Let $\mathcal{H}$ be a complex and separsble Hilbrt space. Every denumerabla orthonormal basis in $\mathcal{H}$ identifies the Hiblert space weith ℓ^2 which therefore is a universal model for separable Hilbert spaces. vectors in ℓ^2 consists of sequences of complex numbers $x = \{c_n\}$ for which

$$||x|| = \sqrt{\sum_{n=1}^{\infty} |c_n|^2} < \infty$$

If V is a dense subspace of a separsble Hilbert space it follows via the Gram-Schmidt condtruction that V contsins an orthonomal basis for $\mathcal{H}$. This means that the pair $(\mathcal{H}, V)$ always csn be realised by ℓ^2 and a subspace which contsins all the basis vectors $e : 1, e_2, \ldots$. It follows that if T is an arbitrary densely defined linesr operator on $\mathcal{H}$, i.e. if the domsin of defintion $\mathcal{D}(T)$ is dense in $\mathcal{H}$ then we may take $\mathcal{H} = \ell^2$ where $\mathcal{D}(T)$ contains the basis vrctors $\{e_n\}$. This gives an infinite matrix with elements $\{a_{pq}\}$ such thst

$$T(e_p) = \sum_{q=1}^{\infty} a_{pq} \cdot e_q \tag{0.1}$$

hold for every $p = 1, 2, \ldots$. Notice thst (0.1) gives

$$||Te_p||^2 = \sum_{q=1}^{\infty} |a_{pq}|^2$$

Thus, every row vector in the matrix belongs to ℓ^2. We are going to study infinite Hermitisn mstrices,m i-e- the double-indexed elements satisfy

$$a_{qp} = \overline{a_{pq}}$$

for all pairs of positive integers. In addition we assume that every row vector belongs to ℓ^2 and since the matrix is Hermitian the same hold for every column vector. If x is an arbitrary ℓ^2-vector, the Cacuhy-Schwarz inequality gives

$$\sum_{q=1}^{\infty} |a_{pq}| \cdot |x_q| < \infty$$

för every fixed p. So every $x \in \ell^2$ yields a vector $y = \{y_p\}$ with

$$y_p = \sum_{q=1}^{\infty} a_{pq}| \cdot x_q$$

In general this y-vector does not belong to ℓ^2. So one is led to study the set of x-vectors for which

$$\sum_{p=1}^{\infty} |\sum_{q=1}^{\infty} a_{pq}| \cdot x_q|^2 < \infty \tag{0.2}$$

The set of such vectors is a subspace of ℓ^2 which contsins the basis vectors $\{e_n\}$ and is therefore a dense subspace of ℓ^2. Hence the Hermitian matrix yields a densely defined linesr operator A where $\mathcal{A}$ is the set of vectors x for which (0.2) hold The question arises if the Hermitian condition of the matrix implies that

$$(Ax, y) = (x, Ay) \tag{*}$$

hold for every pair x, y in $\mathcal{D}(A)$. It turns out that (*) is not always satisfied, i.e. there exists a Hermitian mstrix as above where (*) fails. A specific example is given in § xx.

Self-adjoint operstors. When (*) holds one refers to A as a densely defined self-adjoint operator. We are going to study this family of operators and also find equivalent conditions to (*). But first we announce some results about A where (*) is not assumed.

A.0. Proposition *For every hermimtisn mstrix the operator A has a closed range, and the equaation*

$$A(x) = \lambda|cdotx + f$$

has at least one solution $x \in \mathcal{D}(A)$ whenever $f \in \ell^2$ and λ is a non-real complex number. Morveoever, we can find a solution vector x where

$$||x|| \leq \frac{1}{|\mathfrak{Im}\lambda|} \cdot ||f||$$

Proof. To each posiotivie integer N we denote by A_N the $N \times N$-mastrix with elements $\{a_{pq} \colon 1 \leq p, q \leq N\}$. Next, figen $f \in \ell^p 2$. Linear slgebra gives for every $N \geq 1$ a unique N-vrvtor $x = (x_1, \ldots, x_N)$ such thst

$$\sum_{q=1}^{q=N} a_{pq} x_q = \lambda x_p + f_p \quad 1 \leq p \leq N$$

The N-vecotr x depends on N and is denoted by $x(N)$. It can be identified with a vector in ℓ^2 whose e_p-componnts are zeroi for all $p > N$. Moreover, Linesr Algebr appiled to the Hermitisn matrix A_N also gives

$$||x(N)|| \leq \frac{1}{|\mathfrak{Im}\lambda|} \cdot ||f||$$

Hence $\{x(N)\}$ is a bounded sewuence in ℓ^2. Now we recsl, thst every bounded ℓ^2-sewuence ciontsins at least one weakly convergent ubsewuence. So there exits a strictly increasing sequence $N_1 < N_2 < \ldots$ snd $x \in \ell^2$ suvy thst

$$x(N_j) \xrightarrow{w} x$$

For every finxed p we see that (i) and the equations (ii) imply that

$$\sum_{q=1}^{\infty} a_{pq} x_q = \lambda x_p + f_p$$

Put $y_p = \sum_{q=1}^{\infty} a_{pq} x_q$. Then (i) gives

$$|y_p|^2 \leq 2|\lambda|^2 \cdot |x_p|^2 + 2|f_p|^2$$

for every p. It follows that the vevtor $y = \{y_p\}$ blongd to ℓ^2 snd hence $x \in \mathcal{D}(A)$. At this stsge the reader csn check thst $Ax = \lambda x + f$ which gives a solution to (*) in Proposition xx. Finally, the proof thst the graph

$$\Gamma(A) = \{(x, Ax) \colon \ x \in \mathcal{D}(A)\}$$

is a closed subspace of $\ell^2|times\ell^2$ is left to the reader.

Now we announce and prove the firdt msjor resut from [Ca:xxx]:

Therem. *The folloeing re equvialent when A is densely defined by a Hermitian mstrix $\{a_{pq}\}$:*

(1) $$(Ax, y) = (x, Ay) \quad : \quad x, y \in \mathcal{D}(A)$$

(2) $$Ax = ix \quad \text{or} \quad Ax = -iximplyx = 0$$

(3) $$\overline{\Gamma_*(A)} = \Gamma(A)$$

PROOF

In [Ca] a Hermitsin mstrix for which (1)-(3) hold for the associated densely defined operstor A is referred to as *Case 1.* Here we use the contemporary vocabulsriy and say that A is self-djoint when (1.3) above hold. Notoce that (3) means thst A can be approximated by the operators $\{A_N\}$ in a rsther strong sense. This approximation leads to the existence of other limits which culminate in a spectral resolution of A. More precisely, the jaoro result from [Ca] goes as follows.

Assume thst A is self-adjoint. Then the folloeing exist. Firt, to every $t \geq 0$ one finds a bounded self.adjoint projection E_t^+ where

$$s > t \implies E_s E_t = E_t$$

Moroever, the rsnge of E_t is contsin in $\mathcal{D}(A)$ for every t and

$$E_t(Ax) = A(E_t x) \quad : \quad x \in \mathcal{D}(A)$$

Mrover $AE_t\pounds$ is an everywhere bounded linear operstor on ℓ^2 whose sepcttum satisfied the in- cluduin

$$\sigma(AE_t) \subset [0, t]$$

Next, for every finxed vector $x \in \ell^2$ the funvtion

$$t \mapsto (E_t x, x)$$

is non-decreading and the fsmily $\{E_t\}$ can be chosen so that these functions are left continuous, i.e.

$$\lim_{\delta \to 0} (E_{t-\delta} x, x) = (E_t x, x)$$

hold for every $t > 0$. Next, for every $x \in \ell^2$ and each $T >=$ there exists the Stieltjes' integral

$$\int_0^T t \cdot \frac{d}{dt}(E_t, x, x)$$

If $x \in \mathcal{D}(A)$ then the limit of these integrals is finite as $T \to \infty$, i.e.

$$\int_0^\infty t \cdot \frac{d}{dt}(E_t, x, x) < \infty \quad : \quad x \in \mathcal{D}(T)$$

Finslly there exists a self-adjoint operstor $A^?$ where $\mathcal{D}(A^*?$ contsins $\mathcal{D}(A)$ snd for eavh psir x, y in $\mathcal{D}(A)$ one has

$$(Ax, y) = \int_0^\infty t \cdot \frac{d}{dt}(E_t x, y)$$

The operator A_*. Denote by ℓ_*^2 the subspace of ℓ^2 whose vectors are fiunite $\mathbf{C}$-linesr combintiond of the e-vectors. The given matrix yields a linesr operstor A_* whose domain of definition is ℓ_*^2. If x and y is a pair of vectors in ℓ_*^2 we find an integer N so thst both belong to the subspace generated by $e - 1, \ldots, e_N$ snd then

$$(A_* x, y_*) = \sum_{p=1}^{q=N} a_{pq} x_q \overline{y_p}$$

Since the matrix is Hermitian one sees thst (i) is equal ton $(x, A_* y)$. So the densely defined operator A_* is symmetric. Next, consider its graph

$$\Gamma(A_*) = \{(x, A_* x) : \, x \in \ell_*^2\}$$

and take its closure in the product space $\ell^2 \times \ell^2$. This closure is graphic, i.e. there exists a linesr operstor $\overline{A_*}$ such that

$$\overline{\Gamma(A_*)} = \Gamma(\overline{A_*})$$

The easy verification is left to the reader. So here $\overline{A_*}$ is another densely defined operator and agsin we leave the esy verificstion thst it is symmetric, i.e.

$$\overline{A_*} x, y) = (x, \overline{A_*} y)$$

hold for every pair x, y in $\mathcal{D}(\overline{A_*})$. We shall also need a description $\mathcal{D}(A)$. Namely, let $y \in \ell^2$ and suppose there exists a constsnt C such thst

$$|(Ax, y)| \leq C||x|| \quad : \, x \in \ell_*^2$$

It means thst $x \mapsto (Ax, y)$ is a bounded linesr functionsl on ℓ_*^2 which by density extends to a boundd linesr funvionsl on ℓ^2. Now this Hilbert space ios self-dusl and hence there exists a uniwue vector ξ such thst

$$(x, \xi) = (Ax, y) \quad : \quad x \in \ell_*^2$$

Applied to the case when x run over the basis vctors one chscks thst $\xi = Ay$ and that (x) gives a *necessary and sufficient* confition in order that y belong to $\mathcal{D}(A)$.

Exercise. Show also thst A is closed, i.e. its graph $\Gamma(A)$ is closed in $\ell^2 \times \ell^2$.

Since we have the trivisl inclsuion $\gamma(A_*) \subset \Gamma(A)$ it followd from the exercise that A is abn extension of $\overline{A_*}$, i.e. one has the inclsuuon

$$\Gamma(A_*) \subset \Gamma(A)$$

At this stage we estsblöish a crucial result.

Theorem. *The equality (*) holds if and only if the null spaces of $iE + A$ and $iE - A$ are both zero.*

Proof. Suppose first that (*) holds. If for example $iE - A$ has a non-trivsl nullspace we find $0 \neq x$ and

$$ix = Ax \implies i \cdot (Ax, x) = (x, Ax) = (x, ix) = -i(x, x)$$

which gives a contradiction. In the same way one checks thst $iE + A$ has a trivial null-space. Conversely, assume that both $iE + A$ snd $iE - A$ have trivial null-spaces. Now we prove thst this gives the equality

$$\Gamma(A_*) = \Gamma(A)$$

Namely, if the inclsuion from (xx)nis strict we find a vector $y \in \mathcal{D}(A)$ such that (y, Ay) is $\perp$ to the closed graph of $\overline{A_*}$. In particular

$$(x, y) + (Ax, Ay) = 0 \quad : \forall\, x \in \ell^2_*$$

Now (i) implies that

$$|(Ax, Ay)| \leq ||y|| \cdot |x||$$

so by the criterion in xxx we condlude thst Ay belongs to $\mathcal{D}(A)$ and (xx) gives

$$(x, y) + (x, A^2y) = 0 \quad : \forall\, x \in \ell^2_*$$

Since ℓ^2_* is dense in ℓ^2 thid entails thst $y + A^2y =$ which means that

$$(E + iA) \cdot (A - iA)(y) = 0$$

But then thr nullspace of $E + iA$ or $E - iA$ must be non.-trivial whivh finishes the proof of Theorem xx.

Conclusion. We have alreay seen thst $\overline{A_*}$ is symmetric. when its graph is equal to $\Gamma(A)$ it follows that A also is symmetric, i.e.

$$(Ax, y) = (x, Ay) \quad : x, y \in \mathcal{D}(A)$$

Moreover, when (xx) holds it follows exactly as in (xx) that the null-spaces of $E + iA$ snd $E - iA$ are both trivial.

A surjectivity theorem. Consider as aobe a Hermitisn mstrix which gives the densely defined and symmetric operator $\overline{A_*}$. Here we do not assume that the null-spaces of $E + iA$ and $iE - A$ both are trivisl. However, it turns out the range of $iE + \overline{A_*}$ and of $iE - \overline{A_*}$ both are equal to ℓ^2. In fat, more generslly we consider an inhomogrnoueus equstion

$$\overline{A_*}x = \lambda \cdot \overline{A_*} + y$$

where y is a vector in ℓ^2 and λ is a complex number eith non-zxero imsginsry part,. We shall prove thst (xx) has at leat one solution x whose norm satisfies

$$||x|| \leq \frac{1}{\mathfrak{Im}\lambda} \cdot ||y||$$

DO IT !!!

The normal resolvent operators. Now we assume that A is self-adjoint. From (xx) it follows that the operstopr

$$x \mapsto ix - Ax$$

its inverse operstor is bounded eith whose operstor norm is ≤ 1. Put

$$R = (iE - A)^{-1}$$

As exposed in § xx it means thst R is a resolvent in the sense of Neumann attwched to the resolvent vslue i of the densely defined operstor A. In the same way we find thr bounded operstor

$$S = (iE + A)^{-1}$$

Next, thr bnounded operstor R has an adjoint R^* snd we shall prove the equslity

$$R^* = S$$

To attain this we consider a pair of vectors x, y in ℓ^2_*. Now

$$iR(x) + A \circ R(x) = x \implies$$

$$(x, y) = i(Rx, y) + (A \circ Rx, y) = i(x, R^*y) + (Rx, Ay) = -(x, i \cdot R^*y) + (Rx, R^Ay)$$

Since this hold for every $x \in \ell^*2_*$ it follows that

$$y = -iR(y) + R \cdot A(y)$$

and here $y \in \ell^2_*$ is arbitrary which proves that $R^* = S$.

Conclusion. Recal,l from Neumann's calculus in § xx thst reovents of an rbitasry densely defined oopertor commute. Applied to the two resovlents R snd S it follows in partioculsr that R commutes eith its adjoint, i.e. R is a normal operator. Stsrting from this the spectral resolution of the self-adjoint operator follows via Hilbert's spectral theorem for bounded normal operator in § xx.

The proof rewuires some preliminary obervations. Denote by ℓ^2_* the subspace of ℓ^2 whose vectors are fiunite **C**-linesr combintiond of the e-vectors. The given matrix now defines a linesr operstor A_* whose domain of definition is ℓ^2_*. If x and y is a pair of vectors in ℓ^2_* we find an integer N so thst both belong to the subspace genersted by $e-1,\ldots,e_N$ snd thrn

$$(A_*x,y_*)=\sum_{p=1}^{q=N} a_{pq}x_q\overline{y_p}$$

Since the matrix is Hermitian one sees thst (i) is equal ton (x,A_*y). So the densely defined operator A_* is symmetric. Next, we can consider its graph

$$\Gamma(A_*)=\{(x,A_*x):\,x\in\ell^2_*\}$$

and then take its closure in the product space $\ell^2\times\ell^2$. This closure is grsphic, i.e. there exists a linesr operstor $\overline{A_*}$ such that

$$\overline{\Gamma(A_*)}=\Gamma(\overline{A_*})$$

The easy verification is left to the reader. So here $\overline{A_*}$ is another densely defined operator and agsin we leave the esy verificstion thst it is symmetric, i.e.

$$\overline{A_*}x,y)=(x,\overline{A_*}y)$$

hold for every pair x,y in $\mathcal{D}(\overline{A_*})$.

Glimpses from work by Carleman

Let ρ be a probsbliity measure on the real t-line. We assume that

$$\int_{-\infty}^{\infty} t^{2k}\,d\rho(t) < \infty$$

hold for every positive integer k. Then ther exists thr monents

$$c_\nu = \int_{-\infty}^{\infty} t^\nu\,d\rho(t)$$

for each non-negative integer ν. The sequence $\{c_\nu\}$ is not arbitrary because ρ is a probability measure. Indeed, let $x_0, x_1, \dots x_M$ be a a finite sequence of real numbers. Then

$$\sum c_{p+q}x_px_q = \int_{-\infty}^{\infty} (\sum x_\nu t^\nu)^2\,d\rho(t)$$

and the last term is > 0. It can be expressed by the positive Hankel determinants (page 207). Conversely, start with a positive Hankel sequence $\{c_p\}$. Now there exists the formal expansion into continued fractions:

$$-[\frac{1}{\mu} + \frac{c_1}{\mu^2} + \frac{c_2}{\mu^3} + \dots] = xxx(page207)$$

where (*) entials that eqch β-number is > 0. set $b_\nu = \sqrt{\beta_\nu}$. We obtin the symmetric and infinite matrix

$$A = ***$$

We are gonig to prove

Theorem. *For every positive Hankel sequence $\mathcal{C} = \{c_p\}$ there exists at least one probability measure ρ such that $\mathcal{C}$ gives the sequnce of ρ-moments.*

Keeping $\mathcal{C}$ fixed we denote by $\Pi(\mathcal{C})$ the family of all probability measures ρ for whose moments are given by $\mathcal{C}$. With these nottions the following criterion for uniqueness holds.

Theorem. *The followinig are equivalent for a positive Hankel sequence: A: The famliyl $\Pi(\mathcal{C})$ is reduced to a single probability measure. B: The symmetric matrix A is of Class I.*

To prove this we will use the truncated $n \times n$-matrices

$$A_n = ****$$

Let μ be a complex number in the upper half-plane. The symmetry of A_n gives a unique ciomplex n-vector $\xi(\mu)$ such that

$$A_n(\xi(\mu)) = \mu \cdot \xi(\mu) + f_1$$

Set

$$phi_n(\mu) = \xi_1(\mu)$$

Caremer's rule gives

$$\phi_n(\mu) = \frac{1}{\det(\mu E_n - A_n)} \cdot P_n(\mu)$$

where $P_n(\mu)$ is a polynomial of dgeree $n-1$. Moreover, by simply inspecting the solution of (*) one finds that

$$\phi_n(\mu) = \mathfrak{c}_n$$

where the right hand side is the continued fraction up to order n. Moreover, there exists a probability measure ρ_n such that

$$\phi_n(\mu) = \int \frac{d\rho_n(t)}{t - \mu}$$

and we have also the standard inequality

$$|\phi_n(\mu)| \leq \frac{1}{\mathfrak{Im}\mu}$$

Now (xx) entials that $\{\phi_n(\mu)\}$ is a normal family of analytic functions in the upper half-plane. So by Montel's theorem we can find convergent subsequences. Consider one such limit function

$$\phi_*(\mu) = \lim\, \phi_{n_j}(\mu)$$

By weak star compactness of probability measures it follows that we also have

$$\phi_*(\mu) = \int \frac{d\rho_*(t)}{t-\mu}$$

where ρ_* is a probability measure. It turns out that one has the inclusion

$$\rho_* \in \Pi(\mathcal{C})$$

for every limit measure as above. To prove this we shall perform another construction.

The ψ-polynomials. Consider the infinite A-matrix A. When μ is given in the upper half-plane we find a unique infinite vector γ such that

$$A(\gamma) = \mu \cdot \gamma$$

Rgearding tghe exlitiy eqautions we see that

$$\gamma_n(\mu) = \psi_n(\mu) \cdot \gamma_1(\mu)$$

hold for each $n|geq2$ where $\psi_n(\mu)$ is a polynomial of degree $n-1$.

Glimpses from work by Carleman

Contents

Introduction. The material is foremost is devoted to spectral properties of second order elliptic PDE-operators. To illustrate the methods we are content to treat a special case while elliptic operators with variable coefficients are treated in separate notes devoted to mathematics by Carleman. However, § E in the appendix contains a construction of fundamental solutions to elliptic second order operators in $\mathbf{R}^3$ based upon Carleman's lectures at Institute Mittag Leffler in 1935 which might be of interest to some readers even though it will not be covered during my lecture. As background the appendix contains a section which explains Gustav Neumann's fundamental construction from 1879 of resolvents to densely defined linear operators, and at the end of § A we also recall Hilbert's spectral theorem for bounded normal operators on Hilbert spaces.

Let us now announce a major result which will be exposed in the lecture. In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact established by G. Neumann and H. Poincaré during the years 1879-1895 gives the following: First there exists the Greens' function

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$ with the property that the operator $\mathcal{G}$ defined on $L^2(\Omega)$ by

$$f \mapsto \mathcal{G}_f(p) = \frac{1}{2\pi} \iint G(p,q) f(q)\, dq$$

satisfies

$$\Delta \circ \mathcal{G}_f = -f \quad : f \in L^2(\Omega)$$

Moreover, $\mathcal{G}$ is a compact operator on the Hilbert space $L^2(\Omega)$ and there exists a sequence $\{f_n\}$ in $L^2(\Omega$ such that $\{\phi_n = \mathcal{G}_{f_n}\}$ is an orthonormal basis in $L^2(\Omega$ and

$$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : \ n = 1, 2, \ldots$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$. When eigenspaces have dimension ≥ 2, the eigenvalues are repeated by their multiplicity. The result below was presented by Carleman at the Scandinavian Congress in mathematics held in Stockholm 1934:

0. Theorem. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N \to \infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

Remark. The strategy in the proof is to consider the function of a complex variable s defined by

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

and show that it is a meromorophic function in the whole complex s-plane with a simple pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. For the reader's convenience we insert details of the proof in § D of the appendix which illustrates the "spirit" of the lecture.

Self-adjoint extensions of $\Delta + c(p)$. Here we consider $\mathbf{R}^3$ with points $p = (x, y, z)$ and Δ is the Laplace operator, while $c(p)$ is a real-valued and locally square integrable function. The linear operator

$$u \mapsto L(u) = \Delta(u) + c \cdot u$$

is defined on test-functions and hence densely defined on the Hilbert space $L^2(\mathbf{R}^3)$. In the monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923], Carleman established spectral resolutions for unbounded self-adjoint operators on a Hilbert space together with conditions that densely defined symmetric operators have self-adjoint extensions, The lecture will describe the major steps of a result due to Carleman which asserts that the operator L has a self-adjoint extension under the condition that

$$(*) \qquad \limsup_{p \to \infty} c(p) \leq M$$

hold for some constant M. A special case occurs when $c(p)$ is a Newtonian potential

$$(**) \qquad c(p) = \sum \frac{\alpha_\nu}{|p - q_\nu|} + \beta$$

where $\{q_\nu\}$ is a finite set of points in $\mathbf{R}^2$ while $\{\alpha_\nu\}$ and β are positive real numbers. So here one encounters the Bohr-Schrödinger equation which stems from quantum mechanics. With c as in (**) the requested self-adjoint extension is easily verified while the existence of a self-adjoint extension when (*) holds requires a rather involved proof.

An asymptotic expansion. Consider the Schrödinger equation

$$i \cdot \frac{\partial u}{\partial t} = \Delta(u) + c \cdot u$$

where we assume that $L^\Delta + c$ has a self-adjoint extension. One seeks solutions $u(x, t)$ defined when $t \geq 0$ and $x \in \mathbf{R}^3$ with an initial condition $u(x, 0) = f(x)$ for some $f \in L^2(\mathbf{R}^3)$. The solution is given via the spectral function associated with the L-operator. So the main issue is to get formulas for the spectral function of $\Delta + c$. In Carleman's cited lecture from 1934 an asymptotic expansion is given for this spectral function which merits further study since one nowadays can investigate approximative solutions numerically by computers.

§ A. Linear operators and spectral theory.

Let X be a Banach space and $T\colon X \to X$ a linear and densely defined operator whose domain of definition is denoted by $\mathcal{D}(T)$. In general T is unbounded:

$$\max_{x\in\mathcal{D}(T)} ||T(x)|| = +\infty$$

where the maximum is taken over unit vectors in $\mathcal{D}(T)$. The graph is defined by

$$\Gamma(T) = \{(x, Tx) : x \in \mathcal{D}(T)$$

If $\Gamma(T)$ is closed in $X \times X$ one says that T has a closed graph.

A.1 Invertible operators. A densely defined operator T has a bounded inverse if the range $T(\mathcal{D}(T))$ is equal to X and there exists a positive constant c such that

(i) $$||T(x)|| \geq c \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

Since $T(\mathcal{D}(T)) = X$, (i) gives for each $x \in X$ a unique vector $R(x) \in \mathcal{D}(T)$ such that

(ii) $$T \circ R(x) = x$$

Moreover, the inequality (i) gives

(iii) $$||R(x)|| \leq c^{-1} \cdot ||x|| \quad : x \in X$$

and when R is applied to the left on both sides in (ii), it follows that

(iv) $$R \circ T(x) = x \quad : x \in \mathcal{D}(T)$$

A.2 The spectrum $\sigma(T)$. Let E be the identity operator on X. Each complex number λ gives the densely defined operator $\lambda \cdot E - T$. If it fails to be invertible one says that λ is a spectral point of T and denote this set by $\sigma(T)$. If $\lambda \in \mathbf{C} \setminus \sigma(T)$ the inverse to $\lambda \cdot E - T$ is denoted by $R_T(\lambda)$ and called a Neumann resolvent to T. By the construction in (A.1) the range of every Neumann resolvent is equal to $\mathcal{D}(T)$ and one has the equation:

(A.2.1) $$T \circ R_T(\lambda)(x) = R_T(\lambda) \circ T(x) \quad : x \in \mathcal{D}(T)$$

A.3 Neumann's equation. *Assume that $\sigma(T)$ is not the whole complex plane. For each pair $\lambda \neq \mu$ outside $\sigma(T)$ the operators $R_T(\lambda)$ and $R_T(\mu)$ commute and*

(*) $$R_T(\mu)R_T(\lambda) = \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu}$$

Proof. Notice that

$$(\mu \cdot E - T) \cdot \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} =$$

(i) $$\frac{E}{\lambda - \mu} - (\mu - \lambda) \cdot \frac{R_T(\lambda)}{\lambda - \mu} - (\lambda \cdot E - T) \cdot \frac{R_T(\lambda)}{\lambda - \mu} = R_T(\lambda)$$

Multiplying to the left by $R_T(\mu)$ gives (*) which at the same time this shows that the resolvent operators commute.

A.4 The position of $\sigma(T)$**.** Assume that $\mathbf{C} \setminus \sigma(T)$ is non-empty. We can write (*) in the form

(1) $$R_T(\lambda)(E + (\lambda - \mu)R_T(\mu)) = R_T(\mu)$$

Keeping μ fixed we conclude that $R_T(\lambda)$ exists if and only if $E + (\lambda - \mu)R_T(\mu))$ is invertible which implies that

(A.4.1) $$\sigma(T) = \{\lambda : \frac{1}{\mu - \lambda} \in \sigma(R_T(\mu))\}$$

Hence one recovers $\sigma(T)$ via the spectrum of any given resolvent operator. Notice that (A.4.1) holds even when the open component of $\sigma(T)$ has several connected components.

A.4.2 Example. Suppose that $\mu = i$ and that $\sigma(R_T(i))$ is contained in a circle $\{|\lambda + i/2| = 1/2\}$. If $\lambda \in \sigma(T)$ the inclusion (A.4.1) gives some $0 \leq \theta \leq 2\pi$ such that

$$\frac{1}{i-\lambda} = -i/2 + 1/2 \cdot e^{i\theta} \implies 1 - i \cdot e^{i\theta} = \lambda(e^{i\theta} - i) \implies$$

$$\lambda = \frac{2 \cdot \cos\theta}{|e^{i\theta} - i|^2} \in \mathbf{R}$$

A.4.3 Neumann series. Let λ_0 be outside $\sigma(T)$ and construct the operator valued series

$$S(\zeta) = R_T(\lambda_0) + \sum_{n=1}^{\infty} (-1)^n \cdot \zeta^n \cdot R_T(\lambda_0)^{n+1} \tag{1}$$

It is clear that (1) converges in the Banach space of bounded linear operators when

$$|\zeta| < \frac{1}{||R_T(\lambda_0)||} \tag{2}$$

Moreover, the series expansion (1) gives

$$(\lambda_0 + \zeta - T) \cdot S(\zeta) = (\lambda_0 - T) \cdot S(\zeta) + \zeta S(\zeta) = E \tag{3}$$

Hence $S(\zeta) = R_T(\lambda_0 + \zeta)$ and the locally defined series in (1) entail the complement of $\sigma(T)$ is open where $\lambda \mapsto R_T(\lambda)$ is an analytic operator-valued function. Finally (*) in (A.3) and a passage to the limit as $\mu \to \lambda$ shows that this analytic function satisfies the differential equation

$$\frac{d}{d\lambda}(R_T(\lambda) = -R_T^2(\lambda) \tag{**}$$

B. Bounded normal operators on Hilbert spaces.

The result below stems from Hilbert's work on integral equations from 1904. Let $\mathcal{H}$ be a complex Hilbert space. The inner product of a pair of vectors x, y is denoted by (x, y) and we recall that

$$(y, x) = \overline{(x, y)}$$

If T is a bounded linear operator on $\mathcal{H}$ its adjoint T^* satisfies $(Tx, y) = (x, T^*y)$. A bounded linear operator R is normal if it commutes with its adjoint, i.e. if $RR^* = R^*R$.

B.1 The spectral measure of normal operators. Let R be bounded and normal. Now $\sigma(R)$ is a compact set in the complex plane and denote by $\mathfrak{M}$ the family of all Riesz measures supported by $\sigma(R)$.

Theorem. *There exists a map from $\mathcal{H} \times \mathcal{H}$ to $\mathfrak{M}$ which to each pair of vectors x, y assigns $\mu_{\{x,y\}}$ in $\mathfrak{M}$ such that the following hold for every pair of non-negative integers*

$$(R^m x, R^k y) = \int \lambda^m \cdot \bar{\lambda}^k \cdot d\mu_{\{x,y\}}(\lambda) \tag{*}$$

Remark. If ϕ is a bounded Borel function on $\sigma(R)$ it can be integrated in the sense of Stieltjes with respect to the Riesz measures $\{d\mu_{\{x,y\}}$ which yields a bounded linear operator $\widehat{\phi}$ such that

$$(\widehat{\phi}(x), y) = \int \phi(\lambda) \cdot d\mu_{\{x,y\}}(\lambda) \tag{**}$$

hold for every pair x, y in $\mathcal{H}$. Moreover, (**) in an isometry which means that the operator norm $\widehat{\phi}$ is the maximum norm of $|\phi|$ and the spectrum of the bounded operator $\widehat{\phi}$ is the closure of the range $\phi(\sigma(R))$. It means that $\phi \mapsto \widehat{\phi}$ is an isomorphism of the commutative Banach algebra of bounded Borel functions on $\sigma(R)$ and a commutative and closed subalgebra of $L(\mathcal{H})$. In particular every Borel set $e \subset \sigma(R)$ yields the operator $\widehat{\chi_e}$ which is an idempotent and has a spectrum contained in the compact closure of the Borel set e.

C. Symmetric operators.

A densely defined linear operator S on a Hilbert space is symmetric if

$$(Sx, y) = (x, Sy) : x, y \in \mathcal{D}(S)$$

Now there exists the subspace $\mathcal{D}^*$ of vectors y for which there exists a constant $C(y)$ such that

$$|(Sx, y)| \leq C(y) \cdot ||x|| : x \in \mathcal{D}(S)$$

Since Hilbert spaces are self-dual and $\mathcal{D}(S)$ is dense, each $y \in \mathcal{D}^*$ gives a unique vector S^*y such that

$$(Sx, y) = (x, S^*y) : x \in \mathcal{D}(S)$$

So S^* is a new linear operator where $\mathcal{D}(S^*) = \mathcal{D}^*$. The symmetry of S entails that

$$\Gamma(S) \subset \Gamma(S^*)$$

One shows easily that S^* has a closed graph and hence

(i) $$\overline{\Gamma(S)} \subset \Gamma(S^*)$$

Definition. *A densely defined and symmetric operator S is of type I if equality holds in (i).*

Exercise. Let S be symmetric and suppose in addition that it has a closed graph. Show that S is of type I if and only if the two eigenvalue equations

$$Sx = i \cdot x : Sx = -i \cdot x$$

have no non-zero solutions.

The Cayley transform. Let S be symmetric of Type 1. Using the exercise above one shows that there exists Neumann's resolvent

$$R = R_S(i) = (iE - S)^{-1}$$

The equality $S = S^*$ entails that R is a normal operator and we can apply Hilbert's spectral theorem to R. This gives a spectral measure of S-operator. More precisely, there exists map from $\mathcal{H} \times \mathcal{H}$ into the space of complex-valued Riesz measure on the real line. Here the total variations satisfy

$$||\mu_{x,y}|| = \int_{-\infty}^{\infty} |d\mu_{\{x,y\}}(s)| \leq ||x|| \cdot ||y||$$

Moreover, $\mu_{\{x,x\}}$ are non-negative measures for each $x \in \mathcal{H}$ and a vector x belongs to $\mathcal{D}(S)$ if and only if

$$\int_{-\infty}^{\infty} s^2 \cdot d\mu_{\{x,x\}}(s) < \infty$$

Finally one has the equations

$$(Sx, Sy) = \int_{-\infty}^{\infty} s^2 \cdot d\mu_{\{x,y\}}(s) \quad : \; x, y \in \mathcal{H}$$

Remark. In many applications S is a PDE-operator defined on a suitable family of square integrable functions in a domain Ω of $\mathbf{R}^n$ for some positive integer n. To determine if S is symmetric is an easy task since it suffices to consider its restriction to the dense subspace of test-functions in Ω. But to verify that a symmetric PDE-operator is of Type I can be quite involved, and when S if of Type I there remains to investigate the normal operator $R_S(i)$ above and proceed to determine the spectral measure of S expressed by the μ-map above. For a quite extensive family of elliptic operators the spectrum of S discrete and one is led to analyze its asymptotic behaviour. In such studies the symmetry condition can often be relaxed, i.e. it suffices that the leading part of the PDe-operator is symmetric,

Example. Let $n = 3$ and consider a PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p} + a_0(x)$$

The a-functions are real-valued and defined in a neighborhood of the closure of a bounded domain Ω with a C^1-boundary. Here one has the symmetry $a_{pq} = a_{qp}$, and $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x \in \Omega$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under these conditions, a result which goes back to work by Neumann and Poincaré, gives a positive constant κ_0 such that if $\kappa \geq \kappa_0$ then the inhomogeneous equation

$$L(u) - \kappa^2 \cdot u = f \quad : f \in L^2(\Omega)$$

has a unique solution u which is a C^2.-function in Ω and extends to the closure where it is zero on $\partial\Omega$. Moreover, there exists some κ_0 and for each $\kappa \geq \kappa_0$ a Green's function $G(x, y; \kappa)$ such that

$$\text{(i)} \qquad (L - \kappa^2)(\frac{1}{4\pi} \cdot \int_\Omega G(x, y; \kappa)\, f(y)\, dy) = -f(x) \quad : f \in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

$$\text{(ii)} \qquad f \mapsto -\frac{1}{4\pi} \cdot \int_\Omega G(x, y; \kappa)\, f(y)\, dy$$

is Neumann's resolvent to the densely defined operator $L - \kappa^2$ on the Hilbert space $L^2(\Omega)$. After a detailed study of these G-functions, Carleman established an asymptotic formula for the discrete sequence of eigenvalues $\{\lambda_n\}$. In general they are complex but arranged so that the absolute values increase. To begin with one proves rather easily that they are "almost real" in the sense that there exist positive constants C and c such that

$$|\mathfrak{Im}(\lambda_n)| \leq C \cdot (\mathfrak{Re}(\lambda_n) + c)$$

hold for every n. Next, the elliptic hypothesis means that the determinant function

$$D(x) = \det(a_{p,q}(x))$$

is positive in Ω. With these notations one has

Theorem. *The following limit formula holds:*

$$(*) \qquad \lim_{n\to\infty} \frac{\mathfrak{Re}(\lambda_n)}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_\Omega \frac{1}{\sqrt{D(x)}}\, dx$$

Remark. The formula above is due to Courant and Weyl when P is symmetric and was extended to non-symmetric operators during Carleman 's lectures at Institute Mittag-Leffler in 1935. Weyl and Courant used calculus of variation in the symmetric case while Carleman employed different methods which have the merit that the passage to the non-symmetric case does not cause any trouble. As pointed out by Carleman the methods in the proof give similar asymptotic formulas in other boundary value problems such as those considered by Neumann where one imposes boundary value conditions on outer normals, and so on. A crucial step during the proof of the theorem above is to construct a fundamental solution $\Phi(x, \xi; \kappa)$ to the PDE-operators $L - \kappa^2$ which is exposed in § E.

§ D. Proof of Theorem 0.

Let Ω be a bounded and Dirichlet regular domain. Let $p \in \Omega$ be kept fixed and consider the continuous function on $\partial\Omega$ defined by

$$q \mapsto \log \frac{1}{|p-q|}$$

We find the harmonic function $u_p(q)$ in Ω such that $u_p(q) = \log \frac{1}{|p-q|} \, : \, q \in \partial\Omega$. Green's function is defined for pairs $p \neq q$ in $\Omega \times \Omega$ by

$$G(p,q) = \log \frac{1}{|p-q|} - u_p(q) \tag{1}$$

Keeping if $p \in \Omega$ fixed, the function $q \mapsto G(p,q)$ extends to the closure of Ω where it vanishes if $q \in \partial\Omega$. If $f \in L^2(\Omega$ we set

$$\mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot f(q)\, dq \tag{2}$$

where $q = (x,y)$ so that $dq = dxdy$ when the double integral is evaluated. From (1) we see that

$$\iint_{\Omega \times \Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence $\mathcal{G}$ is of the Hilbert-Schmidt type and therefore a compact operator on $L^2(\Omega)$. Next, recall that $\frac{1}{2\pi} \cdot \log \sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator. From this one can deduce the following:

D.1 Theorem. *For each $f \in L^2(\Omega)$ the Lapacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

$$\Delta(\mathcal{G}_f) = -f \tag{*}$$

The equation (*) means that the composed operator $\Delta \circ \mathcal{G}$ is minus the identity on $L^2(\Omega)$. We are led to introduce the linear operator S on $L^2(\Omega)$ defined by Δ, where $\mathcal{D}(S)$ is the range of $\mathcal{G}$. If $g \in C_0^2(\Omega)$, i.e. twice differentiable and with compact support, it follows via Greens' formula that

$$\frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot \Delta(g)(q)\, dq = -g(p)$$

In particular $C_0^2(\Omega) \subset \mathcal{D}(S)$ which implies that S is densely defined and we leave it to the reader to verify that

$$\mathcal{G}(\Delta(f)) = -f \quad : f \in \mathcal{D}(S)$$

Remark. By the construction of resolvent operators in § 1 this means that $-\mathcal{G}$ is Neumann's inverse of S.

Exercise. Show that S has a closed range and in addition it is self-adjoint, i.e. $S = S^*$.

The spectrum of S. A wellknown result asserts that there exists an orthonormal basis $\{\phi_n\}$ in $L^2(\Omega)$ where each $\phi_n \in \mathcal{D}(S)$ is an eigenfunction. More preciesely there is a non-decreasing sequence of positive real numbers $\{\lambda_n\}$ and

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots \tag{i}$$

Let us remark that (i) means that

$$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n \tag{ii}$$

So above $\{\lambda_n^{-1}\}$ are eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. Eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times.

Now we begin the proof of Theorem 0, i.e. we will show that the following hold for each point $p \in \Omega$:

$$(*) \qquad \lim_{n\to\infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi}$$

To prove (*) we consider the Dirichlet series for each fixed $p \in \Omega$:

$$(**) \qquad \Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

It is easily seen that $\Phi_p(s)$ is analytic in a half-space $\mathfrak{Re}s > b$ for a large b. Less trivial is the following:

D.2 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem D.2 gives (*) via a result due to Wiener in the article *Tauberian theorem* [Annals of Math.1932]. Wiener's theorem asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s\to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n\to\infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Proof of Theorem D.2

Since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\text{(i)} \qquad \sum \lambda_n^{-2} < \infty$$

This convergence entails that various constructions below are defined. For each λ outside $\{\lambda_n\}$ we set

$$\text{(ii)} \qquad G(p,q;\lambda) = G(p,q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda - \lambda_n)}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\text{(iii)} \qquad \mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p,q;\lambda) \cdot f(q)\, dq$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p, \lambda)$. Set

$$F(p,q,\lambda) = G(p,q;\lambda) - G(p,q)$$

Keeping p fixed we see that (ii) gives

$$\text{(B.1)} \qquad \lim_{q\to p} F(p,q,\lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)}$$

Set

$$F(p,\lambda) = \lim_{q\to p} F(p,q,\lambda)$$

From (i) and (B.1) it follows that it is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Show via residue calculus that one has the equality below in a half-space $\mathfrak{Re}\, s > 2$:

$$\Phi(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a-i\infty}^{a+i\infty} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\,\lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\Phi(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta}\, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} F(p,-x) \cdot x^{-s}\, dx \tag{E.1}$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} F(p,-x) \cdot x^{-s}\, dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p,-x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^{\infty} \frac{e^{-zt}}{\sqrt{t^2 - 1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$I_0(z) = \sum_{m=0}^{\infty} \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \tag{i}$$

$$I_1(z) = \sum_{m=0}^{\infty} \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma \tag{ii}$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p,q;\kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p,q;\kappa) = K(\sqrt{\kappa}|p-q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p,q;-\kappa) = K(\sqrt{\kappa}\cdot|p-q|) - H(q;\kappa) \quad : \kappa > 0$$

Next, the construction of $G(p,q)$ gives

$$F(p,-\kappa) = \lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|) + \log\,|p-q|] + \lim_{q\to p}\,[u_p(q) + H(p,q,\kappa)] \tag{G.1}$$

The last term above has the "nice limit" $u_p(p) + H(p,p,\kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|) + \log\,|p-q|] = -\frac{1}{2}\cdot\log\kappa + \log 2 - \gamma \tag{G.2}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p,-\kappa) = -\frac{1}{2}\cdot\log\kappa + A + H(p,p;-\kappa)$$

With $x = \kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty x^{-s}\,dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2}\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty \log x\cdot x^{-s}\,dx$$

Notice that the complex derivative

$$\frac{d}{ds}\int_a^\infty x^{-s}\,dx = -\int_a^\infty \log x\cdot x^{-s}\,dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem D.2 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin\,\pi s}{2\pi^2}\cdot\int_a^\infty H(p,p,\kappa)\cdot\kappa^{-s}\,d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

$$0 \leq H(p,q;\kappa) \leq \max_{q\in\partial\Omega}\,K(\kappa|p-q|) \tag{i}$$

With $p\in\Omega$ fixed there is a positive number δ such that $|p-q|\geq\delta : q\in\partial\Omega$ which gives positive constants B and α such that

$$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0 \tag{ii}$$

The reader may now check that this exponential decay gives Lemma H.2.

§ E. Fundamental solutions to second order Elliptic operators.

In $\mathbf{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ we consider a second order PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x)\cdot\frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p} + a_0(x)$$

where a-functions are real-valued and one has the symmetry $a_{pq} = a_{qp}$. To ensure existence of a globally defined fundamental solutions we suppose the the following limit formulas hold as $(x, y, z) \to \infty$:

$$\lim a_\nu(x,y,z) = 0\colon 0 \leq p \leq 3 \quad : \ \lim a_{pq}(x,y,z) = \text{Kronecker's delta function}$$

Thus, L approaches the Laplace operator as (x, y, z) tends to infinity. Moreover L is elliptic which means that the eigenvalues of the symmetric matrix with elements $\{a_{pq}(x)\}$ are positive for every x. Recall the notion of fundamental solutions. First we consider the adjoint operator:

$$\text{(0.1)} \qquad L^*(x,\partial_x) = P - 2\cdot(\sum_{p=1}^{p=3}(\sum_{q=1}^{q=3}\frac{\partial a_{pq}}{\partial x_q})\cdot\frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3}\frac{\partial a_p}{\partial x_p} + 2\cdot\sum\sum\frac{\partial^2 a_{pq}}{\partial x_p \partial x_q}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ, ψ in $\mathbf{R}^3$ with compact support:

$$\text{(0.2)} \qquad \int L(\phi)\cdot\psi\,dx = \int \phi\cdot L^*(\psi)\,dx$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $L(x,\partial_x)$ if

$$\text{(0.3)} \qquad \psi(0) = \int \Phi\cdot L^*(\psi)\,dx$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $L - \kappa^2$ and a function $x \mapsto \Phi(x;\kappa)$ is a fundamental solution to $L - \kappa^2$ if

$$\text{(0.4)} \qquad \psi(0) = \int \Phi(x:\kappa)\cdot(L^* - \kappa^2)(\psi(x))\,dx$$

hold for compactly supported C^2-functions ψ. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x,\xi;\kappa)$ with the property that

$$(*) \qquad \psi(\xi) = \int \Phi(x,\xi;\kappa)\cdot(L^*(x,\partial_x) - \kappa^2)(\psi(x))\,dx$$

hold for all $\xi \in \mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x,\xi;\kappa)$ is a function of six variables defined in $\mathbf{R}^3 \times \mathbf{R}^3$. Theorem 1.9 below gives sharp estimate for fundamental solutions. The subsequent constructions are based upon a classic formula due to Newton and specific solutions to integral equations found by a convergent Neumann series.

