nu=n-\lambda} \lambda_\nu \cdot \nu^{1+\epsilon}}$$

Next, we have

$$\log^+ u(z) \geq \nu : |, z \in U_\nu\}$$

Since the sets $\{U_\nu\}$ are disjoint it follows that

$$\sum_{\nu=n-\lambda} \lambda_\nu \cdot \nu^{1+\epsilon} \leq \iint_\Omega [\log^+ u(x,y)|^{1+\epsilon}\, dxdy \leq J(F)$$

where the last inequality follows since $u \leq F$. integral in the theorem whose value is denoted by $J(F)$. Hence (*) gives

$$\text{dist}(z_n, \partial\Omega) \leq C \cdot \sqrt{J(F)} \cdot \tau(n)$$

Finally, if K is a compact subset of Ω its distance to $\partial\Omega$ is a positive number a_K and since $\tau(n) \to 0$ we find a large integer N such that $C \cdot \sqrt{J(F)} \cdot \tau(n) \leq a_K$ and by the above it follows that $u \leq e^n$ holds on K. Since $u \in \mathcal{F}_*$ was arbitrary we have proved that $\mathcal{S}_F$ is bounded on K and Theorem 2 is proved.

D. An inequality for harmonic functions.

The result below appears in Beuling's article [xxx]. Denote by U_+ the upper half-plane $\{\mathfrak{Im}\, z > 0\}$.

13.4 Theorem. *For each positive harmonic function $u(x,y)$ in U_+ we put*

$$E_u = \{x+iy \in U_+ \quad : \; u(x,y) \geq y\}$$

Then one has the implication

$$(*) \qquad U_+ \setminus E_u \neq \emptyset \implies \int_{E_u} \frac{dxdy}{1+x^2+y^2} < \infty$$

Proof. The impliction (*) is obvious when $u(x,y) = ay$ for some positive real number a, i.e. regard the cases $a < 1$ and $a \geq 1$ separately. From now on we assume that $u(x,y)$ is not a linear function of y and the wellknown Harnack-Hopf inequality gives the strict inequality

$$(1) \qquad \frac{\partial u}{\partial y}(x,y) < \frac{u(x,y)}{y} \quad : \; (x,y) \in U_+$$

Set $u_* = y - u$ which entails that

$$(2) \qquad \{u_* > 0\} = U_+ \setminus E_u$$

3. A simply connected domain. We assume that the set $U_+ \setminus E_u$ is non-emtpy and let Ω be some connected component of this open set. Replacing u by the harmonic function $u(x+a,y)$ for a suitable real a we may assume that Ω contains a point on the imaginary axis and after a scaling that $i \in \Omega$. Next, (1) gives

$$(4) \qquad \frac{\partial u_*}{\partial y} = 1 - \frac{\partial u}{\partial y} > 1 - \frac{u}{y}$$

Starting at a point $(x,y) \in \Omega$ it follows that $s \mapsto u_*(x, y+is)$ increases strictly on $\{s \geq 0\}$ and hence the open set Ω is placed above a graph, i.e. there exists an interval (a,b) on the real x-axis and a non-negative continuous function $g(x)$ on (a,b) such that

$$(5) \qquad \Omega = \{(x,y) : a < x < b \; : \; y > g(x)\}$$

In (5) it may occur that the interval (a,b) is unbounded and may even be the whole x-axis in which case Ω is the sole component of $U_+ \setminus E_u$. In any case, since $i \in \Omega$ one has $a < 0 < b$ and Ω contains the the set $\{ai\colon\; a \geq 1\}$. Next, for each $r > 2$ we put

$$(6) \qquad C_r = U_+ \cap \{|z+i| < r\}$$

Notice that C_r contains the point $(r-1)i$ which also belongs to Ω. Denote by γ_r be the largest open interval on the circle $\{|z+i| = r$ which contains $(r-1)i$ and at the same time is contained in $\partial\Omega$. With the aid of a figure it is clear that the set $\Omega_r = \Omega \cap C_r$ is simply connected where γ_r is a part of ∂D_r and $u_* = 0$ on $\partial D_r \setminus \gamma_r$.

The use of harmonic measure. Let $h_r(z)$ be the harmonic function in Ω_r which is 1 on γ_r and zero on the rest of the boundary. By the above and the maximum principle for harmonic functions we have the inequality

$$u_*(i) \leq h_r(i) \cdot \max_{z \in \partial D_r} u_*(z) \leq r \tag{7}$$

where the last inequality follows since $u_*(x,y) \leq y \leq r$. To estimate the function $h_r(z)$ with resoect to both r and z we shall use the harmonic function

$$\Psi(z) = \log \big| \frac{z+i}{2i} \big| \implies |\nabla(\Psi)|^2 = \Psi_x^2 + \Psi_y^2 = \frac{1}{|z+i|^2} \tag{8}$$

Set

$$m(r) = \frac{1}{\pi} \iint_{E_u \cap C_r} |\nabla(\Psi)|^2 \, dxdy = \frac{1}{\pi} \iint_{E_u \cap C_r} \frac{1}{|z+i|^2} | \, dxdy \tag{9}$$

Notice that

$$\frac{1}{\pi} \iint_{C_r} \frac{1}{|z+i|^2} | \, dxdy < \log r \tag{10}$$

Together (9-10) give the inequality

$$\frac{1}{\pi} \iint_{\Omega_r} |\nabla(\Psi)|^2 \, dxdy < \log r - m(r) \tag{11}$$

Next, we notice that

$$\Psi(z) = \log r - \log 2 \quad \text{on} \quad \gamma_r \quad \text{and} \quad \Psi(i) = 0 \tag{12}$$

Now we can apply the general inequality from § XX in [Beurling-Special Topics] and conclude that (11-12) give the inequality

$$h_r(i) \leq e^{-\frac{\pi L^2}{A}} \quad \text{where} \quad L = \log r - \log 2 \tag{13}$$

Next, the reader may verify that there exists a constant C which is independent of $r \geq 2$ such that

$$\frac{\pi L^2}{A} \geq \frac{\lg r - \log 2)^2}{\log r - \log m(r)} \geq \log r - 2\log 2 + m(r) \cdot \big[1 - \frac{C}{\log r}\big] \tag{14}$$

Then (13-14) gives a constant C_1 such that

$$h_r(i) \leq e^{-\frac{\pi L^2}{A}} \leq C_1 \cdot \frac{1}{r} \cdot e^{-m(r)} \quad : \ r \geq 2 \tag{15}$$

Then (7) and (15) give

$$u_*(i) \leq C_1 \cdot \cdot e^{-m(r)} \tag{*}$$

This inequality hold for all $r > 2$ and since $u_*(i) > 0$ there must exist a finite limit

$$\lim_{r \to \infty} m(r) = m^* \tag{**}$$

The requested implication in Theorem 15.2 follows since (9) this means that

$$\iint_{E_u} \frac{1}{|z+i|^2} \, dxdy = \lim_{r \to \infty} m(r)$$

E. Carleman's differential inequality and the Denjoy conjecture

Introduction. Let ρ be a positive integer and $f(z)$ is an entire function such that there exists some $0 < \epsilon < 1/2$ and a constant A_ϵ such that

$$|f(z)| \leq A_\epsilon \cdot e^{|z|^{\rho+\epsilon}} \tag{0.1}$$

hold for every z. Then we say that f has integral order $\leq \rho$. Next, the entire function f has an asymptotic value a if there exists a Jordan curve Γ parametrized by $t \mapsto \gamma(t)$ for $t \geq 0$ such that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and

$$\lim_{t \to +\infty} f(\gamma(t)) = a \tag{0.2}$$

In 1907 Denjoy conjectured that (0.1) implies that the entire function f has at most 2ρ many different asymptotic values. Examples show that this upper bound is sharp. In an artice from 1920 Carelan proved that the number of asumptotic values cannot exceed $5 \cdot \rho$. The sharp bound which settled the Denjoy conjecture was proved in 1929 by Ahlfors in [Ahl] and independently by Beulring the same year while he prepared his Ph.D thesis. A few years later T. Carleman found an alternative proof based upon a certain differential inequality which has applications beyond the proof of the Denjoy conjecture for estimates of harmonic measures. See [Ga-Marsh].

A. The differential inequality.

The conclusive result appears in Theorem XX below. We begin to describe the geometric situation. Let Ω be a connected open set in $\mathbf{C}$ whose intersection S_x between a vertical line $\{\mathfrak{Re}\, z = x\}$ is a bounded set on the real y-line for every x. When $S_x \neq \emptyset$ it is the disjoint union of open intervals $\{(a_\nu, b_\nu)\}$ and we set

$$\ell(x) = \max_\nu (b_\nu - a_\nu) \tag{*}$$

Next, let $u(x,y)$ be a positive harmonic function in Ω which extends to a continuous function on the closure $\bar{\Omega}$ with the boundary values identical to zero. Define the function ϕ by:

$$\phi(x) = \int_{S_x} u^2(x,y) \cdot dy \tag{1}$$

The Federer-Stokes theorem gives the following formula for the derivatives of ϕ:

$$\phi'(x) = 2\int_{S_x} u_x \cdot u(x,y)dy \tag{2}$$

$$\phi''(x) = 2\int_{S_x} u_{xx} \cdot u(x,y)dy + 2\int_{S_x} u_x^2 \cdot dy \tag{3}$$

Since $\Delta(u) = 0$ when $u > 0$ we have

$$2\int_{S_x} u_{xx} \cdot u(x,y)dy = -2\int_{S_x} u_{yy} \cdot u(x,y)dy = 2\int u_y^2 dy \tag{4}$$

The Cauchy-Schwarz inequality applied in (2) gives

$$\phi'(x)^2 \leq 4 \cdot \int_{S_x} u_x^2 \cdot \int_{S_x} u^2(x,y)dy = 4 \cdot \phi(x) \cdot \int_{S_x} u_x^2 dy \tag{5}$$

Hence (4) and (5) give:

$$\phi''(x) \geq 2\int_{S_x} u_y^2(x,y) \cdot dy + \frac{1}{2} \cdot \frac{\phi'^2(x)}{\phi(x)} \tag{6}$$

Next, since $u(x,y) = 0$ at the end-points of all intervals of S_x, *Wirtinger's inequality* and the definition of $\ell(x)$ give:

$$\int_{S_x} u_y^2(x,y) \cdot dy \geq \frac{\pi^2}{\ell(x)^2} \cdot \phi(x) \tag{7}$$

Inserting (7) in (6) we have proved

A.1 Proposition *The ϕ-function satisfies the differential inequality*

$$\phi''(x) \geq \frac{2\pi^2}{\ell(x)^2} \cdot \phi(x) + \frac{\phi'^2(x)}{2\phi(x)}$$

Proof continued. The maximum principle for harmonic functions implies that the $\phi(x) > 0$ when $x > 0$ and hence there exists a ψ-function where $\phi(x) = e^{\psi(x)}$. It follows that

$$\phi' = \psi' e^{\psi} \quad \text{and} \quad \phi'' = \psi'' e^{\psi} + \psi'^2 e^{\psi}$$

Now Proposition A.1 gives

$$\psi'' + \frac{\psi'^2}{2} \geq \frac{2\pi^2}{\ell(x)^2} \tag{*}$$

A.2 An integral inequality. From (*) we obtain

$$\frac{2\pi}{\ell(x)} \leq \sqrt{\psi'(x)^2 + 2\psi''(x)} \leq \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

Taking the integral we get

$$2\pi \cdot \int_0^x \frac{dt}{\ell(t)} \leq \psi(x) + \log\, \psi'(x) + O(1) \leq \psi(x) +, \psi'(x) + O(1) \tag{**}$$

where $O(1)$ is a remainder term which is bounded independent of x. Taking the integral once more we obtain:

A.3 Theorem. *The following inequality holds:*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + O(x)$$

where the remainder term $O(x)$ is bounded by Cx for a fixed constant.

B. Solution to the Denjoy conjecture

B.1 Theorem. *Let $f(z)$ be entire of some integral order $\rho \geq 1$. Then f has at most 2ρ many different asymptotic values.*

Proof. Suppose f has n different asymptotic values $a_1, \ldots, a_n$. To each a_ν there exists a Jordan arc Γ_ν as described in the introduction. Since the a-values are different the n-tuple of Γ-arcs are separated from each other when $|z|$ is large. So we can find some R such that the arcs are disjoint in the exterior disc $|z| > R$. We may also consider the tail of each arc, i.e. starting from the last point on Γ_ν which intersects the circle $|z| = R$. So now we have an n-tuple of disjoint Jordan curves in $|z| \geq R$ where each curve intersects $|z| = R$ at some point p_ν and after the curves moves to the point at infinity. See figure. Next, we take one of these curves, say Γ_1. Let D_R^* be the exterior disc $|\zeta| > R$. In the domain $\Omega = \mathbf{C} \setminus \Gamma_1 \cup D_R^*$ we can choose a single-valued branch of $\log \zeta$ and with $z = \log \zeta$ the image of Ω is a simply connected domain Ω^* where S_x for each x has length strictly less than 2π The images of the Γ-curves separate Ω^* into n many disjoint connected domains denoted by $D_1, \ldots, D_n$ where each D_ν is bordered by a pair of images of Γ-curves and a portion of the vertical line $x = \log R$.

Let $\zeta = \xi + i\eta$ be the complex coordinate in Ω^*. Here we get the analytic function $F(\zeta)$ where

$$F(\log(z)) = f(z)$$

We notice that F may have more growth than f. Indeed, we get

$$(1) \qquad |F(\xi + i\eta)| \leq \exp\left(e^{(\rho+\epsilon)\xi}\right)$$

With $u = \text{Log}^+ |F|$ it follows that

$$(2) \qquad u(\xi, \eta) \leq e^{(\rho+\epsilon)\xi}$$

Hence the ϕ-function constructed during the proof of Theorem A.3 satisfies

$$\phi(\xi) \leq e^{2(\rho+\epsilon)\xi}$$

It follows that the ψ-function satisfies

$$(3) \qquad \psi(\xi) = 2 \cdot (\rho + \epsilon)\xi + O(1)$$

Now we apply Theorem A.3 in each region D_ν where we have a function $\ell_\nu(\xi)$ constructed by (0) in section A. This gives the inequality

$$(4) \qquad 2\pi \cdot \int_R^\xi \frac{\xi - s}{\ell_\nu(s)} \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1) \quad : \quad 1 \leq \nu \leq n$$

Next, recall the elementary inequality which asserts that if $a_1, \ldots, a_n$ is an arbitrary n-tuple of positive numbers then

$$(5) \qquad \sum a_\nu \cdot \sum \frac{1}{a_\nu} \geq n^2$$

For each s we apply this to the n-tuple $\{\ell_\nu(s)\}$ where we also have

$$\sum \ell_\nu(s) \leq 2\pi$$

So a summation in (4) over $1 \leq \nu \leq n$ gives

$$n \cdot \int_R^\xi (\xi - s) \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1) \tag{6}$$

Another integration gives:

$$n \cdot \frac{\xi^2}{2} \leq (\rho + \epsilon) \cdot \xi^2 + O(\xi) \tag{7}$$

This inequality can only hold for large ξ if $n \leq 2(\rho + \epsilon)$ and since $\epsilon < 1/2$ is assumed it follows that $n \leq 2\rho$ which finishes the proof of the Denjoy conjecture.

F. Ganelius' inequality for conjugate harmonic functions.

Introduction. We prove a result due to Ganelius from the article *Sequences of analytic functions and their zeros* Arkiv för matematik. Vol 3 (1953). Here is the situation: Let u be a harmonic function defined in some open neighborhood of the closed unit disc where $u(0) = 0$. Its harmonic conjugate v is normalised so that $v(0)$. The constants H, K are defined by:

$$(*) \qquad \max_\theta u(e^{i\theta}) = H \quad : \quad \max_\theta \frac{\partial v}{\partial \theta}(e^{i\theta}) = K$$

where the maximum is taken over $0 \leq \theta \leq 2\pi$.
14.1 Theorem. *There exists an absolute constant C such that*

$$\max_{z \in D} |v(z)| \leq C \cdot \sqrt{HK}$$

Before the proof starts we notice that the mean-value property for harmonic functions and the equality $u(0) = 0$ give

$$\min_\theta u(e^{i\theta}) = -\max_\theta u(e^{i\theta})$$

Replacing u by $-u$ it therefore suffices to get an upper bound for the maximum of v, and by a rotation it suffices to get the inequality when $\theta = 0$, i.e. there remains to show that

$$v(1) \leq C \cdot \sqrt{HK}$$

where the subsequent proof will provide the constant C.

Proof. To each real $\lambda > 1/2$ we put

$$G_\lambda(z) = \log |\frac{1 + z^\lambda}{1 - z^\lambda}|$$

which gives a harmonic function of z in the sector of the open unit disc where

$$-\frac{\pi}{2\lambda} < \arg z < \frac{\pi}{2\lambda}$$

When $z = e^{i\theta}$ we see that

$$G_\lambda e^{i\theta}) = \log \frac{\cos(\lambda\theta/2)}{|\sin(\lambda\theta/2)|}$$

1. Exercise. Apply Green's formula to domains $S \cap \{|z - 1| > \epsilon\}$ and show that the equation below for each $0 \leq \phi \leq 2\pi$:

$$2\lambda \cdot \int_0^1 \frac{u(re^{i(\phi - \frac{\pi}{2\lambda})} + u(re^{i(\phi + \frac{\pi}{2\lambda})}}{1 + r^2}\, r^{\lambda - 1}\, dr - \pi u(e^{i\phi}) =$$

$$(1) \qquad \int_{\frac{\pi}{2\lambda}}^{-\frac{\pi}{2\lambda}} \log \frac{\cos(\lambda\theta/2)}{|\sin(\lambda\theta/2)|} \cdot \frac{\partial u}{\partial r}(e^{i(\phi + \theta)}\, d\theta$$

Next, the Cauchy-Riemann equations entail that

$$\frac{\partial u}{\partial r}(e^{i(\phi+\theta}) = \frac{\partial v}{\partial \theta}(\phi+\theta)$$

Choose some $0 < \xi < \pi$ and integrate (1) with respect to ϕ over $(-\xi,\xi)$. The integral of the last term in (1) becomes

$$\int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log\frac{\cos(\lambda\theta/2)}{\sin(\lambda\theta/2)}\cdot\big[v(\xi+\theta)-v(-\xi+\theta)\big]\,d\theta \tag{2}$$

Exercise. By assumption $\frac{dv}{d\theta} \leq K$ holds for the θ-periodic v-function. Show via Rolle's theorem that

$$v(\xi+\theta)-v^*(-\xi+\theta) \leq v(\xi-\frac{\pi}{2\lambda})-v(-\xi+\frac{\pi}{2\lambda})+K\cdot\frac{\pi}{\lambda} \quad : \quad -\frac{\pi}{2\lambda} < \theta < \frac{\pi}{2\lambda} \tag{3}$$

Set

$$I(\lambda) = \int_{-\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log\frac{\cos(\lambda\theta/2)}{\sin(\lambda\theta/2)}\,d\theta \tag{4}$$

Then (3) entails that (2) is majorised by

$$I(\lambda)\cdot[v(\xi-\frac{\pi}{2\lambda})-v(-\xi+\frac{\pi}{2\lambda})+K\cdot\frac{\pi}{\lambda}] \tag{5}$$

Exercise. Show that

$$I(\lambda) = \frac{4}{\lambda}\int_1^\infty \frac{\log t}{1+t^2}\,dt = \frac{4}{\lambda}\cdot\mathcal{C} \quad : \mathcal{C} = 1+\sum_{m=1}^\infty (-1)^m\cdot\frac{1}{(m+1)^2} \simeq 0,916\ldots \tag{6}$$

7. A lower bound. Recall that u is harmonic with $u(0)=0$ and $-H \leq u(\theta) \leq H$ hold for all θ. This gives

$$\int_a^b u(\theta)\,d\theta \geq -2\pi H$$

for all intervals $(a,b)($ on the unit circle and from this the reader can check that thr integral over $-\xi < \phi < \xi)$ of the first two term in (1) is bounded below by $-4\pi H$ for each $0 < \xi \leq \pi$.

From the above it follows that

$$\frac{4}{\lambda}\cdot\mathcal{C}\big[v(\xi+\theta)-v(-\xi+\theta)\big] \geq -4\pi H - \frac{4K\pi\mathcal{C}}{\lambda^2}) \tag{8}$$

Since v is 2π-periodic we see that (8) gives

$$v(2\kappa+\theta)-v(\theta) \geq -\pi\cdot(\lambda\cdot H+\frac{K\mathcal{C}}{\lambda}) \quad : \; 0 \leq \kappa \leq 2\pi \tag{9}$$

Next, since $v(0)=0$ and v is harmonic the boundary value function $v(\theta)$ has at least one zero κ and with $\theta=0$ in (9) we get

$$v(1) \leq \pi\cdot(\lambda\cdot H+\frac{K\mathcal{C}}{\lambda}) \tag{11}$$

12. The choice of λ. Above (11) hold for all $\lambda > 1/2$.

and we choose λ to minimize

$$K \cdot \frac{\pi}{\lambda} + \frac{4}{\lambda} \cdot k \cdot H$$

It means that we take $\lambda = \sqrt{\frac{Kk}{\pi H}}$ and (7) gives the inequality

$$v^*(\xi - \frac{\pi}{2\lambda}) - v^*(-\xi + \frac{\pi}{2\lambda}) \geq -2\pi \cdot \sqrt{\frac{\pi}{k} HK} \tag{8}$$

Since (8) hold for all ξ and v^* is 2π-periodic we obtain

$$v^*(2\kappa + \xi) - v^*(\xi) \geq -2\pi \cdot \sqrt{\frac{\pi}{k} HK} \quad \forall\ 0 \leq \xi, \kappa \leq 2\pi \tag{9}$$

Finally, since $v(0) = 0$ and v is harmonic the boundary value function v^* has at least one zero ξ_0 and we can choose κ so that

$$v^*(2\kappa + \xi_0) = \min_\theta\ V^*(\theta)$$

Then (9) implies that the minimum of v^* is $\geq -2\pi \cdot \sqrt{\frac{\pi}{k} HK}$. In the same way the reader can verify that the maximum of v^* cannot exceed $2\pi \cdot \sqrt{\frac{\pi}{k} HK}$. Hence (*) in Theorem 14.1 holds with

$$C = 2\pi \cdot \sqrt{\frac{\pi}{k}} \tag{10}$$

where k is the constant from Exercise 4.

14.2 Applications to zeros of polynomials.

Let $n \geq 1$ and consider a polynomial

$$p(z) = \prod (1 - ze^{-i\theta_\nu})$$

where $\{\theta_\nu\}$ is some n-tuple in the periodic interval $[0, 2\pi]$. Let (α, β) be an interval on the unit circle. Denote by $N(\alpha, \beta)$ the number of θ_ν satisfying

$$\alpha < \theta_\nu < \beta$$

and $|p|_D$ is the maximun norm of p over the unit disc.

14.3 Theorem. *With the same absolute constant as in Theorem 14.1 one has*

$$[\frac{N(\alpha, \beta)}{n} - \frac{\beta - \alpha}{2\pi}| \leq C \cdot \sqrt{n \cdot \log |p|_D}$$

Proof. Consider the harmonic function $u(z)$ in the unit disc defined by

$$u(z) = \frac{1}{\pi} \log\ |p(z)| = \frac{1}{\pi} \cdot \sum \log\ |1 - ze^{-i\theta_\nu}|$$

It follows that the conjugate harmonic function becomes

$$v(z) = \frac{1}{\pi} \cdot \sum \arg\,(1 - z \cdot e^{-i\theta_\nu})$$

From this the reader can verify the inequality

$$\frac{\partial v}{\partial\theta}(re^{i\theta}) \leq \frac{n}{2\pi} \quad : \ 0 < r < 1$$

From the above Theorem 14.1 gives

$$\max_\theta v(\theta) \leq C\sqrt{\frac{n}{2\pi}\cdot\log|p|_D} = \sqrt{\frac{2\pi}{k}}\cdot\sqrt{n\cdot\log|p|_D}$$

where k is Catalani's constant from Exercise 4.

NOW DONE by generalformula !

14.4 Schur's inequality.

A polynomial $P(z)$ of degree n with constant term $a_0 \neq 0$ can be written as

$$P(z) = a_0 \cdot \prod (1 - \frac{z}{\alpha_k})$$

which entails that the absolute value of coefficient a_n of z^n becomes

$$(1) \qquad |a_n| = \frac{|a_0|}{\prod|\alpha_k|}$$

Put $\alpha_k = \rho_k \cdot e^{i\theta_k}$ and set

$$p(z) = \prod(1 - e^{-i\theta_k}z)$$

14.5 Theorem. *One has the inequality*

$$|p|_D \leq \frac{|P|_D}{\sqrt{|a_0 a_n|}}$$

Proof. For each k one we set $|\alpha_k| = r_k$ and notice that

$$r_k \cdot |1 - \frac{e^{i\theta}}{\alpha_k}|^2 = r_k + \frac{1}{r_k} - 2\cos(\theta - \theta_k) \geq 2 - 2\cos(\theta - \theta_k) = |1 - e^{i(\theta-\theta_k)}|^2$$

This holds for each θ we choose θ where $|p|$ takes its maximum and conclude that

$$|p|_D^2 \leq \frac{\prod r_k}{|a_0|^2}\cdot|P|_D^2$$

Taking the square root and using (1) above we get Theorem 14.5

14.6 Carlson's formula. Studies of zeros of polynomials, and more generally of analytic functions, in sectors were treated by F. Carlson in the article *Sur quelques suites de polynomes*. We present a specific formula by Carlson. Let $P(z)$ be a polynomial of some degree n where $P(x) \neq 0$ for every real $x \leq 0$. The argument of each root α_ν of P is taken in the interval $(-\pi, \pi)$. If $0 < \phi < \pi$ we set

$$J(\phi) = \sum(\phi - |\arg\alpha_\nu)|$$

where the sum is taken over roots of P with their arguments in $(-\phi, \phi)$.

14.6.1 Theorem. *For each ϕ one has the equality*

$$J(\phi) = \frac{n\phi^2}{2\pi} + \frac{1}{2\pi}\int_0^\infty \frac{\log|P(re^{i\phi})| \cdot \log|P(re^{-i\phi})|}{|\log P(r)|^2}\,\frac{dr}{r}$$

Proof By a factorisation of P the proof is reduced to the case $n = 1$ with $P(z) = 1 - \frac{1}{\alpha}$ for some non-zero complex number α and here the verification is left as an exercise to the reader.

The formula above is special and yet instructive. Let us finally mention that one can study distributions of zeros for Dirichlet series and exponential polynomials. This requires a further analysis and will not be exposed here. The reader may consult the cited article by Ganelis for results about complex zeros of exponential polynomials.

G. Beurling's minimum principle for positive harmonic functions

Introduction. We expose a result from the article [Beurling:Ann.sci, Fennica]. Let Ω be a simply connected domain. Let $g(z,\zeta)$ be the Green function for Ω. If $p \in \partial\Omega$ a Martin function with respect to p is a positive harmonic function ϕ in Ω such that

$$\lim_{z \to q} \phi(z) = 0 \quad : \forall \ q \in \partial\Omega \setminus \{p\}$$

The class of these Martin functions is denoted by $\mathcal{M}(p)$. Following Beurling we give

15.0 Definition. *A sequence $\mathcal{S} = \{z_n\}$ in Ω which convergs to the boundary point p is an equivalence sequence at p if the following implication hold for every positive harmonic function u in D, every real $\lambda > 0$ and each $\phi \in \mathcal{M}(p)$*

$$u - \lambda \cdot \phi \big| S \geq 0 \implies u \geq \lambda \cdot \phi \quad \text{holds in the whole domain } \Omega$$

15.1 Theorem. *A sequence $\mathcal{S} = \{z_n\}$ which converges to p is an equivalence at p if and only if it constains a subsequence $\{\xi_k = z_{n_k}\}$ such that*

$$\sup_{k \neq j} g(\xi_j, \xi_k) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} g(z, \xi_k) \cdot \phi(\xi_k) = +\infty \quad : \ z \in \Omega$$

Remark. The necessity in Theorem 15.1 proved rather easily. See § xx below. From now on we treat the non-trivial part, i.e.we prove that the two conditions in Theorem 15.1 entail that $\mathcal{S}$ is an equivalence at p. By Riemann's mapping theorem it suffices to prove this sufficiency when Ω is the unit disc D. Since every positive harmonic function u in D is the Poisson extension of a non-negative Riesz measure on T, it follows that every $\phi \in \mathcal{M}(p)$ is the Poisson extension of the unit point mass at p times a positive constant λ, i.e. of the form

$$\phi_\lambda(z) = \frac{\lambda}{2\pi} \cdot \frac{1 - |z|^2}{|1 - z|^2}$$

where the mean-value property gives $\phi_\lambda(0) = \lambda$. Moreover, if u is a positive harmonic function given by the Poisson extension P_μ of some non-negative Riesz measure μ in D. then the inequality $P_\mu \geq \phi_\lambda$ holds everywhere in D if and only if the measure μ has an atom at $z = 1$ of mass $\geq \lambda$. Up to scaling it suffices to consider the case $\lambda = 1$ and hence $\mathcal{S}$ is an equivalence sequence if and only if the following hold for every non-negative Riesz measure on T:

$$(*) \qquad P_\mu(z_n) \geq \phi_1(z_n) \quad : \ n = 1, 2, \ldots \implies \mu(\{1\}) \geq 1$$

To prove (*) we prefer to work in the upper half-plane where p is taken as ∞. Here we get Martin function $\phi(x,y) = y$ and Green's function is

$$g(\zeta, z) = \log \frac{|z - \bar{\zeta}|}{|z - \zeta)}$$

When $z = re^{i\theta}$ where r is large and $0 < \theta < \pi$ one has

$$\frac{|re^{i\theta}+i|^2}{|re^{i\theta}-i|^2} = \frac{r^2+1+2r\sin\theta}{r^2+1-2r\sin\theta}$$

It follows that

$$g(i, re^{i\theta}) = \frac{2\sin\,\theta}{r} + O(r^{-2})$$

in polar coordinates we write $\{\zeta_k = r_k e^{i\theta_k}\}$ where $\{0 < \theta_k = \arg\xi_k < \pi\}$ and from the above it is clear that the two conditions in Theorem 15.1 are equivalent to

$$(**) \qquad \inf_{j\neq k} \frac{|\xi_j - \xi_k|}{|\xi_j - \bar{\xi}_k|} \quad : \quad \sum_{k=1}^{\infty} \sin^2\,\theta_k = +\infty$$

At this stage we need a result which has independent interest. In the final part of the proof of Theorem 15.2 below we derive the sufficiency in Theorem 15.1.

