§ xx. Fundamental solutions to second order Elliptic operators.

Introduction. We shall expose material from Carleman's article *xxx*. Elliptic partial differential operators appear frequently and in order to solve inhomogeneous equations one often employs a fundamental solution. It is therefore of interest to contruct fundamental solutions with "best possible regularity" conditions. For PDE-operators with constant coefficients one employs Fouriers' inversion fourmula and we refer to Chapter X in vol.2 in Hörmander's text-book series on linear partial differential operators for a detailed account about constructions of fundamental solutions with optimal regularity in the case of constant coefficients. Passing to the case of variable coefficients one profits upon the constant case and following Carleman we will show that one can obtain fundamental solutions to second order elliptic operators in a canonical fashion. In contrast to the csae of constant coefficients the subsequent constructions do not use the Fourier transform. Insetad one finds fundemental solutions by solving integral equations of the Neumann-Fredholm type. Let us remark that one does not need any concepts from distribution theory since we will find fundamental solutions are locally integrable and in such situations the notion of fundamental solutions were well understood at an early stage after pioneering work by Weyl and Zeilon prior to 1925. In his article Carleman refers to *Grundlösungen* and their requested properties are derived via Green's formula.

Remark. For students interested in PDE-theory the material below offers an instructive lesson and suggests further investigations. We restrict the study to $\mathbf{R}^3$ and remark only that similar constructions can be performed when $n \geq 4$ starting from Newton's potential $|x-\xi|^{-n+2}$. Here it would be interesting to clarify the precise estimates when $n \geq 4$ and establish similar inequalities as in the Main Theorem One can also try to extend the whole condtructions to elliptic operators of order ≥ 3. In the case of even order $2m$ the method would be to emply the canonical fundamental solutions for elliptic operators of even order with constant coefficents which were given by F. John. So here replaces Newton's fundamental solutions which appear in § 1 below by those of John and after it is tempting to perform similar constructions as those by Carleman. This appears to be a "profitable research problem" for ph.d-students. Let us also remark that one does not assume that the elliptic operators are symmetric, i.e. both the constructions as well as estimates for the fundamental solutions do not rely upon symmetry conditions. Recall finally that fundamental solutions are used to construct various Greens' functions and here a priori estiamates are valuable We illustrate this in § xx where we expose some further results by Carleman concerned with asymptotic distributions of eigenvalues to elliptic boundary value problems which we describe below.

An asymptotic formula for the spectrum. Let $n = 3$ and consider a second order PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p} + a_0(x)$$

The a-functions are real-valued and defined in a neighborhood of the closure of a bounded domain Ω in $\mathbf{R}^3$ with a C^1-boundary. Here one has the symmetry $a_{pq} = a_{qp}$, and $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x \in \Omega$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under these conditions, a result which goes back to work by Neumann and Poincaré, gives a positive constant κ_0 such that if $\kappa \geq \kappa_0$ then the inhomogeneous equation

$$L(u) - \kappa^2 \cdot u = f \quad : f \in L^2(\Omega)$$

has a unique solution u which is a C^2-function in Ω and extends to the closure where it is zero on $\partial\Omega$. Moreover, there exists some κ_0 and for each $\kappa \geq \kappa_0$ a Green's function $G(x, y; \kappa)$ such that

$$\text{(i)} \qquad (L - \kappa^2)(\frac{1}{4\pi} \cdot \int_\Omega G(x, y; \kappa)\, f(y)\, dy) = -f(x) \quad : f \in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

$$f \mapsto -\frac{1}{4\pi} \cdot \int_\Omega G(x,y;\kappa)\, f(y)\, dy \tag{ii}$$

is Neumann's resolvent to the densely defined operator $L-\kappa^2$ on the Hilbert space $L^2(\Omega)$. Next, one seeks pairs (u_n, λ_n) where u_n are $L2$-functions in Ω which extend to be zero on $\partial\Omega$ and satisfy

$$L(u_n) + \lambda_n \cdot u_n = 0$$

It turns out that the set of eigenvalues is discrete and moreover their real parts tend to $+\infty$. They are arranged with non-decreasing absolute values and in § xx we prove that there exist positive constants C and c such that

$$|\mathfrak{Im}(\lambda_n)| \leq C \cdot (\mathfrak{Re}(\lambda_n) + c)$$

hold for every n. Next, the elliptic hypothesis means that the determinant function

$$D(x) = \det(a_{p,q}(x))$$

is positive in Ω. With these notations one has

Theorem. *The following limit formula holds:*

$$\lim_{n\to\infty} \frac{\mathfrak{Re}(\lambda_n)}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_\Omega \frac{1}{\sqrt{D(x)}}\, dx \tag{*}$$

Remark. The formula above is due to Courant and Weyl when P is symmetric and extended to non-symmetric operators during Carleman 's lectures at Institute Mittag-Leffler in 1935. Weyl and Courant used calculus of variation in the symmetric case while Carleman employed different methods which have the merit that the passage to the non-symmetric case does not cause any trouble. A crucial step during the proof of the theorem above is to construct a fundamental solution $\Phi(x,\xi;\kappa)$ to the PDE-operators $L-\kappa^2$ which done in § 1 while § 2 treats the asymptotic formula above. As pointed out by Carleman the methods in the proof give similar asymptotic formulas in other boundary value problems such as those considered by Neumann where one imposes boundary value conditions on outer normals. As an example we consider an elliptic operator of the form

$$L = \Delta + \sum_{p=1}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + a_0(x)$$

where Δ is the Laplace operator. Given a positive real-valued continuous function $\rho(x)$ on $\partial\Omega$ we obtain the Neumann-Poincaré operator $\mathcal{NP}$ which sends each $u \in C^0(\partial\Omega)$ to

$$\mathcal{NP}(u) = \frac{\partial u^*}{\partial \mathbf{n}_i} - \rho \cdot u$$

Here u^* is the Dirichlet extension of u to Ω which is equal to u on $\partial\Omega$ and satisfies $L(u) = 0$ in Ω, while $\frac{\partial u^*}{\partial \mathbf{n}_i}$ is the inner normal along the boundary. In the special case when $L = \Delta$ this boundary value problem has unique solutions, i.e. for every $f \in C^0(\partial\Omega)$ there exists a unique u such that

$$\mathcal{NP}(u) = f$$

This was proved by Poincaré in 1897 for domains in $\mathbf{R}^3$ whose boundaries are of class C^2 and the extension to domains with a C^1-boundary is also classic. Passing to general operators L as above which are not neccesarily symmetric one encounters spectral problems, i.e. above $\mathcal{NP}$ regared as a linear operator on the Banach space $C^0(\partial\Omega)$ is densely defined and one seeks its spectrum, i.e, compex numbers λ for which there exists a non-zero u such that

$$\mathcal{NP}(u) + \lambda \cdot u = 0$$

I do not know if there exists an analytic formula for these eigenvalues. Notice that a new feauture is that the ρ-function affects the spectrum.

Fundamental solutions.

In $\mathbf{R}^3$ with coordinates $x=(x_1,x_2,x_3)$ we consider a second order PDE-operator

$$L=\sum_{p=1}^{p=3}\sum_{q=1}^{q=3}a_{pq}(x)\cdot\frac{\partial^2}{\partial x_p\partial x_q}+\sum_{p=1}^{p=3}a_p(x)\frac{\partial}{\partial x_p}+a_0(x)$$

where a-functions are real-valued and one has the symmetry $a_{pq}=a_{qp}$. To ensure existence of a globally defined fundamental solutions we suppose the the following limit formulas hold as $(x,y,z)\to\infty$:

$$\lim a_\nu(x,y,z)=0\colon 0\le p\le 3\quad:\ \lim a_{pq}(x,y,z)=\text{Kronecker's delta function}$$

Thus, L approaches the Laplace operator as (x,y,z) tends to infinity. Moreover L is elliptic which means that the eigenvalues of the symmetric matrix with elements $\{a_{pq}(x)\}$ are positive for every x. Recall the notion of fundamental solutions. Consider the adjoint operator:

$$L^*(x,\partial_x)=P-2\cdot(\sum_{p=1}^{p=3}(\sum_{q=1}^{q=3}\frac{\partial a_{pq}}{\partial x_q})\cdot\frac{\partial}{\partial x_p}-\sum_{p=1}^{p=3}\frac{\partial a_p}{\partial x_p}+2\cdot\sum\sum\frac{\partial^2 a_{pq}}{\partial x_p\partial x_q}\tag{0.1}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ,ψ in $\mathbf{R}^3$ with compact support:

$$\int L(\phi)\cdot\psi\,dx=\int\phi\cdot L^*(\psi)\,dx\tag{0.2}$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $L(x,\partial_x)$ if

$$\psi(0)=\int\Phi\cdot L^*(\psi)\,dx\tag{0.3}$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $L-\kappa^2$ and a function $x\mapsto\Phi(x;\kappa)$ is a fundamental solution to $L-\kappa^2$ if

$$\psi(0)=\int\Phi(x:\kappa)\cdot(L^*-\kappa^2)(\psi(x))\,dx\tag{0.4}$$

hold for compactly supported C^2-functions ψ. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x,\xi;\kappa)$ with the property that

$$\psi(\xi)=\int\Phi(x,\xi;\kappa)\cdot(L^*(x,\partial_x)-\kappa^2)(\psi(x))\,dx\tag{*}$$

hold for all $\xi\in\mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x,\xi;\kappa)$ is a function of six variables defined in $\mathbf{R}^3\times\mathbf{R}^3$. With these notations we announce the main result:

Main Theorem. *There exists a constant κ_* such that for every $\kappa\ge\kappa_*$ one can find a fundaemtal solution $\Phi(x,\xi;\kappa)$ which is locally integrable in the 6-dimensional (x,ξ)-space. Moreover, there exist positive constants C and k and for each $0<\gamma\le 2$ a constant C_γ such that*

$$|\Phi(x,\xi;\kappa)|\le C\cdot\frac{e^{-k\kappa|x-\xi|}}{|x-\xi|}+\frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for all pairs (x,ξ) in $\mathbf{R}^3$ and every where the constants k and C do not depend upon κ.

1. The construction of $\Phi(x,\xi;\kappa)$.

The subsequent constructions are based upon a classic formula due to Newton and specific solutions to integral equations found by a convergent Neumann series. When L has constant coefficients the construction of fundamental solutions was (at least essentially) given by Newton in his famous text-books from 1666 and goes as follows: Consider a positive and symmetric 3×3-matrix $A=\{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix which gives the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p x_q$$

Put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq} a_p a_q - a_0}$$

where κ is chosen so large that the term under the square-root is > 0. Finally, put

$$\Delta = \det(A)$$

With these notations we get a function:

$$H(x;\kappa) = \frac{1}{4\pi \cdot \sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq} a_p \cdot x_q} \tag{1.1}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ yields a fundamental solution to the PDE-operator $L(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients. Now L has variable coefficients. For each $\xi \in \mathbf{R}^3$ the elements of the inverse matrix to $\{a_{pq}(\xi)$ are denoted by $\{b_{pq}(\xi)\}$. Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi)} \tag{i}$$

Following Newton's construction in (1.1) we put:

$$H(x,\xi;\kappa) = \frac{1}{4\pi} \cdot \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) \cdot x_q} \tag{ii}$$

When ξ is kept fixed this function of x is real analytic outside the origin and $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. We are going to construct a fundamental solution which takes the form

$$\Phi(x,\xi;\kappa) = H(x-\xi,\xi;\kappa) + \int_{\mathbf{R}^3} H(x-y,\xi;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy \tag{iii}$$

where the Ψ-function is the solution to an integral equation which we construct in (1.5). But first we need another construction.

1.3 The function $F(x,\xi;\kappa)$. For every fixed ξ we get the differential operator in the x-space:

$$L_*(x, \partial_x, \xi; \kappa) =$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi)) \frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

Apply L_* to the function $x \mapsto H(x-\xi,\xi;\kappa)$ and put

$$F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)) \tag{1.3.1}$$

1.4 Two estimates. The limit conditions in (0.0) give positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \quad : \ [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|^2} \tag{1.4.1}$$

The verification of (1.4.1) is left as an exercise.

1.5 An integral equation. We seek $\Psi(x,\xi;\kappa)$ which satisfies the equation:

$$\Psi(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa)\cdot\Psi(y,\xi;\kappa)\,dy + F(x,\xi;\kappa) \tag{1.5.1}$$

To solve (1.5.1) we construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$F^{(k)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa)\cdot F^{(k-1)}(y,\xi;\kappa)\,dy \quad : \quad k\geq 2 \tag{1.5.2}$$

Then (1.4.1) gives the inequality

$$|F^{(2)}(x,\xi;\kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|}}{|x-y|^2\cdot|\xi-y|^2}\cdot dy \tag{i}$$

To estimate (i) we notice that the triple integral after the substitution $y-\xi\to u$ becomes

$$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2\cdot|u|^2}\cdot du \tag{ii}$$

In (ii) the volume integral can be integrated in polar coordinates and becomes

$$C_1^2\cdot\int_0^\infty\int_{S^2}\frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2}\cdot dwdr \tag{iii}$$

where S^2 is the unit sphere and dw the area measure on S^2. It follows that (iii) becomes

$$2\pi C_1^2\cdot\int_0^\infty\int_0^\pi \frac{e^{-k\kappa r}}{(x-\xi)^2+r^2-2r\cdot|x-\xi|\cdot\sin\theta}\cdot d\theta dr =$$

$$\frac{2\pi C_1^2}{|x-\xi|}\cdot\int_0^\infty e^{-k\kappa|x-\xi|t}\cdot\log|\frac{1+t}{1-t}|\cdot\frac{dt}{t} \tag{iv}$$

where the last equality follows by a straightforward computation.

