# XI. The Denjoy conjecture

**Introduction.** Let $\rho$ be a positive integer and $f(z)$ is an entire function such that there exists some $0 < \epsilon < 1/2$ and a constant $A_\epsilon$ such that

$$|f(z)| \leq A_\epsilon \cdot e^{|z|^{\rho+\epsilon}} \tag{0.1}$$

hold for every $z$. Then we say that $f$ has integral order $\leq \rho$. Next, the entire function $f$ has an asymptotic value $a$ if there exists a Jordan curve $\Gamma$ parametrized by $t \mapsto \gamma(t)$ for $t \geq 0$ such that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and

$$\lim_{t \to +\infty} f(\gamma(t)) = a \tag{0.2}$$

In 1920 Denjoy raised the conjecture that (0.1) implies that the entire function $f$ has at most $2\rho$ many different asymptotic values. Examples show that this upper bound is sharp. The Denjoy conjecture was proved in 1930 by Ahlfors in [Ahl]. A few years later T. Carleman found an alternative proof based upon a certain differential inequality. Theorem A.3 below has applications beyond the proof of the Denjoy conjecture for estimates of harmonic measures. See [Ga-Marsh].

## A. The differential inequality.

Let $\Omega$ be a connected open set in $\mathbf{C}$ whose intersection $S_x$ between a vertical line $\{\mathfrak{Re}\, z = x\}$ is a bounded set on the real $y$-line for every $x$. When $S_x \neq \emptyset$ it is the disjoint union of open intervals $\{(a_\nu, b_\nu)\}$ and we set

$$\ell(x) = \max_\nu (b_\nu - a_\nu) \tag{*}$$

Next, let $u(x,y)$ be a positive harmonic function in $\Omega$ which extends to a continuous function on the closure $\bar{\Omega}$ with the boundary values identical to zero. Define the function $\phi$ by:

$$\phi(x) = \int_{S_x} u^2(x,y) \cdot dy \tag{1}$$

The Federer-Stokes theorem gives the following formula for the derivatives of $\phi$:

$$\phi'(x) = 2 \int_{S_x} u_x \cdot u(x,y) dy \tag{2}$$

$$\phi''(x) = 2 \int_{S_x} u_{xx} \cdot u(x,y) dy + 2 \int_{S_x} u_x^2 \cdot dy \tag{3}$$

Since $\Delta(u) = 0$ when $u > 0$ we have

$$2 \int_{S_x} u_{xx} \cdot u(x,y) dy = -2 \int_{S_x} u_{yy} \cdot u(x,y) dy = 2 \int u_y^2 dy \tag{4}$$

The Cauchy-Schwarz inequality applied in (2) gives

$$\phi'(x)^2 \leq 4 \cdot \int_{S_x} u_x^2 \cdot \int_{S_x} u^2(x,y) dy = 4 \cdot \phi(x) \cdot \int_{S_x} u_x^2 dy \tag{5}$$

Hence (4) and (5) give:

$$\phi''(x) \geq 2 \int_{S_x} u_y^2(x,y) \cdot dy + \frac{1}{2} \cdot \frac{\phi'^2(x)}{\phi(x)} \tag{6}$$

Next, since $u(x,y) = 0$ at the end-points of all intervals of $S_x$, *Wirtinger's inequality* and the definition of $\ell(x)$ give:

$$\int_{S_x} u_y^2(x,y) \cdot dy \geq \frac{\pi^2}{\ell(x)^2} \cdot \phi(x) \tag{7}$$

Inserting (7) in (6) we have proved

**A.1 Proposition** *The $\phi$-function satisfies the differential inequality*

$$\phi''(x) \geq \frac{2\pi^2}{\ell(x)^2} \cdot \phi(x) + \frac{\phi'^2(x)}{2\phi(x)}$$

*Proof continued.* The maximum principle for harmonic functions implies that the $\phi(x) > 0$ when $x > 0$ and hence there exists a $\psi$-function where $\phi(x) = e^{\psi(x)}$. It follows that

$$\phi' = \psi' e^{\psi} \quad \text{and} \quad \phi'' = \psi'' e^{\psi} + \psi'^2 e^{\psi}$$

Now Proposition A.1 gives

$$\psi'' + \frac{\psi'^2}{2} \geq \frac{2\pi^2}{\ell(x)^2} \tag{*}$$

**A.2 An integral inequality.** From (*) we obtain

$$\frac{2\pi}{\ell(x)} \leq \sqrt{\psi'(x)^2 + 2\psi''(x)} \leq \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

