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I. The disc algebra $A(D)$

Contents

Introduction.

Denote by $A(D)$ the algebra of continuous functions on the closed unit disc $\bar{D}$ which are analytic in the open disc. One refers to $A(D)$ as the *disc-algebra*. Since polynomials in z is a dense subalgebra of $A(D)$ it follows that a Riesz measure μ on T is $\perp$ to $A(D)$ if and only if

$$(*) \qquad \int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = 0 \quad : \quad n = 0, 1, 2, \ldots$$

In § 1 we show that (*) implies that μ is absolutely continuous and deduce some facts about boundary values of analytic functions in the open disc. The maximum principle for analytic functions shows that $A(D)$ can be identifies with a closed subalgebra of $C^0(T)$. A result due to J. Wermer asserts that this yields a maximal closed subalgebra. If means that if $f \in C^0(T) \setminus A(D)$, then polynomials in f and $e^{i\theta}$ is a dense subalgebra of $C^0(T)$. In § 3 we prove an extension of this result which goes as follows: Let K be a closed subset of D whose planar Lebegue measure is zero and there exists an open interval ω in T such that $K \cap \omega = \emptyset$. Then, if $f \in C^0(K \cup T)$ is such that its restriction to T does not belong to $A(D)$, it follows that polynomials in z and f is a dense subalgebra of $C^0(K)$.

In § 4 we study subsets on T given zeros of functions in $A(D)$. With no extra regularity than continuity, each closed null set in T can be realised as the zero set of some f in $A(D)$. This result was established by F. and M-Ruesz and is proved in § 1. If one imposes extra regularity the eventual zero sets are more restricted. If $\alpha > 0$ we denote by $A^\alpha(D)$ the functions in the disc-algebra which are Hölder continuous of order α at least on t. A result due to Beurling asserts that zero sets of functions in $A^\alpha(D)$ must be rather thin. More prcisely, denote by $\mathcal{N}*$ the family of closed subsets E on the unit circle for which the integral

$$(*) \qquad \int_0^1 \frac{\phi_E(t)}{t}\, dt = +\infty$$

where $\phi_E(t)$ denotes the Lebesgue measure of the set of points $e^{i\theta}$ whose distance to E is $\leq t$. With this notation the following uniqueness result holds:

Theorem. *Each set $E \in \mathcal{N}^*$ is a set of uniquness for the set of Höder continuous functions in $A(D)$, i.e. if $\alpha > 0$ and $f \in A^\alpha(D)$ is zero on E, then f is identically zero.*

The class $\mathcal{N}_*$. It consists of closed null sets E for which

$$(**) \qquad \int_0^1 \frac{\phi_E(t)}{t}\, dt < \infty$$

In § 4 we give a construction due to Carleson which shows that if $E \in \mathcal{N}_*$ and m is a positive integer, then there exists a function $f \in A^m(D)$ whose set of zeros contains E and while f is not identically zero. A more delicate analysis occurs in § 5 where we expose some further results from Carleson's article *Sets of uniqueness for functions regular in a disc.* Here one studies analytic functions in ther open unit disc with a finite Dirichlet integral, i.e. here

$$\iint_{|z|<1} |f'(z)|^2\, dxdy < \infty$$

We refer to the introduction in § 5 for the results which will be proved in connection with this family of analytic functions. Finally § 6 is devoted to the Wiener algebra $W(T)$ which consists of functions in $A(D)$ whose Taylor series is absolutely convergent. Here further results from [Carelson] are exposed which deal with sets of uniqueness for $W(T)$.

1. Theorem of the Brothers Riesz

At the 4:th Scandinavian Congres held in Stockholm 1916, Friedrich and Marcel Riesz proved the following:

1.1 Theorem *Let $E \subset T$ be a closed null set. Then there exists $\phi \in A(D)$ such that $\phi(e^{i\theta}) = 1$ when $e^{i\theta} \in E$ while $|\phi(z)| < 1$ for every $z \in \bar{D} \setminus E$.*

Before the construction of such functions we draw a consequence.

1.2. Theorem *Let μ be a Riesz-measure on T such that*

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = \quad : n = 1, 2, \ldots$$

Then μ is absolutely continuous.

Proof. Assume the contrary. Then there exists a closed null set E in T such that

$$\text{(i)} \qquad \int_E d\mu(\theta) \neq 0$$

Theorem 1.1 gives $\phi \in A(D)$ which is a peak function for E. The hypothesis in Theorem 1.2 gives

$$\text{(ii)} \qquad \int_0^{2\pi} \phi^m(e^{i\theta}) \cdot d\mu(\theta) = 0 \quad : m = 1, 2, \ldots$$

Next, since ϕ is a peak function for E we have

$$\lim_{m \to \infty} \phi^m(e^{i\theta}) \to \chi_E$$

where χ_E is the characteristic function of E. Hence the dominated convergence theorem in general measure theory applied to $L^1(\mu)$ gives $\int_E d\mu = 0$. But this was not the case by (i) above and this contradiction proves Theorem 1.2.

Proof of Theorem 1.1

Let $E \subset T$ be a closed null set and $\{(\alpha_\nu, \beta_\nu)\}$ is the family of open intervals in $T \setminus E$. Since $b_\nu - a_\nu \to 0$ as ν increases, we can choose a sufficiently spare sequence of positive numbers $\{p_\nu\}$ such that

$$\sum p_\nu(\beta_\nu - \alpha_\nu) < \infty \quad \text{and} \quad \lim_{\nu \to \infty} p_\nu = +\infty$$

To each ν we define a function $g_\nu(\theta)$ on the open interval (α_ν, β_ν) by

$$\text{(1)} \qquad g_\nu(\theta) = \frac{p_\nu(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} : \quad : \ell_\nu = \frac{\beta_\nu - \alpha_\nu}{2} \quad : \gamma_\nu = \frac{\beta_\nu + \alpha_\nu}{2}$$

For each ν a variable substitution gives:

$$\text{(2)} \qquad \int_{\alpha_\nu}^{\beta_\nu} \frac{d\theta}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} = \int_0^1 \frac{ds}{\sqrt{\frac{1}{4} - (s - \frac{1}{2})^2}} = C$$

where C is a positive constant which the reader may compute. Next, (2) and the convergence of $\sum p_\nu(\beta_\nu - \alpha_\nu)$ imply the function

$$\text{(3)} \qquad F(\theta) = \sum g_\nu(\theta)$$

has a finite L^1-norm. Here F is defined outside the null set E and since each single g_ν-function restrics to a real analytic function on (α_ν, β_ν) the same holds for F. Next, we notice that

$$\text{(4)} \qquad \theta \mapsto \frac{(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} \geq 2 \quad \text{for all} \quad \alpha_\nu < \theta < \beta_\nu$$

In addition to this the reader can verify that

$$(5) \qquad \frac{(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\alpha + s - \gamma_\nu)^2}} \geq \frac{\beta_\nu - \alpha_\nu}{\sqrt{s \cdot (\beta_\nu - \alpha_\nu - s)}} \quad : \quad 0 < s < \beta_\nu - \alpha_\nu$$