1.6 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)| \leq \frac{2\pi\cdot C_1^2\cdot C_1^*}{\kappa\cdot|x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

$$|F^{(n)}(x,\xi;\kappa)| \leq \frac{C_1}{|x-\xi|^2}\cdot[\frac{2\pi C_1^2\cdot C_1^*}{\kappa}]^{n-1} \quad \text{for every} \quad n\geq 2 \tag{*}$$

1.6 Conclusion. Choose κ_0^* so large that

$$2\pi C_1^2\cdot C_1^* < \kappa_0^* \tag{1.6.1}$$

Then (*) implies that the Neumann series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $\kappa\geq\kappa_0^*$ and gives the requested solution $\Psi(x,\xi;\kappa)$ in (1.5.1).

1.7 Exercise. We have found Ψ which satisfies the integral equation in § 1.5.1. Next, since the H-function in (ii) from § 1.2 is everywhere positive the integral equation (iii)in § 1.2 has a unique solution $\Phi(x,\xi;\kappa)$. Using Green's formula the reader can check that $\Phi(x,\xi;\kappa)$ yields a fundamental solution of $L(x,\partial_x)-\kappa^2$.

1.8 Some estimates. The constructions above show that the functions

$$x\mapsto\Phi(x,\xi;\kappa) \quad \text{and} \quad x\mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x = \xi$. Consider the difference

$$Q(x, \xi; \kappa) = \Phi(x, \xi; \kappa) - H(x - \xi, \xi; \kappa) \tag{1.8.1}$$

1.8.2 Exercise. Use the previous constructions to show that for every $0 < \gamma \leq 2$ there is a constant C_γ such that

$$\big|Q(x, \xi; \kappa)\,\big| \leq \frac{C_\gamma}{(\kappa|x - \xi|)^\gamma}$$

hold for every pair (x, ξ) and every $\kappa \geq \kappa_0$. Finally, the reader can apply the inequality for the H-function in (1.4.1) to conclude the results in the Main Theorem.

§ 2. Green's functions.

Let Ω be a bounded domain in $\mathbf{R}^3$, and L an elliptic differential operator as in § xx. Let $\kappa > 0$ and suppose we have found a function $G(x, y; \kappa)$ defined when $(x, y) \in \Omega \times \Omega$ with the property that $G(x, y) = 0$ if $x \in \partial\Omega$ and $y \in \Omega$. Moreover

$$(L(x, \partial_x) - \kappa^2)(G(x, y; \kappa)) = \delta(x - y)$$

With G as kernel we get the integral operator

$$\mathcal{G}(f)(x) = \int_\Omega G(x, y; \kappa)\, f(y)\, dy$$

Then $\mathcal{G}(f)(x) = 0$ on $\partial\Omega$ and the composed operator

$$(L(x, \partial_x) - \kappa^2) \circ \mathcal{G} = E$$

To construct G we use the fundamental solution $\Phi(x, y; \kappa)$ from § xx which satisfies

$$(L(x, \partial_x) - \kappa^2)(\Phi(x, y; \kappa)) = \delta(x - y)$$

Next, with $y \in \Omega$ kept fixed we have the continuous boundsry functon

$$x \mapsto \Phi(x, y; \kappa)$$

Solving the Dirchlet problem we find $w(x)$ such that $w(x) = \Phi(x, y; \kappa)$ on the boundary while $(L(x, \partial_x) - \kappa^2)(w) = 0$ holds in Ω. Then we can take

$$G(x, y; \kappa) = \Phi(x, y; \kappa) - w(x)$$

Using the estimates for the Φ-function from § 1 we get estimates for the G-function above. We choose a sufficiently large κ_0 so that $\Phi(x, \xi; \kappa_0)$ is a positive function of (x, ξ). Then the following hold:

2.1 Theorem. *One has*

$$G(x, \xi; \kappa_0) = \frac{1}{\sqrt{\Delta(x)} \cdot \sqrt{\Phi(x, \xi; \kappa_0)}} + R(x, \xi)$$

where the remainder function satisfies the following for all pairs (x, ξ) in Ω:

$$|R(x, \xi)| \leq C \cdot |x - \xi|^{-\frac{1}{4}}$$

and the constant C only depends on the domain Ω and the PDE-operator L.

Remark. Above the negative power of $|x - \xi|$ is a fourth-root which means that the remainder term R is more regular compared to the first term which behaves like $|x - \xi|^{-1}$ on the diagonal $x = \xi$.

2.2 Exercise. Prove Theorem 2.1 If necessary, consult [Carleman: page xx-xx9 for details.

2.3. Almost reality of eigenvalues.

Consider the set of eigenvalues λ for which there exists a function u in Ω which is zero on $\partial\Omega$ while

$$L(u) + \lambda \cdot u = 0$$

holds in Ω.

2.3.1 Proposition. *There exist positive constants C_* and c_* such that every eigenvalue λ above satisfies*

$$|\mathfrak{Im}\, \lambda|^2 \leq C_*(\mathfrak{Re}\, \lambda) + c_*)$$

Proof. Let u be an eigenfunction where $L(u) + \lambda \cdot u = 0$. Stokes theorem and the vanishing of $u|\partial\Omega$ give:

$$0 = \int_\Omega \bar{u} \cdot (L+\lambda)(u)\,dx = -\int_\Omega \sum_{p,q} a_{pq}(x) \cdot \frac{\partial u}{\partial x_p}\frac{\partial \bar{u}}{\partial x_q}\,dx + \int_\Omega \bar{u} \cdot (\sum a_p(x)\frac{\partial u}{\partial x_p}\,)\,dx +$$

$$\int_\Omega |u(x)|^2 \cdot b(x)\,dx + \lambda \cdot \int |u(x)|^2\,dx$$

Write $\lambda = \xi + i\eta$. Separating real and imaginary parts we find the two equations:

$$\text{(i)} \qquad \xi \int |u|^2\,dx = \int \sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} \cdot \frac{\partial \bar{u}}{\partial x_q}\,dx + \int \big(\frac{1}{2} \cdot \sum \frac{\partial a_p}{\partial x_p} - b\big) \cdot |u|^2\,dx$$

$$\text{(ii)} \qquad \eta \int |u|^2\,dx = \frac{1}{2i} \int \sum a_p\big(u\frac{\partial \bar{u}}{\partial x_p} - \bar{u}\frac{\partial u}{\partial x_p}\,\big)\,dx$$

Set

$$A = \int |u|^2\,dx \quad : \quad B = \int |\nabla(u)|^2\,dx$$

Since L is elliptic there exists a positive constant k such that

$$\sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} > k \cdot |\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants c_1, c_2, c_3 such that

$$\text{(iii)} \qquad A\xi > c_1 B - c_2 B \quad : \quad A|\eta| < c_3 \cdot \sqrt{AB}$$

Here (iii) implies that $\xi > -c_2$ and the reader can also confirm that

$$\text{(iv)} \qquad B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A \cdot c_2 \cdot \sqrt{\frac{\xi + c_2}{c_1}} \quad : \quad |\eta| < c_3 \cdot \sqrt{\frac{\xi + c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 2.3.1.

2.4. Asymptotic formula for eigenvalues

Consider a function f which satisfies

$$\mathcal{G}(f) = -\frac{1}{\lambda} \cdot f$$

for some non-zero complex number λ. With $u = \mathcal{G}(f)$ it follows from (xx) that

$$(L - \kappa^2)(u) = f = -\lambda \cdot u$$

Hence

$$L(u) + (\lambda - \kappa^2)u = 0$$

About the proof of Theorem xx. From the above the asymptotic formula in Theorem xx can be derived from asymptotic properties of eigenvalues to the integral operator $|mathcalG$. Using Theorem 2.1 and the estimates for the fundsemtnal solution Φ in § 1, one can proceed as in the the next section where a Tauberian theorem is employed to finish the proof of Theorem xx. The reader may try to supply details or consult [Carleman: page xx-xx] for details.

§ 3. A study of $\Delta(\phi) + \lambda \cdot \phi$.

Introduction. We expose material from Carleman's article *xxx* whose contents were presented at the Scandinavian Congress in Stockholm 1934. In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact established by G. Neumann and H. Poincaré during the years 1879-1895 gives the following: First there exists the Greens' function

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$ with the property that the operator $\mathcal{G}$ defined on $L^2(\Omega)$ by

$$f \mapsto \mathcal{G}_f(p) = \frac{1}{2\pi} \dot{\iint} G(p,q) f(q)\, dq$$

satisfies

$$\Delta \circ \mathcal{G}_f = -f \quad : f \in L^2(\Omega)$$

Moreover, $\mathcal{G}$ is a compact operator on the Hilbert space $L^2(\Omega)$ and there exists a sequence $\{f_n\}$ in $L^2(\Omega$ such that $\{\phi_n = \mathcal{G}_{f_n}\}$ is an orthonormal basis in $L^2(\Omega$ and

$$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : n = 1, 2, \ldots$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$. When eigenspaces have dimension ≥ 2, the eigenvalues are repeated by their multiplicity.

Main Theorem. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N \to \infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

The strategy in the proof is to consider the function of a complex variable s defined by

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

and show that it is a meromorphic function in the whole complex s-plane with a simple pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. More precisely we shall prove:

0.1 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem 0.1 gives ther main theorem by a result due to Wiener in the article *Tauberian theorem* [Annals of Math.1932]. Wiener's theorem asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s \to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n \to \infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Exercise. Derive the main theorem from Wiener's result and Theorem 0.1.

About Wiener's result. It is a version of an famous Tauberian theorem proved by Hardy and Littlewood in 1913 which goes as follows:

0.2 The Hardy-Littelwood theorem. *Let $\{a_n\}$ be a sequence of non-negative real numbers such that*

$$(*) \qquad A = \lim_{r \to 1} (1-r) \cdot \sum a_n r^n$$

exists. Then there also exists the limit

$$(**) \qquad A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Notice that no growth condition is imposed on the sequence $\{a_n\}$, i.e. the sole assumption is the existing limit (*). The proof is quite demanding and does not follow by "abstract nonsense"from functional analysis. For the reader's convenience we include details of the proof in a separate appendix since courses devoted to series rarely appear in contemporary education.

§ 1. Proof of Theorem 0.

Let Ω be a bounded and Dirichlet regular domain. For ech fixed point $p \in \Omega$ we get the continuous function on $\partial\Omega$ defined by

$$q \mapsto \log \frac{1}{|p-q|}$$

We find the harmonic function $u_p(q)$ in Ω such that $u_p(q) = \log \frac{1}{|p-q|} \, : \, q \in \partial\Omega$. Green's function is defined for pairs $p \neq q$ in $\Omega \times \Omega$ by

$$(1) \qquad G(p,q) = \log \frac{1}{|p-q|} - u_p(q)$$

Keeping if $p \in \Omega$ fixed, the function $q \mapsto G(p,q)$ extends to the closure of Ω where it vanishes if $q \in \partial\Omega$. If $f \in L^2(\Omega$ we set

$$(2) \qquad \mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot f(q)\, dq$$

where $q = (x,y)$ so that $dq = dxdy$ when the double integral is evaluated. From (1) we see that

$$\iint_{\Omega\times\Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence $\mathcal{G}$ is of the Hilbert-Schmidt type and therefore a compact operator on $L^2(\Omega)$. Next, recall that $\frac{1}{2\pi} \cdot \log \sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator. From this the reader can deduce the following:

1.1 Theorem. *For each $f \in L^2(\Omega)$ the Lapacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

$$(*) \qquad \Delta(\mathcal{G}_f) = -f$$

The equation (*) means that the composed operator $\Delta \circ \mathcal{G}$ is minus the identity on $L^2(\Omega)$. We are led to introduce the linear operator S on $L^2(\Omega)$ defined by Δ, where $\mathcal{D}(S)$ is the range of $\mathcal{G}$. If $g \in C_0^2(\Omega)$, i.e. twice differentiable and with compact support, it follows via Greens' formula that

$$\frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot \Delta(g)(q)\, dq = -g(p)$$

In particular $C_0^2(\Omega) \subset \mathcal{D}(S)$ which implies that S is densely defined and we leave it to the reader to verify that

$$\mathcal{G}(\Delta(f)) = -f \quad : f \in \mathcal{D}(S)$$

Remark. By Carl Neumann's classic construction of resolvent operators from 1879, the result above means that $-\mathcal{G}$ is Neumann's inverse of S. Since $-\mathcal{G}$ is compact it follows by Neumann's formula for spectra that S has a discrete spectrum,and we recall the following wellknown fact which goes back to work by Poincaré:

1.2 Proposition. *There exists an orthonormal basis* $\{\phi_n\}$ *in* $L^2(\Omega)$ *where each* $\phi_n \in \mathcal{D}(S)$ *is an eigenfunction, and a non-decreasing sequence of positive real numbers* $\{\lambda_n\}$ *such that*

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots \tag{1.2.1}$$

Remark. Above (1.2.1) means that

$$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n$$

This, $\{\lambda_n^{-1}\}$ are eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. As usual eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times.

