XVI.. Beurling-Wiener algebras

Contents

Introduction. The results in this section are foremost due to Wiener and Beurling. The cornerstone is Wiener's general Tauberian Theorem which we apply a class called Beurling-Wiener algebras where the convolution algebra $L^1)\mathbf{R})$ is replaced by various weight algebras introduced by Beurling in the article (Beurling: 1938]. Here follows the set-up in this section.

Consider the Banach space $L^1(\mathbf{R})$ of Lebesgue measurable and absolutely integrable functions whose product is defined by convolutions:

$$f * g(x) = \int f(x-y)g(y)dy$$

A.1 The space $\mathcal{F}_0^\infty$. On the ξ-line we have the space C_0^∞ of infintely differentiable functions with compact support. Each $g(\xi) \in C_0^\infty$ yields an L^1-function on the real x-line defined by

$$(*) \qquad \mathcal{F}(g)(x) = \frac{1}{2\pi}\int e^{ix\xi}\, g(\xi)\cdot d\xi$$

The resulting subspace of L^1 is denoted by $\mathcal{F}_0^\infty$.

A.2 Beurling-Wiener algebras. A subalgebra B of L^1 is called a Beurling-Wiener algebra - for short a $\mathcal{BW}$-algebra - if the following two conditions hold:

Condition 1. B is equipped with a complete norm denoted by $||\cdot||_B$ such that

$$||f * g||_B \leq ||f||_B \cdot ||g||_B \quad : \quad f, g \in B \quad \text{and} \quad ||f||_1 \leq ||f||_B$$

Condition 2. $\mathcal{F}_0^\infty$ is a dense subalgebra of B.

A.3 Theorem *Let B be a $\mathcal{BW}$-algebra. For each multiplicative and continuous functional λ on B which is not identically zero there exists a unique $\xi \in \mathbf{R}$ such that*

$$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

Proof. Suppose that there exists some ξ such that

$$\text{(i)} \qquad \lambda(f) = 0 \implies \widehat{f}(\xi) = 0$$

This means that the linear form $f \mapsto \widehat{f}(\xi)$ has the same kernel as λ and hence there exists some constant c such that

$$\text{(ii)} \qquad \lambda(f) = c \cdot \hat{f}(\xi) \quad \text{for all } f \in B\,.$$

Since λ is multiplicative it follows that $c = c^n$ for every positive integer n and then $c = 1$. Next, since B contains $\mathcal{F}_0^\infty$ and test-functions on the ξ-line separate points, it is clear that ξ is uniquely determined. There remains to prove the existence of some ξ for which (i) holds.

To prove this we use the density of $\mathcal{F}_0^\infty$ in B which by the continuity of λ gives some $g \in \mathcal{F}_0^\infty$ such that $\lambda(g) \neq 0$. Let K be the compact support of the test-function $\widehat{g}(\xi)$ and suppose that (i) fails for each point $\xi \in K$. The density of $\mathcal{F}_0^\infty$ gives some $f_\xi \in \mathcal{F}_0^\infty$ such that

$$\text{(iii)} \qquad \widehat{f}(\xi) \neq 0 \quad \text{and } \lambda(f) = 0$$

Heine-Borel's Lemma yields a finite set of points $\xi_1,\ldots,\xi_N$ in K such that family $\{\widehat{f}_{\xi_k}\}$ have no common zero on K. To simplify notations we set $f_k = f_{\xi_k}$. The complex conjugates of $\{\widehat{f}_k\}$ are again test-functions. So for each k we find $h_k \in B$ such that $\widehat{h}_k$ is the s complex conjugate of $\widehat{f}_k$. Set

$$\phi(\xi) = \sum_{k=1}^{k=N} \widehat{h}_k(\xi)\cdot \widehat{f}_k(\xi)$$

This test-function is > 0 on the support of $\widehat{g}$ and hence there exists the test-function

$$\text{(iv)} \qquad Q(\xi) = \frac{\widehat{g}}{\phi}$$

By Condition 2, Q is the Fourier transform of some B-element q. Since $L^1(\mathbf{R})$-functions are uniquely determined by their Fourier transforms, it follows from (iv) that

$$\text{(v)} \qquad \sum_{k=1}^{k=N} q * h_k * f_k = g$$

Now we get a contradiction since $\lambda(f_k) = 0$ for each k while $\lambda(g) \neq 0$.

A.4 The algebra B_a.

