6. Interpolation and solutions to the $\bar\partial$-equation.

Contents.

I. Carleson's Interpolation Theorem

Introduction. Let $U = \mathfrak{Im}(z) > 0$ be the upper half-plane. Denote by $\mathbf{c}_*$ the family of sequences of complex numbers $\{c_\nu\}$ where every $|c_\nu| \leq 1$. A sequence $z_\bullet = \{z_\nu\}$ in U has a finite interpolation norm if there exists a constant K such that for every sequence $\{c_\nu\}$ in $\mathbf{c}_*$ we can find an analytic function $f(z)$ in U where

$$(*) \qquad f(z_\nu) = c_\nu \quad : \quad \nu = 1, 2, \ldots \quad \text{and} \quad |f|_U \leq K$$

The least constant K above is denoted by $\mathfrak{int}(z_\bullet)$ and called the interpolation norm of $\{z_\nu\}$.

0.1 Theorem. *A sequence $z_\bullet$ is interpolating if*

$$\min_\nu \prod_{k \neq \nu} \left|\frac{z_\nu - z_k}{z_\nu - \bar z_k}\right| > 0$$

Moreover, if $\delta(z_\bullet)$ denotes the minimum above then

$$(2) \qquad \mathfrak{int}(z_\bullet) \leq \frac{4A}{\delta(z_\bullet)} \cdot \text{Log}\, \frac{1}{\delta(z_\bullet)}\,.$$

where A is an absolute constant.

Remark. Theorem 0.1 gives a sufficient condition in order that a sequence is interpolating. That the condition (1) also is *necessary* is easily verified. See Exercise XX below. In [Ca] the proof is carried out in the unit disc D where the companion to Theorem 0.1 is that a sequence $\{z_\nu\}$ is interpolating if and only

$$(3) \qquad \min_\nu \prod_{k \neq \nu} \frac{|z_\nu - z_k|}{|1 - \bar z_k \cdot z_\nu|} > 0$$

for every ν. After a conformal map one verifies that (1) and (3) give equivalent conditions. Here we prove Carleson's result in the upper half-plane since various constructions below become a bit more transparent as compared to the unit disc.

Let $\{z_\nu\}$ be a sequence with a positive δ-number. Since a set of analytic functions in U with a uniform upper bound for the maximum norm is a normal family it is sufficient to prove the requested interpolation by bounded functions for every finite subsequence. The Nevanlinna-Pick theorem assigns to each finite sequence $\{z_\nu\}$ and every sequence $\{c_\nu\}$ a unique interpolating analytic function $F(z)$ with smallest maximum norm. So Carleson's result gives a uniform bound in the Nevannlinna-Pick interpolation.

0.1 Carleson measures.

The main ingredient in the proof is to consider a certain class of non-negative measures in the upper half-plane $\mathfrak{Im}\, z > 0$. For every $h > 0$ we denote by $\mathfrak{square}(h)$ the family of squares of the form

$$\square = \{(x, y) \quad : \quad x_0 - h/2 < x < x_0 + h/2 \quad : 0 < y < h\} \quad : x_0 \in \mathbf{R}$$

0.2. Definition. *A non-negative measure μ in U is called a Carleson measure if there exists a constant Ksuch that*

$$\mu(\square) \leq K \cdot h \quad : \quad \square \in \mathfrak{square}(h) \quad : 0 < h < \infty$$

The least constant K is denoted by $\mathfrak{car}(\mu)$ and called the Carleson norm of μ.

An essential step during the proof Theorem 0.1 is the following inequality:

0.3 Theorem. *Let $\{z_\nu\}$ be a sequence with a positive δ-number. Then*

$$\mathfrak{car}\big(\sum_{\nu=1}^{\nu=\infty} \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu}\big) \leq 2 \cdot \text{Log}\, \frac{1}{\delta)z_\bullet)}$$

where $\{\delta_{z_\nu}\}$ denote Dirac measures.

Use of duality. Once Theorem 0.3 is established the interpolation theorem follows by a duality argument where the Hardy space $H^1(\mathbf{R})$ appears. Namely, to each $h \in H^1(\mathbf{R})$ we associate the maximal function h^* and the following inequality is proved in Section 2:

0.4 Theorem. *Let μ be a Carleson measure in the upper half-plane. Then*

$$\int_U |h(z)| \cdot d\mu(z) \leq \mathfrak{car}(\mu) \cdot ||h^*||_1 \quad : h \in H^1(\mathbf{R})$$

Armed with Theorem 0.3 and 0.4 the interpolation theorem is derived in section 3 below.

1. Proof of Theorem 0.3

First we establish an inequality which is attributed to L. Hörmander.