After these preliminaries we embark upon the proof of Theorem 0.1. First, since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\sum \lambda_n^{-2} < \infty \tag{i}$$

This convergence entails that various constructions below are defined. For each complex number λ outside $\{\lambda_n\}$ we set

$$G(p, q; \lambda) = G(p, q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda - \lambda_n)} \tag{ii}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p, q; \lambda) \cdot f(q)\, dq \tag{iii}$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p, \lambda)$. Set

$$F(p, q, \lambda) = G(p, q; \lambda) - G(p, q)$$

Keeping p fixed we see that (ii) gives

$$\lim_{q \to p} F(p, q, \lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)} \tag{B.1}$$

Set

$$F(p, \lambda) = \lim_{q \to p} F(p, q, \lambda)$$

From (i) and (B.1) it follows that it is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Show via residue calculus that one has the equality below in a half-space $\mathfrak{Re}\, s > 2$:

$$\Phi(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a - i\infty}^{a + i\infty} F(p, \lambda) \cdot \lambda^{-s}\, d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\, \lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p, \lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\Phi(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta} \, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p, -x) \cdot x^{-s} \, dx \tag{E.1}$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p, -x) \cdot x^{-s} \, dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p, -x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^\infty \frac{e^{-zt}}{\sqrt{t^2 - 1}} \, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

(i)
$$I_0(z) = \sum_{m=0}^\infty \frac{2^{-2m}}{(m!)^2} \cdot z^{2m}$$

(ii)
$$I_1(z) = \sum_{m=0}^\infty \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p, q; \kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p, q; \kappa) = K(\sqrt{\kappa}|p - q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p, q; -\kappa) = K(\sqrt{\kappa} \cdot |p - q|) - H(q; \kappa) \quad : \kappa > 0$$

Next, the construction of $G(p, q)$ gives

$$F(p, -\kappa) = \lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log |p - q|] + \lim_{q \to p} [u_p(q) + H(p, q, \kappa)] \tag{G.1}$$

The last term above has the "nice limit" $u_p(p) + H(p, p, \kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log |p - q|] = -\frac{1}{2} \cdot \log \kappa + \log 2 - \gamma \tag{G.2}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = + \log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p, -\kappa) = -\frac{1}{2} \cdot \log \kappa + A + H(p, p; -\kappa)$$

With $x = \kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty x^{-s}\, dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2} \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty \log x \cdot x^{-s}\, dx$$

Notice that the complex derivative

$$\frac{d}{ds} \int_a^\infty x^{-s}\, dx = -\int_a^\infty \log x \cdot x^{-s}\, dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem 0.1 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin\, \pi s}{2\pi^2} \cdot \int_a^\infty H(p,p,\kappa) \cdot \kappa^{-s}\, d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \leq H(p,q;\kappa) \leq \max_{q \in \partial\Omega} K(\kappa|p-q|)$$

With $p \in \Omega$ fixed there is a positive number δ such that $|p-q| \geq \delta : q \in \partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.

Appendix. Theorems by Abel, Tauber, Hardy and Littlewood

Introduction. Consider a power series $f(z) = \sum a_n z^n$ whose radius of convergence is one. If $r < 1$ and $0 \leq \theta \leq 2\pi$ we are sure that the series

$$f(re^{i\theta}) = \sum a_n r^n e^{in\theta}$$

is convergent. In fact, it is even absolutely convergent since the assumption implies that

$$\sum |a_n| \cdot r^n < \infty \quad \text{for all} \quad r < 1$$

Passing to $r = 1$ it is in general not true that the series $\sum a_n e^{in\theta}$ is convergent. An example arises if we consider the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

This leads to the following problem where we without loss of generality can take $\theta = 0$. Consider as above a convergent power series and assume that there exists the limit

$$(*) \qquad \lim_{r \to 1} \sum a_n r^n$$

When can we conclude that the series $\sum a_n$ also is convergent and that one has the equality

$$(**) \qquad \sum a_n = \lim_{r \to 1} \sum a_n r^n$$

The first result in this direction was established by Abel in a work from 1823:

A. Theorem *Let $\{a_n\}$ be a sequence such that $\frac{a_n}{n} \to 0$ as $n \to \infty$ and there exists*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Then $\sum a_n$ is convergent and the sum is A.

An extension of Abel's result was established by Tauber in 1897.

B. Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists the limit*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Set

$$\omega_n = a_1 + 2a_2 + \ldots + na_n \quad : \, n \geq 1$$

If $\lim_{n \to \infty} \omega_n = 0$ it follows that the series $\sum a_n$ is convergent and the sum is A.

C. Results by Hardy and Littlewood. In their joint article *xxx* from 1913 the following extension of Abel's result was proved by Hardy and Littlewood:

C. Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a constant C so that $\frac{a_n}{n} \leq C$ for all $n \geq 1$. Assume also that the power series $\sum a_n z^n$ converges when $|z| < 1$. Then the same conclusion as in Abel's theorem holds.*

Remark. In addition to this they proved a result about positive series from the cited article which has independent interest.

D. Theorem. *Assume that each $a_n \geq 0$ and that there exists the limit:*

$$(*) \qquad A = \lim_{r \to 1} (1-r) \cdot \sum a_n r^n$$

Then there exists the limit

$$(**) \qquad A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Remark. The proofs of Abel's and Tauber's results are easy while C and D require more effort and rely upon results from calculus in one variable. So before we enter the proofs of the theorems above insert some preliminaries.

1. Results from calculus

Below $g(x)$ is a real-valued function defined on $(0,1)$ and of class C^2 at least.

1.1 Lemma *Assume that there exists a constant $C>0$ such that*

$$g''(x) \le C(1-x)^{-2} \quad : 0<x<1 \quad \text{and} \quad \lim_{x\to 1} g(x)=0$$

Then one has the limit formula:

$$\lim_{x\to 1} (1-x)\cdot g'(x)=0$$

1.2 Lemma *Assume that the second order derivative $g''(x)>0$. Then the following implication holds for each $\alpha>0$:*

$$\lim_{x\to 1} (1-x)^{\alpha}\cdot g(x)=1 \implies \lim_{x\to 1} (1-x)^{\alpha+1}\cdot g'(x)=\alpha$$

Remark. If $g(x)$ has higher order derivatives which all are >0 on $(0,1)$ we can iterate the conclusion in Lemma 1.2 where we take α to be positive integers. More precisely, by an induction over ν the reader may verify that if

$$\lim_{x\to 1} (1-x)\cdot g(x)=1$$

exists and if $\{g^{(\nu)}(x)>0\}$ for all every $\nu\ge 2$ then

(*)
$$\lim_{x\to 1} (1-x)^{\nu+1}\cdot g^{(\nu)}(x)=\nu\,! \quad : \nu\ge 2$$

Next, to each integer $\nu\ge 1$ we denote by $[\nu-\nu^{2/3}]$ the largest integer $\le (\nu-\nu^{2/3})$. Set

$$J_*(\nu)=\sum_{n\le[\nu-\nu^{2/3}]} n^{\nu}e^{-\nu} \quad : \quad J^*(\nu)=\sum_{n\ge[\nu+\nu^{2/3}]} n^{\nu}e^{-\nu}$$

1.3 Lemma *There exists a constant C such that*

$$\frac{J^*(\nu)+J_*(\nu)}{\nu\,!}\le\delta(\nu) \quad : \quad \delta(\nu)=C\cdot\exp\big(-\frac{1}{2}\cdot\nu^{\frac{1}{3}}\big) \quad : \nu=1,2,\ldots$$

Proofs

We prove only Lemma 1.1 which is a bit tricky while the proofs of Lemma 1.2 and 1.3 are left as exercises to the reader. Fix $0<\theta<1$. Let $0<x<1$ and set

$$x_1=x+(1-x)\theta$$

The mean-value theorem in calculus gives

(i)
$$g(x_1)-g(x)=\theta(1-x)g'(x)+\frac{\theta^2}{2}(1-x)^2\cdot g''(\xi) \quad \text{for some} \quad x<\xi<x_1$$

By the hypothesis

$$g''(\xi)\le C(1-\xi)^{-2}\le C)1-x_1)^{-2}$$

Hence (i) gives

$$(1-x)g'(x)\ge\frac{1}{\theta}(g(x_1)-g(x))-C\cdot\frac{\theta}{2}\frac{(1-x)^2}{1-x_1)^2}=$$

$$\frac{1}{\theta}(g(x_1)-g(x))-\frac{C\cdot\theta}{2(1-\theta)^2}$$

Keeping θ fixed we have by assumption

$$\lim_{x\to 1} g(x)=0$$

Notice also that $x\to 1 \implies x_1\to 1$. It follows that

$$\liminf_{x\to 1}\ (1-x)g'(x)\ge -\frac{C\cdot\theta}{2(1-\theta)^2}$$

Above $0 < \theta < 1$ is arbitrary, i.e. we can choose small $\theta > 0$ and hence we have proved that

$$(*) \qquad \liminf_{x \to 1} (1-x)g'(x) \geq 0$$

Next we prove the opposed inequality

$$(**) \qquad \limsup_{x \to 1} (1-x)g'(x) \leq 0$$

To get (**) we apply the mean value theorem in the form

$$\text{(ii)} \qquad g(x_1) - g(x) = \theta(1-x)g'(x_1) - \frac{\theta^2}{2}(1-x)^2 \cdot g''(\eta) \quad : \ x < \eta < x_1$$

Since $(1-x_1) = \theta(1-x)(1-\theta)$ we get

$$\text{(iii)} \qquad (1-x_1)g'(x_1) = \frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + \frac{(1-\theta)\theta}{2} \cdot (1-x)^2 g''(\eta)$$

Now $g''(\eta) \leq C(1-\eta)^{-2} \leq C(1-x_1)^{-2}$ so the right hand side in (iii) is majorized by

$$\frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{(1-\theta)\theta}{2} \cdot (1-x)^2(1-x_1)^2 =$$

$$\text{(iv)} \qquad \frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{\theta}{2(1-\theta}$$

Keeping θ fixed while $x \to 1$ we obtain:

$$\liminf_{x \to 1} (1-x)g'(x) \leq C \cdot \frac{\theta}{2(1-\theta}$$

Again we can choose arbitrary small θ and hence (**) holds which finishes the proof of Lemma 1.1.

2. Proof of Abel's theorem.

Without loss of generality we can assume that $a_0 = 0$ and set $S_N = a_1 + \ldots + a_N$. Given $0 < r < 1$ we let $f(r) = \sum a_n r^n$. For every positive integer N the triangle inequality gives:

$$\big|S_N - f(r)\big| \leq \sum_{n=1}^{n=N} |a_n|(1-r^n) + \sum_{n \geq N+1} |a_n| r^n$$

Set $\delta(N) = \max_{n \geq N} \frac{|a_n|}{n}$. Since $1 - r^n) = (1-r)(1 + \ldots + r^{n-1} \leq (1-r)n$ the last sum is majorised by

$$(1-r) \cdot \sum_{n=1}^{n=N} n \cdot |a_n| + \delta(N+1) \cdot \sum_{n \geq N+1} \frac{r^n}{n}$$

Next, the obvious inequality $\sum_{n \geq N+1} \frac{r^n}{n} \leq \frac{1}{N+1} \cdot \frac{1}{1-r}$ gives the new majorisation

$$(1) \qquad (1-r) \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \frac{\delta(N+1)}{N+1} \cdot \frac{1}{1-r}$$

This hold for all pairs N and r. To each $N \geq 2$ we take $r = 1 - \frac{1}{N}$ and hence the right hand side in (1) is majorised by

$$\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \delta(N+1) \cdot \frac{N}{N+1}$$

Here both terms tend to zero as $N \to \infty$. Indeed, Abel's condition $\frac{a_n}{n} \to 0$ implies that $\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n}$ tends to zero as $N \to \infty$. Hence we have proved the limit formula:

$$(*) \qquad \lim_{N\to\infty} \big|s_N - f(1-\frac{1}{N})\big| = 0$$

Finally it is clear that (*) gives Abel's result.

3. Proof of Tauber's theorem.

We may assume that $a_0 = 0$. Notice that

$$a_n = \frac{\omega_n - \omega_{n-1}}{n} \quad : \, n \geq 1$$

It follows that

$$f(r) = \sum \frac{\omega_n - \omega_{n-1}}{n} \cdot r^n = \sum \omega_n (\frac{r^n}{n} - \frac{r^{n+1}}{n+1})$$

Using the equality $\frac{1}{n} = \frac{1}{n+1} = \frac{1}{n(n+1)}$ we can rewrite the right hand side as follows:

$$\sum \omega_n (\frac{r^n - r^{n+1}}{n+1} + \frac{r^n}{n(n+1)})$$

Set

$$g_1(r) = \sum \omega_n \cdot \frac{r^n - r^{n+1}}{n+1} = (1-r) \cdot \sum \frac{\omega_n}{n+1} \cdot r^n$$

By the hypothesis $\lim_{n\to\infty} \frac{\omega_n}{n+1} = 0$ and then it is clear that we get

$$\lim_{r\to 1} g_1(r) = 0$$

Since we also have $f(r) \to 0$ as $r \to 1$ we conclude that

$$(1) \qquad \lim_{r\to 1} \sum \frac{\omega_n}{n(n+1)} \cdot r^n = 0$$

Next, with $b_n = \frac{\omega_n}{n(n+1)}$ we have $nb_n = \frac{\omega_n}{n+1} \to 0$. Hence Abel's theorem applies so (1) gives convergent series

$$(2) \qquad \sum \frac{\omega_n}{n(n+1)} = 0$$

If $N \geq 1$ we have the partial sum

$$S_N = \sum_{n=1}^{n=N} \frac{\omega_n}{n(n+1)} = \sum_{n=1}^{n=N}, \omega_n \cdot (\frac{1}{n} - \frac{1}{n+1})$$

The last term becomes

$$\sum_{n=1}^{n=N} \frac{1}{n}(\omega_n - \omega_{n-1}) - \frac{\omega_N}{N+1} = \sum_{n=1}^{n=N} a_n - \frac{\omega_N}{N+1}$$

Again, since $\frac{\omega_N}{N+1} \to 0$ as $N \to \infty$ we conclude that the convergent series from (2) implies that the series $\sum a_n$ also is converges and has sum equal to zero. This finishes the proof of Tauber's result.