Let $a > 0$ be a positive real number. Given a Beurling-Wiener algebra B we set

$$J_a = \{f \in B \quad : \ \widehat{f}(\xi) = 0 \text{ for all } -a \leq \xi \leq a\}$$

Condition 1 and the continuity of the Fourier transform on L^1-functions imply that J_a is a closed ideal in B. Hence we get the Banach algebra $\frac{B}{J_a}$ which we denote by B_a. Let $g \in \mathcal{F}_0^\infty$ be such that $\widehat{g}(\xi) = 1$ on $[-a,a]$. For every $f \in B$ it follows that $g * f - f$ belongs to J_a which means that the image of f in B_a is equal to the image of $g * f$. We conclude that the g-image yields an identity in the algebra B_a and hence B_a is a Banach algebra with a unit element.

A.5 Theorem. *The Gelfand space of B_a is equal to the compact interval $[-a,a]$.*

A.6 Exercise. Prove this using Theorem A.3

A.7. Examples of $\mathcal{B}W$-algebras

Let B be the space of all continuous functions $f(x)$ on the real x-line such that the positive series below is convergent:

$$(*) \qquad \sum_{-\infty}^{\infty} ||f||_{[\nu,\nu+1]}$$

where $||f||_{[\nu,\nu+1]}$ is the maximum norm of f on the closed interval $[\nu.\nu+1]$ and the sum extends over all integers. The norm on B-elements is defined by the sum of the series above. It is obvious that this norm dominates the L^1-norm. Moreover, one easily verifies that

$$\text{(i)} \qquad ||f * g||_B \leq ||f||| \cdot ||g||_B$$

for pairs in B. Hence B satisfies Condition 1 from B.

Exercise. Show that the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on the real x-line is a dense subalgebra of B.

Next, since $\mathcal{F}_0^\infty \subset \mathcal{S}$ we have the inclusion

$$\text{(ii)} \qquad \mathcal{F}_0^\infty \subset B$$

There remains to see why $\mathcal{F}_0^\infty$ is dense in B. To prove this we construct some special functions on the x-line whose Fourier transforms have compact support. If $b > 0$ we set

$$f_b(x) = \frac{1}{2\pi}\int_{-b}^{b} e^{ix\xi}\cdot(1 - \frac{|\xi|}{b})\cdot d\xi$$

By Fourier's inversion formula this means that

$$\widehat{f_b}(\xi) = 1 - \frac{|\xi|}{b} \qquad -b \leq \xi \leq b \quad \text{and zero if } |\xi| > b$$

A computation which is left to the reader gives

$$f_b(x) = \frac{1}{\pi} \cdot \frac{1 - \cos bx}{bx^2}$$

From this expression it is clear that $f_b(x)$ belongs to B and we leave it to the reader to verify that

(iii) $$\lim_{b \to +\infty} ||f_b * g - g||_B = 0 \quad \text{for all } g \in B$$

Next, the functions $\widehat{f_b}(\xi)$ have compact support but they are not smooth, i.e. they do not belong to $\mathcal{F}_0^\infty$. However, we can perform a smoothing of these functions as follows: Let $\phi(\xi)$ be an even and non-negative C_0^∞-function with support in $-1 \leq \xi \leq 1$ such that the integral

$$\int \phi(\xi) \cdot d\xi = 1$$

With $\delta > 0$ we set $\phi_\delta(\xi) = \frac{1}{\delta} \cdot \phi(\xi/\delta)$ and for each pair δ, b we get the test-function on the ξ-line defined by

$$\psi_{\delta,b}(\xi) = \int_{-b}^{b} \phi_\delta(\xi - \eta) \cdot (1 - \frac{|\eta|}{b}) \cdot d\eta$$

The inverse Fourier transforms

$$f_{\delta,b}(x) = \frac{1}{2\pi} \int e^{ix\xi} \cdot \psi_{\delta,b}(\xi) \cdot d\xi$$

yield functions in $\mathcal{F}_0^\infty$ for all pairs δ, b. Next, if $g \in B$ then the Fourier transform of the B-element $f_{\delta,b} * g$ is equal to the *convolution* of $\phi_\delta(\xi)$ and the Fourier transform of $f_b * g$. This implies that

$$f_{\delta,b} * g \in \mathcal{F}_0^\infty \,.$$

At this stage we leave it to the reader to verify that

$$\lim_{(\delta,b) \to (0,0)} f_{\delta,b} * g = g$$

holds for every $g \in B$. Hence the required density of $\mathcal{F}_0^\infty$ is proved and B is a Beurling-Winer algebra.

