Chapter 7. Residue calculus.

Contents

Introduction

The examples in 0.A - 0.J describe general methods which often appear in residue calculus. A more extensive discussion about zeros of polynomials appears in section 0.K. The second part, listed by sections A-M and are devoted to specific examples. We shall often study integrals where multi-valued functions appear as integrands and refer to elementary text-books for more standard examples. Let us remark that the first extensive use of multi-valued integrands appeared in Abel's pioneering work *xxx* from 1827. The interested reader may consult [Abel legacy] where several articles give for an account about Abel's discoveries and the usefulness of complex line integrals in other areas such as algebraic geometry.

Solutions to differential equations. One application of residue calculus is the construction of solutions to ordinary differential equations. Prior to Abel's work specific multi-valued integrands were used by Cauchy, Legendre and Laplace to obtain solutions of differential equations. An example due to Laplace goes as follows: Consider a differential operator of the form

$$P(x,\partial) = x\cdot\partial^m + \sum_{\nu=0}^{\nu=m-1}(a_\nu + b_\nu x)\partial^\nu$$

where $\partial = \frac{d}{dx}$ and $m \geq 2$, while $\{a_\nu\}$ and $\{b_\nu\}$ are complex numbers. To find solutions we consider the polynomials

$$Q(t) = t^m + \sum_{\nu=0}^{\nu=m-1} b_\nu t^\nu \quad : \quad P(t) = \sum_{\nu=0}^{\nu=m-1} a_\nu t^\nu$$

Assume that Q has simple zeros $\beta_1,\ldots,\beta_m$ which in general are complex numbers. Newton's fractional decomposition gives

$$\frac{P}{Q} = \sum \frac{c_\nu}{t-\beta_\nu} \quad : \; c_\nu = \frac{P(\beta_\nu)}{Q'(\beta_\nu)}$$

Set

(i)
$$Z(t) = \frac{1}{Q(t)}\cdot e^{\int \frac{P}{Q}} = \frac{1}{Q(t)}\cdot \prod (t-\beta_\nu)^{c_\nu}$$

Let γ be a simple and rectifiable curve in the complex t-plane which avoids the zeros of Q. Along γ there exists a single-valued branch of Z and we try to define the function

$$y(x) = \int_\gamma Z(t)\cdot e^{xt}\,dt$$

Let t_* and t^* be the end-points of the oriented curve γ. Now

(ii)
$$P(x,\partial)(y) = \int_\gamma (xQ(t) + P(t))Z(t)\cdot e^{xt}\,dt$$

Since the t-derivative of e^{xt} is xe^{xt}, a partial integration identifies (ii) with

$$Q(t)Z(t)\cdot e^{xt}\big|_{t_*}^{t^*} + \int_\gamma (-\frac{d}{dt}(Q(t)Z(t)) + P(t))Z(t))\cdot e^{xt}\,dt$$

From (i) it is clear that $\frac{d}{dt}(Q(t)Z(t)) = P(t))Z(t)$ and hence (ii) is zero if

(iii)
$$Q(t_*)Z(t_*)\cdot e^{xt_*} = Q(t^*)Z(t^*)\cdot e^{xt^*}$$

This gives a method to find solutions to the equation (*), i.e. one seeks γ-curves such that (iii) hold. Here one can take unbounded and non-closed γ-curves. For example, if $x < 0$ we notice that if $t_* = t^*$ and this common number has a large postive real parti, then the absolute value of e^{xt_*} is small compared to $Q(t_*)Z(t_*)) + P(t_*))Z(t_*))$. In particular (iii) holds if γ is an infinite simple curve which starts at $+\infty$ and moves for a while in the negative direction on the positive real t-line to some number t_0,and then continues via a simple closed curve γ_0 and the final piece of γ is $[t_0,+\infty)$. Thus, every such γ-curve gives a solution $y(x)$ defined for $x < 0$.

If we instead seek solutions defined when $x > 0$ one picks γ-curves which start at $-\infty$ and move to $-t_0$ and continues along a closed curve γ_0, and return to $-\infty$ in the negative diretion on the interval $t \leq -t_0$. In the first consstruction we notice that the solutions $y(x)$ actually become analytic functions of the complex variable $z = x+iy$ which are defined in the half-plane $\mathfrak{Re}(z) < 0$. Since P has order m it follows from general facts that the number of $\mathbf{C}$-linearly independent solutions which are analytic in this half-space is an m-dimensional complex vector space. So in spite of the general constructons via a choice of γ-curves above, the resuting y-functions can only generate an m-dimensional vector space. The merit of the Laplace construction is that there actually exists an $m$tuple of γ -curves as above whose the resulting y-functions give a basis for the solution space in the half-plane $\mathfrak{Re}(z) < 0$. A similar result holds for solutions in the left half-plane. Of course, after this has been achieved there still remains to find global solutions to (*). It turns out that they exist but since x appears as a factor for the highest differential ∂^m, the solutions are not always entire functions, i.e one should also allow distribution solutions. We shall discuss this in more detail in § xx and mention only that the space of distribution solutions to (*) is a complex vecotr space of dimension $m+1$ and there exist distribution solutions supported by the half-line $x \leq 0$. So the Laplace constructions give a start, but not the whole story about solutions to (*). A more favourable case occurs when $P(x,\partial) = \partial^m + P_*(x,\partial)$ where P_* has order $\leq m-1$. In this case the solutions extend to entire functions and form an m-dimensional complex vector space. See § xx for some specifix exampes where we shall employ the Laplace construction.

A calculation by Riemann. In the complex z-plane we consider the line

$$\Gamma = \{z = a + se^{3\pi i/4}\}$$

where $0 < a < 1$ and $-\infty < s < \infty$. Along Γ we have

$$z^2 = a^2 + 2as \cdot e^{3\pi i/4} - is^2$$

We get an entire function $\Phi(w)$ of a new complex variable defined by

$$(*) \qquad \Phi(w) = \int_\Gamma \frac{e^{-\pi iz^2 + 2\pi iws}}{e^{\pi iz} - e^{-\pi iz}}\, dz$$

In fact, this holds since the integrand contains e^{-s^2} as a factor which ensures the convergence for all complex w. In § xx we dexribe how Riemann obtained the equation:

$$(**) \qquad \Phi(w) = \frac{e^{\pi iw} - e^{\pi iw^2}}{e^{\pi iw} - e^{-\pi iw}}$$

More generally, if τ is another complex variable one has the analytic function

$$(***) \qquad \Theta(w,\tau) = \int_\Gamma \frac{e^{\pi i\tau z^2 + 2\pi iws}}{e^{\pi iz} - e^{-\pi iz}}\, dz$$

which is analytic when $\mathfrak{Re}\,\tau < 0$ and $w \in \mathbf{C}$. Residue calculus is used while the integration contour is changed to discover certain functional equations and attain formulas for the Θ-function. Let us remark that formulas such as (**) led Riemann to his famous hypothesis about zeros of the zeta-function. See § x in Chapter VI for further comments. Studies of more involved integrals were later carried out by Weierstrass.

Multi-valued integrands. The complex log-function and fractional powers z^α where α is not an integer appear often as integrands and during integrations. A typical case is when $\sqrt{1-x^2}$ appears in the integrand and integration is over the real interval $[-1,1]$. Here one starts from the single-valued analytic function $g(z)$ defined in $\mathbf{C} \setminus [-1,1]$ by

$$g(z) = z \cdot \sqrt{1 - z^{-2}}$$

To be precise, consider the extended complex plane where the point at infinity is added and then $\Omega = \mathbf{C} \cup \infty \setminus [-1,1]$ is simply connected which implies that there exists the single-valued branch of the square root of $1 - z^{-2}$, i.e.

$$g(z) = z \cdot \sqrt{1 - z^{-2}}$$

is defined in the whole of Ω where g has a simple pole at ∞. If $z = iy$ is purely imaginary then

$$g(iy) = iy \cdot \sqrt{1 + y^{-2}}$$

This implies that

$$g(iy) = i \cdot \sqrt{1 + y^2} \quad \text{when} \quad y > 0$$

while $g(iy) = -i\sqrt{1+y^2}$ when $y < 0$. The change of sign is crucial when reside calculus is employed. For example, consider the integral

$$(*) \qquad J = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

Take the g-function above and construct the complex line integral over the closed curve γ which consists of the two line segments from $-1 + i\epsilon$ to $1 + i\epsilon$ respectively $-i\epsilon$ to $1 - i\epsilon$ together with two small half-circles. See figure XX. Using the change of sign and passing to the limit as $\epsilon \to 0$ it follows that

$$2J = -i \cdot \int_{|z|=R} \frac{dz}{g(z)}$$

where we have taken a line integral over circles of radius $R > 1$. Passing to the limit as $R \to +\infty$ one easily verifies that

$$(**) \qquad J = \pi$$

Let us remark that the last equality can be proved directly since

$$(***) \qquad 2J = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

and here (***) is computed via the variable substitution $x \to \sin\,\theta$ which gives (**). This example has served to illustrate a method. In more involved cases residue calculus is needed to attain exact formulas.

Exercise. Let $0 < a < 1$ and consider the integral

$$J = \int_0^1 \frac{dx}{(1-x)^a \cdot x^{1-a}}$$

In the domain $\mathbf{C} \setminus [0,1]$ there exists the analytic function

$$g(z) = z(1 - \frac{1}{z})^a$$

Use that $(-1)^a = e^{\pi i a}$ to show that

$$J = \frac{\pi}{\sin(\pi a)}$$

Next, let $m \geq 3$ be an integer and put

$$J_m = \int_0^1 \frac{dx}{1 - x^m)^{1/m}}$$

The substitution $x \mapsto t^{1/m}$ gives

$$J_m = \frac{1}{m} \int_0^1 \frac{dt}{t^{1/m-1} \cdot (1-t)^{1/m}} = \frac{\pi}{m \cdot \sin(\pi/m)}$$

Notice that the last term converges to 1 as $m \to +\infty$. The reader should discover that this limit is a consequence of Neper's limit formula for e.

Complex line integrals and homotopy. Remove the three points $0, 1, \infty$ from the Riemann sphere. The open complement Ω is not simply connected and has a fundamental group generated by a pair of closed and simple curves Γ_1, Γ_0 where $\Gamma_1 = \{|z-1| = 1/2$ and $\Gamma_2 = \{|z| = 1/2\}$. In algebraic topology one learns that the fundamental group $\pi_1(\Omega$ is a free group generated by the homotopy classes of these two curves. It means for example that the composed curve

$$(*) \qquad \Gamma_1^{-1} \circ \Gamma_0^{-1} \circ \Gamma_1 \circ \Gamma_0$$

is not homotopic to a trivial curve. Even though this assertion is intuitively clear the formal proof is not so easy. While algebraic topology was developed by Poincaré and his contemporaries, a convincing method to prove that (*) is not homotopic to a trivial curve is to perform a complex line integral of a suitable mult-valued analytic function and pursue its analytic continuation by the Weierstrass' procedure. In § xx we expose how this is done by an explicit calculation of a complex line integral along (*) using a special multi-valued function as integrand.

Example from physics. Historically many residue formulas were established via equations derived by physical laws. So here one encounters important situations where residue calculus is needed. For example, Gauss used residue calculus to establish equations in electro-magnetic fields governed by the Biot-Savart Law. Another area where residue calculus appears naturally is hydromechanics. The interested reader should consult the excellent text-book [XXX] by Horace

Lamb which contains many instructive examples with physical background. For more advanced material related to quantum mechnaics the reader will find applications of residue calculus in the text-book series by L.D Landau and E.M. Lifschitz.

The simple pendelum. A fundamental function appears during the motion of a simple pendelum. Consider a particle p of unit mass attached at the end-point of a rigid bar of some length ℓ whose other end is suspended at a fixed point while the bar and p oscillates in a vertical plane where gravity is the sole external force. The system has one degree of freedom expressed by the angle θ between the bar and the vertical line which is directed downwards. The kinetic energy of the one-point system becomes

$$T = \frac{\ell^2}{2} \cdot \dot{\theta}^2$$

The equation of motion becomes

$$\ell^2 \ddot{\theta} = -g\ell \cdot \sin\theta$$

With initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = v > 0$ the time-dependent θ-function satisfies the differential equation

$$\dot{\theta}^2 = \frac{2g}{\ell} \cdot (\cos\theta - 1) + v^2$$

We assume that v us not to large, i.e.

$$v^2 < \frac{2g}{\ell}$$

Then there exists $0 < \theta^* < \pi/2$ such that

$$\cos\theta^* = 1 - \frac{\ell v^2}{2g}$$

Now $t \mapsto \theta(t)$ oscillates between $-\theta^*$ and θ^*. The time for a quarter of the whole period, i.e. the time to reach θ^* becomes

$$T = \int_0^{\theta^*} \frac{d\theta}{\sqrt{\frac{2g}{\ell} \cdot (\cos\theta - 1) + v^2}}$$

This formula shows that the determination of exact T-values with varying intial velocity v boils down to study the function

$$\theta^* \mapsto \int_0^{\theta^*} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta^*}}$$

Admitting the inverse arccos-function as "elementary" and using the substitution $\cos\theta \to x$ we are led to consider integrals of the form

$$J(a) = \int_a^1 \frac{dx}{\sqrt{(1-x^2) \cdot (x-a)}}$$

where $0 < a < 1$. This J-function belongs to a class of functions which were investigated by Legendre and Jacobi. Let us also remark that not only the numerical value of $J(a)$ as $0 < a < 1$ is of interest here. It turns out that this real-analytic function defined on $(0, 1)$ extends to a multi-valued analytic function in $\mathbf{C} \setminus \{0, 1\}$ where it satisfies a Fuchsian differential equation which gives a further motivation for including $J(a)$ in a class of "elementary functions".