4. Proof of Theorem D.

Set $f(x) = \sum a_n x^n$ which is defined when $0 < x < 1$. Notice that

$$(1-x)f(x) = \sum s_n x^n \quad \text{where} \quad s_n = a_1 + \ldots + a_n$$

Set $g(x) = \sum s_n x^n$ which is defined when $0 < x < 1$. Since $s_n \geq 0$ for all n all the higher order derivatives

$$g^{(p)}(x) = \sum_{n=p}^{\infty} n(n-1)\cdots(n-p+1)a_n x^{n-p} > 0$$

when $0 < x < 1$. The hypothesis that $\lim_{x\to 1} g(x) = A$ and Lemma 1.1 and the inductive result in the remark after Lemma 1.2 give:

(1)
$$\lim_{x\to 1}(1-x)^{\nu+2}\cdot\sum s_n\cdot n^\nu x^n = (\nu+1)! \quad : \nu\geq 1$$

We shall use the substitution $e^{-t} = x$ where $t > 0$. Since $t \simeq 1-x$ when $x \to 1$ we see that (1) gives

(2)
$$\lim_{t\to 0} t^{\nu+2}\cdot\sum s_n\cdot n^\nu e^{-nt} = (\nu+1)! \quad : \nu\geq 1$$

Let us put

$$J_*(\nu,t) = \frac{t^{\nu+2}}{(\nu+1)!}\cdot\sum_{n=1}^{\infty} s_n\cdot n^\nu e^{-nt}$$

So for each fixed ν one has

(3)
$$\lim_{t\to 0} J_*(\nu,t) = 1$$

Next, for each pair $\nu\geq 2$ and $0 < t < 1$ we define the integer

(*)
$$N = [\frac{\nu-\nu^{2/3}}{t}]$$

Since the sequence $\{s_n\}$ is non-decreasing we get

(i)
$$s_N\cdot\sum_{n\geq N} n^\nu e^{-nt} \leq \sum_{n\geq N} s_n\cdot n^\nu e^{-nt} \leq \frac{(\nu+1)!\cdot J_*(\nu,t)}{t^{\nu+2}}$$

Next, the construction of N and Lemma 1.3 give:

(ii)
$$\sum_{n\geq N} n^\nu e^{-nt} \geq \frac{\nu!}{t^{\nu+1}}\cdot(1-\delta(\nu))$$

where the δ function is independent of ν and tends to zero as $\nu\to\infty$. Hence (i-ii) give

(iii)
$$s_N \leq \frac{(\nu+1)}{t}\cdot\frac{1}{1-\delta(\nu)}\cdot J_*(\nu,t)$$

Next, by the construction of N one has

$$N+1 \geq \frac{\nu-\nu^{2/3}}{t} = \frac{\nu}{t}\cdot(1-\nu^{-1/3})$$

It follows that (iii) gives

(iv)
$$\frac{s_N}{N+1} \leq \frac{\nu+1}{\nu}\cdot\frac{1}{1-\nu^{-1/3}}\cdot\frac{1}{1-\delta(\nu)}\cdot J_*(\nu,t)$$

Since $\delta(\nu)\to 0$ it follows that for any $\epsilon > 0$ there exists some ν_* such that

(v)
$$\frac{\nu_*+1}{\nu_*}\cdot\frac{1}{1-\nu_*^{-1/3}}\cdot\frac{1}{1-\delta(\nu_*)} < 1+\epsilon$$

Keeping ν_* fixed we now consider pairs t_n, N such that (*) above hold with $\nu=\nu_*$. Notice that

(vi)
$$N\to+\infty \implies t_N\to 0$$

It follows from (iv) and (v) that we have:

(vii)
$$\frac{s_N}{N+1} < (1+\epsilon)\cdot J_*(\nu_*, t_N) \quad : N\geq 2$$

Now (vi) and the limit in (3) which applies with ν_* while $t_N\to 0$ entail that

$$\lim_{N\to\infty} J(\nu_*, t_N) = 1$$

We have also that $\frac{N}{N+1} \to 1$ and since $\epsilon > 0$ was arbitrary we see that (vii) proves the inequality

(1)
$$\limsup_{N\to\infty} \frac{s_N}{N} \leq 1$$

So Theorem 2 follows if we also prove that

(2)
$$\liminf_{N\to\infty} \frac{s_N}{N} \geq 1$$

The proof of (II) is similar where we now define the integers N by:

$$N = [\frac{\nu + \nu^{2/3}}{t}]$$

Then we have

$$S_N \cdot \sum_{n\leq N} n^\nu e^{-nt} \geq \frac{(\nu+1)! \cdot J_*(\nu,t)}{t^{\nu+2}} - \sum_{n>N} s_n \cdot n^\nu e^{-nt}$$

Here the last term can be estimated above since the Lim.sup inequality (I) gives a constant C such that $s_n \leq Cn$ for all n and then

$$\sum_{n>N} s_n \cdot n^\nu e^{-nt} \leq C \cdot \sum_{n>N} n^{\nu+1} e^{-nt} \leq C \cdot \delta^*(\nu) \cdot \frac{(\nu+1)!}{t^{\nu+2}}$$

where Lemma 1.3 entails that $\delta^*(\nu) \to 0$ as ν increases. At the same time Lemma 1.3 also gives

$$\sum_{n\leq N} n^\nu \cdot e^{-nt} = \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta_*(\nu)$$

where $\delta(\nu_*) \to 0$. At this stage the reader can verify that (2) by similar methods as in the proof of (I).

5. Proof of Theorem C

Set $f(x) = \sum a_n x^n$. Notice that it suffices to prove Theorem C when the limit value

$$\lim_{x\to 1} \sum a_n x^n = 0$$

Next, the assumption that $a_n \leq \frac{c}{n}$ for a constant c gives

$$f''(x) = \sum n(n-1)a_n x^{n-2} \leq c \sum (n-1)x^{n-2} = \frac{c}{1-x)^2}$$

The hypothesis $\lim_{x\to 1} f(x) = 0$ and Lemma xx therefore gives

(i)
$$\lim_{x\to 1} (1-x)f'(x) = 0$$

Next, notice the equality

(ii)
$$\sum_{n=1}^{\infty} \frac{na_n}{c} x^n = \frac{x}{c} \cdot f'(x)$$

At the same time $\sum_{n=1}^\infty x^n = \frac{x}{1-x}$ and hence (i-ii) together give:

$$\lim_{x\to 1} (1-x) \cdot \sum (1 - \frac{na_n}{c}) \cdot x^n = 1$$

Here $1 - \frac{na-n}{c} \geq 0$ so Theorem 2 gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n=N} (1 - \frac{na_n}{c}) = 1$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{n=N} na_n = 0$$

This means precisely that the condition in Tauber's Theorem holds and hence $\sum a_n$ converges and has series sum equal to 0 which finishes the proof of Theorem C.

BRA

A Non-Linear PDE-equation

Introduction. In the article *Über eine nichtlineare Randwertaufgabe bei der Gleichung* $\Delta u = 0$ (Mathematisches Zeitschrift vol. 9 (1921), Carleman considered the following equation: Let Ω be a bounded domain in $\mathbf{R}^3$ with C^1-boundary and $\mathbf{R}^+$ the non-negative real line where t is the coordinate. Let $F(t,p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$t \mapsto F(t,p) \tag{0.1}$$

is strictly increasing for every $p \in \partial\Omega$ and that $F(0,p) \geq 0$. Moreover,

$$\lim_{u\to\infty} F(t,p) = +\infty \tag{0.2}$$

holds uniformly with respect to p. For a given point $q_* \in \Omega$ we seek a function $u(x)$ which is harmonic in $\Omega \setminus \{q_*\}$ and at q_* it is locally $\frac{1}{|x-q_*|}$ plus a harmonic function. Moreover, it is requested that u extends to a continuous function on $\partial\Omega$ and that $u \geq 0$ in $\overline{\Omega}$. Finally, along the boundary the inner normal derivative $\partial u/\partial n$ satisfies the equation

$$\frac{\partial u}{\partial n}(p) = F(u(p),p) \quad : p \in \partial\Omega \tag{*}$$

Remark. The case when $F(t,p) = kt^4$ for some positive constant k means that we regard the Stefan-Boltzmann equation whose physical interpretation ensures that (*) has a unique non-negative solution u.

Theorem. *For each F satisfying (0.1-0.2) the boundary value problem has a unique solution u.*

The strategy in the proof is to consider a family of boundary value problems where one for each $0 \leq h \leq 1$ seeks u_h to satisfy

$$\frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p),p) \quad : p \in \partial\Omega \tag{*}$$

and u_h has the same pole as u above. Let us begin with

0.1 The case $h = 0$. Here we seek u_0 so that

$$\frac{\partial u_0}{\partial n}(p) = u_0 \tag{i}$$

If $G(p)$ is the Greens' function with a pole at q_* we seek a harmonic function h in Ω such that

$$u_0 = G - h \tag{ii}$$

Since $G(p) = 0$ on $\partial\Omega$, the equation (i) holds if

$$\frac{\partial h}{\partial n}(p) = h(p) + \frac{\partial G}{\partial n}(p) \quad : p \in \partial\Omega \tag{iii}$$

This is a classic linear boundary value problem which has a unique solution h. See § xx for further details.

0.2 Properties of u_0. The construction in (ii) entails that u_0 is superharmonic in Ω and therefore attains its minimum on the boundary. Say that

$$u_0(p_*) = \min_{p\in\overline{\Omega}} u_0(p)$$

It follows that $\frac{\partial u_0}{\partial n}(p_*) \geq 0$ and the equation (i) gives

$$u_0(p_*) \geq 0$$

Hnece our unique solution u_0 is non-negative. We can say more. For consider the harmonic function h in (ii) which takes a maxium at some $p^* \in \partial\Omega$. Then $\frac{\partial h}{\partial n}(p^*) \leq 0$ so that (iii) gives

$$h(p^*) + \frac{\partial G}{\partial n}(p^*) \leq 0$$

Hence

$$\max_{p\in\partial\Omega} h(p) \leq -\frac{\partial G}{\partial n}(p^*)$$

which entails that

$$\min_{p\in\partial\Omega} u(p) = -\max_{p\in\partial\Omega} h(p) \geq \frac{\partial G}{\partial n}(p^*) \tag{0.2.1}$$

Here the function

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is continuous and positive on $\partial\Omega$ and if γ_* is the minimum value we conclude that

$$\min_{p\in\partial\Omega} u(p) \geq \gamma_* \tag{0.2.2}$$

Next, let h attain its minimum at some $p_* \in \partial\Omega$ which entails that $\frac{\partial h}{\partial n}(p_*) \geq 0$ and then (iii) gives

$$h(p_*) + \frac{\partial G}{\partial n}(p^*) \geq 0$$

It follows that

$$\max_{p\in\partial\Omega} u_0(p) = \min_{p\in\partial\Omega} h(p) = -h(p_*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^* \tag{0.2.3}$$

where

$$\gamma^* = \max_{p\in\partial\Omega} \frac{\partial G}{\partial n}(p) \tag{0.2.4}$$

So the unique solution u_0 in (i) satisfies

$$\gamma_* \leq u(p) \leq \gamma^* \quad : p \in \partial\Omega \tag{0.2.5}$$

where the positive constants γ_* and γ^* depend on the point $q_* \in \Omega$ and the given domain Ω.

The homotopy method. To proceed from $h = 0$ to $h = 1$ the idea is to use a "homotopy argument" which can be handled via precise estimates of solutions to Neumann's linear boundary value problem which are presented in § B. Thanks to this and some uniqueness properties in § A below, the reduction to the case when F is real-analytic is relatively easy. The ciucial steps during the proof appear in § C where we carry out a "homotoy method" to get solutions in (*) as h increases from zero to one.

A.0. Proof of uniqueness.

Suppose that u_1 and u_2 are two solutions to the equation in the main theorem. Notice that $u_2 - u_1$. is harmonic in Ω. If $u_1 \neq u_2$ we may without loss of generality we may assume that the maximum of $u_2 - u_1$ is > 0. The maximum is attained at some $p_* \in \partial\Omega$ and the strict maximum principle for harmonic functions gives:

$$u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*) \tag{i}$$

for all $x \in \Omega$.With $v = u_2 - u_1$ we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p), p) - F(u_1(p), p)$$

Here (0.1) entails that $\frac{\partial v}{\partial n}(p_*) > 0$ and since we have an inner normal derivative this violates (i) which proves the uniqueness.

A.1 Montonic properties.

Let F_1 and F_2 be two functions which both satisfy (0.1) and (0.2) where

$$F_1(u,p) \leq F_2(u,p)$$

hold for all $(u,p) \in \mathbf{R}^+ \times \partial\Omega$. If u_1, respectively u_2 solve (*) for F_1 and F_2 it follows that $u_2(q) \leq u_1(q)$ for all $q \in \Omega$. To see this we set $v = u_2 - u_1$ which is harmonic in Ω. If $p \in \partial\Omega$ we get

$$\text{(i)} \qquad \frac{\partial v}{\partial n}(p) = F_2(u_2(p),p) - F_1(u_1(p),p) \geq 0$$

Suppose that the maximum of v is > 0 and let the maximum be attained at some point p_*. Since (i) is an inner normal it follows that we must have $0 = \frac{\partial v}{\partial n}(p)$ which would entail that

$$F_2(u_2(p_*)p_*) > F_2(u_1(p_*),p_*) \geq F_1(u_1(p_*),p_*) \implies$$

and this contradicts the strict inequality $u_2(p_*) > u_1(p_*)$ since we have an increasing function in (0.1).

A.2. A bound for the maximum norm. Let G be the Green's function which has a pole at q_* while $G = 0$ on $\partial\Omega$. Then

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is a continuous and positive function on $\partial\Omega$. Set

$$m_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p) \quad : m^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

Next, let $0 \leq h \leq 1$ and suppose that u_h is a solution to (*). Put

$$(*) \qquad m(h) = \min_{p \in \partial\Omega} u_h(p) \quad : M(h) = \max_{p \in \partial\Omega} u_h(p)$$

To estimate these numbers we proceed as follows. Choose $p^* \in \partial\Omega$ such that

$$(1) \qquad u_h(p^*) = M(h)$$

Now the function

$$H = u - G - M(h)$$

is harmonic function in Ω and non-negative on the boundary. Hence H is positive in Ω and since $H(p^*) = 0$ we have

$$\frac{\partial H}{\partial n}(p^*) \leq 0 \implies$$

which via the equation (*) give

$$(2) \qquad (1-h)M(h) + h \cdot F(M(h),p^*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^*$$

Next, the hypothesis on F entails that

$$(3) \qquad t \mapsto (1-h)t + h \cdot F(t,p^*)$$

is a strictly increasing function for each fixed $0 \leq h \leq 1$ and the hypothesis (0.2) together with the inequality (2) above, give a positive constant A^* which is independent of h such that

$$(3) \qquad M(h) \leq A^* \quad : 0 \leq h \leq 1$$

Next, let $m(h)$ be the minimum of u_h on $\partial\Omega$ and this time we consider the harmonic function

$$H = u - m(h) - G$$

Here $H \geq 0$ on $\partial\Omega$ and if $u_h(p_*) = m(h)$ we have $H(p_*) = 0$ p_* is a minimum for H. It follows that

$$\frac{\partial H}{\partial n}(p_*) \geq 0 \implies F(u(p_*),p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$\gamma_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

$$F(m(h), p^*) \geq \gamma_* \tag{4}$$

Above γ^* is the constant from (xx) and the properties of F give a positive constant A_* such that

$$m(h) \geq A_*$$

Conclusion. Above $0 < A_* < A^*$ are constants which are independent of h. Hence the maxima and the minima of u_h stay in a fixed interval $[A_*, A^*]$ as soon as u_h exists.

