II. The Jensen-Nevanlinna class and Blaschke products.

Introduction.

If μ is a real Riesz measure on the unit circle there exist the harmonic function in the disc D defined by

(0.1)
$$H_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} \cdot d\mu(\theta)$$

For each $0 < r < 1$ one has the inequality

(0.2)
$$\int_0^{2\pi} \left| H_\mu(re^{i\theta}) \right| \cdot d\theta \leq ||\mu||$$

where $||\mu||$ is the total variation of μ. Moreover, there exists a weak limit, i.e.

(0.3)
$$\lim_{r\to 1} \int_0^{2\pi} g(\theta) \cdot H_\mu(re^{i\theta}) = \int_0^{2\pi} g(\theta) \cdot d\mu(\theta)$$

holds for every continuous function $g(\theta)$ on T. Conversely we proved in XX that if $H(z)$ is a harmonic function in D for which there exists a constant C such that

(0.4)
$$\int_0^{2\pi} \left| H(re^{i\theta}) \right| \cdot d\theta \leq C$$

hold for all $r < 1$, then there exists a unique Riesz measure μ on T where $H = H_\mu$. Hence there is a 1-1 correspondence between the space of harmonic functions in D satisfying (0.4) and the space of real Riesz measures on T. There also exist radial limits almost everywhere. More precisely, define the μ-primitive function

$$\psi(\theta) = \int_0^\theta d\mu(s)$$

Fatou's Theorem asserts that for each Riesz measure μ there exists a radial limit

(0.5)
$$H^*_\mu(\theta) = \lim_{r\to 1} H(re^{i\theta})$$

for each θ where ψ has an ordinary derivative. Since ψ has a bounded variation this holds almost everywhere by Lebegue's Theorem in [Measure].

0.6 The case when μ is singular. If μ is singular the radial limit (0.5) is zero almost everywhere. If the singular measure μ is non-negative with total mass 2π we have $H_\mu(0) = 1$ and the mean-value property for harmonic functions gives:

$$\int_0^{2\pi} H_\mu(re^{i\theta}) \cdot d\theta = 1$$

for all $0 < r < 1$. At the same time the boundary function $H^*_\mu(\theta)$ is almost everywhere zero which means that no dominated convergence occurs.

0.7 Exercise. Let μ be singular with a Hahn-decomposition $\mu = \mu_+ - \mu_-$. Assume that the positive part $\mu_+(T) = a > 0$. Now there exists a closed null set E such that $\mu_+(E) \geq a - \epsilon$ while $\mu_-(E) = 0$. The last equation gives a small $\delta > 0$ such that if $E_{2\delta}$ is the open 2δ-neighborhood of E then $\mu_-(E_{2\delta}) < \epsilon$. Set

$$H_*(z) = \frac{1}{2\pi} \int_E \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu_+(\theta)$$

Since $\mu_+(E) \geq a - \epsilon$ we get

$$\text{(ii)} \qquad \int_0^{2\pi} H_*(re^{i\theta}) \cdot d\theta \geq a - \epsilon$$

Next, for each pair $\phi \in E_\delta$ and $e^{i\theta} \in T \setminus E_{2\delta}$ we have:

$$\frac{1 - r^2}{|e^{i\theta} - re^{i\phi}|^2} \leq \frac{2(1-r)}{1 + r^2 - 2r\cos(\delta)}$$

So with

$$H_\delta(z) = \frac{1}{2\pi} \int_{T \setminus E_{2\delta}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu_(\theta)$$

it follows that

$$\text{(iii)} \qquad |H_\delta(re^{i\phi})| \leq \frac{1}{2\pi} \cdot \frac{2(1-r)}{1 + r^2 - 2r\cos(\delta)} \cdot \int_{T \setminus E_{2\delta}} |d\mu(\theta)|$$

for each $\phi \in E_\delta$. Since H_* is constructed via the restriction of μ_+ to E, a similar reasoning gives:

$$\text{(iv)} \qquad |H_*(re^{i\phi})| \leq \frac{1}{2\pi} \frac{2(1-r)}{1 + r^2 - 2r\cos(\delta)} \cdot \mu_+(E)$$

when $e^{i\phi} \in T \setminus E_\delta$. Next, by the constructions above we have

$$H = H_* + H_\delta + H_\nu$$

where ν is the measure given by the restriction of μ_+ to $E_{2\delta} \setminus E$ minus μ_- restricted to to $E_{2\delta}$. So by the above the total variation $||\nu|| \leq 2\epsilon$ which gives

$$\int_0^{2\pi} |H_\nu(re^{i\theta})| \cdot d\theta \leq 2\epsilon$$

Deduce from the above that one has an inequality

$$(*) \qquad \int_{E_\delta} H(re^{i\phi}) \cdot d\phi \geq a - \big[2\epsilon + \frac{1}{\pi} \frac{2(1-r)}{1 + r^2 - 2r\cos(\delta)} \cdot ||\mu||\big]$$

Since E is a null-set this shows that mean-value integrals of H behave in an "irregular fashion" when $r \to 1$.