Example by Huyghens. Let p be a particle of unit mass which moves on the horizontal (x, y)-plane where no friction is present and gravity does not affect the motion. The particle slides on an infinite bar suspended at the origin which can rotate and the bar has no mass. So we have a particle systems with two degrees of freedom where the position of the mass point is given in polar coordinates (r, θ). Here θ is the angle between the bar and the positive x-axis. At time $t = 0$ we suppose the initial conditions are

$$\theta(0) = 0 \quad : \dot{\theta}(0) = \omega \quad : r(0) = A \quad : \dot{r}(0) = 0$$

where ω and A are positive. in this situation we have Kepler's identity

$$r^2 \cdot \dot{\theta} = A^2\omega \tag{1}$$

We also get the differential equation

$$\dot{r}^2 + \frac{A^4\omega^2}{r^2} = A^2\omega^2 \tag{2}$$

Exercise. Express θ as a function of r and use this to prove that the increasing time dependent function $\theta(t)$ has a limit as $t \to +\infty$. More precisely a calculation gives the formula

$$\lim_{t\to\infty} \theta(t) = \int_1^\infty \frac{ds}{s \cdot \sqrt{s-1}} \tag{3}$$

Huyghens, Newton and Wallis performed used series expansions to show that (3) is equal to π. The reader is invited to prove this equality using residue calculus. Thus, as time increases the bar moves from the position along the x-axis to positions which come closer to the positive $y$axis. Notice that the limit formula is independent of the pair ω and A. The reader should contemplate upon this by reflecting over daily life experience of the centrifugal force.

Fourier transforms. On the real x line we have the function $\frac{1}{x-i}$. It is not integrable but defines a tempered distribution μ whose Fourier transform is defined when $\xi \neq 0$ by the integral

$$\widehat{\mu}(\xi) = \int_{-\infty}^\infty \frac{e^{-i\xi x}}{x-i} \cdot dx \tag{1}$$

If $\xi < 0$ and $z = x + iy$ with $y > 0$ the absolute value $|e^{-i\xi z}| = e^{\xi y}$ decreases when $y \to +\infty$. Using this fact the reader can verify that

$$\widehat{\mu}(\xi) = 2\pi i \cdot e^{\xi}$$

holds when $\xi < 0$ and reversing the sign verify that $\widehat{\mu}(\xi) = 0$ when $\xi > 0$. On the reader ξ-line we have the tempered distribution defined by e^ξ when $\xi < 0$ and zero if $\xi \geq 0$. Its inverse Fourier transform becomes

$$\frac{1}{2\pi} \cdot 2\pi i \int_{-\infty}^0 e^{ix\xi} \cdot e^{\xi} \cdot d\xi = \frac{i}{ix+1} = \frac{1}{x-i}$$

This confirms the calculations using residues via Fourier's inversion formula.

Fourier transforms on $\mathbf{R}^+$. Let $0 < a < 1$ and β is a complex number outside the non-negative real line. When $\zeta = \alpha + is$ is complex we set

$$J(\zeta) = \int_0^\infty x^\zeta \cdot \frac{x^a}{x-\beta} \cdot \frac{dx}{x} \tag{*}$$

Sincc $|x^{\alpha+is}| = x^\alpha$ when x is real and positive we see that (*) converges when

$$a < \alpha < 1 - a \tag{1}$$

After the reader has become familiar with this calculus it is an easy exercise to verify the equation:

$$J \cdot \left[1 - e^{2\pi i(\zeta+a-1)}\right] = 2\pi i \cdot \beta^{\zeta+a-1}$$

Specific examples. Let $\beta = b$ where $b > 0$ which gives

$$\beta^{\zeta+a-1} = b^{\zeta+a-1} \cdot e^{\pi i(\zeta+a-1)}$$

where we used that $-1 = e^{\pi i}$. Suppose also that $\zeta = \alpha$ is real which gives

$$J \cdot \left[1 - e^{2\pi i(\alpha+a-1)}\right] = 2\pi i \cdot b^{\alpha+a-1} \cdot e^{\pi i(\zeta+a-1)}$$

Using the formula for the complex sine-function the reader may verify that

$$J = \pi \cdot \frac{b^{\alpha+a-1}}{\sin\, \pi(1-a-\alpha)}$$

Since $1 - a - \alpha > 0$ is assumed the formula shows that J is real and positive which it should be since the choice gives

$$J = \int_0^\infty \frac{x^{\alpha+a}}{x+b} \cdot \frac{dx}{x}$$

where the integrand is real and positive. Next, consider the case where $\zeta = is$ is purely imaginary and $\beta = b$ with $b > 0$ while $a = 1/2$. The general formula (xx) gives

$$J \cdot [1 - e^{-2\pi s} \cdot e^{-2\pi i/2}] = 2\pi i \cdot b^{is-1/2} \cdot e^{\pi s} \cdot e^{-\pi i/2}$$

Since $e^{-\pi i} = 1$ and $e^{-\pi/2} = i$ while $i^2 = -1$ it follows that

$$J(1 + e^{-2\pi s}) = e^{-\pi s} \cdot 2\pi \cdot b^{is-1/2}$$

Introducing the complex cosine-function we get the formula

$$\int_0^\infty x^{is} \cdot \frac{\sqrt{x}}{x+b} \cdot \frac{dx}{x} = \pi \cdot b^{is-1/2} \frac{1}{\cos(\pi is)}$$

Remark. The last equation yields a formula for the Fourier transform of the L^1-function $\frac{\sqrt{x}}{x+b}$ on the multiplicative line $\mathbf{R}^+$ where $\frac{dx}{x}$ is the Haar measure. Replace is by the complex variable ζ and set

$$J(\zeta) = \int_0^\infty x^\zeta \cdot \frac{\sqrt{x}}{x+b} \cdot \frac{dx}{x}$$

Then the computations above show that

$$J(\zeta) = \frac{\pi \cdot b^{\zeta - 1/2}}{\cos(\pi\zeta)}$$

Notice that the right hand side is an analytic function in the strip domain $1/2 < \mathfrak{Re}\zeta < 1/2$ while we encounter poles when $\zeta = 1/2$ or $-1/2$ whose appearance is clear from the integral which defines $J(\zeta)$ because we get divergent integrals in these two cases. At the same time (xx) gives a meaning to the integral (xx) for al complex ζ, i.e. the result is a globally defined meromorphic functions with simple poles at the zeros of the complex cosine-function. This illustrates the usefulness of residue calculus since it was needed to get the precise formula (xx) above.

Principal values. Consider the integral

(*) $$J(a) = \int_0^1 \frac{dx}{x-a}$$

where $0 < a < 1$. The principal value is defined by:

(**) $$\lim_{\epsilon \to 0} \Big[\int_0^{a-\epsilon} \frac{dx}{x-a} + \int_{a+\epsilon}^1 \frac{dx}{x-a} \Big]$$

When $0 < \epsilon < a$ and $a + \epsilon < 1$ we can evaluate both integrals and get:

$$\int_0^{a-\epsilon} \frac{dx}{x-a} = -\log\epsilon + \log a \quad : \quad \int_{a+\epsilon}^1 \frac{dx}{x-a} = \log\epsilon - \log(1-a)$$

where it is not even necessary to perform a limit since (**) takes the same value for all $0 < \epsilon < a$. In particular we get the formula

(***) $$J(a) = \log\frac{1-a}{a}$$

The construction (**) can be understood by complex integrals. Namely, for any real number $a > 0$ there exists the complex log-function

$$\log(z-a)$$

with a single valued branch in the upper half-plane $\mathfrak{Im}(z) > 0$ whose complex derivative is $\frac{1}{z-a}$. For $\epsilon > 0$ we can take the line integral on the horizontal line from $i\epsilon$ to $1 + i\epsilon$ which gives:

(1) $$\int_0^1 \frac{dx}{x-a+i\epsilon} = \log 1 - a + i\epsilon - \log(-a + i\epsilon)$$

Passing to the limit as $\epsilon \to 0$ the right hand side becomes

(2) $$\log(1-a) - \log a - \pi i = \log\frac{1-a}{a} - \pi i$$

To clarify this limit formula we rewrite the left hand side in (1) which amounts to compute

$$\int_0^1 \frac{(x-a-i\epsilon)\cdot dx}{(x-a)^2+\epsilon^2} \tag{3}$$

Separating real and imaginary parts it follows from (1) that one has the two limit formulas:

$$\lim_{\epsilon\to 0}\int_0^1 \frac{(x-a)\cdot dx}{(x-a)^2+\epsilon^2} = \log\frac{1-a}{a} \quad \text{and} \quad \lim_{\epsilon\to 0}\int_0^1 \frac{\epsilon\cdot dx}{(x-a)^2+\epsilon^2} = \pi \tag{4}$$

Exercise. Clarify why the first formula in (4) agrees with the previously defined principal value integral. Prove also the second formula using the arctan-function.

Another example. Consider the integral:

$$J = \int_0^\infty \frac{1}{x}\cdot \log\Bigl(\frac{|x+1|}{|x-1|}\Bigr)\cdot dx \tag{1}$$

The reader may verify that the integrand in (1) is a continuous function whose value at $x=0$ is equal to 2 and when $|x|\to\infty$ the integrand decays as x^{-2}. So we have an absolutely convergent integral. Residue calculus is used to compute the integral. The idea is to consider the function

$$g(z) = \frac{1}{z}\cdot\log\frac{z+1}{z-1}$$

which is analytic in the upper half-plane.

Exercise. For a large R and a small $\epsilon > 0$ we take the complex line integral of g along the closed curve Γ which consists of the real interval $[\epsilon, R]$, the quarter circle $\{z = Re^{i\theta}$ where $0 \leq \theta \leq \pi/2$ and the imaginary interval $[i\epsilon, iR]$ and finally the small quarter circle of radius ϵ. Since $g(z)$ is analytic the complex line integral over Γ is zero. The reader may verify that

$$\lim_{R\to\infty}\int_0^{\pi/2} g(Re^{i\theta})\cdot iRe^{i\theta}\cdot d\theta = 0 \tag{2}$$

Next, the integral along the imaginary line $\epsilon \leq y \leq R$ where the line integral taken in the opposite direction becomes

$$-\int_\epsilon^R \log\Bigl(\frac{iy+1}{iy-1}\Bigr)\cdot\frac{dy}{y}$$

Since $|iy+1| = |iy-1|$ this integral is purely imaginary. Regarding the real part and using (2) above the reader should verify that:

$$J = \lim_{R,\epsilon}\int_\epsilon^R \frac{1}{x}\cdot\log\Bigl(\frac{|x+1|}{|x-1|}\cdot dx = \lim_{\epsilon\to 0}\mathfrak{Re}\int_0^{\pi/2} g(\epsilon e^{i\theta})\cdot i\epsilon e^{i\theta}\cdot d\theta$$

Notice that

$$i\epsilon e^{i\theta}\cdot g(\epsilon e^{i\theta}) = i\cdot\log\frac{\epsilon e^{i\theta}+1}{\epsilon e^{i\theta}-1} = i\cdot\log\, 1+\epsilon e^{i\theta}) - i\cdot\log(-1+\epsilon e^{i\theta}) =$$

Now

$$\lim_{\epsilon\to 0}\log\,(-1+e^{i\theta}) = \pi\cdot i \quad \text{and} \quad \lim_{\epsilon\to 0}\log\,\epsilon(1+e^{i\theta}-1) = 0$$

It follows that

$$\lim_{\epsilon\to 0}\int_0^{\pi/2} g(\epsilon e^{i\theta})\cdot i\epsilon e^{i\theta} = \pi^2/2$$

Hence the integral from (1) has the value:

$$J = \frac{\pi^2}{2}$$

0:A Four examples of residue calculcus.

0.1 Example. Let $P(z)$ and Q be two polynomials where $\deg(P) \geq \deg(Q)+1$ and $-1 < a < 0$ is a real number. Assume that P has no zeros on the non-negative real axis and set:

$$J = \int_0^\infty \frac{x^a \cdot Q(x)}{P(x)} \cdot dx$$

To find J we consider the function $g(z) = \frac{z^a \cdot Q(z)}{P(z)}$ which is multi-valued outside the origin. The trick is to integrate g over a contour starting from $x = \epsilon > 0$ until $x = R$ is reached, followed by an integral taken over the circle $|z| = R$ and after one returns from R to ϵ on the x-axis and finish by an integral over the circle $|z| = \epsilon$ which is performed clock-wise, i.e. in the negative direction. Passing to the limit as $R \to +\infty$ and $\epsilon \to 0$ one uses the multi-valued behaviour of z^α and get

$$(*) \qquad (1 - e^{2\pi i a}) \cdot J = 2\pi i \cdot \sum \mathfrak{res}(g : \alpha_\nu)$$

where the sum is taken over the zeros of P.

