8. Series and analytic functions.

Contents

1. A theorem by Kronecker.

Introduction. We seek necessary and sufficient condition in order that a sequence $c_0, c_1, c_2, \ldots$ of complex numbers yield the coefficients in the Taylor series at the origin of a rational function, i.e. that

$$(*) \qquad \sum c_\nu z^\nu = \frac{a_0 + a_1 z + \ldots + a_m z^m}{b_0 + b_1 z + \ldots + b_n z^n} \quad \text{where } b_0 \neq 0\,,$$

Here $A(z) = a_0 + a_1 z + \ldots + a_m z^m$ and $B(z) = b_0 + b_1 z + \ldots + b_n z^n$ are polynomials and we say that $\{c_\nu\}$is of *rational type* when (*) holds. A necessary condition for $\{c_\nu\}$ to be of rational type follows via euclidian division. Namely, let $f(z) = \sum c_\nu z^\nu$ and expand the product $f(z) \cdot B(z)$ into a power series. For each integer $M \geq n$ the coefficient of z^M becomes

$$(1) \qquad c_{M-n} b_n + c_{M-n+1} b_{n-1} + \ldots + c_M \cdot b_0 = 0$$

When (*) holds it follows that (1) is zero for every $M \geq m+1$. It means precisely that if λ is an integer which is $\geq \max(0, n-m+1)$ then

$$(2) \qquad c_\lambda b_n + c_{\lambda+1} b_{n-1} + \ldots + c_{\lambda+n} \cdot b_0 = 0$$

Here $(b_n, \ldots, b_0)$ is a fixed non-zero $(n+1)$-vector and hence (2) implies that the vectors

$$(c_\lambda + k, \ldots, c_{\lambda+n+k}) \quad : \quad 0 \leq k \leq n$$

are linearly dependent which entails that the determinant of the following matrix must be zero:

$$\begin{pmatrix} c_\lambda & \ldots & c_{\lambda+n} \\ c_{\lambda+1} & \ldots & c_{\lambda+1+n} \\ & & \\ c_{\lambda+n} & \ldots & c_{\lambda+2n} \end{pmatrix}$$

Kronecker proved that the vanishing of similar matrices also yields a sufficient condition in order that $\{c_\nu\}$ is of rational type. More precisely, for each pair of integers $\lambda \geq 0$ and $\mu \geq 1$ we set

$$C_\lambda(\mu) = \det \begin{pmatrix} c_\lambda & \dots & c_{\lambda+\mu} \\ c_{\lambda+1} & \dots & c_{\lambda+1+\mu} \\ & & \\ & & \\ c_{\lambda+\mu} & \dots & c_{\lambda+2\mu} \end{pmatrix}$$

1.1 Theorem. *The sequence $\{c_\nu\}$ is of rational type if if there exist a pair of integers $\lambda_* \geq 0$ and $\mu_* \geq 1$ such that*

$$(*) \qquad C_\lambda(\mu_*) = 0 \quad \text{for all} \quad \lambda \geq \lambda_*$$

Proof. For each $\lambda \geq \lambda_*$ we consider the vectors

$$\text{(i)} \qquad \xi_\lambda = (c_\lambda, c_{\lambda+1} \dots, c_{\lambda+\mu_*})$$

If the family $\{\xi_\lambda\}_{\lambda_*}^\infty$ span $\mathbf{C}^{\mu_*+1}$ we find the smallest integer w_* for which there exist

$$\lambda_* \leq w_0 < \dots < w_{\mu_*-1} < w_{\mu_*} \quad \text{and} \quad \xi_{w_0}, \dots, \xi_{w_{\mu_*-1}}, \xi_{w_*} \quad \text{are linearly independent}$$

But this gives a contradiction because the vectors $\{\xi_{w_*-\mu_*}, \dots, \xi_{w_*}\}$ appear as row vectors in the matrix $C_{w_*-\mu_*}(\mu_*)$ whose determinant by hypothesis is zero because $w_* - \mu_* \geq \lambda_*$.

Notice that $w_* \geq M_*$ must hold and (ii) applied with $\lambda = w_* - M_*$ implies that ξ_{w_*} belongs to the linear hull of the vectors $\xi_{w-1}, \dots, \xi_{w-M_*}$. But this contradicts the minimal choice of w_*. Hence the linear hull of the vectors $\{\xi_\lambda\}_0^\infty$ must be a proper subspace of $\neq \mathbf{C}^{M_*+1}$. This gives a non-zero vector $(b_0, \dots, b_{M_*})$ such that

$$\text{(iv)} \qquad c_\lambda \cdot b_0 + \dots + c_{\lambda+M_*} \cdot b_{M_*} = 0 \quad \text{for all } \lambda \geq 0\,.$$

But these relations obviously imply that the sequence $\{c_\nu\}$ is of rational type and Kronecker's theorem is proved.

Sublemma. For each $\mu \geq 2$ and every $\lambda \geq 0$ one has the equality

$$C_\lambda(\mu) \cdot C_{\lambda+2}(\mu) - C_\lambda(\mu+1) \cdot C_{\lambda+2}(\mu+1) = C_{\lambda+1}(\mu) \cdot C_{\lambda+1}(\mu)\,.$$

Proof continued. Notice that the Kronecker matrix $\mathcal{K}_M = \mathcal{C}_0(M)$. Assume that there exists M_* such that

$$\text{(i)} \qquad \det \mathcal{K}_M = 0 \quad \text{for all } M \geq M_*$$

With the notations above (i) means that $C_0(\nu) = 0$ when $\nu \geq M_*$. With $\lambda = 0$ in the Sublemma we conclude that $C_1(\nu) = 0$ for all $\nu \geq M_*$. We can proceed by an induction over λ which gives:

$$\text{(ii)} \qquad C_\lambda(M_*) = 0 \quad \text{for all } \lambda \geq 0\,.$$

Let us then consider the $M_* + 1$-vectors

$$\xi_\lambda = (c_\lambda, c_{\lambda+1} \dots, c_{\lambda+M_*}) \quad : \lambda = 0, 1, \dots$$

The vanishing of the determinants in (i) means that the $(M_* + 1)$-tuple of vectors

$$\text{(iii)} \qquad \xi_\lambda, \xi_{\lambda+1} \dots, \xi_{\lambda+M_*}$$

are linearly dependent for every $\lambda \geq 0$. Suppose now that the family $\{\xi_\lambda\}_0^\infty$ span $\mathbf{C}^{M_*+1}$. Choose the *smallest* integer w_* for which there exist

$$0 \leq w_0 < \dots < w_{M_*-1} < w_* \quad \text{and} \quad \xi_{w_0}, \dots, \xi_{w_{M_*-1}}, \xi_{w_*} \quad \text{are linearly independent}$$

Notice that $w_* \geq M_*$ must hold and (ii) applied with $\lambda = w_* - M_*$ implies that ξ_{w_*} belongs to the linear hull of the vectors $\xi_{w-1}, \dots, \xi_{w-M_*}$. But this contradicts the minimal choice of w_*.

