I. Open Riemann surfaces

Introduction. A Riemann surface is a 1-dimensional and connected complex manifold X. We shall consider open surfaces, i.e. the topological space X is non-compact. Proofs of the announced results below rely upon analytic function theory in one complex variable, i.e. complex analysis on open Riemann surfaces is based upon the study of analytic functions on planar domains. For example, the complex analytic structure on an open Rieann surface X yields an intrinsic notion of harmonic, respectively subharmonic functions om X. This will be explained in § xx and will be used to extend solutions to the Dirichlet prolelm for planar domains to X where one regards relatively compact open sets Y for which the boundary ∂Y locally satisfies similar conditions as in the planar case, to ensure that evety continouos boundary function $f \in C^0(\partial Y)$ has a harmonic extension to Y.

We shall use sheaf-theory without hesitation. Let us remark that original work by Riemann and Weierstrass gave the first exanples about sheaves. So various sheaves on Riemann surfaces provide excellent examples which illustrate and has motivated general sheaf theory. We also employ differential calculus. An open Rieann surface X is in particular an oriented C^∞-manifold whose real dimension is two. The complex analytic structure yields a direct sum decompostiion

$$\mathcal{E}^1(X) = \mathcal{E}^{1,0}(X) \oplus \mathcal{E}^{0,1}(X)$$

where the left hand siude is the space of globally defined differentia1-forms, while $\mathcal{E}^{1,0}(X)$ are 1-forms of bidegree $(1,0)$, and .... If (U,z) is a chart in X then the restriction of a (1,0)-form to U is given by $\phi(z) \cdot dz$ where $\phi \in C^\infty(U)$. Similarly a (0.1)-form restricts in the chart to $\psi(z) \cdot dz$ where $\psi \in C^\infty(U)$. The $\bar{\partial}$-operator sends $f \in C^\infty(X)$ to the $(1,0)$-form $\bar{\partial}(f)$.

In analytic function theory a classic result due to Ponpieu from 1810 asserts that if U is an open subset of $\mathbf{C}$ and $w \in \mathcal{E}^{0,1}(U)$, then there exist $f \in \mathcal{C}^\infty(U)$ such that

$$(*) \qquad \bar{\partial}(f) = w$$

Thus, the inhomogenous $\bar{\partial}$-equation is solvable on planar domains. It turns out that this $\bar{\partial}$-equation is solvable on every open Riemann surface ! The proof requires several steps and is given in § xx. A first step is to establish the existence of non-constant meromorphic functions. Let us already in this introductiuon describe a special construction which is the starting point towards more general results. Conisder a poiunt $p \in X$ and choose a chart Δ arond p, i.e. Δ is biholomorphiuc with the open unit disc where p corresponds to the origin. Now X is covered by the open sets $U_0 = \Delta$ and $U_1 = X \setminus \{p\}$. Notice that the intersection $U_0 \cap U_1$ is a punctured open disc where we let z denote the complex coordinate. With $\mathfrak{U} = (U_0, U_1$ one construct the Cech complex

$$0 \to C^0(\mathfrak{U}, \mathcal{O}X)$$

By definition the Cech cohomology of order one is the quoitent

$$\frac{C^1(\mathfrak{U}, \mathcal{O}X)}{\partial(C^0(\mathfrak{U}, \mathcal{O}_X)}$$

and since the covering only consists of two sets we notice that

$$C^1(\mathfrak{U}, \mathcal{O}_X = \mathcal{O}(\Delta^*)$$

Next, let Y be a relatively compact open subset of X. If $0 < a < 1$ we consider rhe disc $a \cdot \Delta$ of radius a centered at the origin. Put

$$V_0 = Y \cap a \cdot \Delta \quad \& \quad V_1 = Y \setminus a/2 \cdot \overline{\Delta}$$

Now $\mathfrak{V} = (V_0, V_1)$ is an open covering of Y and we get the ceh complex $C^\bullet(\mathfrak{V}, \mathcal{O}X)$. Since $\overline{V_i}$ is a compact subsrt of U_i for evety i, it follows from *Montels theorem* in analytic function theory that the resriction maps

$$C^k(\mathfrak{U}, \mathcal{O}_X) \to$$

are compact for every k. Hence a wellkonown result from the Fredholm theoryentails that the induced map

$$\rho\colon\ H^1(\mathfrak{U} \to$$

has a finite dimensional range. Let N be the dimenson of the ρ-image. Now we consider thr holomorphic functions $1, z^{-1}, \ldots z^{-N}$ in the punctured disc Δ^* and conclude that there exist complex numbers $c_0, c-1, \ldots c_{N-1}$ such that

$$\rho(z^{-N} =$$

This menas that if

$$g = z^{-N} - (c_0 + c_1 z + \ldots + c_{N-1} z^{N-1})$$

then the restrcition of g to $V_0 \cap V_1$ has a zero iimage in the cohomology space (xx) above, and hence there exist holomorphic funvtions $f_i \in \mathcal{O}(V_i)$ such that

$$f_1 - f_0 = g$$

holds in $V_0 \cap V_1$. This impkies thst there exists a meromorphic function ϕ in Y with a pole of order N at p, while ϕ is holomorphic in $Y \setminus \{p\}$.

0.1 The Behnke.Stein theory.

Major result about open Riemann surfaces were established by Behnke and Stein in their article *Entwicklungen analytischer Funktionen auf Riemannschen Flächen* [Math. Ann. 120 (1948)]. They proved that if $\mathcal{O}_X$ is the sheaf of holomorphic functions on an open Riemann surface, then the cohomology group

(0.1.1) $$H^1(X, \mathcal{O}_X) = 0$$

More generally, in [ibid] it is also proved that

(0.1.2) $$H^1(X, \mathcal{L}) = 0$$

hold for every holomorphic line bundle. We establish this in § xx, and also expose classic results due to Riemann and Weierstrass. In § xx we prove a theorem due to Weierstrass which goes as follows:

On an open Riemann surface X the space $\mathfrak{M}(X)$ of globally defined meromorphic functions is ample in the following sense. Let $\{p_k\}$ be a discrete set of points in X and to every p_k we assign an non-zero integer μ_k which can be positive or negative. Then there exists $f \in \mathfrak{M}(X)$ which is holomorphic and zero-free in $X \setminus \{p_k\}$, and for every k it has a pole of order $-\mu_k$ if $\mu_k < 0$, and a zero of multiplicity μ_k if $\mu_k > 0$.

0.2 Additive representations of $\pi_1(X)$. An open Riemann surface X has a fundamental group $\pi_1(X)$ and we consider the family $\mathcal{F}$ of group homomorphisms

$$\rho\colon \pi_1(X) \to \mathbf{Z}$$

where $\mathbf{Z}$ is the abelian group of integers. Identify $\pi_1(X)$ with homotopy classes of closed curves which start and terminate at some point $p \in X$. Denote by $\mathcal{O}^*(X)$ the family of globally defined and zero-free holomorphic functions on X.

0.2.1 Theorem. *For each $\rho \in \mathcal{F}$ there exists $\phi \in \mathcal{O}^*(X)$ such that*

$$\rho(\{\gamma\}) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{\partial \phi}{\phi} \quad : \ \{\gamma\} \in \pi_1(X)$$

where $\{\gamma\}$ is the homotopy class of a closed curve.

The proof of Theorem 0.2.1 relies upon the exact sheaf sequence

$$0 \to \mathbf{Z}_X \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}^*_X \to 0$$

where $\mathbf{Z}_X$ is the sheaf whose sections are ocally constant integer-valued functions and we have used the exponential map which sends holomorphic function f defined in open subsets U of X to

$e^{2\pi i f}$ which are sections of the multiplicative sheaf $\mathcal{O}^*$ of zero-free holomorphic functions. Passing to the long exact sequence of cohomology one gets an exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O}(X) \overset{\exp}{\to} \mathcal{O}^*(X) \to H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}) \to \dots$$

Admitting the vanishing theorem by Behnke and Stein in (0.1) this entails that

(i) $$H^1(X, \mathbf{Z}) \simeq \frac{\mathcal{O}^*(X)}{\exp(\mathcal{O}(X))}$$

Next, we have the sheaf Ω_X whose sections are holomorphic 1-forms and an exact sheaf sequence

$$0 \to \mathbf{C}_X \to \mathcal{O}_X \overset{\partial}{\to} \Omega_X \to 0$$

where $\mathbf{C}_X$ is the sheaf whose sections are locally constant complex-valued functions. Here the Behnke-Stein theorem gives the isomorphism

(ii) $$H^1(X, \mathbf{C}) \simeq \frac{\Omega(X)}{\partial(\mathcal{O}(X))}$$

Next, if $g \in \mathcal{O}^*(X)$ one easily verifies that the holomorphic 1-form $\frac{\partial g}{g}$ is ∂-exact if and only if $g = e^{2\pi i f}$ for some $f \in \mathcal{O}(X)$. Hence (i-ii) give a canonically defined and injective map

(iii) $$H^1(X, \mathbf{Z}) \to H^1(X, \mathbf{C})$$

From this we shall deduce Theorem 0.1.1 via a representation for the family $\mathcal{F}_{\mathbf{C}}$ of group homomorphisms from $\pi_1(X)$ into the additive group of complex numbers. More precisely, to each $\rho \in \mathcal{F}_{\mathbf{C}}$ we shall prove that there exists $\omega \in \Omega(X)$ such that

(iv) $$\rho(\{\gamma\}) = \frac{1}{2\pi i} \cdot \int_\gamma \omega$$

Here ω is unique up to a ∂-exact holomorphic 1-form and it turns out that ω can be represented as the logarithmic differential of a zero-free holomorphic function if and only if the ρ-function is integer-valued.

0.3 The uniformisation theorem. In § xx we construct the universal covering space $\widehat{X}$ of a given open Riemann surface and show first that this yields another open Riemann surface. It is special since it is *simply connected* which means that every closed curve on $\widehat{X}$ is homotopic to a constant curve. In § xx we prove that every simply connected open Riemann surface is bihololomorphic to the open unit disc D or the complex plane $\mathbf{C}$.

0.4 The Riemann-Schwarz inequality for hyperbolic distances. The uniformisation theorem has several consequences. Let X be an open Riemann surface. Conider a hyperbolic metri σ on X which means that σ has a negative curvature. An equivaent condition is that in each open chart (U, z) of X where U is identified with an open subset of the complex z-plane, there exists a subharmonic function $u(z)$ such that the σ-distance in U is defined by

$$e^{u(z)} \cdot |dz|$$

where $|dz|$ the ordinary euclidan length. Assume that σ is hyperbolic and consider a pair of points p, q in X. Denote by $\mathcal{C}(p, q)$ the family of curves in X with end-points at p and q.