1. The construction of $\Phi(x,\xi;\kappa)$.

When L has constant coefficients the construction of fundamental solutions was given by Newton in his famous text-books from 1666. We have the positive and symmetric 3×3-matrix $A = \{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix and put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2}\sum_{p,q} b_{pq}a_p a_q - a_0}$$

where κ is chosen so large that the term under the square-root is > 0. Define the quadratic form

$$B(x) = \sum_{p,q} b_{pq}a_p x_q$$

With these notations Newton's fundamental solution taken at $x = 0$ becomes

$$H(x;\kappa) = \frac{1}{4\pi \cdot \sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq} a_p \cdot x_q} \tag{1.1}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ indeed yields a fundamental solution to the PDE-operator $L(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients. For each $\xi \in \mathbf{R}^3$ the elements of the inverse matrix to $\{a_{pq}(\xi)$ are denoted by $\{b_{pq}(\xi)\}$. Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi)} \tag{i}$$

Following Newton's construction in (1.1) we put:

$$H(x,\xi;\kappa) = \frac{1}{4\pi} \cdot \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) \cdot x_q} \tag{ii}$$

When ξ is kept fixed this function of x is real analytic outside the origin and we also notice that $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. We are going to find a fundamental solution which takes the form

$$\Phi(x,\xi;\kappa) = H(x-\xi,\xi;\kappa) + \int_{\mathbf{R}^3} H(x-y,\xi;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy \tag{iii}$$

where the Ψ-function is the solution to an integral equation which we construct in (1.5).

1.3 The function $F(x,\xi;\kappa)$. For every fixed ξ we consider the differential operator in the x-space:

$$L_*(x,\partial_x,\xi;\kappa) =$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi)) \frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

With ξ fixed we apply L_* to the function $x \mapsto H(x-\xi,\xi;\kappa)$ and put

$$F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)) \tag{1.3.1}$$

1.4 Two estimates. The limit conditions in (0.0) give positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \quad : \; [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|^2} \tag{1.4.1}$$

The verification of (1.4.1) is left as an exercise.

1.5 An integral equation. We seek $\Psi(x,\xi;\kappa)$ which satisfies the equation:

$$\Psi(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy + F(x,\xi;\kappa) \tag{1.5.1}$$

To solve (1.5.1) we construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$F^{(k)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot F^{(k-1)}(y,\xi;\kappa)\, dy \quad : \quad k \geq 2 \tag{1.5.2}$$

Then (1.4.1) gives the inequality

(i) $$|F^{(2)}(x,\xi;\kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|}}{|x-y|^2\cdot|\xi-y|^2}\cdot dy$$

To estimate (i) we notice that the triple integral after the substitution $y-\xi\to u$ becomes

(ii) $$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2\cdot|u|^2}\cdot du$$

In (ii) the volume integral can be integrated in polar coordinates and becomes

(iii) $$C_1^2\cdot\int_0^\infty\int_{S^2}\frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2}\cdot dwdr$$

where S^2 is the unit sphere and dw the area measure on S^2 and we see that (iii) becomes

$$2\pi C_1^2\cdot\int_0^\infty\int_0^\pi\frac{e^{-k\kappa r}}{(x-\xi)^2+r^2-2r\cdot|x-\xi|\cdot\sin\theta}\cdot d\theta dr =$$

(iv) $$\frac{2\pi C_1^2}{|x-\xi|}\cdot\int_0^\infty e^{-k\kappa|x-\xi|t}\cdot\log|\frac{1+t}{1-t}|\cdot\frac{dt}{t}$$

where the last equality follows by a straightforward computation.

1.6 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)| \leq \frac{2\pi\cdot C_1^2\cdot C_1^*}{\kappa\cdot|x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

(*) $$|F^{(n)}(x,\xi;\kappa)| \leq \frac{C_1}{|x-\xi|^2}\cdot\left[\frac{2\pi C_1^2\cdot C_1^*}{\kappa}\right]^{n-1} \quad \text{hold for every} \quad n\geq 2$$

1.6 Conclusion. With κ_0^* so large that $2\pi C_1^2\cdot C_1^* < \kappa_0^*$ it follows from (*) that the Neumann series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $\kappa\geq\kappa_0^*$ and gives the requested solution $\Psi(x,\xi;\kappa)$ in (1.5.1).

1.7 Exercise. Above we have found Ψ which satisfies the integral equation in § 1.5.1 Use Green's formula to show that the function $\Phi(x,\xi;\kappa)$ defined in (1.2.1) gives a fundamental solution of $L(x,\partial_x)-\kappa^2$.

1.8 A final estimate. The constructions above show that the functions

$$x\mapsto\Phi(x,\xi;\kappa) \quad \text{and} \quad x\mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x=\xi$. Consider the difference

(1.8.1) $$G(x,\xi;\kappa)=\Phi(x,\xi;\kappa)-H(x-\xi,\xi;\kappa)$$

1.8.2 Exercise. Use the previous constructions to show that for every $0<\gamma\leq 2$ there is a constant C_γ such that

$$\big|G(x,\xi;\kappa)\big| \leq \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for every pair (x,ξ) and every $\kappa\geq\kappa_0$. Together with the the inequality for the H-function in (1.4.1) this gives an estimate for the fundamental solution Φ. More precisely we have proved:

1.9 Theorem. *With κ_0^* as above there exist positive constants C and k and for each $0 < \gamma \leq 2$ a constant C_γ such that*

$$|\Phi(x,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} + \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for all pairs (x,ξ) in $\mathbf{R}^3$ and every $\kappa \geq \kappa_0^*$.

Remark. Above C and k are independent of κ as soon as κ_0^* has been chosen as above. The size of these constant depend on the C^2-norms of the functions $\{a_{pq}(x)\}$ and as well as the C^1-norms of $\{a_1, a_2, a-3\}$ and the maximum norm of a_0. Notice that the whole construction is "canonical", i.e. the resulting fundaemtnal ssolutions $\{\Phi(x,\xi;\kappa)\}$ are uniquely determined. We remark that simiular constructions can be made for ellptic operators of even degree $2m$ with $m \geq 2$. Here Newton's solution for constant coefficients is replaced by those of Fritz John which arise via the wave deompostion of the Dirac measure. It would be interesting tanalyze the resulting version of Theorem 1.9. Of course, one can also extend everything to elliptic operators of n variables were $n \geq 4$ in which case the denominator $|x-\xi|^{-1}$ is replaced by $|x-\xi|^{-n+2}$.

The proof of Theorem 1 relies upon the construction of fundamental solutions which is given in § 1 below. After this has been achieved, the asymptotic formula (*) in Theorem 2 is derived via Tauberian theorems for Dirichlet series which goes as follows: Let $\{a_\nu\}$ and $\lambda_\nu\}$ be two sequences of positive numbers where $\lambda_\nu \to +\infty$ and the series

$$f(x) = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\lambda_\nu + x}$$

converges when $x > x_*$ for some positive number x_*. Next, for every $x > 0$ we define the function

$$\mathcal{A}(x) = \sum_{\{\lambda_\nu < x\}} a_\nu$$

In other words, with $x > 0$ we find the largest integer $\nu(x)$ such that $\lambda - \nu(x) < x$ and then $\mathcal{A}(x)$ is the sum over the a-numbers up to this index. With these notations the following implication holds for every pair $A > 0$ and $0 < \alpha < 1$

3. Theorem. *Suppose there exists a constant $A > 0$ and some $0 < \alpha < 1$ such that*

$$\lim_{x\to\infty} x^\alpha \cdot f(x) = A \implies \lim_{x\to\infty} \mathcal{A}(x) = \frac{A}{\pi} \cdot \frac{\sin \pi\alpha}{1-\alpha} \cdot x^{1-\alpha}$$

0 Preliminary constructions.

We are given an ellipitic operator L as above and assume that the coefficients are defined in the whole space $\mathbf{R}^3$. To ensure convergence of volume integrals taken over the whole of $\mathbf{R}^3$ we add the conditions that

$$\lim_{|x|\to\infty} a_{pp}(x) = 1 \quad : \quad 1 \leq p \leq 3$$

while $\{a_{pq}\}$ for $p \neq q$ and a_1, a_2, a_3, b tend to zero as $|x| \to +\infty$. This means that P approaches the Laplace operator when $|x|$ is large. Let us recall the notion of a fundamental solution which prior to the general notion of distributions introduced by L. Schwartz, was referred to as a *Grundlösung.* First, the regularity of the coefficients of a PDE-operator P enable us to construct the adjoint operator:

$$P^*(x, \partial_x) = P - 2 \cdot (\sum_{p=1}^{p=3} (\sum_{q=1}^{q=3} \frac{\partial a_{pq}}{\partial x_q}) \cdot \frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3} \frac{\partial a_p}{\partial x_p} + 2 \cdot \sum\sum \frac{\partial^2 a_{pq}}{\partial x_p \partial x_q}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ, ψ in $\mathbf{R}^3$ with compact support:

$$\int P(\phi) \cdot \psi \, dx = \int \phi \cdot P^*(\psi) \, dx$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $P(x, \partial_x)$ if

$$\psi(0) = \int \Phi \cdot P^*(\psi) \, dx$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $P - \kappa^2$ and a function $\Phi(x; \kappa)$ is a fundamental solution to $P - \kappa^2$ if

$$\psi(0) = \int \Phi(x : \kappa) \cdot (P^* - \kappa^2)(\psi(x)) \, dx \tag{1}$$

hold for compactly supported C^2-functions ψ. Above κ appears as an index of Φ, i.e. for each fixed κ we have the locally integrable function $x \mapsto \Phi(x : \kappa)$. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x, \xi; \kappa)$ with the property that

$$\psi(\xi) = \int \Phi(x, \xi : \kappa) \cdot (P^* - \kappa^2)(\psi(x)) \, dx \tag{*}$$

hold for all $\xi \in \mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x, \xi; \kappa)$ is a function of six variables defined in $\mathbf{R}^3 \times \mathbf{R}^3$. Fundamental solutions are in general not unique. However, when P is an elliptic operator as above we shall give an explicit construction of fundamental solutions $\Phi(x, \xi; \kappa)$ for all sufficiently large κ in § 1.

1. The construction of $\Phi(x, \xi; \kappa)$.

1.1 The case when P has constant coefficients. Here the fundamental solution is given by a formula which goes back to Newton's work in his classic text-books from 1666. We have the positive and symmetric 3×3-matrix $A = \{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix and recall that they are found via Cramér's rule:

$$b_{pq} = \frac{A_{pq}}{\Delta}$$

where $\Delta = \det(A)$ and $\{A_{pq}\}$ are the cofactor minors of the A-matrix. Put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq} a_p a_q - b}$$

where κ is chosen so large that the term under the square-root is > 0. Next, define the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p x_q$$

With these notations Newton's fundamental solution taken at $x = 0$ becomes

$$(*) \qquad H(x; \kappa) = \frac{1}{\sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2} \sum_{p,q} b_{pq} a_p \cdot x_q}$$

Exercise. Verify by Stokes formula that $H(x; \kappa)$ indeed yields a fundamental solution to the PDE-operator $P(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients.

Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\text{(i)} \qquad \alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi)}$$

Following Newton's construction for the case of constant coefficients we put:

$$\text{(ii)} \qquad H(x, \xi; \kappa) = \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) \cdot x_q}$$

When ξ is kept fixed this function of x is real analytic outside the origin and the singularity at $x = 0$ is of Newton's type. In particular $x \to H(x, \xi; \kappa)$ is locally integrable as a function of x in a neighborhood of the origin. Next, for every fixed ξ we consider the differential operator in the x-space:

$$L_*(x, \partial_x, \xi; \kappa) =$$

$$\sum_{p=1}^{p=3} \sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi)) \frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

For each fixed ξ there exists the function $x \mapsto H(x - \xi, \xi; \kappa)$ and we apply the L_*-operator on this x-dependent function and put:

$$\text{(iii)} \qquad F(x, \xi; \kappa) = \frac{1}{4\pi} \cdot L_*(x, \partial_x, \xi; \kappa)(H(x - \xi, \xi, \kappa)$$

1.3 Two estimates. The hypothesis that $\{a_{pq}(x)\}$ are of class C^2 and$\{a_p(x)\}$ of class C^1, together with the limit conditions (*) in § XX give the existence of positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \tag{1.3.1}$$

$$[F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|^2}}{|x-\xi|^2} \tag{1.3.2}$$

The verification is left as an exercise.

1.4 An integral equation. We seek $\Phi(x,\xi;\kappa)$ which solves the equation:

$$\Phi(x,\xi;\kappa) = \iiint F(x,y;\kappa)\cdot\Phi(y,\xi;\kappa)\,dy + F(x,\xi;\kappa) \tag{1}$$

where the integral is taken over $\mathbf{R}^3$. To solve (1) we construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$F^{(k)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa)\cdot F^{(k-1)}(y,\xi;\kappa)\,dy \quad : \quad k \geq 2 \tag{1.4.1}$$

Then (1.3.2) gives the inequality

$$|F^{(2)}(x,\xi;\kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|^2}}{|x-y|^2\cdot|\xi-y|^2}\cdot dy \tag{i}$$

To estimate (i) we first notice that the triple integral after the substitution $y-\xi \to u$ becomes

$$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2\cdot|u|^2}\cdot du \tag{ii}$$

In (ii) the volume integral is integrated in polar coordinates and becomes

$$C_1^2 \cdot \int_0^\infty \int_{S^2} \frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2}\cdot dwdr \tag{iii}$$

where S^2 is the unit sphere and dw the area measure on S^2 and we see that (iii) becomes

$$2\pi C_1^2 \cdot \int_0^\infty \int_0^\pi \frac{e^{-k\kappa r^2}}{(x-\xi)^2 + r^2 - 2r\cdot|x-\xi|\cdot\sin\theta}\cdot d\theta dr =$$

$$\frac{2\pi C_1^2}{|x-\xi|}\cdot\int_0^\infty e^{-k\kappa|x-\xi|t}\cdot\log|\frac{1+t}{1-t}|\cdot\frac{dt}{t} \tag{iv}$$

where the last equality follows by a straightforward computation.

1.5 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)| \leq \frac{2\pi C_1^2\cdot C_1^*}{\kappa\cdot|x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

$$|F^{(n)}(x,\xi;\kappa)| \leq \frac{C_1}{|x-\xi|^2}\cdot[\frac{2\pi C_1^2\cdot C_1^*}{\kappa}]^{n-1} \quad \text{hold for every} \quad n \geq 2 \tag{*}$$

1.6 Conclusion. With κ so large that $2\pi C_1^2\cdot C_1^* < \kappa$ it follows form (*) that the series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $x \neq \xi$ and this Neumann series gives the requested solution $\Phi(x,\xi;\kappa)$. Moreover, $\Phi(x,\xi;\kappa)$ satisfies a similar estimate as in (1.3.2) above with another constant than C_2 instead of C_1.

1.7 Exercise. Above we have found Φ which satisfies the integral equation in § 1.4 Use Green's formula to show that $\Phi(x,\xi;\kappa)$ gives a fundamental solution of $P(x,\partial_x)-\kappa^2$ with a pole at ξ.

1.8 Some final estimates. The constructions above show that the functions

$$x \mapsto \Phi(x,\xi;\kappa) \quad \text{and} \quad x \mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x=\xi$. Consider the difference

$$\Psi(x,\xi;\kappa) = \Phi(x,\xi;\kappa) - H(x-\xi,\xi;\kappa) \tag{1.8.1}$$

1.8.2 Exercise. Use the previous constructions to show that for every $0<\gamma\leq 2$ there is a constant C_γ such that

$$\big|\Psi(x,\xi;\kappa)\big| \leq \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for every pair (x,ξ) and every $\kappa \geq \kappa_0$. Together with (1.3.1) this gives an estimate for the fundamental solution Φ.

2. Green's functions.

Let Ω be a bounded domain in $\mathbf{R}^3$. A Green's function $G(x,y;\kappa)$ attached to this domain and the PDE-operator $P(x,\partial_x;\kappa)$ is a function which for fixed κ is a function in $\Omega\times\Omega$ with the following properties:

$$G(x,y;\kappa)=0 \quad \text{when} \quad x\in\partial\Omega \quad \text{and} \quad y\in\Omega \tag{*}$$

$$\psi(y)=\int_\Omega (P^*(x,\partial_x)-\kappa^2)(\psi(x))\cdot G(x,y;\kappa)\,dx \quad : \quad y\in\Omega \tag{**}$$

hold for all C^2-functions ψ with compact support in Ω. To find G we solve Dirchlet problems. With $\xi\in\Omega$ kept fixed one has the continuous function on $\partial\Omega$:

$$x\mapsto \Phi^*(x,\xi;\kappa)$$

Solving Dirchlet's problem gives a unique C^2-function $w(x)$ which satisfies:

$$P(x,\partial_x)(w)+\kappa^2\cdot w=0 \quad \text{holds in} \quad \Omega \quad \text{and} \quad w(x)=\Phi(x,\xi;\kappa) \quad : x\in\partial\Omega=0$$

From the above it is clear that this gives the requested G-function, i.e. one has:

2.1 Proposition. *The the function*

$$G(x,\xi;\kappa)=\Phi(x,\xi;\kappa)-w(x) \quad \text{satisfies} \quad (*-**)$$

Using the estimates for the ϕ-function from § 1 we get estimates for the G-function above. Start with a sufficiently large κ_0 so that $\Phi^*(x,\xi;\kappa_0)$ is a positive function of (x,ξ). Then the following hold:

2.2 Theorem. *One has*

$$G(x,\xi;\kappa_0)=\frac{1}{\sqrt{\Delta(x)}\cdot\sqrt{\Phi(x,\xi;\kappa_0)}}+R(x,\xi)$$

where the remainder function satisfies the following for all pairs (x,ξ) in Ω:

$$|R(x,\xi)|\leq C\cdot|x-\xi|^{-\frac{1}{4}}$$

and the constant C only depends on the domain Ω and the PDE-operator P.

Remark. Above the negative power of $|x-\xi|$ is a fourth-root which means that the remainder term R is more regular compared to the first term which behaves like $|x-\xi|^{-1}$ on the diagonal $x=\xi$.

2.3 Exercise. Prove Theorem 2.3 If necessary, consult [Carleman: page xx-xx9 for details.

2.4 The integral operator $\mathcal{J}$

.

With κ_0 chosen as above we consider the integral operator which sends a function u in Ω to

$$\mathcal{J}_u(x) = \int_\Omega G(x,\xi;\kappa_0)\cdot u(\xi)\,d\xi$$

The construction of the Green's function gives:

$$(P-\kappa_0^2)(\mathcal{J}_u)(x) = u(x) \quad : \quad x\in\Omega \tag{2.4.1}$$

In other words, if E denotes the identity we have the operator equality

$$P(x,\partial_x)\circ\mathcal{J}_u = \kappa_0^2\cdot\mathcal{J} + E \tag{2.4.2}$$

Consider pairs (u,γ) such that

$$u(x) + \gamma\cdot\mathcal{J}_u(x) = 0 \quad : \quad x\in\Omega \tag{2.4.3}$$

The vanishing from (*) for the G-function implies that $J_u(x) = 0$ on $\partial\Omega$. Hence every u-function which satisfies in (2.4.3) for some constant γ vanishes on $\partial\Omega$. Next, apply P to (2.4.3) and then the operator formula (2.4.2) gives

$$0 = P(u) + \gamma\kappa_0^2\cdot\mathcal{J}_u + \gamma\cdot u \implies P(u) + (\gamma-\kappa_0^2)u = 0$$

2.4.4 Conclusion. Hence the boundary value problem (*) from 0.B is equivalent to find eigenfunctions of $\mathcal{J}$ via (2.4.3) above.

3. Almost reality of eigenvalues.

Consider the set of eigenvalues λ to (*) in (0.B). Then we have:

3.1 Proposition. *There exist positive constants C_* and c_* such that every eigenvalue λ to (*) in (0.B) satisfies*

$$|\mathfrak{Im}\,\lambda|^2 \leq C_*(\mathfrak{Re}\,\lambda) + c_*)$$

Proof. Let u be an eigenfunction where $P(u) + \lambda\cdot u = 0$. Stokes theorem and the vanishing of $u|\partial\Omega$ give:

$$0 = \int_\Omega \bar{u}\cdot(P+\lambda)(u)\,dx = -\int_\Omega \sum_{p,q} a_{pq}(x)\cdot\frac{\partial u}{\partial x_p}\frac{\partial\bar{u}}{\partial x_q}\,dx + \int_\Omega \bar{u}\cdot(\sum a_p(x)\frac{\partial u}{\partial x_p}\,)\,dx +$$
$$\int_\Omega |u(x)|^2\cdot b(x)\,dx + \lambda\cdot\int |u(x)|^2\,dx$$

Write $\lambda = \xi + i\eta$. Separating real and imaginary parts we find the two equations:

$$\xi\int |u|^2\,dx = \int\sum_{p,q} a_{p,q}(x)\frac{\partial u}{\partial x_p}\cdot\frac{\partial\bar{u}}{\partial x_q}\,dx + \int\big(\frac{1}{2}\cdot\sum\frac{\partial a_p}{\partial x_p} - b\big)\cdot|u|^2\,dx \tag{i}$$

$$\eta\int |u|^2\,dx = \frac{1}{2i}\int\sum a_p\big(u\frac{\partial\bar{u}}{\partial x_p} - \bar{u}\frac{\partial u}{\partial x_p}\,\big)\,dx \tag{ii}$$

Set

$$A = \int |u|^2\,dx \quad : \quad B = \int |\nabla(u)|^2\,dx$$

Since P is elliptic there exists a positive constant k such that

$$\sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} > k \cdot |\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants c_1, c_2, c_3 such that

(iii) $$A\xi > c_1 B - c_2 B \quad : \quad A|\eta| < c_3 \cdot \sqrt{AB}$$

Here (iii) implies that $\xi > -c_2$ and the reader can also confirm that

(iv) $$B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A \cdot c_2 \cdot \sqrt{\frac{\xi + c_2}{c_1}} \quad : \quad |\eta| < c_3 \cdot \sqrt{\frac{\xi + c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 3.1.

4. The asymptotic formula.

Using the results above where we have found a good control of the integral operator $\mathcal{J}$ and the identification of eigenvalues to | and those from (*) in (0.B), one can proceed and apply Tauberian theorems to derive the asymptotic formula in Theorem 1 using similar methods as described in § XX where we treated the Laplace operator. For details the reader may consult [Carleman:p age xx-xx].

A.5. Operators with closed graph.

Let T be a densely defined operator. In the product $X \times X$ we get the graph:

(A.5.1) $$\Gamma(T) = \{(x, Tx) \quad : x \in \mathcal{D}(T)\}$$

If $\Gamma(T)$ is a closed subspace of $X \times X$ we say that T is closed.

A.5.1 Exercise. Let T be densely defined and assume that $\sigma(T)$ is not the whole complex plane. Show that T is automatically closed.

In the study of spectra we shall foremost restrict the attention to closed operators. Assume that T is densely defined and closed. Let λ be a complex number such that (i) holds in (A.1) for some constant c and the range of $\lambda \cdot E - T$ is dense.

A.5.2 Exercise. Show that the closedness of T implies that the range of $\lambda \cdot E - T$ is equal to X so that $R_T(\lambda)$ exists. A hint is that if $y \in X$, then the density gives a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $\xi_n = \lambda \cdot x_n - T(x_n) \to y$. In particular $\{\xi_n\}$ is a Cauchy- sequence By (i) in (A.1) we have

$$||x_n - x_m|| \leq c^{-1} \cdot ||\xi_n - \xi_m||$$

Hence $\{x_n\}$ is a Cauchy sequence and since X is a Banach space there exists $x \in S$ such that $x_n \to x$. But then $\{(x_n, T(x_n)) = (x_n, \lambda \cdot x_n - \xi_n)$ converges to $(x, \lambda \cdot x - y)$ and since T is closed it follows that $(x, \lambda \cdot x - y) \in \Gamma(T)$ which gives the requested surjectivity since

$$T(x) = \lambda \cdot x - y \implies y = (\lambda \cdot E - T)(x)$$

A.5.3 Adjoints. Let T be densely defined but not necessarily closed. In the dual space X^* we get the subspace of vectors y for which there exists a constant $C(y)$ such that

(i) $$|y(Tx)| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

When (i) holds the density of $\mathcal{D}(T)$ gives a unique vector $T^*(y)$ in X^* such that

(ii) $$y(Tx) = T^*(y)(x) \quad : x \in \mathcal{D}(T)$$

One refers to T^* as the adjoint operator of T whose domain of definition is denoted by $\mathcal{D}(T^*)$.

Exercise. Show that T^* has a closed graph.

A.5.4 Closed extensions. Let T be densely defined but not closed. The question arises when the closure of $\Gamma(T)$ is the graph of a linear operator $\widehat{T}$ and then we refer to $\widehat{T}$ as a closed extension of T. A sufficient condition for the existence of a close extension goes as follows:

A.5.5 Theorem. *If $\mathcal{D}(T^*)$ is dense in X^* then T has a closed extension.*

Proof. Consider the graph $\Gamma(T)$ and let $\{x_n\}$ and $\{\xi_n\}$ be two sequences in $\mathcal{D}(T)$ which both converge to a point $p \in X$ while $T(x_n) \to y_1$ and $T(\xi_n) \to y_2$ hold for some pair y_1, y_2. We must rove that $y_1 = y_2$. To achieve this we take some $x^* \in \mathcal{D}(T^*)$ which gives

$$x^*(y_1) = \lim\, x^*(Tx_n) = \lim\, T^*(x^*)(x_n) = T^*(x^*)(p)$$

In the same way we get $x^*(y_2) = T^*(x^*)(p)$. Now the density of $\mathcal{D}(T^*)$ gives $y_1 = y_2$ which proves that the closure of $\Gamma(T)$ is a graphic subset of $X \times X$ and gives the closed operator $\widehat{T}$ with

$$\Gamma(\widehat{T}) = \overline{\Gamma(T)}$$

A.5.6 Remark. In general, let T be closed and densely defined. There may exist several closed operators S with the property that

$$\Gamma(T) \subset \Gamma(S)$$

Passing to adjoint operators the reader may verify that the density of $\mathcal{D}(T)$ implies that

$$T^* = S^*$$

hold for every closed extension S.

A.6 Operational calculus.

Let T be densely defined and closed. To each pair (γ, f), where γ is a rectifiable Jordan arc contained in $\mathbf{C} \setminus \sigma(T)$ and $f \in C^0(\gamma)$, there exists the bounded linear operator

$$T_{(\gamma,f)} = \int_\gamma f(z) R_T(z)\, dz \tag{A.6.1}$$

The integrand has values in the Banach space of bounded linear operators on X and (A.6.1) is calculated by Riemann sums. Next, Neumann's equation (A.3) entails that $R_T(z_1)$ and $R_T(z_2)$ commute for all pairs z_1, z_2 on γ. From this it is clear that if g is another function in $C^0(\gamma)$, then the operators $T_{f,\gamma}$ and $T_{g,\gamma}$ commute. Moreover, for each $f \in C^0(\gamma)$ the reader may verify that the closedeness of T implies that the range of $T_{f,\gamma}$ is contained in $\mathcal{D}(T)$ and

$$T_{f,\gamma} \circ T(x) = T \circ T_{f,\gamma}(x) \quad : x \in \mathcal{D}(T)$$

Next, let Ω be an open set of class $\mathcal{D}(C^1)$, i.e. $\partial\Omega$ is a finite union of closed differentiable Jordan curves. When $\partial\Omega \cap \sigma(T) = \emptyset$ we construct the line integrals (A.6.1) for continuous functions on the boundary. Consider the algebra $\mathcal{A}(\Omega)$ of analytic functions in Ω which extend to be continuous on the closure. Each $f \in \mathcal{A}(\Omega)$ gives the operator

$$T_f = \int_{\partial\Omega} f(z) R_T(z)\, dz \tag{A.6.2}$$

A.6.3 Theorem. *The map $f \mapsto T_f$ is an algebra homomorphism from $\mathcal{A}(\Omega)$ into a commutative algebra of bounded linear operators on X whose image is a commutative algebra of bounded linear operators denoted by $T(\Omega)$.*

Proof. Let f, g be a pair in $\mathcal{A}(\Omega)$. We must show the equality

$$T_{gf} = T_f \circ T_g \tag{*}$$

To attain this we choose a slightly smaller open set $\Omega_* \subset \Omega$ which again is of class $\mathcal{D}(C^1)$ and each of it bounding Jordan curve is close to one boundary curve in $\partial\Omega$ and $\Omega \setminus \Omega_*$ does not intersect $\sigma(T)$. By Cauchy's theorem we can shift the integration to $\partial\Omega_*$ and get

$$T_g = \int_{\partial\Omega_*} g(z) R_T(z_*)\, dz_* \tag{i}$$

where we use z_* to indicate that integration takes place along $\partial\Omega_*$. Now

$$T_f \circ T_g = \iint_{\partial\Omega_* \times \partial\Omega} f(z) g(z_*) R_T(z) \circ R_T(z_*)\, dz_* dz \tag{ii}$$

Neumann's equation (*) from (A.3) entails that the right hand side in (ii) becomes

$$\iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z_*)}{z - z_*}\, dz_* dz + \iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z)}{z - z_*}\, dz_* dz = A + B \tag{iii}$$

Here A is evaluated by first integrating with respect to z and Cauchy's theorem gives

$$f(z_*) = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega} \frac{f(z)}{z - z_*} \quad : z_* \in \partial\Omega_*\, dz$$

It follows that

$$A = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega_* \times \partial\Omega} f(z_*) g(z_*) R_T(z_*)\, dz_* = T_{fg}$$

Next, B is evaluated when we first integrate with respect to z_*. Here

$$\iint_{\partial\Omega} \frac{g(z_*)}{z - z_*} \quad : z \in \partial\Omega$$

which entails that $B = 0$ and the theorem follows.

A.7 Spectral gap sets.

Let K be a compact subset of $\sigma(T)$ such that $\sigma(T) \setminus K$ is a closed set in $\mathbf{C}$. This implies that if V is an open neighborhood of K, then there exists a relatively compact subdomain $U \in \mathcal{D}(C^1)$ which contains K as a compact subset while $\partial U \cap \sigma(T) = \emptyset$. To every such domain U we can apply Theorem A.6.3. If $U_* \subset U$ for a pair of such domains we can restrict functions in $\mathcal{A}(U)$ to U_*. This yields an algebra homomorphism

$$\mathcal{T}(U) \to \mathcal{T}(U_*)$$

Next, denote by $\mathcal{O}(K)$ the algebra of germs of analytic functions on K. So each $f \in \mathcal{O}(K)$ comes from some analytic function in a domain U as above. The resulting operator $T_U(f)$ depends on the germ f only. In fact, this follows because if $f \in \mathcal{A}(U)$ and $U_* \subset U$ is a similar $\mathcal{D}(C^1)$-domain which again contains K, then Cauchy's vanishing theorem in analytic function theory applies to $f(z)R_T(z)$ in $U \setminus \bar{U}_*$ and entails that

$$\int_{\partial U_*} f(z)R_T(z)\,dz = \int_{\partial U} f(z)R_T(z)\,dz$$

Hence there exists an algebra homorphism from $\mathcal{O}(K)$ into a commutative algebra of bounded linear operators on X denoted by $\mathcal{T}(K)$. The identity in $\mathcal{T}(K)$ is denoted by E_K and called the spectral projection operator attached to the compact set K in $\sigma(T)$. By this construction one has

$$E_K = \frac{1}{2\pi i} \cdot \int_{\partial U} R_T(z)\,dz$$

for every open domain U surrounding K as above.

A.7.1 The operator T_K. When K is a compact spectral gap set of T we set

$$T_K = TE_K$$

This is a bounded linear operator given by

$$\frac{1}{2\pi i} \cdot \int_{\partial U} z \cdot R_T(z)\,dz$$

where U is a domain as above which contains K.

A.7.2 Theorem. *Identify T_K with a linear operator on the subspace $E_K(X)$. Then one has the equality*

$$\sigma(T_K) = K$$

Proof. If λ_0 is outside K we can choose U so that λ_0 is outside $\bar{U}$ and construct the operator

$$S = \frac{1}{2\pi i} \cdot \int_{\partial U} \frac{1}{\lambda_0 - z} \cdot R_T(z)\,dz$$

The operational calculus gives

$$S(\lambda_0 E_K - T) = E_K$$

here E_K is the identity operator on $E_K(X)$ which shows that $\sigma(T_K) \subset K$.

A.7.3 Discrete spectra. Consider a spectral set reduced to a singleton set $\{\lambda_0\}$, i.e. λ_0 is an isolated point in $\sigma(T)$. The associated spectral projection is denoted by $E_T(\lambda_0)$ and given by

$$E_T(\lambda_0) = \frac{1}{2\pi i} \cdot \int_{|\lambda - \lambda_0| = \epsilon} R_T(\lambda)\,d\lambda$$

for all sufficiently small ϵ. Now $R_T(\lambda)$ is an analytic function defined in some punctured disc $\{0 < \lambda - \lambda_0| < \delta\}$ with a Laurent expansion

$$R_T(\lambda) = \sum_{-\infty}^{\infty} (\lambda - \lambda_0)^k \cdot B_k$$

where $\{B_k\}$ are bounded linear operators obtained by residue formulas:

$$B_k = \frac{1}{2\pi i} \cdot \int_{|\lambda| = \epsilon} \frac{R_T(\lambda)}{(\lambda - \lambda_0)^{k+1}}\,d\lambda \quad : \epsilon < \delta$$

Exercise. Show that $R_T(\lambda)$ is meromorphic, i.e. $B_k = 0$ hold when $k << 0$, if and only if there exists a constant C and some integer $M \geq 0$ such that the operator norms satisfy

$$||R_T(\lambda)|| \leq C \cdot |\lambda - \lambda_0)^{-M}$$

Suppose now that R_T has a pole of some order $M \geq 1$ at λ_0 which gives a series expansion

(i) $$R_T(\lambda) = \sum_1^M \frac{B_{-k}}{(\lambda - \lambda_0)^k} + \sum_0^\infty (\lambda - \lambda_0)^k \cdot B_k$$

Residue calculus gives

(ii) $$B_{-1} = E_T(\lambda_0)$$

The case of a simple pole. Suppose that $M = 1$. Then it is clear that Operational calculus gives

(iii) $$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - \lambda_0) R(\lambda)\, d\lambda = 0$$

This vanishing and operational calculus entail that

$$(\lambda_0 E - T) E_T(\lambda_0) = 0$$

which means that the ragne of the projection operator $E_T(\lambda_0)$ is equal to the kernel of $\lambda_0 \cdot E - T$, i.e. the set of eigenvectors x for which

$$Tx = \lambda_0 \cdot x$$

The case $M \geq 2$. To begin with residue calculus identifies (iii) with B_2 and at the same time operational calculus which aftera reversed sign gives

$$(\lambda_0 E - T) E_T(\lambda_0) = -B_2$$

Since $E_T(\lambda_0)$ is a projection which commutes with T it follows that

$$E_T(\lambda_0) \cdot B_2 = B_2 \cdot E_T(\lambda_0) = B_2$$

Exercise. Show that if $M \geq 3$ then

$$(\lambda_0 E - T)^k \cdot E_T(\lambda_0) = (-1)^k \cdot B_{k+1} \quad : 2 \leq k \leq M$$

Consider also the subspaces

$$\mathcal{N}_k(\lambda_0) = \{x \colon (\lambda_0 E - T)^k(x) = 0\} \quad : 1 \leq k \leq M$$

and show that they are non-decreasing and for every $k > M$ one has

$$\mathcal{N}_M(\lambda_0) = \{x \colon (\lambda_0 E - T)^k(x) = 0\}$$

A.7.4 The case when $E_T(\lambda_0)$ has a finite dimensional range. Here the operator $T(\lambda_0) = TE_T(\lambda_0)$ acts on this finite dimensional vector space which entails that the nullspaces $\{\mathcal{N}_k(\lambda_0)\}$ above are finite dimensional and if M is the order of the pole one has

$$(T(\lambda_0) - \lambda_0)^M = 0$$

So λ_0 is the sole eigenvalue of $T(\lambda_0)$. Moreover, the finite dimensional range of $E_T(\lambda_0)$ has dimension equal to that of $\mathcal{N}_M(\lambda_0)$.

Exercise. Recall l that finite dimensional subspaces appear as direct sum compoinetns. So if $E_T(\lambda_0)$ is finite dimensional there exists a direct sum decomposition

$$X = E_T(\lambda_0)(X) \oplus E - E_T(\lambda_0)$$

where $V = E - E_T(\lambda_0)$ is a closed subspace of X. Show that V is T-invariant and that there exists some $c > 0$ such that

$$||\lambda_0 x - Tx|| \geq ||x|| \quad x \in V \cap \mathcal{D}(T)$$

Tbe spectral theorem for ubounded self-adjoint operators.

The lecture will expose results from Carleman's monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923]. Recall that a separable Hilbert space is isomorphic to ℓ^2 whose vectors are sequences of complex numbers $\{x_p\}$ indexed by integers and $\sum |x_p|^2 < \infty$. A doubly indexed sequence $\{c_{pq}\}$ is Hermitian if:

$$c_{q,p} = \bar{c}_{p,q}$$

Impose the condition that each column of this infinite matrix belongs to ℓ^2, i.e.

(1)
$$\sum_{q=0}^{\infty} |c_{pq}|^2 < \infty \quad : \ p = 1, 2, \ldots$$

The Cauchy-Schwarz inequality entails that if $x \in \ell^2$ then the series

(2)
$$\sum_{q=0}^{\infty} c_{p,q} \cdot x_q$$

converges absolutely for each p and let y_p denote the sum. But (1) does not imply that $\{y_p\}$ belongs to ℓ^2. So we get a subspace $\mathcal{D}$ of ℓ^2 which consists of vectors x such that

(3)
$$\sum_{p=0}^{\infty} \Big|\sum_{q=0}^{\infty} c_{p,q} \cdot x_q\Big|^2 < \infty$$

Notice that (1) implies that $\mathcal{D}$ contains the ℓ^2-vectors $\{x_p\}$ for which only finitely many x_p are non-zero. It follows that $\mathcal{D}$ is a dense subspace of ℓ^2 which gives the densely defined linear operator S sending a vector x to the vector $Sx = y$ where

$$y_p = \sum_{q=0}^{\infty} c_{qp} \cdot x_q \quad : p = 0, 1, 2 \ldots$$

By definition the domain of definition of S is the subspace $\mathcal{D}$ of ℓ^2. In an article from 1920 Carleman contructed an "ugly example" of a doubly indexed sequence $\{c_{pq}\}$ of real numbers satisfying (1) and the symmetri condition $c_{pq} = c_{qp}$, and yet there exists a non-zero complex vector $\{x_p = a_p + ib_p\}$ in ℓ^2 such that

(4)
$$S(x) = ix$$

This should be compared with the finite dimensional case where the spectral theorem due to Cauchy and Weierstrass asserts that if A is a real and symmetric $N \times N$-matrix for some positive integer N, then there exists an orthogonal $N \times N$-matrix U such that UAU^* is a diagonal matrix with real elements. Carleman extended this finite dimensional result to infinite Hermitian matrices for which the densely defined linear operator S has no eigenvectors with eigenvalue i or $-i$. More precisely, one says that the densely defined operator S is of Type I if the equations

(*)
$$S(z) = iz \quad : S(\zeta) = -i\zeta$$

do not have non-zero solutions with complex vectors z or ζ in ℓ^2. The major result in the cited monograph is as follows:

Theorem. *Each densely Hermitian operator S of type I has a unique adapted resolution of the identity.*

Remark. The notion spectral resolutions is given in (0.2) below. For a bounded Hermitian operator the existence of an adapted resolution of the identity was proved by Hilbert in 1904 whose pioneering work together with Fredholm's previous studies of integral equations was put forward in Carleman's lecture at the IMU-congress in 1932:

La théorie, crée par Hilbert, des formes quadratiques (ou hermitiennes) à une infinité de variables en connexion avec la théorie des équations intégrales à noyeau symmetrique est certainement la plus importante découverte qui ait été faite dans la théorie des équations intégrales aprés les travaux fondamentaux de Fredholm.

0.1 Applications to quantum mechanics.

The work by Fredholm, Hilbert and Carleman was foremost motivated by applications to integral equations and resolvents of various PDE-equations which stem from physics prior to quantum mechanics. It was therefore quite exciting when Niels Bohr in a lecture held at the Scandinavian Congress in Copenhagen 1925, talked about the new quantum mechanics and adressed new problems to the mathematical community. Recall that a crucial point in quantum mechanics is the hypothesis on energy levels which correspond to orbits in Bohr's theory of atoms. For this physical background the reader should consult Bohr's plenary talk when he received the Nobel Prize in physics 1923. In the "new-born" quantum mechanics the following second order PDE-equation plays a crucial role:

$$(*) \qquad \Delta\phi + 2m\cdot\big(E-U\big)\Big(\frac{2\pi}{h}\Big)^2\cdot\phi = 0$$

Here Δ is the Laplace operator in the 3-dimensional (x,y,z)-space, m the mass of a particle and h Planck's constant while $U(x,y,z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Leaving physics aside, the mathematical problem amounts to study second order PDE-operators:

$$(**) \qquad L = \Delta + c(x,y,z) \quad : \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

where $c(x,y,z)$ is a real-valued and Lebesgue measurable function which is locally square integrable in $\mathbf{R}^3$. Above L is defined on the dense subspace of $L^2(\mathbf{R}^3)$ which consists of test-functions. Moreover, Greens' formula entails that it is symmetric, i.e. for each pair f,g in $C_0^\infty(\mathbf{R}^3)$ one has:

$$\iint L(f)\cdot g\,dxdydz = \iint L(g)\cdot f\,dxdydz$$

So the study of the operators in (*) amounts precisely to determine under which conditions on the potential function c the favourable Case I holds. Complete answers were given by Carleman in his lectures *La théorie des équations intégrales singulières* held at Institut Poincaré in 1930. In physical applications one is foremost concerned with the case when c is a potential function

$$W(p) = \sum \frac{\alpha_k}{|p-q_k|} + \beta$$

where $\{q_k\}$ is a finite subset of $\mathbf{R}^3$ and $\{\alpha_k\}$ and β are real and positive numbers. For this special c-function the results from Carleman's cited monograph imply that the densely defined operator L is of type I and that the real spectrum of L is a discrete sequence of real numbers whose absolute values tend to $+\infty$. We shall review Carleman's account about this case from his lectures in Paris in the subsequent sections. A general result appears in his article *Sur la théorie mathématique de l'équation de Schrödinger* [Arkiv för matematik och fysik: 1934] and goes as follows:

Theorem *If there exists if there is a constant M such that*

$$\limsup_{x^2+y^2+z^2\to\infty} c(x,y,z) \leq M$$

Then the densely defined operator L is of type 1.

Remark. Hundreds - or rather thousands - of articles have later exposed the Bohr-Schrödinger equation with various variants where one for example introduce boundary value conditions and a time variable which leads to Schrödinger's equation

$$i\cdot\frac{\partial u}{\partial t}(x,y,z,t) = L(u)(x,y,z,t)$$

where t is a time variable. In all these studies the spectral theorem for unbounded self-adjoint operators plays a significant role. In the subsequent sections we expose Carleman's solution to (*) when $L = \Delta + c$ is of Case 1 and has a spectrum confined to an interval $[\ell, +\infty)$ for some positive real number ℓ.

0.2 Resolutions of the identity.

Following Hilbert we recall the notion spectral resolutions. A resolution of the identity on ℓ^2 consists of a family $\{E(\lambda)\}$ of self-adjoint projections, indexed by real numbers λ which satisfies (A-C) below.

A. Each $E(\lambda)$ is an orthogonal projection from ℓ^2 onto the range $E(\lambda)(\ell^2)$ and these operators commute pairwise, i.e.

(i)
$$E(\lambda) \cdot E(\mu) = E(\mu) \cdot E(\lambda)$$

hold for pairs of real numbers. Moreover, for each $x \in \ell^2$ one has

(ii)
$$\lim_{\lambda + \infty} ||E(\lambda)(x) - x|| = 0 \quad : \lim_{\lambda \to -\infty} ||E(\lambda)(x)|| = 0$$

B. To each pair of real numbers $a < b$ we set

$$E_{a,b} = E(b) - E(a)$$

Then

(iii)
$$E_{a,b} \cdot E_{c,d} = 0$$

for each pair of disjoint interval $[a, b]$ and $[b, c]$.

C. Notice first that if $\mu < \lambda$ for a pair of real numbers and $x \in \ell^2$ then (ii) gives

$$E(\mu)[E - E(\lambda)](x) = \lim_{s \to +\infty} (E\mu) - E(-s))(E(s) - E(\lambda)(x)$$

By (iii) the last product is zero as soon as $s > \lambda$ and $-s < \mu$ and hence we have

$$E(\mu) = E(\mu)E(\lambda)$$

It means that the range $E(\mu)(\ell^2)$ increases with μ and from this it is clear that when $x \in \ell^2$ then the real-valued function

(c)
$$\lambda \mapsto \langle E(\lambda)(x), x \rangle$$

is a real-valued and non-decreasing function. Hilbert's last condition is that the non-dcreasing function in (c) is right continuous for every fixed x.

S-adapted resolutions.

Let S be a densely defined linear operator on ℓ^2. A spectral resolution $\{E(\lambda)\}$ of the identity is S-adapted if the following hold:

A.1 For each interval bounded $[a, b]$ the range of $E_{a,b}$ from (B) above is contained in $\mathcal{D}(S)$ and

$$E_{a,b}(Sx) = S \circ E_{a,b}(x) \quad : x \in \mathcal{D}(S)$$

B.1 By (C) each $x \in \ell^2$ gives the non-decreasing function $\lambda \mapsto \langle E(\lambda)(x), x \rangle$ on the real line. Together with the right continuity in (c) there exist Stieltjes' integrals

$$\int_a^b \lambda \cdot \langle dE(\lambda)(x), x \rangle$$

for each bounded interval. Carleman's second condition for $\{E(\lambda)\}$ to be S-adapted is that a vector x belongs to $\mathcal{D}(S)$ if and only if

(b.1)
$$\int_{-\infty}^{\infty} |\lambda| \cdot \langle dE(\lambda)(x), x \rangle < \infty$$

C.1 The last condition is that

$$\langle Sx, y\rangle = \int_{-\infty}^{\infty} \lambda \cdot \langle dE(\lambda)(x), y\rangle \quad : x, y \in \mathcal{D}(S) \tag{c.1}$$

where (b.1) and the Cauchy-Schwarz inequality entail that the Stieltjes' integral in (c.1) is absolutely convergent.

Glimpses from work by Torsten Carleman

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Contents.