Hence the linear hull of the vectors $\{\xi_\lambda\}_0^\infty$ must be a proper subspace of $\neq \mathbf{C}^{M_*+1}$. This gives a non-zero vector $(b_0, \ldots, b_{M_*})$ such that

$$\text{(iv)} \qquad c_\lambda \cdot b_0 + \ldots + c_{\lambda+M_*} \cdot b_{M_*} = 0 \quad \text{for all } \lambda \geq 0\,.$$

But these relations obviously imply that the sequence $\{c_\nu\}$ is of rational type and Kronecker's theorem is proved.

Remark. Kronecker's theorem can be used to establish conditions in order that a meromorphic function is rational. One has for example the following result which is due to Polya in [Pol]:

1.2 Theorem. *Let $\{c_n\}$ be a sequence of integers. Suppose that the power series*

$$f(z) = \sum c_n \cdot z^n$$

converges in some open disc centered at the origin and that $f(z)$ extends to an analytic function in a simply connected domain Ω. whose mapping radius with respect to $z = 0$ is strictly greater than one Then $f(z)$ is a rational function.

Remark. For the definition and various results about the *mapping radius* of simply connected domains the reader may consult Chapter X in [Po-Szegö] where other results based upon Kronecker's theorem appear.

II. Newton polynomials and the disc algebra $A(D)$

Introduction. Let $A(D)$ be the disc algebra. If $f(z) \in A(D)$ then its Taylor series a $z = 0$ give the partial sum polynomials $\{s_n^f(z)\}$. Denote by $A^*(D)$ the unit ball, i.e. funtions f with maximum norm $|f|_D \leq 1$ and set

$$(*) \qquad \mathcal{M}_n = \max_{f \in A*(D)} |s_n^f|_D$$

We are going to determine these $\mathcal{M}$-numbers. In his text-books from 1666, Isaac Newton studied the funcion $\sqrt{1-z}$ whose series expansion becomes:

$$(1) \qquad \sqrt{1-z} = q_0 + q_1 z \ldots \quad : \quad q_n = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \cdots 2n}$$

Notice that these positive coefficients decrease, i.e.

$$(2) \qquad 1 = q_0 > q_1 > q_2 > \ldots$$

To each $n \geq 1$ we get the Newton polynomial

$$(3) \qquad Q_n(z) = q_0 + q_1 z + \ldots + q_n z^n$$

By (2) and Kakeya's result from XXX, $Q_n(z)$ has no zeros in the closed unit disc. Put

$$(4) \qquad \mathcal{G}_n = 1 + q_1^2 + \ldots + q_n^2$$

1. Theorem. *For each integer $n \geq 1$ one has the equality $\mathcal{M}_n = \mathcal{G}_n$ and the maximum in (1) is attained by the $A^*(D)$-function*

$$(2) \qquad f_n^*(z) = \frac{z^n \cdot Q_n(\frac{1}{z})}{Q_n(z)}$$

2. Remark. Using *Stirling's formula* one can easily show that

$$\lim_{n \to \infty} \frac{\mathcal{G}_n}{\log n} = \frac{1}{\pi}$$

Before Theorem 1 is proved we need some preliminary observations about partial sum functions. Let $f \in A_*(D)$. Cauchy's formula gives

$$(\text{i}) \qquad s_n^f(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f(z)}{z^{n+1}} \cdot (1 + z + \ldots + z^n) \cdot dz \quad : \; n = 0, 1, \ldots$$

Since $\int_{|z|=1} f(z) z^k dz = 0$ for every $k \geq 0$ we see that if $Q(z)$ is any polynomial of the form

$$(\text{ii}) \qquad Q(z) = 1 + z + \ldots + z^n + q_{n+1} z^{n+1} + \ldots \implies$$

$$(\text{iii}) \qquad s_n^f(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f(z)}{z^{n+1}} \cdot Q(z) \cdot dx$$

Proof of Theorem 1. For each $n \geq 1$ the squared Newton polynomial $Q_n^2(z)$ satisfies (ii) above. So if $f \in A * (D)$ we have

$$(\text{iv}) \qquad s_n^f(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f(z)}{z^{n+1}} \cdot Q_n^2(z) \cdot dx$$

Since the maximum norm of $|f|_D \leq 1$, the triangle inequality gives:

$$(\text{v}) \qquad |s_n^f(1)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \left|Q_n(e^{i\theta})\right|^2 \cdot d\theta$$

By *Parseval's formula* the last integral is equal to $\mathcal{G}_n$. Hence (v) gives the inequality

$$(\text{vi}) \qquad \mathcal{M}_n \leq \mathcal{G}_n$$

Next, with n kept fixed we consider the function

(vii)
$$f^*(z) = \frac{z^n \cdot Q_n(\frac{1}{z})}{Q_n(z)} \implies$$

(viii)
$$s_n^{f^*}(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f_n^*(z)}{z^{n+1}} \cdot Q_n^2(z) \cdot dx$$

where $\implies$ follows from (iii) above. Notice that

(ix)
$$\frac{f_n^*(z)}{z^{n+1}} \cdot Q^2(z) = \frac{1}{z} Q_n(z) \cdot Q_n(\frac{1}{z}) \implies$$

(5)
$$s_n^{f^*}(1) = \frac{1}{2\pi} \cdot \int_0^{2\pi} Q_n(e^{i\theta}) \cdot Q(e^{-i\theta}) \cdot d\theta = \frac{1}{2\pi} \cdot \int_0^{2\pi} \left|Q_n(e^{i\theta})\right|^2 \cdot d\theta = \mathcal{G}_n$$

Since $f^* \in A^*(D)$ we conclude that (vi) above is an equality and Theorem 1 is proved.