0.4.1 Theorem. *For every pair γ_1, γ_2 in $\mathcal{C}(p, q)$ and each point $\xi \in \gamma_1$, there exist a pair of curves α, β, where $\alpha \in \mathcal{C}(p, q)$ while β has end-points at ξ and some point $\eta \in \gamma_2$ such that*

(*) $$\ell(\alpha)^2 + \ell(\beta)^2 \leq \frac{1}{2}\big(\ell(\gamma_1)^2 + \ell(\gamma_2)^2\big)$$

where $\ell(\gamma)$ denotes the σ-length of a curve in X. Moreover, the inequality (*) is strict with the exception for special rhombic configurations.

Remark. Theorem 0.4.1 gives in particular existence of unique geodesic curves on X which join a pair of points. The proof relies upon the uniformisation theorem which reduces the proof to show the following:

Let D be the unit disc and $0 < \theta < \pi/2$ and consider the unique circle C_θ which contains $e^{i\theta}$ and $e^{-i\theta}$ and intersects the unit circle at right angles at these points. Let α denote the circular subarc of C_θ contained in D. Then the following inequality holds for every subharmonic function u in D:

$$\Big(\int_{-1}^{1} e^{u(x)}\,dx\Big)^2 + \Big(\int_\alpha e^{u(z)}\,|dz|\Big)^2 \leq \frac{1}{4}\cdot\Big(\int_0^{2\pi} e^{u(e^{i\phi})}\,d\phi\Big)^2 \tag{0.4.2}$$

We prove this result in § xx and how one derives Theorem 0.4.1.

delete .....

Next, let $\mathbf{C}_X$ be the sheaf whose sections are locally constant complex-valued functions and consider the cohomology group $H^1(X, \mathbf{C}_X)$. If it vanishes we shall prove that the open Riemann surface is biholomorphic with the open unit disc D or the complex plane $\mathbf{C}$. One refers to this fact as the *Uniformisation Theorem* for open Riemann surfaces. The proof is given in § 5.

A crucial result which is used to prove (*) is the Pompieu theorem which asserts thst the $\bar{\partial}$-operator is sirjhective on planar domains, i.e. if $\Omega$is an open subset of $\mathbf{C}$ and ϕ is a complex-valued C^∞-function on Ω then there exists $g \in C^\infty(\Omega)$ such that

$$\bar{\partial}g/\partial\bar{z} = \phi$$

The uniformisation theorem has several consequences. An example is *The Riemann-Schwarz inequality* when an open Riemann surface X is equipped with a hyperbolic metric, This result goes as followss: Let σ be a hyperbolic metric. This means that σ has a negative curvature. An equivaent condition is that in each open chart (U, z) of X where U is identified with an open subset of the complex z-plane, there exists a subharmnic function $u(z)$ such that the σ-distance in U is defined by

$$e^{u(z)} \cdot |dz|$$

where $|dz|$ the ordinary euclidan length. Assume that σ is hyperbolic and consider a pair of points p, q in X. Denote by $\mathcal{C}(p, q)$ the family of curves in X with end-points at p and q.

0.1 Theorem. *For every pair γ_1, γ_2 in $\mathcal{C}(p, q)$ and each point $\xi \in \gamma_1$, there exist a pair of curves α, β, where $\alpha \in \mathcal{C}(p, q)$ while β has end-points at ξ and some point $\eta \in \gamma_2$ such that*

$$(*) \qquad \ell(\alpha)^2 + \ell(\beta)^2 \leq \frac{1}{2}\big(\ell(\gamma_1)^2 + \ell(\gamma_2)^2\big)$$

where $\ell(\gamma)$ denotes the σ-length of a curve in X. Moreover, the inequality (*) is strict with the exception for special rhombic configurations.

Remark. The theorem gives in particular existence of unique geodesic curves on X which join a pair of points. The proof relies upon the uniformisation theorem which reduces the proof to show the following:

0.2 *Let D be the unit disc and $0 < \theta < \pi/2$ and consider the unique circle C_θ which contains $e^{i\theta}$ and $e^{-i\theta}$ and intersects the unit circle at right angles at these points. Let α denote the circular subarc of C_θ contained in D. Then the following inequality holds for every subharmonic function u in D:*

$$\Big(\int_{-1}^{1} e^{u(x)}\, dx\Big)^2 + \Big(\int_\alpha e^{u(z)}\, |dz|\Big)^2 \leq \frac{1}{4} \cdot \Big(\int_0^{2\pi} e^{u(e^{i\phi})}\, d\phi\Big)^2$$

We prove this result in § xx and how one derives Theorem 0.1 from (0.2).

A comment. Basic facts about sheaves and calculus on 2-dimensional manifolds will be used without hesitation. Let us remark that the study of open Riemann surfaces gives an excellent illustration to general theory related to analysis on manifolds. For example, sheaf theory has emerged from Weierstrass' constructions of analytic continuations where the "total sheaf space" $\widehat{O}$ in § x below is an instructive example of a sheaf. During our whole study the $\bar{\partial}$-operator plays a crucial role. On a Riemann surface it assigns to every complex valued C^∞-function f the differential form $\bar{\partial}(f)$ of bi-degree $(0, 1)$. The Cauchy-Riemann equations show that $\bar{\partial}(f) = 0$ if and only if f is holomorphic. The *Pompieu Theorem* for planar domains is crucial. This classic result asserts that if Ω is an open subset of $\mathbf{C}$ and $f \in C^\infty(\Omega)$ then there exists $g \in C^\infty(\Omega)$ such that

$$(0.3.1) \qquad \frac{\partial g}{\partial \bar{z}} = f$$

Passing to an open Riemann surface X we shall prove that a similar result holds, i.e. if $\phi \in \mathcal{E}^{0,1}(X)$ is a smooth differential form of bi-degree $(0,1)$, then there exists $f \in C^\infty(X)$ such that

$$\bar\partial(f) = \phi \tag{0.3.2}$$

One can express this by saying that every smooth differential form of bi-degree $(0,1)$ is $\bar\partial$-exact on an open Riemann surface. A result due to Dolbeault implies that the solvability in (0.3.2) for every ϕ is equivalent to the Behnke-Stein Theorem in (*) from the introduction. A notsble factis that the inhomogeneus equation (0.3.2) can be solved for non-smooth ϕ-froms of bi-degree $(0,1)$. For example, given a smooth form ϕ as in (0.3.2) we also take a bounded Lebesgue measurable function ρ on X and get the current $\rho\cdot\phi$ of bi-degree $(0,19$ on X. Then it turnd out that there exists a continuous function f on X such that

$$\bar\partial(f) = \rho\cdot\phi \tag{0.3.4}$$

where the left hand side is the $(0,1)$-current founf via charts where distribution derivatives of f appear.

0.4 The Dirichlet problem. A result due to Rado asserts that every open Riemann surface has a countable topology. The proof relies upon a theorem due to Poincaré and Volterra from the appendix about topology, together with solutions to Dirichlet problems. Here one profits upon Perron's solution for planar domains which has the merit that it extends verbatim to open Riemann surfaces, using the observation that the complex analytic structure on a Riemann surface X leads to an intrinsic notion of subharmonic functions. From this we shall describe how to solve the Dirichlet problem exactly as in the planar case. More precisely, let Ω be a relatively compact open set in X. A point $p \in \partial\Omega$ is regular in the sense of Perron if there exists some open chart D in X which contains p and a continuous subharmonic function u in D such that

$$u(x) < 0 : \ x \in U \cap D \quad : u(p) = 0$$

0.4.1 Theorem. *If Ω is a relatively compact open set and each boundary point is Perron regular, then the Dirchlet problem has a solution for every $g \in C^0(\partial\Omega)$, i.e. there exists a harmonic function G in Ω with continuous boundary values equal to g.*

Let us finish this introduction by with some preliminary considerations which illustrate the general theory.

0.5. Additive representations of fundamental groups in the planar case. Each open and connected set U in $\mathbf{C}$ yields an open Riemann surface. A special case is to take $U = \mathbf{C}\setminus(a_1,\ldots,a_k)$ where $a_1,\ldots,a_k$ is a finite set of points which have been removed from the complex plane. Residue calculus can be used to describe some topological properties of U. For each $1 \leq \nu \leq k$ the zero-free holomorphic function $\phi_\nu(z) = z - a_\nu$ in U has a logarithmic differential

$$\frac{\partial\phi_\nu}{\phi_\nu} = \frac{dz}{z-a_\nu}$$

This yields a holomorphic 1-form on U which is not ∂-exact, i.e. we cannot find $g \in \mathcal{O}(U)$ such that $\partial(g) = g'(z)dz = \frac{dz}{z-a_\nu}$. The obstruction stems from the multi-valued behaviour of $\log(z-a_\nu)$ and can also be seen by residue calculus. Namely, choose a closed Jordan curve γ which surrounds a_ν, i.e. its winding number is $+1$ as explained in § xx. Then

$$\int_\gamma \frac{dz}{z-a_\nu} = 2\pi i$$

On the other hand the closedness of γ implies that

$$\int_\gamma \partial g = 0 \quad : g \in \mathcal{O}(U)$$

Next, the fundamental group $\pi_1(U)$ can be identified with homotopy classes of closed curves with a common intial and terminal point $p \in U$. Topology teaches that $\pi_1(U)$ is a free group of rank k with generators $\gamma_1,\ldots,\gamma_k$ where each γ_ν is a closed Jordan curve whose winding number with respec to a_ν is $+1$ while its winding number with respect to other a-points are zero. Consider the family of group homomorphisms

$$\rho\colon \pi_1(U) \to \mathbf{Z} \tag{1}$$

where $\mathbf{Z}$ is the additive group of integers. Denote by $\mathcal{O}^*(U)$ the family of zero-free holomorphic functions in U. If γ is a closed curve at p and $\phi \in \mathcal{O}^*(U)$ residue calculus entails that the line integral

$$\frac{1}{2\pi i} \cdot \int_\gamma \frac{\partial\phi}{\phi}$$

is an integer. Recall that elementary integration theory from Chapter II implies that this line integral only depends upon the homotopy class of γ, i.e. upon the image $\{\gamma\}$ in $\pi_1(X)$. Keeping ϕ fixed it follows that

$$\{\gamma\} \mapsto \frac{1}{2\pi i} \cdot \int_\gamma \frac{\partial\phi}{\phi} \tag{2}$$

is a group homomorphism from $\pi_1(X)$ into $\mathbf{Z}$. It turns out that every such group homomorphism is realised by a zero-free ϕ. To see this we take the free basis $\{\gamma_1\}, \ldots, \{\gamma_k\}$ in $\pi_1(U)$ and then every ρ in (1) is determined by a k-tuple of integers $e_\nu = \rho(\{\gamma_\nu\})$. With

$$\phi(z) = \prod (z - a_\nu)^{e_\nu}$$

we see that ρ is equal to the group homomorphis given by (2).

