Course in advanced analysis

Special topics: Subharmonic and plurisubharmonic functions in the complex domain.

Time and location. The first lecture takes place January 20 : $10^{15} - 12^{00}$ in Lecture room 306 at the department of mathematics SU.

Welcome

Jan-Erik Björk

Contents. The course is foremost adressed to Phd-students interested in complex analysis. A proposed examination procedure is that participants deliver a lecture about a selected topic. The headline *Lectures 4-9* describe the general spirit (or level) of the lectures. Much attention is given to harmonic and subharmonic functions where probabilistic considerations also appear. Here is an example: Denote by Π the family of open and simply connected domains in $\mathbf{C}$ which contain the origin and have area π. Let Ω be in this family and suppose that ℓ is an open interval of the vertical line $\mathfrak{Re}(z) = \rho$ for some $\rho > 0$. Then there exists the harmonic measure $\mathfrak{m}_0^\Omega(\ell)$. which is the probability that a Brownian motion hits a point in ℓ at the first moment when it has reached $\partial\Omega$. Or equivalently, $\mathfrak{m}_0^\Omega(\ell)$ is the value at $z = 0$ of the harmonic function in Ω with boundary value 1 on ℓ and zero on $\partial\Omega \setminus \ell$.

Theorem. *For every simply connected domain $\Omega \in \Pi$ and each pair ρ and ℓ as above one has the inequality*

$$\mathfrak{m}_0^\Omega(\ell) \leq \frac{2}{\pi} \cdot \text{arcsin} e^{-\rho^2}$$

Remark. This theorem is due to Otto Frostman and appears in his thesis devoted to equilibrium measures in potential theory, together with a description of a simply connected domain Ω_ρ which together with an interval ℓ give the equality above. Personally I find a result of this kind both striking and elegant, especially since it is not "intuitively clear" how to choose an extremal domain Ω_ρ in spite of the probabilistic interpretation. A proof will be given during a session devoted to Dirichlet integrals based upon an article by Beurling where more general results appear which can be used to establish upper bounds for harmonic measures and therefore has independent interest in probability theory - such as the "pop-subject" mathematics of finance where evaluations of barrier options rely upon formulas for harmonic measures. In any case, the lectures during the course are adressed to those who are interested to learn about results as above where complex analysis plays a central role.

Functional analysis. Some general facts will be used without hesitation from basic measure theory and functional analysis. In complex analysis one encounters various Frechet spaces and Theorem 1.1 below is often used. Recall that a Frechet space E is a complex vector space equipped with a sequence of semi-norms $\{||\cdot||_n \colon n = 1, 2, \ldots\}$ which become stronger with n, i.e. $||x||_n \leq ||x||_{n+1}$ hold for every n. Frechet's condition is that these semi-norms give a complete metric defined by the distance function

$$d(x, y) = \sum_{n=1}^{\infty} \frac{||x - y||_n}{1 + ||x - y||_n}$$

A subset W of E is bounded if there to each n exists a constant $C(n)$ such that

$$\max_{x \in W} ||x||_n \leq C(n)$$

A linear operator $T \colon E_1 \to E_2$ from one Frechet space to another is compact if it maps every bounded subset in E_1 to a relatively compact set in the metric space E_2.

1.1 Theorem. *Let $S \colon E_1 \to E_1$ be a bounded and surjective linear operator. Then $S + T$ has a closed range with finite codimension for every compact operator $T \colon E_1 \to E_2$.*

A variant goes as follows: Let E_1, E_2, F be three Frechet spaces and consider a pair of linear operators

$$S_1 \colon E_1 \to F \quad : S_2 \colon E_2 \to F$$

where S_1 is continuous and S_2 compact and suppose that the map

$$(e_1, e_2) \mapsto S_1(e_1) + S_2(e_2)$$

is surjective. Then the quotient space $\frac{F}{S_1(E_1)}$ is finite dimensional.

Exercise. Prove the results above, The hint is to apply Baire's category theorem for complete metric spaces.

Spectral theory for linear operators. Tbe pioneering work by Gustav Neumann from 1879 about solutions ot the Dirichlet problem, followed by some further studies by Poincaré, leads to the extension of matrix calculus where matrices with complex elements are replaced by general densely defined linear operators acting on a complex Banach space X. More precisely, the spectrum $\sigma(T)$ of a densely defined linear operator $T \colon X \to X$ is by definition the closed complement of the open set of $\mathbf{C}$ which consists of complex numbers λ for which Neumann's resolvent operator $R_T(\lambda)$ exists. Here $R_T(\lambda)$ is a uniquely determined bounded linear operator on X which satisfies

$$R_T(\lambda)(\lambda \cdot E - T)(x) = x \quad : x \in \mathcal{D}(T)$$

Moreover, the range of $R_T(\lambda)$ is equal to $\mathcal{D}(T)$ and

$$(\lambda \cdot E - T)(R_T R(\lambda)(x) = x \quad : x \in X$$

Above $\mathcal{D}(T)$ is the domain of definition of T and E the identity operator on X. A major result, which stems from Neumann's equation for resolvents, is the operational calculus which is proved by Cauchy's integral formula. For the reader's convenience we have inserted an appendix about Neumann's classic theory since text-books devoted to functional analysis have tendency to "delay" Neumann's theory until a late stage. My personal opinion is that this theory should be exposed immediately after one has introduced the notion of norms on vector spaces in order to illustrate the power of complex analysis.

Contents.

Below follows an outline of the contents during the first 9 lectures which are divided in 3 separate parts. Lecture 1-3 deals with Riemnan surfaces and Lectures 4-5 with the Laplace operator in the complex domain, while Lecture 5-9 are devoted to subharmonic functions of one complex variable. The later part of the course treats several complex variables with special attention about currents and plurisubharmonic functions. We give a rather detailed account about Riemann surfaces, but begin with a survey about material in lecture 4-9.

2nd part. A more detailed account about material related to Riemann surfaces appears at the end of these notes which include proofs of the Behnke-Stein theorem and the uniformisation of open Riemann surfaces. Material about compact Riemann surfaces appear in separate notes starting with an account about doubly period meromorphic functions in the complex plane which serves as an introduction. Here the results about elliptic curves appear which go back to work by Abel, Jacobi and Weierstrass.

Lectures 4-5

.

Here is the set-up for Lecture 4. Let Ω be a bounded and connected domain in $\mathbf{C}$ whose boundary consists of a finite number of pairwise disjoint and differentiable closed Jordan curves. With $p \in \Omega$ kept fixed we consider the continuous function on $\partial\Omega$ defined by

$$q \mapsto \log \frac{1}{|p-q|}$$

Solving the Dirichlet problem we get the harmonic function $u_p(q)$ in Ω where

$$u_p(q) = \log \frac{1}{|p-q|} \,:\, q \in \partial\Omega$$

Greens' function is defined for pairs $p \neq q$ in $\Omega \times \Omega$ by

$$G(p,q) = \log \frac{1}{|p-q|} - u_p(q) \tag{1}$$

It is wellknown that G is non-negative in $\Omega \times \Omega$ and symmetric, i.e. $G(p,q) = G(q,p)$. Next, let $f \in L^2(\Omega)$, i.e f is complex-valued and square integrable over Ω. Set

$$\mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot f(q)\, dq \tag{2}$$

Since

$$\iint_{\Omega\times\Omega} |G(p,q)|^2\, dpdq < \infty$$

the linear operator $f \mapsto \mathcal{G}_f$ is of the Hilbert-Schmidt type and therefore compact on the Hilbert space $L^2(\Omega)$. Moreover, since $\frac{1}{2\pi} \cdot \log \sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator one has:

A.1 Theorem. *For each $f \in L^2(\Omega)$ the Laplacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

$$\Delta(\mathcal{G}_f) = -f \tag{*}$$

Using this one proves that there exists an orthonormal basis $\{\phi_n\}$ in $L^2(\Omega)$ and a non-decreasing sequence of positive real numbers $\{\lambda_n\}$ such that

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots \tag{i}$$

holds in Ω where each ϕ_n extends to a continuous function on the closure with boundary value zero on $\partial\Omega$. Let us remark that (i) means that

$$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n \tag{ii}$$

So $\{\lambda_n^{-1}\}$ are the eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. Eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times. With these notations the result below was proved by Carleman in a lecture during the Scandinavian Congress at Stockholm in 1934:

A.2 Theorem. *For each point $p \in \Omega$ one has the limit formula*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi} \tag{*}$$

Remark. Notice that (*) holds for every domain Ω as above, i.e. independent if its degree of connectivity. Similar asymptotic formulas hold for derivatives of the ϕ-functions. One has for example

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \frac{\partial \phi_k}{\partial x}(p)^2 = \frac{1}{16\pi^2} \tag{**}$$

The proof relies heavily upon analytic function theory where one considers the Dirichlet series:

$$\Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s} \tag{0.4.3}$$

The major result to be proved in Lecture 4 goes as follows:

A.3 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Carleman's proof of Theorem A.3 is very instructive. It relies upon a change of complex contour integrals similar to those employed by Riemann in his studies about the ζ-function, together with a sharp asymptotic estimates of Greens' kernel functions $G(x;y;-\kappa)$ when $\kappa \to +\infty$. The fact that Theorem A.3 gives Theorem A.2 follows via a result from N. Wiener's article *Tauberian theorem* [Annals of Math.1932 which asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers tending to infinity, and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s\to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then

$$\lim_{n\to\infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Time will prevent me from giving detailed proofs of this Tauberian theorem but details appear in my notes devoted to analytic function theory and Wiener's result could be an example for an "examination lecture".

Lecture 5: The heat equation.

In probability theory one considers stochastic processes in continuous time of the Markov type. The time dependent distributions densitites appear as solutions to parabolic PDE-equations. An example is the diffusion process expressed by the stochastic differential equation

$$dX_t = b(X(t),t) \cdot dW$$

where dW is a "white noise" which for small dt is a normally distributed variable with variance $\sqrt{dt}$ and independent of the present stage $X(t)$, i.e. one has independent increments. It means that the random variabe $X(t+dt)$ is infinitesmally equal to the sum of $X(t)$ and $b(X(t),t)\cdot W(\sqrt{dt})$ where $X(t)$ and $W(\sqrt{dt})$ are independent random variables. Taking the Fourier transform of the distribution function $f(x,t)$ of $X(t)$ we get

$$\widehat{f}(\xi;t+dt) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{dt}} \cdot \iint e^{-i\xi(x+b(x,t)u)}) \cdot f(x;t) \cdot e^{-u^2/2dt}\, dxdu + o(dt)$$

where the remainder is small ordo of dt. The variable substitution $u \mapsto \sqrt{dt}\cdot s$ identifies the double integral above with

$$\frac{1}{\sqrt{2\pi}} \cdot \iint e^{-i\xi(x+b(x,t)\sqrt{dt}\cdot s}) \cdot f(x;t) \cdot e^{-s^2/2}\, dxds$$

A Taylor expansion of the complex exponential function $\zeta \to e^{i\zeta}$ at $\zeta = 0$ gives

$$\lim_{dt\to 0} \frac{\widehat{f}(\xi;t+dt).\widehat{f}(\xi,t)}{dt} = -b(x,t)^2/2 \cdot \int \xi^2 \cdot \widehat{f}(\xi;t)\, d\xi$$

Fourier's inversion formula entails that $f(x,t)$ satisfies the PDE

$$\frac{\partial f}{\partial t} = \frac{b(x,t)^2}{2} \cdot \partial_x^2(f) \tag{*}$$

At time $t = 0$ we suppose that X_0 has a given distribution function $f_0(x)$ which therefore gives the initial condition

$$f(x; 0) = f_0(x)$$

where one from assumes that X_0 has a finite variance, i.e.

$$\int x^2 \cdot f_0(x)\, dx < \infty$$

The PDE in (*) is parabolic and basic results for such equations were established by Gevrey, Hadamard and Holmgren around 1900. The prototype is the equation

$$\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2}$$

defined in a strip domain $B = \{-\infty < x < \infty\} \times \{-A < y < A\}$. The favourable case occurs if there exists a constant C such that

$$(**) \qquad |u(x, y)| \leq C \cdot e^{x^2}$$

Then $u(x, y)$ is determined by its restriction $x \mapsto u(x, a)$ for every fixed $-A < a < A$ by the integral formula

$$\text{(i)} \qquad u(x, y) = \frac{1}{2\sqrt{\pi(y - a)}} \int_{-\infty}^{\infty} u(\xi, a) \cdot U(x - \xi, y - a)\, d\xi \quad : a < y < A$$

where $U(x, y)$ is the function which is zero when $y \leq 0$ and

$$U(x, y) = e^{-x^2/4y} \,:\, y > 0$$

When (**) fails one cannot represent u in the trivial fashion above but the question remains if uniqueness hold under a less restrictive condition than (**). A positive answer was given by Erik Holmgren in the article *Sur les solution quasi-analytique d l'équation de chaleur* [Arkiv för matematik. 1924] where he proved that the solution u is uniquely determined by a restriction to some $\{y = a\}$ if there exists a constant $k \geq 0$ such that

$$(***) \qquad |u(x, y]| \leq e^{kx^2 \cdot \log(e + |x|)} \quad : \ (x, y) \in B$$

Leter work by Täcklund demonstrated that Holmgren's uniqueness theorem is essentially sharp. A suggestion for an "examination lecture" is to expose constructions by Beurling who considered solutions satisfying

$$(****) \qquad u(x, y)| \leq e^{\epsilon(|x|) \cdot x^2 \cdot \log(e + |x|)}$$

where $\epsilon(r)$ tends to zero as $r \to +\infty$. When (****) holds Beurling found an integral kernel U^* which serves as a substitute for the standard kernel U in (i). The construction of U_* gives an instructive lesson about the role of analytic functions in PDE-theory.

