Glimpses from work by Carleman

Contents

Self-adjoint operators on Hilbert spaces.

Introduction. The theory about unbounded self-adjoint operators on separable Hilbert spaces was developed by Carleman in the monograph *Sur les équations singulières à noyau reèl et symmétrique* [Uppsala University. 1923]. Actually this work goes further since families of spectral functions are constructed from a densely defined and hermitian operator on a separable Hilbert space. This general construction is for example used to study non-determined moment problems where Carleman established several conclusive results. See § x below for further comments. Apart from thr moment problem, special attention in [ibid] is given to operators on L^2-spaces defined by kernel functions. As pointed out by Carleman in the introduction to [ibid], the solution to the Dirichlet problem for a double layer potential where the boundary is not smooth leads to singular operators where the earlier theory by Fredholm does not apply. It goes without saying that Carleman was inspired by earlier work, foremost by Fredholm and Hilbert. The need for extending the specteal theorem for bounded self-adjoint operators was put forward at an early stage by Schmidt and Weyl. Several results in [ibid] treat extensions of results by the authors above. In addition to the "pute operator theory", the whole analysis in [ibid] also contains instructive facts in measure theory. An example is the notion of absolutely continous linear operators on $L^2[0,1]$ which was introduced by Hilbert via certain measure theoretic assumptions on the integral kernel. As we explain in § xx this can be interpreated via the specteral theorem for unbounded self-adjoint operators where the conclusive result relates Hilbert's conditions with properties of their spectra.

The abstract version of the spectral theorem. Here one starts with a restricted class of unbounded hermitian operators which enjoy the self-adjoint property. Using Hilbert's spectral theorem for bounded self-adjoint operators and the classic calculus of resolvent operators by Carl Neumann, one easily gets the spectral theorem for unbounded self-adjoint operators. A proof is given in my notes devoted to functional analysis. A merit in the abstract proof is that it works on non-separable Hilbert spaces. But in most applications to PDE-theory one is content to dispose the spectral theorem for separable Hilbert spaces, and here the constructive ingredients in Carleman's work are helpful when one tries to decide if a given densely defined hermitian operator is self-adjoint or not. Every separable Hilbert space is isomorphic to ℓ^2 whose vectors are complex sequences $\{c_p\}$ indexed by positive integers and $\sum |c_p|^2 < \infty$. Moreover, up to unitary equivalence every densely defined and hermitian operator A on ℓ^2 is represented by an infinite matrix with elements $\{a_{pq}\}$ such that $a_{qp} = \overline{a_{pq}}$ and

$$\text{(i)} \qquad \sum_{q=1}^{\infty} |a_{pq}|^2 < \infty$$

hold for each p. The Cauchy-Schwarz inequality implies that if $x \in \ell^2$ then the series

$$\sum_{q=1}^{\infty} a_{pq} \cdot x_q$$

is absolutely convergent for each p and if y_p is the sum we obtain a vector $y = (y_1, y_2, \ldots)$. However, (i) need not imply that y belongs to ℓ^2. So in general we find a densely defined but unbounded operator A whose domain of definition consists of vectors $x \in \ell^2$ such that

$$\sum_{p=1}^{\infty} |\sum_{q=1}^{\infty} a_{pq} \cdot x_q|^2 < \infty$$

The favourable self-adjoint case occurs if the equations

$$\text{(ii)} \qquad Ax = i \cdot x \quad : \ Ax = -i \cdot x$$

have no non-zero solutions. Under this hypothesis there exists a spectral resolution of A with similar properties as for bounded self-adjoint operators. In Carleman's work one refers to A as a Class I-operator when (ii) have no non-zero solutions. Let us remark that one does not know a general criterion in order that a Hermitian matrix A satisfying (i) is of Class I. It appears to be a very difficult problem and I suspect that no general necessary and sufficient condition is available. The difficulty can already be explained via the moment problem by Stieltjes, as well as the extension by Hamburger. Here one considers a pair of real sequences $\{a_p\}$ and $\{b_p\}$ where each $b_p > 0$ and the matrix is given by

$$A = xxxx$$

A major result in Carleman's theory is that this symmetric matrix gives a Class I-operaotre if and only if the associated moment problem is determined. We refer to § xx for an account about th moment problem and here we only mention that a sufficient condition for A to be self-adjoint is that

$$\sum_{p=1}^{\infty} \frac{1}{b_p} = +\infty$$

However, examples show that the series may converge and while A still is self-adjoint. So conditions for a matrix in (x) to be self-adjoint relies upon subtle arithmetic condtions during exansions into continued fractions where we recal,that a major theorem due to Stieltjes assets that the moment problem is determined if and only if the expansion in continued fractions is convergent. But in spite of this elegant fact, it is in practice not easy to decide this convergence.

Passing to PDE-operators the spectral theorem is very useful. Again one encounters an initial problem, i.e. to decide when a densely defined PDE-operator is self-adjoint or not. An example is the Bohr-Schrödinger operator

$$P = \delta + c(x)$$

where Δ is the Laplce operator in $\mathbf{R}^3$ and $c(x)$ a real-valued and locally square integrable function. To begin with P is defined on test-functions and here the symmtriy holds by Green's equation, i.e.

$$\int P(f) \cdot \bar{g}\, dx = \int f \cdot \overline{P(g)}\, dx$$

for every pair of test-functions. The question arises when P extends to a densely defined self-adjoint operator. The following sufficiency result was proved by Carleman in his lectures at Sorbonne 1930 and appears to be the sharpest known result up to the present date:

Theorem. *P has a self-adjoint extension if there exists a constant M such that*

$$\limsup_{|x| \to \infty}\ c(x) \le M$$

The construction of spectral functions.

We shall expose material from Chapter I in [ibid]. Here various measure theoretic considerations appear. Sime readers may prefer an "abstract theory" but in reality the condtrucutve ingredients below are useful when one wants to apply the general theory to special integral operators which for example appear in double layer potentials where irregular boundaries occur. To make the presentstion more teransparent we shall restrict the atrnetion to a case where certain "ugly nullsets" are given in a rather concrete way. Let us recall that every separable Hilbert space is isomorphic to $L^2[0,1)$ whose vectors are complex-valued and square integrable functions on the unit interval. To avoid too many notations we consider real and symmetric kernels below and remark only that the whole analysis can be carried out in a similar fashion for hermitian kernels. To begin with we have the class of symmetric Hilbert-Schmidt operators which arise as follows. Let $G(x,y)$ be a real-valued and Lebesgue measurable function on $[0,1]^2$ such that

$$\text{(i)} \qquad G(y,x) = G(x,y) \quad : \iint |G(x,y)^2 \, dxdy < \infty$$

When (i) holds it follows from basic Lebesgue theory that $G(x,y)$ yields a bounded linear operator $\mathcal{G}$ defined on $L^2[0,1)$ by

$$\mathcal{G}(f)(x) = \int_0^1 G(x,y) \cdot f(y) \, dy$$

More precisely, the Cauchy-Schwarz inequality entails that the operator norm of $\mathcal{G}$ is bounded above by

$$\sqrt{\iint G(x,y)^2 \, dxdy}$$

Moreover, there exists a sequence of eigenfunctions $\{\phi_\nu(x)\}$ which are pairwise orthogonal and normalised with L^2-norms equal to one and

$$\phi_\nu = \lambda_\nu \cdot \mathcal{G}(\phi_\nu)$$

Baic Lebesgue theory shows that each eigenfuntion is continuos on $[0,1]$. The symmetry of G implies that $\{\lambda_\nu\}$ are non-zero real numbers which may be > 0 or < 0. Moreover, it is wellknown that $\mathcal{G}$ is a compact operator which implies that the set of eigenvalues over each bounded interval $[-\ell, \ell]$ is finite. The spectral function associated to $\mathcal{G}$ is defined for pairs x, y in $[0,1]^2$ and each real λ by:

$$\theta(x,y;\lambda) = \sum_{0<\lambda_\nu<\lambda} \phi_\nu(x)\phi_\nu(y) \quad : \lambda > 0$$

$$\theta(x,y;\lambda) = -\sum_{\lambda<\lambda_\nu<0} \phi_\nu(x)\phi_\nu(y) \quad : \lambda < 0$$

Keeping x and y fixed we have the function

$$\lambda \mapsto \theta(x,y;\lambda)$$

It has jump discontinuites at the spectral values but we can control it total variation in a uniform way. For let $(-\ell, \ell)$ be a bounded open interval. The total variation of (i) over this interval becomes

$$\sum_{-\ell<\lambda<\ell} |\phi_\nu(x)\phi_\nu(y)| = \sum_{-\ell<\lambda<\ell} |\lambda_\nu|^2 \sqrt{\int_0^1 G(x,s)\phi_\nu(s)\, ds} \cdot \sqrt{\int_0^1 G(y.s)\phi_\nu(s)\, ds}$$

Above $\lambda_| nu^2 \leq \ell^2$ hold for each ν. Moreover, since the eigenfun ctons are orthonormal, Bessel's inequality gives

$$\sum_{-\ell<\lambda<\ell} \sqrt{\int_0^1 G(x,s)\phi_\nu(s)\, ds} \leq \sqrt{\int_0^1 G(x,s)^2 \, ds}$$

and similary with x replaced by y. Hence the total variation of (*) over $(-\ell, \ell)$ is majorised by

$$\ell^2 \cdot \sqrt{\int_0^1 G(x,s)^2\, ds} \cdot \sqrt{\int_0^1 G(y,s)^2\, ds}$$

Unbounded symmetric kernels. Let $K(x,y)$ be a real-valued Lebegue measurable function on $[0,1]^2$ which is symmetric and

$$\text{(ii)} \qquad \int_0^1 K(x,y)^2\, dy < \infty$$

hold for all x outside a nullset $\mathcal{N}$. Notice that (ii) implies that if $g \in L^2[0,1]$ then the integrals

$$\int_0^1 K(x,y) \cdot g(y)\, dy$$

converge absolutely for all x outside $\mathcal{N}$. Indeed, this is clear from the Cauchy-Schwarz inequality. So (ii) yield an almost everywhere defined function denoted by $\mathcal{K}(\phi)$. We shall restrict the attention to the case when $\mathcal{N}$ is a denumerable set $\{\xi_k\}$ whose number of cluster points in $[0,1]$ is finite. In addition we assume that there exists a finite subset $\eta_1, \ldots, \eta_m$ of these ξ-numbers such that for every $|delta > 0$ it holds that

$$J(\delta) = \iint_\delta K(x,y)^2\, dxdy < \infty$$

where the integration takes place over the product of $0 \leq x \leq 1$ and the part of the unit y-interval for which $|y - \eta_k| \geq \delta$ hold for every k. But no uniform bound is imposed on these integrals, i.e. $J(\delta)$ may increase to $+\infty$ when $\delta \to 0$. Denote by $I_*(\delta)$ the part of $[0,1]$ where δ-intervals around $\eta_1, \ldots, \eta_m$ have been removed. define the kernel function $K_\delta(x,y)$ which is zero outside $I_*(\delta) \times I_*(\delta)$ and equal to $K(x,y)$ on this product set. Then it is clear that $K_\delta(x,y)$ is a Hilbert-Schmidt kernel and we get its associated spectral function to be denoted by $\theta_\delta(x,y;\lambda)$.

Spectral functions of K. In the $[0,1]^2$ we remove the union of the lines $\{x = \xi_\nu\}$ and $\{y = \xi_\nu\}$ and obtain a subset denoted by D. Since the ξ-points only have a finite set of cluster points there exists a finite family of open and disjoint rectangles which together with their boundaries cover $[0,1]^2$ and inside each rectangle the number of removed lines is locally finite. Using this one get the following:

Proposition. *Let D_* be a compact subset of D and $\ell > 0$. Then there exists some $\delta_* > 0$ such that family $\{\theta_\delta(x,y;\lambda) : 0 < \delta \leq \delta_* : -\ell < \lambda < \ell\}$ restrict to an equicontinuous family of functions on D_*. Moreover, the total variations of*

$$\lambda \mapsto \theta_\delta(x,y;\lambda)$$

over $(-\ell, \ell)$ are bounded by a constant M which is independent of the points $(x,y) \in D_*$ and $\delta \in (0, \delta_*)$.

Exercise. Prove this via the previous results. If necessary, consult page 27-29 in [ibid] for details. Next, using the Arzela-Ascolli theorem for equicontoinuous families it follows from the above that there exist sequences $\{\delta_\nu\}$ which tends to zero and a limit function $\theta(x, y : \lambda)$ where

$$\lim \theta_{\delta_\nu}(x,y;\lambda) = \theta(x,y:\lambda)$$

holds when $(x,y) \in D$ and every real $|lambda$. Moreover, this convergence holds uniformly over compact subsets of D while λ stays in a bounded interval. The limit funftion $\theta(x,y:\lambda)$ in D is symmetric in x and y and

$$\lambda \mapsto \theta(x,y:\lambda$$

has bounded variation over each finite λ-interval.