1.1 Lemma *Let $z_1, \ldots, z_N$ be a finite sequence in U and put $\delta = \delta(z_\bullet)$. Then*

(i)
$$\sum_{\nu \neq k} \mathfrak{Im}(z_k) \cdot \frac{\mathfrak{Im}(z_\nu)}{|z_k - \bar{z}_\nu|^2} \leq \frac{1}{2} \cdot \text{Log}\, \frac{1}{\delta} \quad : \quad 1 \leq k \leq N$$

Proof. The left hand side as well as the δ-norm of the z-sequence are unchanged if we translate all points to $z_\nu + a$ where a is a real number. Similarly, the δ-norm and the left hand side in (i) are both unchanged when the sequence is replaced by $\{A \cdot z_\nu\}$ for some $A > 0$. To prove (i) for a fixed k which we may take $k = N$ and it suffices to consider the case when $z_N = i$. Put $z_\nu = a_\nu + ib_\nu$ when $1 \leq \nu \leq N - 1$. Then we must show

(i)
$$\sum_{\nu=1}^{\nu=N-1} \frac{b_\nu}{(1 + b_\nu)^2 + a_\nu^2} \leq \frac{1}{2} \cdot \text{Log}\, \frac{1}{\delta}$$

Notice that

(iii)
$$\frac{|i - \bar{z}_\nu|^2}{|i - z_\nu|^2} = \frac{(1 + b_\nu)^2 + a_\nu^2}{(1 - b_\nu)^2 + a_\nu^2}$$

Next, by inverting the δ-we have:

(iii)
$$\prod_{\nu=1}^{\nu=N-1} \frac{(1 + b_\nu)^2 + a_\nu^2}{(1 - b_\nu)^2 + a_\nu^2} \leq \delta^{-2}$$

Passing to the Log-functions it follows that

$$\text{(iv)} \qquad \sum_{\nu=1}^{\nu=N-1} \log\Big[\,\frac{(1+b_\nu)^2+a_\nu^2}{(1-b_\nu)^2+a_\nu^2}\,\Big] \leq 2\cdot\log\frac{1}{\delta}$$

Next, for each ν we have the integral formula

$$\log\,\frac{(1+b_\nu)^2+a_\nu^2}{(1-b_\nu)^2+a_\nu^2} = \int_{-b_\nu}^{b_\nu} \frac{2(1+s)\cdot ds}{(1+s)^2+a_\nu^2}$$

Apply this with (a_ν, b_ν) and after a summation over ν the inequality (iv) gives (i) in Lemma 1.1.

Final part of the proof of Theorem 0.3

If $z_\bullet \in \mathcal{S}(\delta)$ and a is any real number then the translated sequence $z_\bullet + a = \{z_\nu + a\}$ also belongs to $\mathcal{S}(\delta)$. Since Theorem 0.3 asserts an a priori estimate we may assume that $\square$ is centered at $x = 0$, i.e.

$$\square = \{(x,y): \; -h/2 < x < h/2 \text{ and } 0 < y < h\}$$

There remains to estimate

$$\text{(i)} \qquad \sum_{z_\nu \in \square} \mathfrak{Im}\, z_\nu$$

Set

$$y^* = \max\{\mathfrak{Im}(z_\nu): \; z_\nu \in \square\}$$

Let k give the equality $y^* = \mathfrak{Im}(z_k)$. With $z_k = x_k + iy^*$ and $z_\nu = x_\nu + iy_\nu \in \square$ we have

$$|z_k - \bar{z}_\nu|^2 = (x_k - x_\nu)^2 + (y^* - y_\nu)^2 \leq h^2 + (y^*)^2 \implies$$

$$\frac{\mathfrak{Im}(z_k)}{|z_k - \bar{z}_\nu|^2} \geq \frac{y^*}{h^2 + (y^*)^2} \quad : \nu \neq k$$

Next, notice that

$$y^* \geq h/2 \implies \frac{y^*}{h^2 + (y^*)^2} \geq \frac{1}{5h}\,.$$

Lemma 1.1. applied with $\nu = k$ gives therefore

$$\text{(ii)} \qquad \sum_{z_\nu \in \square} \mathfrak{Im}(z_\nu) \leq y^* + \frac{5h}{2}\cdot \text{Log}\,\frac{1}{\delta} \leq h\cdot\big(1 + \frac{5}{2}\cdot\text{Log}\,\frac{1}{\delta}\big)$$

So if $y^* \geq h/2$ we are done. Suppose now that $y^* < h/2$ and regard the cubes:

$$\square_1 = \{-h/2 < x < 0 \,\text{and}\; 0 < y < h/2\} \quad \square_2 = \{0 < x < h/2 \text{ and } 0 < y < h/2\}$$

We want to estimate

$$S_1 + S_2 = \sum_{z_\nu \in \square_1} \mathfrak{Im}(z_\nu) + \sum_{z_\nu \in \square_2} \mathfrak{Im}(z_\nu)$$

We have also two sequences:

$$\{z_\nu : \; z_\nu \in \square_1\} \quad \text{and} \quad \{z_\nu : \; z_\nu \in \square_2\}$$

Since all factors defining the δ-norm are ≤ 1 these two smaller sequences both belong to $\mathcal{S}(\delta)$. Suppose that:

$$y_1^* = \max_{z_\nu \in \square_1} \mathfrak{Im}(z_\nu) \geq \frac{h}{4}$$

When this holds we obtain exactly as above:

$$S_1 \leq \frac{h}{2}\cdot 2\cdot \text{Log}\,\frac{1}{\delta}$$

If $y_1^* < \frac{h}{4}$ we continue to split the cube $\square_1$. In a similar fashion we treat the sequence which stays in $\square_2$. After a finite number of steps we get the required inequality in Theorem 0.3.