From (4-5) we can show that $F(\theta)$ gets large when we approach E. Namely, let N be an arbitrary positive integer. Then we find ν_* such that

$$\text{(i)} \qquad \nu > \nu_* \implies p_\nu > N$$

Next, let $\delta > 0$ and consider the open set E_δ of points with distance $< \delta$ to E. If $\theta \in E_\delta$ we have $\alpha_\nu < \theta < \beta_\nu$ for some ν. If $\nu > \nu*$ then (i) and (4) give

$$\text{(ii)} \qquad F(\theta) > 2N$$

Next, set

$$\text{(iii)} \qquad \gamma = \min_{1 \leq \nu \leq \nu_*} \rho_\nu \cdot \sqrt{\beta_\nu - \alpha_\nu}$$

Let us now consider some $1 \leq \nu \leq \nu_*$ and a point $\theta \in E_\delta$. which belongs to (α_ν, β_ν). Since $E \cap (\alpha_\nu, \beta_\nu = \emptyset$ we see that

$$\text{(iv)} \qquad \theta - \alpha_\nu < \delta \quad \text{or} \quad \beta_\nu - \theta < \delta$$

must hold. In both cases (4) gives:

$$\text{(v)} \qquad g_\nu(\theta) \geq \frac{\rho_\nu \cdot \sqrt{(\beta - \nu - \alpha - \nu}}{\sqrt{\delta}} \geq \frac{\gamma}{\sqrt{\delta}}$$

With γ fixed we find a small δ such that the right hand side is $> N$ and together with (ii) it follows that

$$\text{(vi)} \qquad \theta \in E_\delta \setminus E \implies F(\theta) > N$$

The construction of ϕ. The Poisson kernel gives the harmonic function:

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 + \cos(\theta - t)} \cdot F(t) dt \quad : \quad re^{i\theta} \in D$$

Since $F \geq 0$ we have U it is ≥ 0 in D and by (vi) $U(z)$ increases to $+\infty$ as z approaces E. More precisely, the following companion to (vi) holds:

Sublemma To every positive integer N there exists $\delta > 0$ such that

$$U(z) > N \quad : \quad z \in D \cap E_\delta^*$$

where $E_\delta^* = \{z \in D : \ dist(z, E) < \delta\}$.

Now we construct the harmonic conjugate:

$$V(re^{i\theta}) = \frac{1}{\pi} \int_0^{2\pi} \frac{r \cdot \sin(\theta - t)}{1 + r^2 + \cos(\theta - t)} \cdot F(t) dt \quad : \quad re^{i\theta} \in D$$

We have no control for the limit behaviour of $V(re^{i\theta})$ as $r \to 1$ and $e^{i\theta} \in E$. But on the open intervals (α_ν, β_ν) where F restricts to a real analytic function there exists a limit function V^*:

$$\lim_{r \to 1} V(re^{i\theta}) = V^*(e^{i\theta)}) \quad : \quad \alpha_\nu < \theta < \beta_\nu$$

Thus, V^* is a function defined on $T \setminus E$. Similarly, $U(re^{i\theta})$ has a limit function $U^*(e^{i\theta})$ defined on $T \setminus E$. Now we set

$$(*) \qquad \phi(z) = \frac{U(z) + iV(z)}{U(z) + 1 + iV(z)} \quad : \quad z \in D$$

This is an analytic function in D. Outside E we get the boundary value function

$$\lim_{r \to 1} \phi(re^{i\theta}) = \frac{U^*(e^{i\theta}) + iV^*(e^{i\theta})}{U^*(e^{i\theta}) + 1 + iV^*(e^{i\theta})}$$

The limit on E. Concerning the limit as $z \to E$ we have:

$$[1 - \phi(z)| = \frac{1}{|1 + U(z) + iV(z)|} \leq \frac{1}{1 + U(z)}$$

By the Sublemma the last term tends to zero as $z \to E$. We conclude that $\phi \in A(D)$ and here $\phi = 1$ on E while $|\phi(z)| < 1$ for al $z \in \bar{D} \setminus E$ which gives the requested peak function.

Remark. Above we have followed the orighinal proof by F. and M. Riesz. It has the merit that it is quite constructive. For alternative proofs using functional analysis and the Hilbert space $L^2(d\mu)$ attached to a Riesz measure on T we refer to the text-book [Koosis: p. 40-47].

1.3 An application of Theorem 1.1

Let $f(z)$ be analytic in the open unit disc and assume there exists a constant M such that

$$\int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta \leq M \quad : \quad 0 < r < 1$$

Consider the family of measures on the unit circle defined by

$$\{\mu_r = f(re^{i\theta}) \cdot d\theta \,:\, r < 1\}$$

The uniform upper bound for their total variation implies by compactness in the weak topology that there exists a sequence $\{r_\nu\}$ with $r_\nu \to 1$ and a Riesz measure μ such that $\mu_{r_\nu} \to \mu$ holds *weakly*. In particular we have

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = \lim_{r_\nu \to 1} \int_0^{2\pi} e^{in\theta} f(r_\nu e^{i\theta}) \cdot d\theta$$

for every integer n. Since f is analytic the right hand side integrals vanish whenever $n \geq 1$ and hence μ is absolutely continuous by Theorem 1.2. So we have $\mu = f^*(\theta) d\theta$ for an L^1-function f^*. Now we construct the analytic function

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(\theta) \cdot e^{i\theta} d\theta}{e^{i\theta} - z}$$

When $z \in D$ is fixed the *weak* convergence applies to the θ-continuous function $\theta \mapsto \frac{e^{i\theta}}{e^{i\theta} - z}$ and hence

$$F(z) = \lim_{\nu \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r_\nu e^{i\theta}) e^{i\theta} d\theta}{e^{i\theta} - z}$$

At the same time, as soon as $|z| < r_\nu$ one has Cauchy's formula:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r_\nu e^{i\theta}) \cdot r_\nu e^{i\theta} \cdot d\theta}{r_\nu \cdot e^{i\theta} - z}$$

Since this hold for every large ν we can pass to the limit and conclude that $F(z) = f(z)$ olds in D. Hence $f(z)$ is represented by the Cauchy kernel of the $L^1(T)$-function $f^*(\theta)$. At this stage we apply *Fatou's theorem* to conclude that

$$\lim_{r \to 1} f(re^{i\theta}) = f^*(\theta) \quad \text{holds almost everywhere}$$

Moreover, one has convergence in the L^1-norms:

$$\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta} - f^*(\theta)| = 0$$

Thus, thanks to Theorem 1.2 the $L^1(T)$- sequence defined by the functions $\theta \mapsto f(re^{i\theta})$ converges almost everywhere to a unique limit function $f^*(\theta) \in L^1(T)$.