0.1.1 Exercise. Take $Q = 1$ and $P = z - i$ above. Then we get

$$(1 - e^{2\pi i a}) \cdot J = 2\pi i \cdot i^a = 2\pi i \cdot e^{\pi i a/2}$$

Consider the case $a = -\epsilon$ with a positive ϵ. Since $\frac{1}{z-i} = \frac{z+i}{z^2+1}$ we get

$$J = \int_0^\infty \frac{x^{-\epsilon}(x+i)}{1+x^2} \cdot dx$$

It is instructive to check the equation (*) via (xx). Separating the real and imaginary part the reader may verify the formula

$$\int_0^\infty \frac{x^{1-\epsilon}}{1+x^2} \cdot dx = \pi \cdot \frac{1 + \cos(\pi\epsilon)}{(2 + 2\cos(\pi\epsilon) \cdot \sin(\pi\epsilon/2)}$$

To check this formula we consider a limit as $\epsilon \to 1$. Since $\cos \pi = 0$ the reader may verify that the limit in the right hand side becomes $\frac{\pi}{2}$ which is okay since we know from calculus that

$$\lim_{\epsilon \to 0} \int_0^\infty \frac{x^{1-\epsilon}}{1+x^2} \cdot dx = \int_0^\infty \frac{1}{1+x^2} \cdot dx = \frac{\pi}{2}$$

0.2 Example. Let P, Q be a pair of polynomials where $\deg(P) \geq \deg(Q) + 2$ and consider the integral

$$J = \int_0^\infty \frac{Q(x)}{P(x)} \cdot dx$$

To overcome the lack of a multi-valued integrand we use the complex log-function $\log z$ and define

$$g(z) = \frac{Q(z)}{P(z)} \cdot \log z$$

Perform an integral of g over the same contour as in the previous example. After one turn around $|z| = R$ $\log z$ has changed its branch with the constant $2\pi i$ and since the "home run" is the integral from R back to the origin we get:

$$(*) \qquad J = -\sum \mathfrak{res}(g : \alpha_\nu)$$

where the sum is taken over the zeros of P. For example, take $Q = 1$ and $P(z) = (z+a)(z+b)$ where $a > b > 0$ are positive real numbers. Then the right hand side in (*) becomes

$$-[\frac{\log a + \pi i}{b - a} + \frac{\log b + \pi i}{a - b}] = \frac{\log(a/b)}{a - b}$$

It is always good to confirm a general formula. Take $b = 1$ while $a = 1 + \epsilon$ and pass to the limit as $\epsilon \to 0$ which gives:

$$\int_0^\infty \frac{dx}{(x+1)^2} = \lim_{\epsilon \to 0} \frac{\log(1+\epsilon)}{\epsilon} = 1$$

Next, consider the integral

$$J=\int_0^\infty \frac{dx}{1+x^3}$$

The polynomial $1+z^3$ has simple roots $j_1=e^{\pi/3}, j_2=1, j_3=e^{5\pi/3}$. The formula (*) above gives

$$J=-\frac{1}{3}\cdot\sum\frac{\log j_\nu}{j_\nu^2}=\frac{1}{3}\cdot\sum j_\nu\cdot\log j_\nu$$

where the last equality holds since $j_\nu^3=1$ for each ν. The reader should check that the last sum becomes

$$\frac{i}{3}(j_1\cdot\pi/3+j_2\cdot\pi+j_3\cdot 5\pi/3)=\frac{i}{3}\cdot(i\sqrt{3}/2\cdot\pi/3-i\sqrt{3}/2\cdot 5\pi/3)=\frac{2\pi}{3\cdot\sqrt{3}}$$

0.3 Example. Let $a>0$ be real and put

$$J=\int_{-1}^1\frac{dx}{(1+ax^2)\sqrt{1-x^2}} \tag{0.1}$$

To compute this integral we use the analytic function defined in the complement of the real interval $-1\le x\le 1$ by

$$g(z)=z\cdot\sqrt{1-z^{-2}}$$

The reader should verify the two limit formulas:

$$\lim_{\epsilon\to 0} g(x+i\epsilon)=\cdot\sqrt{1-x^2}\quad:\quad 1<x<1 \tag{2}$$

$$\lim_{\epsilon\to 0} g(x-i\epsilon)=-i\cdot\sqrt{1-x^2}\quad:\quad 1<x<1 \tag{3}$$

Now we consider the function

$$f(z)=\frac{1}{(1+az^2)\cdot g(z)}$$

It has to simple poles when $1+az^2=0$, i.e. we find purely imaginary poles at plus and minus $i\cdot a^{-1/2}$. Notice also that

$$\lim_{R\to\infty}\int_{|z|=R} f(z)\cdot dz=0 \tag{4}$$

We use (4) and apply residue calculus while we integrate over closed curves around $[-1,1]$ as illustrated by figure XX. Using (2-3) it follows that

$$\frac{2}{i}\cdot J=2\pi\cdot i\cdot\frac{2}{2a\alpha\cdot g(\alpha)} \tag{5}$$

where $\alpha=i\cdot a^{-1/2}$ and we used that g is odd when the two residues are added. At this stage the reader can verify that

$$J=\frac{\pi}{\sqrt{1+a}}$$

0.4 Example. Consider the integral

$$J=\int_{-\infty}^\infty\frac{e^{ax}}{e^x-iA}\cdot dx \tag{0.4}$$

where $0<a<1$ and $A>0$ is real. To find J we use the meromorphic function $g(z)=\frac{e^{az}}{e^z-iA}$ which has simple poles when $e^z=iA$. Consider the complex line integral taken over the boundary of a rectangle

$$\square=\{-R\le x\le R\}\times\{0\le y\le 2\pi\}$$

The reader should verify that $e^z - iA$ has a simple zero at the point $\log A + i\pi/2$ which therefore gives a simple pole of $g(z)$. We have also $g(x+2\pi i) = e^{2\pi ia} \cdot g(x)$ when x is real. When $R \to +\infty$ we get a limit where residue calculus gives:

$$(*) \qquad (1 - e^{2\pi ia}) \cdot J = 2\pi i \cdot \mathfrak{res}\big(g(z) : \log A + i\pi/2\big) = 2\pi i \cdot \frac{A^a \cdot i^a}{iA} = 2\pi A^{a-1} \cdot e^{a\pi i/2}$$

0:B Summation formulas.

Various sums are often computed using the meromorphic function:

$$(*) \qquad g(z) = \frac{\cos \pi z}{\sin \pi z}$$

It has simple poles at all integers with the common residue $\frac{1}{\pi}$. Consider a pair of polynomials P, Q where $\deg P \geq 2 + \deg(Q)$ and $P(n) \neq 0$ hold at all integers. Residue calculus gives the summation formula

$$(**) \qquad \frac{1}{\pi} \cdot \sum_{-\infty}^{\infty} \frac{Q(k)}{P(k)} = \sum \mathfrak{res}(g(z) \cdot \frac{Q(z)}{P(z)} : \alpha_\nu)$$

where the right hand is the sum of residues over all zeros of P. As an illustration we take $P(z) = z^2 + 1$ which has simple zeros at i and $-i$. Since

$$g(z) = i \cdot \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}$$

a computation shows that the right hand side in (**) becomes $\frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}$. Hence

$$\sum_{-\infty}^{\infty} \frac{1}{1+k^2} = \pi \cdot \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}$$

0.B.1 Exercise. Let α be a complex number which is not an integer. Show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{k+\alpha)^2} = \frac{\pi^2}{\sin^2 \pi\alpha}$$

0.B.2 Exercise. Certain summation formulas can be established directly without residue calculus. Consider the meromorphic function

$$g_*(z) = \frac{\pi}{\sin \pi z}$$

It has simple poles at all integers and we can write out an infinite sum of rational functions which will match these poles. Namely, consider

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (\frac{1}{z-2n} + \frac{1}{z+2n}) - \sum_{n=0}^{\infty} (\frac{1}{z-2n-1} + \frac{1}{z+2n+1})$$

It is easily seen that $g_* - g$ has no poles and the reader should verify that this entire is bounded and hence a constant and finally that this constant is zero. We can express g via a series where the convergence for z-values outside the set of integers is expressed more directly, i.e. one has

$$(***) \qquad \frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - 4n^2} - \sum_{n=0}^{\infty} \frac{2z}{z^2 - (2n+1)^2}$$

For example, with $z = 1/4$ one gets

$$\sqrt{2} \cdot \pi = 4 - \sum_{n=1}^{\infty} \frac{8}{48n^2 - 1} + \sum_{n=0}^{\infty} \frac{8}{48(2n+1)^2 - 1}$$

The right hand side is an infinite sum of rational numbers while the trancendental number π appears in the left hand side. So the formula is quite remarkable.

0.B.3 Wallis limit formula. There exists a meromorphic function with simple poles at all integers defined by the series

$$g_*(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - j^2}$$

At the same time we have the function from (*) which also has simple poles at the integers with residues $\frac{1}{\pi}$.

0.B.4 Exercise. Show that

$$\text{(i)} \qquad \frac{\cos \pi z}{\sin \pi z} = \frac{1}{\pi} \cdot g_*(z)$$

Next, use that the derivative of $\sin \pi z$ is equal to $\cos \pi z$ and deduce the product formula

$$\text{(ii)} \qquad \sin \pi z = \pi z \cdot \prod_{n=1}^{\infty} \bigl(1 - \frac{z^2}{n^2}\bigr)$$

Next, take $z = 1/2$ in (ii). If $N \geq 2$ we consider partial products in the right hand side which gives the limit formula

$$\text{(iii)} \qquad \lim_{N \to \infty} \prod_{n=1}^{n=N} \bigl(1 - \frac{1}{4 \cdot n^2}\bigr) = \frac{2}{\pi}$$

0.B.5 Exercise. Rewrite the product and show that (iii) entails Wallis' limit formula:

$$\sqrt{\frac{\pi}{2}} = \lim_{N \to \infty} \frac{2 \cdot 4 \cdots 2N}{1 \cdot 3 \cdot 5 \cdots (2N-1)} \cdot \frac{1}{\sqrt{2N+1}}$$

0.C Asymptotic expansions.

Residue calculus is often used to find asymptotic formulas. We describe a result of this nature. Let $\{\lambda_\nu\}$ be a strictly increasing sequence of positive real numbers and $\{a_\nu\}$ some sequence of positive real numbers. Assume that there exists some positive number r_* such that

$$f(x) = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\lambda_\nu + x}$$

is convergent for all $x > r_*$. To each $x > 0$ we denote by $\omega(x)$ the largest integer ν such that $\lambda_\nu < x$.

0.C.1 Theorem. *Suppose that the following limit formula holds for some $0 < \alpha < 1$ and a constant A:*

$$\text{(1)} \qquad \lim_{x \to +\infty} x^{-\alpha} \cdot f(x) = A$$

Then it follows that

$$\text{(2)} \qquad \lim_{x \to +\infty} x^{1-\alpha} \cdot \sum_{\nu=1}^{\nu=\omega(x)} a_\nu = \frac{A}{\pi} \cdot \frac{\sin \pi\alpha}{1-\alpha}$$

The proof requires Fourier analysis and Wiener's general Tauberian theorem. So here more advanced methods are needed but residue calculus is used to compute the value of this limit.

0.D Ugly examples.

There are situations where an integral cannot be expressed in an elementary fashion even if it is defined by elementary functions. For example, consider the integral

$$\text{(1)} \qquad \int_0^1 \frac{e^x}{1+x} \cdot dx$$

With $z = x + iy$ the function $g(z) = \frac{e^z}{z+1}$ is analytic in the half space $\mathfrak{Re}\, z > -1$. The line integral of g along rectangles $\{0 \leq x \leq 1\} \times 0 \leq y \leq R\}$ is zero and after a passage to the limit when $R \to +\infty$ we see that (1) is equal to

$$\int_0^\infty \frac{e^{is}}{1+is} \cdot ids - \int_0^\infty \frac{e^{1+is}}{2+is} \cdot ids \tag{2}$$

The conclusion is that (1) can be calculated by an "exact formula" if we can handle the integrals:

$$J(a) = \int_0^\infty \frac{e^{is}}{s - ia} \cdot ds \quad : a > 0 \tag{3}$$

Here one encounters an annoying fact. If we instead consider the integral

$$J^*(a) = \int_{-\infty}^\infty \frac{e^{is}}{s - ia} \cdot ds \quad : a > 0 \tag{4}$$

then there is no problem to compute it. In fact, we shall learn that the value of (4) is found by ordinary residue calculus and becomes $2\pi i \cdot e^{-a}$, obtained from a residue at $z = ia$ when we consider the function $g(z) = \frac{e^{iz}}{z-ia}$ in the upper half-plane where it has a simple pole at $z = ia$. But (3) cannot be found in this simple fashion. After the substitution $s \mapsto a\xi$ we see that (3) is equal to

$$J(a) = \int_0^\infty \frac{e^{ia\xi}}{\xi - i} \cdot d\xi \tag{5}$$

Apart from the factor $\frac{1}{2\pi}$ this is an inverse Fourier transform of the tempered distribution on the real ξ-line which is supported by $\{\xi \geq 0\}$ given by the density $\frac{1}{\xi - i}$ on $\xi > 0$. This illustrates a close interplay between Fourier transforms and the calculations of various integrals. Let a be replaced by x to indicate that $J(x)$ is a function of x which is defined on $x > 0$ but becomes a tempered distribution on the real x-line via Fourier's inversion formula. It follows for example that the distribution J satisfies the differential equation

$$\partial_x(J) + J = 2\pi i \cdot H^* \tag{6}$$

where H^* is the inverse Fourier transform of the Heaviside distribtution on the ξ-line which is 1 if $\xi \geq 0$ and zero on $\{\xi < 0\}$. In XX we return to a study of the J-distribution and get a certain formula for the evaluation of the integral in (4).

Let us now turn to "nice" situations and begin with some general formulas which are used in residue calculus.

0:E Fractional decompostion

The vanishing below holds for every pair of polynomials p, q if $\deg(p) \geq \deg(q) + 2$:

$$\lim_{R \to \infty} \int_{|z|=R} \frac{q(z) \cdot dz}{p(z)} = 0 \tag{*}$$

A second useful formula is the fractional decomposition:

$$\frac{1}{p(z)} = \sum_{k=1}^{k=n} \frac{c_k}{z - \alpha_k} \quad \text{where} \quad c_k = \frac{1}{p'(\alpha_k)}$$

where

$$p(z) = \prod_{k=1}^{k=n} (z - \alpha_k) \quad \text{has simple zeros}$$

0.E.1 Exercise. Show that (*) applied with $q(z) = 1$ gives:

$$c_1 + \ldots + c_n = 0 \tag{1}$$

Next, let $1 \le \nu \le n-1$ and show that one has the fractional decomposition

$$\frac{z^\nu}{p(z)} = \sum_{k=1}^{k=n} \frac{\alpha_k^\nu}{p'(\alpha_k)} \cdot \frac{1}{z-\alpha_k} \tag{2}$$

0:F Computing local residues.

