Hörmander's L^2-estimate in dimension one

Following original work by Lars Hörmander we establish a result about the $\bar{\partial}$-operator in planar domains. Thus we restrict the attention to $\mathbf{C}$ where $z = x + iy$ is the complex coordinate.

Remark. The full strength of L^2-estimates appears in dimension $n \geq 2$ where one works with *plurisubharmonic functions* and impose conditions on strictly pesudo-convex subsets of $\mathbf{C}^n$ where one seeks solutions of inhomogeneous $\bar{\partial}$-equations for differential forms of every bi-degree (p, q) where $0 \leq p, q \leq n$. See Hörmander's text-boook in several complex variables for details.

The Cauchy-Riemann operator sends a differentiable function f into

$$\bar{\partial}(f) = \frac{1}{2}(\partial_x(f) + i \cdot \partial_y(f))$$

The Hilbert space $\mathcal{H}_\phi(\Omega)$. Let Ω be an open set in $\mathbf{C}$. A real-valued continuous and non-negative function ϕ on Ω gives the Hilbert space $\mathcal{H}_\phi$ whose elements are complex-valued Lebesgue measurable functions f in Ω such that

$$\int_\Omega |f|^2 \cdot e^{-\phi}\, dxdy < \infty \tag{1}$$

The square root yields norm denoted by $||f||_\phi$. Let ψ be another continuous and non-negative function which gives the Hilbert space $\mathcal{H}_\psi$ where the norm of an element g is denoted by $||g||_\psi$. We consider the $\bar{\partial}$-operator which sends a function $f \in \mathcal{H}_\phi$ to $\bar{\partial}(f) = df/d\bar{z}$ and study the equation

$$\bar{\partial}(f) = g \quad : g \in H_\psi \tag{2}$$

One seeks conditions for the pair (ϕ, ψ) in order that there exists a constant C such that (2) has a solution f for every g where

$$||f||_\phi \leq C \cdot ||g||_\psi \tag{3}$$

Notice that (2) does not have a unique solution since f can be replaced by $f + a(z)$ where a is a holomorphic function which belongs to $\mathcal{H}_\phi$. For example, non-uniquneness fails when Ω is a bounded open set and the function $e^{-\phi}$ is bounded in Ω. For then f can be replaced by $f + p$ for an arbitrary polynomial $p(z)$. We shall find a sufficient condition in order that (2-3) above hold.

Hörmander's condition. The pair ϕ, ψ satisfies the Hörmander condition if ψ is a C^2-function and ϕ is at least a C^1-function, and there exists a positive constant c_0 such that the following pointwise inequality holds in Ω:

$$\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y \geq 2 \cdot c_0^2 \cdot e^{\psi(z)-\phi(z)} \tag{*}$$

where we have put $|\nabla(\psi)|^2 = \psi_x^2 + \psi_y^2$.

Main Theorem. *If the pair (ϕ, ψ) satisfies (*) the equation $\bar{\partial}(f) = g$ has a solution for every $g \in \mathcal{H}_\psi$ where*

$$||f||_\phi \leq \frac{1}{c_0} \cdot ||g||_\psi$$

Before the proof starts we recall some facts sbout linear operators between Hilbert spaces. In general, let $\mathcal{H}_0$ and $\mathcal{H}_1$ be a pair of complex Hilbert spaces and $T \colon \mathcal{H}_0 \to \mathcal{H}_1$ is a densely defined linear operator. Following Torsten Carlemsn's famous monograph about unbounded operators on Hilbert spaces published by Uppsala university in 1923, we recall the construction of an adjoint. Namely, a vector $y \in \mathcal{H}_1$ belongs to the domain of definition for the adjoint operator T^* if and only if there exists a constant C such that

$$\big|\langle Tx, y\rangle_1\big| \leq C \cdot |x|_0 \quad : x \in \mathcal{D}(T) \tag{i}$$

where $|x|_0$ is the norm of the vector x taken in $\mathcal{H}_0$, and in the left hand side we considered the hermitiain inner product on $\mathcal{H}_1$. Since $\mathcal{D}(T)$ is dense and Hilbert spaces are self-dual, each y for which (i) holds yields a unique vector $T^*(y) \in \mathcal{H}_0$ such that

$$\langle Tx, y\rangle_1 = \langle x, T^*y\rangle_0 \tag{ii}$$

In general $\mathcal{D}(T^*)$ is not a dense subspace of $\mathcal{H}_1$. But let us add this as an hypothesis on T. Moreover, assume that the two densely defined operators T and T^* both are closed, i.e. their graphs are closed in the product of the two Hilbert spaces.