0.6 Line integrals of multi-valued functions. Before the passage to calculus on Reimann surfaces where one contructs complex line integrals, an intermediate step is to evaluate complex line integrals of multi-valued functions. Consider as an example some complex number α which is not an integer and the function $f = z^\alpha$ which is multi-valued in outside the origin. A local branch at $z = 1$ is chosen so that $f(1) = 1$. If $g(z)$ is an analytic function defined in a neighbrhood of the closed unit disc $\{|z| \leq 1\}$ we seek the line integral

$$\int_T g(z) \cdot z^\alpha \, dz \tag{1}$$

where T is the unit circle oriented in the positive sense, i.e. integration moves counter clockwise. It means that (1) is equal to

$$i \cdot \int_0^{2\pi} g(e^{i\theta}) e^{i\alpha\theta} \, e^{i\theta} \, d\theta$$

Here g has a series expnasion $\sum c_n z^n$ which converges when $|z| \leq 1$ and for each non-negative integer one has

$$i \cdot \int_0^{2\pi} e^{in\theta} e^{i\alpha\theta} \, e^{i\theta} \, d\theta = \frac{i}{n+1+\alpha} \cdot (e^{2\pi i\alpha} - 1)$$

Taking a sum over n we evaluate (1). Wirh $g = 1$ we have for example

$$\int_T z^\alpha \, dz = \frac{i}{1+\alpha} \cdot (e^{2\pi i\alpha} - 1) \tag{2}$$

If $0 < r < 1$ and we instead integrate along the circle $T(r) = \{|z| = r\}$ in the counter clockwise sense, the reader can verify that

$$\int_{T(r)} z^\alpha \, dz = \frac{i \cdot r^{1+\alpha}}{1+\alpha} \cdot (e^{2\pi i\alpha} - 1) \tag{3}$$

Notice that (2) and (3) differ even though the closed sucrves T and $T(r)$ are homotopic in the punctured complex plane. So the usual invariance for line integrals of single-valued analytic functions fails. Following original contructions by Riemann one can explain this by regarding the universl covering space of $\widehat{X} = \mathbf{C} \setminus \{0\}$ where z^α becomes a single valued holomoråphic functions but when the closed curves T and $T(r)$ are lifted to $\widehat{X}$ they are no longer closed and have different end-points on $\widehat{X}$ where z^α takes distinict values and this explains why (2) $\neq$ (3).

0.7 An extension to Riemann surfaces. An open Riemann surface X has a fundamental group $\pi_1(X)$ and we consider the family $\mathcal{F}$ of group homomorphisms $\rho\colon \pi_1(X) \to \mathbf{Z}$. Identify $\pi_1(X)$ with homotopy classes of closed curves which start and terminate at some point $p \in X$. Denote by $\mathcal{O}^*(X)$ the family of globally defined and zero-free holomorphic functions on X.

0.7.1 Theorem. *For each $\rho \in \mathcal{F}$ there exists $\phi \in \mathcal{O}^*(X)$ such that*

(*) $$\rho(\{\gamma\}) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{\partial \phi}{\phi} \quad : \{\gamma\} \in \pi_1(X)$$

where $\{\gamma\}$ is the homotopy class of a closed curve.

Remark. The proof of Theorem 0.7.1 relies upon the exact sheaf sequence

$$0 \to \mathbf{Z}_X \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0$$

where $\mathbf{Z}_X$ is the sheaf whose sections are ocally constant integer-valued functions and we have used the exponential map which sends holomorphic function f defined in open subsets U of X to $e^{2\pi i f}$ which are sections of the multiplicative sheaf $\mathcal{O}^*$ of zero-free holomorphic functions. Passing to the long exact sequence of cohomology one gets an exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O}(X) \overset{\exp}{\to} \mathcal{O}^*(X) \to H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}) \to \ldots$$

Admitting the vanishing theorem by Behnke and Stein this entails that

(i) $$H^1(X, \mathbf{Z}) \simeq \frac{\mathcal{O}^*(X)}{\exp(\mathcal{O}(X))}$$

Next, we have the sheaf Ω whose sections are holomorphic 1-forms and an exact sheaf sequence

$$0 \to \mathbf{C}_X \to \mathcal{O} \overset{\partial}{\to} \Omega \to 0$$

where $\mathbf{C}_X$ is the sheaf whose sections are locally constant complex-valued functions. Here the Behnke-Stein theorem gives the isomorphism

(ii) $$H^1(X, \mathbf{C}) \simeq \frac{\Omega(X)}{\partial(\mathcal{O}(X)}$$

Next, if $g \in \mathcal{O}^*(X)$ one easily verifies that the holomorphic 1-form $\frac{\partial g}{g}$ is ∂-exact if and only if $g = e^{2\pi i f}$ for some $f \in \mathcal{O}(X)$. Hence (i-ii) give a canonically defined and injective map

(iii) $$H^1(X, \mathbf{Z}) \to H^1(X, \mathbf{C})$$

From this one gets Theorem 0.7.1 via a similar representation for the family $\mathcal{F}_{\mathbf{C}}$ of group homomorphisms from $\pi_1(X)$ into the additive group of complex numbers. More precisely, to each $\rho \in \mathcal{F}_{\mathbf{C}}$ we shall prove that there exists $\omega \in \Omega(X)$ such that

$$\rho(\{\gamma\}) = \frac{1}{2\pi i} \cdot \int_\gamma \omega$$

Here ω is unique up to a ∂-exact holomorphic 1-form and it turns out that ω can be represented as the logarithmic differential of a zero-free holomorphic function if and only if the ρ-function in (*) is integer-valued.

§ 1. Summary of some major constructions and results.

1.1 The sheaf $\widehat{\mathcal{O}}$. An extensive family of open Riemann surfaces arises via a construction due to Weierstrass. For each point $z \in \mathbf{C}$ one has the space $\mathcal{O}(z)$ of germs of analytic functions at z. Weierstrass total sheaf space is defined by

$$\widehat{\mathcal{O}} = \bigcup_{z \in \mathbf{C}} \mathcal{O}(z)$$

One has the projection $\pi\colon \widehat{\mathcal{O}} \to \mathbf{C}$ with inverse fibers $\pi^{-1}(z) = \mathcal{O}(z)$. Now $\widehat{\mathcal{O}}$ becomes a complex manifold where π is locally biholomorphic. Charts consist of pairs (Δ, f) where Δ is an open disc in $\mathbf{C}$ and $f \in \mathcal{O}(\Delta)$ whose corresponding open set in $\widehat{\mathcal{O}}$ are the points $\{(z, f_z)\colon z \in \Delta\}$ where $f_z \in \mathcal{O}(z)$ denotes the germ determined by f at a point $z \in \Delta$.

Every open and connected subset of $\widehat{\mathcal{O}}$ gives an open Riemann surface. As an example one has the Riemann surface attached to the multi-valued function $\log z$. At $z = 1$ we start with the local branch of $\log z$. whose value is zero. Now $\log z$ extends to a multi-valued function in $\mathbf{C} \setminus \{0\}$ and in $\widehat{\mathcal{O}}$ we get the connected open set $\mathcal{L}$ whose points are pairs (p, f) where $p \in \mathbf{C} \setminus \{0\}$ and f is a local branch of $\log z$ at p. Here $\pi\colon \mathcal{L} \to \mathbf{C} \setminus \{0\}$ is surjective where each inverse fiber is a copy of the set of integers which reflects the property that local branches of the log-function differ by integer-multiples of 2π.

Exercise. Show that $\log z$ yields a single-valued holomorphic function $\widehat{\log}$ on the open Riemann surface $\mathcal{L}$ and the map

$$p \to \widehat{\log}(p)$$

from $\mathcal{L}$ into the complex plane $\mathbf{C}$ is biholomorphic. In topology it means that $\mathcal{L}$ can be taken as the universal covering space of the punctured complex plane and (*) gives a first example of the uniformisation theorem applied to the simply connected Riemann surface $\mathcal{L}$.

1.1.1 The space $M_\Omega(z_0)$. Let Ω be a open and connected subset of $\mathbf{C}$ and consider a point $z_0 \in \Omega$. In § xx we introduced the family of germs $M_\Omega(z_0)$ which can be extended in the sense of Weierstrass along every curve γ in Ω with initial point at z_0 and some terminal point z. For a pair (f, γ) with $f \in M_\Omega(z_0)$, the analytic continuation along γ gives the germ the germ $T_\gamma(f) \in \mathcal{O}(z)$ wihch by definition is a point in $\widehat{\mathcal{O}}$. The union of these points as γ varies yields a connected open subset of $\widehat{O}$ denoted by $\widehat{X}_f$ and called the associated Riemann surface of f. Here $\pi\colon \widehat{X}_f \to \Omega$ is surjective and the construction gives a *single-valued* holomorphic function $\widehat{f}$ on $\widehat{X}_f$ where

$$\widehat{f}(T_\gamma(f)) = T_\gamma(f)(z)$$

when γ is a curve with terminal point at z.

Example. Remove 0 and 2 from the complex plane and in $\Omega = \mathbf{C} \setminus \{0, 2\}$ we have the multi-valued function

$$f(z) = \sqrt{z} + \sqrt{2 - z}$$

where one starts with the germ at $z = 1$ for which $\sqrt{1} = \sqrt{2 - 1} = 1$. Under analytic continuation the germs of the square root functions can change sign and from this the reader may verify that $\pi\colon \widehat{X}_f \to \Omega$ is a four-sheeted covering map, i.e. the inverse fibers consist of four points.