1. The Herglotz integral

Let μ be a real Riesz measure on the unit circle T. Set

$$(*) \qquad g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot d\mu(\theta)$$

This analytic function is called the Herglotz extension of the Riesz measure. Since μ is real it follows that

$$\mathfrak{Re}\, g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z[^2}{|e^{i\theta} - z|^2} \cdot d\mu(\theta) = H_\mu(z)$$

In particular the L^1-norms from (0.2) are uniformly bounded with respect to r when we integrate the absolute value of $\mathfrak{Re}\, g_\mu$. But the conjugate harmonic function representing $\mathfrak{Im}\, g_\mu$ does not satisfy (0.2) in general. However the following holds:

1.1 Theorem. *For almost every θ there exists a radial limit*

$$\lim_{r\to 1} g_\mu(re^{i\theta})$$

To prove Theorem 1.1 we shall use some tricks. The Hahn-decomposition $\mu = \mu_+ - \mu_-$ enables us to express g_μ as a difference $g_1 - g_2$ where g_1, g_2 both are Herglotz extensions of non-negative Riesz measures and hence $\mathfrak{Re}\, g_\nu(z) > 0$ in D. Let us now discuss analytic functions with a positive real part.

1.2 Exercise. Let $f \in \mathcal{O}(D)$ where $\mathfrak{Re}\, f(z) > 0$ and $\mathfrak{Im} f(0) = 0$. Set $f = u + iv$ which gives the analytic function

$$\phi(z) = \log(1 + u + iv)$$

Here

$$\mathfrak{Re}\, \phi = \log\, |1 + u + iv| = \frac{1}{2} \log[(1+u)^2 + v^2]$$

In particular $\mathfrak{Re}\, \phi > 0$ so this harmonic function has a radial limit almost everywhere. We also know that u has a radial limit almost everywhere and from this the reader may conclude that there almost everywhere exist finite radial limits

$$\lim_{r\to 1} v^2(re^{i\theta}) \tag{1}$$

In order to determine the sign of these radial limits we consider the analytic function

$$\psi = e^{-u-iv}$$

Since $u > 0$ we have $|\psi(z)| = e^{-u(z)} \leq 1$ and hence $\psi(z)$ is a bounded analytic function in D. The Brothers Riesz theorem shows that ψ has a radial limit almost everywhere. Finally, when we have a radial limit

$$\lim_{r\to 1} e^{-u(re^{i\theta})-iv(re^{i\theta})}$$

and in addition suppose that u has a radial limit, then it is clear that v has a radial limit too.

Proof of Theorem 1.1 By the Hahn-decomposition of μ the proof is reduced to the case $\mu \geq 0$ and Exercise 1.2 applies.

1.3 The case when μ is singular. When this holds the radial limits of $\mathfrak{Re}\, g_\mu$ are almost everywhere zero. With $v = \mathfrak{Im}\, g_\mu$ there remains to study the almost everywhere defined function

$$v^*(\theta) = \lim_{r\to 1} v(re^{i\theta})$$

It turns out that this Lebesgue-measurable function never is integrable when μ is singular. In fact, the Brothers Riesz theorem shows that if there exists a constant C such that

$$\int_0^{2\pi} |v(re^{i\theta})| \cdot d\theta \leq C$$

hold for all $r < 1$, then the analytic function g_μ belongs to the Hardy space and its radial limits give an L^1-function $g^*(\theta)$ on the unit circle which would entail that σ is equal to the absolutely continuous measure defined by g^*. Thus, for every singular measure μ one has

$$\lim_{r\to 1} \int_0^{2\pi} \big| \mathfrak{Im}\, g_\mu(re^{i\theta})\big| \cdot d\theta = +\infty \tag{*}$$

1.4 Example. Take the case where μ is 2π times the Dirac measure at $\theta = 0$ which gives the analytic function

$$g(z) = \frac{1+z}{1-z}$$

It follows that

$$v(re^{i\theta}) = -2r \cdot \frac{\sin\theta}{1 + r^2 - 2r\cos\theta}$$

and radial limits exist except for $\theta = \pi/2$ or $-\pi/2$, i.e.

$$v^*(\theta) = -2 \cdot \frac{\sin\theta}{2 - 2\cos\theta}$$

when θ is $\neq \pi/2$ and $-\pi/2$. At the same time the reader may verify that $v^*(\theta)$ does not belong to $L^1(T)$ and that

$$\int_0^{2\pi} |v(re^{i\theta})| \cdot d\theta \simeq \log \frac{1}{1-r}$$

as $r \to 1$.

2. The Jensen-Nevanlinna class

Every Riesz measure μ on T gives the zero-free analytic function

(*) $$G_\mu(z) = e^{g_\mu(z)}$$

Here $\log |G_\mu(z)| = \mathfrak{Re}\, g_\mu(z)$ which gives the inequality

$$\log^+ |G_\mu(z)| \leq \big|\mathfrak{Re}\, g_\mu(z)\big|$$

Applying (0.2) we obtain:

(**) $$\int_0^{2\pi} \log^+ |G_\mu(re^{i\theta})| \cdot d\theta \leq ||\mu||$$

for each $r < 1$.

2.1 A converse. Let $F(z)$ be a zero-free analytic function in D where $F(0) = 1$. Assume that there exists a constant C such that

(i) $$\int_0^{2\pi} \log^+ |F(re^{i\theta})| \cdot d\theta \leq C$$

hold for each $r < 1$. The mean-value property applied to the harmonic function $H = \log |F|$ gives

(ii) $$\int_0^{2\pi} |H(re^{i\theta})| \cdot d\theta = 2 \cdot \int_0^{2\pi} \log^+ |F(re^{i\theta})| \cdot d\theta$$

Hence (i) entails that H satisfies (0.4) and now the reader can settle the following:

2.2 Exercise. Show that (i) above entails that there exists a Riesz measure μ such that $F = G_\mu$ where the normalisation $F(0) = 1$ gives $\mu(T) = 2\pi$.

