9. Neumann-Poincaré boundary value problems

Introduction. Several fundamental results were achieved by Carl Neumann in his pioneering article emphxxx from 1877. Apart from an elegant solution to the Dirichlet problem, Neumann solved boundary valued problems where double-layer potentials occur. His results were carried out in dimension 3. The strategy was to use resolvents and prove that certain poles are absent while meromorphic extensions of operator-valued series are constructed which after has become a widespread tool to study elliptic differential equations. Neumann's studies were confined to convex domains where certain majorizations become straighforward since one deals with positive integral kernels. The extension to general domains with a C^2-boundary was achieved by Poincaré in the article *xxx* from 1897 where new methods were introduced to overcome the failure of positivity for the non-symmetric kernel defining the double-layer potential. In § xx we expose Poincaré's solution to the Neumann problem in dimension two and mention only that a similar proof is available in higher dimension. In these proof analytic function theory appears which to establish existence results for elliptic boundary value problems. For example, a crucial point during the proof of Theorem xx is that one first has found a certain meromorphic operator-valued function and following Poincaré one shows that uniqueness entails that a pole is absent at the critical value $\lambda = 1$.

Here we shall restrict the discussion to planar domains Ω borded by a finite union of closed Jordan arcs of class C^2. Let Γ denote Ω and following Poincaré in his article *La méthode de Neumann et le problème de Dirhclet* one introduces the certain linear operators acting on continuous functions on Γ as follows:

Denote by $\mathbf{n}$ the *inner normal* to Γ, i.e. the unit normal along $\partial\Omega$ which is directed into the domain, and let ds be the arc-length. When ψ and ϕ are continuous functions on Γ we put:

$$(*) \qquad \mathcal{N}_\psi(x) = \int_\Gamma \frac{\langle x-p, \mathbf{n}(x)\rangle}{|x-p|^2} \cdot \psi(p)\, ds(p) \quad \& \quad \mathcal{D}_\phi(x) = \int_\Gamma \frac{\langle x-p, \mathbf{n}(p)\rangle}{|x-p|^2} \cdot \phi(p)\, ds(p)$$

We refer to $\mathcal{N}$ as the Neumann operator and $\mathcal{D}$ as the Dirichlet operator. Poincaré proved that both these linear operators acting on the Banach space $C^0(\Gamma)$ have a real spectrum whose sole cluster point is at the origin and the eigenspace at every spectral point is 1-dimensional. This can be expressed by saying that the operator valued functions

$$(**) \qquad \lambda \mapsto (\mathcal{N} - \lambda \cdot E) \quad \& \quad \lambda \mapsto (\mathcal{D} - \lambda \cdot E)$$

are holomorphic in a disc centered at the origin and extend to meromorphic functions in the complex parameter λ with a discrete set of simple real poles whose absolute values tend to ∞.

The case when Γ has corner points.

Certain existence theorems for planar domains where isolated corner points are allowed were established by Zarmela in 1904. Further results which led to quite precise facts about the spectra for operators as above were obtained in Carleman's thesis *xxx* from 1916. A novelty in this work was that solutions to equations of the form

$$(0.1) \qquad \mathcal{N}_\psi = \lambda \cdot \psi \quad \& \quad \mathcal{D}_\phi = \lambda \cdot \phi$$

were stided for functions which only are integrable on the boundary. This leads to new phenomena for the spectra of the operators in (*), i.e. they need not be confined to discrete subsets of the complex λ-plane since the equations (0.1) can have L^1-solutions for values of λ where no continous function appears. The reason is that the oprecence of corner pints means that the operators in (*) are unbounded when they act on a space like $L^1(\Gamma)$. Let me recall some results from Carleman's thesis which I personally find to be worth studying even today by readers interested in elliptic boundary value problems. As far as I know the sharp results from [ibid] for planar domains have not been improved in more recent literature and methods initiated in [ibid] have benn adopted by generations of authors later n.

Suppose that Γ has a finite set of corner points where interior angles $\{\alpha_1, \ldots, \alpha_M$ appear and give numbers

$$\text{(i)} \qquad R_\nu = \frac{\pi}{|\pi - \alpha_\nu|} > 1$$

Notice that a small R-value means that the corresponding corner point is rather accute. When the corner points appear a first result in [ibid] goes as follows.