B. Lecture 6-9

The logarithmic potentials and the Cauchy transform play a central role. Here "pure analysis" intervenes with more geometric constructions while harmonic and subharmonic functions are studied. Let us begin with a typical result which will be covered during lecture 6-9:

A. A theorem by Nevanlinna .

Let Ω be a connected and bounded domain in $\mathbf{C}$ whose boundary consists of p many pairwise disjoint and closed Jordan curves $\Gamma_1, \ldots, \Gamma_p$. On each Γ_ν we consider a function ϕ_ν which takes the value $+1$ on a finite union of intervals and zero on the complementary intervals. So unless ϕ happens to be constant on a single curve Γ_ν we get jumps at common boundary points of intervals where ϕ_ν is constant. The number of such points is an even integer denoted by $2 \cdot k_\nu$ where $k_\nu = 0$

means that ϕ_ν is identically 1 or 0 on Γ_ν. Solving Dirichlet's problem we get a harmonic function $\omega(z)$ whose boundary function is ϕ and in Ω there exists the analytic function

$$g(z) = \frac{\partial \omega}{\partial x} - i \cdot \frac{\partial \omega}{\partial y}$$

Theorem. *The number of zeros of g counted with multiplicities is equal to*

$$k_1 + \ldots + k_p + p - 2$$

Nevanlinna's proof gives an instructive lesson how the argument principle intervenes with level sets of harmonic functions.

B. Subharmonic configurations

Let μ be a Riesz measure in $\mathbf{C}$ supported by a compact null set. Assume that its Cauchy transform

$$\widehat{\mu}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

satisfies an algebraic equation $p_m(z) \cdot \widehat{\mu}^m(z) + \ldots p_1(z) \cdot \widehat{\mu}(z) + p_0(z) = 0$ in $\mathbf{C} \setminus \mathrm{Supp}(\mu)$ where $\{p_\nu(z)\}$ are polynomials. In a joint article by J. Borcea, R. Bögvad and myself, it is proved that the support of μ is a finite union of real-analytic compact Jordan arcs. The proof relies upon a study of piecewise harmonic subharmonic functions. For example, given a finite family of harmonic functions $h_1, \ldots, h_k$ in some open connected domain Ω there exists the subharmonic function u defined as the maximum of this k-tuple. Conversely, suppose that u is subharmonic in Ω where its Riesz measure is supported by some null-set while the restriction of u to every connected component of $\Omega \setminus \mathrm{Supp}(\Delta(u)$ is equal to one of the given h-functions. An example discovered by Borcea and Bögvad shows that this in general need not imply that $u = \max(h_1, \ldots, h_k)$. However, one can prove that this equality holds under the condition that the k-tuple of gradient vectors of the h-functions are extreme points in the convex set they generate in $\mathbf{C}$. When this holds one says that $h_1, \ldots, h_k$ admit a unique subharmonic configuration. Without this condition on the gradient vectors one seeks the family of all subharmonic configurations, i.e. the set of all subharmonic functions u in Ω for which $\mathrm{Supp}(\Delta u)$ is a null set and the restriction of u to every connected component of $\Omega \setminus \mathrm{Supp}(\Delta u)$ is equal to one of the given harmonic h-functions. This leads to some open and problems and it is also natural to try to extend this study to higher dimensions where one instead consider finite families of pluirsubharmonic functions and seek there plurisubharminic configurations. This will be a topic during the second half of the course.

C. Zero sets of subharmonic functions.

Some of the topics are described in § B-E below where logarithmic potentials and the Cauchy transform play a central role. Here "pure analysis" intervenes with more geometric constructions while harmonic and subharmonic functions are studied. Let us describe a typical result which will be covered during lecture 6-9:

A theorem by Nevanlinna. Let Ω be a connected and bounded domain in $\mathbf{C}$ whose boundary consists of p many pairwise disjoint and closed Jordan curves $\Gamma_1, \ldots, \Gamma_p$. On each Γ_ν we consider a function ϕ_ν which takes the value $+1$ on a finite union of intervals and zero on the complementary intervals. So unless ϕ happens to be constant on a single curve Γ_ν we get jumps at common boundary points of intervals where ϕ_ν is constant. The number of such points is an even integer denoted by $2 \cdot k_\nu$ where $k_\nu = 0$ means that ϕ_ν is identically 1 or 0 on Γ_ν. Solving Dirichlet's problem we get a harmonic function $\omega(z)$ whose boundary function is ϕ and in Ω there exists the analytic function

$$g(z) = \frac{\partial \omega}{\partial x} - i \cdot \frac{\partial \omega}{\partial y}$$

Theorem. *The number of zeros of g counted with multiplicities is equal to*

$$k_1 + \ldots + k_p + p - 2$$

Nevanlinna's proof gives an instructive lesson how the argument principle intervenes with level sets of harmonic functions.

Let Ω in $\mathbf{C}$ be a bounded open set and denote by $\mathrm{SH}_0(\Omega)$ the set of subharmonic functions in Ω whose Laplacian is a Riesz measure supported by a compact null set. Every such function u is locally the sum of a harmonic function and a logarithmic potential

$$u(z) = \frac{1}{2\pi} \cdot \int \log|z-\zeta| \cdot d\mu(\zeta)$$

where μ is a non-negative Riesz measure with compact support. This entails that the distribution derivatives $\partial u/\partial x$ and $\partial u/\partial y$ belong to $L^1_{\mathrm{loc}}(\Omega)$. Before the theorem below is announced we need a geometric construction. If U is a bounded open subset of Ω and $\zeta \in U$ we denote by $s(\zeta) > 0$ the largest positive number such that the line segment

$$\ell_\zeta = \{\zeta + x : \; 0 \leq x < s(\zeta)\} \subset \Omega$$

Put

$$(*) \qquad \mathfrak{s}(U) = \bigcup_{\zeta \in U} \ell_\zeta$$

This yields an open set called the forward star domain of U.

Theorem. *Let $g \in SH_0(\Omega)$ and put $K = Supp(\Delta(g))$. Let U be an open subset of $\Omega \setminus K$ with the property that $g = 0$ in U and*

$$\partial g/\partial x(z) < 0, \quad : z \in \Omega \setminus (K \cup U)$$

Then $g = 0$ in $\mathfrak{s}(U)$.

This result is a slight extension of a theorem which appears in a joint article by T. Bergquist and H. Rullgård where it was used to prove that if $Q(z)$ is a monic polynomial of some degeee n, then there exists a unique compact and connected null-set Γ in $\mathbf{C}$ which contains the zeros of Q and the extended complex plane minus Γ is connected. Moreover, and this is the crucial point, there exists a probability measure μ supported by Γ such that the Cauchy transform

$$\widehat{\mu}(z) = \int \frac{d\mu(\zeta)}{z-\zeta}$$

satisfies the equation

$$\widehat{\mu}(z)^n = Q(z)^{-1} \quad : z \in \mathbf{C} \setminus |\Gamma$$

Another Example. Consider an algebraic equation

$$(1.1.4) \qquad y^2 = \frac{p(x)}{q(x)}$$

where q is a monic polynomial with simple zeros of some degree $m+2$ and p is monic with degree m and the polynomials p and q have no common zero. On the attached compact Riemann surface X we get a local branch of y at a point p^* above $x = \infty$ whose series expansion starts with x^{-1}.

Conjecture There exists a unique tree Γ in the complex z-plane such that the single-valued branch of $\sqrt{\frac{p}{q}}$ in $\mathbf{P}^1 \setminus \Gamma$ with the Laurent expansion as above at $z = \infty$ is equal to the Cauchy transform of a probability measure supported by Γ.

Remark. Affirmative answers to this conjecture have been established in special cases by J. Borcea and B. Shapiro and applied to study asymptotic distributions of roots to Lamé equations. To settle the conjecture amounts to contruct a specific branch-cut on the associated hyperelliptic curve. For an eventual proof is its tempting to employ potential-theoretic methods where results by Beurling might be useful, i.e. employ subharmonic principles to settle a free boundary value problem where one minimizes certain energy integrals. To achieve eventual solutions may be "too hard" but at least one may settle specific cases which might be a reasonable and attractive subject for Phd-students interested in this kind of analysis. Let us remark that the proposed problem amounts to construct special triangulations on compact Riemann surfaces X whose (eventual) solutions rely upon interesting variational problems where the Riesz mass of certain subharmonic functions are minimized.

D. The Riemann-Schwarz inequality.

Let X be a non-compact 1-dimensional complex manifold and σ a hyperbolic metric, i.e. in a local chart (U, z) the metric is $e^{w(z)} \cdot d|z|$ where w is subharmonic in U. To each pair of points p, q in X one has the family $\mathcal{C}(p, q)$ of rectifiable curves in X with end-points at p and q. The Riemann-Schwarz inequality asserts that for every pair γ_1, γ_2 in $\mathcal{C}(p, q)$ and each point $\xi \in \gamma_1$ there exists a curve $\alpha \in \mathcal{C}(p, q)$ and a curve β with end-points at ξ and some point $\eta \in \gamma_2$ such that

$$(*) \qquad \sigma(\alpha)^2 + \sigma(\beta)^2 \leq \frac{1}{2}\big(\sigma(\gamma_1)^2 + \sigma(\gamma_2)^2\big)$$

Moreover, the inequality (*) is strict with the exception for a special rhombic situation when X is the unit disc. This inequality gives the existence of unique geodesic curves on X with respect to the σ-metric. The proof relies upon the uniformisation theorem for Riemann surfaces and calculations based upon geometric properties of Möbius transforms in the unit disc.

E. Special topics about subharmonic functions.

E.1. Largest subharmonic minorants. Let F be a real-valued function with values in $[1, +\infty]$ defined in a bounded open set Ω in $\mathbf{C}$. The function is assumed to be upper semi-continuous, i.e. the sets $\{F < a\}$ are open for every $1 \leq a \leq +\infty$. Notice that $F(z) = +\infty$ is allowed on some non-empty and closed subset of Ω. One associates Sjöberg's minorant function

$$\mathcal{S}_F(z) = \max_u u(z)$$

where the maximum is taken over all subharmonic functions u in Ω which are everywhere $\leq F$. In general $\mathcal{S}_F(z)$ fails to be subharmonic but a result due to Beurling gives the following sufficient condition in order that $\mathcal{S}_F$ is subharmonic. Namely, subharmonicity holds if there to every relatively compact set Ω_0 of Ω exists some $\epsilon > 0$ such that

$$\iint_{\Omega_0} [\log F(z)]^{1+\epsilon}\, dxdy < \infty$$

Beurling's proof teaches a good lesson about estimates of mass distributions of $\Delta(u)$ for subharmonic functions which also will appear in the second half of the lectures when plurisubharmonic functions are studied.

E.2. Dirichlet integrals. Let $h(z)$ be a harmonic function in the open unit disc $D = \{|z| < 1\}$ with a finite Dirichlet integral

$$D(h) = \sqrt{\iint_D (h_x^2 + h_y^2)\, dxdy}$$

Using polar coordinates (r, θ) we set

$$M_h(re^{i\theta}) = \int_0^r |\frac{\partial h(se^{i\theta})}{\partial s}|\, ds \quad : M_h^*(\theta) = \lim_{r \to 1} M_h(re^{i\theta})$$

Let h^* be the harmonic extension to D of the boundary value function M_h^*. With these notations a result due to Beurling asserts that

$$D(h^*) \leq D(h)$$

Just as in E.1, Beurling's ingenious proof offers an instructive lesson about subharmonic functions.