B. The linear equation.

Let $f(p)$ and $W(p)$ be a pair of continuous functions on the boundary $\partial\Omega$ where W is positive, i.e. $W(p) > 0$ for every boundary point. Set

$$w_* = \min_p W(p)$$

So by the assumption on W we have $w_* > 0$. The classical Neumann theorem asserts that there exists a unique function U which is harmonic in Ω, extends to a continuous function on the closed domain and its inner normal derivative satisfies:

$$\partial U/\partial n(p) = W(p) \cdot U(p) + f(p) \quad p \in \partial\Omega \tag{1}$$

For the unique solution in (1) some estimates hold. Namely, set

$$M^* = \max_p U(p) \quad \text{and} \quad m_* = \min_p U(p)$$

Since U is harmonic in Ω the maximum and the minimum are both taken on the boundary. If $U(p^*) = M^*$ for some $p^* \in \partial\Omega$ we have $\partial U/\partial n(p^*) \leq 0$ which together with (1) entails that

$$M^* \cdot W(p^*) + f(p^*) \leq 0 \implies M^* \leq \frac{|f|_{\partial\Omega}}{w_*}$$

where $|f|_{\partial\Omega}$ is the maximum norm of f on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial\Omega}}{w_*}$$

Hence the following inequality holds for the the maximum norm $|U|_{\partial\Omega}$:

$$|U|_{\partial\Omega} \leq \frac{|f|_{\partial\Omega}}{w_*} \tag{B.0}$$

Notoice that (B.0) and the equation (1) entails that Suppose that $W \in C^0(\partial\Omega)$ satisfies

$$w_* \leq W(p) \leq w^*$$

for a pair of positive constants. If $|f|_{\partial\Omega}$ is the maximum norm of f it follows from (B.0) that

$$\big|W(p) \cdot U(p) + f(p)\big| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega}$$

Hence the equation (1) gives

$$\max_{p \in \partial\Omega} |\frac{\partial U}{\partial n}(p)| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega} \tag{B.1}$$

B.2 An estimate for first order derivatives. Let $p \in \partial\Omega$ and denote by N the inner normal at p. Since $\partial\Omega$ is of class C^1 a sufficiently small line segment from p along N stays in Ω. So for small positive ℓ we have points $q = p + \ell \cdot N$ in Ω and take the directional derivative of U along N_p. This gives a function

$$\ell \mapsto \partial U/\partial N(p + \ell \cdot N)$$

Since the boundary is C^1 these functions are defined on a fixed interval $0 \leq \ell \leq \ell^*$ for all boundsry points p. A classic result which appears in *Der zweite Randwertaufgabe* gives a constant B such that

$$\big|\partial U/\partial N(p + \ell \cdot N)\big| \leq B \cdot \max_{p \in \partial\Omega} |\frac{\partial U}{\partial n}(p)|$$

hold for all $p \in \partial\Omega$ and $0 \leq \ell \leq \ell^*$.

C. Proof of Theorem when $t \mapsto F(t,p)$ is analytic.

Assume that $t \mapsto F(t,p)$ is a real-analytic function on the positive real axis for each $p \in \partial\Omega$ where local power series converge uniformly with respect to p. In this situation we shall prove the *existence* of a solution u in the Theorem. To attain this we proceed as follows. To each real number $0 \leq h \leq 1$ we seek a solution u_h where

$$\frac{\partial u_h}{\partial n}(p) = h \cdot F(u_h, p) + (1-h) \cdot u_h(p) \tag{1}$$

When $h = 0$ we found the solution u_0 in § xx. Next, suppose that $0 \leq h_0 < 1$ and that we have found the solution u_{h_0} to (1). By the result in § B there exists a pair of positive constants $A_* < A^*$ such that

$$A_* \leq u_{h_0}(p) \leq A^* \tag{*}$$

which are independent of h_0 and of p.

Set $u_0 = u_{h_0}$ and with $h = h_0 + \alpha$ for some small $\alpha > 0$ we shall find u_h by a series

$$u_h = u_{h_0} + \sum_{\nu=1}^{\infty} \alpha^\nu \cdot u_\nu \tag{2}$$

The pole at q_* occurs already in u_0. So $u_1, u_2, \ldots$ is a sequence of harmonic functions in Ω and there remains to find them so that u_h solves (1). We will show that this can be achieved when α is sufficiently small. Keeping h_0 fixed we set

$$u_0 = u_{h_0}$$

The analyticity of F with respect to t gives for every $p \in \partial\Omega$ a series expansion

$$F(u_0(p) + \alpha, p) = F(u_0(p), p) + \sum_{k=1}^{\infty} c_k(p) \cdot \alpha^\nu \tag{3}$$

where $\{c_k(p)\}$ are continuous functions on $\partial\Omega$. Here (*) and the hypothesis on F entail that the radius of convergence has a uniform bound below, i.e. there exists $\rho > 0$ which is independent of $p \in \partial\Omega$ and a constant K such that

$$\sum_{k=1}^{\infty} |c_k(p)| \cdot \rho^k \leq K \tag{4}$$

Now the equation (1) can be solved via a system of equations where the harmonic functions $\{u_\nu\}$ are determined inductively while α-powers are identified. The linear α-term gives the equation

$$\frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1-h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p) \tag{i}$$

For u_2 we find that

$$\frac{\partial u_2}{\partial n} = (1-h_0)u_2 - u_1 + h_0 c_1(p) u_2 + c_1(p) u_1 + c_2(p) u_1^2 \tag{ii}$$

In general we have

$$\frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \ldots, u_{\nu-1}, p) \quad : \nu \geq 1 \tag{iii}$$

where $\{R_\nu\}$ are polynomials in the preeceding u-functions whose coefficients are continuous functions obtained from the c-functions. The function $c_1(p)$ is given by

$$c_1(p) = \frac{\partial F}{\partial t}(u_0(p), p)$$

which by the hypothesis on F is a positive continuous function on $\partial\Omega$. It follows that the function

$$\text{(iv)} \qquad W(p) = (1 - h_0) + h_0 \cdot c_1(p)$$

also is positive on $\partial\Omega$ and in the recursion above we have

$$\text{(v)} \qquad \frac{\partial u_\nu}{\partial n} = W(p) \cdot u_\nu(p) + R_\nu(u_0, \ldots, u_{\nu-1}, p) \quad : \ \nu = 1, 2, \ldots$$

Above we encounter linear equations exactly as in (B.0) where the the f-functions are the R-polynomials. Put

$$w_* = \min_{p \in \partial\Omega} W(p)$$

From § B.XX we get

$$\text{(vi)} \qquad |u_\nu|_{\partial\Omega} \leq w_*^{-1} \cdot |R_\nu(u_0, \ldots, u_{\nu-1}, p)|_{\partial\Omega}$$

Finally, (vi) and a majorising positive series expressing maximum norms imply that if α is sufficiently small then the series (2) converges and gives the requested solution for (1). Moreover, α can be taken *independently* of h_0. Together with the established uniqueness of solutions u_h whenever they exist, it follows that we can move from $h = 0$ until $h = 1$ and arrive at the requested solution in Theorem 1.

Remark. The reader may consult page 106 in [Carleman] where the existence of a uniform constant $\alpha > 0$ for which the series (2) converge for every h is demonstrated by an explicit majorant series.

9. Neumann-Poincaré boundary value problems

Introduction. Several fundamental results were achieved by Carl Neumann in the article emphxxx from 1877 which in particular solved boundary valued problems where double-layer potentials occur. The results were carried out in dimension 3 which for physical reasons is the most relevant part. Here we are content to expose Neumann's theory in dimension two. The crucial steategy is to use Neumann's analytic series expansions which reduces the proof of existence to show that certain poles are absent while meromorphic extensions of Neumann series are constructed. We remark that Neumann's existence results were confined to convex domains where certain majorizations become straighforward since this gives rise to positive integral kernels. The extension of Neumann's results to general domains was achieved by Poincaré in the article *xxx* from 1897 where some ingenoius new methods were introduced to overcome the failure of positivity for the non-symmetric kernel defining the double-layer potential.

The smoothness of boundaries was relaxed in later work. Existence results for planar domains where isolated corner points are allowed were established by Zarmela in 1904. Further studies of Neumann's problem for planar domains with non-regular boundary appear in Carleman's thesis from 1916. A novelty in this work is that solutions to the Neumann's boundary value problem also are exhibited for functions which only are integrable on the boundary. This leads to new phenomena for the spectrum of ïntegral operators, i.e the spectrum is not always confined to discrete subsets of the complex λ-plane. It will take us too far to go into details, especially in the delicate analysis from Part 3 in [ibid]. The interested reader can also consult the expositary article *xxxx* by Holmgen which describes how non-discrete spectral can occur for integral kernels associated to the Neumann problem which in those days was a new phenomenon in operator theory. Let us now expose the methods introduced by Neumann and Poincaré.

Preliminaries. Let $\mathcal{C}$ be a closed Jordan curve of class C^2 whose arc-lengt measure is denoted by σ. If g is a continuous function on $\mathcal{C}$ the logarithmic potential

$$U_g(z) = \frac{1}{\pi}\int_{\mathcal{C}} \log\,\frac{1}{|z-q|}\cdot g(q)\,d\sigma(q)$$

yields a harmonic function in open the complement of $\mathcal{C}$. Since $\log|z|$ is locally integrable in $\mathbf{C}$ and $U_g(z)$ the convolution of this log-function and the compactly supported Riesz measure $g\cdot\sigma$. By elementary measure theory this implies that U_g extends to a continuous function. In particular the pair of harmonic functions in the inner respectively outer component of $\mathcal{C}$ are equal on $\mathcal{C}$. Moreover, the Laplacian of U_g taken in the distribution sense is equal to the measure $g\cdot\sigma$. Now we consider partial derivatives of U and study the inner normal derivative as z approaches points $p\in\mathcal{C}$ from the inside. Let $\mathbf{n}_*$ denote the inner normal derivative along $\mathcal{C}$ which gives the function on $\mathcal{C}$ defined by:

$$p\mapsto \frac{\partial U_g}{\partial \mathbf{n}_*}(p)$$

This function is recaptured via an integral kernel function $K(p,q)$ defined on the product $\mathcal{C}\times\mathcal{C}$. With $p\neq q$ we consider the vector $p-q$ and the unit vector $\mathbf{n}_*(q)$ and constructing an inner product we set

$$K(p,q) = \frac{\langle p-q,\mathbf{n}_*(q)\rangle}{|p-q|^2} \tag{*}$$

Let analyze the behaviour of K close to a point on the diagonal. Working in local coordinates we can take $p=q=(0,0)$ and close the this boundary point the C^2-curve $\mathcal{C}$ is locally defined by a function

$$y=f(x)$$

where $\phi(x)$ is a C^1-function and the (x,y) belong to the bounded Jordan domain when $y>f(x)$. By drawing a figure the reader can verify that

$$\mathbf{n}_*(x,f(x))\cdot d\sigma = (-f'(x),1)\dot{d}x$$

So with $p = (t, f(t))$ and $q = (x, f(x))$ we have

$$K(p, q) \cdot d\sigma(q) = \frac{f(t) - f(x) - f'(x)(t - x)}{(t - x)^2 + (f(t) - f(x))^2} \cdot dx$$

By hypothesis f is of class C^2 which implies that the right hand side stays bounded as y and x independently of each other approach zero. This eanble us to construct integrals and Green's formula yields:

Theorem. *For each* $p \in \mathcal{C}$ *one has*

$$\frac{\partial U_g}{\partial \mathbf{n}_*}(p) = g(p) + \int_{\mathcal{C}} K(p, q) \cdot g(q)\, d\sigma(q)$$

Exercise. Prove this equality. A hint is by additivity it suffices to take g-functions with supports confined to small sub-intervals of $\mathcal{C}$ and profit upon local coordinates and parametrizations as above for $\mathcal{C}$ close to the support of g.

1. Neumann's boundary value problem.

Let Ω be a bounded domain where $\partial\Omega$ consists of a finite set of closed Jordan curves of class C^2. Let h and f be a pair of real-valued continuous functions on $\partial\Omega$ where h is positive. We seek a function U which is harmonic in Ω and on the boundary satisfies

$$\frac{\partial U}{\partial \mathbf{n}_*}(p) = h(p)U(p) + f(p)$$

1.1 Theorem. *The boundary value problem above has a unique solution* U.

The uniquenss amonuts to show that if V is harmonic in Ω and

$$\frac{\partial V}{\partial \mathbf{n}_*}(p) = h(p)V(p)$$

holds on $\partial\Omega$, then $V = 0$. Since h is positive this follows from § XX: Chapter V.