3. Convergence of Fourier series

Let $f(z) = \sum c_n z^n$ be in the disc algebra $A(D)$. With $c_n = a_n + ib_n$ and $f = u + iv$ we get series for the real and the imaginary part respectively:

$$u(e^{i\theta}) = \sum_{n=0}^{n=N} a_n \cdot \cos n\theta - \sum_{n=0}^{N} b_n \cdot \sin n\theta$$

$$v(e^{i\theta}) = \sum_{n=-N}^{n=N} a_n \cdot \sin n\theta + \sum_{n=-N}^{N} b_n \cdot \cos n\theta$$

Continuous boundary values of f certainly exist if

(*)
$$\sum |a_n| + |b_n| < \infty$$

We shall give a sufficient condition for the validity of (*) expressed by the modulos of continuity of u:

$$\omega_u(\delta) = \max |u(e^{i\theta}) - u(e^{i\phi})| \quad : \text{ maximum over pairs } \; |\theta - \phi| \leq \delta$$

1. Theorem. *The series (*) is convergent if*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot \omega_u(\frac{1}{n}) < \infty$$

Theorem 1 is due to S. Bernstein in [1]. One may ask for other convergence criteria. Denote by $\mathcal{F}$ the class of continuous functions $F(s)$ defined for $s \geq 0$ which are increasing and concave and $F(0) = 0$ while $F(s) > 0$ for every $s > 0$. Then the following result is in [Salem: P. 39]:

2. Theorem. *Let $F \in \mathcal{F}$ be such that*

(*)
$$\sum_{n=1}^{\infty} F\big(\frac{1}{n} \cdot \omega_u^2(\frac{1}{n})\big) < \infty$$

Then it follows that

(**)
$$\sum_{n=1}^{\infty} F(a_n^2 + b_n^2) < \infty$$

Notice that Bernstein's theorem is the case $F(s) = \sqrt{s}$. The proof of Theorem 2 relies upon a result due to La Vallée Poussin who established a *lower bound* for the modulus of continuity. Recall first the general L^2-equality:

(*)
$$2a_0^2 + \sum_{n \geq 1} a_n^2 + b_n^2 = \frac{1}{\pi} \int_0^{2\pi} u(e^{i\theta})^2 \cdot d\theta$$

Now we consider the tail sums

$$V_N = \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2)$$

3. Theorem. *For every real valued and continuous function u on the unit circle one has*

$$\omega_u^2(\frac{1}{N}) > \frac{1}{72} \cdot V_N \quad : \quad N = 1, 2, \ldots$$

Remark. See Vallée Poussin's text-book *Lecons sur l'approximation des fonctions d'une variable réelle* for the proof.

Proof of Theorem 2. Put $\rho_n^2 = a_n^2 + b_n^2$. Since $F(s)$ is concave and increasing we have for every $N \geq 1$:

$$\text{(i)} \qquad \frac{1}{N} \sum_{n=N+1}^{n=2N} F(\rho_n^2) \leq F(\frac{1}{N} \sum_{n=N+1}^{n=2N} \rho_n^2) < 72 \cdot F\big(\frac{1}{N}\omega^2(\frac{1}{N})\big)$$

where the last inequality follows from Theorem 3 above. Apply (i) with $N = 2^k$ as $k = 0, 1, \ldots$. Then (i) obviously gives the implication:

$$\text{(ii)} \qquad \sum_{k=0}^{\infty} 2^k \cdot F\big(2^{-k} \cdot \omega^2(2^{-k})\big) < \infty \implies \sum_{n=1}^{\infty} F(\rho_n^2) < \infty$$

Now we are almost done. Namely, the sequence $2^{-k} \cdot \omega^2(2^{-k})$ decreases with k and since F increases we have

$$\sum_{n=2^{k-1}+1}^{2^k} F(\frac{1}{n}\omega^2(\frac{1}{n})) \geq 2^k \cdot F(2^{-k} \cdot \omega^2(2^{-k})) \quad : \; k = 1, 2, \ldots$$

Hence Salem's convergence condition (*) in Theorem 2 gives (ii) above and the proof is finished.

4. Harald Bohr's inequality.

Let $A^*(D)$ be the unit ball in $A(D)$. When $f \in A^*(D)$ and $0 < r < 1$ we set

$$(*) \qquad \mathfrak{M}_f(r) = \sum |a_n| r^n$$

The question arises for which r it holds that

$$(**) \qquad \mathfrak{M}_f(r) \leq 1 \quad : \forall\, f \in A^*(D)$$

1. Theorem. *One has (**) if and only if $r \leq \frac{1}{3}$.*

Proof. Given f in $A_*(D)$ we set

$$\text{(i)} \qquad \phi(z) = \frac{f(z) - a_0}{1 - \bar{a}_0 \cdot f(z)}$$

Then the maximum norm $|\phi|_D \leq 1$ and its derivative at $z = 0$ becomes

$$\text{(ii)} \qquad \phi'(0) = \frac{f'(0)}{1 - |a_0|^2}$$

Since $g(0) = 0$ we have $|g'(0)| \leq 1$ by the inequality of Schwarz. It follows that

$$\text{(iii)} \qquad |a_1| \leq (1 - |a_0|^2) \leq 2 \cdot [1 - |a_0|)$$

where the last inequality holds since $1 + |a_0| \leq 2$. Next, put $\rho = e^{2\pi i/n}$ for every $n \geq 2$ and regard the function

$$(*) \qquad F_n(z) = \frac{f(z) + f(\rho z) + \ldots + f(\rho^{n-1} z)}{n}$$

Since $1+1+\rho^\nu+\ldots+\rho^{\nu(n-1)}=0$ whenever ν is not a multiple of n we conclude that

(**) $$F_n(z)=a_0+a_nz^n+a_{2n}z^{2n}+\ldots$$

Notice that the maximum norm $|F_n(z)|_D\leq 1$. Now we regard the analytic function

$$g(z)=a_0+a_nz+a_{2n}z^2+\ldots$$

Since $|F_n(z)|_D\leq 1$ it is clear that we also have $|g|_D\leq 1$. Then (*) applied to g gives