1.4 Exercise. Show that for every Lebesgue point θ_0 of $f^*(\theta)$ there exists a radial limit:

$$\lim_{r\to 1} f(re^{i\theta_0}) = f^*(\theta_0)$$

1.5 Exercise. In general, let K be a compact subset of D and μ a Riesz measure supported by K which is $\perp$ to analytic polynomials, i.e.

$$\int z^n \cdot d\mu(z) = 0$$

hold for all $n \geq 0$. Use the existence of peaking functions in $A(D)$ to conclude that if $E \subset T$ is a null-set for linear Lebesgue measure $d\theta$, then E is a null-set for μ. In particular, if K contains a relatively open set given by an arc α on the unit circle, then the restriction of μ to α is absolutely continuous

1.6 Principal ideals in the disc algebra.

Let $A(D)$ be the disc algebra. The point $z = 1$ gives a maximal ideal in $A(D)$:

$$\mathfrak{m} = \{f \in A(D) \quad : f(1) = 0\}$$

Let $f \in A(D)$ be such that $f(z) \neq 0$ for all z in the closed disc except at the point $z = 1$. The question arises if the principal ideal generated by f is dense in $\mathfrak{m}$. This is not always true. A counterexample is given by the function

$$f(z) = e^{\frac{z+1}{z-1}}$$

Following the appendix in [Carleman: Note 3] we shall give a sufficient condition on f in order that its principal ideal is dense in $\mathfrak{m}$. To begin with there exists the analytic function in ther open disc defined by

$$f^*(z) = \exp\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \log|\frac{1}{f(e^{i\theta})}| \cdot d\theta\}$$

Since the continuous boundary function of f is $\neq 0$ except at $z = 1$, it follows by a wellknoen limit formula that f^* extends to $D \setminus \{1\}$ a where it is equal to f. We say that f has no logarithmic residue a $z = 1$ if $f = f^*$ holds everywhere on D.

1.6.1 Theorem. *If f has no logarithmic residue then $A(D)f$ is dense in $\mathfrak{m}$.*

Proof. With $\delta > 0$ we choose a continuous function $\rho_\delta(\theta)$ on T which is equal to $\log|\frac{1}{f(e^{i\theta})|}$ outside the interval $(-\delta, \delta)$ while

(i)
$$0 < \rho_\delta(\theta) < \log|\frac{1}{f(e^{i\theta})}| \quad : -\delta < \theta < \delta$$

Next, let $\phi \in \mathfrak{m}$ and set

(ii)
$$\omega_\delta(z) = \phi(z) \cdot \exp\{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \rho_\delta(\theta) \cdot d\theta\}$$

It follows that

(iii)
$$\big|\omega_\delta(z) \cdot f(z) - \phi(z)\big| = |f(z)| \cdot |\phi(z)| \cdot \big| 1 - \exp\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \big[\log\frac{1}{|f(e^{i\theta})|} - \rho_\delta(\theta)\big] \cdot d\theta\}|$$

Exercise. Show that the limit of the right hand side is zero when $\delta \to 0$ and conclude that ϕ belongs to the closure of the principal ideal generated by f.

§ 2. Wermer's maximality theorem.

The disc algebra $A(D)$ is a uniform algebra, where the spectral radius norm is equal to the maximum over the closed disc. By the maximum principle for analytic functions in D one has $|f|_D = |f|_T$. One therefore calls T the *Shilov boundary* of $A(D)$. A notable point is that $A(D)$ is a Dirichlet algebra which means that the linear space of real parts of functions restricted to T is a dense subspace of all real-valued and continuous functions on T. In fact, using the Herglotz integral A $\rho(\theta)$ is real-valued function ρ in $C^0(T)$ is equal to $\mathfrak{Re}(f)$ on T for some $f \in A(D)$ if and only if the function

$$z \mapsto \int_0^{2\pi i} \frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta} - z|^2} \cdot \rho(\theta)\, d\theta$$

extends to a continuous function on the closed disc. For example, every C^1-function on T belongs to $\mathfrak{Re}(A(D))$.

Now we prove Wermer's result which asserts that $A(D)$ is a maximal uniform algebra. It means that if $f \in C^0(T)$ is such that the closed subalgebra of $C^0(T)$ generated by f and z is not equal to $C^0(T)$, then f must belong to $A(D)$.

Proof. Consider the closed algebra $B = [z, f]_T$ of $C^0[T]$. This is a commutative Banach algebra which gives the maximal ideal space $\mathfrak{M}_B$ whose points correspond to multiplicative functionals on B. If $p \in \mathfrak{M}_B$ and p^* is the corresponding multiplicative functional it is clear that there exists a unique point $z(p) \in D$ such that

$$p^*(g) = g(z(p))$$

hold for every g in the subalgebra $A(D)$ of B. If $z(p) \in T$ holds for every p in the maximal ideal space then the B-element z is invertible. This means that B regarded as a subalgebra of $C^0(T)$ contains both $e^{i\theta}$ and $e^{-i\theta}$ which by the Weierstrass approximation theorem generate a dense subalgebra of $C^0(T)$. Hence, since $B \neq C^0(T)$ is assumed there must exist at least one point $p \in \mathfrak{M}_B$ such that $z(p)$ belongs to the open unit disc. In fact, *every point* $z_0 \in D$ is of the form $z(p)$ for some p for otherwise $\frac{1}{z-z_0}$ belongs to B and one verifies easily that the two functions on T given by $e^{i\theta}$ and $\frac{1}{e^{i\theta}-z_0}$ also generate a dense subalgebra of $C^0(T)$. Hence $p \mapsto z(p)$ sends $\mathfrak{M}_B$ onto the closed disc.

At this stage one employs a general result from uniform algebras. Namely, since every multiplicative functional has norm one it follows that that for every $p \in \mathfrak{M}_B$ there exists a probability measure μ_p on the unit circle such that

$$(*) \qquad p^*(g) = \int_T g(e^{i\theta}) \cdot d\mu_p(\theta) \quad \text{hold for all} \quad g \in B$$

Now we use that $A(D)$ is a Dirichlet algebra. Namely, (*) holds in particular for $A(D)$-functions and since μ_p is a real measure we conclude that it must be equal to the Poisson kernel of the point $z(p)$. This proves to begin with that the map $p \to z(p)$ is *bijective*, and each function $g \in B$ gives a continuous function on the closed unit disc defined by

$$g^*(z(p) = p^*(g)$$

Here (*) shows that g^* is the harmonic extension to the open unit disc of the boundary function g on T. since B is an algebra of functiond, this easily entails that each such har,onic extension actually is an analytic function. This means precisely that $B = A(D)$. In particul ar the B-element f belongs to $A(D)$ which is the assertion in Wermer's theorem.

3. Relatively maximal algebras

Introduction. An extension of Wermer's maximality theorem was proved in [Björk] and goes as follows. Let K be a closed subset of $\bar{D}$ whose planar Lebesgue measure is zero and there exists an open interval ω in T such that $K \cap \omega = \emptyset$.

3.1. Theorem. *Let $f \in C^0(K \cup T)$ be such that the uniform algebra B generated by z and f on $K \cup T$ is a proper subalgebra of $C^0(K \cup T)$. Then the restriction of f to T belongs to $A(D)$.*

Remark. The case when K is the union of T and a finite set of Jordan arcs where each arc has one end-point on T and the other in the open disc D is of special interest. If these Jordan arcs are not too fat, then f extends analytically across each arc which means that the restriction of f to T must belong to the disc-algebra. This case was a motivation for Theorem 3.1 since it is connected to the problem of finding conditions on a Jordan arc J in order that it is locally a removable singularity for continuous functions g which are analytic in open neighborhoods of J. The interested reader may consult [Björk:x] for a further discussion about this problem where comments are given by Harold Shapiro about the connection to between Theorem 3.1 and results by Privalov concerning analytic extensions across a Jordan arc.