2.3 Radial limits. Whenever g_μ has a radial limit for some θ it is clear that G_μ also has a radial limit in this direction. So Theorem 1.1 implies that there exists an almost everywhere defined boundary function

$$G^*_\mu(\theta) = \lim_{r \to 1} G_\mu(re^{i\theta})$$

The material above suggests the following:

2.4 Definition. *An analytic function f in D belongs to the Jensen-Nevanlinna class if there exists a constant C such that*

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \cdot d\theta \leq C$$

hold for all $r < 1$. The family of Jensen-Nevannlina functions is denoted by JN(D).

Above we described zero-free functions in JN(D). Now we shall study eventual zeros of functions in JN(D). Recall that if $f \in \mathcal{O}(D)$ where $f(0) = 1$ then Jensen's formula gives:

(*)
$$\sum_{|\alpha_\nu|<r} \text{Log}\, \frac{r}{|\alpha_\nu|} = \frac{1}{2\pi} \int_0^{2\pi} \log \left|f(re^{i\theta})\right| \cdot d\theta \quad : \quad 0 < r < 1$$

where the left hand side is the sum of zeros of f in the disc D_r.

A notation. If $f \in \mathcal{O}(D)$ and $r < 1$ we set

$$\mathcal{T}_f(r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \cdot d\theta$$

Since $\log \left|f(re^{i\theta})\right| \leq \log^+ |f|$ it follows that

$$\sum_{|\alpha_\nu|<r} \text{Log}\, \frac{r}{|\alpha_\nu|} \leq \mathcal{T}_f(r)$$

So if $f \in \text{JN}(D)$ we can pass to the limit as $r \to 1$ and conclude that the positive series

(**)
$$\sum \text{Log}\, \frac{1}{|\alpha_\nu|} < \infty$$

where the sum is taken over all zeros in D. Next, recall form XX that the positive series (**) converges if and only if

(***)
$$\sum (1 - |\alpha_\nu| < \infty$$

When (***) holds we say that the sequence $\{\alpha_\nu\}$ satisfies the Blaschke condition. Hence we have proved:

2.5 Theorem. *Let f be in JN(D). Then its zero set satisfies the Blaschke condition.*

3. Blaschke products.

Consider an infinite sequence $\{\alpha_\nu\}$ in D where $|\alpha_1| \leq |\alpha_2| \leq \ldots$ and the Blaschke condition holds. For every $N \geq 1$ we put:

$$B_N(z) = \prod_{\nu=1}^{\nu=N} \frac{|\alpha_\nu|}{\alpha_\nu} \cdot \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z}$$

We are going to prove that the sequence of analytic function $\{B_N\}$ converge in D to a limit function $B(z)$ expressed by the infinite product

(3.1)
$$B(z) = \prod_{\nu=1}^{\infty} \frac{|\alpha_\nu|}{\alpha_\nu} \cdot \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z}$$

To prove this we first analyze the individual factors. For each non-zero $\alpha \in D$ we set

$$B_\alpha(z) = \frac{|\alpha|}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}$$

Exercise. Show that

(i)
$$B_\alpha(z) = |\alpha| \cdot \frac{1 - z/\alpha}{1 - \bar{\alpha} z} = |\alpha| + \frac{|\alpha|^2 - 1}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha} \cdot z$$

and conclude that

(ii)
$$B_\alpha(z) - 1 = (|\alpha| - 1) \cdot [1 + \frac{|\alpha| + 1}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha} \cdot z]$$

Finally, use the triangle inequality to show the inequality

(iii) $$\max_{|z|=r} |B_\alpha(z) - 1| \leq (1 - |\alpha|) \cdot (1 + \frac{2r}{1-r}) = \frac{1+r}{1-r} \cdot (1 - |\alpha|)$$

The convergence of (3.1) From (iii) and general results about product series the requested convergence in (3.1) follows from the assumed Blaschke condition. In fact, when $|z| \leq r < 1$ stays in a compact disc the Blaschke condition and (iii) entail that

$$\sum_{\nu=1}^{\infty} \max_{|z|=r} |B_\alpha(z) - 1| < \infty$$

which implies that (3.1) converges uniformly on $|z| \leq r$ to an analytic function and since $r < 1$ is arbitrary we get a limit function $B(z) \in \mathcal{O}(D)$.

3.2 Exercise. The rate of convergence in $|z| \leq r$ can be described as follows: For each $N \geq 1$ we set

$$G_N(z) = \prod_{\nu=N+1}^{\infty} B_{\alpha_\nu}(z) \quad : \quad \Gamma_N = \sum_{\nu=N+1}^{\infty} 1 - |\alpha_\nu|$$

With $r < 1$ kept fixed we choose n so large that

$$\frac{1+r}{1-r} \cdot (1 - |\alpha_\nu|) \leq \frac{1}{2} \quad : \quad \nu > N$$

Show that this gives:

$$\max_{|z|=r} |G_N(z) - 1| \leq 8 \cdot \frac{1+r}{1-r} \cdot \Gamma_N$$

Since the Blaschke condition implies that $\Gamma_N \to 0$ as $N \to \infty$ this gives a control for the rate of convergence in $|z| \leq r$.