(***) $$|a_n|\leq 2(1-|a_0|)\quad : n\geq 1$$

Armed with this we can finish the proof of Theorem 1. First we show

2. The inequality $\mathfrak{M}_f(\frac{1}{3})\leq 1$. From (***) we obtain

$$\mathfrak{M}_f(\frac{1}{3})=|a_0|+\sum_{n=1}^\infty 3^{-n}|\cdot|a_n|\leq|a_0|+\sum_{n=1}^\infty 3^{-n}\cdot 2(1-|a_0|)=1$$

There remains to prove that the upper bound $\frac{1}{3}$ is sharp in Theorem 1. To see this we take a real number $0<a<1$ and consider the Möbius function

$$f(z)=\frac{z-a}{1-az}=-a+(1-a^2)z+(a-a^2)z^2+(a^2-a^3)z^3+\ldots\implies$$

$$\mathfrak{M}_f(r)=+(1-a^2)r+(a-a^2)r?2+\ldots=a+\frac{(1-a^2)r}{1-ar}$$

The last term is ≤ 1 if and only if

$$a(1-ar)+(1-a^2)r\leq 1-ar\implies(1+a-2a^2)r\leq 1-a$$

With $a=1-s$ and $s>0$ small this gives

$$s+2s-2s^2)r\leq s\implies r\leq\frac{1}{3}+2s$$

Since s can be arbitrary small the upper bound $\frac{1}{3}$ in Theorem 1 is best possible.

5. A theorem by Fatou and M. Riesz.

Introduction. We prove a result due to Fatou and M. Riesz. See the article [M. Rie] from 1911. Let

(1) $$f(z)=\sum c_nz^n$$

be an analytic function in the open unit disc. We shall consider the situation when f extends analytically along some arc of the unit circle. For example, the analytic function $\frac{1}{1-z}$ extends analytically outside the boundary point $z=1$ and the series

$$\sum e^{in\theta}$$

converges for all $0<\theta<2\pi$. Let us now consider some $f\in\mathcal{O}(D)$ for which there exists some $0<\theta^*<\pi/2$ such that f extends to an analytic function in the union of D and the sector

(ii) $$S=\{z=re^{i\theta}\quad : 1\leq r<R\quad : -\theta^*<\theta<\theta^*\}$$

Moreover, we suppose that f extends to a continuous function on the closed union of $D\cup S$. See figure XXX. With these notations one has

1. Theorem. *Assume that $c_n\to 0$. Then the partial sums $\{s_n(e^{i\theta})\}$ converge uniformly to $f(e^{i\theta})$ one every compact interval of $(-\theta^*,\theta^*)$.*

Proof. To each $n \geq 1$ we consider the function

(i) $$g_n(z) = \frac{f(z) - (c_0 + c_1 z + \ldots + c_n z^n)}{z^{n+1}} \cdot (z + e^{i\theta^*})((z - e^{i\theta^*})$$

This is an analytic function in the domain $D \cup S$. Consider a closed circular interval $\ell = [-\theta_* \leq \theta \leq \theta_*]$ for some $0 < \theta_* < \theta^*$. It appears as a compact subset of $S \cup D$ and it is clear that the required uniform convergence of $\{s_n\}$ holds on ℓ if the g-functions converge uniformly to zero on ℓ,. In fact, this follows since the absolute values

$$|(z + e^{i\theta^*}| \cdot |z - e^{i\theta^*}| \geq (\theta^* - \theta_*)^2 \quad : \quad z \in \ell$$

To prove that the maximum norms $|g_n|_\ell \to 0$ it suffices by the maximum principle for analytic functions to show that g_n converges uniformly to zero on the boundary of the sector S which by the construction contains ℓ. Here ∂S contains the outer circular arc

(i) $$\Gamma = \{Re^{\imath\theta} \quad : \theta_* \leq \theta \leq \theta_*$$

In addition ∂S contains two rays. Let us regard the two pieces of ∂S given by

(ii) $$\Gamma_* = \{z = re^{i\theta^*} \,:\, 0 \leq r \leq 1\} \quad : \quad \Gamma^* = \{z = re^{i\theta^*} \,:\, 1 \leq r \leq R\}$$

There remains to estimate the maximum norms of g_n over each of these pieces of ∂S. Of course, in addition to (ii) we have the contribution when $z = re^{-i\theta^*}$ but by symmetry the subsequent estimates are valid here too. Before we establish the required estimates we introduce a notation. To each integer $m \geq 1$ we set:

(1) $$A_m = M + |c_0| + |c_1| \cdot R + \ldots + |c_m| \cdot R^m \quad : \epsilon_m = \max_{\nu > m} |c_\nu|$$

By hypothesis we have

(2) $$\lim_{m \to \infty} \epsilon_m = 0$$

2. The estimate of $|g_n|_\Gamma$. By assumption f extends continuously to the closure of S so the maximum norm $|f|_S = M$ is finite. If $z \in \Gamma$ we have $|z| = R$ and the triangle inequality gives for each pair $1 \leq M < N$:

(i) $$\big|f(z) - (c_0 + c_1 z + \ldots + c_n z^n)\big| \leq A_m + \epsilon_m(R^{m+1} + \ldots + R^n) \leq A_m + \epsilon_m \cdot \frac{R^{n+1}}{R-1}$$

With the constant

$$K = \max_{z \in \Gamma} |z - e^{i\theta^*}| \cdot |z + e^{i\theta^*}|$$

we therefore obtain

(ii) $$|g_n|_\Gamma \leq \frac{K}{R^{n+1}} \cdot \big(A_m + \epsilon_m \cdot \frac{R^{n+1}}{R-1}\big) = \frac{K \cdot A_m}{R^{n+1}} + \frac{K \cdot \epsilon_m}{R_1}$$

If $\delta > 0$ we use (2) above and find m so that $\epsilon_m < \delta$. Once m is fixed we use that $R > 1$ and hence $\frac{K \cdot A_m}{R^{n+1}} < \delta$ if n is large. Since $\delta > 0$ is arbitrary we conlcude that $|g_n|_\Gamma \to 0$ as required.