A limit function $\theta(x,y;\lambda)$ found as above is called a specteal function associated to K. We remark that in general there exists several spectral functions which arise from different δ-sequences which tend to zero. However, it turns out that every spectral function can be used to recover action by the unbounded kernel K.

Properties of spectral functions. A number of results appear in [ibid: page 28-51] which can be seen as generalisations of the Hilbert-Schmidt theory. To begin eith we notice that while we stsy with Hilbert-Schmidt kernels then

$$|\int_0^1 \theta_\delta(x,s;\lambda)\theta_\delta(s,y;\lambda)|, ds\,| = |\theta_\delta(x,y;\lambda)|$$

Wirh $x = y$ it follows that

$$\int_0^1 \theta_\delta(x,s;\lambda)^2\, ds \leq |\theta_\delta(x,y;\lambda)| \leq |\lambda|^2 \cdot K^*(x)^2$$

Since thi hold for every $\delta > 0$ it follows after a passag to thr,omit that each spectral function satisfies

$$\int_0^1 \theta(x,y;\lambda)^2\, dy \leq |\theta(x,x;\lambda)| \leq |\lambda|^2 \cdot K^*(x)^2$$

Hence the function $y \mapsto \theta(x,y;\lambda)$ is square integrable with respect to y for every x outside the denumerable set of ξ-points.

Now one gets the subspace of $L^2[0,1]$ for which $\mathcal{K}(\phi)$ is square integrable. It is denoted by $\mathcal{D}(\mathcal{K})$ and a crucial fact is that it is dense in the Hilbert space $L^2[0,1]$. To see this we define for each positive integer N the set

$$E_N = \{x : \int_0^1 K(x,y)^2\, dy \leq N$$

They increase with N and their union is $[0.1] \setminus \mathcal{N}$. For each $N \geq 1$ we obtain a Hilbert-Schmidt kernel $G_N(x,y)$ which is zero if x or y is outside E_N while

$$G_N(x,y) = K(x,y) \quad : (x,y) \in E \times E$$

Notice that $\{G_N\}$ converge weakly to K in the sense that

$$\lim_{N\to\infty} \int_0^1 [G_N(x,y) - K(x,y))^2 = 0 \quad : x \in [0,1] \setminus \mathcal{N}$$

Let us resume how this goes before we discuss further material in [ibid]. Let $\mathcal{H}$ be a complex Hilbert space and $A\colon \mathcal{H} \to \mathcal{H}$ is densely defined and hermitian. It means that

(i) $$\langle Ax, y\rangle = \langle x, Ay\rangle$$

hold for each pair of vectors in $\mathcal{D}(A)$. Now A has a graph

$$\Gamma(A) = \{(x, Ax) : x \in \mathcal{D}(A)\}$$

Using (i) one easily verifes that if the graph is not closed, then we one construct an extended operator B whose graph is the closure of $\Gamma(A)$ and b satsifies (i). Replacing A by B if necessary we restrict the attentiont to densely defined hermitian operators with a closed graph. Guven a vector $x \in \mathcal{H}$ we consider the equation

(ii) $$Ay = i \cdot y + x$$

where i is the imaginary unit. It turns out that (ii) has at least one solution given by a vector $y \in \mathcal{D}(A)$ and Bessel's inequality gives

$$||y|| \leq ||x||$$

for every such y. In § xx we give an example of an operator A for which the equation

$$Ay = i \cdot y$$

has a non-zero solution. When this occurs there exist seveeral different spectral functions associated to A and the whole study becomes quite involved. For the moment we ignore this and assume that (ii) has a unique solution for every x. Then (xx) gives a bounded linear operator R such that $Rx = y$. It means that the range of R is equal to $\mathcal{D}(A)$ and the composed operator

$$R \circ (A - iE)(y) = y \quad : y \in \mathcal{D}(A)$$

We also notice that

$$(A - iE) \circ R(x) = x \quad : x \in \mathcal{H}$$

Next, we can consider the equation

$$Ay = -i \cdot y$$

It it has no non-zero solution we find as above a bounded linear operator S such that

$$S \circ (A + iE)(y) = y \quad : y \in \mathcal{D}(A)$$

$$(A + iE) \circ S(x) = x \quad : x \in \mathcal{H}$$

From these equations one easily derives that the adjoint R^* is equal to S. Moreover, by Neumann's calculus for resolvent opertors it follows that R and R^* commute, i.e. R is a normal operator. Moreover, one shows via general results from § xx that the spectrum of A is confined to the real line while $|sigma(R)$ is contined in the circle

$$\mathcal{C} = \{\lambda : \ |\lambda + i/2| = 1/2\}$$

At this stage we apply Hilbert's spectral theorem for bounded normal operators. It gives a map

$$\mu \colon \mathcal{H} \times \mathcal{H} \to \mathfrak{M}(\mathcal{C})$$

where $\mathfrak{M}(\mathcal{C})$ is the space of complex-valued Riesz measures on the compact set $\mathcal{C}$. So to each pair x, y in $\mathcal{H}$ one assigns a measure $\mu_{x,y}$. Moreover, Neumann's operational calculus gives for each bounded Borel function g on $\mathcal{C}$ a bounded linear operator G on $\mathcal{H}$ such that

$$\langle Gx, y \rangle = \int_{\mathcal{C}} g(\lambda)\, \mu_{x,y}(\lambda)$$

Moreover, the spectrum $\sigma(G)$ is contained in the closure of the range of g. In particular we consider a closed Borel set e in $\mathcal{C}$ which does not contain $\lambda = 0$. Now

$$g(\lambda) = \chi_e(\lambda) \cdot \frac{1 - i\lambda}{\lambda}$$

is a bounded Borel function which yields a bounded opertor G as above. With $\lambda = -i/2 + \zeta$ where the absolute value $|\zeta| = 1/2$ we get

$$\frac{1 - i\lambda}{\lambda} = \frac{1/2 - i\zeta}{-i/2 + \zeta} = \frac{(1/2 - i\zeta)(i/2 + \bar{\zeta})}{|\zeta - i/2|^2} = \frac{\mathfrak{Re}(\zeta)}{|\zeta - i/2|^2}$$

Since e deos not contsin zero we see that (x) gives a bounded and real-valued function which via Neumann's operational calculus correspnids to a normal and bounded operator G whose spectrum is real. Since a normal operatoer with real spectrum is self-adjoint, it follows that G is self-adjoint.

Let us first remark that every separable Hilbert space is isomorphic to $L^2[0,1)$ whose vectors are complex-valued and square integrable functions on the unit interval. The starting point in [ibid] was Hilbert's earlier studies of Hermitian and bounded operators on $L^2[0,1)$. Among these occur the class of Hilbert-Schmidt operators which arise when $G(x,y)$ is a Lebesgue measurable function on $[0,1]^2$ such that

$$\text{(i)} \qquad G(y,x) = \overline{G(x,y)} \quad : \quad \iint |G(x,y)|^2 \, dxdy < \infty$$

A special case occurs when G us real-valued and then the first equation above means that this kernel is symmetric, i.e. $G(y,x) = G(x,y)$. When (i) holds it follows from basic Lebesgue theory that $G(x,y)$ yields a bounded linear operator $\mathcal{G}$ defined on $L^2[0,1)$ by

$$\mathcal{G}(f)(x) = \int_0^1 G(x,y) \cdot f(y)\, dy$$

More precisely, the Cauchy-Schwarz inequality entails that the operator norm of $\mathcal{G}$ is bounded above by

$$\sqrt{\iint |G(x,y)|^2 \, dxdy}$$

More generally, let $K(x,y)$ be a real-valued Lebegue measurable function on $[0,1]^2$ which is symmetric and

$$\text{(ii)} \qquad \int_0^1 K(x,y)^2 \, dy < \infty$$

hold for all x outside a nullset $\mathcal{N}$. Notice that (ii) implies that if $g \in L^2[0,1]$ then the integrals

$$\int_0^1 K(x,y) \cdot g(y) \, dy$$

converge absolutely for all x outside $\mathcal{N}$. Indeed, this is clear from the Cauchy-Schwarz inequality. So (ii) yield an almost everywhere defined function denoted by $\mathcal{K}(\phi)$. Now one gets the subspace of $L^2[0,1]$ for which $\mathcal{K}(\phi)$ is square integrable. It is denoted by $\mathcal{D}(\mathcal{K})$ and a crucial fact is that it is dense in the Hilbert space $L^2[0,1]$. To see this we define for each positive integer N the set

$$E_N = \{x : \int_0^1 K(x,y)^2 \, dy \leq N$$

They increase with N and their union is $[0.1] \setminus \mathcal{N}$. For each $N \geq 1$ we obtain a Hilbert-Schmidt kernel $G_N(x,y)$ which is zero if x or y is outside E_N while

$$G_N(x,y) = K(x,y) \quad : (x,y) \in E \times E$$

Notice that $\{G_N\}$ converge weakly to K in the sense that

$$\lim_{N \to \infty} \int_0^1 [G_N(x,y) - K(x,y))^2 = 0 \quad : x \in [0,1] \setminus \mathcal{N}$$

Next, for each fixed N the Hilbert-Schmidt theory entails that $\mathcal{G}_N$ is a compact operator whose spectrum is real and discrete outside $\lambda = 0$. More precisely, there exists a sequence of real numbers $\{\mu_\nu^{(N)}\}$ arranged with non-decreasing absolute values and a sequence of pairwise orthonormal and real-valued functions $\{\phi_\nu^{(N)}\}$ in $L^2[0,1]$ such that

$$\mathcal{G}_N(\phi_\nu^{(N)}) = \frac{1}{\mu_\nu^{(N)}} \cdot \phi_\nu^{(N)}$$

Here $|\mu_\nu^{(N)}| \to +\infty$ as ν increases. Following Hilbert one introduces spectral kernels for bounded intervals as follows: If $0 < a < b$ we set

$$\theta_N(a,b;x.y) = \sum_{a \leq \mu_\nu^{(N)} < b} \phi_\nu^{(N)}(x) \phi_\nu^{(N)}(y)$$

Similarly, if $0 < c < d$ then

$$\theta_N(-c,-d;x.y) = \sum_{-d \leq \mu_\nu^{(N)} < -c} \phi_\nu^{(N)}(x) \phi_\nu^{(N)}(y)$$

Keeping an interval $[a,b]$ fixed with $0 < a < b$ we get a linear operator which sends $h \in L^2[0,1]$ to

$$E_N[a,b](h) = \int_0^1 \theta_N(a,b;x.y) \cdot h(y) \, dy$$

This operator s bounded because h can be written as

$$h(y) = \sum_{a \leq \mu_\nu^{(N)} < b} a_\nu \cdot \phi_\nu^{(N)}(y) + h^*(y)$$

where $h^*(y)$ is $\perp$ to the linear space Π_N generated by the finite set of eigenfunctions $\{\phi_\nu^{(N)} : a \leq \mu_\nu^{(N)} < b\}$ while

$$a_\nu = \int_0^1 \phi_\nu^{(N)}(y) \cdot h(y)\, dy$$

Bessel's inequality entails that the operator in (x) has norm ≤ 1. Notice also that $E_N[a,b]$ is the self-adjoint projection operator from $L^2[0,1]$ onto Π_N. The question arises if these E-operators converge. More precisely, keeping $[a,b]$ fixed we ask if there exists a bounded linear operator $E_*[a,b]$ such that

$$\lim_{N\to\infty} E_N[a,b](h) = E_*[a,b](h)$$

hold for each $h \in L^2[0,1]$, where the limit is taken in the weak sense, i.e.

$$\lim_{N\to\infty} \int_0^1 E_N[a,b](h)(y) \cdot g(y)\, dy = \int_0^1 E_*[a,b](h)(y) \cdot g(y)\, dy$$

hold for each pair of L^2-functions. A similar limit can be impoed when we instead consider intervals $[-d,-c]$.