2. Proof of Theorem 0.4

Let $h \in H^1(\mathbf{R})$ and recall that its maximal function is defined by

(i) $$h^*(t) = \max\, |h(x+iy)| \quad : \quad |x-t| < y$$

To each $\lambda > 0$ we consider the open subset on the real line defined by $\{h^* > \lambda\}$. It is some union of disjoint intervals $\{(a_j, b_j)\}$ and (i) gives the set-theoretic inclusion:

(ii) $$\{|h(x+iy)| > \lambda\} \subset \cup\, T_j \quad :$$

where T_j is the triangle side standing on the interval (a_j, b_j) as explained in XXX. (Hardy space). In particular we have the inclusion:

(iii) $$T_j \subset \square(a_j, b_j) = \{x + iy \,:\, |x - \frac{a_j + b_j}{2}| < b_j - a_j \;:\; 0 < y < b_j - a_j\}$$

See figure XXX. So if μ is a positive measure in U we obtain:

(iv) $$\mu(\{|h| > \lambda\}) \leq \sum \mu(T_j) \leq \sum \mu(\square(a_j, b_j))$$

If μ is a Carleson measure the right hand side is estimated by

(v) $$\mathfrak{car}(\mu) \cdot \sum (b_j - a_j) = \mathfrak{car}(\mu) \cdot \mathfrak{m}(\{h^* > \lambda\})$$

where $\mathfrak{m}$ refers to the 1-dimensional Lebesgue measure. Here (v) holds for every $\lambda > 0$. So by the general inequality for distribution functions from XXX we get:

$$\int_U |h| \cdot d\mu \leq \mathfrak{car}(\mu) \cdot ||h^*||_1$$

This finishes the proof of Theorem 0.4.

3. Proof of Theorem 0.1.

As explained in XX the Banach space $H^1(\mathbf{R})$ contains a dense subspace of functions $h(z)$ with polynomial decay at infinite, i.e. functions in the Hardy space for which

$$|h(z)| \leq C_N \cdot (1 + |z|)^{-N} \quad : \text{hold for some constant } C_N \quad : N = 1, 2, \ldots$$

This is used below to ensure that various integrals are defined where it suffices to to use "nice" functions while an *a priori* inequality is established. Consider a finite sequence $z_1, \ldots, z_N$ in U and a finite sequence $c_1, \ldots, c_N$ in $\mathbf{c}_*$. Newton's interpolation gives a unique polynomial $P(z)$ of degree $N-1$ such that:

(i) $$P(z_k) = c_k \quad : \quad 1 \leq k \leq N$$

Let $B(z)$ be the Blascke product associated to the z-sequence:

(ii) $$B(z) = \prod_{\nu=1}^{\nu=N} \frac{z - z_\nu}{z - \bar{z}_\nu}$$

Let $h \in H^1(\mathbf{R})$ have the polynomial decay $\geq N+2$. Residue calculus gives

(iii) $$\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx = \sum_{k=1}^{k=N} \frac{c_k}{B'(z_k)} \cdot h(z_k)$$

If k is fixed we have

(iv) $$\frac{1}{B'(z_k)} = \prod_{\nu \neq k} \frac{z_k - \bar{z}_\nu}{z_k - z_\nu} \cdot 2 \cdot \mathfrak{Im}(z_k)$$

It follows that

(v) $$\Big|\frac{1}{B'(z_k)}\Big| \leq \frac{2}{\delta(z_\bullet)} \cdot \mathfrak{Im}(z_k)$$

Since $\{c_\nu\} \in \mathbf{c}_*$ we see that (v) and the triangle inequality applied to (iii) give:

$$(*) \qquad \Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx \Big| \leq \frac{2}{\delta(z_\bullet)} \cdot \sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k)$$

Now Theorem 0.4 gives the inequality

$$(5) \qquad \sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k) \leq \mathfrak{car}(\sum \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu} \cdot ||h^*||_1$$

Use of duality. Let us put

$$C_\delta = \frac{2}{\delta(z_\bullet)} \cdot 2 \cdot \log \frac{1}{\delta(z_\bullet)}$$

Then (5) and Theorem 0.3 Let C_δ be the constant from Theorem 0.3 together with (*) give

$$\Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx \Big| \leq C \cdot ||h^*||_1$$

Next, the result in (Hardy Chapter) gives an absolute constant A such that

$$||h^*||_1 \leq A \cdot ||h||_1$$

Hence the densely defined linear functional

$$h \mapsto \int_{-\infty}^{\infty} \frac{P(x)}{B(x)} \cdot h(x) \cdot dx$$

has norm $\leq C \cdot A$. The Duality Theorem from XXX implies that if $\epsilon > 0$, then there exists some $G(z) \in \mathcal{O}(U)$ such that the maximum norm

$$(6) \qquad \Big|\frac{P(x)}{B(x)} - G(x)\Big|_U < A \cdot C_\delta + \epsilon$$

Since $B(x)$ is a Blaschke product we have $|B(x)| = 1$ almost everywhere and hence (6) gives:

$$\Big|P(x) - B(x) \cdot G(x)\Big| < A \cdot C_\delta + \epsilon$$

Now $f(z) = P(z) - B(z)G(z)$ is analytic in U and since $B(z_\nu) = 0$ for every ν we have

$$f(z_\nu) = P(z_\nu) = c_\nu$$

So the bounded analytic function $f(z)$ interpolates and since $\epsilon > 0$ can be arbitrary small and $c_1, \ldots, c_N$ was an arbitrary sequence in $\mathbf{c}_*$ we conclude that the interpolation norm of the finite sequence $z_1, \ldots, z_N$ is at most $A \cdot C_\delta$. Since this uniform bound holds for all N we get

$$\mathfrak{int}(z_\bullet) \leq A \cdot C_\delta$$

which finishes the proof of Theorem 0.1.