3.3 Radial limits of $B(z)$

When $z = e^{i\theta}$ the absolute value $|B_\alpha(e^{i\theta}| = 1$. So $B(z)$ is the product of analytic functions where every term has absolute value ≤ 1 and hence the maximum norm

$$\max_{z \in D} |B(z)| \leq 1$$

Since the analytic function $B(z)$ is bounded, Fatou's Theorem from Section XX gives an almost everywhere defined limit function

(1) $$B^*(e^{i\theta}) = \lim_{r \to 1} B(re^{i\theta})$$

where the radial convergence holds almost everywhere. Moreover, the Brothers Riesz theorem gives:

(2) $$\lim_{r \to 1} \int_0^{2\pi} |B^*(e^{i\theta}) - B(re^{i\theta})| d\theta = 0$$

3.4 Theorem. *The equality*

(*) $$|B^*(e^{i\theta})| = 1 \quad \text{holds almost everywhere}$$

Proof. Since $|B^*| \leq 1$ it is clear that (*) follows if we have proved that

(i) $$\int_0^{2\pi} |B^*(e^{i\theta})| \cdot d\theta = 1$$

Using (2) above and the triangle inequality we get (i) if we prove the limit formula

(ii) $$\lim_{r \to 1} \int_0^{2\pi} |B(re^{i\theta})| \cdot d\theta = 1$$

To show (ii) we will apply Jensen's formulas to $B(z)$ in discs $|z| \leq r$. The convergent product which defines $B(z)$ gives

$$B(0) = \prod \log |\alpha_\nu|$$

Next, for $0 < r < 1$ Jensen's formula gives

$$\log B(0) = \sum_{\nu=1}^{\rho(r)} \log \frac{|\alpha_\nu|}{r} + \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta$$

where $\rho(r)$ is the largest ν for which $|\alpha_\nu| = r$. It follows that

$$\text{(1)} \qquad \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta \geq \sum_{\nu=1}^{\rho(r)} \log \frac{r}{|\alpha_\nu|} - \sum_{\nu=1}^{\infty} \log \frac{1}{|\alpha_\nu|}$$

Next, with $\epsilon > 0$ we find an integer N such that

$$\text{(2)} \qquad \sum_{\nu=1}^{\nu=N} \log \frac{1}{|\alpha_\nu|} < \epsilon$$

Since $|\alpha_\nu| \to 1$ here exists r_* such that

$$\text{(3)} \qquad r \geq r_* \implies \rho(r) \geq N$$

When (3) holds it follows from (1-2) that

$$\text{(4)} \qquad \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta \geq \sum_{\nu=1}^{\rho(r)} \log \frac{r}{|\alpha_\nu|} - \sum_{\nu=1}^{\rho(r_*)} \log \frac{1}{|\alpha_\nu|} - \epsilon$$

In the first sum every term is ≥ 1 so we get a better inequality when the sum is restricted to $\nu \leq \rho(r_*)$, i.e. we have

$$\text{(5)} \qquad \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta \geq \sum_{\nu=1}^{\rho(r_*)} \log \frac{r}{\alpha_\nu|} - \sum_{\nu=1}^{\rho(r_*)} \log \frac{1}{\alpha_\nu|} - \epsilon$$

Here (5) hold for every $r_* < r < 1$ and a passing to the limit as $r \to 1$ where we only have a finite sum $1 \leq \nu \leq \rho(r_*)$ above we conclude that

$$\lim_{r \to 1} \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta > -\epsilon$$

Since $\epsilon > 0$ is arbitrary we have proved (ii) and hence also Theorem 3.4.

3.5 Division by Blaschke products.

Let $F \in \mathcal{O}(D)$ and assume that its zero set in D is a Blaschke sequence $\{\alpha_\nu\}$. Then we obtain the analytic function

$$G(z) = \frac{F(z)}{B(z)}$$

Here G has no zeros in D and we can construct the analytic function $\operatorname{Log} G(z)$. Set

$$\mathcal{I}_G^+(r) = \int_0^{2\pi} \log^+ |\, G(re^{i\theta})| \cdot d\theta$$

Since $\log^+ [ab| \leq \log^+ |a| + \log^+ |b|$ for every pair of complex numbers we get:

$$\text{(1)} \qquad \mathcal{I}_G^+(r) \leq \mathcal{I}_F^+(r) + \int_0^{2\pi} \log^+ \frac{1}{|\, B(re^{i\theta})|} \cdot d\theta$$

The last nondecreasing function is $\leq \log^+ \frac{1}{|B(0)|}$ for every r which gives

$$\mathcal{I}^+_G(r) \leq \mathcal{I}^+_F(r) + \log^+ \frac{1}{|B(0)|} \tag{2}$$

for every $r < 1$. When $F \in \text{JN}(D)$ this implies that G also belongs to $\text{JN}(D)$. Hence we have proved

3.6 Theorem. *For each $f \in JN(D)$ the function $\frac{f}{B_f}$ also belongs to $JN(D)$, where $B_f(z)$ is the Blaschke product formed by zeros of f outside the origin.*

3.7 Conclusion. Theorem 3.6 and the material in section 2 about zero-free Jensen-Nevanlinna functions give the following factorisation formula:

3.8 Theorem. *For each $f \in JN(D)$ there exists a unique real Riesz measure μ on T with $\mu(T) = 0$ such that*

$$f(z) = az^k \cdot B_f(z) \cdot e^{g_\mu(z)}$$

where $k \geq 0$ is the order of zero of f at $z = 0$ and $a \neq 0$ a constant. Moreover

$$\mu = \log |f(e^{i\theta})| \cdot d\theta + \sigma$$

where σ is the singular part of μ.