Class I kernels. Returing to the kerenl $K(x,y)$ one considers the integral equation

$$\int_0^1 K(x,y) \cdot \phi(y)\, dy = i \cdot \phi(x)$$

where i is the imaginry unit. If it has no non-zero solution ϕ one says that $K(x,y)$ is of Class I. A major result in Carlemans work goes as follows

Theorem. *If $K(x,y)$ is of class I then the opertor valued seences $\{E_N[a,b]\}$ and $\{E_N[-d,-c]\}$ converge weakly for all bounded intervals. Moreover, for each $h \in \mathcal{D}(\mathcal{K})$ one has the integral formula*

$$\mathcal{K}(h) = \lim_{\delta\to 0} \big[\int_{-1/\delta}^{-\delta} \lambda^{-1} \cdot \frac{d}{d\lambda}(E_*(h) + \lim \int_\delta^{1/\delta} \lambda^{-1} \cdot \frac{d}{d\lambda}(E_*(h)\,\big]$$

Remark. Above appears Stieltjes' integrals. For example,

$$\int_\delta^{1/\delta} \frac{d}{d\lambda}(E_*(h) = \lim \sum_{\nu=0}^{\nu=M} E_*[s_\nu, s_{\nu+1})](h)$$

where the limit is taken over sequences

$$\delta = s_0 < s_1 < \ldots < s_M = 1/\delta \quad : \max(s_{\nu+1} - s_\nu) \to 0$$

The domain of definition for $\mathcal{K}$ is also fully described by the spectral resolution defined by the E_*-operators. More precisely, an L^2-function h belongs to $\mathcal{D}(\mathcal{K})$ if and only if there exists a constant C such that

$$\sum_{\nu=0}^{\nu=M} \big[\,||E_*[-s_{\nu+1}, -s_\nu)](h)|| + \sum_{\nu=0}^{\nu=M} ||E_*[s_\nu, s_{\nu+1})](h)||\,\big] \leq C$$

hold for every finite sequence $0 < s_< s_1 < \ldots s_M$, where the norms are taken in $L^2[0,1]$. So the integrals which appear in the right hnd sid in Theorem xx are absolutely convergent with respect to the norm in $L^2[0,1]$.

The closure property. Let $K(x,y)$ be of class I. For each $h \in L^2[0,1]$ we putconsider the integrals

$$h_\delta = \int_{-1/\delta}^{-\delta} \frac{d}{d\lambda}(E_*(h) = + \int_\delta^{1/\delta} \frac{d}{d\lambda}(E_*(h)$$

The question arises if

$$\lim_{\delta\to 0} h_\delta = h$$

Here one may refer to a weak limit in $L^2[0,1]$ or impose the stronger condition that the L^2-norms $||h_\delta - h|| \to 0$. One says that K has the closure property when (xx) holds. In most applicstions one encounters Class i operators which enjoy the closure property. But in specific situations one is obliged to check if (x) is valid or not.

Further results. A central issue is to determine when a given symmetric kernel K is of Class I. No general necessary and sufficient condition exists, and it is unlikely that such a criterion exists. But is is of interest to search for sufficient conditions. several results which give sufficient conditions appear in Carleman's work. Among these occur criteria via iterated kernels. Given K we set

$$K^{(2)}(x,y) = \int_0^1 K(x,s)K(s,y)\,ds$$

Since K is symmetric this is the same as

$$\int_0^1 K(x,s)K(y,s)\,ds$$

and we notice that the Cauchy-Schwarz inequality entails that this integral taken with respect to s is absolutely convergent when both x and y are outside the nullset $\mathcal{N}$. For a further study of these iterated kernels and how they are used to determine when K is of Class I we refer to § xx.

Further examples which emerge from this work appears in his lecture from [Comptes rendus du VI^e Congrés des Mathématiques Scandinavies. Copenhagen 1925].

After Carleman's pioneering work, hundreds of text-books and thousands of articles have treated spectral theory for unbounded self-adjoint operators. But the basic material is covered in [ibid] and to this I would like to add a personal comment. Even though "abstract methods" in mathematics often are useful, one only becomes truly familiar with the "source" of a subject after executing explicit computations. From this point of view [ibid] appears as an outstanding text about unbounded linear operators on Hilbert spaces which in addition to theoretic results contain several concrete applications exposed with detailed and very elegant proofs. At the same time specific situations illustrate how to perform the pasage to limits which are taken in various weak topologies. We shall describe some of the major results from [ibid] and begin with a specific case which illiustrates the flavour of the general theory.

Propagation of sound. With (x,y,z) as space variables in $\mathbf{R}^3$ and a time variable t, the propagation of sound in the infinite open complement $U = \mathbf{R}^3 \setminus \overline{\Omega}$ of a bounded open subset Ω is governed by solutions $u(x,y,z,t)$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \tag{1}$$

where Δ is the Laplace operator in x,y,z. So here (1) holds when $p=(x,y,z) \in U$ and $t>0$. We assume that $\partial\Omega$ is of class C^1, i.e. given as a union of pairwise disjoint closed surfaces of class C^1 along which normal vectors are defined. We seek solutions such that $p \mapsto u(p,t)$ belong to $L^2(U)$ for each t, and the outer normal derivatives taken along $\partial\Omega$ are zero, i.e. for every $t>0$

$$\frac{\partial u}{\partial n}(p,t) = 0 \quad : p \in \partial\Omega \tag{2}$$

In addition to this u satisfies initial conditions expressed by a pair of C^2-functions $f_1 x,y,z)$ and $f_2(x,y,z)$ defined in U such that the four functions $f_1, f_2, \Delta(f_1, \Delta(f_2)$ belong to $L^2(U)$, and the outer normal derivatives of f_1 and f_2 are zero.

Theorem. *To each pair f_1, f_2 satisfyinfg the conditions above there exists a unique solution u which satisfies (1-2) and the initial conditions $u(p,0) = f_0(p)$ and $\frac{\partial u}{\partial t}(p,0) = f_1(p)$.*

This theorem is proved in [ibid: page 174-185]. Apart from the conclusive result the methods in the proof are very instructive and relies upon the construction of a spectral function to a certain densely defined and self-adjoint operator A acting on the Hilbert space $L^2(U)$. In this way the proof boils down to solve the ordinary Dirchlet problem in the exterior domain U and after A

is found via a kernel given by a Greens function. In § xx we explain in detail that the unique solution u for a given pair f_1, f_2 is is given by

$$(*) \qquad u = \int_0^\infty \cos(\sqrt{\lambda}\cdot t)\cdot \frac{d}{d\lambda}\Theta(\lambda)(f_1) + \int_0^\infty \frac{\sin(\sqrt{\lambda}\cdot t)}{\sqrt{\lambda}}\cdot \frac{d}{d\lambda}\Theta(\lambda)(f_2)$$

where $\Theta(\lambda)$ is the spectral function of A. In addition to this, Carleman proved that the spectral function is absolutely continuous with respect to the parameter λ andused this to prove that the first order partial derivatives of u with respect to the space variables x, y, z, tend to zero as $t \to +\infty$. This was already expected by physical reasoning. But the confirmation via the theorem above is of course satisfactory. So Carleman's pioneering work pointed out the usefulness of spectral theory for unbounded self-adjoint operators on Hilbert spaces. More precisely, the merit is that one often can attain integral formulas for solutions and properties of the spectral function can in favourable cases be reduced to specific problems which are setrled by ordinary calculus. For example, the absolute continuity of the spectral function above is established in [ibid] by a result which goes as follows

Let $\{a \leq s \leq b\}$ be a compact interval on the real s-line and $s \mapsto G_s$ is a function with values in the Hilbert space $L^2(U)$ which is continuous in the sense that

$$\text{(i)} \qquad \lim_{s \to s_0} ||G_s - G_{s_0}||_2 = 0$$

hold for each s_0, where we introduced the L^2-norms in $L^2(U)$. In addition we assume that there exists a constant M such that

$$\text{(ii)} \qquad \sum_{\nu=0}^{\nu=N-1} ||G_{s_{\nu+1}} - G_{s_\nu}||_2 \leq M$$

hold for every partition $a = s_0 < s_1 < \ldots < s_N = b$. Using (ii) one construct Stieltjes integrals and for every subinterval $[\alpha, \beta]$ there exists the L^2-function in U

$$\Phi_{[\alpha,\beta]} = \int_\alpha^\beta s \cdot \frac{dG_s}{ds}$$

which arise via limits of Steiltjes' sums:

$$\sum_{\nu=0}^{\nu=N-1} s_\nu (G_{s_{\nu+1}} - G_{s_\nu})$$

where the limit exists as $\max(s_{\nu+1} - s_\nu) \to 0$. We impose the extra conditions that the normal deriviatives $\frac{\partial G_s}{\partial n}$ exist and vanish $\partial\Omega$ for every $a \leq s \leq b$ and the following differential equation holds for every sub-interval $[\alpha, \beta]$ of $[a, b]$:

$$\Delta(G_\beta - G_\alpha) + \Phi_{[\alpha,\beta]} = 0$$

2.1.1 Theorem. *The equations above imply that $s \mapsto G_s$ is absolutely continuous.*

Remark. The absolute continuity of $s \mapsto G_s$ means that whenever $\{\ell_1, \ldots, \ell_M\}$ a finite family of disjoint intervals in $[a, b]$ where the sum of their lengths is $< \delta$, then the sum of the total variations over these intervals is bounded by $\rho(\delta)$ where ρ is a function of δ which tends to zero as $\delta \to 0$. We refer to § xx for an account of the proof of this result.

0.0. Densely defined hermitian operators.

Let A be a linear operator on a separable Hilbert space $\mathcal{H}$ whose domain of definition $\mathcal{D}(A)$ is dense and satisfies the hermitian equation:

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

for each pair x, y in $\mathcal{D}(A)$. When thid holds we first notice that if $\{x_n\}$ is a sequence in $\mathcal{D}(A)$ which converges the zero vector, i.e. the norms $||x_n||$ tend to zero, then $\{A(x_n)\}$ cannot converge to a non-zero vector y in $mathcalH$. To see this we take some $\xi \in \mathcal{D}(A)$ and since $||x_n|| \to 0$ we get

$$0 = \lim_{n\to\infty} \langle A(\xi), x_n\rangle$$

Since A is hermitian this entails that

$$0 = \lim_{n\to\infty} \langle \xi, A(x_n)\rangle\langle \xi, y\rangle$$

where the last equality holds since $||A(x_n) - y|| \to 0$ was assumed. But then y is $\perp$ to the dense subspace $\mathcal{D}(A)$ and must be zero. From this observation it follows that A can be extended to an operator whose graph is closed and the reader can check that this alos yields a hermitian oeprator. So from now on it sufficies to consider densey defined and closed hermitain operators.

0.1 An inhomogenous equation. A first result from Chapter 1 in [ibid] asserts that the inhomogeneous equation

$$x = \lambda \cdot A(x) + \xi \qquad (*)$$

has at least one solution $x \in \mathcal{D}(A)$ for every vector $\xi \in \mathcal{H}$ and non-real complex number λ. To prove this Carleman used weak approximations of A by hermitian Hilbert-Schmidt operators. Moreover, Bessel's inequality entails that every solution in (*) satisfies

$$||x|| \leq |\mathfrak{Im}(\lambda)|^{-1} \cdot ||\xi|| \qquad (**)$$

0.2 Class I operators. If the equations (*) have unique solutions for every non-real λ, then e say that A is of Class I. A notsable fact which is proved in [ibdi] asserts that if (*) has a uniquw solution for some fixed λ^* in the upper hlf-plane, then the same uniquness hold for all λ with positive real part. A siumilar conclusion hold for complex numbers in the lower half-plane. So in order to check when A is of class 1 or not, it suffices to consider the inhomogeneous equations in (*) when λ is i or $-i$.

0.3. The spectral function. Let A be of Class I. a densely defined self-adjoint operator on a separable Hilbert space $\mathcal{H}$.

Then there exists an operator valued function

$$\lambda \mapsto \Theta_+(\lambda) \tag{i}$$

of finite total variation with respect to the operator norms on the Banach space of bounded linear operators on $\mathcal{H}$ defined when $0 < \lambda < +\infty$, and another operator valued function

$$\lambda \mapsto \Theta_-(\lambda) \tag{ii}$$

defined when $-\infty < \lambda < 0$, These Θ-operators enjoy similar properties as those in Hilbert's spectral resolution of bounded self-adjoint operators. More precisely, for the Θ_+-operators the following hold: For each interval $[a, b]$ with $0 < a < b$ the operator

$$E_+[a, b] = \Theta_+(b)\Theta_+(a)$$

is a self-adjoint projection, i.e. its range is a closed subspace in $\mathcal{H}$ while the kernel of $E[a, b]$ is the orthogonal complement and finally $E[a, b]^2 = E(a, b]$. Moreover, if $[a, b]$ and $[c, d]$ is a pair of disjoint intervals in $(0, +\infty)$ then $E[a, b] \circ E[c, d] = 0$, which means that the subspaces $E[a, b](\mathcal{H})$ and $E[c, d](\mathcal{H})$ are orthogonal. In addition, for each pair of vector x, y in $\mathcal{H}$ the complex-valued function

$$\langle \Theta_+(\lambda)(x), y \rangle$$

is continuous on $(0, +\infty)$.

From the above there exist Stieltjes' integrals, i.e. if $g(\lambda) \in C^0[a, b]$ for some bounded and closed interval $[a, b]$ in $(0. + \infty)$ and x is a vector in $\mathcal{H}$, then there exists a vector

$$\int_a^b g(\lambda) \cdot \frac{d}{d\lambda}(\Theta(\lambda)(x))$$

found as the limit of Stieltjes' sums

$$\sum_{\nu=1}^{\nu=N-1} g(\lambda_\nu) \cdot E_+(\lambda_\nu, \lambda_{\nu-1})(x))$$

where $\delta = \lambda_0 < \lambda_1 < \ldots < \lambda_N = 1/\delta$ is a partition of the interval $[a, b]$ and the limit is taken as $\max(\lambda_\nu - \lambda_{\nu-1}) \to 0$. Keeping $[a, b]$ and g fixed this yields a bounded linear operator $\mathcal{G}[a, b]$ whose operator norm is bounded by the maximum norm of g over $[a, b]$.