Theorem. *Let R_* be the smallest number from (i), Then each of the operator-valued functions in (**) acting on $C^0(\Gamma)$ are meromorphic in the open disc $|\lambda| < R_*$ with a finite number of poles which are all real and simple.*

A more involved analysis appears in the second chapter from [ibid] where Carleman studies the spectrum in the closed complement of the disc of radius R_*. Consider for example a given continuous function f on Γ and the equation

$$\text{(i)} \qquad \psi - \lambda \cdot \mathcal{N}_\psi = f$$

Under the hypothesis that f is Hölder continuous of some order $\delta > 0$ close to the corner point where the minimal R-value is attained, the solutions to (i) as λ is close to R_* can be expressed via an algebraic function of λ with a branch at $\lambda = R_*$. For the precise expansions which occur we refer to [ibid].

One may also consider the homogenous equations

$$\text{(ii)} \qquad \psi - \lambda \cdot \mathcal{N}_\psi = 0 \quad \& \quad \phi - \lambda \cdot \mathcal{D}_\phi = 0$$

Carleman proved non-zero continuous solutions only occurs at a discrete set of real numbers which tends to ∞ and the spectra are the same for Neumann's and Dirichlet's operators. Moreover, the dimesion of the solution spaces in (ii) are equal at every spectral λ-value.

One can also solutions to (ii) where ψ or ϕ only are integrable with respect to the arc-length meaure,. In that case Carleman proved that the spectrum becomes quite extensive, i.e. it has positive area and contains an assigned geometrically define planar domain which is described explicityl in [ibid]. It would briung us too far to discuss this further since it requires familarity about unbounded linear operstors. But readers interested to learn about examples where peculiar spectra of linear operators appear should consult Carleamn's thesis which provides "concrete" and yet highly non-trivial examples. A survey of (ibid] appears in the article *Sur les recherches de M. Carelab relatives fonctions harmoniques* (Comptes Rendus vol. 43 1920) by Erik Holmgen.

PDE-equations in unbounded domains. At the Scandinavian congress in matheamtics 1925, Niles Bohr put forward the phyciual signiciance of solutions in $L^(\mathbf{R}^3)$ to equations of th form

$$\Delta(u) + W \cdot u + \lambda \cdot u = 0$$

where $W(p)$ is a potentia, function of the form

$$W(p) = \sum \frac{m_\nu}{|p - q_\nu|}$$

with the sum taken over a finite set of pints $\{q_\nu\}$ in $\mathbf{R}^3$ and $\{m_\nu\}$ are postivie real numbers. It turns out that (*) has non-zero L^2-solutions for a discrete set of real numbers which tend to $+\infty$. The first demonstration of this result is due to carelan who gave an affirative answer to Bohr's raides questi on via an immediate appication of this general theory about *unbounded self-adjoint operators on Hilbert spaces.* which was created in his monograph *Sur les équations singuliers à noyeau réel et symmetrique* published by Uppsala University is 1923, a work which can be seen as a historic landmark in operator theory. Here one finds the contructions of spectral resolutions and their properties assigned to an unbounded self-ajoint operator whuch has a wide range of applications. Hundreds - ot reven thousands - of articles have later exploited Carleman's original work. Concerning the equation (*) it is a special case of more general equations where one seeks L^2-functions u which satisfy

$$\Delta(u) + c(x, y, z) \cdot u + \lambda \cdot u = 0$$

where c is a real-valued and locally square integrable function in $\mathbf{R}^3$. The operator

$$L = \Delta + c(x, y, z)$$

is ab example of a denseyl defined and symmetric operator on the Hilbert space $L^2(\mathbf{R}^3)$. There remains to find conditions on c in order that L is self-adjoint. In Carelan's cited work this is equivspent to the condition that for every pair u, v in L^2 for which $L(u)$ and $L(v)$ also are square integrable over $\mathbf{R}^3$, one has the equation

$$\int u \cdot L(v)\, dm = \int L(u) \cdot v\, dm$$

where dm is the Lebesgue measure in $\mathbf{R}^3$. No general *sufficient and necessary* condfitions for L to be self-adjoin t are known. However, one can seek for sufficient con ditions on the c-functions in order that (xx) holds. The following result of was presented by Carelan in his lectures at instuut Pincaré helt in May 1930. It assets that (xx) holds under the condition that the locally suare integrable function c satisfies

$$\limsup c(x, y, z) < +\infty$$

where the limes superior is taken as $|x|^2 + |y|^2 + |z|^2 \to +\infty$. Actually the more precise condition to be imposed upon c is that for everu u siuch that both u and $L(u)$ belong to $L^2(\mathbf{R}^3$, it follows that

$$\liminf_{r \to \infty} \int_{B(r)} c \cdot u^2\, dm < \infty$$

where $B(r)$ denote the open balls of radius r centered at the origin. As far as I know this is still the shapest known sufficency resut in order that L is self-adjoint. As expected Carelan's sufficency theorem cannot be derived by "abstract reasoning". However, the proof is not too cumeersome and is presented with all details in Carleman's article where systemtic use of Newton's fundaemtnal solution to the Laplace operator in $\mathbf{R}^3$ is used.