1.2 The $\bar\partial$-operator.

The complex analytic structure on a Riemann surface arises via charts (Δ, z) where Δ are open discs in a complex z-plane and equips also X with a structure as an oriented C^∞-manifold whose real dimension is two. Denote by $\mathcal{E}(X)$ the space of complex-valued C^∞-functions on X and $\mathcal{E}^1(X)$ is the space of smooth differential 1-forms. The exterior differential $d\colon \mathcal{E}(X) \mapsto \mathcal{E}^1(X)$ is a sum $\partial + \bar\partial$. More precisely, one has a direct sum decomposition

$$\mathcal{E}^1(X) = \mathcal{E}^{1,0}(X) \oplus \mathcal{E}^{0,1}(X)$$

and the two differentials:

$$\partial\colon \mathcal{E}(X) \mapsto \mathcal{E}^{1,0}(X) \quad : \bar{\partial}\colon \mathcal{E}(X) \mapsto \mathcal{E}^{0,1}(X)$$

Here a differential 1-form ϕ has bi-degree $(1,0)$ if its restriction to a chart (Δ, z) is $a(z)dz$ with $a \in C^\infty(\Delta)$. Similarly, a 1-form ψ has bi-degree $(0,1)$ if $\psi|\Delta = b(z)d\bar{z}$ with $b \in C^\infty(\Delta)$. The Cauchy-Riemann equations entail that the $\bar{\partial}$-kernel on $\mathcal{E}(X)$ consists of holomorphic functions. If U is an open subset of X we denote by $\mathcal{O}(U)$ the set of holomorphic functions in U and by $U \mapsto \mathcal{O}(U)$ one gets the sheaf $\mathcal{O}$. As already announced in the introduction a major result about non-compact Riemann surfaces is the following:

1.2.1 Theorem. *Every $(0,1)$-form on X is $\bar{\partial}$-exact.*

Runge's theorem.

Let Y be an open subset of X. For each compact subset K of Y we get connected open components of $X \setminus K$ and put

$$\mathcal{C}^*(K) = \cup\, \Omega_\alpha$$

where the union is taken over all components Ω_α which are not relatively compact, i.e. the closure $\overline{\Omega}_\alpha$ is not a compact set in X. One says that K is a compact Runge set in Y if

$$\mathfrak{h}(K) = X \setminus \mathcal{C}^*(K)$$

is a compact subset of Y.

1.2.2 Theorem. *For every pair (Y, K) where K is a compact Runge set in Y and each $f \in \mathcal{O}(Y)$ there exists a sequence $\{g_n\}$ in $\mathcal{O}(X)$ such that*

$$\lim_{n\to\infty} |g_n - f|_K = 0$$

where we have taken maximum norms over K.

The two theorems above are proved in § xx and give rise to further results which we describe below.

Weierstrass' Theorem. A divisor on X consists of a discrete sequence of points $\{p_\nu\}$ to which one assigns integers $\{k_\nu\}$ which may be positive or negative. A globally defined and non-constant meromorphic function g gives the principal divisor $D(g)$ where one assigns a positive integer at each zero of g given by its multiplicity. Similarly one assigns negative integers at poles given by their orders.

1.2.3 Theorem. *Every divisor on a non-compact Riemann surface is principal.*

The Mittag-Leffler Theorem. On the Riemann surface X one has the sheaf $\mathcal{O}$ of holomorphic functions and the sheaf $\mathfrak{M}$ whose sections are meromorphic functions. Now there exists the quotient sheaf

$$\mathcal{M} = \frac{\mathfrak{M}}{\mathcal{O}}$$

Sections in $\mathcal{M}$ are special since one only gets contributions at the set of poles of a meromorphic function which over an open subset in X is discrete. Theorem 0.2.1 and the long exact sequence of cohomology entails that one has an exact sequence

$$0 \to \mathcal{O}(X) \to \mathfrak{M}(X) \to \mathcal{M}(X) \to 0 \tag{1.2.4}$$

This surjective maps means that one can assign a Mittag-Leffler distribution in an arbitrary fashion on disrete subsets of X which extends the classic result for planar domains.

1.3 Harmonic functions and the Dirichlet problem.

A real-valued function u on X is harmonic if its restriction to every chart satisfies the Laplace equation. An equivalent condition is that the $(1,0)$-form $\partial(u)$ is $\bar{\partial}$-closed, i.e. a holomorphic 1-form. The space of harmonic functions on X is denoted by $\mathfrak{h}(X)$. We have also the sheaf $\mathfrak{h}_X$ whose sections over open sets U is $\mathfrak{h}(U)$. Let us then consider an open subset Y of X whose boundary ∂Y is compact. For each $h \in C^0(\partial Y)$ one seeks a continuous function H on $\overline{Y}$ where

H is harmonic in Y and equal to h on the boundary. To find H one proceeds exactly as in the planar case using Perron's method. This gives a sufficiency for solutions to the Dirichlet problem for arbitrary continuous boundary value functions. More precisely, a point $p \in \partial Y$ satisfies the Perron condition if we can find a pair (Δ, u) where Δ is a chart with the origin at p and u is a continuous function in $\overline{Y} \cap \Delta$ with the property that u is subharmonic in $Y \cap \Delta$ where it takes strictly negative values, while $u(p) = 0$.

1.3.1 Theorem. *Let Y be an open set in X such that every $p \in \partial Y$ is Perron regular. Then the Dirichlet problem has a solution for every $h \in C^0(\partial Y)$.*

1.3.2 Remark. For brevity we say that Y is Dirichlet regular when the Dirichlet problem has a solution. The proof of Theorem 1.3.1 is verbatim to the planer case and left as an exercise to the reader.

1.4 The uniformisation theorem.

Denote by $\Omega(X)$ the space of globally defined holomorphic 1-forms. We can impose the condition that

(1.4.1)
$$\partial(\mathcal{O}(X)) = \Omega(X)$$

1.4.2 Theorem. *Let X be a non-compact Riemann surface for which (1.4.1) holds. Then X is biholomorphic with the unit disc D or the complex plane* $\mathbf{C}$.

Remark. In the special case when X is a planar domain the result above is Riemann's ordinary mapping theorem for simply connected domains. The extension to an arbitrary open Riemann surface requires extra work which is given in § 5.

1.5 Further results.

The Behnke-Stein theorem extends to the case when $\mathcal{O}$ is replaced by a holomorphic line bundle $\mathcal{L}$ which by definition is a locally free sheaf of $\mathcal{O}$-modules with rank one. Thus

(1.5.1)
$$H^1(X, \mathcal{L}) = 0$$

There also exists the multiplicative sheaf $\mathcal{O}^*$ whose sectios are zero-free holomorphic functions. The exponential map yields an exact sequence of sheaves

$$0 \to \mathbf{Z}_X \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0$$

In § x we prove that $H^1(X, \mathcal{O}^*) = 0$ and the long exact sequence of cohomology gives

(1.5.2)
$$H^2(X, \mathbf{Z}_X) = 0$$

Next, let Ω be the sheaf of holomorphic 1-forms which gives the exact sheaf sequence

$$0 \to \mathbf{C}_X \to \mathcal{O} \overset{\partial}{\to} \Omega \to 0$$

The vanishing of $H^1(X, \mathcal{O})$ gives the exact sequence

(1.5.3)
$$0 \to \mathbf{C}_X \to \mathcal{O}(X) \overset{\partial}{\to} \Omega(X) \to H^1(X, \mathbf{C}_X) \to 0$$

Hence one has the isomorphism

(*)
$$H^1(X, \mathbf{C}_X) \simeq \frac{\Omega(X)}{\partial(\mathcal{O}(X)}$$

When (1.4.1) fails one is led to analyze when a globally defined holomorphic 1-form ω is ∂-exact, i.e. equal to $\partial(f)$ for some $f \in \mathcal{O}(X)$. To analyze when this holds we fix a point $x_0 \in X$ and denote by $\mathcal{F}$ the family of closed and differentiable curves γ with common start and terminal points at x_0. If $\omega = \partial(f)$ it is clear that

(i)
$$\int_\gamma \omega = 0 \quad : \gamma \in \mathcal{F}$$

Conversely, assume that (i) holds for some $\omega \in \Omega(X)$. If x is a point in X we find a differentiable curve $\gamma(x)$ which starts at x_0 and has x as terminal point. Now (i) entails that

(ii) $$\int_{\gamma(x)} \omega$$

only depends on x and not on the specific choice of $\gamma(x)$. This gives $f \in \mathcal{O}(X)$ such that $f(x)$ equals (ii) for every x and $\omega = \partial(f)$. So the vanishing in (i) is a necessary and sufficient condition in order that ω is ∂-exact.

1.5.4 Additive representations of $\pi_1(X)$. Recall that the fundamental group of the manifold X can be identified with homotopy classes of closed curves in the family $\mathcal{F}$. Exactly as in the planar case one verifies that the integrals in (i) only depend upon the homotopy classes. So if ω is kept fixed we get a group homomorphism from $\pi_1(X)$ into the abelian group $\mathbf{C}$ defined by

(i) $$\{\gamma\} \mapsto \int_\gamma \omega$$

Conversely one applies (1.5.3) to show that *every* group homomorphism $\rho\colon \pi_1(X) \to \mathbf{C}$ is given by (i) for some holomorphic 1-form ω.

§ 1.6. About the proofs.

Theorem 1.3.1 plays a crucial role where Perron's solution for planar domains can be carried out directly on a manifold by repeating the proof for planar domains. To begin with Theorem 1.3.1 is used to prove that the topology of X is countable which means that there exists denumerable increasing sequence of compact subsets $\{K_n\}$ where K_n is contained in the interior of K_{n+1} for every n. The proof is given in tbe Appendix where the crucial point is to apply Theorem 1.3.1 when Y is the open complement in X when one has removed a pair of closed discs in separate charts.

Another very useful fact is Montel's theorem for planar domains. Namely, let V and U be two open sets in $\mathbf{C}$ where V is relatively compact in U, i.e $\overline{V}$ is a compact subset of U. Then the restriction operator from $\mathcal{O}(U)$ into $\mathcal{O}(V)$ is compact where both are Frechet spaces under uniform convergence over compact subsets of V and U respectively. In § xx we show that this compactness and Pompieu's theorem for planar domains give:

1.6.1 Dolbeault's Lemma. *Let Y be a relatively compact open subset of X. Then, for every open set Y^* which contains $\overline{Y}$ and each $\phi \in \mathcal{E}^{0,1}(Y^*)$, the restriction of ϕ to Y is $\bar\partial$-exact, i.e. one has the inclusion*

$$\phi|Y \in \bar\partial(\mathcal{E}(Y))$$

Remark. We shall see that this result constitutes the essential step to get the Behnke-Stein theorem where the relative compactness is relaxed.