3. Estimate of $|g_n|_{\Gamma^*}$. With the same notations as above we consider some $z = re^{i\theta^*}$ with $1 < r < R$ and obtain:

$$\big|f(z) - (c_0 + c_1 z + \ldots + c_n z^n)\big| \leq$$

$$A_m + \epsilon_m(r^{m+1} + \ldots + r^n) \leq A_m + \epsilon_m \cdot \frac{r^{n+1}}{r-1} \implies$$

(i) $$|g_n(re^{i\theta}| \leq \big(A_m + \epsilon_m \cdot \frac{r^{n+1}}{r-1}\big) \cdot \frac{1}{r^{n+1}} \cdot |re^{i\theta^*} - e^{i\theta^*}| \cdot |re^{i\theta^*} - e^{-i\theta^*}|$$

Here $|re^{i\theta^*} - e^{i\theta^*}| = r - 1$ and $|re^{i\theta^*} - e^{-i\theta^*}| \leq 2R$ for all $1 \leq r \leq R$. So with $K = (R^2 - 1)$ we see that (i) gives

(ii) $$|g_n(re^{i\theta}| \leq \frac{K \cdot A_m}{r^{n+1}} \cdot (r-1) + R \cdot \epsilon_m$$

At this stage we use the obvious inequality for $r > 1$:

$$\frac{r-1}{r^{n+1}} < \frac{r-1}{r^{n+1}-1} = \frac{1}{1+r+\ldots+r^n} < \frac{1}{n} \implies$$

$$|g_n|_{\Gamma_*} \leq \frac{KA_m}{n} + 2R \cdot \epsilon_m$$

Since $\epsilon_m \to 0$ the reader concludes that $|g_n|_{\Gamma_*} \to 0$ as $n \to \infty$.

Estimate of $|g_n|_{\Gamma_*}$. With $x = re^{i\theta^*}$ and $0 < r < 1$ the triangle inequality gives

(i) $$\big|f(z) - (c_0 + c_1 z + \ldots + c_n z^n)\big| \leq |c_{n+1}| \cdot |z|^{n+1} + |c_{n+2}| \cdot |z|^{n+2} + \ldots$$

Recall that $\epsilon_n = \max_{\nu > n} |c_\nu|$. Hence (i) is majorized by

$$\epsilon_n \cdot \big(|z|^{n+1} + |z|^{n+2} + \ldots\big) = \epsilon_n \cdot \frac{|z|^{n+1}}{1-|z|}$$

Now $z = re^{i\theta^*}$ and we get as before with $K = \max |re^{i\theta^*} - e^{-i\theta^*}|$:

$$|g_n(z)| \leq \frac{\epsilon_n \cdot \frac{|z|^{n+1}}{1-|z|}}{|z|^{n-1}} \cdot (1-|z|) \cdot K = K \cdot \epsilon_n$$

Again, since ϵ_m can be chosen arbitrary small we conclude that $|g_n|_{\Gamma_*} \to 0$ as $n \to \infty$. This finishes the proof of the Theorem 1.

6. On Laplace transforms.

Let $f(t)$ be a bounded function defined on the real t-line. Consider its Laplace transform

$$L(z) = \int_0^\infty f(t)e^{-zt} \cdot dt$$

which is analytic in the open half-plane $\mathfrak{Re}\, z > 0$. *Assume* that there exists some open subset Ω of $\mathbf{C}$ which contains the closed half-plane $\mathfrak{Re}\, z \geq 0$ such that $L(z)$ extends to an analytic function in Ω. Under this assumption one has

1. Theorem. *There exists the limit*

$$\lim_{T \to \infty} \int_0^T f(t) \cdot dt$$

Moreover, the limit value is equal to $L(0)$ *where* L *under its analytic extension to* Ω *has been evaluated at* $z = 0$.

Proof. To each $T > 0$ we have the entire function

(i) $$L_T(z) = \int_0^T f(t)e^{-zt} \cdot dt$$

Theorem 1 amounts to prove that

(ii) $$\lim_{T \to \infty} L_T(0) = L(0)$$

To prove (ii) we consider certain complex line integrals. If $R > 0$ the assumption on L gives some $\delta > 0$ such that Ω contains the closed set given by the union of the half disc $\bar{D}_R^+ = \bar{D}_R \cap \mathfrak{Re}\, z \geq 0$ and the rectangle

$$\square = \{x + iy \quad : \; -\delta \leq x \leq 0 \quad : \; -R \leq y \leq R\}$$

Let Γ be the boundary of $\bar{D}_R^+ \cup \square$ and introduce the function

$$\text{(iii)} \qquad g(z) = \big[L(z) - L_T(z)\big] \cdot (1 + \frac{z^2}{R^2}) \cdot \frac{e^{zT}}{z}$$

Here $g(z)$ is a meromorphic function in Ω with a simple pole at $z = 0$ whose residue is $L(0) - L_T(0)$. Hence residue calculus applied to g and $\square$ gives:

$$\text{(iv)} \qquad L(0) - L_T(0) = \frac{1}{2\pi i} \cdot \int_\Gamma g(z) \cdot dz$$

Put $B = \max_t\, |f(t)|$ which for every $T > 0$ gives the inequality:

$$|L(z) - L_T(z)| = \big|\int_T^\infty f(t) e^{-zt} dt\,\big| \leq$$

$$\text{(v)} \qquad B \cdot \int_T^\infty \big|e^{-zT}\big| \cdot dt = B \cdot \int_T^\infty e^{-\mathfrak{Re}\, z \cdot T} \cdot dt = B \cdot \frac{e^{-\mathfrak{Re}(z)\dot{T}}}{\mathfrak{Re}(z)} \quad : \quad \mathfrak{Re}(z) > 0$$

Now we begin to estimate the line integral in (iv). Consider first the part of Γ given by the half circle ∂D_R^+. Here we notice that

$$\text{(vi)} \qquad |1 + \frac{R^2 e^{2i\theta}}{R^2}| = |1 + e^{2i\theta}| = 2 \cdot \cos\theta$$

Next, $\frac{dz}{z} = R \cdot d\theta$ holds during the integration on ∂D_R^+ and we also have

$$\frac{1}{\mathfrak{Re}(R \cdot e^{i\theta})} = \frac{1}{R \cdot \cos\theta}$$

Hence (v) and (vi) give

$$(*) \qquad \big|\int_{\partial D^+ R} g(z) \cdot \frac{dz}{z}\big| \leq 2 \cdot B \cdot \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{R \cdot \cos\theta} \cdot d\theta \leq 2 \cdot B \cdot \frac{\pi}{R}$$