Exercise. To each $\rho > 0$ and every pair of points p, q in $\mathbf{R}^3$ we set

$$A_\rho(p, q) = \frac{1}{|p - q|} + \frac{|p - q|}{\rho^2} - \frac{2}{\rho}$$

Consider an open domain Ω in $\mathbf{R}^3$ and a Lebesgue measurable function ϕ in Ω with the property that for every point $p \in \Omega$ there exists a small ball centered at p such that

$$\int_{B_p(\epsilon)} \frac{\phi(q)}{|p - q|}\, dm < \infty$$

So in particular ϕ is locally integrable. Apply Newton's classic formulas to show that a function u satisfies the equation

$$\Delta(u) = \phi$$

in Ω if and only if the following is valid: For each point $p \in \Omega$ and every $\rho > 0$ such that the ball $B_p(\rho)$ is a relatively compact subset of Ω, one has the equation

$$u(p) - \frac{1}{2\pi \cdot \rho^2} \cdot \int_{B_p(\rho)} \frac{u(q)}{|p - q|}\, dm(q) + \frac{1}{4\pi} \cdot \int_{B_\rho(p)} A_\rho(p.q) \cdot u(q)\, dm(q) = 0$$

Let us remark that Newton's criterion for solutions to the equation (xx) is a gateway to analyze operators such that $L = \delta + c$. See also § xxx where we expose Carelan's proof of the sufficency in order that the L-operator above is self-adjoint. For another intructive lesson dealing with spectral resolutions of unbounded self-adjoint operators we refer to § xx which treats Carleman's solution to equations related to propagation of sound in from the cited monograph [xxx]. Here one is led to more refined questions about the spectral resolution, namely if they are exhibited by *aboslutely continuous* spectral functions. In § xx we follow Carleamns' original wojk and prove the absolute contoinuity for a class of boundsry value probelms of the Neumann type where solutions are L^2.functions defined over unbounded domains in $\mathbf{R}^3$ which leads to spectra whch in general contain unbounded real intervals.

Solutions for regukar boundary curves.

Let $\mathcal{C}$ be a closed Jordan curve of class C^2 whose arc-lengt measure is denoted by σ. If g is a continuous function on $\mathcal{C}$ the logarithmic potential

$$L_g(z) = \frac{1}{\pi} \int_{\mathcal{C}} \log\, \frac{1}{|z-q|} \cdot g(q)\, d\sigma(q)$$

yields a harmonic function in open the complement of $\mathcal{C}$. Notice that we one gets a pair of harmonic functions, defined in the bounded Jordan domain and the the exterior domain respectively. Since $\log\,|z|$ is locally integrable in $\mathbf{C}$ and $L_g(z)$ the convolution of this log-function and the compactly supported Riesz measure $g{\cdot}\sigma$, it follows that L_g extends to a continuous function. In particular the pair of harmonic functions in the inner respectively outer component are equal on $\mathcal{C}$. Moreover, by the result in § xx the Laplacian of L_g taken in the distribution sense is equal to the measure $g \cdot \sigma$. Of course, this was known to Poincaré even if one did not speak about distriobutins in those dys, but rather "Grundlösungen" and the fact that the *log*-function is a findfsem,tnal solution to the Lpace operator in $\mathbf{R}^2$ is classic and already used by Isaac Newton,