1.6.2 The elliptic property of $\bar\partial$. Recall that if μ is a distribution in some open subset U of $\mathbf{C}$ such that $\partial(\mu) = 0$ then it is a holomorphic density. Passing to a Riemann surface X we consider a current S of bi-degree $(1,0)$, i.e a continuous linear form on test-forms of bi-degree $(0,1)$. The current S is $\bar\partial$-closed in an open subset U of X if

(i) $$S(\bar\partial(g)) = 0 \quad : g \in C_0^\infty(U)$$

The planar result entails that (i) gives a holomorphic 1-form σ in U such that

(ii) $$S(\phi^{0,1}) = \int \phi \wedge \sigma$$

for every $\phi \in \mathcal{E}^{0,1}(X)$ having compact support in U. That $(i) \implies (ii)$ plays a crucial role in the proof of Theorem 1.2.2.

1.6.3 Counting zeros. A substitute for the argument principle in the complex plane to count zeros of analytic functions exists on Riemann surfaces. Let f be holomorphic in an open subset Y of X and suppose that the zeros in Y is a finite set $p_1, \ldots, p_k$, where each p_ν is zero of some multiplicity e_ν. Now $\log|f|^2$ is locally integrable in Y which gives a current defined on test-forms ω of bi-degree $(1,1)$ with compact support in Y which is defined by

(i) $$\int_Y \log|f|^2 \cdot \omega^{1,1}$$

To evaluate (i) we apply the calculus with currents. To begin with we get the current

(ii) $$\partial(\log|f|^2) = \frac{\partial f}{f}$$

whiuch is a holomorphic density outside the zeros and if $(|Delta, z)$ is a chart around a zero p_ν of multiplicity e_ν then the current in (ii) restricted to this chart is $e_\nu \cdot \frac{dz}{z}$. Now we can apply $\bar\partial$ and residue calculus shows that the (1,1)-current

$$\bar\partial\partial(\log|f|^2) = 2\pi i \cdot \sum e_\nu \cdot \delta(p_\nu)$$

where $\{\delta(p_\nu)\}$ are Dirac measures. More precisely, for every test-function $g \in C_0^\infty(Y)$ one has the equality

(iii) $$2\pi i \cdot \sum e_\nu \cdot g(p_\nu) \int_Y \log|f|^2 \cdot \bar\partial\partial(g)$$

1.6.4 A continuity principle. Let $f \in \mathcal{O}(Y)$ and suppose it extends to a continuous function on $\overline{Y}$ and there is some $\delta > 0$ such that $|f(p)| \geq \delta$ hold for each $p \in \partial Y$. For each complex number a with $|a| < \delta$ it follows that $f - a$ has a finite number of zeros in Y. Denote by $e_f(a)$ be the number of zeros counted with multiplicity. Exactly as above we get

(1) $$2\pi i \cdot e(a) = \int \log|f-a|^2 \cdot \bar\partial\partial(g)$$

when g is a test-function in Y which take values one at the zeros of $f - a$. Let $0 < \delta_0 < \delta$ so that the set $\{|f| \leq \delta_0\}$ is compact in Y and fix $g \in C_0^\infty(Y)$ so that $g = 1$ over this compact set. Now we get ther integer-valued function $a \mapsto e(a)$ which for each a is evaluated via (1). At the same time we have the L^1-functions $\{\log|f-a|\}$ and elementary measure theory shows that $a \mapsto \log|f-a|$ is continuous with values in $L^1(Y)$. Passing to their distribution derivatives give a weak-star continuous current-valued function $a \mapsto \bar\partial\partial(\log|f-a|)$ which entails that (1) is a continuous integer-valued function defined in the disc $\{|a| < \delta\}$. Hence this function is constant, i.e. the number of a-values taken by f and counted with multiplicities is constant.

Remark. It is possible to prove the result above by regarding proper maps with finite fibers from one manifold into another. However, the proof given above has the merit that it can be extended to considerably more general situations since we only used the fact that passage to higher order derivatives of distributions is a weak-star continuous process.

Appendix: Topological results.

We restrict the attention to topological spaces X which are Hausdorff. A Hausdorff space X is arc-wise connected if there to every pair of points p, q in X exists a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Next, one says that the Hausdorff space X has a countable topology if there exista a denumerable family $\mathfrak{U}$ of open subsets $\{U_n\}$ such that every open set in X is the union of sets from $\mathfrak{U}$. The following remarkable result is due to Poincaré and Volterra:

A.1 Theorem. *Let X be arc-wise connected and Hausdorff which in addition is locally compact. Suppose there exists a continuous map $f\colon X \to Y$ where Y is a Hausdorff space with countable topology and the fibers $f^{-1}(y)$ are discrete subsets of X for every $y \in Y$. Then X has countable topology.*

Proof. Denote by $\mathfrak{B}$ the family of open sets V in X which satisfy the following:

(i) $\quad$ V has a countable topology

(ii) $\quad$ V is a connected component of $f^{-1}(U)$ for some open set U in Y

Theorem A.1 follows if we prove that $\mathfrak{B}$ is a countable family and every open set in X is a union of $\mathfrak{B}$-sets.

Proof that $\mathfrak{B}$ is countable. Let V_0 be a set in $\mathfrak{B}$ and consider an open set U in Y. The connected components of $f^{-1}(U)$ are disjoint by (i) V_0 has countable topology. It follows that at most countably many of the connected components of $f^{-1}(U)$ have a non-empty intersection with V. Next, since every $V \in \mathfrak{B}$ is of the form $f^{-1}(U)$ for an open set in Y and Y has a countable topology we conclude that

(iii) $\quad$ $V_0 \cap V \neq \emptyset$

only can hold for countably many V in $\mathfrak{B}$. Next, fix $V_* \in \mathfrak{B}$ and set

$$\mathfrak{B}_0 = \{V\colon V_* \cap V \neq \emptyset\}$$

By (iii) this is a countable set in $\mathfrak{B}$. If $n \geq 1$ we define inductively

$$\mathfrak{B}_n = \{V\colon \exists\, W \in \mathfrak{B}_{n-1} : W \cap V \neq \emptyset\}$$

An induction over n entails that each $\mathfrak{B}_n$ is countable and since X is connected it is clear that their union is equal to $\mathfrak{B}$ which therefore is a countable family.

Proof that $\mathfrak{B}$ is a basis of the topology. Let $\Omega \subset X$ be open and take some $x \in \Omega$. We must prove that there exists $V \in \mathfrak{B}$ such that

$$x \in V \subset \Omega$$

To show this we consider the discrete fiber $f^{-1}(f(x))$. Since X is Hausdorff and locally compact there exists a small open and relatively compact neighborhood W of x such that $W \subset \Omega$ and $\partial W \cap f^{-1}(f(x)) = \emptyset$. Now $f(\partial W)$ is a compact set in Y which does not contain $f(x)$ which gives $U \in \mathfrak{U}$ such that

$$f(x) \in U \quad : f(\partial W) \cap U = \emptyset$$

Let V be the connected component of $f^{-1}(U)$ which contains x. From the above $V \cap \partial W = \emptyset$ and we conclude that

$$x \in V \subset W \subset \Omega$$

which finishes the proof.

A.2 Covering spaces.

Let X be arc-wise connected and Hausdorff. A pair (Y, ρ) is a covering of X if $\rho\colon Y \to X$ is a continuous map and for each point $p \in X$ there exists some open neighborhood U such that

$$\rho^{-1}(U) = \cup V_\alpha$$

where the right hand side is a disjoint union of open sets in Y and $\rho\colon V_\alpha \to U$ are homeomorphisms for each α.

A.2.1 Exercise. Let $\gamma\colon [0,1] \to X$ be a continuous map and let $y_0 \in Y$ be a point such that

$$\rho(y_0) = x_0 = \gamma(0)$$

Show that there exists a unique continuous map $\gamma^*\colon [0,1] \to Y$ such that $y_0 = \gamma^*(0)$ and

(i) $$\rho(\gamma^*(t)) = \gamma(t) : \ 0 \leq t \leq 1$$

which means that there exist unique liftings of curves.

Next, let $\pi_1(X)$ be the fundamental group of X. Since X is arc-wise connected it can be identified with homotoy classes of closed curves γ which have a given $x_0 \in X$ as a common initial and terminal point. Let us also fix a point y_0 in the fiber $\rho^{-1}(x_0)$. To each closed γ-curve as above the exercise gives a unique lifted curve γ^* where (i) entials that

(ii) $$\rho(\gamma(1)) \in \rho^{-1}(x_0)$$

A.2.2 Exercise. Show that the end-point $\gamma(1)$ in (ii) only depends upon the homotopy class of γ and conclude that one has a map

(*) $$\rho_*\colon \pi_1(X) \to \rho^{-1}(x_0)$$

Show also that ρ_* is surjective if Y is connected.

A.2.3 The universal covering property. A covering $\phi\colon Z \to X$ has a universal property if the following holds: For every covering $\rho\colon Y \to X$ and each pair of points $z_0 \in \phi^{-1}(x_0)$ and $y_0 \in \rho^{-1}(x_0)$ there exists a unique fiber preserving continuous map $f\colon Z \to Y$ with $f(z_0) = y_0$ and

(**) $$\rho \circ f = \phi$$

Notice that (**) means that f is fiber preserving.

A.2.4 Exercise. Show that a covering with the universal property is unique in the sense that if (Z_1, ϕ_1) and $Z_2, \phi_2)$ is such a pair then there exists a homeomorphism $g\colon Z_1 \to Z_2$ with

$$\phi_1 = \phi_2 \circ g$$

A.2.5 Existence. Show that an arc-wise connected Hausdorff space X has a universal covering $\widehat{X}$ and that the fundamental group $\pi_1(\widehat{X}) = 0$.

A.2.6 The Galois property. A covering $\rho\colon Y \to X$ where Y is connected has the Galois property if there to each $p \in X$ and every pair of points y_1, y_2 in $\rho^{-1}(p)$ exista a fiber presering homeomorphism $f\colon Y \to Y$ with $f(y_1) = y_2$.

A.2.7 Exercise. Verify that the universal covering $\widehat{X}$ has the Galois property.

A.3 Runge sets on Riemann surfaces.