Proof of existence. For each $g \in C^0(\partial\Omega)$ we construct U_g which by Theorem 0.1 solves the Neumann problem if the g-function satisfies the integral equation

$$g(p) + \int_{\mathcal{C}} K(p, q) \cdot g(q)\, d\sigma(q) = h(p) \cdot \frac{1}{\pi} \cdot \int_{\partial\Omega} \log \frac{1}{|p - q|} \cdot g(q)\, d\sigma(q) + f(p) \tag{1}$$

With h kept fixed we introduce the kernel

$$K_h(p, q) = h(p) \cdot \frac{1}{\pi} \cdot \log \frac{1}{|p - q|} - K(p.q)$$

and (1) reduces to the equation

$$g(p) - \int_{\partial\Omega} K_h(p, q) g(q)\, d\sigma(q) = f(p) \tag{2}$$

Next, introduce the linear operator on the Banach space $C^0(\partial\Omega)$ defined by

$$\mathcal{K}_h(f) = \int_{\partial\Omega} K_h(p, q) f(q)\, d\sigma(q) \quad : \quad f \in C^0(\partial\Omega) \tag{3}$$

With this notation a g-function satisfies (2) if

$$(E - \mathcal{K}_h)(g) = f \tag{4}$$

where E is the identity operator on $C^0(\partial\Omega)$. Next, from the general result in §§ $\mathcal{K}_h$ is a compact linear operator and this entails by another general result from § xx that each $f \in C^0(\partial\Omega)$ yields a meromorphic function of the complex parameter λ given by

$$N_f(\lambda) = f + \sum_{n=1}^{\infty} \lambda^n \cdot \mathcal{K}_h^n(f)$$

If $\delta > 0$ is so small that $||\mathcal{K}_h|| < \delta^{-1}$ it is clear that

$$(E - \lambda\mathcal{K}_h)(N_f(\lambda) = f$$

If $N_f(\lambda)$ has no pole at $\lambda = 1$ it follows by analyticity that

$$(E - \mathcal{K}_h)(N_f(1) = f$$

which means that $g = N_f(1)$ solves (4) and the existence part follows. So there remains only to show:

The absence of a pole at $\lambda = 1$. Suppose that $N_f(\lambda)$ has a pole at $\lambda = 1$ which entails that there is a positive integer m such that

$$N_f(\lambda) = \sum_{k=1}^{k=m} \frac{a_k}{(1-\lambda)^k} + b(\lambda)$$

hold when $|\lambda - 1|$ is small where $a_m \neq 0$ in $C^0(\partial\Omega)$ and $b(\lambda$ is analytic in some disc centered at $\lambda = 1$. It follows that

$$(1-\lambda)^m N_f(\lambda) = a_m + (1-\lambda)\beta(\lambda)$$

where $\beta(\lambda)$ again is an analytic $C^0(\partial\Omega)$-valued function close to 1. Apply $E - \mathcal{K}_h$ on both sides which gives

$$(1-\lambda)^m (E - \mathcal{K}_h)(N_f(\lambda)) = (E - \mathcal{K}_h)(a_m) + (1-\lambda) + (E - \mathcal{K}_h)(\beta(\lambda))$$

Now $\lambda = 1$ gives

$$(E - \mathcal{K}_h)(a_m) = 0 \implies a_m = \mathcal{K}_h(a_m)$$

This contradicts the uniqueness part which already has been proved.

2. The case when $\mathcal{C}$ has corner points.

In the preceeding section we found a unique solution to Neumann's boundary problem where the inner normal derivative of U along $\partial\Omega$ is a continuous function. If corner points appear this will no longer be true. But stated in an appropriate way we can extend Theorem 1.1. Let us analyze the specific case when the boundary curves are piecewise linear, i.e. each closed Jordan curve in $\partial\Omega$ is a simple polygon with a finite number of corner points. Given one of these we begin to study the K-function. Let $\xi_1, \ldots, \xi_N$ be the corner points on $\mathcal{C}$. On the linear interval ℓ_i which joints two succescive corner points ξ_i and ξ_{i+1} we notice that $\mathbf{n}_*$ is constant and it is even true that

$$K(p,q) = 0 \quad : \quad p, q \in \ell_i$$

Indeed, this is obvious for if p and q both belong toℓ_i then the vector $p - q$ is parallell to ℓ_i and hence $\perp$ to the normal of this line. Next, keeping q fixed on the open interval ℓ_i while p varies on $\mathcal{C} \setminus \ell_i$ the behaviour of the function

$$p \mapsto \langle p - q, \mathbf{n}_*(q) \rangle$$

is can be understood via a picture and it is clear that (x) is a continuous function. By a picture the reader should discover the different behaviour in the case when $\mathcal{C}$ is convex or not. For example, in the non-convex case it is in general not true that $\mathcal{C} \setminus \ell_i$ stays in the half-space bordered by the line passing ℓ_i and then (*) can change sign, i.e. take both positive and negative values. In the special case when $\mathcal{C}$ is a convex polygon the reader should confirm that (x) is a positive function of p because we have taken the *inner* normal $\mathbf{n}_*(q)$.

2.1 Local behaviour at a corner point. After a linear change of coordinates we take a corner point ξ_* placed at the origin and one ℓ-line is defined by the equation $\{y = 0\}$ to the left of ξ_* where $x < 0$ while $y = Ax$ hold to the right for some $A \neq 0$. If $A > 0$ it means that the angle α at the corner point is detemined by

$$\alpha = \pi - \text{arctg}(A)$$

If $A < 0$ the inner angle is between 0 and $\pi/2$ which the reader should illustrate by a picture. Next, consider a pair of points $p = (-x, 0)$ and $q = (t, At)$ where $x, t > 0$. So p and q belong to opposite sides of the corner point. To be specific, suppose that $A > 0$ which entails that

$$\mathbf{n}_*(q) = \frac{(-A, 1)}{\sqrt{1 + A^2}} \implies$$

$$K(p,q) \cdot d\sigma(q) = \frac{Ax + t}{(x+t)^2 + A^2t^2}$$

When x and t decrease to the origin the order of magnitude is $\frac{1}{x+t}$ so the kernel function is unbounded and the order of magnitude is $\frac{1}{x+t}$. If ℓ_+ denotes the boundsry interval to the right of the origin where q are placed we conclude that

$$\int_{\ell_+} K(p,q) \cdot d\sigma(p) \simeq \int_0^1 \frac{dt}{x+t} \simeq \log \frac{1}{x}$$

The last function is integrable with respect to x. This local computation shows that the kernel function K is not too large in the average. In particular

$$\iint_{\mathcal{C} \times \mathcal{C}} |K(p,q)| \cdot d\sigma(p) d\sigma(q) < \infty$$

But the growth of K near corner points prevail a finite L^2-integral, i.e. the reader may verify that

$$\iint_{\mathcal{C} \times \mathcal{C}} |K(p,q)|^2 \cdot d\sigma(p) d\sigma(q) = +\infty$$

2.2 The integral operator $\mathcal{K}_h$. Let h be a positive continuous function on $\partial\Omega$. Now we define the kernel function $K_h(p,q)$ exactly as in § xx and obtain the corresponding linear operator

$$g \mapsto \int_{\partial\Omega} K_h(p,q) g(q) d\sigma(q)$$

It has a natural domain of definition. Namely, introduce the space L^1_* which consists of functions on g on $\partial\Omega$ for which

$$(*) \qquad \iint \log \frac{R}{|p-q|} \cdot |g(p)| \cdot d\sigma(q) d\sigma(p) < \infty$$

where $R > 0$ is so large that $\frac{R}{|p-q|} > 1$ hold for pairs p, q on $\partial\Omega$. Return to the local situation in (xx) and consider a g-function in L^1_*. Locally we encounter an integral of the form

$$\iint_{\square_+} \frac{1}{x+t} \cdot |g(t, At))| \, dt$$

where $0 \leq x, t \leq 1$ hold in $\square_+$. In this double integral we first perform integration with respect to x which is finite since the inclusion $g \in L^1_*$ entails that

$$\int_0^1 \log \frac{1}{t} \cdot |g(t, At)| \, dt < \infty$$

From the above we obtain the following:

2.3 Theorem. *The kernel function K_h yields a continuous linear operator from L^1_* into $L^1(\partial\Omega)$, i.e. there exists a constant C such that*

$$\int_{\partial\Omega} |\mathcal{K}_h(g)| \cdot d\sigma \leq C \cdot \iint_{\partial\Omega \times \partial\Omega} \log \frac{R}{|p-q|} \cdot |g(p)| \cdot d\sigma(q) d\sigma(p)$$

Armed with Theorem 2.3 we can solve Neumann's boundary value problem for domains whose boundary curves are polygons.

2.4 Theorem. *For each $f \in L^1(\partial\Omega$ there exists a unique harmonic function U in Ω such that*

$$\frac{\partial U}{\partial \mathbf{n}_*}(p) = h(p)U(p) + f(p)$$

holds on $\partial\Omega$. Moreover, $U = U_g$ where $g \in L^1_*$ solves the integral equation

$$g - \mathcal{K}_h(g) = f$$

The uniquenss part. At corner points the inner normal of U has no limit and to establish the uniqueness part we use instead an integral formula:

2.5 Proposition. *For each $g \in L^1_*$ the potential function $U = U_g$ satisfies*

$$\iint_\Omega [(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2]\, dxdy + \int_{\partial\Omega} U \cdot \frac{\partial U}{\partial \mathbf{n}_*}\, d\sigma = 0$$

Exercise. Prove this result.

The requested uniqueness follows. For if $\frac{\partial U}{\partial \mathbf{n}_*} = h \cdot U$ holds on the boundary we get

$$0 = \iint_\Omega [(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2]\, dxdy = \int_{\partial\Omega} h \cdot U^2\, d\sigma \implies g = 0$$

2.6 Proof of existence.

It is carried out by the same mehod as in § X. The crucial point is that the kernel function K_h is sufficiently well-behaved in order that every $f \in L^1(\Omega)$ yields a meromorphic function $N_f(\lambda$ where $\mathcal{K}_h$-powers are applied to f exactly as in XX.

Exercise. Supply details which prove that $N_f(\lambda)$ is meromorphic.

Remark. In [Carleman: Part 3] it is proved that the unique solution g to the integral equation is represented in a canonical fashion using a certain orthonormal family of functions with respect to the L^2-function $\log \frac{1}{|p-q|}$ wtih respect to the product measure $\sigma \times \sigma$. Moreover, there exists a representation formula expressed by convergent series for the inhomogenous equation

$$g + \lambda \cdot \mathcal{K}_h(g) = f$$

where poles of $N_f(\lambda)$ are taken into the account.

A uniqueness theorem for an elliptic boundary value problem

Introduction. We shall work in $\mathbf{R}^2$ with coordinates (x, y). Let $n = 2m$ be an even positive integer and consider two $n \times n$-matrices $\mathcal{A} = \{A_{pq}\}$ and $\mathcal{B} = \{B_{pq}\}$ whose elements are real-valued functions of x and y where the B-functions are continuous and the A-functions of class C^2. Eigenvalues of the $\mathcal{A}$-matrix when (x, y)-varies give an n-tuple of roots $\lambda_1(x, y), \ldots, \lambda_n(x, y)$ which solve

$$\det\big(\lambda \cdot E_n - \mathcal{A}(x, y)\big) = 0 \tag{1}$$

Next, we have a system of first order PDE-equations whose solutions are vector valued functions $(f_1, \ldots, f_n)$ defined in a half-disc

$$D_+(\rho) = \{x^2 + y^2 < \rho^2 \quad : \; x > 0\}$$

where the f-functions satisfy the system:

$$\frac{\partial f_p}{\partial x} + \sum_{q=1}^{q=n} A_{pq}(x, y) \cdot \frac{\partial f_p}{\partial y} + \sum_{q=1}^{q=n} B_{pq}(x, y) \cdot f_q(x, y) = 0 \tag{*}$$

together with the boundary conditions:

$$f_p(0, y) = 0 \quad \text{for all} \quad 1 \leq p \leq n \tag{**}$$

If the λ-roots are non-real in (1) we say that (*) is an elliptic system. When this holds one exocets that the vanishing Cauchy data in (**) entails that the solution f is identically zero. This uniqueness was proved by Erik Holmgren in the article [Holmgren] under the assumption that the A-functions and the B-functions are real analytic. The question remained if the uniqueness still holds under less regularity on the coefficient functions. An affirmative answer was proved by Carleman in the article *xxx*.

1. Theorem. *Assume that the λ-roots are all simple and non-real as (x, y) varies in the open half-disc. Then every solution f to (**) with vanishing Cauchy-data is identically zero.*

The proof requires several steps and the methods which occur below have inspired more recent work where Carleman estimates are used to handle boundary value problems in PDE-theory.

A. First part of the proof

The system in (*) is equivalent to a system of m-many equations where one seeks complex-valued functions $g_1, \ldots, g_m$ satisfying:

$$\frac{\partial g_p}{\partial x} + \sum_{q=1}^{q=m} \lambda_p(x, y) \cdot \frac{\partial g_p}{\partial y} =$$

$$\sum_{q=1}^{q=m} a_{pq}(x, y) \cdot g_q(x, y) + b_{pq}(x, y) \cdot \bar{g}_q(x, y) = 0 \; : \; 1 \leq p \leq m \tag{**}$$

Above $\{a_{pq}\}$ and $\{b_{pq}\}$ are complex-valued,and by the elliptic hyptheis the complex-valued λ-functions can be chosen so that their imaginary parts are positive functions of (x, y). The reduction of the originalsystem to to this complex family of equations is left to the reader. From now on we study the system (**) and Theorem 1 amounts to prove that if the g-functions satisfy (**) in a half-disc $D_+(\rho)$ and

$$g_p(0, y) = 0 \quad : \quad 1 \leq p \leq m$$

then there exists some $0 < \rho_* \leq \rho$ such that the g-functions are identically zero in $D_+(\rho_*)$. To attain this we introduce domains as follows: For a pair $\alpha > 0$ and $\ell > 0$ we put

$$D_\ell(\alpha) = \{x + y^2 - \alpha x^2 < \ell^2\} \cap \{x > 0\} \tag{1}$$

Notice that the boundary

$$\partial D_\ell(\alpha) = \{0\} \times [-\ell, \ell] \cup \{x + y^2 - \alpha x^2 = \ell^2\}$$

Above α and ℓ are small so the the g-functions satisfy (**) in $D_\ell(\alpha)$. For each $t>0$ we define the m-tuple of functions by

(2) $$\phi_p(x,y)=g_p(x,y)\cdot e^{-t(x+y^2-\alpha x^2)}$$

Since the g-functions satisfy (**) one verifies easily that the ϕ-functions satisfy the system

(3) $$\frac{\partial\phi_p}{\partial x}+\frac{\partial}{\partial y}\big(\lambda_p\cdot\phi_p\big)+t(1-2\alpha x+2y\lambda_p)\cdot\phi_p=H_p(\phi)$$

where

$$H_p(\phi)=\sum_{q=1}^{q=n}a_{pq}(x,y)\cdot\phi_q(x,y)+b_{pq}(x,y)\cdot\bar{\phi}_q(x,y)=0\ :\ 1\leq p\leq m$$

Next, we set

(4) $$\Phi(x,y)=\sum_{p=1}^{p=m}|\phi_p(x,y)|$$

The crucial step in the proof of Theorem 1 is to establish the following inequality.