Mathematics by Carleman

Introduction. Carleman's collected work covers fifty articles of high standard together with several monographs. He became professor at Stockholm University in 1924 when he replaced the chair by Helge von Koch. From 1927 he also served director at Institute Mittag-Leffler where he delivered frequent seminars during the period 1928-1938. His text-book for undergraduate mathematics published in 1926 was used for several decades. Personally I find it outstanding as a beginner's text for students and during my early education Carleman's presentation of basic material in analysis and algebra served as a "veritable bible". His last major publication was *Sur un probléme d'unicité pour les systèmes d'équations aux dérivees partielles à deux variables indépendantes* [Arkiv för matematik 1938] concerned with a uniqueness theorem for elliptic PDE-systems where the variable coefficients of the PDE-operators are non-analytic which extended an earlier result by Erik Holmgren when the coefficients are real-analytic. Let us recall that Holmgren served as supervisor while Carleman prepared his thesis entitled *Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, presented at Uppsala University in 1916 when he was 23 years old. To this one can add that Carleman got much inspiration from Ivar Fredholm's work on integral equations. At that time Fredholm was professor in mathematical physics at Stockholm University and considered as the leading mathematician in Sweden.

Carleman was aware of the interplay between "pure mathematics" and experimental sciences which inspired his own research. After World War I he studied a year at an engineering school in Paris. The article entitled *Sur les équations différentielles de la mecanique d'avion* published in [La Technique Aéronautique, vol. 10 1921), inspired by Lanchester's pioneering work *Le vol aérien* which played a significant role while airplanes were designed at an early stage. Carlemans article end with the following conclusion after an investigation of integral curves to a certain non-linear differential system: *Quelle que soit la vitesse initiale, l'avion, après avoir éxecuté s'il y lieu, un nombre fini des loopings, prend un mouvement qui s'approche indéfinement du régime de descente rectiligne ert uniforme.*

The reader may also consult his lecture held 1944 at the Academy of Science in Sweden entitled *Sur l'action réciproque entre les mathématiques et les sciences expérimentales exactes* which underlines Carleman's concern about applictions of mathematics.

During the last years in life Carleman suffered from health problems which caused his decease on January 11 1949 at the age of 56 years. A memorial article about his scientific achievement appears in [Acta. Math. 1950] by Fritz Carlson who was Carleman's collegue at the department of mathematics at Stockholm university for several decades. See also his collected work published by Institute Mittag-Leffler in 1960.

In this lecture we pay attention to Carleman's monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923] where the existence of spectral resolutions for unbounded and self-adjoint operators on a Hilbert space was established. The expositary article *La theorie des equations intégrales singuliérs* [Ann. l'Institut Poincaré Vol. 1 (1931)] from his lectures in Paris during the spring 1930 gives a good introduction to the more involved material in the cited work above and contains also instructive examples. In § 5 we establish an "abstract version" of the spectral theorem for unbounded self-adjoint operators, but remark that in many applications it is convenient to adapt the rather explicit contructions which occur in the above mentioned references, The material in § 0.3 about the Schrödinger equation illustrates this.

An example.. A basic equation which emerges from quantum mechanics is to find solutions $u(p,t)$ defined in $\mathbf{R}^3 \times \mathbf{R}^+$ where t is a time variable and $p = (x, y, z)$ which satisfies the PDE-equation

$$i \cdot \frac{\partial u}{\partial t} = \Delta(u)(p,t) - c(p) \cdot u(p,t) = 0 \quad t > 0$$

and the initial condition

$$u(p,0) = f(p)$$

Here $f(p)$ belongs to $L^2(\mathbf{R}^3)$ and $c(p)$ is a real-valued and locally square integrable function. Carleman proved that this equation has solutions under the condition that

$$\limsup_{p|\to\infty} c(p) \leq M$$

holds for some constant M and the spectrum of the densely defined sefl-adjoint operator $\Delta - c$ on the Hilbert space $L^2(\mathbf{R}^3)$ is confined to an interval $[\lambda_1, +\infty)$ on the positive real line, i.e. here $\lambda_1 > 0$. When this holds the solution u is given by an equatiuon

$$u(p,t) = \int_{\mathbf{R}^3} [\int_{\lambda_1}^{\infty} e^{i\lambda t} \cdot d\theta(p,q,\lambda)\,] \cdot f(q)\, dq$$

where $\lambda \mapsto \theta(p,q,\lambda)$ is a non-decreasing function on $[\lambda_1, +\infty)$ for each fixed pair $p.q$. An asymptotic expansion which recaptures the θ-function will be exposed in § xx and gives an illustration of Carleman's vigour in analysis.

A second example. An early contribution by Carleman is the article *Sur le genere du denominateur $D(\lambda)$ de Fredholm* [Arkiv för matematik 1917]. Here one starts with a continuous function $K(x,y)$ defined on the square $\{0 \leq x, y \leq 1\}$ which yields the integral operator

$$f \mapsto \mathcal{K}_f(x) = \int_0^1 K(x,y)f(y)\, dy$$

There exists the discrete sequence of spectral values $\{\lambda_\nu\}$ for which $E - \lambda_\nu \cdot \mathcal{K}_f$ are not invertible where E is the identity operator on the Hilbert space $L^2[0,1]$. Here $\{0 < |\lambda_1| \leq |\lambda_2| \leq \ldots\}$ are arranged with non-decreasing absolute values and eventual eigenspaces of diemension $e \geq 1$ means that the corresponding λ-value is repated e times in the sequence. With these notations Carleman proved that

$$(*) \qquad \mathcal{D}(\lambda) = e^{a\lambda} \cdot \prod_{\nu=1}^{\infty} (1 - \frac{\lambda}{\lambda_\nu})$$

Here $\mathcal{D}(\lambda)$ is the Fredholm resolvent defined for each complex λ by the series

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\,!} \cdot \int_{[0,1]^n} \det(K_n(s_1, \ldots, s_n))\, ds_1 \cdots ds_n$$

where $K_n(s_1, \ldots, s_n)$ is the $n \times n$-matrix with elements $\{K(s_i, s_k) : 1 \leq i, k \leq n\}$. A notable point is that (*) holds even when the kernel function $K(x,y)$ is not symmetric and (*) means that $\mathcal{D}(\lambda)$ is an entire function whose order in the sense of Hadamard is one. The proof in the cited article gives an excellent lesson about spectral properties of linear operators where a crucial point is to overcome the lack of symmetry of $K(x,y)$. This was achieved by Carleman by analytic function theory where a result due to Arne Wiman plays a crucial role. Recall that Wiman also was professor at Uppsala during this time. So Carleman's career started in a an "optimal enviroment" guided by Holmgren and Wiman.

About the contents.

Some familarity with basic results about linear operators and their spectra is needed to follow the lecture. For the less experienced readers we insert an appendix about this classic material which relies upon Gustaf Neumann's constructions in his pioneering article about the Dirichlet problem from 1877, together with an application of Cauchy's integral formula in § A.6 which leads to the operational calculs for densely defined operators whose Neumann resolvents exist in non-empty open subsets of the complex plane. Since our main concern is to expose contributions by Carleman, we include in § A.8 a proof of his inequality for resolvent operators from the article *Zur Theorie der linearen Intergralgleichungen* [Math. Zeitschrift 1921]. Since this result is not used in the spectral theorem it is not necessary to pursue details of the rather involved proof at a first instant.

Readers familiar with basic spectral theory can turn directly to § 1 where we begin to study operators on Hilbert spaces. The first result is Hilbert's spectral theorem for bounded self-adjoint operators, including spectral resolutions for bounded normal operators. In § 2 we expose facts about densely defined unbouned operators on Hilbert spaces where the construction of adjoints plays a crucial role. After this we embark upon material in Carleman's cited monograph. Here densely defined symmetric operators and conditions in order that they have self-adjoint extensions are described in § 3 and in the final part of § 3 we construct symmetrizations of linear operators which are used to analyze spectra of non-symmetric operators. Finally § 4-5 presents the "high points" which culminate in the spectral theorem for unbounded self-adjoint operators.

Remark. From a historic point of view we recall that Weyl already in 1910 considered unbounded self-adjoint operators which appear for densely defined ordinary differential operators. See § 3.4 for an example. Extensive studies of spectra associated to ODE-operators also occur in pionering articles by Birkhoff's from the same time. The "ugly example" in § 5.3.1 discovered by Carleman in 1920 gave rise to a general investigation about self-adjoint extensions of densely defined and symmetric operators on Hilbert spaces which was carried out in his cited monograph. A crucial result in the spectral theory for unbounded operators on Hilbert spaces goes as follows:

Theorem. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a densely defined operator whose graph $\Gamma(T) = \{(x, Tx)\}$ is closed in $\mathcal{H} \times \mathcal{H}$. Then the adjoint operator T^* as well as the composed operator T^*T are densely defined and every vector $h \in \mathcal{H}$ has a unique representation*

$$h = x + T^*(Tx) \quad : x \in \mathcal{D}(T^*T)$$

where the last space is contained in $\mathcal{D}(T)$.

Remark. Special cases of the result above was known by Weyl at an early stage and also in Carleman's cited work when T is expressed by an integral operator. A conclusive result - known as the polarisation theorem - due to F. Riesz and Nagy shows that with T as above, there exists a densely defined and self-adjoint operator A such that $A^2 = T^*T$, and T can be expressed as

$$T = P \circ A$$

where P is a bounded operator whose kernel is the orthogonal complement of the range of A while

$$\langle P \circ A(x), P \circ A(x) \rangle = \langle x, x \rangle : \ x \in \mathcal{D}(A)$$

About the mathematics by Carleman.

Personally I find that few mathematical texts (if any) superseed Carleman's fundamental approach to many problems. Several of his articles appear as veritable "classics" which merit a study up to the present date. We refrain from exposing work by Carleman devoted to quasi-analytic functions and Bolzmann's kinetic gaz theory. But we present some results with proofs of a more self-contained character. The material below is independent of the main topics in this lecture and has been inserted to illustrate the vigour in Carleman's work. For the student intersted in analysis the proofs below are self-contained which personlly think offer more valuable lessons compared to digesting "general theories".

§ 0. An inversion formula. A fundamental result due to Abel gives an inversion formula for the potential function $U(x)$ in a conservative field of forces. More precisely, let $U(x)$ be an even function of x with $U(0) = 0$ and $x|toU(x)$ is stricitly increasing and convex on $x \geq 0$. A particle of unit mass which moves on the real x-line satisfies Newton's equation

$$\ddot{x}(t) = -U'(x(t))$$

where the initial conditions are $x(0) = 0$ and $\dot{x}(0) = v > 0$. It follows that

$$\frac{\dot{x}(t)^2}{2} + U(x(t)) = \frac{v^2}{2} \implies$$

$$\dot{x}(t) = \sqrt{v^2 - 2U(x(t))} \tag{1}$$

during a time interval $[0, T]$ where $\dot{x}(t) > 0$ when $0 \leq t < T$ and $\dot{x}(T) = 0$. From (1) we get the equation

$$T = \int_0^{x(T)} \frac{dx}{\sqrt{v^2 - 2U(x)}} \quad : \ 2 \cdot U(x(T)) = v^2$$

With U given this means that $T = T(v)$ is a function of v. In a work from 1823, Abel established an inversion formula which recaptures U when the function $v \mapsto T(v)$ is known. The reader should consult the literature for this famous and very important result.

In the article *Abelsche Intergalgleichung mot konstanten Integralgrenzen* [Mathematische Zeitschrift 1922], Carleman established inversion formulas in Abel's spirit. An example is as follows: For every fixed real $0 \leq x \leq 1$

$$t \mapsto \log |x - t|$$

is integrable on the unit interval $[0, 1]$ and yields a bounded linear operator on the Banach space $C^0[0, 1]$ sending every $g \in C^0[0, 1]$ to

$$T_g(x) = \int_0^1 \log |x - t| \cdot g(t)\, dt$$

It is not difficult to show that T_g is injective. It turns out that there is an inversion formula for the T-operator which at the same time gives a description of its range.

0.1 Theorem. *With $f = T_g$ one has the inversion formula*

$$\sqrt{x(1-x)} \cdot g(x) = \frac{1}{\pi^2} \cdot \int_0^1 \frac{f'(t) \cdot \sqrt{t(1-t)}}{x - t}\, dt + \frac{1}{\pi} \cdot \int_0^1 g(t)\, dt \tag{*}$$

Remark. This inversion formula shows that a function f in the range of T must satisfy certain regularity properties. More precisely, in (*) the first order derivative $f'(t)$ appears in an integral where we have taken a principal value. Let us also remark that one has the equation

$$\int_0^1 g(t)\, dt = -\frac{1}{2\pi \cdot \log 2} \cdot \int_0^1 \frac{f(x)}{\sqrt{(1-x)x}}\, dx \tag{**}$$

Proof of Theorem 0.1. The complex log-function

$$z \mapsto \log\, z - t$$

is defined when $z \in \mathbf{C} \setminus (-\infty, 1]$ for each $0 \leq t \leq 1$. The single-valued branches are chosen so that the argument of these log-functions stay in $(-\pi, \pi)$ and $\log x - t$ is real if $x > t$. It follows that

(1) $$\lim_{\epsilon \to 0} \log(x + i\epsilon - t) = \log|x - t| + \pi i \quad : x < t$$

where the limit is taken as $\epsilon > 0$ decrease to zero. Let $g(t)$ be a continuous function on $[0, 1]$ and put

(2) $$G(z) = \int_0^1 \log(z - t) \cdot g(t)\, dt$$

A. Exercise. Show that (1) gives:

(i) $$G(x + i0) = T_g(x) + \pi i \cdot \int_x^1 g(t)\, dt \quad : 0 < x < 1$$

where $G(x + i0)$ is the limit as $z = x + i\epsilon$ and $\epsilon > 0$ decrease to zero. Show in a similar way that

(ii) $$G(x - i0) = T_g(x) - \pi i \cdot \int_x^1 g(t)\, dt \quad : 0 < x < 1$$

Next, outside $[0, 1]$ the complex derivative of G becomes

$$G'(z) = \int_0^1 \frac{g(t)}{z - t}\, dt$$

Then (i-ii) give

(iii) $$G'(x + i0) + G'(x - i0) = 2 \cdot \frac{T_g(x)}{dx} \quad : \quad G'(x + i0) - G'(x - i0) = -2\pi \cdot g(x)$$

B. The Φ-function. In $\mathbf{C} \setminus [0, 1]$ we have the analytic function $h(z) = \sqrt{z(z-1)}$ whose branch is chosen so that it is real and positive when $z = x > 1$. It follows that

(b.1) $$h(x + i0) = i \cdot \sqrt{x(1-x)} \quad : \quad h(x - i0) = i \cdot \sqrt{x(1-x)} \quad : 0 < x < 1$$

Consider the analytic function

$$\Phi(z) = \sqrt{z(z-1)} \cdot G'(z)$$

With $f = T_g$ we see that (iii-iv) give the two equations

(b.2) $$\Phi(x + i0) + \Phi(x - i0) = 2\pi \cdot \sqrt{x(1-x)} \cdot g(x)$$

(b.3) $$\Phi(x + i0) - \Phi(x - i0) = 2i \cdot f'(x) \cdot \sqrt{x(1-x)}$$

C. The Ψ-function. Set

(c.1) $$\Psi(z) = \int_0^1 \frac{1}{z - t} \cdot f'(t) \cdot \sqrt{t(1-t)}\, dt$$

The equation (b.3) and the general formula from § XX give

(c.2) $$\Phi(x + i0) - \Phi(x - i0) = \Psi(x + i0) - \Psi(x - i0) \quad : \ 0 < x < 1$$

D. Exercise. Deduce from (c.2) that

$$\Phi(z) = \Psi(z) + \int_0^1 g(t)\, dt$$

where the equality holds when $z \in \mathbf{C} \setminus (-\infty, 1]$. Conclude from the above that

$$2\pi \cdot \sqrt{x(1-x)} \cdot g(x) = \Psi(x+i0) + \Psi(x-i0) + 2 \cdot \int_0^1 g(t)\,dt \tag{d.1}$$

Next, from (c.1) and the general formula in § XX we have

$$\Psi(x+i0) + \Psi(x-i0) = \frac{2}{\pi} \cdot \int_0^1 \frac{f'(t) \cdot \sqrt{t(1-t)}}{x-t}\,dt \tag{d.2}$$

where the last integral is taken as a principal value. Together (d.1-2) give (*) in Theorem 0.1.

1. An approximation theorem.

The result below was proved in the article *Sur un théorème de Weierstrass* [Arkiv för mathematik och fysik. vol 20 (1927]:

Theorem. *Let f be a continuous and complex valued function on the real x-line. To each $\epsilon > 0$ there exists an entire function $\phi(z) = \phi(x+iy)$ such that*

$$\max_{x \in \mathbf{R}} |f(x) - \phi(x)| < \epsilon$$

In the cited article Carleman gave an elementary proof using Cauchys integral formula. But his constructions can be extended to cover a more general situation which goes as follows. Let K be an unbounded closed null-set in $\mathbf{C}$. If $0 < R < R^*$ we put

$$K[R, R^*] = K \cap \{R \leq |z| \leq R^*\}$$

and if $R > 0$ we put $K_R = K \cap \bar{D}_R$ where $\bar{D}_R = \{|z| \leq R\}$.

1.1 Theorem. *Suppose there exists a strictly increasing sequence $\{R_\nu\}$ where $R_\nu \to +\infty$ such that $\mathbf{C} \setminus K_{R_1}$ and the sets*

$$\Omega_\nu = \mathbf{C} \setminus \bar{D}_{R_\nu} \cup K[R_\nu, R_{\nu+1}]$$

are connected for each $\nu \geq 1$. Then every continuous function on K can be uniformly approximated by entire functions.

To prove this result we first establish the following.

1.2 Lemma. *Consider some $\nu \geq 1$ a continuous function ψ on $S = \bar{D}_{R_n} \cup K[R_\nu, R_{\nu+1}]$ where ψ is analytic in the open disc D_{R_ν}. Then ψ can be uniformly approximated on S by polynomials in z.*

Proof. If we have found a sequence of polynomials $\{p_k\}$ which approximate ψ uniformly on $S_* = \{|z| = R_\nu\} \cup K[R_\nu, R_{\nu+1}]$ then this sequence approximates ψ on S. In fact, this follows since ψ is analytic in the disc D_{R_ν} so by the maximum principle for analytic functions in a disc we have

$$||p_k - \psi||_S = ||p_k - \psi||_{S_*}$$

for each k. Next, if uniform approximation on S_* fails there exists a Riesz-measure μ supported by S_* which is $\perp$ to all analytic polynomials while

$$\int \psi \cdot d\mu \neq 0 \tag{1}$$

To see that this cannot occur we consider the Cauchy transform

$$\mathcal{C}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

Since $\int \zeta^n \cdot d\mu(\zeta) = 0$ for every $n \geq 0$ we see that $\mathcal{C}(z) = 0$ in the exterior disc $|z| > R_{\nu+1}$. The connectivity hypothesis implies that $\mathcal{C}(z) = 0$ in the whole open complement of S. Now K was a null set which means that the L^1_{loc}-function $\mathcal{C}(z)$ is zero in the exterior disc $|z| > R_\nu$ and hence its distribution derivative $\bar{\partial}(\mathcal{C}_\nu)$ also vanishes in this exterior disc. At the same time we have the equality

$$\bar{\partial}(C_\nu) = \mu$$

We conclude that the support of μ is confined to the circle $\{|z| = R_\nu\}$. But then (1) cannot hold since the restriction of ψ to this circle by assumption extends to be analytic in the disc D_{R_ν} and therefore can be uniformly approximated by polynomials on the circle.

Proof of Theorem A.2. Let $\epsilon > 0$ and $\{\alpha_\nu\}$ is a sequence of positive numbers such that $\sum \alpha_\nu < \epsilon$. Consider some $f \in C^0(K)$. Starting with the set K_{R_1} we use the assumption that its complement is connected and using Cauchy transforms as in Lemma A.3 one shows that the restriction of f to this compact set can be uniformly approximated by polynomials. So we find $P_1(z)$ such that

$$\text{(i)} \qquad ||P_1 - f||_{K_{R_1}} < \alpha_1$$

From (i) one easily construct a continuous function ψ on $\bar{D}_{R_1} \cup K[R_1, R_2]$ such that $\psi = P_1$ holds in the disc $\bar{D}_{R_1}$ and the maximum norm

$$||\psi - f||_{K[R_1,R_2]} \leq \alpha_1$$

Lemma A.3 gives a polynomial P_2 such that

$$||P_2 - P_1||_{D_{R_1}} < \alpha_2 \quad \text{and} \quad ||P_2 - f||_{K[R_1,R_2]} \leq \alpha_1 + \alpha_2$$

Repeat the construction where Lemma A.3 is used as ν increases. This gives a sequence of polynomials $\{P_\nu\}$ such that

$$||P_\nu - P_{\nu-1}||_{D_{R_\nu}} < \alpha_\nu \quad \text{and} \quad ||P_\nu - f||_{K[R_{\nu-1},R_\nu]} < \alpha_1 + \ldots + \alpha_\nu$$

hold for all ν. From this it is easily seen that we obtain an entire function

$$P^*(z) = P_1(z) + \sum_{\nu=1}^{\infty} P_{\nu+1}(z) - P_\nu(z)$$

Finally the reader can check that the inequalities above imply that the maximum norm

$$||P^* - f||_K \leq \alpha_1 + \sum_{\nu=1}^{\infty} \alpha_\nu$$

Since the last sum is $\leq 2\epsilon$ and $\epsilon > 0$ was arbitrary we have proved Theorem A.3.

1.3 Exercise. Use similar methods as above to show that if $f(z)$ is analytic in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$ and has continuous boundary values on the real line, then f can be uniformly approximated by an entire function, i.e. to every $\epsilon > 0$ there exists an entire function $F(z)$ such that

$$\max_{z \in U^+} |F(z) - f(z)| \leq \epsilon$$

2. An inequality for differentiable functions.

A fundamental result was proved by Carleman in the article *Sur un théorème de M. Denjoy* [C.R. Acad. Sci. Paris 1922]

2.1 Theorem. *There exists an absolute constant $\mathcal{C}$ such that the inequality below holds for every pair (f, n), where n is a positive integer and f a non-negative real-valued C^∞- function defined on the closed unit interval $[0, 1]$ whose derivatives up to order n vanish at the two end points.*

$$(*) \qquad \sum_{\nu=1}^{\nu=n} \frac{1}{[\beta_\nu]^{\frac{1}{\nu}}} \leq \mathcal{C} \cdot \int_0^1 f(x)dx \quad : \quad \beta_\nu = \sqrt{\int_0^1 [f^{(\nu)}(x)]^2 \cdot dx}$$

Remark. The proof below shows that one can take

$$(*) \qquad C \leq 2e\pi \cdot (1 + \frac{1}{4\pi^2 e^2 - 1})$$

The best constant $\mathcal{C}_*$ which would give

$$\text{(i)} \qquad \sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \leq \mathcal{C}^*$$

for all n and every $f \in \mathcal{S}_n$ is not known. Let us also remarkl that the inequality (*) is sharp in the sense that there exists a constant C_* such that for every $n \geq 2$ there exists a function $f_n(x)$ as above so that the opposed inequality (*) holds with C_*. Hence (*) demonstrates that the standard cut-off functions which are used in many applications to keep maximum norms of derivatives small up to order n small, are optimal up to a constant. So the theoretical result in (*) plays a role in numerical analysis where one often uses smoothing methods. The proof of (*) employs estimates for harmonic measures applied to the subharmonic Log-function of the absolute value of the Laplace transform of f, i.e. via a "detour into the complex domain" which in 1922 appeared as a "revolutionary method".

Proof of Theorem 2.1.

Let $n \geq 1$ and keeping f fixed we put $\beta_p = \beta_p(f)$ to simplify notations. Using partial integrations and the Cauchy-Schwarz inequality one shows that the β-numbers are non-decreasing. i.e.

$$(*) \qquad 1 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_{n+1}$$

Define the complex Laplace transform

$$\Phi(z) = \int_0^1 e^{-zt} f(t)dt$$

Since f by assumption is n-flat at the end-ponts, integration by parts p times gives:

$$\Phi(z) = z^{-p} \int_0^1 e^{-zt} \cdot \partial^p(f^2)(t)dt \quad : \quad 1 \leq p \leq n+1$$

where $\partial^p(f^2)$ is the derivative of order p of f^2. We have

$$(1) \qquad \partial^p(f^2) = \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot f^{(\nu)} \cdot f^{(p-\nu)} \quad : \ 1 \leq p \leq n+1$$

Now we study the absolute value of Φ on the vertical line $\mathfrak{Re}(z) = -1$. Since $|e^{t-iyt}| = e^t$ for all y, the triangle inequality gives

$$(2) \qquad |-1+iy|^p \cdot |\Phi(-1+iy)| \leq \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \int_0^1 e^t \cdot |f^{(\nu)}(t)| \cdot |f^{(p-\nu)}(t)| \cdot dt$$

Since $e^t \leq e$ on $[0,1]$, the Cauchy-Schwarz inequality and the definition of the β-numbers give:

$$(3) \qquad |-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu}$$

From (*) it follows that $\beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu} \leq \beta_p^p$ for each ν and since $\sum_{\nu=0}^{\nu=p} \binom{p}{\nu} = 2^p$ we obtain

$$(4) \qquad |-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot 2^p \cdot \beta_p^p$$

Passing to the logarithm we get

$$(5) \qquad \log |\Phi(-1+iy)| \leq 1 + p \cdot \log \frac{2\beta_p}{|-1+iy|}$$

Here (5) holds when $1 \leq p \leq n+1$ and the assumption that $\beta_0 = 1$ also gives

$$(6) \qquad \log |\Phi(-1+iy)| \leq 1$$

The ω-function. To each $1 \leq p \leq n+1$ we find a positve number y_p such that

$$|-1+iy_p| = 2e\beta_p$$

Now we define a function $\omega(y)$ where $\omega(y)=0$ when $y<y_1$ and

$$\omega(y)=p \quad : \quad y_p\le y<y_{p+1}$$

and finally $\omega(y)=n+1$ when $y\ge y_{n+1}$. Then (5-6) give the inequality

$$\log|\phi(-1+iy)|\le 1-\omega(y) \quad : \quad -\infty<y<+\infty \tag{7}$$

A harmonic majorisation. With $1-\omega(y)$ as boundary function in the half-plane $\mathfrak{Re}(z)>-1$ we construct the harmonic extension $H(z)$ which by Poisson's formula is given by:

$$H(0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\omega(y)}{1+y^2}\cdot dy$$

Now $\log|\Phi(z)|$ is subharmonic in this half-plane and hence (7) gives:

$$0=\log|\Phi(0)|\le H(0)$$

We conclude that

$$\int_{-\infty}^{\infty}\frac{\omega(y)}{1+y^2}\cdot dy\le\pi \tag{8}$$

Since $\omega(y)=0$ when $y\le y_1$ we see that (8) gives the inequality

$$\int_{y_1}^{\infty}\frac{\omega(y)}{y^2}\cdot dy\le\frac{y_1^2}{1+y_1^2}\cdot\pi \tag{9}$$

The construction of the ω-function gives the equation

$$\int_{y_1}^{\infty}\frac{\omega(y)}{y^2}\cdot dy=\frac{1}{y_1}+\ldots+\frac{1}{y_{n+1}} \tag{10}$$

Next, the construction of the y_p-numbers entail that $y_p\le 2e\beta_p$ so (9-10) give

$$\frac{1}{\beta_1}+\ldots+\frac{1}{\beta_{n+1}}\le 2e\pi\cdot\frac{1}{1+\frac{1}{y_1^2}} \tag{11}$$

Finally, we have $1+y_1^2=4e^2\beta_1^2$ and recall that Writinger's inequality implies that $\beta_1\ge\pi$. Hence

$$\frac{1}{1+\frac{1}{y_1^2}}\le 1+\frac{1}{4\pi^2e^2-1} \tag{12}$$

and then (11-12) give the requested inequality in Theorem 2.1.

3. An inequality for inverse Fourier transforms in $L^2(\mathbf{R}^+)$.

By Parsevel's theorem the Fourier transform sends L^2-functions on the ξ-line to L^2-functions on the x-line. Now one can seek the class of non-negative L^2-functions $\phi(x)$ such that there exists an L^2-function $F(\xi)$ supported by the half-line $\xi\ge 0$ and

$$\phi(x)=\Big|\int_0^{\infty}e^{ix\xi}\cdot F(\xi)\cdot d\xi\Big| \tag{*}$$

The theorem below was proved in [Carleman] which apart from applications to quasi-analytic functions has several other consequences which are put forward by Paley and Wiener in their text-book [Pa-Wi]. Carleman's proof employs a detour in the complex domain where Jensen's formula plays a crucial role. Here is the result:

3.1 Theorem. *An L^2-function $\phi(x)$ satisfies (*) if and only if*

$$\int_{-\infty}^{\infty}\log^+\Big[\frac{1}{\phi(x)}\Big]\cdot\frac{dx}{1+x^2}<\infty \tag{i}$$

Moreover, when (*) holds and $F(\xi)$ satisfies the weighted mean-value equality

$$\int_0^{\infty}F(\xi)\cdot e^{-\xi}\,d\xi=1 \tag{ii}$$

then

$$\text{(iii)} \qquad \int_{-\infty}^{\infty} \log^+\Big[\frac{1}{\phi(x)}\Big] \cdot \frac{dx}{1+x^2} \leq \int_{-\infty}^{\infty} \frac{\phi(x)^2}{1+x^2} \cdot dx$$

Proof. First we prove the suffiency. Let $\phi(x)$ be a non-negative L^2-function where the integral (i) is finite. The harmonic extension of $\log \phi(x)$ to the upper half-plane is given by:

$$\text{(1)} \qquad \lambda(x+iy) = \frac{y}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\log\ \phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0$$

Let $\mu(z)$ be the conjugate harmonic function of λ and set

$$\text{(2)} \qquad h(z) = e^{\lambda(z)+i\mu(z)}$$

Fatou's theorem gives for almost every x a limit

$$\text{(3)} \qquad \lim_{y\to 0} \lambda(x+iy) = \log\ \phi(x)$$

Or, equivalently

$$\text{(4)} \qquad \lim_{y\to 0} |h(x+iy)| = \phi(x)$$

From (1) and the fact that the geometric mean value of positive numbers cannot exceed their arithmetic mean value, one has

$$\text{(5)} \qquad \big|h(x+iy)\,\big| = e^{\lambda(x+iy)} \leq \frac{y}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0$$

Then (5) the Schwarz inequality give:

$$\text{(6)} \qquad \int_{-\infty}^{\infty} \big|h(x+iy)\,\big|^2\, dx \leq \int_{-\infty}^{\infty} \big|\phi(x)\,\big|^2\, dx \quad : y > 0$$

Here $h(z)$ is analytic in the upper half-plane so that (6) and Cauchy's formula entail that if $\xi < 0$, then the integrals

$$\text{(7)} \qquad J(y) = \int_{-\infty}^{\infty} h(x+iy) \cdot e^{-ix\xi+y\xi} \cdot dx \quad : \ y > 0$$

are independent of y. Passing to the limit as $y \to \infty$ and using the uniform upper bounds on the L^2-norms of the functions $h_y(x) \mapsto h(x+iy)$, it follows that $J(y)$ vanishes identically. So the Fourier transforms of $h_y(x)$ are supported by $\xi \geq 0$ for all $y > 0$. Passing to the limit as $y \to 0$ the same holds for the Fourier transform of $h(x)$. Finally (4) gives

$$\text{(8)} \qquad \phi(x) = |h(x)|$$

By Parseval's theorem $\widehat{h}(\xi)$ is an L^2-function and hence $\phi(x)$ has the requested form (*).

Necessity. Since F is in L^2 there exists the Plancherel limit

$$\text{(9)} \qquad \psi(x) = \lim_{N\to\infty} \frac{1}{2\pi} \cdot \int_0^N e^{ix\xi} \cdot F(\xi)\, d\xi$$

and in the upper half plane we get the analytic function

$$\text{(10)} \qquad \psi(x+iy) = \frac{1}{2\pi} \cdot \int_0^{\infty} e^{ix\xi-y\xi} \cdot F(\xi)\, d\xi$$

Suppose that $F(\xi)$ satisfies (ii) in the Theorem which gives

$$\psi(i) = 1$$

Consider the conformal map from the upper half-plane into the unit disc where

$$w = \frac{z-i}{z+i}$$

Here $\psi(x)$ corresponds to a function $\Phi(e^{is})$ on the unit circe $|w|=1$ and:

$$(11) \qquad \int_{-\pi}^{\pi} |\Phi(e^{is})|^2\, ds = 2\cdot \int_{-\infty}^{\infty} \frac{|\phi(x)|^2}{1+x^2}\, dx$$

Similarly let $\Psi(w)$ be the analytic function in $|w|<1$ which corresponds to $\psi(z)$. From (10-11) it follows that $\Psi(w)$ is the Poisson extension of Φ, i.e.

$$(12) \qquad \Psi(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|e^{is}-w|^2}\cdot \Phi(e^{is})\cdot ds$$

If $0<r<1$ it follows that

$$(13) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}|\cdot ds \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\Psi(re^{is}\right|^2\cdot ds \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(e^{is}|^2\cdot ds$$

Now (12) gives:

$$(14) \qquad \lim_{r\to 1} \Psi(re^{is}) = \Phi(e^{is}) \quad : \text{almost everywhere} \quad 0\leq s\leq 2\pi$$

Next, since $\psi(i)=1$ we have $\Psi(0)=1$ which gives the inequality

$$(15) \qquad \int_{-\pi}^{\pi} \log^+ \frac{1}{|\Psi(re^{is}|}\cdot ds \leq \int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}|\cdot ds \quad : 0<r<1$$

By (13-15) a passage to the limit as $r\to 1$ gives

$$(16) \qquad \int_{-\pi}^{\pi} \log^+ \frac{1}{|\Phi(e^{is}|}\cdot ds \leq \int_{-\pi}^{\pi} \left|\, \Phi(e^{is}\right|^2\cdot ds$$

Returning to the real x-line the inequality (iii) follows which at the same time finishes the proof of the theorem.

4. The Bergman kernel.

Let Ω be a bounded and simply connected domain in $\mathbf{C}$. If $a\in\Omega$ a famous result due to Stefan Bergman gives the conformal mapping function $f_a\colon\Omega\to D$ via the kernel function of the Hilbert space $H^2(\Omega)$.

A. Bergman's Theorem. *The conformal map f_a is given by*

$$f_a(z) = \sqrt{\frac{\pi}{K(a,a)}}\cdot \int_a^z K(z,a)\, dz$$

Next, the Gram-Schmidt construction gives an orthonormal basis $\{P_n(z)\}$ in $H^2(\Omega)$ where P_n has degree n and

$$\iint_\Omega P_k\cdot \bar{P}_m\cdot dxdy = \text{Kronecker's delta function}$$

From Bergman's result one expects that these polynomials are related to a conformal mapping function. We shall consider the case when Ω is a Jordan domain whose boundary curve Γ is *real-analytic.* Let ϕ be the conformal map from the *exterior domain* $\Omega^* = \Sigma\setminus\bar{\Omega}$ onto the exterior disc $|z|>1$. Here ϕ is normalised so that it maps the point at infinity into itself. The inverse conformal mapping function ψ is defined in $|z|>1$ and has a series expansion

$$(*) \qquad \psi(z) = \tau\cdot z + \tau_0 + \sum_{\nu=1}^{\infty} \tau_\nu\cdot \frac{1}{z^\nu}$$

where τ is a positive real number. The assumption that Γ is real-analytic gives some $\rho_1<1$ such that ψ extends to a conformal map from the exterior disc $|z|>\rho_1$ onto a domain whose compact complement is contained in Ω.

It turns out that the polynomials $\{P_n\}$ are approximated by functions expressed by ϕ and is complex derivative on $\partial\Omega$. Inspired by Faber's article *Über Tschebyscheffsche Polynome* [Crelle.

J. 1920], Carleman proved an asymptotic result in the article *Über die approximation analytischer funktionen durch linearen aggregaten von vorgegebenen potenzen* [Arkiv för matematik och fysik. 1920].

B. Theorem. *There exists a constant C which depends upon Ω only such that to every $n \geq 1$ there is an analytic function $\omega_n(z)$ defined in Ω^* and*

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \cdot \phi'(z) \cdot \phi(z)^n \cdot \big(1 + \omega_n(z)\big) \quad : z \in \partial\Omega$$

where

$$\max_{z \in \partial\Omega} |\omega_n(z)| \leq C \cdot \sqrt{n} \cdot \rho_1^n \quad : \quad n = 1, 2, \ldots$$

Proof.

For each $n \geq 2$ we denote by $\mathcal{M}_n$ the space of monic polynomials of degree n:

$$Q(z) = z^n + b_{n-1}z^{n-1} + \ldots + b_0$$

Put

$$I(Q) = \iint_\Omega |Q(z)|^2 \, dxdy$$

and with n kept fixed we set

$$I_*(n) = \min_{Q \in \mathcal{M}_n} I(Q)$$

To each Q we introduce the primitive polynomial

$$\widehat{Q}(z) = \frac{z^{n+1}}{n+1} + \frac{b_{n-1}}{n} z^n + \ldots + b_0 z$$

1. Exercise. Use Green's formula to show that

$$I(Q) = \frac{1}{4} \int_{\partial\Omega} |\partial_n(\widehat{Q})|^2 \, ds$$

where ds is the arc-length measure on $\partial\Omega$ and we have taken the outer normal derivative of $\widehat{Q}$. Next, take the inverse conformal map $\psi(\zeta)$ in (*) and set

$$F(\zeta) = \widehat{Q}(\psi(\zeta))$$

Then F is analytic in the exterior disc $|\zeta| > 1$ and by (*) above, F has a series expansion

$$F(\zeta) = \tau^{n+1} \Big[\frac{\zeta^{n+1}}{n+1} + A_n \zeta^n + \ldots + A_1 \zeta + A_0 + \sum_{\nu=1}^{\infty} \alpha_\nu \cdot \zeta^{-\nu}\Big] \tag{1.1}$$

2. Exercise. Use a variable substitution via ψ to show that

$$I(Q) = \int_{|\zeta|=1} \frac{d}{dr} (|F(e^{i\theta})|^2) \, d\theta$$

Show also that the series expansion (1.1) identifies the right hand side with

$$\pi \cdot \tau^{2n+2} \cdot \Big[\frac{1}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 - \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \Big] \tag{2.1}$$

3. An upper bound for $I_*(n)$. In (2.1) the coefficients $A_1, \ldots, A_n$ are determined via Q and the reader may verify that there exists $Q \in \mathcal{M}_n$ such that $A_1 = \ldots = A_n = 0$. It follows that

$$I_*(n) \leq \pi \cdot \tau^{2n+2} \cdot \Big[\frac{1}{n+1} - \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \Big] \leq \pi \cdot \tau^{2n+2} \cdot \frac{1}{n+1} \tag{3.1}$$

4. A lower bound for I_*. The upper bound (3.1) did not use that $\partial\Omega$ is real- analytic, i.e. (3.1) is valid for every Jordan domain whose boundary curve is of class C^1. To get a lower bound we use the constant $\rho - 1 < 1$ from the above and choose $\rho_1 < \rho < 1$. Now ψ maps the exterior disc

$|\zeta| > \rho$ conformally to an exterior domain $U^* = \Sigma \setminus \bar{U}$ where U is a relatively compact Jordan domain inside Ω. Choose $Q_n \in \mathcal{M}_n$ so that

$$I(Q_n) = I_*(n)$$

Since $\Omega \setminus \bar{U} \subset \Omega$ we have

$$I_* > \iint_{\Omega \setminus \bar{U}} |Q_n(z)|^2 \, dx dx \tag{4.1}$$

5. Exercise. Show that (4.1) is equal to

$$\int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot d\theta - \int_{|\zeta|=\rho} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot \rho \cdot d\theta =$$

$$\pi \cdot \tau^{2n+2} \cdot [\frac{1-\rho^{2n+2}}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1)\,]$$

and conclude that one has the lower bound

$$I_*(n) \geq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot (1-\rho^{2n+2}) \tag{5.1}$$

Together (4.1) and (5.1) give the inequality

$$\sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1) \leq \frac{\pi}{n+1} \cdot \rho^{2n+2} \tag{5.2}$$

Since $1-\rho^2 \leq 1-\rho^{2\nu}$ for every $\nu \geq 1$ it follows that

$$\sum_{k=1}^{k=n} k \cdot |A_k|^2 + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \leq \frac{\pi}{(1-\rho^2) \cdot n+1} \cdot \rho^{2n+2} \tag{5.3}$$

6. Conclusion. Recall that $F(\zeta) = \widehat{Q_n}(\psi(\zeta)$. So after a derivation we get

$$F'(\zeta) = \psi'(\zeta) \cdot Q_n(\psi(\zeta))$$

Hence the series expansion of $F(\zeta)$ gives

$$Q_n(\psi(\zeta)) = \frac{\tau^{n+1}}{\psi'(\zeta)} \cdot \big[\zeta^n + \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}\,\big] \tag{6.1}$$

where the equality holds for $|\zeta| > \rho$. Put

$$\omega^*(\zeta) = \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}$$

When $|\zeta| = 1$ the triangle inequality gives

$$|\omega^*(\zeta)| \leq \sum_{k=1}^{k=n} k \cdot |A_k| + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu| \tag{6.2}$$

7. Exercise. Notice that (5.3) holds for every $\rho > \rho_1$ and use this together with suitable Cauchy-Schwarz inequalities to show that (6.1) gives a constant C which is independent of n such that

$$|\omega^*(\zeta)| \leq C \cdot \sqrt{n} \cdot \rho_1^{n+1} \tag{7.1}$$

Final part of the proof. Since ψ is the inverse of ϕ we have

$$\psi'(\phi(z)) \cdot Q_n(\psi(\phi(z) = \frac{Q_n(z)}{\phi'(z)}$$

Define the function on $\partial\Omega$ by

(i) $$\omega_n(z) = \frac{\omega^*(\phi(z))}{\phi'(z)}$$

Then (6.2) gives

(ii) $$Q_n(z) = \tau^{n+1} \cdot \phi'(z) \cdot [\phi(z)^n + \omega_n(z)\,]$$

where Exercise 7 shows that $|\omega_n(z)|$ satisfies the estimate in Theorem 2. Finally, the polynomial Q_n minimized the L^2-norm under the constraint that the leading term is z^n and for this variational problem the upper and the lower bounds in (4.1-5.1) imply that

$$|I_*(n) - \frac{\pi}{n+1} \cdot \tau^{2n+2}| \leq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot \rho^{2n+2}$$

If we normalise Q_n so that its L^2-norm is one gets a polynomial $P_n(z)$ where the factor τ^{n+1} is replaced by $\frac{\sqrt{n+1}}{\sqrt{\pi}}$ which finishes the proof of Theorem B.

5. Partial sums of Fourier series.

Let $f(x)$ be a continuous function on $0 \leq x \leq 2\pi$ with $f(0) = f(2\pi)$ and consider the Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

Gibbs gave examples which show that the partial sums

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{k=n} a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

can fail to converge to $f(x)$ for certain x-values. To analyze the situation we introduce the maximum norms

$$\rho_f(n) = \max_{0\leq x\leq 2\pi} |s_n(x) - f(x)| \quad : n = 1, 2, \ldots$$

In general they behave in an irregular fashion as $n \to +\infty$ and Hardy constructed examples of continuous functions f such that

$$\limsup_{n\to\infty} \rho_f(n) = +\infty$$

This led Carleman to study the behaviour in the mean. For each $n \geq 1$ we set

$$\mathcal{C}_n(f) = \sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} \rho_f(\nu)^2}$$

Next, put

$$\omega_f(\delta) = \max_{x,y} |f(x) - f(y)| : |x - y| \leq \delta$$

Since f is uniformly continuous $\omega_f(\delta) \to 0$ as $\delta \to 0$. The following result is proved in the article *A theorem concerninig Fourier series* [London math.soc. 1923]:

5.1 Theorem. *There exists an absolute constant K such that the following hold for every continuous function f with maximum norm ≤ 1:*

$$\mathcal{D}_n(f) \leq K \cdot [\frac{1}{\sqrt{n}} + \omega_f(\frac{1}{n})] \quad : \; n = 1, 2, \ldots$$

Remark. This result shows that Gibbs' phenomenon for a given continuous function f only arises for a sparse sequence of n-values. In the joint article *Fourier's series and analytic functions* [Proc. of the Royal Soc. 1923] with Hardy, pointwise convergence was studied where the limit is unrestricted as $n \to \infty$. However, these results were rather incomplete and it was not until 1965 when Lennart Carleson proved that for every continuous function $f(x)$ there exists a null set $\mathcal{N}$ in $[0,2\pi]$ such that

$$(*) \qquad \lim_{n\to\infty} s_n(x) = f(x) \quad : \; x \in [0,2\pi] \setminus \mathcal{N}$$

Carleson's result constitutes one of the greatest achievements ever in mathematical analysis and we shall not try to enter details of the proof. Recall that Carleson also proved the almost everywhere convergence when $f \in L^2[0,2\pi]$. When pointwise convergence holds in (*) is obvious that

$$(**) \qquad \lim_{n\to\infty} \sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} |s_n(x) - f(x)|^2 = 0}$$

Here (**) is much weaker than (*). On the other hand, the null-set which appears in (*) for a given L^2-function f is in general strictly larger than the set of Lebesgue points for f, i.e. those x for which

$$f(x) = \lim_{\delta\to 0} \frac{1}{2\delta} \cdot \int_{x-\delta}^{x+\delta} |f(s) - f(x)|\, ds = 0$$

In the cited article from 1922, Carleman proved that (**) holds for every Lebesgue point of the L^2-function f which for "ugly Lebegue points" which belong to $\mathcal{N}$ gives an averaged substitute to Carleson's result. Carleman's proof of the theorem above employs some special estimates of the Dini kernel which have independent interest.

5.2 A convergence result for L^2-functions. Let $f(x)$ be a real-valued and square integrable function on $(-\pi,\pi)$, i.e.