15.2 Theorem. *For each positive harmonic function $u(x,y)$ in U_+ we put*

$$E_u = \{x+iy \in U_+ \quad : \; u(x,y) \geq y\}$$

Then one has the implication

$$(15.2) \qquad U_+ \setminus E_u \neq \emptyset \implies \int_{E_u} \frac{dxdy}{1+x^2+y^2} < \infty$$

Proof. The impliction (15.2) is obvious when $u(x,y) = ay$ for some positive real number a, i.e. regard the cases $a < 1$ and $a \geq 1$ separately. From now on we assume that $u(x,y)$ is not a linear function of y and the Harnack-Hopf inequality from § XX gives the strict inequality

$$(1) \qquad \frac{\partial u}{\partial y}(x,y) < \frac{u(x,y)}{y} \quad : \; (x,y) \in U_+$$

Set $u_* = y - u$ which entails that

$$(2) \qquad \{u_* > 0\} = U_+ \setminus E_u$$

3. A simply connected domain. Assume that the set $U_+ \setminus E_u$ is non-emtpy and let Ω be a connected component of this open set. Replacing u by the harmonic function $u(x+a,y)$ for a suitable real a we may assume that Ω contains a point on the imaginary axis and after a scaling that $i \in \Omega$. Next, (1) gives

$$(4) \qquad \frac{\partial u_*}{\partial y} = 1 - \frac{\partial u}{\partial y} > 1 - \frac{u}{y}$$

Starting at a point $(x,y) \in \Omega$ it follows that $s \mapsto u_*(x, y+is)$ increases strictly on $\{s \geq 0\}$ and hence the open set Ω is placed above a graph, i.e. there exists an interval (a,b) on the real x-axis and a non-negative continuous function $g(x)$ on (a,b) such that

$$(5) \qquad \Omega = \{(x,y) : a < x < b \; : \; y > g(x)\}$$

In (5) it may occur that the interval (a,b) is unbounded and may even be the whole x-axis in which case Ω is the sole component of $U_+ \setminus E_u$. In any case, since $i \in \Omega$ one has $a < 0 < b$ and Ω contains the the set $\{ai\colon\ a \geq 1\}$. Next, for each $r > 2$ we put

$$C_r = U_+ \cap \{|z+i| < r\} \tag{6}$$

Notice that C_r contains the point $(r-1)i$ which also belongs to Ω. Denote by γ_r be the largest open interval on the circle $\{|z+i| = r$ which contains $(r-1)i$ and at the same time is contained in $\partial\Omega$. With the aid of a figure it is clear that the set $\Omega_r = \Omega \cap C_r$ is simply connected where γ_r is a part of ∂D_r and $u_* = 0$ on $\partial D_r \setminus \gamma_r$.

The use of harmonic measure. Let $h_r(z)$ be the harmonic function in Ω_r which is 1 on γ_r and zero on the rest of the boundary. By the above and the maximum principle for harmonic functions we have the inequality

$$u_*(i) \leq h_r(i) \cdot \max_{z \in \partial D_r} u_*(z) \leq r \tag{7}$$

where the last inequality follows since $u_*(x,y) \leq y \leq r$. To estimate the function $h_r(z)$ with respect to r and z we use the harmonic function

$$\Psi(z) = \log|\frac{z+i}{2i}| \implies |\nabla(\Psi)|^2 = \Psi_x^2 + \Psi_y^2 = \frac{1}{|z+i|^2} \tag{8}$$

Set

$$m(r) = \frac{1}{\pi}\iint_{E_u \cap C_r} |\nabla(\Psi)|^2\,dxdy = \frac{1}{\pi}\iint_{E_u \cap C_r} \frac{1}{|z+i|^2}|\,dxdy \tag{9}$$

Notice that

$$\frac{1}{\pi}\iint_{C_r} \frac{1}{|z+i|^2}|\,dxdy < \log r \tag{10}$$

Together (9-10) give the inequality

$$\frac{1}{\pi}\iint_{\Omega_r} |\nabla(\Psi)|^2\,dxdy < \log r - m(r) \tag{11}$$

Next, we notice that

$$\Psi(z) = \log r - \log 2 \quad \text{on} \quad \gamma_r \quad \text{and} \quad \Psi(i) = 0 \tag{12}$$

Now we can apply the general inequality from § XX in [Beurling-Special Topics] and conclude that (11-12) give the inequality

$$h_r(i) \leq e^{-\frac{\pi L^2}{A}} \quad \text{where} \quad L = \log r - \log 2 \tag{13}$$

Next, the reader may verify that there exists a constant C which is independent of $r \geq 2$ such that

$$\frac{\pi L^2}{A} \geq \frac{\lg r - \log 2)^2}{\log r - \log m(r)} \geq \log r - 2\log 2 + m(r) \cdot \big[1 - \frac{C}{\log r}\big] \tag{14}$$

Then (13-14) gives a constant C_1 such that

$$h_r(i) \leq e^{-\frac{\pi L^2}{A}} \leq C_1 \cdot \frac{1}{r} \cdot e^{-m(r)} \quad : \; r \geq 2 \tag{15}$$

Then (7) and (15) give

$$u_*(i) \leq C_1 \cdot \cdot e^{-m(r)} \tag{*}$$

This inequality hold for all $r > 2$ and since $u_*(i) > 0$ there must exist a finite limit

$$\lim_{r \to \infty} m(r) = m^* \tag{**}$$

The requested implication in Theorem 15.2 follows since (9) this means that

$$\iint_{E_u} \frac{1}{|z+i|^2}\, dxdy = \lim_{r \to \infty} m(r)$$

H. An L^1-inequality for inverse Fourier transforms.

Theorem 15.1 below is due to Beurling in [Beurling]. Let $g(t)$ be a function defined on $t \geq 0$ where the inverse Fourier transform of $tg(t)$ is integrable, i.e. the function defined on the x-axis by

$$(*) \qquad f(x) = \int_0^\infty e^{itx} \cdot tg(t)\, dt$$

belongs to $L^1(\mathbf{R})$.

15.1 Theorem. *When $f \in L^1(\mathbf{R})$ it follows that $g(t)$ is integrable and one has the inequality*

$$\int_0^\infty |g(t)|\, dt \leq \frac{1}{2} \int_{-\infty}^\infty |f(x)|\, dx$$

Proof. Since (*) is taken over $t \geq 0$, $f(x)$ is the boundary value function of the analytic function defined in $\mathfrak{Im}(z) > 0$ by

$$(1) \qquad f(z) = \int_0^\infty e^{itz} \cdot tg(t) \cdot dt$$

We first prove the inequality in Theorem 15.1 when $f(z)$ is zero-free in the upper halfplane and consider the normalised situation where the L^1-integral of $|f(x)|$ is one. Now the complex square root of $f(z)$ exists in $\mathfrak{Im}(z) > 0$ and gives an analytic function $F(z)$ such that $F^2 = f$. Since $|F(x)|^2 = |f(x)|$ it follows that F belongs to the Hardy space $H^2(\mathbf{R})$ and Plancherel's theorem gives a function $h(t)$ on $t \geq 0$ where

$$(2) \qquad F(z) = \int_0^\infty e^{itz} \cdot h(t) dt$$

Parseval's equality gives

$$(*) \qquad 1 = \int |F(x)|^2\, dx = 2\pi \cdot \int_0^\infty |h(t)|^2\, dt$$

The Fourier transform of the convolution $h * h$ is equal to $F^2(x) = f(x)$. This gives

$$(**) \quad t \cdot g(t) = \int_0^t h(t-s)h(s)\, ds \implies |t \cdot g(t)| \leq H(t) = \int_0^t |h(t-s)| \cdot |h(s)|\, ds$$

Put

$$(4) \qquad F_*(x) = \int_0^\infty e^{itz} \cdot |h(t)|\, dt$$

Parseval's formula applied to the pair $|h|$ and F_* gives

$$\int |F^2(x)|\, dx = 2\pi \cdot \int_0^\infty |h^2(t)|\, dt$$

We conclude that the L^2-norm of F^* also is one and here

$$(5) \qquad F_*(x)^2 = \int_0^\infty e^{itz} \cdot H(t)\, dt$$

At this stage we use a result from XXX which shows that the function

$$\theta \mapsto \log \Big[\int_0^\infty |F_*(re^{i\theta})|^2 \cdot dr\,\Big]$$

is a convex function of θ where $-\pi \leq \theta \leq 0$. Apply this when $\theta = \pi/2$ with end-values 0 and π. This gives

$$\int_0^\infty |F_*(iy)|^2 \cdot dy \leq \sqrt{\int_{-\infty}^0 |F_*(x)|^2 \cdot dx} \cdot \sqrt{\int_0^\infty |F_*(x)|^2 \cdot dx}$$

Since $x \mapsto |F_*(x)|$ is an even function of x the equality in (*) entails that the product above is equal to one. Hence we obtain:

$$\int_0^\infty \Big[\int_0^\infty e^{-ty} \cdot H(t) \cdot dt\Big] \cdot dy \leq \frac{1}{2}$$

Integration by parts shows that the left hand side is equal to

$$\int_0^\infty \frac{H(t)}{t} \cdot dt$$

Finally, by (**) $|g(t)| \leq \frac{H(t)}{t}$ and hence the L^1-norm of g is bounded by $\frac{1}{2}$ as requested.

Removing zeros. If f is not zero-free we let $B(z)$ be the Blaschke product of its zeros and write

$$f = B(z)\phi(z)$$

Here ϕ is zero-free and we do not change the L^1-norm on the x-line since $|B(x)| = 1$ holds almost everywhere. Notice that we can write

$$f = \phi\Big(\frac{1+B}{2}\Big)^2 + \phi\Big(\frac{1-B}{2}\Big)^2 = F_1^2 - F_2^2$$

where F_1 and F_2 as above are zero-free in the Hardy space H^2. Since we have

$$\Big|\frac{1+B}{2}\Big|^2 + \Big|\frac{1-B}{2}\Big|^2 \leq 1$$

it follows that

$$|F_1|^2 + |F_2|^2 \leq |\phi|$$

Now the established zero-free case gives the inequality in Theorem 15.1

I. On functions with spectral gap

Introduction. A fore-runner to distribution theory appears in work by Beurling where spectral gaps of functions f on the real x-line are analyzed. We expose a result from a seminar by Beurling at at Uppsala University in March 1942.

16.1 Theorem. *Let $f(x)$ be a bounded and continuous function on the real x-line such that $\widehat{f}(\xi)$ is zero on $\{-1 \leq \xi \leq 1\}$ and*

$$f(x+h) - f(x) \leq h$$

hold for all $h > 0$ and every x. Then its maximum norm is at most π.

The proof is postponed until § 16.3. First we establish some preliminary results where an essential ingredient is the entire function:

$$2 \cdot H(z) = \big(2 \sin \frac{z}{2}\big)^2 \cdot [\sum_{n=1}^{\infty} \frac{1}{(z-2\pi n)^2} - \sum_{n=0}^{\infty} \frac{1}{(z+2\pi n)^2} + \frac{1}{\pi z}]$$

A. Exercise. Verify the identity

$$(*) \qquad \frac{1}{\big(2\sin\frac{z}{2}\big)^2} = \sum_{-\infty}^{\infty} \frac{1}{(z-2\pi n)^2}$$

The hint is to consider the meromorphic function

$$\phi(z) = \frac{\cos z/2}{2\sin z/2}$$

It has simple poles at $\{2\pi n\}$ where n runs over all integers and we notice that

$$\psi(z) = \phi(z) - \sum_{-\infty}^{\infty} \frac{1}{z-2\pi n}$$

is entire. Hence the derivative $\psi'(z)$ is also entire. Since $\cos^2 z/2 + \sin^2 z/2 = 1$ we get:

$$\psi'(z) = -\frac{1}{\big(2\sin\frac{z}{2}\big)^2} + \sum_{-\infty}^{\infty} \frac{1}{(z-2\pi n)^2}$$

At the same time the reader may verify that the right hand side is bounded so this entire function must be identically zero which gives (*).

B. Exercise. Use (*) to show that if $x > 0$ then

$$2H(x) = 1 - \big(2\sin\frac{x}{2}\big)^2 \cdot [\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} - \frac{1}{\pi x}]$$

The θ-function. It is defined for all real x by:

$$\theta(x) = \frac{1}{2} \cdot \operatorname{sign}(x) - H(x)$$

where $\operatorname{sign}(x)$ is -1 if $x < 0$ and $+1$ if $x > 0$.

16.2 Proposition. *The θ-function is everywhere ≥ 0 and*

$$(*) \qquad \int_{-\infty}^{\infty} \theta(x) \cdot dx = \pi$$

Proof. Exercise B gives for every $x > 0$:

$$\theta(x) = \frac{1}{2}\big(2\sin\frac{x}{2}\big)^2 \cdot \Big[\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} - \frac{1}{\pi x}\Big]$$

Next, notice the two inequalities

$$(1) \qquad \sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} \leq \int_0^{\infty} \frac{2dt}{(x+2\pi t)^2} = \frac{1}{\pi x}$$

$$(2) \qquad \sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} \geq \int_0^{\infty} \frac{2dt}{(x+2\pi t)^2} = \frac{1}{\pi x}$$

Here (2) entails that $\theta(x) \geq 0$ on $x > 0$ and (1) obviously implies that the integral

$$(3) \qquad \int_0^{\infty} \theta(x) \cdot dx < \infty$$

We leave as an exercise to the reader to verify the similar result for $x < 0$, i.e. that $\theta(x) \geq 0$ hold for $x < 0$ and that its integral over $(-\infty, 0)$ is finite. To establish the equality (*) in Proposition 16.2 we notice that the function

$$\text{sign}(x) + (2\sin\frac{x}{2})^2 \cdot \frac{1}{\pi x}$$

is odd so its integral over the real line is zero. The reader may also check the equation

$$\int_{-\infty} \frac{(\sin\frac{x}{2})^2}{(x-2\pi n)^2} \cdot dx = \int_{-\infty} \frac{(\sin\frac{x}{2})^2}{(x+2\pi n)^2} \cdot dx \quad \text{for every} \quad n \geq 1$$

and then verify the equality

$$\int_{-\infty}^{\infty} \theta(x) \cdot dx = \int_0^{\infty} \frac{(2\sin\frac{x}{2})^2}{x^2} \cdot dx$$

where residue calculus shows that the last integral is π.

16.3 Proof of Theorem 16.1

Let $\mathcal{H}$ be the Heaviside function which is one on $x > 0$ and zero on $\{(x \leq 0\}$. Recall that the distribution derivative $\partial_x(\mathcal{H}) = \delta_0$. Regarding f as a temperate distribution this gives the equation

$$(\text{i}) \qquad f = \partial_x(f * (\mathcal{H} - \frac{1}{2}))$$

As explained in § XX the Fourier transform $\widehat{H}(\xi)$ is supported by $[-1, 1]$. Since the support of $\widehat{f}$ is disjoint from $[-1, 1]$ it follows that $f * H = 0$ and hence the distribution derivative

$$(\text{ii}) \qquad \partial_x(f * H) = 0$$

Next, notice that

$$\text{(iii)} \qquad \frac{1}{2}\cdot \operatorname{sign}(x) = \mathcal{H} - \frac{1}{2}$$

The construction of θ in (B) and (i-iii) therefore give

$$f = \partial_x(f * \theta)$$

This means that one has the equation

$$\text{(iv)} \qquad f(x) = \int_{-\infty}^{\infty} \theta(x-y)\cdot f'(y)\cdot dy$$

By assumption $f'(y) \leq 1$ for all y and since $\theta \geq 0$ the right hand side is bounded above by

$$\int_{-\infty}^{\infty} \theta(x-y)\cdot dy = \pi \implies f(x) \leq \pi$$

To get $f(x) \geq -\pi$ we consider the function $g(x) = -f(-x)$ which again is a bounded continuous function and the reader easily verifies that $g(x+h) - g(x) \leq h$ for all $h > 0$. Moreover, $\widehat{g}(\xi)$ is minus the complex conjugate of $\widehat{f}$ so g has the same spectral gap as f and just as above we get the upper bound $g(x) \leq \pi$ which entails that $f(x) \geq -\pi$ hold for all x. Hence its maximum norm is bounded by π which finishes the proof of Theorem 16.1

16.4 Question. Investigate if Theorem 16.1 is sharp, i.e. try to use the proof above in order to construct f whose maximum norm is close to π.

J. A theorem about limits

The space of complex-valued bounded and uniformly continuous functions on the real x-line is a Banach space $\mathcal{C}_*$ where we use the maximum norm over the whole line. A subspace arises as follows: On the ξ-line we have the space $\mathfrak{M}$ of complex Riesz measures γ with a finite total variation and to each γ we get the function

$$\mathcal{F}_\gamma(x) = \int_{-\infty}^{\infty} e^{ix\xi} \cdot d\gamma(\xi)$$

It is clear that $\mathcal{F}_\mu$ belongs to C_*. Denote by $\mathcal{A}$ the subspace of C_* given by $\mathcal{F}_\gamma$-functions as γ varies over $\mathfrak{M}$. Before we announce Theorem 17.1 below we recall the notion of weak-star limits in $\mathfrak{M}$. Let $\{\mu_n\}$ be a bounded sequence of Riesz measures, i.e. there exists a constant such that

$$||\mu_n|| \leq M$$

hold for all n. The sequence $\{\mu_n\}$ converges weakly to zero if

$$\lim_{n\to\infty} \int e^{ix\xi} \cdot d\mu_n(x) = 0$$

holds pointwise for every ξ.

17.1 Theorem. *A function $\psi \in C_*$ belongs to the closure of $\mathcal{A}$ if and only if*

$$(*) \qquad \lim_{n\to\infty} \int \psi(x) \cdot d\mu_n(x) = 0$$

whenever $\{\mu_n\}$ is a sequence in $\mathfrak{M}$ which converges weakly to zero.

The sufficiency part is easy. For suppose that ψ belongs to the closure of $\mathcal{A}$ and let $\{\mu_n\}$ converge weakly to zero. Since the total variations in this weakly convergent sequence of measures is uniformly bounded, it suffices to show that (*) holds when $\psi \in \mathcal{A}$. So let $\psi = \mathcal{F}_\gamma$ for some $\gamma \in \mathfrak{M}$. Since γ and μ_n both have a finite total variation it is clear that

$$\int \psi(x) \cdot d\mu_n(x) = \int \mathcal{F}_{\mu_n}(\xi) \cdot d\gamma(\xi)$$

Here $\{\mathcal{F}_{\mu_n}(\xi)\}$ is a sequence of uniformly bounded continuous functions on the real ξ-line which by assumption converges pointwise to zero. Since the Riesz measure γ has a finite total variation, the Borel-Riesz convergence result in [Measure] shows that (*) tends to zero with n.

Proof of necessity.

There remains to show that if $\psi \in C_*$ is outside the closure of $\mathcal{A}$, then there exists a sequence $\{\mu_n\}$ which converges weakly to zero while $\{\int \psi \cdot d\mu_n\}$ stay away from zero. To attain this we shall consider a class of variational problem and extract a certain sequence of measures which does the job.

A class of variational integrals. Let a, b, s be positive numbers and $q > 2$. With p chosen so that

$$\frac{1}{p} + \frac{1}{q} = 1$$

we have the space $L^p[-a,a]$ where $[-a,a]$ is an interval on the ξ-line. To each function $g(\xi) \in L^p[-a,a]$ we get the function

$$\mathcal{F}_g(x) = \int e^{ix\xi} \cdot g(\xi) \cdot d\xi$$

This gives a continuous function which is restricted to $[-b,b]$ and we set

$$||\psi - \mathcal{F}_g||_q^b = \Big[\int_{-b}^{b} [\psi(x) - \mathcal{F}_g(x)|^q \cdot dx\Big]^{1/q}$$

where the upper index b indicates that we compute a L^q-norm on the bounded interval $[-b,b]$. To each $g \in L^p[-a,a]$ we set

$$(*) \qquad \mathcal{J}(g;q,b,a,s) = ||\psi - \mathcal{F}_g||_q^b + ||g||_p$$

where the last term is the L^p-norm of g taken over $[-a,a]$. Since the Banach space $L^p[-a,a]$ is strictly convex one easily verifies:

17.2 Proposition. *The variational problem where $\mathcal{J}$ is minimized over g while a, b, s are fixed has a unique extremal solution.*

17.3 Exercise. Regarding infinitesmal variations via the classic device due to Euler and Lagrange, the reader can verify that there exists a unique extremal solution g which satisfies

$$(*) \qquad ||g||_p)^{1-p} \cdot \frac{|g(\xi)|^p}{g(\xi)} = M^{-1/p} \cdot \int_{-b}^{b} e^{i\xi x} \cdot \frac{|\psi(x) - \mathcal{F}_g(x)|^q}{|\psi(x) - \mathcal{F}_g(x)} \cdot dx$$

where we have put

$$M = \int_{b}^{b} |\psi - \mathcal{F}_g|^q \cdot dx$$

Consider the absolutely continuous measure on the x-line defined by the density

$$d\mu = M^{-1/p} \cdot \frac{\psi(x) - \mathcal{F}_g(x)|^q}{|\psi(x) - \mathcal{F}_g(x)} \quad : \quad -b \leq x \leq b$$

This gives

$$\int_{-b}^{b} |d\mu(x)| = M^{-1/p} \cdot \int_{-b}^{b} |\psi(x) - \mathcal{F}_g(x)|^{q-1} \cdot dx$$

Hölder's inequality applied to the pair of functions $|\psi(x) - \mathcal{F}_g(x)|^{q-1}$ and 1 on $[-b,b]$ gives the inequality below for the total variation:

$$(**) \qquad ||\mu|| \leq (2b)^{1/q}$$

17.4 Lemma. *The following two formulas hold:*

$$\int \psi \cdot d\mu = \mathcal{J}(g;q,b,a,s)$$

$$\int \mathcal{F}_g \cdot d\mu = s \cdot ||g||_p$$

Proof. To begin with we have

$$\int (\psi - \mathcal{F}_g) \cdot d\mu = M^{-1/p} \cdot \int_b^b |\psi - \mathcal{F}_g| \cdot dx = M^{1-1/p} \cdot M = M^{1/q}$$

Next Fubini's theorem gives

$$\int \mathcal{F}_g \cdot d\mu = \int_a^a [\int e^{ix\xi} \cdot \mu(x)] \cdot g(\xi) \cdot d\xi = s \cdot \int_a^a ||g||_p)^{1-p} \cdot \frac{|g(\xi)|^p}{g(\xi)} \cdot g(\xi) d\xi = s \cdot ||g||_p$$

This proves formula (2) and (1) follows via (i) and the equality

$$\mathcal{J}(g; q, b, a, s) = M + s \cdot ||g||_p$$

17.5 Passage to limits. Following [Beurling] we now consider certain limits where we first let $q \to +\infty$ and after $b \to \infty$, and finally use pairs $a = 2^m$ and $s = 2^{-m}$ where m will be large positive integers. To begin with, the measure μ depends on q, b, a, s and let us denote it by $\mu_q(b, a, s)$. The uniform bound (*) from Exercise 17.3 entails that while a, b, s are kept fixed, then there is a sequence $\{q_\nu\}$ which tends to $+\infty$ and give a weak limit measure

$$\mu_*(b, a, s) = \lim \mu_{q_\nu}(b, a, s)$$

The extremal g-functions depend on q and are denoted by g_q. Their L^q-norms remain bounded and passing to a subsequence we get an L^∞-function g_* on $[-a, a]$ where $g_{q_\nu} \to g_*$. Here g_* depends on b, a, s and is therefore indexed as $g_*(b, a, s)$. We have also a limit:

$$\lim_{\nu \to \infty} \mathcal{J}(g_g; q_\nu; b, a, s) = \mathcal{J}_*(g_*(a, b, s))$$

Moreover

$$\mathcal{J}_*(g_*(a, b, s)) = \max_{-b \le x \le b} |\psi(x) - \mathcal{F}_{g_*(b,a,s)}| + s \cdot \int_{-a}^a |g_*(b, a, s)(\xi)| \cdot d\xi$$

At this stage we use the hypothesis that ψ does not belong to the closure of $\mathcal{A}$ which entails that with a and s fixed, then here is a constant $\rho > 0$ such that

$$\liminf_{b \to \infty} \max_{-b \le x \le b} |\psi(x) - \mathcal{F}_{g_*(b,a,s)}| \ge \rho$$

At this stage proof is easily FINISHED.

L. Dagerholm's system of infinite linear equations.

Introduction. The main issue in this section is the construction of a unique solution to the system

$$\sum_{q\neq p} \frac{x_q}{p-q} = 0 \tag{*}$$

where (*) hold for all positive integers p and $\{x_q\}$ is a sequence of real numbers for which the series

$$\sum_{q=1}^{\infty} \frac{x_q}{q} \tag{**}$$

is convergent. The fact that (*) has a non-trivial solution is far from evident. Before the study of (*) in Section 1 we discuss a general situation which was described by Carleman in his major lecture at the international congress in Zürich 1932. A homogeneous system of linear equations with an inifinite number of variables takes the form

$$\sum_{q=1}^{\infty} c_{pq}x_q = a_p \quad : \quad p = 1, 2, \ldots \tag{***}$$

where $\{c_{pq}\}$ is a matrix with an infinite number of elements. A sequence $\{x_q\}$ of complex numbers is a solution to (***) if the sum in the left hand side converges for each p and has value a_p. Notice that one does not require that the series are absolutely convergent.

The generic case. To every $p \geq 1$ we have the linear form L_p defined on finite sequences $\{x_1, x_2, \ldots\}$, i.e. where $x_q = 0$ when $q >> 0$ by:

$$L_p(x_\bullet) = \sum_{q\geq 1} c_{pq} \cdot x_q$$

Follownig Carleman the generic case occurs when $c_{1q} \neq 0$ for every q and the linear forms $\{L_p\}_1^\infty$ are linearly independent. The last condition means that for every positive integer M there exists a strictly increasing sequence $q_1 < \ldots < q_M$ so that the $M \times M$-matrix with elements $a_{p\nu} = c_{pq_\nu}$ has a non-zero determinant.

The R-functions. Assume that the system (***) is generic. Given a sequence $\{a_p\}$ we seek a solution $\{x_q\}$. The necessary and sufficient condition for its existence goes as follows: Consider n-tuples of positive integers $m_1, \ldots, m_n$ where $n \geq 2$. For every such n-tuple and $1 \leq k \leq n-1$ we set

$$\mathcal{D}(k) = \{\nu\colon\ m_1 + \ldots + m_k < \nu \leq m_1 + \ldots + m_{k+1}\}$$

Next, with $M = m_1 + \ldots + m_N$ we denote by $\mathcal{F}(m_1, \ldots, m_n)$ the family of sequences $(x_1, \ldots, x_M)$ such that the following inequalities hold for every $1 \leq k \leq n-1$:

$$\Big|\sum_{q=1}^{q=\nu} c_{pq}x_q - a_q\Big| \leq \frac{1}{k} \quad : \ \nu \in \mathcal{D}(k) \quad \text{and} \quad 1 \leq p \leq k$$

The generic assumption implies that the set $\mathcal{F}(m_1,\ldots,m_n)$ is non-empty provided that we start with a sufficiently large m_1 and for every such n-tuple we set

$$R(m_1,\ldots,m_n)=\min\sum_{\nu=1}^{m_1} x_\nu^2$$

where the minimum is taken over sequences $x_1,\ldots,x_{m_1}$ which give the starting terms of some sequence $x_1,\ldots,x_M\in\mathcal{F}(m_1,\ldots,m_n)$.

Theorem. *The necessary and sufficient condition in order that (***) has a least one solution is that there exists a constant K and an infinite sequence of positive integers $\mu_1,\mu_2,\ldots$ such that*

$$R(\mu_1,\mu_2,\ldots,\mu_r)\leq K \quad \text{hold for every } r$$

Remark. The reader may consult [Carleman] for further remarks about this result and also comments upon the more involved criterion for non-generic linear systems. From now on we study linear systems which arise as follows: Consider a rational function of two variables x,y:

$$f(x,y)=\frac{a_0(x)+a_1(x)y+\ldots a_n(x)y^n}{b_0(x)+b_1(x)y+\ldots b_m(x)y^m}$$

Here n and m are positive integers and $a_\nu(x)$ and $b_j(x)$ polynomials in x. No special assumptions are imposed on these polynomials except that $b_m(x)$ and $a_n(x)$ are not identically zero. For example, it is not neccesary that the degree of b_m is $\geq\deg(b_j)$ for all $0\leq j\leq m-1$.

0.1 Proposition *Let $b_m^{-1}(0)$ be the finite set of zeros of b_m. Let $\zeta_0\in\mathbf{C}\setminus b_m^{-1}(0)$ be such that*

$$\sum_{j=0}^{j=m} b_j(\zeta_0)\cdot q^j=0$$

holds for some finite set of positive integers, say $1\leq q_1<\ldots<q_k$. Then there exists $\delta>0$ such that

$$\sum_{j=0}^{j=m} b_j(\zeta)q^j\neq 0 \quad : \text{for all positive integers } q \quad : \quad 0<|\zeta-\zeta_0|<\delta$$

Exercise. Prove this result.

Next, consider the sequence of polynomials of the complex ζ-variable given by:

$$B_q(\zeta)=b_0(\zeta)+b_1(\zeta)q+\ldots+b_m(\zeta)q^m \quad : q=1,2,\ldots$$

0.2 Proposition *Put*

$$W=b_m^{-1}(0)\bigcup_{q\geq 1}B_q^{-1}(0)$$

Then W is a discrete subset of $\mathbf{C}$, i.e its intersection with any bounded disc is finite.

Exercise. Prove this result where a hint is to apply Rouche's theorem.

Next, put $W^*=W\cup a_n^{-1}(0)$, i.e. add the zeros of the polynomial a_n to W.

0.3 Proposition. *Let $\zeta_0 \in C \setminus W^*$ and suppose that $\{x_q\}$ is a sequence such that the series*

$$\text{(i)} \qquad \sum_{q=1}^{\infty} f(\zeta_0, q) \cdot x_q$$

is convergent. Then the series

$$\text{(ii)} \qquad \sum_{q=1}^{\infty} f(\zeta, q) \cdot x_q \quad \text{converges for every } \zeta \in C \setminus W^*$$

Moreover, the series sum is a meromorphic function of ζ whose poles are contained in W^*.

Remark. Proposition 0.3 gives a procedure to find solutions $\{x_q\}$ which is not the trivial null solution to a homogeneous system:

$$(*) \qquad \sum_{q \neq p} f(p, q) \cdot x_q = 0 \quad : \quad p = 1, 2, \ldots$$

More precisely, assume that the rational function $f(x, y)$ is such that $f(p, q) \neq 0$ when p and q are distinct positive integers. To get a solution $\{x_q\}$ to (iii) it suffices to begin with to verify (i) in Proposition 0.3 for some ζ_0 and then also try to find $\{x_q\}$ so that the meromorphic function

$$(**) \qquad \zeta \mapsto \sum_{q=1}^{\infty} x_q \cdot f(\zeta, q)$$

has zeros at every positive integer. Using this criterium for a solution we can show the following:

Theorem. *For every complex number $a \in \mathbf{C} \setminus (-\infty, 0]$ the system*

$$\sum_{q=1}^{\infty} \frac{x_q}{p + aq} = 0 \quad : \quad p = 1, 2, \ldots$$

has no non-trivial solution $\{x_q\}$.

There remains to analyze the case when a is real and < 0. In this case complete answers about possible when f is the rational function were established by K. Dagerholm in his Ph.D-thesis at Uppsala University from 1938 with Beurling as supervisor. The hardest case occurs when $a = 1$ which will be studied in the next section.