E.3 Carleman's differential inequality. Let Ω be a connected open set in $\mathbf{C}$ with the property that for every $x > 0$ the intersection S_x between Ω and a vertical line $\{\mathfrak{Re}\, z = x\}$ is a non-empty bounded set on the real y-line. Each S_x is the disjoint union of open intervals $\{(a_\nu, b_\nu)\}$ and we set

(*) $$\ell(x) = \max_\nu (b_\nu - a_\nu)$$

Next, let $u(x, y)$ be an arbitrary positive harmonic function in Ω which extends to a continuous function on the closure $\bar\Omega$ where $u = 0$ on $\partial\Omega$. Notice that this class can be non-empty since Ω is unbounded. A typical case is that Ω is a sector bordered by a pair of disjoint simple curves which both issue from the origin and tend to inifinity. Set

(1) $$\psi(x) = \log\left(\int_{S_x} u^2(x, y) \cdot dy\right)$$

With these notations the following result was proved by Carleman in 1933.

Theorem. *Given a pair Ω and u as above there exists a constant $C \geq 0$ such that*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + Cx \quad : x > 0$$

The proof is instructive since it teaches how to use subharmonic majorisations and suitable L^2-inequalities of Wirtinger.

E.4 Weighted Runge approximations. Put $z = x + iy$ and $\square = \{-a < x < a\} \times \{-b < y < b\}$. This rectangle contains $\square_+ = \{-a < x < a\} \times \{0 < y < b\}$ and the lower part $\square_- = \square \cap \{y < 0\}$. Let $\phi(z)$ be an analytic function in $\square_+$ which extends continuously to its closure. A classic result, known as Runge's theorem, entails that for every $\delta > 0$ there exists a sequence of polynomials $\{P_n(z)$ such that the maximum norms $|P_n - \phi|_{\square_+}$ tend to zero and at the same time the maximum norms of $|P_n|$ taken over the rectangle $\square(-\delta) = \{-a < x < a\} \times \{-b < y < -\delta\}$ tend to zero. While such an approximation takes place the maximum norms of $\{P_n\}$ taken over $\square$ cannot remain uniformly bounded when $\delta \to 0$. For if this holds, another classic result, known as Montel's theorem, implies that there exists a sequence of polynomials which converge to an analytic function g in $\square$, where this convergence is uniform over compact subsets of $\square$. Here g is zero in $\square(-\delta)$ which by analyticity entails that g is identically zero in $\square$. At the same time $g = \phi$ in $\square_+$ which contradicts the assumption that ϕ is not identically zero.

To compensate for this one introduces the family $\mathcal{W}$ of real-valued continuous functions $\omega(y)$ which are even and positive $y \neq 0$, and decrease to zero as y approaches zero through positive or negative values. An example is $\omega(y) = y^2$ and another is $\omega(y) = e^{-1/y^2}$ which decreases quite rapidly to zero with y. With $\phi(z)$ given as above and $\omega \in \mathcal{W}$ we say that a weighted Runge approximation holds if there exists a sequence of polynomials $\{P_n(z)\}$ such that

(*) $$\lim_{n\to\infty} ||\omega(y)(P_n(z) - \phi(z)||_{\square_+} + ||\omega(y)(P_n(z)||_{\square_-} = 0$$

It turns out that such a weighted approximation exists if $\omega(y)$ tends to zero sufficiently fast. The following was established by Beurling ín the article *Analytic continuations along a linear boundary: Acta mathematica 1953.*

E.4.1 Theorem *The weighted Runge approximation (*) exists if and only if*

(*) $$\int_0^b \log\log\frac{1}{\omega(y)}\, dy = +\infty$$

The proof gives an instructive lesson how to employ subharmonic estimates.

Several complex variables.

Complex analysis in several variables is more involved compared with the 1-dimensional case. However, it is tempting to try to extend 1-dimensinal results to higher dimensions. A basic result of this nature due to Deval and Sibony goes as follows:

Jensen's formula in dimension $n \geq 2$. Let Ω be a bounded and strictly pseudo-convex domain in $\mathbf{C}^n$ with smooth boundary. When $z_0 \in \Omega$ a probability measure μ supported by $\partial\Omega$ is called a Poisson-Jensen measure with respect to z_0 if

$$f(z_0) = \int_{\partial\Omega} f(z) \cdot d\mu(z)$$

hold for all analytic functions f in Ω which extend to be continuous on the closure $\bar{\Omega}$. Given the pair (z_0, μ) the *defect of value* taken at z_0 by pluri-subharmonic functions $u(z)$ of class C^2 is expressed by the difference

$$\text{def}(u) = \int_{\partial\Omega} u(z) \cdot d\mu(z) - u(z_0)$$

In dimension one this defect is an integral of $\Delta(u)$ times the positive density in Ω given by Green's function of the domain. If $n \geq 2$ the result by Deval and Sibony goes as follows:

Theorem. *To each pair (z_0, μ) there exists a current T_Ω of bi-degree $(n-1, n-1)$ such that*

$$(*) \qquad \text{def}(u) = \int_{\Omega} T_\Omega \wedge \bar{\partial}\partial(u)$$

hold for every pluri-subharmonic function u of class C^2 where the right hand side in (*) is the integral of the volume form $T_\Omega \wedge \bar{\partial}\partial(u)$.

Currents in $\mathbf{C}^n$. This is a major issue during the second half of the course. A typical result goes as follows: Let V be a complex analytic set in a complex manifold X with positive codimesion and T a current of bi-degree (p, p) defined in $X \setminus V$ where $1 \leq p \leq \dim(X) = -1$. If T in addition is positive. closed and has locally finite Hausdorff mass of degree $2p$ in neighborhoods of V, it follows that T extends to a positive and closed current in X. During the lectures basic facts about analytic sets in complex manifolds of dimension ≥ 2 with emphasis upon their metric properties will be recalled, including special results for algebraic sets. As an example I will expose a beautiful theorem due to Christer Lech which asserts that if V is an algebraic set in $\mathbf{C}^n$ for some $n \geq 2$, then there exists a polynomial $P(z)$ which is zero on V and a constant C such that

$$\text{dist}(x, V) \leq C \cdot \text{dist}(x, P^{-1}(0) \quad : x \in \mathbf{R}^n$$

where one measures ordinary euclidian distances and $\mathbf{R}^n$ is the set of points in $\mathbf{C}^n$ with real coordinates. The lectures will also cover results from work by Y.T. Siu which lead to the existence of Lelong numbers attached to plurisubharmonic functions. Siu's theorem is exposed in Chapter 2 from Hörmander's text-book in several complex variables and gives an example of a proposed "examination lecture".

Riemann surfaces

.

The first three lectures are devoted to the study of both open and closed Riemann surfaces, i.e connected 1-dimensional complex manifolds. In contrast to later lectures this subject is more oriented towards algebra and the requested complex analysis is "soft". The esssential result is the Pompieu formula which gives solutions to inhomogeneous $\bar{\partial}$-equations together with two residue formulas which are as follows. Let $z = x + iy$ be the complex coordinate in $\mathbf{C}$, and k a positive integer. Then the following two limit formulas hold when $g(z)$ is a test-function of the real variables (x, y):

$$\text{(i)} \qquad \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{g(z)}{z^k}\, d\bar{z} = 0$$

$$\text{(ii)} \qquad \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{g(z)}{z^k}\, dz = 2\pi i \cdot (-1)^{k-1} \cdot \frac{\partial^{k-1}(g)(0)}{(k-1)!}$$

where

$$\partial(g) = \frac{1}{2}(\frac{\partial g}{\partial x} - i \cdot \frac{\partial g}{\partial y})$$

and if $k \geq 2$ we have taken the ∂-derivative of order $k-1$ in (ii). A proof of (i-ii) occurs in a separate appendix A:xx. Calculus based upon currents plays a crucial role during the study of Riemann surfaces. In appendix § Xx we expose the theory about doubly periodic meromorphic functions which can be seen as a modest introduction to the general theory. Here no manifolds appear, i.e. everything relies upon analytic function theory. On the other hand one does not really grasp this special material until general facts about Riemann surfaces have been developed.

Example. Denote by $\mathcal{P}$ the class of meromorphic functions $f(z)$ in $\mathbf{C}$ which are doubly periodic in the sense that

$$f(z) = f(z+1) = f(z+i)$$

Let $\mathcal{G}$ be the set of Gaussian integers, i.e. complex numbers $a+ib$ where a, b are integers, A result due to Abel asserts that for every pair of complex numbers $\{z_\nu = x_\nu + iy_\nu\}$ where $0 < x_\nu < 1$ and $0 < y_\nu < 1$ both hold, there exists $f \in \mathcal{P}$ with a double pole at $z = 0$ and holomorphic outside the Gaussian integers while f has simple zeros of z_1 and z_2 if and only if the sum $z_1 + z_2$ is a Gaussian integer. Even though it is possible to write down an explicit series for the requested meromorphic function when $z_1 + z_2 \in \mathcal{G}$, the full contents of Abel's result which in addition asserts that this condition also is necessary, is not fully understood until one has developed a calculus which involves line integrals of holomorphic 1-forms. Abel's theorem on compact Riemann surfaces which expresses a necesssary and sufficient condition in order that a divisor is principal is a veritable high-light in the whole theory and we shall give a proof which foremost is based upon the calculus with currents on complex manifolds.

Comments about Lecture 1-3.

A gateway to understand the concept of Riemann surfaces is Weierstrass' construction of multi-valued analytic functions. Let Ω be a connected open subset of $\mathbf{C} \setminus \{0, 1\}$ and z_0 a point in Ω. Denote by $M_\Omega(z_0)$ the family of germs of analytic functions $f(z)$ at z_0 which extend analytically along every curve in Ω with z_0 as initial point while its end-point in Ω can be arbitrary. If $f_0 \in M_\Omega(z_0)$ and $z \in \Omega$ we put

$$\widehat{f(z)} = \{T_\gamma(f)(z) \quad : \gamma \in \mathcal{C}(z_0, z)\}$$

Thus, one considers the values at z attained by all analytic continuations of the given germ f_0. Now $M_\Omega(z_0)$ contains a subfamily $M^*_\Omega(z_0)$ characterized by the property that

$$\widehat{f(z_1)} \cap \widehat{f(z_2)} = \emptyset$$

for each pair of distinct points in Ω.

Theorem. *There exists a unique $f_0 \in M^*_\Omega(z_0)$ whose complex derivative $f_0'(z_0)$ is real and positive and*

$$\bigcup_{z\in\Omega} \widehat{f(z)} = D \tag{*}$$

where D is the open unit disc.

Remark.. A consequence is that the universal covering space of Ω is biholomorphic with the open unit disc. This is a special case of the uniformisation theorem for open Riemann surfaces. We remarr that this theorem can be proved without the passage to manifolds, i.e. one employs similar methods as for simply connected domains. Details are presented in my notes on analytic function thery in a chapter devoted to Riemann's mapping theorem.

Currents and sheaves. Some relevant background material appears in § 0-4 below. Let us remark that sheaf theory finds excellent illustrations within complex analysis. The first relevant sheaf was introduced by Weiertrass, namely the total sheaf space $\widehat{\mathcal{O}}$ whose points are pairs (p, f) where $p \in \mathbf{C}$ and f is a germ of an analytic function at p. One has the projection $\pi(p, f) = p$ from $\widehat{\mathcal{O}}$ onto $\mathbf{C}$ which is a local homemorphism and equips $\widehat{\mathcal{O}}$ with a structure as a complex manifold of dimension one. If Ω is a conected subset of $\mathbf{C}$ one has a 1-1 correspondence between open and connected subsets of $\widehat{\mathcal{O}}$ whose π-image is Ω and the familly of multi-valued analytic functions in Ω. So every connected component in $\widehat{\mathcal{O}}$ is an open Riemann surface.