Exercise. Prove that (1) in the Interpolation Theorem is necessary.

II. Wolff's Theorem.

Introduction. The Pompieu formula solves the inhomogeneous $\bar\partial$-equation in the unit disc D. So if $h(z)$ is a C^∞-function defined in some open neighborhood of the closed disc there exists a C^∞-function v such that

(*) $$\bar\partial(v)(z)=h(z)\quad:\quad z\in D$$

We seek conditions in order that (*) has a solution v whose maximum norm over D is controlled by some extra properties of h. Conditions of this kind were imposed in [Wolff] which we begin to explain. To every C^∞-function h on $\bar D$ we define a pair of non-negative functions:

(**) $$\mu_h(z)=\text{Log}\,\frac{1}{|z|}\cdot|\partial(h)(z)|\quad:\quad\nu_h(z)=\text{Log}\,\frac{1}{|z|}\cdot|h(z)|^2$$

Wolff's condition is expressed in terms of Carleson norms on μ_h and ν_h. Before we announce Theorem 0.4 we recall the following.

0.1 Carleson measures in D**.** Consider the family of sector domains defined for all pairs $0<h<1$ and $0\leq\theta\leq 2\pi$ by:

$$S_h(\theta)=\{\,z=r\cdot e^{i\phi}\;:1-h<r<1:|e^{i\phi}-e^{i\theta}|\leq\frac{h}{2}\,\}$$

0.2 Definition. *A non-negative measure μ in D is called a Carleson measure if there exists a constant K such that*

$$\iint_{S_h(\theta)}d\mu\leq K\cdot h\quad:\quad 0<h>1\quad:\quad 0\leq\theta<2\pi$$

The least constant K is denoted by $\mathfrak{car}(\mu)$ and called the Carleson norm of μ.

0.3 An inequality. Exactly as in the upper half-plane there exists an absolute constant A such that the following holds for every pair of a Carleson measure μ in D and a function $f(z)$ in the Hardy space $H^1(T)$

$$\iint_D\big|f(z)\big|\cdot d\mu(z)\leq A\cdot\mathfrak{car}(\mu)\cdot||f||_1$$

0.4 Theorem. *Let A be as in (0.3). For every C^∞-function $h\in C^\infty(\bar D)$ the equation (*) has a C^∞-solution v_* where*

$$\max_\theta|v_*(e^{i\theta})|\leq 2\cdot A\cdot\mathfrak{car}(\mu_h)+2\cdot\sqrt{A\cdot\mathfrak{car}(\nu_h)}$$

For the proof we need an integral formula due to Jensen.

0.5 The Fourier-Jensen formula. *Let $F(z)$ be an analytic function in D with a simple zero at $z=0$ and otherwise it is $\neq 0$. Then one has the equality:*

(*) $$\iint_D\text{Log}\,\frac{1}{|z|}\cdot\frac{|F'(z)|^2}{|F(z)|}\cdot dxdy=\int_0^{2\pi}|F(e^{i\theta})|\cdot d\theta$$

To prove (*) we set $F(z)=z\cdot G(z)$ where the hypothesis means that G is zero-free so we can construct a square root function and write

(i) $$F(z)=z\cdot\Psi^2(z)\quad:\;\Psi\in\mathcal{O}(D)$$

This implies that

$$\frac{|F'(z)|^2}{|F(z)|}=\frac{|\Psi(z)+2z\cdot\Psi'(z)|^2}{|z|}$$

Hence the left hand side in (*) becomes:

(ii) $$\iint_D\log\frac{1}{|z|}\cdot\big|\Psi(z)+2z\cdot\Psi'(z)\big|^2\cdot\frac{1}{|z|}\cdot dxdy$$

To evaluate this integral we consider the series expansion $\Psi(z) = \sum a_n z^n$. In polar coordinates the double integral becomes

(iii) $$\int_0^1 \int_0^{2\pi} \log \frac{1}{r} \cdot \big|\sum (2n+1) \cdot a_n \cdot r^n \cdot e^{in\theta}\big|^2 \cdot dr d\theta$$

Exercise. Show that (iii) is equal to

$$2\pi \cdot \sum |a_n|^2 = \int_0^{2\pi} |\Psi(e^{i\theta})|^2 \cdot d\theta = \int_0^{2\pi} |F(e^{i\theta})| \cdot d\theta$$

which gives the requested equality (*).