1. The Dagerholm series.

Let $\mathcal{F}$ be the family of all sequences of real numbers $x_1, x_2, \ldots$ such that the series

$$\text{(i)} \qquad \sum_{q=1}^{\infty} \frac{x_q}{q} < \infty$$

We only require that the series is convergent, i.e. it need not be absolutely convergent.

1.1 Theorem. *Up to a multiple with a real constant there exists a unique sequence* $\{x_q\}$ *in* $\mathcal{F}$ *such that*

$$\sum_{q\neq p} \frac{x_q}{p-q} = 0 \quad : \quad p = 1, 2, \ldots$$

The proof of uniquenss relies upon Jensen's formula and the solution to a specific Wiener-Hopf equation. We begin to describe the strategy of the proof. For each $\{x_q\} \in \mathcal{F}$ there exists the meromorphic function

$$\text{(ii)} \qquad h(z) = \sum_{q=1}^{\infty} \frac{x_q}{z-q}$$

To see that $h(z)$ is defined we notice that if s_* is the series sum in (i) then

$$\text{(iii)} \qquad h(z) + s_* = \sum_{q=1}^{\infty} x_q \cdot [\frac{1}{z-q} + \frac{1}{q}] = z \cdot \sum_{q=1}^{\infty} \frac{x_q}{q(z-q)}$$

It is clear that the right hand side is a meromorphic function with poles confined to the set of positive integers. Hence we obtain the entire function:

$$H(z) = \frac{1}{\pi} \cdot \sin(\pi z) \cdot h(z)$$

1.2 Proposition. *The following hold for each positive integer:*

$$H(p) = (-1)^p \cdot x_p \quad : H'(p) = (-1)^q \cdot \sum_{q\neq p} \frac{x_q}{p-q} = 0$$

Proof. Let $p \geq 1$ be an integer. With ζ small we have

$$H(p+\zeta) = \frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) \cdot [\frac{x_p}{\zeta} + \sum_{q\neq p} \frac{x_q}{p+\zeta-q}]$$

A series expansion of the complex sine-function at πp gives

$$\frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) = [\zeta \cdot \cos(\pi p) + O(\zeta^3)] \cdot [\frac{x_p}{\zeta} + \sum_{q\neq p} \frac{x_q}{p+\zeta-q}]$$

Proposition 1.2 follow since $\cos \pi p = (-1)^p$.

Remark. Proposition 1.2 shows that $\{x_p\}$ solves the homogeneous system in Theorem 1.1 if the complex derivative of the entire H-function has zeros on all positive integers. This observation is the gateway towards the proof of Dagerholm's Theorem. But let us first establish the uniqueness.

2. Proof of uniqueness

Let $\{x_q\}$ be a sequence in $\mathcal{F}$. From the constructions in above it is clear that the meromorphic function $h(z)$ satisfies the following in the left half-plane $\mathfrak{Re}(z) \leq 0$:

$$\text{(i)} \qquad \lim_{x\to-\infty} h(x) = 0 : \quad |h(x+iy)| \leq C_* \quad : \ x \leq 0$$

where C_* is a constant. Moreover, in the right half-plane there exists a constant C^* such that

(ii) $$|h(x+iy)| \leq C^* \cdot \frac{|x|}{1+|y|} \quad : \ |x-q| \geq \frac{1}{2} \text{ for all positive integers}$$

To $h(z)$ we get the entire function $H(z)$ and (i-ii) above give the two the estimates below in the right half-plane:

(iii) $$|H(x+iy)| \leq Ce^{\pi|y|} \ : \ x \leq 0 \quad : \quad |H(x+iy)| \leq C\frac{|x|}{1+|y|} \cdot e^{\pi|y|}$$

Moreover, the first limit formula in (i) gives

(iv) $$\lim_{x \to -\infty} H(x) = 0$$

It is easily seen that the same upper bounds hold for the entire function $H'(z)$ and a straightforward application of the Phragmén-Lindelöf theorem gives:

2.1 Proposition. *The complex derivative of $H(z)$ satisfies the growth condition:*

$$\lim_{r \to \infty} e^{-\pi r \cdot |\sin \phi|} \cdot |H'(re^{i\theta} | = 0 \quad : \text{ holds uniformly when } \ 0 \leq \theta \leq 2\pi$$

Now we are prepared to prove the uniqueness part in Theorem 0.1. For suppose that we have two sequences $\{x_q\}$ and $\{x_q^*\}$ which both give solutions to (*) and are not equal up to a constant multiple of each other. The two sequences give entire functions H_1 and H_2. Since both are constructed via real sequences their Taylor coefficients are real and there exists a linear combination

$$G = aH_1 + bH_2$$

where a, b are real numbers and the complex derivative $G'(0) = 0$. The hypothesis that there exists two **R**-linearly independent solutions to (*) leads to a contradiction once we have proved the following

2.2 Lemma *The entire function $G'(z)$ is identically zero.*

Proof. To simplify notations we set $g(z) = G'(z)$ and consider the series expansion

$$g(z) = a_p z^p + a_{p+1} z^{p+1} + \ldots$$

where a_p ia the first non-vanishing coefficient. Since $g(0) = G'(0) = 0$ we have $p \geq 1$ and since the two x-sequences both are solutions to (*), the second equation in Proposition 0.2 gives

(i) $$g(p) = 0 \quad : \ p = 1, 2, \ldots$$

Next, G is real-valued on the x-axis and since the H-functions are zero for every integer ≤ 0 the same holds for G. *Rolle's theorem* implies that for every $n \geq 1$ there exists

(i) $$-n < \lambda_n < -n+1 \quad : \ g(\lambda_n) = 0$$

So if $\mathcal{N}$ is the counting function for the zeros of the entire g-function one has the inequality

(iii) $$\mathcal{N}(r) \geq [2r]$$

where $[2r]$ is the largest integer $\leq 2r$. Next, recall that a_p is the first non-zero term in the series expansion of g. Hence Jensen's formula gives:

$$(*) \qquad \log|a_p| + p\cdot \log\, r + \int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Log}|g(re^{i\theta}|\,\cdot d\theta$$

Proposition 2.1 applied to $g(z)$ gives:

$$\text{(iv)} \qquad \int_0^{2\pi} \operatorname{Log}|g(re^{i\theta}|\,\cdot d\theta = 2r - m(r) \quad \text{where } \lim_{r\to\infty} m(r) = +\infty$$

At this stage we get the contradiction as follows. First (iii) gives

$$\int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} \geq 2r - \operatorname{Log}(r) - 1$$

Now (*) and (iii) give the inequality

$$\text{(vi)} \qquad \log|a_p| + p\cdot \log\, r + 2r - 1 - \log r \leq 2r - m(r) \quad : \; r \geq 1$$

Here $p \geq 1$ which therefore would give:

$$\log|a_p| - 1 + m(r) \leq 0$$

But this is impossible since we have seen that $m(r) \to +\infty$.

3. Proof of existence

We start with a general construction. Let $\phi(z)$ be analytic in the unit disc D which extends to a continuous function on T except at the point $z = 1$. We also assume that there exists some $0 < \beta < 2$ and a constant C such that

$$(1) \qquad |\phi(\zeta)| \leq C|1-\zeta|^{-\beta}$$

This implies that the function

$$\theta \mapsto \theta\cdot\phi(e^{i\theta})$$

is integrable on the unit circle. Hence there exists the entire function

$$(2) \qquad f(z) = \frac{-i}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta z}\cdot\theta\cdot\phi(e^{i\theta})\cdot d\theta$$

Next, with $\epsilon > 0$ small we let γ_ϵ be the interval of the circle $|z-1| = \epsilon$ with end-points at the intersection with $|z| = 1$. So on γ_ϵ we have

$$z = 1 + \epsilon\cdot e^{i\theta} \quad : \; -\frac{\pi}{2} + \epsilon_* < \theta < \frac{\pi}{2} - \epsilon_*$$

where ϵ_* is small with ϵ. We obtain the entire function

$$F(z) = \frac{1}{2\pi}\int_\epsilon^\pi e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi}\int_{-\pi}^{-\epsilon} e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{e^{-z\cdot\operatorname{Log}\zeta}\cdot\phi(\zeta)d\zeta}{\zeta}$$

If $z = n$ is an integer we have

$$e^{-in\theta} = \zeta^{-n} \quad : \quad e^{-n\cdot\operatorname{Log}\zeta} = \zeta^{-n}$$

Hence we get

$$F(n) = \frac{1}{2\pi i} \cdot \int_{\Gamma_\epsilon} \frac{\phi(\zeta) \cdot d\zeta}{\zeta^{n+1}} \tag{*}$$

where Γ_ϵ is the closed curve given as the union of γ_ϵ and the interval of T where $|\theta| \geq \epsilon$. Cauchy's formula applied to ϕ gives:

2.1 Proposition. *Let* $\phi(\zeta) = \sum c_n \zeta^n$. *Then*

$$F(n) = c_n \quad : n \geq 0 \quad \text{and} \quad F(n) = 0 \quad n \leq -1$$

Next, using (i) above we also have:

2.2 Proposition. *The complex derivative of* F *is equal to* f.

Proof. With $\epsilon > 0$ the derivative of the sum of first two terms from the construction of $F(z)$ above become

$$\frac{1}{2\pi} \int_{|\theta| \geq \epsilon} -i\theta \cdot e^{-iz\theta} \phi(e^{i\theta}) d\theta \tag{i}$$

In the last integral derivation with respect to z gives

$$-\frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{e^{-z \cdot \mathrm{Log}\zeta} \cdot \phi(\zeta) d\zeta}{\zeta} \tag{ii}$$

Now $\zeta = 1 + \epsilon \cdot e^{i\theta}$ during the integration along γ_ϵ which gives:

$$|\mathrm{Log}\,(1 + \epsilon \cdot e^{i\theta})| \leq \epsilon$$

At the same time the circle interval γ_ϵ has length $\leq \epsilon$ and hence the growth condition (i) shows that the integral (iii) tends to zero when $\epsilon \to 0$. Finally, since we assumed that the function $\theta \mapsto \theta \cdot \phi(e^{i\theta})$ is absoutely integrable on T a passage to the limit as $\epsilon \to 0$ gives $F' = f$ as requested.

2.3 Conclusion. If n is a positive integer in Proposition 2.3 we have:

$$F'(n) = \frac{-i}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \cdot \theta \cdot \phi(e^{i\theta}) \cdot d\theta \tag{**}$$

These integrals are zero for every $n \geq 1$ if and only if $\theta \cdot \phi(e^{i\theta})$ is the boundary value function of some $\psi(z)$ which is analytic in the exterior disc $|z| > 1$. In 2.X we will show that this is true for a specific ϕ-function satisfying the growth condition (1) above and in addition the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{c_n}{n}$$

converges.

2.4 How to deduce a solution $\{x_p\}$. Suppose we have found ϕ satisfying the conditions above which gives the entire function $F(z)$ whose derivatives are zero for all $n \geq 1$. Now we set

$$x_p = (-1)^p \cdot c_p$$

By (***) this sequence belongs to $\mathcal{F}$ and we construct the associated entire function $H(z)$. From (i) in Proposition 0.1 and Proposition 2.1 we get

$$H(p) = (-1)^p \cdot x_p = c_p = F(p)$$

In addition both H and F have zeros at all integers ≤ 0. Next, by the construction of F it is clear that this is an entire function of exponential type and by the above the entire function $G = H - F$ has zeros at all integers. We leave as an exercise to the reader to show that G must be identically zero. The hint is to use similar methods as in the proof of the uniqueness. It follows that

$$H'(q) = F'(q) = 0$$

for all $q \geq 1$. By (ii) in Proposition 0.2 this means precisely that $\{x_p\}$ is a solution to the requested equations in (*) which gives the existence in Dagerholm's Theorem.

2.5 The construction of ϕ.

There remains to find ϕ such that the conditions above hold. To obtain ϕ we start with the integrable function on T defined by:

$$u(\theta) = \frac{1}{2} \cdot \log \frac{1}{|\theta|} \quad : \; -\pi < \theta < \pi$$

We get the analytic function

$$g(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} - \zeta}{e^{i\theta} + \zeta} \cdot u(\theta) \cdot d\theta$$

In the exterior disc we find the analytic function

$$\psi(\zeta) = \exp \; -\bar{g}(\frac{1}{\bar{\zeta}})$$

Let us also put $\phi_*(\zeta) = e^{g(\zeta)}$. Now we have

$$\log |\phi(e^{i\theta})| = \mathfrak{Re}\, g(e^{i\theta}) = u(\theta)$$

In the same way we see that

$$\log |\psi(e^{i\theta})| = -\mathfrak{Re}\, g(e^{i\theta}) = -u(\theta)$$

Since $2u(\theta) = -\log |\theta|$ it follows that

$$\log |\theta| + \log |\phi_*(e^{i\theta})| = \log |\psi(e^{i\theta})|$$

Taking exponentials we obtain

$$|\theta| \cdot |\phi_*(e^{i\theta})| = |\psi(e^{i\theta})|$$

Exercise. Check also arguments and verify that we can remove absolute values in the last equality to attain

$$(*) \qquad |\theta| \cdot \phi_*(e^{i\theta}) = \psi(e^{i\theta})$$

Here (*) is not precisely what we want since our aim was to construct ϕ so that $\theta| \cdot \phi(e^{i\theta})$ is equal to the boundary value of an analytic function in $|z| > 1$. So in order to get rid of the absolute value for θ in (*) we modify ϕ_* as follows: Set

$$\rho(\theta) = \frac{\pi i}{2} \cdot \operatorname{sign}\theta \cdot e^{-i\theta} \quad : \; -\pi < \theta < \pi$$

Next, consider the two analytic functions in D, respectively in $|z| > 1$ defined by:

$$\phi_1(z) = \frac{1}{\sqrt{1-z^2}} \quad \text{and} \quad \psi_1(z) = \frac{1}{\sqrt{1-z^{-2}}}$$

Exercise. Show that one has the equality

$$\rho(\theta) = \frac{\phi_1(e^{i\theta})}{\psi_1(e^{i\theta})}$$

when $-\pi < \theta < \pi$ and $\theta \neq 0$.

The ϕ-function. it is defined by

$$\phi(z) = \frac{z}{\sqrt{1-z^2}} \cdot \phi_*(z)$$

From (*) above and the construction of ρ it follows that

$$\theta \cdot \phi(e^{i\theta}) = \frac{\pi}{2} \cdot \psi_1(e^{i\theta}) \cdot \psi(e^{i\theta})$$

The right hand side is the boundary function of an analytic function in $|z| > 1$ and hence ϕ satisfies (**) from XX. Consider its Taylor expansion

$$\phi(z) = \sum c_n \cdot z^n$$

There remains to verify that the series (**) converges and that ϕ satisfies the growth condition in XX. To prove this we begin to analyze the function

(i) $$\phi_*(z) = e^{g(z)}$$

Rewrite the u function as a sum

(ii) $$u(\theta) = \frac{1}{2}\log\Big|\frac{1}{1-e^{i\theta}}\Big| + k(\theta) \quad \text{where} \quad k(\theta) = \frac{1}{2}\log\Big|\frac{1-e^{i\theta}}{\theta}\Big|$$

When θ is small we have an expansion

(iii) $$\frac{1-e^{i\theta}}{\theta} = -i + \theta/2 + \ldots$$

From this we conclude that the k-function is at least twice differentiable as a function of θ. So the Fourier coefficients in the expansion

(iv) $$k(e^{i\theta}) = \sum b_\nu e^{i\nu\theta}$$

have a good decay. For example, there is a constant C such that

(v) $$|b_\nu| \leq \frac{C}{\nu^2} \quad : \; \nu \neq 0$$

This implies that the analytic function

$$\text{(vi)} \qquad \mathcal{K}(z) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot k(e^{i\theta}) \cdot d\theta$$

yields a bounded analytic function in the unit disc. Next, the construction of the g-function gives:

$$\text{(vii)} \qquad g(z) = \frac{1}{2} \cdot \log \frac{1}{1-z} + \sum b_\nu z^\nu \implies$$

$$\phi_*(z) = \frac{1}{\sqrt{1-z)}} \cdot e^{\mathcal{K}(z)}$$

We conclude that

$$\phi(z) = \frac{z}{1-z} \cdot \frac{1}{\sqrt{1+z)}} \cdot e^{\mathcal{K}(z)}$$

Since $\mathcal{K}(z)$ extends to a continuous function on the closed disc it follows that ϕ satisfies the growth condition (1) with $\beta = 1$. Moreover, the function $\theta \cdot \phi(e^{i\theta})$ belongs to $L^1(T)$ since $\frac{1}{\sqrt{1+e^{i\theta}}}$ is integrable. There remains only to prove:

Lemma. The series

$$\sum_{p=1}^{\infty} (-1)^n \cdot \frac{c_n}{n}$$

is convergent.

Proof. let us put

$$A(z) = \frac{z}{\sqrt{1+z)}} \cdot e^{\mathcal{K}(z)}$$

This gives

$$\phi(z) = \frac{A(1)}{1-z} + \frac{A(z) - A(1)}{1-z}$$

From (v) it follows that $K(z)$, and hence also $e^{K(z)}$ is differentiable at $z01$ which gives the existence of a constant C such that

$$|\frac{A(z)-1}{1-z}| \leq C \cdot \frac{1}{|\sqrt{1+z}|}$$

Here the function $\theta \mapsto \frac{1}{|\sqrt{1+e^{i\theta}}|}$ belongs to $L^p(T)$ for each $p < 2$ which by the inequality for L^p-norms between functions and their Fourier coefficients in XX for example implies that if $\{c^*_\nu\}$ give the Taylor series for $\frac{A(z)-1}{1-z}$ then

$$\sum |c^*_\nu|^3 < \infty$$

Now Hölder's inequality gives

$$\sum \frac{|c^*_\nu|}{\nu} \leq \left(\sum |c^*_\nu|^3\right)^{\frac{1}{3}} \cdot \left(\sum \nu^{-3/2}\right)^{\frac{2}{3}} < \infty \tag{8}$$

We conclude that the Taylor series for ϕ becomes

$$A(1) \cdot (1 + z + z^2 + \ldots) + \sum c^*_\nu z^\nu$$

Hence $c_n = A(1) + c^*_\nu$ and now Lemma xx follows since the alternating series $\sum (-1)^n \frac{1}{n}$ is convergent and we have the absolute convergence in XX above.

§ M. A theorem by Kjellberg.

Introduction. We expose an article by Bo Kjellberg - former Ph.D-student of Beurling - which deals with a comparison between integrals and certain sums of analytic functions of exponential type in a half-space. Here is the situation: Let $f(z)$ be analytic in the closed half-spae $\mathfrak{Re}\, z \geq 0$. Assume that the function $x \mapsto |f(x)|$ is bounded on the real x-axis and there exists a positive real number c such that

$$\limsup_{|z|\to\infty} \frac{\log |f(z)|}{|z|} = c \tag{1}$$

Next, let $\phi(t)$ be a C^2-functionon $t \geq$ with $\phi(0) = 0$ for which

$$\phi'(t) \geq 0 \quad : \quad t \cdot \phi''(t) + \phi'(t) \geq 0$$

Thus, ϕ is non-decreasing and the last inequality means that it is convex as a function of $\log t$.

Gap sequences. A strictly increasing sequence $\{\lambda_n\}$ has gaps of order $\geq \delta$ if

$$\lambda_{n+1} - \lambda_n \geq 2 \cdot \delta \quad : n = 1, 2, \ldots$$

0.1 Theorem. *Let f and ϕ be as above. Then, for every $\delta > 0$ and each λ-sequence with gaps of order $\geq \delta$ one has the implication*

$$\int_0^\infty \phi(|f(x)|)\, dx < \infty \implies \sum_{n=1}^\infty \phi(e^{-\delta \cdot c} \cdot |f(\lambda_n)|) < \infty \tag{*}$$

Remark. The result above gave an affirmative anser to a question posed by Boas in the article *Inequalities between series and integrals involving entire functions.* The proof of Theorem 0.1 has two ingredients. First one has a uniqueness result which goes as follows:

0.2 Theorem. *Let f and ϕ be as above where (1) holds for f and the integral in the left hand side of Theorem 0.1 is finite. Then, if*

$$\int_0^{1/2} \frac{\phi(t)}{t \cdot \log \frac{1}{t}}\, dt = +\infty$$

it follows that f is identically zero.

Remark. We shall first prove Theorem 0.2 and notice that it then is suffcient to prove Theorem 0.1 under the *additional assumption* that the integral in Theorem 0.2 is finite. In both theorems the convexity of ϕ with respect to $\log t$ will be used. More precisely we first notice that if $H(x, y)$ is a harmonic function defined in some open set in $\mathbf{C}$ then

$$u(x, y) = \phi(e^{H(x,y)})$$

is subharmonic. Indeed , this follows since

$$u_x = H_x \cdot e^H \cdot \phi'(e^H) \implies u_{xx} = H_{xx} e^H \cdot \phi'(e^H) + H_x^2 e^H (e^H \cdot \phi''(e^H) + \phi'(e^H))$$

A similar equation holds for u_{yy} and adding the result we use that $\Delta(H) = 0$ and see that (2) above entails that $\Delta(u) \geq 0$. We shall need the following prelimninary result:

1.1 Proposition. *Let $u(x)$ be a continuous and non-negtive function on $x \geq 0$ where*

$$\limsup_{x \to +\infty} u(x) = +\infty$$

and g is a stricty increasing function on $x > 0$. Then the implication below holds:

$$\int_0^\infty g(u(x))\,dx < \infty \quad \text{and} \quad \int_1^\infty \frac{g(u)}{u}\,du = +\infty \implies \int_1^\infty \frac{u(x)}{x^3}\,dx = +\infty$$

Proof. Let $v(x)$ be the non-decreasing funftion which is equi-distributed with u as explained in §§. Then we have

$$\int_1^\infty \frac{v(x)}{x^3}\,dx \leq \int_1^\infty \frac{u(x)}{x^3}\,dx$$

while the left and side integrals are unchanged when u is replaced by v. Hence it suffices to prove the result for v, i.e. from now on we assume that u is non-decreasing. If $0 < a < A$ a partial integration gives

$$\int_a^A g(u(x))\,dx = \quad \text{and}$$

Easy finish ....

1.2 Majorisations in quarter-planes. We are given the constnst c which implies that if $c' > c$ then there is a constant B such that the inequality below holds in the right half-plane:

$$|f(x+iy)| \leq B\dot{e}^{c'|y|}$$

Let us then choose $c^* > c'$ snd set

$$\psi(z) = \phi(e^{ic^* z} \cdot f(z)|)$$

By (xx) ψ is a subharmonic function in the quarter plane $U_* = \{x > 0\} \times \{y > 0\}$ and here

$$\psi(x+iy) = \phi(e^{-c^* \cdot y} \cdot |f(x+iy)|)$$

The subharmonicity of ψ entails that

$$\psi(z) \leq h_1(z) + h_2(z)$$

where $h_1(z)$ is the harmonic extension of the boundary value function which is $\phi|f(x)|$ on the positive real axis and zero on the positive y-axis, while $h_2(x,0) = 0$ and

$$h_2(0,y) = \phi(e^{-c^* \cdot y} \cdot |f(iy)|)$$

Since $c^* > c'$ it follows from (xx) that $h_2(0,y)$ is bounded and tends to zero as $y \to \infty$ The growth properties of the two h-functions on the boundsry of U_* entail that both are represented as in § XX. Thus, one has

$$h_1(x+iy) = \frac{1}{\pi}\int_0^\infty y\cdot\psi(\xi)\cdot\big[\frac{1}{(\xi-x)^2+y^2} - \frac{1}{(\xi+x)^2+y^2}\big]\,d\xi$$
$$h_2(x+iy) = \frac{1}{\pi}\int_0^\infty x\cdot\psi(i\eta)\cdot\big[\frac{1}{(\xi-x)^2+y^2} - \frac{1}{(\xi+x)^2+y^2}\big]\,d\eta$$

From (i) we obtain

$$(1)\qquad \int_0^\infty h_1(x+iy)\,dx = \frac{2}{\pi}\int_0^\infty \psi(\xi)\cdot\arctan\frac{\xi}{y}\,d\xi \leq \int_0^\infty \psi(\xi)\,d\xi$$

For h_2 we obtain

$$(2)\qquad \int_0^\infty h_1(x+iy)\,dx = \frac{1}{\pi}\int_0^\infty \psi(i\eta)\cdot\log\big|\frac{\eta+y}{\eta-y}\big|\,d\eta \leq C + A\cdot\int_{2\delta}^\infty \frac{\psi(i\eta)}{\eta}\,d\eta$$

where A and C are two constants which the reader may derive from (x-x) above, Let us now estaimte the last integral in (2). Since ϕ is non-decreasing we see that (xx) and (xx) give have

$$\psi(i\eta) \leq \phi(B\cdot e^{(c'-c^*)\eta})$$

Consider the variable substitution

$$t = B\cdot e^{(c'-c^*)\eta}) \implies \frac{dt}{t} = -Bc^*\cdot d\eta$$

Notice that it gives

$$\eta = \frac{1}{c^*-c'}\cdot\log\frac{B}{t}$$

It follows that

$$\int_{2\delta}^\infty \frac{\phi(B\cdot e^{(c'-c^*)\eta}}{\eta})\,d\eta = (c^*-c')\cdot\int_{t_0}^\infty \frac{\phi(t)}{t\cdot\log\frac{B}{t}}\,dt$$

Conclusion. Under the hypothesis that the integral (**) in Theorem 0.2 converges and integral in the left hand side in Theorem 0.1 also converges, it follows that for every $\delta > 0$ there exis constants C_1, C_2 which may depend upon δ such that

$$\int_{C_1}^\infty (h_1(x+iy) + h_2(x+iy))\,dy \leq C_2 \quad : \quad 0 \leq y \leq \delta$$

Together with the majorization in (xx) it follows that with another constant C_3 we have

$$\iint_{\square_\delta} \psi(x+iy)\,dxdy \leq C_3$$

where $\square_\delta = \{0 < x < \infty\}\times\{0 < y < \delta\}$.

NOW easy to finish the proof of theorems....

§ N. Carleson's Interpolation Theorem

Introduction. Let $U = \mathfrak{Im}(z) > 0$ be the upper half-plane. Denote by $\mathbf{c}_*$ the family of sequences of complex numbers $\{c_\nu\}$ where every $|c_\nu| \leq 1$. A sequence $z_\bullet = \{z_\nu\}$ in U has a finite interpolation norm if there exists a constant K such that for every sequence $\{c_\nu\} \in \mathbf{c}_*$ one can find an analytic function $f(z)$ in U such that

$$(*) \qquad f(z_\nu) = c_\nu \quad : \quad \nu = 1, 2, \ldots \quad \text{and} \quad |f|_U \leq K$$

The smallest constant K for which (*) holds is denoted by $\mathfrak{int}(z_\bullet)$.

0.1 Theorem. *A sequence $z_\bullet$ has a finite interpolating norm if and only if*

$$(1) \qquad \min_\nu \prod_{k \neq \nu} |\frac{z_\nu - z_k}{z_\nu - \bar{z}_k}| > 0$$

Moreover, if $\delta(z_\bullet)$ denotes the minimum above then

$$(2) \qquad \mathfrak{int}(z_\bullet) \leq \frac{A}{\delta(z_\bullet)} \cdot \text{Log}\, \frac{1}{\delta(z_\bullet)}\,.$$

where A is an absolute constant.

Remark. That the condition (1) is necessary is easily verified and left to the reader as an exercise. The proof of sufficieny is more involved. In [Ca] the proof is carried out in the unit disc D where (1) in Theorem 0.1 means that a sequence $\{z_\nu\}$ in D should satisfy

$$(3) \qquad \min_\nu \prod_{k \neq \nu} \frac{|z_\nu - z_k|}{|1 - \bar{z}_k \cdot z_\nu|} > 0$$

Carleson's result will be proved in the upper half-plane where certain constructions become a bit more transparent compared to the unit disc. Let $\{z_\nu\}$ be a sequence in U where (1) holds in Theorem 0.1. Since a family of analytic functions in U with a uniform upper bound for the maximum norm is a normal in Montel's sense, it is sufficient to prove the requested interpolation by bounded functions for every finite subsequence of $\{z_\nu\}$. The Nevanlinna-Pick theorem assigns to each finite sequence $\{z_\nu\}$ and every sequence $\{c_\nu\}$ a unique interpolating analytic function $F(z)$ with smallest maximum norm. So Carleson's result gives a uniform bound in the Nevannlinna-Pick interpolation expressed via the numbers $\delta(z_\bullet)$.

From the above there remains to find an absolute constant A such that (2) in Theorem 0.1 hold for every finite set $E = \{z_1, \ldots, z_N\}$ in the upper half-plane. The proof has several ingredients. We shall use the Hardy space $H^1(\mathbf{R})$ and apply some fundamental results due to Hardy and Littlewood concerned with maximal functions. In addition we use the Borthers Riesz theorem which yields a describes the dual space of $H^1(\mathbf{R})$. A crucial step in Carleson's proof is to tge construction of a certain norm on non-negative measures in the upper half-plane and via the Hardy-Littlewood theory one gets Theorem 0.1 after a duality argument.

0.1 Carleson measures.

For every $h > 0$ we denote by $\mathcal{S}(h)$ the family of squares of the form

$$\square = \{(x,y) \quad : \quad x_0 - h/2 < x < x_0 + h/2 \quad : 0 < y < h\} \quad : \; x_0 \in \mathbf{R}$$

0.2. Definition. *A non-negative measure μ in U is called a Carleson measure if there exists a constant K such that*

$$\max_{\square \in \mathcal{S}(h)} \mu(\square) \leq K \cdot h$$

hold for each $h > 0$. The least constant K is denoted by $\mathfrak{car}(\mu)$ and called the Carleson norm of μ.

In § xx we prove the following inequality:

0.3 Theorem. *For eqch finite set $\{z_\nu\}$ in the upper half-plane one has*

$$\mathfrak{car}\Big(\sum_{\nu=1}^{\nu=\infty} \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu}\Big) \leq (1 + \frac{5}{2}) \cdot \text{Log}\, \frac{1}{\delta(z_\bullet)}$$

where $\{\delta_{z_\nu}\}$ denote Dirac measures.

Use of duality. We are going to use the Hardy space $H^1(\mathbf{R})$. To each $h \in H^1(\mathbf{R})$ we associate the maximal function h^* as explained in § xx.

0.4 Theorem. *For each Carleson measure μ one has the inequality*

$$\int_U |h(z)| \cdot d\mu(z) \leq \mathfrak{car}(\mu) \cdot ||h^*||_1 \quad : \; h \in H^1(\mathbf{R})$$

Armed with Theorems 0.3-0.4 we derive Theorem 0.1 in § 3.

1. Proof of Theorem 0.3

First we establish an inequality which is attributed to L. Hörmander.