Counting zeros of analytic functions. Here is an example which illustrates how one can use measure theory to grasp results on manifolds. Let f be a holomorphic function on a Riemann surface X and assume it has a finite number of zeros $p_1, \ldots, p_k$ with multiplicities $e_1, \ldots, e_k$. Stokes theorem entails that

$$2\pi i \cdot \sum e_\nu g(p_\nu) = \int_X \log|f|^2 \cdot \bar\partial(\partial(g) \tag{*}$$

for every $g \in C_0^\infty(X)$. This integral formula serves as a substitute for the argument principle which counts zeros of analytic functions in the complex plane. Suppose in addition that there exists some $\delta > 0$ such that the set $\{|f| \leq \delta\}$ is a compact subset of X and fix $g \in C_0^\infty$ so that $g = 1$ on this compact set. For each complex number a with $|a| < \delta$ we apply (*) to $f - a$ and conclude that

$$2\pi i \cdot \sum E(a) = \int_X \log|f - a|^2 \cdot \bar\partial(\partial(g)$$

where $E(a)$ is the number of zeros of $f - a$ counted with multiplicity. At the same time Lebesgue theory shows that the right hand side is a continuous function of a and since $a \to E(a)$ is integer-valued this E-function is constant while $|a| < \delta$. This ressembles Rouche's theorem in the planar case and illustrates the efficiency to employ currents. So one major aim in the lectures is to bring distribution theory together with complex analysis.

Open Riemann surfsces.

A result due to Behnke and Stein asserts that if X is a non-compact Riemann surface then every differential $(0, 1)$-form is $\bar\partial$-exact, i.e. the inhomogeneous $\bar\partial$-equation

$$\omega^{0,1} = \bar\partial(g) \tag{*}$$

has a solution $g \in C^\infty(X)$ for every globally defined and smooth differential form ω of bi-degree $(0, 1)$. After this has been proved, we show that if X is an open Riemann surface where every globally defined holomorphic 1-form is ∂-exact, then X is biholomorphic with the open unit disc or the complex plane. This is the uniformisation theorem for open Riemann surfaces.

About the proof of (*). To begin with, Pompieu's formula in analytic function theory implies that (*) holds for planar domains where open subsets of $\mathbf{C}$ are regarded as open Riemann surfaces. Starting from this, the next step is to prove that (*) also holds for relatively compact open subsets of an open Riemann surfaces and finally an extended version of Runge's theorem gives the Behnke-Stein result above.

The steps during the whole proof are instructive and teaches the lesson that systematic use of currents is very useful. Concerning the uniformisation theorem it is deduced via (*) and solutions to the Dirichlet problems where the existence of solutions is proved just as for planar domains using Perron's method.

Subharmonic functions. A crucial fact is that the class of subharmonic functions on a Riemann surface X is defined. More precisely, if u is a real-valud function of class C^2 on X there exists the 2-current $\bar\partial\partial(u)$ which by defintion is a linear functional on the Frechet space of C^∞-functions g on X defined by

$$\text{(i)} \qquad g \mapsto \int_X g \cdot \bar\partial\partial(u)$$

In a chart Δ with local coordinate $z = x + iy$ we have

$$\bar\partial\partial(u) = \frac{1}{2} \cdot \bar\partial((u'_x - iu'_y) \cdot dz) = \frac{1}{4} \cdot (u''_{xx} + u''_{yy}) \cdot d\bar z \wedge dz$$

Recall that

$$d\bar z \wedge dz = 2i \cdot dx \wedge dy$$

So if g is a test-function with compact support in the chart the right hand side in (i) becomes

$$\frac{i}{2} \int_\Delta g \cdot \Delta(u)\, dxdy$$

where $\Delta(u) = u''_{xx} + u''_{yy}$. The planar condition for u to be subharmonic is that $\Delta(u)$ is a non-negative function. This leads to

Definition. *A real-valued C^2-function u on X is subharmonic if*

$$\mathfrak{Im} \int_X g \cdot \bar\partial\partial(u) \geq 0$$

for every real-valued and non-negative g-function.

Remark. A two-fold application of Stokes Theorem gives

$$\int_X g \cdot \bar\partial\partial(u) = -\int_X \bar\partial(g) \wedge \partial(u) = \int_X \partial(u) \wedge \bar\partial(g) =$$

$$-\int_X u \cdot \partial\bar\partial(g) = \int_X u \cdot \bar\partial\partial(g)$$

This leads to an extension of the class of subharmonic function just as in the planar case. More precisely, a real-valued and locally integrable function u is subharmonic if

$$(*) \qquad \mathfrak{Im} \int_X u \cdot \bar\partial\partial(g) \geq 0$$

for every non-negative and real-valued test-function g.

Compact Riemann surfaces

.

If X is a compact Riemann surface it is not difficult to show that

$$\text{(i)} \qquad \frac{\mathcal{E}^{0,1}(X)}{\bar\partial(\mathcal{E}(X))}$$

is a finite dimensional vector space whose dual space consists of globally defined holomorphic 1-form, i.e. the space $\Omega(X)$. Set

$$\text{(ii)} \qquad k(X) = \dim_{\mathbf{C}} \Omega(X)$$

A fundamental result for compact Riemann surfaces asserts that $k(X)$ is equal to the topological genus number g, i.e. when X is identified with a sphere where a finite number of handles is attached, then this number is equal to $k(X)$. This result goes back to original work by Riemann. The proof which employs currents has the merit that no specific triangulations or constructions of a canonical basis for the homology of X are neeeded. Instead one disposes of a "two-line proof" using general facts due to Leray and Dolbeault which also are used to study complex manifolds in dimension ≥ 2. So in the lectures we prove the equality $k(X) = g$ using the Dolbeault-Leray methods. Let us now describe two major results about compact Riemann surfaces which go back to work by Abel and Weierstrass.

Abel's theorem.

On a compact Riemann surface X the zeros and the poles of a meromorphic function f cannot be assigned in an arbitrary fashion. First f can be identified with a holomorphic map from X onto $\mathbf{P}^1$. This entails that the number of zeros counted with multiplicities is equal to the number of poles counted with their orders. Conversely, let $\{q_1, \ldots, q_s\}$ and $\{p_1, \ldots p_r\}$ be two finite and disjoint sets in X. To the q-points we assign positive integers $m_1, \ldots, m_s$ and to the p-points negative integers $-k_1, \ldots, -k_r$ where

$$m_1 + \ldots + m_s = k_1 + \ldots + k_r$$

One seeks conditions in order that there exists $f \in \mathfrak{M}(X)$ whose associated principal divisor is given by the pair above. To analyze this we introduce the family $\mathcal{C}$ of all currents $\gamma \in \mathfrak{c}^{0,1}$ with the property that

$$(*) \qquad \gamma(\partial(g)) = 2\pi i \cdot \left[\sum m_\nu g(q_\nu) - \sum k_\nu g(p_\nu)\right] \quad : g \in C^\infty(X)$$

With these notations one has:

Theorem. *A pair $\{q_\nu; m_\nu\}$ and $\{p_\nu; -k_\nu\}$ as above is a principal divisor of a meromorphic function f if and only if there exists some $\gamma \in \mathcal{C}$ such that*

$$(*) \qquad \gamma(\omega) = 0 \quad : \omega \in \Omega(X)$$

Remark. In one direction the result follows from residue calculus For suppose that $f \in \mathfrak{M}(X)$ is given. Now $\log |f|^2$ is a locally integrable function on X and therefore defines a distribution. Hence there exists the current

$$\gamma = \bar\partial(\log |f|^2)$$

If $g \in C^\infty(X)$ it follows that

$$\gamma(\partial g) = -\int_X \log |f|^2 \cdot \bar\partial\partial(g)$$

and residue calculus shows that the current γ belongs to $\mathcal{C}$ associated to the principal divisor of f. At the same time $\gamma(\omega) = 0$ hold for each $\Omega \in \Omega(X)$ since these holomorphic forms are $\bar\partial$-closed. The non-trivial part is therefore the converse, i.e. if $\mathcal{C}$ contains some γ such that the vanishing (*) holds in the theorem above, then the divisor expressed as

$$\sum m_\nu \cdot \delta(q_\nu) - \sum k_\nu \cdot \delta(p_\nu)$$

is the principal divisor of a meromorphic function.

Weierstrass points.

Let X be a compact Riemann surface and consider a point $p \in X$. Now one seeks meromorphic functions f on X which are holomorphic in $X\setminus\{p\}$. As we shall see later on there exists such f if we allow a sufficiently high order of the pole at p. To analyze the polar part at p of such meromorphic functions we choose a chart Δ around p with a local coordinate z where p correspoinds to $z=0$ in the z-disc. For small $\epsilon>0$ we get the open complement

$$X_\epsilon = X \setminus \{|z| \geq \epsilon\}$$

If $\omega \in \Omega(X)$ then Stokes Theorem applied to the open domain X_ϵ gives

$$\int_{|z|=\epsilon} f\cdot\omega = 0 \tag{1}$$

In the chart we can write

$$f(z) = f_*(z) + c_1 z^{-1} + \ldots + c_k z^{-k} \tag{2}$$

where f_* is holomorphic in Δ and $\{c_\nu\}$ are complex numbers where $c_k \neq 0$ and k is the order of the ple at p. At the same time one has a series expansion

$$\omega|\Delta = \sum_{\nu=0}^{\infty} a_\nu(\omega)\cdot z^\nu \cdot dz \tag{3}$$

By residue calculus (1) implies that

$$\sum_{\nu=1}^{\nu=k} a_{\nu-1}(\omega)\cdot c_\nu = 0 \tag{*}$$

In § xx we prove the following converse:

Theorem. *Let $c_1,\ldots,c_k$ be complex numbers such that (*) holds for every $\omega \in \Omega(X)$. Then there exists $f \in \mathfrak{M}(X)$ which is holomorphic in $X\setminus\{p\}$ and its negative Laurent series in (2) is given by the c-sequence.*

Conclusion. Suppose that the genus number g of X is ≥ 2 and $\omega_1,\ldots,\omega_g$ is a basis in the complex vector space $\Omega(X)$. This yield a $g\times g$-matrix W_p with elements

$$\xi_{j,\nu} = a_{j-1}(\omega_\nu)\colon 1 \leq j,\nu \leq g$$

If $\det(W_p)=0$ we can solve the homogenous system of linear equations in (*) and together with the theorem it follows that there exists a meromorphic function f whose negative Laurent series at p is of the form $c_1 z^{-1}+\ldots+c_k z^{-k}$ for some $k \leq g$ which means that the order of the pole at p is at most g. To analyze this more closely we shall construct determinant functions in charts.

The divisor $\mathfrak{w}$. Let Δ be a chart in X with local coordinate z. For each $1\leq\nu\leq g$ we can write

$$\omega_\nu|\Delta = h_\nu(z)\cdot dz \quad : h_\nu \in \mathcal{O}(\Delta)$$

In the chart we also have the holomorphic differential operator ∂_z and get new holomorphic functions

$$\rho_{j,\nu}(z) = \frac{1}{(j-1)\,!}\cdot \partial_z^{j-1}(h_\nu) \quad : 1 \leq j,\nu \leq g$$

where we have put $0\,!=1$. Since $h_1,\ldots,h_g$ are $\mathbf{C}$-linearly independent, the holomorphic function

$$\mathcal{W}(z) = \det(\{\rho_{j,\nu}(z)\})$$

is not identically zero in Δ. We refer to $\mathcal{W}(z)$ as Weierstrass determinant function in the chart.

Next, let ζ be a local coordinate in another chart Δ^* where $\Delta^*\cap\Delta\neq\emptyset$. Now

$$d\zeta = \frac{d\zeta}{dz}\cdot dz$$

where $\frac{d\zeta}{dz}$ is holomorphic and zero-free in $\Delta^* \cap \Delta$. In Δ^* we have the holomorphic determinant function $\mathcal{W}(\zeta)$ and matrix calculus gives

(i) $$\mathcal{W}(\zeta) = (\frac{d\zeta}{dz})^{g(g+1)/2} \cdot \mathcal{W}(z)$$

The transition formula (i) implies that the zeros counted with multiplicities of the locally defined Weierstrass determinants are intrinsic, i.e. we obtain a non-negative divisor denoted by $\mathfrak{w}$ which is called the Weierstrass divisor on X. Its degree is the sum of zeros counted with multiplicities and denoted by $\deg(\mathfrak{w})$. The support, i.e. points where Weierstrass determinants are zero is the set of Weierstrass points. The question arises if this set is non-empty and if it holds we would also like to determine the number of points. Details are given in § xx after we have established the Riemann-Roch theorem in § x and a conclusive fact which goes back to work by Weierstrass asserts the following:

Theorem. *For every compact Riemann surface X with genus number $g \geq 2$ one has the equation*

$$\deg(\mathfrak{w}) = g(g+1)(g-1)$$

In particular the support is non-empty if $g \geq 2$ which gives at least one point $p \in X$ and $f \in \mathfrak{M}(X)$ with a pole of order $\leq g$ at p and otherwise it is holomorphic. Moreover, f yields a map from X onto $\mathbf{P}^1$ which is generically k sheeted for some $k \leq g$.