1. Proof of Theorem 0.4

The Pompieu formula gives a solution v to the $\bar{\partial}$-equation

(i) $$\bar{\partial}(v) = h$$

We get new solutions to (i) by $v_* = v - G$ when $G(z)$ are analytic functions in D. So in order to minimize the maximum norm of a solution to (i) we seek a bounded analytic function G_* such that

(ii) $$|v - G_*|_D = \min_G \ |v - G|_D \quad : G \in H^\infty(T)$$

Let m_* be the minimum value in (ii). To estimate m_* we use the duality between $H^\infty(T)$ and $H_0^1(T)$ where $H_0^1(T)$ is the space of functions $F(z)$ in the Hardy space $H^1(T)$ for which $F(0) = 0$. Denote by $S_*^(T)$ the set of functions $F \in H_0^1(T)$ such that

(1) $$\int_0^{2\pi} \big|F(e^{i\theta})\big| \cdot d\theta = 1$$

The duality result from (XX) gives:

(2) $$m_* = \max_F \Big| \int_0^{2\pi} v(e^{i\theta}) \cdot F(e^{i\theta}) \cdot d\theta \Big| \quad : F \in S_*(T)$$

Since $F(0) = 0$ Green's formula shows that (2) becomes

(3) $$\iint_D \operatorname{Log} \frac{1}{|z|} \cdot \Delta(vF) \cdot dxdy$$

Since $\Delta = \partial\bar{\partial}$ and v solves (i) while $\bar{\partial}(F) = 0$, we get

(4) $$\Delta(vF) = 4 \cdot \partial(hF) = 4 \cdot F \cdot \partial(h) + 4 \cdot h \cdot F'$$

Hence we have proved the following

1. Lemma. *One has the equality*

$$m_* = \max_F \Big| \iint_D \operatorname{Log} \frac{1}{|z|} \cdot \big[F \cdot \partial(h) + h \cdot F'\big] \cdot dxdy \Big| \quad : \quad F \in S_*^1(T)$$

To profit upon this expression for m_* we use the Jensen-Nevanlinna factorisation and reduce the estimate to the case when $F(z)$ has a simple zero at $z = 0$ while it is $\neq 0$ in the punctured disc $D \setminus \{0\}$. Thus, consider some F in $S_*(T)$. Since $F(0) = 0$ we there exists the Jensen-Nevanlinna factorisation:

(i) $$F(z) = z \cdot B(z) \cdot G(z)$$

where $B(z)$ is a Blaschke product and G has no zeros in D. Moreover, since $|B| = 1$ holds almost everywhere on T it follows that G belong to $S_*(T)$. Set:

(ii) $$F_1(z) = \frac{z}{2}\big(B(z)-1\big)G(z) \quad \text{and} \quad F_2(z) = \frac{z}{2}\big(B(z)+1\big)G(z)$$

It follows that $F = F_1 + F_2$ and since the maximum norms of $B(z)-1$ and $B(z)+1$ are at most 2 we have

(iii) $$||F_\nu||_1 \leq 1 \quad : \quad \nu = 1,2$$

From (ii) we see that F_1 and F_2 both have a simple zero at the origin and are otherwise $\neq 0$ in the punctured disc. Hence we can apply the Fourier-Jensen formula from (0.4) which gives

(iv) $$[\iint_D \operatorname{Log}\frac{1}{|z|} \cdot \frac{|F_\nu'(z)|^2}{|F_\nu(z)|} \cdot dxdy = \int_0^{2\pi} |F_\nu(e^{i\theta})| \cdot d\theta \leq 1 \quad : \nu = 1,2$$

Final part of the proof. For each $\nu = 1,2$ we set

(1) $$V(F_\nu) = \iint_D \operatorname{Log}\frac{1}{|z|} \cdot |\partial(h)| \cdot |F_1(z)| dxdy + \iint_D \operatorname{Log}\frac{1}{|z|} \cdot |h(z)| \cdot |F_1'(z)| \cdot dxdy$$

By the triangle inequality the right hand side in Lemma 1 is $\leq V(F_1) + V(F_2)$. Let us for example estimate $V(F_1)$. By the inequality (0.3) the first integral in (1) is estimated by

(2) $$A \cdot \mathfrak{car}(\operatorname{Log}\frac{1}{|z|} \cdot |\partial(h)|) \cdot ||F_1||_1$$

Since $||F_1||_1 \leq 1$ the definition of μ_h means that (2) is majorised by

(*) $$A \cdot \mathfrak{car}(\mu_h)$$

To estimate the second integral in (1) we insert $\sqrt{|F_1|}$ as a factor and by the Cauchy-Schwartz inequality this second integral is estimated by the square root of

(3) $$[\iint_D \operatorname{Log}\frac{1}{|z|} \cdot \frac{|F_1'(z)|^2}{|F_1(z)|} \cdot dxdy] \cdot [\iint_D \operatorname{Log}\frac{1}{|z|} \cdot |h(z)|^2 \cdot |F_1(z)| \cdot dxdy$$

In this product the first factor is given by the formula (iv) and is therefore $\leq ||F_1||_1 \leq 1$. Finally, by the definition of the Caelson norm the last factor is majorised by $A \cdot \mathfrak{car}(\nu_h)$. Taking the square root together with (*) above we have proved that

(4) $$V(F_1) \leq +A \cdot \mathfrak{car}(\mu_h) + \sqrt{A \cdot \mathfrak{car}(\nu_h)}$$

The same holds for F_2 and thanks to the factor 2 the requested inequality in Wolff's theorem follows.