Proof of Theorem 3.1. Let π be the projection from the maximal ideal space $\mathfrak{M}_B$ into D which means that when z is regarded as an element in B then its Gelfand transform $\widehat{z}$ satisfies

$$\widehat{z}(p) = \pi(p) \quad : \, p \in \mathfrak{M}_B$$

As usual K is identified with a compact subset of $\mathfrak{M}_B$ and contains the Shilov boundary. If $e^{i\theta} \in T$ we use that it is a peak point for $A(D)$ and hence also for B. This entails that the fiber $\pi^{-1}(e^{i\theta})$ is reduced to the corresponding point $e^{i\theta} \in K$. Next, since we assume that K has planar measure zero we know from XX that the uniform algebra on K generated by rational functions with poles outside K is equal to $C^0(K)$. Since $z \in B$ and $B \neq C^0(K)$ is assumed, it follows that $\pi^{-1}(D \setminus K) \neq \emptyset$. Now we shall prove:

Sublemma. *The fiber above every point in $D \setminus K$ is reduced to a single point.*

To prove this we choose a non-zero Riesz measure μ on K which annihilates B. We get two analytic functions in the open set $D \setminus K$:

$$(*) \qquad W(z) = \int_K \frac{f(\zeta) \cdot d\mu(\zeta}{\zeta - z} \quad \text{and} \quad R(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z}$$

The crucial step in the proof of the sublemma is to show that if $z \in D \setminus K$ and $\xi \in \pi^{-1}(z)$, then the Gelfand transform $\widehat{f}$ satisfies:

$$(**) \qquad \widehat{f}(\xi) \cdot R(z) = W(z) \quad : \, \forall \, \xi \in \pi^{-1}(z)$$

To prove (**) we proceed as follows. First, since μ annihilates the functions z^N and $z^N \cdot f(z)$ for every $N \geq 0$ we have

$$\int_K \frac{\bar{z} \cdot d\mu(\zeta)}{1 - \bar{z} \cdot \zeta} = \int_K \frac{\bar{z} \cdot f(\zeta) \cdot d\mu(\zeta)}{1 - \bar{z} \cdot \zeta} = 0 \quad \text{for every} \quad z \in D$$

Adding these zero-functions in (*) it follows that

$$(1) \qquad W(z) = \int_K \frac{(1 - |z|^2| \cdot f(\zeta) \cdot d\mu(\zeta)}{(\zeta - z)(1 - \bar{z}\zeta)} \quad \text{and} \quad R(z) = \int_K \frac{(1 - |z|^2 \cdot d\mu(\zeta)}{(\zeta - z)(1 - \bar{z}\zeta)}$$

The assumption that the closure of $K \setminus T$ does not contain T gives some open arc $\alpha = (\theta_0, \theta_1)$ on T which is disjoint from the closure of $K \setminus T$, and the local version of the Brother's Riesz theorem from Exercise 1.5 implies that the restriction of μ to α is absolutely continuous. Hence, by Fatou's theorem there exist the two limits

$$(2) \qquad \lim_{r \to 1} W(re^{i\phi}) = W(e^{i\phi}) \quad : \, \lim_{r \to 1} R(re^{i\phi}) = R(e^{i\phi})$$

almost every on $\theta_0 < \theta < \theta_1$. Let us fix a pair (θ_0^*, θ_1^*) where $\theta_0 < \theta_0^* < \theta_1^* < \theta_1$ and the radial limits in (2) exist for θ_0^* and θ_1^*.

Next, consider a point $z_0 \in D \setminus K$ and choose a closed Jordan curve Γ which is the union of the interval $[\theta_0^*, \theta_1^*]$ on T a Jordan arc γ which is disjoint to the closure of $K \setminus T$ while z_0 belongs to the Jordan domain Ω bordered by Γ. We can always choose a nice arc Γ which is of class C^1 and hits T at $e^{i\phi_0}$ and $e^{i\phi_1}$ at right angles. Since Γ has a positive distance from $K \setminus T$ there exists $r_* < 1$ such that if $r_* < r < 1$ then the functions

$$W_r(z) = W(rz) \quad : R_r(z) = R(rz) \tag{3}$$

are analytic in a neighborhood of the closure of Ω. Now we consider the set $\pi^{-1}(\Omega)$ in $\mathcal{M}_B$ whose boundary in $\mathcal{M}_B$ is contained in $\pi^{-1}(\Gamma)$. If $Q(z)$ is an arbitrary polynomial the *Local Maximum Principle* gives

$$|Q(z_0)| \cdot [\widehat{f}(\xi) \cdot R_r(z_0) - W_r(z_0)| \leq \big|Q \cdot (\widehat{f} \cdot R_r - W_r)\big|_{\Gamma^*} \tag{4}$$

Recall that $\pi^{-1}(T)$ is a copy of T Identifying the subinterval $[\theta_0^*, \theta_1^*]$ with a closed subset of $\mathcal{M}_B$ we can write

$$\pi^{-1}(\Gamma) = \gamma^* \cup [\theta_0^*, \theta_1^*] \quad : \gamma^* = \pi^{-1}(\Gamma) \setminus (\theta_0^*, \theta_1^*) \tag{5}$$

Now (4) and the continuity of the Gelfand transform $\widehat{f}$ give a constant M which is independent of r such that the maximum norms

$$m(r) = \big|\widehat{f} \cdot R_r - W_r)\big|_{\pi^{-1}(\Gamma)} \leq M \quad : r_* < r < 1 \tag{6}$$

Next, since we already know that $\widehat{f}(e^{i\theta}) = f(e^{i\theta})$ holds on T it follows from (1) that the maximum norms:

$$m(r) = \big|\widehat{f} \cdot R_r - W_r|_{[\theta_0^*, \theta_1^*]} = 0 \tag{7}$$

tend to zero as $r \to 1$. Next, for each $\epsilon > 0$, Runge's theorem gives a polynomial $Q(z)$ such that

$$Q(z_0) = 1 \quad : |Q|_\gamma < \frac{\epsilon}{M} \tag{8}$$

When $\xi \in \pi^{-1}(z_0)$ it follows from (6) that

$$|\widehat{f}(\xi) R_r(z_0) - W_r(z_0)| \leq \text{Max}\,(\epsilon, \big|Q\big|_{[\theta_0^*, \theta_1^*]} \cdot \delta(r)) \tag{9}$$

Passing to the limit as $r \to 1$ we have seen that $m(r) \to 0$, and together with the obvious limit formulas $R_r(z_0) \to R(z_0)$ and $W_r(z_0) \to W(z_0)$ we conclude that that

$$\big|\widehat{f}(\xi) \cdot R(z_0) - W(z_0)\big| \leq \epsilon \tag{10}$$

Since we can choose ϵ arbitrary small we get

$$\widehat{f}(\xi) \cdot R(z_0) = W(z_0) \quad : \xi \in \pi^{-1}(z_0) \tag{11}$$

To profit upon (11) we first notice that the $R(z)$ cannot be identically zero in $D \setminus K$ for then the Riesz measure μ would be identically zero by the observation in § xx. If $R(z) \neq 0$ for some $z \in D \setminus K$ then (11) entails that the fiber $\pi^{-1}(z)$ is reduced to a single point. So this hold for all points in $D \setminus K$ with an eventual exception of a discrete subset. Applying the oocal maximum principal for points in a fiber $\pi^{-1}(z)$ where $R(z) = 0$ the reader may verify that it again is reduced to a single point and the Sublemma follows.