Let us now consider the inner normal derivative as z approaches points $p \in \mathcal{C}$ from the inside to be denoted by $\mathbf{n}_*$. We get the function on $\mathcal{C}$ defined by:

$$p \mapsto \frac{\partial L_g}{\partial \mathbf{n}_*}(p) \tag{1.1}$$

It turns out that (1.1) is recaptured by an integral kernel function $K(p,q)$ defined on the product $\mathcal{C} \times \mathcal{C}$. With $p \neq q$ we consider the vector $p-q$ and the unit vector $\mathbf{n}_*(q)$ and put

$$K(p,q) = \frac{\langle p-q, \mathbf{n}_*(q)\rangle}{|p-q|^2} \tag{*}$$

Let analyze the behaviour of K close to a point on the diagonal. Working in local coordinates we can take $p = q = (0,0)$ and close the this boundary point the C^2-curve $\mathcal{C}$ is locally defined by a function

$$y = f(x)$$

where a point (x,y) belongs to the bounded Jordan domain when $y > f(x)$. By drawing a figure the reader can verify that

$$\mathbf{n}_*(x, f(x)) \cdot d\sigma = (-f'(x), 1)\dot{d}x$$

So with $p = (t, f(t))$ and $q = (x, f(x))$ we have

$$K(p,q) \cdot d\sigma(q) = \frac{f(t) - f(x) - f'(x)(t-x)}{(t-x)^2 + (f(t) - f(x))^2} \cdot dx$$

By hypothesis f is of class C^2 which implies that the right hand side stays bounded as t and x independently of each other approach zero. Now one has:

Theorem. *For each $p \in \mathcal{C}$ one has the equality*

$$\frac{\partial U_g}{\partial \mathbf{n}_*}(p) = g(p) + \int_{\mathcal{C}} K(p,q) \cdot g(q)\, d\sigma(q) \tag{0.2}$$

Exercise. Prove (0.2). A hint is that by additivity it suffices to take g-functions with supports confined to small sub-intervals of $\mathcal{C}$ and profit upon local coordinates and parametrizations as above when we study $\mathcal{C}$ close to the support of g.

1. Neumann's boundary value problem.

Let Ω be a bounded domain where $\partial\Omega$ consists of a finite set of closed Jordan curves of class C^2. Let h and f be a pair of real-valued continuous functions on $\partial\Omega$ where h is positive. We seek a function U which is harmonic in Ω and on the boundary satisfies

$$\frac{\partial U}{\partial \mathbf{n}_*}(p) = h(p) \cdot U(p) + f(p)$$

Following Poincaré we announce and prove the following result.

1.1 Theorem. *The boundary value problem above has a unique solution U.*

The uniqueness amounts to show that if V is harmonic in Ω and

$$\frac{\partial V}{\partial \mathbf{n}_*}(p) = h(p)V(p)$$

on $\partial\Omega$, then $V = 0$. Since h is positive this follows from the maximum porinciple for harmonic functions.

Proof of existence. For each $g \in C^0(\partial\Omega)$ we construct L_g which by Theorem 0.1 solves the Neumann problem if g satisfies the integral equation

$$(1) \qquad g(p) + \int_{\mathcal{C}} K(p,q) \cdot g(q)\, d\sigma(q) = h(p) \cdot \frac{1}{\pi} \cdot \int_{\partial\Omega} \log \frac{1}{|p-q|} \cdot g(q)\, d\sigma(q) + f(p)$$

With h kept fixed we introduce the kernel

$$K_h(p,q) = h(p) \cdot \frac{1}{\pi} \cdot \log \frac{1}{|p-q|} - K(p.q)$$

and (1) reduces to the equation

$$(2) \qquad g(p) - \int_{\partial\Omega} K_h(p,q) g(q)\, d\sigma(q) = f(p)$$

To solve (2) we regard the linear operator on the Banach space $C^0(\partial\Omega)$ defined by

$$(3) \qquad \mathcal{K}_h(f) = \int_{\partial\Omega} K_h(p,q) f(q)\, d\sigma(q) \quad : \quad f \in C^0(\partial\Omega)$$

With this notation a g-function satisfies (2) if

$$(4) \qquad (E - \mathcal{K}_h)(g) = f$$

where E is the identity operator on $C^0(\partial\Omega)$. Next, from the general result in §§ $\mathcal{K}_h$ is a compact linear operator and by another general result from § xx each $f \in C^0(\partial\Omega)$ yields a meromorphic function of the complex parameter λ given by

$$N_f(\lambda) = f + \sum_{n=1}^{\infty} \lambda^n \cdot \mathcal{K}_h^n(f)$$

If $\delta > 0$ is so small that $||\mathcal{K}_h|| < \delta^{-1}$ it is clear that

$$(E - \lambda\mathcal{K}_h)(N_f(\lambda) = f$$

If $N_f(\lambda)$ has no pole at $\lambda = 1$ it follows by analyticity that

$$(E - \mathcal{K}_h)(N_f(1) = f$$

which means that $g = N_f(1)$ solves (4) and the existence part follows. So there remains only to show:

The absence of a pole at $\lambda = 1$. Suppose that $N_f(\lambda)$ has a pole at $\lambda = 1$ which entails that there is a positive integer m such that

$$N_f(\lambda) = \sum_{k=1}^{k=m} \frac{a_k}{(1-\lambda)^k} + b(\lambda)$$

hold when $|\lambda - 1|$ is small where $a_m \neq 0$ in $C^0(\partial\Omega)$ and $b(\lambda)$ is analytic in some disc centered at $\lambda = 1$. It follows that

$$(1-\lambda)^m N_f(\lambda) = a_m + (1-\lambda)\beta(\lambda)$$

where $\beta(\lambda)$ again is an analytic $C^0(\partial\Omega)$-valued function close to 1. Apply $E - \mathcal{K}_h$ on both sides which gives

$$(1-\lambda)^m (E - \mathcal{K}_h)(N_f(\lambda)) = (E - \mathcal{K}_h)(a_m) + (1-\lambda) + (E - \mathcal{K}_h)(\beta(\lambda))$$

Now $\lambda = 1$ gives

$$(E - \mathcal{K}_h)(a_m) = 0 \implies a_m = \mathcal{K}_h(a_m)$$

This contradicts the uniqueness part which already has been proved and hence the proof of Theiorem 1 is finished.

2. The case when $\mathcal{C}$ has corner points.

In the previous section we found a unique solution to Neumann's boundary problem where the inner normal derivative of U along $\partial\Omega$ is a continuous function. If corner points appear this will no longer be true. But stated in an appropriate way one can extend Theorem 1.1. Let us analyze the specific case when the boundary curves are piecewise linear, i.e. each closed Jordan curve in $\partial\Omega$ is a simple polygon with a finite number of corner points. Given one of these curves $\mathcal{C}$ we shall study the K-function locally. Let $\xi_1, \ldots, \xi_N$ be the corner points on $\mathcal{C}$. On the linear interval ℓ_i which joints two succescive corner points ξ_i and ξ_{i+1} we notice that $\mathbf{n}_*$ is constant and

$$K(p,q) = 0 \quad : \quad p, q \in \ell_i$$

Indeed, this is obvious for if p and q both belong to ℓ_i then the vector $p - q$ is parallell to ℓ_i and hence $\perp$ to the normal of this line. Next, keeping q fixed on the open interval ℓ_i while p varies on $\mathcal{C} \setminus \ell_i$ the behaviour of the function

(*) $$p \mapsto \langle p - q, \mathbf{n}_*(q) \rangle$$

is can be understood via a picture and it is clear that (i) is a continuous function. By a picture the reader should discover the different behaviour in the case when $\mathcal{C}$ is convex or not. For example, in the non-convex case it is in general not true that $\mathcal{C} \setminus \ell_i$ stays in the half-space bordered by the line passing ℓ_i and then (*) can change sign, i.e. take both positive and negative values. In the special case when $\mathcal{C}$ is a convex polygon the reader should confirm that (*) is a positive function of p because we have taken the *inner* normal $\mathbf{n}_*(q)$.

2.1 Local behaviour at a corner point. After a linear change of coordinates we take a corner point ξ_* placed at the origin and one of the ℓ-lines with end-point at the origin is defined by the equation $\{y = 0\}$ to the left of ξ_* where $x < 0$ while $y = Ax$ hold to the right for some $A \neq 0$. If $A > 0$ it means that the angle α at the corner point is detemined by

$$\alpha = \pi - \operatorname{arctg}(A)$$

If $A < 0$ the inner angle is between 0 and $\pi/2$ which the reader should illustrate by a picture. Next, consider a pair of points $p = (-x, 0)$ and $q = (t, At)$ where $x, t > 0$. So p and q belong to opposite sides of the corner point. To be specific, suppose that $A > 0$ which entails that

$$\mathbf{n}_*(q) = \frac{(-A, 1)}{\sqrt{1 + A^2}} \implies K(p,q) \cdot d\sigma(q) = \frac{Ax + t}{(x+t)^2 + A^2 t^2}$$

When x and t decrease to the origin the order of magnitude is $\frac{1}{x+t}$ so the kernel function is unbounded and the order of magnitude is $\frac{1}{x+t}$. If ℓ_+ denotes the boundary interval to the right of the origin where q are placed we conclude that

$$\int_{\ell_+} K(p,q)\cdot d\sigma(p) \simeq \int_0^1 \frac{dt}{x+t} \simeq \log\frac{1}{x}$$