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx < \infty$$

We say that f has a determined value $A = f(0)$ at $x = 0$ if the following two conditions hold:

$$\text{(i)} \qquad \lim_{\delta\to 0} \frac{1}{\delta} \cdot \int_0^{\delta} |f(x) + f(-x) - 2A|\, dx = 0$$

$$\text{(ii)} \qquad \int_0^{\delta} |f(x) + f(-x) - 2A|^2\, dx \le C \cdot \delta \quad \text{holds for some constant} \quad C$$

Remark. In the same way we can impose this condition at every point $-\pi < x_0 < \pi$. To simplify the subsequent notations we take $x = 0$. If $x = 0$ is a Lebesgue point for f and A the Lebesgue value we have (i). Hence Lebesgue's Theorem entails that (i) holds almost everywhere when $x = 0$ is replaced by other points x_0. We leave it to the reader to show that the second condition also is valid almost everywhere when f is square integrable. Next, expand f in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

and with $x = 0$ we consider the partial sums

$$s_n(0) = \frac{a_0}{2} + a_1 + \ldots + a_n + b_1 + \ldots + b_n$$

The result below is proved in Carelan's cited article.

5.3 Theorem. *Assume that f has a determined value A at $x = 0$. Then the following hold for every positive integer k*

$$(*) \qquad \lim_{n\to\infty} \frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} |s_\nu - A|^k = 0$$

Proof. Set $A = f(0)$ and $s_n = s_n(0)$. Introduce the function:

$$\phi(x) = f(x) + f(-x) - 2A \implies$$

$$(0) \qquad s_n - A = \int_0^{\pi} \frac{\sin(n+1/2)x}{\sin x/2} \cdot \phi(x) \cdot dx$$

where Dini's kernel was used. By trigonometric formulas (0) is expressed by three terms for each $0 < \delta < \pi$:

$$\alpha_n = \frac{1}{\pi} \cdot \int_0^\delta \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx \tag{1}$$

$$\beta_n = \frac{1}{\pi} \cdot \int_\delta^\pi \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx \tag{2}$$

$$\gamma_n = \frac{1}{\pi} \cdot \int_0^\pi \cos nx \cdot \phi(x) \cdot dx \tag{3}$$

By Hölder's inequality it suffices to show Theorem 5.3 if $k = 2p$ is an even integer. Minkowski's inequality gives

$$\Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq \Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p} \tag{4}$$

Denote by $o(\delta)$ small ordo and $O(\delta)$ is big ordo. When $\delta \to 0$ we shall establish the following:

$$\Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} = n^{1+1/2p)} \cdot o(\delta) \tag{i}$$

$$\Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} \leq K \cdot p \cdot \delta^{-1/2p} \tag{ii}$$

$$\Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p} \leq K \tag{iii}$$

In (ii-iii) K is an absolute constant which is independent of p, n and δ. Let us first see why (i-iii) give Theorem 5.3. Write $o(\delta) = \epsilon(\delta) \cdot \delta$ where $\epsilon(\delta) \to 0$. With these notations (4) gives:

$$\Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq n^{1+1/2p} \cdot \delta \cdot \epsilon(\delta) + Kp \cdot \delta^{-1/2p} + K \tag{5}$$

Next, let $\rho > 0$ and choose b so large that

$$pKb^{-1/2p} < \rho/3$$

Take $\delta = b/n$ and with n large it follows that $\epsilon(\delta)$ is so small that

$$b \cdot \epsilon(b/n) < \rho/3$$

Then right hand side in (5) is majorized by

$$\frac{2\rho}{3} \cdot n^{1/2p} + K$$

When n is large we also have

$$K \leq \frac{\rho}{3} \cdot n^{1/2p}$$

Hence the left hand side in (*) is majorized by $\rho \cdot n^{1/2p}$ for all sufficiently large n. Since $\rho > 0$ was arbitrary we get Theorem 5.3 when the power is raised by $2p$.

Proof of (i-iii)

To obtain (i) we use the triangle inequality which gives the following for every integer $\nu \geq 1$:

$$|a_\nu| \leq \frac{2}{\pi} \cdot \max_{0 \leq x \leq \delta} |\sin \nu x \cdot \cot x/2| \cdot \int_0^\delta |\phi(x)|\, dx = \nu \cdot o(\delta)$$

where the small ordo δ-term comes from the hypothesis expressed by (*) in the introduction. Hence the left hand side above is majorized by

$$[\sum_{\nu=1}^{\nu=n} \nu^{2p}\,]^{\frac{1}{2p}} \cdot o(\delta) = n^{1+1/2p} \cdot o(\delta)$$

which was requested to get (i). To prove (iii) we notice that

$$\gamma_0^2 + 2 \cdot \sum_{\nu=1}^{\infty} \gamma_\nu^2 = \frac{1}{\pi} \int_0^\pi |\phi(x)|^2\, dx$$

Next, we have

$$\sum_{\nu=1}^{\infty} |\gamma_\nu|^{2p} \leq [\sum_{\nu=1}^{\infty} |\gamma_\nu|^2\,]^{1/2p} \leq K$$

where K exists since ϕ is square-intergable on $[0, \pi]$ and (iii) follows.

Proof of (ii). Here several steps are required. For each $0 < s < \pi$ we define the function $\phi_s(x)$ by

$$\phi_s(x) = \phi(x) \quad : \quad 0 < x < s$$

and extend it to an odd function, i.e. $\phi_s(-x) = -\phi_s(x)$ while $\phi_s(x) = 0$ when $|x| > s$. This odd function has a sine series

(A) $$\phi_s(x) = \sum_{\nu=1}^{\infty} c_\nu(s) \cdot \sin x$$

Let us also introduce the functions

(B) $$\rho(s) = \int_0^s |\phi(x)| \cdot dx \quad \text{and} \quad \Theta(s) = \int_0^s |\phi(x)|^2 \cdot dx$$

The crucial step towards the proof of (ii) is the following:

Sublemma. *One has the inequality*

$$\sum_{|nu=1}^{\infty} |c_\nu(s)|^{2p} \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot \rho(s)^{2p-2}$$

Proof. We employ convolutions and define inductively a sequence of functions $\{\phi_{n,s}(x)\}$ where $\phi_{1,s}(x) = \phi_s(x)$ and

$$\phi_{n+1,s}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_{n,s}(y)\phi_s(x+y) \cdot dy$$

Since convolution yield products of the Fourier coefficients and $2p$ is an even integer we have the standard formula:

(1) $$\sum_{\nu=1}^{\infty} c_n(s)^{2p} = \phi_{2p,s}(0)$$

Next, using the Cauchy-Schwarz inequality the reader may verify that

$$|\phi_{2,s}(x)| \leq \frac{2}{\pi} \cdot \Theta(x)$$

This entails that

$$\phi_{3,s}(x) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |\phi_{2,s}(y)| \cdot |\phi_s(x+y)| \cdot dy \leq \frac{2}{\pi^2} \cdot \Theta(s) \cdot \int_{-\pi}^{\pi} |\phi_s(x+y)| \cdot dy = (\frac{2}{\pi})^2 \cdot \Theta(s) \cdot \rho(s)$$

Proceeding in this way it follows by an induction that

$$\phi_{2p,s}(x) \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot (\rho(s))^{2p-2}$$

This holds in particular when $x = 0$ and then (1) above gives the sublemma.

Proof continued. We have by definition

$$\beta_\nu = \frac{2}{\pi}\int_\delta^\pi \sin \nu x \cdot \frac{1}{2}\cot(\frac{x}{2}) \cdot \phi(x) \cdot dx$$

An integration by parts and the construction of the Fourier coefficients $\{c_\nu(s)\}$ which applies with $s = \delta$ give:

$$\text{(C)} \qquad \beta_\nu = -\frac{1}{2} \cdot \cot \delta/2 \cdot c_\nu(\delta) + +\frac{1}{4}\int_\delta^\pi c_\nu(x) \cdot \operatorname{cosec}^2(\frac{x}{2}) \cdot dx$$

Now we profit upon Minkowski's inequality. Let q be the conjugate of $2p$, i.e $\frac{1}{q} + \frac{1}{2p} = 1$ and choose $\{\xi_\nu\}$ to be the sequence in ℓ^q of unit norm such that

$$|\sum \xi_\nu \cdot \beta_\nu| = ||\beta_\bullet||_{2p}$$

where the last term is the left hand side in (ii). At the same time (*) above and the triangle inequality give

$$||\beta_\bullet||_{2p} \leq -\frac{1}{2} \cdot \cot(\delta/2) \cdot \sum |c_\nu(\delta)| \cdot |\xi_\nu| + \frac{1}{4}\int_\delta^\pi \operatorname{cosec}^2(\frac{x}{2}) \cdot \sum |c_\nu(x) \cdot \xi_\nu| \cdot dx \leq$$

$$\text{(D)} \qquad \frac{1}{2} \cdot \cot(\delta/2) \cdot ||c_\bullet(\delta)||_{2p} + \frac{1}{4}\int_\delta^\pi \operatorname{cosec}^2(\frac{x}{2}) \cdot ||c_\bullet(x)||_{2p} \cdot dx$$

At this stage we apply the sublemma and the assumption which give a constant K such that

$$\text{(E)} \qquad \Theta(s) \leq K \quad \text{and} \quad \rho(s) \leq K \cdot s$$

The last estimate actually is weaker than the hypothesis but it will be sufficient to get the requested estimate of the ℓ^{2p}-norm in (ii). Namely, the sublemma gives a constant K_1 such that

$$||c_\bullet(\delta)||_{2p} \leq K_1 \cdot \delta^{1-1/p}$$

At the same time we have a constant K_2 such that

$$\cot(\delta/2) \leq \frac{K_2}{\delta}$$

The product in the first term from (D) is therefore majorized by $K_1 K_2 \cdot \delta^{-1/2p}$ as requested in (ii). For the second term we use again the sublemma which first gives

$$||c_\bullet(x)||_p \leq K \cdot x^{-1/2p}$$

At this stage we leave it to the reader to verify that we get a constant K so that

$$\int_\delta^\pi x^{-1/2p} \cdot \operatorname{cosec}^2(\frac{x}{2}) \cdot dx \leq K \cdot \delta^{-1/2p}$$

which finishes the proof of (ii).

The case when f is continuous. Under the normalisation that the L^2-integral of f is ≤ 1 the inequalities (ii-iii) hold for an absolute constant K. In (i) we notice that the construction of ϕ and the definition of ω_f give the estimates

$$|a_\nu| \leq \nu \cdot \delta \cdot \omega_f(\delta)$$

With $p = 2$ the proof of Theorem 5.3 entails that (i) is majorised by

$$n^{1+1/2} \cdot \delta \cdot \omega_f(\delta)$$

This holds for every $0 \leq x \leq 2\pi$ and from the previous proof we conclude that the following hold for each $n \geq 2$ and every $0 < \delta < \pi$:

$$\text{(i)} \qquad \mathcal{D}_n(f) \leq \frac{1}{\sqrt{n+1}} \cdot [n^{1+1/2} \cdot \delta \cdot \omega_f(\delta) + 2K\delta^{-1/2} + K]$$

With $n \geq 2$ we take $\delta = n^{-1}$ and see that (i) gives a requested constnt in Theorem 5.1

6. The Denjoy conjecture

Introduction. Let ρ be a positive integer and $f(z)$ is an entire function such that there exists some $0 < \epsilon < 1/2$ and a constant A_ϵ such that

$$|f(z)| \leq A_\epsilon \cdot e^{|z|^{\rho+\epsilon}} \tag{0.1}$$

hold for every z. Then we say that f has integral order $\leq \rho$. Next, the entire function f has an asymptotic value a if there exists a Jordan curve Γ parametrized by $t \mapsto \gamma(t)$ for $t \geq 0$ such that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and

$$\lim_{t \to +\infty} f(\gamma(t)) = a \tag{0.2}$$

In 1920 Denjoy raised the conjecture that (0.1) implies that the entire function f has at most 2ρ many different asymptotic values. Examples show that this upper bound is sharp. The Denjoy conjecture was proved in 1930 by Ahlfors in [Ahl]. A few years later T. Carleman found an alternative proof based upon a certain differential inequality. Theorem A.3 below has applications beyond the proof of the Denjoy conjecture for estimates of harmonic measures. See [Ga-Marsh].

A. The differential inequality.

Let Ω be a connected open set in $\mathbf{C}$ whose intersection S_x between a vertical line $\{\mathfrak{Re}\, z = x\}$ is a bounded set on the real y-line for every x. When $S_x \neq \emptyset$ it is the disjoint union of open intervals $\{(a_\nu, b_\nu)\}$ and we set

$$\ell(x) = \max_\nu \, (b_\nu - a_\nu) \tag{*}$$

Next, let $u(x,y)$ be a positive harmonic function in Ω which extends to a continuous function on the closure $\bar{\Omega}$ with the boundary values identical to zero. Define the function ϕ by:

$$\phi(x) = \int_{S_x} u^2(x,y) \cdot dy \tag{1}$$

The Federer-Stokes theorem gives the following formula for the derivatives of ϕ:

$$\phi'(x) = 2 \int_{S_x} u_x \cdot u(x,y) dy \tag{2}$$

$$\phi''(x) = 2 \int_{S_x} u_{xx} \cdot u(x,y) dy + 2 \int_{S_x} u_x^2 \cdot dy \tag{3}$$

Since $\Delta(u) = 0$ when $u > 0$ we have

$$2 \int_{S_x} u_{xx} \cdot u(x,y) dy = -2 \int_{S_x} u_{yy} \cdot u(x,y) dy = 2 \int u_y^2 dy \tag{4}$$

The Cauchy-Schwarz inequality applied in (2) gives

$$\phi'(x)^2 \leq 4 \cdot \int_{S_x} u_x^2 \cdot \int_{S_x} u^2(x,y) dy = 4 \cdot \phi(x) \cdot \int_{S_x} u_x^2 dy \tag{5}$$

Hence (4) and (5) give:

$$\phi''(x) \geq 2 \int_{S_x} u_y^2(x,y) \cdot dy + \frac{1}{2} \cdot \frac{\phi'^2(x)}{\phi(x)} \tag{6}$$

Next, since $u(x,y) = 0$ at the end-points of all intervals of S_x, *Wirtinger's inequality* and the definition of $\ell(x)$ give:

$$\int_{S_x} u_y^2(x,y) \cdot dy \geq \frac{\pi^2}{\ell(x)^2} \cdot \phi(x) \tag{7}$$

Inserting (7) in (6) we have proved

A.1 Proposition *The ϕ-function satisfies the differential inequality*

$$\phi''(x) \geq \frac{2\pi^2}{\ell(x)^2} \cdot \phi(x) + \frac{\phi'^2(x)}{2\phi(x)}$$

Proof continued. The maximum principle for harmonic functions implies that the $\phi(x) > 0$ when $x > 0$ and hence there exists a ψ-function where $\phi(x) = e^{\psi(x)}$. It follows that

$$\phi' = \psi' e^{\psi} \quad \text{and} \quad \phi'' = \psi'' e^{\psi} + \psi'^2 e^{\psi}$$

Now Proposition A.1 gives

$$(*) \qquad \psi'' + \frac{\psi'^2}{2} \geq \frac{2\pi^2}{\ell(x)^2}$$

A.2 An integral inequality. From (*) we obtain

$$\frac{2\pi}{\ell(x)} \leq \sqrt{\psi'(x)^2 + 2\psi''(x)} \leq \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

Taking the integral we get

$$(**) \qquad 2\pi \cdot \int_0^x \frac{dt}{\ell(t)} \leq \psi(x) + \log \psi'(x) + O(1) \leq \psi(x) +, \psi'(x) + O(1)$$

where $O(1)$ is a remainder term which is bounded independent of x. Taking the integral once more we obtain:

A.3 Theorem. *The following inequality holds:*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + O(x)$$

where the remainder term $O(x)$ is bounded by Cx for a fixed constant.

B. Solution to the Denjoy conjecture

B.1 Theorem. *Let $f(z)$ be entire of some integral order $\rho \geq 1$. Then f has at most 2ρ many different asymptotic values.*

Proof. Suppose f has n different asymptotic values $a_1, \ldots, a_n$. To each a_ν there exists a Jordan arc Γ_ν as described in the introduction. Since the a-values are different the n-tuple of Γ-arcs are separated from each other when $|z|$ is large. So we can find some R such that the arcs are disjoint in the exterior disc $|z| > R$. We may also consider the tail of each arc, i.e. starting from the last point on Γ_ν which intersects the circle $|z| = R$. So now we have an n-tuple of disjoint Jordan curves in $|z| \geq R$ where each curve intersects $|z| = R$ at some point p_ν and after the curves moves to the point at infinity. See figure. Next, we take one of these curves, say Γ_1. Let D_R^* be the exterior disc $|\zeta| > R$. In the domain $\Omega = \mathbf{C} \setminus \Gamma_1 \cup D_R^*$ we can choose a single-valued branch of $\log \zeta$ and with $z = \log \zeta$ the image of Ω is a simply connected domain Ω^* where S_x for each x has length strictly less than 2π The images of the Γ-curves separate Ω^* into n many disjoint connected domains denoted by $D_1, \ldots, D_n$ where each D_ν is bordered by a pair of images of Γ-curves and a portion of the vertical line $x = \log R$.

Let $\zeta = \xi + i\eta$ be the complex coordinate in Ω^*. Here we get the analytic function $F(\zeta)$ where

$$F(\log(z)) = f(z)$$

We notice that F may have more growth than f. Indeed, we get

$$(1) \qquad |F(\xi + i\eta)| \leq \exp\left(e^{(\rho+\epsilon)\xi}\right)$$

With $u = \text{Log}^+ |F|$ it follows that

(2) $$u(\xi,\eta) \leq e^{(\rho+\epsilon)\xi}$$

Hence the ϕ-function constructed during the proof of Theorem A.3 satisfies

$$\phi(\xi) \leq e^{2(\rho+\epsilon)\xi}$$

It follows that the ψ-function satisfies

(3) $$\psi(\xi) = 2 \cdot (\rho+\epsilon)\xi + O(1)$$

Now we apply Theorem A.3 in each region D_ν where we have a function $\ell_\nu(\xi)$ constructed by (0) in section A. This gives the inequality

(4) $$2\pi \cdot \int_R^\xi \frac{\xi - s}{\ell_\nu(s)} \cdot ds \leq \int_R^\xi (\rho+\epsilon)s \cdot ds + (\rho+\epsilon)\xi + O(1) \quad : \quad 1 \leq \nu \leq n$$

Next, recall the elementary inequality which asserts that if $a_1, \ldots, a_n$ is an arbitrary n-tuple of positive numbers then

(5) $$\sum a_\nu \cdot \sum \frac{1}{a_\nu} \geq n^2$$

For each s we apply this to the n-tuple $\{\ell_\nu(s)\}$ where we also have

$$\sum \ell_\nu(s) \leq 2\pi$$

So a summation in (4) over $1 \leq \nu \leq n$ gives

(6) $$n \cdot \int_R^\xi (\xi - s) \cdot ds \leq \int_R^\xi (\rho+\epsilon)s \cdot ds + (\rho+\epsilon)\xi + O(1)$$

Another integration gives:

(7) $$n \cdot \frac{\xi^2}{2} \leq (\rho+\epsilon) \cdot \xi^2 + O(\xi)$$

This inequality can only hold for large ξ if $n \leq 2(\rho+\epsilon)$ and since $\epsilon < 1/2$ is assumed it follows that $n \leq 2\rho$ which finishes the proof of the Denjoy conjecture.

PDE-equations.

Introduction. We expose Carleman's article *Über eine nichtlineare Randwertaufgabe bei der Gleichung* $\Delta u = 0$ (Mathematisches Zeitschrift vol. 9 (1921). Here is the equation to be considered: Let Ω be a bounded domain in $\mathbf{R}^3$ with C^1-boundary and $\mathbf{R}^+$ the non-negative real line where t is the coordinate. Let $F(t,p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$(0{,}1) \qquad t \mapsto F(t,p)$$

is strictly increasing for every fixed $p \in \partial\Omega$ and that $F(0,p) \geq 0$. Moreover,

$$(0.2) \qquad \lim_{t\to\infty} F(t,p) = +\infty$$

holds uniformly with respect to p. For a given point $q_* \in \Omega$ we seek a function $u(x)$ which is harmonic in $\Omega \setminus \{q_*\}$ and at q_* it is locally $\frac{1}{|x-q_*|}$ plus a harmonic function and on $\partial\Omega$ the inner normal derivative $\partial u/\partial n$ satisfies the equation

$$(*) \qquad \frac{\partial u}{\partial n}(p) = F(u(p),p) \quad : p \in \partial\Omega$$

Finally it is required that u extends to a continuous function on $\partial\Omega$.

0.1 Theorem. *For each F as above the boundary value problem has a unique solution.*

Remark. The strategy in Carleman's proof is to consider the family of boundary value problems where we for each $0 \leq h \leq 1$ seek u_h to satisfy

$$(**) \qquad \frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p),p) \quad : p \in \partial\Omega$$

and u_h has the same pole as u above. Before we start to consider the equations in (**) we establish some preliminary results about uniqueness in (*).

A.0. Proof of uniqueness.

Let u_1 and u_2 be two solutions to the equation (*). Then $u_2 - u_1$. is harmonic in Ω and if $u_1 \neq u_2$ we may assume that the maximum of $u_2 - u_1$ is > 0. The maximum is attained at some $p_* \in \partial\Omega$ and the strict maximum principle for harmonic functions gives:

$$\text{(i)} \qquad u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*)$$

for all $x \in \Omega$. With $v = u_2 - u_1$ we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p),p) - F(u_1(p),p)$$

Now (0.1) and (*) entail that $\frac{\partial v}{\partial n}(p_*) > 0$ and since we have taken an inner normal derivative this violates (i) which proves the uniqueness.

A.1 Montone properties.

Let F_1 and F_2 be a pair of functions which both satisfy (0.1) and (0.2) and suppose that

$$F_1(t,p) \leq F_2(t,p) \quad : (t,p) \in \mathbf{R}^+ \times \partial\Omega$$

If u_1, respectively u_2 solve (*) for F_1 and F_2 it follows that $u_2(q) \leq u_1(q)$ for all $q \in \Omega$. To see this we set $v = u_2 - u_1$ which is harmonic in Ω. If $p \in \partial\Omega$ we have

$$\text{(i)} \qquad \frac{\partial v}{\partial n}(p) = F_2(u_2(p),p) - F_1(u_1(p),p)$$

Suppose that the maximum of v is > 0 and taken at some $p_* \in \partial\Omega$. Then

$$\text{(ii)} \qquad u_2(p_*) > u_1(p_*)$$

Next, since v attains the maximum at p_* the inner nortmal derivative $\frac{\partial v}{\partial n}(p_*)$ must be ≤ 0 and (i) gives

$$F_2(u_2(p_*)p_*) > F_2(u_1(p_*), p_*) \geq F_1(u_1(p_*), p_*) \tag{iii}$$

Now we get a contradiction since the hypothesis (0.1) implies that (ii) and (iii) cannot hold at the same time.

A.2. A bound for the maximum norm. Let u be a solution to (*) and M_u denotes the maximum norm of its restriction to $\partial\Omega$. Choose $p^* \in \partial\Omega$ such that

$$u(p^*) = M_u \tag{1}$$

Let G be the Green's function which has a pole at Q_* while $G = 0$ on $\partial\Omega$. Now

$$h = u - M_u - G$$

is a harmonic function in Ω. On the boundary we have $h \leq 0$ and $h(p^*) = 0$. So p^* is a maximum point for this harmonic function in the whole closed domain $\bar{\Omega}$. It follows that

$$\frac{\partial h}{\partial n}(p^*) \leq 0 \implies$$

$$F(u(p^*), p^*) = \frac{\partial u}{\partial n}(p^*) \leq \frac{\partial G}{\partial n}(p^*)$$

Set

$$A^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

Then we have

$$F(M_u, p^*) \leq A^* \tag{*}$$

Hence the assumption (0.2) for F this gives a robust estimate for the maximum norm M_u. Next, let m_u be the minimum of u on $\partial\Omega$ and consider the harmonic function

$$h = u - m_u - G$$

This time $h \geq 0$ on $\partial\Omega$ and if $u(p_*) = m_u$ we have $h(p_*) = 0$ so here p_* is a minimum for h. It follows that

$$\frac{\partial h}{\partial n}(p_*) \geq 0 \implies F(u(p_*), p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$A_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

$$F(m_u, p^*) \geq A_* \tag{**}$$

Remark. Above $0 < A_* < A^*$ are constants which are independent of F. Hence the maximum norms of solutions $u = u_F$ are controlled if the F-functions stay in a family where (0.2) holds uniformly.

B. Neumann's linear equation.

Let $f(p)$ and $W(p)$ be a pair of continuous functions on the boundary $\partial\Omega$ where W is positive, i.e. $W(p) > 0$ for every boundary point. The classical Neumann theorem asserts that there exists a unique function U which is harmonic in Ω, extends to a continuous function on the closed domain and its inner normal derivative satisfies:

$$\partial U/\partial n(p) = W(p) \cdot U(p) + f(p) \quad p \in \partial\Omega \tag{1}$$

The uniqueness is a consequence of Green's formula. For suppose that U_1 and U_2 are two solutions to (1) and set $v = U_1 - U_2$. Since v is harmonic in Ω it follows that:

$$\iiint_\Omega |\nabla(v)|^2 dxdydz + \iint_{\partial\Omega} v \cdot \partial v/\partial n \cdot dS = 0$$

Here $\partial v/\partial n = W(p)v$ and since $W(p) > 0$ holds on $\partial\Omega$ we conclude that v must be identically zero. For the unique solution to (1) some estimates hold. Namely, set

$$M_U = \max_p U(p) \quad \text{and} \quad m_U = \min_p U(p)$$

Since U is harmonic in Ω the the maximum and the minimum are taken on the boundary. If $U(p^*) = M_U$ for some $p^* \in \partial\Omega$ we have $\partial U/\partial n(p^*) \leq 0$. Set

$$W_* = \min_p W(p)$$

By assumption $W_* > 0$ and we get

$$M_U \cdot W(p^*) + f(p^*) = \partial U/\partial n(p^*) \leq 0 \implies M_U \leq \frac{|f|_{\partial\Omega}}{W_*}$$

where $|f|_{\partial\Omega}$ is the maximum norm of f on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial\Omega}}{W_*}$$

Hence the following inequality holds for the the maximum norm $|U|_{\partial\Omega}$:

$$(*) \qquad |U|_{\partial\Omega} \leq \frac{|f|_{\partial\Omega}}{W_*}$$

B.1 Estimates for first order derivatives. Let $p \in \partial\Omega$ and denote by N the inner normal at p. Since $\partial\Omega$ is of class C^1 a sufficiently small line segment from p along N stays in Ω. So at points $q = p + \ell \cdot N$ we can take the directional derivative of U along N_p This gives a function

$$\ell \mapsto \partial U/\partial N(p + \ell \cdot N)$$

Since the boundary is C^1 these functions are defined on a fixed interval $0 \leq \ell \leq \ell^*$ for all p. With these notations there exists a constant B such that

$$(**) \qquad \big|\partial U/\partial N(p + \ell \cdot N)\big| \leq B \cdot ||\partial U/\partial n||_{\partial\Omega} \quad : \ p \in \partial\Omega \ : \ 0 \leq \ell \leq \ell^*$$

where the size of B is controlled by the maximum norm of f on $\partial\Omega$ and the positive constant W_* above.

C. Proof of Theorem 1

It suffices to prove the theorem when $F(t,p)$ is an analytic function with respect to t. For if we have an arbitrary F-function satisfying (0.1) and (0.2), then F can be uniformly approximated by a sequence $\{F_n\}$ of analytic functions which again satisfy (0.1-0.2). If $\{u_n\}$ are the unique solutions to $\{F_n\}$ the estimates in (B) show that there exists a limit function $\lim_{n\to\infty} u_n = u$ where u solves (*) for the given F-function. So from now on we assume that $t \mapsto F(t,p)$ is a real-analytic function on the positive real axis for each $p \in \partial\Omega$ where local power series converge uniformly with respect to p. It remains to prove the existence of a solution to the PDE in (*) above Theorem 1.

C.1 The successive solutions $\{u_h\}$. To each real number $0 \leq h \leq 1$ we seek a solution u_h where

$$(1) \qquad \frac{\partial u_h}{\partial n}(p) = h \cdot F(u_h(p), p) + (1-h) \cdot u_h(p)$$

C.2 The case $h = 0$. The linear Neumann problem

$$\frac{\partial u_0}{\partial n}(p) = u_0(p)$$

has a unique solution given by

$$u_0 = G + \phi$$

where G is Green's function with a pole at Q_* and ϕ is the harmonic function in Ω satisfying the boundary equation

$$\frac{\partial \phi}{\partial n}(p) + \frac{\partial G}{\partial n}(p) = \phi(p) \tag{i}$$

Recall that G is a super-harmonic function in Ω where $\frac{\partial G}{\partial n}$ is a continuous and positive function on $\partial\Omega$. This gives a pair of positive constants $0 < \gamma_* < \gamma^*$ such that

$$\gamma_* \leq \frac{\partial G}{\partial n}(p) \leq \gamma^* \quad : \quad p \in \partial\Omega \tag{ii}$$

Let ϕ attain its maximum at some $p^* \in \partial\Omega$. Then its inner normal derivative at p^* is ≤ 0. Hence (i-ii) and the maximum principle for harmonic functions entail that

$$\max_{p \in \bar{\Omega}} \phi(p) \leq \gamma^*$$

In a similar fashion one proves that

$$\min_{p \in \bar{\Omega}} \phi(p) \geq \gamma_*$$

C.3 Local series expansions. Let $0 \leq h_0 < 1$ and suppose we have found the solution u_{h_0} in (1) above. Set $u_0 = u_{h_0}$ and with $h = h_0 + \alpha$ for some small $\alpha > 0$ we shall find u_h by a series

$$u_h = u_0 + \sum_{\nu=1}^{\infty} \alpha^\nu \cdot u_\nu \tag{C.3.1}$$

The pole at q_* occurs already in u_0 so $u_1, u_2, \ldots$ are harmonic functions in Ω. There remains to determine $\{u_\nu\}$ so that the series (3) converges and u_h yields a solution to (1). We will show that this can be achieved when α is sufficiently small. To begin with the results from § B give positive constants $0 < c_1 < c_2$ such that

$$0 < c_1 \leq u_0(p) \leq c_2 \quad : p \in \partial\Omega \tag{C.3.2}$$

Next, the analyticity of F with respect to t enables us to write:

$$F(u_h(p), p) = F(u_0(p) + \sum_{k=1}^{\infty} c_k(p) \cdot \big[\sum_{\nu=1}^{\infty} \alpha^\nu u_\nu(p)\big]^k \tag{C.3.3}$$

where $\{c_k(p)\}$ are continuous functions on $\partial\Omega$ which appear in an expansion

$$F(u_0(p) + \xi, p) = F(u_0(p), p) + \sum_{k=1}^{\infty} c_k(p) \cdot \xi^k \tag{C.3.4}$$

Here (C.3.2) and the hypothesis on F give a pair of positive constants $\rho > 0$ and C which are independent of h such that

$$\max_{p \in \partial\Omega} |c_k(p)| \leq C \cdot \rho^k \quad : \quad k = 0, 1, \ldots \tag{C.3.5}$$

Moreover, the hypothesis (0.2) from the introduction gives a positive constant C_* which also is independent of h such that

$$\min_{p \in \partial\Omega} |c_1(p)| \geq C_* \tag{C.3.6}$$

D. Inductive solutions. To solve (1) we determine the harmonic functions $\{u_\nu\}$ inductively while α-powers are identified. The linear α-term gives the equation

$$\frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1 - h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p) \tag{D.1}$$

For u_2 we find that

$$\frac{\partial u_2}{\partial n} = \big(1 - h_0 + h_0 c_1(p)\big)u_2 - u_1 + c_1(p)u_1 + c_2(p)u_1^2 \tag{D.2}$$

In general, for $\nu \geq 3$ one has

$$\frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \ldots, u_{\nu-1}, p) \tag{D.3}$$

where $\{R_\nu\}$ are polynomials in the preeceding u-functions whose coefficients are determined via the c-functions above. Notice that (C.3.2) gives a positive constant C_* which again is independent of h such that

$$\min_{p \in \partial\Omega} 1 - h_0 + h_0 \cdot c_1(p) \geq C_* \tag{D.4}$$

Next, the equations in (D.3) can be expressed as follows:

$$\frac{\partial u_m}{\partial n} = \big(1 - h_0 + h_0 \cdot c_1(p)\big) \cdot u_\nu(p) + \alpha \cdot \big\{ \sum_{k=1}^{\infty} c_k(p) \big[\sum \alpha^\nu u_\nu(p) \big]^k \big\}_{m-1} \tag{D.5}$$

where the index $m-1$ indicates that one takes out the coefficient of α^{m-1} when the double sum inside the bracket is expanded as a series in α. Now (D.5) and the estimates for the inhomogeneous linear equation in § B give a constant C^* which is independent of h such that

$$\max_{p \in \Omega} |u_m(p)| \leq \alpha \cdot \max_{p \in \Omega} \big| \big\{ \sum_{k=1}^{\infty} c_k(p) \big[\sum \alpha^\nu u_\nu(p) \big]^k \big\}_{m-1} \big| \leq$$

$$C \cdot \alpha \cdot \sum_{k=1}^{\infty} \rho^k \cdot \max_{p \in \Omega} \big| \big\{ \big[\sum \alpha^\nu u_\nu(p) \big]^k \big\}_{m-1} \big| \tag{D.6}$$

where the last inequality used (C.3.5).

E. Majorant series. Finally, (D.6) and a suitable majorant series give $\alpha_* > 0$ such that the series (C.3.1) converges if $\alpha \leq \alpha_*$ and gives a solution u_h. Moreover, α_* can be chosen independently of h_0 and from this the reader can conclude that thr imductive condstructions enable us to attain a solution with $h = 1$ which proves Theorem 0.1.

Elliptic operators and their eigenvalues.

We expose material from Carleman's article *Über die asymptotische Verteilung der Eigenwerthe partieller Differentialgleichungen* [xxx 1938]. As pointed out by Carleman the proof of Theorem xx below verbatim to give similar asymptotic expansions (but with other contants) for elliptic operators of even order in any number of space variables $x_1, \ldots, x_n$. As pointed out in [Carleman] similar asymptotic formulas as in Theorem 0.B.1 hold for elliptic operators of arbitrary even order $2m$ where m is a positive integer. Here one replaces Newton's fundamental solution for ellpitic second order operators with constant coefficients in § 1 below by those of the canonical construction by Fritz John which are found in a canonical way for elliptic PDE:s with constant coefficients of arbitrary high order. Here we are content to treat the case $n = 3$ and a second order elliptic operator expressed in $\mathbf{R}^3$ with coordinates $x = (x_1, x_2, x_3)$:

$$L = \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} a_{pq}(x) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + a_0(x)$$

The a-functions are real-valued and in the double sum one has the symmetry $a_{pq} = a_{qp}$. Moreover we assume that the a-functions are defined in a neighborhood of the closure of a bounded domain Ω with a C^1-boundary, where $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x \in \Omega$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under this condition a result which goes back to work by Neumann and Poincaré, asserts that there exists a positive constant κ_0 such that if $\kappa \geq \kappa_0$ then the inhomogeneous equation

$$L(u) - \kappa^2 \cdot u = f \quad : f \in L^2(\Omega)$$

has a unique solution u given by a function in Ω which extends to the closure where it is zero on $\partial\Omega$. This gives the Green's function $G(x,y;\kappa)$, which for each $\kappa \geq \kappa_0$ is defined outside the diagonal in $\Omega \times \Omega$ and satisfies the equation

$$\text{(i)} \qquad (L-\kappa^2)(\frac{1}{4\pi} \cdot \int_\Omega G(x,y;\kappa)\, f(y)\, dy) = -f(x) \quad : f \in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

$$\text{(ii)} \qquad f \mapsto \frac{1}{4\pi} \cdot \int_\Omega G(x,y;\kappa)\, f(y)\, dy \quad : f \in L^2(\Omega)$$

is the inverse to the densely defined operator $L - \kappa^2$ on the Hilbert space $L^2(\Omega)$ for each $\kappa \geq \kappa_0$. Now one studies the Green's functions as κ varies. A crucial inequality was established by Carleman in lectures at Institute Mittag-Leffler during the winter semester 1935-36 which goes as follows. Let $B(x) = A^{-1}(x)$ be the inverse of the symmetric and positive A-matrix and $\{b_{pq}(x)\}$ are the elements of $B(x)$. Set

$$\phi(x,\xi) = \sqrt{\sum\sum b_{pq}(x)(\xi_p - x_p)(\xi_q - x_q)}$$

1. Theorem. *For each $x \in \Omega$ there exists a constant $C(x)$ such that*

$$\left|\, G(x,\xi;\kappa_0) - \frac{1}{\phi(x,\xi)\cdot\sqrt{\det A(x)}}\,\right| \leq \frac{C(x)}{|x-\xi|^{\frac{1}{4}}} \quad : \xi \in \Omega$$

Notice that we have not assumed that L is symmetric. So in general $G(x,y;\kappa) \neq G(y,x;\kappa)$ can hold. Using this result Carleman proved the following asymptotic formula for the eigenvalues of L which in the non-symmetric case may be complex.

2. Theorem. *Let $\{\rho_n\}$ be the absolute values of the eigenvalues of L arranged in a non-decreasing order. Then*

$$(*) \qquad \lim_{n\to\infty} \frac{\rho_n}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_\Omega \frac{1}{\sqrt{\det A(x)}}\, dx$$

Remark. The formula above is due to Courant and Weyl when P is symmetric and extended to non-symmetric operators in the cited article above based upon Carleman 's lectures at Institute Mittag-Leffler in 1935. The proof of Theorem 1 relies upon the construction of fundamental solutions which is given in § 1 below. After this has been achieved, the asymptotic formula (*) in Theorem 2 is derived via Tauberian theorems for Dirichlet series which goes as follows: Let $\{a_\nu\}$ and $\lambda_\nu\}$ be two sequences of positive numbers where $\lambda_\nu \to +\infty$ and the series

$$f(x) = \sum_{\nu=1}^\infty \frac{a_\nu}{\lambda_\nu + x}$$

converges when $x > x_*$ for some positive number x_*. Next, for every $x > 0$ we define the function

$$\mathcal{A}(x) = \sum_{\{\lambda_\nu < x\}} a_\nu$$

In other words, with $x > 0$ we find the largest integer $\nu(x)$ such that $\lambda - \nu(x) < x$ and then $\mathcal{A}(x)$ is the sum over the a-numbers up to this index. With these notations the following implication holds for every pair $A > 0$ and $0 < \alpha < 1$

3. Theorem. *Suppose there exists a constant $A > 0$ and some $0 < \alpha < 1$ such that*

$$\lim_{x\to\infty} x^\alpha \cdot f(x) = A \implies \lim_{x\to\infty} \mathcal{A}(x) = \frac{A}{\pi} \cdot \frac{\sin \pi\alpha}{1-\alpha} \cdot x^{1-\alpha}$$

0 Preliminary constructions.

We are given an ellipitic operator L as above and assume that the coefficients are defined in the whole space $\mathbf{R}^3$. To ensure convergence of volume integrals taken over the whole of $\mathbf{R}^3$ we add the conditions that

$$\lim_{|x|\to\infty} a_{pp}(x) = 1 \quad : \quad 1 \leq p \leq 3$$

while $\{a_{pq}\}$ for $p \neq q$ and a_1, a_2, a_3, b tend to zero as $|x| \to +\infty$. This means that P approaches the Laplace operator when $|x|$ is large. Let us recall the notion of a fundamental solution which prior to the general notion of distributions introduced by L. Schwartz, was referred to as a *Grundlösung*. First, the regularity of the coefficients of a PDE-operator P enable us to construct the adjoint operator:

$$P^*(x, \partial_x) = P - 2 \cdot (\sum_{p=1}^{p=3} (\sum_{q=1}^{q=3} \frac{\partial a_{pq}}{\partial x_q}) \cdot \frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3} \frac{\partial a_p}{\partial x_p} + 2 \cdot \sum\sum \frac{\partial^2 a_{pq}}{\partial x_p \partial x_q}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ, ψ in $\mathbf{R}^3$ with compact support:

$$\int P(\phi) \cdot \psi \, dx = \int \phi \cdot P^*(\psi) \, dx$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $P(x, \partial_x)$ if

$$\psi(0) = \int \Phi \cdot P^*(\psi) \, dx$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $P - \kappa^2$ and a function $\Phi(x; \kappa)$ is a fundamental solution to $P - \kappa^2$ if

$$\psi(0) = \int \Phi(x : \kappa) \cdot (P^* - \kappa^2)(\psi(x)) \, dx \tag{1}$$

hold for compactly supported C^2-functions ψ. Above κ appears as an index of Φ, i.e. for each fixed κ we have the locally integrable function $x \mapsto \Phi(x : \kappa)$. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x, \xi; \kappa)$ with the property that

$$\psi(\xi) = \int \Phi(x, \xi : \kappa) \cdot (P^* - \kappa^2)(\psi(x)) \, dx \tag{*}$$

hold for all $\xi \in \mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x, \xi; \kappa)$ is a function of six variables defined in $\mathbf{R}^3 \times \mathbf{R}^3$. Fundamental solutions are in general not unique. However, when P is an elliptic operator as above we shall give an explicit construction of fundamental solutions $\Phi(x, \xi; \kappa)$ for all sufficiently large κ in § 1.

1. The construction of $\Phi(x, \xi; \kappa)$.

1.1 The case when P has constant coefficients. Here the fundamental solution is given by a formula which goes back to Newton's work in his classic text-books from 1666. We have the positive and symmetric 3×3-matrix $A = \{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix and recall that they are found via Cramér's rule:

$$b_{pq} = \frac{A_{pq}}{\Delta}$$

where $\Delta = \det(A)$ and $\{A_{pq}\}$ are the cofactor minors of the A-matrix. Put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq} a_p a_q - b}$$

where κ is chosen so large that the term under the square-root is > 0. Next, define the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p x_q$$

With these notations Newton's fundamental solution taken at $x = 0$ becomes

$$(*) \qquad H(x;\kappa) = \frac{1}{\sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq} a_p \cdot x_q}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ indeed yields a fundamental solution to the PDE-operator $P(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients.

Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\text{(i)} \qquad \alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi) - b(\xi)}$$

Following Newton's construction for the case of constant coefficients we put:

$$\text{(ii)} \qquad H(x,\xi;\kappa) = \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)\cdot x_q}$$

When ξ is kept fixed this function of x is real analytic outside the origin and the singularity at $x = 0$ is of Newton's type. In particular $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. Next, for every fixed ξ we consider the differential operator in the x-space:

$$L_*(x,\partial_x,\xi;\kappa) =$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3}(a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3}(a_p(x) - a_p(\xi))\frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

For each fixed ξ there exists the function $x \mapsto H(x-\xi,\xi;\kappa)$ and we apply the L_*-operator on this x-dependent function and put:

$$\text{(iii)} \qquad F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)$$

1.3 Two estimates. The hypothesis that $\{a_{pq}(x)\}$ are of class C^2 and$\{a_p(x)\}$ of class C^1, together with the limit conditions (*) in § XX give the existence of positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$\text{(1.3.1)} \qquad |H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|}$$

$$\text{(1.3.2)} \qquad [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|^2}}{|x-\xi|^2}$$

The verification is left as an exercise.

1.4 An integral equation. We seek $\Phi(x,\xi;\kappa)$ which solves the equation:

$$\text{(1)} \qquad \Phi(x,\xi;\kappa) = \iiint F(x,y;\kappa) \cdot \Phi(y,\xi;\kappa)\,dy + F(x,\xi;\kappa)$$

where the integral is taken over $\mathbf{R}^3$. To solve (1) we construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$\text{(1.4.1)} \qquad F^{(k)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot F^{(k-1)}(y,\xi;\kappa)\,dy \quad : \quad k \geq 2$$

Then (1.3.2) gives the inequality

(i) $$|F^{(2)}(x,\xi;\kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|^2}}{|x-y|^2 \cdot |\xi-y|^2} \cdot dy$$

To estimate (i) we first notice that the triple integral after the substitution $y-\xi \to u$ becomes

(ii) $$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2 \cdot |u|^2} \cdot du$$

In (ii) the volume integral is integrated in polar coordinates and becomes

(iii) $$C_1^2 \cdot \int_0^\infty \int_{S^2} \frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2} \cdot dwdr$$

where S^2 is the unit sphere and dw the area measure on S^2 and we see that (iii) becomes

$$2\pi C_1^2 \cdot \int_0^\infty \int_0^\pi \frac{e^{-k\kappa r^2}}{(x-\xi)^2 + r^2 - 2r\cdot|x-\xi|\cdot\sin\theta} \cdot d\theta dr =$$

(iv) $$\frac{2\pi C_1^2}{|x-\xi|} \cdot \int_0^\infty e^{-k\kappa|x-\xi|t} \cdot \log|\frac{1+t}{1-t}| \cdot \frac{dt}{t}$$

where the last equality follows by a straightforward computation.

1.5 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)| \leq \frac{2\pi C_1^2 \cdot C_1^*}{\kappa \cdot |x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

(*) $$|F^{(n)}(x,\xi;\kappa)| \leq \frac{C_1}{|x-\xi|^2} \cdot \left[\frac{2\pi C_1^2 \cdot C_1^*}{\kappa}\right]^{n-1} \quad \text{hold for every} \quad n \geq 2$$

1.6 Conclusion. With κ so large that $2\pi C_1^2 \cdot C_1^* < \kappa$ it follows form (*) that the series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $x \neq \xi$ and this Neumann series gives the requested solution $\Phi(x,\xi;\kappa)$. Moreover, $\Phi(x,\xi;\kappa)$ satisfies a similar estimate as in (1.3.2) above with another constant than C_2 instead of C_1.

1.7 Exercise. Above we have found Φ which satisfies the integral equation in § 1.4 Use Green's formula to show that $\Phi(x,\xi;\kappa)$ gives a fundamental solution of $P(x,\partial_x) - \kappa^2$ with a pole at ξ.

1.8 Some final estimates. The constructions above show that the functions

$$x \mapsto \Phi(x,\xi;\kappa) \quad \text{and} \quad x \mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x=\xi$. Consider the difference

(1.8.1) $$\Psi(x,\xi;\kappa) = \Phi(x,\xi;\kappa) - H(x-\xi,\xi;\kappa)$$

1.8.2 Exercise. Use the previous constructions to show that for every $0 < \gamma \leq 2$ there is a constant C_γ such that

$$\big|\Psi(x,\xi;\kappa)\big| \leq \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for every pair (x,ξ) and every $\kappa \geq \kappa_0$. Together with (1.3.1) this gives an estimate for the fundamental solution Φ.