When multiple zeros occur local calculations are needed to find residues. The typical case is as follows: We have an analytic function $f(z)$ defined in disc centered at $\{z=0\}$ and with a zero of order $k \ge 2$ at the origin. Now

$$\frac{1}{f(z)} = c_k z^{-k} + \ldots + c_1 z^{-1} + d_0 + d_1 z + \ldots$$

Here c_1 is the residue coefficient. In practice an expansion

$$f = bz^k(1-(b_1z+b_2z^2+\ldots))$$

is known from the start. To find c_1 therefore amounts to find the coefficient of z^{k-1} in the power series

$$\frac{1}{1-(b_1z+b_2z^2+\ldots)} = 1 + w_1z + w_2z^2 + \ldots \tag{*}$$

In (*) we can take $\{b_\nu\}$ to be arbitrary and seek for algebraic expressions of the w-numbers. This leads to every integer $k \ge 1$ to a certain polynomial R_k of the b-variables. We see for example that

$$b_1 = w_1 \quad : \quad b_1^2 + b_2 = w_2 \quad : \quad b_1^3 + b_1b_2 + b_3 = w_3 \tag{1}$$

0.F.1 Exercise. Show that for every $k \ge 1$ there exists a polynomial of the form

$$R_k(b_\bullet) = \sum \rho_{i_1\ldots i_m} b_1^{i_1} \cdot b_m^{i_m} \tag{**}$$

where $1 \le m \le k$ holds in each term and

$$i_1 + 2i_2 + \ldots + ki_k = k$$

hold for every k-tuple of the non-negative i-numbers. Use also (**) to continue the computations in (1) above for higher k-values. One has for example

$$b_1^4 + 3b_1^2b_2 + b_1b_3 + b_2^2 + b_4 = w_4$$

Employ a computer to extend the result to get exact formulas for a set of positive integers, say up to $k = 50$. Notice that all the ρ-coefficients in (**) are positive integers.

0.F.2 Exercise. Let $g(z)$ be a meromorphic function with a pole of order k at $z=0$. Then $z^k \cdot g(z)$ is holomorphic at the origin. Show that the residue of g given by the coefficient c_1 of z^{-1} in the Laurent expansion is given by

$$\frac{1}{(k-1)!} \cdot \partial^{k-1}(z^k \cdot g)(0) \tag{1}$$

Take for example $g(z) = \frac{1}{\sin^3 z}$ which has a triple pole a $z=0$. We write

$$\sin z = z(1 - z^2/3! + z^4/5! - \ldots))z \cdot \rho(z)$$

By (1) the residue becomes

$$\frac{1}{2}\partial^2(\frac{1}{\rho(z)} = -\frac{1}{2} \cdot \partial(\frac{\rho'(z)}{\rho^2(z)})$$

Now the reader can verify that the residue becomes $\frac{1}{6}$.

0.G Line integrals of multi-valued functions.

Let Ω be a connected domain in $\mathbf{C}$ and f_* is a germ of a multi-valued analytic function at some point $z_* \in \Omega$. Let γ be a curve which starts at z_* and stays in Ω. The end-point z^* of γ can be equal, to z_*, i.e. we do not exclude the case when γ is closed. Now f_* can be extended in the sense of Weierstrass along γ and using its analytic continuation along γ the line integral below is defined:

$$\int_\gamma f \cdot dz \tag{1}$$

Exercise. Use the monodromy theorem to show that if γ and γ^* are curves starting at z_* with the same end-point z^* and homotopic in a family of curves which stay in Ω and have end-points at z_* and z^*, then the integral (1) taken over γ or γ^* are equal.

0.G.1 An application. Remove the to points -1 and $+1$ from $\mathbf{C}$ which gives the domain $\Omega = \mathbf{C} \setminus \{-1, 1\}$ and consider the multi-valued function

$$f(z) = \sqrt{1+z} \cdot (1-z)^a \quad \text{where} \quad 0 < a < 1$$

whose local branch f_* at the origin is chosen so that $f_*(0) = 1$. Let γ_1 be the closed curve at the origin which follows the circle $\{z - 1| = 1\}$ and is oriented in the counter-clockwise direction. Similarly, γ_2 is the closed curve which now follows the circle $\{|z+1| = 1\}$ in the counter clock-wise direction. We have also the closed curves γ_1^{-1} and γ_2^{-1} with reversed orientation. Now we get the composed closed curve

$$\gamma = \gamma_2^{-1} \circ \gamma_1^{-1} \circ \gamma_2 \circ \gamma_1 \tag{1}$$

Along γ_1 we have $z = 1 + e^{i\theta}$ and the line integral becomes

$$J_1 = \int_0^{2\pi} \sqrt{2 + e^{i\theta}} \cdot e^{ia\theta} \cdot ie^{i\theta} \cdot d\theta \tag{i}$$

Next, when the integral over γ_2 is computed we have performed an analytic continuation of f along γ_1 which means that we have a new local branch of f at $z = 0$ which takes the value $e^{2\pi ia}$. So along γ_2 we get the contribution

$$J_2 = e^{2\pi ia} \cdot \int_0^{2\pi} e^{ia\theta} \cdot \sqrt{-2 - e^{i\theta}} \cdot ie^{i\theta} \cdot d\theta =$$

$$i \cdot e^{2\pi ia} \cdot \int_0^{2\pi} e^{ia\theta} \cdot \sqrt{2 + e^{i\theta}} \cdot ie^{i\theta} \cdot d\theta = i \cdot e^{2\pi ia} \cdot J_1 \tag{ii}$$

For the integral along γ_1^{-1} we start with a local branch where $f(0) = -e^{2\pi ia}$ and since this line integral performed in the clockwise direction the contribution becomes

$$J_3 = e^{2\pi ia} \cdot J_1$$

Finally, the local branch of f at $z = 0$ when we start integration along γ_2^{-2} is minus one and we see the contribution of the last line integral becomes

$$J_4 = e^{-2\pi ia} J_2 = i \cdot J_1$$

From this we conclude that the line integral of f taken over γ is equal to

$$(1+i)(1 + e^{2\pi i\alpha}) \cdot J_1 \tag{*}$$

The reader may verify that $J_1 \neq 0$ and hence the line integral of f over the closed curve γ is non-zero. The exercise above therefore shows that γ cannot be homotopic to the trivial curve which stays at the origin in Ω. This means that the image $\{\gamma\}$ in the fundamental group $\pi_1(\Omega)$ is non-zero, i.e. the homotopy classes $\{\gamma_1\}$ and $\{\gamma_2\}$ do not commute in this group.

Remark. The example above shows how to a classic result in topology using complex line integrals. The topological result is that the fundamental group $\pi_1(\Omega)$ is a free group generated by the homotopy classes of γ_1 and γ_2. That this indeed holds can be proved integrating multi-valued functions of the form along composed closed γ-curves.

$$f(z) = (z-1)^a \cdot (z+1)^b$$

where a, b can be arbitrary pairs of complex numbers.

0.G.2 Exercise. Let $0 < a < 1$ be a real number. Consider the multi-valued function $f(z) = z^a \cdot \log z$ defined outside the two points 0 and 1. Let $R > 1$ and at $z = R$ we choose the local branch f_* where $f_*(R) = R^a \cdot \log R$ is real and positive. Calculate the line integral

$$\int_\gamma f \cdot dz$$

where γ is the circle $\{|z| = R\}$ oriented in the counter-clock wise sense.

0.G.3 Example. Let $a > 0$ be real and consider the integral:

$$(*) \qquad J = \int_0^1 \frac{dx}{(1+ax^2)\sqrt{1-x^2}}$$

To evaluate this integral we use the fact that there exists a *single-valued* analytic function $\sqrt{1-z^2}$ in $\mathbf{C} \setminus [0,1]$. Choose a closed contour formed by a the line segment where $y = \epsilon$ and $0 \leq x \leq 1$ plus a small half circle around 1 and return along the line $y = -\epsilon$ while x moves from 1 to zero and finish with a small half-cirlce from $-i\epsilon$ to $i\epsilon$. Notice that the line integral over large circles $|z| = R$ of $\frac{1}{(1+az^2)\sqrt{1-z^2}}$ tend to zero. Now Cauchy's residue formula gives

$$(1-i) \cdot J = 2\pi i \cdot \mathfrak{res}((1+az^2)\sqrt{1-z^2} : \frac{i}{a}) + 2\pi i \cdot \mathfrak{res}((1+az^2)\sqrt{1-z^2} : \frac{-i}{a})$$

Here a computation gives the equality:

$$J = \frac{\pi}{2\sqrt{1+a}}$$

0.H Solving a differential equation.

Line integrals of multi-valued functions are also used in other situations. Here follows an example from chapter VII in the text-book [Cartan] where we remark that the original constructions are due to Laplace. Let $n \geq 2$ and consider the differential equation

$$(*) \qquad (a_n z + b_n) \cdot y^{(n)}(z) + \ldots + (a_1 z + b_1) \cdot y'(z) + (a_0 z + b_0) \cdot y(z) = 0$$

Here $\{a_k\}$ and $\{b_k\}$ are complex constants with $a_n \neq 0$. Define the two polynomials

$$(1) \qquad A(z) = \sum a_k z^k \quad \text{and} \quad B(z) = \sum b_k z^k$$

Assume that the zeros of A are simple and denote them by $c_1, \ldots, c_n$. Under this assumption the set of entire functions which solve (*) is a complex vector space of dimension $(n-1)$. To find these solutions we use the fractional decomposition and write

$$(2) \qquad \frac{B(z)}{A(z)} = \alpha + \frac{\alpha_1}{z - c_1} + \ldots + \frac{\alpha_n}{z - c_n}$$

Next, define the function

$$(3) \qquad U(z) = e^{\alpha z} \cdot \prod (z - c_k)^{\alpha_k}$$

Since $\{\alpha_k\}$ in general are not integers this U-function is multi-valued. Outside the zeros of A we notice that one has the equality

$$(4) \qquad \frac{U'(z)}{U(z)} = \frac{B(z)}{A(z)}$$

This will be used to construct solutions to (*). Namely, fix a point $z_0 \in \mathbf{C} \setminus \{c_k\}$ and in a small disc centered at z_0 we choose a local branch of U. Next, let γ be a closed curve which stays in

$\mathbf{C}\setminus\{c_k\}$ and has z_0 as a common start and end-point. For each complex number z we can evaluate the line integral and get a function

$$f(z)=\int_\gamma e^{z\zeta}\cdot\frac{U(\zeta)}{A(\zeta)}\cdot d\zeta \tag{5}$$

It is clear that f is an entire function of z and each complex derivative is given by:

$$f^{(k)}(z)=\int_\gamma e^{z\zeta}\cdot\zeta^k\cdot\frac{U(\zeta)}{A(\zeta)}\cdot d\zeta$$

So the construction of the polynomials A and B show that f is a solution to the differential equation (*) if

$$\int_\gamma e^{z\zeta}\cdot[z\cdot A(\zeta)+B(\zeta)]\cdot\frac{U}{A}(\zeta)\cdot d\zeta=0 \tag{**}$$

where the equality holds for all z.

A partial integration. Since $\partial_\zeta(e^{z\zeta})=z\cdot e^{z\zeta}$ it follows that

$$\int_\gamma e^{z\zeta}\cdot z\cdot U(\zeta)\cdot d\zeta=e^{z\zeta}\cdot U(\zeta)|_{\gamma_*}^{\gamma^*}-\int_\gamma e^{z\zeta}\cdot U'(\zeta)\cdot d\zeta \tag{6}$$

At the same time (4) above gives the equality $U'=\frac{BU}{A}$ and we conclude that (**) holds if and only if

$$e^{z\zeta}\cdot U(\zeta)|_{\gamma_*}^{\gamma^*}=0 \tag{7}$$

By the construction of the line integral where the multi-valued function U appears, (7) means that the after analytic continuation along γ one has the equality

$$T_\gamma(U)(z_0)=U(z_0) \tag{***}$$

If we want that the f-function in (5) is not identically zero we must choose closed curves γ which are not trivial, i.e. homotopic to the constant curve at z_0, and at the same time (***) should hold. To achieve this we consider for each $1\leq k\leq n$ a simple closed curve γ_k at z_0 whose winding number with respect to c_k is equal to one, while the winding number with respect to the remaining c-rots are zero. This means that the homotopy classes of $\gamma_1,\dots,\gamma_n$ generate the free group $\pi_1(\mathbf{C}\setminus\{c_k\})$. Notice also that

$$T_{\gamma_k}(U)(z_0)=e^{2\pi i\cdot\alpha_k}\cdot U(z_0) \tag{8}$$

hold for every k. To satisfy (***) we introduce the following $(n-1)$-tuple of closed curves:

$$\gamma_k^*=\gamma_k^{-1}\circ\gamma_1^{-1}\circ\gamma_k\circ\gamma_1\quad:\ 1\leq k\leq n \tag{9}$$

With this choice we have

$$T_{\gamma_k^*}(U)(z_0)=e^{-2\pi i\cdot\alpha_k}\cdot e^{-2\pi i\cdot\alpha_1}\cdot e^{2\pi i\cdot\alpha_k}\cdot e^{2\pi i\cdot\alpha_1}\cdot U(z_0)=U(c_0) \tag{10}$$

Hence the differential equation is solved by the functions

$$f_k(z)=\int_{\gamma_k^*} e^{z\zeta}\cdot\frac{U(\zeta)}{A(\zeta)}\cdot d\zeta\quad:2\leq k\leq n \tag{11}$$

There remains to prove that the functions above are $\mathbf{C}$-linearly independent and give a basis for the entire solutions to (*). The fact that the complex vector space of entire solutions to (*) has dimension $n-1$ at most can be proved in several ways. One is to apply results from $\mathcal{D}$-module theory. See (xx). Less obvious that $f_2,\dots,f_n$ are $\mathbf{C}$-linearly independent. To see this we suppose that one has a relation $q_2f_2(z)+\ldots+q_nf_n(z)=0$ where $\{q_k\}$ are complex constants. This is an identity for all z and by expanding $e^{z\zeta}$ it would follow that:

$$\sum_{k=2}^{k=n} q_k\cdot\int_{\gamma_k^*}\zeta^m\cdot\frac{U(\zeta)}{A(\zeta)}\cdot d\zeta=0\quad\text{for all}\quad m=0,1,\dots \tag{12}$$

Now the homotopy classes of the γ^*-curves are different in the fundamental group and using this one can show that (12) implies that all the q-numbers are zero which gives the requested $\mathbf{C}$-linear independence of the f-functions. See also § xx where we go further and construct distribution solutions to (*) which in partocular calrifies the calim that the space of entire solutions is $(m-1)$-dimensional.