III. A class of Carleson measures

Let $f(z)$ be a bounded analytic function in D and associate the non-negative measure in D by:

$$\mu_f = |f'(z)|^2 \cdot \operatorname{Log}\frac{1}{|z|}$$

3.1 Theorem. *There exists an absolute constant A_* such that*

$$\sqrt{\mathfrak{car}(\mu_f)} \leq A_* \cdot |f|_D \quad : \quad f \in H^\infty(D)$$

Proof. By the Heine-Borel Lemma it suffices to prove this for small sectors. Notice also that

$$\operatorname{Log}\frac{1}{[z[} \simeq |1-z|$$

when z approaches the unit circle. By a conformal mapping the proof is therefore reduced to the case when we have a bounded analytic function $f(z)$ defined in a square

$$\square = \{z = x + iy \quad : \quad -1 < x < 1 \quad : 0 < y < 1\}$$

where it suffices to get an absolute constant such that

$$\text{(i)} \qquad \frac{1}{h}\int_{\square_h} y \cdot |f'(x+iy)|^2 \cdot dxdy \leq A \cdot |f|^2_\square \quad : 0 < h < \frac{1}{2}$$

Set $f = u + iv$ which gives $|f'(|^2 = u_x^2 + u_y^2$. The left hand side in (i) becomes:

$$\text{(ii)} \qquad \frac{1}{h}\int_{\square_h} y \cdot (u_x^2 + u_y^2) \cdot dxdy$$

It remains to find an absolute constant A such that (ii) is majorised by $A \cdot |u|^2_\square$. To achive this we replace $\square_h$ by the larger semi-disc

$$D_h = \{z = x + iy \quad : |z| < h \quad : y > 0\}$$

which only with increase the left hand side in (ii). Next, since $4h^2 - |z|^2 \geq 3h^2$ when $z \in D_h$ we get a larger contribution by integrating over the larger semi-disc D_{2h}. Hence it suffices to get an absolute constant A such that

$$(*) \qquad J(h) = \int_{D_{2h}} y(4h^2 - |z|^2) \cdot (u_x^2 + u_y^2) dxdy \leq A \cdot h^3 \cdot |u|^2_\square$$

To get A in (*) we use the equality

$$\Delta(u^2) = 2(u_x^2 + u_y^2)$$

Next, the function $g(x, y) = y(4h^2 - |z|^2)$ is zero on the boundary of D_{2h} and Green's formula gives

$$2 \cdot J(h) = \int_{D_{2h}} u^2 \cdot \Delta(y(4h^2 - |z|^2) \cdot dxdy - \int_{\partial D_{2h}} u^2 \cdot \partial_{\mathbf{n}}(y(4h^2 - |z|^2) \cdot ds$$

Notice that $\Delta(y(4h^2 - |z|^2) = -8y < 0$ in D_{2h} and an easy computation gives

$$-\int_{\partial D_{2h}} u^2 \cdot \partial_{\mathbf{n}}(y(4h^2 - |z|^2) \cdot ds =$$

$$\int_{-2h}^{2h} u^2(x, 0) \cdot (4h^2 - x^2) \cdot dx + \int_0^\pi u^2(2he^{i\theta}) \cdot \sin\theta \cdot \left[-4h^2 + 3 \cdot (2h)^2\right] \cdot h \cdot d\theta$$

Introducing the maximum norm $|u|_\square$ we conclude that

$$2 \cdot J(h) \leq |u|^2_\square \cdot \Big[\int_{-2h}^{2h} (4h^2 - x^2) \cdot dx + \int_0^\pi \sin\theta \cdot \left[-4h^2 + 3 \cdot (2h)^2\right] \cdot h \cdot d\theta\Big]$$

At this stage the reader can evaluate the requested constant A which estimates the last factor by $2A \cdot h^3$.

IV. Berndtsson's $\bar{\partial}$-solution

We announce an inequality due to Bo Berndtsson in [Bern] which has the merit that it is valid for a quite extensive family of domains in $\mathbf{C}$. Here is the set-up : Let $\mathcal{B}$ denote the family of bounded open sets Ω defined by

$$\Omega = \{\rho(z) < 0\}$$

where $\rho(z)$ is a real-valued C^2-function defined in some neighborhood of $\bar{\Omega}$ which satisfies

$$\Delta(\rho)(z) > 0 : \ z \in \Omega \quad : \quad \nabla(\rho)(z) \neq 0 \ z \in \partial\Omega$$

Next, let $\Omega \in \mathcal{B}$ be given together with a bounded and subharmonic function $\phi(z)$ in Ω. Denote by $\mathfrak{Bernt}(\Omega,\phi)$ the family of C^∞-functions $f(z)$ in Ω which satisfy:

$$(*) \qquad |f(z)| \leq -\rho(z)\cdot\Delta\,\phi(z) \quad : \quad z\in\Omega$$

4.1 Theorem. *To each $f \in \mathfrak{Bernt}(\Omega,\phi)$ the inhomogeneous equation*

$$(*) \qquad \bar\partial(u) = f$$

has a solution $u(z)$ which satisfies

$$(**) \qquad \max_{z\in\partial\Omega} \frac{\big||u(z)|\cdot e^{-\phi(z)/2}\big|}{\big|\nabla\rho(z)\big|} \leq \max_{z\in\Omega} \frac{\big||f(z)|\cdot e^{-\phi(z)/2}\big|}{\big|-\rho(z)\cdot\Delta\,\phi(z)+\nabla\rho(z)\big|}$$