1.1 Lemma *Let $z_1, \ldots, z_N$ be a finite sequence in U and put $\delta = \delta(z_\bullet)$. Then*

$$(*) \qquad \sum_{\nu \neq k} \mathfrak{Im}(z_k) \cdot \frac{\mathfrak{Im}(z_\nu)}{|z_k - \bar{z}_\nu|^2} \leq \frac{1}{2} \cdot \log \frac{1}{\delta} \quad : \quad 1 \leq k \leq N$$

Proof. The left hand side as well as the δ-norm of the z-sequence are unchanged if we translate all points to $z_\nu + a$ where a is a real number. Similarly, the δ-norm and the left hand side in (*) are unchanged when the sequence is replaced by $\{A \cdot z_\nu\}$ for some $A > 0$. To prove (*) for a fixed k which we may therefore take $k = N$ and $z_N = i$. Put $z_\nu = a_\nu + ib_\nu$ when $1 \leq \nu \leq N-1$. Then we must show

$$\text{(i)} \qquad \sum_{\nu=1}^{\nu=N-1} \frac{b_\nu}{(1+b_\nu)^2 + a_\nu^2} \leq \frac{1}{2} \cdot \log \frac{1}{\delta}$$

To prove (i) we first notice that

$$\text{(iii)} \qquad \frac{|i - \bar{z}_\nu|^2}{|i - z_\nu|^2} = \frac{(1+b_\nu)^2 + a_\nu^2}{(1-b_\nu)^2 + a_\nu^2}$$

By inverting the $\delta = \delta(z_\bullet)$ we also have:

$$\text{(iii)} \qquad \prod_{\nu=1}^{\nu=N-1} \frac{(1+b_\nu)^2+a_\nu^2}{(1-b_\nu)^2+a_\nu^2} \leq \delta^{-2} \implies \sum_{\nu=1}^{\nu=N-1} \log\Big[\frac{(1+b_\nu)^2+a_\nu^2}{(1-b_\nu)^2+a_\nu^2}\Big] \leq 2\cdot \log\frac{1}{\delta}$$

Next, for each ν we have the integral formula

$$\log\frac{(1+b_\nu)^2+a_\nu^2}{(1-b_\nu)^2+a_\nu^2} = \int_{-b_\nu}^{b_\nu} \frac{2(1+s)}{(1+s)^2+a_\nu^2}\, ds$$

Next, for every pair (a,b) of real numbers with $b>0$ the reader may verify that

$$\text{(iv)} \qquad \frac{b}{(1+b)^2+a^2} \leq \frac{1}{2}\cdot \int_{-b}^{b} \frac{1+s}{(1+s)^2+a^2}\, ds$$

Now (iii-iv) and a summation over ν gives (i).

Final part of the proof of Theorem 0.3

If $z_\bullet \in \mathcal{S}(\delta)$ and a is a real number the sequence $z_\bullet + a = \{z_\nu + a\}$ also belongs to $\mathcal{S}(\delta)$. Since Theorem 0.3 asserts an a priori estimate we may assume that $\square$ is centered at $x=0$, i.e.

$$\square = \{(x,y):\ -h/2 < x < h/2 \text{ and } 0 < y < h\}$$

There remains to estimate

$$\text{(i)} \qquad \sum_{z_\nu \in \square} \mathfrak{Im}\, z_\nu$$

Set

$$y^* = \max\{\mathfrak{Im}(z_\nu):\ z_\nu \in \square\}$$

Let k give the equality $y^* = \mathfrak{Im}(z_k)$. With $z_k = x_k + iy^*$ and $z_\nu = x_\nu + iy_\nu \in \square$ we have

$$|z_k - \bar z_\nu|^2 = (x_k - x_\nu)^2 + (y^* - y_\nu)^2 \leq h^2 + (y^*)^2 \implies$$

$$\frac{\mathfrak{Im}(z_k)}{|z_k - \bar z_\nu|^2} \geq \frac{y^*}{h^2+(y^*)^2} \quad :\ \nu \neq k$$

Next, notice that

$$y^* \geq h/2 \implies \frac{y^*}{h^2+(y^*)^2} \geq \frac{1}{5h}\,.$$

Lemma 1.1. applied with $\nu = k$ gives therefore

$$\text{(ii)} \qquad \sum_{z_\nu \in \square} \mathfrak{Im}(z_\nu) \leq y^* + \frac{5h}{2}\cdot \mathrm{Log}\,\frac{1}{\delta} \leq h\cdot\big(1 + \frac{5}{2}\cdot \mathrm{Log}\,\frac{1}{\delta}\big)$$

So if $y^* \geq h/2$ we are done. Suppose now that $y^* < h/2$ and regard the cubes:

$$\square_1 = \{-h/2 < x < 0 \text{ and } 0 < y < h/2\} \quad \square_2 = \{0 < x < h/2 \text{ and } 0 < y < h/2\}$$

We want to estimate

$$S_1 + S_2 = \sum_{z_\nu \in \square_1} \mathfrak{Im}(z_\nu) + \sum_{z_\nu \in \square_2} \mathfrak{Im}(z_\nu)$$

We have also two sequences:

$$\{z_\nu : z_\nu \in \square_1\} \quad \text{and} \quad \{z_\nu : z_\nu \in \square_2\}$$

Since all factors defining the δ-norm are ≤ 1 these two smaller sequences both belong to $\mathcal{S}(\delta)$. Suppose that:

$$y_1^* = \max_{z_\nu \in \square_1} \mathfrak{Im}(z_\nu) \geq \frac{h}{4}$$

When this holds we obtain via an obviouos scaling and the same argument as above:

$$S_1 \leq \frac{h}{2} \cdot (1 + \frac{5}{2} \cdot \text{Log}\,\frac{1}{\delta}$$

If $y_1^* < \frac{h}{4}$ we continue to split the cube $\square_1$. In a similar fashion we treat the sequence which stays in $\square_2$. After a finite number of steps the requested inequality in Theorem 0.3 follows.

2. Proof of Theorem 0.4

Let $h \in H^1(\mathbf{R})$ and recall that its maximal function is defined by

(i) $$h^*(t) = \max |h(x+iy)| \quad : \quad |x-t| < y$$

To each $\lambda > 0$ we consider the open subset on the real line defined by $\{h^* > \lambda\}$. It is some union of disjoint intervals $\{(a_j, b_j)\}$ and (i) gives the set-theoretic inclusion:

(ii) $$\{|h(x+iy)| > \lambda\} \subset \cup\, T_j \quad :$$

where T_j is the triangle side standing on the interval (a_j, b_j) as explained in XXX. (Hardy space). In particular we have the inclusion:

(iii) $$T_j \subset \square(a_j, b_j) = \{x+iy : |x - \frac{a_j+b_j}{2}| < b_j - a_j \ : \ 0 < y < b_j - a_j\}$$

See figure XXX. So if μ is a positive measure in U we obtain:

(iv) $$\mu(\{|h| > \lambda\}) \leq \sum \mu(T_j) \leq \sum \mu(\square(a_j, b_j))$$

If μ is a Carleson measure the right hand side is estimated by

(v) $$\mathfrak{car}(\mu) \cdot \sum (b_j - a_j) = \mathfrak{car}(\mu) \cdot \mathfrak{m}(\{h^* > \lambda\})$$

where $\mathfrak{m}$ refers to the 1-dimensional Lebesgue measure. Here (v) holds for every $\lambda > 0$ and the general inequality for distribution functions from XXX gives:

$$\int_U |h| \cdot d\mu \leq \mathfrak{car}(\mu) \cdot ||h^*||_1$$

This finishes the proof of Theorem 0.4.

3. Proof of Theorem 0.1.

As explained in § xx, the Banach space $H^1(\mathbf{R})$ contains a dense subspace denoted by H^1_* of functions $h(z)$ with rapid decay at infinity, i.e.

$$|h(z)| \leq C_N \cdot (1 + |z|)^{-N} \quad : \text{hold for some constant } C_N \quad : N = 1, 2, \ldots$$

Consider a finite sequence $z_1, \ldots, z_N$ in U and a finite sequence $c_1, \ldots, c_N$ in $\mathbf{c}_*$. Newton's interpolation gives a unique polynomial $P(z)$ of degree $N-1$ such that:

(i) $$P(z_k) = c_k \quad : \quad 1 \leq k \leq N$$

Let $B(z)$ be the Blascke product associated to the z-sequence:

(ii) $$B(z) = \prod_{\nu=1}^{\nu=N} \frac{z - z_\nu}{z - \bar{z}_\nu}$$

For each $h \in H^1_*$ we have the absolutely convergent integral

$$L(h) = \int_{-\infty}^{\infty} \frac{P(x)}{B(x)} \cdot h(x) \cdot dx$$

Residue calculus gives the equation

(iii) $$L(h) = 2\pi i \cdot \sum_{k=1}^{k=N} \frac{c_k}{B'(z_k)} \cdot h(z_k)$$

Next, for each $1 \leq k \leq N$ we notice that

(iv) $$\frac{1}{B'(z_k)} = \prod_{\nu \neq k} \frac{z_k - \bar{z}_\nu}{z_k - z_\nu} \cdot 2 \cdot \mathfrak{Im}(z_k)$$

The definition of $\delta(z_\bullet)$ and the triangle inequality give

(v) $$\Big|\frac{1}{B'(z_k)}\Big| \leq \frac{2}{\delta(z_\bullet)} \cdot \mathfrak{Im}(z_k)$$

Since $\{c_\nu\} \in \mathbf{c}_*$ we see that (v) and the triangle inequality applied to (iii) give:

(vi) $$\Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx\,\Big| \leq \frac{4\pi}{\delta(z_\bullet)} \cdot \sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k)$$

Next, Theorem 0.4 gives the inequality

(vii) $$\sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k) \leq \mathfrak{car}(\sum \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu}) \cdot ||h^*||_1$$

Put

$$C_\delta = \frac{4\pi}{\delta(z_\bullet)} \cdot 2 \cdot \log \frac{1}{\delta(z_\bullet)}$$

Then (vi-vii) and Theorem 0.3 give

(*) $$\Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx\,\Big| \leq C_\delta \cdot ||h^*||_1$$

Next, the Hardy-Littlewood inequality from §§ (Hardy Chapter) gives an absolute constant A such that

$$||h^*||_1 \leq A \cdot ||h||_1$$

Hence the norm of the densely defined linear functional

$$h \mapsto \int_{-\infty}^{\infty} \frac{P(x)}{B(x)} \cdot h(x) \cdot dx$$

is $\leq C_\delta \cdot A$. The Duality Theorem from XXX implies that if $\epsilon > 0$, then there exists $G(z) \in \mathcal{O}_*(U)$ such that

$$\max_x \Big| \frac{P(x)}{B(x)} - G(x) \Big| < A \cdot C_\delta + \epsilon \tag{6}$$

where the maximum is taken over all real x. Since B is a finite Blaschke product we have $|B(x)| = 1$ for each real x and hence (6) gives:

$$\max_x \big| P(x) - B(x) \cdot G(x) \big| < A \cdot C_\delta + \epsilon$$

Set

$$f(z) = P(z) - B(z)G(z)$$

Since $B(z_\nu) = 0$ for every ν we have

$$f(z_\nu) = P(z_\nu) = c_\nu$$

So the analytic function $f(z)$ interpolates the c-values on the given finite set $Z_\bullet$. Moreover, by (xx) and the maximum principle from (*) for functions in $\mathcal{O}_*(U)$ it follows that

$$|f|_U \leq A \cdot C_\delta + \epsilon$$

Above $\epsilon > 0$ can be arbitrary small and since $c_1, \ldots, c_N$ was an arbitrary sequence in $\mathbf{c}_*$ we conclude that the interpolation norm of the finite sequence $z_1, \ldots, z_N$ is at most $A \cdot C_\delta$. As observed in § xx this uniform estimate finishes the proof of Theorem 0.1.

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Appendix 1: Invariant subspaces of $H^2(T)$

0. Introduction.

1. The Herglotz integral.

2. The class JN(D)

3. Blaschke products

4. Invariant subspaces of $H^2(T)$

5. Beurling's closure theorem

6 The Helson-Szegö theorem.

Introduction.

A fundamental result was discovered by Jensen in 1899 which leads to the familiar expression for counting numbers of zeros of analytic functions. Together with factorisations with Blascke products and Heglotz' integral formula which serves as the holomorphic version of th Poisson formula in discs, one arrives at some powerful factorisation theorems for analytic functions in the unit disc which belong to the Jensen-Nevanlinna class which is described below. Starting from this Beuliung investigaged decompositions of the boundary measure of analytic functions and and defined inner, respectively outer factors. Using these results in analytic function theory, he then studied invarant subspaces of the Hardy space $H^2(T)$ where the closure theorem in § 4 is fundamental. It was later applied by Helson and Szegö and leads to certain results concerened with polynomial approximation which are exposed in § 5-6.

Remark. For the reader's convenicnce the subsequent material is quite detailed and essentially self-contained.

§ 0. Poisson extensions.

Let μ be a real Riesz measure on the unit circle T. It gives the harmonic function in the unit disc D defined by

$$(0.1) \qquad H_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} \cdot d\mu(\theta)$$

For each $0 < r < 1$ one has the inequality

$$(0.2) \qquad \int_0^{2\pi} \left|H_\mu(re^{i\theta})\right| \cdot d\theta \leq ||\mu||$$

where $||\mu||$ is the total variation of μ. Moreover, there exists a weak limit, i.e.

$$(0.3) \qquad \lim_{r\to 1} \int_0^{2\pi} g(\theta) \cdot H_\mu(re^{i\theta}) = \int_0^{2\pi} g(\theta) \cdot d\mu(\theta)$$

holds for every continuous function $g(\theta)$ on T. Recall the converse result due to F. Riesz. Namely, if $H(z)$ is a harmonic function in D for which there exists a constant C such that

$$\int_0^{2\pi} \left|H(re^{i\theta})\right| \cdot d\theta \leq C \tag{0.4}$$

hold for all $r < 1$, then there exists a unique Riesz measure μ on T and $H = H_\mu$. In other words, there is a 1-1 correspondence between the space of harmonic functions in D satisfying (0.4) and the space of real Riesz measures on T. Next, by a result due to Fatou there exist radial limits almost everywhere. More precisely, define the μ-primitive function

$$\psi(\theta) = \int_0^\theta d\mu(s)$$

Fatou's Theorem asserts that for each Riesz measure μ there exists a radial limit

$$H_\mu^*(\theta) = \lim_{r\to 1} H(re^{i\theta}) \tag{0.5}$$

for each θ where ψ has an ordinary derivative. Since ψ has a bounded variation this holds almost everywhere by a famous result due to Lebesgue.

0.6 The case when μ is singular. If μ is singular the radial limit (0.5) is zero almost everywhere. If the singular measure μ is non-negative with total mass 2π we have $H_\mu(0) = 1$ and the mean-value property for harmonic functions gives:

$$\int_0^{2\pi} H_\mu(re^{i\theta}) \cdot d\theta = 1$$

for all $0 < r < 1$. At the same time the boundary function $H_\mu^*(\theta)$ is almost everywhere zero which means that one cannot have any kind of dominated convergence.

0.7 Exercise. Let μ be singular with a Hahn-decomposition $\mu = \mu_+ - \mu_-$. Assume that the positive part $\mu_+(T) = a > 0$. Now there exists a closed null set E such that $\mu_+(E) \geq a - \epsilon$ while $\mu_-(E) = 0$. The last equation gives a small $\delta > 0$ such that if $E_{2\delta}$ is the open 2δ-neighborhood of E then $\mu_-(E_{2\delta}) < \epsilon$. Set

$$H_*(z) = \frac{1}{2\pi} \int_E \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu_+(\theta)$$

Since $\mu_+(E) \geq a - \epsilon$ we get

$$\int_0^{2\pi} H_*(re^{i\theta}) \cdot d\theta \geq a - \epsilon \tag{ii}$$

Next, for each pair $\phi \in E_\delta$ and $e^{i\theta} \in T \setminus E_{2\delta}$ we have:

$$\frac{1 - r^2}{|e^{i\theta} - re^{i\phi}|^2} \leq \frac{2(1-r)}{1 + r^2 - 2r\cos(\delta)}$$

So with

$$H_\delta(z) = \frac{1}{2\pi} \int_{T\setminus E_{2\delta}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu_(\theta)$$

it follows that

$$\text{(iii)} \qquad |H_\delta(re^{i\phi})| \leq \frac{1}{2\pi} \cdot \frac{2(1-r)}{1+r^2-2r\cos(\delta)} \cdot \int_{T\setminus E_{2\delta}} |d\mu(\theta)|$$

for each $\phi \in E_\delta$. Since H_* is constructed via the restriction of μ_+ to E, a similar reasoning gives:

$$\text{(iv)} \qquad |H_*(re^{i\phi})| \leq \frac{1}{2\pi} \frac{2(1-r)}{1+r^2-2r\cos(\delta)} \cdot \mu_+(E)$$

when $e^{i\phi} \in T \setminus E_\delta$. Next, by the constructions above we have

$$H = H_* + H_\delta + H_\nu$$

where ν is the measure given by the restriction of μ_+ to $E_{2\delta} \setminus E$ minus μ_- restricted to to $E_{2\delta}$. So by the above the total variation $||\nu|| \leq 2\epsilon$ which gives

$$\int_0^{2\pi} |H_\nu(re^{i\theta})| \cdot d\theta \leq 2\epsilon$$

Deduce from the above that one has an inequality

$$(*) \qquad \int_{E_\delta} H(re^{i\phi}) \cdot d\phi \geq a - \big[2\epsilon + \frac{1}{\pi} \frac{2(1-r)}{1+r^2-2r\cos(\delta)} \cdot ||\mu||\big]$$

Since E is a null-set, the mean-value integrals of H behave in an "irregular fashion" when $r \to 1$.

1. The Herglotz integral

Let μ be a real Riesz measure on the unit circle T. Set

$$(*) \qquad g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot d\mu(\theta)$$

This analytic function is called the Herglotz extension of the Riesz measure. Since μ is real it follows that

$$\mathfrak{Re}\, g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z[^2}{|e^{i\theta}-z|^2} \cdot d\mu(\theta) = H_\mu(z)$$

In particular the L^1-norms from (0.2) are uniformly bounded with respect to r when we integrate the absolute value of $\mathfrak{Re}\, g_\mu$. But the conjugate harmonic function representing $\mathfrak{Im}\, g_\mu$ does not satisfy (0.2) in general. However the following holds:

1.1 Theorem. *For almost every θ there exists a radial limit*

$$\lim_{r\to 1} g_\mu(re^{i\theta})$$

To prove Theorem 1.1 we shall use some tricks. The Hahn-decomposition $\mu = \mu_+ - \mu_-$ enables us to express g_μ as a difference $g_1 - g_2$ where g_1, g_2 both are Herglotz extensions of non-negative Riesz measures and hence $\mathfrak{Re}\, g_\nu(z) > 0$ in D. Let us now discuss analytic functions with a positive real part.

1.2 Exercise. Let $f \in \mathcal{O}(D)$ where $\mathfrak{Re}\, f(z) > 0$ and $\mathfrak{Im} f(0) = 0$. Set $f = u + iv$ which gives the analytic function

$$\phi(z) = \log(1 + u + iv)$$

Here

$$\mathfrak{Re}\, \phi = \log\, |1 + u + iv| = \frac{1}{2} \log[(1+u)^2 + v^2]$$

In particular $\mathfrak{Re}\, \phi > 0$ so this harmonic function has a radial limit almost everywhere. We also know that u has a radial limit almost everywhere and from this the reader may conclude that there almost everywhere exist finite radial limits

$$\lim_{r \to 1} v^2(re^{i\theta}) \tag{1}$$

In order to determine the sign of these radial limits we consider the analytic function

$$\psi = e^{-u-iv}$$

Since $u > 0$ we have $|\psi(z)| = e^{-u(z)} \leq 1$ and hence $\psi(z)$ is a bounded analytic function in D. The Brothers Riesz theorem shows that ψ has a radial limit almost everywhere. Finally, when we have a radial limit

$$\lim_{r \to 1} e^{-u(re^{i\theta}) - iv(re^{i\theta})}$$

and in addition suppose that u has a radial limit, then it is clear that v has a radial limit too.

Proof of Theorem 1.1 By the Hahn-decomposition of μ the proof is reduced to the case $\mu \geq 0$ and Exercise 1.2 applies.

1.3 The case when μ is singular. When this holds the radial limits of $\mathfrak{Re}\, g_\mu$ are almost everywhere zero. With $v = \mathfrak{Im}\, g_\mu$ there remains to study the almost everywhere defined function

$$v^*(\theta) = \lim_{r \to 1} v(re^{i\theta})$$

It turns out that this Lebesgue-measurable function never is integrable when μ is singular. In fact, the Brothers Riesz theorem shows that if there exists a constant C such that

$$\int_0^{2\pi} |v(re^{i\theta})| \cdot d\theta \leq C$$

hold for all $r < 1$, then the analytic function g_μ belongs to the Hardy space and its radial limits give an L^1-function $g^*(\theta)$ on the unit circle which would entail that σ is equal to the absolutely continuous measure defined by g^*. Thus, for every singular measure μ one has

$$\lim_{r \to 1} \int_0^{2\pi} \left| \mathfrak{Im}\, g_\mu(re^{i\theta}) \right| \cdot d\theta = +\infty \tag{*}$$

1.4 Example. Take the case where μ is 2π times the Dirac measure at $\theta = 0$ which gives the analytic function

$$g(z) = \frac{1+z}{1-z}$$

It follows that

$$v(re^{i\theta}) = -2r \cdot \frac{\sin\theta}{1 + r^2 - 2r\cos\theta}$$

and radial limits exist except for $\theta = \pi/2$ or $-\pi/2$, i.e.

$$v^*(\theta) = -2 \cdot \frac{\sin\theta}{2 - 2\cos\theta}$$

when θ is $\neq \pi/2$ and $-\pi/2$. At the same time the reader may verify that $v^*(\theta)$ does not belong to $L^1(T)$ and that

$$\int_0^{2\pi} |v(re^{i\theta})| \cdot d\theta \simeq \log \frac{1}{1-r}$$

as $r \to 1$.

2. The Jensen-Nevanlinna class

Every Riesz measure μ on T gives the zero-free analytic function

$$G_\mu(z) = e^{g_\mu(z)} \tag{*}$$

Here $\log |G_\mu(z)| = \mathfrak{Re}\, g_\mu(z)$ which gives the inequality

$$\log^+ |G_\mu(z)| \leq \big|\mathfrak{Re}\, g_\mu(z)\big|$$

Applying (0.2) we obtain:

$$\int_0^{2\pi} \log^+ |G_\mu(re^{i\theta})| \cdot d\theta \leq ||\mu|| \tag{**}$$

for each $r < 1$.

2.1 A converse. Let $F(z)$ be a zero-free analytic function in D where $F(0) = 1$. Assume that there exists a constant C such that

$$\int_0^{2\pi} \log^+ |F(re^{i\theta})| \cdot d\theta \leq C \tag{i}$$

hold for each $r < 1$. The mean-value property applied to the harmonic function $H = \log |F|$ gives

$$\int_0^{2\pi} |H(re^{i\theta})| \cdot d\theta = 2 \cdot \int_0^{2\pi} \log^+ |F(re^{i\theta})| \cdot d\theta \tag{ii}$$

Hence (i) entails that H satisfies (0.4) and now the reader can settle the following:

2.2 Exercise. Show that (i) above entails that there exists a Riesz measure μ such that $F = G_\mu$ where the normalisation $F(0) = 1$ gives $\mu(T) = 2\pi$.

2.3 Radial limits. Whenever g_μ has a radial limit for some θ it is clear that G_μ also has a radial limit in this direction. So Theorem 1.1 implies that there exists an almost everywhere defined boundary function

$$G^*_\mu(\theta) = \lim_{r\to 1} G_\mu(re^{i\theta})$$

The material above suggests the following:

2.4 Definition. *An analytic function f in D belongs to the Jensen-Nevanlinna class if there exists a constant C such that*

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \cdot d\theta \leq C$$

hold for all $r < 1$. The family of Jensen-Nevannlina functions is denoted by $JN(D)$.

Above we described zero-free functions in $\text{JN}(D)$. Now we shall study eventual zeros of functions in $\text{JN}(D)$. Recall that if $f \in \mathcal{O}(D)$ where $f(0) = 1$ then Jensen's formula gives:

$$(*) \qquad \sum_{|\alpha_\nu|<r} \text{Log}\,\frac{r}{|\alpha_\nu|} = \frac{1}{2\pi}\int_0^{2\pi} \log\,\big|f(re^{i\theta})\big| \cdot d\theta \quad : \quad 0 < r < 1$$

where the left hand side is the sum of zeros of f in the disc D_r.

A notation. If $f \in \mathcal{O}(D)$ and $r < 1$ we set

$$\mathcal{T}_f(r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \cdot d\theta$$

Since $\log\,\big|f(re^{i\theta})\big| \leq \log^+ |f|$ it follows that

$$\sum_{|\alpha_\nu|<r} \text{Log}\,\frac{r}{|\alpha_\nu|} \leq \mathcal{T}_f(r)$$

So if $f \in \text{JN}(D)$ we can pass to the limit as $r \to 1$ and conclude that the positive series

$$(**) \qquad \sum \text{Log}\,\frac{1}{|\alpha_\nu|} < \infty$$

where the sum is taken over all zeros in D. Next, recall form XX that the positive series (**) converges if and only if

$$(***) \qquad \sum (1 - |\alpha_\nu| < \infty$$

When (***) holds we say that the sequence $\{\alpha_\nu\}$ satisfies the Blaschke condition. Hence we have proved:

2.5 Theorem. *Let f be in JN(D). Then its zero set satisfies the Blaschke condition.*

3. Blaschke products.

Consider an infinite sequence $\{\alpha_\nu\}$ in D which satisfies $|\alpha_1| \leq |\alpha_2| \leq \ldots$. For every $N \geq 1$ we put:

$$B_N(z) = \prod_{\nu=1}^{\nu=N} \frac{|\alpha_\nu|}{\alpha_\nu} \cdot \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z}$$

We are going to prove that the sequence of analytic function $\{B_N\}$ converge in D to a limit function $B(z)$ expressed by the infinite product

$$B(z) = \prod_{\nu=1}^{\infty} \frac{|\alpha_\nu|}{\alpha_\nu} \cdot \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z}$$

To prove this we first analyze the individual factors. For each non-zero $\alpha \in D$ we set

$$B_\alpha(z) = \frac{|\alpha|}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}$$

Exercise. Show that

(i) $$B_\alpha(z) = |\alpha| \cdot \frac{1 - z/\alpha}{1 - \bar{\alpha} z} = |\alpha| + \frac{|\alpha|^2 - 1}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha} \cdot z$$

and conclude that

(ii) $$B_\alpha(z) - 1 = (|\alpha| - 1) \cdot [1 + \frac{|\alpha| + 1}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha} \cdot z]$$

Finally, use the triangle inequality to show the inequality

(iii) $$\max_{|z|=r} |B_\alpha(z) - 1| \leq (1 - |\alpha|) \cdot (1 + \frac{2r}{1-r}) = \frac{1+r}{1-r} \cdot (1 - |\alpha|)$$

.1 The convergence of (**) From (iii) and general results about product series we get the requested convergence in (**). In fact, when $|z| \leq r < 1$ stays in a compact disc the Blaschke condition and (iii) entail that

$$\sum_{\nu=1}^{\infty} \max_{|z|=r} |B_\alpha(z) - 1| < \infty$$

which implies that (**) converges uniformly on $|z| \leq r$ to an analytic function and since $r < 1$ is arbitrary we get a limit function $B(z) \in \mathcal{O}(D)$.

3.2 Exercise. The rate of convergence in $|z| \leq r$ can be described as follows: For each $N \geq 1$ we set

$$G_N(z) = \prod_{\nu=N+1}^{\infty} B_{\alpha_\nu}(z) \quad : \quad \Gamma_N = \sum_{\nu=N+1}^{\infty} 1 - |\alpha_\nu|$$

With $r < 1$ kept fixed we choose n so large that

$$\frac{1+r}{1-r} \cdot (1 - |\alpha_\nu|) \leq \frac{1}{2} \quad : \quad \nu > N$$

Show that this gives:

$$\max_{|z|=r} |G_N(z) - 1| \leq 8 \cdot \frac{1+r}{1-r} \cdot \Gamma_N$$

Since the Blaschke condition implies that $\Gamma_N \to 0$ as $N \to \infty$ this gives a control for the rate of convergence in $|z| \leq r$.

3.3 Radial limits of $B(z)$

When $z = e^{i\theta}$ the absolute value $|B_\alpha(e^{i\theta}| = 1$. So $B(z)$ is the product of analytic functions where every term has absolute value ≤ 1 and hence the maximum norm

$$\max_{z \in D} |B(z)| \leq 1$$

Since the analytic function $B(z)$ is bounded, Fatou's Theorem from Section XX gives an almost everywhere defined limit function

$$B^*(e^{i\theta}) = \lim_{r \to 1} B(re^{i\theta}) \tag{1}$$

where the radial convergence holds almost everywhere. Moreover, the Brothers Riesz theorem gives:

$$\lim_{r \to 1} \int_0^{2\pi} |B^*(e^{i\theta}) - B(re^{i\theta})| d\theta = 0 \tag{2}$$

3.4 Theorem. *The equality*

$$|B^*(e^{i\theta})| = 1 \quad \text{holds almost everywhere} \tag{*}$$

Proof. Since $|B^*| \leq 1$ it is clear that (*) follows if we have proved that

$$\int_0^{2\pi} |B^*(e^{i\theta})| \cdot d\theta = 1 \tag{i}$$

Using (2) above and the triangle inequality we get (i) if we prove the limit formula

$$\lim_{r \to 1} \int_0^{2\pi} |B(re^{i\theta})| \cdot d\theta = 1 \tag{ii}$$

To show (ii) we will apply Jensen's formulas to $B(z)$ in discs $|z| \leq r$. The convergent product which defines $B(z)$ gives

$$B(0) = \prod \log |\alpha_\nu|$$

Next, for $0 < r < 1$ Jensen's formula gives

$$\log B(0) = \sum_{\nu=1}^{\rho(r)} \log \frac{|\alpha_\nu|}{r} + \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta$$

where $\rho(r)$ is the largest ν for which $|\alpha_\nu| = r$. It follows that

(1) $$\frac{1}{2\pi}\int\int_0^{2\pi} \log|B(re^{i\theta})|\cdot d\theta \geq \sum_{\nu=1}^{\rho(r)} \log\frac{r}{|\alpha_\nu|} - \sum_{\nu=1}^{\infty} \log\frac{1}{|\alpha_\nu|}$$

Next, with $\epsilon > 0$ we find an integer N such that

(2) $$\sum_{\nu=1}^{\nu=N} \log\frac{1}{|\alpha_\nu|} < \epsilon$$

Since $|\alpha_\nu| \to 1$ here exists r_* such that

(3) $$r \geq r_* \implies \rho(r) \geq N$$

When (3) holds it follows from (1-2) that

(4) $$\frac{1}{2\pi}\int\int_0^{2\pi} \log|B(re^{i\theta})|\cdot d\theta \geq \sum_{\nu=1}^{\rho(r)} \log\frac{r}{|\alpha_\nu|} - \sum_{\nu=1}^{\rho(r_*)} \log\frac{1}{|\alpha_\nu|} - \epsilon$$

In the first sum every term is ≥ 1 so we get a better inequality when the sum is restricted to $\nu \leq \rho(r_*)$, i.e. we have

(5) $$\frac{1}{2\pi}\int\int_0^{2\pi} \log|B(re^{i\theta})|\cdot d\theta \geq \sum_{\nu=1}^{\rho(r_*)} \log\frac{r}{\alpha_\nu|} - \sum_{\nu=1}^{\rho(r_*)} \log\frac{1}{\alpha_\nu|} - \epsilon$$

Here (5) hold for every $r_* < r < 1$ and a passing to the limit as $r \to 1$ where we only have a finite sum $1 \leq \nu \leq \rho(r_*)$ above we conclude that

$$\lim_{r\to 1}\frac{1}{2\pi}\int\int_0^{2\pi} \log|B(re^{i\theta})|\cdot d\theta > -\epsilon$$

Since $\epsilon > 0$ is arbitrary we have proved (ii) and hence also Theorem 3.54.