Exercise. Suppose that both T and T^* are closed with dense domains of definition. Assume in addition that there exists a positive constant c such that

$$|T^*y|_0 \geq |y|_1 \quad : \quad y \in \mathcal{D}(T^*)$$

Show that this implies that the range $T^*(\mathcal{D}(T^*))$ is a closed subspace of $\mathcal{H}_0$ which is equal to the orthogonal complement of the nullspace $\text{Ker}(T)$ Moreover, show that for each $y \in \mathcal{H}_1$ we can find $x \in \mathcal{D}(T)$ such that

$$Tx = y \quad \& \quad |x|_0 \leq c^{-1} \cdot |y|_1$$

Proof of the Main Theorem

Since $C_0^\infty(\Omega)$ is a dense subspace of $\mathcal{H}_\phi$ the linear operator $T\colon f \mapsto \bar\partial(f)$ from $\mathcal{H}_\phi$ to $\mathcal{H}_\psi$ is densely defined and we leave as an exercise to the reader to check that T is closed. In fact, this relies upon a general fact about closedness of operators defined by differential operators. The reader may also check that Stokes Teorem entails that test-functions in Ω belong to $\mathcal{D}(T^*)$ and since $C_0^\infty(\Omega)$ is dense in the Hilbert space $\mathcal{H}_\psi$ the adjoint is also densely defined. Let us then consider some $g \in \mathcal{D}(T^*)$. For each $f \in C_0^\infty(\Omega)$ Stokes theorem gives

$$\text{(i)} \qquad \langle T(f), g\rangle = \int \bar\partial(f) \cdot \bar g \cdot e^{-\psi}\, dxdy = -\int f \cdot \big[\bar\partial(\bar g) - \bar g \cdot \bar\partial(\psi)\big] \cdot e^{-\psi}\, dxdy$$

Since ψ is real-valued, $\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)$ is equal to the complex conjugate of $\partial(w) - w \cdot \partial(\psi)$. We conclude that (i) gives

$$\text{(ii)} \qquad T^*(g) = -\big[\partial(g) - g \cdot \partial(\psi)\big] \cdot e^{\phi-\psi}$$

In particular T^* is defined via a differential operator and has therefore a closed graph. Taking the squared L^2-norm in $\mathcal{H}_\phi$ we obtain

$$||T^*(g)||_\phi^2 = \int |\partial(g) - g \cdot \partial(\psi)|^2 \cdot e^{\phi-2\psi} =$$

$$\text{(iii)} \qquad \int \big(|\partial(g)|^2 + |g|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi-2\psi} - 2 \cdot \mathfrak{Re}\Big(\int \partial(g) \cdot \bar g \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}\Big)$$

By partial integration the last integral in (iii) is equal to

$$\text{(iv)} \qquad 2 \cdot \mathfrak{Re}\Big(\int g \cdot [\partial(\bar w) \cdot \bar\partial(\psi) + \bar g \cdot \partial\bar\partial(\psi) - 2\bar w \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar g \cdot \bar\partial(\psi) \cdot \partial(\phi)] \cdot e^{\phi-2\psi}\Big)$$

Next, the Cauchy-Schwarz inequality gives

$$\text{(v)} \qquad \Big|2 \cdot \int g \cdot \partial(\bar g) \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}\Big| \leq \int \big(|\partial(g)|^2 + |g|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi-2\psi}$$

Together (iii-v) give

$$||T^*(g)||_\phi^2 \geq 2 \cdot \mathfrak{Re} \int |g|^2 \cdot \big[\,\partial\bar\partial(\psi) - 2 \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar\partial(\psi) \cdot \partial(\phi)\,\big] \cdot e^{\phi-2\psi} =$$

$$\text{(vi)} \qquad 2 \cdot \mathfrak{Re} \int |g|^2 \cdot \frac{1}{4}\big[\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y\,\big] \cdot e^{\phi-2\psi}$$

where the last equality follows since ϕ and ψ are real-valued. Funally, since (4.1) is assumed it follows that

$$\text{(vi)} \qquad ||T^*(g)||_\phi^2 \geq c_0^2 \cdot \int |g|^2 \cdot e^{\psi-\phi} \cdot e^{\phi-2\psi} = c_0^2 \cdot ||g||_\psi^2$$

Now we apply the Exercise above and the proof of the Main Theorem is finished.

Remark. Let Ω be an open subset of a disc $\{|z| < r\}$ for somne $r < 1$ which is centered at the origin. Consider the function

$$\phi(z) = \log(1 - |z|^2) = \log(1 - x^2 - y^2)$$

Now we can take $\psi = \phi$ and Hörmander's condition (*) is valid. To see this we notice that

(i)
$$\Delta(\psi) = \frac{4}{(1 - x^2 - y^2)^2}$$

(ii)
$$\psi_x^2 + \psi_y^2 = \frac{4x^2 + 4y^2}{(1 - x^2 - y^2)^2}$$

Since $\phi = \psi$ we see that the right hand side in (*) becomes

(iii)
$$\frac{4}{(1 - x^2 - y^2)^2} - \frac{4x^2 + 4y^2}{(1 - x^2 - y^2)^2}$$

Inside the disc of radius $r < 1$ we notice that (iii) is

$$4 \cdot \frac{1 - r^2}{(1 - r^2)^2}$$

which can be taken as c_0 in the Main Theorem.