Remark. A special case occurs when $\Omega = D$ and $\rho(z) = |z|^2 - 1$. Then (*) means that a function f in the Berndtsson class satisfies

$$|f(z)| \leq 2(1-|z|)\cdot\Delta(\phi)(z)$$

So here $|f|$ decays as $|z|\to 1$ and when $\Delta(\phi)$ is bounded this inequality estimates the Carleson norm of f. So in this situation Theorem 4.1 resembles Wolf's theorem. The solution u in Theorem 4.1 is found by solving an extremal problem in a Hilbert space. Namely, given the ϕ-function, Berndtsson considered the Hilbert space of functions in D which are square integrable with respect to $e^{-\phi}$, i.e. functions g for which

$$(1) \qquad \iint_D |g(z)|^2\cdot e^{-\phi(z)}\cdot dxdy < \infty$$

Now there exists the unique extremal solution u to the equation $\bar\partial(u) = f$ whose norm in $L^2(e^{-\phi})$ is minimal among all functions ψ in D satisfying $\bar\partial\psi = f$. In [Berndtsson] it is proved that his extremal solution u satisfies the inequality (**) in Theorem 4.1.

V. Hörmander's L^2-estimate

Let Ω be an open set in $\mathbf{C}$. If ϕ is a real-valued continuous and non-negative function we get the Hilbert space $\mathcal{H}_\phi$ whose elements are Lebesgue measurable functions f in Ω such that

$$(*) \qquad \int_\Omega |f|^2\cdot e^{-\phi}\cdot dxdy < \infty$$

The square root of (*) yields norm and is denoted by $||f||_{2,\phi}$. Let ψ be another real-valued continuous and non-negative function which gives the Hilbert space $\mathcal{H}_\psi$ where the norm of an element g is denoted by $||g||_{2,\psi}$. We are interested in the inhomogenous $\bar\partial$-equation, i.e. given $w \in \mathcal{H}_\psi$ we seek $f \in \mathcal{H}_\phi$ such that $\bar\partial(f) = w$. In addition we want to solve this equation with a bound for the L^2-norms. To attain this we impose

5.1 Hörmander's condition. The pair ϕ,ψ satisfies the Hörmander condition if there exists some positive constant c_0 such that the following pointwise inequality holds in Ω:

$$(*) \qquad \Delta(\psi) - 2\cdot|\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y \geq 2\cdot c_0^2\cdot e^{\psi(z)-\phi(z)}$$

where we have put $|\nabla(\psi]|^2 = \psi_x^2 + \psi_y^2$.

5.2 Theorem. *If (ϕ,ψ) satisfies (*) for some $c_0 > 0$ then the equation $\bar\partial(f) = w$ has a solution for every $w \in \mathcal{H}_\psi$ where*

$$||f||_\phi \leq \frac{1}{c_0}\cdot||w||_\psi$$

Proof. Since C_0^∞ is a dense subspace of $\mathcal{H}_\phi$ the linear operator T from $\mathcal{H}_\phi$ to $\mathcal{H}_\psi$ given by $T(f) = \bar{\partial}(f)$ is densely defined. Let w be in the domain of definition for the adjoint operator T^*. If $f \in C_0^\infty(\Omega)$ we get

$$(*) \qquad \langle T(f), w\rangle = \int \bar{\partial}(f) \cdot \bar{w} \cdot e^{-\psi} dxdy = -\int f \cdot \big[\bar{\partial}(\bar{w}) - \bar{w} \cdot \bar{\partial}(\psi)\big] \cdot e^{-\psi} dxdy$$

Since ψ is real-valued it follows that $\bar{\partial}(\bar{w}) - \bar{w} \cdot \bar{\partial}(\psi)$ is the complex conjugate of $\partial(w) - w \cdot \partial(\psi)$ which gives

$$(**) \qquad T^*(w) = -\big[\partial(w) - w \cdot \partial(\psi)\big] \cdot e^{\phi-\psi}$$

Taking the squared L^2-norm we obtain

$$(1) \qquad ||T^*(w)||_\phi^2 = \int |\partial(w) - w \cdot \partial(\psi)|^2 \cdot e^{\phi-2\psi}$$

Expanding the integrand it follows that (1) is equal to

$$(2) \qquad \int \big[|\partial(w)|^2 + |w|^2 \cdot |\partial(\psi)|^2\big] \cdot e^{\phi-2\psi} - 2 \cdot \mathfrak{Re}\Big(\int \partial(w) \cdot \bar{w} \cdot \bar{\partial}(\psi) \cdot e^{\phi-2\psi}\Big)$$

