Contractions

A bounded linear operator A on the Hilbert space $\mathcal{H}$ is called a contraction if its operator norm is ≤ 1, i.e. if

(1) $$||Ax|| \leq ||x|| \quad : \quad x \in \mathcal{H}$$

This means that if $x \in \mathcal{H}$ then

(2) $$\langle Ax, Ax \rangle \leq ||x||^2 = \langle x, x \rangle$$

Let E be the identity operator on $\mathcal{H}$. Now $E - A^*A$ is a self-adjoint operator and we see that (2) entails

$$\langle x - A^*Ax, x \rangle = ||x||^2 - ||Ax||^2 \geq 0$$

So this self-adjoint operator is non-negative and by § xx it has a square root, i.e. there exists the self-adjoint operator

$$B_1 = \sqrt{E - A^*A}$$

Next, recall that the operator norms of A and its adjoint A^* are the same so A^* is also a contraction and the bidaiöity formula $A^{**} = A$ entails that we get another self-adjoint operator

$$B_2 = \sqrt{E - AA^*}$$

Since we have not assumed that $AA^* = A^*A$ the two self-adjoint operators B_1, B_2 need not be equal. However, the follwonig hold:

Propostion. *One has the equalities*

$$AB_1 = B_2A \quad \text{and} \quad A^*B_2 = B_1A^*$$

Proof. If n is a positive integer we notice that

(i) $$A(A^*A)^n = (AA^*)^nA$$

Now A^*A is a self-adjoint operator whose compact spectrum is confined to the closed unit interval $[0,1]$. if $f \in C^0[0,1]$ is a real-valued continuous function it can be approximated uniformly by a sequence of polynomials $\{p_n\}$ and the operational calculus from § XX yields an operator $f(A^*A)$ such that the perator norms tend to zero, i.e.

$$\lim ||p_n(A^*A) - f(A^*A)|| = 0$$

Since the spectrum of AA^* also is confined to $[0,1]$ it follows that when ewe take the same polynomial sequence $\{p_n\}$ then we get an operator $f(AA^*)$ and

$$\lim ||p_n(AA^*) - f(AA^*)|| = 0$$

Now (i) and the teo limit formulas above entail that

(ii) $$Af(A^*A) = f(AA^*)A$$

In particular we can use the continuous function $f(t) = \sqrt{1-t}$ and then Proposition XX follows.

The unitary operator U_A. On the Hilbert space $\mathcal{H}\times\mathcal{H}$ we define a linear operator U_A represented by the block matrix

$$U_A = \begin{pmatrix} A & B_2 \\ B_1 & -A^* \end{pmatrix}$$

Proposition. *U_A is a unitary operator on $\mathcal{H} \times \mathcal{H}$.*

Proof. For a pair of vectors x, y in $\mathcal{H}$ we must prove the equality

$$\text{(i)} \qquad ||U_A(x \oplus y)||^2 = ||x||^2 + ||y||^2$$

To prove this we first notice that for every vector $h \in \mathcal{H}$ the self-adjointness of B_1 gives

$$\text{(ii)} \qquad ||B_1 h||^2 = \langle B_1 h, B_1 h\rangle = \langle B_1^2 h, h\rangle = \langle h - A^*Ah, h\rangle = ||h||^2 - ||Ah||^2$$

Above the last equality holds since $\langle A^*Ah, h\rangle = \langle Ah, A^{**}h\rangle = ||Ah||^2$ where we used the biduality formula $A = A^{**}$. In the same way we find that

$$\text{(iii)} \qquad ||B_2 h||^2 = ||h||^2 - ||A^*h||^2$$

Next, by the construction of U_A the left hand side in (i) becomes

$$\text{(iv)} \qquad ||Ax + B_2 y||^2 + ||B_1 x - A^* y||^2$$

Using (iii) the first term in (iv)becomes

$$||Ax + B_2 y||^2 = ||Ax||^2 + ||y||^2 - ||A^*y||^2 + \langle Ax, B_2 y\rangle + \langle B_2 y, Ax\rangle$$

By (ii) the second term becomes

$$||B_1 x - A^* y||^2 = ||x||^2 - ||Ax||^2 + ||A^*y||^2 - \langle B_1 x, A^* y\rangle - \langle A^* y, B_x\rangle$$

Adding this we conclude that (i) follows from the equality

$$\text{(v)} \qquad \langle Ax, B_2 y\rangle + \langle B_2 y, Ax\rangle = \langle B_1 x, A^* y\rangle + \langle A^* y, B_x\rangle$$

To get (v) we use Proposition XX which for example gives

$$\langle Ax, B_2 y\rangle = \langle x, A^* B_2 y\rangle = \langle x, B_1 A^* y\rangle = \langle B_1 x, A^* y\rangle$$

where the last equality used that B_1 is self-adjoint. In the same way one verifies that

$$\langle B_2 y, Ax\rangle = \langle A^* y, B_x\rangle$$

and (v) follows.

The Nagy-Szegö theorem. The constructions above yield the following result which is due to Nagy and Szegö

Theorem *For every bounded linear operator A on a Hilbert space $\mathcal{H}$ there exists a Hilbert space $\mathcal{H}^*$ which contains $\mathcal{H}$ and a unitary operator U on $\mathcal{U}_1$ such that*

$$A^n = \mathcal{P} \cdot U^n \quad : \quad n = 1, 2, \ldots$$

where $\mathcal{P}$ is the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}$.

Proof. Take $\mathcal{H}_1 = \mathcal{H} \times \mathcal{H}$ where we have the unitary operator U_A above and let $\mathcal{P}(x, y) = x$ be the projection onto the first factor. By the boock from of U_A we have $A = \mathcal{P}U_A$ and we leave it to the reader to show that the previous constructions imply that $A^n = \mathcal{P} \cdot U^n$ hold for every $n \geq 1$.

A general norm inequality. The Nahy-Szegö resut has an important conseqe-unce. Let A as above be a contraction. If $p(z) = c_0 + c_1 < + \ldots + c_n z^n$ is an arbitrary polynomial with complex coefficients we get the operator $p(A) = \sum c_\nu A^\nu$ and with these notations one has:

Theorem *For every pair $A, p(z)$ as above one has*

$$||p(A)|| \leq \max_{z \in D} |p(z)|$$

where the the maximum in the right hand side is taken on the unit disc.

Proof. Theroem X gives $p(A) = \mathcal{P} \cdot p(U_A)$. Since the orthogonal $\mathcal{P}$-projection is norm decreasing we get

$$||p(A)(\xi)||^2 \leq ||p(U_A)(\xi, 0)||^2$$

Now the operational calculus from § 7 is applied to the unitaty operator U_A which yields a probablity measure μ_ξ on the unit circle such that

$$||p(U_A)(\xi, 0)||^2 = \int_0^{2\pi} |p(e^{i\theta})|^2 \cdot d\mu_\xi(\theta)$$

The right hand side is majorized by $|p|_D^2$ and Theorem XX follows.

Corollary. Let $A(D)$ be the disc algebra. Since each $f \in A(D)$ can be uniformly approximated by analytic polynomials, Theorem X entails that there exists a bounded linear operator $f(A)$ for every contraction A.