Final part of the proof. The Sublemma and (**) show that when the Gelfand transform of f is restricted to $\bar{D} \setminus K$ then it is analytic outside the eventual zeros of $R(z)$. At the same time the Gelfand transform is a continuous function and since isolated points are removable singularities for bounded analytic functions, it follows that the Gelfand transform restricted to $\bar{D} \setminus K$ is an analytic function. The same holds of course for every $g \in B$. At this stage we apply Wermer's theorem. For if $f|T$ does not belongs to $A(D)$ every continuous function on T can be approximated uniformly by polynomials in $f|T$ and $e^{i\theta}$.

§ 5. Sets of uniqueness for analytic functions in the unit disc.

Introduction. We shall begin with some measure theoretic considerations. If E is a closed subset of T then $\phi_E(t)$ is the linear measure of the set of points on T whose distance to E is $\leq t$. Denote by $\mathcal{N}_*$ the family of closed subsets E of T for which the integral

$$\int_0^1 \frac{\phi_E(t)}{t}\,dt < \infty \tag{*}$$

Similarly , we have the family $\mathcal{N}^*$ of closed null sets for the integral is divergent.

4.1 Some facts about the classes $\mathcal{N}_*$ and $\mathcal{N}^*$

In general, let E be a closed null set and put

$$E_n = \{\theta \in T \colon 2^{-n-1} < \text{dist}(\theta, E) \leq 2^{-n}\} \quad : n = 0, 1, 2, \ldots$$

It means that the Lebesgue measure $|E_n|$ satisfies

$$|E_n| = \phi(2^{-n}) - \phi(2^{-n-1})$$

Since

$$\int_{2^{-n-1}}^{2^{-n}} \frac{dt}{t} = \log 2$$

hold for every n we get

$$\int_0^1 \frac{\phi_E(t)}{t}\,dt = \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \frac{\phi_E(t)}{t}\,dt \leq \log 2 \cdot \sum_{n=0}^{\infty} \phi_E(2^{-n})$$

where the last inequality holds since the function $\phi_E(t)$ is increasing. The reader may also verify the inequality

$$\log 2 \cdot \sum_{n=1}^{\infty} \phi_E(2^{-n-1}) \leq \int_0^1 \frac{\phi_E(t)}{t}\,dt$$

Hence tghe integral in (*) is divergent if and only if

$$\sum_{n=0}^{\infty} \phi_E(2^{-n}) = +\infty \tag{4.1.1}$$

Exercise. Conclude from the above that (*) diverges if and only if

$$\sum n \cdot |E_n| = +\infty \tag{4.1.2}$$

Next, let $\{\omega_\nu = (\alpha_\nu, \beta_\nu)\}$ be the open intervals in $T \setminus E$. Adding a finite set of points to E if necessary we can assume that each interval has length $\ell_\nu \leq 1$ for every ν.

Exercise. Show that if E is a closed null set as above then the integral (*) is finite if and only if

$$\sum \ell_\nu \cdot \log \frac{1}{\ell_\nu} < \infty \tag{4.1.3}$$

Using the results above we can prove Beurling's uniqueness theorem from the introduction.

Proof of Theorem 0.1. Let $\alpha > 0$ and consider a function $f \in A^\alpha(D)$ which vanishes on a closed null set E in T. The Hölder continuity entails that

$$|f(t) \leq C \cdot \text{dist}(t, E)^\alpha$$

for some constant C. Replacing f by $C^{-1}f$ we may assume that $C = 1$. With $\{E_n\}$ as above it follows that

$$\int_{E_n} \log|f(t)|\,dt \leq -\alpha \cdot \log 2 \cdot n \cdot |E_n| \tag{1}$$

By (4.1.2) the divergence in (*) implies that

$$\text{(i)} \qquad \sum_{n=0}^{\infty} \int_{E_n} \log|f(t)|\,dt = -\infty$$

This cannot hold unless f is identically zero. In fact, when f is not identically zero it belongs to the Jensen-Nevannlina class which implies that

$$\int_0^{2\pi} \log|f(t)|\,dt > -\infty$$

This finishes the proof of Theorem 0.1.

The case when $E \in \mathcal{N}_*$. When this holds we are going to construct smooth functions in the disc algebra which are zero on E. More precisley we prove the following result which is due to Carleson.

4.1.4 Theorem. *If $E \in \mathcal{N}_*$ there exists for every positive integer m a non-zero function $f \in A^m(D)$ such that $f = 0$ on E while f is not identically zero.*

The proof relies upon a number of constructions. We are given $E \in \mathcal{N}_*$ and let $\{\omega_\nu\}$ denote the family of open intervals in $T \setminus E$. Adding a finite set of points to E if necessary we can assume that each interval has length ≤ 1. Define a function $h(\theta)$ outside the null set E as follows:

$$\text{(i)} \qquad h(\theta) = \log\frac{1}{\beta_\nu - \theta} + \log\frac{1}{\theta - \alpha_\nu} \quad : \ \alpha_\nu < \theta < \beta_\nu$$

Notoce that h is a non-negative function.We notice that

$$\text{(ii)} \qquad \int_{\omega_\nu} h(\theta)\,d\theta = xxx$$

From (ii) and the hypothesis that $E \in \mathcal{N}_*$ it follows that the almost everywhere defined and non-negative h-function is integrable on T, i.e.

$$\text{(iii)} \qquad \int_0^{2\pi} h(\theta)\,d\theta < \infty$$

If K is a positive integer we construct the zero-free analytic function in ther open disc defined by the exppnential Herglotz integral:

$$\text{(iv)} \qquad f(z) = xxx$$

From the construction of h it follows that when $\theta \in \omega_\nu$ for some ν, then

$$\text{(v)} \qquad \log|f(e^{i\theta})| = -K \cdot h(\theta) \implies |f(e^{i\theta})| = |\beta_\nu - \theta|^K \cdot |\theta - \alpha_\nu|^K$$

Next, from the construction in (iv) the analytic function $f(z)$ is bounded in D so its radial oimits exist almost evertywhere and yields the boundary value function $f^*(\theta)$. On the open intervals the radial limits exist and nicely since h is a real-analytic function on the ω-intervals. In particular the absolute value $|f^*|$ is equal to the right hand side in (v) for every ω interval. Since it vanishes at the end-points we can extend f^* to T where it is zero on E. Moreover, since E is a nullset we have

$$\text{(vi)} \qquad f(z) = Cauchy formula with f^*$$

In (xx) we have the positive integer K which obviously entials that the boundary function $f^*(\theta)$ is Lipschitz continuous and then the result in § xx implies that $f(z)$ is Lipschitz continuous in the whole disc. In particular f belongs to $A(D)$ and the complex derivative $f'(z)$ is in the open disc is a bounded analytic function. It turns out that we get more regularity when K is large. To prove this we use the expression of the logarithmic derivative of f which comes from (xx). More precisely, by the general result in § xx we have

(vii) $$\frac{f'(z)}{f(z)} == xxx \cdot xxx$$

Let θ belong to an interval ω_ν. We shall estimate the abolute values of $|f(re^{i\theta}]|$ as $r \to 1$. To acihve this we consider the interval on T given by

$$\omega_* = \{x : |x - \theta| \leq \frac{1}{8} \cdot \rho_\nu(\theta)\}$$

Exercise. Show with the aid of a figure that ω_* is a closed subinterval of ω_ν.