0:I. More involved integrals.

Even though standard residue calculus settles a quite extensive family of integrals there remain integrals where the evaluation is more cumbersome and eventually force us to employ numerical calculations. Consider for example the integral

$$J(a)=\int_0^1 \frac{dx}{x^a\cdot(1+x)^a} \tag{*}$$

where $0<a<1$. We find a series for the solution using the expansion

$$(1+x)^{-a}=\sum_{n=0}^\infty c_n(a)\cdot x^n \quad : 0<x<1 \implies$$

$$J(a)=\sum_{n=0}^\infty c_n(a)\cdot\frac{1}{n+1-a} \tag{**}$$

Recall also that

$$c_n(a)=(-1)^n\cdot\frac{a(a+1)\cdots(a+n-1)}{n!} \quad : n=1,2,\ldots$$

So above we have an explicit series and it is a matter of taste if one includes the J-function which evaluates the integral among the "elementary functions". The J-integral above is related to integrals of the form

$$I(a)=\int_{-\infty}^\infty \frac{dx}{(1+x^2)^a} \tag{***}$$

which exist when a is real and $>1/2$. To solve (***) it is tempting to consider the multi-valued analytic function $g(z)=(1+z^2)^{-a}$. If $R>1$ we get the simply connected domain Ω_R which is the upper disc $\{|z|<R\}$ where $\mathfrak{Im}(z)>0$ and the imaginary interval $[0,i]$ is removed. In this domain there exists a single-valued branch of the g-function which admits a factorisation

$$g(z)=(z+i)^{-a}\cdot(z-i)^{-a}$$

The whole line integral

$$\int_{\partial\Omega_R} g(z)\cdot dz=0$$

On the portion of $\partial\Omega_R$ given by the half-circle of radius R we get a vanishing integral as $R\to+\infty$. On the portion on the real x-axis we must take into the account that the restriction of g to the negative real axis has changed. More precisely we have performed an analytic continuation of $(z-i)^{-a}=e^{-a\log(z-i)}$ and the effect is that

$$g(x)=e^{-\pi ai}\cdot\frac{1}{1+x^2} \quad : x<0$$

0.I.1 Exercise. Conclude from the above that

$$(1+e^{-\pi ai})\cdot\frac{I(a)}{2}=\int_\Gamma g(z)\cdot dz \tag{1}$$

where Γ is the contour give by two copies of the imaginary interval $[0,i]$. As illustrated by a picture the portion of the complex line integral of $g(z)\cdot dz$ on the "positive side" is taken in the negative direction and therefore contributes with the term:

$$-\int_0^1 \frac{idy}{(1-y^2)^a} \tag{2}$$

On the "negative side" the g-function has a new branch and taking the negative direction into the account the contribution to the complex line integral $\int_\Gamma g(z)\cdot dz$ becomes

$$e^{-\pi ai}\cdot\int_0^1 \frac{idy}{(1-y^2)^a} \tag{3}$$

Conclusion. From (1-3) above we obtain

$$(1+e^{-\pi ai})\cdot\frac{I(a)}{2}=(1-e^{-\pi ai})\cdot i\cdot\int_0^1\frac{dy}{(1-y^2)^a}$$

Multiply both sides with $e^{\pi ia/2}$. Then the reader can verify that

$$\cos(\pi a/2)\cdot I(a)=2\cdot\sin(\pi a/2)\cdot\int_0^1\frac{dy}{(1-y^2)^a} \tag{*}$$

0.I.2 Exercise. To each $0<a<1$ we have the integral

$$L(a)=\int_0^1\frac{dy}{(1-y^2)^a}$$

By the substitution $y\mapsto\cos\theta$ the integral becomes

$$\int_0^1 (\cos\theta)^{(1-a}\cdot d\theta$$

From a numerical point of view this is a robust integral since the integrand is a continuous function and the reader should check numerical values with a computer as $0<a<1$ varies.

0.J Abel integrals and equations

Residue calculus can be used to solve integral equations. Given $0<a<1$ we consider the equation

$$\int_0^1\frac{\phi(y)}{|x-y|^a}\cdot dy=f(x) \tag{*}$$

Here $f(x)$ is a given function which at least is continuous and one seeks ϕ. A complete solution was given by Carleman in an article entitled *Über die Abelsche Integralgleichung mit konstanter Integrationsgrenzen* from 1922. To begin with f yields the function

$$F(t)=-\frac{1}{\pi}\cos\frac{a\pi}{2}\cdot\int_0^1\frac{1}{s-t}\cdot f(s)\big[(s(1-s)\big]^{\frac{a-1}{a}}\cdot ds \tag{1}$$

The unique solution $\phi(x)$ is found given via a complex contour integral where Γ_x for each $0<x<1$ is a simple closed curve whose intersection with the positive real axis is the singleton set $\{x\}$ and one has;

$$\phi(x)=\frac{1}{2\pi i}\int_{\Gamma_x}\frac{1}{(t-x)^{1-a}}\cdot\big[t(t-1)\big]^{\frac{1-a}{2}}\cdot F(t)dt \tag{2}$$

Another integral equation is

$$\int_0^1\log|x-y|\cdot\phi(y)\cdot dy=f(x) \tag{**}$$

In Carleman's article is is proved the ϕ-solution is unique and determined by the formula:

$$\frac{1}{\pi^2}\cdot\frac{1}{\sqrt{x(1-x)}}\cdot\int_0^1\frac{f'(s)\sqrt{s(1-s)}}{s-x}\cdot ds-\frac{1}{2\pi^2\log 2\cdot\sqrt{x(1-x)}}\cdot\int_0^1\frac{f(s)}{\sqrt{s(1-s)}}\cdot ds$$

In Chapter XX we shall prove the inversion formula above using results about distributions and boundary values of analytic functions.

0.K Location of zeros of polynomials.

In the article [xx] from 1876, Eduard Routh applied Cauchy's residue calculus to analyze positions of roots to polynomials. For applications to dynamical systems the main concern is to determine zeros in a half-plane such as $\mathfrak{Re}(z) > 0$. We shall discuss some examples. Consider a polynomial

$$P(z) = z^{2m} + c_{2m-1}z^{2m-1} + \ldots + c_1 z + c_0$$

whose coefficients are real numbers. We assume also that P has no zeros in the imaginay axis and the zeros of

$$y \mapsto \mathfrak{Re}\, P(iy)$$

are simple. When $|y|$ is large we have

$$P(iy) \simeq (-1)^m \cdot y^{2m}$$

Exercise. Assume that m is even and $\mathfrak{Re}(P(iy))$ has at least one zero. Show that there exists a positive integer k and a strictly increasing sequence

$$\alpha_1 < \beta_1 < \ldots < \alpha_k < \beta_k$$

where $\{i\alpha_\nu\}$ and $\{i\beta_\nu\}$ are the zeros of $\mathfrak{Re}(P)$ on the imaginary axis. By assumption $\mathfrak{Im}\, P \neq 0$ hold at these simple zeros of the real part.. With these notations Routh's formula gives:

K.1 Theorem. *The number of zeros of P counted with multiplicity in the right half-plane is equal to*

$$m - \frac{1}{2}\sum_{\nu=1}^{\nu=k} \big[\text{sign}(\mathfrak{Im}\, P(i\alpha_\nu) - \text{sign}(\mathfrak{Im}\, P(i\beta_\nu)\big]$$

Proof. We use the argument principle and study the the function

$$y \mapsto \arg P(iy)$$

as y decreases from $+\infty$ to $-\infty$ on the real y-line. An induction over k gives:

$$\lim_{R\to\infty} [\arg P(iR) - \arg P(-iR)] = -\pi \cdot \sum_{\nu=0=}^{\nu=k} \big[\text{sign}\, \mathfrak{Im}\, P(i\alpha_\nu) - \text{sign}\, \mathfrak{Im}\, P(i\beta_\nu)\big]$$

At the same time the argument of P along a half-circle $z = Re^{i\theta}$ with $-\pi/2 < \theta < \pi/2$ increases by the term $\simeq 2m\pi$ when R is large. Now Routh's theorem follows from the argument principle in Chapter IV.

K.2 Example. Consider the case $m = 2$ and a polynomial of the form

$$P(z) = z^4 + 2Az^2 + Bz + Cz^3 - 1$$

where A, B, C are real. We get

$$\mathfrak{Re} P(iy) = y^4 - 2Ay^2 - 1 = (y^2 - A)^2 - A^2 - 1$$

Here two real roots appear via the equation

$$y^2 = A + \sqrt{A^2 + 1}$$

More precisely, we get two roots $-\rho$ and ρ where

$$\rho = \sqrt{\sqrt{A^2+1} + A}$$

At the same time

$$\mathfrak{Im}\, P(iy) = By - Cy^3$$

With the notations in Theorem K.1 we have $\alpha_1 = -\rho$ and $\beta_1 = \rho$. Here the difference

$$\text{sign}(B \cdot -\rho + C\rho^3 - \text{sign}(B \cdot \rho - C\rho^3 = 2 \cdot \text{sign}(C\rho^3 - B\rho)$$

Taking the minus sign into the account in Routh's formula we conclude that P has one zero in the right half-plane if

(i) $$C\rho^2 > B$$

while it has 3 zeros in this half-plane when

(ii) $$C\rho^2 < B$$

where $\rho = A + \sqrt{A^2+1}$. Notice that when P is restricted to the real axis then it is < 0 when $|x|$ is small and > 0 when $|x|$ is large so P has always at least one real zero on $x > 0$ and one on $x < 0$ while (i-ii) determine the real part of the two remaining zeros which are conjugate since the coefficients of P are real.

The case when P has odd degree. Consider the case of a cubic polynomial

$$P(z) = z^3 + a_2z^2 + a_1z + a_0$$

where each a-number is real and positive. Now

$$P(iy) = -iy^3 - a_2y^2 + ia_1y + a_0$$

We assume in addition that P has no zeros on the imaginary axis. The real part has two zeros $\rho > 0$ and $-\rho$ where

$$\rho = \sqrt{\frac{a_0}{a_2}}$$

Let us pursue the variation of $\arg P(iy)$ while y descreases from $+\infty$ to $-\infty$. On the positive real axis $P(x)$ is real and positive and it follows that

$$\lim_{R\to\infty} \arg P(iR) = 3\pi/2$$

Next, the real part of $P(iy)$ is < 0 as long as $y > \rho$ and when $y = \rho$ the imaginary part becomes

$$\rho(a_1 - \rho^2)$$

Suppose that this term is > 0. Then a figure shows that the argument has decreased from $3\pi/2$ to $\pi/2$. Next, while $-\rho < y < \rho$ the real part is > 0 so $P(iy)$ moves in the right half-plane and when $y = -\rho$ the imaginary part gets a reversed sign to (1) which means that the argument now has decreased from $\pi/2$ to $-\pi/2$. So up to $y = -\rho$ the decrease of the argument is -2π. Finally, when $y < -\rho$ then the real part is again < 0 while $\mathfrak{Im}\, P(-i\rho) < 0$ and after we see that the imaginary part increases when $y \to -\infty$ which means that the argument of P continues to decrease and the total effect is that

$$\lim_{R\to\infty} [\arg P(iR) - \arg P(-iR)] = -3\pi$$

At the same time $P(Re^{i\theta} \simeq R^3e^8i\theta$ when $R >> 1$ so the argument increases by 3π along the half-circle of radius R in the right half-plane. From this we conclude that P has no zeros in half-discs $\{|z| < R \cap \mathfrak{Re}\, z > 0\}$ which mens that the zero of the cubic polynomial are confined to the left half-plane. hence we have proved

Theorem. *Let $P(z)$ be a cubic polynomial where $\{a_k\}$ are real and positive and P has no zeros on the imaginary axis. Then all roots belong to the left half-plane if*

$$a_1a_2 > a_0$$

Exercise. Show that if $a_1a_2 < a_0$ then P has exactly one root in the right half-plane.

Some other examples. Consider a polynomial of degree 3:

$$P(z) = z^3 + Az - 1$$

where $A > 0$. Hence the derivative $P'(x) = 3x^2 + A > 0$ on the real x-axis so $P(x)$ has at most one real zero and since $P(0) = -1$ this zero ρ must be > 0. The two remaining roots appear in a conjugate pair $\xi, \bar{\xi}$ and since z^2 is missing in P we have

$$\xi + \bar{\xi} + \rho = 0$$

Since $\rho > 0$ we conclude that $\mathfrak{Re}\, \xi < 0$, i.e, P has one root in the right half-plane. It is instructive to check this via the argument principle. On the imaginary axis we notice that $\mathfrak{Re}\, P(iy) = -1$ is constant so $y \mapsto P(iy)$ moves in the left half-plane and when $R >> 0$ we see that

$$\arg P(iR) \simeq -\pi/2$$

A figure shows that the argument of $P(iy)$ decreases from $-\pi/2$ to $-3\pi/2$. At the same time the argument increases by the factor 3π as we move on a circle from $-iR$ to iR. The total variation of the argument along a large half-circle becomes $3\pi - \pi = 2\pi$ which reflects the fact that P has one root in the right half-plane.