3.5 Division by Blaschke products.

Let $F \in \mathcal{O}(D)$ and assume that its zero set in D is a Blaschke sequence $\{\alpha_\nu\}$. Then we obtain the analytic function

$$G(z) = \frac{F(z)}{B(z)}$$

Here G has no zeros in D and we can construct the analytic function $\operatorname{Log} G(z)$. Set

$$\mathcal{I}_G^+(r) = \int_0^{2\pi} \log^+|\,G(re^{i\theta})|\cdot d\theta$$

Since $\log^+[ab| \leq \log^+|a| + \log^+|b|$ for every pair of complex numbers we get:

(1) $$\mathcal{I}_G^+(r) \leq \mathcal{I}_F^+(r) + \int_0^{2\pi} \log^+\frac{1}{|\,B(re^{i\theta})|}\cdot d\theta$$

The last nondecreasing function is $\leq \log^+ \frac{1}{|B(0)|}$ for every r which gives

$$\mathcal{I}_G^+(r) \leq \mathcal{I}_F^+(r) + \log^+ \frac{1}{|B(0)|} \tag{2}$$

for every $r < 1$. When $F \in \text{JN}(D)$ this implies that G also belongs to $\text{JN}(D)$. Hence we have proved

3.6 Theorem. *For each $f \in JN(D)$ the function $\frac{f}{B_f}$ also belongs to $JN(D)$, where $B_f(z)$ is the Blaschke product formed by zeros of f outside the origin.*

3.7 Conclusion. Theorem 3.6 and the material in section 2 about zero-free Jensen-Nevanlinna functions give the following factorisation formula:

3.8 Theorem. *For each $f \in JN(D)$ there exists a unique real Riesz measure μ on T with $\mu(T) = 0$ such that*

$$f(z) = az^k \cdot B_f(z) \cdot e^{g_\mu(z)}$$

where $k \geq 0$ is the order of zero of f at $z = 0$ and $a \neq 0$ a constant. Moreover

$$\mu = \log |f(e^{i\theta})| \cdot d\theta + \sigma$$

where σ is the singular part of μ.

3.9 Outer factors. In Theorem 3.8 we get the analytic function

$$\mathfrak{O}_f(z) = e^{g_{\log |f|}(z)}$$

We refer to $\mathfrak{O}_f$ as the outer part of f.

3.10 A division result. Consider a pair f, h in $\text{JN}(D)$ which gives the analytic function in D defined by

$$k(z) = \frac{\mathfrak{O}_h(z)}{\mathfrak{O}_f(z)}$$

By (2.3) there exists the almost everywhere defined quotient on T

$$k^*(\theta) = \frac{\mathfrak{O}_h^*(\theta)}{\mathfrak{O}_f^*(\theta)}$$

3.11 Theorem. *Assume that $k^* \in L^1(T)$. Then k^* belongs to the Hardy space $H^1(T)$.*

Proof. In D there exists the harmonic function

$$k(z) = \log |\mathfrak{O}_h(z)| - \log |O_f(z)|$$

The two harmonic functions in the right hasnd side have by definition boundary functions in $L^1(T)$ and Poisson's formula gives for each point $z = re^{i\theta}$:

$$\log |k(re^{i\theta})| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)} \cdot \log |k^*(\phi)| \cdot d\phi$$

By the general mean-value inequality from (xx) the left hand side is majorized by:

$$\leq \log\Big[\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}\cdot|k^*(\phi)|\cdot d\phi\Big]$$

Taking exponentials on both sides we get

$$|k(re^{i\theta})|\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}\cdot|k^*(\phi)|\cdot d\phi$$

Now we integrate both sides with respect to θ. Since

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}\cdot d\theta=1$$

for every ϕ, it follows that

$$\int_0^{2\pi}|k(re^{i\theta})|\cdot d\theta\leq\int_0^{2\pi}|k^*(e^{i\phi})|\cdot d\phi$$

This proves that the L^1-norms of $\theta\to k(re^{i\theta})$ are bounded which means that k belongs to $H^1(T)$. Moreover, by the Brother's Riesz theorem there exist radial limits almost everywhere so we have also the equality

$$\lim_{r\to 1}k(re^{i\theta})=k^*(\theta)$$

almost everywhere. This proves that k^* is the boundary value function of the $H^1(T)$-function k.

3.12 Exercise. Show by a similar technique that if we instead assume that k^* is square-integrable, i.e. if $k^*\in L^2(T)$ then $k(z)$ belongs to the Hardy space $H^2(T)$.

3.13 Inner functions. If σ is a non-negative and singular measure on T we get the bounded analytic function

$$G_{-\sigma}(z)=e^{-g_\sigma(z)} \tag{1}$$

Keeping σ fixed we denote this function with f. Here

$$\lim_{r\to 1}|f(re^{i\theta})|=1$$

holds almost everywhere. So the boundary function $f^*(\theta)$ has absolute value almost everywhere. The class of analytic functions obtained via (1) is denoted by $\mathfrak{I}_*(D)$ and are called zero-free inner functions. In general a bounded analytic function f in D whose boundary values have absolute value almost everywhere is called an inner function and this class is denoted by $\mathfrak{I}(D)$.

3.14 Exercise.. Use the factorisation in Theorem 3.8 to show that every $f\in\mathfrak{I}(D)$ is a product

$$f=B_f\cdot f_*$$

where f_* is a zero-free inner function.

3.15 The case of signed singular measures. Let $\mu=\mu_+-\mu_-$ be a signed singular measure where $\mu_+\neq 0$. We get the analytic function G_μ and from the above we know that it has radial limits almost everywhere and since μ is singular the boundary function G^*_μ has absolute value almost everywhere. Here the presence of μ_+ implies that the analytic function G_μ is unbounded. In fact, its maximum modules function

$$M(r) = \max_{|z|=r|} |G_\mu(z)|$$

has a quite rapid growth as $r \to 1$. Moreover one always has

$$(*) \qquad \lim_{r\to 1} \int_0^{2\pi} |G_\mu(re^{i\theta})| \cdot d\theta = +\infty$$

in other words, G_μ-functions constructed by signed measures with non-zero negative part never belongs to $H^1(T)$.

3.16 Exercise. Prove (*) above using the divergence in (*) from 1.3.

4. Invariant subspaces of $H^2(T)$

The Hilbert space $L^2(T)$ of square integrable functions on T contains the closed subspace $H^2(T)$ whose elements are boundary values of analytic functions in D. If f $f \in H^2(T)$ it is expanded as

$$\sum_{n=0}^\infty a_n \cdot e^{in\theta}$$

and Parseval's theorem gives the equality

$$\sum_{n=0}^\infty |a_n|^2 = \frac{1}{2\pi} \cdot \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$$

Moreover, in D we get the analytic function $f(z) = \sum a_n z^n$ where radial limits

$$\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta})$$

exist almost everywhere.in fact, this follows via the Brothers Riesz theorem and the inclusion $H^2(T) \subset H^1(T)$. We shall study subspaces of $H^2(T)$ which are invariant under multiplication by $e^{i\theta}$.

4.2 Definition. *A closed subspace V of $H^2(T)$ is called invariant if $e^{i\theta}V \subset V$.*

4.3 Theorem *Let V be an invariant subspace of $H^2(T)$. Then there exists $w(\theta) \in H^2(T)$ whose absolute value is one almost everywhere and*

$$V = H^2(T) \cdot w$$

Proof. First we show that that $e^{i\theta}V$ is a proper subspace of V. For an equality $e^{i\theta}V = V$ gives $e^{in\theta}V = V$ for every $n \geq 1$ which entails that if $0 \neq f \in V$ then $f = e^{in\theta} \cdot g_n$ for some $g_n \in H^2(T)$. This means that the Taylor series of f at $z = 0$ starts with order $\geq n$ which cannot hold for every n unless f is identically zero. So now $e^{i\theta}V$ is a proper closed subspace of V which gives some $0 \neq w \in V$ which is $\perp$ to $e^{i\theta}V$. It follows that

$$\langle w, e^{in\theta} \cdot w\rangle \int_0^{2\pi} w(e^{i\theta})\bar{w}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0$$

hold for every $n \geq 1$. Since $w \cdot \bar{w} = |w|^2$ is real-valued we conclude that this function is constant and we can normalize w so that $|w(\theta)| = 1$ holds almost everywhere. There remains to prove the equality

$$\text{(i)} \qquad V = H^2(T) \cdot w$$

Since $|w| = 1$ almost everywhere the right hand side is a closed subspace of V. If it is proper we find $0 \neq u \in V$ where $u \perp H^2(T)w$ which gives

$$\text{(ii)} \qquad \int_0^{2\pi} u(e^{i\theta})\bar{w}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0 \quad : \quad n \geq 0$$

Taking complex conjugates we get

$$\text{(iii)} \qquad \int_0^{2\pi} w(e^{i\theta})\bar{u}(e^{i\theta}) \cdot e^{in\theta} \cdot d\theta = 0 \quad : \quad n \geq 0$$

At the same time $w \perp e^{i\theta}V$ which entails that

$$\text{(iv)} \qquad \int_0^{2\pi} w(e^{i\theta})\bar{u}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0 \quad : \quad n \geq 1$$

Together (iiii-iv) imply that $w\bar{u}$ has vanishing Fourier coefficients and is therefore identically zero which gives $u = 0$ and proves that $V = H^2(T) \cdot w$ must hold.

4.4 Examples. Let $B(z)$ be a non-constant Blaschke product. Now $|B(e^{i\theta})| = 1$ holds almost everywhere and the presence of zeros of $B(z)$ in D show that $H^2(T) \cdot B$ is a proper and invariant subspace of $H^2(T)$. Next, let σ be a singular Riesz measure on T which is real and non-negative. We get the analytic function

$$f(z) = e^{-g_\sigma(z)}$$

Here

$$|f(z)| = e^{-H_\sigma(z)}$$

and since $\sigma \geq 0$ we have $H_\sigma(z) \geq 0$ and hence $|f(z)| \leq 1$. So f is a bounded analytic function in D and in particular it belongs to $H^2(T)$. Moreover we know from XX that the boundary function $f(e^{i\theta})$ has absolute value one almost everywhere. So $H^2(T) \cdot f$ is an invariant subspace of $H^2(T)$ and the question arises if it is proper or not. In contrast to the case for Blaschke functions B above this is not obvious since f has no zeros in D. However it turns out that one has

4.5 Theorem. *Let σ be a singular and non-negative Riesz measure which is not identically zero. Then $H^2(T) \cdot e^{-g_\mu}$ is a proper subspace of $H^2(T)$.*

Proof. Set $w(\theta) = e^{-g_\mu(e^{i\theta})}$. For the analytic function $w(z)$ in the disc its value at $z = 0$ becomes

$$w(0) = e^{-g_\mu(0} = e^{-\sigma(T)/2\pi}$$

Next, if $P(z)$ is a polynomial we have

$$\frac{1}{2\pi}\int_0^{2\pi} |P(\theta)w(\theta)-1|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} |P(\theta)|^2\cdot d\theta + 1 + 2\mathfrak{Re}\,[\int \frac{1}{2\pi}\int_0^{2\pi} P(\theta)\cdot w(\theta)\cdot d\theta]$$

By Cauchy's formula the last term becomes

$$2\mathfrak{Re}(P(0)w(0)) = 2w(0)\cdot\mathfrak{Re}(P(0))$$

By (i) we have $0 < w(0) < 1$ and if $||P||_2$ is the L^2-norm of P the right hand side majorizes

$$||P||_2^2 + 1 - 2w(0)\cdot|P(0)|$$

We have also the inequality

$$|P(0)| \leq ||P||_2$$

So if we set $\rho = ||P||_2$ then we have shown that

$$\frac{1}{2\pi}\int_0^{2\pi} |P(\theta)w(\theta)-1|^2 d\theta \geq \rho^2 + 1 - 2w(0)\cdot\rho$$

Now we notice that the right hand side is $\geq 1 - w(0)^2$ for every ρ. Since P is an arbitrary polynomial we conclude that the L^2-distance of 1 to the subspace $H^2(T)\cdot e^{-g_\mu}$ is at least

(*) $$1 - w(0)^2 = 1 - e^{-2\sigma(T)}$$

5. Beurling's closure theorem.

A zero-free function $f \in H^2(T)$ is of outer type when

$$f(z) = G_\mu(z)$$

where μ is the absolutely continuous Riesz measure $\log|f(e^{i\theta}|$. The following result is due to Beurling in [Beur]:

5.1 Theorem. *For every nonzero $f \in H^2(T)$ of outer type the closed invariant subspace generated by analytic polynomials $P(z)$ times f is equal to $H^2(T)$.*

Proof. If the density fails we find $0 \neq g \in H^2(T)$ such that

(i) $$\int_0^{2\pi} e^{in\theta} f(e^{i\theta})\cdot\bar{g}(e^{i\theta})\cdot d\theta = 0$$

for every $n \geq 0$. By Cauchy-Schwarz the product $f\cdot\bar{g}$ belongs to $L^1(T)$ and (i) implies that this function is of the form $e^{i\theta}\cdot h(\theta)$ where $h \in H^1(T)$. So on T we have almost everywhere:

(ii) $$\bar{g}(e^{i\theta}) = e^{i\theta}\cdot\frac{h(e^{i\theta})}{f(e^{i\theta})}$$

Now we take the outer factor $\mathfrak{O}_h$ whose absolute value is equal to $|k|$ almost everywhere on T. It follows that

(iii) $$|g^*(\theta)| = \frac{\mathfrak{O}_h^*(\theta)}{\mathfrak{O}_f^*(\theta)}$$

Since $g \in H^2(T)$ Exercise 3.12 shows that the quotient in (ii) is the boundary value of an analytic function in $H^2(T)$ which implies that the conjugate function $\bar{g}$ also belongs to $H^2(T)$. But then g must be a constant and this constant is zero because the factor $e^{i\theta}$ appears in (ii). So g must be zero which gives a contradiction and the requested density is proved.

5.2 Szegö's theorem.

Let $w(\theta)$ be real-valued and non-negative function in $L^1(T)$ and $\mathcal{P}_0$ is the space of analytic polynomials $P(z)$ where $P(0) = 0$. Put

$$\rho(w) = \frac{1}{2\pi} \inf_{P \in \mathcal{P}_0} \int_0^{2\pi} \big|1 - P(e^{i\theta})\big| \cdot w(\theta) \cdot d\theta$$

5.3 Theorem. *One has the equality*

$$\rho(w) = \exp\,[\frac{1}{2\pi} \int_0^{2\pi} \log w(\theta) \cdot d\theta]$$

Proof. First we consider the case when $\log|w| \in L^1(T)$. Multiplying w with a positive constant we may assume that

$$\text{(i)} \qquad \int_0^{2\pi} \log w(\theta) \cdot d\theta = 0$$

Now we must show that $\rho(w) = 1$. To prove this we use that $\log w \in L^1(T)$ and construct the analytic function

$$f(z) = G_{\log w(z)}$$

So f is an outer function where on T one has

$$\text{(ii)} \qquad |f(e^{i\theta})| = e^{\log|w(\theta)|} = w(\theta)$$

Hence $f \in H^1(T)$ and (1) gives $f(0) = 1$. Let us now consider some $P(z) \in \mathcal{P}_0$ and set

$$F(z) = (1 - P(z))f(z)$$

Again $F(0) = 1$ and $F \in H^1(T)$ which gives the inequality

$$\text{(iii)} \qquad 1 \leq \int_0^{2\pi} |F(e^{i\theta})| \cdot d\theta$$

By (ii) this means that

$$1 \leq \int_0^{2\pi} |1 - P(e^{i\theta})| \cdot w(\theta) \cdot d\theta$$

Since this hold for every $P \in \mathcal{P}_0$ we have proved the inequality

$$\text{(iv)} \qquad \rho(w) \geq 1$$

To prove the reverse inequality we apply Beurling's theorem to the outer function f. This gives a sequence of polynomials $\{Q_n(z)\}$ such that

$$\text{(v)} \qquad \lim_{n\to\infty} ||Q_n \cdot f - 1||_1 = 0$$

where we use the norm on $H^1(T)$. Since $f(0) = 1$ it follows that $Q_n(0) \to 1$ and we can normalize the approximating sequence so that $Q_n(0) = 1$ for every n and write $Q_n = 1 - P_n$ with $P_n \in \mathcal{P}_0$. Finally using (ii) we get

$$\lim_{n\to\infty} \int_0^{2\pi} |1 - P(e^{i\theta})| \cdot w(\theta) \cdot d\theta = 1$$

This gives $\rho(w) \geq 1$ and Szegö's theorem is proved for the case A above.

B. The case when $\log^+ \frac{1}{|w|}$ *is not integrable.* Here we must show that $\rho(w) = 0$ and the proof of this is left as an exercise to the reader.

6. The Helson-Szegö theorem

A trigonometric polynomial on the unit circle is of the form

$$P(\theta) = \sum a_n \cdot e^{in\theta}$$

where $\{a_n\}$ are complex numbers and only a finite family is $\neq 0$. The conjugation operator $\mathcal{C}$ is defined by

$$(*) \qquad \mathcal{C}(P) = i \cdot \sum_{n<0} a_n \cdot e^{in\theta} - i \cdot \sum_{n>0} a_n \cdot e^{in\theta}$$

Let $w(\theta)$ be a non-negative function in $L^1(T)$ and assume also that $\big|\log|w|\big| \in L^1(T)$.

6.1 Definition. *A w-function as above is of Helson-Szegö type if there exists a constant C such that*

$$(*) \qquad \int_0^{2\pi} |\mathcal{C}(P)(e^{i\theta})|^2 \cdot w(\theta) \cdot d\theta \leq C \cdot \int_0^{2\pi} |P(e^{i\theta})|^2 \cdot w(\theta) \cdot d\theta$$

hold for all trigonometric polynomials.

Notice that if (*) holds for some w then it holds for every function of the form $\rho \cdot w$ where $0 < c_0 \leq \rho(\theta) \leq c_1$ for some pair of positive constants. Or equivalently, with w replaced by $e^u \cdot w$ for some bounded function $u(\theta)$. With this in mind we announce the result below which is due to Helson and Szegö in [HS]:

6.2 Theorem. *A function $w(\theta)$ is of the Helson-Szegö type if and only if there exists a bounded function u and a function $v(\theta)$ for which the maximum norm of $|v|$ over T is < 1 and*

$$w(\theta) = e^{u(\theta) + v^*(\theta)}$$

where v^* is the harmonic conjugate of v.

The proof requires several steps. The first part is an exercise on norms on the Hilbert space $L^2(w)$ which is left to the reader.

Exercise. Show that w is of the Helson-Szegö type if and only if there exists a constant $\rho < 1$ such that

$$(*) \qquad \Big| \int_0^{2\pi} P(\theta) \cdot e^{-i\theta} \cdot Q(\theta) \cdot w(\theta) \cdot d\theta \Big| \leq \rho \cdot ||P||_w \cdot ||Q||_w$$

hold for all pairs P, Q in $\mathcal{P}_0$.

6.3 The outer function ϕ. We define the analytic function $\phi(z)$ by

$$\phi(z) = \exp\Big[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot \log(\sqrt{w(\theta)}) \cdot d\theta \Big]$$

Since $\log \sqrt{w(\theta)} = \frac{1}{2} \cdot \log w(\theta)$ is in $L^1(T)$ it means that $\phi(z)$ is an outer function and on the unit circle we have the equality

$$(1) \qquad |\phi(\theta)|^2 = w(\theta)$$

Using (1) we find a real-valued function $\gamma(\theta)$ such that

$$(2) \qquad w(\theta) = \phi^2(\theta) \cdot e^{i\gamma(\theta)}$$

Next, (1) implies that the weighted L^2-norm $||P||_w$ is equal to the standard L^2-norm of $\phi \cdot P$ on T. Hence (1) holds if and only if

$$(3) \qquad \Big| \int_0^{2\pi} \phi(\theta) P(\theta) \cdot e^{-i\theta} \cdot \phi(\theta) Q(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||\phi \cdot P||_2 \cdot ||\phi \cdot Q||_2$$

hold for all pairs P, Q in $\mathcal{P}_0$. Now we use that ϕ is outer which by Beurling's closure theorem means that $\mathcal{P}_0 \cdot \phi$ is dense in $H^2_0(T)$. Hence (3) is equivalent to

$$(4) \qquad \Big| \int_0^{2\pi} F(\theta) \cdot e^{-i\theta} \cdot G(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||F||_2 \cdot ||G||_2$$

for all pairs F, G in $H^2_0(T)$.

Next, in XX we prove that every $f \in H^1_0(T)$ admits a factorization $f = F \cdot G \cdot e^{-i\theta}$ for a pair F, G where $||f||_1 = ||F||_2 \cdot ||G||_2$. So (4) is equivalent to

$$(5) \qquad \Big| \int_0^{2\pi} f(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||f||_1$$

for each $f \in H^1_0(T)$. At this stage we use the duality between $H^\infty(T)$ and $H^1_0(T)$ from Section XX. It follows that (5) is equivalent to the following

6.4 Approximation condition. One has

$$\min_h ||e^{i\gamma(\theta)} - h(\theta)||_\infty = \rho$$

where the minimum is taken over h-functions in $H^\infty(T)$.

Final part of the proof.

Since $w \geq 0$ and > 0 outside a set of measure zero, the approximation condition is equivalent with the existence of some $h \in H^\infty(T)$ and some $\rho < 1$ such that

$$(*) \qquad |w(\theta) - \phi^2(\theta) \cdot h(\theta)| \leq \rho \cdot w(\theta)$$

hold on T. It remains to show that (*) is equivalent to the existence of a pair u, v in Theorem 6.2. Let us begin with

Proof that (*) gives the pair u, v. Since $\log w$ is in $L^1(T)$ we have $w > 0$ almost everywhere and (*) entails that $\phi^2(\theta) \cdot h(\theta)$ stay in the sector

$$Z = \{z\colon \ -\pi/2 + \delta \leq \arg(z) \leq \pi/2 - \delta\}$$

where we have put $\delta = \arccos(\rho)$. This inclusion of the range of $\phi^2 \cdot h$ implies that it is outer. See XX above. Hence we can find a harmonic function V such that

$$\phi^2 \cdot h = e^{ia} \cdot e^{V + iV^*}$$

where a is some real constant. The inclusion of the range implies that

$$|a + V^*(\theta)| \leq \pi/2 - \delta$$

Next, define the harmonic function

$$v(\theta) = -(a + V^*(\theta))$$

It follows that

$$\phi^2(\theta) \cdot h(\theta) = e^{v(\theta) + iv^*(\theta) + c}$$

for some constant c. Finally, since $w = |\phi|^2$ we obtain

$$w(\theta) = e^{v(\theta)} \cdot \frac{e^a}{|h(\theta)|}$$

By (xx) above the last factor is bounded both below and above and hence e^u for some bounded function. Together with the bound (xx) for the harmonic conjugate of v we get the requested form for $w(\theta)$ in Theorem 6.2.

Proof that a pair (u, v) gives (*). Consider the special case when $w = e^v$ and

$$|v^*(\theta)| \leq \pi/2 - \epsilon$$

holds for some $\epsilon > 0$. It is clear that the corresponding ϕfunction obtained via (xx) above satisfies

$$\phi^2(\theta) = e^{v(\theta) + iv^*(\theta)}$$

This gives

$$e^{i\gamma(\theta)} = e^{-iv^*(\theta)}$$

and we notice that if we take the constant function $h(\theta) = \epsilon$ then the maximum norm

$$||e^{i\gamma(\theta)} - \epsilon||_\infty < 1$$

which proves that (*) holds.

Appendix II: Beurling-Wiener algebras

Contents

Introduction. The results in this section are foremost due to Wiener and Beurling. The cornerstone is Wiener's general Tauberian Theorem which we apply a class called Beurling-Wiener algebras where the convolution algebra $L^1)\mathbf{R})$ is replaced by various weight algebras introduced by Beurling in the article (Beurling: 1938]. Here follows the set-up in this section.

Consider the Banach space $L^1(\mathbf{R})$ of Lebesgue measurable and absolutely integrable functions whose product is defined by convolutions:

$$f * g(x) = \int f(x-y)g(y)dy$$

A.1 The space $\mathcal{F}_0^\infty$. On the ξ-line we have the space C_0^∞ of infintely differentiable functions with compact support. Each $g(\xi) \in C_0^\infty$ yields an L^1-function on the real x-line defined by

$$(*) \qquad \mathcal{F}(g)(x) = \frac{1}{2\pi} \int e^{ix\xi}\, g(\xi) \cdot d\xi$$

The resulting subspace of L^1 is denoted by $\mathcal{F}_0^\infty$.

A.2 Beurling-Wiener algebras. A subalgebra B of L^1 is called a Beurling-Wiener algebra - for short a $\mathcal{BW}$-algebra - if the following two conditions hold:

Condition 1. B is equipped with a complete norm denoted by $||\cdot||_B$ such that

$$||f * g||_B \leq ||f||_B \cdot ||g||_B \quad : \quad f, g \in B \quad \text{and} \quad ||f||_1 \leq ||f||_B$$

Condition 2. $\mathcal{F}_0^\infty$ is a dense subalgebra of B.

A.3 Theorem *Let B be a $\mathcal{BW}$-algebra. For each multiplicative and continuous functional λ on B which is not identically zero there exists a unique $\xi \in \mathbf{R}$ such that*

$$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

Proof. Suppose that there exists some ξ such that

$$\text{(i)} \qquad \lambda(f) = 0 \implies \widehat{f}(\xi) = 0$$

This means that the linear form $f \mapsto \widehat{f}(\xi)$ has the same kernel as λ and hence there exists some constant c such that

$$\text{(ii)} \qquad \lambda(f) = c \cdot \hat{f}(\xi) \quad \text{for all } f \in B\,.$$

Since λ is multiplicative it follows that $c = c^n$ for every positive integer n and then $c = 1$. Next, since B contains $\mathcal{F}_0^\infty$ and test-functions on the ξ-line separate points, it is clear that ξ is uniquely determined. There remains to prove the existence of some ξ for which (i) holds.

To prove this we use the density of $\mathcal{F}_0^\infty$ in B which by the continuity of λ gives some $g \in \mathcal{F}_0^\infty$ such that $\lambda(g) \neq 0$. Let K be the compact support of the test-function $\widehat{g}(\xi)$ and suppose that (i) fails for each point $\xi \in K$. The density of $\mathcal{F}_0^\infty$ gives some $f_\xi \in \mathcal{F}_0^\infty$ such that

$$\text{(iii)} \qquad \widehat{f}(\xi) \neq 0 \quad \text{and } \lambda(f) = 0$$

Heine-Borel's Lemma yields a finite set of points $\xi_1, \ldots, \xi_N$ in K such that family $\{\widehat{f}_{\xi_k}\}$ have no common zero on K. To simplify notations we set $f_k = f_{\xi_k}$. The complex conjugates of $\{\widehat{f}_k\}$ are again test-functions. So for each k we find $h_k \in B$ such that $\widehat{h}_k$ is the s complex conjugate of $\widehat{f}_k$. Set

$$\phi(\xi) = \sum_{k=1}^{k=N} \widehat{h}_k(\xi) \cdot \widehat{f}_k(\xi)$$

This test-function is > 0 on the support of $\widehat{g}$ and hence there exists the test-function

$$\text{(iv)} \qquad Q(\xi) = \frac{\widehat{g}}{\phi}$$

By Condition 2, Q is the Fourier transform of some B-element q. Since $L^1(\mathbf{R})$-functions are uniquely determined by their Fourier transforms, it follows from (iv) that

$$\text{(v)} \qquad \sum_{k=1}^{k=N} q * h_k * f_k = g$$

Now we get a contradiction since $\lambda(f_k) = 0$ for each k while $\lambda(g) \neq 0$.

A.4 The algebra B_a.

Let $a > 0$ be a positive real number. Given a Beurling-Wiener algebra B we set

$$J_a = \{f \in B \quad : \; \widehat{f}(\xi) = 0 \text{ for all } -a \leq \xi \leq a\}$$

Condition 1 and the continuity of the Fourier transform on L^1-functions imply that J_a is a closed ideal in B. Hence we get the Banach algebra $\frac{B}{J_a}$ which we denote by B_a. Let $g \in \mathcal{F}_0^\infty$ be such that $\widehat{g}(\xi) = 1$ on $[-a, a]$. For every $f \in B$ it follows that $g * f - f$ belongs to J_a which means that the image of f in B_a is equal to the image of $g * f$. We conclude that the g-image yields an identity in the algebra B_a and hence B_a is a Banach algebra with a unit element.

A.5 Theorem. *The Gelfand space of B_a is equal to the compact interval $[-a, a]$.*

A.6 Exercise. Prove this using Theorem A.3

A.7. Examples of $\mathcal{B}W$-algebras

Let B be the space of all continuous functions $f(x)$ on the real x-line such that the positive series below is convergent:

$$(*) \qquad \sum_{-\infty}^{\infty} ||f||_{[\nu,\nu+1]}$$

where $||f||_{[\nu,\nu+1]}$ is the maximum norm of f on the closed interval $[\nu.\nu+1]$ and the sum extends over all integers. The norm on B-elements is defined by the sum of the series above. It is obvious that this norm dominates the L^1-norm. Moreover, one easily verifies that

$$\text{(i)} \qquad ||f * g||_B \leq ||f||| \cdot ||g||_B$$

for pairs in B. Hence B satisfies Condition 1 from B.

Exercise. Show that the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on the real x-line is a dense subalgebra of B.