A.1 Proposition. *Provided that α from the start is sufficiently large there exists some $0<\ell_*\leq\ell$ and a constant C which is independent of t such that*

$$\iint_{D_{\ell_*}}\Phi(x,y)\cdot dxdy\leq C\cdot\int_{T_{\ell_*}}\sum_{p=1}^{p=n}|\phi_p|\cdot|dy-\lambda_p\cdot dx|$$

How to deduce Theorem 1. Let us show why Proposition A.1 gives Teorem 1. In addition to ℓ_* we fix some $0<\ell_{**}<\ell_*$. In (2) above we have used the function

$$w(x,y)=e^{-t(x+y^2-\alpha x^2)}\implies$$

(i) $$w(x,y)=e^{-t\ell_*^2}\quad:\quad\{x+y^2-\alpha x^2=\ell_*^2\}\quad:\quad w(x,y)\geq e^{-t\ell_{**}^2}\quad:\quad(x,y)\in D_{\ell_{**}}$$

Next, we have $|\phi_p|=|g_p|\cdot w$ for each p. Replacing the left hand side in Proposition A.1 by the area integral over the smaller domain $D_{\ell_{**}}$ we obtain the inequality;

(ii) $$\iint_{D_{\ell_{**}}}\sum_{p=1}^{p=m}|g_p(x,y)|\cdot dxdy\leq C\cdot e^{t(\ell_{**}^2-\ell_*^2)}\cdot\int_{T_{\ell_*}}\sum_{p=1}^{p=n}|g_p|\cdot|dy-\lambda_p\cdot dx|$$

Here (ii) holds for every $t>0$. When $t\to+\infty$ we have $e^{t(\ell_{**}^2-\ell_*^2)}\to 0$ and conclude that

$$\iint_{D_{\ell_{**}}}\sum_{p=1}^{p=m}|g_p(x,y)|\cdot dxdy=0$$

This means that the g-functions are all zero in $D_{\ell_{**}}$ and Theorem 1 follows.

B. Proof of Proposition A.1

The proof relies upon the construction of certain ψ-functions. More precisely, when $t>0$ and a point $(x_*,y_*)\in D_\ell$ are given we shall construct an m-tuple of ψ-functions satisfying the following:

Condition 1. Each ψ_p is defined in the punctured domain $D_\ell\setminus\{(x_*,y_*)\}$ where ψ_p for a given $1\leq p\leq m$ satisfies the equation

(i) $$\frac{\partial\psi}{\partial x}+\lambda_p\cdot\frac{\partial\psi}{\partial y}-t(1-2\alpha x+2y\lambda_p)\psi_p=0$$

Condition 2. For each p the singularity of ψ_p at (x_*,y_*) is such that the line integrals below have a limit:

(ii) $$\lim_{\epsilon\to 0}\int_{[z-z_*|=\epsilon}\psi_p\cdot(dx-\lambda_p\cdot dy)=2\pi$$

Condition 3. There exists a constant K which is independent both of (x_*, y_*) and of t such that

$$\text{(iii)} \qquad |\psi_p(z)| \leq \frac{K}{|z - z_*|}$$

Notice that the ψ-functions depend on the parameter t, i.e. they are found for each t but the constant K in (3) is independent of t.

The deduction of Proposition A.1

Before the ψ-functions are constructed in Section C we show how they give Proposition A.1. Consider a point $z_* \in D_+(\ell)$. We get the associated ψ-functions from § B at this particular point. Remove a small disc γ_ϵ centered at z_* and consider some fixed $1 \leq p \leq m$. Now ϕ_p satisfies the differential equation (3) from section A and ψ_p satisfies (i) in Condition 1 above. Stokes theorem gives:

$$\int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) = \iint_{D_\ell \setminus \gamma_\epsilon} H_p(\phi) \cdot \psi_p \cdot dxdy + \int_{|z - z_*| = \epsilon} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big)$$

Passing to the limit as $\epsilon \to 0$, Condition 2 gives

$$\text{(1)} \qquad \phi_p(x_*, y_*) = \frac{1}{2\pi} \int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) - \frac{1}{2\pi} \cdot \iint_{D_\ell} H_p(\phi) \cdot \psi_p \cdot dxdy$$

Let L be the maximum over D_ℓ of the coefficient functions of ϕ and $\bar{\phi}$ which appear in $H_p(\phi)$ from (3) i § A. We have also the constant K from Condition 3 for ψ_p. The triangle inequality gives:

$$(*) \qquad \big|\phi_p(x_*, y_*)\big| \leq \frac{K}{2\pi} \int_{T_\ell} \frac{|\phi_p| \cdot |dy - \lambda_p \cdot dx|}{z - z_*|} + \frac{LK}{\pi} \cdot \sum_{q=1}^{q=m} \iint_{D_\ell} \frac{|\phi_q|}{|z - z_*|} \cdot dxdy$$

Next, we use the elementary inequality

$$(**) \qquad \iint_\Omega \frac{dxdy}{\sqrt{x - a)^2 + (y - b)^2}} \leq 2 \cdot \sqrt{\pi} \cdot \sqrt{\text{Area}(\Omega)}$$

where Ω is an arbitrary bounded domain and $(a, b) \in \Omega$. Apply (**) with $\Omega = D_\ell$ and set $S = \text{area}(D_\ell)$. Integrating both sides in (*) over D_ℓ for every p and taking the sum we get

$$\iiint_{D_\ell} \Phi \cdot dxdy \leq$$

$$K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |\phi_p| \cdot |dy - \lambda_p \cdot dx| + 2\pi m LK \cdot \sqrt{\frac{S}{\pi}} \iint_{D_\ell} \Phi \cdot dxdy$$

This inequality hold for all small ℓ. Choose ℓ so small that

$$2\pi m LK \cdot \sqrt{\frac{S}{\pi}} \leq \frac{1}{2}$$

Then the inequality above gives

$$(***) \qquad \iiint_{D_\ell} \Phi \cdot dxdy \leq 2 \cdot K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |\phi_p| \cdot |dy - \lambda_p \cdot dx|$$

Finally, consider some relatively compact domain Δ in D_ℓ. Then there exists $0 < \ell_* < \ell$ such that

$$\Delta \subset D_{\ell_*}$$

Now we notice that

$$|\phi_p(z)| \geq e^{-t\ell_*^2} \cdot |u_p(z)| \quad : \quad z \in \Delta \quad : \quad |\phi_p(z)| \geq e^{-t\ell^2} \cdot |u_p(z)| \quad : \quad z \in T_\ell$$

We conclude that

$$(****) \qquad e^{-t\ell_*^2} \iiint_\Delta \sum_{p=1}^{p=m} |u_p(z)| \cdot dxdy \leq e^{-t\ell^2} \cdot 2 \cdot K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |u_p| \cdot |dy - \lambda_p \cdot dx|$$

Here (****) hold for every $t > 0$. Passing to the limit as $t \to +\infty$ it follows that

$$\cdot \iiint_\Delta \sum_{p=1}^{p=m} |u_p(z)| \cdot dxdy \leq$$

Since Δ was any relatively compact subset of D_ℓ, we conclude that the u-functions are zero in D_ℓ and Theorem 1 follows.

C. Construction of the ψ-functions.

Before we embark upon specific constructions we investigate the whole family of solutions to a first order differential operators of the form

$$(*) \qquad Q = \partial_x + \lambda(x, y) \cdot \partial_y$$

where $\lambda(x, y)$ is a complex valued C^2-function whose imaginary part is > 0. Set

$$\lambda(x, y) = \mu(x, y) + i \cdot \tau(x, y) \quad : \tau(x, y) > 0$$

Now we look for solutions $h(x, y)$ to the equation $Q(h) = 0$. With $h(x, y) = \xi(x, y) + i \cdot \eta(x, y)$ where ξ and η are real-valued C^2-functions this gives the differential system:

$$\frac{\partial \xi}{\partial x} + \mu_p \cdot \frac{\partial \xi}{\partial y} - \tau_p \cdot \frac{\partial \eta}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial x} + \mu_p \cdot \frac{\partial \eta}{\partial y} + \tau_p \cdot \frac{\partial \xi}{\partial y} = 0$$

Suppose we have found one solution $h = \xi + i \cdot \eta$ where the Jacobian $\xi_x \eta_y - \xi_y \eta_x$ is $\neq 0$ at the origin. Then $(x, y) \mapsto (\xi, \eta)$ is a local C^2-diffeomorphism. With $\zeta = \xi + i\eta$ we have the usual Cauchy-Riemann operator.

$$\frac{1}{2}(\frac{\partial}{\partial \xi} + i \cdot \frac{\partial}{\partial \eta})$$

Let $g(\xi + i\eta)$ be a holomorphic function in the complex ζ-space with $\zeta = \xi + i\eta$ and put

$$g_*(x, y) = g(\xi(x, y) + i\eta(x, y))$$

Then one easily verifies that $Q(g_*) = 0$ and conversely, every solution to this equation is expressed by a $g - *$function derived from an analytic function in the complex ζspace.satisfies $Q(g_*)$.

Conclusion. *If a non-degenerate solution $h = \xi + i\eta$ has been found then the homogenous solutions to Q is in a 1-1 correspondence to analytic functions in the ζ-variable.*

Remark. The effect of a coordinate transformation as above is that the Q-operator is transported to the Cauchy-Riemann operator in the complex ζ-space where $\zeta = \xi + i\eta$. Later we employ such (ξ, η)-transformations to construct solutions to an inhomogeneous equation of the form

$$Q(\psi) = (t - \alpha x + 2y\lambda(x, y)) \cdot \psi(t, x, y)$$

where t is a positive parameter and the ψ-functions will have certain specified properties. Notice that it suffices to construct the ψ-functions separately, i.e. we no longer have to bother about a differential system. With a fixed p fixed $\lambda_p(x, y) = \mu_p + \tau_p$ and from now on we may drop the index p and explain how to obain ψ-functions satisfying the three conditions from § B. So we consider the first order differential operator

$$(1) \qquad Q = \frac{\partial}{\partial x} + (\mu(x, y) + i\tau(x, y)) \cdot \frac{\partial}{\partial y}$$

where $\tau(x,y) > 0$.

C.1 A class of (ξ,η)-functions. Let $V(x,y)$ and $W(x,y)$ be two quadratic forms, i.e. both are homogeneous polynomials of degree two. Given a point (x_*,y_*) and with $z = x+iy$ we seek a coordinate transformation $(x,y) \mapsto (\xi,\eta)$ of the form:

$$\xi(z) = \tau_p(z_*) \cdot (x - x_*) + V(x - x_*, y - y_*) + \gamma_1(z) \cdot |z - z_*|^2$$

$$\eta(z) = (y - y_*) - \mu_p(z_*) \cdot (x - x_*) + W(x - x_*, y - y_*) + \gamma_2(z) \cdot |z - z_*|^2$$

Lemma. *There exists a pair of quadratic forms V and W whose coefficients depend on (x_*,y_*) and a pair of γ-functions which both vanish at (x_*,y_*) up to order one such that the complex-valued function $\xi + i\eta$ solves the homogeneous equation $Q(\xi + i\eta) = 0$.*

A solution above gives a change of variables so that Q is expressed in new real coordinates (ξ,η) by the operator

$$\frac{\partial}{\partial \xi} + i \cdot \frac{\partial}{\partial \eta} \tag{2}$$

There exist many coordinate transforms $(x,y) \to (\xi,\eta)$ which change Q into (2). This *flexible choice* of coordinate transforms is used to construct the required ψ-functions. Notice that Condition (2) in § B is of a pointwise character, i.e. it suffices to find a ψ-function for a given point $z_* = x_* + iy_*$. With this in mind the required construction in § B boils down to perform a suitable coordinate transformation adapated to z_*, and after use the existence of a ψ-function which to begin with is expressed in the (ξ,η)-variables where the Q-operator is replaced by the Cauchy-Riemann operator. In this special case the required ψ-function is easy to find, i.e. see the remark in § B.0. So all that remains is to exhibit suitable coordinate transformations which send Q to the $\bar{\partial}$-operator. We leave it to the reader to carry out such coordinate transformations. If necessary, consult Carleman's article where a very detailed construction appears.

The Schrödinger equation.

We work in $\mathbf{R}^3$ with the coordinates (x, y, z). Let $c(x, y, z)$ be a real-valued function in $L^2_{\text{loc}}(\mathbf{R}^3$. In order that the subsequent formulas can be stated in a precise manner we also assume that $c$is almost everywhere continuous which of course is a rather weak condition and in any case satisfied in applications. Next, let Δ be the Laplace operator and define the operator L by

$$(*) \qquad L(u) = \Delta(u) + c \cdot u$$

Denote by $E_L(\mathbf{R}^3)$ the set of functions u such that both u and $L(u)$ belong to $L^2(\mathbf{R}^3)$. Given a pair (f, λ) where $f \in L^2(\mathbf{R}^3$ and λ is a complex number we seek solutions $u \in E_L(\mathbf{R}^3)$ such that

$$(**) \qquad L(u) + \lambda \cdot u = f$$

The case $\mathfrak{Im}(\lambda \neq 0$. By a classic result about solutions to the Neumann boundary value problem in open balls in $\mathbf{R}^3$ one proves that (1) has at least one solution u whenever λ is not real. The remains to investigate the uniqueness, i.e, when one has the implication

$$(***) \qquad \mathfrak{Im}(\lambda \neq 0 \quad \text{and} \quad L(u) + \lambda \cdot u = 0 \implies u = 0$$

This uniqueness property depends on the c-function. A sufficient condition is the following:

Theorem. *Assume that there exists a constant M and some $r_* > 0$ such that*

$$c(x, y, z) \leq M \quad \text{when} \quad x^2 + y^2 + z^2 \geq r_*^2$$

Then (***) above holds.

