On minimal surfaces.

Introduction. To avoid possible confusion we remark that the subsequent material deals with an isoperimetric problem with a fixed boundary curve. Considerably more involved situations appear when the boundary value problem for minimal surfaces do not have entirely fixed Jordan curves, but are free to move on prescribed manifolds. Here one encounters the problems by Plateau and Douglas. For an account about the general case we refer to Courant's article *The existence of minimal surfaces of given toplogical structure under prescribed boundary conditions.* (Acta. Math. Vol 72 1940]) where the reader also finds an extensive references to relevant literature.

From now on we discuss the problem when the boundary curve is fixed in $\mathbf{R}^3$. Let (x, y, z) be the coordinates in $\mathbf{R}^3$. Consider a rectifiable closed Jordan curve C and denote by $\mathcal{S}(C)$ the family of surfaces which are bordered by C. A surface M in this family is minimal if it has smallest possible area. To find such a minimal surface corresponds to a problem in the calculus of variation and was studied by Weierstrass in a series of articles starting from *Untersuchungen über die Flächen deren mittlere Krümmung überall gleich null ist* from 1866. A revised version appears in volume I of his collected work. To begin with a local study shows that minimal surface M in $\mathcal{S}(C)$ has vanishing imean curvature. Moreover, Weierstrass proved that M has no singular points and is simply connected. These properties for a minimal surface are natural and rather easy to prove. After a more careful analysis, Weierstrass proved that there exists a homeomorphic parametrization of M above an open disc in the complex u-plane which can be achieved via complex analytic functions, or as expressed by Weierstrass in the introduction to [Wei]: *Ich habe mich mit der Theorie die Flächen, deren mittlere Krümmung überall gleich null ist, besonders auf dem grunde eingehender beschäftigt, weil sie, wie ich zeigen werde, auf das Innigste mit der Theorie der analytischen funktionen einer komplexen Argumentz zusammenhhängt.* Or shortly phrased: *The theory about minimal surfaces is closely linked to the theory of analytic functions in one complex variable.*

The isoperimetric inequality. Using Weierstrass' description of minimal surfaces the following was proved by Carleman in the article *Zur Theorie der Minimalflächen* in 1920:

Theorem 1. *For every rectifiable simple closed curve C the area A of the minimal surface in $\mathcal{S}(C)$ satisfies the inequality*

$$A \leq \frac{\ell(C)^2}{4\pi}$$

where $\ell(C)$ is the arc-length of C.

Remark. For historic reasons one may wonder why this result was not already discovered by Weierstrass. The reason might be that analytic function theory was not developed enough in 1890. Carleman's proof relies upon the Jensen-Blascke factorisation of analytic functions which was not know prior to 1900. Another obstacle was the discovery by Hermann Schwarz that the minimal surface in the family $\mathcal{S}(C)$ is not determined by vanishing mean curvarute alone. See Volume II, page 264 and 151-167 in the collected work of Hermann Schwarz for this "ugly phenomenon" which was one reason why Weierstrass paid much attention to existence problems in the calculus of variation. As remarked by Carleman at the end of his article, the alternative proof by Blaschke which was given after [Carleman[had been published, is restricted to a special class of minimal surfaces. The case when C is piecewise linear and consists of n many line segements $L_1, \ldots, L_n$ means in the words of Weierstrass that one regards the following problem: *Es soll ein einfach zusamenhängenden Minimalflächenstück M analytish bestimmt werden, dess vorgeschiebinen begrenzung C aus n geradlinigen strecken bestecht, welche eine einfache, geschlossene, nicht verknotete Linie bilden.* In [Weierstrass] appears a far reaching study of this problem. The main result shows that M is determined via a speciai pair of analytic functions $G(u)$ and $H(u)$ defined in the lower half-plane $\mathfrak{Im}\, u < 0$ for which the three functions defined by

$$\phi_1(u) = \det \begin{pmatrix} G(u) & H'(u) \\ G'(u) & H'(u) \end{pmatrix}$$

$$\phi_2(u) = \det \begin{pmatrix} G(u) & H'(u) \\ G''(u) & H''(u) \end{pmatrix}$$

$$\phi_3(u) = \det \begin{pmatrix} G'(u) & H'(u) \\ G''(u) & H''(u) \end{pmatrix}$$

become rational functions of u. Moreover, [ibid] exhibits second order differential equations of the Fuchsian type satisfied by the rational functions which appear above and the position of their poles are described in terms of the geometric configuration of C. It would lead us too far to enter the material in [Weierstrass] so its rich contents is left to the interested reader for further studies.