A.8 Adding discrete measures

Let $M_d(\mathbf{R})$ be the Banach algebra of discrete measures of finite total variation, i.e. measures of the form

$$\mu = \sum c_\nu \cdot \delta_{x_\nu} \quad : \ ||\mu|| = \sum |c_\nu| < \infty$$

As explained in XX the Gelfand space is the compact Bohr group $\mathfrak{B}$, where the real ξ-line via the Fourier transform appears as a dense subset. Now we adjoin some $\mathcal{BW}$-algebra B and obtain a Banach algebra B_d which consists of measures of the form

$$f + \mu \quad : \quad f \in B \text{ and } \mu \in M_d(\mathbf{R})$$

where the norm of $f + \mu$ is the sum of the B-norm of f and the total variation of μ. Since B is a subspace of L^1 one easuly checks that this yields a complete norm. next, by condition (2) in A.2 it follows that if $f \in b$ and $\mu \in M_d(\mathbf{R})$ then the convolution $f * \mu$ belongs to B. This means that B appears as a closed ideal in B_d.

A.9 The Gelfand space $\mathcal{M}_{B_d}$. Let λ is a multiplicative functional on B_d which is not identically zero on B. Theorem A.3 gives a unique ξ such that

(i) $$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

If a is a real number then $\delta_a * f$ has the Fourier transform becomes $e^{ia\xi} \cdot \widehat{f}(\xi)$. It follows that

(ii) $$\lambda(\delta_a) \cdot \widehat{f}(\xi) = \lambda(\delta_a * f) = e^{-ia\xi} \cdot \widehat{f}(\xi)$$

We conclude that $\lambda(\delta_a) = e^{-ia\xi}$ and hence the restriction of λ is the evaluation of the Fourier transform at ξ on the whole algebra B_d. In this way the real ξ-line is embedded in $\mathcal{M}_B$ where a point $\lambda \in \mathcal{M}_B$ belongs to this subset if and only if $\lambda(f) \neq 0$ for some $f \in B$. The construction of the Gelfand topology shows that this copy of the real ξ-line appears as an *open* subset of $\mathcal{M}_{B_d}$ denoted by $\mathbf{R}_\xi$.

A.10 The set $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$. If λ belongs to this closed subset it is identically zero on the ideal B and its restriction to $M_d(\mathbf{R})$ corresponds to a point γ in the Bohr group $\mathfrak{B}$. Conversely, every point in $\mathfrak{B}$ yields a $\lambda \in \mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ since the quotient algebra

$$\frac{B_d}{B} \simeq M_d(\mathbf{R})$$

Hence we have the set-theoretic equality

(*) $$\mathcal{M}_{B_d} = \mathbf{R}_\xi \cup \mathfrak{B}$$

A.11 Proposition. *The open subset $\mathbf{R}_\xi$ is dense in $\mathcal{M}_B$.*

Proof. Let λ be a point in $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ which therefore corresponds to a point $\gamma \in \mathfrak{B}$. By the result in XX we know that for every finite set $\mu_1, \ldots, \mu_N$ of discrete measures, there exists a sequence $\{\xi_\nu\}$ such that

$$\lim_{\nu \to \infty} \widehat{\mu}_i(\xi_\nu) = \gamma(\mu_i) \quad \text{and } |\xi_\nu| \to \infty$$

At the same time the Riemann-Lebesgue Lemma entails that

$$\lim_{\nu \to \infty} \widehat{f}(\xi_\nu) = 0$$

for every $f \in B$. Hence the construction of the Gelfand topology on $\mathcal{M}_{B_d}$ gives the requested density in Proposition A.11

A.12 An inversion formula. Let $f \in B$ and μ is some discrete measure. Suppose that there exists $\delta > 0$ such that the Fourier transform of $f + \mu$ has absolute value $\geq \delta$ for all ξ. Proposition A.11 implies that its Gelfand transform has no zeros and hence this B_d-element is invertible, i.e. there exist $g \in B$ and a discrete measure γ such that

(i) $$\delta_0 = (f + \mu) * (g + \gamma)$$

Notice that the right hand side becomes

$$f * g + f * \gamma + g * \mu + \mu * \gamma$$

Here $f * g + f * \gamma + g * \mu$ belongs to B while $\mu * \gamma$ is a discrete measure. So (i) implies that γ must be the inverse of μ in $M_d(\mathbf{R})$ and hence (i) also gives the equality:

(ii) $$f * g + f * \mu^{-1} + g * \mu = 0$$

B. A Tauberian Theorem.

Consider the Banach algebra B above. The dual space B^* consists of Riesz measures μ on the real line for which there exists a constant A such that

$$\int_\nu^{\nu+1} |d\mu(x)| \leq A \quad \text{for all integers } \nu\,.$$

The smallest A above is the norm of μ in B^* and duality is expressed by:

$$\mu(f) = \int f(x) \cdot d\mu(x) \quad : \quad f \in B \text{ and } \mu \in B^*$$

Let $f \in B$ be such that $\widehat{f}(\xi) \neq 0$ for all ξ. For each $a > 0$ it follows from Theorem A.5 that the f-image in B_a generates the whole algebra. Since this hold for every $a > 0$ it follows that each $\phi \in \mathcal{F}_0^\infty$ belongs to the principal ideal generated by f in B, i.e. there exists some $g \in B$ such that

(*) $$\phi = g * f$$

Since $\mathcal{F}_0^\infty$ is dense in B we conclude that $B \cdot f$ is dense in B. Using this density we have:

B.1 Theorem *Let $\mu \in B^*$ be such that*

$$\lim_{y \to +\infty} \int f(y-x) \cdot d\mu(x) = A \text{ exists.}$$

Then, for each $g \in B$ it follows that

$$\lim_{y \to +\infty} \int g(y-x) \cdot d\mu(x) = B \quad \text{where} \quad B = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

Proof. Let $g \in B$. If $\epsilon > 0$ we find $h_\epsilon \in B$ such that $||g - f * h_\epsilon||_B < \epsilon$. When $y > 0$ we get:

(i) $$\int (f * h_\epsilon)(y-x) \cdot d\mu(x) =$$

$$\int \big[f(y-s-x) h_\epsilon(s) \cdot ds \big] \cdot d\mu(x) = \int h_\epsilon(s) \cdot \big[\int f(y-s-x)\mu(x) \big] \cdot ds$$

By the hypothesis the inner integral converges to A when $y \to +\infty$ every fixed s. Since h belongs to B it follows easily that the limit of (i) when $y \to +\infty$ is equal to

(ii) $$A \cdot \int h_\epsilon(s) \cdot ds = A \cdot \widehat{h}_\epsilon(0)$$

Next, since the B-norm is stronger than the L^1-norm it follows that

(iii) $$|\widehat{g}(0) - \hat{h}_\epsilon(0) \cdot \widehat{f}(0)| < \epsilon$$

Moreover, since the B-norm is invariant under translations we have

(iv) $$\Big| \int g(y-x) d\mu(x) - \int (f * h_\epsilon)(y-x) \cdot d\mu(x) \Big| \leq \epsilon \cdot ||\mu|| \quad \text{for all } y$$

where $||\mu||$ is the norm of μ in the dual space B^*. Notice also that (iii) gives:

$$\lim_{\epsilon \to} \hat{h}_\epsilon(0) = \frac{\hat{g}(0)}{\hat{f}(0)}$$

Finally, since $\epsilon > 0$ is arbitrary we see that the limit formula for (i) when $y \to +\infty$ expressed by (ii) and (iv) above together imply that

$$\lim_{y \to +\infty} \int g(y-x) d\mu(x) = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

This finishes the proof of Theorem A.9

B.2 The multiplicative version

Let $\mathbf{R}^+$ be the multiplicative group of positive real numbers. To each function $f(t)$ on $\mathbf{R}^+$ we get the function $E_f(x) = f(e^x)$ on the real x-line. Since $dt = e^x dx$ under the exponential map we have

$$\int_0^\infty f(t) \frac{dt}{t} = \int_{-\infty}^\infty E_f(x) dx$$

provided that f is integrable. On $\mathbf{R}^+$ we get the convolution algebra $L^1(\mathbf{R}^+)$ where

$$f * g(t) = \int_0^\infty f(\frac{t}{s}) \cdot g(s) \cdot \frac{ds}{s}$$

This convolution commutes with the E map from $L^1(\mathbf{R}^+$ into $L^1(\mathbf{R}^1)$, i.e.

$$E_{f*g} = E_f * E_g$$

Next, recall that the Fourier transform on $L^1(\mathbf{R}^+)$ is defined by

$$\widehat{f}(\xi) = \int_0^\infty t^{-i\xi} \cdot f(t) \cdot \frac{dt}{t}$$

B.3 The Banach algebra B_m. The companion to B on $\mathbf{R}^+$ consists of continuous functions $f(t)$ for which

$$\sum ||f||_{[2^\nu,2^{\nu+1}]} < \infty$$

where the is taken over all integers. Notice that with $\nu < 0$ one takes small intervals approaching $t = 0$. Just as in Theorem A.9 we obtain a Tauberian Theorem for functions $f \in B_m$ whose Fourier transform is everywhere $\neq 0$. Here we the dual space B_m^* consists of Riesz measures μ on $\mathbf{R}^+$ for which there exists a constant C such that

$$\int_{2^m}^{2^{m+1}} |d\mu(t)| \leq C$$

for all integers m.