Another case. Consider a polynomial of the form

$$P(z) = z^3 + Az + 1$$

where A is real. Here $y \to P(iy)$ moves in the right half-plane and we have

$$\arg P(-iR) \simeq \pi/2 \quad \text{and} \quad \arg P(iR) \simeq -\pi/2$$

So the variation of the argument as y moves from R to $-R$ increases by the factor π. From this we can conclude that P has two zeros in the right half-plane. Notice that $P(0) = B > 0$ which implies that $P(x)$ has a real zero ρ on the negative axis. The two remaining roots appear in a conjugate pair $\xi, \bar{\xi}$ and since z^2 is missing we have

$$2\xi + \rho = 0$$

which implies that $\mathfrak{Re}\, \rho > 0$ in accordance with the previous verification via the argument principle.

The case $P(z) = z^5 + z + 1$**.** Here we get

$$P(iy) = i(y^5 + y) + 1$$

The variation of $\arg(P)(iy)$ as y moves from R to $-R$ is now π while that along the half-circle is 5π. The conclusion is that $P(z)$ has two roots in the right half-plane. Notice that in this example

$$P'(x) = 5x^4 + 1 > 0$$

So $x \mapsto P(x)$ is strictly increasing on the real x-line where it has a simple zero $x_* < 0$ since $P(0) = 1 > 0$. We have seen that two complex roots α and $\bar{\alpha}$ appear with a common real part > 0 while the two other complex roots β and $\bar{\beta}$ have a negative real part. The reader should find numerical values for these complex roots and confirm the assertion that two roots appear in the right half-plane. Let us remark that the polynomial above appears in Abel's article [Abel] where he demonstrated that his specific algebraic equation of degree five cannot be solved by roots and radicals.

K.3 The Hurwitz-Routh theorem. Consider a polynomial

$$P(z) = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

whose coefficients are real numbers. We assign the 4×4-matrix

$$A = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 1 & a_2 & a_4 \end{pmatrix}$$

The matrix has four principal minors. The first is just a_1, the second $a_1 a_2 - a_3$ and the third

$$a_4(a_1(a_2 a_3 - a_1 a_4) - a_3^2)$$

The last minor is $\det(A)$. With these notations the Hurwitz-Routh theorem for polynomials of degree 4 asserts that the roots of $P(z)$ all belong to the left half-plane if and only if the four minors above are all > 0. A similar criterion holds for polynomials of arbitrary high degree. More precisely, consider a polynomial

$$P(z) = z^n + a_1 z^{n-1} + \ldots a_{n-1} z + a_n$$

with real coefficients. The necessary and sufficient condition in order that all roots belong to the half-plane $\mathfrak{Re}(z) < 0$ is expressed by minors of an $n \times n$-matrix A with elements

$$\alpha_{ik} = a_{2k-i} \tag{*}$$

where $a_0 = 01$ while $a_\nu = 0$ when $\nu < 0$ or $\nu > n$. For example, with $k = n$ we get a non-zero element in the n:th column if and only if $2n - i \leq n$ which means that only $i = n$ is possible, i.e. $\alpha_{nn} = 1$ while $\alpha_{in} = 0$ when $1 \leq i \leq n-1$. The Hurwitz-Routh theorem asserts that the roots of P belong to the right half-plane if and only if all principal minors of the A-matrix are positive. For a proof we refer to Chapter 11 in [xx] which in addition to this criterion expressed by signs of minors contains w ealth of other results and also an extensive historic account where one major contributions in addition to those of Routh and Hurwitz are due to Sturm. Let us also mention that instead of using the argument principle which involves complex computations, one can consider the so called Cauchy index which arises when one pursues the real-valued function

$$\frac{\mathfrak{Re}\, P(iy)}{\mathfrak{Im}\, P(iy}$$

where jumps at zeros of the imaginary part leads to sign-chains and makes it possible to apply the rule of Descartes for zeros of real-valued functions. here a very efficient algorithm was discovered and developed by Sturm which can be used to determine the number of zeros of a polynomial with real coefficients in the right and the left half-plane. The interested reader should consult chapter 10 in the excellent text-book [XX] for further details and which in addition gives a very complete account of the extensive theory dealing with positions of zeros of polynomials which in general can have complex coefficients.

Special Integrals.

A. The integral of e^{-x^2}

.

Recall that one can use a trick in calculus to evaluate:

(1) $$J = \int_0^\infty e^{-x^2} \cdot dx$$

Namely, use polar coordinates in the first quadrant which gives

$$J^2 = \int_0^{\pi/2} [\int_0^\infty r \cdot e^{-r^2} dr] d\theta = \pi/4$$

Hence we get $J = \sqrt{\pi}/2$. If we instead take the substitution $x^2 \to t$ one gets:

$$J = \frac{1}{2} \cdot \int_0^\infty t^{-1/2} \cdot e^{-t} \cdot dt$$

In general we consider an integral of the form

(1) $$J_a = \int_0^\infty t^{-a} \cdot e^{-t} \quad : \quad 0 < a < 1$$

In (1) we recognize the Γ-function and conclude that:

(*) $$J_a = \Gamma(1-a)$$

This is admitted as an "analytic formula". More precisely one should include the Γ-functions in the family of "elementary functions". Of course a computer is needed for numerical values as a varies.

B. Integrals of rational functions.

Let $P(z)$ be a polynomial of degree $n \geq 2$ and assume that it has no real zeros which implies that the integral below exists:

(1) $$J = \int_{-\infty}^\infty \frac{dx}{P(x)}$$

Consider the case when the roots of $P(z)$ are simple. Newton's formula gives:

(2) $$\frac{1}{P(z)} = \sum_{k=1}^{k=n} \frac{1}{P'(\alpha_k)} \cdot \frac{1}{z - \alpha_k}$$

where the sum extends over the roots $\alpha_1, \ldots, \alpha_n$. The absolute convergence of (1) implies that we have a limit as we integrate over $-R \leq x \leq R$.

Exercise. Show that

$$J = 2\pi i \cdot \sum^* \frac{1}{P'(\alpha_k)} = -2\pi i \cdot \sum_* \frac{1}{P'(\alpha_k)}$$

where the sum is taken over the zero of P in the upper, respectively the lower half-plane. Conclude that one always has

(*) $$\sum_{k=1}^{k=n} \frac{1}{P'(\alpha_k)} = 0$$

Hint. Let $\alpha = a + ib$ be a complex number with $b \neq 0$. If $b > 0$ there exists a single-valued branch of $\log(z - \alpha)$ along the real axis where

(1) $$-\pi < \arg(x - \alpha) < 0$$

while $-\alpha$ moves in the lower half-plane. It follows that

$$\int_{-R}^{R} \frac{dx}{x-\alpha} = \log(R-\alpha) - \log(-R-\alpha) = \log \frac{|R-\alpha|}{(R+\alpha|} + i \cdot (\arg(R-\alpha) - \arg(-R-\alpha))$$

From (1) the reader may verify that

$$\lim_{R\to\infty} \arg(R-\alpha) = 0 \quad \text{and} \quad \lim_{R\to\infty} \arg(-R-\alpha) = -\pi$$

At the same time we notice that

$$\lim_{R\to\infty} \log \frac{|R-\alpha|}{(R+\alpha|} = 0$$

It follows that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x-\alpha} = \pi i$$

In the case $\alpha = a + ib$ where $b < 0$ the reader can verify that the limit integral instead takes the value $-\pi i$.

C. The integral $\int_0^\infty \frac{dx}{P(x)}$

Assume as above that the zeros of P are simple and non-real. Removing the half-line $0 \leq x < \infty$ from the complex plane we obtain a single value branch of $\log z$ whose imaginary part stays in $(0, 2\pi)$. Now we apply residue calculus to the function

$$g(z) = \log z \cdot \frac{1}{P(z)}$$

Exercise. Explain how to choose suitable contours and use that $\log z$ after one positive turn around the origin changes the branch by adding $2\pi i$ to get the formula:

$$(*) \qquad -2\pi i \cdot \int_0^\infty \frac{dx}{P(x)} = 2\pi i \cdot \sum \mathfrak{res}_{z=\alpha_k} (\log z \cdot \frac{1}{P(z)})$$

where the residue sum is taken over all zeros of P.

C.1 Example. Consider the special case $P(z) = z^2 + 1$. Now

$$\frac{1}{xz^2+1} = \frac{1}{2i} \cdot [\frac{1}{z-i} - \frac{1}{z+i}]$$

It follows tha the residue sum becomes

$$\frac{1}{2i} \cdot (\log i - \log(-i)) = \frac{1}{2i} \cdot (\pi i/2 - 3\pi i/2) = -\pi/2$$

Taking the minus sign into the account in (*) we conclude that

$$\int_0^\infty \frac{dx}{1+x^2} = \pi/2$$

which confirms a wellknown formula in calculus.

D. The integrals $\int_{-\infty}^\infty \frac{e^{iax} \cdot dx}{P(x)}$

Assume that P has no real zeros and of degree ≥ 2. When a is a real number it is clear that the integral above exists. Let us consider the case $a > 0$. The entire analytic function e^{iaz} is small in the upper half-plane since

$$|e^{ia(x+iy)}| = e^{-ay}$$

Assume that the complex roots of P are simple and consider its fractional decomposition. Then the integral in (D) becomes

$$(1) \qquad \lim_{R\to\infty} \sum \frac{1}{P'(\alpha_k)} \cdot \int_{-R}^{R} \frac{e^{iax} \cdot dx}{x-\alpha_k}$$

D.1 Exercise. Show that (1) is equal to

$$2\pi i \cdot \sum^{*} \frac{e^{ia\cdot\alpha_k}}{P'(\alpha_k)} \tag{2}$$

where $\sum^*$ extends over those k for which $\mathfrak{Im}(\alpha_k) > 0$.

D.2 Example. Consider the case $P(x) = x^2 + 1$. Then $\alpha_1 = i$ is the sole simple root in the upper half-plane and we get

$$\int_{-\infty}^{\infty} \frac{e^{iax} \cdot dx}{1 + x^2} = 2\pi i \cdot c_1 \cdot e^{-a} = \pi \cdot e^{-a}$$

D.4 Exercise. Let $a < 0$ and assume that the zeros of $P(z)$ which belong to the lower half-plane are all simple. Show that in this case the integral (1) becomes

$$-2\pi i \cdot \sum_{*} \frac{e^{ia\cdot\alpha_k}}{P'(\alpha_k)}$$

where the sum extends over zeros with $\mathfrak{Im}(\alpha_k) < 0$. The reader should explain why a minus sign occurs by the aid of a figure and the orientation of the continuous which is used when the residue formula is applied.

E. Principal value integrals.

Outside $x = 0$ we have the odd function $\frac{1}{x}$ which entails that

$$\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} = 0$$

From this it is clear that the limit below exists for every C^1-function $g(x)$ on $[0, 1]$:

$$\lim_{\epsilon \to 0} \int_{\epsilon < |x| < 1} \frac{g(x) \cdot dx}{x}$$

In general, let $P(x)$ be a polynomial of some degree $k \geq$ whose zeros are all real and simple. Using principal values we can define the integral

$$\text{PV} \int_{-\infty}^{\infty} \frac{dx}{P(x)} \tag{1}$$

To compute (1) we consider the function $g(z) = \frac{1}{P(z)}$ which is analytic in the upper half-plane and integrated along a large half-circle of radius R, and along the real axis where small intervals around each real root are replaced by half-circles of radius ϵ.