Next, since $\mathcal{F}_0^\infty \subset \mathcal{S}$ we have the inclusion

$$\text{(ii)} \qquad \mathcal{F}_0^\infty \subset B$$

There remains to see why $\mathcal{F}_0^\infty$ is dense in B. To prove this we construct some special functions on the x-line whose Fourier transforms have compact support. If $b > 0$ we set

$$f_b(x) = \frac{1}{2\pi} \int_{-b}^{b} e^{ix\xi} \cdot (1 - \frac{|\xi|}{b}) \cdot d\xi$$

By Fourier's inversion formula this means that

$$\widehat{f_b}(\xi) = 1 - \frac{|\xi|}{b} \qquad -b \leq \xi \leq b \quad \text{and zero if } |\xi| > b$$

A computation which is left to the reader gives

$$f_b(x) = \frac{1}{\pi} \cdot \frac{1 - \cos bx}{bx^2}$$

From this expression it is clear that $f_b(x)$ belongs to B and we leave it to the reader to verify that

$$\text{(iii)} \qquad \lim_{b \to +\infty} ||f_b * g - g||_B = 0 \quad \text{for all } g \in B$$

Next, the functions $\widehat{f_b}(\xi)$ have compact support but they are not smooth, i.e. they do not belong to $\mathcal{F}_0^\infty$. However, we can perform a smoothing of these functions as follows: Let $\phi(\xi)$ be an even and non-negative C_0^∞-function with support in $-1 \leq \xi \leq 1$ such that the integral

$$\int \phi(\xi) \cdot d\xi = 1$$

With $\delta > 0$ we set $\phi_\delta(\xi) = \frac{1}{\delta} \cdot \phi(\xi/\delta)$ and for each pair δ, b we get the test-function on the ξ-line defined by

$$\psi_{\delta,b}(\xi) = \int_{-b}^{b} \phi_\delta(\xi - \eta) \cdot (1 - \frac{|\eta|}{b}) \cdot d\eta$$

The inverse Fourier transforms

$$f_{\delta,b}(x) = \frac{1}{2\pi} \int e^{ix\xi} \cdot \psi_{\delta,b}(\xi) \cdot d\xi$$

yield functions in $\mathcal{F}_0^\infty$ for all pairs δ, b. Next, if $g \in B$ then the Fourier transform of the B-element $f_{\delta,b} * g$ is equal to the *convolution* of $\phi_\delta(\xi)$ and the Fourier transform of $f_b * g$. This implies that

$$f_{\delta,b} * g \in \mathcal{F}_0^\infty \,.$$

At this stage we leave it to the reader to verify that

$$\lim_{(\delta,b) \to (0,0)} f_{\delta,b} * g = g$$

holds for every $g \in B$. Hence the required density of $\mathcal{F}_0^\infty$ is proved and B is a Beurling-Winer algebra.

A.8 Adding discrete measures

Let $M_d(\mathbf{R})$ be the Banach algebra of discrete measures of finite total variation, i.e. measures of the form

$$\mu = \sum c_\nu \cdot \delta_{x_\nu} \quad : \ ||\mu|| = \sum |c_\nu| < \infty$$

As explained in XX the Gelfand space is the compact Bohr group $\mathfrak{B}$, where the real ξ-line via the Fourier transform appears as a dense subset. Now we adjoin some $\mathcal{BW}$-algebra B and obtain a Banach algebra B_d which consists of measures of the form

$$f + \mu \quad : \quad f \in B \text{ and } \mu \in M_d(\mathbf{R})$$

where the norm of $f + \mu$ is the sum of the B-norm of f and the total variation of μ. Since B is a subspace of L^1 one easuly checks that this yields a complete norm. next, by condition (2) in A.2 it follows that if $f \in b$ and $\mu \in M_d(\mathbf{R})$ then the convolution $f * \mu$ belongs to B. This means that B appears as a closed ideal in B_d.

A.9 The Gelfand space $\mathcal{M}_{B_d}$. Let λ is a multiplicative functional on B_d which is not identically zero on B. Theorem A.3 gives a unique ξ such that

(i) $$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

If a is a real number then $\delta_a * f$ has the Fourier transform becomes $e^{ia\xi} \cdot \widehat{f}(\xi)$. It follows that

(ii) $$\lambda(\delta_a) \cdot \widehat{f}(\xi) = \lambda(\delta_a * f) = e^{-ia\xi} \cdot \widehat{f}(\xi)$$

We conclude that $\lambda(\delta_a) = e^{-ia\xi}$ and hence the restriction of λ is the evaluation of the Fourier transform at ξ on the whole algebra B_d. In this way the real ξ-line is embedded in $\mathcal{M}_B$ where a point $\lambda \in \mathcal{M}_B$ belongs to this subset if and only if

$\lambda(f) \neq 0$ for some $f \in B$. The construction of the Gelfand topology shows that this copy of the real ξ-line appears as an *open* subset of $\mathcal{M}_{B_d}$ denoted by $\mathbf{R}_\xi$.

A.10 The set $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$. If λ belongs to this closed subset it is identically zero on the ideal B and its restriction to $M_d(\mathbf{R})$ corresponds to a point γ in the Bohr group $\mathfrak{B}$. Conversely, every point in $\mathfrak{B}$ yields a $\lambda \in \mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ since the quotient algebra

$$\frac{B_d}{B} \simeq M_d(\mathbf{R})$$

Hence we have the set-theoretic equality

$$(*) \qquad \mathcal{M}_{B_d} = \mathbf{R}_\xi \cup \mathfrak{B}$$

A.11 Proposition. *The open subset $\mathbf{R}_\xi$ is dense in $\mathcal{M}_B$.*

Proof. Let λ be a point in $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ which therefore corresponds to a point $\gamma \in \mathfrak{B}$. By the result in XX we know that for every finite set $\mu_1, \ldots, \mu_N$ of discrete measures, there exists a sequence $\{\xi_\nu\}$ such that

$$\lim_{\nu \to \infty} \widehat{\mu}_i(\xi_\nu) = \gamma(\mu_i) \quad \text{and } |\xi_\nu| \to \infty$$

At the same time the Riemann-Lebesgue Lemma entails that

$$\lim_{\nu \to \infty} \widehat{f}(\xi_\nu) = 0$$

for every $f \in B$. Hence the construction of the Gelfand topology on $\mathcal{M}_{B_d}$ gives the requested density in Proposition A.11

A.12 An inversion formula. Let $f \in B$ and μ is some discrete measure. Suppose that there exists $\delta > 0$ such that the Fourier transform of $f + \mu$ has absolute value $\geq \delta$ for all ξ. Proposition A.11 implies that its Gelfand transform has no zeros and hence this B_d-element is invertible, i.e. there exist $g \in B$ and a discrete measure γ such that

$$\text{(i)} \qquad \delta_0 = (f + \mu) * (g + \gamma)$$

Notice that the right hand side becomes

$$f * g + f * \gamma + g * \mu + \mu * \gamma$$

Here $f*g+f*\gamma+g*\mu$ belongs to B while $\mu*\gamma$ is a discrete measure. So (i) implies that γ must be the inverse of μ in $M_d(\mathbf{R})$ and hence (i) also gives the equality:

$$\text{(ii)} \qquad f * g + f * \mu^{-1} + g * \mu = 0$$

B. A Tauberian Theorem.

Consider the Banach algebra B above. The dual space B^* consists of Riesz measures μ on the real line for which there exists a constant A such that

$$\int_\nu^{\nu+1} |d\mu(x)| \leq A \quad \text{for all integers } \nu \,.$$

The smallest A above is the norm of μ in B^* and duality is expressed by:

$$\mu(f) = \int f(x) \cdot d\mu(x) \quad : \quad f \in B \text{ and } \mu \in B^*$$

Let $f \in B$ be such that $\widehat{f}(\xi) \neq 0$ for all ξ. For each $a > 0$ it follows from Theorem A.5 that the f-image in B_a generates the whole algebra. Since this hold for every $a > 0$ it follows that each $\phi \in \mathcal{F}_0^\infty$ belongs to the principal ideal generated by f in B, i.e. there exists some $g \in B$ such that

(*) $$\phi = g * f$$

Since $\mathcal{F}_0^\infty$ is dense in B we conclude that $B \cdot f$ is dense in B. Using this density we have:

B.1 Theorem *Let $\mu \in B^*$ be such that*

$$\lim_{y \to +\infty} \int f(y-x) \cdot d\mu(x) = A \text{ exists.}$$

Then, for each $g \in B$ it follows that

$$\lim_{y \to +\infty} \int g(y-x) \cdot d\mu(x) = B \quad \text{where} \quad B = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

Proof. Let $g \in B$. If $\epsilon > 0$ we find $h_\epsilon \in B$ such that $||g - f * h_\epsilon||_B < \epsilon$. When $y > 0$ we get:

(i) $$\int (f * h_\epsilon)(y-x) \cdot d\mu(x) =$$

$$\int \big[f(y-s-x) h_\epsilon(s) \cdot ds \big] \cdot d\mu(x) = \int h_\epsilon(s) \cdot \big[\int f(y-s-x) \mu(x) \big] \cdot ds$$

By the hypothesis the inner integral converges to A when $y \to +\infty$ every fixed s. Since h belongs to B it follows easily that the limit of (i) when $y \to +\infty$ is equal to

(ii) $$A \cdot \int h_\epsilon(s) \cdot ds = A \cdot \widehat{h}_\epsilon(0)$$

Next, since the B-norm is stronger than the L^1-norm it follows that

(iii) $$|\widehat{g}(0) - \hat{h}_\epsilon(0) \cdot \widehat{f}(0)| < \epsilon$$

Moreover, since the B-norm is invariant under translations we have

(iv) $$\Big| \int g(y-x) d\mu(x) - \int (f * h_\epsilon)(y-x) \cdot d\mu(x) \Big| \leq \epsilon \cdot ||\mu|| \quad \text{for all } y$$

where $||\mu||$ is the norm of μ in the dual space B^*. Notice also that (iii) gives:

$$\lim_{\epsilon \to} \hat{h}_\epsilon(0) = \frac{\hat{g}(0)}{\hat{f}(0)}$$

Finally, since $\epsilon > 0$ is arbitrary we see that the limit formula for (i) when $y \to +\infty$ expressed by (ii) and (iv) above together imply that

$$\lim_{y \to +\infty} \int g(y-x)d\mu(x) = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

This finishes the proof of Theorem A.9

B.2 The multiplicative version

Let $\mathbf{R}^+$ be the multiplicative group of positive real numbers. To each function $f(t)$ on $\mathbf{R}^+$ we get the function $E_f(x) = f(e^x)$ on the real x-line. Since $dt = e^x dx$ under the exponential map we have

$$\int_0^\infty f(t)\frac{dt}{t} = \int_{-\infty}^\infty E_f(x)dx$$

provided that f is integrable. On $\mathbf{R}^+$ we get the convolution algebra $L^1(\mathbf{R}^+)$ where

$$f * g(t) = \int_0^\infty f(\frac{t}{s}) \cdot g(s) \cdot \frac{ds}{s}$$

This convolution commutes with the E map from $L^1(\mathbf{R}^+$ into $L^1(\mathbf{R}^1)$, i.e.

$$E_{f*g} = E_f * E_g$$

Next, recall that the Fourier transform on $L^1(\mathbf{R}^+)$ is defined by

$$\widehat{f}(\xi) = \int_0^\infty t^{-i\xi} \cdot f(t) \cdot \frac{dt}{t}$$

B.3 The Banach algebra B_m. The companion to B on $\mathbf{R}^+$ consists of continuous functions $f(t)$ for which

$$\sum ||f||_{[2^\nu, 2^{\nu+1}]} < \infty$$

where the is taken over all integers. Notice that with $\nu < 0$ one takes small intervals approaching $t = 0$. Just as in Theorem A.9 we obtain a Tauberian Theorem for functions $f \in B_m$ whose Fourier transform is everywhere $\neq 0$. Here we the dual space B_m^* consists of Riesz measures μ on $\mathbf{R}^+$ for which there exists a constant C such that

$$\int_{2^m}^{2^{m+1}} |d\mu(t)| \leq C$$

for all integers m.

C. Ikehara's theorem.

Let ν be a non-negative Riess measure supported on $[1, +\infty)$ and assume that

$$\int_1^\infty x^{-1-\delta} \cdot d\nu(x) < \infty \quad \text{for all } \delta > 0$$

This gives an analytic function $f(s)$ of the complex variable s defined in the right half plane $\mathfrak{Re}(s) > 1$ by

$$f(s) = \int_1^\infty x^{-s} \cdot d\nu(x)$$

D.1 Theorem. *Assume that there exists a constant A and a locally integrable function $G(u)$ defined on the real u-line such that*

$$(*) \quad \lim_{\epsilon \to 0} \int_{-b}^{b} \left| f(1+\epsilon+iu) - \frac{A}{1+\epsilon+iu} - G(u) \right| \cdot du = 0 \quad \text{holds for each } b > 0$$

Then it follows that

$$(**) \qquad \lim_{x \to +\infty} \frac{1}{x} \int_1^x d\nu(t) = A$$

Proof. Define the measure ν^* on the non-negative real ξ-line by

$$(1) \qquad d\nu^*(\xi) = e^{-\xi} \cdot d\nu(e^\xi) - A \cdot d\xi \quad : \ \xi \geq 0$$

If $\eta > 1$ we notice that

$$\int_0^\eta e^{-\eta+\xi} \cdot d\nu^*(\xi) = e^{-\eta} \int_0^\eta d\nu(e^\xi) - A(1 - e^{-\eta}) = e^{-\eta} \int_0^{e^\eta} d\nu(t) - A(1 - e^{-\eta})$$

hence (**) holds if and only if

$$(2) \qquad \lim_{\eta \to \infty} \int_0^\eta e^{-\eta+\xi} \cdot d\nu^*(\xi) = 0$$

It is also clear that condition (xx) for ν entails that

$$(3) \qquad \int_0^\infty e^{-\delta \cdot \xi} \cdot d\nu^*(\xi) < \infty \quad \text{for all } \delta > 0$$

Moreover, a variable substitution gives

$$(4) \qquad f(s) - \frac{A}{s-1} = \int_0^\infty e^{(1-s)\xi} d\nu^*(\xi)$$

C.1 A reformulation of Ikehara's theorem.

From (1-4) we can restate Ikehara's theorem. Let ν^* be a non-negative measure on $0 \leq \xi < +\infty$ such that

$$(1.1) \qquad \int_0^\infty e^{-\delta \cdot \xi} \cdot d\nu^*(\xi) < \infty \quad \text{for all } \delta > 0$$

Let $A > 0$ be some positive constant and define the measure μ by

$$(1.2) \qquad d\mu(\xi) = d\nu^*(\xi) - A \cdot d\xi$$

Then (1.1) gives the analytic function $g(s)$ defined in $\mathfrak{Re}(s) > 0$ by

$$g(s) = \int_0^\infty e^{-s\cdot\xi} \cdot d\mu(\xi) \tag{1.3}$$

C.1.1. Definition. *We say that the measure μ is of the Ikehara type if there exists a locally integrable function $G(u)$ defined on the real u-line such that*

$$\lim_{\epsilon\to 0} \int_{-b}^{b} \bigl|g(\epsilon + iu) - G(u)\bigr| \cdot du = 0 \quad \text{holds for each } b > 0$$

C.1.2. The space $\mathcal{W}$. Let $\mathcal{W}$ be the space of continuous functions $\rho(\xi)$ defined on $\xi \geq 0$ which satisfy:

$$\sum_{n\geq 0} ||\rho||_n < \infty \quad \text{where } ||\rho||_n = \max_{n\leq u\leq n+1} |\rho(u)|$$

The dual space $\mathcal{W}^*$ consists of Riesz measures γ on $[0, +\infty)$ such that

$$\max_{n\geq 0} \int_n^{n+1} |d\gamma(\xi)| < \infty$$

With these notations we shall prove:

C.1.3. Theorem. *Let ν^* be a non-negative meausure on $[0, +\infty)$ and $A \geq 0$ some constant such that the measure $\mu = \nu^* - A \cdot d\xi$ is of Ikehara type. Then $\mu \in \mathcal{W}^*$ and for every function $\rho \in \mathcal{W}$ one has*

$$\lim_{\eta\to+\infty} \int_0^\eta \rho(\eta - \xi) \cdot d\mu(\xi) = 0$$

C.1.4 Exercise. Use the material above to show that Theorem C.1.3 gives Theorem C.0 where a hint is to use the function $\rho(s) = e^{-s}$ above.

C.2 Proof of Theorem C.1.3.

Let $b > 0$ and define the function $\omega(u)$ by

(i) $\quad \omega(u) = 1 - \frac{|u|}{b}, \quad -b \leq u \leq b \quad$ and $\omega(u) = 0$ outside this interval

Set

$$J_b(\epsilon, \eta) = \int_{-b}^{b} e^{i\eta u} \cdot g(\epsilon + iu) \cdot \omega(u) \cdot du \tag{ii}$$

From Definition C.1.1 we have the L^1_{loc}-function $G(u)$ and since $\omega(u)$ is a continuous function on the compact interval $[-b, b]$ we have

$$\lim_{\epsilon\to 0} J_b(\epsilon, \eta) = J_b(0, \eta) = \int_{-b}^{b} e^{i\eta u} \cdot G(u) \cdot \omega(u) \cdot du \tag{iii}$$

With b kept fixed the right hand side is a Fourier transform of an L^1-function. So the Riemann-Lebesgue theorem gives:

$$\text{(iv)} \qquad \lim_{\eta \to +\infty} J_b(0,\eta) = 0$$

Moreover, the triangle inequality gives the inequality:

$$\text{(v)} \qquad |J_b(0,\eta)| \leq \int_{-b}^{b} \bigl|G(u)\bigr| \cdot du$$

Some integral formulas. From the above it is clear that

$$\text{(1)} \qquad J_b(\epsilon,\eta) = \int_0^\infty \bigl[\int_{-b}^{b} e^{i\eta u - i\xi u} \cdot \omega(u) \cdot du\bigr] \cdot e^{-\epsilon\cdot\xi} \cdot d\mu(\xi)$$

Next, notice that

$$\text{(2)} \qquad \int_{-b}^{b} e^{i\eta u - i\xi u} \cdot \omega(u) \cdot du = 2 \cdot \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}$$

Hence we obtain

$$\text{(3)} \qquad J_b(\epsilon,\eta) = 2 \cdot \int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \cdot e^{-\epsilon\xi} \cdot d\mu(\xi)$$

From (iii) above it follows that (3) has a limit as $\epsilon \to 0$ which is equal to the integral in the right hand side in (iii) which is denoted by $J_b(0,\eta)$. Next, it is easily seen that there exists the limit

$$\text{(4)} \qquad \lim_{\epsilon \to 0} 2 \cdot \int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \cdot e^{-\epsilon\xi} \cdot A d\xi = 2\pi \cdot A$$

Hence (3-4) imply that there exists the limit

$$\text{(5)} \qquad \lim_{\epsilon \to 0} 2 \cdot \int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \cdot e^{-\epsilon\xi} \cdot d\nu^*(\xi) = J_b(0,\eta) + 2\pi \cdot A$$

Next, the measure $\nu^* \geq 0$ and the function $\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \geq 0$ for all ξ. So the existence of a finite limit in (5) entails that there exists the convergent integral

$$\text{(6)} \qquad \int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \cdot d\nu^*(\xi) = J_b(0,\eta) + 2\pi \cdot A$$

Proof that $\mu \in \mathcal{W}^*$. Since $A \cdot d\xi$ obviously belongs to $\mathcal{W}^*$ it suffices to show that $\nu^* \in \mathcal{W}^*$. To prove this we consider some integer $n \geq 0$ and with $b = 1$ it is clear that (6) gives

$$\Bigl|\int_n^{n+1} \frac{1-\cos(\eta-\xi)}{(\eta-\xi)^2} \cdot d\nu^*(\xi)\Bigr| \leq |J_1(0,\eta)| + 2\pi = \int_{-1}^{1} |G(u)| \cdot du + 2\pi \cdot A$$

Apply this with $\eta = n + 1 + \pi/2$ and notice that

$$\frac{1-\cos(n+1+\pi/2-\xi)}{(n+1+\pi/2-\xi)^2} \geq a \quad \text{for all } n \leq \xi \leq n+1$$

This gives a constant K such that

$$\int_n^{n+1} d\nu^*(\xi) \leq K \quad n = 0, 1, \ldots$$

Final part of the proof. We have proved that $\mu \in \mathcal{W}^*$. Moreover, from (iv) above and the integral formula (6) we get

$$(*) \qquad \lim_{\eta \to +\infty} \int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \cdot d\mu(\xi) = 0 \quad \text{for all } b > 0$$

Next, for each fixed $b > 0$ we notice that the function

$$\rho_b(\xi) = 2 \cdot \frac{1-\cos(b\xi)}{b \cdot \xi^2}$$

belongs to $\mathcal{W}$ and its Fourier is $\omega_b(u)$. Here $\omega_b(u) \neq 0$ when $-b < u < b$. So the family of these ω-functions have no common zero on the real u-line. By the Remark in XX this means that the linear subspace of $\mathcal{W}$ generated by the translates of all ρ_b -functions with arbitrary large b is dense in $\mathcal{W}$. Hence (*) above implies that we get a zero limit as $\eta \to +\infty$ for every function $\rho \in \mathcal{W}$. But this is precisely the assertion in Theorem C.1.3.

E. The algebra $L^1(\mathbf{R}^+)$

Consider the family of L^1-functions on the real x-line which are supported by the half-line $x \geq 0$. This yields a closed subalgebra of $L^1(\mathbf{R})$ denoted by $L^1(\mathbf{R}^+)$. Indeed, if $f(x)$ and $g(x)$ are two such functions in $L^1(\mathbf{R}^+)$. the support of the convolution $g * f$ stays in $[0, +\infty)$. Adding the unit point mass δ_0 we obtain a commutative Banach algebra

$$B = \mathbf{C} \cdot \delta_0 + L^1(\mathbf{R}^+)$$

1. The Gelfand space $\mathfrak{M}_B$. To obtain this space we consider some $f(x) \in L^1(\mathbf{R}^+)$ and set:

$$\widehat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot f(x) \cdot dx, \quad \text{where} \quad \mathfrak{Im}(\zeta) \geq 0$$

With $\zeta = \xi + i\eta$ we get

$$|\widehat{f}(\xi + i\eta)| \leq \int_0^\infty |e^{i\xi x - \eta x}| \cdot |f(x)| \cdot dx = \int_0^\infty |e^{-\eta x} \cdot |f(x)| \cdot dx \leq ||f||_1$$

We conclude that for every point $\zeta = \xi + i\eta$ in the closed upper half-plane corresponds to a point in $\zeta^* \in \mathfrak{M}_B$ defined by

$$\zeta^*(f) = \widehat{f}(\zeta) \quad \text{and} \quad \zeta^*(\delta_0) = 1$$

In addition to this $L^1(\mathbf{R}^+)$ is a maximal ideal in B and there is the special point $\zeta^\infty \in \mathfrak{M}_B$ such that

$$\zeta^\infty(f) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

2. Theorem. *The Gelfand space $\mathfrak{M}_B$ can be identified with the union of ζ^∞ and the closed upper half-plane.*

Remark. The theorem asserts that every multiplicative functional on B is either ζ^∞ or determined by a point $\zeta = \xi + i\eta$ where $\eta \geq 0$. Concerning the topological identification ζ^∞ corresponds to the one point compactification of the closed upper half-plane. Thus, whenever $\{\zeta_\nu\}$ is a sequence in $\mathfrak{Im}(\zeta) \geq 0$ such that $|\zeta_\nu| \to \infty$ then $\{z_\nu^*\}$ converges to ζ^* in $\mathfrak{M}_B$. In fact, this follows via the Riemann-Lebegue Lemma which gives

$$\lim_{|\zeta|\to\infty} \widehat{f}(\zeta) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

By the general result in XX Theorem 2 holds if we have proved if the following:

3. Proposition. *Let $\{g_\nu = \alpha_\nu \cdot \delta_0 + f_\nu\}_1^k$ be a finite family in B such that the k-tuple $\{\hat{g}_\nu\}$ has no common zero in $\bar{U}_+ \cup \{\infty\}$. Then the ideal in B generated by this k-tuple is equal to B.*

The proof requires some preliminary constructions. We use the conformal map from the upper half-plane onto the unit disc defined by

$$w = \frac{\zeta - i}{\zeta + i}$$

So here w is the complex coordinate in D. Next, consider the disc algebra $A(D)$. Via the conformal map each transform $\widehat{f}(\zeta)$ of a function $f \in L^1(\mathbf{R}^+)$ yields an element of $A(D)$ defined by:

$$F(w) = \widehat{f}(\frac{i + iw}{1 - w})$$

It is clear that $F(w) \in A(D)$. Moreover, we notice that

$$w \to 1 \implies |\frac{i + iw)}{1 - w}| \to \infty$$

It follows that the $A(D)$-function $F(w)$ is zero at $w = 1$ and we can conlcude:

4. Lemma. *By $f \mapsto F$ we have an algebra homomorphism from $L^1(\mathbf{R}^+)$ to functions in $A(D)$ which are zero at $w = 1$.*

Next, let $\mathcal{H}$ denote the algebra homomorphism in Lemma 4 and consider the function $1 - w$ in $A(D)$. We claim this it belongs to the image under $\mathcal{H}$. To see this we consider the function

$$f(x) = e^{-x} \quad x \geq 0 \quad \text{and } f(x) = 0 \quad \text{when } x < 0$$

Then we see that

$$\hat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot e^{-x} \cdot dx = \frac{1}{1 - i\zeta}$$

It follows that

$$F(w) = \frac{1}{1 - i(\frac{iw+i}{1-w})} = \frac{1-w}{1-w+w+1} = \frac{1-w}{2}$$

Using $2f$ we conclude that $1-w$ belongs to the $\mathcal{H}$-image. Next, the identity element δ_0 is mapped to the constant function on D. So via $\mathcal{H}$ we have an algebra homomorphism from B into a subalgebra of $A(D)$ which contains $1-w$ and the identity function and hence all w-polynomials. Returning to the special B-element f we notice that the convolution

$$f * f(x) = x \cdot e^{-x}$$

We can continue and conclude that the subalgebra of B generated by f and δ_0 contains L^1-functions of the form $p(x) \cdot e^{-x}$ where $p(x)$ are polynomials.

5. Exercise. Prove that the linear space $\mathbf{C}[x] \cdot e^{-x}$ is a dense subspace of $L^1(\mathbf{R}^+)$.

From the result in the exercise it follows that the polynomial algebra $\mathbf{C}[w]$ appears as a dense subalgebra of $\mathcal{H}(B)$ when it is equipped with the B-norm. At this stage we are prepared to give:

Proof of Proposition 3. In $A(D)$ we have the functions $\{G_\nu = \mathcal{H}(g_\nu)\}$. By assumption $\{G_\nu\}$ have no common zero in the closed disc D. Since D is the maximal ideal space of the disc algebra and $\mathbf{C}[w]$ a dense subalgebra, it follows that for every $\epsilon > 0$ there exist polynomials $\{p_\nu(w)\}$ such that the maximum norm

$$|p_1 \cdot G_1 + \ldots + p_k \cdot G_k - 1|_D < \epsilon| \tag{1}$$

where 1 is the identity function. Now $p_\nu = \mathcal{H}(\phi_\nu)$ for B-elements $\{\phi_\nu\}$. So in B we get the element

$$\psi = \phi_1 g_1 + \ldots + \phi_k \cdot g_k \tag{2}$$

Moreover we have $|\mathcal{H}(\psi) - 1|_D < \epsilon$ and here we can choose $\epsilon < 1/4$ and by the previous identifications it follows that

$$|\widehat{\psi}(\xi)| \geq 1/4 \quad \text{for all} \quad -\infty < \xi < \infty \tag{3}$$

The proof of Proposition 3 is finished if we can show that (3) entails that the B-element ψ is invertible. Multiplying ψ with a non-zero scalar we may assume that

$$\psi = \delta_0 - g \quad : \quad g \in L^1(\mathbf{R}^+)$$

and the Fourier transform $\widehat{\psi}(\xi)$ satisfies

$$|\widehat{\psi}(\xi) - 1| \leq 1/2$$

for all ξ. It means that $|\widehat{g}(\xi)| \leq 1/2$. The spectral radius formula applied to L^1-functions shows that if N is a sufficiently large integer then

$$||g^{(N)}||_1 \leq (3/4)^N \tag{4}$$

where $g^{(N)}$ is the N-fold convolution of g. Now we have

$$(1 + g + \ldots + g^{N_1}) \cdot \psi = 1 - g^{(N)} \tag{5}$$

By (4) the norm of the B-element $g^{(N)}$ is strictly less than one and hence the right hand side is invertible where the inverse is given by a Neumann series, i.e. with $g_* = g^{(N)}$ the inverse is

$$\delta_0 + \sum_{\nu=1}^{\infty} g_*^{\nu}$$

Since convolutions of $L^1(\mathbf{R}^+)$-functions still are supported by $x \geq 0$, it follows from the above that ψ is invertible in B and Proposition 3 is proved.

Appendix III. The Hardy space H^1

0. Introduction.

At several occasions we have met the situation where $F(z) \in \mathcal{O}(D)$ has bounded L^1-norms over circles of radius $r < 1$. The Brothers Riesz theorem in section I shows that if there is a constant M such that

$$\int_0^{2\pi} |F(re^{i\theta})| d\theta \leq M \quad : \quad 0 < r < 1$$

then there exists an L^1-function $F(e^{i\theta})$ on the unit circle and

$$(*) \qquad \lim_{r \to 1} \int_0^{2\pi} |F(re^{i\theta}) - F(e^{i\theta}| \cdot d\theta = 0$$

The class of analytic functions F with boundary function in $L^1(T)$ is denoted by $H^1(T)$ and called the *Hardy space*. It is tempting to start with a real valued L^1-function $u(\theta)$ on the unit circle and apply the Herglotz integral formula which produces both the harmonic extension of u and its conjugate harmonic function by the equation:

$$(**) \qquad g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot u(\theta) d\theta$$

It turns out that g_μ in general does not belong to $H^1(T)$, i.e. the condition that $u \in L^1(T)$ does not imply that $g_\mu \in H^1(T)$. Theorem 0.1 below is due to Zygmund and gives a necessary and sufficient condition for the inclusion $F \in L^1(T)$ when u is non-negative.

0.1 Theorem. *Let $u(\theta)$ be a non-negative L^1-function on T. Then $g_\mu(z) \in H^1(T)$ if and only if*

$$\int_0^{2\pi} u(\theta) \cdot \log^+ |u(\theta)| \cdot d\theta < \infty$$

Remark. That the condition is necessary is proved in section 1. The proof of sufficiency relies upon study of linear operators satisfying weak type estimates where a result due to Kolmogorov is the essential point. To profit upon Kolmogorov's result in section 3 we need a weak-type estimate for the harmonic conjugation functor which is proved in section 2.

0.2 The dual space of $H^1(T)$. On the unit circle the Banach space $C^0(T)$ of continuous complex-valued functions contains the closed subspace $A_*(D)$ which consists of those continuous function $f(e^{i\theta})$ on T which extend to analytic functions in the open disc $|z| < 1$ and vanish at $z = 0$. In Theorem 4.3 we prove that $H^1(T)$ is the dual of the quotient space

$$B = \frac{C^0(T)}{A_*(D)}$$

The proof uses the Brothers Riesz theorem. We shall also consider the subspace $H_0^1(T)$ of those functions in the Hardy space for which $f(0) = 0$. Here we find that

(1)
$$H_0^1(T) \simeq \Big[\frac{C^0(T)}{A(D}\Big]^*$$

Next, we seek the dual space of $H_0^1(T)$. Using the Brothers Riesz theorem one finds that

(2)
$$H_0^1(T)^* \simeq \frac{L^\infty(T)}{H^\infty(T)}$$

where $H^\infty(T)$ is the space of boundary values of bounded analytic functions in D.