If K is a compact subset of the Riemann surface X we get the conncted components $\{\Omega_\alpha\}$ of the open complement $X \setminus K$. Since X is connected we notice that

(i) $$\partial\Omega_\alpha \subset K$$

hold for every connected component in $X \setminus K$. Next, choose an open set U which contains K where $\overline{U}$ is compact. This is of course possible since K can be covered by a finite number of charts where each chart has a compact closure in X. If a component Ω_α is such that

(ii) $$\partial U \cap \Omega_\alpha = \emptyset$$

the connectivity of X and (i) entail that

(iii) $$\Omega_\alpha \subset U$$

Next, the Heine-Borel Lemma a finite family of $\Omega_{\alpha_1}, \ldots, \Omega_{\alpha_N}$ whose union cover the compact set ∂U. Among these occur a finite family of non-compact sets, say $\Omega_{\alpha_1}, \ldots, \Omega_{\alpha_k}$ where $1 \leq k \leq N$. Set

(iv) $$\mathfrak{h}(K) = K \cup \bigcup_{*} \Omega_\alpha$$

where the *-marked union is taken over all components for which (iii) hold together with the finite family of relatively compact components which intersect ∂U, or equivalently

$$\mathfrak{h}(K) = X \setminus \Omega_{\alpha_1} \cup \ldots \cup \Omega_{\alpha_k}$$

A.3.1 Exercise. Verify that $\mathfrak{h}(K)$ is compact and that

$$\mathfrak{h}(\mathfrak{h}(K)) = \mathfrak{h}(K)$$

A.3.2 Open Runge sets. An open subset Y in X is Runge if

$$K \subset Y \implies \mathfrak{h}(K) \subset Y$$

hold for every compact set K, i.e. the Runge hull of a compact subset of Y stays in Y.

A.3.3 Proposition. *Let K be a compact Runge set in X with a non-empty interior. Let $\Delta_1, \ldots, \Delta_N$ be a finite number of open discs whose union cover ∂K. Then the open set below is Runge:*

$$Y = K \setminus \cup \overline{\Delta}_\nu$$

The easy proof is left as an exercise.

§ 1. Proofs of the main results.

Preliminary results are established in § 2 where Dolbeaulit's Lemma from § xx is crucial. Here the proof requires several steps which starts by a certain finiteness result in Propostion 2.3.2 and after one derives Dolbeault's Lemma via a "tricky construction" which is due to Cauchy and based upon the general fact that if $A\colon V|toV$ is a linear operator on a finite dimensional vector space of some dimension N and

$$\det(\lambda \cdot E_V - A) = \lambda^N + c_{N-1}\lambda^{N-1} + \ldots + c_0$$

then the linear operator

$$A^N + c_{N--1}A^{N-1} + \ldots + c_0 E_V$$

is identically zero. In § 3 we first establish an approximation result where calculis with currents are used. The crucial point is the elliptic property of $\bar\partial$ which implies that if S is a current of bi-degree (1,0) in some open set U of X which is $\bar\partial$-closed, then there exists a holomorphic 1-form σ in U such that

$$S(\omega^{0,1}) = \int_U \omega^{0,1} \wedge \sigma$$

for every test-form $\omega^{0,1}$ of bi-degree $(0,1)$ having compact support in U. At the end of § 3 we deduce the Behnke-Stein theorem from the previous results. In § 4 we derive further results such as Weierstrass' theorem which asserts that every divisor on an open Riemann surface is principal. Another crucial fact is the existence of a globally defined holomorphic 1-form which never is zero.

Armed with previous results we prove the uniformisation theorem in § 5 which is based upon repeated solutions to certain Dirchlet problems in an exhaustion of the simply connected Riemann surface X by relatively compact Perron regular Runge domains. A crucial point during the proof is that the hypothesis

$$\Omega(X) = \partial(\mathcal{O}(X))$$

entails that the same holds for every open Runge set in X.

§ 2. Dolbeault's Lemma.

Let X be an open Riemann surface and Y a relatively compact open subset. Heine-Borel's Lemma gives a finite covering of $\overline{Y}$ by open charts $\{\Delta_\alpha\}$ and hence a finite open covering $\mathfrak{U} = \{U_\alpha\}$ of Y where

$$U_\alpha = \Delta_\alpha \cap Y$$

Construct the Cech complex

$$C^0(\mathfrak{U}, \mathcal{O}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathcal{O}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathcal{O}) \xrightarrow{\delta} \ldots$$

and consider the cohomology group

$$H^1(\mathfrak{U}, \mathcal{O}) = \frac{Z^1(\mathfrak{U}, \mathcal{O})}{\delta(C^0(\mathfrak{U}, \mathcal{O})} \tag{2.1}$$

Pompieu's theorem and Leray's acyclicity theorem for sheaf cohomology entail that (2.1) is independent of the chosen covering and equal to the cohomology group $H^1(Y; \mathcal{O}_X)$. Moreover, the Dolbeault isomorphism from § XX gives:

$$H^1(\mathfrak{U}, \mathcal{O}_X)) \simeq \frac{\mathcal{E}^{0,1}(Y)}{\bar{\partial}(\mathcal{E}(Y))} \tag{2.2}$$

2.3 A finiteness result. Let Y and Y^* be a pair of relatively compact subsets of X where $\overline{Y}$ is a compact subset of Y^*. Let $\{\Delta_\alpha\}$ be a finite covering of $\overline{Y^*}$ by charts and set $U_\alpha = \Delta_\alpha \cap Y^*$ so that $\mathfrak{U}^* = \{U_\alpha\}$ is a finite covering of Y^*. Since $\overline{Y}$ is compact in Y^* there exists some $0 < a < 1$ such that $\{a \cdot \Delta_\alpha\}$ cover $\overline{Y}$ and give a covering $\mathfrak{U}_*$ of Y where

$$V_\alpha = a \cdot \Delta_\alpha \cap Y$$

It is clear that V_α is a compact subset of U_α for every α. Montel's theorem entails that the restriction operators

$$C^p(\mathfrak{U}, \mathcal{O}) \to C^p(\mathfrak{U}_*, \mathcal{O})$$

are compact for each p. These restrictions commute with the Cech coboundary mappings and give a $\mathbf{C}$-linear operator

$$H^1(\mathfrak{U}, \mathcal{O}) \to H^1(\mathfrak{U}_*, \mathcal{O}) \tag{2.3.1}$$

A general result in functional analysis about compact operators between Frechet spaces entails that the range of (2.3.1) is finite dimensional. By (2.2) it follows that the restriction map

$$H^1(Y^*, \mathcal{O}) \to H^1(Y, \mathcal{O}) \tag{2.3.2}$$

has a finite dimensional range. Above Y^* was relatively compact. More generally, let $U \subset X$ be open where $\overline{Y}$ is a compact subset of U. We can find Y^* as above where $Y \subset Y^* \subset U$. Let N be the dimension of the range from (2.3.2). It means that there exists an N-tuple $\{\phi_\nu\}$ in $\mathcal{E}^{1,0}(Y^*)$ and for every other $\phi \in \mathcal{E}^{1,0}(Y^*)$ there exist complex numbers $\{c_k\}$ such that

$$\phi|Y - \sum c_k \cdot \phi_k|Y \in \bar{\partial}\mathcal{E}(Y)$$

This applies when we start from some $\psi \in \mathcal{E}^{1,0}(U)$ and take the restriction $\phi = \psi|Y^*$. Hence the range of the map

$$H^1(U, \mathcal{O}) \to H^1(Y, \mathcal{O})$$

is contained in the N-dimensional complex vector space generated by the images of $\{\phi_k|Y\}$ in Dolbeault's quotient space from (2.2) which defines $H^1(Y, \mathcal{O})$. Summing up we have proved

2.3.3 Proposition. *Let Y be a relatively compact open set in X. Then the range of*

$$H^1(U, \mathcal{O}) \to H^1(Y, \mathcal{O})$$

is finite dimensional for every open set U which contains $\overline{Y}$.

2.4 Another finiteness result. Consider some finite open covering $\mathfrak{U} = \{U_\alpha\}$ of X. No assumption is imposed on the open sets U_α. Let Y be a relatively compact subset of X and

$\mathfrak{U}_* = \{V_\alpha\}$ a finite open covering of Y such that $\overline{V}_\alpha$ is a compact subset of U_α for every α. Restrictions give a morphism from the Cech complex $C^\bullet(\mathfrak{U}, \mathcal{O})$ to $C^\bullet(\mathfrak{U}_*, \mathcal{O})$, and exactly as above Montel's theorem implies that the map

$$H^1(\mathfrak{U}; \mathcal{O}) \to H^1(\mathfrak{U}_*; \mathcal{O}) \tag{2.4.1}$$

has a finite dimensional range.

2.4.2 A special case. Let p be a point in X and Y a relatively compact subset. Here p may or may not belong to Y. Choose a chart Δ centered at p. Now $U_0 = X \setminus \{p\}$ and $U_1 = \Delta$ is an open covering X. Let $0 < a < 1$ and put

$$V_0 = Y \setminus \frac{a}{2} \cdot \overline{\Delta} \quad : V_1 = Y \cap a \cdot \Delta$$

where $\frac{a}{2} \cdot \overline{\Delta}$ is the closed disc of radius $a/2$. Here V_0, V_1 cover Y and from the above the restriction map

$$\rho\colon H^1(U_0, U_1; \mathcal{O}) \to H^1(V_0, V_1; \mathcal{O}) \tag{2.4.3}$$

has a finite dimensional range. Notice that

$$C^1(U_0, U_1; \mathcal{O}) = \mathcal{O}(\Delta^*)$$

where Δ^* is the punctured disc wih $z = 0$ removed. To each non-negative integer we have $z^{-j} \in \mathcal{O}(\Delta^*)$. Since $C^2(U_0, U_1; \mathcal{O}) = 0$ each of these holomorphic functions in the punctured disc yields a vector in $H^1(U_0, U_1; \mathcal{O})$ and we apply ρ to get

$$\rho(z^{-j}) \in H^1(V_0, V_1; \mathcal{O})$$

If N is the dimension of the range of ρ we find complex numbers $c_0, \ldots, c_{N-1}$ such that

$$\rho(z^{-N}) = \sum c_j \cdot \rho(z^{-j})$$

This means that there exists a pair $f_0 \in \mathcal{O}(V_0)$ and $f_1 \in \mathcal{O}(V_1)$ such that

$$f_1 = f_0 + \sum c_j \cdot z^{-j} - z^{-N} \quad : z \in V_0 \cap V_1 \cap Y \tag{2.4.4}$$

Hence f_1 and $g_0 = f_0 + \sum c_j \cdot z^{-j} - z^{-N}$ are glued in $V_0 \cap V_1$ and give together a meromorphic function g in Y which is holomorphic in $Y \setminus \{p\}$ and has a pole of order N at p when $p \in Y$.