There remains to estimate the integral over the part of Γ which belongs to $\partial\square$. Here we simply perform estimates for the two functions $L(z)$ and $L_T(z)$ separately. First, since $L_T(z)$ is entire we can just as well integrate over the half-circle D_R^- where $\mathfrak{Re}(z) < 0$. We notice that

$$|L_T(z)| \leq B \int_0^T e^{-\mathfrak{Re}\, z \cdot t} \cdot dt \leq B \cdot \frac{e^{-\mathfrak{Re}\, z \cdot T}}{\big|\mathfrak{Re}\, z\big|} \quad : \quad \mathfrak{Re}\, z < 0$$

Here $e^{-\mathfrak{Re}\, z \cdot T}$ is large when $z \in D_R^-$ but this factor is cancelled by the absolute value of e^{zT} which appears in the g-function. Hence we obtain

$$(**) \qquad \big|\int_{D_R^-} L_T(z) \cdot (1 + \frac{z^2}{R^2}) \cdot e^{zT} \cdot \frac{dz}{z}\big| \leq B \cdot \int_{\pi/2}^{3\pi/2} \frac{\big|1 + e^{2i\theta}\big|}{R \cdot \big|\cos\theta\,\big|} \cdot d\theta \leq \frac{2\pi \cdot B}{R}$$

Finally, consider the line integral along $\Gamma \cap \partial\square$ where the analytic function $L(z)$ appears. First we regard the line integral along the vertical line where $\mathfrak{Re}(z) = -\delta$ whose absolute value becomes:

$$\text{(vii)} \qquad \big|\int_R^R L(-\delta + iy) \cdot (1 + \frac{(-\delta + iy)^2}{R^2}) \cdot e^{-\delta T} \cdot e^{iyT} \cdot \frac{i \cdot dy}{(-\delta + iy}\big|$$

Notice that we have not imposed any growth condition Here $e^{-\delta T}$ appears and at the same time e^{iyT} has absolute value one. Hence (vii) is estimated by

$$(***) \qquad \max_{-R \leq y \leq R} \big|\frac{L(-\delta + iy) \cdot (1 + \frac{(-\delta+iy)^2}{R^2})}{(-\delta + iy)}\big| \cdot 2R \cdot e^{-\delta T} = M^*(R) \cdot e^{-\delta T}$$

where $M^*(R)$ depends on R only.

For the integrals on the two intervals where $z = -s + iR$ and $z = -s - iR$ with $0 \leq s \leq \delta$ we also get a constant $M^{**}(R)$ which is independent of T while the sum of absolute values of the line integrals over these two lines is estimated by

$$(****) \qquad M^{**}(R) \cdot \int_0^\delta e^{-sT} \cdot ds = M^{**}(R) \cdot \frac{1 - e^{-\delta T}}{T}$$

Now the requested limit formula (ii) follows from the (*)-inequalities above. Namely, for a given $\epsilon > 0$ we first choose R so large that the sum of (*) and (**) is $\leq \epsilon/2$. With R kept fixed we can then choose T so large that (***) and (****) both are $\leq \epsilon/2$ which finishes the proof of Theorem 1.

7. The Lagrange series and the Kepler equation

Let $f(w)$ be an analytic function of the complex variable w defined in some disc of radius R centered at $w = 0$. We assume that $f(0) = 0$ and with another complex variable z we seek an analytic function $w = w(z)$ such that

$$(*) \qquad w(z) = z \cdot f(w(z))$$

We will use residue calculus and Rouche's theorem to find $w(z)$. Let z be fixed for a while and consider some $0 < r < R$ such that

$$\max_{|w|=r} |z \cdot f(w)| < r$$

This means that the analytic function $g(w) = z \cdot f(w)$ has absolute value $< |w|$ on the circle $|w| = r$. Rouche's theorem implies that the analytic function $w - zf(w)$ has a unique simple zero in the disc $|w| < r$. Moreover, by the formula in XX this zero is given by

$$w(z) = \frac{1}{2\pi i} \cdot \int_{|w|=r} \frac{1 - zf'(w)}{w - zf(w)} \cdot w \cdot dw$$

We can evaluate the integral using the series expansion

$$\frac{1}{1 - \frac{z^k \cdot f(w)^k}{w}} = 1 + \sum_{k=1}^{\infty} \frac{(zf(w))^k}{w^k}$$

More precisely, we see that $w(z)$ becomes

$$\frac{1}{2\pi i} \cdot \int_{|w|=r} \sum_{k=1}^{\infty} \frac{(z(f(w))^k}{w^k} - \frac{1}{2\pi i} \cdot \int_{|w|=r} \sum_{k=2}^{\infty} \frac{z^k \cdot f'(w) \cdot f(w))^{k-1}}{w^{k-1}}$$

If $k \geq 1$ residue calculus gives

$$\frac{1}{2\pi i} \cdot \int_{|w|=r} \frac{(z(f(w))^k}{w^k} - \cdot dw = z^k \cdot \frac{f^k)^{(k-1)}(0)}{(k-1)!}$$

Similarly we find

$$\frac{1}{2\pi i} \cdot \int_{|w|=r} \sum_{k=2}^{\infty} \frac{z^k \cdot f'(w) \cdot f(w))^{k-1}}{w^{k-1}} \cdot dw = z^k \cdot \frac{f' \cdot f)^{k-1})^{(k-2)}(0)}{(k-2)!}$$