0.3 The dual of $\mathfrak{Re}\, H_0^1(T)$. The real part determine functions in $H_0^1(T)$ which means that we can identify $H_0^1(T)$ with a real subspace of $L^1_{\mathbf{R}}(T)$ whose elements consist of those real-valued and integrable functions $u(\theta)$ for which the the Riesz transform also is integrable. Or equivalently, if we take the harmonic extension H_u then the harmonic conjugate has a boundary function in $L^1_{\mathbf{R}}(T)$ which we denote by u^*. The norm of such a u-function is defined as

(*)
$$||u|| = ||u||_1 + ||u^*||_1$$

The norm in (3) is not equivalent to the L^1-norm so we cannot conclude that the dual space is reduced to real-valued functions in $L^\infty(T)$. To exhibit elements in the dual space we first consider some real-valued function $F(\theta)$ on T. Let H_F be its harmonic extension to D. For each $0 < r < 1$ we define the linear functional on $\mathfrak{Re}\, H_0^1(T)$ by:

(**)
$$u \mapsto \int_0^{2\pi} H_F(re^{i\theta}) \cdot u(\theta) \cdot d\theta$$

If the limit (*) exists for every u when $r \to 1$ and the absolute value of this limit is $\leq C \cdot ||u||$ for a constant C, then we have produced a continuous linear functional on $\mathfrak{Re}\, H^1_0(T)$. This leads to a description of the dual space which goes as follows. The definition of the norm in (*) and the Hahn-Banach theorem yields for each Λ in the dual space a pair (ϕ,ψ) in $L^\infty(T)$ such that when $f = u + iu^*$ is in $H^1_0(T)$ then

$$\Lambda(u+iu^*) = \int_0^{2\pi} u(\theta)\cdot\phi(\theta)\cdot d\theta + \int_0^{2\pi} u^*(\theta)\cdot\psi(\theta)\cdot d\theta$$

Let ψ^* be the harmonic conjugate of ψ which gives the analytic function $H_\psi + iH^*_\psi$ in D. Since $f = u + iu^*$ vanishes at $z = 0$ we get

$$\int_0^{2\pi} (u+iu^*)(\psi+i\psi^*)\cdot d\theta = 0$$

Regarding the imaginary part it follows that

$$\int_0^{2\pi} u^*\cdot\psi\cdot d\theta = -\int_0^{2\pi} u\cdot\psi^*\cdot d\theta$$

We conclude that Λ is expressed by

$$\Lambda(u) = \lim_{r\to 1}\int_0^{2\pi} H_F(re^{i\theta})\cdot u(\theta)\cdot d\theta$$

where

$$(***) \qquad F(\theta) = \phi(\theta) - \psi^*(\theta)$$

Above ψ^* is the harmonic conjugate of a bounded ψ-function where an arbitrary $\psi \in L^\infty(T)$ can be chosen. Next, recall from XXX that if $\psi \in L^\infty(T)$ then its conjugate ψ^* belongs to BMO(T). Hence (***) identifies the of $\mathfrak{Re}\, H^1_0(T)$ with a subspace of BMO(T). It turns out that one has equality. More precisely, Theorem 0.4 below which is due to C. Fefferman and E. Stein asserts that F yields such a continuous linear form if and only if F has a bounded mean oscillation in the sense of F. John and L. Nirenberg.

0.4 Theorem. *A real-valued L^1-function F on T yields a continuous linear functional on $H^1_0(T)$ as above if and only if $F \in BMO(T)$. Moreover, there exists an absolute constant C such that*

$$\Big|\int_0^{2\pi} H_F(re^{i\theta})\cdot u(\theta)\cdot d\theta\Big| \leq C\cdot ||F||_{\text{BMO}}\cdot ||u||_1$$

for all $r < 1$ and $u \in H^1_0(T)$.

We refer to Section 6 for details of the proof which involves several steps where the essential step is to exhibit certain Carleson measures. The space of real-valued functions of bounded mean oscillation is denoted by BMO(T) and studied in Section 5 where Theorem 5.5 is an important result which clarifies many properties of functions in BMO(T).

0.5 The Hardy space on R. It consists of analytic functions $F(z)$ in the upper half-plane for which there exists a constant C such that

$$(*)\qquad \int_{-\infty}^{\infty} |F(x+i\epsilon)\cdot dx \leq C$$

hold for every $\epsilon > 0$. This space is denoted by $H^1(\mathbf{R})$. Let us remark that it differs from $H^1(\mathbf{T})$ even if we employ the conformal map

$$\text{(i)}\qquad w = \frac{z-i}{z+i}$$

onto the unit disc. More precisely, with $F(z)$ given in the upper half-plane we set

$$\text{(ii)}\qquad f(w) = F(\frac{i+iw}{1-w})$$

Then the reader can verify that

$$\text{(iii)}\qquad \lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|\cdot d\theta = \int_{-\infty}^{\infty} \frac{|F(x)|}{1+x^2}\cdot dx$$

where $F(x)$ is the almost everywhere defined limit of F on the real x-line which by (*) identifies $F(x)$ with an element in $H^1(\mathbf{R})$. Since $\frac{1}{1+x^2}$ is bounded it follows that the right hand side is finite in (iii) and hence f belongs to $H^1(\mathbf{T})$. However, the map $F \to f$ is not bijective because the convergence in (iii) need not imply that (*) is finite. In other words, the Hardy space on the real line is more constrained and via $F \mapsto f$ it appears s a proper subspace of $H^1(\mathbf{T})$ and the corresponding norms are not equivalent.

Sections 7 and 9 study $H^1(\mathbf{R})$ and at the end of section 9 we introduce Carleson norms on non-negative Riesz measures in $\mathfrak{Im}(z) > 0$ which will be used for interpolation of bounded analytic functions in Chapter XXX.

1. Zygmund's inequality

Let $u(\theta)$ be a non-negative real-valued function on T such that

$$(*)\qquad \frac{1}{2\pi}\int_0^{2\pi} u(\theta)d\theta = 1$$

Put

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\cdot u(\theta)d\theta$$

We write

$$F = u + iv$$

where u is the harmonic extension of $u(\theta)$ from T to D and v is the harmonic conjugate which by the Herglotz formula is normalised so that $v(0) = 0$ The sufficency part in Zygmund's theorem follows from the general inequality below:

1.1 Theorem. *When $u(\theta)$ is non-negative and (*) holds we have*

$$(*)\qquad \int_0^{2\pi} u(\theta)\cdot \text{Log}^+\,|u(\theta)|\cdot d\theta \leq \frac{\pi}{2}\cdot \int_0^{2\pi} |v(\theta)|\cdot d\theta + \int_0^{2\pi} \text{Log}^+|F(e^{i\theta}]|\cdot d\theta$$

Proof. Since $\mathfrak{Re}(F) > 0$ holds in D we can write

(i) $$\log F(z) = \log |F(z)| + i\gamma(z) \quad : \quad -\pi/2 < \gamma(z) < \pi/2$$

Set $G(z) = F(z) \cdot \log F(z)$. Since $F(0) = 1$ we have $G(0) = 0$ and the mean value formula for harmonic functions gives

(iii) $$\int_0^{2\pi} u(e^{i\theta}) \cdot \text{Log}\,|F(e^{i\theta})| \cdot d\theta = \int_0^{2\pi} \gamma(e^{i\theta}) \cdot v(e^{i\theta}) \cdot d\theta$$

By (i) the absolute value of the right hand side is majorised by

(iii) $$\frac{\pi}{2} \cdot \int_0^{2\pi} |v(\theta)| \cdot d\theta$$

Now we use the decomposition

$$\log |F(e^{i\theta})| = \text{Log}^+ |F(e^{i\theta})| + \text{Log}^+ \frac{1}{|F(e^{i\theta})|}$$

Then (ii) and (iii) give the inequality

$$\int_0^{2\pi} u(e^{i\theta}) \cdot \text{Log}^+ |F(e^{i\theta})| \cdot d\theta \leq$$

(iv) $$\frac{\pi}{2} \cdot \int_0^{2\pi} |v(\theta)| \cdot d\theta + \int_0^{2\pi} u(e^{i\theta}) \cdot \text{Log}^+ \frac{1}{|F(e^{i\theta})|} \cdot d\theta$$

Since $\text{Log}^+ \frac{1}{|F(e^{i\theta})|} \neq 0$ entails that $|F| \leq 1$ and hence also $u \leq 1$, it follows that the last integral above is majorised by

(v) $$\int_0^{2\pi} \text{Log}^+ \frac{1}{|F(e^{i\theta})|} \cdot d\theta$$

Next, $\log |F(z)|$ is a harmonic function whose value at $z = 0$ is zero. So the mean-value formula for harmonic functions in D gives the equality:

(vi) $$\int_0^{2\pi} \text{Log}^+ \frac{1}{|F(e^{i\theta})|} \cdot d\theta = \int_0^{2\pi} \text{Log}^+ |F(e^{i\theta})| \cdot d\theta$$

Using this and (iv) we get the requested inequality in Theorem 1.1 since we also have the trivial estimate

(vii) $$\text{Log}^+ u(e^{i\theta}) \leq \text{Log}^+ |F(e^{i\theta})|$$

2. The weak type estimate.

Let $u(\theta)$ be non-negative and denote by $v(\theta)$ its harmonic conjugate function which is obtained via Herglotz formula.If E is a subset of T we denote its linear Lebesgue measure by $\mathfrak{m}(E)$. With these notations the following weak.type estimate hods:

2.1 Theorem. *For each non-negative u-function on T with mean-value one the following holds:*

$$\mathfrak{m}(\{|v|) > \lambda\} \leq \frac{4\pi}{1+\lambda} \quad : \quad \lambda > 0$$

Proof. For a given $\lambda > 0$ we set

$$\phi(z) = 1 + \frac{F(z) - \lambda}{F(z) + \lambda} \tag{1}$$

where $F(z)$ the analytic function constructed as in section 1. Here $F(0) = u(0) = 1$ which gives:

$$\phi(0) = \frac{2}{\lambda + 1} \tag{2}$$

Next, since $\mathfrak{Re}\, F = u \geq 0$ it follows that

$$|\frac{F(z) - \lambda}{F(z) + \lambda}| \leq 1 \tag{3}$$

Hence (1) gives $\mathfrak{Re}(\phi) \geq 0$ and mean value formula for the harmonic function $\mathfrak{Re}(\phi)$ gives:

$$\frac{4\pi}{1+\lambda} = \int_0^{2\pi} \mathfrak{Re}\, \phi(e^{i\theta}) \cdot d\theta \geq \int_{\mathfrak{Re}\, \phi \geq 1} \mathfrak{Re}\, \phi(e^{i\theta}) \cdot d\theta \geq \mathfrak{m}(\{\mathfrak{Re}\, \phi \geq 1\}) \tag{4}$$

Rewriting the last inequality we get:

$$\mathfrak{m}(\{\mathfrak{Re}\, \phi \geq 1\}) \leq \frac{4\pi}{1+\lambda} \tag{5}$$

Next, the construction of ϕ yields the following equality of sets:

$$\{\mathfrak{Re}\, \phi(e^{i\theta}) \geq 1\} = \{\mathfrak{Re}\, \frac{F(e^{i\theta}) - \lambda}{F(e^{i\theta}) + \lambda} \geq 0\} \tag{6}$$

Finally, with $F(e^{i\theta}) = u(\theta) + iv(\theta)$ one has

$$\mathfrak{Re}\, [\frac{F(e^{i\theta}) - \lambda}{F(e^{i\theta}) + \lambda}] = \frac{u^2 + v^2 - \lambda^2}{(u+\lambda)^2 + v^2}$$

The right hand side is ≥ 0 when $|v| \geq \lambda$ which gives the set-theoretic inclusion:

$$\{|v| > \lambda\} \subset \{\mathfrak{Re}\, \phi \geq 1\}). \tag{7}$$

Then (5) above gives Theorem 2.1.

3. Kolmogorov's inequality

3.1 Notations. Consider a measure space equipped with a probability measure μ. Let f be a complex-valued and μ-measurable function. For each $\lambda > 0$ we get the μ-measurable set $\{|f| > \lambda\}$ and then

$$\lambda \mapsto \mu(\{|f| > \lambda\})$$

is a decreasing function. Construct the differential function defined for every $\lambda > 0$ by:

$$d\rho_f(\lambda) = \lim_{\delta \to 0} \frac{\mu(\{|f| > \lambda - \delta\}) - \mu(\{|f| > \lambda\})}{\delta} \tag{*}$$

For an arbitrary continuous function $Q(\lambda)$ defined when $\lambda \geq 0$ the formula in XX gives the equality:

$$(**) \qquad \int_0^\infty Q(|f|)d\mu = \int_0^\infty Q(\lambda) \cdot d\rho_f(\lambda)$$

Recall also from XX the formula

$$(***) \qquad \int_0^\infty \mu[\{|f| > \lambda\}] \cdot d\lambda = \int |f| \cdot d\mu$$

3.2 Operators of Weak type (1,1). Let γ be a probability measure on another sample space and T is some linear map from μ-measurable functions into γ-measurable functions.

3.3 Definition. *We say that T satisfies a weak-type estimate of type (1,1) if there is a constant K such that the inequality below holds for every $\lambda > 0$:*

$$\gamma(\{|Tf| > \lambda\}\,) \leq \frac{K}{\lambda} \cdot \int |f| \cdot d\mu \quad \text{when} \quad f \ \text{ is}\, \mu - \text{measurable}$$

We can also regard L^2-spaces. The operator T is L^2-continuous if there exists a constant K_2 such that one has the inequality

$$\int |T(f)|^2 \cdot d\gamma \leq K_2^2 \cdot \int |f|^2 \cdot d\mu$$

Taking square roots it means that the L^2-norm is K_2.

3.4 Theorem. *Let T be a linear operator whose L^2-norm is 1 and with finite weak-type norm K. Then the following holds for each μ-measurable function f:*

$$\int |T(f)| \cdot d\gamma \leq 1 + 4 \cdot \int |f| \cdot d\mu + 2K \cdot \int |f| \cdot \text{Log}^+ |f| \cdot d\mu$$

Proof. When $\lambda > 0$ we decompose f as follows:

$$\text{(i)} \qquad f = f_* + f^* \quad : f_* = \chi_{\{|f| \leq \lambda\}} \cdot f \quad : \quad f^* = \chi_{\{|f| > \lambda\}} \cdot f$$

For the lower f_*-function we use that T has L^2-norm ≤ 1 and get

$$\text{(ii)} \qquad \gamma[\{|Tf_*| > \lambda/2\}\,] \leq \frac{4}{\lambda^2} \int_0^\lambda s^2 \cdot d\rho_f(s)$$

For Tf^* we apply the weak-type estimate which gives

$$\text{(iii)} \qquad \gamma[\{|Tf^*| > \lambda/2\}\,] \leq \frac{2K}{\lambda} \cdot \int_\lambda^\infty s \cdot d\rho_f(s)$$

where we used that $\int_\lambda^\infty s \cdot d\rho_f(s)$ is the L^1-norm of f^*. The set-theoretic inclusion

$$\{|Tf| > \lambda\} \subset \{|Tf_*| > \lambda/2\} \cup \{|Tf^*| > \lambda/2\} \implies$$

$$\text{(iv)} \qquad \gamma[\{|Tf| > \lambda\}\,] \leq \frac{4}{\lambda^2} \int_0^\lambda s^2 \cdot d\rho_f(s) + \frac{2K}{\lambda} \cdot \int_\lambda^\infty s d\rho_f(s)$$

Next, since γ has total mass one the inequality:

$$\text{(v)} \qquad \int_0^\infty |Tf|\cdot d\gamma \leq 1 + \int_{\{|Tf|>1\}} |Tf|\cdot d\gamma$$

Now (***) in (3.1) is applied to Tf and the measure γ which gives

$$\int_{\{|Tf|>1\}} |Tf|\cdot d\gamma = \int_1^\infty \gamma[\{|Tf|>\lambda\}\cdot d\lambda$$

By (iv) the last integral in (v) is majorised by:

$$\text{(vi)} \qquad 4\cdot\int_1^\infty \big[\frac{1}{\lambda^2}\int_0^\lambda s^2\cdot d\rho_f(s)\big]\cdot d\lambda + 2K\int_1^\infty \frac{1}{\lambda}\cdot\big[\int_\lambda^\infty s d\rho_f(s)\big]\cdot d\lambda$$

Next, from (XX) one has the equality:

$$\text{(vii)} \qquad \int_0^\infty \big[\frac{1}{\lambda^2}\int_0^\lambda s^2\cdot d\rho_f(s)\big]\cdot d\lambda = \int |f|\cdot d\mu$$

The left hand side is only smaller if the λ-integration starts at 1. It follows that the first term in (vi) above is majorised by $4\cdot\int |f|\cdot d\mu$ and together with (v) we conclude that

$$\text{(viii)} \qquad \int_0^\infty |Tf| d\gamma \leq 1 + 4\int |f| d\mu + 2K\int_1^\infty \big[\frac{1}{\lambda}\cdot\int_\lambda^\infty s d\rho_f(s)\big]\cdot d\lambda$$

Finally,

$$\int_1^\infty \big[\frac{1}{\lambda}\cdot\int_\lambda^\infty s d\rho_f(s)\big]\cdot d\lambda = \iint_{1\leq\lambda\leq s} \frac{1}{\lambda}\cdot s\rho_f(s) ds = \int_1^\infty s\cdot \operatorname{Log} s\cdot d\rho_f(s)$$

The last integral is equal to $\int f\cdot \operatorname{Log}^+|f|\cdot d\mu$ by the general formula XX. Inserting this in (viii) we get Theorem 3.4.

3.5. Final part of Theorem 0.1

There remains to show that if u is non-negative and if $u\cdot\operatorname{Log}^+ u$ is integrable so is v. To prove this we use $d\mu = d\gamma = \frac{d\theta}{2\pi}$ on the unit circle. Theorem 2.1 which shows that the harmonic conjugation operator $T\colon u\mapsto v$ is of weak-type (1,1) and it is continuous on $L^2(T)$ by Parseval's formula. Hence Kolomogorv's Theorem gives $v\in L^1(T)$ which proves the necessity in Theorem 0.1.

Remark. Notice that Theorem 3.4 applies when we start from any real-valued function $u(\theta)$. So have the following supplement to Theorem 0.1.

3.6 Theorem. *There exists an absolute constant A such that*

$$\int_0^{2\pi} |v(\theta)|\cdot d\theta \leq A\cdot\Big[\int_0^{2\pi} |u(\theta)|\cdot d\theta + \int_0^{2\pi} |u(\theta)|\cdot\operatorname{Log}^+|u(\theta)|\cdot d\theta\Big]$$

4. The Dual space of $H^1(T)$

On the unit circle T we have the Banach space $L^1(T)$ where $H^1(T)$ is a closed subspace. Next, let $C^0(T)$ be the Banach space of continuous functions on T equipped with the maximum norm. It contains the closed subspace $A(D)$ whose functions can be extended as analytic functions in the open disc D. We have also the subspace $A_*(D)$ which consists of the functions in $A(D)$ whose analytic extensions are zero at the origin. As explained in XXX a continuous function f on T belongs to $A_*(D)$ if and only if

$$(*) \qquad \int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad n = 0, 1, \ldots$$

From (*) it follows that

$$(**) \qquad \int_0^{2\pi} g(e^{i\theta}) \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad f \in A_*(D) \quad \text{and} \quad g \in H^1(T)$$

Let us now regard the Banach space

$$B = \frac{C^0(T)}{A_*(D}$$

Riesz representation formula identifies the dual space of $C^0(T)$ with Riesz measures on T. Since B is a quotient space its dual space becomes

$$\text{(i)} \qquad B^* = \{ \mu \in M(T) \quad : \mu \perp A_*(D \}$$

Now a Riesz measure μ is $\perp A_*(D$ if and only if

$$\text{(ii)} \qquad \int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = 0 \quad : \quad n = 1, 2 \ldots$$

The Brothers Riesz theorem means that (ii) holds if and only if μ is absolutely continuous, i.e. μ is given by some L^1-function f which satisfies:

$$\text{(iii)} \qquad \int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad n = 1, 2 \ldots$$

This is precisely the condition that $f \in H^1(T)$. Hence the whole discussion gives:

4.1 Theorem. *The Hardy space $H^1(T)$ is the dual of B.*

4.2 The dual of $H^1(T)$. Recall that $L^\infty(T)$ is the dual space of $L^1(T)$. So by a general formulafrom Appendix: Functional analysis we get:

$$H^1(T)^* = \frac{L^\infty(T)}{H^1(T)^\perp}$$

Next, an L^∞-function f is $\perp H^1(T)$ if and only if

$$\int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad n = 0, 1, 2 \ldots$$

But this means precisely that f is the boundary value of an analytic function in D which vanishes at the origin. Let us identify $H^\infty(D)$ with a subalgebra of $L^\infty(T)$ which is denoted by $H^\infty(T)$. Then we also get the subspace $H_0^\infty(T)$ of those functions which are zero at the origin. Hence we have proved

Theorem 4.3 *The dual space of $H^1(T)$ is equal to the quotient space*

$$\frac{L^\infty(T)}{H_0^\infty(T)}$$

5. BMO

Introduction. Functions of bounded mean oscillation were introduced by F. John and L. Nirenberg in [J-N]. This class of Lebesgue measurable functions can be defined in $\mathbf{R^n}$ for every $n \geq 1$. Here we are content to study the case $n = 1$ and restrict the attention to periodic functions which is adapted to the class BMO on the unit circle. So let $F(x)$ be a locally integrable function on the real x-line which is 2π-periodic, i.e. $F(x + 2\pi) = F(x)$. If $J = (a, b)$ is an interval we get the mean value

$$F_J = \frac{1}{b-a} \cdot \int_a^b F(x)dx$$

To every interval J we put

$$|F|_J^* = \int_J |F(x) - F_J| \cdot dx$$

5.1 Definition. *The function F has a bounded mean oscillation if there exists a constant C such that $|F|_J \leq C$ for all intervals J. When this holds the smallest constant is denoted by $|F|_{BMO}$.*

5.2 The case $n \geq 2$. Even though these notes are devoted to complex analysis in dimension one, we cannot refrain from mentioning a result which illustrates how the class BMO enters in Fourier analysis. Namely, let $F(x)$ be an L^1-function with compact support in $\mathbf{R^n}$. Assume that there exists a constant C such that its Fourier transform $\hat{F}(\xi)$ satisfies the decay condition

$$(*) \qquad |\widehat{F}(\xi) \leq C \cdot (1 + |\xi|)^{-n} \quad : \quad \xi \in \mathbf{R^n}$$

This is not quite enough for $\widehat{F}$ to be integrable. So we cannot expect that (*) implies that $F(x)$ is a bounded function. However, its belongs to BM0 and more precisely one has:

5.3 Theorem. *To each $M > 0$ there exists a constant C_M such that if $F(x)$ has support in the ball $\{|x| \leq M\}$ then*

$$||F||_{\text{BMO}} \leq C_M \cdot \max_\xi \left[1 + |\xi|)^n \cdot |\widehat{F}(\xi)|\right]$$

For the proof we refer to [Björk]. See also [Sjölin] for an improved result that F belongs to BMO under less restrictive conditions expressed by certain L^2-integrals of $\widehat{F}$ over dyadic grids.

5.4 The John-Nirenberg inequality.

Now we turn to the main topic in this section and prove an inequality due to F. John and L. Nirenberg which is presented for the 1-dimensional periodic case. See [J-N] for higher dimensional results.

5.5 Theorem *Let $F(x)$ be a 2π-periodic function on the real x-line which belongs to BMO on T. For every interval J on* **R** *and every positive integer n one has*

$$\mathfrak{m}[\{x \in J : \big|F(x) - F_J\big| \geq 4n \cdot |F|_{\text{BMO}}\}] \leq 2^{-n} \cdot |J|$$

The proof requires several steps. To begin with we make some trivial observations. The BMO-norm of F is unaffected when we add a constant to F and also under a translation, i.e. when we regard $F_a(x) = F(x+a)$ for some real number a. Moreover, the BMO-norm is unchanged under dilations, i.e. when $t > 0$ and $F_t(x) = F(tx)$. Before we enter the proof we need a preliminary result.

5.6 Lemma. *Let F belong to BMO. Let $I \subset J$ be two intervals with the same mid-point. Then*

$$|F_J - F_I| \leq 2 \cdot [\text{Log}_2 \frac{|J|}{|I|} + 1] \cdot |F|_{\text{BMO}}$$

Exercise. Prove this result.

Proof of Theorem 5.5. Replacing F by cF for some positive constant we may assume that its BMO-norm is 1/2. and that $F_J = 0$. Moreover, by the invariance under dilations and translations we may assume that J is the unit interval. Thus, there remains to consider the set

(i) $$E_n = \{x \in [0,1] \ : \ F(x) > 2n\}$$

and show that

(ii) $$\mathfrak{m}(E_n) \leq 2^{-n}$$

Let us begin with the case $n = 1$. For every $x \in E_1$ which is a Lebesgue point for F we find the unique largest dyadic interval $J(x)$ such that

(iii) $$x \in J(x) \subset [0,1] \quad : \quad \frac{1}{\mathfrak{m}(J(x))} \int_{J(x)} F(t)dt > 1$$

Up to measure zero, i.e. ignoring the null set where F fails to have Lebesgue points, we have the inclusion

(iv) $$E_1 \subset \cup_{x \in E_1} J(x)$$

Next, suppose we have a *strict* inclusion $J(x) \subset J(y)$ for a pair of dyadic intervals in this family which means that $\mathfrak{m}(J(y)) > \mathfrak{m}(J(x))$. But this is impossible for then $x \in J(y)$ which contradicts the maximal choice of $J(x))$ as the dyadic interval of largest possible length containing x. Hence the family $\{J(x_\nu)\}$ consists of dyadic intervals which either are equal or disjoint. We can therefore pick a disjoint family where the corresponding x-points are denoted by x_ν^* and obtain the set-theoretic inclusion

(v) $$E_1 \subset \cup\, (J(x_\nu^*)$$

Next, put $\mathcal{E} = \cup\, J(x_\nu^*)$. Since the mean value of F over each $J(x_\nu^*)$ is ≥ 1 we obtain

$$\mathfrak{m}(\mathcal{E}) \leq \sum \int_{J(x_\nu^*)} F(x)dx = \int_{\mathcal{E}} F(x)dx \leq \int_{\mathcal{E}} |F(x)|dx \leq \int_0^1 |F(x)|dx \leq |F|_{\mathrm{BMO}}$$

where the last inequality follows from the definition of the BMO-norm and the condition that the mean-value of F over the unit interval was zero. Since the BMO-norm of F was $1/2$ the inclusion (v) gives:

(*) $$\mathfrak{m}(E_1) \leq \mathfrak{m}(\mathcal{E}) \leq 1/2$$

This proves the case $n = 1$ and we proceed by an induction over n. Let us first regard one of the dyadic intervals $J(x_\nu^*)$ from the family covering E_1. If 2^{-N} is the length of $J(x_\nu^*$ the dyadic exhaustion of $[0,1]$ gives a dyadic interval J' of length 2^{-N+1} which contain $J(x_\nu^*)$. The maximal choice of $J(x_\nu^*)$ gives:

(vi) $$\frac{1}{\mathfrak{m}(J')} \int_{J'} F(t)dt \leq 1$$

Apply Proposition XX to the pair $J(x_\nu^*)$ and J'. Since $|F|_{\mathrm{BMO}} = 1/2$ is assumed and $\mathrm{Log}_2(2) = 0$ we obtain

(vii) $$F_{J(x_\nu^*)} = \frac{1}{\mathfrak{m}(J(x_\nu^*))} \int_{J(x_\nu^*)} F(t)dt \leq 2$$

Let $n \geq 2$ and for every ν we set:

(viii) $$E_n(\nu) = E_n \cap J(x_\nu^*)$$

Since $F(x) > 2n$ holds on E_n we get

(ix) $$F(x) - F_{J(x_\nu^*)} > 2(n-1) \quad : \quad x \in E_n(\nu)$$

Hence we have the inclusion

(x) $$E_n(\nu) \subset W_n(\nu) = \{x \in J(x_\nu^*) \quad : \; F(x) - F_{J(x_\nu^*)} > 2(n-1)\}$$

By a change of scale we can use the interval $J(x_\nu^*)$ instead of the unit interval and by an induction assume that the inequality in Theorem xx holds for $n-1$. It follows that the set in right hand side in (x) is estimated by:

(xi) $$\mathfrak{m}(W_n(\nu)) \leq 2^{-n+1} \cdot \mathfrak{m}(J(x_\nu^*))$$

The set-theoretic inclusion (x) therefore gives

(xii) $$\mathfrak{m}(E_n(\nu)) \leq 2^{-n+1} \cdot \mathfrak{m}(J(x_\nu^*))$$

Finally, since $E_n \subset E_1$ and we already have the inclusion (iv) we obtain

$$\mathfrak{m}(E_n) = \sum \mathfrak{m}(E_n(\nu)) \leq 2^{-n+1} \cdot \sum \mathfrak{m}(J(x_\nu^*)) = 2^{-n+1}\mathfrak{m}(\mathcal{E}) \leq 2^{-n+1} \cdot \frac{1}{2} = 2^{-n}$$

This proves the induction step and Theorem 5.5 follows.

5.7 An L^2-inequality

Let $F \in \text{BMO}(T)$ be given. Given some interval $J \subset T$ and $\lambda > 0$ we set

$$\mathfrak{m}_J(\lambda) = \{\theta \in J : |F(\theta) - F_J| > \lambda\}$$

Consider the integral

$$(*) \qquad I = \frac{1}{|J|} \cdot \int_0^\infty \lambda \cdot \mathfrak{m}_J(\lambda) \cdot d\lambda$$

Set $A = 4 \cdot ||F||_{\text{BMO}}$. Theorem 5.5. gives

$$I = \frac{1}{|J|} \cdot \sum_{n=0}^\infty \int_{nA}^{(n+1)A} \lambda \cdot \mathfrak{m}_J(\lambda) \cdot d\lambda \leq \frac{1}{|J|} \cdot \sum_{n=0}^\infty (n+1)A \cdot |J| \cdot \cdot 2^{-n} = C||F||_{\text{BMO}}$$

where $C = 4 \cdot \sum_{n=0}^\infty (n+1)\, 2^{-n}$ is an absolute constant. Next, by the general result in XX (*) is equal to

$$(**) \qquad \frac{1}{|J|} \cdot \int_J |F(x) - F_J|^2 \cdot dx$$

So by the above (**) is majorized by an absolute constant times the BMO-norm of F.

5.8 BMO and the Garsia norm.

Using the L^2-inequality in (5.7) an elegant description of BMO(T) was discovered by Garsia which we shall use in Section 6. First we give:

5.9 Definition. *To each real-valued $u \in L^1(T)$ we define a function in D by*

$$\mathcal{G}_u(z) = \frac{1}{8\pi^2} \cdot \iint \frac{(1-|z|^2)^2}{|e^{i\theta} - z|^2 \cdot |e^{i\phi} - z|^2} \cdot [u(\theta) - u(\phi)]^2 \cdot d\theta d\phi$$

If this function is bounded we set

(*) $$\mathcal{G}(u) = \max_{z \in D} \sqrt{\mathcal{G}_u(z)}$$

and say that u has a finite Garsia norm.

Remark. Notice that constant functions have zero-norm. So just as for BMO the $\mathcal{G}$-norm is defined on the quotient of functions modulu constants.

5.10 Exercise. Expanding the square $[u(\theta) - u(\phi)]^2$ the reader can verify that

(*) $$\mathcal{G}_u = H_{u^2} - H_u^2$$

where H_{u^2} is the harmonic extension of u^2. and H_u^2 the square of the harmonic extension H_u.