C. Ikehara's theorem.

Let ν be a non-negative Riess measure supported on $[1, +\infty)$ and assume that

$$\int_1^\infty x^{-1-\delta} \cdot d\nu(x) < \infty \quad \text{for all } \delta > 0$$

This gives an analytic function $f(s)$ of the complex variable s defined in the right half plane $\mathfrak{Re}(s) > 1$ by

$$f(s) = \int_1^\infty x^{-s} \cdot d\nu(x)$$

D.1 Theorem. *Assume that there exists a constant A and a locally integrable function $G(u)$ defined on the real u-line such that*

(*) $$\lim_{\epsilon \to 0} \int_{-b}^{b} \big| f(1+\epsilon+iu) - \frac{A}{1+\epsilon+iu} - G(u) \big| \cdot du = 0 \quad \text{holds for each } b > 0$$

Then it follows that

(**) $$\lim_{x \to +\infty} \frac{1}{x} \int_1^x d\nu(t) = A$$

Proof. Define the measure ν^* on the non-negative real ξ-line by

(1) $$d\nu^*(\xi) = e^{-\xi} \cdot d\nu(e^\xi) - A \cdot d\xi \quad : \ \xi \geq 0$$

If $\eta > 1$ we notice that

$$\int_0^\eta e^{-\eta+\xi} \cdot d\nu^*(\xi) = e^{-\eta} \int_0^\eta d\nu(e^\xi) - A(1 - e^{-\eta}) = e^{-\eta} \int_0^{e^\eta} d\nu(t) - A(1 - e^{-\eta})$$

hence (**) holds if and only if

(2) $$\lim_{\eta \to \infty} \int_0^\eta e^{-\eta+\xi} \cdot d\nu^*(\xi) = 0$$

It is also clear that condition (xx) for ν entails that

(3) $$\int_0^\infty e^{-\delta\cdot\xi}\cdot d\nu^*(\xi)<\infty \quad \text{for all } \delta>0$$

Moreover, a variable substitution gives

(4) $$f(s)-\frac{A}{s-1}=\int_0^\infty e^{(1-s)\xi}d\nu^*(\xi)$$

C.1 A reformulation of Ikehara's theorem.

From (1-4) we can restate Ikehara's theorem. Let ν^* be a non-negative measure on $0\leq\xi<+\infty$ such that

(1.1) $$\int_0^\infty e^{-\delta\cdot\xi}\cdot d\nu^*(\xi)<\infty \quad \text{for all } \delta>0$$

Let $A>0$ be some positive constant and define the measure μ by

(1.2) $$d\mu(\xi)=d\nu^*(\xi)-A\cdot d\xi$$

Then (1.1) gives the analytic function $g(s)$ defined in $\mathfrak{Re}(s)>0$ by

(1.3) $$g(s)=\int_0^\infty e^{-s\cdot\xi}\cdot d\mu(\xi)$$

C.1.1. Definition. *We say that the measure μ is of the Ikehara type if there exists a locally integrable function $G(u)$ defined on the real u-line such that*

$$\lim_{\epsilon\to 0}\int_{-b}^{b}\big|g(\epsilon+iu)-G(u)\big|\cdot du=0 \quad \text{holds for each } b>0$$

C.1.2. The space $\mathcal{W}$. Let $\mathcal{W}$ be the space of continuous functions $\rho(\xi)$ defined on $\xi\geq 0$ which satisfy:

$$\sum_{n\geq 0}||\rho||_n<\infty \quad \text{where } ||\rho||_n=\max_{n\leq u\leq n+1}|\rho(u)|$$

The dual space $\mathcal{W}^*$ consists of Riesz measures γ on $[0,+\infty)$ such that

$$\max_{n\geq 0}\int_n^{n+1}|d\gamma(\xi)|<\infty$$

With these notations we shall prove:

C.1.3. Theorem. *Let ν^* be a non-negative meausure on $[0,+\infty)$ and $A\geq 0$ some constant such that the measure $\mu=\nu^*-A\cdot d\xi$ is of Ikehara type. Then $\mu\in\mathcal{W}^*$ and for every function $\rho\in\mathcal{W}$ one has*

$$\lim_{\eta\to+\infty}\int_0^\eta \rho(\eta-\xi)\cdot d\mu(\xi)=0$$

C.1.4 Exercise. Use the material above to show that Theorem C.1.3 gives Theorem C.0 where a hint is to use the function $\rho(s)=e^{-s}$ above.