The planar case. If C is a simple closed curve in the complex z-plane the isoperimetric inequality follows easily via analytic function theory. Namely, let M be the Jordan domain bordered by C. By Riemann's theorem there exists a conformal mapping $\phi\colon D \to M$ and we have

$$\ell(C) = \int_0^{2\pi} |\phi'(e^{i\theta})|\, d\theta \quad : \quad A = \text{area}(M) = \iint_D |\phi'(z)|\; 2\, dxdy$$

We leave as an exercise to verify the isoperimetric inequality in Theorem 1 and that equality holds if and only if the complex derivative satisfies

$$\phi'(z) = \frac{a}{(1-qz)^2}$$

for a pair of constants a, b. This means that ϕ is is a Möbius transform and hence C must be a circle, i.e. equality in Theorem 1 for a planar curve holds if and only if C borders a disc,

B. Proof of Theorem 1.

The crucial step in the proof relies upon the following:

B.1 Proposition. *Let M be a minimal surface in $\mathcal{S}(C)$. Then there exists an analytic function $F(u)$ in the open unit disc such that points $(x, y, z) \in M$ are given by the equations:*

$$x = \mathfrak{Re} \int (1-u^2)F(u)\, du \quad : \quad y = \mathfrak{Re} \int i(1+u^2)F(u)\, du \quad : \quad z = \mathfrak{Re} \int 2F(u)\, du$$

The proof is quite involved and occupies more than five pages in [Weierstrass]. Let us at least indicate some steps in Weierstrass' constructions which lead to Proposition B.1. To begin with the one finds a planar domain Σ with real coordinates (p, q) and a diffeomorphism between M and Σ which is conformal, i.e. M is defined by the equations

$$x = x(p,q) \quad : y = y(p,q) \quad : z = z(p,q)$$

where the vectors $\mathbf{v} = (\frac{\partial x}{\partial p}, (\frac{\partial y}{\partial p}, (\frac{\partial x}{\partial p},)$ and $\mathbf{w} = (\frac{\partial x}{\partial q}, (\frac{\partial y}{\partial q}, (\frac{\partial z}{\partial q})$ are pairwise orthogonal unit vectors. Since the mean curvature of M vanishes the three functions in (1) are harmonic, i.e.

(i) $$\Delta(x) = \frac{\partial^2 x}{\partial p^2} + \frac{\partial^2 x}{\partial q^2} = 0$$

and similarly for y and z. In fact, this follow from classical calculus of variation and basic differential geometry on surfaces which is exposed in many text-books such as [Darbroux]. The harmonic functions above are real parts of analytic functions which yields a triple f, g, h in $\mathcal{O}(\Sigma)$ such that

$$x = \mathfrak{Re}\, f(u)$$

The orthogonality of the vectors $\mathbf{v}$ and $\mathbf{w}$ above entails via the Cauchy Riemann equations that

$$(f'(u))^2 + (g'(u))^2 + (h'(u))^2 = 0$$

Starting from this, Weierstrass used sterographic projections and Riemann's conformal mapping theorem to construct an analytic function $F(u)$ which gives the equations in Proposition B.1. Admitting this one we can prove the following:

B.2 Propostion. *One has the formulas*

$$\text{area}(M) = \iint_D (1+|u|^2)^2 \cdot |F(u)|^2 \, d\xi d\eta \quad : \quad \ell(C) = 2 \cdot \int_0^{2\pi} |F(e^{i\theta})| \, d\theta$$

Proof. With $u = \alpha + i\beta$ this amounts to show that

(i) $$dx^2 + dy^2 + dz^2 = (1+|u|^2)|F(u)|^2 \cdot (d\alpha^2 + d\beta^2)$$

To prove (i) we consider some point $u \in D$. Set $F(u) = |F(u)| \cdot e^{i\theta}$ and $u = se^{i\alpha}$. With $du = d\alpha$ real we have

$$dx = \mathfrak{Re}(1-u^2)F(u)) \cdot d\alpha = |F(u)| \cdot \big(\cos\theta - |u|^2 \cos\theta \cdot \cos 2\alpha - |u|^2 \sin\theta \cdot \sin 2\alpha\big) \cdot d\alpha$$

By trigonometric forulas it follows that

(i) $$(dx)^2 = |F(u)| \cdot \big[\cos^2\theta + |u|^4 \cos^2(2\alpha - \theta) - 2|u|^2 \cos\theta \cdot \cos(2\alpha - \theta)\,\big] \cdot (d\alpha)^2$$