2. Green's functions.

Let Ω be a bounded domain in $\mathbf{R}^3$. A Green's function $G(x, y; \kappa)$ attached to this domain and the PDE-operator $P(x, \partial_x; \kappa)$ is a function which for fixed κ is a function in $\Omega \times \Omega$ with the following properties:

$$(*) \qquad G(x, y; \kappa) = 0 \quad \text{when} \quad x \in \partial\Omega \quad \text{and} \quad y \in \Omega$$

$$(**) \qquad \psi(y) = \int_\Omega (P^*(x, \partial_x) - \kappa^2)(\psi(x)) \cdot G(x, y; \kappa)\, dx \quad : \quad y \in \Omega$$

hold for all C^2-functions ψ with compact support in Ω. To find G we solve Dirchlet problems. With $\xi \in \Omega$ kept fixed one has the continuous function on $\partial\Omega$:

$$x \mapsto \Phi^*(x, \xi; \kappa)$$

Solving Dirchlet's problem gives a unique C^2-function $w(x)$ which satisfies:

$$P(x, \partial_x)(w) + \kappa^2 \cdot w = 0 \quad \text{holds in} \quad \Omega \quad \text{and} \quad w(x) = \Phi(x, \xi; \kappa) \quad : x \in \partial\Omega = 0$$

From the above it is clear that this gives the requested G-function, i.e. one has:

2.1 Proposition. *The the function*

$$G(x, \xi; \kappa) = \Phi(x, \xi; \kappa) - w(x) \quad \text{satisfies} \quad (* - **)$$

Using the estimates for the ϕ-function from § 1 we get estimates for the G-function above. Start with a sufficiently large κ_0 so that $\Phi^*(x, \xi; \kappa_0)$ is a positive function of (x, ξ). Then the following hold:

2.2 Theorem. *One has*

$$G(x, \xi; \kappa_0) = \frac{1}{\sqrt{\Delta(x)} \cdot \sqrt{\Phi(x, \xi; \kappa_0)}} + R(x, \xi)$$

where the remainder function satisfies the following for all pairs (x, ξ) in Ω:

$$|R(x, \xi)| \leq C \cdot |x - \xi|^{-\frac{1}{4}}$$

and the constant C only depends on the domain Ω and the PDE-operator P.

Remark. Above the negative power of $|x - \xi|$ is a fourth-root which means that the remainder term R is more regular compared to the first term which behaves like $|x - \xi|^{-1}$ on the diagonal $x = \xi$.

2.3 Exercise. Prove Theorem 2.3 If necessary, consult [Carleman: page xx-xx9 for details.

2.4 The integral operator $\mathcal{J}$

.

With κ_0 chosen as above we consider the integral operator which sends a function u in Ω to

$$\mathcal{J}_u(x) = \int_\Omega G(x, \xi; \kappa_0) \cdot u(\xi)\, d\xi$$

The construction of the Green's function gives:

$$(2.4.1) \qquad (P - \kappa_0^2)(\mathcal{J}_u)(x) = u(x) \quad : \quad x \in \Omega$$

In other words, if E denotes the identity we have the operator equality

$$(2.4.2) \qquad P(x, \partial_x) \circ \mathcal{J}_u = \kappa_0^2 \cdot \mathcal{J} + E$$

Consider pairs (u, γ) such that

$$(2.4.3) \qquad u(x) + \gamma \cdot \mathcal{J}_u(x) = 0 \quad : \quad x \in \Omega$$

The vanishing from (*) for the G-function implies that $J_u(x) = 0$ on $\partial\Omega$. Hence every u-function which satisfies in (2.4.3) for some constant γ vanishes on $\partial\Omega$. Next, apply P to (2.4.3) and then the operator formula (2.4.2) gives

$$0 = P(u) + \gamma\kappa_0^2 \cdot \mathcal{J}_u + \gamma \cdot u \implies P(u) + (\gamma - \kappa_0^2)u = 0$$

2.4.4 Conclusion. Hence the boundary value problem (*) from 0.B is equivalent to find eigenfunctions of $\mathcal{J}$ via (2.4.3) above.

3. Almost reality of eigenvalues.

Consider the set of eigenvalues λ to (*) in (0.B). Then we have:

3.1 Proposition. *There exist positive constants C_* and c_* such that every eigenvalue λ to (*) in (0.B) satisfies*

$$|\mathfrak{Im}\,\lambda|^2 \leq C_*(\mathfrak{Re}\,\lambda) + c_*)$$

Proof. Let u be an eigenfunction where $P(u) + \lambda \cdot u = 0$. Stokes theorem and the vanishing of $u|\partial\Omega$ give:

$$0 = \int_\Omega \bar{u} \cdot (P + \lambda)(u)\,dx = -\int_\Omega \sum_{p,q} a_{pq}(x) \cdot \frac{\partial u}{\partial x_p}\frac{\partial \bar{u}}{\partial x_q}\,dx + \int_\Omega \bar{u} \cdot (\sum a_p(x)\frac{\partial u}{\partial x_p}\,)\,dx +$$

$$\int_\Omega |u(x)|^2 \cdot b(x)\,dx + \lambda \cdot \int |u(x)|^2\,dx$$

Write $\lambda = \xi + i\eta$. Separating real and imaginary parts we find the two equations:

(i) $$\xi \int |u|^2\,dx = \int \sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} \cdot \frac{\partial \bar{u}}{\partial x_q}\,dx + \int \big(\frac{1}{2} \cdot \sum \frac{\partial a_p}{\partial x_p} - b\big) \cdot |u|^2\,dx$$

(ii) $$\eta \int |u|^2\,dx = \frac{1}{2i} \int \sum a_p\big(u\frac{\partial \bar{u}}{\partial x_p} - \bar{u}\frac{\partial u}{\partial x_p}\,\big)\,dx$$

Set

$$A = \int |u|^2\,dx \quad : \quad B = \int |\nabla(u)|^2\,dx$$

Since P is elliptic there exists a positive constant k such that

$$\sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} > k \cdot |\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants c_1, c_2, c_3 such that

(iii) $$A\xi > c_1 B - c_2 B \quad : \quad A|\eta| < c_3 \cdot \sqrt{AB}$$

Here (iii) implies that $\xi > -c_2$ and the reader can also confirm that

(iv) $$B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A \cdot c_2 \cdot \sqrt{\frac{\xi + c_2}{c_1}} \quad : \quad |\eta| < c_3 \cdot \sqrt{\frac{\xi + c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 3.1.

4. The asymptotic formula.

Using the results above where we have found a good control of the integral operator $\mathcal{J}$ and the identification of eigenvalues to | and those from (*) in (0.B), one can proceed and apply Tauberian theorems to derive the asymptotic formula in Theorem 1 using similar methods as described in § XX where we treated the Laplace operator. For details the reader may consult [Carleman:p age xx-xx].

Unbounded operators and quantum mechanics.

Two years after Carleman's cited monograph published 1923, the mathematical community was confronted with new problems via the new-born quantum mechanics. Some examples are exposed in § 0.3 under the heading *Schrödinger equations*. But let us begin with a question which was raised in a joint article by Born, Heisenberg and Jordan [Zeitschrift der Physik vol 24. 1925]:

Determine pairs of linear operators P and Q on the Hilbert space $L^2(\mathbf{R})$ of complex-valued and square integrable functions satisfying the equation

$$(*) \qquad PQ - QP = \frac{h}{2\pi i}\cdot E$$

where h is a positive constant and E the identity operator.

In 1926 Schrödinger proposed a solution to (*) using the unbounded operators

$$P(f) = \frac{h}{2\pi i}\cdot\frac{df}{dq} \quad : Q(f) = q\cdot f(q)$$

where q is the coordinate on the real line. Here P and Q are densely defined on test-functions and (*) follows from derivation rules. A notable point is that P and Q are symmetric on the subspace of test-functions in the complex Hilbert space $L^2(\mathbf{R})$, i.e. one verifies easily that

$$\int P(f)\cdot\bar{g}\,dq = \int f\cdot\overline{P(g)}\,dq \quad : f,g\in C_0^\infty(\mathbf{R})$$

and similarly for Q.

Weyl's unitary operators. In an article [Zeitschrift für physik: Vol. 46 1928] the search of solutions to (*) was reformulated by Weyl into a problem about unitary operators. Let us first describe this for Schrödinger's solution. If β is a complex number we get the unitary operator defined on $L^2(\mathbf{R})$ by:

$$\text{(i)} \qquad V(\beta)(f)(q) = e^{\frac{2\pi i}{h}\cdot\beta q}\cdot f(q)$$

Then

$$V(\beta_1+\beta_2) = V(\beta_1)V(\beta_2)$$

and we also notice that

$$\lim_{\beta\to 0}\frac{V(\beta)(f)-f}{\beta} = \frac{2\pi i}{h}\cdot Q(f)$$

For the P-operator the inverse Fourier transform gives

$$P(f)(q) = \frac{h}{2\pi i}\cdot\frac{1}{2\pi}\int e^{iq\xi}\cdot(i\xi)\widehat{f}(\xi)\,d\xi$$

Plancherel's formula yields unitary operators for real numbers α defined on test-functions f by

$$U(\alpha)(f) = e^{\frac{2\pi i}{h}\alpha P}(f) = E + \sum_{n=1}^{\infty}\frac{1}{n!}\cdot(\frac{2\pi i}{h})^n\cdot\alpha^n\cdot P^n(f)$$

It means that on the ξ-line the corresponding unitary operator $\widehat{U}(\alpha)$ is multiplication with $e^{i\alpha\xi}$ and hence

$$\text{(ii)} \qquad U(\alpha)(f)(q) = f(q+\alpha)$$

It is clear that (i-ii) give

$$(**) \qquad U(\alpha)V(\beta) = e^{\frac{2\pi i}{h}\cdot\alpha\beta}\cdot V(\beta)U(\alpha)$$

One is therefore led to determine pairs of unitary groups $\{U(\alpha)\}$ and $\{V(\beta)\}$ for which (**) hold. It turns out that Schrödinger's solution is unique up to unitary equivalence under the assumption that one has an irreducible situation. More precisely one has:

Theorem. *Let $\{U_*(\alpha)\}$ and $\{V_*(\beta)\}$ be a pair of unitary groups which satisfies (**). Suppose in addition that the sole non-zero closed subspace of $L^2(\mathbf{R})$ which is invariant under all U_*- and V_*-operators is $L^2(\mathbf{R})$. Then there exists a unitary operator S on $L^2(\mathbf{R})$ such that*

$$U_*(\alpha) = SU(\alpha)S^* \quad : V_*(\beta) = SV(\beta)S^*$$

where the pair $\{U(\alpha)\}, \{V(\beta)\}$ is Schrödinger's solution.

Remark. This theorem was proved by J. von Neumann in the article *Die eindutigkeit der Schrödingerschen Operatoren* [xxx: 1934]. Actually von Neumann went further and proved a general result for pairs of unitary groups which need not be irreducible. In this case decompositions into invariant and irreducible subspaces of $L^2(\mathbf{R})$ appear. More generally one can consider n-tuples of pairs satisfying (*) for $n \geq 2$ where the equations are related to those in the n-dimensional Weyl algebra of differential operatos with polynomial coefficients which leads to the theory about von Neumann algebras. For this we refer to articles by von Neumann which illustrate the importance of studying unbounded self-adjoint operators.

Poincaré's Ergodic Hypothesis

A problem which played a crucial role while the theory about unbounded linear operators was developed stems from the lecture *L'avénir des Mathematiques* held at the congress in Rome 1908 by Poincaré. He put forward that non-linear systems of differential equations can be expressed by an infinite system of equations with constant coefficients. Consider the case where $n \geq 2$ and $A_1(x), \ldots, A_n(x)$ are polynomials in $x_1, \ldots, x_n$ and the non-linear system

$$\frac{dx_\nu}{dt} = A_\nu(x_1, \ldots, x_n) \tag{1}$$

where t is a time variabe. Taking monomials in the x-variabes give further equations. For example

$$\frac{d}{dt}(x_\nu \cdot x_j) = A_\nu(x_1, \ldots, x_n) \cdot x_j + A_j(x_1, \ldots, x_n) \cdot x_\nu$$

In this way (1) can be expressed via a system of linear equations with constant coefficients in several ways. Following Poincaré we take a positive function $\mu(x)$ in $\mathbf{R}^n$ satisfying a growth condition

$$\mu(x) \leq e^{\alpha \cdot |x|^2}$$

for some $0 < \alpha < 1$ where $|x|^2 = x_1^2 + \ldots + x_n^2$. The polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$ has a basis formed by polynomials $\{P_\nu(x)\}$ such that the integrals

$$\int_{\mathbf{R}^n} P_\nu(x) \cdot P_k(x) \cdot e^{-2|x|^2} \cdot \mu(x)\, dx = \delta(\nu, k)$$

where the right hand side is Kronecker's delta-function. With $\phi_\nu(x) = P_\nu(x) \cdot e^{-|x|^2}$ the ϕ-functions satisfy an infinite system

$$\frac{d\phi_\nu}{dt} = \sum c_{\nu,j} \cdot \phi_j \tag{2}$$

where $\{c_{\nu j}\}$ are constants and for each fixed ν one has $c_{\nu j} \neq 0$ for finitely many j. A case of special interest occurs when solutions to (1) stay in a compact hypersurface S in $\mathbf{R}^n$. Picard's uniqueness theorem for ODE-equations give transformations $\{T_t\}$ from S onto itself where a point $p \in S$ is mapped to $T_t(P)$, determined by the values at time t taken by a solution to (1) with initial condition $x(0) = p$. Suppose in addition that there exists a positive measure μ on S which is invariant for the T-operators. In most applications μ is absolutely continuous with respect to the surface measure $d\sigma$ on S and by a suitable change of variables one can rewrite the original system (1) so that $d\sigma$ is T-invariant. When these conditions hold we consider a pair U, V of complex-valued C^1-functions on S. The invariance entails that

$$t \mapsto \int_S U(T_t(p)) \cdot \overline{V(T_t(p))}\, d\sigma(p)$$

is a constant function of t. Taking the derivative at $t=0$ we get

(i) $$\int_S \sum_{j=1}^{\text{J}=n} A_j(p)\cdot\frac{\partial U}{\partial x_j}(p)\cdot\overline{Vp)}\,d\sigma(p)+\int_S U(p)\cdot\sum_{j=1}^{\text{J}=n} A_j(p)\cdot\overline{\frac{\partial V}{\partial x_j}}(p)\,d\sigma(p)$$

Define the operator Ω on the vector space $C^1(S)$ by

$$\Omega(U)(p)=i\cdot\sum_{j=1}^{\text{J}=n} A_j(p)\cdot\frac{\partial U}{\partial x_j}(p)$$

Then (i) entails that

(ii) $$\langle\Omega(U),V\rangle=\int_S \Omega(U)\cdot\overline{V}\,d\sigma=\langle U,\Omega(V)\rangle$$

Hence Ω is a densely defined and Hermitian operator on the Hilbert space $L^2(S)$.

Remark. The first "serious attempt" to apply Poincaré's device in a general set-up was done by Fredholm in a Comptes Rendus Note from August 1920. He pointed out new obstacles compared to previous studies of integral equations, caused by the "inevetible unboundedness" of Poincaré's associated linear systems. Fredholm's article was a major inspiration when Carleman developed the theory in [1923]. Concerning the specific case above where the invariant surface measure σ exists, the theory about unbounded densely defined Hermitian operators were later used to confirm Poincaré's Ergodic Hypothesis. We shall not enter a detailed discussion about this but mention that the first rigorous proof was given by Carleman in lectures at Mittag-Leffler Institute in 1931. See the article *Application de la théorie des l'équations intègrales linéaires aux systèmes d'équations differéntielles non-linéaires* [Acta Math.1932] for an account. To this one should add that Birkhoff soon after Carelan's lectures in the spring 1931, established deeper results which lead to almost everywhere properties in the ergodic theorem. His article [Birk] constitutes the fundamental account which has become a very extensive subject dealing with ergodic properties and the search for invariant measures in different contexts.

§ 0. About the spectral theorem for self-adjoint operators

A separable Hilbert space is isomorphic to ℓ^2 whose vectors are sequences of complex numbers $\{x_p\}$ indexed by integers where $\sum |x_p|^2 < \infty$. A doubly indexed sequence $\{c_{pq}\}$ is Hermitian if:

$$c_{q,p} = \bar{c}_{p,q}$$

We impose the condition that each column of this infinite matrix belongs to ℓ^2, i.e.

$$\sum_{q=0}^{\infty} |c_{pq}|^2 < \infty \quad : p = 1, 2, \ldots \tag{1}$$

The Cauchy-Schwarz inequality entails that if $x \in \ell^2$ then the series

$$\sum_{q=0}^{\infty} c_{p,q} \cdot x_q \tag{2}$$

converges absolutely for each p and let y_p denote the sum. Condition (1) does not imply that $\{y_p\}$ belongs to ℓ^2. So we get a subspace $\mathcal{D}$ of ℓ^2 which consists of vectors x such that

$$\sum_{p=0}^{\infty} \Big|\sum_{q=0}^{\infty} c_{p,q} \cdot x_q\Big|^2 < \infty \tag{3}$$

If q is fixed the Hermitian condition gives

$$\sum_{p=0}^{\infty} |c_{pq}|^2 = \sum_{p=0}^{\infty} |c_{qp}|^2$$

and the last sum is finite by (1). Hence $\mathcal{D}$ contains the ℓ^2-vector e_q where $x_q = 1$ while $x_\nu = 0$ for $\nu \neq q$. Taking linear combinations, $\mathcal{D}$ contains the subspace ℓ^2_* of vectors x for which $x_\nu \neq 0$ for a finite set of ν. Since ℓ^2_* is a dense subspace of ℓ^2 there exists the densely defined linear operator S which sends a vector x to the vector $Sx = y$ where

$$y_p = \sum_{q=0}^{\infty} c_{qp} \cdot x_q \quad : p = 0, 1, 2 \ldots$$

Thus, $\mathcal{D}$ is the space of vectors $x \in \ell^2$ such that $y \in \ell^2$. Reversing the role between rows and columns we get the subspace $\mathcal{D}^*$ of vectors $y \in \ell^2$ for which

$$\sum_{q=0}^{\infty} \Big|\sum_{p=0}^{\infty} c_{p,q} \cdot y_p\Big|^2 < \infty \tag{4}$$

A pairing via inner products. Let y be a vector in $\mathcal{D}^*$. For each $x \in \mathcal{D}$ we get the Hermitian inner product

$$\langle Sx, y\rangle = \sum\sum c_{pq} x_q \bar{y}_p$$

The Cauchy-Schwarz inequality entails that

$$|\langle Sx, y\rangle|^2 \leq ||x||^2 \cdot \sum_{q=0}^{\infty} \Big|\sum_{p=0}^{\infty} c_{p,q} \cdot y_p\Big|^2 \tag{5}$$

where $||x||^2 = \sum |x_p|^2$ is the squared norm of the ℓ^2-vector x. This can be expressed by saying that if $y \in \mathcal{D}^*$ then the linear functional on $\mathcal{D}$ defined by

$$x \mapsto \langle Sx, y\rangle$$

has a norm bounded by the square root of (4). Since $\mathcal{D}$ is dense in ℓ^2 and the Hilbert space ℓ^2 is self-dual this gives a unique vector y^* such that

$$\langle Sx, y\rangle = \langle x, y^*\rangle \quad : x \in \mathcal{D} \tag{6}$$

Conversely, if y is a vector in ℓ^2 for which there exists a vector y^* as above then one varies x in ℓ^2_* and conclude that $y \in \mathcal{D}^*$ where (4) is equal to $||y^*||^2$.

The inclusion $\mathcal{D} \subset \mathcal{D}^*$. let $\xi \in \mathcal{D}$. For every $x \in \mathcal{D}$ we have

$$\langle Sx, \xi \rangle = \sum \sum c_{pq} x_q \bar{\xi}_p = \sum \sum x_q \bar{c}_{qp} \bar{\xi}_p = \langle x, S\xi \rangle$$

This shows that $\mathcal{D} \subset \mathcal{D}^*$ and one has the equality

(7) $$S\xi = \xi^* \quad : \xi \in \mathcal{D}$$

Following Carleman we give

Definition. *The Hermitian matrix $\{c_{pq}\}$ is of type I if one has the equality $\mathcal{D} = \mathcal{D}^*$.*

Remark. Suppose that the matrix is of type I and let S be the densely defined linear operator above with $\mathcal{D}(S) = \mathcal{D}$. Let $\lambda = a + ib$ be a non-real complex number, i.e. $b \neq 0$ If $x \in \mathcal{D}(S)$ we get

(i) $$||\lambda x - S||^2 = \lambda x - Sx\rangle = ||\lambda|^2 ||x||^2 + ||Sx||^2 + \lambda \cdot \langle x, Sx \rangle + \bar{\lambda} \cdot \langle Sx, x \rangle$$

Since we are in Type I we have

$$\lambda \cdot \langle x, Sx \rangle + \bar{\lambda} \cdot \langle Sx, x \rangle = (\lambda + \bar{\lambda}) \cdot \langle Sx, x \rangle = 2a \cdot \langle Sx, x \rangle$$

Hence the right hand side in (i) becomes

$$b^2 ||x||^2 + \left(a^2 ||x||^2 + ||Sx||^2 + 2a \cdot \langle Sx, x \rangle \right)$$

Tbe Cauchy-Schwarz inequality entails that the sum inside the last bracket is ≥ 0 and taking a square root we get the inequality

(ii) $$||\lambda x - Sx|| \geq |\mathfrak{Im}(\lambda)|| \cdot ||x||$$

This inequality implies that there cannot exist a non-zero x in $\mathcal{D}(S)$ such that

(iii) $$Sx = ix$$

In § 5.3.1 we a construct Hermitian matrix whose associated S-operator has a non-zero eigenvector with eigenvalue i. From the above it means that there exist c-matrices where Case I fails.

A sufficiency for Case I. The condition in the definition above is rather implicit. The following necessary and sufficient condition i order that Case I occurs was established [ibid]:

Theorem. *Case I holds for S if and only if the equations*

$$Sx = ix \quad : Sx = -ix$$

have no non-zero solutions $x \in \mathcal{D}(S)$.

In most applications one uses this criterion to decide when Case I holds. The proof is given in § xx.

0.1 Spectral resolutions.

Before Theorem § 0.1.1 is announced below we recall a construction due to Hilbert. A resolution of the identity on ℓ^2 consists of a family $\{E(\lambda)\}$ of self-adjoint projections, indexed by real numbers λ which satisfies (A-C) below.

A. Each $E(\lambda)$ is an orthogonal projection from ℓ^2 onto the range $E(\lambda)(\ell^2)$ and these operators commute pairwise, i.e.

(i) $$E(\lambda) \cdot E(\mu) = E(\mu) \cdot E(\lambda)$$

hold for pairs of real numbers. Moreover, for each $x \in \ell^2$ one has

(ii) $$\lim_{\lambda + \infty} ||E(\lambda)(x) - x|| = 0 \quad : \lim_{\lambda \to -\infty} ||E(\lambda)(x)|| = 0$$

B. To each pair of real numbers $a < b$ we set

$$E_{a,b} = E(b) - E(a)$$

Then

(iii) $$E_{a,b} \cdot E_{c,d} = 0$$

for each pair of disjoint interval $[a,b]$ and $[b,c]$.

C. Notice first that if $\mu < \lambda$ for a pair of real numbers and $x \in \ell^2$ then (ii) gives

$$E(\mu)[E - E(\lambda)](x) = \lim_{s\to+\infty} (E\mu) - E(-s))(E(s) - E(\lambda)(x)$$

By (iii) the last product is zero as soon as $s > \lambda$ and $-s < \mu$ and hence we have

$$E(\mu) = E(\mu)E(\lambda)$$

It means that the range $E(\mu)(\ell^2)$ increases with μ and from this it is clear that when $x \in \ell^2$ then the real-valued function

$$\text{(c)} \qquad \lambda \mapsto \langle E(\lambda)(x), x\rangle$$

is a real-valued and non-decreasing function. Hilbert's last condition is that (c) is right continuous for every fixed x.

***S*-adapted resolutions.** Let S be a densely defined linear operator on ℓ^2. A spectral resolution $\{E(\lambda)\}$ of the identity is S-adapted if the following hold:

A.1 For each interval bounded $[a,b]$ the range of $E_{a,b}$ is contained in $\mathcal{D}(S)$ and the everywhere defined linear operator

$$S_{a,b} = S \circ E_{a,b}$$

is bounded and in addition

$$E_{a,b}(Sx) = S_{a,b}(x) \quad : x \in \mathcal{D}(S)$$

B.1 By (C) each $x \in \ell^2$ gives the non-decreasing function $\lambda \mapsto \langle E(\lambda)(x), x\rangle$ on the real line. Together with the right continuity in (c) there exist Stieltjes' integrals

$$\int_a^b \lambda \cdot \langle dE(\lambda)(x), x\rangle$$

for each bounded interval. The second condition for $\{E(\lambda)\}$ is that a vector x belongs to $\mathcal{D}(S)$ if and only if

$$\text{(b.1)} \qquad \int_{-\infty}^{\infty} |\lambda| \cdot \langle dE(\lambda)(x), x\rangle < \infty$$

C.1 The last condition is that

$$\text{(c.1)} \qquad \langle Sx, y\rangle = \int_{-\infty}^{\infty} \lambda \cdot \langle dE(\lambda)(x), y\rangle \quad : x, y \in \mathcal{D}(S)$$

where (b.1) and the Cauchy-Schwarz inequality entail that the Stieltjes' integral in (c.1) is absolutely convergent.

Now we can announce a major result from [Carleman].

0.1.1 Theorem. *Each Hermitian operator S of type I has a unique adapted resolution of the identity.*

Remark. For a bounded Hermitian operator the existence of an adapted resolution of the identity was proved by Hilbert in 1904 whose pioneering work was put forward in Carleman's lecture at the IMU-congress in 1932:

La théorie, crée par Hilbert, des formes quadratiques (ou hermitiennes) à une infinité de variables en connexion avec la théorie des équations intégrales à noyeau symmetrique est certainement la plus importante découverte qui ait été faite dans la théorie des équations intégrales aprés les travaux fondamentaux de Fredholm.

In practice a crucial point is to check when a densely defined symmetric operator S_* can be extended to an operator S of type I. From the start S_* need not be expressed by a Hermitian matrix. The reason is that on a separable Hilbert space many orthonormal bases exist. So an important issue in Carleman's work was to find sufficient conditions in order that Type I extensions exist for a given symmetric operator, which from the start is not expressed by a Hermitian matrix. The next section gives a criterion in order that symmetric integral operators are of Type I.

0.2 Integral operators.

Let $K(x,y)$ be a complex-valued Lebesgue measurable function on the product of the unit interval $[0,1]$ which satisfies $K(y,x) = \overline{K(x,y)}$ and there exists a null set $\mathcal{N}$ in $[0,1]$ such that

$$(*) \qquad \int_0^1 |K(x,y)|^2 \, dy < \infty \quad : x \in [0,1] \setminus \mathcal{N}$$

This gives a non-decreasing sequence of measurable set $\{E_1, E_2, \ldots\}$ whose union is $[0,1] \setminus \mathcal{N}$ and

$$(1) \qquad \int_0^1 |K(x,y)|^2 \, dy \leq n \quad : x \in E_n$$

To each n we define the truncated kernel $K_n(x,y)$ which is equal to $K(x,y)$ if both x and y belong to E_n, and is otherwise zero. Here (1) gives

$$(2) \qquad \iint_{[0,1]\times[0,1]} |K_n(x,y)|^2 \, dxdy \leq n$$

Hence each K_n yields a Hilbert-Schmidt operator on the Hilbert space $\mathcal{H} = L^2[0,1]$. Since $K_n(y,x) = \overline{K_n(x,y)}$ the spectrum of the associated integral operator is real. In particular, if $\mathfrak{Im}(\lambda) \neq 0$ and $f \in \mathcal{H}$ there exists a unique $\phi_n \in \mathcal{H}$ such that

$$(3) \qquad \phi_n(x) = \lambda \cdot \int_0^1 K_n(x,y) \cdot \phi_n(y) \, dy + f(x)$$

By a similar reasoning as in the remark from § 0.0 one has

$$(4) \qquad ||\phi_n||_2 \leq \frac{1}{|\mathfrak{Im}(\lambda)|} \cdot ||f||_2$$

Starting from this, Carleman applied results from the article *Über Systeme integrierbarer Funktionen* [Math. Annalen 1910] by F. Riesz. This gives the existence of at least one subsequence $\{\psi_\nu = \phi_{n_\nu}\}$ which converges weakly to an L^2-function ψ, i.e.

$$\lim_{\nu \to \infty} \int_0^1 \psi_\nu \cdot g \, dx = \int_0^1 \psi \cdot g \, dx$$

hold for every $g \in \mathcal{H}$. Now (2) and the weak convergence entail that

$$(*) \qquad \psi(x) = \lambda \cdot \int_0^1 K_n(x,y) \cdot \psi(y) \, dy + f(x)$$

Hence (*) has a solution for every pair $f \in \mathcal{H}$ and non-real λ. The solution ψ was found via a weakly convergent sequence and there reamins to analyze if it is unique. To ensure this we impose the condition that the homogeneous equation

$$(**) \qquad \phi(x) = \lambda \cdot \int_0^1 K_n(x,y) \cdot \phi(y) \, dy$$

has no non-zero solution $\phi \in \mathcal{H}$ when λ is non-real. In addition Carleman imposed the similar condition for the adjoint equation:

$$(***) \qquad \psi(x) = \lambda \cdot \int_0^1 K_n(x,y) \cdot \psi(y) \, dy \implies \psi = 0$$

When (**-***) hold it is proved in [1923] that the spectral measures associated to the Hilbert-Schmidt kernels $\{K_n\}$ converge and give an adapted spectral resolution of the densely defined integral operator K. Let us also remark that (**-**) is generic in the sense that it suffices to be satisfied when λ is any given non-real complex number. For example one can take $\lambda = i$ to check if (**-***) holds or not. An example of a symmetric and densely defined kernel function

$K(x,y)$ where (**-***) fails is given in § xx. Let us finish by some other examples which illustrate phenomena which can occur for unbounded integral operators.

0.2.1 Some examples.

Picard's equation. Here one takes $K(x,y) = e^{-|x-y|}$ on the product of the real line. For a complex number λ one seeks functions ϕ such that

$$\phi(x) = \lambda \cdot \int_{-\infty}^{\infty} e^{-|x-y|} \cdot \phi(y)\, dy \tag{1}$$

Exercise. Show that for every real number α the exponential function $e^{i\alpha x}$ solves (1) with the eigenvalue

$$\lambda = \frac{1+\alpha^2}{2}$$

Hence the spectrum of Picard's operator contains the real half-line $[1/2, +\infty)$. Notice that we employ eigenfunctions which are bounded but no in $L^2(\mathbf{R})$. So Picard's integral operator is not confined to L^2-spaces. The reason is that the exponential decay of the kernel allows an extensive family of functions for which the convolution integral in the right hand side converges.

Weyl's equation. Here one seeks functions ϕ such that

$$\phi(x) = \lambda \cdot \int_0^{\infty} \sin(xy) \cdot \phi(y)\, dy$$

In this case the special eigenvalue $\lambda = \sqrt{\frac{2}{\pi}}$ yields an *infinite* family of eigenfunctions of the form

$$\phi(x) = \frac{x}{x^2+a^2} + \sqrt{\frac{2}{\pi}} \cdot e^{-ax} \quad : a > +0$$

The kernel $K(z,\zeta) = \frac{1}{(1-z\bar{\zeta})^2}$. Let z and ζ be complex numbers and D the unit disc. If $g(z)$ is square integrable in D we set

$$\mathcal{K}_g(z) = \frac{1}{\pi} \cdot \int_D \frac{1}{(1-z\bar{\zeta})^2} \cdot g(\zeta)\, d\xi d\eta$$

It is clear that $\mathcal{K}_g(z)$ is an analytic function in D. Moreover, if $g(z)$ is analytic in D then the reader may verify that

$$\mathcal{K}_g(z) = g(z)$$

This means that $\mathcal{K}_g$ is a projection from $L^2(D)$ onto the subspace $H^2(D)$ of square integrable and analytic functions. So $H^2(D)$ is the eigenspace for the eigenvalue $\lambda = \pi^{-1}$

Exercise. Show that π^{-1} is the sole eigenvalue, i.e.

$$g \mapsto \mathcal{K}_g - \lambda \cdot g$$

is injective on $L^2(D)$ for every $\lambda \neq \pi^{-1}$. Show also the ineuality below for every $g \in L^2(D)$:

$$\frac{1}{\pi} \cdot \iint_{D\times D} \frac{1}{(1-z\bar{\zeta})^2} \cdot \bar{g}(z) \cdot g(\zeta)\, d\xi d\eta dx dy \leq \int_D |g(z)|^2\, dx dy$$

with equality if and only if g is analytic in D. Hence analytic functions in D are characterized by an extremal property of weighted integrals via the kernel function K.

§ 0.3 The Schrödinger equation.

In 1923 Carleman was concerned with extensions of Fredholm's theory to unbounded kernels in connection with boundary value problems. Two years later it turned out that the theory in [ibid] could be applied to the "new-born" quantum mechanics ! In a lecture at the Scandinavian congess in mathematics held at Copenhagen in 1925, Niels Bohr talked about quantum mechanics and adressed new problems for the mathematical communtity where the following second order PDE-equation plays a crucial role:

$$(*) \qquad \Delta\phi + 2m\cdot\big(E-U\big)\big(\frac{2\pi}{h}\big)^2\cdot\phi = 0$$

Here Δ is the Laplace operator in the 3-dimensional (x,y,z)-space, m the mass of a particle and h Planck's constant while $U(x,y,z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Recall that the fundamental point in Schrödinger's theory is the hypothesis on energy levels which correspond to orbits in Bohr's theory of atoms. For further physical background the reader should consult Bohr's plenary talk when he received the Nobel Prize in physics 1923, as well as Schrödinger's plenary tal in 1932. Leaving physics aside, the mathematical problem amounts to study second order PDE-operators:

$$(*) \qquad L = \Delta + c(x,y,z) \quad : \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

where $c(x,y,z)$ is a real-valued function which is almost everywhere continuous in the sense of Lebesgue, and in addition locally square integrable in $\mathbf{R}^3$. Consider the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^3)$ where the L-operator is defined on the dense subspace of test-functions in $\mathbf{R}^3$. Greens' formula entails that it is symmetric, i.e. for each pair f,g in $C_0^\infty(\mathbf{R}^3)$ one has:

$$\iint L(f)\cdot g\,dxdydz = \iint L(g)\cdot f\,dxdydz$$

A major problem is to decide if there exists a subspace $\mathcal{H}_*$ of $\mathcal{H}$ which contains the test-functions and has the property that $f \in \mathcal{H}_*$ entails that $L(f) \in L^2(\mathbf{R}^3)$ where symmetry persists:

$$\iint L(f)\cdot g\,dxdydz = \iint L(g)\cdot f\,dxdydz \quad : f,g \in \mathcal{H}_*$$

Moreover, L^2-functions $\phi \in \mathcal{H}_*$ are characterized by the property that there exists a constant $C(\phi)$ such that

$$\Big|\iint L(f)\cdot g\,dxdydz\,\Big| \leq ||f||_2 \quad : f \in C_0^\infty(|bfR^3)$$

where $||f||_2$ is the L^2-norm. In general such extensions of densely defined symmetric operators do not exist. A counter-example is described in § 5.3. But with L as in (*) the following sufficiency result is proved in Carleman's article *Sur la théorie mathématique de l'équation de Schrödinger* [Arkiv för math. 1934] whose proof is given in § 6.

Theorem *The subspace $\mathcal{H}_*$ exists if there is a constant M such that*

$$\limsup_{x^2+y^2+z^2\to\infty} c(x,y,z) \leq M$$

Example. In physical applications one is foremost concerned with the case when c is a potential function

$$W(p) = \sum \frac{\alpha_k}{|p-q_k|} + \beta$$

where $\{q_k\}$ is a finite subset of $\mathbf{R}^3$ and $\{\alpha_k\}$ and β are real and positive numbers. For this special c-function Carleman solved the problem raised by Bohr and Schrödinger in his lectures *La théorie des équations intégrales singulières* held at Institut Poincaré in 1930. The solution is quite direct and one does not need the more involved result in the theorem above. Namely, for each $\kappa > 0$ we define a function $H(p,q)$ in $\mathbf{R}^3 \times \mathbf{R}^3$ by

$$H(p,q) = \frac{e^{-|\kappa|p-q|}}{|p-q|}$$

where $|p-q|$ is the euclidian distance between a pair of points in $\mathbf{R}^3$. Choose κ so large that the volume integrals

$$\text{(i)} \qquad \iiint H(p,q) \cdot W(q)\, dq < 4\pi$$

for every $p \in \mathbf{R}^3$. From now on κ is fixed so that (i) holds. Set

$$\text{(ii)} \qquad \Omega(p,q) = \sqrt{W(p)} \cdot H(p,q) \cdot \sqrt{W(q)}$$

By (i) we find a positive constant $k < 4\pi$ such that

$$\text{(iii)} \qquad \iiint \Omega(p,q) \cdot \sqrt{W(q)}\, dq \leq k \cdot \sqrt{W(p)}$$

for every p in $\mathbf{R}^3$. This entails that the integral operator on $\mathcal{H}$ defined by

$$g \mapsto T_g(p) = \frac{1}{4\pi} \cdot \iiint \Omega(p,q) \cdot g(q)\, dq$$

has operator norm $\leq \frac{k}{4\pi} < 1$. Hence Neumann's resolvent series gives a unique solution $L(p,q)$ to the integral equation

$$\text{(iv)} \qquad L(p,q) = \frac{1}{4\pi} \iiint \Omega(p,x) \cdot L(p,x)\, dx + \Omega(p,q)$$

Set

$$G(p,q) = \frac{L(p,q)}{\sqrt{W(p)W(q)}}$$

From the above the G-kernel gives a bounded and compact linear operator on $\mathcal{H}$ defined by

$$g \mapsto \iiint G(p,q) \cdot g(q)\, dq$$

Hilbert's theory for bounded self-adjoint operators implies that non-zero solutions $\psi \in \mathcal{H}$ to the equation

$$\text{(v)} \qquad \psi(p) = \frac{\lambda + \kappa^2}{4\pi} \cdot \iiint G(p,q) \cdot \psi(q)\, dq$$

only exist for positive real numbers λ which form a discrete sequence tending to $+\infty$. Finally, using and Greens formula, one verifies that solutions in (v) are in a 1-1 correspondence to ϕ-functions satisfying

$$\text{(vi)} \qquad \Delta(\phi) + (W + \lambda)\phi = 0$$

Conclusion. From the above the densely defined operator $L = \Delta + W$ has a self-adjoint extension to $\mathcal{H}$ and for each complex number λ outside the discrete real spectrum above the inhomogeneous equation

$$\Delta(\phi) + (W + \lambda)\phi = f$$

has a unique solution $\phi \in \mathcal{H}$ for every $f \in \mathcal{H}$.

§ 0.3.1 The equation $\Delta(u) - c(p)u = i\frac{\partial u}{\partial t}$

Let $p = (x,y,z)$ denote points in $\mathbf{R}^3$. We seek functions $u = u(p,t) = u(x,y,z,t)$ of three real variables satisfying the PDE-equation above where $c(p)$ is a real-valued function which at least is locally square integrable. Under the condition that the densely defined operator $\Delta - c$ has a self-adjoint extension with a spectrum confined to $[\ell,+\infty)$ for some $\ell>0$ one finds a spectral function $\theta(p,q,\lambda)$ such that if $\in L^2(\mathbf{R}^3)$ then the function

$$u(p,t)=\int_{\mathbf{R}^2}\int_\ell^\infty e^{i\lambda t}\cdot\frac{d}{d\lambda}(\theta(p,q,\lambda)\cdot f(q)\,dq$$

satisfies the PDE-equation above with the inital condition $u(p,0)=f(p)$. The main issue is to find expressions for

$$\theta(p,q;\lambda+\lambda_1)-\theta(p,q;\lambda) \tag{0.1.0}$$

for pairs $\lambda\geq\ell$ and $\lambda_1>0$. To attain this one introduces the associated Greens function which satisfies the equation

$$G(p,q;\lambda)-G(p,q;0)=4\pi\lambda\cdot\int_\ell^\infty\frac{1}{\mu(\mu-\lambda)}\frac{d}{d\mu}(\theta(p,q,\mu)$$

Passing to Dirichlet series, residue calculus entails that the equation below hold for complex numbers s whose real part is sufficiently large and each $0<a<\ell$:

$$\int_\ell^\infty\frac{d}{d\mu}(\theta(p,q,\mu)\cdot\mu^{-s}\,d\mu=\frac{1}{8\pi^2 i}\cdot\int_{a-i\infty}^{a+i\infty}[G(p,q;\lambda)-G(p,q;0)]\cdot\lambda^{-s}\,d\lambda$$

The Neumann-Poincaré series for G. For each $\kappa>0$ we set

$$K(p,q;-\kappa)=\frac{e^{-\sqrt{\kappa}[p-q|}}{|p-q|}$$

Inductively we define functions $K^{(n)}(p,q;-\kappa)$ by

$$K^{(n+1)}(p,q;-\kappa)=\int_{\mathbf{R}^2}K^{(n)}(p,\xi;-\kappa)\cdot c(\xi)K(\xi,q;-\kappa)\,d\xi\quad: n=1,2,\ldots$$

With these notations one has the Neumann-Poincaré equation

$$G(p,q;-\kappa)=K(p,q;-\kappa)+\sum_{n=1}^\infty\frac{(-1)^n}{(4\pi)^n}\cdot K^{(n)}(p,q;-\kappa)$$

Using this and complex contour integrals similar to those which occur in (0.4.2) below, one finds (at least formally) an expansion of the differences in (0.1.0) which goes as follows: With fixed pair (λ,λ_1) as in (0.1.0) we set

$$J(p,q)=\frac{1}{4\pi^2}\int_\lambda^{\lambda+\lambda_1}\frac{\sin(\sqrt{x}\cdot|p-q|)}{|p-q|}\,dx$$

$$J^{(1)}(p,q)=-\frac{1}{16\pi^2}\int_\lambda^{\lambda+\lambda_1}\big[\int_{\mathbf{R}^2}\frac{\sin(\sqrt{x}\cdot|p-\xi|+|\xi-q|)}{|p-\xi|+|\xi-q|}\cdot c(\xi)\,d\xi\big]\,dx$$

If $n\geq 2$ one has

$$J^{(n)}(p,q)=$$

$$\frac{(-1)^n}{(4\pi)^{n+1}}\cdot\int_\lambda^{\lambda+\lambda_1}\int_{\mathbf{R}^n}\frac{\sin(\sqrt{x}\cdot|p-\xi_1|+|\xi_1-\xi_2|+\ldots+|\xi_n-q|)}{|p-\xi_1|+|\xi_1-\xi_2|+\cdots+|\xi_n-q|}\cdot c(\xi_1)\cdots c(\xi_n)\,d\xi_1\ldots d\xi_n dx$$

With these notations one has a formal expansion

$$\theta(p,q;\lambda+\lambda_1)-\theta(p,q;\lambda)=J(p,q)+J^{(1)}(p,q)+\ldots$$

Remark. At the end of the article [xx] Carleman raises the question to what extent the formal J-sum gives an asymptotic series for the left hand side. I do not know if more recent work provides extra information about this question. In any case it is tempting to analyze the expansion numerically for given c-functions.

§ 0.4 The Laplace operator in the complex domain.

Tbe flavour about densely defined and self-adjoint operators on Hilbert spaces is illustrated by the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$ and the Hilbert space $L^2(D)$ of square integrable complex valued functions in the unit disc $D = \{x^2 + y^2 < 1\}$. A dense subspace of $L^2(D)$ consists of test-functions $f \in C_0^\infty(D)$ which gives the densely defined operator on $L^2(D)$ defined by

$$f \mapsto \Delta(f)$$

By Green's formula the following holds for each pair of test-functions:

$$\langle \Delta(f), g\rangle = \iint_D \Delta(f) \cdot \bar{g}\, dxdy = \iint_D f \cdot \overline{\Delta(g)}\, dxdy = \langle f, \Delta(g)\rangle$$

The question arises if we can extend the domain of definition of Δ to a larger subspace $\mathcal{D}$ of $L^2(D)$ where the symmetry above holds for pairs f, g in $\mathcal{D}$. It turns out that a specific choice of $\mathcal{D}$ exists. Namely, introduce the bounded linear operator $\mathcal{L}$ on $L^2(D)$ defined by

$$\mathcal{L}(f)(p) = \frac{1}{2\pi} \cdot \iint_D L(p,q) \cdot f(q)\, dq$$

where points in D are denoted by p and q so that $dq = dxdy$ and

$$L(p,q) = \log \frac{|p-q|}{1 - \bar{q}p|}$$

It is easily seen that $\mathcal{L}$ sends each $f \in L^2(D)$ to a continuous function which is zero on ∂D. Next, recall that $\frac{1}{2\pi} \log |z|$ is a fundamental solution to the Laplace operator. From this one easily shows that

$$\Delta(\mathcal{L}(f)) = f \quad : f \in L^2(D)$$

So if the domain of definition for Δ is taken as the range of $\mathcal{L}$, then the composed operator

$$(*) \qquad \Delta \circ \mathcal{L} = E$$

where E is the identity operator on $L^2(D)$. It turns out that this is "optimal choice" of $\mathcal{D}$ yields a densely defined self-adjoint operator. The crucial fact is the following:

Theorem. *A function $g \in L^2(D)$ belongs to the range of $\mathcal{L}$ if and only if there exists a constant $C(g)$ such that*

$$|\langle \Delta(f), g\rangle| = \Big|\iint_D \Delta(f) \cdot \bar{g}\, dxdy\Big| \le C(g) \cdot ||f||_2 \quad : f \in \mathcal{L}(L^2(D))$$

Remark. Notice that ordinary harmonic functions "disappear" above. The reason is that if we take a function $g = \mathcal{L}(f)$ then g is zero on ∂D so by the maximum principle for harmonic functions it cannot satisfy $\Delta(g) = 0$ in the open disc. Moreover (*) means that Δ yields a *bijective* map from $\mathcal{L}(L^2(D))$ to $L^2(D)$. In § 1 we shall learn that $\mathcal{L}$ is Neumann's resolvent of the densely defined operator Δ. Tbe spectrum of the unbounded operator Δ consists of a non-decreasing sequence of real numbers $\{0 < \lambda_0 \le \lambda_1 \le \ldots\}$ where one for each n has an eigenfunction ϕ_n in the range of $\mathcal{L}$ with unit L^2-norm such that

$$\Delta(\phi_n) + \lambda \cdot \phi_n = 0$$

A crucial point is that these eigenvalues and corresponding eigenfunctions are equally well found via the compact operator $\mathcal{L}$ on $L^2(D)$ whose spectrum is the discrete set $\{-\frac{1}{\lambda_n}\}$ with a sole cluster point at $\{\lambda = 0\}$ and here

$$\mathcal{L}(\phi_n) = -\frac{1}{\lambda_n} \cdot \phi_n$$

The lesson from this example which stems from work by Poincaré, is that the study of the unbounded Δ-operator can be recovered via the compact operator $\mathcal{L}$ on $L^2(D)$. The reader should keep this example in mind while the general theory about unbounded self-adjoint operators is presented.