Next, notice the equality

$$(f^k)^{(k-1)}(0) = k \cdot (f' \cdot f^{k-2})^{(k-2)}(0)$$

Since $\frac{1}{(k-1)!} - \frac{1}{k \cdot (k-2)!} = \frac{1}{k!}$ we conclude that one has the series formula:

$$(*) \qquad w(z) = \sum_{k=1}^{\infty} \frac{(f^k)^{(k-1)}(0)}{k!} \cdot z^k$$

Radius of convergence. The analytic function $w(z)$ has the expansion by the Lagrange series above. The determination of the radius of convergence depends on the given function $f(w)$. A *lower bound* for the radius of convergence is found by the use of Rouche's theorem above. Assume for simplicity that $f(w)$ is an entire function. If $r > 0$ is given we find the positive number $\rho(r)$ for which

$$\rho(r) \cdot \max_{|w|=r} |f(w)| = r$$

By (*) above and Rouche's theorem we have seen that the Lagrange series converges in the disc $|z| < r$. Here we have a *free choice* of r. But each time r is chosen we must take into the account the maximum of $f(w)$ on $|w| = r$. More precisely, put

$$M_f(r) = \max_{|w|=r} |f(w)|$$

Then the discussion above gives

Theorem. *The Lagrange series converges in the disc of the complex z-plane whose radius is*

$$\rho^* = \max_r \frac{r}{M_f(r)}$$

Example. In his far reaching studies of the motion of orbits of those planets which astronomers were able to watch before 1600, Kepler's work contains a study of the equation

$$(*) \qquad \zeta = a + z \cdot \sin \zeta$$

where $a > 0$ is a real constant. We shall determine the series expansion of $\zeta(z)$. Notice that if $w = \zeta - a$ then (*) becomes

$$w = z \cdot \sin(w + a)$$

So with the entire function $f(w) = \sin(w + a)$ we encounter the general case above and conclude that the series becomes

$$\zeta(z) = a + z \cdot \sin a + \sum_{k=2}^{\infty} \frac{z^k}{k!} \cdot \frac{d^{k-1}(\sin^k a)}{da^{k-1}}$$

Exercise. Let $r > 0$ and show that the series $\zeta(z)$ converges when

$$|z| \cdot \frac{e^r + e^{-r}}{2r} < 1 \implies |z| < \frac{2r}{e^r + e^{-r}}$$

and for z in this disc we get $|\zeta(z)| < r$. To obtain a largest possible disc we seek

$$\max_r \frac{2}{e^r + e^{-r}}$$

The reader is invited to calculate the maximum numerically and in this way find a *lower bound* for the radius of convergence of the Kepler series. In contrast to all "heroic computations" by Kepler carried out in the years 1600-1620 and the subsequent refined studies of series expansions by Lagrange around 1760 described above, today's student can use a computer to determine the radius of convergence numerically. This, it is an instructive exercise to determine numerically the radius of convergence of the Lagrange series for each real a. Here it is of course interesting to analyze hos the radius of convergence depends on a.

8. An example by Bernstein.

Let $n \geq 1$ and consider a polynomial $P(z) = a_0 + \ldots + a_n z^n$ of some degree n. We have the equality

$$\sum |a_n|^2 = \frac{1}{2\pi} \cdot \int_0^{2\pi} \left|P(e^{i\theta})\right|^2 \cdot d\theta$$

So if we consider the maximum norm over the unit disc D:

$$||P||_D = \max_\theta |P(e^{i\theta})|$$

then the Cauchy-Schwarz inequality gives

$$(*) \qquad \sum |a_k| \leq \sqrt{n+1} \cdot \sqrt{||P||_D}$$

It turns out that (*) is sharp, i.e for arbitrary large n we can find a polynomial $P_n(z)$ such that

$$(**) \qquad \frac{\left(\sum |a_k|\right)^2}{n+1} \simeq ||P_n||_D$$

The first example of this kind comes from a construction by S. Bernstein from 1914. He considered a prime number p of the form $4\mu+1$. For each integer $1 \leq k \leq p-1$ there exists the Legendre symbol $\binom{k}{p}$which is $+1$ if k is a quadratic remained to p and otherwise -1. Now we get the trigonometric cosine-polynomial

$$B_p(\theta) = \frac{2}{p^{\frac{3}{2}}} \cdot \sum_{k=1}^{p-1} (p-k) \binom{k}{p} \cdot \cos(k\theta) = \sum_{k=1}^{p-1} a_k^{[p)} \cdot \cos(k\theta)$$

Bernstein proved that

$$\max_\theta |B_p(\theta) \leq 1 \quad \text{and} \quad \sum_{k=1}^{p-1} a_k^{[p)} = \frac{p-1}{\sqrt{p}}$$

With $n = 4\mu$ we get the polynomial $Q_n(z)$ where

$$\mathfrak{Re}(Q_n(e^{i\theta}) = B_p(\theta) \quad \text{and} \quad \mathfrak{Im}(Q_n(0)) = 0$$

The maximum norm for $\mathfrak{Im}(P(e^{i\theta})$ is estimated above by the Exercise below. It follows that

$$Q_n(z) = \sum_{k=1}^{p-1} a_k^{[p)} \cdot z^k \quad \text{and} \quad |Q_n|_D \leq C \cdot \log p$$

where C is the absolute constant from Exercise XX below. So for this polynomial the left hand side in (**) becomes $\frac{(p-1)^2}{p^2}$ which is close to 1 when p is large. At the same time $\log p$ is considerably smaller than the degree $n = p-1$. So with

$$P_n(z) = \frac{1}{|Q_n|_D} \cdot Q_n(z)$$

we get a polynomial whose maximum norm is one while the left hand side gets close to one as p increases.

Exercise. Let $u(\theta) = \sum_{k=0}^n a_k \cdot \cos\theta + \sum_{k=1}^n b_k \cdot \sin\theta$ be a trigonometric polynomial of degree n where $\{a_k\}$ and $\{b_k\}$ are real. The conjugate trigonometric polynomial is defined by

$$v(\theta) = \sum_{k=1}^n \left[-b_k \cdot \cos\theta + a_k \cdot \sin\theta\right]$$

Show the integral formula

$$(1) \qquad v(\phi) = \frac{1}{\pi} \cdot \int_0^{2\pi} \frac{\sin\frac{n(\phi-\theta)}{2} \cdot \sin\frac{(n+1)(\phi-\theta)}{2}}{\sin\frac{(\phi-\theta)}{2}} \cdot u(\theta) \cdot d\theta$$

From (1) the reader should verify that if $M = \max_\theta |u(\theta)|$ then

$$|v(\phi)| \leq \frac{M}{\pi} \int_0^{2\pi} \Big|\frac{\sin\frac{n(\phi-\theta)}{2}}{\sin\frac{(\phi-\theta)}{2}}\Big| \cdot d\theta$$

Finally, show that there is an absolute constant C such that

$$\frac{1}{\pi}\cdot\int_0^{2\pi}\Big|\frac{\sin\frac{n(\phi-\theta)}{2}}{\sin\frac{(\phi-\theta)}{2}}\Big|\cdot d\theta\leq C\cdot\operatorname{Log} n\quad:\quad n\geq 2$$

Hence the maximum norm for the conjugate v-function satisfies

$$(*)\qquad \max_{\theta}\,|v(\theta)|\leq C\cdot\operatorname{Log} n\cdot\max_{\theta}\,|u(\theta)|$$

Remark. The inequality above was first demonstrated by Fekete in his article (Journal für mathematik 146).