3.9 Outer factors. In Theorem 3.8 we get the analytic function

$$\mathfrak{O}_f(z) = e^{g_{\log|f|}(z)}$$

We refer to $\mathfrak{O}_f$ as the outer part of f.

3.10 A division result. Consider a pair f, h in $\text{JN}(D)$ which gives the analytic function in D defined by

$$k(z) = \frac{\mathfrak{O}_h(z)}{\mathfrak{O}_f(z)}$$

By (2.3) there exists the almost everywhere defined quotient on T

$$k^*(\theta) = \frac{\mathfrak{O}^*_h(\theta)}{\mathfrak{O}^*_f(\theta)}$$

3.11 Theorem. *Assume that $k^* \in L^1(T)$. Then k^* belongs to the Hardy space $H^1(T)$.*

Proof. In D there exists the harmonic function

$$k(z) = \log |\mathfrak{O}_h(z)| - \log |O_f(z)|$$

The two harmonic functions in the right hasnd side have by definition boundary functions in $L^1(T)$ and Poisson's formula gives for each point $z = re^{i\theta}$:

$$\log |k(re^{i\theta})| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)} \cdot \log |k^*(\phi)| \cdot d\phi$$

By the general mean-value inequality from (xx) the left hand side is majorized by:

$$\leq \log\Big[\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)} \cdot |k^*(\phi)| \cdot d\phi\Big]$$

Taking exponentials on both sides we get

$$|k(re^{i\theta})| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)} \cdot |k^*(\phi)| \cdot d\phi$$

Now we integrate both sides with respect to θ. Since

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)} \cdot d\theta = 1$$

for every ϕ, it follows that

$$\int_0^{2\pi} |k(re^{i\theta})| \cdot d\theta \leq \int_0^{2\pi} |k^*(e^{i\phi})| \cdot d\phi$$

This proves that the L^1-norms of $\theta \to k(re^{i\theta})$ are bounded which means that k belongs to $H^1(T)$. Moreover, by the Brother's Riesz theorem there exist radial limits almost everywhere so we have also the equality

$$\lim_{r \to 1} k(re^{i\theta}) = k^*(\theta)$$

almost everywhere. This proves that k^* is the boundary value function of the $H^1(T)$-function k.

3.12 Exercise. Show by a similar technique that if we instead assume that k^* is square-integrable, i.e. if $k^* \in L^2(T)$ then $k(z)$ belongs to the Hardy space $H^2(T)$.

3.13 Inner functions. If σ is a non-negative and singular measure on T we get the bounded analytic function

$$G_{-\sigma}(z) = e^{-g_\sigma(z)} \tag{1}$$

Keeping σ fixed we denote this function with f. Here

$$\lim_{r \to 1} |f(re^{i\theta})| = 1$$

holds almost everywhere. So the boundary function $f^*(\theta)$ has absolute value almost everywhere. The class of analytic functions obtained via (1) is denoted by $\mathfrak{I}_*(D)$ and are called zero-free inner functions. In general a bounded analytic function f in D whose boundary values have absolute value almost everywhere is called an inner function and this class is denoted by $\mathfrak{I}(D)$.

3.14 Exercise.. Use the factorisation in Theorem 3.8 to show that every $f \in \mathfrak{I}(D)$ is a product

$$f = B_f \cdot f_*$$

where f_* is a zero-free inner function.

3.15 The case of signed singular measures. Let $\mu = \mu_+ - \mu_-$ be a signed singular measure where $\mu_+ \neq 0$. We get the analytic function G_μ and from the above we know that it has radial limits almost everywhere and since μ is singular the boundary function G^*_μ has absolute value almost everywhere. Here the presence of μ_+ implies that the analytic function G_μ is unbounded. In fact, its maximum modules function

$$M(r) = \max_{|z|=r|} |G_\mu(z)|$$

has a quite rapid growth as $r \to 1$. Moreover one always has

$$\lim_{r \to 1} \int_0^{2\pi} |G_\mu(re^{i\theta})| \cdot d\theta = +\infty \tag{*}$$

in other words, G_μ-functions constructed by signed measures with non-zero negative part never belongs to $H^1(T)$.

3.16 Exercise. Prove (*) above using the divergence in (*) from 1.3.

4. Invariant subspaces of $H^2(T)$

The Hilbert space $L^2(T)$ of square integrable functions on T contains the closed subspace $H^2(T)$ whose elements are boundary values of analytic functions in D. If f $f \in H^2(T)$ it is expanded as

$$\sum_{n=0}^{\infty} a_n \cdot e^{in\theta}$$

and Parseval's theorem gives the equality

$$\sum_{n=0}^{\infty} |a_n|^2 = \frac{1}{2\pi} \cdot \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$$

Moreover, in D we get the analytic function $f(z) = \sum a_n z^n$ where radial limits

$$\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})$$

exist almost everywhere.in fact, this follows via the Brothers Riesz theorem and the inclusion $H^2(T) \subset H^1(T)$. We shall study subspaces of $H^2(T)$ which are invariant under multiplication by $e^{i\theta}$.