If $e^{ix} \in T \setminus \omega_*$ we have

$$|e^{ix} - re^{i\theta}| \geq \frac{1}{8} \cdot \rho_\nu(\theta)\}$$

The triangle inequality entails that

$$|\int_{T \setminus \omega_*} xxxx| \leq \frac{64}{\rho_\nu(\theta)^2} \cdot \int_{T \setminus \omega_*} |h|$$

Now we estimate the integral over ω_*. Let us for exampe consider the integral

$$\int_{\omega_*} \log \frac{1}{x - \alpha_\nu} \cdot \frac{e^{ix}}{(e^{ix} - re^{i\theta})^2}\, dx$$

Partial integration estimates this integral by an absolute constant times $\rho_\nu(\theta)^{-2}$.

From the above we conclude that the boundsry value function of the complex derivtive $f'(z)$ satisfies

$$|f'(e^{i\theta})| \leq C \cdot \rho_\nu(\theta)^{K-2}$$

So if $K \geq 3$ the derivative f' is Lipschits continuous on T and hence the second order derivative $f''(z)$ is bounded in D.

Exercise. Proceed as above and show that if $K \geq 4$ then the third order derivative is bounded and so on. So if m is an arbitrary positive integer we can take K so large that the f-function in (xx) belongs to $A^m(D)$ and at the same time it vanishes on E which proves Theorem 4.1.4

5. Functions with finite Dirichlet integrals

We shall consider closed sets E in T with a positive logarithmic capacity. Recall from § xx that it means that there exists a probablity measure μ on E whose energy intergal

$$\iint \log \frac{1}{|e^{i\theta} - e^{i\phi}|} \, d\mu(\theta) \cdot d\mu(\phi) < \infty$$

Next, denote by $\mathcal{D}$ the class of analytic functions f in the open unit disc with a finite Dirichlet integral, i.e.

$$\iint_D |f'(z)|^2 \, dxdy < \infty$$

In § XX we proved a result due to Beurling which asserts that if $f \in \mathcal{D}$ then the radial limits

$$\lim_{r \to 1} f(re^{i\theta}) \tag{1}$$

exists for all θ outside a set whose logartihmic capacity is zero.

5.1 The class $\mathcal{D}_E$. Let E be a closed subset of T with positive logarithmic capacity and consider some $f \in \mathcal{D}$. Beurling's result gives a set $\mathcal{N}_f$ of logarthmic capactiy zero such that the radial limits exist in $T \setminus \mathcal{N}_f$. We say that $f = 0$ holds almost everywhere on E if the set of non-zero Beurling limits in $E \setminus \mathcal{N}_f$ has logarithmic capacity zero. The class of these functions is denoted by $\mathcal{D}_E$.

Now we can announce a major results from [ibid] where E is a closed set with positive capacity and in addition belongs to $\mathcal{E}$. Since smooth functions in $A(D)$ have finite Dirichlet integrals, it follows from Theorem 4.1.4 that the class $\mathcal{D}_E$ contains functions which are not identically zero. Let $\mathcal{D}_E^*$ be the family of functions in $\mathcal{D}_E$ which are normalised so that $f(0) = 1$.

5.2 Theorem. *When E is as above the variational problem*

$$\min_{f \in \mathcal{D}_E^*} D(f)$$

has a unique solution. Moreover, the extremal function f_E extends to a continuous function in $\bar{D} \setminus E$ without zeros and the complex derivative $f'(z)$ extends to an analytic function in $\mathbf{C} \setminus E$.

About the proof. That the variational problem has a unique solution f_E which is zero-free in the open disc is is fairly easy to establish. See § xx below. The remaining parts of the proof are more involved and require several steps. First we will show that f_E extends continuously to $\bar{D} \setminus E$ and the boundary value function $f(e^{i\theta})$ is locally Lipschitz continuous on the open set $T \setminus E$. Using this regularity the next step is to show that f_E has no zeros on $T \setminus E$. To prove this one uses the extremal property which entails that if $\tau(z)$ is an arbitrary function in $A^2(D)$ which is zero at the origin, then

$$t \mapsto D(f \cdot e^{t\tau})$$

achieves its minimum when t varies over real numbers. The final step in § xx below shows that the previous facts imply that the derivative f' extends to be analytic in $\mathbf{C} \setminus E$.

§ 5.3 Preliminaries.

We expose some general facts which are used in the subseqent proofs. Let μ be a non-negative Riesz measure on the unit circle which has complex Fouirer coefficients, as well as the trigonometric coefficients:

$$\widehat{\mu}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \, d\mu(\theta)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \int \cos(n\theta) \, d\mu(\theta) \quad : \quad b_n = \frac{1}{\pi} \int_0^{2\pi} \int \sin(n\theta) \, d\mu(\theta)$$

where $\{a_n\}$ and $\{b_n\}$ are defined for positive integers. Next we have the analytic function in D defined by

$$\mathcal{H}_\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)$$

It has the Taylor series expansion

$$\mathcal{H}_\mu(z) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^\infty \widehat{\mu}(n)\cdot z^n$$

where the constant term appears since μ is a probability measure. A computation which is left to the resder shows that

$$\iint_D |\mathcal{H}_\mu(z)|^2\, dxdy = ++++$$

Next, the logarithmic potential defined by

$$L_\mu(\theta) = \int \log\frac{1}{|e^{i\theta}-e^{is}|}\, d\mu(s)$$

The energy integral

$$J(\mu) = \int \log\frac{1}{|e^{i\theta}-e^{is}|}\, d\mu(s)\cdot d\mu(\theta)$$

can be computed via the Fouier series expansion of μ. The crucial point is the Fourier formula

$$\log\frac{1}{|e^{i\theta}-e^{is}|} \simeq \sum_{n=1}^\infty n^{-1}\cdot\cos(\theta-s)$$

The trigonometric identity

$$\cos(\theta-s) = \cos(\theta)\cos(s) + \sin(\theta)\sin(s)$$

entails that

$$J(\mu) = \pi\cdot\sum_{n=1}^\infty \frac{a_n^2+b_n^2}{n}$$

So the condition that μ has a finite energy integral is expressed by the convergence in the right hand side.

Next, let $f(z) = \sum\, c_n z^n$ be the Taylor series of an analytic function with finite Diriehlet integral. Now

$$\frac{1}{2\pi}\cdot\int_0^{2\pi} f(e^{i\theta})\cdot d\mu(\theta) = f(0) + \sum_{n=1}^\infty c_n\cdot\widehat{\mu}(-n)$$

The Cauchy-Schwarz inequality majorizes the absolute value of the last sum by

$$\sqrt{\sum_{n=1}^\infty n|c_n|^2}\cdot\sqrt{\sum_{n=1}^\infty n^{-1}|\widehat{\mu}(-n)|^2}$$

Now

$$|\widehat{\mu}(-n)|^2 = 4(a_n^2+b_n^2)$$

Majorize....