Similar properties as above hold for the Θ_--operators, and here

$$E_+[a, b] \circ E_-[-d, -c] = 0$$

for all pairs of intervals when $0 < a < b$ and $0 < c < d$.

0.4 Representation of A. For each vector $x \in \mathcal{H}$ and $\delta > 0$ we have the non-negative integral

$$\int_{-1/\delta}^{-\delta} |\lambda|^{-1} \cdot ||\frac{d}{d\lambda}(\Theta(\lambda)(x))|| + \int_\delta^{1/\delta} |\lambda|^{-1} \cdot ||\frac{d}{d\lambda}(\Theta(\lambda)(x))|| \tag{1}$$

Above the integrals are taken in the sense of Stieltjes. For example, the the last integral in (1) is given by

$$\max \sum_{\nu=1}^{\nu=N-1} \lambda_\nu^{-1} \cdot ||\Theta(\lambda_\nu)(x)) - ||\Theta(\lambda_{\nu-1})(x))|| \tag{2}$$

taken over every partition $\delta = \lambda_0 < \lambda_1 < \ldots \lambda_N = 1/\delta$ of the interval $[\delta, 1/\delta]$.

With these notations, Carleman proved that a vector x belongs to $\mathcal{D}(A)$ if and only (1) is bounded by a constant C which may depend on the given vector x but not upon δ. Moreover,

$$A(x) = \lim_{\delta \to 0} \Big[\int_{-1/\delta}^{-\delta} \lambda^{-1} \cdot \frac{d}{d\lambda}(\Theta_-(\lambda)(x)) + \int_\delta^{1/\delta} \lambda^{-1} \cdot \frac{d}{d\lambda}(\Theta_+(\lambda)(x)) \Big] \tag{***}$$

where the integrals as usual are taken in the sense of Stieltjes and from the above they converge absolutely when x belongs to $\mathcal{D}(A)$.

0.4.1 Remark. From the above it follows that if $0 < a < b$, then the self-adjoint projection $E[a,b]$ has a range contained in $\mathcal{D}(A)$ and

$$A \circ E_+[a,b](x) = \int_a^b \lambda^{-1} \cdot \frac{d}{d\lambda}(\Theta_+(\lambda)(x))$$

hold for every $x \in \mathcal{H}$. Moreover, this yields a bounded linear operator $A[a,b]$ whose kernel is the orthogonal $E[a,b](\mathcal{H})$, while

$$A[a,b](x) = \int_a^b \lambda^{-1} \cdot \frac{d}{d\lambda}(\Theta_+(\lambda)(x)) \quad : x \in E[a,b](\mathcal{H})$$

Similar bounded operators $A[-d,-c]$ arise when $0 < c < d$.

0.4.2 The closure property. For every vector $x \in \mathcal{H}$ there exists a limit vector

$$\text{(xx)} \qquad x_* = \lim_{\delta \to 0} \Big[\int_{-1/\delta}^{-\delta} \frac{d}{d\lambda}(\Theta_-(x)) + \int_\delta^{1/\delta} \frac{d}{d\lambda}(\Theta_+(x)) \Big]$$

Following [ibid. page 136] one says that A has the closure property when $x = x_*$ for every $x \in \mathcal{H}$. In [ibid: Chapire 4] it is proved that A has the closure property if it is is injective, i.e. $Ax \neq 0$ for every non-zero vector in $\mathcal{D}(A)$. We shall refrain from a more subtle study about the closure property which appears in [ibid:page 138-142] where the closure property is derived from the existence of certain weak approximations of A.

0.5 Unitary groups. Given a spectral resolution of a Class I operator A which hjas the closure property there exists a unitary group $\{U_t\}$ indexed by real numbers where

$$U_t(x) = i \cdot \int_{-\infty}^{\infty} e^{i\frac{t}{\lambda}} \cdot \frac{d}{d\lambda}(\Theta(\lambda)(x)) \quad : x \in \mathcal{H}$$

To be precise, one verifies easily that each U_t is a unitary operator on $\mathcal{H}$ where the closure property entails that U_0 is the identity. From calculus one has the limit formula

$$i \cdot \lim_{t \to 0} t^{-1}(e^{i\frac{t}{\lambda}} - 1) = \lambda^{-1} \quad : \lambda \neq 0$$

In § xx we explain that this implies that A is the infinitesmal generator of $\{U_t\}$ to be defined in § xx. The converse also holds and is proved in the section devoted to the Hille-Phillips-Yosida theorem. The conclusion is that there exists a 1-1 corespondence between the family of strongly continuous unitary groups and the family of Ckass I operators with the closure property.

0.5.1 Example. In $\mathbf{R}^3$ the Laplace operator Δ yields a densely defined linear operstor on the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^3)$. Introducing the Fourier transform, Parseval's equality gives the unitary group $\{U_t\}$ where each $f \in \mathcal{H}$ is sent to

$$U_t(f)(x) = i \cdot (2\pi)^{-3} \cdot \int_{\mathbf{R}^3} e^{i\langle x,\xi\rangle} \cdot e^{\frac{it}{|\xi|^2}} \cdot \widehat{f}(\xi)\, d\xi$$

When f is such that $\widehat{f}(\xi) \in C_0^\infty(\mathbf{R}^3)$ we have

$$\lim_{t \to 0} \frac{U_t(f) - f}{t} = -(2\pi)^{-3} \cdot \int_{\mathbf{R}^3} |\xi|^{-2} \cdot e^{i\langle x,\xi\rangle} \cdot \widehat{f}(\xi)\, d\xi = \Delta(f)(x)$$

It follows that the infinitesmal generator of $\{U_t\}$ is the densely defined and hermitian operator A where $\mathcal{D}(A)$ consists of L^2-functions f for which $\Delta(f)$ taken in the sense of distribution theory is square integrable. As explained in § xx, the densely defined operator A has a spectrum in Neumann's sense. Fourier's inversion formula entails that $\sigma(A) = (-\infty, 0])$, and the resolvents are given by the bounded operators

$$R_A(\lambda)(f)(x) = -(2\pi)^{-3} \cdot \int_{\mathbf{R}^3} \frac{e^{i\langle x,\xi\rangle}}{\lambda + |\xi|^2} \cdot \widehat{f}(\xi)\, d\xi$$

when λ is outside $\mathbf{C} \setminus (-\infty, 0]$.

0.5.2 The closure property. To check this we use that Δ is an elliptic PDE-operator and there reamins to show that if H is a harmonic funtion in $\mathbf{R}^3$ which is square integrable, then $H = 0$. We leave this proof as an exercsie to the reader. See also § xx for a further study of closure properties related to the Laplace operator in $\mathbf{R}^3$.

0.6 The case of integral operators. Here one starts with a real-valued function $K(x, y)$ defined in a product $\Omega \times \Omega$ where Ω is an open set in $\mathbf{R}^n$ for some $n \geq 1$. Assume the symmetry $K(x, y) = K(y, x)$ and that there exists a nullset $\mathcal{N}$ in Ω such that

$$\int_\Omega K(x, y)^2 \, dy < \infty \quad : x \in \Omega \setminus \mathcal{N} \tag{0.6.1}$$

Let $f(x)$ is a continuous and square integrable function in Ω. Now there exist the non-empty family of L^2-functions ϕ which satisfy

$$\phi(x) = \lambda \cdot \int_\Omega K(x, y) \cdot \phi(y) \, dy + f(x) \tag{0.6.2}$$

To begin with this equality hold for almost every $x \in \Omega$, i.e. outside a null-set which in general contains $\mathcal{N}$. Suppose that E is a compact subset of $\Omega \setminus \mathcal{N}$ such that

$$\max_{|x_1| - x_2| \leq \delta} \int_\Omega (K(x_1, y) - K(x_2, y))^2 \, dy$$

tends to zero as $\delta \to 0$ while x_1, x_2 stay in E. Then it is easily seen that each ϕ-function is continuous on E and one is led to introduce the sets

$$\mathcal{D}(x : \lambda) = \bigcup \phi(x)$$

with the union taken over all solutions to (3.1). Properties of these sets are studied in [ibid] and in the section devoted to the moment problem we shall give some precise results. Returning to (0.6-1) one has a densely defined and symmetric operator. In specific situations one is led to determine when this gives a Class I operator which in many applications is a major issue.

0.7. An ugly example. Following [Carleman - page 62-66] we give examples which show that the existence of self-adjoint extensions of densely defined symmetric operator is not automatic. Every separable Hilbert space can be identified with $L^2[0, 1]$ whose vectors are complex-valued square integrable functions. Let us then consider a real-valued Lebegue measurable function $K(x, y)$ defined on the square $\{0 \leq x \leq 1\} \times \{0 \leq y \leq 1\}$ which is symmetric, i.e. $K(y, x) = K(x, y)$. Assume that there exists a null set $\mathcal{N}$ such that

$$\int_0^1 K(x, y)^2 \, dy < \infty \quad : x \in [0, 1] \setminus \mathcal{N} \tag{1}$$

If $f \in L^2[0, 1]$ the Cauchy-Schwarz inequality entails that $y \mapsto K(x, y) \cdot f(y)$ is absolutely integrable when x is outside $\mathcal{N}$. Hence there exists the almost everywhere defined function

$$\mathcal{K}(f)(x) = \int_0^1 K(x, y) \cdot f(y) \, dy \tag{2}$$

However, (1) does not imply that $\mathcal{K}(f)$ is square integrable. So we consider the subspace $\mathcal{D}$ of $L^2[0.1]$ where $f \in \mathcal{D}$ gives $\mathcal{K}(f)$ in $L^2[0.1]$. To analyze this subspace we consider positive integers N and denote by E_N the subset of the unit x-interval such that

$$\int_0^1 K(x, y)^2 \, dy \leq N \quad : x \in E_N \tag{3}$$

If $f \in L^2[0, 1]$ is supported by E_N and g is another L^2-function, the symmetry of K and Fubini's theorem give

$$\int_0^1 g(x) \mathcal{K}(f)(x) \, dx = \int_0^1 g(x) \cdot (\int K(x, y) f(y) \, dy) \, dx = \int_0^1 f(x) \cdot (\int K(x, y) g(y) \, dy) \, dx$$

The Cauchy-Schwartz inequality and (3) entail that the absolute value of the last term is majorised by

$$\sqrt{N} \cdot ||g||_2 \cdot \int_{E_N} |f(x)|\, dx \leq \sqrt{N} \cdot ||g||_2 \cdot ||f||_2$$

Since this hold for every L^2-function g, a wellknown fact from integration theory implies that $\mathcal{K}(f)$ is square integrable and its L^2-norm is majorised by $\sqrt{N} \cdot ||f||_2$. Hence $\mathcal{D}$ contains every L^2-function supported by E_N. Since $\cup E_N = [0,1] \setminus \mathcal{N}$ it follows that $\mathcal{D}$ is a dense subspace. The question arises if there exists some complex-valued function $\phi \in \mathcal{D}$ which satisfies the eigenvalue equation

$$(*) \qquad \phi = i \cdot \mathcal{K}(\phi)$$

When $\mathcal{K}$ is a bounded linear operator on the Hilbert space $L^2[0,1]$ such non-zero solutions do not exist. In fact, this follows from Hilbert's theorem for bounded symmetric operators. So if a non-trivial solution to (*) exists, then $\mathcal{K}$ must be unbounded. To exhibit examples of such "ugly operators", Carleman introduced the an orthonormal basis for $L^2[0,1]$ given by a sequence $\{\psi_n\}$ where $\psi_0(x) = 1$ and $\psi_1(x) = -1$ on $(0,1/2)$ and $+1$ on $(1/2,1)$. Finally, for each $n \geq 2$ we set

$$\psi_n(x) = -2^{\frac{n-1}{[}2} : 1 - 2^{-n+1} \leq x < 1 - 2^{-n} \quad \text{and} \quad \psi_n(x) = 2^{\frac{n-1}{2}} : 1 - 2^{-n} < x < 1$$

while

$$\psi_n(x) = 0 \quad : 0 < x < 1 - 2^{-n+1}$$

It is easily seen that this gives an orthonormal basis. Next, for every sequece $\{a_p\}$ of real numbers we define the kernel function on $[0,1) \times [0,1)$ by

$$\text{(i)} \qquad K(x,y) = \sum a_p \cdot \psi_p(x)\psi_q(x)$$

To this symetric function we associate the operator

$$\mathcal{K}(u)(x) = \int_0^1 K(x,y)u(y)\, dy$$

The construction of the ψ-functions show that $\mathcal{D}$ contains L^2-functions u supported by $0 \leq x \leq x_*$ for every $x_* < 1$. In § xx we shall prove the following:

Theorem. *The equation (*) has a non-trivial L^2-solution if and only if*

$$\sum_{p=0}^{\infty} \frac{2^p}{1 + a_p^2} < \infty$$

1. The case of non-separable Hilbert spaces.

Carleman's studies were restricted to separable Hilbert spaces. Using the Cayley transform and Hilbert's spectral theorem for bounded normal operators which is valid for non-separble Hilbert spaces one gets a rather short proof of the spectral theorem for densely defined self-adjoint operators on non-separable Hilbert spaces. This was carried out in J. von Neumann's article *Allgemeine Eigenwerttheorie Hermitscher Funktionaloperatoren* [Math. Annalen, vol. 102 (1929)]. See § 9 in my notes on functional analysis for details. During the search of self-adjoint extensions of densely defined hermitian operators, the notion of hypermaximality introduced by Schmidt is useful. It stems from Schmidt's early contributions in the article *Auflösing der allgemeinen linearen Integralgleichung* [Math. Annalen, vol. 64 (1907].