Algebraic function fields

A result due to Weyl shows that every compact Riemann surface X is algebraic, i.e. equal to the family of valuation rings of an algebraic function field K in one variable, which means that K is a finitely generated field extension of $\mathbf{C}$ whose degree of trancendency is one. Riemann started with K and constructed X so that $K = \mathfrak{M}(X)$ where $\mathfrak{M}(X)$ is the field of meromorphic functions on X. The construction of local charts in X relies upon a result due to Puiseux from 1854 which goes as follows: Let Δ be an open disc centered at $\{z = 0\}$ and consider a y-polynomial

$$\phi(z, y) = y^m + q_{m-1}(z)y^{m-1} + \ldots + q_1(z)y + q_0(z) \tag{1}$$

where $\{q_\nu(z)\}$ are holomorphic in an open disc Δ of some radius r centered at the origin and $q_\nu(0) = 0$ for every ν. Assume in addition that ϕ is irreducible in the polynomial ring $\mathcal{O}(\Delta)[y]$. Then there exists a holomorphic function $A(\zeta)$ in a disc $D^* = \{|\zeta| < r^{1/m}\}$ such that

$$\phi(\zeta^m, A(\zeta)) = 0 \quad : \zeta \in D^*$$

We give the proof in the lectures and explain how it is used to contruct local coordinates on the Riemann surface attached to a given algebraic function field. More precisely, one starts wtih an irreducible polynomial $P(x, y)$ and in $\mathbf{C}^2$ we get the affine algebraic curve $S = \{P = 0\}$. Singukar points on S occur when the gradient vector $(P'_x, P'_y) = (0, 0)$ and then one employs the local Puiseux condtruction to obtain local analytic coordinates on the associated Riemann surface X. An example is the cusp curve $\{y^2 = x^3\}$, i.e $P(x, y) = y^2 - x^3$. Here $t \mapsto (t^2, t^3)$ yields a bijective map from the complex t-plane to S. One refers to S as a monomial curve and t serves as a gobal analytic coordinate on S. In general the construction of local coordinastes on X is a bit more involved, especially when one encounters singular points on S where the germ of P is not irreducible. An example is the curve defined by the equation

$$y^8 - (y^2 - x^3)y^3 - x^4) = 0$$

It is singular at the origin and here one must emplly two different Puiseux charts around the origin which entails that the map $X \to S$ no longer is bijective.

§ 0 Intersection numbers.

Let $P(x, y)$ and $Q(x, y)$ be a distinct pair of irreducible polynomials in $\mathbf{C}[x, y]$. The common zero set $\{P = Q = 0\}$ in $\mathbf{C}^2$ is finite and one seeks the number of intersection points counted with multiplicities which gives an integer $\mathfrak{i}(P, Q)$ called the intersection number. Under the hypothesis that

$$\lim_{|x|+|y| \to +\infty} |P(x, y)| + |Q(x, y]| = +\infty \tag{i}$$

one obtains $\mathfrak{i}(P, Q)$ in a "robust manner". To illustrate the computations we suppose that

$$P(x, y) = \prod_{\nu=1}^{\nu=N} (y - \alpha_\nu x) + P_*(x, y) \quad : \deg(P_*) \leq N - 1 \tag{ii}$$

where $\{\alpha_1, \ldots, \alpha_N\}$ are distinct complex numbers. Let X be the compact Riemann surface attached to $\{P = 0\}$. We shall learn that (ii) entails that if x is considered as a meromorphic function on X, then it has N simple poles which appear after we have taken the projective closure of the affine curve $\{P = 0\}$ in $\mathbf{C}^2$. More precisely, in the coordinates of $\mathbf{P}^2$ these simple poles appear at the points $\{p_\nu = (1, \alpha_\nu, 0)\}$. Next, the polynomial $Q(x, y)$ is identified with a meromorphic function on X and we shall learn that $\mathfrak{i}(P, Q)$ is equal to the number of zeros of Q in X counted with multiplicities. Moreover, this integer is equal to the number of poles counted with their orders which often is easier to find. Suppose for example that

$$Q(x, y) = \prod_{j=1}^{j=M} (y - \beta_j x) + Q_*(x, y) \quad : \deg(Q_*) \leq M - 1 \tag{ii}$$

for some integer $M \geq 2$ where $\beta_1, \ldots, \beta_M$ are distinct. If $\alpha_\nu \neq \beta_j$ hold for all pairs, then one easily verifies that Q has a pole of order M at each point p_ν above. This entials that the intersection number is NM. If some equality occurs, say that $\beta_1 = \alpha_1$, then the order of the pole of Q at p_1 is at most $M-1$. In the case when the polynomial

$$x \mapsto P_*(x, \alpha_1 x)$$

has degree $N-1$ while $\deg(Q_*) \leq M-2$, an easy computation shows that Q has a pole of order $M-1$ at p_1 which entails that the "generic intersection number" $N \cdot M$ has decreased with one unit. One can pursue these calculations at "points at inifinity" to recover intersection numbers in quite general situations.

Next, Stokes Theorem implies that the intersectioin number is robust. When (i) holds it turns out that

$$\mathfrak{i}(P, Q) = \mathfrak{i}(P-a, Q-b)$$

for every pair of complex numbers a, b. Moreover, this common number can be computed via multi-residue calculus, i.e.

$$\mathfrak{i}(P, Q) = \lim_{(\epsilon,\delta)\to(0,0)} \int_{\{|P|=\epsilon \,\cap\, Q|=\delta\}} \frac{\partial P}{P} \wedge \frac{\partial Q}{Q}$$

0.1 The curve $y^2 = q(x)$.

Let $q(x)$ be a polynomial of odd degree $2k+1$ with simple zeros $\alpha_1, \ldots, \alpha_{2k+1}$. Suppose also that $q(0) \neq 0$. Now $P(x,y) = y^2 - q(x)$ is an irreducible polynomial and let X be the attached Riemann surface. The local Puiseux constructions show that x viewed as a meromorphic function on X has simple zeros at the poinits $(0, \sqrt{q(0)})$ and $(0, -\sqrt{q(0)})$. Hence the meromorphic function x on X yields a holomorphic map from X onto $\mathbf{P}^1$ which is generically two-sheeted and x has a double pole above the point at infinity in $\mathbf{P}^1$ which corresponds to the point $p_\infty = (1,0,0)$ in $\mathbf{P}^2\}$ where X is imbedded. The meromorphic function y has simple zeros at the points $\{(\alpha_\nu, 0, 1)$ and a pole of order $k+1$ at $(1,0,0)$. Now one determines the topological genus number g of the compact manifold X. It turns out that

(*) $$g = k$$

To prove this by a "direct vizualisation" is not so easy. Instead it can be established via the equation (ii) above which means that we only have to determine the dimension of the complex vector space $\Omega(X)$. The crucial point is that the differential 1-form dx has simple zeros at the points $\{\alpha_\nu, 0, 1)\}$ which entails that

$$\omega_0 = \frac{dx}{y}$$

is a holomorphic 1-form outside the point p_∞. Now x has a pole of order two at this point and hence dx has a pole of order 3 while $\frac{1}{y}$ has a zero of multiplicitiy $2k+1 \geq 3$. It follows that $\omega_0 \in \Omega(X)$ and if $k \geq 2$ we see that

$$\omega_\nu = \frac{x^\nu \cdot dx}{y} \quad : 1 \leq \nu \leq k-1$$

also are holomorphic at p_∞. Another rather easy computation shows that this k-tuple $\{\omega_\nu\}$ is a basis for the complex vector space $\Omega(X)$ and (*) follows.

0.2 The curve $y^k = x^k + 1$.

Let $k = 2m+1$ be odd which gives the irreducible polynomial $P = y^k - x^k - 1$ and X is the associated compact Riemann surface. When x is regarded as a meromorphic function on X it has simple poles at the points $\{p_\nu = (1, e^{2\pi i\nu/k}, 0) : 0 \leq \nu \leq 2m\}$. Next, one verifies by Puiseux that x has a zero of multiplicity k at the points $\{q_\nu = (e^{2\pi i\nu/k}, 0, 1)\}$ This entails that dx vanishes up to order $k-1$ at these k many points. Hence the sheaf Ω_X of holomorphic 1-forms is equal to

$$\mathcal{O}_D \cdot dx$$

where $\mathcal{O}_D$ is the sheaf of meromorphic functions on X whose sections have poles up to order $k-1$ at the points $\{q_\nu\}$ and zeros of multiplicity ≥ 2 at $\{p_\nu\}$. It means that the divisor D has degree

$$(k-1)k - 2k = (k-3)k$$

In § xx we show that the genus number g of X therefore is given by the equation

$$2g - 2 = (k-3)k \implies g = 1 + (2m+1)(m-1)$$

With $k = 3$ one has $g = 1$ which means that X is a torus while the Riemann surface attached to $y^5 = x^5 + 1$ has genus two.

Remark. The results above illustrate the flavour in the calculus on compact Riemann surfaces where results related to sheaves such as Ω_X play a central role.

§ 1 Work by Niels Henrik Abel (1802-1829)

In 1826 Abel presented the article *Remarques sur quelques propriétes génerales d'une classe de fonctions trancendentes*. In addition to a wealth of results, this ingeniuos work paved the way to the contemporary theory about Riemann surfaces. Before we give comments about this we recall that Abel's first major contribution to mathematics was the proof that the general equation of degree 5 cannot be solved by roots and radicals. His proof from 1823 was published in the first volume of Crelle's journal. In addition to the actual result this article includes foundational results about algebraic field extensions. Abel studied the symmetric group with 120 elements and constructed an over-determined system of linear equations which in general fails to have solutions and explains why the general algebraic equation of degree 5 cannot be solved by roots and radicals. An example is

$$x^5 + x + 1 = 0$$

For an exposition of Abel's proof the reader can consult the book *The Abel Legacy* published at the innaguration of the Abel Prize in Oslo. Two years after Abel's decease in 1829, Galois discovered that systematic use of group theory leads to criteria for non-solvability. Let us also remark that Abel's original proof has the merit that today's use of computers can be applied to analyze extensive linear systems and in this way one can decide when an specific algebraic equation has solutions by roots and radicals in situations where the its associated Galois group is not easy to handle.

Abelian differentials and their integrals. Now we discuss integrals on Riemann surfaces from the cited article above. We are given an affine curve $\{F(x,y) = 0\}$ and let $G(x,y)$ be a polynomial of some degree n whose coefficients depend upon indepenent parameters $a_1, \ldots, a_s$. To be specific

$$G(x,y;a) = \sum_{j,k} c_{j,k}(a_1, \ldots, a_s) \cdot x^j y^k$$

where $\{c_{j,k}\}$ are polynomials in the a-variables for each pair (j,k) with $j + k \leq n$. Suppose that F has some degree m wehich implies that the generic intersection $\{F(x,y = G(x,y;a) = 0\}$ consists of $n \cdot m$ many simple points on the Riemann surface attached to F. These points depend on a and consists of an unordered mn-tuple $\{p_\nu(a)\}$ on $\{F = 0\}$. With a fixed point $p_* \in \{F = 0\}$ we construct nm many simple curves $\{\gamma_\nu\}$ where γ_ν has end-points at p_* and $p_\nu(a)$. If ω is a holomorphic 1-form on the Riemann surface of $\{F = 0\}$ we consider the function

$$J(a_1, \ldots, a_s) = \sum \int_{\gamma_\nu} \omega$$

The crucial point is that the right hand side is symmetric in the a-variables. At the same time we can consider small variations of the a-parameters and find that J is locally holomorphic as a function of the a-variables and since the J-function is bounded while $a_1, \ldots, a_s$ varies in $\mathbf{C}^s$ it must be reduced to a constant function. This was Abel's first observation and he also considered more general cases where one integrates a meromorphic 1-form ω while the a-parameters initially are chosen so that the points of intersection above do not meet poles of ω. In this case the $J-$function need not be constant but takes a special form form a general result by Abel which we describe below. Let us first remark that with the language used in "contempoary mathematics" Abel's result asserts that if $T \subset \mathbf{C}^n$ is a hypersurface with k irreducible components defined by irreducible polynomials $P_1, \ldots, P_k$ in $\mathbf{C}[z_1, \ldots, z_n]$, then the integral cohomology of order one over the open set $\mathbf{C}^N \setminus T$ is a free abelian group generated by the k-tuple of ∂-closed 1-forms $\{\frac{\partial P_\nu}{P_\nu}\}$.