Green's formulas. Let f and g be a pair in $\mathcal{D}$. We write $f = u+iv$ and $g = \xi + i\eta$. The Cacuhy-Riemann equations give

$$\iint_D f'(z)\cdot\overline{g'(z)}\, dxdy = \iint_D (u_x - iu_y)(\xi_x + i\xi_y)\, dxdy$$

Hence the real part becomes

$$\iint_D (u_x\xi_x + u_y\xi_y)\, dxdy$$

Using Stokes formua we recal from § xx that (i) is equal to

$$\lim_{r\to 1}\int_0^{2\pi} u(re^{i\theta})\cdot\frac{\partial v}{\partial r}(re^{i\theta})\,d\theta$$

Hence one has the equation

$$(*)\qquad \mathfrak{Re}\iint_D f'(z)\cdot\overline{g'(z)}\,dxdy=\lim_{r\to 1}\int_0^{2\pi} u(re^{i\theta})\cdot\frac{\partial v}{\partial r}(re^{i\theta})\,d\theta$$

Integrals of $\Delta(|f|^2)$. As above f is a function in $\mathcal{D}$. With $f=u+iv$ one has

$$\Delta(|f|^2)=\Delta(u^2)+\Delta(v^2)=2(u_x^2+u_y^2+v_x^2+v_y^2)=4(u_x^2+u_y^2)$$

Hence we have

$$4\cdot D(f)=\iint_D \Delta(|f|^2)\,dxdy$$

Keeping f fixed while g is a function in $A^1(D)$ and t is a real number we get the analytic function $f\cdot e^{tg}$. Write $g=\xi+i\eta$ which gives $|fe^{tg}|^2=|f|^2\cdot e^{2t\xi}$. With t small one has the expansion

$$|f|^2\cdot e^{2t\xi}=|f|^2+2t\cdot|f|^2\cdot\xi+O(t^2)$$

Here ξ is a harmonic function which entails that

$$\Delta(|f|^2\cdot\xi)=\Delta(|f|^2\cdot\xi+2\partial_x(|f|^2)\cdot\xi_x+2\partial_y(|f|^2)\cdot\xi_y$$

Exercise. Set $U=\Delta(|f|^2$ and use Green-s formula to show that (x) is equal to

$$\lim_{r\to 1}\int_0^{2\pi}\big[\frac{\partial U}{\partial r}(re^{i\theta})\cdot\xi(re^{i\theta})+\frac{\partial\xi}{\partial r}(re^{i\theta})\cdot U(re^{i\theta})\big]\,d\theta$$

Now we can consider the boundsry value function of U and solving Dirichlet's problem we find it harmonic extension in D denoted by u.

Exercise. Use Green's formula to show that

$$\lim_{r\to 1}\int_0^{2\pi}\frac{\partial\xi}{\partial r}(re^{i\theta})\cdot U(re^{i\theta})\big]\,d\theta=\lim_{r\to 1}\int_0^{2\pi}\frac{\partial u}{\partial r}(re^{i\theta})\cdot\xi(re^{i\theta})\,d\theta$$

Conclusion. With $g=\xi+i\eta$ one has

$$4D(fe^{tg})=4D(f)+2t\cdot\lim_{r\to 1}\int_0^{2\pi}(\frac{\partial U}{\partial r}+\frac{\partial u}{\partial r})\cdot\xi\,d\theta+O(t^2)$$

Proof of Theorem 5.2.

The proof requires several steps. First we prove that the minimum in (*) form Theorem 5.2 is a posotive number denoted by $D_*(E)$. to see rthis we use the assumtion that e has positive capacity which gives a probbility measure μ on E with a finite energy integral

$$J(\mu) =$$

Recall from (xx) that one has the equality

$$J(\mu) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot |\widehat{\mu}(-n)|^2 \quad : \widehat{\mu}(-n) = \int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) \tag{i}$$

If $f = 1 + \sum_{n=1}^{\infty} a_n z^n$ is a function in $\mathcal{D}_E$ and $r < 1$ we have

$$\int_0^{2\pi} f(re^{i\theta} \cdot d\mu(\theta) = 1 + \sum a_n r^n \cdot \int e^{in\theta} d\mu(\theta) \tag{ii}$$

The Cauchy-Schwarz inequality and (i) imply that the absolute value of the last sum is majorized by

$$\sqrt{\sum n|a_n|^2 \cdot r^{2n}} \cdot \sqrt{J(\mu)} \tag{iii}$$

Next, since $f \in \mathcal{D}_E$ its radial limits are zero almost everywhere with respect to μ on the closed set E, i.e. the left hand side in (ii) tends to zero. A passage to the limit as $r \to 1$ together with (ii) and (iii) therefore give

$$\mathcal{D}(f) \cdot \mathcal{J}(\mu) \geq 1 \tag{iv}$$

Since $f \in \mathcal{D}_E$ was arbitrary we get the lower bound

$$D_*(E) \geq \frac{1}{J(\mu)}$$

1.A Existence of an extremal function. To prove that there exists some $f \in \mathcal{D}_E$ for which $\mathcal{D}(f) = D_*(E)$ it suffices - via the strict inequality of the inner product norm - to verify that $\mathcal{D}_E$ appears as a closed subset of $\mathcal{D}$. So consider a sequence $\{f_n\}$ in $\mathcal{D}_E$ with limit f taken in the $\mathcal{D}$-norm. We must prove that the boundary values of f are zero almost everywhere with respect to an arbitrary positive measure μ on E whose energy integral is finite.

PROVE ...+ZERO FREE

1.B Properties of the extremal function.

We have proved the existence of a unique $f \in \mathcal{D}_E$ for which $\mathcal{D}(f) = D_*(E)$ and in § x we have shown that f is zero-free in D. We will show that f extends to a continuous function on $\bar{D} \setminus E$ and that the continuous extension has no zeros in $T \setminus E$. Let us for a while admit these properties of f and use it to establish the analytic extension of the derivative f' in Theorem XX. For this purpose we construct another analytic function in D.

1.C The function $F(z)$. If p is a positive integer and λ is a complex number then $f + \lambda \cdot z^p f$ belongs to $\mathcal{D}_E$. This gives

$$\mathcal{D}(f + \lambda \cdot z^p f) \geq \mathcal{D}(f)$$

Since it holds for small non-zero λ a standard argument in the calculus of variation and the formuoa (xx) give

$$0 = pa_0\bar{a}_p + (p+1)a_1\bar{a}_{p+1} + \ldots \tag{1}$$

Put

$$c_p = \sum_{n=1}^{\infty} n \cdot \bar{a}_n a_{n+p} \quad : p = 0, 1, 2, \ldots \tag{2}$$

Since $\sum n \cdot |a_n|^2 < \infty$ the right hand side is an absolutely convergent series for each p and there exists the analytic function $F(z)$ in the unit disc defined by

$$F(z) = \sum_{p=1}^{\infty} c_p \cdot z^p \tag{4}$$

We shall express F as a limit of analytic functions defined in smaller discs than D. To each $0 < r < 1$ we get the analytic function $F_r(z)$ in $\{|z| < r\}$ defined by

$$F_r(z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \bar{f}'(\zeta) \cdot f(\zeta) \cdot \frac{1}{\zeta - z} \cdot \frac{d\zeta}{\zeta} \tag{5}$$