Non-separable Hilbert spaces arise in many situations. An example is the Hilbert space of square integrable functions on the compact Bohr group which by definition is the dual of the discrete abelian group of real numbers. The theory of almost periodic functions was created by Harald Bohr (brother to the phycisist Niels Bohr) and presented in the articles *Zur Theorie der fastperiodischen Funktionen I-III* [Acta. Math. vol 45-47 (1925-26]. This led in a natural way to consider of operators on non-separable Hilbert spaces and motivated a more "abstract account" where unbounded operators are not described via infinite matrices indexed by pairs of integers. Other examples emerge from quantum mechanics. The uncertainty principle led to questions concerned with actions by pairs of non-commutating densely defined operators on a Hilbert space. In 1928 this was reformulated by Weyl to specific problems about non-commuting families of unitary groups. See § for further details where we present von Neumann's proof of a result which settled original problems posed by de Broigle, Heisenberg and Schrödinger.

1.2 The Cayley transform.

Recall a classic result about matrices which goes back to work by Cayley and Hamilton. If $N \geq 2$ is an integer we have the family $\mathfrak{h}(N)$ of Hermitian $N \times N$-matrices. Next, a matrix R is normal if it commutes with the adjoint matrix R^*. The Cayley transform sends a Hermitian matrix A to the normal matrix

$$R_A = (iE_N - A)^{-1}$$

where E_N is the identity matrix in $M_N(\mathbf{C})$. If $\{\alpha_\nu\}$ is the real spectrum of A it is readily seen that

$$\sigma(R_A) = \{\frac{1}{i - \alpha_\nu}\}$$

Next, when a is a real number we notice that

$$\frac{1}{i-a} + \frac{i}{2} = \frac{1}{2(i-a)}(2 + i^2 - ia) = \frac{1-ai}{2i(1+ai)}$$

The last quotient is a complex number with absolute value 1/2. Hence $\sigma(R_A)$ is contained in the circle

$$\mathcal{C} = \{|\lambda + \frac{i}{2}| = \frac{1}{2}\}$$

Conversely, let R be a normal operator such that $\sigma(R)$ is contained in $\mathcal{C}$ and does not contain the origin. Now there exists the inverse operator R^{-1} and

$$\text{(i)} \qquad A = iE_N - R^{-1}$$

is a normal operator whose spectrum $\sigma(A)$ is real, and since A also is normal it follows that it is Hermitian. Notice also that $R = R_A$ in (i). *Summing up*, the Cayley transform gives a bijective map between Hermitian matrices and the family of invertible normal operators whose spectra are contained in $\mathcal{C}$.

1.3 The passage to Hilbert spaces. Consider a complex Hilbert space $\mathcal{H}$ and denote by $\mathcal{N}$ the family of bounded normal operators R with the property that $\sigma(R)$ is contained in the circle $\mathcal{C}$, and in addition R is injective and has a dense range. To each such R we find a densely defined operator S_R on $\mathcal{H}$ which for each $y = R(x)$ in the dense range assigns the vector x, i.e. the composed operator $S_R \circ R = E$, where E is the identity on $\mathcal{H}$. Now there also exists the densely defined operator

$$(*) \qquad A = iE - S_R$$

We shall learn that the densely operator A is self-adjoint, and conversely every densely defined and self-adjoint operator A is of the form (*) for a unique R in $\mathcal{C}$. So this gives 1-1 correspondence which extends the previous result for matrices. Let us remark that bounded self-adjoint operators appear in (*) when the normal operator R is invertible.

2. Two examples from PDE-theory.

The study of unbounded self-adjoint operators is especially relevant while one regards PDE-operators. We present two examples below.

2.1. Propagation of sound. With (x, y, z) as space variables in $\mathbf{R}^3$ and a time variable t, the propagation of sound in the infinite open complement $U = \mathbf{R}^3 \setminus \overline{\Omega}$ of a bounded open subset Ω is governed by solutions $u(x, y, z, t)$ to the wave equation

$$(1) \qquad \frac{\partial^2 u}{\partial t^2} = \Delta u$$

where Δ is the Laplace operator in x, y, z. So here (1) holds when $p = (x, y, z) \in U$ and $t \geq 0$. We assume that $\partial\Omega$ is of class C^1, i.e. given as a union of pairwise disjoint closed surfaces of class C^1 along which normal vectors are defined. A well-posed Cauchy problem arises when we seek solutions such that $p \mapsto u(p, t)$ belong to $L^2(U)$ for each t, and the outer normal derivatives taken along $\partial\Omega$ are zero, i.e. for every t

$$(2) \qquad \frac{\partial u}{\partial n}(p, t) = 0 \quad : p \in \partial\Omega$$

Initial conditions are expressed by a pair of C^2-functions $f_1 x, y, z)$ and $f_2(x, y, z)$ defined in U such that f_1, f_2 together with $\Delta(f_1$ and $\Delta(f_2)$ belong to $L^2(U)$, and their outer normal derivatives along $\partial\Omega$ are zero. Then there exists a unique solution u which satisfies (1-2) and the initial conditions $u(p, 0) = f_0(p)$ and $\frac{\partial u}{\partial t}(p, 0) = f_1(p)$. A proof of existence and uniqueness relies upon the construction of a time-independent self-adjoint operator A acting on $L^2(U)$ whose kernel is expressed via a Green's function which is found by Neumann's standard elliptic boundary value problem in the domain U. After this has been done one gets an integral representation of the solution u expressed via the pair f_0, f_1 and the spectral function associated to A. For physical reaons one expects that if u is a solution, then the first order partial derivatives of $u(p, t)$ with respect to the space variables tend to zero as $t \to +\infty$. In [Carleman 1923] this is proved by analyzing the spectral function associated with A where the crucial point is that its associated spectral function is absolutely continuous with respect to the λ-parameter. In §§ we show that this absolute continuity is derived from a general result which goes as follows:

Let $\{a \leq s \leq b\}$ be a compact interval on the real s-line and $s \mapsto G_s$ is a function with values in the Hilbert space $L^2(U)$ which is continuous in the sense that

$$\lim_{s \to s_0} ||G_s - G_{s_0}||_2 = 0$$

hold for each s_0, where we introduced the L^2-norms. The function has a finite total variation if there exists a constant M such that

$$\sum ||G_{s_{\nu+1}} - G_{s_\nu}||_2 \leq M$$

hold for every partition $a = s_0 < s_1 < \ldots < s_M = b$. When this holds one construct Stieltjes integrals and for every subinterval $[\alpha, \beta]$ there exists the L^2-function in U

$$\Phi_{[\alpha,\beta]} = \int_\alpha^\beta s \cdot \frac{dG_s}{ds}$$

We impose the extra conditions that the normal deriviatives $\frac{\partial G_s}{\partial n}$ exist and vanish $\partial\Omega$ for every $a \leq s \leq b$ and the following differential equation holds for every sub-interval $[\alpha, \beta]$ of $[a, b]$:

$$\Delta(G_\beta - G_\alpha) + \Phi_{[\alpha,\beta]} = 0$$

2.1.1 Theorem. *The equations above imply that $s \mapsto G_s$ is absolutely continuous.*

Remark. The absolute continuity of $s \mapsto G_s$ means that whenever $\{\ell_1, \ldots, \ell_M\}$ a finite family of disjoint intervals in $[a, b]$ where the sum of their lengths is $< \delta$, then the sum of the total variations over these intervals is bounded by $\rho(\delta)$ where ρ is a function of δ which tends to zero as $\delta \to 0$. We refer to § xx for an account of the proof of this result.

2.2 The Bohr-Schrödinger equation.

In 1923 quantum mechanics had not yet appeared so the studies in [Carleman: 1923] were concerned with singular integral equations, foremost inspired from previous work by Fredholm, Hilbert, Weyl and Volterra. The creation of quantum mechanics gave new challenges for the mathematical community,. The interested reader should consult the lecture held by Niels Bohr at the Scandianavian congress in mathematics in Copenhagen 1925 where he speaks about the interplay between the new physics and mathematics. Bohr's lecture presumably inspired Carleman when he some years later resumed work from [Car 1923]. Recall that the fundamental point in

Schrödinger's theory is the hypothesis on energy levels which correspond to orbits in Bohr's theory of atoms. For an account about the physical background the reader may consult Bohr's plenary talk when he received the Nobel Prize in physics 1923. Mathematically the Bohr-Schrödinger theory leads to the equation

$$(*) \qquad \Delta\phi + 2m \cdot \big(E - U\big)\big(\frac{2\pi}{h}\big)^2 \cdot \phi = 0$$

Here Δ is the Laplace operator in the 3-dimensional (x, y, z)-space, m the mass of a particle and h Planck's constant while $U(x, y, z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Let us cite an excerpt from Carlemans lectures in Paris at Institut Henri Poincaré held in 1930:

Dans ces dernières années l'ínteret de la question qui nous occupe a considérablement augmenté. C'est en effet un instrument mathématique indispensable pour development de la mechanique moderne crée par M.M. de Brogile, Heisenberg et Schrödinger. Etude de l'équation integrale:

$$\phi(x) = \lambda \cdot \int_a^b K(x, y)\phi(y)dy + f(x) \quad : \lambda \in \mathbf{C} \setminus \mathbf{R}$$

The theory from [Carleman:1923] applies to the following PDE-equations attached to a second order differential operator

$$(**) \qquad L = \Delta + c(x, y, z) \quad : \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

where $c(x, y, z)$ is a real-valued function. The L-operator is densely defined and symmetric on the subspace of test-functions in $\mathbf{R}^3$. The problem is to find conditions on the c-function in order that L is self-adjoint on the Hilbert space $L^2(\mathbf{R}^3)$ with a real spectrum confined to $[a, +\infty)$ for some real number a. The following sufficiency result was presented by Carleman's during his lectures at Sorbonne in the spring 1930:

Theorem *Let $c(x, y, z)$ be a continuous and real-valued function such that there is a constant M for which*

$$\limsup_{x^2+y^2+z^2 \to \infty} c(x, y, z) \leq M$$

Then $\Delta + c(x, y, z)$ is self-adjoint.

Example. The result applies when c is given by a potential function:

$$W(p) = \sum \frac{\alpha_k}{|p - q_k|} + \beta$$

where $\{q_k\}$ is a finite subset of $\mathbf{R}^3$ and the α-numbers and β are real and positive. Here the requested self-adjointness is easy to prove and solutions to the Bohr-Schrödinger equation are found via robust limit formulas. See §§ xx for details.

Further comments. The literature about the Schrödinger equation and other equations which emerge from quantum mechanics is extensive. For the source of quantum mechanics the reader should first of all consult the plenary talks by Heisenberg, Dirac and Shrödinger when they received the Nobel prize in physics. Apart from physical considerations the reader will find expositions where explanations are given in a mathematical framework. Actually Heisenberg was sole winner 1931 while Dirac and Schrödinger shared the prize in 1932. But they visited Stockholm together in December 1932.

For mathematician who wants to become acquainted with quantum physics the eminent textbooks by Lev Landau are recommended. Especially *Quantum mechanics: Non-relativistic theory* in Vol. 3. Here Landau exposes Heisenberg's matrix representation and Dirac's equations are used to study radiation phenomena. In the introduction to [ibid: Volume 3] Landau inserts the following remark: *It is of interest to note that the complete mathematical formalisim of quantum mechanics was constructed by W. Heisenberg and E. Schrödinger in 1925-26, before the discovery of the uncertainty principle which revealed the physical contents of this formalism.*

Introduction. The material is foremost is devoted to spectral properties of second order elliptic PDE-operators. To illustrate the methods we are content to treat a special case while elliptic operators with variable coefficients are treated in separate notes devoted to mathematics by Carleman. However, § E in the appendix contains a construction of fundamental solutions to elliptic second order operators in $\mathbf{R}^3$ based upon Carleman's lectures at Institute Mittag Leffler in 1935 which might be of interest to some readers even though it will not be covered during my lecture. As background the appendix contains a section which explains Gustav Neumann's fundamental construction from 1879 of resolvents to densely defined linear operators, and at the end of § A we also recall Hilbert's spectral theorem for bounded normal operators on Hilbert spaces.