The spectral θ-function. When (***) holds it was proved in [Carleman] that classical solutions to the Neumanns boundary value problem in open balls yield a θ-function which enable us to describe solutions to (*) for real λ-values. More precisely, there exists two increasing sequence of positive real numbers $\{\lambda^*(\nu)\}$ and $\lambda_*(\nu)\}$ and two sequence of pairwise orthogonal functions $\{\phi_\nu(p))\}$ and $\{\psi_\nu(p)\}$ in $L^(\mathbf{R})^3$ where all these functions have L^2-norm equal to one such that the following hold. First, set

$$\theta(p.q, \lambda) = \sum_{0 < \lambda^*(\nu) \leq \lambda} \phi_\nu(p) \cdot \phi_\nu(q) \quad : \quad \lambda > 0$$

$$\theta(p.q, \lambda) = -\sum_{\lambda \leq \lambda_*(\nu) \leq < 0} \psi_\nu(p) \cdot \psi_\nu(q) \quad : \quad \lambda < 0$$

such that the following hold:

$$(1) \qquad v(p) = \lim_{R \to \infty} \sum_{[\lambda_\nu < R} \theta(p, q, \lambda) \cdot v(q) \cdot dq \quad \text{for all} \quad v \in L^2(\mathbf{R}^3)$$

$$(2) \qquad v \in E_L(\mathbf{R})^{\mathbf{3}} \quad \text{if and only if} \quad xxxx$$

$$(3) \qquad L(v)(p) = \lim_{R \to \infty} \sum_{[\lambda_\nu < R} \lambda \cdot [\int_{\mathbf{R}^3} \theta(p, q, \lambda) \cdot v(q) \cdot dq] \cdot d\lambda \quad \text{for all} \quad v \in E_L(\mathbf{R}^3)$$

Here the equaliy holds in L^2, i.e, in the sense of a Plancherel's limit.

Remark. Here the equaliy holds in L^2, i.e, in the sense of a Plancherel's limit.

Construction of the ϕ-functions. For each finite r we have the ball B_r and consider the space $E_L(B_r)$ of functions u in B_r such that both u and $L(u)$ also belong to $L^2(B_r)$. By a classical result in the Fredholm theory that exist discrete sequences of real numbers $\{\lambda * (\nu)$ and $\lambda_*(\nu)$ as above and two families of orthonormal functions $\{\phi_\nu^{(r)}\}$ and $\{\psi_\nu^{(r)}\}$ satisfying (xx)

and here a classical result shows that the real eigenvalues to the equation $L(u) + \lambda \cdot u = 0$ xxx

$$xxx$$

The proofs of the assertions above rely on a systematic use of Green's formula. To begin with we recall how to express solutions to an inhomogeneous the Laplace equation by an integral formula.

A. The equation $\Delta(u) = \phi$. let D be a domain in $\mathbf{R}^3$ and ϕ a function in $L^2(D)$. Then a function u for which both u and $\Delta(u)$ belong to $L^2(D)$ gives $\Delta(u) = \phi$ if and only if the following hold for every $p \in D$ and every $\rho < \text{dist}(p, \partial D)$:

$$\text{(i)} \qquad u(p) = \frac{1}{2\pi\rho^2} \cdot \int_{B_p(\rho)} \frac{1}{|p-q|} \cdot u(q) \cdot dq + \frac{1}{4\pi\rho^2} \cdot \int_{B_p(\rho)} A(p,q) \cdot \phi(q) \cdot dq$$

where we have put

$$\text{(ii)} \qquad A(p,q) = \frac{2}{\rho} - \frac{1}{|p-q|} - \frac{|p-q|}{\rho^2}$$

Exercise. Prove this result.The hint is to apply Green's formula while ϕ is replaced by $\Delta(u)$ in the last integral.

Remark. Let us also recall also that when $\Delta(u)$ is in L^2, then u is automatically a continuous function in D.

The class $\mathfrak{Neu}(B_r)$. Let B_r be the open ball of radius r centered at the origin. The class of functions u which are continuous on the closed ball and whose interior normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ is continuous on the boundary $S^2[r]$ is denoted by $\mathfrak{Neu}(B_r)$.

The Neumann equation. Let $c(x,y,z)$ be a function in $L^2(\mathbf{R}$ and consider also a pair a, H where a be a continuous function on $S^2[r]$ and $H(p,q)$ a continuous hermitian function on $S^2[r] \times S^2[r]$, i.e. $H(q,p) = \bar{H}(p,q)$ hold for all pairs of point p, q on the sphere $S^2[r]$. With these notations the following hold:

Theorem For each $f \in L^2(B_r)$ and every non-real complex number λ there exists a unique $u \in \mathfrak{Neu}(B_r)$ such that u satisfies the two equations:

$$L(u + \lambda \cdot u = f \quad \text{holds in} \quad B_r$$

$$\partial u / \partial \mathbf{n}(p) = xx$$

Moreover, one has the L^2-estimate

$$\int_{B_r} |u|^2 \cdot dxdydz \leq \left| \frac{1}{\mathfrak{Im}(\lambda)} \right| \leq \int_{B_r} |f|^2 \cdot dxdydz$$

Classic result: $R > 0$ we have unit ball B_R and unit sphere S_R. Let $\mathfrak{Neu}(R)$ set uf u -functions where $\Delta(u)$ in L^2, continuous on closed ball and limit of interior derivative as a continuous funtion. Given $c \in L^2(B_R)$ define

$$L(u) = \delta(u) + c \cdot u$$

Theorem For each pair (a, H) in (*) there exists a unique $u \in \mathfrak{Neu}(R)$ such that

$$L(u + \lambda \cdot u = f \quad \text{holds in} \quad B_R$$

and u saisfies th boundsry condition

$$\partial u / \partial \mathbf{n}(p) = xx$$

from that L^2-esteimate as well.

Do if for $R = m$ running over positive integers. Catch up sequence with L^2-convergence bounded uniformly. $u_m \to u_*$ weak sense and see tha u_* is a solution to (*) on al over space.

Second point about eventual uniquenss. Class I type. Euquialent condition-

In an article from 1920 Carleman constructed an "ugly example" of a doubly indexed sequence $\{c_{pq}\}$ of real numbers satisfying (1) and the symmetry condition $c_{pq} = c_{qp}$, and yet there exists a non-zero complex vector $\{x_p = a_p + ib_p\}$ in ℓ^2 such that

(4) $$S(x) = ix$$

This should be compared with the finite dimensional case where the spectral theorem due to Cauchy and Weierstrass asserts that if A is a real and symmetric $N \times N$-matrix for some positive integer N, then there exists an orthogonal $N \times N$-matrix U such that UAU^* is a diagonal matrix with real elements. Carleman extended this finite dimensional result to infinite Hermitian matrices for which the densely defined linear operator S has no eigenvectors with eigenvalue i or $-i$. More precisely, one says that the densely defined operator S is of Class I if the equations

(*) $$S(z) = iz \quad : S(\zeta) = -i\zeta$$

do not have non-zero solutions with complex vectors z or ζ in ℓ^2. The major result in the cited monograph is as follows:

0.0.1 Theorem. *Each densely Hermitian operator S of Class I has a unique adapted resolution of the identity.*

0.0.2 Resolutions of the identity. In order to digest Theorem 0.0.1 we recall the notion of spectral resolutions. To begin with, a resolution of the identity on ℓ^2 consists of a family $\{E(\lambda)\}$ of self-adjoint projections, indexed by real numbers λ which satisfies (A-C) below.

A. Each $E(\lambda)$ is an orthogonal projection from ℓ^2 onto the range $E(\lambda)(\ell^2)$ and these operators commute pairwise, i.e.

(i) $$E(\lambda) \cdot E(\mu) = E(\mu) \cdot E(\lambda)$$

hold for pairs of real numbers.

B. To each pair of real numbers $a < b$ we set

$$E_{a,b} = E(b) - E(a)$$

Then

(iii) $$E_{a,b} \cdot E_{c,d} = 0$$

for each pair of disjoint interval $[a, b]$ and $[b, c]$.

C. For each $x \in \ell^2$ the real-valued function

(c) $$\lambda \mapsto \langle E(\lambda)(x), x \rangle$$

is a non-decreasing and right continuous function.

0.0.3 ***S*-adapted resolutions.** Let S be a densely defined and hermitian linear operator on ℓ^2. A spectral resolution $\{E(\lambda)\}$ of the identity is S-adapted if the following three conditions hold:

A.1 For each interval bounded $[a, b]$ the range of $E_{a,b}$ from (B) above is contained in $\mathcal{D}(S)$ and

$$E_{a,b}(Sx) = S \circ E_{a,b}(x) \quad : x \in \mathcal{D}(S)$$

B.1 By (C) each $x \in \ell^2$ gives the non-decreasing function $\lambda \mapsto \langle E(\lambda)(x), x\rangle$ on the real line. Together with the right continuity in (c) there exist Stieltjes' integrals

$$\int_a^b \lambda \cdot \langle dE(\lambda)(x), x\rangle$$

for each bounded interval. Carleman's second condition for $\{E(\lambda)\}$ to be S-adapted is that a vector x belongs to $\mathcal{D}(S)$ if and only if

$$\int_{-\infty}^{\infty} |\lambda| \cdot \langle dE(\lambda)(x), x\rangle < \infty \tag{b.1}$$

C.1 The last condition is that

$$\langle Sx, y\rangle = \int_{-\infty}^{\infty} \lambda \cdot \langle dE(\lambda)(x), y\rangle \quad : x, y \in \mathcal{D}(S) \tag{c.1}$$

where (b.1) and the Cauchy-Schwarz inequality entail that the Stieltjes' integral in (c.1) is absolutely convergent.

2. An example from PDE-theory.

2.1. Propagation of sound. With (x, y, z) as space variables in $\mathbf{R}^3$ and a time variable t, the propagation of sound in the infinite open complement $U = \mathbf{R}^3 \setminus \overline{\Omega}$ of a bounded open subset Ω is governed by solutions $u(x, y, z, t)$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \tag{1}$$

where Δ is the Laplace operator in x, y, z. So here (1) holds when $p = (x, y, z) \in U$ and $t \geq 0$. We assume that $\partial\Omega$ is of class C^1, i.e. given as a union of pairwise disjoint closed surfaces of class C^1 along which normal vectors are defined. A boundary value problem arises when we seek solutions such that $p \mapsto u(p, t)$ belong to $L^2(U)$ for each t, and the outer normal derivatives taken along $\partial\Omega$ are zero, i.e. for every t

$$\frac{\partial u}{\partial n}(p, t) = 0 \quad : p \in \partial\Omega \tag{2}$$

Initial conditions are expressed by a pair of C^2-functions $f_1 x, y, z)$ and $f_2(x, y, z)$ defined in U such that f_1, f_2 together with $\Delta(f_1$ and $\Delta(f_2)$ belong to $L^2(U)$, and their outer normal derivatives along $\partial\Omega$ are zero. So here $u(p, 0) = f_0(p)$ and $\frac{\partial u}{\partial t}(p, 0) = f_1(p)$ hold for each $p \in U$.

That this boundary value problem has a unique solution u can be established via variational methods. In the cited monograph, Carleman gave a proof using the spectral function θ attached to a Class I operator A constructed via solutions to the Dirichlet problem where no time variable appears. A merit of thisproof is that it confirms the physically expected result expressed by:

$$\lim_{t \to \infty} (\frac{\partial u}{\partial x})^2(p, t) + (\frac{\partial u}{\partial y})^2(p, t) + (\frac{\partial u}{\partial z})^2(p, t) = 0 \tag{*}$$

with uniform convergence when p stays in a relatively compact subset of U. More precisely, Carleman proved (*) by proving that the spectral function of A is *absolutely continuous* with respect to the λ-parameter. In §§ we expose how Carelan derived this from a general result which goes as follows:

Let $\{a \leq s \leq b\}$ be a compact interval on the real s-line and $s \mapsto G_s$ is a function with values in the Hilbert space $L^2(U)$ which is continuous in the sense that

$$\lim_{s \to s_0} ||G_s - G_{s_0}||_2 = 0$$

hold for each s_0, where we introduced the L^2-norms. The function has a finite total variation if there exists a constant M such that

$$\sum ||G_{s_{\nu+1}} - G_{s_\nu}||_2 \leq M$$

hold for every partition $a = s_0 < s_1 < \ldots < s_M = b$. When this holds one construct Stieltjes integrals and for every subinterval $[\alpha, \beta]$ there exists the L^2-function in U

$$\Phi_{[\alpha,\beta]} = \int_\alpha^\beta s \cdot \frac{dG_s}{ds}$$

Impose the extra conditions that the normal deriviatives $\frac{\partial G_s}{\partial n}$ exist and vanish $\partial\Omega$ for every $a \leq s \leq b$ and the following differential equation holds for every sub-interval $[\alpha, \beta]$ of $[a, b]$:

$$\Delta(G_\beta - G_\alpha) + \Phi_{[\alpha,\beta]} = 0$$

Theorem. *The equations above imply that $s \mapsto G_s$ is absolutely continuous which means that whenever $\{\ell_1, \ldots, \ell_M\}$ a finite family of disjoint intervals in $[a, b]$ where the sum of their lengths is $< \delta$, then the sum of the total variations over these intervals is bounded by $\rho(\delta)$ where ρ is a function of δ which tends to zero as $\delta \to 0$.*