In the same way we find that

(ii) $$(dy)^2 = |F(u)|^2 \cdot \big[\sin^2\theta + |u|^4 \sin^2(2\alpha - \theta) + 2|u|^2 \sin\theta \cdot \sin(2\alpha - \theta)\big] \cdot d\alpha$$

(iii) $$(dz)^2 = 4|F(u)|^2 \cdot |u|^2 \big(\cos^2(\theta - \alpha)\big] \cdot (d\alpha)^2$$

Adding (i-ii) we get

$$(dx)^2 + (dy)^2 = |F(u)|^2 \cdot \big[1 + |u|^4 - 2 \cdot |u|^2 \cos(2\theta - 2\alpha)\big] \cdot (d\alpha)^2$$

Finally, the trigonometric formula

$$4\cos^2\phi = 2 - 2\cos 2\phi$$

shows that

$$(dx)^2 + (dy)^2 + (dz)^2 = |F(u)|^2 \cdot (1+|u|^2)^2 \cdot (d\alpha)^2$$

The same hold when $u = id\beta$ and this infinitesmal equations give Proposition B.2.

Proof of Theorem 0.1

With F as above we put

$$f_1(u) = F(u)u^2 \quad \text{and} \quad f_2(u) = F(u) \implies$$

(i) $$A = \text{area}(M) = \iint_D \big[|f_1(u)|^2 + |f_2(u)|^2\,\big] \, d\xi d\eta + 2 \cdot \iint_D |f_1(u)| \cdot |f_2(u)| \, d\xi d\eta$$

where the implication follows from Proposition B.2. Since $|f_1| = |f_2| = |F|$ holds on the unit circle we also get

(ii) $$\ell(C)^2 = \Big[\int_0^{2\pi} |f_1(e^{i\theta})| \, d\theta\Big]^2 + \Big[\int_0^{2\pi} |f_2(e^{i\theta})| \, d\theta\Big]^2 + 2 \cdot \int_0^{2\pi} |f_1(e^{i\theta})| \, d\theta \cdot \int_0^{2\pi} |f_2(e^{i\theta})| \, d\theta$$

At this stage we shall need:

B.3 Lemma. *For each pair of analytic functions g, h in the unit disc one has*

$$\iint_D [g(u)| \cdot |h(u)|\, d\xi d\eta \leq \frac{1}{4\pi} \cdot \int_0^{2\pi} |g(e^{i\theta})|\, d\theta \cdot \int_0^{2\pi} |h(e^{i\theta})|\, d\theta$$

Let us first notice that Lemma B.3 applied to the pairs $g = h = f_1$, $g = h = f_2$ and the pair $g = f_1$ and $h = f_2$ together with (i-ii) give Theorem 1. So there remains to prove Lemma B.3. We can write

$$g = B_1 \cdot g^* \quad : \quad h = B_2 \cdot h^*$$

where B_1, B_2 are Blaschke products and the analytic functions g^* and h^* are zero free in the unit disc. Since $|B_1| = |B_2| = 1$ hold on the unit circle it suffices to prove Lemma B.3 for the pair g^*, h^*, i.e. we may assume that both g and h are zero-free. Then they posses square roots so we can find analytic functions G, H in the unit disc where

$$g = G^2 \quad : \quad h = H^2$$

Consider the Taylor series

$$G(z) = \sum a_k u^k \quad : \quad H(z) = \sum b_k u^k$$

Now $GH = \sum c_k u^k$ where

(i) $$c_k = a_0 b_k + \ldots + a_k b_0$$

Using polar coordintes to perform double integrals it follows that

$$\iint_D |G^2(u)| \cdot |H^2(u)|\, d\xi d\eta = \pi \cdot \sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1}$$

At the same time one has

$$\int_0^{2\pi} |G^2(e^{i\theta})|\, d\theta = 2\pi \cdot \sum_{k=0}^{\infty} |a_k|^2$$

with a similar formula for the integral of H^2. Hence Lemma B.2 follows if we have proved the inequality

(ii) $$\sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1} \leq \sum_{k=0}^{\infty} |a_k|^2 \cdot \sum_{k=0}^{\infty} |b_k|^2$$

To get (ii) we use (i) which for every k gives:

$$|c_k|^2 \leq (|a_0||b_k| + \ldots + |a_k||b_0|)^2 \leq (k+1) \cdot (|a_0|^2 |b_k|^2 + \ldots + |a_k|^2 |b_0|^2)$$

Finally, a summation over k entails (ii) and Lemma B.3 is proved.