Let us now announce a major result which will be exposed in the lecture. In $\mathbf{R}^3$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact established by G. Neumann and H. Poincaré during the years 1879-1895 gives the following: First there exists the Greens' function

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$ with the property that the operator $\mathcal{G}$ defined on $L^2(\Omega)$ by

$$f \mapsto \mathcal{G}_f(p) = \frac{1}{2\pi} \dot{\iint} G(p,q) f(q)\, dq$$

satisfies

$$\Delta \circ \mathcal{G}_f = -f \quad : f \in L^2(\Omega)$$

Moreover, $\mathcal{G}$ is a compact operator on the Hilbert space $L^2(\Omega)$ and there exists a sequence $\{f_n\}$ in $L^2(\Omega$ such that $\{\phi_n = \mathcal{G}_{f_n}\}$ is an orthonormal basis in $L^2(\Omega$ and

$$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : n = 1, 2, \ldots$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$. When eigenspaces have dimension ≥ 2, the eigenvalues are repeated by their multiplicity. The result below was presented by Carleman at the Scandinavian Congress in mathematics held in Stockholm 1934:

0. Theorem. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N \to \infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

Remark. The strategy in the proof is to consider the function of a complex variable s defined by

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

and show that it is a meromorphic function in the whole complex s-plane with a simple pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. The proof is given in § xx.

0.1 Self-adjoint extensions of $\Delta + c(p)$. Here we consider $\mathbf{R}^3$ with points $p = (x, y, z)$ and Δ is the Laplace operator, while $c(p)$ is a real-valued and locally square integrable function. The linear operator

$$u \mapsto L(u) = \Delta(u) + c \cdot u$$

is defined on test-functions and hence densely defined on the Hilbert space $L^2(\mathbf{R}^3)$. In the monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923], Carleman established spectral resolutions for unbounded self-adjoint operators on a Hilbert space together with conditions that densely defined symmetric operators have self-adjoint extensions, The lecture will describe the major steps of a result due to Carleman which asserts that the operator L has a self-adjoint extension under the condition that

$$(*) \qquad \limsup_{p\to\infty} c(p) \leq M$$

hold for some constant M. A special case occurs when $c(p)$ is a Newtonian potential

$$(**) \qquad c(p) = \sum \frac{\alpha_\nu}{|p-q_\nu|} + \beta$$

where $\{q_\nu\}$ is a finite set of points in $\mathbf{R}^2$ while $\{\alpha_\nu\}$ and β are positive real numbers. So here one encounters the Bohr-Schrödinger equation which stems from quantum mechanics. With c as in (**) the requested self-adjoint extension is easily verified while the existence of a self-adjoint extension when (*) holds requires a rather involved proof.

0.3 An asymptotic expansion. Consider the Schrödinger equation

$$i\cdot\frac{\partial u}{\partial t} = \Delta(u) + c\cdot u$$

where we assume that $L^\Delta + c$ has a self-adjoint extension. One seeks solutions $u(x,t)$ defined when $t\geq 0$ and $x\in\mathbf{R}^3$ with an initial condition $u(x,0)=f(x)$ for some $f\in L^2(\mathbf{R}^3)$. The solution is given via the spectral function associated with the L-operator. So the main issue is to get formulas for the spectral function of $\Delta + c$. In Carleman's cited lecture from 1934 an asymptotic expansion is given for this spectral function which merits further study since one nowadays can investigate approximative solutions numerically by computers.

0.4 Asymptotic formula for eigenvalues. Let $n=3$ and consider a PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x)\cdot\frac{\partial^2}{\partial x_p\partial x_q} + \sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p} + a_0(x)$$

The a-functions are real-valued and defined in a neighborhood of the closure of a bounded domain Ω with a C^1-boundary. Here one has the symmetry $a_{pq}=a_{qp}$, and $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x\in\Omega$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under these conditions, a result which goes back to work by Neumann and Poincaré, gives a positive constant κ_0 such that if $\kappa\geq\kappa_0$ then the inhomogeneous equation

$$L(u) - \kappa^2\cdot u = f \quad : f\in L^2(\Omega)$$

has a unique solution u which is a C^2.-function in Ω and extends to the closure where it is zero on $\partial\Omega$. Moreover, there exists some κ_0 and for each $\kappa\geq\kappa_0$ a Green's function $G(x,y;\kappa)$ such that

$$\text{(i)} \qquad (L-\kappa^2)(\frac{1}{4\pi}\cdot\int_\Omega G(x,y;\kappa)\,f(y)\,dy) = -f(x) \quad : f\in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

$$\text{(ii)} \qquad f\mapsto -\frac{1}{4\pi}\cdot\int_\Omega G(x,y;\kappa)\,f(y)\,dy$$

is Neumann's resolvent to the densely defined operator $L-\kappa^2$ on the Hilbert space $L^2(\Omega)$. After a detailed study of these G-functions, Carleman established an asymptotic formula for the discrete sequence of eigenvalues $\{\lambda_n\}$. In general they are complex but arranged so that the absolute values increase. To begin with one proves rather easily that they are "almost real" in the sense that there exist positive constants C and c such that

$$|\mathfrak{Im}(\lambda_n)| \leq C\cdot(\mathfrak{Re}(\lambda_n)+c)$$

hold for every n. Next, the elliptic hypothesis means that the determinant function

$$D(x) = \det(a_{p,q}(x))$$

is positive in Ω. With these notations one has

Theorem. *The following limit formula holds:*

$$(*) \qquad \lim_{n\to\infty} \frac{\mathfrak{Re}(\lambda_n)}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_\Omega \frac{1}{\sqrt{D(x)}}\, dx$$

Remark. The formula above is due to Courant and Weyl when P is symmetric and was extended to non-symmetric operators during Carleman 's lectures at Institute Mittag-Leffler in 1935. Weyl and Courant used calculus of variation in the symmetric case while Carleman employed different methods which have the merit that the passage to the non-symmetric case does not cause any trouble. As pointed out by Carleman the methods in the proof give similar asymptotic formulas in other boundary value problems such as those considered by Neumann where one imposes boundary value conditions on outer normals, and so on. A crucial step during the proof of the theorem above is to construct a fundamental solution $\Phi(x, \xi; \kappa)$ to the PDE-operators $L - \kappa^2$ which is exposed in § E.

§ D. Proof of Theorem 0.

Let Ω be a bounded and Dirichlet regular domain. Let $p \in \Omega$ be kept fixed and consider the continuous function on $\partial\Omega$ defined by

$$q \mapsto \log \frac{1}{|p-q|}$$

We find the harmonic function $u_p(q)$ in Ω such that $u_p(q) = \log \frac{1}{|p-q|} \,:\, q \in \partial\Omega$. Green's function is defined for pairs $p \neq q$ in $\Omega \times \Omega$ by

(1) $$G(p,q) = \log \frac{1}{|p-q|} - u_p(q)$$

Keeping if $p \in \Omega$ fixed, the function $q \mapsto G(p,q)$ extends to the closure of Ω where it vanishes if $q \in \partial\Omega$. If $f \in L^2(\Omega$ we set

(2) $$\mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot f(q)\, dq$$

where $q = (x,y)$ so that $dq = dxdy$ when the double integral is evaluated. From (1) we see that

$$\iint_{\Omega\times\Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence $\mathcal{G}$ is of the Hilbert-Schmidt type and therefore a compact operator on $L^2(\Omega)$. Next, recall that $\frac{1}{2\pi} \cdot \log \sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator. From this one can deduce the following:

D.1 Theorem. *For each $f \in L^2(\Omega)$ the Lapacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

(*) $$\Delta(\mathcal{G}_f) = -f$$

The equation (*) means that the composed operator $\Delta \circ \mathcal{G}$ is minus the identity on $L^2(\Omega)$. We are led to introduce the linear operator S on $L^2(\Omega)$ defined by Δ, where $\mathcal{D}(S)$ is the range of $\mathcal{G}$. If $g \in C_0^2(\Omega)$, i.e. twice differentiable and with compact support, it follows via Greens' formula that

$$\frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot \Delta(g)(q)\, dq = -g(p)$$

In particular $C_0^2(\Omega) \subset \mathcal{D}(S)$ which implies that S is densely defined and we leave it to the reader to verify that

$$\mathcal{G}(\Delta(f)) = -f \quad : f \in \mathcal{D}(S)$$

Remark. By the construction of resolvent operators in § 1 this means that $-\mathcal{G}$ is Neumann's inverse of S.

Exercise. Show that S has a closed range and in addition it is self-adjoint, i.e. $S = S^*$.

The spectrum of S. A wellknown result asserts that there exists an orthonormal basis $\{\phi_n\}$ in $L^2(\Omega)$ where each $\phi_n \in \mathcal{D}(S)$ is an eigenfunction. More preciesely there is a non-decreasing sequence of positive real numbers $\{\lambda_n\}$ and

(i) $$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots$$

Let us remark that (i) means that

(ii) $$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n$$

So above $\{\lambda_n^{-1}\}$ are eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. Eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times.

Now we begin the proof of Theorem 0, i.e. we will show that the following hold for each point $p \in \Omega$:

$$(*) \qquad \lim_{n\to\infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi}$$

To prove (*) we consider the Dirichlet series for each fixed $p \in \Omega$:

$$(**) \qquad \Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

It is easily seen that $\Phi_p(s)$ is analytic in a half-space $\mathfrak{Res} > b$ for a large b. Less trivial is the following:

D.2 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem D.2 gives (*) via a result due to Wiener in the article *Tauberian theorem* [Annals of Math.1932]. Wiener's theorem asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s\to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n\to\infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Proof of Theorem D.2

Since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\text{(i)} \qquad \sum \lambda_n^{-2} < \infty$$

This convergence entails that various constructions below are defined. For each λ outside $\{\lambda_n\}$ we set

$$\text{(ii)} \qquad G(p,q;\lambda) = G(p,q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda - \lambda_n)}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\text{(iii)} \qquad \mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p,q;\lambda) \cdot f(q)\, dq$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p,\lambda)$. Set

$$F(p,q,\lambda) = G(p,q;\lambda) - G(p,q)$$

Keeping p fixed we see that (ii) gives

$$\text{(B.1)} \qquad \lim_{q\to p} F(p,q,\lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)}$$

Set

$$F(p,\lambda) = \lim_{q\to p} F(p,q,\lambda)$$

From (i) and (B.1) it follows that it is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Show via residue calculus that one has the equality below in a half-space $\mathfrak{Re}\, s > 2$:

$$\Phi(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a-i\infty}^{a+i\infty} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\, \lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p,\lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\Phi(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta}\, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx \tag{E.1}$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p,-x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^\infty \frac{e^{-zt}}{\sqrt{t^2 - 1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$I_0(z) = \sum_{m=0}^\infty \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \tag{i}$$

$$I_1(z) = \sum_{m=0}^\infty \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma \tag{ii}$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p,q;\kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p,q;\kappa) = K(\sqrt{\kappa}|p-q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p,q;-\kappa) = K(\sqrt{\kappa}\cdot|p-q|) - H(q;\kappa) \quad : \kappa > 0$$

Next, the construction of $G(p,q)$ gives

$$F(p,-\kappa) = \lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|) + \log\,|p-q|] + \lim_{q\to p}\,[u_p(q) + H(p,q,\kappa)] \tag{G.1}$$

The last term above has the "nice limit" $u_p(p) + H(p,p,\kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|) + \log\,|p-q|] = -\frac{1}{2}\cdot\log\kappa + \log 2 - \gamma \tag{G.2}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p,-\kappa) = -\frac{1}{2}\cdot\log\kappa + A + H(p,p;-\kappa)$$

With $x=\kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty x^{-s}\,dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2}\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty \log x\cdot x^{-s}\,dx$$

Notice that the complex derivative

$$\frac{d}{ds}\int_a^\infty x^{-s}\,dx = -\int_a^\infty \log x\cdot x^{-s}\,dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem D.2 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s\mapsto \frac{\sin\,\pi s}{2\pi^2}\cdot\int_a^\infty H(p,p,\kappa)\cdot\kappa^{-s}\,d\kappa$$

Proof. When $\kappa>0$ the equation (F.1) shows that $q\mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

$$0 \le H(p,q;\kappa) \le \max_{q\in\partial\Omega}\,K(\kappa|p-q|) \tag{i}$$

With $p\in\Omega$ fixed there is a positive number δ such that $|p-q|\ge\delta : q\in\partial\Omega$ which gives positive constants B and α such that

$$H(p,p;\kappa) \le e^{-\alpha\kappa} \quad : \kappa > 0 \tag{ii}$$

The reader may now check that this exponential decay gives Lemma H.2.

§ E. Fundamental solutions to second order Elliptic operators.