4.2 Definition. *A closed subspace V of $H^2(T)$ is called invariant if $e^{i\theta} V \subset V$.*

4.3 Theorem *Let V be an invariant subspace of $H^2(T)$. Then there exists $w(\theta) \in H^2(T)$ whose absolute value is one almost everywhere and*

$$V = H^2(T) \cdot w$$

Proof. First we show that that $e^{i\theta} V$ is a proper subspace of V. For an equality $e^{i\theta} V = V$ gives $e^{in\theta} V = V$ for every $n \geq 1$ which entails that if $0 \neq f \in V$ then $f = e^{in\theta} \cdot g_n$ for some $g_n \in H^2(T)$. This means that the Taylor series of f at $z = 0$ starts with order $\geq n$ which cannot hold for every n unless f is identically zero. So now $e^{i\theta} V$ is a proper closed subspace of V which gives some $0 \neq w \in V$ which is $\perp$ to $e^{i\theta} V$. It follows that

$$\langle w, e^{in\theta} \cdot w \rangle \int_0^{2\pi} w(e^{i\theta}) \bar{w}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0$$

hold for every $n \geq 1$. Since $w \cdot \bar{w} = |w|^2$ is real-valued we conclude that this function is constant and we can normalize w so that $|w(\theta)| = 1$ holds almost everywhere. There remains to prove the equality

(i) $$V = H^2(T) \cdot w$$

Since $|w| = 1$ almost everywhere the right hand side is a closed subspace of V. If it is proper we find $0 \neq u \in V$ where $u \perp H^2(T) w$ which gives

(ii) $$\int_0^{2\pi} u(e^{i\theta}) \bar{w}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0 \quad : \quad n \geq 0$$

Taking complex conjugates we get

(iii) $$\int_0^{2\pi} w(e^{i\theta}) \bar{u}(e^{i\theta}) \cdot e^{in\theta} \cdot d\theta = 0 \quad : \quad n \geq 0$$

At the same time $w \perp e^{i\theta} V$ which entails that

(iv) $$\int_0^{2\pi} w(e^{i\theta}) \bar{u}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0 \quad : \quad n \geq 1$$

Together (iiii-iv) imply that $w\bar{u}$ has vanishing Fourier coefficients and is therefore identically zero which gives $u = 0$ and proves that $V = H^2(T) \cdot w$ must hold.

4.4 Examples. Let $B(z)$ be a non-constant Blaschke product. Now $|B(e^{i\theta})| = 1$ holds almost everywhere and the presence of zeros of $B(z)$ in D show that $H^2(T) \cdot B$ is a proper and invariant subspace of $H^2(T)$. Next, let σ be a singular Riesz measure on T which is real and non-negative. We get the analytic function

$$f(z) = e^{-g_\sigma(z)}$$

Here

$$|f(z)| = e^{-H_\sigma(z)}$$

and since $\sigma \geq 0$ we have $H_\sigma(z) \geq 0$ and hence $|f(z)| \leq 1$. So f is a bounded analytic function in D and in particular it belongs to $H^2(T)$. Moreover we know from XX that the boundary function $f(e^{i\theta})$ has absolute value one almost everywhere. So $H^2(T) \cdot f$ is an invariant subspace of $H^2(T)$ and the question arises if it is proper or not. In contrast to the case for Blaschke functions B above this is not obvious since f has no zeros in D. However it turns out that one has

4.5 Theorem. *Let σ be a singular and non-negative Riesz measure which is not identically zero. Then $H^2(T) \cdot e^{-g_\mu}$ is a proper subspace of $H^2(T)$.*

Proof. Set $w(\theta) = e^{-g_\mu(e^{i\theta})}$. For the analytic function $w(z)$ in the disc its value at $z = 0$ becomes

$$w(0) = e^{-g_\mu(0} = e^{-\sigma(T)/2\pi}$$

Next, if $P(z)$ is a polynomial we have

$$\frac{1}{2\pi}\int_0^{2\pi} |P(\theta)w(\theta) - 1|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} |P(\theta)|^2 \cdot d\theta + 1 + 2\mathfrak{Re}\,[\int \frac{1}{2\pi}\int_0^{2\pi} P(\theta) \cdot w(\theta) \cdot d\theta]$$

By Cauchy's formula the last term becomes

$$2\mathfrak{Re}(P(0)w(0)) = 2w(0) \cdot \mathfrak{Re}(P(0))$$

By (i) we have $0 < w(0) < 1$ and if $||P||_2$ is the L^2-norm of P the right hand side majorizes

$$||P||_2^2 + 1 - 2w(0) \cdot |P(0)|$$

We have also the inequality

$$|P(0)| \leq ||P||_2$$

So if we set $\rho = ||P||_2$ then we have shown that

$$\frac{1}{2\pi}\int_0^{2\pi} |P(\theta)w(\theta) - 1|^2 d\theta \geq \rho^2 + 1 - 2w(0) \cdot \rho$$

Now we notice that the right hand side is $\geq 1 - w(0)^2$ for every ρ. Since P is an arbitrary polynomial we conclude that the L^2-distance of 1 to the subspace $H^2(T) \cdot e^{-g_\mu}$ is at least

$$(*) \qquad 1 - w(0)^2 = 1 - e^{-2\sigma(T)}$$

5. Beurling's closure theorem.

A zero-free function $f \in H^2(T)$ is of outer type when

$$f(z) = G_\mu(z)$$

where μ is the absolutely continuous Riesz measure $\log|f(e^{i\theta}|$. The following result is due to Beurling in [Beur]:

5.1 Theorem. *For every nonzero $f \in H^2(T)$ of outer type the closed invariant subspace generated by analytic polynomials $P(z)$ times f is equal to $H^2(T)$.*

Proof. If the density fails we find $0 \neq g \in H^2(T)$ such that

$$\text{(i)} \qquad \int_0^{2\pi} e^{in\theta} f(e^{i\theta}) \cdot \bar{g}(e^{i\theta}) \cdot d\theta = 0 \quad \text{for every} \quad n \geq 0$$