With $\zeta = re^{i\theta}$ we use the expansion

$$\frac{1}{re^{i\theta} - z} = \sum_{p=0}^{\infty} r^{-p-1} \cdot e^{-i(p+1)\theta} z^p$$

The coefficient of z^p in the Taylor series of $F_r(z)$ becomes (5) becomes

$$\frac{1}{2\pi} \cdot \sum \sum \int_0^{2\pi} \bar{a}_n r^{n-1} e^{-i(n-1)\theta} \cdot a_m \cdot r^m e^{im\theta} \cdot r^{-p-1} \cdot e^{-i(p+1)\theta}\, d\theta = \sum r^{2n+p}\, \bar{a}_n \cdot a_{n+p}$$

Denote the last sum by $c_p(r)$ so that

$$F_r(z) = \sum c_p(r) \cdot z^p \tag{6}$$

Then it is clear that

$$\lim_{r \to 1} F_r(z) = F(z) \tag{7}$$

with uniform convergence in compact subsets of D. So far we have not used the vanishing in (1). With $|z| < r$ we set

$$G_r(z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \bar{f}'(\zeta) \cdot f(\zeta) \cdot \frac{\bar{z}}{\bar{\zeta} - \bar{z}} \cdot \frac{d\zeta}{\zeta^2} \tag{7}$$

Exercise. Use (1) to show that (7) is identically zero.

Now we can add the zero function $G_r(z)$ to $F_r(z)$ and use that

$$\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}} = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$$

This gives

$$F_r(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \bar{f}'(\zeta) \cdot f(\zeta) \cdot \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} \cdot \frac{d\zeta}{\zeta^2} = \frac{1}{2\pi} \int_0^{2\pi} \bar{f}'(re^{i\theta}) \cdot f(re^{i\theta}) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \cdot \frac{d\theta}{re^{i\theta}}$$

In the last integral the Poisson kernel appears. Consider the radial limit function:

$$g^*(\theta) = \lim_{r \to 1} \frac{1}{re^{i\theta}} \cdot \bar{f}'(re^{i\theta}) \cdot f(re^{i\theta}) \tag{8}$$

By the above $F(z)$ is Poisson's extension of $g^*(\theta)$ and hence $g^*(\theta)$ is the boundary value of an analytic function. If we for the moment admit that f extends to a continuous function on $T \setminus E$ and if ω is an open subinterval where f has no zeros, then the boundary value of the complex conjugate function $\bar{f}'(z)$ taken on ω coincides with those of

$$\frac{z \cdot F(z)}{f(z)}$$

Schwarz's reflection principle implies that $f'(z)$ extends analytically across ω whose the extension to the exterior disc becomes

$$z \mapsto \frac{\bar{F}(\frac{1}{\bar{z}})}{z \cdot \bar{f}(\frac{1}{\bar{z}})} \tag{9}$$

Since we already know that $f(z)$ is zero-free in D, the function in (9) is analytic in the whole exterior disc $\{|z| > 1\}$. This proves that if f is zero-free on $T \setminus E$ then the complex derivative $f'(z)$ extends to an analytic function in $\mathbf{C} \setminus E$.

1.8 Proof of the continuous extension lemma

Theorem 0.1 enable us to construct an ample family of smooth functions which vanish on E. Applied with $m = 2$ the reader should verify the following:

1.8.1 Exercise. For each point $\xi \in T \setminus E$ there exists some open inteval ω centered at ξ and $h \in A^2(D)$ where $h(0) = 0$ and $\mathfrak{Re}\, h = 0$ on ω while $\mathfrak{Im}\, h \neq 0$ on ω.

Next, let ω_* be a compact subinterval of ω centered at ξ and ψ is a real-valued C^2-function on ω_* which vanishes up to order two at the end-points. Solving the Dirichlet problem we find a function $p(z) \in A(D)$ such that

$$\mathfrak{Im}(p) = \frac{\psi}{\mathfrak{Im}\, h} \quad \text{holds on } \omega_* \quad \text{and} \quad \mathfrak{Im}(p)|T \setminus \omega_* = 0 \tag{1.8.2}$$

Set

$$g = hp$$

Since the real part of h is zero on ω it follows that

$$\mathfrak{Re}\, g = -\psi \quad \text{holds on} \quad \omega \tag{1.8.3}$$

Next, since $h(0) = 0$ we also have $g(0) = 0$ and therefore $f + \lambda \cdot g$ belongs to $\mathcal{D}_E$ for each complex number λ. Since f is an extremal in the variational problem, a standard argument shows that the inner product $\langle f, g \rangle = 0$. Let us write

$$f = u + iv \quad : \quad g = \sigma_1 + i\sigma_2 \tag{1.8.3}$$

The inner product formula (xx) gives the equation

$$\iint_D (\frac{\partial v}{\partial x} \cdot \frac{\partial \sigma_2}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial \sigma_2}{\partial y}\, dxdy = 0 \tag{1.8.4}$$

Above v and σ_2 are harmonic functions and the integral is the limit of double integrals taken over discs $\{|z| \leq r\}$ as $r \to 1$. Using Green's formula we therefore get

$$\lim_{r \to 1} \int_0^{2\pi} v(re^{i\theta}) \cdot \frac{\partial \sigma_2}{\partial r}(re^{i\theta})\, d\theta = 0 \tag{1.8.5}$$

Since the harmonic functions σ_1 and σ_2 are conjugate one has:

$$\frac{\partial \sigma_2}{\partial r}(e^{i\theta}) = -\frac{\partial \sigma_1}{\partial \theta}(e^{i\theta}) \tag{1.8.6}$$

Hence (1.8.5) and (1.8.3) give

$$\int_{\omega_*} v(e^{i\theta}) \cdot \frac{\partial \psi}{\partial \theta}(e^{i\theta})\, d\theta = \int_{T \setminus \omega} v(e^{i\theta}) \cdot \frac{\partial \sigma_1}{\partial \theta}(e^{i\theta})\, d\theta \tag{1.8.7}$$

where

$$\sigma_1 = \mathfrak{Re}\, h \cdot p_1 \quad \text{holds in } T \setminus \omega_*$$

Here p_1 is the harmonic extension of ψ which is supported by ω_* and $\mathfrak{Re}\, h = 0$ in ω. It follows from the generalinequality § xx that there exists a constant C which only depends upon h and the pair ω_*, ω such that

$$\max_{\theta \in T \setminus \omega} |\frac{\partial \sigma_1}{\partial \theta}(e^{i\theta})| \leq C \cdot \int_{\omega_*} |\psi(\theta)|\, d\theta \tag{1.8.8}$$

Hnece we have the inequality

$$|\int_{\omega_*} v(e^{i\theta})\cdot\frac{\partial\psi}{\partial\theta}(e^{i\theta})\,d\theta| \leq C\cdot\int_{\omega_*}|\psi(\theta)|\,d\theta \tag{1.8.9}$$

Above ψ was an arbitrary C^2-function which vanishes up to order two at the end-points of ω_*. By general distribution theory this entaills that the restriction of the boundsry value function v to the open interval ω_* is Lipschitz continuous. As explained in § xx this entials that v has a continuous extension to this interval. Finally, starting exactly as above with a function h where we now let $\mathfrak{Im}\,h = 0$ on ω the reader may check that a similar conclusion holds for the real part of f which finishes the proof of the Continuity Lemma.