In $\mathbf{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ we consider a second order PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + a_0(x)$$

where a-functions are real-valued and one has the symmetry $a_{pq} = a_{qp}$. To ensure existence of a globally defined fundamental solutions we suppose the the following limit formulas hold as $(x, y, z) \to \infty$:

$$\lim a_\nu(x, y, z) = 0 \colon 0 \leq p \leq 3 \quad : \ \lim a_{pq}(x, y, z) = \text{Kronecker's delta function}$$

Thus, L approaches the Laplace operator as (x, y, z) tends to infinity. Moreover L is elliptic which means that the eigenvalues of the symmetric matrix with elements $\{a_{pq}(x)\}$ are positive for every x. Recall the notion of fundamental solutions. First we consider the adjoint operator:

$$(0.1) \qquad L^*(x, \partial_x) = P - 2 \cdot (\sum_{p=1}^{p=3} (\sum_{q=1}^{q=3} \frac{\partial a_{pq}}{\partial x_q}) \cdot \frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3} \frac{\partial a_p}{\partial x_p} + 2 \cdot \sum\sum \frac{\partial^2 a_{pq}}{\partial x_p \partial x_q}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ, ψ in $\mathbf{R}^3$ with compact support:

$$(0.2) \qquad \int L(\phi) \cdot \psi \, dx = \int \phi \cdot L^*(\psi) \, dx$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $L(x, \partial_x)$ if

$$(0.3) \qquad \psi(0) = \int \Phi \cdot L^*(\psi) \, dx$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $L - \kappa^2$ and a function $x \mapsto \Phi(x; \kappa)$ is a fundamental solution to $L - \kappa^2$ if

$$(0.4) \qquad \psi(0) = \int \Phi(x : \kappa) \cdot (L^* - \kappa^2)(\psi(x)) \, dx$$

hold for compactly supported C^2-functions ψ. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x, \xi; \kappa)$ with the property that

$$(*) \qquad \psi(\xi) = \int \Phi(x, \xi; \kappa) \cdot (L^*(x, \partial_x) - \kappa^2)(\psi(x)) \, dx$$

hold for all $\xi \in \mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x, \xi; \kappa)$ is a function of six variables defined in $\mathbf{R}^3 \times \mathbf{R}^3$. Theorem 1.9 below gives sharp estimate for fundamental solutions. The subsequent constructions are based upon a classic formula due to Newton and specific solutions to integral equations found by a convergent Neumann series.

1. The construction of $\Phi(x, \xi; \kappa)$.

When L has constant coefficients the construction of fundamental solutions was given by Newton in his famous text-books from 1666. We have the positive and symmetric 3×3-matrix $A = \{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix and put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq} a_p a_q - a_0}$$

where κ is chosen so large that the term under the square-root is > 0. Define the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p x_q$$

With these notations Newton's fundamental solution taken at $x = 0$ becomes

$$H(x;\kappa) = \frac{1}{4\pi \cdot \sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq} a_p \cdot x_q} \tag{1.1}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ indeed yields a fundamental solution to the PDE-operator $L(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients. For each $\xi \in \mathbf{R}^3$ the elements of the inverse matrix to $\{a_{pq}(\xi)$ are denoted by $\{b_{pq}(\xi)\}$. Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi)} \tag{i}$$

Following Newton's construction in (1.1) we put:

$$H(x,\xi;\kappa) = \frac{1}{4\pi} \cdot \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) \cdot x_q} \tag{ii}$$

When ξ is kept fixed this function of x is real analytic outside the origin and we also notice that $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. We are going to find a fundamental solution which takes the form

$$\Phi(x,\xi;\kappa) = H(x-\xi,\xi;\kappa) + \int_{\mathbf{R}^3} H(x-y,\xi;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy \tag{iii}$$

where the Ψ-function is the solution to an integral equation which we construct in (1.5).

1.3 The function $F(x,\xi;\kappa)$. For every fixed ξ we consider the differential operator in the x-space:

$$L_*(x,\partial_x,\xi;\kappa) =$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi))\frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

With ξ fixed we apply L_* to the function $x \mapsto H(x-\xi,\xi;\kappa)$ and put

$$F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)) \tag{1.3.1}$$

1.4 Two estimates. The limit conditions in (0.0) give positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \quad : \; [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|^2} \tag{1.4.1}$$

The verification of (1.4.1) is left as an exercise.

1.5 An integral equation. We seek $\Psi(x,\xi;\kappa)$ which satisfies the equation:

$$\Psi(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy + F(x,\xi;\kappa) \tag{1.5.1}$$

To solve (1.5.1) we construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$F^{(k)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot F^{(k-1)}(y,\xi;\kappa)\, dy \quad : \quad k \geq 2 \tag{1.5.2}$$

Then (1.4.1) gives the inequality

(i) $$|F^{(2)}(x,\xi;\kappa)| \le C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|}}{|x-y|^2\cdot|\xi-y|^2}\cdot dy$$

To estimate (i) we notice that the triple integral after the substitution $y-\xi \to u$ becomes

(ii) $$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2\cdot|u|^2}\cdot du$$

In (ii) the volume integral can be integrated in polar coordinates and becomes

(iii) $$C_1^2\cdot\int_0^\infty\int_{S^2}\frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2}\cdot dwdr$$

where S^2 is the unit sphere and dw the area measure on S^2 and we see that (iii) becomes

$$2\pi C_1^2\cdot\int_0^\infty\int_0^\pi\frac{e^{-k\kappa r}}{(x-\xi)^2+r^2-2r\cdot|x-\xi|\cdot\sin\theta}\cdot d\theta dr =$$

(iv) $$\frac{2\pi C_1^2}{|x-\xi|}\cdot\int_0^\infty e^{-k\kappa|x-\xi|t}\cdot\log|\frac{1+t}{1-t}|\cdot\frac{dt}{t}$$

where the last equality follows by a straightforward computation.

1.6 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)| \le \frac{2\pi\cdot C_1^2\cdot C_1^*}{\kappa\cdot|x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

(*) $$|F^{(n)}(x,\xi;\kappa)| \le \frac{C_1}{|x-\xi|^2}\cdot\left[\frac{2\pi C_1^2\cdot C_1^*}{\kappa}\right]^{n-1}\quad\text{hold for every}\quad n\ge 2$$

1.6 Conclusion. With κ_0^* so large that $2\pi C_1^2\cdot C_1^* < \kappa_0^*$ it follows from (*) that the Neumann series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $\kappa \ge \kappa_0^*$ and gives the requested solution $\Psi(x,\xi;\kappa)$ in (1.5.1).

1.7 Exercise. Above we have found Ψ which satisfies the integral equation in § 1.5.1 Use Green's formula to show that the function $\Phi(x,\xi;\kappa)$ defined in (1.2.1) gives a fundamental solution of $L(x,\partial_x)-\kappa^2$.

1.8 A final estimate. The constructions above show that the functions

$$x\mapsto\Phi(x,\xi;\kappa)\quad\text{and}\quad x\mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x=\xi$. Consider the difference

(1.8.1) $$G(x,\xi;\kappa)=\Phi(x,\xi;\kappa)-H(x-\xi,\xi;\kappa)$$

1.8.2 Exercise. Use the previous constructions to show that for every $0<\gamma\le 2$ there is a constant C_γ such that

$$\big|G(x,\xi;\kappa)\big| \le \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for every pair (x,ξ) and every $\kappa\ge\kappa_0$. Together with the the inequality for the H-function in (1.4.1) this gives an estimate for the fundamental solution Φ. More precisely we have proved:

1.9 Theorem. *With κ_0^* as above there exist positive constants C and k and for each $0 < \gamma \leq 2$ a constant C_γ such that*

$$|\Phi(x,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} + \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for all pairs (x,ξ) in $\mathbf{R}^3$ and every $\kappa \geq \kappa_0^*$.

Remark. Above C and k are independent of κ as soon as κ_0^* has been chosen as above. The size of these constant depend on the C^2-norms of the functions $\{a_{pq}(x)\}$ and as well as the C^1-norms of $\{a_1, a_2, a-3\}$ and the maximum norm of a_0. Notice that the whole construction is "canonical", i.e. the resulting fundaemtnal ssolutions $\{\Phi(x,\xi;\kappa)\}$ are uniquely determined. We remark that simiular constructions can be made for ellptic operators of even degree $2m$ with $m \geq 2$. Here Newton's solution for constant coefficients is replaced by those of Fritz John which arise via the wave deompostion of the Dirac measure. It would be interesting tanalyze the resulting version of Theorem 1.9. Of course, one can also extend everything to elliptic operators of n variables were $n \geq 4$ in which case the denominator $|x-\xi|^{-1}$ is replaced by $|x-\xi|^{-n+2}$.

The proof of Theorem 1 relies upon the construction of fundamental solutions which is given in § 1 below. After this has been achieved, the asymptotic formula (*) in Theorem 2 is derived via Tauberian theorems for Dirichlet series which goes as follows: Let $\{a_\nu\}$ and $\lambda_\nu\}$ be two sequences of positive numbers where $\lambda_\nu \to +\infty$ and the series

$$f(x) = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\lambda_\nu + x}$$

converges when $x > x_*$ for some positive number x_*. Next, for every $x > 0$ we define the function

$$\mathcal{A}(x) = \sum_{\{\lambda_\nu < x\}} a_\nu$$

In other words, with $x > 0$ we find the largest integer $\nu(x)$ such that $\lambda - \nu(x) < x$ and then $\mathcal{A}(x)$ is the sum over the a-numbers up to this index. With these notations the following implication holds for every pair $A > 0$ and $0 < \alpha < 1$

3. Theorem. *Suppose there exists a constant $A > 0$ and some $0 < \alpha < 1$ such that*

$$\lim_{x \to \infty} x^\alpha \cdot f(x) = A \implies \lim_{x \to \infty} \mathcal{A}(x) = \frac{A}{\pi} \cdot \frac{\sin \pi\alpha}{1-\alpha} \cdot x^{1-\alpha}$$

0 Preliminary constructions.

We are given an ellipitic operator L as above and assume that the coefficients are defined in the whole space $\mathbf{R}^3$. To ensure convergence of volume integrals taken over the whole of $\mathbf{R}^3$ we add the conditions that

$$\lim_{|x| \to \infty} a_{pp}(x) = 1 \quad : \quad 1 \le p \le 3$$

while $\{a_{pq}\}$ for $p \neq q$ and a_1, a_2, a_3, b tend to zero as $|x| \to +\infty$. This means that P approaches the Laplace operator when $|x|$ is large. Let us recall the notion of a fundamental solution which prior to the general notion of distributions introduced by L. Schwartz, was referred to as a *Grundlösung*. First, the regularity of the coefficients of a PDE-operator P enable us to construct the adjoint operator:

$$P^*(x, \partial_x) = P - 2 \cdot (\sum_{p=1}^{p=3} (\sum_{q=1}^{q=3} \frac{\partial a_{pq}}{\partial x_q}) \cdot \frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3} \frac{\partial a_p}{\partial x_p} + 2 \cdot \sum \sum \frac{\partial^2 a_{pq}}{\partial x_p \partial x_q}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ, ψ in $\mathbf{R}^3$ with compact support:

$$\int P(\phi) \cdot \psi \, dx = \int \phi \cdot P^*(\psi) \, dx$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $P(x, \partial_x)$ if

$$\psi(0) = \int \Phi \cdot P^*(\psi) \, dx$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $P - \kappa^2$ and a function $\Phi(x; \kappa)$ is a fundamental solution to $P - \kappa^2$ if

(1) $$\psi(0) = \int \Phi(x : \kappa) \cdot (P^* - \kappa^2)(\psi(x)) \, dx$$

hold for compactly supported C^2-functions ψ. Above κ appears as an index of Φ, i.e. for each fixed κ we have the locally integrable function $x \mapsto \Phi(x : \kappa)$. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x, \xi; \kappa)$ with the property that

(*) $$\psi(\xi) = \int \Phi(x, \xi : \kappa) \cdot (P^* - \kappa^2)(\psi(x)) \, dx$$

hold for all $\xi \in \mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x, \xi; \kappa)$ is a function of six variables defined in $\mathbf{R}^3 \times \mathbf{R}^3$. Fundamental solutions are in general not unique. However, when P is an elliptic operator as above we shall give an explicit construction of fundamental solutions $\Phi(x,\xi;\kappa)$ for all sufficiently large κ in § 1.