The case $n = 2$. Here is a result whose proof is left as an exercise. Consider a pair $R(x, y)$ and $Q(x, y)$ of rational functions in two variables such that the differential 1-form $Rdx + Qdy$ is closed which means that

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial x}$$

Since the polynomial ring $\mathbf{C}[x, y]$ is a unique factorisation domain each rational funtion is the quotient of two polynomials with no common factor. Suppose that the denominators of R and Q are non-zero at the origin in $\mathbf{C}^2$ and set

$$\Phi(x, y) = \int_0^x R(\zeta, y)\, d\zeta + \int_0^y Q(x, \eta)\, d\eta$$

which is defined when (x, y) stays close to the origin.

Theorem *There exists a unique pair F, P where P is a polynomial with $P(0, 0) = 1$ and F a rational function such that the following hold in a neighborhood of the origin in $\mathbf{C}^2$:*

$$\Phi(x, y) = F(x, y) + \log P(x, y)$$

Abel's original work is presented in a truly "modern way" and therefore gives an "up-to-date account of the whole theory. Of course the reader may consult other text-books such as *Théorie functions algébriques et leurs intégrales* by P. Appel E. Goursat (2nd. edition 1929). Here the reader will find several different proofs of Abel's results where algebraic, geometric or analytic considerations are put forward, together with many instructive examples.

Work by Jacobi. Abel's original work was continued by Jacobi where one should mention his famous article *Mémoire sur les fonctions quadruplement périodiques de deux variables* (Creelle. J. vol 13). Here the issue is to find inversion formulas for hyperelliptic integrals which in his work were restricted to the case of genus 2. Jacobi's inversion formulas were later extended to every genus number by Weierstrass in the articles *Zur Theorie der Abelschen Funktionen* (Crelle vol. 47 and vol. 52]

Elliptic integrals.

Let us discuss a specific example in Abel's work. Consider the equation

$$(*) \qquad y^2 = x(x-1)(x-\beta)$$

where the complex number β differs from 0 and 1. This yields a Riemann surface X whose genus number $g = 1$, i.e. an elliptic curve where a global holomorphic 1-form is given by

$$\omega = \frac{dx}{y}$$

Let γ_1^*, γ_2^* be a canonical basis for the homology on X which gives the two period numbers

$$w_k = \int_{\gamma_k^*} \frac{dx}{y}$$

Next, consider a complex line ℓ in the 2-dimensional complex (x, y) space defined by some equation

$$\alpha x + \beta y + \gamma = 0$$

where $\beta \neq 0$. It is clear that $X \cap \ell$ consists of three points whose x-coordinates are roots of the cubic equation

$$(1) \qquad x(x-1)(x-\beta) = \frac{1}{\beta^2}(\alpha x + \gamma)^2$$

One can solve this equation by Cardano's formula. Abel invented another point of view to describe the three intersection points. Suppose the line is chosen so that (1) has three simple roots $\{x_\nu\}$ which determine three points p_1, p_2, p_3 on X where

$$p_\nu = (x_\nu, -\frac{1}{\beta}(\alpha x_\nu + \gamma)$$

Next, starting at the point $p_* = (0,0)$ on X we choose simple curves $\{\gamma_\nu\}$ which join p_* with $\{p_\nu\}$ and integrate ω along each such γ-curve. With these notations, Abel proved the following:

Theorem. *There exists a constant C which depends on X only such that for every line ℓ as above one has an equation*

$$(*) \qquad \sum_{\nu=1}^{\nu=3} \int_{\gamma_\nu} \frac{dx}{dy} = C + m_1 \cdot w_1 + m_2 \cdot m_2$$

where m_1, m_2 is a pair of integers.

Remark. One refers to this as an addition theorem and C is called the Abel constant of the given curve, i.e. here the teo period numbers ω_1, ω_2 together weith C are invariants of the given curve. A notable fact is that the equation (*) implies that two intersection points determine the third. Let us for example explain why p_3 is determined by p_1 and p_2. Choose the line ℓ^* which passes through p_1 and p_2 and let p_3^* be the third point in $\ell^* \cap X$. The equation applied to ℓ^* gives a pair of integers m_1', m_2' and an equation

$$\sum_{\nu=1}^{\nu=2} \int_{\gamma_\nu} \frac{dx}{dy} + \int_{\gamma} \frac{dx}{dy} = C + m_1' + m_2'$$

where γ is a curve from $(0,0)$ to p_3^*. The composed curve $\gamma_3^{-1} \circ \gamma$ joins p_3 with p_3^* and subtracting the equations for the two lines the constant C disappears and we get

$$\int_{p_3}^{p_3^*} \frac{dx}{dy} = (m_1' - m_1)w_1 + (m_2' - m_2)w_2$$

In § xx we explain that this implies that $p_3^* = p_3$ which shows that two of the intersection points determine the third via Abel's addition theorem. Moreover, Abel proved that the equation (*) is not only is necessary, but gives also a *sufficient* condition in order that three given points on the elliptic curve stay on a line. This can be established by algebraic computations based upon Cardano's formula, but Abel's method has a wider range since it applies to intersection points of the elliptic curve above with algebraic curves of arbitrary high degree. For example, Abel's general theory gives the following remarkable result:

Abel's addition theorem. *Let $s \geq 3$ and $p_1, \ldots, p_{3s}$ are $3s$ distinct points on the elliptic curve from (*). Then they all belong to an algebraic curve of degree s if and only if there exists a pair of integers m_1, m_2 such that*

$$(*) \qquad \sum_{\nu=1}^{\nu=3s} \int_{\gamma_\nu} \frac{dx}{dy} = C \cdot s + m_1 \cdot w_1 + m_2 \cdot m_2$$

Where $\{\gamma_\nu\}$ are curves on X which join $(0,0)$ to $\{p_\nu\}$ and C the constant of the preceeding theorem.

A result about hyperelliptic curves. Let $g \geq 1$ and consider a Riemann surface X defined by

$$y^2 = p(x)$$

where $p(x)$ is a monic polynomial of degree $2g+2$ with simple zeros $\{c_\nu\}$. As we shall explain later, X has genus g and a basis for $\Omega(X)$ consists of the 1-forms

$$\frac{x^k dx}{y} \quad : \quad 0 \leq k \leq g-1$$

Let us now consider polynomials $Q(x)$ of the form

$$Q(x) = x^{g+1} + a_g x^g + \ldots + a_1 x + a_0$$

where $\{a_\nu\}$ is an arbitary $(g+1)$-tuple of complex numbers. As will be explained in § x the number of intersection of the two curves $\{Q = 0\}$ and $\{u^2 - p(x) = 0\}$ when x is finite consists of $2g+1$ points on X counted with multiplicity. Next, let $\{\xi_\nu = (x_\nu, y_\nu)\}$ be some $g+1$-tuple of points on X where the x-coordinates are finite. Above we dispose $g+1$ many a-parameters. They can be choosen so that the intersection above contains $\xi_1, \ldots, \xi_{g+1}$ together with another p-tuple of points $\{\xi_{g+2}, \ldots, \xi_{2g+1}\}$ on X. For the generic choice of the first $(g+1)$-tuple of ξ-points the a-tuple is unique and the last g-tuple depends only upon the assigned $(g+1)$-tuple. Set

$$\eta_j = \xi^*_{g+1+j} \quad : \quad 1 \leq j \leq g$$

where the *-sign means that when $\xi_{g+1+j} = (x, u)$ in $\mathbf{C}^2$ then we take the *-marked point changes u to $-u$, i.e. the η–-point becomes $(x, -u)$. With these notations Abel proved the following:

Theorem. *There exists a constant C such that the following hold for each $\omega \in \Omega(X)$ and every tuple $(\xi_1, \ldots, \xi_{g+1})$*

$$\sum_{k=1}^{k=g+1} \int_{\xi_*}^{\xi_\nu} \omega = C + \sum_{j=1}^{j=g} \int_{\xi_*}^{\eta_j} \omega$$

Remark. As explained above the η-points are algebraic functions of $\xi_1, \ldots, \xi_{g+1}$. Since we have taken a sum of integrals in the right hand side, it is determined by the unordered g-tuple of η-points on X. The theorem constitutes an addition formula since it asserts that the sum of line integrals over a family of $g+1$ curves is equal to a sum over g curves. When $g = 1$ the theorem above corresponds to an equation which was established by Euler and presumably inspired Abel while he developed the theory of general hyperelliptic equations. The case $g = 2$ can solved rather explicitly and the reader may consult [Appel-Goursat: page 423-425] for detailed formulas expressing Abel's addition theorem for hyper-elliptic curves.

§ 2 Root functions and Cauchy transforms.

Tbe theory about Riemann surfaces is quite exploited. However, many open questions related to roots of algebraic functions remain. Here is an example which leads to a proposed "research problem": Let $P(y) = y^n + c_{n-1}y^{n-1} + \ldots + c_2 y^2 + y$ be a polynomial without constant term where $n \geq 3$. Outside a finite set Σ in the complex z-plane which contains $\{z = 0\}$ the roots of the equation

(i) $$P(y) = \frac{1}{z}$$

are simple. Let Γ be a tree which contains Σ. It means that Γ is a connected compact set given as the union of a finite number of real-analytic Jordan arcs and the open complement $\Omega = \mathbf{P}^1 \setminus \Gamma$ is simply connected. Now there exists a distinguished single-valued root function $\alpha(z)$ to the equation (i) which is holomorphic in Ω and has a simple zero at the point at infinity, i.e.

(ii) $$\alpha(z) \simeq \frac{1}{z}$$

when the absolute value $|z|$ is large. Since Γ has planar Lebegue measure zero we can identify α with a locally integrable function in $\mathbf{C}$ and take its $\bar{\partial}$-derivative. This gives a complex-valued Riesz measure μ_Γ supported by Γ such that

$$\alpha(z) = \int \frac{d\mu_\Gamma(\zeta)}{z - \zeta}$$

where (ii) entails that $\int d\mu_\Gamma(\zeta) = 1$.

Question. Does there exists a unique tree Γ such that μ_Γ is real-valued and non-negative. Let us remark that this is related to asymptotic properties of roots of eigen-polynomials to ODE-equations after one has taken the Bergquist-scaling into the account. So numerical experiments from the Phd-thesis by T. Bergquist indicate that the answer is affirmative. However, no proof - or an eventual counter example - is known. The search for Γ relies on a variational problem, i.e. the issue is to find extremals where free boundaries expressed by trees appear. So this is an example of an open problem where the course does not provide an answer but at least will give background and tools related to the problem.

§ 0. Sheaf theory.

We restrict the study to sheaves whose sections over open sets are complex vector spaces and assume the reader is familiar with the axiomatic definition of sheaves which can be digested during a short coffee-break. However, the general notion of sheaves is extremely useful. Especially when it is combined with constructions of cohomology which in the general context is due to Leray from his pioneering work during the years 1943-1949. For the student in complex analysis basic familarity with sheaves and constructions of various cohomology groups should be considered as equally basic as integral formulas related to Stokes theorem. The generality in Leray's theory makes many proofs trivial, while the applications often are quite remarkable. We give examples of this in § 0.1. In our applications we consider a paracompact manifold X of some real dimension $n \geq 1$. It means that every open covering has a locally finite subcovering, and whenever $\{U_\alpha\}$ is a locally finite covering of X there exists a family $\{\phi_\alpha \in C_0^\infty(U_\alpha)\}$ such that $\sum \phi_\alpha = 1$ on X. Next, if $\mathcal{F}$ is a sheaf one constructs the Cech complex

(i) $$0 \to C^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta^0} C^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta^1} \dots$$

To each $p \geq 1$ we set

$$Z^p(\mathfrak{U}, \mathcal{F}) = \mathrm{Ker}(\delta^p)$$

which are called Cech cocycles. The cohomology groups

$$H^p(\mathfrak{U}, \mathcal{F}) = \frac{Z^p(\mathfrak{U}, \mathcal{F})}{\delta^{p-1}(C^{p-1}(\mathfrak{U}, \mathcal{F}))}$$

are called Cech cohomology groups with values in the sheaf $\mathcal{F}$.