9. Almost periodic functions and additive number theory.

Introduction. We expose a result presented by Beurling at a seminar at Uppsala University in April 1948. Let $2 \leq m_1 < m_2 < \ldots$ be a strictly increasing sequence of integers. Denote by S the even set given by the union of $\{m_\nu\}$ and $\{-m_\nu\}$. Assume that the additive group generated by the integers in S is equal to $\mathbf{Z}$ which means that the sequence $\{m_\nu\}$ has no common prime number ≥ 2 as factor. Next, consider some non-negative and even function ϕ defined on S. By hypothesis every integer n can be represented by a finite sum of integers from S where repetitions are allowed. Hence we can define the function $\mathfrak{p}_\phi$ on $\mathbf{Z}$ by

$$(1) \qquad \mathfrak{p}_\phi(n) = \min \sum \phi(m_\nu) \quad \text{such that} \quad n = \sum m_\nu$$

where the minimum is taken over finite subsets of S. It is obvious that this function is even and subadditive:

$$\mathfrak{p}_\phi(n_1+n_2) \leq \mathfrak{p}_\phi(n_1) + \mathfrak{p}(n_2)$$

In particular $\mathfrak{p}_\phi(n) = 0$ for all $n \neq 0$ if and only if $\mathfrak{p}_\phi(1) = 0$ and this vanishing holds if and only if for every $\delta > 0$ there exists a finite set $\{m_\nu\}$ in S such that

$$(*) \qquad \sum m_\nu = 1 \quad \text{and} \quad \sum \phi(m_\nu) < \delta$$

We seek conditions on ϕ in order that (*) holds, or equivalently that $\mathfrak{p}_\phi(1) = 0$. To get such a criterion Beurling restricted the attention to a class of ϕ-functions satisfying the following extra condition. An even subset W of $\mathbf{Z}$ is called relatively dense if the additive group generated by W is equal to $\mathbf{Z}$.

9.1 Definition. *Given the even set S above we denote by $AP(S)$ the set of even functions ϕ defined on S such that for every $\epsilon > 0$ the set*

$$S_\epsilon(\phi) = \{m \in S : \phi(m) < \epsilon\}$$

is relatively dense.

The zig-zag function $\rho(x)$. Before Theorem 9.2 is announced we introduce the periodic ρ-function on the real x-line where

$$\rho(x) = |x| \quad : \ -1/2 \leq x \leq 1/2$$

and extended so that $\rho(x) = \rho(x+1)$ hold for every x.

9.2 Theorem. *For each $\phi \in AP(S)$ the necessary and sufficient condition in order that (*) holds is that*

$$(**) \qquad \max_{m \in S} \ \rho(\alpha m) - \eta \cdot \phi(m) \geq 0$$

hold for all pairs $0 < \alpha < 1$ and $\eta > 0$.

Before we enter the proof we recall some facts about almost periodic functions. A bounded complex-valued function f on $\mathbf{Z}$ is almost periodic if there to every $\epsilon > 0$ exists a relatively dense set W such that

$$\max_{n \in \mathbf{Z}} |f(n+w) - f(n)| < \epsilon \quad \text{for all} \quad w \in W$$

From this it follows easily that there exists the mean-value defined by

$$\mathcal{M}(f) = \lim_{b-a \to +\infty} \frac{f(a) + f(a+1) + \ldots f(b)}{b-a+1}$$

Next, for each real number α the exponential function E_α defined by $E_\alpha(n) = e^{2\pi i \alpha n}$ is almost periodic on $\mathbf{Z}$. It follows that when f is almost periodic then there exists the function

$$\mathcal{C}_f(\alpha) = \mathcal{M}(E_\alpha \cdot f)$$

A result due to Harald Bohr asserts that if f is almost periodic and $f(1) \neq 0$ then the $\mathcal{C}_f$-function is not identically zero on $(0,1)$, i.e. there exists some $0 < \alpha < 1$ such that $\mathcal{C}_f(\alpha)$. For the proof of Theorem 9.2 we shall alos need the following:

9.3 Proposition *If ϕ belongs to $AP(S)$ it follows that $\mathfrak{p}_\phi$ is an almost periodic function on* $\mathbf{Z}$.

Exercise. Prove this assertion.

Proof of Theorem 9.2 Suppose first that $\mathfrak{p}_\phi(1) \neq 0$ which means that (*) has no solution for small δ. To show that the inequalitites (**) in Theorem 9.2 cannot hold for all pairs $\alpha.\eta$ we proceed as follows: Since $\mathfrak{p}_\phi$ by defintion is periodic it is in particular almost periodic and by a general formula for $\mathcal{M}$-functions attached to almost periodic functions we get for each integer $m \in S$:

$$|e^{2\pi i\alpha m} - 1| \cdot \mathcal{C}_{\mathfrak{p}_\phi}(\alpha)| \leq \max_n |\mathfrak{p}_\phi(n+m) - p(n)\big| \leq \mathfrak{p}_\phi(m)$$

where the last inequality follows since that $\mathfrak{p}_\phi$ is subadditive. Introducing the sine-function we get

$$2 \cdot |\sin(\pi\alpha m)| \cdot \mathcal{C}_{\mathfrak{p}_\phi}(\alpha)| \leq \mathfrak{p}_\phi(m) \leq \phi(m) \quad : \quad m \in S$$

Since $\mathfrak{p}_\phi(1) \neq 0$ is assumed we know from Bohr's theory that there exists some $0 < \alpha < 1$ such that $\mathcal{C}_{\mathfrak{p}_\phi}(\alpha) \neq 0$. At the same time the zig-zag function satisfies:

$$\rho(x) \leq \frac{\pi}{2} \cdot |\sin \pi \cdot x|$$

for every real x. Hence we get

$$\rho(\alpha \cdot m) \cdot \mathcal{C}(\alpha) \leq \frac{4}{\pi}\phi(m)$$

FINISH ....