C.2 Proof of Theorem C.1.3.

Let $b>0$ and define the function $\omega(u)$ by

(i) $$\omega(u)=1-\frac{|u|}{b}\,, \quad -b\leq u\leq b \quad \text{and } \omega(u)=0 \text{ outside this interval}$$

Set

(ii) $$J_b(\epsilon,\eta)=\int_{-b}^{b}e^{i\eta u}\cdot g(\epsilon+iu)\cdot\omega(u)\cdot du$$

From Definition C.1.1 we have the L^1_{loc}-function $G(u)$ and since $\omega(u)$ is a continuous function on the compact interval $[-b, b]$ we have

(iii)
$$\lim_{\epsilon\to 0} J_b(\epsilon,\eta) = J_b(0,\eta) = \int_{-b}^{b} e^{i\eta u} \cdot G(u)\cdot\omega(u)\cdot du$$

With b kept fixed the right hand side is a Fourier transform of an L^1-function. So the Riemann-Lebesgue theorem gives:

(iv)
$$\lim_{\eta\to+\infty} J_b(0,\eta) = 0$$

Moreover, the triangle inequality gives the inequality:

(v)
$$|J_b(0,\eta)| \leq \int_{-b}^{b} \big|G(u)\big|\cdot du$$

Some integral formulas. From the above it is clear that

(1)
$$J_b(\epsilon,\eta) = \int_0^\infty \big[\int_{-b}^{b} e^{i\eta u - i\xi u}\cdot\omega(u)\cdot du\big]\cdot e^{-\epsilon\cdot\xi}\cdot d\mu(\xi)$$

Next, notice that

(2)
$$\int_{-b}^{b} e^{i\eta u - i\xi u}\cdot\omega(u)\cdot du = 2\cdot\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}$$

Hence we obtain

(3)
$$J_b(\epsilon,\eta) = 2\cdot\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\mu(\xi)$$

From (iii) above it follows that (3) has a limit as $\epsilon\to 0$ which is equal to the integral in the right hand side in (iii) which is denoted by $J_b(0,\eta)$. Next, it is easily seen that there exists the limit

(4)
$$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot A d\xi = 2\pi\cdot A$$

Hence (3-4) imply that there exists the limit

(5)
$$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\nu^*(\xi) = J_b(0,\eta) + 2\pi\cdot A$$

Next, the measure $\nu^*\geq 0$ and the function $\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\geq 0$ for all ξ. So the existence of a finite limit in (5) entails that there exists the convergent integral

(6)
$$\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot d\nu^*(\xi) = J_b(0,\eta) + 2\pi\cdot A$$

Proof that $\mu\in\mathcal{W}^*$. Since $A\cdot d\xi$ obviously belongs to $\mathcal{W}^*$ it suffices to show that $\nu^*\in\mathcal{W}^*$. To prove this we consider some integer $n\geq 0$ and with $b=1$ it is clear that (6) gives

$$\big|\int_n^{n+1} \frac{1-\cos(\eta-\xi)}{(\eta-\xi)^2}\cdot d\nu^*(\xi)\big| \leq |J_1(0,\eta)| + 2\pi = \int_{-1}^{1} |G(u)|\cdot du + 2\pi\cdot A$$

Apply this with $\eta = n+1+\pi/2$ and notice that

$$\frac{1-\cos(n+1+\pi/2-\xi)}{(n+1+\pi/2-\xi)^2}\geq a \quad \text{for all } n\leq\xi\leq n+1$$

This gives a constant K such that

$$\int_n^{n+1} d\nu^*(\xi)\leq K \quad n=0,1,\ldots$$

Final part of the proof. We have proved that $\mu \in \mathcal{W}^*$. Moreover, from (iv) above and the integral formula (6) we get

$$(*) \qquad \lim_{\eta \to +\infty} \int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \cdot d\mu(\xi) = 0 \quad \text{for all } b > 0$$