Soft resolutions. In the applications during the lectures these cohomology groups can be obtained via soft resolutions. It means that if $\mathcal{E}_X$ is the sheaf of complex-valued C^∞-functions then there exists a sheaf resolution

(ii) $$0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \dots$$

where $\{\mathcal{J}^\nu\}$ are sheaves of $\mathcal{E}_X$-modules. Using C^∞-partitions of the unity one proves that

$$H^p(X, \mathcal{J}) = 0$$

hold for every $p \geq 1$ and every sheaf of C^∞-modules. Moreover, when we stsrt from an arbitrary sheaf $\mathcal{F}$ and takes a soft resolution (ii), then one verifies by "diagram chasing" that the cohomology of the complex

(iii) $$0 \to \mathcal{J}^0(X) \to \mathcal{J}^1(X) \to \dots$$

depend on the sheaf $\mathcal{F}$ only, i.e. not upon the particular choice of a resolution in (ii). These intrinsically defined cohomology groups are denoted by $\{H^p(X, \mathcal{F}) : \ p = 0, 1, \dots\}$. More generally one can restrict $\mathcal{F}$ to an open subset Y and consider a soft resolution over Y which give intrinsically defined cohomology groups $\{H^p(Y, \mathcal{F})\}$.

Exercise. The restriction of a sheaf $\mathcal{J}$ of C^∞-modules to an open subset of Y of X is again a sheaf of C^∞-modules. Use this to explain how one gets natural complex linear maps

$$H^p(X, \mathcal{F}) \to H^p(Y; \mathcal{F})$$

for every $p \geq 1$. Find also examples where such maps fail to be injective or surjective.

0.0 An Edge Map. Assume that $\mathcal{F}$ has a soft resolution and let $\mathfrak{U} = \{U_\alpha\}$ be a finite family of open sets. Put $Y = \cup U_\alpha$. Leray's construction of spectral sequences gives an exact sequence of complex vector spaces

(0.0.1) $$0 \to H^1(\mathfrak{U}, \mathcal{F}) \to H^1(Y, \mathcal{F}) \to \Gamma \to 0$$

where Γ is a direct sum of vector spaces

(0.0.2) $$\bigoplus_\alpha \mathrm{Ker}\big[H^1(U_\alpha, \mathcal{F}) \to H^2(\mathfrak{U}, \mathcal{F})\big]$$

Hence one has the isomorphism

$$H^1(\mathfrak{U},\mathcal{F}) \simeq H^1(Y,\mathcal{F}) \tag{0.0.3}$$

under the condition that $H^1(U_\alpha,\mathcal{F}) = 0$ for each α. One refers to this as Leray's acyclicity theorem.

0.1 Construction of meromorphic functions.

Let X be a Riemann surface and consider the sheaf $\mathcal{O}$ whose sections are holomorphic functions. Suppose that an open subset Y of X is such that $H^1(Y,\mathcal{O})$ is finite dimensional. Fix a point $p \in Y$ and Δ is a chart centered at p with local coordinate z and consider the open covering $\mathfrak{U} = (\Delta, Y \setminus \{p\}$. Here $C^1(\mathfrak{U})$ is reduced to $\mathcal{O}(\Delta \setminus \{p\})$ and since $\mathfrak{U}$ only consists of two sets we have $C^2(\mathfrak{U},\mathcal{O}) = 0$. It follows that

$$H^1(\mathfrak{U},\mathcal{O}) = \frac{\mathcal{O}((\Delta \setminus \{p\}}{\mathrm{Im}(\delta^0)}$$

It means that if ϕ is a holomorphic function in the punctured disc $\Delta \setminus \{p\}$ then its cohomology class is zero if and only if there exists a pair $f \in \mathcal{O}(\Delta)$ and $g \in \mathcal{O}(Y \setminus \{p\})$ such that

$$\phi = g - f$$

Let M the dimension of the complex vector space $H^1(Y,\mathcal{O})$. To each $1 \leq n \leq M+1$ we have the function $z^{-n} \in O(\Delta, Y \setminus \{p\})$. By (0.0.1) the map $H^1(\mathfrak{U},\mathcal{O}) \to H^1(Y,\mathcal{O})$ is injective. It follows from the above that there exists complex numbers $c_1,\ldots,c_{N+1}$ which are not all identically zero such that

$$c_1 z^{-1} + \ldots + c_{N+1} z^{-N-1} = g - f$$

From this it is clear that g extends to a meromorphic function in Y with a pole of order $N+1$ at most at p, while g is holomorphic in $Y \setminus \{p\}$. This shows how Leray's injective map in (0.0.1) can be used to construct non-trivial meromorphic functions provided one has a finite dimensional cohomology group as above.

0.1.1 Exercise. Use the fact that if $Y \subset X$ is a relatively compact open set in X then its compact closure can be covered by a finite family of charts and apply the construction above to show that this gives the existence of a holomorphic function f defined in some open neighborhood of $\overline{Y}$ which does not vanish identically in any non-empty open subset of Y.

0.2 Surjective cohomology maps.

Now X is an arbitrary paracompact manifold and $\mathcal{F}$ is a sheaf on X. An open set U in X is said to be fully $\mathcal{F}$-acyclic if $H^p(U \cap \Omega,\mathcal{F}) = 0$ for every open set Ω in X and each $p \geq 1$. If $\mathfrak{U} = \{U_\alpha\}$ is a finite family of open and fully $\mathcal{F}$-acyclic sets then Leray's acyclicity result in (0.0.3) gives

$$H^1(\cup U_\alpha,\mathcal{F}) = H^1(\mathfrak{U},\mathcal{F})$$

More generally, if Y_0 is an open subset of $Y = \cup U_\alpha$ we have the open covering $\mathfrak{U}_0 = \{Y_0 \cap U_\alpha\}$ and the isomorphism

$$H^1(Y_0,\mathcal{F}) = H^1(\mathfrak{U}_0,\mathcal{F})$$

0.2.1. Theorem. *With $\mathfrak{U} = \{U_\alpha\}$ and $\mathcal{F}$ as above the restriction map*

$$\mathrm{res}\colon H^1(Y,\mathcal{F}) \to H^1(Y_0,\mathcal{F})$$

is surjective for every open subset Y_0 of Y.

Proof. Let N be the number of sets in $\mathfrak{U}$ which can be indexed with open sets $U_1,\ldots,U_N$. To each $1 \leq k \leq N$ we put

$$Y_k = Y_0 \bigcup_{1 \leq \nu \leq k} U_\nu$$

It suffices to show that

$$\mathrm{res}\colon H^1(Y_k,\mathcal{F}) \to H^1(Y_{k-1},\mathcal{F}) \tag{i}$$

are surjective for each $1 \leq k \leq N$. Here $\mathfrak{U}^* = \{U_1 \cap Y_{k-1}, \ldots U_N \cap Y_{k-1}, U_k)\}$ is an open covering of Y_k while $\mathfrak{U}_* = \{U_1 \cap Y_{k-1}, \ldots U_N \cap Y_{k-1}\}$ is an open covering of Y_{k-1} and by Leray's theorem

$$H^1(\mathfrak{U}^*, \mathcal{F}) = H^1(Y_k, \mathcal{F}) \quad : H^1(\mathfrak{U}_*, \mathcal{F}) = H^1(Y_{k-1}, \mathcal{F})$$

Notice that

$$(U_\nu \cap Y_{k-1}) \cap U_k = (U_\nu \cap Y_{k-1}) \cap (U_k \cap Y_{k-1}) \quad : 1 \leq \nu \leq N$$

This entails that

$$C^1(\mathfrak{U}^*, \mathcal{F}) = C^1(\mathfrak{U}_*, \mathcal{F})$$

from which the surjectivity in (i) is obvious.

Applications to Riemann surfaces.

Let X be a Riemann surface. Pompieu's theorem for planar domains implies that every open set in X which is biholomorphic to a planar open set is fully $\mathcal{O}$-acylic. This basic result is used to analyze cohomology over open subsets of X. Let $Y \subset X$ be a relatively compact open subset and consider in addition a relatively compact subset Y_0 of Y. By Heine-Borel's Lemma we can cover the compact closure $\overline{Y}$ with a finite family of open charts $\{\Delta_\alpha\}$, i.e. each Δ_α is biholomorphic with the open unit disc. If $0 < a < 1$ is sufficiently close to one, the discs $\{a \cdot \Delta_\alpha\}$ is a covering of Y_0. Put

$$U_\alpha^* = Y \cap \Delta_\alpha : U_\alpha = Y_0 \cap a \cdot \Delta_\alpha$$

It is clear that V_α is a relatively compact subset of U_α for each α and the reader can also check that each non-empty intersection $V_\alpha \cap V_\beta$ is relatively compact in $U_\alpha \cap U_\beta$. Montel's theorem entails that the restriction map

(i) $$\text{res}\colon Z^1(\mathfrak{U}^*, \mathcal{O}) \to Z^1(\mathfrak{U}, \mathcal{O})$$

is a compact operator, where we the Frechet toplogy on $\mathcal{O}(\Omega)$ for each open set in X equips the two Cech cocylce spaces above with a Frechet space topology. Recall from the above that every open set in X which is biholomorphic to a planar open set is fully $\mathcal{O}$-acylic. Theorem 0.2.1 and Leray's acyclicity theorem applied to Y and Y_0 imply that the map

(ii) $$C^0(\mathfrak{U}, \mathcal{O}) \oplus Z^1(\mathfrak{U}^*, \mathcal{O}) \overset{\delta^0 + \text{res}}{\to} Z^1(\mathfrak{U}, \mathcal{O})$$

is surjective. Now one applies the result in functional analysis from the introduction and conclude that the range of δ^0 is finite dimensional which means that

(iii) $$H^1(Y_0, \mathcal{O}) \simeq \frac{Z^1(\mathfrak{U}, \mathcal{O})}{\delta^0(C^0(\mathfrak{U}, \mathcal{O}))}$$

is finite dimensional.

Starting with a relatively compact open set Y in X we can always construct another relatively compact open set Y^* which contains the compact closure of Y and hence $H^1(Y, \mathcal{O})$ is finite dimensional.

0.2.1 A vanishing result. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules with the property that every open chart Δ in X is fully $\mathcal{F}$-acyclic. Let $Y \subset X$ be relatively compact, i.e. the closure is a compact subset of X By (0.1.1) there exists a holomorphic function f defined in a neighborhood of the compact closure $\bar{Y}$ which is not constant on any connected component of the open set Y. We shall use this function to prove the following:

0.2.2 Theorem. *If $Y_0 \subset Y$ is such that $H^1(Y_0, \mathcal{F})$ is finite dimensional, then this cohomology group is zero.*

Proof. Suppose that $H^1(Y_0, \mathcal{F})$ has some dimension M. Theorem 0.2.1 give vectors $\xi_1, \ldots, \xi_M$ in $H^1(Y, \mathcal{F})$ such that $\{\text{res}(\xi_k)\}$ is a basis for $H^1(Y_0, \mathcal{F})$. Since $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, $H^1(Y, \mathcal{F})$

is a module over the ring $\mathcal{O}(Y)$ which gives an $M \times M$-matrix $B = \{b_{jk}\}$ with complex elements such that

$$\text{(ii)} \qquad \text{res}(f \cdot \xi_k) = \sum_{j=1}^{j=M} b_{jk} \cdot \text{res}(\xi_k)$$

Set

$$F = \det(f \cdot E_M - B)$$

Next, (ii) and the Cayley-Hamilton theorem in matrix calculus entail that

$$\text{(iii)} \qquad \text{res}(F \cdot \xi_k) = 0 \quad : k = 1, \ldots, M$$

Moreover, F is a holomorphic function of the form

$$F = f^M + c_{M-1} f^{M-1} + \ldots + c_1 f + c_0$$

and by analyticity the condition on f implies that F is not identically zero on any connected component of Y. Now $\overline{Y}$ can be covered by a finite family $\{\Delta_\alpha\}$ of open charts. It is clear that the condition on F implies that there exists a finite family such that if $\mathfrak{U} = \{U_\alpha = \Delta_\alpha \cap Y\}$, then the restriction of F to each non-empty intersection $U_\alpha \cap U_\beta$ is zero-free. Now the assumption on $\mathcal{F}$ and leray's acyclicity result gives

$$H^1(Y, \mathcal{F}) = H^1(\mathfrak{U}, \mathcal{F})$$

Consider a Cech cocycle

$$\xi = \{\xi_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \mathcal{F})$$

Since F is zero-free on $U_\alpha \cap U_\beta$ and $\mathcal{F}(U_\alpha \cap U_\beta)$ is a module over $\mathcal{O}(U_\alpha \cap U_\beta)$, there exists the 1-cocycle

$$\eta = \{F^{-1} \cdot \xi_{\alpha\beta}\}$$

Now (iii) gives $\text{res}(F \cdot \eta) = 0$ and at the same time

$$\xi = F \cdot \eta$$

Hence $\text{res}(\xi) = 0$ and together with the surjectivity in (0.2.1) we get Theorem 0.2.2.