By Cauchy-Schwarz the product $f \cdot \bar{g}$ belongs to $L^1(T)$ and (i) implies that this function is of the form $e^{i\theta} \cdot h(\theta)$ where $h \in H^1(T)$. So on T we have almost everywhere:

(ii) $$\bar{g}(e^{i\theta}) = e^{i\theta} \cdot \frac{h(e^{i\theta})}{f(e^{i\theta})}$$

Now we take the outer factor $\mathfrak{O}_h$ whose absolute value is equal to $|k|$ almost everywhere on T. It follows that

(iii) $$|g^*(\theta)| = \frac{\mathfrak{O}_h^*(\theta)}{\mathfrak{O}_f^*(\theta)}$$

Since $g \in H^2(T)$ Exercise 3.12 shows that the quotient in (ii) is the boundary value of an analytic function in $H^2(T)$ which implies that the conjugate function $\bar{g}$ also belongs to $H^2(T)$. But then g must be a constant and this constant is zero because the factor $e^{i\theta}$ appears in (ii). So g must be zero which gives a contradiction and the requested density is proved.

5.2 Szegö's theorem.

Let $w(\theta)$ be real-valued and non-negative function in $L^1(T)$ and denote by $\mathcal{P}_0$ the space of analytic polynomials $P(z)$ where $P(0) = 0$. Put

$$\rho(w) = \frac{1}{2\pi} \inf_{P \in \mathcal{P}_0} \int_0^{2\pi} \left|1 - P(e^{i\theta})\right| \cdot w(\theta) \cdot d\theta$$

5.3 Theorem. *One has the equality*

$$\rho(w) = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \log w(\theta) \cdot d\theta\right]$$

Proof. First we consider the case when $\log |w| \in L^1(T)$. Multiplying w with a positive constant we may assume that

(i) $$\int_0^{2\pi} \log w(\theta) \cdot d\theta = 0$$

Now we must show that $\rho(w) = 1$. To prove this we use that $\log w \in L^1(T)$ and construct the analytic function

$$f(z) = G_{\log w(z)}$$

So f is an outer function where on T one has

(ii) $$|f(e^{i\theta})| = e^{\log |w(\theta)|} = w(\theta)$$

Hence $f \in H^1(T)$ and (1) gives $f(0) = 1$. Let us now consider some $P(z) \in \mathcal{P}_0$ and set

$$F(z) = (1 - P(z))f(z)$$

Again $F(0) = 1$ and $F \in H^1(T)$ which gives the inequality

(iii) $$1 \leq \int_0^{2\pi} |F(e^{i\theta})| \cdot d\theta$$

By (ii) this means that

$$1 \leq \int_0^{2\pi} |1 - P(e^{i\theta})| \cdot w(\theta) \cdot d\theta$$

Since this hold for every $P \in \mathcal{P}_0$ we have proved the inequality

(iv) $$\rho(w) \geq 1$$

To prove the reverse inequality we apply Beurling's theorem to the outer function f. This gives a sequence of polynomials $\{Q_n(z)\}$ such that

(v) $$\lim_{n \to \infty} ||Q_n \cdot f - 1||_1 = 0$$

where we use the norm on $H^1(T)$. Since $f(0)=1$ it follows that $Q_n(0)\to 1$ and we can normalize the approximating sequence so that $Q_n(0)=1$ for every n and write $Q_n = 1-P_n$ with $P_n\in\mathcal{P}_0$. Finally using (ii) we get

$$\lim_{n\to\infty}\int_0^{2\pi} |1-P(e^{i\theta})|\cdot w(\theta)\cdot d\theta = 1$$

This gives $\rho(w)\geq 1$ and Szegö's theorem is proved for the case A above.

B. The case when $\log^+\frac{1}{|w|}$ *is not integrable.* Here we must show that $\rho(w)=0$ and the proof of this is left as an exercise to the reader.

6. The Helson-Szegö theorem

A trigonometric polynomial on the unit circle is of the form

$$P(\theta)=\sum a_n\cdot e^{in\theta}$$

where $\{a_n\}$ are complex numbers and only a finite family is $\neq 0$. The conjugation operator $\mathcal{C}$ is defined by

$$(*)\qquad \mathcal{C}(P)=i\cdot\sum_{n<0}a_n\cdot e^{in\theta}-i\cdot\sum_{n>0}a_n\cdot e^{in\theta}$$

Let $w(\theta)$ be a non-negative function in $L^1(T)$ and assume also that $\big|\log|w|\,\big|\in L^1(T)$.

6.1 Definition. *A w-function as above is of Helson-Szegö type if there exists a constant C such that*

$$(*)\qquad \int_0^{2\pi}|\mathcal{C}(P)(e^{i\theta})|^2\cdot w(\theta)\cdot d\theta\leq C\cdot\int_0^{2\pi}|P(e^{i\theta})|^2\cdot w(\theta)\cdot d\theta$$

hold for all trigonometric polynomials.

Notice that if (*) holds for some w then it holds for every function of the form $\rho\cdot w$ where $0<c_0\leq\rho(\theta)\leq c_1$ for some pair of positive constants. Or equivalently, with w replaced by $e^u\cdot w$ for some bounded function $u(\theta)$. With this in mind we announce the result below which is due to Helson and Szegö in [HS]:

6.2 Theorem. *A function $w(\theta)$ is of the Helson-Szegö type if and only if there exists a bounded function u and a function $v(\theta)$ for which the maximum norm of $|v|$ over T is <1 and*

$$w(\theta)=e^{u(\theta)+v^*(\theta)}$$

where v^* is the harmonic conjugate of v.