In the last integral we perform a partial integration and conclude that the last term is the real part of

$$2 \cdot \int w \cdot [\partial(\bar{w}) \cdot \bar{\partial}(\psi) + \bar{w} \cdot \partial\bar{\partial}(\psi) - 2\bar{w} \cdot \bar{\partial}(\psi) \cdot \partial(\psi) + \bar{w} \cdot \bar{\partial}(\psi) \cdot \partial(\phi)] \cdot e^{\phi-2\psi}$$

Next, the Cauchy-Schwarz inequality shows that the absolute value of

$$2 \cdot \int w \cdot \partial(\bar{w}) \cdot \bar{\partial}(\psi) \cdot e^{\phi-2\psi}$$

is majorized by the left hand integral in (2). It follows that

$$(3) \qquad ||T^*(w)||_\phi^2 \geq 2 \cdot \mathfrak{Re} \int |w|^2 \cdot \big[\,\partial\bar{\partial}(\psi) - 2 \cdot \bar{\partial}(\psi) \cdot \partial(\psi) + \bar{\partial}(\psi) \cdot \partial(\phi)\,\big] \cdot e^{\phi-2\psi}$$

Since ϕ and ψ are real-valued the real part of the inner bracket above becomes

$$(4) \qquad \frac{1}{4}\big[\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y\,\big]$$

So when (4) majorizes $\frac{c_0^2}{2} \cdot e^{\psi-\phi}$ it follows that

$$(5) \qquad ||T^*(w)||_\phi^2 \geq c_0^2 \cdot \int |w|^2 \cdot e^{\psi-\phi} \cdot e^{\phi-2\psi} = c_0^2 \cdot ||w||_\psi^2$$

This lower bound implies that the norm of T is bounded by c_o and Theorem XX follows.

5.3 Remark. There exist different pairs (ϕ, ψ) for which Hörmander's condition (*) applies. and We refer to the article [Kiselman] by C. Kiselman for some specific applications of L^2-estimates in $\mathbf{C}$ applied to study carriers of Borel transforms. The full strength of L^2-estimate appears in dimension $n \geq 2$ where one works with *plurisubharmonic functions* and impose the condition that Ω is a strictly pesudo-convex set in $\mathbf{C}^n$ and solve inhomogeneous $\bar{\partial}$-equations for differential forms of bi-degree (p, q). As expected the proofs are more involved where various Hermitian forms appear. In addition to Hörmander's original article [Hörmander] we refer to his text-book [Hörmander] and Chapter XX in [Hömander XX-Vol 2] where certain bounds for $\bar{\partial}$-equations are established with certain relaxed assumptions which are used to settle the fundamental principle for over-determined systems of PDE-equations in the smooth case. Working on complex manifolds with various metric properties L^2-estimates is a powerful tool. For a wealth of results of this nature we refer to the notes by Demailly in [Dem].

VI. The Corona Theorem.

Introduction. In the unit disc D we have the Banach algebra $H^\infty(D)$ of bounded analytic functions. Let $\mathfrak{M}_\infty(D)$ denote its maximal ideal space. Via point evaluations in D we get a map

$$i\colon D \mapsto \mathfrak{M}_\infty(D)$$

The Corona problem asked if $i(D)$ is dense in $\mathfrak{M}_\infty(D)$. The affirmative answer to this question was found by Carleson. It is easily seen that the density of the i-image is equivalent to the following result which was proved in [Carleson]:

6.1 Theorem. *The ideal generated by a finite family $f_1, \ldots, f_n$ in $H^\infty(D)$ is equal to $H^\infty(D)$ if and only if there exists $\delta > 0$ such that*

$$|f_1(z)| + \ldots + |f_n(z)| \geq \delta \quad \text{hold for all } \; z \in D$$

Remark. Just as in the proof of the Interpolation Theorem an essential ingredient of the proof relies upon Carleson measures. An alternative to Carleson's original proof was given by Wolff and relies upon his result for the inhomogeneous $\bar{\partial}$-equation from XX. The deduction from Wolff's Theorem in XX to the solution of the Corona problem is exposed a several places. Se for example Chapter XX in [Narasimhan] and also the article [Gamelin] by T.Gamelin. So here we refrain from giving further details. Let us only remark that one may consider related problems where the boundedness of the f-function is relaxed. For example, using L^2-estimates with weight functions one can solve a problem where $f_1, \ldots, f_n$ are analytic functions in D with moderate growth, i.e there is an integer m and a constant A such that

$$(1 - |z|)^m \cdot |f_\nu(z)| \leq K$$

holds in D for every ν. Assume also that there is an integer m_* and $\delta > 0$ such that

$$|f_1(z)| + \ldots + |f_n(z)| \geq \delta \cdot (1 - |z|)^{m_*} \quad \text{hold for all } \; z \in D$$

Then one can show that there exists an n-tuple $g_1, \ldots g_n$ of analytic functions with moderate growth such that

$$g_1 \dot{f}_1 + \ldots + g_n \cdot f_n = 1 \tag{1}$$

holds in D.

Question. Find the smallest possible number $\rho = \rho(m, m_*)$ such that (1) has a solution where the g-functions satisfy

$$|g_\nu(z(| \leq C \cdot (1 - |z|)^{-\rho}$$

for some constant C. It appears that the best constant ρ is not known. However, upper bounds for ρ can be established using L^2-estimates for the $\bar{\partial}$-equation. But there remains to investigate how sharp such bounds are.