5.11 Theorem. *An L^1-function u has finite Garsia norm if and only if it belongs to BMO. Moreover, there exists a constant $C \geq 1$ such that*

$$\frac{1}{C} \cdot ||u||_{\text{BMO}} \leq \mathcal{G}(u) \leq C \cdot ||u||_{\text{BMO}}$$

5.12 Exercise. The reader is invited to prove this result using the previous facts about BMO and also straightforward properties of the Poisson kernel. if necessary, consult [Koosis p. xxx-xxx] for details.

5.13 The Garsia norm and Carleson measures. Let f be a real-valued continuous function on T. We get the two harmonic functions H_f and H_{f^2} and recall from (5.10) that

$$\mathcal{G}_f = H_{f^2} - (H_f)^2$$

In XX we introduced the family of Carleson sectors in D and now we prove an important inequality.

5.14 Theorem. *For every Carleson sector S_h with $0 < h < 1/2$ and each $f \in C^0(T)$ one has the inequality*

$$\frac{1}{h} \cdot \iint_{S_h} |z| \cdot \log \frac{1}{|z|} \cdot \left|\nabla(H_f)\right|^2 \cdot dxdy \leq 96 \cdot \mathcal{G}(f)^2$$

Proof. We use the conformal map where $z = \frac{\zeta - i}{\zeta + i}$. If $\phi(z)$ is a function in D we get the function $\phi^*(\zeta)$ in the upper half-plane where

$$\phi(\frac{\zeta - i}{\zeta + i}) = \phi^*(\zeta)$$

One easily verifies that

$$\text{(i)} \qquad (|z| \cdot \log \frac{1}{|z|})^*(\xi + i\eta) \leq 8 \cdot \eta$$

Set $w(\zeta) = H_f^*(\zeta)$. Then (i) implies that the double integral which appears in the Theorem 5.14 is majorised by

$$\text{(ii)} \qquad J^* = 8 \cdot \iint_{S_h^*} \eta \cdot |\nabla(w)|^2 \cdot d\xi d\eta$$

where S_h^* is the image of S_h under the conformal map and $|\nabla(w)|^2 = w_\xi^2 + w_\eta^2$. Next, from (*) in Exercise 5.10 we have

$$w^2 = H_{f^2}^* - \mathcal{G}_f^*$$

Since $H_{f^2}^*$ is harmonic we obtain

$$\text{(iii)} \qquad 2 \cdot |\nabla(w)|^2 = \Delta(w^2) = -\Delta(\mathcal{G}_f^*)$$

where the first easy equality follows since w is harmonic. As explained by figure XX the set S_h^* is placed above an interval on the real ξ-line and and since the subsequent estimates are invariant under the center of this interval we therefore may take it as $\xi = 0$. Let us introduce the half-disc

$$D_{2h} = \{|\zeta| < 2h\} \cap \{\eta > 0\}$$

Then a figure shows that

$$\text{(iv)} \qquad S_h^* \subset D_{2h}$$

Next, consider the function $1 - \frac{|\zeta|}{2h}$ and notice that it is $\geq 1/4$ in D_{2h}. Recall from the above that

$$\Delta(\mathcal{G}_f^*) = -2 \cdot |\nabla(w)|^2 \leq 0$$

From the inclusion (iv) and taking the minus signs into the account it follows from (ii) that

$$\text{(v)} \qquad J^* \leq -16 \cdot \iint_{D_{2h}} \eta(1 - \frac{|\zeta|}{2h}) \cdot \Delta(\mathcal{G}_f^*) \cdot d\xi d\eta$$

Apply Green's formula to the pair $\mathcal{G}_f^*$ and $\rho = \eta(1 - \frac{|\zeta|}{2h})$. Here $\rho = 0$ on the boundary of D_{2h} and it is easily checked that the outer normal $\partial_n(\rho) \leq 0$. At the same time $\mathcal{G}_f^* \geq 0$ and from this it follows that (v) gives:

$$\text{(iv)} \qquad J^* \leq -16 \cdot \iint_{D_{2h}} \Delta(\eta(1 - \frac{|\zeta|}{2h})) \cdot \mathcal{G}_f^* \cdot d\xi d\eta$$

Next, using polar coordinates (r,ϕ) an easy computation gives

$$\Delta(\eta(1-\frac{|\zeta|}{2h}))=-\frac{3}{2h}\cdot\sin\phi$$

It follows that

$$J^*\leq\frac{24}{h}\cdot\iint_{D_{2h}}\mathcal{G}_f^*\cdot\sin\phi\cdot rdrd\phi$$

Finally, by definition $\mathcal{G}(f)^2$ is the maximum norm of $\mathcal{G}_f$ in D which is $\geq$ the maximum norm of $\mathcal{G}_f^*$ in D_{2h}. So the last integral is majorised by

$$\frac{24\mathcal{G}(f)^2}{h}\cdot\iint_{D_{2h}}\sin\phi\cdot rdrd\phi=96\cdot\mathcal{G}(f)^2\cdot h$$

After a division with h we get Theorem 5.14.

5.15 Remark. Since $|z|\geq 1/2$ holds in sectors S_h with $0<h<1/2$ we can remove the factor $|z|$ and hence Theorem 5.14 shows that if $\mathcal{G}_f(z)$ is bounded in D then we obtain a Carleson measure in D defined by

(*) $$\mu_f=\log\frac{1}{|z|}\cdot\big|\nabla(H_f)\big|^2$$

Moreover, its Carleson norm is estimated via Theorem 5.11 and Theorem 5.24 by an absolute constant times $|F|_{\mathrm{BMO}}$.

6. Proof of Theorem 0.4

By the observations before Theorem 0.4 here remains to prove that if $F\in\mathrm{BMO}(T)$ then (*) holds in Theorem 0.4 for some constant C. To obtain this we need some preliminary results derived via Green's formula.

6.1 Some integral formulas. To simplify notations we set

$$\int_0^{2\pi}g(e^{i\theta})\cdot d\theta=\int_T g\cdot d\theta$$

for integrals over the unit circle. Now follow some results which are left as exercises and proved by Green's formula.

A. Exercise. For every C^2-function W in the closed unit disc with $W(0)=0$ we have

(1) $$\int_T W\cdot d\theta=\iint_D\log\frac{1}{|z|}\cdot\Delta(W)\cdot dxdy$$

Next, if

(2) $$W=|z|\cdot W_1$$

B. Exercise. Let u,v is a pair of C^2-functions which both are harmonic in the open disc. Show that

(i) $$\Delta(uv)=2\cdot\langle\nabla(u),\nabla(v)\rangle$$

and use (A) to prove the equality

$$\text{(ii)} \qquad \int_T uv \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \langle \nabla(u), \nabla(v) \rangle \cdot dxdy$$

C. Exercise. Let $f = u + iv$ be analytic in D. Verify that

$$\text{(i)} \qquad \Delta(|f|) = \frac{1}{|f|} \cdot |\nabla(u)|^2$$

holds outside the zeros of f. Show also that if

$$\text{(ii)} \qquad f = z \cdot g$$

where g is zero-free in D then

$$\text{(iii)} \qquad \int_T |f| \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \frac{|\nabla(u)|^2}{|f|} \cdot dxdy$$

Finally, let f be as in (ii) and F a real-valued C^2-function in D. Show that

$$\text{(iii)} \qquad \frac{1}{2} \int_T u \cdot F \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \langle \nabla(u), \nabla H_F \rangle \cdot dxdy$$

6.2 Proof of Theorem 0.4

Let $f \in H_0^1(T)$. Then one finds a Blaschke product B such that

$$f(z) = z \cdot B(z) \cdot g(z)$$

where g is zero-free in D. It follows that

$$2f = z(B+1) \cdot g + z(B-1) \cdot g = f_1 + f_2$$

where $||f_\nu||_1 \leq 2 \cdot ||f||_1$ hold for each ν. Using this trick we conclude that it suffices to estasblish Theorem 0.4 for $H^1(T)$-functions of the form $f(z) = z \cdot g(z)$ with a zero-free function g in D. We write $f = u + iv$ and for each real-valued C^2-function $F(\theta)$ on T we have by (iii) from Exercise C:

$$\text{(1)} \qquad \frac{1}{2} \cdot \int_0^{2\pi} F(\theta) \cdot u(\theta) \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \langle \nabla(u), \nabla(H_F) \rangle \cdot dxdy$$

Insert the factor $1 = \sqrt{|f|} \cdot \frac{1}{\sqrt{|f|}}$ and apply the Cauchy-Schwarz inequality which estimates the absolute value of (i) by

$$\text{(2)} \quad J = \sqrt{\iint_D \log \frac{1}{|z|} \cdot \frac{|\nabla(u)|^2}{|f(z)|} \cdot dxdy} \cdot \sqrt{\iint_D \log \frac{1}{|z|} \cdot |\nabla(H_F)|^2 \cdot |f(z)| \cdot dxdy}$$

The equality (iii) in Exercise C shows the first factor is equal to $\sqrt{||f||_1}$. In the second factor appears the density function $\log \frac{1}{|z|} \cdot |\nabla(H_F)|^2$ in D.

Finally, by the Remark in (5.15) the Carleson norm of the density $\log \frac{1}{|z|} \cdot |\nabla(H_F)|^2$ is bounded by an absolute constant C times the BMO-norm of F. Together with the result in XXX in Section XXX we get an absolute constant C such that the last factor in (2) above is bounded by

$$C \cdot |F|_{\text{BMO}} \cdot \sqrt{||f||_1} \tag{3}$$

which finishes the proof of Theorem 0.4.

7. A theorem by Gundy Silver

Introduction. Let $U(x)$ be in $L^1(\mathbf{R})$ and construct its harmonic extension to the upper half plane:

$$U(x+iy) = \frac{1}{\pi} \cdot \int \frac{y}{(x-t)^2 + y^2} \cdot U(t) \cdot dt$$

The harmonic conjugate of $U(x+iy)$ is given by:

$$V(x+iy) = \frac{y}{\pi} \int \frac{U(t) \cdot dt}{(x-t)^2 + y2} \tag{0.1}$$

Next, to each real x_0 the Fatou sector in the upper half-plane is defined by

$$\{x+iy) \quad \text{such that} \quad |x-x_0| \leq y\} \tag{0.2}$$

and the maximal function U^* over Fatou sectors is defined on the real x-axis by

$$U^*(x_0) = \max\, |U(x+iy)|: \quad : |x-x_0| \leq y \tag{0.3}$$

In XXX we proved that if $V \in L^1(R)$ then $U^*(x) \in L^1(\mathbf{R})$ and there exists an absolute constant C_0 such that

$$||U^*||_1 \leq \int_{-\infty}^{\infty} \big(\, |U(x)| + |V(x)|\,\big) dx \tag{*}$$

A reverse inequality is due to Burkholder, Gundy and Silverstein.

Theorem 7.1. *One has the inequality*

$$\int |V(x)| dx \leq 4 \int U^*(x) dx$$

Remark. Hence U^* belongs to L^1 if and only if the boundary value function $V(x)$ belongs to L^1. The original proof in [BGS] used probabilistic methods. Here we give a proof based upon methods from [Feff-Stein]. Since we shall establish an *a priori estimate*, it suffices to assume that $U(x)$ from the start is a nice function. In particular we may assume that both $U(x+iy)$ and $V(x+iy)$ have rapid decay when $y \to +\infty$ in the upper half-plane. This assumption is used below to ensure that a certain complex line integral is zero.

Proof of Theorem 7.1

Given $\lambda > 0$ we put

$$J_\lambda = \{x \colon U^*(x) > \lambda\}$$

The closed complement $\mathbf{R} \setminus J_\lambda$ is denoted by E. Let $\{(a_\nu, b_\nu)\}$ be the disjoint intervals of J_λ. Construct the piecewise linear Γ-curve which stays on the real x-line on E while it follows the two sides of the triangle T_ν standing on (a_ν, b_ν) for each ν. So the corner point of T_ν in the upper half-plane is:

$$p_\nu = \frac{1}{2}(a_\nu + b_\nu)(1+i)$$

Set $\partial T = \Gamma \setminus E$ and notice that the construction of Fatou sectors gives

$$U^*(x) \leq \lambda \quad : \quad x \in T \tag{1}$$

In $\mathfrak{Im}(z) > 0$ we have the analytic function $G(z) = (U + iV)^2$. By hypothesis U $y \mapsto G(x+iy)$ decreases quite rapidly which gives a vanishing complex line integral:

$$\int_\Gamma G(z)dz = 0$$

Now Γ is the union of E and the union of of the broken lines which give the two sides of the T_ν-triangles. Let ∂T denote the union of these broken lines. Since the complex differential $dz = dx + idy$ the real part of the complex line integral is zero which gives

$$\int_E (U^2 - V^2) \cdot dx + \int_{\partial T} (U^2 - V^2) \cdot dx - 2 \cdot \int_{\partial T} U \cdot V dy \tag{2}$$

On the sides of the T-triangles the slope is plus or minus $\pi/4$ and hence $|dy| = |dx|$ where $|dx| = dx$ is positive. Hence the he inequality $2ab \leq a^2 + b^2$ for any pair of non-negative numbers gives:

$$2 \cdot \Big| \int_{\partial T} UV dy \Big| \leq \int_{\partial T} U^2 \cdot dx + \int_{\partial T} V^2 \cdot dx \tag{3}$$

Since (2) is zero we see that (3) and the triangle inequality give:

$$\int_E V^2 \cdot dx \leq \int_E U^2 \cdot dx + 2 \cdot \int_{\partial T} U^2 \cdot dx \tag{4}$$

Next, put

$$V_\lambda^+ = \{x : |V(x)| > \lambda\}$$

Then (4) gives:

$$\mathfrak{m}(V_\lambda^+ \cap E) \leq \frac{1}{\lambda^2} \cdot \int_E V^2 \cdot dx \leq \frac{1}{\lambda^2} \cdot \int_E U^2 \cdot dx + \frac{2}{\lambda^2} \cdot \int_{\partial T} U \cdot dx \tag{5}$$

Next, Since the integral $\int_{T_\nu} dx = (b_\nu - a_\nu)$ for each ν and (1) holds we have

$$\frac{2}{\lambda^2} \cdot \int_{\partial T} U^2 \cdot dx \leq 2 \cdot \sum (b_\nu - a_\nu) = 2 \cdot \mathfrak{m}(J_\lambda) \tag{6}$$

Using the set-theoretic inclusion $V_\lambda^+ \subset (V_\lambda^+ \cap E_\lambda) \cup J_\lambda$ it follows after adding $\mathfrak{m}(J_\lambda)$ on both sides in (5):

$$\mathfrak{m}(V_\lambda^+) \leq 3 \cdot \mathfrak{m}(J_\lambda) + \frac{1}{\lambda^2} \cdot \int_E U^2 \cdot dx \tag{6}$$

Finally, $U \leq U^*$ holds on E and since E is the complement of J_λ we have $E = \{x : U^*(x) \leq \lambda\}$. Now we apply general integral formulas which after integration over $\lambda \geq 0$ gives

$$\int |V(x) \cdot dx = 3 \cdot \int U^*(x) \cdot dx + \int_0^\infty \frac{1}{\lambda^2} \Big[\int_{(U^* \leq \lambda} (U^*)^2 \cdot dx \Big] \cdot d\lambda$$

By the integral formula from XX the last term is equal to $\int U^*(x) \cdot dx$ and Theorem 7.1 follows.

8. The Hardy space on R

Consider an analytic function $F(z)$ in the upper half-plane whose boundary value function $F(x)$ on the real line is integrable. This class of analytic functions in $\mathfrak{Im}\, z > 0$ is denoted by $H^1(\mathbf{R})$. To each such F we introduce the non-tangential maximal function

$$(*) \qquad F^*(x) = \max_{z \in \mathcal{F}(x)} |F(z)|$$

where $\mathcal{F}(x)$ is the Fatou sector of points $z = \xi + i\eta$ for which $|\xi - x| \leq \eta$. With these notations one has

8.1 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty}^\infty |F^*(x)| \cdot dx \leq C \cdot \int_{-\infty}^\infty |F(x)| \cdot dx$$

To prove this we shall first study harmonic functions and reduce the proof of Theorem 8.1 to a certain L^2-inequality. To begin with, let $u(x)$ is a real-valued function on the x-axis such that the integral

$$\int_{-\infty}^\infty \frac{|u(x)|}{1+x^2} \cdot dx < \infty$$

The harmonic extension to the upper half-plane becomes:

$$U(x+iy) = \frac{1}{\pi} \cdot \int_{-\infty}^\infty \frac{y}{(x-t)^2 + y^2} \cdot u(t) \cdot dt$$

The non-tangential maximal function is defined by:

$$(*) \qquad U^*(x) = \max_{z \in \mathcal{F}(x)} |U(z)|$$

When $u(x)$ belongs to $L^2(\mathbf{R})$ it turns out that one there is an L^2-inequality.

8.2 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty}^\infty (U^*(x))^2 \cdot dx \leq \int_{-\infty}^\infty u^2(x) \cdot dx$$

for every L^2-function on the x-axis.

In 8.X below we show how Theorem 8.2 gives Theorem 8.1. The proof of Theorem 8.2 relies upon a point-wise estimate of U via the Hardy-Littlewood maximal function of u. Let us first consider a function $u(x)$ supported by $x \geq 0$ such that the function

$$t \mapsto \frac{1}{t} \int_0^t |u(x)| \cdot dx$$

is bounded on $(0, +\infty)$. Let $u^M(0)$ denote this supremum over t. Then one has

8.3 Proposition. *For each $z = x + iy$ in the upper half-plane one has*

$$|U(x+iy)| \leq (1 + \frac{|x|}{2y}) \cdot u^M(0)$$

Proof. Since the absolute values $|U(x+iy)|$ increase when u is replaced by $|u|$ we may assume that $u \geq 0$ from the start. Put

$$\Phi(t) = \int_0^t u(x) \cdot dx$$

which yields a primitive of u and a partial integration gives

$$U(x+iy) = \lim_{A \to \infty} \frac{1}{\pi} \cdot \Big| \frac{y}{(x-t)^2 + y^2} \cdot \Phi(t) \Big|_0^A + \lim_{A \to \infty} \frac{2}{\pi} \cdot \int_0^A \frac{y(t-x)}{((x-t)^2 + y^2)^2} \cdot \Phi(t) \cdot dt$$

With (x, y) kept fixed the finiteness of $u^M(0)$ entails that $t^{-2} \cdot \Phi(t)$ tends to zero with A and there remains

$$U(x+iy) = \frac{2}{\pi} \cdot \int_0^\infty \frac{y(t-x)}{((x-t)^2 + y^2)^2} \cdot \Phi(t) \cdot dt$$

Now $\Phi(t) \leq u^M(0) \cdot t$ gives the inequality

$$U(x+iy) = \frac{2u^M(0)}{\pi} \cdot \int_0^\infty \frac{y(t-x) \cdot t}{((x-t)^2 + y^2)^2} \cdot dt$$

To estimate the integrand we notice that it is equal to

$$\frac{y}{((x-t)^2 + y^2)} + \frac{y(t-x)x}{((x-t)^2 + y^2)^2}$$

The Cauchy-Schwarz inequality gives

$$\Big| \frac{2y(t-x)x}{((x-t)^2 + y^2)^2} \Big| \leq \frac{|x|}{(x-t)^2 + y^2}$$

It follows that

$$|U(x+iy)| \leq \frac{2u^M(0)}{\pi} \cdot \int_0^\infty \frac{y}{(x-t)^2 + y^2)} + \frac{u^M(0) \cdot |x|}{\pi} \cdot \int_0^\infty \frac{1}{(x-t)^2 + y^2} \cdot dt$$

The last sum of integrals is obviously majorised by $u^M(0)(1 + \frac{|x|}{2y})$ and Proposition XX is proved.

8.4 General cae. If no constraint is imposed on the support of u it is written as $u_1 + u_2$ where u_1 is supported by $x \leq 0$ and u_2 b $x \geq 0$. Here we consider the maximal function

$$u^M(0) = \max_t \frac{1}{2t} \int_{-t}^{t} |u(x)| \cdot dx$$

Exactly as above the reader may verify that

$$\text{(i)} \qquad |U(x,y)| \leq u^M(0)(2 + \frac{|x|}{y})$$

In the Fatou sector at $x = 0$ we have $x \leq |y|$ and hence (i) gives

$$U^*(0) \leq\leq 3 \cdot u^M(0)$$

After a translation with respect to x a similar inequality holds. More precisely, put

$$u^M(x) = \max_t \frac{1}{2t} \int_{-t}^{t} |u(x+s)| \cdot ds$$

for every x, Then we have

$$U^*(x) \leq 3\dot{u}^M(x)$$

Now we apply the Hardy-Littlewood inequality from XX for the L^2-case and obtain the conclusive result:

8.5 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty>}^{\infty} U^*(x)^2 \cdot dx \leq C \cdot \int_{-\infty>}^{\infty} u^2(x) \cdot dx$$

for every L^2-function u on the real line.

8.6 Proof of Theorem 8.1 We use a factorisation via Blaschke products which enable us to write

$$F(z) = B(z) \cdot g^2(z)$$

where $g(z)$ is a zero-free analytic function in the upper half-plane. Since $|B(z)| \leq 1$ holds in $\mathfrak{Im}(z) > 0$ we have trivially

$$F^*(x) \leq g^*(x)^2$$

On the real axis we have $|F(x)| = |g(x)|^2$ almost everywhere so the L^1-norm of F is equal to the L^2-norm of g. Next, with $g = U + iV$ we have a pair of harmonic functions and since $|g|^2 = U^2 + V^2$ we can apply Theorem 8.5 to each of these harmonic functions and at this stage we leave it to the reader to confirm the assertion in Theorem 8.1

8.7 Carleson measures

Let $F(z)$ be in the Hardy space $H^1(\mathbf{R})$. If $\lambda > 0$ we put

$$J_\lambda = \{F^*(x) > \lambda\}$$

We assume that the set is non-empty and hence this open set is a union of disjoint intervals $\{(a_k,b_k)\}$. To each interval we construct the triangle T_k with corners at the points a_k,b_k and $p_k=\frac{1}{2}(a_k+b_k)+\frac{i}{2}(b_k-a_k)$. Put

$$\Omega=\cup T_k$$

Exercise. Use the construction of Fatou sectors and the definition of F^* to show that

$$\{|F(z)|>\lambda\}\subset\Omega \tag{1}$$

Let us now consider a non-negative Riesz measure μ in the upper half-plane. For the moment we assume that μ has compact support and that $F(z)$ extends to a continuos function on the closed upper half-plane This is to ensure that various integrals exists but does not affect the final a priori inequality in Theorem X below. General formulas for distribution functions give:

$$\int|F(z)|\cdot d\mu(z)=\int_0^\infty\lambda\cdot\mu(\{|F(z)|>\lambda\})\cdot d\lambda \tag{2}$$

To profit upon (1) we impose a certain norm on μ. To each x and every h we construct the triangle $T_x(a)$ standing on the interval $(x-a/2,x+a/2]$.

8.8 Definition. *The Carleson norm of μ is defined as smallest constant C such that*

$$\mu(T_x(a)\leq C\cdot a$$

hold for all pairs $x\in\mathbf{R}$ and $a>0$ and is denoted by $\mathfrak{car}(\mu)$.

8.9 Application. Given μ with its Carleson norm the inclusion (1) gives

$$\mu(\{|F(z)|>\lambda\})\leq\sum\mu(T_k)\leq\mathfrak{car}(\mu)\cdot\sum(b_k-a_k) \tag{i}$$

The last sum is the Lebesgue measure of $\{F^*>\lambda\}$ and hence the right hand side in (i) is estimated above by

$$\mathfrak{car}(\mu)\cdot\int_0^\infty\lambda\cdot\mathfrak{m}(\mu(\{F^*>\lambda\})\cdot d\lambda=\mathfrak{car}(\mu)\cdot\int_{-\infty}^\infty F^*(x)\cdot dx \tag{ii}$$

Together with Theorem 8.1 we arrive at the conclusive result:

8.10 Theorem. *There exists an absolute constant C such that*

$$\int|F(z)|\cdot d\mu(z)\leq C\cdot\mathfrak{car}(\mu)\cdot\int_{-\infty}^\infty|F(x)|\cdot dx$$

hold for each $F\in H^1(\mathbf{R})$ and every non-negative Riesz measure μ in the upper half-plane.

9. BMO and radial norms of measures

Theorem 0.4 together with the preceeding description of the dual space of $\mathfrak{Re}\,H_0^1(T)$ implies that every BMO-function F can be written as a sum

$$F=\phi+v^* \tag{i}$$

where ϕ is bounded and v^* is the harmonic conjugate of a bounded function. However, this decomposition is not unique. A *constructive* procedure to find a pair u, v in for a given BMO-function F was given by P. Jones in [Jones]. See also the article [Carleson] from 1976.

9.1 Radial norms on measures. Let D be the unit disc. An L^1-function $u(z)$ in D is radially bounded if there exists a constant C such that

$$(*) \qquad \frac{1}{\pi} \cdot \iint_{S_h} |u(z)| \cdot dxdy \leq C \cdot h$$

for each sector

$$S_h = \{z \colon \theta - h/2 < \arg z\theta + h/2\} \quad : \quad h > 0$$

The smallest C for which (*) holds is denoted by $|u|^*$. Notice that $|u|^*$ in general is strictly larger than the L^1-norm over D which occurs when we take $h = \pi$ above. If u satisfies (*) we define a function P_u on the unit circle by

$$P_u(\theta) = \frac{1}{\pi} \cdot \iint_D \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot u(z) \cdot dxdxy$$

With these notations Fefferman proved:

9.2 Theorem *There exists an absolute constant C such that*

$$|P_u|_{\text{BMO}} \leq C \dot{|} u|^*$$

Thus, $u \mapsto P_u$ sends radially bounded $L^1(D)$-functions to BMO(T). The proof of Theorem 8.1 relies upon Theorem 0.4 and the following observation:

9.3 Exercise. Show that when u is radially bounded and $H(z)$ is a harmonic function in D with continuous boundary values on T then

$$\iint_D H(z) \cdot u(z) \cdot dxdy = \int_0^{2\pi} H(e^{i\theta}) \cdot P_u(\theta) \cdot d\theta$$

The following result is also due to Fefferman:

9.4 Theorem. *Let $F(\theta) \in BMO(T)$. Then there exists a radially bounded $L^1(D)$-function u and some $s(\theta) \in H^\infty(T)$ such that*

$$F(\theta) = s(\theta) + P_u(\theta)$$

For detailed proofs of the results above we refer to Chapter XX in [Koosis].

Analytic extensions across a boundary.

Proof. First we show that the divergence in (*) entails that $A_\omega(\Box)$ is dense in $A^*_\omega(\Box)$. To prove this we consider a Riesz measure μ supported by $\overline{\Box}$ where $\omega(y)\cdot\mu$ is $\perp$ to $A_\omega(\Box)$. It remains to show that

$$(1) \qquad \int f\,d\mu = 0 \quad : f \in A^*_\omega(\Box)$$

To get this we proceed as follows. For each complex number ζ $e^{i\zeta z} \in A^*_\omega(\Box)$. It follows that

$$0 = \int_{y\geq 0} e^{i\zeta z}\cdot\omega(y)\,d\mu(z) + \int_{y<0} e^{i\zeta z}\cdot\omega(y)\,d\mu(z) = F_+(\zeta) + F_-(\zeta)$$

Here F_+ and F_- are entire functions of exponential type. If $\zeta = \xi$ is real and non-negative we have

$$\text{(i)} \qquad |F_+(\xi)| \leq \int_{y\geq 0} e^{-\xi y}\cdot\omega(y)\,|d\mu(z)| \leq \max_{0\leq y\leq b}\,\omega(y)\cdot||\mu||$$

Hence $F_+(\xi)$ is bounded when $\xi \geq 0$, and in the same way $F_-(\xi)$ is bounded when $\xi \geq 0$. Since $F_+(\xi) + F_-(\xi) = 0$ for all real ξ we conclude that $F_+(\xi)$ is bounded on the whols real ξ-line. Consider the function

$$h(y) = -\log\omega(y) \quad : y > 0$$

and introduce the lower Legendre envelope defined for $\xi > 0$ by

$$k(\xi) = \min_{0<y\leq b}\, h(y) + \xi y$$

Then we have

$$\text{(ii)} \qquad e^{-\xi y}\cdot\omega(y) = e^{-\xi y + \log\omega(y)} \leq e^{-k(\xi)} \quad : 0 < y \leq 1$$

Hence (i) gives

$$\text{(iii)} \qquad |F_+(\xi) \leq e^{-k(\xi)}\cdot||\mu|| \quad : \xi > 0$$

At this stage we use a wellknown general result about Legendre envelopes which entails that

$$\text{(iv)} \qquad \int_1^\infty \frac{k(\xi)}{\xi^2}\,d\xi = +\infty$$

Hence (iii) gives

$$\log^+\frac{1}{|F_+(\xi)|} \geq k(\xi) - \log\,||\mu||$$

hence (ii) entials that

$$\text{(iv)} \qquad \int_1^\infty \log^+\frac{1}{|F_+(\xi)|}\cdot\frac{d\xi}{\xi^2} = +\infty$$

At the same time F_+ is an entire function of exponential type which is bounded on the real axis and therefore belongs to the Carleman class $\mathcal{N}$ defined in § xx and Thoerem § xx implies that F_+ is identically zero. This means that the restriction of $\omega(y)\cdot\mu$ to the closed rectangle $\overline{\Box}_+$ is $\perp$ to functions of the form $z \mapsto e^{i\zeta z}$: $\zeta \in \mathbf{C}$.

In the same way one verifies the restriction to the lower rectangle has this property. There remains to show that this gives the vanishing in (1). To achieve this we consider first the upper triangle and when $0 < \epsilon < 1$ we define

$$T_\epsilon(z) = -i/2 + (1-\epsilon)(z - i/2)$$

If $f \in A^*_\omega(\square)$, a picture shows that $f(T_\epsilon(z)$ is analytic in the closure of $\square_+$. By Runge's theorem it can be uniformly approximated by exponential polynomials $\{e^{iz\zeta}\}$ on $\overline{\square^*}$ and the vanishing of F_+ entails that

$$\text{(v)} \qquad \int_{\square^*} f(T_\epsilon(z)) \cdot \omega(y)\, d\mu(z) = 0$$

Next, we notice that there exists a constant C which is independent of ϵ such that

$$\text{(v)} \qquad \max_{x+iy \in \square^*} \omega(y) \cdot |f(T_\epsilon(x+iy))| \leq C \cdot ||f||_\omega$$

Finally,

$$\lim_{\epsilon \to 0} f(T_\epsilon(z)) = f(z)$$

holds pointwise in the open square $\square^*$, and by (v) we can apply dominated convergence while we take μ-integrals. Hence the vanishing integrals in (v) imply that

$$\text{(vi)} \qquad \int_{\square^*} f(z)) \cdot \omega(y)\, d\mu(z)$$

In the same way one proves that the integral over $\square_*$ is zero which proves (1).

The case when (*) is $> -\infty$. There remains to show that when this holds, then there exist some $f \in A^*_\omega(\square)$ which cannot be uniformly approximated by functions from $A_\omega(\square)$. CONTINUE proof ....