Further comments. In view of the fundaemtnal character of the example above we insert some further remarks related to the theorem above where we constructed the operator $\mathcal{L}$.

Exercise. Verify that for each $\phi \in L^2(D)$, $\mathcal{L}(\phi)$ is a continuous function in D which vanishes on the boundary $T = \{|z| = 1\}$ and show the equality

$$\Delta(\mathcal{L}(\phi)) = \phi \tag{i}$$

where the left hand side is the Laplacian of $\mathcal{L}(\phi)$ taken in the distribution sense.

By (i) the restriction of Δ to the range of $\mathcal{L}$ gives a linear operator with values in $L^2(D)$. Identifying $\mathcal{L}(L^2(D))$ with a subspace of $L^2(D)$, this means that we have constructed a densely defined operator A where $\mathcal{D}(A) = \mathcal{L}(L^2(D))$ and one has a surjective map

$$A\colon \mathcal{D}(A) \to L^2(D) \tag{ii}$$

Notice that

$$\mathcal{L} \circ A(\mathcal{L}(\phi)) = \mathcal{L}(\phi) \quad : \phi \in L^2(D) \tag{iii}$$

Hence the composed operator $\mathcal{L} \circ A$ is the identity on $\mathcal{D}(A)$ which means that $\mathcal{L}$ is Neumann's inverse sense to the densely defined operator A.

1. The adjoint A^*. To each pair u, v in $L^2(D)$ we set $\phi = \mathcal{L}(u)$ and $\psi = \mathcal{L}(v)$ and take the inner product in the Hilbert space $L^2(D)$. This gives

$$\langle\, A(\phi), \psi\rangle = \int_D u(p) \cdot \overline{\mathcal{L}(v)(p)}\, dp = \iint_{D\times D} u(p) \cdot L(p,q) \cdot \overline{v(q)}\, dpdq \tag{1.1}$$

where we have put

$$L(p,q) = \frac{1}{2\pi} \cdot \log \frac{|p-q|}{1-\bar{p}q|}$$

Fubini's theorem entails that the last term in (1.1) is equal to $\langle \phi, A(\psi)\rangle$ which proves the symmetry

$$\langle\, A(\phi), \psi\rangle = \langle \phi, A(\psi)\rangle \quad : \phi, \psi \in \mathcal{D}(A) \tag{1.2}$$

Next, the construction of adjoints gives the linear operator $\mathcal{D}(A^*)$ whose domain of definition consists of $L^2(D)$-functions g for which there exists a constant $C(g)$ such that

$$|\langle\, A(\phi), g\rangle| \leq C(g) \cdot ||\phi||_2 \quad : \phi \in \mathcal{D}(A) \tag{1.3}$$

By the construction of $\mathcal{D}(A)$ this means that

$$\iint_{D\times D} u(p) \cdot L(p,q) \cdot \overline{g(q)}\, dpdq \leq C(g) \cdot ||\mathcal{L}(u)||_2 \quad : u \in L^2(D) \tag{1.4}$$

Exercise. Using the symmetry of $L(p,q)$ to show that (1.4) holds if and only if g belongs to the range of $\mathcal{L}$. Hence $\mathcal{D}(A^*) = \mathcal{D}(A)$ which means that A is a densely defined and self-adjoint operator.

2. The spectrum of A**.** Neumann's construction of spectra associated to densely defined operators presented in § 0 yields a closed set $\sigma(A)$. From the above $\mathcal{L}$ is an inverse to A so $\lambda = 0$ is outside the spectrum. It turns out that $\sigma(A)$ is a real and discrete set. Given a real λ the operator $\lambda \cdot E - A$ fails to be injective if we find a real-valued function $\phi \in \mathcal{D}(A)$ such that

$$\Delta(\phi) = \lambda \cdot \phi \tag{2.1}$$

Each eigenvalue λ is strictly negative. For suppose that (2.1) holds with $\lambda > 0$. Since $\phi \in \mathcal{D}(A)$ it is a continous function which is not identically zero and replacing ϕ by $-\phi$ if necessary we may assume that the maximum is > 0 and taken at some point $p \in D$. A Taylor expansion of ϕ at p gives $\Delta(\phi)(p) \leq 0$ which contradicts (2.1) when $\lambda > 0$.

Remark. One sometimes rewrites the eigenvalues and make the assertion that

$$\phi \mapsto \Delta(\phi) + \lambda \cdot \phi \tag{*}$$

is injective with the exclusion of a discrete set $\{\lambda_n\}$ of positive real numbers. To get a *lower bound* for positive λ where (*) has a non-zero solution one employs Green's formula which entails that

$$\int_D \Delta(\phi)\cdot\phi\,dxdy = -\int_D (\phi_x^2+\phi_y^2)\cdot\phi\,dxdy$$

It follows that

$$\lambda\cdot\int\phi^2\,dxdy = \int_D (\phi_x^2+\phi_y^2))\cdot\phi\,dxdy$$

Since $\phi=0$ on the boundary of D, the Dirichlet integral in the right hand side majorises the squared L^2-norm of ϕ by an absolute constant $c>0$, i.e.

(**) $$\int\phi^2\,dxdy \leq c\cdot\int_D (\phi_x^2+\phi_y^2))\cdot\phi\,dxdy$$

Exercise. Find the smallest c for which (**) holds and consult the literature for a description of the eigenvalues of A and the corresponding eigenfunctions.

0.4.1 The operator $\Delta + c(x,y)$

.

A real-valued function $c\in L^2(D)$ gives the densely defined operator $P=\Delta+c$. We seek a domain of definition which entails that P is self-adjoint. To attain this we introduce the kernel function:

$$R(p,q) = L(p,q)\cdot c(q)$$

Keeping p fixed we notice that the Cauchy-Schearz ineqwuality entials that the L^1-integral of

$$q\mapsto L(p,q)\cdot c(q)$$

is finite. So if g is a continuous function with compact support in D we get the function

$$p\mapsto \mathcal{R}(g)(p) = \int_D L(p,q)\cdot c(q)\cdot g(q)\,dq$$

Next, let $\psi(p)\in L^2(D)$. Since $L(p,q)$ is a hiolbert-Schmidt kernel there exists a constant C such that

$$\max_{q\in D}\int_D |\psi(p)L(p,q)|\,dp \leq C\cdot||\psi||_2$$

Then Fubini's theorem and the Cauchy-Schwarz inequality give

$$|\langle\psi,\mathcal{R}(g)\rangle| \leq C\cdot||\psi||_2\cdot||c||_2\cdot||g||_2$$

This entails that $\mathcal{R}$ extends to a bounded linear operator on $L^2(D)$ whose operator norm is $\leq C\cdot||c||_2$.

Exercise. Above $\mathcal{R}=\mathcal{R}_c$ depends upon c. Show that if c from the start is a test function in D, then $\mathcal{R}_c$ is a compact operator. Use that a linear operator which can be apprximated by compact operators in the operator norm also is compact and conclude from the density of $C_0^\infty(D)$ in $L^2(D)$ that R_c is a compact operator on $L^2(D)$ for every $c\in L^2(D)$.

The equation $P(g)=f$. Keeping c fixed we set $\mathcal{R}=\mathcal{R}_c\ldots$ If $g\in L^2(D)$ is such that $c\cdot g\in L^2(D)$ we have

$$\mathcal{R}(g) = \int L(p,q)c(q)g(q)\,dq = \mathcal{L}(cg)$$

It follows from the previous section that

(i) $$\Delta(\mathcal{R}(g)) = cg$$

Hence

(ii) $$P(g) = \Delta\circ(E+\mathcal{R})(g)$$

Let us then consider the inhomogenous equation

$$P(g) = f$$

If the linear operator $E+\mathcal{R}$ has an inverse we set

(iii) $$g=(E+\mathcal{R})^{-1}\circ\mathcal{L}(f)$$

Then (ii-iii) give

$$P(g)=\Delta\circ\mathcal{L}(f)=f$$

A self-adjoint operator. If g belongs to the range of $\mathcal{L}$ we have seen that it is a continuous function which is zro on ∂D. It follows that $c\cdot g$ stays in $L^2(D)$ and hence there exists the densely defined operator

$$g\mapsto P(g) \quad : g\in\mathcal{L}(L^2(D))$$

Exercise. Shows that with $\mathcal{D}(P)$ tsken as the range of $\mathcal{L}$, it follows that one has a self-adjoint operator whose spectrum is a discrete subset of the real λ-line. Concerning the points in the spetrum $\sigma(P)$ and the associated eigenfunctions they depend upon c and here one is obliged to perform numerical calculations to get further information.

§ 0.4.2 The Δ-operator on general domains.

Above we considered the unit disc. More generally, let Ω be a bounded and connected domain in **C** whose boundary consists of a finite number of pairwise disjoint and differentiable closed Jordan curves. It is wellknown that the Dirichlet problem is solvable for such domains. In particular, let $p\in\Omega$ be kept fixed and consider the continuous function on $\partial\Omega$ defined by

$$q\mapsto\log\frac{1}{|p-q|}$$

We find the harmonic function $u_p(q)$ in Ω such that $u_p(q)=\log\frac{1}{|p-q|}\,:\,q\in\partial\Omega$. Green's function is defined for pairs $p\neq q$ in $\Omega\times\Omega$ by

(1) $$G(p,q)=\log\frac{1}{|p-q|}-u_p(q)$$

This entails that if $p\in\Omega$ is kept fixed, then $q\mapsto G(p,q)$ extends to the closure of Ω where it vanishes if $q\in\partial\Omega$. If $f\in L^2(\Omega$ we set

(2) $$\mathcal{G}_f(p)=\frac{1}{2\pi}\cdot\int_\Omega G(p,q)\cdot f(q)\,dq$$

where $q=(x,y)$ so that $dq=dxdy$ when the double integral is evaluated. It is easily seen that

$$\iint_{\Omega\times\Omega}|G(p,q)|^2\,dpdq<\infty$$

Hence $\mathcal{G}$ is of the Hilbert-Schmidt type and therefore a compact operator on $L^2(\Omega)$. Next, recall that $\frac{1}{2\pi}\cdot\log\sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator. From this one can deduce the following:

0.4.2.1 Theorem. *For each $f\in L^2(\Omega)$ the Lapacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

(*) $$\Delta(\mathcal{G}_f)=-f$$

The equation (*) means that the composed operator $\Delta\circ\mathcal{G}$ is minus the identity on $L^2(\Omega)$. We are theorefore led to introduce the linear operator S on $L^2(\Omega)$ defined by Δ, where its domain of definition $\mathcal{D}(S)$ is the range of $\mathcal{G}$. If $g\in C_0^2(\Omega)$, i.e. twice differentiable and with compact support, it follows via Greens' formula that

$$\frac{1}{2\pi}\cdot\int_\Omega G(p,q)\cdot\Delta(g)(q)\,dq=-g(p)$$

In particular $C_0^2(\Omega) \subset \mathcal{D}(S)$ which implies that S is densely defined and we leave it to the reader to verify that

$$\mathcal{G}(\Delta(f)) = -f \quad : f \in \mathcal{D}(S)$$

Remark. By the construction of resolvent operators in § 1 this means that $-\mathcal{G}$ is Neumann's inverse of S.

Exercise. Show that S has a closed range and in addition it is self-adjoint, i.e. $S = S^*$.

The spectrum of S. A wellknown result asserts that there exists an orthonormal basis $\{\phi_n\}$ in $L^2(\Omega)$ where each $\phi_n \in \mathcal{D}(S)$ is an eigenfunction. More preciesely there is a non-decreasing sequence of positive real numbers $\{\lambda_n\}$ and

(i) $$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots$$

Let us remark that (i) means that

(ii) $$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n$$

So above $\{\lambda_n^{-1}\}$ are eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. Eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times. Now we announce a result about the values taken by the eigenfunctions.

0.4.2.2 Theorem. *For each point $p \in \Omega$ one has the limit formula*

(*) $$\lim_{n \to \infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi}$$

To prove this we consider the Dirichlet series for each fixed $p \in \Omega$:

(0.4.3) $$\Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

It is easily seen that $\Phi_p(s)$ is analytic in a half-space $\mathfrak{Re}\, s > b$ for a large b. Less trivial is the following:

0.4.2.3 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem 0.4.2.3 gives Theorem 0.4.2.2 Namely, a result due to Wiener in the article *Tauberian theorem* [Annals of Math.1932 asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s \to 1}\, , (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n \to \infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Proof of Theorem 0.4.2.3

Since $\mathcal{G}$ is a Hilbert-Schmidt operator one easily verifies that

(i) $$\sum \lambda_n^{-2} < \infty$$

This convergence entails that various constructions below are defined. To begin with, for each λ outside $\{\lambda_n\}$ we set

$$\text{(ii)} \qquad G(p,q;\lambda) = G(p,q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda-\lambda_n)}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\text{(iii)} \qquad \mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p,q;\lambda) \cdot f(q)\, dq$$

A. Exercise. Use that the eigenfunctions$\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p,\lambda)$. Set

$$F(p,q,\lambda) = G(p,q;\lambda) - G(p,q)$$

Keeping p fixed we see that (ii) gives

$$\text{(B.1)} \qquad \lim_{q\to p} F(p,q,\lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda-\lambda_n)}$$

Set

$$F(p,\lambda) = \lim_{q\to p} F(p,q,\lambda)$$

From (i) and (B.1) it follows that it is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Show via residue calculus that one has the equality below in a half-space $\mathfrak{Re}\, s > b$ provided that b is large enough:

$$\text{(C.1)} \qquad \Phi(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a-i\infty}^{a+i\infty} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda$$

where the line integral is taken on the vertical line $\mathfrak{Re}\lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\text{(E.1)} \qquad \Phi(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta}\, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

$$\text{(E.2)} \qquad s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p,-x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^\infty \frac{e^{-zt}}{\sqrt{t^2-1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

(F.1) $$K(z) = -I_0(z) \cdot \log z + I_1(z)$$

where I_0 and I_1 are entire functions with series expansions

(i) $$I_0(z) = \sum_{m=0}^\infty \frac{2^{-2m}}{(m!)^2} \cdot z^{2m}$$

(ii) $$I_1(z) = \sum_{m=0}^\infty \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a a function $q \mapsto H(p, q; \kappa)$ which satisfies the equation

(F.2) $$\Delta(H) - \kappa \cdot H = 0$$

in Ω with boundary values

$$H(p, q; \kappa) = K(\sqrt{\kappa}|p - q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p, q; -\kappa) = K(\sqrt{\kappa} \cdot |p - q|) - H(q; \kappa) \quad : \kappa > 0$$

Next, recalling the construction of $G(p, q)$ we get

(G.1) $$F(p, -\kappa) = \lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q|] + \lim_{q \to p} [u_p(q) + H(p, q, \kappa)]$$

The last term above has a "nice limit" $u_p(p) + H(p, p, \kappa)$ and from (F.1) the reder can verify the limit formula:

(G.2) $$\lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q|] = -\frac{1}{2} \cdot \log \kappa + \log 2 - \gamma$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p, -\kappa) = -\frac{1}{2} \cdot \log \kappa + A + H(p, p; -\kappa)$$

With $x = \kappa$ in (E.2) we proceed as follows. To begin with the reader may verify that

$$s \mapsto A \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty x^{-s}\, dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2} \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty \log x \cdot x^{-s}\, dx$$

Notice that the complex derivative

$$\frac{d}{ds} \int_a^\infty x^{-s}\, dx = -\int_a^\infty \log x \cdot x^{-s}\, dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem 0.4.4 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty H(p,p,\kappa) \cdot \kappa^{-s}\, d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \leq H(p,q;\kappa) \leq \max_{q \in \partial\Omega} K(\kappa|p-q|)$$

With $p \in \Omega$ fixed there is a positive number δ such that $|p-q| \geq \delta : q \in \partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.

§ 1. Bounded self-adjoint operators.

Introduction. A bounded linear operator S on a complex Hilbert space $\mathcal{H}$ is self-adjoint if $S = S^*$ which means that

$$(*) \qquad \langle x, Sy\rangle = \langle Sx, y\rangle \quad : \quad x, y \in \mathcal{H}$$

If S is self-adjoint one has the equality of operator norms:

$$(1) \qquad ||S||^2 = ||S^2||$$

To see this we notice that if $x \in \mathcal{H}$ has norm one then

$$(2) \qquad \langle Sx, Sx\rangle = \langle x, S^*Sx\rangle = \langle x, S^2x\rangle$$

By the Cauchy-Schwarz inequality the last term is $\leq ||x| \cdot ||S^2||$ and since (2) holds for every x of unit norm we conclude that

$$||S||^2 \leq ||S^2||$$

Next, the multiplicative inequality for operator norms and an induction over n give

$$||S||^{2n} = ||S^n||^2 \quad : \; n \geq 1$$

Taking the n:th root and passing to the limit the spectral radius formula gives

$$(*) \qquad ||S|| = \max_{|z \in \sigma(S)} |z|$$

1.1 Theorem. *The spectrum of a bounded self-adjoint operator is a compact real interval.*

Proof. Let λ be a complex number and for a given x we set $y = \lambda x - Sx$. It follows that

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 + \lambda \cdot \langle x, Sx\rangle + \bar{\lambda} \cdot \langle Sx, x\rangle$$

Since S is self-adjoint we get

$$\lambda \cdot \langle x, Sx\rangle + \bar{\lambda} \cdot \langle Sx, x\rangle = 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle Sx, x\rangle$$

Now $|\langle Sx, x\rangle| \leq ||Sx|| \cdot ||x||$ so the triangle inequality gives

$$\text{(i)} \qquad ||y||^2 \geq |\lambda|^2 \cdot ||x||^2 + ||Sx||^2 - 2|\mathfrak{Re}(\lambda)|| \cdot ||Sx|| \cdot ||x||$$

With $\lambda = a + ib$ the right hand side becomes

$$b^2||x||^2 + a^2||x||^2 + ||Sx||^2 - 2a \cdot ||Sx|| \cdot ||x|| \geq b^2||x||^2$$

Hence we have proved that

$$\text{(ii)} \qquad ||\lambda x - Sx||^2 \geq (\mathfrak{Im}\,\lambda)^2 \cdot ||x||^2$$

This implies that $\lambda E - S||$ is invertible for every non-real λ which proves Theorem 1.1.

Now the operational calculus from § 0.6 and Theorem 1.1 give

1.2 Theorem *Denote by* $\mathbf{S}$ *the closed subalgebra of* $L(\mathcal{H}, \mathcal{H})$ *generated by* S *and the identity operator. Then* $\mathbf{S}$ *is isomorphic to the sup-norm algebra* $C^0(\sigma(S))$.

1.3 Normal operators.

A bounded linear operator S is normal if it commutes with its adjoint S^*. Let S be normal and put $A = S^*S$ which is self-adjoint. From (1) above one has

$$(1.3.1) \qquad ||A||^2 = ||A^2|| = ||S^2 \cdot (S^*)^2|| \leq ||S^2|| \cdot ||(S^*)^2||$$

Here $(S^*)^2$ is the adjoint of S^2 and recall that the norms of an operator and its adjoint are equal. Hence the right hand side in (1.3.1) is equal to $||S^2||^2$ and since $||A|| = ||S||^2$ we get the equality

$$(1.3.2) \qquad ||S||^2 = ||S^2||$$

Exactly as in the self-adjoint case we can take higher powers and obtain the equality

$$(1.3.3) \qquad ||S|| = \max_{z \in \sigma(S)} |z|$$

Since every polynomial in S again is a normal operator for which (1.3.3.) holds we have proved the following:

1.4 Theorem *Let S be a normal operator. Then the closed subalgebra* $\mathbf{S}$ *generated by S in $L(\mathcal{H},\mathcal{H})$ is a sup-norm algebra.*

Remark. The spectrum $\sigma(S)$ is some compact subset of $\mathbf{C}$ and in general analytic polynomials restricted to $\sigma(S)$ do not generate a dense subalgebra of $C^0(\sigma(A))$. To get a more extensive algebra we consider the closed subalgebra $\mathcal{B}$ of $L(\mathcal{H},\mathcal{H})$ generated by S and S^*. Since every polynomial in S and S^* again is a normal operator it follows that $\mathcal{B}$ is a sup-norm algebra and the following holds:

1.5 Theorem. *The sup-norm algebra $\mathcal{B}$ is via the Gelfand transform isomorphic with $C^0(\sigma(S))$.*

Proof. If $A \in \mathcal{B}$ is self-adjoint then we know from the previous section that its Gelfand transform is real-valued. Next, let $Q \in \mathcal{B}$ be arbitrary. Now $A = Q + Q^*$ is self-adjoint. So if $p \in \mathfrak{M}_{\mathcal{B}}$ it first follows that the Gelfand transform $\widehat{Q}(p) + \hat{Q}^*(p)$ is real, i.e. with $\widehat{Q}(p) = a + ib$ we must have $\hat{Q}^* = a_1 - ib$ for some real number a_1. Next, the product QQ^* is also self-adjoint and hence $(a+ib)(a_1 - ib)$ is real. This gives $a = a_1$ which shows that the Gelfand transform of Q^* is the complex conjugate function of $\widehat{Q}$. Hence the Gelfand transforms of $\mathcal{B}$-elements is a self-adjoint algebra and the Stone-Weierstrass theorem implies that the Gelfand transforms of $\mathcal{B}$-elements is equal to the whole algebra $C^0(\mathfrak{M}_{\mathcal{B}})$. Finally, since $\widehat{S}^*$ is the complex conjugate function of $\widehat{S}$, it follows that the Gelfand transform $\widehat{S}$ alone separates points on $\mathfrak{M}_{\mathcal{B}}$ which means that $\mathfrak{M}_{\mathcal{B}}$ can be identified with $\sigma(S)$.

1.6 Spectral measures.

Let S be a bounded normal operator which give $\mathcal{B}$ and $\sigma(S)$ in Theorem 1.5. Now we construct Riesz measures on $\sigma(S)$ as follows: Each pair of vectors x, y in $\mathcal{H}$ yields a linear functional on $\mathcal{B}$ defined by

$$T \mapsto \langle Tx, y\rangle$$

Riesz representation formula gives a unique Riesz measure $\mu_{x,y}$ on $\sigma(S)$ such that

$$\langle Tx, y\rangle = \int_{\sigma(S)} \widehat{T}(z) \cdot d\mu_{x,y}(z) \tag{1.6.1}$$

hold for every $T \in \mathcal{B}$. Since $\widehat{S}(z) = z$ we have

$$\langle Sx, y\rangle = \int z \cdot d\mu_{x,y}(z)$$

Similarly one has

$$\langle S^*x, y\rangle = \int \bar{z} \cdot d\mu_{x,y}(z)$$

1.7 The operators $E(\delta)$. By (1.6.1) the map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures on $\sigma(S)$ is bi-linear. We have for example: $\mu_{x_1+x_2,y} = \mu_{x_1,y} + \mu_{x_2,y}$. Moreover, since $\mathcal{B}$ is the sup-norm algebra $C^0(\sigma(S))$ the total variations of the μ-measures satisfy:

$$||\mu_{x,y}|| \le \max_{T \in \mathcal{B}_*} |\langle Tx, y\rangle| \tag{1.7.1}$$

where $\mathcal{B}_*$ is the unit ball in $\mathcal{B}$. From this we obtain

$$||\mu_{x,y}|| \le ||x|| \cdot ||y|| \tag{1.7.2}$$

Next, let δ be a Borel subset of $\sigma(S)$. Keeping $y \in \mathcal{H}$ fixed gives a linear functional on $\mathcal{H}$ defined by

$$x \mapsto \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta)$$

By (1.7.2) it has norm $\leq ||y||$ and is represented by a vector $E(\delta)x$ in $\mathcal{H}$. More precisely

$$\langle E(\delta)x, y\rangle = \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta) \tag{1.7.3}$$

1.8 Exercise. Show that $x \mapsto E(\delta)x$ is linear and that the resulting linear operator $E(\delta)$ commutes with all operators in $\mathcal{B}$. Moreover, show that it is a projection, i.e.

$$E(\delta)^2 = E\delta)$$

Finally, show that

$$E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$$

holds for every pair of Borel subsets and with $\delta = \sigma(S)$ one gets the identity operator.

1.9 Resolution of the identity. If $\delta_1, \dots, \delta_N$ is any finite family of disjoint Borel sets whose union is $\sigma(S)$ then

$$1 = E(\delta_1) + \ldots + E(\delta_N)$$

At the same tine we get a decomposition of the operator S:

$$S = S_1 + \ldots + S_N \quad \text{where} \quad S_k = E(\delta_k) \cdot S$$

For each k the spectrum $\sigma(S_k)$ is equal to the closure of δ_k. Hence the normal operator is represented by a sum of normal operators where the individual operators have small spectra when the δ-partition is fine.

§ 2. Unbounded operators on Hilbert spaces

Let T be a densely and unbounded linear operator on $\mathcal{H}$.

2.1 The adjoint T^*. Each vector $y \in \mathcal{H}$ gives a linear functional on $\mathcal{D}(T)$ defined by

$$x \mapsto \langle Tx, y\rangle \tag{i}$$

If there exists a constant $C(y)$ such that the absolute value of (i) is $\leq C(y) \cdot ||x||$ for every $x \in \mathcal{D}(T)$, then (i) extends to a continuous linear functional on $\mathcal{H}$. The extension is unique because $\mathcal{D}(T)$ is dense and since $\mathcal{H}$ is self-dual there is a unique vector T^*y such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad : \quad x \in \mathcal{D}(T) \tag{2.1.1}$$

This gives a linear operator T^* where $\mathcal{D}(T^*)$ is characterised by the above. Next, the product $\mathcal{H} \times \mathcal{H}$ is a Hilbert space with the inner product

$$\langle (x, y), (x_1, y_1)\rangle = \langle x, x_1\rangle + \langle y, y_1\rangle$$

On $\mathcal{H} \times \mathcal{H}$ we define the linear operator J by

$$J(x, y) = (-y, x)$$

2.2 Proposition. *For every densely defined operator T one has the equality*

$$\Gamma(T^*) = J(\Gamma(T))^\perp$$

Proof. Let (y, T^*y) be a vector in $\Gamma(T^*)$. If $x \in \mathcal{D}(T)$ the equality (2.1.1) and the construction of J give

$$\langle -Tx, y\rangle + \langle x, T^*y\rangle = 0$$

This proves that $\Gamma(T^*) \perp J(\Gamma(T))$. Conversely, if $(y, z) \perp J(\Gamma(T))$ we have

$$\langle -Tx, y\rangle + \langle x, z\rangle = 0 \quad : x \in \mathcal{D}(T) \tag{i}$$

This shows that $y \in \mathcal{D}(T^*)$ and $z = T^*y$ which proves Proposition 2.2.

2.3 Consequences. Since the orthogonal complement of a subspace in a Hilbert space is closed, Proposition 2.2 implies that T^* has a closed graph. Taking the closure of $\Gamma(T)$ the decompostion of a Hilbert space into a direct sum of a closed subpace and its orthogonal complement gives

$$\mathcal{H} \times \mathcal{H} = J(\overline{\Gamma(T)}) \oplus \Gamma(T^*) \tag{2.3.1}$$

Notice also that

$$\Gamma(T^*)^\perp = J(\overline{\Gamma(T)}) \tag{2.3.2}$$

2.4 Closed extensions of operators. Let T be densely defined. A closed operator S such that $\Gamma(T)$ is a subspace of $\Gamma(S)$ is called a closed extension of T.

2.4.1 Exercise. Show that if S is a closed extension of T then

$$S^* = T^*$$

2.4.2 Theorem. *A densely defined operator T has a closed extension if and only if $\mathcal{D}(T^*)$ is dense. Moreover, if T is closed one has the biduality formula $T = T^{**}$.*

Proof. Suppose first that T has a closed extension. By (2.1) it suffices to show that $\mathcal{D}(T^*)$ is dense when T is closed. If $\mathcal{D}(T^*)$ is not dense there exists a non-zero vector $0 \neq h \perp \mathcal{D}(T^*)$ and (2.3.2) gives

$$(h,0) \in \Gamma(T^*)^\perp = J(\Gamma(T)) \tag{ii}$$

By the construction of J this would give $x \in \mathcal{D}(T)$ such that $(h,0) = (-Tx, x)$ which cannot hold since this equation first gives $x =$ and then $h = T0 = 0$. Hence closedness of T implies that $\mathcal{D}(T^*)$ is dense. Conversely, assume that $\mathcal{D}(T^*)$ is dense. Starting from T^* we construct its adjoint T^{**} and (2.3.2) applied with T^* gives

$$\Gamma(T^{**}) = J(\Gamma(T^*)^\perp \tag{i}$$

At the same time $J(\Gamma(T^*)^\perp$ is equal to the closure of $\Gamma(T)$ so (i) gives

$$\overline{\Gamma(T)} = \Gamma(T^{**}) \tag{ii}$$

which proves that T^{**} is a closed extension of T.

2.4.3 The biduality formula. If T from the start is closed then its graph equals that of T^{**} which gives the equality:

$$T = T^{**} \tag{2.4.3}$$

2.5 Inverse operators.

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Denote by $\mathfrak{I}(\mathcal{H})$ the family of closed and densely defined operators T such that T is injective on $\mathcal{D}(T)$ and the range $T(\mathcal{D}(T))$ is dense in $\mathcal{H}$. If $T \in \mathfrak{I}(\mathcal{H})$ there exists the densely defined operator S where $\mathcal{D}(S)$ is the range of T and

$$S(Tx) = x \quad : \quad x \in \mathcal{D}(T)$$

Next, on $\mathcal{H} \times \mathcal{H}$ we have the isometry defined by $I(x,y) = (y,x)$, i.e we interchange the pair of vectors. The construction of S gives

$$\Gamma(S) = I(\Gamma(T)) \tag{i}$$

Since $\Gamma(T)$ by hypothesis is closed it follows that S has a closed graph and we conclude that $S \in \mathfrak{I}(\mathcal{H})$. Moreover, since I^2 is the identity on $\mathcal{H} \times \mathcal{H}$ we have

$$\Gamma(T) = I(\Gamma(S)) \tag{ii}$$

We refer to S is as the inverse of T. It is denoted by T^{-1} and (ii) implies that T is the inverse of T^{-1}.

2.5.1 Exercise. Let T belong to $\mathfrak{I}(\mathcal{H})$. Use the description of $\Gamma(T^*)$ in Proposition 2.3 to show that $T^* \in \mathfrak{I}(\mathcal{H})$ and the equality

$$(T^{-1})^* = (T^*)^{-1} \tag{*}$$

2.6 The operator T^*T. Every $h \in \mathcal{H}$ gives the vector $(h,0)$ in $\mathcal{H}\times\mathcal{H}$ and (2.3.1) a pair $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$. such that

$$(h,0) = (x,Tx) + (-T^*y, y) = (x - T^*y, Tx + y)$$

With $u = -y$ we get $Tx = u$ and obtain

$$h = x + T^*(Tx) \tag{2.6.1}$$

2.6.2 Proposition. *The vector x in (2.6.1) is uniquely determined by h.*

Proof. Uniqueness follows if we show that

$$x + T^*(Tx) \implies x = 0$$

But this is clear since the construction of T^* gives

$$0 = \langle x, x)\rangle + \langle x, T^*(Tx)\rangle = \langle x, x)\rangle + \langle Tx, Tx)\rangle \implies x = 0$$

2.7 The density of $\mathcal{D}(T^*T)$. This is a subspace of $\mathcal{D}(T)$ where the extra condition for a vector $x \in \mathcal{D}(T)$ is that $Tx \in \mathcal{D}(T^*)$. To prove that $\mathcal{D}(T^*T)$ is dense we consider some orthogonal vector h. Now (2.6.1) gives $x \in \mathcal{D}(T)$ such that $h = x + T^*(Tx)$ and for every $g \in \mathcal{D}(T^*T)$ we have

$$0 = \langle x, g\rangle + \langle T^*Tx, g\rangle = \langle x, g\rangle + \langle Tx, Tg\rangle = \langle x, g\rangle + \langle x, T^*Tg\rangle \tag{i}$$

Here (i) hold for every $g \in \mathcal{D}(T^*T)$ and by another application (2.6.1) we can find $g_* \in \mathcal{D}(T^*T)$ so that $x = g_* + T^*Tg_*$. Then (i) applied with g_* gives $\langle x, x\rangle = 0$. Hence $x = 0$ which entials that $h = 0$ which shows that $\mathcal{D}(T^*T)^\perp = 0$ and the requested density follows.

2.8 The equality $TT^* = T^*T$. Recall the biduality formula in (2.4.3 and apply Proposition 2.6 starting with T^*. It follows that $\mathcal{D}(TT^*)$ also is dense and every $h \in \mathcal{H}$ has a unique representation

$$h = y + T(T^*y)$$

2.9. Conclusion. From the above T^*T is densely defined. Using the biduality formula (2.4.3) the reader may verify the equality

$$TT^* = T^*T$$

which means that T^*T is equal to its own adjoint. So if $A = T^*T$ then A is closed and $A = A^*$ which entails that A is self-adjoint operator in the sense of § XX below. Notice also that (2.6.1) entails that the densely defined and closed operator $E + T^*T$ is surjective, and by Proposition 2.6.2 it is injective. Hence it has an inverse resolvent in Neumann's sense.

§ 3. Symmetric operators.

A densely defined operator T on a Hilbert space $\mathcal{H}$ is symmetric if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{hold for all pairs} \quad x, y \in \mathcal{D}(T) \tag{*}$$

The symmetry means that the adjoint T^* extends T, i.e. one has the set-theoretic inclusion

$$\Gamma(T) \subset \Gamma(T^*)$$

In particular $\mathcal{D}(T^*)$ contains the dense space $\mathcal{D}(T)$ so Theorem 2.4.2 entials that T has a closed extension $\widehat{T}$ whoe graph is the closure $\overline{\Gamma(T)}$. Moreover, since T^* has a closed graph we have the inclusion

$$\overline{\Gamma(T)} \subset \Gamma(T^*) \tag{**}$$

If equality holds in (**) the closed abnd densely operator $\widehat{T}$ is slef-adjoint, i.e. it is symmstric and equal to its adjoint. When the inclusion in (**) is strict one may ask if there exist self-adjoint operators A whose graphs are contained between $\overline{\Gamma(T)}$ and $\Gamma(T^*)$. To analyze this we introduce the eigenspaces:

$$\mathcal{D}_+ = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = ix\} \quad \text{and} \quad \mathcal{D}_- = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = -ix\}$$

Since T^* has a closed graph these two subspaces of $\mathcal{H}$ are closed. Next, $x \mapsto (x, T^*x)$ is a bijective map between $\mathcal{D}(T^*)$ and $\Gamma(T^*)$. Since $\Gamma(T^*)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$, it follows that $\mathcal{D}(T^*)$ is a Hilbert space equipped with the inner product

$$\{x, y\} = \langle x, y\rangle + \langle T^*x, T^*y\rangle$$

Since $\widehat{T}$ has a closed graph and $\Gamma(T) \subset \Gamma(T^*)$ it follows that $\mathcal{D}(\widehat{T})$ appears as a closed subspace with vectors $(x, \widehat{T}x) : x \in \mathcal{D}(\widehat{T})$.

3.1 Proposition. *The following orthogonal decomposition exists in the Hilbert space $\mathcal{D}(T^*)$:*

$$(*) \qquad \mathcal{D}(T^*) = \mathcal{D}(\widehat{T}) \oplus \mathcal{D}_- \oplus \mathcal{D}_+$$

Proof. The verification that the three subspaces are pairwise orthogonal is left to the reader. There remains to see that the direct sum is equal to $\mathcal{D}(T^*)$ which amounts to show that

$$(1) \qquad \mathcal{D}(\widehat{T})^\perp = \mathcal{D}_- \oplus \mathcal{D}_+$$

To verify (1) we pick $y \in \mathcal{D}(\widehat{T})^\perp$ which gives

$$(2) \qquad \{x, y\} = \langle x, y\rangle + \langle Tx, T^*y\rangle = 0 \quad : \quad x \in \mathcal{D}(\widehat{T})$$

Since (2) in particular hold when $x \in \mathcal{D}(T)$ we see that $T^*y \in \mathcal{D}(T^*)$ and obtain

$$(3) \qquad \langle x, y\rangle + \langle x, T^*T^*y\rangle = 0 \quad : x \in \mathcal{D}(T)$$

The density of $\mathcal{D}(T)$ entails that

$$0 = y + T^*T^*y = (T^* + iE)(T^* - iE)(y) \implies$$
$$\xi = T^*y - iy \in \mathcal{D}_- \quad \text{and} \quad \eta = T^*y + iy \in \mathcal{D}_+ \implies$$
$$y = \frac{1}{2i}(\eta - \xi) \in \mathcal{D}_- \oplus \mathcal{D}_+$$

which proves (1) above.

3.2 The case when $\dim(\mathcal{D}_+) = \dim(\mathcal{D}_-)$**.** When they are finite dimensional with a common dimension $n \geq 1$ self-adjoint extensions of T are found as follows: Let $e_1, \ldots, e_n$ be an ON-basis in $\mathcal{D}_+$ and $f_1, \ldots, f_n$ an ON-basis in $\mathcal{D}_-$. For each n-tuple $e^{i\theta_1}, \ldots, e^{i\theta_n}$ of complex numbers with absolute value one we have the subspace of $\mathcal{H}$ generated by $\mathcal{D}(T)$ and the vectors

$$\xi_k = e_k + e^{i\theta_k} \cdot f_k \quad : \quad 1 \leq k \leq n$$

On this linear space we define a linear operator A_θ where $A_\theta = T$ on $\mathcal{D}(T)$ while

$$A_\theta(\xi_k) = ie_k - ie^{i\theta_k} \cdot f_k$$

3.3 Exercise. Verify that A_θ is self-adjoint and prove the converse, i.e. if A is an arbitrary self-adjoint extension of T with $\Gamma(A) \subset \Gamma(T^*)$, then $A = A_\theta$ for a unique n-tuple of complex unit vectors. Thus, the family of self-adjoint extensions of T can be identified with points on the n-dimensional torus.

3.4 Example.

The following example goes back to Weyl's studies about self-adjoint extensions of symmetric ODE-equations from 1910. Let $\mathcal{H}$ be the Hilbert space $L^2[0,1]$ of square-integrable functions on the unit interval $[0,1]$ with the coordinate t. A dense subspace $\mathcal{H}_*$ consists of functions $f(t) \in C^1[0,1]$ such that $f(0) = f(1) = 0$. On $\mathcal{H}_*$ we define the operator T by

$$T(f) = if'(t)$$

A partial integration gives

$$\langle T(f), g\rangle = i\int_0^1 f'(t) \cdot \bar{g}(t) \cdot dt = \int_0^1 \bar{g}'(t) \cdot f(t)dt = \langle f, T(g)\rangle$$

Hence T is symmetric. Next, an L^2-function h belongs to $\mathcal{D}(T^*)$ if and only if there exists a constant $C(h)$ such that

$$\Big|\int_0^1 if'(t)\cdot \bar{h}(t)dt\Big| \leq C(h)\cdot ||f||_2 \quad : f\in\mathcal{H}_*$$

This means that $\mathcal{D}(T^*)$ consists of all L^2-functions h such that the distribution derivative $\frac{dh}{dt}$ again belongs to L^2.

Exercise. Show that

$$\mathcal{D}_+ = \{h\in L^2 \quad : \frac{dh}{dt} = ih\}$$

is a 1-dimensional vector space generated by the L^2-function e^{ix}. Similarly, $\mathcal{D}_-$ is 1-dimensional and generated by e^{-ix}.

Self-adjoint extensions of T. For each complex number $e^{i\theta}$ we get the linear space $\mathcal{D}_\theta$ of functions $f(t)\in\mathcal{D}(T^*)$ such that

$$f(1) = e^{i\theta}f(0)$$

Exercise. Verify that one gets a self-adjoint operator T_θ which extends T where is $\mathcal{D}(T_\theta) = \mathcal{D}_\theta$. Conversely, show every self-adjoint extension of T is equal to T_θ for some θ. Hence the family $\{T_\theta\}$ give all self-adjoint extensions of T.

3.5 Semi-bounded symmetric operators.

Let T be closed, densely defined and symmetric. It is said to be bounded below if there exists some positive constant k such that

$$(*) \qquad \langle Tx, x\rangle \geq k\cdot ||x|| \quad : \quad x\in\mathcal{D}(T)$$

An example is when $K(x,y)$ is a symmetric positive kernel function on $[0,1]^2$ which gives a densely defined operator on $L^2[0,1]$. That the associated symmetric integral operator is self-adjoint was proved in [Carleman]. Passing to the "abstract case" where one starts from some T for which (*) holds, its self-adjoint extension was later described by Friedrichs and von Neumann in a general set-up which goes as follows: A sequence $\{x_n\}$ in $\mathcal{D}(T)$ is called a strong Cauchy-sequence if

$$\lim_{m,n\to\infty} ||x_n - x_m||^2 + ||T(x_n - x_m)||^2 = 0$$

Denote by $\mathcal{D}_0$ the linear space of vectors y for which there exist a strong Cauchy sequence $\{x_n\}$ with $y = \lim x_n$. Put

$$\mathcal{D}_* = \mathcal{D}_0 \cap \mathcal{D}(T^*)$$

3.5.1 Theorem. *The restriction of the adjoint T^* to $\mathcal{D}_*$ yields a self-adjoint operator. Moreover the range of this self-adjoint operator is equal to $\mathcal{H}$.*

Proof We can assume that $k=1$ in (*) and on $\mathcal{D}(T)$ we have the Hermitian bilinear form:

$$\text{(i)} \qquad \{x,y\} = \langle Tx, y\rangle \quad \text{where (*) entails that} \quad \{x,x\}\geq ||x||^2$$

This Hermitian form extends uniquely $\mathcal{D}_0$ where it becomes complete so that $\mathcal{D}_0$ is a Hilbert space. Next, one has the equality

$$\text{(ii)} \qquad T^*(\mathcal{D}_0) = \mathcal{H}$$

To see this we take an arbitrary vector $x\in\mathcal{H}$ which gives a continuous linear functional on $\mathcal{D}_0$ defined by

$$y\mapsto \langle y, x\rangle$$

Since $\mathcal{H}$ is self-dual we find a vector $z\in\mathcal{D}_0$ such that

$$\text{(iii)} \qquad \langle y, x\rangle = \{y,z\} = \langle Ty, z\rangle$$

This holds in particular for every $y \in \mathcal{D}(T)$ and hence z belongs to $\mathcal{D}(T^*)$ so that (iii) gives

$$\text{(iv)} \qquad \langle y, x\rangle = \langle y, T^*(z)\rangle$$

The density of $\mathcal{D}_0$ in $\mathcal{H}$ therefore gives $x = T^*(z)$ and (ii) is proved. Finally, denote by T_1 the linear operator with $\mathcal{D}(T_1) = \mathcal{D}_0$ and $T_1(x) = T^*(x)$. It is easily seen that T_1 is closed and symmetric. Now the requested self-adjointness follows from (ii) and the result below.

3.5.2 Proposition. *Let S be a densely defined, closed and symmetric operator such that*

$$(*) \qquad S(\mathcal{D}(S)) = \mathcal{H}$$

Then S is self-adjoint.

Proof. Let S^* be the adjoint of S. When $y \in \mathcal{D}(S^*)$ we have by definition

$$\langle Sx, y\rangle = \langle x, S^*y\rangle \quad : \quad x \in \mathcal{D}(S)$$

If $S^*y = 0$ this entails that $\langle Sx, y\rangle = 0$ for all $x \in \mathcal{D}(S)$ so the assumption that $S(\mathcal{D}(S)) = \mathcal{H}$ gives $y = 0$ and hence S^* is injective. Finally, if $x \in \mathcal{D}(S^*)$ the hypothesis (*) gives $\xi \in \mathcal{D}(S)$ such that

$$\text{(i)} \qquad S(\xi) = S^*(x)$$

Since S is symmetric, S^* extends S so that (i) gives $S^*(x-\xi) = 0$. Since we already proved that S^* is injective we have $x = \xi$. This proves that $\mathcal{D}(S) = \mathcal{D}(S^*)$ which means that S is self-adjoint.

3.6 Factorizations of non-symmetric kernels.

In Carleman's thesis [xxx 1916] spectral properties of Neumann-Poincaré kernels are reduced to the symmetric case. Recall that the Neumann-Poincaré kernel $K(p,q)$ of a plane C^1-curve $\mathcal{C}$ is given by

$$K(p,q) = \frac{|\langle p-q, \mathbf{n}_i(p)\rangle|}{|p-q|}$$

where $\mathbf{n_i(p)}$ denote inner normal vectors. Let R be a positive number which exceeds the diameter of $\mathcal{C}$ so that $|p-q| < R :\ p,q \in \mathcal{C}$. Set

$$N(p,q) = \int_{\mathcal{C}} K(p,\xi) \cdot \log \frac{R}{|q-\xi|} \cdot ds(\xi)$$

Exercise. Verify that N is symmetric, i.e. $N(p,q) = N(q,p)$ hold for all pairs p,q in $\mathcal{C}$. Moreover,

$$S(p,q) = \log \frac{R}{|p-q|}$$

is a symmetric and positive kernel function and since $\mathcal{C}$ is of class C^1 the reader should verify that it gives a Hilbert-Schmidt kernel, i.e.

$$\iint_{\mathcal{C}\times\mathcal{C}} S(p,q)^2\, ds(p)ds(q) < \infty$$

Hence the Neumann-Poincaré kernel $\mathcal{K}$ appears in an equation

$$(*) \qquad \mathcal{N} = \mathcal{K} \circ \mathcal{S}$$

where $\mathcal{S}$ is defined via a positive symmetric Hilbert-Schmidt kernel. Following [Carleman: § 11] we describe the spectrum of $\mathcal{K}$.