The last function is integrable with respect to x. This local computation shows that the kernel function K is not too large in the average. In particular

$$\iint_{\mathcal{C}\times\mathcal{C}} |K(p,q)|\cdot d\sigma(p)d\sigma(q) < \infty$$

But the growth of K near corner points prevail a finite L^2-integral, i.e. the reader may verify that

$$\iint_{\mathcal{C}\times\mathcal{C}} |K(p,q)|^2\cdot d\sigma(p)d\sigma(q) = +\infty$$

2.2 The integral operator $\mathcal{K}_h$. Let h be a positive continuous function on $\partial\Omega$. Now we define the kernel function $K_h(p,q)$ exactly as in § 1 and obtain the linear operator

$$g \mapsto \int_{\partial\Omega} K_h(p,q)g(q)d\sigma(q)$$

It has a natural domain of definition. Namely, introduce the space L^1_* which consists of functions on g on $\partial\Omega$ for which

$$(*) \qquad \iint \log\frac{R}{|p-q|}\cdot|g(p)|\cdot d\sigma(q)d\sigma(p) < \infty$$

where $R>0$ is so large that $\frac{R}{|p-q|}>1$ hold for pairs p,q on $\partial\Omega$. Return to the local situation in (2.1) and consider a g-function in L^1_*. Locally we encounter an integral of the form

$$\iint_{\square_+} \frac{1}{x+t}\cdot|g(t,At))|\,dt$$

where $0\le x,t\le 1$ hold in $\square_+$. In this double integral integration with respect to x is finite since the inclusion $g\in L^1_*$ entails that

$$\int_0^1 \log\frac{1}{t}\cdot|g(t,At)|\,dt < \infty$$

From the above we obtain the following:

2.3 Theorem. *The kernel function K_h yields a continuous linear operator from L^1_* into $L^1(\partial\Omega)$, i.e. there exists a constant C such that*

$$\int_{\partial\Omega} |\mathcal{K}_h(g)|\cdot d\sigma \le C\cdot\iint_{\partial\Omega\times\partial\Omega}\log\frac{R}{|p-q|}\cdot|g(p)|\cdot d\sigma(q)d\sigma(p)$$

Armed with Theorem 2.3 one can solve Neumann's boundary value problem for domains whose boundary curves are polygons.

2.4 Theorem. *For each $f\in L^1(\partial\Omega$ there exists a unique harmonic function U in Ω such that*

$$\frac{\partial U}{\partial\mathbf{n}_*}(p) = h(p)U(p) + f(p)$$

holds on $\partial\Omega$. Moreover, $U = L_g$ where $g\in L^1_*(\partial\Omega)$ solves the integral equation

$$g - \mathcal{K}_h(g) = f$$

Proof of uniqueness. At corner points the inner normal of U has no limit and to establish the uniqueness we use instead an integral formula:

2.5 Proposition. *For each $g \in L^1_*$ the potential function $U = L_g$ satisfies*

$$\iint_\Omega [(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2]\, dxdy + \int_{\partial\Omega} U \cdot \frac{\partial U}{\partial \mathbf{n}_*}\, d\sigma = 0 \tag{2.5.1}$$

Exercise. Prove this or consult Carleman's Phd-thesis for details.

The requested uniqueness in Theorem 2..4 follows from (2.5.1). For if $\frac{\partial U}{\partial \mathbf{n}_*} = h \cdot U$ holds on the boundary we get

$$0 = \iint_\Omega [(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2]\, dxdy = \int_{\partial\Omega} h \cdot U^2\, d\sigma \implies g = 0$$

2.6 Proof of existence.

It is carried out by the same mehod as in § X. The crucial point is that the kernel function K_h is sufficiently well-behaved in order that every $f \in L^1(\Omega)$ yields a meromorphic function $N_f(\lambda)$ where $\mathcal{K}_h$-powers are applied to f exactly as in XX.

Exercise. Supply details which prove that $N_f(\lambda)$ is meromorphic.

Remark. In [Carleman: Part 3] it is proved that the unique solution g to the integral equation is represented in a canonical fashion using a certain orthonormal family of functions with respect to the L^2-function $\log \frac{1}{|p-q|}$ wtih respect to the arc-length measure $\sigma \times \sigma$. Moreover, there exists a representation formula expressed by convergent series for the equation

$$g + \lambda \cdot \mathcal{K}_h(g) = f$$

where poles of $N_f(\lambda)$ are taken into the account.