E.1 Exercise. Draw a figure to illustrate the contour integral of $g(z)$ which is used above. Next, the complex line integral of g along this contour is zero. The definition of the principal value integral shows that (1) is equal to the limit

$$\sum \int_0^{\pi} \epsilon \cdot \frac{e^{i\theta} \cdot id\theta}{P(a_k + \epsilon \cdot e^{i\theta})} = \pi \cdot \sum \frac{1}{P'(a_k)} \tag{*}$$

F. The integral $J = \int_0^\infty \frac{T(\sin x)}{P(x)} \cdot dx$

Let $P(x)$ be a polynomial of degree $N \geq 2$ whose zeros are all real and simple. Set

$$T(x) = \sum_{\nu=1}^{\nu=m} c_\nu \cdot \sin(\nu x)$$

where $\{c_\nu\}$ are real and assume that $T(a_k) = 0$ holds for each real zero of P. To compute the integral in (F) we consider the analytic function

$$g(z) = \sum c_\nu \cdot e^{i\nu z} \cdot \frac{1}{P(z)} \tag{1}$$

Take a complex line integral over a contour which consists of a large half-circle $\{|z| = R\}$ in the upper half-plane and on $-R \leq x \leq R$ we replace small intervals around the zeros $\{a_k\}$ by small half-circles. If $\Gamma_{R,\epsilon}$ denotes the contour we get

$$0 = \int_{\Gamma_{R,\epsilon}} g(z) \cdot dz \tag{2}$$

Next, in the upper half plane the imaginary part of g is small which implies that

$$\lim_{R \to +\infty} \int_0^\pi g(Re^{i\theta}) \cdot iRe^{i\theta} \cdot d\theta = 0 \tag{3}$$

F.1 Exercise. Conclude from the above that one has the formula

$$\int_{-\infty}^{\infty} \frac{T(\sin x)}{P(x)} \cdot dx = \lim_{\epsilon \to 0} \mathfrak{Im}\,[\sum_{k=1}^{k=N} \int_0^\pi g(a_k + \epsilon \cdot e^{i\theta}) \cdot i\epsilon \cdot e^{i\theta} \cdot d\theta\,] \tag{1}$$

Show also that for each real zero a_K one has the formula

$$\lim_{\epsilon \to 0} \mathfrak{Im}\,[\int_0^\pi g(a_k + \epsilon \cdot e^{i\theta}) \cdot i\epsilon \cdot e^{i\theta} \cdot d\theta\,] = \mathfrak{Im}\big[\sum c_\nu e^{i\nu a_k} \cdot \frac{1}{P'(a_k)} \cdot \pi i\big] \tag{2}$$

Finally, since $\{c_\nu\}$ are real and $P'(a_k)$ also is real we see that the last term becomes

$$\frac{\pi}{P'(a_k)} \cdot \sum c_\nu \cdot \cos(\nu a_k) \tag{3}$$

Hence we have established the formula

$$\int_{-\infty}^{\infty} \frac{T(\sin x)}{P(x)} \cdot dx = \sum_{k=1}^{k=N} \frac{\pi}{P'(a_k)} \cdot T(\cos a_k) \tag{*}$$

F.2 Exercise. Above we assumed that the degree of P is ≥ 2. Show that the same formula as in (*) holds if P is linear, i.e is of the from $x - a_1$ for some real a_1. In particular consider the case $a_1 = 0$ and the sine-function $\sin x$. Then the general formula (*) gives

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \cdot dx = \pi$$

G. Adding complex zeros to P

Above the zeros of P were real and simple. Suppose now that P has some non-real roots occur which appear in conjugate pairs. In this case the calculation are the similar to those in (F), except that we also get residues from zeros of P in the upper half-plane. More precisely, the line integrals of g over $\Gamma_{R,\epsilon}$ from (F:1) are no longer zero. With R large and ϵ small the contour $\Gamma_{R,\epsilon}$ contains all zeros of $P(z)$ i the open upper half-plane. Let us assume that these zeros are simple and denote them by $\beta_1, \ldots, \beta_M$. So here $\mathfrak{Im}(\beta_\nu) > 0$ for each ν. Each β-root gives a residue

$$2\pi i \cdot \frac{1}{P'(\beta_j)} \cdot \sum c_\nu e^{i\nu \cdot \beta_j} \tag{1}$$

Here we shall take the imaginary part to get a contribution. The result is that in the formula (*) from (F) one adds a term in the right hand side which becomes

$$2\pi \sum_{j=1}^{j=M} \mathfrak{Re}[\frac{1}{P'(\beta_j)} \cdot \sum c_\nu e^{i\nu \cdot \beta_j}] \tag{**}$$

G.1 Example. Consider the case $P(x) = x(x^2 + 1)$ and $T(x) = \sin x$. We get the root $\beta_1 = i$ and notice that $\frac{1}{P'(\beta_1)} \cdot e^{i \cdot \beta_1} = \frac{1}{2i^2} \cdot e^{-1} = -\frac{1}{2e}$ which gives

$$\int_{-\infty}^{\infty} \frac{\sin x \cdot dx}{x(1 + x^2)} = \pi - \frac{\pi}{e}$$

H. The integral $\int_0^\infty \frac{x^a}{1+x^2} \cdot dx$

.

Let $0 < a < 1$. To find the integral above use the function z^a which under analytic continuation in the upper half-plane reaches the negative real axis where we have

$$(1) \qquad (-x)^a = x \cdot e^{\pi i a} \quad : \quad x > 0$$

At the same time $\frac{1}{1+z^2}$ has a simple pole at $z = i$. So if J is the value of the integral in (H:1) we obtain:

$$J(1 - e^{\pi i a}) = 2\pi i \cdot \mathfrak{res}(\frac{z^a}{1+z^2} : i)$$

At $z = i$ we use that $i = e^{\pi i/2}$ so that $z^a = e^{a\pi i/2}$ and find that the right hand side above becomes:

$$2\pi i \cdot \frac{e^{a\pi i/2}}{2i} = \pi \cdot e^{a\pi i/2} \implies$$

$$(*) \qquad J = \pi \cdot \frac{e^{a\pi i/2}}{1 - e^{\pi i a})} = 2 \cdot \pi \cdot \sin{(\pi a/2)}$$

H.1 Exercise. Use the methods from XX and explain a formula for the integral

$$\int_0^\infty \frac{x^a \cdot dx}{P(x)}$$

when P is a polynomial of degree ≥ 2 and without real zeros.

H.2 The case when P has negative real zeros. If this occurs one uses another method. The strategy is to take a complex integral of $g(z) = \frac{z^a}{P(z)}$ which starts with the interval $[\epsilon, R]$ where ϵ is small and R is large. Then one takes the circle $|z| = R$ and after one turn one integrates back from $x = R$ to $x = \epsilon$. Finally a small integral is taken over $|z| = \epsilon$ and the reader should illustrate the whole construction by a figure.

H.3 Exercise. If $\Gamma_{\epsilon,R}$ is the contour described above then it borders a simply connected domain where a single-valued branch of z^a exists. Then we can apply the residue formula and the reader should verify that

$$(1 - e^{2\pi a i}) \cdot J = 2\pi i \sum \mathfrak{res}(g(z) : \alpha_k)$$

where the sum is taken over all zeros of P.

H.4 Exercise Consider the case $P(x) = (x+1)^2$ which has a double zero at $x = -1$. Use the formula above to show that

$$\int_0^\infty \frac{x^a \cdot dx}{(x+1)^2} = \frac{\pi a}{\sin(\pi a)}$$

I. Use of the Log-function.

Consider the integral

$$J = \int_0^\infty \frac{dx}{1 + x + x^5}$$

To compute it we consider the multi-valued analytic function

$$g(z) = \frac{\log z}{1 + z + z^5}$$

Start the integration on the real x-axis from 0 to R and continue the complex line integral over the large circle $|z| = R$ and after one returns from R to $x = 0$ in the negative direction. While this is done we have a new branch of the log-function, i.e. it is now $\log x + 2\pi \cdot i$. Taking the negative direction into the account during the last integration along the non-negative x-axis it follows that

$$(*) \qquad 2\pi i \cdot J = -2\pi i \cdot \sum \mathfrak{res}(\frac{\log z}{1 + z + z^5})$$

Notice the minus sign above !

I.1 A simpler example. Suppose above that we instead take the polynomial $1+z^2$. It has simple roots at i and $-i$. Now

$$\log i = \pi \cdot i/2 \quad \text{and} \quad \log -i = 3\pi \cdot i/2$$

The sum of residues therefore becomes

$$\frac{\pi \cdot i/2}{2i} + \frac{3\pi \cdot i/2}{-2i} = \frac{\pi}{2}$$

Thanks to the minus sign in (*) above we conclude that

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

This reflects a wellknown formula which can be established directly, i.e use that $\frac{1}{1+x^2}$ is the derivative of the arctg-function. But it is illuminating to see that the general procedure using the multi-valued log-function works.

J. Trigonometric integrals.

A trigonometric polynomial is of the form $P(\theta) = \sum c_k \cdot e^{ik\theta}$ where the coefficients $\{c_k\}$ in general are complex numbers and the sum extends over a finite set of integers which may be both positive and negative. Consider a quotient of two such trigonometric polynomials

$$R(\theta) = \frac{P(\theta)}{Q(\theta)}$$

Assume that $Q(\theta) \neq 0$ for all $0 \leq \theta \leq 2\pi$ and put:

$$(*) \qquad J_R = \int_0^{2\pi} R(\theta) \cdot d\theta$$

To find (*) we use the substitutions $e^{ik\theta} \mapsto z^k$ and obtain:

$$(**) \qquad J_P = \int_{|z|=1} \frac{P(z)}{Q(z)} \cdot \frac{dz}{iz}$$

One must not forget $\frac{1}{iz}$ which appears since

$$ie^{i\theta} \cdot d\theta = dz \implies d\theta = \frac{dz}{iz}$$

If M is a sufficiently large integer then

$$(1) \qquad Q(z) = z^{-M} \cdot Q_*(z) \quad \text{and} \quad P(z) = z^{-M} \cdot P_*(z)$$

where P_* and Q_* are polynomials in z. Using (1) there remains to evaluate

$$(2) \qquad \int_{|z|=1} \frac{P_*(z)}{Q_*(z)} \cdot \frac{dz}{iz}$$

Usually one picks residues in the open unit disc D. However, there are cases when Q_* has many zeros in D and then one can use residue calculus in the exterior disc. More precisely, choose a large positive number r so that $\{|z| < r\}$ contains all zeros of P_* and Q_*. Let $m(P_*)$ and $m(Q_*)$ be the degrees of the polynomials. If $m(P_*) < m(Q_*)$ the the integral (*) becomes

$$2\pi \cdot \sum \mathfrak{res}(R(\alpha_k)$$

with the sum taken over zeros of $Q*$ in the exterior disc $|z| > 1$.

J.1 Experiment with a computer. It is instructive to perform a calculation via residues and compare the result by a computer which provides a numerical answer within a fraction of a second. Consider for example the integral

$$J = \int_0^{2\pi} \frac{d\theta}{1 + a \cdot \cos\theta}$$

where a is a complex number with absolute value < 1. Then

$$J = \frac{1}{i} \cdot \int_{|z|=1} \frac{2zdz}{2z + az^2 + a}$$

The quadratic polynomial has a simple zero $\alpha \in D$ and residue calculus gives the formula below which after can be solved numerically for different values of $0 < a < 1$.

$$J = 2\pi \cdot \frac{\alpha}{1 + a\alpha}$$

K. Summation formulas.

Consider the meromorphic function

(1) $$g(z) = \frac{\cos \pi z}{\sin \pi z}$$

It has simple poles at all integers. Notice that g can be written as

(2) $$i \cdot \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}$$

K.1 Exercise. Show that there exists a constant A_1 such that the following holds for all integers N and every real number s:

(*) $$\big| g((N + 1/2)i + is) \big| \leq A_1$$

Show also that there exists a positive constant A_2 such that the following hold when $z = x + iy$ and $|y| \geq 1$:

(**) $$|g(x + iy) \leq A$$

Thus, the g-function is bounded when we stay away a bit from the real axis. Using (*) and (**) we can establish various summation formulas. In general, let p and q be two polynomials where we assume that $\deg(p) \geq \deg(q) + 1$ and that p has no real zeros. If N is a positive integer and $R >\geq 1$ we consider the rectangle

$$\square_{R,A} = \{-N - 1/2 < x < N + 1/2\} \times \{-R < y < R\}$$

Here N and R are chosen so that this rectangle contains all zeros $\{\alpha_\nu\}$ of p. Then Cauchy's residue formula is applied when we integrate $\frac{q}{p} \cdot g$ over the boundary of this rectangle. The result is

(***) $$\frac{1}{2\pi i} \int_{\partial \square_{R,A}} \frac{q(z)}{p(z)} \cdot g(z) \cdot dz = \pi \cdot \sum_{k=-N}^{k=N} \frac{q(k)}{p(k)} + \sum \mathfrak{res}(\frac{q}{p} \cdot g : \alpha_\nu)$$

K.2 Exercise. Assume that $\deg(p) \geq \det(q) + 1$. Show that the line integrals over $\partial \square_{A,R}$ tend to zero when $A >> R >> 1$ and conclude that one has the general summation formula:

(****) $$\pi \sum_{k=-\infty}^{k=\infty} \frac{q(k)}{p(k)} = -\mathfrak{res}(\frac{q}{p} \cdot g : \alpha_\nu)$$

Remark. Consider as an example the case $p(z) = z - i$. Then the left hand side becomes

$$\pi[-\frac{1}{i} + \sum_{k=1}^{\infty} (\frac{1}{k - i} - \frac{1}{k + i})] = -\cot(i) = i \cdot \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}} \implies$$

$$\pi \cdot [1+\sum_{k=1}^{\infty}\frac{2}{1+k^2}] = \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}$$

L. A Fourier integral.

We shall calculate an integral which is used in certain Tauberian theorems. The formula in (**) below is used to calculate certain Fourier transforms and gives rise to highly non-trivial limit formulas in Wiener's study of Tauberian theorems. The computations below illustrate that one is sometimes confronted with extra difficulties in order to handle singular log-functions. Our aim is to find a formula for the integral:

$$(*) \qquad J(s)=\int_0^{\infty}\frac{\log|1-x^2|}{x^2}\cdot x^{is}dt \quad : \quad s>0$$

Since the absolute value $|x^{is}|=1$ and $\log(1-x^2)\simeq -x^2$ when x is small we see that the integral converges. But it is not clear how to compute it via residue calculus. However, we shall see that this can be done after a number of steps. In the upper half-plane there exists an analytic function defined by:

$$(1) \qquad g(z)=\frac{\log(1-z^2)}{z^2}\cdot z^{is}$$

Here the single valued branch of $\log(1-z^2)$ is chosen so that its argument belongs to $(-\pi,0)$. We have also $z^{is}=e^{is\log z}$ where the single valued branch of $\log z$ has an argument in $(0,\pi)$ as usual. On the positive imaginary axis we get

$$(2) \qquad g(iy)=\frac{\log(1+y^2)}{y^2}\cdot(iy)^{is}=\frac{\log(1+y^2)}{y^2}\cdot y^{is}\cdot e^{-\pi s/2}$$

After these preparations we consider the complex line integral of the g-function over the closed curve given by the real interval $0\le x\le R$, the quarter-circle $\{z=Re^{i\theta}\colon\ 0\le\theta\le\pi/2\}$ and the imaginary interval from iR to 0. Along the real axis the argument of the log-function changes. More precisely we notice that the imaginary part of $\log(1-x^2)$ is zero when $0<x<1$ and is $-\pi$ if $x>1$. From this we obtain

$$(3) \qquad \lim_{R\to\infty}\int_0^R g(x)\cdot dx=\int_0^{\infty}\frac{\log|1-x^2|}{x^2}\cdot x^{is}\cdot dx - i\pi\cdot\int_1^{\infty}\frac{x^{is}\cdot dx}{x^2}$$