3.6.1 Spectral properties of non-symmetric kernels. In general, let $K(x,y)$ be a continuous real-valued function on the closed unit square $\square = \{0 \leq x, y \leq 1\}$ and assume that there exists a positive definite Hilbert-Schmidt kernel $S(x,y)$ such that

$$N(x,y) = \int_0^1 S(x,t)K(t,y)\, dy$$

yields a symmetric kernel function. The Hilbert-Schmidt theory gives an orthonormal basis $\{\phi_n\}$ in $L^2[0,1]$ formed by eigenfunctions to $\mathcal{S}$ where

$$\mathcal{S}\phi_n = \kappa_n\phi_n \tag{1}$$

where the positive κ-numbers tend to zero and each $u \in L^2[0,1]$ has a Fourier-Hilbert expansion

$$u = \sum \alpha_n \cdot \phi_n \tag{2}$$

We seek eigenfunctions of the integral operator $\mathcal{K}$. Let u be a function in $L^2[0,1]$ such that:

$$u = \lambda \cdot \mathcal{K}u \tag{3}$$

where λ in general is a complex number. It follows that

$$\lambda \cdot \int N(x,y)u(y)\,dy = \lambda \iint SA(x,t)K(t,y)u(y)\,dtdy = \int S(x,t)u(t)\,dt \tag{4}$$

Multiplying with $\phi_p(x)$ an integration gives

$$\lambda \cdot \int \phi_p(x)N(x,y)u(y)\,dxdy = \iint \phi_p(x)S(x,t)u(t)\,dxdt = \kappa_p \int \phi_p(t)u(t)\,dt \tag{5}$$

Next, the expansion of u from (2) gives the equations:

$$\sum_{q=1}^{\infty} \alpha_q \cdot \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy = \kappa_p\alpha_p \quad : \; p = 1, 2, \ldots \tag{6}$$

Set

$$c_{qp} = \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy$$

It follows that $\{\alpha_p\}$ satisfies the system

$$\kappa_p\alpha_p = \lambda \cdot \sum_{q=1}^{\infty} c_{qp}\alpha_q \tag{7}$$

Since $N(x,y) = N(y,x)$ the doubly indexed c-sequence is symmetric. Set

$$\beta_p = \sqrt{\kappa_p} \cdot \alpha_p \implies \beta_p = \lambda \cdot \sum_{q=1}^{\infty} \frac{c_{pq}}{\sqrt{\kappa_p} \cdot \sqrt{\kappa_q}} \cdot \beta_q \tag{8}$$

Next, put

$$k_{p,q} = \iint K(x,y)\phi_p(x)\phi_q(y)\,dxdy \tag{9}$$

From the above the following hold for each pair p, q:

$$c_{pq} = \iiint \phi_q(x)\phi_p(y)S(x,t)K(t,y)\,dxdydt = \kappa_q k_{p,q} = \kappa_p k_{q,p} \implies$$

$$\frac{c_{p,q}^2}{\kappa_p \cdot \kappa_q} \leq |k_{p,q} \cdot k_{q,p}| \leq \frac{1}{2}(k_{p,q}^2 + k_{q,p}^2) \tag{10}$$

Here $\{k_{p,q}\}$ are the Fourier-Hilbert coefficients of $K(x,y)$ which entails that

$$\sum\sum k_{p,q}^2 \leq \iint K(x,y)^2\,dxdy$$

Hence the symmetric and doubly indexed sequence

$$\frac{c_{p,q}}{\sqrt{\kappa_p \cdot \kappa_q}} \tag{11}$$

is of Hilbert-Schmidt type.

3.6.2 Conclusion. The eigenfunctions u in $L^2[0,1]$ associated to the $\mathcal{K}$-kernel have Fourier-Hilbert expansions via the $\{\phi_n\}$-basis which are determined by α-sequences satisfying the system (7)

3.6.3 Remark. When a plane curve $\mathcal{C}$ has corner points the Neumann-Poincaré kernel is unbounded. Here the reduction to the symmetric case is more involved and leads to quite intricate results which appear in Part II from Carleman's Ph.d-thesis. The interplay between singularities on boundaries in the Neumann-Poincaré equation and corresponding unbounded kernel functions illustrates the general theory densely defined self-adjoint operators. Much analysis remains to be done and open problems about the Neumann-Poincaré equation remain to be settled in dimension three. So far it appears that only the 2-dimensional case is properly understood via results in [Carleman:1916].

§ 4. Unbounded self-adjoint operators.

A densely defined operator A on the Hilbert space $\mathcal{H}$ for which $A = A^*$ is called self-adjoint.

4.1 Proposition *The spectrum of a self-adjoint operator A is contained in the real line. Moreover, if λ is non-real the resolvent satisfies the norm inequality*

$$||R_\lambda|| \leq \frac{1}{|\mathfrak{Im}\,\lambda|}$$

Proof. Set $\lambda = a + ib$ where $b \neq 0$. If $x \in \mathcal{D}(A)$ and $y = \lambda x - Ax$ we have

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Ax||^2 - 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle x, Ax\rangle$$

The Cauchy-Schwarz inequality gives

$$\text{(i)} \qquad ||y||^2 \geq b^2||x||^2 + a^2||x||^2 + ||Ax||^2 - 2|a| \cdot ||Ax|| \cdot ||x|| \geq b^2||x||^2$$

This proves that $x \to \lambda x - Ax$ is injective and since A is closed the range of $\lambda \cdot E - A$ is closed. Next, if y is $\perp$ to this range we have

$$0 = \lambda\langle x, y\rangle - \langle Ax, y\rangle \quad : x \in \mathcal{D}(A)$$

From this we see that y belongs to $\mathcal{D}(A^*)$ and since A is self-adjoint we get

$$0 = \lambda\langle x, y\rangle - \langle x, Ay\rangle$$

This holds for all x in the dense subspace $\mathcal{D}(A)$ which gives $\lambda \cdot y = Ay$ Since λ is non-real we have already seen that this entails that $y = 0$. Hence the range of $\lambda \cdot E - A$ is equal to $\mathcal{H}$ and the inequality (i) entails $R_A(\lambda)$ has norm $\leq \frac{1}{|\mathfrak{Im}\lambda}|$.

4.2 An adjoint formula. Let A be self-adjoint. If λ is non-real we have the adjoint formula:

$$\bar{\lambda} - A = (\lambda \cdot E - A)^*$$

So when we take the complex conjugate of λ it follows that § 9.5 that

$$R_A(\lambda)^* = R_A(\bar{\lambda})$$

4.3 Properties of resolvents. Let A be self-adjoint. By Neumann's resolvent calculus the family $\{(R_\lambda\}$ consists of pairwise commuting bounded operators outside the spectrum of A. In particular there exist that operator-valued analytic functions $\lambda \mapsto R_\lambda$ in the upper- respectively the lower half-plane. Moreover, since the resolvent operators commute, it follows from (9.12) that $R_A(\lambda)$ commutes with its adjoint, i.e. every resolvent is a normal operator.

4.4 A special resolvent operator. Take $\lambda = i$ and set $R = R_A(i)$. So here

$$R(iE - A)(x) = x \quad : \quad x \in \mathcal{D}(A)$$

4.5 Theorem. *The spectrum $\sigma(R)$ is contained in the circle*

$$C_* = \{|\lambda + i/2| = 1/2\}$$

Proof. Since $\sigma(A)$ is confined to the real line, it follows from § 0.0.5.1 that points in $\sigma(R)$ have the form

$$\lambda = \frac{1}{i-a} \quad : a \in \mathbf{R}$$

This gives

$$\lambda + i/2 = \frac{1}{i-a} + i/2 = \frac{1}{2(i-a)}(2+i^2-ia) = \frac{1-ia}{2i(1+ia)}$$

and the last term has absolute value $1/2$ for every real a.

§ 5. The spectral theorem for unbounded self-adjoint operators.

The operational calculus applies to the bounded normal operator R from (4.4). If N is a positive integer we set

$$C_*(N) = \{\lambda \in C_* : \mathfrak{Im}(\lambda) \leq -\frac{1}{N}\} \quad \text{and} \quad \Gamma_N = D_*(N) \cap \sigma(R)$$

Here Γ_N is a a compact subset of $\sigma(R)$. The function $g(\lambda) = \frac{1-i\lambda}{\lambda}$ is analytic in a neighborhood of Γ_N and operational calculus gives the bounded linear operator

$$S_N = g(R)$$

On Γ_N we have $\lambda = -i/2 + \zeta$ where $|\zeta| = 1/2$ which gives

$$(1) \qquad \frac{1-i\lambda}{\lambda} = \frac{1/2 - i\zeta}{-i/2+\zeta} = \frac{(1/2-i\zeta)(i/2+\bar{\zeta})}{|\zeta - i/2|^2} = \frac{\mathfrak{Re}\,\zeta}{|\zeta - i/2|^2}$$

Neumann's the spectral equation implies that $\sigma(S_N)$ is real. Since the operator S_N also is normal it follows from § xx that it is self-adjoint. Moreover, we notice that

$$\lambda \cdot (\frac{1-i\lambda}{\lambda} + i) = 1$$

Hence operational calculus gives the operator equation

$$\text{(i)} \qquad R(S_N + i) = E(\Gamma_N)$$

Notice also that

$$\text{(ii)} \qquad R \cdot S_N = (E - iR) \cdot E(\Gamma_N)$$

Hence (i-ii) entail that

$$\text{(iii)} \qquad E(\Gamma_N) - iRE(\Gamma_N) = (E - iR) \cdot E(\Gamma_N)$$

Next, the equation $RA = E - iR$ gives

$$(*) \qquad RAE(\Gamma_N) = (E - iR)E(\Gamma_N) = R\dot{S}_N$$

5.1 Exercise. Conclude from the above that

$$(*) \qquad AE(\Gamma_N) = SE(\Gamma_N)$$

where the right hand side is a bounded self-adjoint operator. Show also that:

$$(**) \qquad \lim_{N\to\infty} AP_N(x) = A(x) \quad \text{for each} \quad x \in \mathcal{D}(A)$$

5.2 Conclusion. Together (*) and (**) give the spectral resolution of A. Moreover, for each bounded Borel set e on the real line we get a corresponding Borel set on e_* in $\sigma(R)$ as desribed in (XX) above. The operational calculus gives the bounded self-adjoint projection P_{e_*} and the spectrum of the bounded self-adjoint operator AP_{e_*} is confined to the closure of e. More generally, real-valued and bounded Borel function $g(t)$ on the real line with compact support there exists the bounded self-adjoint operator $g(A)$ which commutes with A.

5.3 The case of integral operators.

Restricting the attention to self-adjoint operators on separable Hilbert spaces we can present the spectral theorem in a transparent way since the spectral Riesz measures arise via limits of measures obtained from Hilbert's result for bounded self-adjoint operators. Let us first insert:

A historic comment. In a plenary talk [Carleman] at the Scandinavian Congress held at Copenhagen in 1925, Carleman described the relation to the new spectral theory with Fredholm's pioneering discoveries where Carleman expressed his admiration for his work: *Peu du découvertes mathématiques ont été données par leur auteur sous une forme si achevée que la découverte de la solution de l´équation intégrale*

$$\phi(x) = \lambda \cdot \int_a^b K(x,y)\phi(y)\,dy + f(x)$$

par M. Fredholm. Les nombreuses application de cette théorie admirable ont pourtant améné un grand nombere de recherecheş concerntant le cas óu le noyaue $K(x,y$ est non borneé.

5.3.1 An ugly example.

Following Carleman in [ibid] we give an example which illustrates that new phenomena can occur for unbounded symmetric operators. On the unit interval $[0,1]$ we get an orthonormal family of functions $\psi_0, \psi_1, \ldots$ where $\psi_0 = 1$ is the identity and $\psi_1(x) = -1$ on $(0,1/2)$ and $+1$ on $(1/2,1)$. For each $n \geq 2$ we set:

$$\psi_n(x) = -2^{\frac{n-1}{[}2} : 1 - 2^{-n+1} \leq x < 1 - 2^{-n} \quad \text{and} \quad \psi_n(x) = 2^{\frac{n-1}{2}} : 1 - 2^{-n} < x < 1$$

while $\psi_n(x) =$ when $0 < x < 1 - 2^{-n+1}$. If $\{a_p\}$ is a sequence of real numbers we define the kernel function on $[0,1) \times [0,1)$ by

$$\text{(i)} \qquad K(x,y) = \sum a_p \cdot \psi_p(x)\psi_q(x)$$

It is clear that K is symmetric and the associated operator

$$\mathcal{K}(u)(x) = \int_0^1 K(x,y)u(y)\,dy$$

is defined on L^2-functions u supported by $0 \leq x \leq x_*$ for every $x_* < 1$ and therefore densely defined. Choose a sequence $\{a_p\}$ such that the positive series.

$$\text{(ii)} \qquad \sum_{p=0}^{\infty} \frac{2^p}{1 + a_p^2} < \infty$$

This convergence means that the sequence $\{a_p\}$ must have a high rate of growth so the kernel function (i) becomes turbulent when x and y approach 1.

Exercise. Show that (ii) entails that there exists an L^2-function $\phi(x)$ such that $\mathcal{K}(\phi)$ also is an L^2-function and

$$\phi = i \cdot \mathcal{K}(\phi)$$

If necessary, consult [ibid: page 62-66] for a demonstration.

5.3.2 A favourable case.

The example above shows that new phenomena can occur if one starts with a symmetric linear operator A on a Hilbert space $\mathcal{H}$ which only is densely defined. Let us now expose Carleman's results for "wellbehaved" integral operators. Let $K(x,y)$ be a complex-valued function defined in the open unit square $\{0 < x, y < 1\}$ satisfying the hermitian condition

$$K(y,x) = \bar{K}(x,y)$$

Assume that for all x outside a nullset $\mathcal{N}$ one has a finite integral

$$\text{(1)} \qquad \int_0^1 |K(x,y)|^2\,dy$$

However, the almost everywhere defined values in (1) need not give an integrable function. i.e. it can occur that the double integral

$$\iint_{\square} |K(x,y)|^2 \, dxdy = +\infty \tag{2}$$

But even when (2) occurs we notice that if $\phi(y)$ is an L^2-function, the Cauchy-Schwartz inequality gives:

$$\int_0^1 K(x,y)\phi(y)\, dy \quad \text{exists for all} \quad x \in (0,1) \setminus \mathcal{N}$$

A.1 Definition. *The hermitian kernel K is of type I if the homogeneous equation*

$$\phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\, dy$$

has no non-trivial solution ϕ when $\mathfrak{Im}\,\lambda \neq 0$.

A.2 Theorem. *If K is of type I then the equation below has a unique L^2-solution ϕ for every $f \in L^2$ for every non-real λ:*

$$\phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\, dy + f(x) \tag{*}$$

Moreover one has the estimate

$$\int_0^1 |\phi(x)|^2 \, dx \leq \frac{|\lambda|^2}{\mathfrak{Im}(\lambda)^2} \cdot \int_0^1 |f(x)|^2 \, dx \tag{**}$$

Proof. To prove this we use the truncated kernels $\{K_n(x,y)\}$ where $K_n(x,y) = K(x,y)$ when $K(x,y)| \leq n$ and otherwise zero. Given $f(x) \in L^2[0,1]$ and a non-real λ, the Hilbert-Schmidt theory from § xx gives a unique $\phi_n \in L^2$ such that

$$\phi_n(x) = \lambda \cdot \int_0^1 K_n(x,y)\phi_n(y)\, dy + f(x) \tag{i}$$

Mutltiplying with $\bar{\phi}_n$ it follows that

$$\frac{1}{\lambda} \int_0^1 \bar{\phi}_n(x)(\phi_n(x) - f(x))\, dx = \iint_{\square} K_n(x,y)\bar{\phi}_n(x)\phi_n(y)\, dxdy$$

Since K_n is hermitian the right hand side is real and taking imaginary parts we obtain

$$\left(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right) \cdot \int_0^1 |\phi_n(x|^2 \, dx) = \frac{1}{\lambda} \cdot \int_0^1 \bar{\phi}_n(x) f(x) dx) - \frac{1}{\bar{\lambda}} \cdot \int_0^1 \phi_n(x)\bar{f}(x) dx) \tag{ii}$$

and the Cauchy-Schwarz inequality gives

$$\int_0^1 |\phi_n(x)|^2 \, dx \leq \frac{|\lambda|^2}{\mathfrak{Im}(\lambda)^2} \cdot \int_0^1 |f(x)|^2 \, dx \tag{iii}$$

Hence the L^2-norms of $\{\phi_n\}$ are uniformly bounded and we can find a subsequence which converges weakly to an L^2-function ϕ whose L^2-norm again is bounded by the right hand side in (iii). We leave it to the reader to check that ϕ solves the equation (*) and since K is of type I the limit ϕ is unique which'entails that the whole sequence $\{\phi_n\}$ converges weakly to ϕ which is the requested solution to the inhomogenous equation (*).

A.3 Spectral measures. Let $K(x,y)$ be a hermitian kernel of type I. Hilbert's theorem applies to the bounded kernels $\{K_n\}$ and give for each pair $f,g \in L^2[0,1]$ a Riesz measure $\mu^n_{f,g}$ supported by the compact real spectrum of the kernel operator defined by K_n. A "non-trivial analysis" in Carleman's monograph [ibid] shows that the sequence $\{\mu^n_{f,g}\}$ converges weakly to a Riesz-measure $\mu_{f,g}$ supported by the real line. Now Hilbert's operational calculus applies when one integrates continuous functions $\phi(t)$ with compact support, i.e. every such ϕ yields a bounded linear operator Φ on $L^2[0,1]$ such that

$$\int_0^1 \Phi(f)\cdot\bar{g}\,dx = \int \phi(t)d\mu_{f,g}(t)$$

hold for every pair f, g in $L^2[0,1]$ and from this one can read off the conclusions in § 5.2

§ 6. Proof of Theorem xx.

Recall that we study the operator $L\Delta + c$ which is densely defined on $L^2(\mathbf{R}^3)$. Set

$$A(p,q) = \frac{1}{|p-q|} + \frac{|p-q|}{R^2} - \frac{2}{R}$$

If D is an open domain in $\mathbf{R}^3$ and $\phi \in L^2(D)$, then Newton's classical formula shows that a function u satisfies the inhomogeneous equation $\Delta(u) = \phi$ if and only if the following hold for every open ball of radius R

$$u(p) = \frac{1}{2\pi R^2}\cdot\iiint_{B_R}\frac{\phi(q)}{|p-q|}\,dq + \frac{1}{4\pi}\cdot\iiint_{B_R} A(p,q)\cdot\phi(q)\,dq$$

From this Greens formula gives the equation below for bounded domains D with C^2-boundary:

$$4 + 5onpage464. \tag{6.1}$$

Using (6.1) a straighforward investigation of solutions to L-equations over relatively compact balls which is left as an exercise to the reader, gives a sufficient condition in order that the symmetric operator L which to begin with is defined on test-functions in $\mathbf{R}^3$, has a self-adjoint extension whose the domain of definition precisely consists of those $u \in L^2\mathbf{R}^3$ for which $L(u)$ stays square integrable. More precisely, a sufficient condition is that the following hold for each pair u, v in $\mathcal{D}(L)$:

$$\liminf_{R\to\infty}\int_{S_R}\big|\,u\cdot\frac{\partial v}{\partial\mathbf{n}} - v\cdot\frac{\partial u}{\partial\mathbf{n}}\,\big|\,d\sigma_R = 0 \tag{*}$$

where S_R denote spheres of radius R and σ_R is the area measure. There remains to show that the existence of a constant M such that

$$\limsup_{x^2+y^2+z^2} c(x,y,z) \leq M \tag{**}$$

gives (*). To prove this implication one needs the following:

Lemma. *For each $u \in \mathcal{D}(L)$ one has*

$$\liminf_{R\to\infty}\int_{S_R}\frac{\partial u}{\partial\mathbf{n}}\,d\sigma_R = 0 \tag{6.2}$$

The proof is left as an exercise. If necessary, consult [p. 178 in Carleman] for details.

Next, apply (6.1) with $v = u$. Then (6.2) and (**) entail that the Dirichlet integral

$$\iiint |\nabla(u)|^2\,dq < \infty \tag{6.3}$$

and *a fortiori*

$$\int_0^\infty \big[\int_{S_R}(\frac{\partial u}{\partial\mathbf{n}})^2\,d\sigma_R\,\big]\,dR < \infty \tag{6.4}$$

The same holds for v which entails that

$$\int_0^\infty \big[\int_{S_R}(u^2 + v^2 + (\frac{\partial u}{\partial\mathbf{n}})^2 + (\frac{\partial v}{\partial\mathbf{n}})^2)\,d\sigma_R\,\big]\,dR < \infty \tag{6.5}$$

Set

$$J(R) = \int_{S_R} \left| u \cdot \frac{\partial v}{\partial \mathbf{n}} - v \cdot \frac{\partial u}{\partial \mathbf{n}} \right| d\sigma_R \tag{6.6}$$

Now (6.5) and the Cauchy-Schwarz inequality give

$$\int_0^\infty J(R)\, dR < \infty \tag{6.6}$$

Finally it is clear that (6.6) gives (*) above.

Appendix. Linear operators and spectral theory.

Let X be a Banach space and $T\colon X \to X$ a linear and densely defined operator whose domain of definition is denoted by $\mathcal{D}(T)$. In general T is unbounded:

$$\max_{x\in\mathcal{D}_*(T)} ||T(x)|| = +\infty$$

where the maximum is taken over unit vectors in $\mathcal{D}(T)$.

A.1 Inverse operators. A densely defined operator T has a bounded inverse if the range $T(\mathcal{D}(T))$ is equal to X and there exists a positive constant c such that

(i) $$||T(x)|| \geq c \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

Since $T(\mathcal{D}(T)) = X$, (i) gives for each $x \in X$ a unique vector $R(x) \in \mathcal{D}(T)$ such that

(ii) $$T \circ R(x) = x$$

Moreover, the inequality (i) gives

(iii) $$||R(x)|| \leq c^{-1} \cdot ||x|| \quad : x \in X$$

and when R is applied to the left on both sides in (ii), it follows that

(iv) $$R \circ T(x) = x \quad : x \in \mathcal{D}(T)$$

A.2 The spectrum $\sigma(T)$. Let E be the identity operator on X. Each complex number λ gives the densely defined operator $\lambda \cdot E - T$. If it fails to be invertible one says that λ is a spectral point of T and denote this set by $\sigma(T)$. If $\lambda \in \mathbf{C} \setminus \sigma(T)$ the inverse to $\lambda \cdot E - T$ is denoted by $R_T(\lambda)$ and called a Neumann resolvent to T. By the construction in (A.1) the range of every Neumann resolvent is equal to $\mathcal{D}(T)$ and one has the equation:

(A.2.1) $$T \circ R_T(\lambda)(x) = R_T(\lambda) \circ T(x) \quad : x \in \mathcal{D}(T)$$

Example. Let X be the Hilbert space ℓ^2 whose vectors are complex sequences $\{c_1, c_2, \ldots\}$ for which $\sum\, |c_n|^2 < \infty$. We have the dense subspace ℓ^2_* vectors such that $c_n \neq 0$ only occurs for finitely many integers n. If $\{\xi_n\}$ is an arbitrary sequence of complex numbers there exists the densely defined operator T on ℓ^2 which sends every sequence vector $\{c_n\} \in \ell^2_*$ to the vector $\{\xi_n \cdot c_n\}$. If λ is a complex number the reader may check that (i) holds in (A.1) if and only if there exists a constant C such that

(i) $$|\lambda - \xi_n| \geq C \quad : n = 1, 2, \ldots$$

Thus, $\lambda \cdot E - T$ has a bounded inverse if and ony if λ belongs to the open complement of the closure of the set $\{\xi_n\}$ taken in the complex plane. Moreover, if (i) holds then $R_T(\lambda)$ is the bounded linear operator on ℓ^2 which sends $\{c_n\}$ to $\{\frac{1}{\lambda - \xi_n} \cdot c_n\}$. Since every closed subset of $\mathbf{C}$ is equal to the closure of a denumerable set of points our construction shows that the spectrum of a densely defined operator $\sigma(T)$ can be an arbitrary closed set in $\mathbf{C}$. The equation below is due to G. Neumann:

A.3 Neumann's equation. *Assume that $\sigma(T)$ is not the whole complex plane. For each pair $\lambda \neq \mu$ outside $\sigma(T)$ the operators $R_T(\lambda)$ and $R_T(\mu)$ commute and*

(*) $$R_T(\mu) R_T(\lambda) = \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu}$$

Proof. Notice that

$$(\mu \cdot E - T) \cdot \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} =$$

(i) $$\frac{E}{\lambda - \mu} - (\mu - \lambda) \cdot \frac{R_T(\lambda)}{\lambda - \mu} - (\lambda \cdot E - T) \cdot \frac{R_T(\lambda)}{\lambda - \mu} = R_T(\lambda)$$

Multiplying to the left by $R_T(\mu)$ gives (*) which at the same time this shows that the resolvent operators commute.

A.4 The position of $\sigma(T)$. Assume that $\mathbf{C} \setminus \sigma(T)$ is non-empty. We can write (*) in the form

$$R_T(\lambda)(E + (\lambda - \mu)R_T(\mu)) = R_T(\mu) \tag{1}$$

Keeping μ fixed we conclude that $R_T(\lambda)$ exists if and only if $E+(\lambda-\mu)R_T(\mu))$ is invertible which implies that

$$\sigma(T) = \{\lambda : \frac{1}{\mu - \lambda} \in \sigma(R_T(\mu))\} \tag{A.4.1}$$

Hence one recovers $\sigma(T)$ via the spectrum of any given resolvent operator. Notice that (A.4.1) holds even when the open component of $\sigma(T)$ has several connected components.

A.4.2 Example. Suppose that $\mu = i$ and that $\sigma(R_T(i))$ is contained in a circle $\{|\lambda + i/2| = 1/2\}$. If $\lambda \in \sigma(T)$ the inclusion (A.4.1) gives some $0 \leq \theta \leq 2\pi$ such that

$$\frac{1}{i - \lambda} = -i/2 + 1/2 \cdot e^{i\theta} \implies 1 - i \cdot e^{i\theta} = \lambda(e^{i\theta} - i) \implies$$

$$\lambda = \frac{2 \cdot \cos\theta}{|e^{i\theta} - i|^2} \in \mathbf{R}$$

A.4.3 Neumann series. Let λ_0 be outside $|sigma(T)$ and construct the operator valued series

$$S(\zeta) = R_T(\lambda_0) + \sum_{n=1}^{\infty} (-1)^n \cdot \zeta^n \cdot R_T(\lambda_0)^{n+1} \tag{1}$$

It is clear that (1) converges in the Banach space of bounded linear operators when

$$|\zeta| < \frac{1}{||R_T(\lambda_0)||} \tag{2}$$

Moreover, the series expansion (1) gives

$$(\lambda_0 + \zeta - T) \cdot S(\zeta) = (\lambda_0 - T) \cdot S(\zeta) + \zeta S(\zeta) = E \tag{23}$$

Hence $S(\zeta) = R_T(\lambda_0 + \zeta)$ and the locally defined series in (1) entail the complement of $\sigma(T)$ is open where $\lambda \mapsto R_T(\lambda)$ is an analytic operator-valued function. Finally (*) in (A.3) and a passage to the limit as $\mu \to \lambda$ shows that this analytic function satisfies the differential equation

$$\frac{d}{d\lambda}(R_T(\lambda) = -R_T^2(\lambda) \tag{**}$$

A.5. Operators with closed graph.

Let T be a densely defined operator. In the product $X \times X$ we get the graph:

$$\Gamma(T) = \{(x, Tx) \quad : x \in \mathcal{D}(T)\} \tag{A.5.1}$$

If $\Gamma(T)$ is a closed subspace of $X \times X$ we say that T is closed.

A.5.1 Exercise. Let T be densely defined and assume that $\sigma(T)$ is not the whole complex plane. Show that T is automatically closed.

In the study of spectra we shall foremost restrict the attention to closed operators. Assume that T is densely defined and closed. Let λ be a complex number such that (i) holds in (A.1) for some constant c and the range of $\lambda \cdot E - T$ is dense.

A.5.2 Exercise. Show that the closedness of T implies that the range of $\lambda \cdot E - T$ is equal to X so that $R_T(\lambda)$ exists. A hint is that if $y \in X$, then the density gives a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $\xi_n = \lambda \cdot x_n - T(x_n) \to y$. In particular $\{\xi_n\}$ is a Cauchy- sequence By (i) in (A.1) we have

$$||x_n - x_m|| \leq c^{-1} \cdot ||\xi_n - \xi_m||$$

Hence $\{x_n\}$ is a Cauchy sequence and since X is a Banach space there exists $x \in S$ such that $x_n \to x$. But then $\{(x_n, T(x_n)) = (x_n, \lambda \cdot x_n - \xi_n)$ converges to $(x, \lambda \cdot x - y)$ and since T is closed it follows that $(x, \lambda \cdot x - y) \in \Gamma(T)$ which gives the requested surjectivity since

$$T(x) = \lambda \cdot x - y \implies y = (\lambda \cdot E - T)(x)$$

A.5.3 Adjoints. Let T be densely defined but not necessarily closed. In the dual space X^* we get the subspace of vectors y for which there exists a constant $C(y)$ such that

$$\text{(i)} \qquad |y(Tx)| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

When (i) holds the density of $\mathcal{D}(T)$ gives a unique vector $T^*(y)$ in X^* such that

$$\text{(ii)} \qquad y(Tx) = T^*(y)(x) \quad : x \in \mathcal{D}(T)$$

One refers to T^* as the adjoint operator of T whose domain of definition is denoted by $\mathcal{D}(T^*)$.

Exercise. Show that T^* has a closed graph.

A.5.4 Closed extensions. Let T be densely defined but not closed. The question arises when the closure of $\Gamma(T)$ is the graph of a linear operator $\widehat{T}$ and then we refer to $\widehat{T}$ as a closed extension of T. A sufficient condition for the existence of a close extension goes as follows:

A.5.5 Theorem. *If $\mathcal{D}(T^*)$ is dense in X^* then T has a closed extension.*

Proof. Consider the graph $\Gamma(T)$ and let $\{x_n\}$ and $\{\xi_n\}$ be two sequences in $\mathcal{D}(T)$ which both converge to a point $p \in X$ while $T(x_n) \to y_1$ and $T(\xi_n) \to y_2$ hold for some pair y_1, y_2. We must rove that $y_1 = y_2$. To achieve this we take some $x^* \in \mathcal{D}(T^*)$ which gives

$$x^*(y_1) = \lim\, x^*(Tx_n) = \lim\, T^*(x^*)(x_n) = T^*(x^*)(p)$$

In the same way we get $x^*(y_2) = T^*(x^*)(p)$. Now the density of $\mathcal{D}(T^*)$ gives $y_1 = y_2$ which proves that the closure of $\Gamma(T)$ is a graphic subset of $X \times X$ and gives the closed operator $\widehat{T}$ with

$$\Gamma(\widehat{T}) = \overline{\Gamma(T)}$$

A.5.6 Remark. In general, let T be closed and densely defined. There may exist several closed operators S with the property that

$$\Gamma(T) \subset \Gamma(S)$$

Passing to adjoint operators the reader may verify that the density of $\mathcal{D}(T)$ implies that

$$T^* = S^*$$

hold for every closed extension S.

A.6 Operational calculus.

Let T be densely defined and closed. To each pair (γ, f), where γ is a rectifiable Jordan arc contained in $\mathbf{C} \setminus \sigma(T)$ and $f \in C^0(\gamma)$, there exists the bounded linear operator

$$\text{(A.6.1)} \qquad T_{(\gamma,f)} = \int_\gamma f(z) R_T(z)\, dz$$

The integrand has values in the Banach space of bounded linear operators on X and (A.6.1) is calculated by Riemann sums. Next, Neumann's equation (A.3) entails that $R_T(z_1)$ and $R_T(z_2)$ commute for all pairs z_1, z_2 on γ. From this it is clear that if g is another function in $C^0(\gamma)$, then the operators $T_{f,\gamma}$ and $T_{g,\gamma}$ commute. Moreover, for each $f \in C^0(\gamma)$ the reader may verify that the closedeness of T implies that the range of $T_{f,\gamma}$ is contained in $\mathcal{D}(T)$ and

$$T_{f,\gamma} \circ T(x) = T \circ T_{f,\gamma}(x) \quad : x \in \mathcal{D}(T)$$

Next, let Ω be an open set of class $\mathcal{D}(C^1)$, i.e. $\partial\Omega$ is a finite union of closed differentiable Jordan curves. When $\partial\Omega \cap \sigma(T) = \emptyset$ we construct the line integrals (A.6.1) for continuous functions on the boundary. Consider the algebra $\mathcal{A}(\Omega)$ of analytic functions in Ω which extend to be continuous on the closure. Each $f \in \mathcal{A}(\Omega)$ gives the operator

$$\text{(A.6.2)} \qquad T_f = \int_{\partial\Omega} f(z) R_T(z)\, dz$$

A.6.3 Theorem. *The map $f \mapsto T_f$ is an algebra homomorphism from $\mathcal{A}(\Omega)$ into a commutative algebra of bounded linear operators on X whose image is a commutative algebra of bounded linear operators denoted by $T(\Omega)$.*

Proof. Let f, g be a pair in $\mathcal{A}(\Omega)$. We must show the equality

$$(*) \qquad T_{gf} = T_f \circ T_g$$

To attain this we choose a slightly smaller open set $\Omega_* \subset \Omega$ which again is of class $\mathcal{D}(C^1)$ and each of it bounding Jordan curve is close to one boundary curve in $\partial\Omega$ and $\Omega \setminus \Omega_*$ does not intersect $\sigma(T)$. By Cauchy's theorem we can shift the integration to $\partial\Omega_*$ and get

$$\text{(i)} \qquad T_g = \int_{\partial\Omega_*} g(z) R_T(z_*)\, dz_*$$

where we use z_* to indicate that integration takes place along $\partial\Omega_*$. Now

$$\text{(ii)} \qquad T_f \circ T_g = \iint_{\partial\Omega_* \times \partial\Omega} f(z) g(z_*) R_T(z) \circ R_T(z_*)\, dz_* dz$$

Neumann's equation (*) from (A.3) entails that the right hand side in (ii) becomes

$$\text{(iii)} \qquad \iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z_*)}{z - z_*}\, dz_* dz + \iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z)}{z - z_*}\, dz_* dz = A + B$$

Here A is evaluated by first integrating with respect to z and Cauchy's theorem gives

$$f(z_*) = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega} \frac{f(z)}{z - z_*} \quad : z_* \in \partial\Omega_*\, dz$$

It follows that

$$A = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega_* \times \partial\Omega} f(z_*) g(z_*) R_T(z_*)\, dz_* = T_{fg}$$

Next, B is evaluated when we first integrate with respect to z_*. Here

$$\iint_{\partial\Omega} \frac{g(z_*)}{z - z_*} \quad : z \in \partial\Omega$$

which entails that $B = 0$ and the theorem follows.

A.7 Spectral gap sets.

Let K be a compact subset of $\sigma(T)$ such that $\sigma(T) \setminus K$ is a closed set in $\mathbf{C}$. This implies that if V is an open neighborhood of K, then there exists a relatively compact subdomain $U \in \mathcal{D}(C^1)$ which contains K as a compact subset while $\partial U \cap \sigma(T) = \emptyset$. To every such domain U we can apply Theorem A.6.3. If $U_* \subset U$ for a pair of such domains we can restrict functions in $\mathcal{A}(U)$ to U_*. This yields an algebra homomorphism

$$\mathcal{T}(U) \to \mathcal{T}(U_*)$$

Next, denote by $\mathcal{O}(K)$ the algebra of germs of analytic functions on K. So each $f \in \mathcal{O}(K)$ comes from some analytic function in a domain U as above. The resulting operator $T_U(f)$ depends on the germ f only. In fact, this follows because if $f \in \mathcal{A}(U)$ and $U_* \subset U$ is a similar $\mathcal{D}(C^1)$-domain which again contains K, then Cauchy's vanishing theorem in analytic function theory applies to $f(z) R_T(z)$ in $U \setminus \bar{U}_*$ and entails that

$$\int_{\partial U_*} f(z) R_T(z)\, dz = \int_{\partial U} f(z) R_T(z)\, dz$$

Hence there exists an algebra homorphism from $\mathcal{O}(K)$ into a commutative algebra of bounded linear operators on X denoted by $\mathcal{T}(K)$. The identity in $\mathcal{T}(K)$ is denoted by E_K and called the spectral projection operator attached to the compact set K in $\sigma(T)$. By this construction one has

$$E_K = \frac{1}{2\pi i} \cdot \int_{\partial U} R_T(z)\, dz$$

for every open domain U surrounding K as above.

A.7.1 The operator T_K. When K is a compact spectral gap set of T we set

$$T_K = TE_K$$

This is a bounded linear operator given by

$$\frac{1}{2\pi i}\cdot\int_{\partial U} z\cdot R_T(z)\,dz$$

where U is a domain as above which contains K.

A.7.2 Theorem. *Identify T_K with a linear operator on the subspace $E_K(X)$. Then one has the equality*

$$\sigma(T_K) = K$$

Proof. If λ_0 is outside K we can choose U so that λ_0 is outside $\bar{U}$ and construct the operator

$$S = \frac{1}{2\pi i}\cdot\int_{\partial U}\frac{1}{\lambda_0 - z}\cdot R_T(z)\,dz$$

The operational calculus gives

$$S(\lambda_0 E_K - T) = E_K$$

here E_K is the identity operator on $E_K(X)$ which shows that $\sigma(T_K) \subset K$.

A.7.3 Discrete spectra. Consider a spectral set reduced to a singleton set $\{\lambda_0\}$, i.e. λ_0 is an isolated point in $\sigma(T)$. The associated spectral projection is denoted by $E_T(\lambda_0)$ and given by

$$E_T(\lambda_0) = \frac{1}{2\pi i}\cdot\int_{|\lambda-\lambda_0|=\epsilon} R_T(\lambda)\,d\lambda$$

for all sufficiently small ϵ. Now $R_T(\lambda)$ is an analytic function defined in some punctured disc $\{0 < \lambda - \lambda_0| < \delta\}$ with a Laurent expansion

$$R_T(\lambda) = \sum_{-\infty}^{\infty}(\lambda-\lambda_0)^k\cdot B_k$$

where $\{B_k\}$ are bounded linear operators obtained by residue formulas:

$$B_k = \frac{1}{2\pi i}\cdot\int_{|\lambda|=\epsilon}\frac{R_T(\lambda)}{(\lambda-\lambda_0)^{k+1}}\,d\lambda \quad : \epsilon < \delta$$

Exercise. Show that $R_T(\lambda)$ is meromorphic, i.e. $B_k = 0$ hold when $k << 0$, if and only if there exists a constant C and some integer $M \geq 0$ such that the operator norms satisfy

$$||R_T(\lambda)|| \leq C\cdot|\lambda-\lambda_0)^{-M}$$

Suppose now that R_T has a pole of some order $M \geq 1$ at λ_0 which gives a series expansion

(i)
$$R_T(\lambda) = \sum_{1}^{M}\frac{B_{-k}}{(\lambda-\lambda_0)^k} + \sum_{0}^{\infty}(\lambda-\lambda_0)^k\cdot B_k$$

Residue calculus gives

(ii)
$$B_{-1} = E_T(\lambda_0)$$

The case of a simple pole. Suppose that $M = 1$. Then it is clear that Operational calculus gives

(iii)
$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\cdot\int_{|\lambda-\lambda_0|=\epsilon}(\lambda-\lambda_0)R(\lambda)\,d\lambda = 0$$

This vanishing and operational calculus entail that

$$(\lambda_0 E - T)E_T(\lambda_0) = 0$$

which means that the ragne of the projection operator $E_T(\lambda_0)$ is equal to the kernel of $\lambda_0 \cdot E - T$, i.e. the set of eigenvectors x for which

$$Tx = \lambda_0 \cdot x$$

The case $M \geq 2$. To begin with residue calculus identifies (iii) with B_2 and at the same time operational calculus which aftera reversed sign gives

$$(\lambda_0 E - T)E_T(\lambda_0) = -B_2$$

Since $E_T(\lambda_0)$ is a projection which commutes with T it follows that

$$E_T(\lambda_0) \cdot B_2 = B_2 \cdot E_T(\lambda_0) = B_2$$

Exercise. Show that if $M \geq 3$ then

$$(\lambda_0 E - T)^k \cdot E_T(\lambda_0) = (-1)^k \cdot B_{k+1} \quad : 2 \leq k \leq M$$

Consider also the subspaces

$$\mathcal{N}_k(\lambda_0) = \{x \colon (\lambda_0 E - T)^k(x) = 0\} \quad : 1 \leq k \leq M$$

and show that they are non-decreasing and for every $k > M$ one has

$$\mathcal{N}_M(\lambda_0) = \{x \colon (\lambda_0 E - T)^k(x) = 0\}$$

A.7.4 The case when $E_T(\lambda_0)$ has a finite dimensional range. Here the operator $T(\lambda_0) = TE_T(\lambda_0)$ acts on this finite dimensional vector space which entails that the nullspaces $\{\mathcal{N}_k(\lambda_0)\}$ above are finite dimensional and if M is the order of the pole one has

$$(T(\lambda_0) - \lambda_0)^M = 0$$

So λ_0 is the sole eigenvalue of $T(\lambda_0)$. Moreover, the finite dimensional range of $E_T(\lambda_0)$ has dimension equal to that of $\mathcal{N}_M(\lambda_0)$.

Exercise. Recall l that finite dimensional subspaces appear as direct sum compoinetns. So if $E_T(\lambda_0)$ is finite dimensional there exists a direct sum decomposition

$$X = E_T(\lambda_0)(X) \oplus E - E_T(\lambda_0)$$

where $V = E - E_T(\lambda_0)$ is a closed subspace of X. Show that V is T-invariant and that there exists some $c > 0$ such that

$$||\lambda_0 x - Tx|| \geq ||x|| \quad x \in V \cap \mathcal{D}(T)$$

A.8 Carleman's inequality.

In many applications one is interested to estimate the operator norm of resolvent operators. In this connection a useful result was proved by Carleman in the article *Sur le genre du denominateur $D(\lambda)$ de Fredholm* from 1917. Let us remark that the finite dimensional case in the theorem below easily extends to analogue results for operators on infinite-dimensional Hilbert spaces. See Chapter XI in [Dunford-Schwartz] an account and applications of Carleman's inequality in Theorem A.8.1 below. First we recall:

The Hilbert-Schmidt norm. Let n be a positive integer and $A = \{a_{ik}\}$ some $n \times n$-matrix whose elements are complex numbers. Set

$$||A|| = \sqrt{\sum\sum |a_{ik}|^2}$$

where the double sum extends over all pairs $1 \leq i, k \leq n$. Next, for a linear operator S on $\mathbf{C}^n$ its *operator norm* is defined by

$$\text{Norm}[S] = \max_x ||S(x)|| \quad \text{with the maximum taken over unit vectors.}$$

Identifying a matrix A with a linear operator on the Hermitian vector space $\mathbf{C}^n$, it is clear that

$$\text{Norm}[A] \leq ||A||$$

Examples show that the inequality in general is strict. Let $\lambda_1 \ldots, \lambda_n$ be the unordered n-tuple of roots of $\det(\lambda \cdot E_n - A)$ with eventual multiple zeros repeated. The union of these roots give the spectrum $\sigma(A)$.

A.8.1 Theorem. *For each non-zero λ outside $\sigma(A)$ one has the inequality:*

$$|\prod_{i=1}^{i=n}(1-\frac{\lambda_i}{\lambda})e^{\lambda_i/\lambda}| \cdot \text{Norm}\big[R_A(\lambda)\big] \leq |\lambda| \cdot \exp\big(\frac{1}{2}+\frac{||A||^2}{2\cdot|\lambda|^2}\big)$$

The proof requires some preliminary results. First we recall a wellknown inequality due to Hadamard which goes as follows:

1. Hadamard's inequality. *For every matrix A with a non-zero determinant one has the inequality*

$$\big|\det(A)\big| \cdot \text{Norm}(A^{-1}) \leq \frac{||A||^{n-1}}{(n-1)^{n-1)/2}}$$

2. Traceless matrices. The trace of the $n \times n$-matrix A is given by

(i) $$\text{Tr}(A) = b_{11} + \ldots + b_{nn}$$

Recall that $\text{Tr}(A)$ is equal to the sum of the roots of the characterstic polynomial $\det(\lambda \cdot E - n - A)$. In particular the trace of two equivalent matrices are equal which will be used to prove the following:

3. Lemma. *Let A be an $n \times n$-matrix whose trace is zero. Then there exists a unitary matrix U such that the diagonal elements of U^*AU all are zero.*

Proof. Consider first consider the case $n = 2$. Since A can be transformed to an upper tringular matrix via a unitary transformation in $\mathbf{C}^2$, it suffices to consider the case when the traceless 2×2-matrix A has the form

$$A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$$

where a, b is a pair of complex numbers. If $a = 0$ the diagonal elements are zero and we can take $U = E_2$ to be the identity. If $a \neq 0$ we consider a vector $\phi = (1, z)$ in $\mathbf{C}^2$. Then $A(\phi)$ is the vector $(a + bz, -az)$ and the inner product becomes:

(i) $$\langle A(\phi), \phi\rangle = a + bz - a|z|^2$$

We can write

$$\frac{b}{a} = re^{i\theta}$$

where $r > 0$ and then (i) is zero if

(ii) $$|z|^2 = 1 + se^{i\theta} \cdot z$$

With $z = se^{-i\theta}$ it amounts to find a positive real number s such that $s^2 = 1 + s$ which clearly exists. Now we get the vector

$$\phi_* = \frac{1}{1+s^2}(1, se^{-i\theta})$$

which has unit length and

(ii) $$\langle A(\phi_*), \phi_*\rangle = 0$$

Now we can find another unit vector ψ_* so that ϕ_*, ψ_* is an orthonormal base in $\mathbf{C}^2$ and hence there exists a unitary matrix U such that $U(e_1) = \phi_*$ and $U(e_2) = \psi_*$. If $B = U^*AB$ the vanishing in (ii) gives $b_{11} = 0$. At the same time the trace is unchanged, i.e. $\text{tr}(B) = 0$ holds and hence we also get $b_{22} = 0$. This means that the diagonal elements of U^*AU are both zero as required.