1. The construction of $\Phi(x,\xi;\kappa)$.

1.1 The case when P has constant coefficients. Here the fundamental solution is given by a formula which goes back to Newton's work in his classic text-books from 1666. We have the positive and symmetric 3×3-matrix $A=\{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix and recall that they are found via Cramér's rule:

$$b_{pq}=\frac{A_{pq}}{\Delta}$$

where $\Delta=\det(A)$ and $\{A_{pq}\}$ are the cofactor minors of the A-matrix. Put

$$\alpha=\sqrt{\kappa^2+\frac{1}{2}\sum_{p,q} b_{pq}a_pa_q-b}$$

where κ is chosen so large that the term under the square-root is >0. Next, define the quadratic form

$$B(x)=\sum_{p,q} b_{pq}a_px_q$$

With these notations Newton's fundamental solution taken at $x=0$ becomes

$$(*)\qquad H(x;\kappa)=\frac{1}{\sqrt{\Delta\cdot B(x)}}\cdot e^{-\alpha\sqrt{B(x)}-\frac{1}{2}\sum_{p,q}b_{pq}a_p\cdot x_q}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ indeed yields a fundamental solution to the PDE-operator $P(\partial_x)-\kappa^2$.

1.2 The case with variable coefficients.

Choose $\kappa_0>0$ such that

$$\kappa_0^2+\frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi)-b(\xi)>0\quad\text{hold for all}\quad \xi\in\mathbf{R}^3$$

and for every $\kappa\geq\kappa_0$ we set

$$\text{(i)}\qquad \alpha_\kappa(\xi)=\sqrt{\kappa^2+\frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi)-b(\xi)}$$

Following Newton's construction for the case of constant coefficients we put:

$$\text{(ii)}\qquad H(x,\xi;\kappa)=\frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi)\cdot x_px_q}}\cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)}-\frac{1}{2}\sum_{p,q}b_{pq}(\xi)a_p(\xi)\cdot x_q}$$

When ξ is kept fixed this function of x is real analytic outside the origin and the singularity at $x=0$ is of Newton's type. In particular $x\to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. Next, for every fixed ξ we consider the differential operator in the x-space:

$$L_*(x,\partial_x,\xi;\kappa)=$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3}(a_{pq}(x)-(a_{pq}(\xi))\cdot\frac{\partial^2}{\partial x_p\partial x_q}+\sum_{p=1}^{p=3}(a_p(x)-a_p(\xi))\frac{\partial}{\partial x_p}+(b(x)-b(\xi))$$

For each fixed ξ there exists the function $x\mapsto H(x-\xi,\xi;\kappa)$ and we apply the L_*-operator on this x-dependent function and put:

$$\text{(iii)}\qquad F(x,\xi;\kappa)=\frac{1}{4\pi}\cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)$$

1.3 Two estimates. The hypothesis that $\{a_{pq}(x)\}$ are of class C^2 and$\{a_p(x)\}$ of class C^1, together with the limit conditions (*) in § XX give the existence of positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \tag{1.3.1}$$

$$[F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|^2}}{|x-\xi|^2} \tag{1.3.2}$$

The verification is left as an exercise.

1.4 An integral equation. We seek $\Phi(x,\xi;\kappa)$ which solves the equation:

$$\Phi(x,\xi;\kappa) = \iiint F(x,y;\kappa) \cdot \Phi(y,\xi;\kappa)\,dy + F(x,\xi;\kappa) \tag{1}$$

where the integral is taken over $\mathbf{R}^3$. To solve (1) we construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$F^{(k)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot F^{(k-1)}(y,\xi;\kappa)\,dy \quad : \quad k \geq 2 \tag{1.4.1}$$

Then (1.3.2) gives the inequality

$$|F^{(2)}(x,\xi;\kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|^2}}{|x-y|^2 \cdot |\xi-y|^2} \cdot dy \tag{i}$$

To estimate (i) we first notice that the triple integral after the substitution $y - \xi \to u$ becomes

$$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2 \cdot |u|^2} \cdot du \tag{ii}$$

In (ii) the volume integral is integrated in polar coordinates and becomes

$$C_1^2 \cdot \int_0^\infty \int_{S^2} \frac{e^{-k\kappa r^2}}{|x - r\cdot w - \xi|^2} \cdot dwdr \tag{iii}$$

where S^2 is the unit sphere and dw the area measure on S^2 and we see that (iii) becomes

$$2\pi C_1^2 \cdot \int_0^\infty \int_0^\pi \frac{e^{-k\kappa r^2}}{(x-\xi)^2 + r^2 - 2r \cdot |x-\xi| \cdot \sin\theta} \cdot d\theta dr =$$

$$\frac{2\pi C_1^2}{|x-\xi|} \cdot \int_0^\infty e^{-k\kappa|x-\xi|t} \cdot \log|\frac{1+t}{1-t}| \cdot \frac{dt}{t} \tag{iv}$$

where the last equality follows by a straightforward computation.

1.5 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)| \leq \frac{2\pi C_1^2 \cdot C_1^*}{\kappa \cdot |x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

$$|F^{(n)}(x,\xi;\kappa)| \leq \frac{C_1}{|x-\xi|^2} \cdot [\frac{2\pi C_1^2 \cdot C_1^*}{\kappa}]^{n-1} \quad \text{hold for every} \quad n \geq 2 \tag{*}$$

1.6 Conclusion. With κ so large that $2\pi C_1^2 \cdot C_1^* < \kappa$ it follows form (*) that the series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $x \neq \xi$ and this Neumann series gives the requested solution $\Phi(x,\xi;\kappa)$. Moreover, $\Phi(x,\xi;\kappa)$ satisfies a similar estimate as in (1.3.2) above with another constant than C_2 instead of C_1.

1.7 Exercise. Above we have found Φ which satisfies the integral equation in § 1.4 Use Green's formula to show that $\Phi(x,\xi;\kappa)$ gives a fundamental solution of $P(x,\partial_x) - \kappa^2$ with a pole at ξ.

1.8 Some final estimates. The constructions above show that the functions

$$x \mapsto \Phi(x,\xi;\kappa) \quad \text{and} \quad x \mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x = \xi$. Consider the difference

$$\Psi(x,\xi;\kappa) = \Phi(x,\xi;\kappa) - H(x-\xi,\xi;\kappa) \tag{1.8.1}$$

1.8.2 Exercise. Use the previous constructions to show that for every $0 < \gamma \leq 2$ there is a constant C_γ such that

$$\big|\Psi(x,\xi;\kappa)\big| \leq \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for every pair (x,ξ) and every $\kappa \geq \kappa_0$. Together with (1.3.1) this gives an estimate for the fundamental solution Φ.

2. Green's functions.

Let Ω be a bounded domain in $\mathbf{R}^3$. A Green's function $G(x,y;\kappa)$ attached to this domain and the PDE-operator $P(x,\partial_x;\kappa)$ is a function which for fixed κ is a function in $\Omega \times \Omega$ with the following properties:

$$G(x,y;\kappa) = 0 \quad \text{when} \quad x \in \partial\Omega \quad \text{and} \quad y \in \Omega \tag{*}$$

$$\psi(y) = \int_\Omega (P^*(x,\partial_x) - \kappa^2)(\psi(x)) \cdot G(x,y;\kappa)\,dx \quad : \quad y \in \Omega \tag{**}$$

hold for all C^2-functions ψ with compact support in Ω. To find G we solve Dirchlet problems. With $\xi \in \Omega$ kept fixed one has the continuous function on $\partial\Omega$:

$$x \mapsto \Phi^*(x,\xi;\kappa)$$

Solving Dirchlet's problem gives a unique C^2-function $w(x)$ which satisfies:

$$P(x,\partial_x)(w) + \kappa^2 \cdot w = 0 \quad \text{holds in} \quad \Omega \quad \text{and} \quad w(x) = \Phi(x,\xi;\kappa) \quad : x \in \partial\Omega = 0$$

From the above it is clear that this gives the requested G-function, i.e. one has:

2.1 Proposition. *The the function*

$$G(x,\xi;\kappa) = \Phi(x,\xi;\kappa) - w(x) \quad \text{satisfies} \quad (* - **)$$

Using the estimates for the ϕ-function from § 1 we get estimates for the G-function above. Start with a sufficiently large κ_0 so that $\Phi^*(x,\xi;\kappa_0)$ is a positive function of (x,ξ). Then the following hold:

2.2 Theorem. *One has*

$$G(x,\xi;\kappa_0) = \frac{1}{\sqrt{\Delta(x)} \cdot \sqrt{\Phi(x,\xi;\kappa_0)}} + R(x,\xi)$$

where the remainder function satisfies the following for all pairs (x,ξ) in Ω:

$$|R(x,\xi)| \leq C \cdot |x-\xi|^{-\frac{1}{4}}$$

and the constant C only depends on the domain Ω and the PDE-operator P.

Remark. Above the negative power of $|x-\xi|$ is a fourth-root which means that the remainder term R is more regular compared to the first term which behaves like $|x-\xi|^{-1}$ on the diagonal $x = \xi$.

2.3 Exercise. Prove Theorem 2.3 If necessary, consult [Carleman: page xx-xx9 for details.

2.4 The integral operator $\mathcal{J}$

.

With κ_0 chosen as above we consider the integral operator which sends a function u in Ω to

$$\mathcal{J}_u(x) = \int_\Omega G(x, \xi; \kappa_0) \cdot u(\xi)\, d\xi$$

The construction of the Green's function gives:

$$(P - \kappa_0^2)(\mathcal{J}_u)(x) = u(x) \quad : \quad x \in \Omega \tag{2.4.1}$$

In other words, if E denotes the identity we have the operator equality

$$P(x, \partial_x) \circ \mathcal{J}_u = \kappa_0^2 \cdot \mathcal{J} + E \tag{2.4.2}$$

Consider pairs (u, γ) such that

$$u(x) + \gamma \cdot \mathcal{J}_u(x) = 0 \quad : \quad x \in \Omega \tag{2.4.3}$$

The vanishing from (*) for the G-function implies that $J_u(x) = 0$ on $\partial\Omega$. Hence every u-function which satisfies in (2.4.3) for some constant γ vanishes on $\partial\Omega$. Next, apply P to (2.4.3) and then the operator formula (2.4.2) gives

$$0 = P(u) + \gamma\kappa_0^2 \cdot \mathcal{J}_u + \gamma \cdot u \implies P(u) + (\gamma - \kappa_0^2)u = 0$$

2.4.4 Conclusion. Hence the boundary value problem (*) from 0.B is equivalent to find eigenfunctions of $\mathcal{J}$ via (2.4.3) above.

3. Almost reality of eigenvalues.

Consider the set of eigenvalues λ to (*) in (0.B). Then we have:

3.1 Proposition. *There exist positive constants C_* and c_* such that every eigenvalue λ to (*) in (0.B) satisfies*

$$|\mathfrak{Im}\,\lambda|^2 \leq C_*(\mathfrak{Re}\,\lambda) + c_*)$$

Proof. Let u be an eigenfunction where $P(u) + \lambda \cdot u = 0$. Stokes theorem and the vanishing of $u|\partial\Omega$ give:

$$0 = \int_\Omega \bar{u} \cdot (P + \lambda)(u)\, dx = -\int_\Omega \sum_{p,q} a_{pq}(x) \cdot \frac{\partial u}{\partial x_p} \frac{\partial \bar{u}}{\partial x_q}\, dx + \int_\Omega \bar{u} \cdot (\sum a_p(x) \frac{\partial u}{\partial x_p}\,)\, dx +$$

$$\int_\Omega |u(x)|^2 \cdot b(x)\, dx + \lambda \cdot \int |u(x)|^2\, dx$$

Write $\lambda = \xi + i\eta$. Separating real and imaginary parts we find the two equations:

$$\text{(i)} \qquad \xi \int |u|^2\, dx = \int \sum_{p,q} a_{p,q}(x) \frac{\partial u}{\partial x_p} \cdot \frac{\partial \bar{u}}{\partial x_q}\, dx + \int \big(\frac{1}{2} \cdot \sum \frac{\partial a_p}{\partial x_p} - b\big) \cdot |u|^2\, dx$$

$$\text{(ii)} \qquad \eta \int |u|^2\, dx = \frac{1}{2i} \int \sum a_p \big(u \frac{\partial \bar{u}}{\partial x_p} - \bar{u} \frac{\partial u}{\partial x_p}\,\big)\, dx$$

Set

$$A = \int |u|^2\, dx \quad : \quad B = \int |\nabla(u)|^2\, dx$$

Since P is elliptic there exists a positive constant k such that

$$\sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} > k\cdot|\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants c_1, c_2, c_3 such that

(iii) $$A\xi > c_1B - c_2B \quad : \quad A|\eta| < c_3\cdot\sqrt{AB}$$

Here (iii) implies that $\xi > -c_2$ and the reader can also confirm that

(iv) $$B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A\cdot c_2\cdot\sqrt{\frac{\xi+c_2}{c_1}} \quad : \quad |\eta| < c_3\cdot\sqrt{\frac{\xi+c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 3.1.

4. The asymptotic formula.

Using the results above where we have found a good control of the integral operator $\mathcal{J}$ and the identification of eigenvalues to | and those from (*) in (0.B), one can proceed and apply Tauberian theorems to derive the asymptotic formula in Theorem 1 using similar methods as described in § XX where we treated the Laplace operator. For details the reader may consult [Carleman:p age xx-xx].