2.5 Exercise. Use the above to show that when Y is relatively compact then there exists an open neighborhood U of $\overline{Y}$ and some $f \in \mathcal{O}(U)$ whose restriction to every connected component of Y is non-constant. In particular the number of its zeros is finite and the reader should verify that the same holds if we take

$$F = f^N + c_{N-1} f^{N_1} + \ldots + c_1 f + c_0$$

where $N \geq 1$ and $\{c_k\}$ are complex numbers.

2.6 Proof of Dolbeault's Lemma.

Let $Y \subset X$ be relatively compact and Y^* another relatively compact set which contains $\overline{Y}$ as a compact subset. Choose coverings as in § xx to get the restriction map

$$\rho\colon H^1(Y^*, \mathcal{O} \to H^1(Y, \mathcal{O}) \tag{2.6.1}$$

We are going to prove that the ρ-image is zero. To get this we argue by a contradiction. If the image is $\neq 0$ we already know that it has a finite dimension N. Choose $\xi_1, \ldots, \xi_N$ in $C^1(\mathfrak{U}, \mathcal{O})$ such that $\{\rho(\xi_k)\}$ is a basis of the ρ-image. Here we employ Dolbeault's respresentation of cohomology so that each $\xi_k \in \mathcal{E}^{0,1}(Y^*)$. Next, Exercise 2.5 applied with Y^* gives $f \in \mathcal{O}(Y^*)$. Now $f \cdot \xi_k$ belongs to $\mathcal{E}^{0,1}(Y^*)$ for each k and since $\{\rho(\xi_k)\}$ was a basis we find for each fixed k complex numbers $\{b_{kj}\}$ such that

$$f \cdot \xi_k - \sum_{j=1}^{j=N} b_{kj} \cdot \xi_k \,\big|Y \in \bar{\partial}(\mathcal{E}(Y)) \tag{2.6.2}$$

Here $B = \{b_{kj}\}$ is an $N \times N$-matrix and we construct the holomorphic function F in Y^*

$$F = \det(f \cdot E_N - B)$$

The Cayley-Hamilton theorem from linear algebra gives

$$F \cdot \xi_k | Y \in \bar\partial(\mathcal{E}(Y)) \quad : k = 1, \dots, N \tag{2.6.3}$$

Since $\{\rho(\xi_k)\}$ spans the ρ-image it follows that

$$\rho(F \cdot \eta) = 0 \tag{2.6.4}$$

for every cohomology class $\eta \in H^1(Y^*; \mathcal{O})$. Next, f is non-constant on connected components in Y^* and we notice that the same holds for F. Recall also that f even is holomorphic in a neighborhood of Y^* which implies that the zeros of F in Y^* is a finite set at most. Then it is clear that there exists a finite covering $\mathfrak{U}$ via charts in Y^* such that F has no zeros in every non-empty intersection of $\mathfrak{U}$-sets. Let us now consider an arbitrary cohomology class $\eta \in H^1(Y^*, \mathcal{O})$ represented by a Cech cocycle $\{\phi_{\alpha\beta}\}$. Since F has no zeros in non-empty intersections $U_\alpha \cap U_\beta$ we get another Cech cocycle

$$\psi_{\alpha\beta} = \frac{\phi_{\alpha\beta}}{F}$$

Let η^* be its cohomology class. Now $\eta = F \cdot \eta^*$ and (2.6.4) applied with η^* entails that $\rho(\eta) = 0$. Since η was arbitrary the restriction map from $H^1(Y^*; \mathcal{O})$ into $H^1(Y; \mathcal{O})$ is zero.

§ 3. Proofs of Theorem 0.2.1-0.2.2

Let us first consider a pair (Y, K) from Theorem 0.2.2 and Y^* is another relatively compact subset of X such that $\overline{Y}$ is compact in Y^*. First we establish a weaker variant of Theorem 0.2.2 where the subsequent proof is attributed to Malgrange.

3.1 Proposition. *When $f \in \mathcal{O}(Y)$ there exists a sequence $\{g_n\}$ in $\mathcal{O}(Y^*)$ such that $|g_n - f|_K \to 0$.*

Proof. Let μ be a Riesz measure supported by K such that

$$\int \phi \cdot d\mu = 0 \quad : \phi \in \mathcal{O}(Y^*) \tag{i}$$

The Hahn-Banach theorem gives Proposition 3.1 if we show that (i) entails that

$$\int f \cdot d\mu = 0 \quad : f \in \mathcal{O}(Y) \tag{ii}$$

To prove this we construct a current S on X as follows: Apply Dolbeault's Lemma to the relatively compact set Y^*. It follows that for each $\omega^{0,1} \in \mathcal{E}^{0,1}(X)$ there exists $g \in \mathcal{E}(Y^*)$ such that

$$\omega | Y^* = \bar\partial(g) \tag{iii}$$

Here g is unique up to $\mathcal{O}(Y^*)$ so by (i) the integral $\int g \cdot d\mu$ only depends upon ω. Hence there exists a current S on X defined by

$$S(\omega^{0,1}) = \int g \cdot d\mu \tag{iv}$$

with g determined via (iii). Next, consider an open chart Δ contained in $X \setminus K$. If $g \in C_0^\infty(X)$ has compact support in Δ we take $\omega^{0,1} = \bar\partial(g)$ and since $g = 0$ on K one has

$$S(\bar\partial(g)) = 0$$

Since this hold for every pair (Δ, g) as above, S is $\bar\partial$-closed in $X \setminus K$ and the elliptic property of $\bar\partial$ gives a holomorphic 1-form σ in $X \setminus K$ such that

$$S(\omega^{0,1}) = \int \omega^{0,1} \wedge \sigma \tag{v}$$

holds when $\omega^{0,1}$ has compact support in $X \setminus K$. Let us now consider a point $p \in X \setminus \mathfrak{h}(K)$. By the construction of $\mathfrak{h}(K)$ this gives a non-compact open and connected subset Ω of $X \setminus \mathfrak{h}(K)$ which contains p. Since the current S has compact support it follows that

$$\text{(vi)} \qquad \Omega \setminus \mathrm{Supp}(S) \cup K) \neq \emptyset$$

Let q be a point in this set and choose a small open chart Δ centered at q which stays in Ω and is disjoint from the support of S. Then

$$\int \omega^{0,1} \wedge \sigma = 0$$

for every test-form $\omega^{0,1}$ with compact support in Δ. Hence the restriction of σ to Δ is zero. Since σ is a holomorphic 1-form and Ω is connected, analyticity gives $\sigma|\Omega = 0$. In particular $\sigma = 0$ holds in a neighborhood of p which proves the inclusion

$$\text{(vii)} \qquad \mathrm{Supp}(S) \subset \mathfrak{h}(K)$$

Finally, by assumption $\mathfrak{h}(K)$ is a compact subset of Y. So if $f \in \mathcal{O}(Y)$ we can find a test-function g with compact support in Y such that $g = f$ holds in a neighborhood of $\mathfrak{h}(K)$. Then $\bar{\partial}(g)$ vanishes in a neighborhood of $\mathfrak{h}(K)$ so (vii) and the construction of S entail that

$$\text{(viii)} \qquad 0 = S(\bar{\partial} g) = \int_K g \cdot d\mu = \int_K f \cdot d\mu$$

which proves Proposition 3.1.

3.2 Proof of Theorem 0.2.1

By § xx we can exhaust X by an increasing sequence of Runge domains $\{Y_n\}$ where the closure of Y_n is a compact subset of Y_{n+1} for each n. Consider some $\phi \in \mathcal{E}^{0,1}(X)$. Dolbeault's Lemma applies to each pair Y_n, Y_{n+1} and the restriction of ϕ to Y_{n+1}. Hence there exists $f_n \in \mathcal{E}(Y_n)$ such that

$$\text{(3.2.1)} \qquad \bar{\partial} f_n = \phi | Y_n$$

It follows that the restriction to Y_n of $f_{n+1} - f_n$ is holomorphic and since Y_n is Runge and $\overline{Y}_{n-1}$ is a compact subset, Proposition 2.4 entails that $f_{n+1} - f_n$ can be uniformly approximated on Y_{n-1} by functions from $\mathcal{O}(Y_{n+1})$. Hence there exists $g_{n+1} \in \mathcal{O}(Y_{n+1})$ such that the maximum norm

$$||f_{n+1} + g_{n+1} - f_n||_{Y_{n-1}} < 2^{-n}$$

With $f^*_{n+1} = f_{n+1} + g_{n+1}$ we still have

$$\bar{\partial} f^*_{n+1} = \phi | Y_{n+1}$$

Perfoming these modifications inductively with increasing n we obtain a sequence $\{f^*_n\}$ such that (3.2.1) hold for every n and

$$\text{(3.2.2)} \qquad ||f^*_{n+1} - f^*_n||_{Y_{n-1}} < 2^{-n} \quad : \; n = 1, 2, \ldots$$

It follows that $\{f^*_n\}$ converge uniformly on X to a continuous function F which entails that the distributions $\{\bar{\partial} f^*_n\}$ converge to the distribution derivative $\bar{\partial}(F)$. Moreover, since (3.2.1) hold for every n it is clear that

$$\text{(3.2.3)} \qquad \bar{\partial}(F) = \phi$$

holds everywhere in X. Finally, ϕ is a smooth form so the elliptic property of $\bar{\partial}$ implies that F is a C^∞-function. Hence the mapping $\bar{\partial} \colon \mathcal{E}(X) \to \mathcal{E}^{(0,1)}(X)$ is surjective which proves Theorem 0.2.1.

3.3 Proof of Theorem 0.2.2

TO BE GIVEN

§ 4. Further results.

A locally free sheaf of $\mathcal{O}$-modules of rank one is called a holomorphic line bundle. Let $\mathcal{L}$ be such a sheaf. By definition it has local trivialisations, i.e. we can find a locally finite covering $\mathfrak{U}$ by charts such that the restricted sheaf $\mathcal{L}|U_\alpha \simeq \mathcal{O}$. In each non-empty intersection $U_\alpha \cap U_\beta$ we find a zero free function $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ so that the space $\mathcal{L}(V)$ of sections over an open subset V in X consists of a family $\{\xi_\alpha \in \mathcal{O}(U_\alpha \cap V)\}$ satisfying the glueing conditions

$$(*) \qquad \xi_\alpha | U_\alpha \cap U_\beta \cap V = g_{\alpha\beta} \cdot \xi_\beta | U_\alpha \cap U_\beta \cap V$$

Repeating the methods which were used to prove Theorem 0.2.1 one has the result below whose detailed verification is left as an exercise to the reader.