The case $n \geq 3$. For the induction the following is needed:

Sublemma. For each $n \geq 3$ *there exists some non-zero vector* $\phi \in \mathbf{C}^n$ *such that*

$$\langle A(\phi), \phi \rangle = 0 \tag{*}$$

Proof. If (*) does not hold we get the positive number

$$m_* = \min_{\phi} \big|\langle A(\phi), \phi \rangle\big|$$

where the minimum is taken over unit vectors in $\mathbf{C}^n$. The minimum is achieved by some unit vector ϕ_*. Let $\phi_*^\perp$ be its orthonormal complement and E is the self-adjoint projection from $\mathbf{C}^n$ onto $\phi_*^\perp$. On the $(n-1)$-dimensional inner product space $\phi_*^\perp$ we get the linear operator $B = EA$, i.e.

$$B(\xi) = E(A(\xi)) \quad : \quad \xi \in \phi_*^\perp \tag{i}$$

If $\psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\phi_*^\perp$ then the n-tuple $\phi_*, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and since the trace of A is zero we get

$$0 = \langle A(\phi_*), \phi_* \rangle + \sum_{\nu=1}^{\nu=n-1} \langle A(\psi_\nu), \psi_\nu \rangle = m + \sum_{\nu=1}^{\nu=n-1} \langle B(\psi_\nu), \psi_\nu \rangle \tag{ii}$$

where we used that $E(\psi_\nu) = \psi_\nu$ for each ν and that E is self-adjoint so that

$$\langle A(\psi_\nu), \psi_\nu \rangle = \langle A(\psi_\nu), E(\psi_\nu) \rangle = \langle E(A(\psi_\nu)), \psi_\nu \rangle = \langle B(\psi_\nu), \psi_\nu \rangle$$

Now (ii) gives

$$\mathrm{Tr}(B) = -m$$

Hence the $(n-1) \times (n-1)$-matrix which represents $B + \frac{m}{n-1} \cdot E$ has trace zero. By an induction over n we find a unit vector $\psi \in \phi_*^\perp$ such that

$$\langle B(\psi_*), \psi_* \rangle = -\frac{m}{n-1}$$

Finally, since E is self-adjoint we have already seen that

$$\langle A(\psi_*), \psi_* \rangle = \langle B(\psi_*), \psi_* \rangle \implies \big|\langle A(\psi_*), \psi_* \rangle\big| = \big|\frac{m}{n-1}\big| = \frac{m_*}{n-1}$$

Since $n \geq 3$ the last number is $< m_*$ which contradicts the minimal choice of m_*. Hence we must have $m_* = 0$ which proves lemma 6.5

Final part of the proof. Let $n \geq 3$. The Sublemma gives unit vector ϕ such that $\langle A(\phi), \phi \rangle = 0$. Consider the hyperplane $\phi^\perp$ and the operator B from the Sublemma which now has trace zero on this $(n-1)$-dimensional space. So by an induction over n there exists an orthonormal basis $\psi_1, \ldots, \psi_{n-1}$ in $\phi^\perp$ such that $\langle B(\psi_\nu), \psi_n u \rangle = 0$ for every ν. Now $\phi, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and if U is the unitary matrix which has this n-tuple as column vectors it follows that the diagonal elements of $U^* AU$ all vanish. This finishes the proof of Lemma 3.

Proof Theorem A.8.1

Set $B = \lambda^{-1} A$ so that $\sigma(B) = \{\lambda_i/\lambda\}$ and $\mathrm{Tr}(B) = \sum \frac{\lambda_i}{\lambda}$. We also have

$$||B||^2 = \frac{||A||^2||}{|\lambda|^2} \quad \text{and} \quad \big|\lambda\big| \cdot \mathrm{Norm}[R_A(\lambda)] = \mathrm{Norm}[(E-B)^{-1}]$$

Hence Theorem A.8.1 follows if we prove the inequality

$$\big|e^{\mathrm{Tr}(B)}\big| \cdot \big|\prod_{i=1}^{i=n} \big[1 - \frac{\lambda}{\lambda_i}\big] \cdot \mathrm{Norm}\big[E - B)^{-1}\big] \leq \exp\big[\frac{1 + ||B||^2}{2}\big] \tag{*}$$

To prove (*) we choose an arbitrary integer N such that $N > \big|\text{Tr}(B)\big|$ and for each such N we define the linear operator B_N on the $n+N$-dimensional complex space with points denoted by (x,y) with $y \in \mathbf{C}^N$ as follows:

$$(**) \qquad B_N(x,y) = (Bx\,,\, -\frac{\text{Tr}(B)}{N}\cdot y)$$

The eigenvalues of the linear operator $E - B_N$ is the union of the n-tuple $\{1 - \frac{\lambda_i}{\lambda}\}$ and the N-tuple of equal eigenvalues given by $1 + \frac{\text{Tr}(B)}{N}$. This gives the determinant formula

$$(1) \qquad \det(E - B_N) = \big(1 + \frac{\text{Tr}(B)}{N}\big)^N \cdot \prod_{i=1}^{i=n}(1 - \frac{\lambda_i}{\lambda})$$

The choice of N implies that (1) is $\neq 0$ so the inverse $(E - B_N)^{-1}$ exists. Moreover, the construction of B_N gives for any pair (x,y) in $\mathbf{C}^{N+n}$:

$$(E - B_N)^{-1}(x,y) = \big(E - B)^{-1}(x), \frac{y}{1 + \frac{1}{N}\cdot \text{Tr}(B)}\big)$$

It follows that

$$\text{Norm}\big[(E - B)^{-1})\big] \leq \text{Norm}\big[(E - B_N)^{-1}\big] \implies$$

$$(2) \qquad \big|\det(E - B_N)\big| \cdot \text{Norm}\big[(E - B)^{-1})\big] \leq \big|\det(E - B_N)\big| \cdot \text{Norm}\big[(E - B_N)^{-1}\big]$$

Hadarmard's inequality estimates the hand side in (2) by:

$$(3) \qquad \frac{||E - B_N||^{N+n-1}}{(N+n-1)^{N+n-1)/2}}$$

Next, the construction of B_N implies that its trace is zero. So by Lemma 3 we can find an orthonormal basis $\xi_1, \ldots, \xi_{n+N}$ in $\mathbf{C}^{n+N}$ such that

$$\langle B_N(\xi_k), \xi_k\rangle = 0 \quad : 1 \leq k \leq n+N$$

Relative to this basis the matrix of $E - B_N$ has 1 along the diagonal and the negative of the elements of B_N elsewhere. It follows that the Hilbert-Schmidt norm satisfies the equality:

$$(4) \qquad ||E - B_N||^2 = N + n + ||B_N||^2 = N + n + ||B||^2 + N^{-1}\cdot|\text{Tr}(B)|^2$$

Hence, (1) and the inequalities from (2-3) give:

$$\big(1 + \frac{\text{Tr}(B)}{N}\big)^N \cdot \prod_{i=1}^{i=n}(1 - \frac{\lambda_i}{\lambda}) \cdot \text{Norm}\big[(E - B)^{-1}\big] \leq$$

$$\frac{\big(N + n + ||B||^2 + N^{-1}\cdot|\text{Tr}(B)|^2\big)^{(N+n-1)(2}}{\big(N+n-1\big)^{N+n-1/2}} = \frac{\big(1 + \frac{||B||^2}{N+n} + \frac{|\text{Tr}(B)|^2}{N(N+n)}\big)^{(N+n-1)/2}}{(1 - \frac{1}{N+n}\big)^{N+n-1/2}}$$

This inequality holds for arbitrary large N. Passing to the limit as $N \to \infty$ the definition of Neper's constant e give

$$\lim_{N\to\infty}\big(1 + \frac{\text{Tr}(B)}{N}\big)^N = e^{\text{Tr}(B)}$$

and the reader may also verify that the limit of the last term above is equal to $\exp\big[\frac{1+||B||^2}{2}\big]$ which finishes the proof of (*) above and hence also of Theorem A.8.1.

Hermitian integral operators.

Let us describe some contructions from Carleman's cited monograph. Here is the set-up: Consider the compact set $\square = \{(x,y)\colon a \leq x, y \leq b\}$ in $\mathbf{R}^2$. A Lebesgue measurable and complex-valued function $K(x,y)$ is Hermitian if

$$K(y,x) = \overline{K(x,y)}$$

We impose the condition that there exists a null set $\mathcal{N}$ in $[a,b]$ such that

$$\int_a^b |K(x,y)|^2\,dy < \infty \quad : x \in [a,b] \setminus \mathcal{N}$$

Lebesgue theory gives a non-decreasing sequence of closed subsets $\{E_n\}$ of $[a,b]$ such that

$$\text{(1)} \qquad \int_{E_n} |K(x,y)|^2\,dy \leq n \quad : x \in E_n$$

To each n we define the truncated kernel $K_n(x,y)$ which is equal to K on $E_n \times E_n$ and otherwise zero. It follows that

$$\int_a^b |K_n(x,y)|^2\,dy \leq n \quad : a \leq x \leq b$$

which entails that

$$\text{(2)} \qquad \iint_\square |K_n(x,y)|^2\,dxdy < \infty$$

Hence each K_n yields a Hilbert-Schmidt operator on the Hilbert space $L^2[a,b]$ and is therefore compact. Since K_n also is Hermitian the eigenvalues are real. Keeping n fixed one seeks all real numbers $\lambda \neq 0$ for which there exists a non-zero $\phi \in L^2[a,b]$ such that

$$\text{(3)} \qquad \phi(x) = \lambda \cdot \int_a^b K_n(x,y) \cdot \phi(y)\,dy$$

At each eigenvalue which can be positive or negative, the eigenspace may have dimension $e \geq 2$, but it is always finite dimensional. The Gram-Schmidt construction gives an orthonormal family of eigenfunctions $\{\phi_{\lambda_k}\}$ with eigenvalues $\{\lambda_k\}$, where eigenspaces of dimension $e \geq 2$ means that λ_k is repeated e times. Expanding an arbitrary function $h \in L^2[a,b]$ with respect to the orthormal eigenfunctions and an eventual term which is $\perp$ to all eigenfunctions with non-zero eigenvalues, Bessel's inequality gives

$$\text{(4)} \qquad \sum \Big| \iint_\square \phi_{\lambda_k}(x) \cdot h(x)\,dx \Big|^2 \leq \int_a^b |h(x)|^2\,dx$$

Following Hilbert one associates a spectral function defined for $\lambda > 0$ by

$$\theta_n(x,y;\lambda) = \sum_{0<\lambda_k<\lambda} \phi_{\lambda_k}(x) \cdot \overline{\phi_{\lambda_k}(y)}$$

$$\theta_n(x,y;-\lambda) = \sum_{-\lambda<\lambda_k<0} \phi_{\lambda_k}(x) \cdot \overline{\phi_{\lambda_k}(y)}$$

and finally $\theta_n(x,y;0) = 0$. Next, each $h \in L^2[a,b]$ has an expansion

$$h = \sum c_k \cdot \phi_{\lambda_k}$$

With $\lambda > 0$ one has

$$\iint_\square \theta_n(x,y;\lambda) \cdot h(x) \cdot \overline{h(y)}\,dxdy = \sum_{0<\lambda_k<\lambda} |c_k|^2$$

$$\iint_\square \theta_n(x,y;\lambda) \cdot h(x) \cdot \overline{h(y)}\,dxdy = \sum_{-\lambda<\lambda_k<0} |c_k|^2$$

Moreover, keeping h fixed, weak star convergence applied to a conutable dense set of h-functions gives a subsequence $n_1 < n_2 < \ldots$ such that there exists a limit

$$(5) \qquad \lim_{k\to\infty} \iint_\square \theta_{n_k}(x,y;\lambda)\cdot h(x)\cdot \overline{h(y)}\,dxdy$$

for each $h \in L^2[a,b]$ and every real λ. Next, the construction of the truncated kernels give a limit function defined on $E_* \times E_*$ by

$$(6) \qquad \theta(x,y;\lambda) = \lim_{k\to\infty} \theta_{n_k}(x,y;\lambda)$$

When is $\theta(x,y;\lambda)$ a spectral function for K. So far the constructions were straighforward. There remains to analyze if the θ-function in (6) yields an adapted spectral resolution of the densely defined integral operator attached to $K(x,y)$ where the limit in (5) is unconstrained. The following sufficiency criterion for this to be true goes as follows:

Theorem. *The limit in (5) exists as $n \to +\infty$ and the θ-function yields a spectral resolution of the densely defined K-operator under the condition that the two equations below do not have non-zero L^2-solutions:*

$$(*) \qquad \phi(x) = i\cdot\int_a^b K(x,y)\phi(y)\,dy \quad : \psi(x) = i\cdot\int_a^b K(y,x)\psi(y)\,dy$$

Remark. More generally, Carleman proved the existence of an adapted spectral resolution if the two equations in (*) both give finite dimensional spaces with equal dimension. We refrain from giving details of the proof but refer to [Carleman] and the expositary article *La theorie des equations intégrales singuliérs* [Ann. l'Institut Poincaré Vol. 1 (1931)]. In § xx we establish an "abstract version" of the theorem which therefore includes a proof of the sufficiency result above.

A non-linear equation.

We are given s domain Ω in the family $\mathcal{D}(C^1)$ in the complex plane. Let a and f be continuous functions on $\partial\Omega$ where a is positive. Given $\epsilon > 0$ we seek u such thsat

$$\Delta(u) = \epsilon \cdot u^2$$

holds in Ω while u satisfies the Neumann-Poincaré equation on the boundsry with resepdct to a and f. To find u we try a series expansioin

$$u = u_0 + \epsilon\phi - 1 + \dots$$

where u_0 solves the ordfnisate NP-problem,.i.re. here u_0 is haarmonic in Ω. For ϕ_1 we find thst

$$\text{(i)} \qquad \Delta(\phi_1) = u_0^2$$

and on the boundary

$$\partial_n(\phi_1) = a\phi_1$$

So now one takes

$$\phi_1 = \iint_\Omega G(p,q) \cdot u_0^2(q)\, dq + H$$

where H is harmonic in Ω and G the normalised Greens' function. Then (i) holds and passing to the boundary we have

$$\partial_n(H) + \iint_\Omega \partial_n(G(p,q)) \cdot u_0^2(q)\, dq$$

A uniquneness theorem.

Let $u(x,y,z)$ be defined in an exterior ball $U=\{x^2+y^2+z^2>r_*^2\}$ in $\mathbf{R}^3$. Suppose it satisfies the equation

$$\Delta(u)+\lambda\cdot u=0$$

for some real number λ and that

$$\int_U u^2dxdydx<\infty$$

Then we shall prove that there exists a constant C such that if $S(r)=\{x^2+y^2+z^2=r^2\}$ then

$$\max_{(x,y,z)\in S(r)}|u(x,y,z)|\leq\frac{C}{r^2}\quad : r\geq r_*$$

Proof. We shall use Euler's angular variables θ,ϕ where $0<\theta<\pi$ and $0<\phi<2\pi$. Let $n\geq 1$ and consider a spherical function $Y_n(\theta,\phi)$ on the unit sphere S^2 of some degree $n\geq 1$. For each $r>r_*$ we put

$$Z(r)=\int_0^{2\pi}\int_0^{\pi}Y_n(\theta,\phi)\cdot u(r,\theta,\phi)\cdot\sin\theta\,d\theta d\phi$$

By a classic formula which goes back to Newton and Euler, it followdsfrom (*) that $Z(r)$ satisfies the the second order ODE:

$$r^{-2}\cdot\frac{d}{dr}(r^2\cdot\frac{dZ}{dr})=(\lambda-\frac{n(n+1)}{r^2}\cdot Z=0$$

Its general solution is of the form

$$Z(r)=\frac{c_1\cdot\cos(\sqrt{\lambda}r)+c_2\cdot\sin(\sqrt{\lambda}r)}{r}+\frac{B(r)}{r^2}$$

where c_1,c_2 are constants and $B(r)$ is a bounded function as $r\to+\infty$. Next, the caicvhy-Schwarz inequslity gives

$$Z(r)^2\leq\int_0^{2\pi}\int_0^{\pi}Y_n^2(\theta,\phi)\cdot\cdot\sin\theta\,d\theta d\phi\cdot\int_0^{2\pi}\int_0^{\pi}\sin\theta u^2(r,\theta,\phi)\cdot$$

Since the L^2-integral if Y_n is normalised to be one, it follows by integration that

$$\int_{2r_*}^{\infty}Z(r)^2r^2\,dr\leq\int_{U^*}|,u^2(x,y,z)\,dxdydz$$

where $U^*=\{x^2+y^2+z^2\geq 4r_*^2\}$. Hence the L^2-integrability of u entails that the left integral is finite. But then we see that $c_1=c_2=0$ must hold in (xx) which means that

$$r\mapsto r^2Z(r)$$

is a bounded function on $[2r_*+\infty)$. Morevoer, the function $b8r)$ is indenepdnet of the normalised spherical function Y_n. Hence there is a constant C such that

$$|\int_0^{2\pi}\int_0^{\pi}Y_n(\theta,\phi)\cdot u(r,\theta,\phi)\cdot\sin\theta\,d\theta d\phi\,|\leq\frac{C}{r^2}$$

hold for all Y_n.

Some non-linear equations.

Introduction. We announce and propose results in the 2-dimensional case and remark only that similar results can be posed in higher dimensions where the case $n = 3$ is relevant. The Neumann-Poincaré equation amounts to find a harmonic function u in a domain Ω with C^1-boundary such that

$$\frac{\partial u}{\partial n} = au + f \tag{1}$$

holds on $\partial\Omega$ where a is a positive continuous function while f is arbitrary. The calssic result due to Neumann and Pioncaré shows that (*) has a unique solution $u = u_f$ which for a fixed a depends in a linear fashion on f and is expressed by a continuous linear operator on $C^0(\partial\Omega)$, i.e. here one uses that a harmonic function is determined by its continuous boundary values. In particular the maximun norm of u is bounded above by

$$C \cdot |f|_{\partial\Omega}$$

where C is a constant which depends on the given domain and the a-function. It is tempting to try to solve non-linear equations. A general case goes as follows: Let $\phi(s)$ be a continuous real-valued function with $\phi(0) = 0$ and $\phi(s)$ is strictly increasing when $s \geq 0$. Now we ask if there exists a solution u where the boundary equation (*) holds while

$$\Delta(u) = \phi(u) \tag{2}$$

in Ω. So here u will be subharmonic. One can also try to go further and replace (2) by an equation

$$\Delta(u) = \phi(u) + g \tag{3}$$

where g is a continuous function in Ω. Finally one can replace the linear boundary equation (1) by a non-linear equation. Keeping (2) we can for example ask if there exist solutions u with the boundary value equation

$$\frac{\partial u}{\partial n}(p) = F(u(p), p) + f(p) \tag{3}$$

where $F(s, p)$ is a conti uous functionon defined for $\{s \geq 0\} \times \partial\Omega$ satisfying the conditions in Carleman's non-linear version of the Neumann problem.

Remark. We refrain from discussing the case when $\partial\Omega$ is not smooth but mention only that Carleman's thesis from 1916 contains results which cover cases where corner points appear in Neumann's boundary value problem. The strategy to analyze non-linear PDE-equations is to employ the homotopy method introduced by Poincaré, i.e. the idea is to solve non-linear equations via infinite sytems of linear equations which arise via series dependiing on suitable parameters where one encounters a family of equations which becomes "more and more " non-linear. Whether one can profit upon calculus of variation is less clear. Personally I think that precise estimates for linear equations together with Poincaré's method give the optimal procedure to study non-linear equations.

A specific problem. Let Ω be domain which belongs to the family $\mathcal{D}(C^1)$ in the complex plane. Let a and f be continuous functions on $\partial\Omega$ where a is positive.

Theorem. *Given Ω and a pair (a, f) as above, there exists $\epsilon_* > 0$ such that for each $\epsilon > 0$ one can find a function u_ϵ which satisfies*

$$\Delta(u_\epsilon) = \epsilon \cdot u_\epsilon^2$$

in Ω and the boundary equation (1).

Remark. A special case is to take Ω as the unit disc and $a = 1$ while $f = 0$. So now

$$\partial_n(u) = u$$

is the boundary value equation. If u is a solution it is subharmonic in D and the boundary value equation entails that $u \leq 0$ in D. If u and v are two solutions we set $g = u - v$. Now

$$\Delta(g) = \epsilon(u^2 - v^2) = \epsilon(u + v) \cdot g$$

Here $u+v \leq 0$ so g is superharmonic in D. If g takes a minimum at a point $p \in T$ we have $\partial_n(g)(p) \geq 0$ so the equation $\partial_n(g) = g$ entails that thr minimum is ≥ 0, i.e $g \geq 0$ in D by the minimum principle for superharmonic functions. Hence $u \geq v$ and since we can reverse the role the uniquneness follows. Since the whole system of equations is invariant under rotations, the uniqueness entails that a solution is a radial function if it exists. Hence one is led to analyzie if there exists a function $y = y(r)$ defined for $0 \leq 0 \leq 1$ which satisfies the equations above. It means that

$$y''(r) + 2xx = \epsilon \cdot y^2(r) \quad : 0 \leq r < 1$$

with the boundary condition

$$y'(1) = -y(1)$$

A solution -if it exists - is given by a series

$$y(r) = -A + c_1 r + c_2 r^2 + \ldots$$

To be analyzed, i.e. do we get a convergent solution via a series if ϵ gets large ???

The general situation. To find u in the theorem sbove we try a series expansioin

$$(*) \qquad u = u_0 + \epsilon\phi_1 + \epsilon^2\phi_2 \ldots$$

Above u_0 solves the standard NP-problem, i.e. u_0 is harmonic in Ω, and ϕ_1 satsfies the equation

$$\text{(i)} \qquad \Delta(\phi_1) = u_0^2$$

with the boundary condition

$$\text{(ii)} \qquad \partial_n(\phi_1) = a\phi_1$$

If ϕ_1 has been found then

$$\Delta(\phi_2) = 2u_0\phi_1$$

while

$$\partial_n(\phi_2) = a\phi_2$$

In each step, ϕ_k is found by a recursion which will express $\Delta(\phi_k)$ in Ω while the boundary equations are as in (ii).

Solution. To find ϕ_1 we put

$$\phi_1 = \iint_\Omega G(p,q) \cdot u_0^2(q)\, dq + H$$

where H is harmonic in Ω and G the normalised Greens' function. Then (i) holds and the boundary equation (ii) becones

$$\partial_n(H) + \iint_\Omega \partial_n(G(p,q)) \cdot u_0^2(q)\, dq = aH$$

where we used that $G(p,q) = 0$ when $p \in \partial\Omega$. So here H is found by the ordinary Neumann-Poincaré equation, and similar equations determine ϕ_k for every $k \geq 2$.

Convergence. To check that the series solution (*) converges if ϵ is small enough we need a priori estimates. In general, consider the equation

$$\Delta(\phi) = A$$

where A is a function in Ω and on the boundary ϕ satisfies

$$\partial_n(\phi) = a\phi$$

Then

$$\phi = \iint_\Omega G(p,q) \cdot A(q)\, dq + H$$

where H is harmonic in Ω. With a given A-function we define a function on $\partial\Omega$ by

$$p \mapsto \iint_\Omega \frac{\partial G(p,q)}{\partial n} \cdot A(q)\, dq = A^*(p)$$

where the inner normal derivative is taken with respect to p for each fixed $q \in \partial\Omega$. Set

$$C = \max_{p \in \partial\Omega} \iint_\Omega |\frac{\partial G(p,q)}{\partial n}| \, dq$$

This entails that

$$\max_{p \in \partial\Omega} |A^*(p)| \leq C \cdot |A|_\Omega$$

After H is determined by the Neumann-Poincaré boundary value equation

$$\partial_n(H) = a \cdot H - A^*$$

With

$$a_* = \min_{p \in \partial\Omega} a(p)$$

we know that there is a constant C_1 such that

$$|H|_\Omega \leq C_1 \frac{C \cdot |A|_\Omega}{a_*} = C_2 \cdot |A|_\Omega$$

So with the given lower bound $a_* > 0$ for the positive boundary function a, there exists a constant C^* such that the maximum norm

$$|\phi|_\Omega \leq \frac{C^*}{a_*} \cdot |A|_\Omega$$

Here C^* only depends on the given domain Ω. Applying a suitable majorant series, thiess a priori estimates entail that the series (*) converges when ϵ is sufficiently small.

Spectral theory for linear operstors

Let X be a complex Banach space. We shall consider densely defined linear operators, i.e. linear maps

$$T\colon \mathcal{D}(T) \to X$$

where $\mathcal{D}(T)$ is as a dense subspace of X, and called the domain of definition for T. To each such operator one associates the graph

$$\Gamma(T) = \{(x, Tx) :\ x \in \mathcal{D}(T)\}$$

If this subspace of $X \times X$ is closed one refers to T as a densely defined and closed linear operator on X. In general T is unbounded, i.e.

$$\max_{x \in \mathcal{D}_*(T)} ||T(x)|| = +\infty$$

where the maximum is taken over unit vectors in $\mathcal{D}(T)$. Recall that a straightforward application of Baire's theorem entails that when $\Gamma(T)$ is closed and $\mathcal{D}(T) = X$ then T is a bounded operator, i.e. there exists a constant C such thst

$$||Tx|| \leq C \cdot ||x||$$

In the literature this result is called the closed graph theorem.

A.1 Invertible operators. Let T be closed and densely defined. Suppose that the range

$$\text{(i)} \qquad T(\mathcal{D}(T)) = X$$

and that there exists a positive constant C such that

$$\text{(ii)} \qquad ||T(x)|| \geq c \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

Now (i) gives for each $x \in X$ a unique vector $R(x) \in \mathcal{D}(T)$ such that

$$\text{(iii)} \qquad T(R(x)) = x$$

By this construction R is defined on the whole of X and the inequality (ii) gives

$$\text{(iv)} \qquad ||R(x)|| \leq c^{-1} \cdot ||x|| \quad : x \in X$$

Thus, R is a bounded linear operator whose rsnge is equal to $\mathcal{D}(T)$. Moreover, when R is applied to the left on both sides in (ii) the reader can conclude thst

$$\text{(v)} \qquad R \circ T(x) = x \quad : x \in \mathcal{D}(T)$$

One refers to R as the inverse of T. It is denoted by T^{-1}. Notice thst

$$\Gamma(T^{-1}) = \{(Tx, x)\colon\ x \in \mathcal{D}(T)\}$$

and that the bounded opersator T^{-1} is injective, i.e. its null-space is zero. Conversely, let S be a bounded operator whose kernel is zero and the range $S(X)$ is dense. Then $S = T^{-1}$ where $\mathcal{D}(T) = S(X)$ and $T \circ R(x) = x$ for every $x \in X$. Hence one has a 1-1 correspondence between the family of bounded injective operators with a dense range and the family of closed and densely defined operators which satisfy (i-ii) above.

A.1.1 Remark. When T as above is densely defined and closed the equality $T(\mathcal{D}(T)) = X$ follows from (ii) and the relaxed condition thst T has a dense rsnge. For assume this and consider some vector $y \in X$. By the density of the T-rsnge we find a sewuence $\{x_n\}$ in $\mathcal{D}(T)$ such thsat

$$\lim ||Tx_n - y|| = 0$$

Now (ii) entails thst $\{x_n\}$ is a Cauchy seqrunce and therefore converges to a vector x. Since $\Gamma(T)$ is closed it follows thst $x \in \mathcal{D}(T)$ and $Tx = y$ which proves the requested surjectivity.

A.2 The spectrum $\sigma(T)$. Let T be closed and densely defined and let E denote the identity operator on X. Each complex number λ gives the densely defined operator $\lambda \cdot E - T$. If is is invertible one says thst λ is a resolvent value for T and we shall use the notation

$$\text{(A.2.1)} \qquad (\lambda \cdot E - T)^{-1} = R_T(\lambda)$$

The set of non-resolvent vslues is denoted by $\sigma(T)$ and called the spectrum of T. It turns out that $\sigma(T)$ is a closed subset of $\mathbf{C}$. To prove this we analyze the resolvent operators of T where a major result is the following:

A.3 Neumann's equation. *Assume that $\sigma(T)$ is not the whole complex plane. For each pair $\lambda \neq \mu$ outside $\sigma(T)$ the operators $R_T(\lambda)$ and $R_T(\mu)$ commute and*

$$R_T(\mu)R_T(\lambda) = \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} \tag{A.3.1}$$

Proof. We have

$$(\mu \cdot E - T) \cdot \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} = \frac{E}{\lambda - \mu} + \frac{(T - \mu E)R_T(\lambda)}{\lambda - \mu}$$

$$\frac{E}{\lambda - \mu} - (\mu - \lambda) \cdot \frac{R_T(\lambda)}{\lambda - \mu} + \frac{(T - \lambda E)R_T(\lambda)}{\lambda - \mu} = R_T(\lambda)$$

Multiplying to the left by $R_T(\mu)$ gives (A.3.1) which at the same time this shows that the resolvent operators commute.

We can write (A.3.1) as

$$(E + (\mu - \lambda) \cdot R_T(\lambda)R_T(\mu) = R_T(\lambda)$$

Let us fix a resolvent vslue λ and let z be a complex number such that

$$|z| \cdot ||R_T(\lambda)|| < 1$$

Then the bounded linear operator $E + z \cdot R_T(\lambda)$ is invertible and we get the new bounded linear operator

$$S(z) = (E + z \cdot R_T(\lambda)^{-1} \circ R_T(\lambda)$$

The reader can check that $\lambda + z$ is a resolvent value and one has the equality

$$R_T(\lambda + z) = S(z)$$

Hence the set of resolvent values contain the open disc of radius $||R_T(\lambda)||$ centered at λ. This proves that $\sigma(T)$ is closed.

A.4 Exercise. One can recapture $\sigma(T)$ via the spectrum of a single resolvent operator where we assume that $\mathbf{C} \setminus \sigma(T) \neq \emptyset$. Namely, fix some resolvent value μ. If λ is another resolvent vslue then (A.3.1) gives

$$R_T(\lambda)(E + (\lambda - \mu)R_T(\mu)) = R_T(\mu)$$

The reader can check thst thisimplies that the bounded operator $E + (\lambda - \mu)R_T(\mu)$ is invertible which means that $\frac{1}{\mu - \lambda}$ is a resolvent value for $R_T(\mu)$, and then conclude thst

$$\sigma(T) = \{\lambda : \frac{1}{\mu - \lambda} \in \sigma(R_T(\mu))\} \tag{A.4.1}$$

An Example. Let X be the Hilbert space ℓ^2 whose vectors are complex sequences $\{c_1, c_2, \ldots\}$ for which $\sum |c_n|^2 < \infty$. We have the dense subspace ℓ^2_* vectors such that $c_n \neq 0$ only occurs for finitely many integers n. If $\{\xi_n\}$ is an arbitrary sequence of complex numbers there exists the densely defined operator T on ℓ^2 which sends every sequence vector $\{c_n\} \in \ell^2_*$ to the vector $\{\xi_n \cdot c_n\}$. If λ is a complex number the reader may check that λ is a resolvent vslued if and only if there exists a constant C such that

$$|\lambda - \xi_n| \geq C \quad : n = 1, 2, \ldots \tag{i}$$

Thus, $\lambda \cdot E - T$ has a bounded inverse if and ony if λ belongs to the open complement of the closure of the set $\{\xi_n\}$ taken in the complex plane, and when (i) holds then $R_T(\lambda)$ is the bounded linear operator on ℓ^2 which sends $\{c_n\}$ to $\{\frac{1}{\lambda - \xi_n} \cdot c_n\}$. Since every closed subset of $\mathbf{C}$ is equal to the closure of a denumerable set of points our construction shows that the spectrum of a densely defined operator $\sigma(T)$ can be an arbitrary closed set in $\mathbf{C}$.

A.4.2 Another Example. Suppose that i is a resolvent value and

(i) $$\sigma(R_T(i)) \subset \{|\lambda + i/2| = 1/2\}$$

If $\lambda \in \sigma(T)$ then (A.4.1) gives some $0 \leq \theta \leq 2\pi$ such that

$$\frac{1}{i-\lambda} = -i/2 + 1/2 \cdot e^{i\theta} \implies 1 - i \cdot e^{i\theta} = \lambda(e^{i\theta} - i) \implies$$

$$\lambda = \frac{2 \cdot \cos\theta}{|e^{i\theta} - i|^2} \in \mathbf{R}$$

Hence (i) entails that the spectrum of T is real. Converseley, using (A.4.1) the reader can check that if $\sigma(T)$ is real then the inclusion in (i) holds.

B. Operational calculus

Let λ_0 be a resolvent value of T and construct the operator valued series

(1) $$S(z) = R_T(\lambda_0) + \sum_{n=1}^{\infty} (-1)^n \cdot z^n \cdot R_T(\lambda_0)^{n+1}$$

It is clear that (1) converges in the Banach space of bounded linear operators when

(2) $$|z < \frac{1}{||R_T(\lambda_0)||}$$

Moreover, from (xxx) one has

$$S(z) = R_T(\lambda_0 + z)$$

Hence the resolvents of T yield an operator-valued analytic function

$$\lambda \mapsto R_T(\lambda)$$

defined in the open complement of $\sigma(T)$. Using Neumann's equation the reader can check that this function satisfies the differential equation

(3) $$\frac{d}{d\lambda}(R_T(\lambda) = -R_T^2(\lambda)$$

B.2 Some line integrals. Let γ be a rectifiable Jordan arc contained in $\mathbf{C} \setminus \sigma(T)$ and let $f \in C^0(\gamma)$, where f in general is complex-valued. Now there exists the bounded linear operator

(B.2.1) $$T_{(\gamma,f)} = \int_\gamma f(z) R_T(z)\, dz$$

The integrand has values in the Banach space of bounded linear operators on X, where (B.2.1) is calculated by Riemann sums. Next, recall that a pair of resolvent operators $R_T(z_1)$ and $R_T(z_2)$ commute for all pairs z_1, z_2 on γ. From this it is clear that if g is another function in $C^0(\gamma)$, then the operators $T_{f,\gamma}$ and $T_{g,\gamma}$ commute. Moreover, for each $f \in C^0(\gamma)$ the reader may verify that the closedeness of T implies that the range of $T_{f,\gamma}$ is contained in $\mathcal{D}(T)$ and

(B.2.2) $$T_{f,\gamma} \circ T(x) = T \circ T_{f,\gamma}(x) \quad : x \in \mathcal{D}(T)$$

Next, let Ω be an open set of class $\mathcal{D}(C^1)$, i.e. $\partial\Omega$ is a finite union of closed differentiable Jordan curves. When $\partial\Omega \cap \sigma(T) = \emptyset$ we can construct the line integrals (B.2.1) for each continuous function on the boundary. Consider the algebra $\mathcal{A}(\Omega)$ of analytic functions in Ω which extend to be continuous on the closure. Each $f \in \mathcal{A}(\Omega)$ gives the operator

(B.2.3) $$T_f = \int_{\partial\Omega} f(z) R_T(z)\, dz$$

B.3 Theorem. *The map $f \mapsto T_f$ is an algebra homomorphism from $\mathcal{A}(\Omega)$ into a commutative algebra of bounded linear operators on X whose image is denoted by $\mathcal{T}(\Omega)$.*

Proof. Let f, g be a pair in $\mathcal{A}(\Omega)$. We must show the equality

$$(*) \qquad T_{gf} = T_f \circ T_g$$

To attain this we choose a slightly smaller open set $\Omega_* \subset \Omega$ which again is of class $\mathcal{D}(C^1)$ and each of it bounding Jordan curve is close to one boundary curve in $\partial\Omega$ and $\Omega \setminus \Omega_*$ does not intersect $\sigma(T)$. By Cauchy's theorem we can shift the integration to $\partial\Omega_*$ and get

$$\text{(i)} \qquad T_g = \int_{\partial\Omega_*} g(z) R_T(z_*)\, dz_*$$

where we use z_* to indicate that integration takes place along $\partial\Omega_*$. Now

$$\text{(ii)} \qquad T_f \circ T_g = \iint_{\partial\Omega_* \times \partial\Omega} f(z) g(z_*) R_T(z) \circ R_T(z_*)\, dz_* dz$$

Neumann's equation (*) from (A.3) entails that the right hand side in (ii) becomes

$$\text{(iii)} \qquad \iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z_*)}{z - z_*}\, dz_* dz + \iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z)}{z - z_*}\, dz_* dz = A + B$$

Here A is evaluated by first integrating with respect to z and Cauchy's theorem gives

$$f(z_*) = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega} \frac{f(z)}{z - z_*} \quad : z_* \in \partial\Omega_*\, dz$$

It follows that

$$A = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega_* \times \partial\Omega} f(z_*) g(z_*) R_T(z_*)\, dz_* = T_{fg}$$

Next, B is evaluated when we first integrate with respect to z_*. Here

$$\iint_{\partial\Omega} \frac{g(z_*)}{z - z_*} = 0 \quad : z \in \partial\Omega$$

which entails that $B = 0$ and Theorem B.3 follows.

B.4 Spectral gap sets.

Let K be a compact subset of $\sigma(T)$ such that $\sigma(T) \setminus K$ is a closed set in **C**. This implies that if V is an open neighborhood of K, then there exists a relatively compact subdomain $U \in \mathcal{D}(C^1)$ which contains K as a compact subset while $\partial U \cap \sigma(T) = \emptyset$. To every such domain U we can apply Theorem B.3 If $U_* \subset U$ for a pair of such domains we can restrict functions in $\mathcal{A}(U)$ to U_*. This yields an algebra homomorphism

$$\mathcal{T}(U) \to \mathcal{T}(U_*)$$

Next, denote by $\mathcal{O}(K)$ the algebra of germs of analytic functions on K. So each $f \in \mathcal{O}(K)$ comes from some analytic function in a domain U as above. The resulting operator $T_U(f)$ depends on the germ f only. In fact, this follows because if $f \in \mathcal{A}(U)$ and $U_* \subset U$ is a similar $\mathcal{D}(C^1)$-domain which again contains K, then Cauchy's vanishing theorem in analytic function theory applies to $f(z)R_T(z)$ in $U \setminus \bar{U}_*$ and entails that

$$\int_{\partial U_*} f(z) R_T(z)\, dz = \int_{\partial U} f(z) R_T(z)\, dz$$

Hence there exists an algebra homorphism from $\mathcal{O}(K)$ into a commutative algebra of bounded linear operators on X denoted by $\mathcal{T}(K)$. The identity in $\mathcal{T}(K)$ is denoted by E_K and called the spectral projection operator attached to the compact set K in $\sigma(T)$. By this construction one has

$$\text{(B.4.1)} \qquad E_K = \frac{1}{2\pi i} \cdot \int_{\partial U} R_T(z)\, dz$$

for every open domain U surrounding K as above.

B.4.2 The operator T_K. Let K be a compact spectral gap set of T we put

$$T_K = T E_K$$

This is a bounded linear operator given by

$$\frac{1}{2\pi i}\cdot\int_{\partial U} z\cdot R_T(z)\,dz$$

where U is a domain as above which contains K.

B.4.3 Theorem. *Identify T_K with a linear operator on the subspace $E_K(X)$. Then one has the equality*

$$\sigma(T_K)=K$$

Proof. If λ_0 is outside K we can choose U so that λ_0 is outside $\bar{U}$ and construct the operator

$$S=\frac{1}{2\pi i}\cdot\int_{\partial U}\frac{1}{\lambda_0-z}\cdot R_T(z)\,dz$$

The operational calculus gives

$$S(\lambda_0 E_K-T)=E_K$$

here E_K is the identity operator on $E_K(X)$ which shows that $\sigma(T_K)\subset K$.

B.5 The case of discrete spectra. Suppose we have a spectral set reduced to a singleton set $\{\lambda_0\}$, i.e. λ_0 is an isolated point in $\sigma(T)$. The associated spectral projection is denoted by $E_T(\lambda_0)$ and given by

$$E_T(\lambda_0)=\frac{1}{2\pi i}\cdot\int_{|\lambda-\lambda_0|=\epsilon} R_T(\lambda)\,d\lambda$$

for all sufficiently small ϵ. Here $R_T(\lambda)$ is an analytic function defined in some punctured disc $\{0<\lambda-\lambda_0|<\delta\}$ with a Laurent expansion

$$R_T(\lambda)=\sum_{-\infty}^{\infty}(\lambda-\lambda_0)^k\cdot B_k$$

where $\{B_k\}$ are bounded linear operators obtained via residue formulas:

$$B_k=\frac{1}{2\pi i}\cdot\int_{|\lambda|=\epsilon}\frac{R_T(\lambda)}{(\lambda-\lambda_0)^{k+1}}\,d\lambda\quad:\ \epsilon<\delta$$

B.5.1 Exercise. Show that $R_T(\lambda)$ is meromorphic, i.e. $B_k=0$ hold when $k<<0$, if and only if there exists a constant C and some integer $M\geq 0$ such that the operator norms satisfy

$$||R_T(\lambda)||\leq C\cdot|\lambda-\lambda_0)^{-M}$$

Let us now consider the case when R_T has a pole of some order $M\geq 1$ at λ_0 which gives a series expansion

(i) $$R_T(\lambda)=\sum_1^M\frac{B_{-k}}{(\lambda-\lambda_0)^k}+\sum_0^{\infty}(\lambda-\lambda_0)^k\cdot B_k$$

The reader should check thst residue calculus gives

(ii) $$B_{-1}=E_T(\lambda_0)$$

B.5.2 The case of a simple pole. Suppose that $M=1$. Then it is clesr that

(i) $$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\cdot\int_{|\lambda-\lambda_0|=\epsilon}(\lambda-\lambda_0)R(\lambda)\,d\lambda=0$$

This vanishing and the operational calculus entail that

(ii) $$(\lambda_0 E-T)E_T(\lambda_0)=0$$

which means that the range of the projection operator $E_T(\lambda_0)$ is equal to the kernel of $\lambda_0\cdot E-T$, i.e. the set of eigenvectors x for which

$$Tx=\lambda_0\cdot x$$

B.5.3 The case $M \geq 2$. Now we suppose that a pole of order $M \geq 2$ appesrs. Using residue calculus the reader can check that

$$(\lambda_0 E - T)^k \cdot E_T(\lambda_0) = (-1)^k \cdot B_{k+1} \quad : 1 \leq k \leq M-1$$

while

$$(\lambda_0 E - T)^M \cdot E_T(\lambda_0) = 0$$

Next, consider also the subspaces

$$\mathcal{N}_k(\lambda_0) = \{x \colon (\lambda_0 E - T)^k(x) = 0\} \quad : 1 \leq k \leq M$$

and show that they are non-decreasing and for every $k > M$ one has

$$\mathcal{N}_M(\lambda_0) = \{x \colon (\lambda_0 E - T)^k(x) = 0\}$$

A.7.4 The case when $E_T(\lambda_0)$ **has a finite dimensional range.** Here the operator $T(\lambda_0) = TE_T(\lambda_0)$ acts on this finite dimensional vector space which entails that the nullspaces $\{\mathcal{N}_k(\lambda_0)\}$ above are finite dimensional and if M is the order of the pole one has

$$(T(\lambda_0) - \lambda_0)^M = 0$$

So λ_0 is the sole eigenvalue of $T(\lambda_0)$. Moreover, the finite dimensional range of $E_T(\lambda_0)$ has dimension equal to that of $\mathcal{N}_M(\lambda_0)$.

Exercise. Recall l that finite dimensional subspaces appear as direct sum compoinetns. So if $E_T(\lambda_0)$ is finite dimensional there exists a direct sum decomposition

$$X = E_T(\lambda_0)(X) \oplus E - E_T(\lambda_0)$$

where $V = E - E_T(\lambda_0)$ is a closed subspace of X. Show that V is T-invariant and that there exists some $c > 0$ such that

$$||\lambda_0 x - Tx|| \geq ||x|| \quad x \in V \cap \mathcal{D}(T)$$

C. Adjoints.

Let T be densely defined but not necessarily closed. In the dual space X^* we get the subspace of vectors y for which there exists a constant $C(y)$ such that

(i) $$|y(Tx)| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

When (i) holds the density of $\mathcal{D}(T)$ gives a unique vector $T^*(y)$ in X^* such that

(ii) $$y(Tx) = T^*(y)(x) \quad : x \in \mathcal{D}(T)$$

One refers to T^* as the adjoint operator of T whose domain of definition is denoted by $\mathcal{D}(T^*)$.

C.1 Exercise. Show that T^* has a closed graph.

C.2 Closed extensions. Let T be densely defined but not closed. The question arises when the closure of $\Gamma(T)$ is the graph of a linear operator $\widehat{T}$ and then we refer to $\widehat{T}$ as the closed extension of T. A sufficient condition for the existence of this closed extension goes as follows:

C.3 Theorem. *If $\mathcal{D}(T^*)$ is dense in X^* then T has a closed extension.*

Proof. Consider the graph $\Gamma(T)$ and let $\{x_n\}$ and $\{\xi_n\}$ be two sequences in $\mathcal{D}(T)$ which both converge to a point $p \in X$ while $T(x_n) \to y_1$ and $T(\xi_n) \to y_2$ hold for some pair y_1, y_2. We must rove that $y_1 = y_2$. To achieve this we take some $x^* \in \mathcal{D}(T^*)$ which gives

$$x^*(y_1) = \lim x^*(Tx_n) = \lim T^*(x^*)(x_n) = T^*(x^*)(p)$$

In the same way we get $x^*(y_2) = T^*(x^*)(p)$. Now the density of $\mathcal{D}(T^*)$ gives $y_1 = y_2$ which proves that the closure of $\Gamma(T)$ is a graphic subset of $X \times X$ and gives the closed operator $\widehat{T}$ with

$$\Gamma(\widehat{T}) = \overline{\Gamma(T)}$$

C.4 Remark. Let T be closed and densely defined. A closed operators S is an extension of T if

$$\Gamma(T) \subset \Gamma(S)$$

Passing to adjoint operators the reader should verify that the density of $\mathcal{D}(T)$ implies that

$$T^* = S^* \tag{C.4.1}$$

hold for every S as above.