The proof requires several steps. The first part is an exercise on norms on the Hilbert space $L^2(w)$ which is left to the reader.

Exercise. Show that w is of the Helson-Szegö type if and only if there exists a constant $\rho<1$ such that

$$(*)\qquad \Big|\int_0^{2\pi}P(\theta)\cdot e^{-i\theta}\cdot Q(\theta)\cdot w(\theta)\cdot d\theta\,\Big|\leq\rho\cdot||P||_w\cdot||Q||_w$$

hold for all pairs P,Q in $\mathcal{P}_0$.

6.3 The outer function ϕ. We define the analytic function $\phi(z)$ by

$$\phi(z)=\exp\Big[\,\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\cdot\log(\sqrt{w(\theta)})\cdot d\theta\,\Big]$$

Since $\log\sqrt{w(\theta)}=\frac{1}{2}\cdot\log w(\theta)$ is in $L^1(T)$ it means that $\phi(z)$ is an outer function and on the unit circle we have the equality

$$(1)\qquad |\phi(\theta)|^2=w(\theta)$$

Using (1) we find a real-valued function $\gamma(\theta)$ such that

(2) $$w(\theta) = \phi^2(\theta) \cdot e^{i\gamma(\theta)}$$

Next, (1) implies that the weighted L^2-norm $||P||_w$ is equal to the standard L^2-norm of $\phi \cdot P$ on T. Hence (1) holds if and only if

(3) $$\Big| \int_0^{2\pi} \phi(\theta)P(\theta) \cdot e^{-i\theta} \cdot \phi(\theta)Q(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||\phi \cdot P||_2 \cdot ||\phi \cdot Q||_2$$

hold for all pairs P, Q in $\mathcal{P}_0$. Now we use that ϕ is outer which by Beurling's closure theorem means that $\mathcal{P}_0 \cdot \phi$ is dense in $H_0^2(T)$. Hence (3) is equivalent to

(4) $$\Big| \int_0^{2\pi} F(\theta) \cdot e^{-i\theta} \cdot G(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||F||_2 \cdot ||G||_2$$

for all pairs F, G in $H_0^2(T)$.

Next, in XX we prove that every $f \in H_0^1(T)$ admits a factorization $f = F \cdot G \cdot e^{-i\theta}$ for a pair F, G where $||f||_1 = ||F||_2 \cdot ||G||_2$. So (4) is equivalent to

(5) $$\Big| \int_0^{2\pi} f(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||f||_1$$

for each $f \in H_0^1(T)$. At this stage we use the duality between $H^\infty(T)$ and $H_0^1(T)$ from Section XX. It follows that (5) is equivalent to the following

6.4 Approximation condition. One has

$$\min_h ||e^{i\gamma(\theta)} - h(\theta)||_\infty = \rho$$

where the minimum is taken over h-functions in $H^\infty(T)$.

Final part of the proof.

Since $w \geq 0$ and > 0 outside a set of measure zero, the approximation condition is equivalent with the existence of some $h \in H^\infty(T)$ and some $\rho < 1$ such that

(*) $$|w(\theta) - \phi^2(\theta) \cdot h(\theta)| \leq \rho \cdot w(\theta)$$

hold on T. It remains to show that (*) is equivalent to the existence of a pair u, v in Theorem 6.2. Let us begin with

Proof that (*) gives the pair u, v. Since $\log w$ is in $L^1(T)$ we have $w > 0$ almost everywhere and (*) entails that $\phi^2(\theta) \cdot h(\theta)$ stay in the sector

$$Z = \{z : -\pi/2 + \delta \leq \arg(z) \leq \pi/2 - \delta\}$$

where we have put $\delta = \arccos(\rho)$. This inclusion of the range of $\phi^2 \cdot h$ implies that it is outer. See XX above. Hence we can find a harmonic function V such that

$$\phi^2 \cdot h = e^{ia} \cdot e^{V + iV^*}$$

where a is some real constant. The inclusion of the range implies that

$$|a + V^*(\theta)| \leq \pi/2 - \delta$$

Next, define the harmonic function

$$v(\theta) = -(a + V^*(\theta))$$

It follows that

$$\phi^2(\theta) \cdot h(\theta) = e^{v(\theta) + iv^*(\theta) + c}$$

for some constant c. Finally, since $w = |\phi|^2$ we obtain

$$w(\theta) = e^{v(\theta)} \cdot \frac{e^a}{|h(\theta)|}$$

By (xx) above the last factor is bounded both below and above and hence e^u for some bounded function. Together with the bound (xx) for the harmonic conjugate of v we get the requested form for $w(\theta)$ in Theorem 6.2.

Proof that a pair (u,v) gives (*). Consider the special case when $w = e^v$ and

$$|v^*(\theta)| \leq \pi/2 - \epsilon$$

holds for some $\epsilon > 0$. It is clear that the corresponding ϕfunction obtained via (xx) above satisfies

$$\phi^2(\theta) = e^{v(\theta)+iv^*(\theta)}$$

This gives

$$e^{i\gamma(\theta)} = e^{-iv^*(\theta)}$$

and we notice that if we take the constant function $h(\theta) = \epsilon$ then the maximum norm

$$||e^{i\gamma(\theta)} - \epsilon||_\infty < 1$$

which proves that (*) holds.