4.1 Theorem. *For every holomorphic line bundle $\mathcal{L}$ one has $H^1(X, \mathcal{L}) = 0$.*

4.2 Example. The sheaf Ω of holomorphic 1-forms is an example of a holomorphic line bundle. Another example is the sheaf Θ_X whose sections are holomorphic vector fields. Notice that

$$\Omega = Hom_{\mathcal{O}}(\Theta_X, \mathcal{O})$$

is the dual line bundle. Theorem 4.1 entails that

$$(4.2.1) \qquad H^1(X, \Omega) = H^1(X, \Theta_X) = 0$$

4.3 Proof of Weierstrass' Theorem

Let D be a divisor on X which means that integers $\{k_\nu\}$ are assigned to a discrete sequence of points $\{p_\nu\}$. We must prove that there exists a globally deefined meromorphic function ψ whose principal divisor is D. To begin with one has:

4.3.1 Exercise. Use a C^∞-partition of the unity to find a function ϕ which is zero-free and C^∞ in the open complement of the discrete set $\{p_\nu\}$, and for each p_ν there is a small open chart Δ_ν centered at p_ν with a local coordinate z_ν where

$$\phi | \Delta_\nu = z_\nu^{k_\nu}$$

Next, we find a locally finite covering $\mathfrak{U} = \{U_\alpha\}$ of charts in X which contains the discs $\{\Delta_\nu\}$ while each U_α which differs from these discs does not contain points in $\{p_\nu\}$. For each U_α which does not contain any point in $\{p_\nu\}$ we take $f_\alpha = 1$ while $f_\alpha = z_\nu^{k_\nu}$ when $U_\alpha = \Delta_\nu$. Then we can write

$$\text{(i)} \qquad \phi | U_\alpha = f_\alpha \cdot \psi_\alpha \quad : \ \psi_\alpha \in C^\infty(U_\alpha)$$

Since ϕ is zero-free outside $\{p_\nu\}$ we see that every ψ-function is zero-free. Since every chart U_α is simply connected there exist $\{\rho_\alpha \in C^\infty(U_\alpha)\}$ such that

$$\text{(ii)} \qquad \psi_\alpha = e^{\rho_\alpha}$$

In each non-empty intersection $U_\alpha \cap U_\beta$ it follows that

$$\text{(iii)} \qquad e^{\rho_\alpha - \rho_\beta} = \frac{f_\beta}{f_\alpha}$$

Here each quotient $\frac{f_\beta}{f_\alpha} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and (iii) entails that the restricted differences

$$\rho_\alpha - \rho_\beta | U_\alpha \cap U_\beta) \in \mathcal{O}(U_\alpha \cap U_\beta)$$

So with $g_{\alpha\beta} = \rho_\alpha - \rho_\beta | U_\alpha \cap U_\beta)$ we get a Cech-cocycle in $H^1(\mathfrak{U}, \mathcal{O})$. The Behnke-Stein theorem entails that it is a δ-coboundary. Hence there exist $\{g_\alpha \in \mathcal{O}(U_\alpha)\}$ such that

$$\text{(iv)} \qquad \rho_\alpha - \rho_\beta | U_\alpha \cap U_\beta) = g_\alpha - g_\beta$$

Taking exponentials in (iv) and using (iii) it follows that

(v) $$e^{g_\alpha} \cdot f_\alpha = e^{g_\beta} \cdot f_\beta$$

hold in $U_\alpha \cap U_\beta$. This gives a globally defined meromorphic function ψ such that

(vi) $$\psi|U_\alpha = e^{g_\alpha} \cdot f_\alpha$$

Finally, the construction of the f-functions implies that the principal divisor $D(\psi)$ is equal to the given divisor D which finishes the proof of Weierstrass' theorem.

4.4 The construction of a non-vanishing holomorphic form. By § xx there exists a non-zero global section ω in $\Omega(X)$. If (Δ, z) is a chart we have

$$\omega|\Delta = g(z) \cdot dz$$

where $g \in \mathcal{O}(\Delta)$ is non-constant and hence has a discrete set of zeros. The locally defined divsors of these g-functions match each other and yield a globally defined divisor $D = \{(p_\nu; k_\nu)\}$ in X. Notice that D is positive, i.e. the integers $k_\nu > 0$. Weierstrass's theoren gives $\psi \in \mathcal{O}(X)$ with $D(\psi) = D$. Then it is clear that

$$\omega^* = \psi^{-1} \cdot \omega$$

yields a holomorphic 1-form whose associated g-functions in local charts are zero-free.

4.5 The Mittag-Leffler theorem.

Announce and Prove !

§ 5. The uniformisation theorem

Consider a non-compact Riemann surface X where every holomorphic 1-form is ∂-exact. By § xx there exists $\omega \in \Omega(X)$ which is everywhere non-zero. So if Y is an open set in X and $\mu \in \Omega(Y)$ there exists $g \in \mathcal{O}(Y)$ such that

(i) $$\mu = g \cdot \omega_Y$$

where ω_Y is the restriction of ω to Y. We use this to prove

5.1 Proposition. *If Y is an open Runge set in X one has*

(5.1.1) $$\Omega(Y) = \partial(\mathcal{O}(Y))$$

Proof. Let μ be as in (i). Consider a point $y_0 \in Y$ and a closed curve γ in Y which has y_0 as initial and terminal point. Since Y is Runge, Theorem 0.2.2 gives a sequence $\{g_n\}$ in $\mathcal{O}(X)$ and the maximum norms $|g_n - g|_\gamma \to 0$. It follows that

$$\int_\gamma \mu = \lim_{n\to\infty} \int_\gamma g_n \cdot \omega$$

Here γ is a closed curve in X and since each $g_n \cdot \omega$ belong to $\Omega(X)$ the hypothesis on X implies that the integrals in the right hand side above are all zero. Hence the line integrals of μ are zero for all closed curves at y_0 and then § xx shows that it is ∂-exact in Y which finishes the proof.

5.2 Semi-local conformal mappings. Let Y be a relatively compact open subset of X which is Runge and Dirichlet regular. Let $y_0 \in Y$ and by §§ there exists $g \in \mathcal{O}(X)$ with a simple zero at y_0 and zero-free in $X \setminus \{y_0\}$. Now $\mathfrak{Re}(g)$ restricts to a continuous function on ∂Y and we find a harmonic function u in Y whose boundary value function is $\log|g|$. The equality (5.1.1) and the remark in § xx entail that u has a harmonic conjugate v in Y which gives $u + iv \in \mathcal{O}(Y)$. Set

$$f = g \cdot e^{-(u+iv)}$$

On ∂Y it follows that

$$|f| = |g| \cdot e^{-\log|g|} = 1$$

At the same time f has a simple zero at y_0 and is zero-free in $Y \setminus \{y_0\}$. The result in § xx shows that $f\colon Y \to D$ is biholomorphic. Around y_0 we choose a small chart (Δ, z) and get the non-zero complex derivative $f'(0)$ of f at $z = 0$. With

(4.2.1) $$f^* = \frac{1}{f'(0)} \cdot f$$

we have a biholomorphic mapping from Y into the open disc of radius $|f'(0)|$ and the f^*-derivative at $z = 0$ is one.

5.3 A global construction. By § xx we can exhaust X with an increasing sequence of relatively compact open subsets $\{Y_n\}$ which are Runge and Dirichlet regular. Fix a point $p \in Y_0$ and a local chart (Δ, z) as above. For each $n \geq 0$ (5.2.1) gives some $r_n > 0$ and a biholomorphic map

$$f_n\colon Y_n \to D(r_n)$$

where $f_n(p) = 0$ and $f_n'(0) = 1$. The inverse mapping f_n^{-1} is holomorphic in the open disc $D(r_n)$ and we notice that the complex derivative

(i) $$(f_n^{-1})'(0) = 1$$

Next, to each $n \geq 0$ we have the composed function

$$h_n = f_{n+1} \circ f_n^{-1}$$

which maps $D(r_n)$ into $D(r_{n+1})$ where $h_n(0) = 0$ and (i) gives $h_n'(0) = 1$. Schwarz' inequality implies that

$$r_n \leq r_{n+1}$$

Set

$$R = \lim_{n\to\infty} r_n$$

We shall construct a biholomorphic map $f^*\colon X \to D(R)$ where the case $R = +\infty$ means that $D(+\infty) = \mathbf{C}$. To find f^* we shall pass to certain subsequences of $\{f_n\}$. Consider first the composed functions

$$g_n = \frac{1}{r_0} \cdot f_n \circ f_0^{-1}(r_0 z)$$

which by the choice of r_0 and the inclusion $Y_0 \subset Y_n$ are holomorphic in the open unit disc D of the complex z-plane. It is clear that we have

$$g_n(0) = 0 \quad : g_n'(0) = 1$$

Moreover, g_n is a biholomorphic map from D onto the open set $r_0^{-1} \cdot f_n(Y_0)$. By the result from § xx $\{g_n\}$ is a normal family in the sense lof Montel. Hence we can extract a subsequence $\{g_{n_k}\}$ which converges uniformly on compact subsets of D.

5.4 Exercise. Verify that the convergence of $\{g_{n_k}\}$ above entails that $\{f_{n_k}\}$ converges uniformly over compact subsets of Y_0 to a limit function $f_0^* \in \mathcal{O}(Y)$. Moreover, since each restriction $f_{n_k}|Y_0$ is biholomorphic and the derivatives f_{n_k}' at p all are equal to one, it follows from the result in § xx that $f_0^*\colon Y_0 \to f_0^*(Y_0)$ is biholomorphic.

Next, starting with the sequence

$$g_k = r_1^{-1} \cdot f_{n_k} \circ f_1^{-1}$$

we repeat the arguments above and find a subsequence $\{\phi_{k_j}\}$ which conveges uniformly in the unit disc and returning to the f-sequence, its corresponding subsequence converges uniformly on Y_1 to a limit function f_1^* which yields a biholomorphic map from Y_1 onto $f_1^*(Y_1)$. We continue by an induction over n and obtain biholomorphic maps $\{f_n^*\colon Y_n \to f_n^*(Y_n)\}$ and after the passage to a "diagonal subsequence" we find a subsequen ce $\{f_{n_k}\}$ which converges uniformly on compact subsets of x to a limit function f^* which gives the requested biholomorphic map from X onto $D(R)$.