Taking the integral we get

$$2\pi \cdot \int_0^x \frac{dt}{\ell(t)} \leq \psi(x) + \log\, \psi'(x) + O(1) \leq \psi(x) +, \psi'(x) + O(1) \tag{**}$$

where $O(1)$ is a remainder term which is bounded independent of $x$. Taking the integral once more we obtain:

**A.3 Theorem.** *The following inequality holds:*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + O(x)$$

*where the remainder term $O(x)$ is bounded by $Cx$ for a fixed constant.*

## B. Solution to the Denjoy conjecture

**B.1 Theorem.** *Let $f(z)$ be entire of some integral order $\rho \geq 1$. Then $f$ has at most $2\rho$ many different asymptotic values.*

*Proof.* Suppose $f$ has $n$ different asymptotic values $a_1, \ldots, a_n$. To each $a_\nu$ there exists a Jordan arc $\Gamma_\nu$ as described in the introduction. Since the $a$-values are different the $n$-tuple of $\Gamma$-arcs are separated from each other when $|z|$ is large. So we can find some $R$ such that the arcs are disjoint in the exterior disc $|z| > R$. We may also consider the tail of each arc, i.e. starting from the last point on $\Gamma_\nu$ which intersects the circle $|z| = R$. So now we have an $n$-tuple of disjoint Jordan curves in $|z| \geq R$ where each curve intersects $|z| = R$ at some point $p_\nu$ and after the curves moves to the point at infinity. See figure. Next, we take one of these curves, say $\Gamma_1$. Let $D_R^*$ be the exterior disc $|\zeta| > R$. In the domain $\Omega = \mathbf{C} \setminus \Gamma_1 \cup D_R^*$ we can choose a single-valued branch of $\log \zeta$ and with $z = \log\, \zeta$ the image of $\Omega$ is a simply connected domain $\Omega^*$ where $S_x$ for each $x$ has length strictly less than $2\pi$ The images of the $\Gamma$-curves separate $\Omega^*$ into $n$ many disjoint connected domains denoted by $D_1, \ldots, D_n$ where each $D_\nu$ is bordered by a pair of images of $\Gamma$-curves and a portion of the vertical line $x = \log R$.

Let $\zeta = \xi + i\eta$ be the complex coordinate in $\Omega^*$. Here we get the analytic function $F(\zeta)$ where

$$F(\log(z)) = f(z)$$

We notice that $F$ may have more growth than $f$. Indeed, we get

$$|F(\xi + i\eta)| \leq \exp\bigl(e^{(\rho+\epsilon)\xi}\bigr) \tag{1}$$

With $u = \text{Log}^+ |F|$ it follows that

$$u(\xi,\eta) \leq e^{(\rho+\epsilon)\xi} \tag{2}$$

Hence the $\phi$-function constructed during the proof of Theorem A.3 satisfies

$$\phi(\xi) \leq e^{2(\rho+\epsilon)\xi}$$

It follows that the $\psi$-function satisfies

$$\psi(\xi) = 2 \cdot (\rho+\epsilon)\xi + O(1) \tag{3}$$

Now we apply Theorem A.3 in each region $D_\nu$ where we have a function $\ell_\nu(\xi)$ constructed by (0) in section A. This gives the inequality

$$2\pi \cdot \int_R^\xi \frac{\xi - s}{\ell_\nu(s)} \cdot ds \leq \int_R^\xi (\rho+\epsilon)s \cdot ds + (\rho+\epsilon)\xi + O(1) \quad : \quad 1 \leq \nu \leq n \tag{4}$$

Next, recall the elementary inequality which asserts that if $a_1, \dots, a_n$ is an arbitrary $n$-tuple of positive numbers then

$$\sum a_\nu \cdot \sum \frac{1}{a_\nu} \geq n^2 \tag{5}$$

For each $s$ we apply this to the $n$-tuple $\{\ell_\nu(s)\}$ where we also have

$$\sum \ell_\nu(s) \leq 2\pi$$

So a summation in (4) over $1 \leq \nu \leq n$ gives

$$n \cdot \int_R^\xi (\xi - s) \cdot ds \leq \int_R^\xi (\rho+\epsilon)s \cdot ds + (\rho+\epsilon)\xi + O(1) \tag{6}$$

Another integration gives:

$$n \cdot \frac{\xi^2}{2} \leq (\rho+\epsilon) \cdot \xi^2 + O(\xi) \tag{7}$$

This inequality can only hold for large $\xi$ if $n \leq 2(\rho+\epsilon)$ and since $\epsilon < 1/2$ is assumed it follows that $n \leq 2\rho$ which finishes the proof of the Denjoy conjecture.