Next, with s real and positive the absolute value $|z^{is}|$ is bounded in the upper half-plane and the reader can verify that the line integral of g along the quarter circle tends to zero when $R\to+\infty$. There remains to consider the line integral along the imaginary axis which on the closed contour above is taken in the negative direction. Taking this sign into the account together with (2) and the vanishing of the complex line integral over the whole closed contour we see that (3) is equal to

$$(4) \qquad \int_0^{\infty}\frac{\log(1+y^2)}{(iy)^2}\cdot y^{is}\cdot idy$$

After a partial integration (4) becomes

$$(5) \qquad \frac{1}{is-1}\cdot i\cdot\int_0^{\infty}\frac{2y}{1+y^2}\cdot y^{is-1}\cdot dy=\frac{2}{s+i}\cdot\int_0^{\infty}\frac{y^{is}\cdot dy}{1+y^2}$$

To calculate the last integral we use that $(-1)^{is}=e^{\pi i\cdot is}=e^{-\pi s}$ and conclude that

$$(6) \qquad (1+e^{-\pi s})\cdot\int_0^{\infty}\frac{y^{is}\cdot dy}{1+y^2}=\int_{-\infty}^{\infty}\frac{y^{is}\cdot dy}{1+y^2}$$

The last integral is found by residue calculus and as a consequence the reader may verify that (4) becomes

$$(7) \qquad \frac{2}{s+i}\cdot\frac{1}{1+e^{-\pi s}}\cdot 2\pi i\cdot\frac{i^{is}}{2i}=\frac{2}{s+i}\cdot\frac{1}{1+e^{-\pi s}}\cdot\pi\cdot e^{-\pi s/2}$$

Conclusion. One has the equality

$$(**) \qquad \int_0^\infty \frac{\log|1-x^2|}{x^2}\cdot x^{is}\cdot dx = \frac{i\pi}{1-is}+\frac{2\pi}{s+i}\cdot\frac{1}{e^{\pi s}+e^{-\pi s}}$$

Notice that the right hand side is zero when $s=0$ which gives

$$(**) \qquad \int_0^\infty \frac{\log|1-x^2|}{x^2}\cdot dx = 0$$

M. Multi-valued integrands.

A more involved study arises when the integrands are branches of multi-valued functions and one seeks values which depend upon parameters. Let us give an example.

$$(*) \qquad J(z)=\int_0^1 \frac{dt}{\sqrt{t(z-t)}}$$

When z is real and >1 we can evaluate the integral as in ordinary calculus. In the half-plane $\mathfrak{Re}(z)>1$ we see that $J(z)$ is an analytic function of z whose complex derivative becomes

$$(**) \qquad J'(z)=-\frac{1}{2}\cdot\int_0^1 \frac{dt}{\sqrt{t(z-t)^3}}$$

It turns out that $J(z)$ extends to an analytic function where the sole branch points are 0 and 1. To begin with we can choose a single valued branch of $\sqrt{z-t}$ when $\mathfrak{Im}(z)>0$ so that $J(z)$ is analytic in the upper half-plane. Less obvious is that J extends analytically across the open real interval $0<x<1$. One can prove this using a deformation of the contour which defines J, i.e. replace $[0,1]$ by curves in the complex t-plane which joint 0 and 1. Examples of deformation the contour during the analytic continuation of the J-function appear in the classic literature. It was for example used by Hermite and appears in many text-books devoted to algebraic functions. See for example the excellent material in Paul Appel's books which contains a wealth of examples related to integrals on algebraic curves and especially so called hyper-elliptic integrals.

Use of $\mathcal{D}$-module theory. A method which avoids the rather involved deformation of contours to achieve the analytic continuation goes back to Fuchs, and was later put forward in a much wider context in lectures by Pierre Deligne at Harvard University in 1967, inspired by deep studies by Nils Nilsson from the article [Nilsson-1965] which deals with integrals over algebraic chains in higher dimensions and leads to the notion of Nilsson class functions. Here we stay in dimension one and begin to seek a differential operator $Q(z,\partial_z)$ with polynomial coefficients which annihilates the J-function. Set $\nabla = z\partial_z$ which gives

$$-\nabla(J)=\frac{1}{2}\cdot\int_0^1 \frac{(z-t)dt}{\sqrt{t(z-t)^3}}+\frac{1}{2}\cdot\int_0^1\frac{t\cdot dt}{\sqrt{t(z-t)^3}}=\frac{J}{2}+\frac{1}{2}\int_0^1\frac{\sqrt{t}\cdot dt}{(z-t)^{\frac{3}{2}}}$$

In the last integral we perform a partial integration with respect to t and obtain

$$(*) \qquad \sqrt{t}\cdot\frac{1}{\sqrt{z-t}}|_0^1-\frac{1}{2}\int_0^1\frac{dt}{\sqrt{t\cdot(z-t)}} \implies \nabla(J)=-\frac{1}{\sqrt{z-1}}$$

It follows that J extends to a multi-valued function outside 0 and 1. Since

$$(z-1)\partial(z-1)^a = a(z-1)^a$$

for all a it follows from (*) that

$$[(z-1)\partial+1/2]\cdot\nabla(J)=0 \implies$$

$$(*) \qquad (z-1)z\cdot\partial^2(J)+\frac{3}{2}\cdot\nabla(J)-\partial(J)=0$$

Exercise Investigate the multi-valued behavior of J around 0 and 1. More precisely, J generates a Nilsson class function of rank 2 and as described in Chapter 4 this leads to the local monodromy expressed by 2×2-matrices at each of these ramification points.

L. Multi-valued Laplace integrals.

Let $k \geq 1$ and $a_1, \ldots, a_k$ is some k-tuple of distinct points in $\mathbf{C}$. Let us also consider another k-tuple of non-zero complex numbers $\lambda_1, \ldots, \lambda_k$. In the complex w-plane we have the multi-valued analytic function

$$g(w) = \prod (w - a_\nu)^{\lambda_\nu}$$

defined in the complement of the set of a-points. If say λ_1 is not an integer then analytic continuation of g along a small circle centered at a_1 changes local branch.... Let us now consider a simple curve C in Ω which contains two infinite pieces $w = t + ib^*$ and $w = t + ib_*$ where $-\infty < t \leq a$. Along C we choose some single valued branch of the function in (*) denoted by $g_C(w)$. If z is a new complex variabve whose real part is < 0 we get an absolutely convergent integral cand it is clear that $J(z)$ is analytic in $\mathfrak{Re}(z) > 0$. Moreover, the complex derivative

$$J_C(z) = \int_{C_R} e^{zw} \cdot g_C(w)\, dw \tag{1}$$

$$J'_C(z) = \int_{C_R} e^{zw} \cdot w \cdot g_C(w)\, dw \tag{2}$$

We can also perform a partial integration and the reader may check that

$$-z \cdot J(z) = \int_{C_R} e^{zw} \cdot \partial_w(g_C)(w)\, dw \tag{3}$$

We can also replace the unbounded curves above by closed Jordan curves Γ as long as they do not contain any a-point. Here it is obvious that

$$J_\Gamma(z) = \int_\Gamma e^{zw} \cdot g_\Gamma(w)\, dw$$

is an entire function of z. However, during the analytic continuation of g along Γ, it may occur that it takes distinct values after one full turn. In this case (3) does not hold.

Next, the multi-valued function in (*) is a solution to a first order differential equation. More precisely, in the Weyl algebra of differential operators in the w-variable we set

$$Q(w, \partial_w) = q(w)\partial_w + \sum \lambda_\nu \cdot f_\nu(\lambda)$$

and then $Q(g_C) = 0$ holds. Int the Weyl algebra of the z-variable we associate the differential operator

$$Q^*(z, \partial_z) = -q(\partial_z) \circ z + \sum \lambda_\nu \cdot f_\nu(\partial_z)$$

Then (x-xx) entail that $Q^*(J) = 0$. From the algebraic calculations with differential operators in §xx we can write

$$Q^*(z, \partial_z) = -zq(\partial_z) + q'(-\partial_z) + \sum \lambda_\nu \cdot f_\nu(\partial_z)$$

Example. Let α and γ be a pair of non-zero complex numbers and set

$$Q(w, \partial_w) = (w + w^2)\partial_w + 2w - \gamma w + 1 - \alpha$$

Then we obtain

$$Q^*(z, \partial_z) = z \cdot \partial^2 + (\gamma - z)\partial - \alpha$$

Starting with the z-variable we seek null solutions to the second order differential operator Q^*. To get such solutions we first introduce the g-function which is a null solution to Q. We try

$$g(w) = w^a (1 + w)^b$$

Then we obtain

$$Q(g) = (a(1+w) + bw + 2w - \gamma w + 1 - \alpha)g$$

and this is zero for a unique pair a, b. With this choice it follows that the second order differential operator Q^* annihilates $J(z)$. From the general results in § xx it follows that each choice of C and a branch of (xx) along this curve yields a J- function which from the start is analytic in the left half-plane and after has an analytic contionuation to the punctured z-plane where it satisfies the equation $Q^*(J) = 0$. So one is led to analyze the null solutions of Q^*.

In as complex w-plane we consider the simple curve C_R which consists of the half circle $\{w = Re^{i\theta} \quad -\pi/2 \leq \theta \leq \pi/2\}$ and the two horizontal lines

$$\ell * (R) = w = \{t - iR \quad -\infty < t \leq 0\} \quad \text{and} \quad \ell^*(R) = \{t + iR \quad -\infty < t \leq 0\}$$

Her R is chosen so large that the absolute values $|a_\nu| < R$ for each ν. In a neighborhood of $w = R$ we get local branches of the complex powers w^{λ_ν}, i.e. with $w = R + \zeta$ and ζ small one has

$$(R+\zeta)^{\lambda_\nu} = e^{\lambda_\nu \cdot \log(R+\zeta)}$$

and the branch is chosen so that the value when $\zeta = 0$ is the ordinary complex exponential $e^{\lambda_\nu \cdot \log(R}$ where $\log R$ is real. Now each function $w^{\lambda\nu}$ extends analytically along C_R and if z is another complex variable whose real part is < 0 we see that there exists a convergent integral

$$(*) \qquad J(z) = \int_{C_R} e^{zw} \cdot \prod (w - a_\nu)^{\lambda_\nu} \cdot dw$$

It is clear that this J-function is analytic in the half-plane $\mathfrak{Re}\, z > 0$. In general $J(z)$ does not extend to an entire function. Consider as an example the case $k = 1$ with $a_1 = 0$ and set $\lambda = \lambda_1$. So here

$$J(z) = \int_{C_R} e^{zw} \cdot w^\lambda \cdot dw$$

When $z = x$ is real and positive we can perform the variable substitution $w \mapsto u/x$ and get

$$J(x) = x^{-\lambda-1} \int_{C_{Rx}} e^u \cdot u^\lambda \cdot du$$

The integral above is expressed via the Γ-function, i.e.

$$J(x) = x^{-\lambda-1} \cdot \frac{2\pi i}{\Gamma(-\lambda)}$$

Hence $J(z)$ extends to the function $z^{-\lambda-1}$ times the constant $\frac{2\pi i}{\Gamma(-\lambda)}$. So unless λ is an integer we get a multi-valued J-function. For the general case (*) one has

L.1 Theorem. *The J-integral extends to $\mathbf{C} \setminus (a_1, \ldots, a_k)$ as an analytic function which in general is multi-valued with ramification points or poles at $a_1, \ldots, a_k$.*

To prove this result one finds a differential equation with polynomial coefficients satisfied by J, i.e. the efficient procedure is to use $\mathcal{D}$-module technique. More precisely, suppose we have found a differential operator

$$Q(w, \partial_w) = \sum q_j(w) \cdot \partial_w^j$$

in the Weyl algebra A_1 with respect to the w-variable such that

$$Q(\prod (w - a_\nu)^{\lambda_\nu}) = 0$$

Then we associate the differential operator Q^* in the z-variable given by

$$Q^*(z, \partial_z) = \sum q_j(-\partial_z) \cdot z^j$$

Exercise. Show that we get $Q^*(J(z)) = 0$. Then hint is that a partial integration gives

$$\text{(i)} \qquad zJ(z) = -\int_{C_R} e^{zw} \cdot \partial_w (\prod (w - a_\nu)^{\lambda_\nu}) \cdot dw$$

At the same time we notice that

$$\text{(ii)} \qquad \partial_z(J) = \int_{C_R} e^{zw} \cdot w \cdot \prod (w - a_\nu)^{\lambda_\nu} \,) \cdot dw$$

A computation will show that the differential operator Q^* has order k and that the polynomial coefficient in front of ∂_z^k is given by $\prod(z - a_\nu)$ from which Theorem L.1 follows.

Remark. The computations above can be reversed and lead to integral representations of functions which are solution to a differential equation defined by $Q^*(z, \partial_z)$ for a suitable Q^*. A classic case are the confluent hypergeometric functions which are solutions to certain differential equations. More precisely, let α and γ be a pair of non-zero complex numbers. One seeks solutions $f(z)$ to the second order equation

$$z \cdot \partial^2(f) + (\gamma - z)\dot{\partial}(f) - \alpha \cdot f = 0$$

So here

$$Q^* = z \cdot \partial^2 + (\gamma - z)\partial - \alpha$$

which in the non-commutative Weyl algebra can be written in the form

$$\partial^2 \cdot z - 2\partial + \gamma\partial - \partial \cdot z + 1 - \alpha$$

Hence Q^* is associated to the differential operator

$$Q(w, \partial_w) = w^2\partial_w + 2w - \gamma w + w\partial_w + 1 - \alpha$$