Next, for each fixed $b > 0$ we notice that the function

$$\rho_b(\xi) = 2 \cdot \frac{1-\cos(b\xi)}{b \cdot \xi^2}$$

belongs to $\mathcal{W}$ and its Fourier is $\omega_b(u)$. Here $\omega_b(u) \neq 0$ when $-b < u < b$. So the family of these ω-functions have no common zero on the real u-line. By the Remark in XX this means that the linear subspace of $\mathcal{W}$ generated by the translates of all ρ_b -functions with arbitrary large b is dense in $\mathcal{W}$. Hence (*) above implies that we get a zero limit as $\eta \to +\infty$ for every function $\rho \in \mathcal{W}$. But this is precisely the assertion in Theorem C.1.3.

E. The algebra $L^1(\mathbf{R}^+)$

Consider the family of L^1-functions on the real x-line which are supported by the half-line $x \geq 0$. This yields a closed subalgebra of $L^1(\mathbf{R})$ denoted by $L^1(\mathbf{R}^+)$. Indeed, if $f(x)$ and $g(x)$ are two such functions in $L^1(\mathbf{R}^+)$. the support of the convolution $g * f$ stays in $[0, +\infty)$. Adding the unit point mass δ_0 we obtain a commutative Banach algebra

$$B = \mathbf{C} \cdot \delta_0 + L^1(\mathbf{R}^+)$$

1. The Gelfand space $\mathfrak{M}_B$. To obtain this space we consider some $f(x) \in L^1(\mathbf{R}^+)$ and set:

$$\widehat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot f(x) \cdot dx, \quad \text{where} \quad \mathfrak{Im}(\zeta) \geq 0$$

With $\zeta = \xi + i\eta$ we get

$$|\widehat{f}(\xi + i\eta)| \leq \int_0^\infty |e^{i\xi x - \eta x}| \cdot |f(x)| \cdot dx = \int_0^\infty |e^{-\eta x} \cdot |f(x)| \cdot dx \leq ||f||_1$$

We conclude that for every point $\zeta = \xi + i\eta$ in the closed upper half-plane corresponds to a point in $\zeta^* \in \mathfrak{M}_B$ defined by

$$\zeta^*(f) = \widehat{f}(\zeta) \quad \text{and} \quad \zeta^*(\delta_0) = 1$$

In addition to this $L^1(\mathbf{R}^+)$ is a maximal ideal in B and there is the special point $\zeta^\infty \in \mathfrak{M}_B$ such that

$$\zeta^\infty(f) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

2. Theorem. *The Gelfand space $\mathfrak{M}_B$ can be identified with the union of ζ^∞ and the closed upper half-plane.*

Remark. The theorem asserts that every multiplicative functional on B is either ζ^∞ or determined by a point $\zeta = \xi + i\eta$ where $\eta \geq 0$. Concerning the topological identification ζ^∞ corresponds to the one point compactification of the closed upper half-plane. Thus, whenever $\{\zeta_\nu\}$ is a sequence in $\mathfrak{Im}(\zeta) \geq 0$ such that $|\zeta_\nu| \to \infty$ then $\{z_\nu^*\}$ converges to ζ^* in $\mathfrak{M}_B$. In fact, this follows via the Riemann-Lebegue Lemma which gives

$$\lim_{|\zeta| \to \infty} \widehat{f}(\zeta) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

By the general result in XX Theorem 2 holds if we have proved if the following:

3. Proposition. *Let $\{g_\nu = \alpha_\nu \cdot \delta_0 + f_\nu\}_1^k$ be a finite family in B such that the k-tuple $\{\hat{g}_\nu\}$ has no common zero in $\bar{U}_+ \cup \{\infty\}$. Then the ideal in B generated by this k-tuple is equal to B.*

The proof requires some preliminary constructions. We use the conformal map from the upper half-plane onto the unit disc defined by

$$w = \frac{\zeta - i}{\zeta + i}$$

So here w is the complex coordinate in D. Next, consider the disc algebra $A(D)$. Via the conformal map each transform $\widehat{f}(\zeta)$ of a function $f \in L^1(\mathbf{R}^+)$ yields an element of $A(D)$ defined by:

$$F(w) = \widehat{f}(\frac{i + iw}{1 - w})$$

It is clear that $F(w) \in A(D)$. Moreover, we notice that

$$w \to 1 \implies |\frac{i + iw)}{1 - w}| \to \infty$$

It follows that the $A(D)$-function $F(w)$ is zero at $w = 1$ and we can conlcude:

4. Lemma. *By $f \mapsto F$ we have an algebra homomorphism from $L^1(\mathbf{R}^+)$ to functions in $A(D)$ which are zero at $w = 1$.*

Next, let $\mathcal{H}$ denote the algebra homomorphism in Lemma 4 and consider the function $1 - w$ in $A(D)$. We claim this it belongs to the image under $\mathcal{H}$. To see this we consider the function

$$f(x) = e^{-x} \quad x \geq 0 \quad \text{and } f(x) = 0 \quad \text{when } x < 0$$

Then we see that

$$\hat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot e^{-x} \cdot dx = \frac{1}{1 - i\zeta}$$

It follows that

$$F(w) = \frac{1}{1 - i(\frac{iw+i}{1-w})} = \frac{1 - w}{1 - w + w + 1} = \frac{1 - w}{2}$$

Using $2f$ we conclude that $1 - w$ belongs to the $\mathcal{H}$-image. Next, the identity element δ_0 is mapped to the constant function on D. So via $\mathcal{H}$ we have an algebra homomorphism from B into a subalgebra of $A(D)$ which contains $1 - w$ and the identity function and hence all w-polynomials. Returning to the special B-element f we notice that the convolution

$$f * f(x) = x \cdot e^{-x}$$

We can continue and conclude that the subalgebra of B generated by f and δ_0 contains L^1-functions of the form $p(x) \cdot e^{-x}$ where $p(x)$ are polynomials.

5. Exercise. Prove that the linear space $\mathbf{C}[x] \cdot e^{-x}$ is a dense subspace of $L^1(\mathbf{R}^+)$.

From the result in the exercise it follows that the polynomial algebra $\mathbf{C}[w]$ appears as a dense subalgebra of $\mathcal{H}(B)$ when it is equipped with the B-norm. At this stage we are prepared to give:

Proof of Proposition 3. In $A(D)$ we have the functions $\{G_\nu = \mathcal{H}(g_\nu)\}$. By assumption $\{G_\nu\}$ have no common zero in the closed disc D. Since D is the maximal ideal space of the disc algebra and $\mathbf{C}[w]$ a dense subalgebra, it follows that for every $\epsilon > 0$ there exist polynomials $\{p_\nu(w)\}$ such that the maximum norm

$$|p_1 \cdot G_1 + \ldots + p_k \cdot G_k - 1|_D < \epsilon| \tag{1}$$

where 1 is the identity function. Now $p_\nu = \mathcal{H}(\phi_\nu)$ for B-elements $\{\phi_\nu\}$. So in B we get the element

$$\psi = \phi_1 g_1 + \ldots + \phi_k \cdot g_k \tag{2}$$

Moreover we have $|\mathcal{H}(\psi) - 1|_D < \epsilon$ and here we can choose $\epsilon < 1/4$ and by the previous identifications it follows that

$$|\widehat{\psi}(\xi)| \geq 1/4 \quad \text{for all} \quad -\infty < \xi < \infty \tag{3}$$

The proof of Proposition 3 is finished if we can show that (3) entails that the B-element ψ is invertible. Multiplying ψ with a non-zero scalar we may assume that

$$\psi = \delta_0 - g \quad : \quad g \in L^1(\mathbf{R}^+)$$

and the Fourier transform $\widehat{\psi}(\xi)$ satisfies

$$|\widehat{\psi}(\xi) - 1| \leq 1/2$$

for all ξ. It means that $|\widehat{g}(\xi)| \leq 1/2$. The spectral radius formula applied to L^1-functions shows that if N is a sufficiently large integer then

$$||g^{(N)}||_1 \leq (3/4)^N \tag{4}$$

where $g^{(N)}$ is the N-fold convolution of g. Now we have

$$(1 + g + \ldots + g^{N_1}) \cdot \psi = 1 - g^{(N)} \tag{5}$$

By (4) the norm of the B-element $g^{(N)}$ is strictly less than one and hence the right hand side is invertible where the inverse is given by a Neumann series, i.e. with $g_* = g^{(N)}$ the inverse is

$$\delta_0 + \sum_{\nu=1}^{\infty} g_*^{\nu}$$

Since convolutions of $L^1(\mathbf{R}^+)$-functions still are supported by $x \geq 0$, it follows from the above that ψ is invertible in B and Proposition 3 is proved.