0.2.3 Remark. When $\mathcal{F}$ is the sheaf $\mathcal{O}$ of holomorphic functions on a Riemann surface we have seen that the finiteness assumption in Theorem 0.2.1 holds and hence we get:

0.2.4 Proposition. *For every relatively compact open set Y in a Riemann surface it follows that $H^1(Y, \mathcal{O}) = 0$.*

Starting from this result we shall relax the condition that Y is relatively compact and establish the Behnke-Stein theorem in § xx.

§ 1. Residue formulas.

Let k be a positive integer and $g(z)$ is a C^∞-function of (x, y) defined in an open neighborhood of the origin in the complex z-plane. To each $\epsilon > 0$ we consider the line integral

$$\text{(i)} \qquad \int_{|z|=\epsilon} \frac{g(z)}{z^k} \, d\bar{z}$$

In polar coordinates it takes the form

$$i \cdot \epsilon^{-k+1} \cdot \int_0^{2\pi} g(\epsilon \cdot e^{i\theta}) e^{-i(k+1)\theta} \, d\theta$$

Now g has a Taylor expansion where (x, y)-polynomials are expressed in z and $\bar{z}$. In particular we get

$$\text{(ii)} \qquad g(z) = \sum_{j+m \leq k} c_{j,m} \cdot z^j \cdot \bar{z}^m + \rho(z)$$

where the absolute value $|\rho(z)| \leq C \cdot \epsilon^{k+1}$ for a constant C. Now

$$\int_0^{2\pi} e^{i(j-m-k+1)\theta}\, d\theta = 0 \quad : j+m \leq k$$

From this the reader discovers that there exists a constant C such that the absolute value of (i) is majorized by $C \cdot |z|^2$. in particular the limit is zero as $z \to 0$.

1.1 Exercise. Use (ii) and show that

$$\lim_{z \to 0}, \int_{|z|=\epsilon} \frac{g(z)}{z^k}\, dz = 2\pi i \cdot c_{k-1,0}$$

§ 2. Pompieu's formula.

Let Ω be a connectd open set in $\mathbf{C}$ whose boundary $\partial\Omega$ is a union of pairwise disjoint and differentiable closed Jordan curves, and $f(z)$ is a continuously differentable function defined on the closure of Ω. With $z \in \Omega$ we put

(i) $$g(z) = \frac{1}{\pi} \cdot \iint_\Omega \frac{\frac{\partial f}{\partial \bar\zeta}(\zeta)}{\zeta - z}\, d\xi d\eta - \frac{1}{2\pi i} \cdot \int_{\partial\Omega} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

With $d\zeta = d\xi + i d\eta$ and $d\bar\zeta = d\xi - i d\eta$ one has

(ii) $$d\xi d\eta = \frac{1}{2i} \cdot d\bar\zeta \wedge d\zeta$$

If $\epsilon > 0$ is small we set $\Omega_\epsilon = \Omega \setminus \{|\zeta - z| \leq \epsilon\}$. Since the function $\zeta \mapsto \frac{1}{z-\zeta}$ is integrable over Ω one has

(iii) $$\frac{1}{\pi} \cdot \iint_\Omega \frac{\frac{\partial f}{\partial \bar\zeta}(\zeta)}{\zeta - z}\, d\xi d\eta = \lim_{\epsilon \to 0} \frac{1}{\pi} \cdot \iint_{\Omega_\epsilon} \frac{\frac{\partial f}{\partial \bar\zeta}(\zeta)}{\zeta - z}\, d\xi d\eta$$

When $\epsilon > 0$ the equality (ii) identifies the right hand side in (iii) with

(iv) $$\frac{1}{2\pi i} \cdot \iint_{\Omega_\epsilon} \frac{\frac{\partial f}{\partial \bar\zeta}(\zeta)}{\zeta - z}\, d\bar\zeta \wedge d\zeta = \frac{1}{2\pi i} \cdot \iint_{\Omega_\epsilon} \bar\partial\left(\frac{f(\zeta)}{\zeta - z}\right) \wedge d\zeta$$

Stokes Theorem identifies (iv) with

(v) $$\frac{1}{2\pi i} \cdot \int_{\partial\Omega} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{2\pi i} \cdot \int_{\{|\zeta - z| = \epsilon\}} \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

where the last line integral is taken in the positive sense and with the aid of a figure which keeps trace of orientations the reader should verify the minus sign in (v). Next, one has the residue formula

(vi) $$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{\{|\zeta - z| = \epsilon\}} \frac{f(\zeta)\, d\zeta}{z - \zeta} = f(z)$$

2.1 Conclusion. With f as above one has the equation

(*) $$f(z) = \frac{1}{2\pi i} \cdot \int_{\partial\Omega} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{\pi} \cdot \iint_\Omega \frac{\frac{\partial f}{\partial \bar\zeta}(\zeta)}{\zeta - z}\, d\xi d\eta$$

In the case when f is holomorphic in Ω this recovers Cauchy's integral formula.

2.2 An inhomogeneous equation. Let ϕ be a continuous function with compact support in Ω and put

(**) $$g(z) = \frac{1}{\pi} \cdot \iint_\Omega \frac{\phi(\zeta)}{z - \zeta}\, d\xi d\eta$$

If f is a test-function with compact support in Ω we get

$$\iint_\Omega g(z) \cdot \frac{\partial f}{\partial \bar z}(\zeta)\, dx dy = \iint_\Omega \Big[\frac{1}{\pi} \cdot \iint_\Omega \frac{\frac{\partial f}{\partial \bar z}(\zeta)}{z - \zeta}\, dx dy\Big]\, \phi(\zeta)\, d\xi d\eta = -\iint_\Omega f(\zeta) \cdot \phi(\zeta)\, d\xi d\eta$$

The construction of distribution derivatives entails that

$$\frac{\partial g}{\partial \bar{z}}(z) = \phi(z)$$

Hence the g-function in (**) solves the inhomogenous $\bar{\partial}$-equation.

Remark. The equations above were established by Pompieu around 1820. Starting from (*) where the ϕ-function has compact support one can go further and the following resuklt is proved in my notes in analytic function theory.

2.3 Theorem. *For every open set Ω in $\mathbf{C}$ and each $\phi \in C^0(\Omega)$ there exists a C^1-function g in Ω such that*

$$\frac{\partial g}{\partial \bar{z}}(z) = \phi(z)$$

§ 3. Analysis on compact Riemann surfaces.

A compact Riemann surface X has an underlying real-analytic structure which is an oriented compact real manifold of dimension two. For a while we ignore the complex analytic structure. There exists the sheaf $\mathcal{E}$ whose sections are complex-valued C^∞-functions and the sheaf $\mathcal{E}^1$ of differential 1-forms with C^∞-coefficients. For brevity we refer to smooth 1-forms. Finally there is the sheaf $\mathcal{E}^2$ of smooth 2-forms. Since X is oriented and compact, every smooth 2-form σ can be integrated over X and we write

$$\int_X \sigma$$

If α is a smooth 1-form its exterior differential $d\alpha$ belongs to $\mathcal{E}^2(X)$ and Stokes Theorem gives

$$\int_X d\alpha = 0$$

It means that the integral of an exact 2-form over X is zero. A non-trivial fact is the converse whose proof is given in § 3.10 below.

3.0.Theorem. *A smooth 2-form is exact if and only if its integral over X is zero.*

3.1 Currents. A current of degree zero is a continuous linear form on the Frechet space $\mathcal{E}^2(X)$ and we set

$$\mathfrak{c}^0(X) = \mathcal{E}^2(X)^*$$

Let us remark that $\mathfrak{c}^0(X)$ corresponds to the space of distributions on X, i.e.

$$\mathfrak{c}^0(X) = \mathfrak{Db}(X)$$

The dual $\mathcal{E}^1(X)^*$ is denoted by $\mathfrak{c}^1(X)$ and its elements are called 1-currents. Finally, the dual of $\mathcal{E}(X)$ is denoted by $\mathfrak{c}^2(X)$ and its elements are called 2-currents.

Example. A function $f \in \mathcal{E}(X)$ defines a zero-current via the linear form

$$\sigma \mapsto \int_X f \cdot \sigma \quad : \quad \sigma \in \mathcal{E}^2(X)$$

In this way $\mathcal{E}(X)$ appears as a subspace of $\mathfrak{c}^0(X)$ and such currents are called smooth. In the same way we get an inclusion $\mathcal{E}^1(X) \subset \mathfrak{c}^1(X)$ where $\phi \in \mathcal{E}^1(X)$ yields a continuous linear functional on $\mathcal{E}^1(X)$ defined by

$$w \mapsto \int_X \phi \wedge w \quad : \quad w \in \mathcal{E}^1(X)$$

Exterior differential maps on smooth forms extend to currents. If $w \in \mathcal{E}^1(X)$ is a smooth 1-current then dw is the 2-current defined by

$$f \mapsto \int_X f \cdot dw \quad : \quad f \in \mathcal{E}(X)$$

Stokes theorem gives:

$$0 = \int_X d(fw) = \int_X f \cdot dw + \int_X df \wedge w$$

Since $df \wedge w = -w \wedge df$ we get

$$\int_X f \cdot dw = \int_X w \wedge df$$

This leads to the construction of exterior differentials of 1-currents. Namely, if $\gamma \in \mathfrak{c}^1(X)$ then the 2-current $d\gamma$ is defined by

$$f \mapsto \gamma(df) \quad : \quad f \in \mathcal{E}(X)$$

By the above it follows that if $w \in \mathcal{E}^1(X)$ is regarded as a smooth 1-current then its current differential dw is the smooth 2-current expressed by the differential 2-form dw. Thus, the exterior differentials on smooth forms extend to currents and one has a complex

(3.1.1) $$0 \to \mathfrak{c}^0(X) \xrightarrow{d} \mathfrak{c}^1(X) \xrightarrow{d} \mathfrak{c}^2(X) \to 0$$

3.1.2 Remark about Stokes theorem. Let Ω be an open set in X. The characteristic function χ_Ω is a 0-current and we get the 1-current $d\chi_\Omega$ which is supported by $\partial\Omega$. In the favourable case when $\partial\Omega$ is a piecewise differentiable curve the current $d\chi_\Omega)$ has order zero, i.e. in local charts it is expressed by a Riesz measure given by a bounded functions times a locally constructed arc-length measure of the boundary curve. Recall also that the boundary curve is oriented by the rule of thumbs and therefore one can integrate differential 1-forms along $\partial\Omega$ in an intrinsic fashion without specifying local coordinates.

3.2. d-closed zero currents. A current γ of order zero is d-closed if and only if

(3.2.1)
$$d\gamma(w) = \gamma(dw) = 0 \quad : \quad w \in \mathcal{E}^1(X)$$

To grasp this condition we consider a chart U in X taken as an open disc in the (x,y)-plane. If $f(x,y) \in C_0^\infty(U)$ we have the 1-form $f \cdot dx$ and get $d(fdx) = f'_y \cdot dy \wedge dx$. Hence (2.i) means that

(3.2.2)
$$\gamma(f'_y \cdot dy \wedge dx) = 0 \quad : \quad f \in C_0^\infty(U)$$

The restriction of γ to U is a distribution in the sense of § X and (i) means that the distribution derivative $\frac{\partial\gamma}{\partial y}$ vanishes in U. Similarly, if we start with a 1-form gdx where $g \in C_0^\infty(U)$ we find that $\frac{\partial\gamma}{\partial x} = 0$ in U. It follows that the distribution γ restricts to a constant density function in U. The conclusion is that the d-kernel on $\mathfrak{c}^0(X)$ is reduced to constant functions, i.e.

(3.2.3)
$$\ker_d(\mathfrak{c}^0(X)) = \mathbf{C}$$

3.3 Inverse and direct images under mappings. Let X, Y be a pair of oriented compact two-dimensional manifolds and $F \colon X \to Y$ a real-analytic map. Assume that F is surjective, i.e. $F(X) = Y$ and that the mapping is finite which means that there exists an integer n such that the inverse fiber $F^{-1}(\{y\}$ contains at most n points in X for every $y \in Y$. Now we construct inverse images of smooth forms on Y. First, if $g \in \mathcal{E}(Y)$ the composed function $g \circ F$ belongs to $\mathcal{E}(X)$. Next, let w be a smooth 1-form on Y. Using local coordinates one verifies that there exists a pull-back denoted by $F^*(w)$ which yields a smooth 1-form on X. More precisely, let $p \in X$ be a given point and U is some chart around p while V is a chart around $F(p)$. Let (ξ, η) be local coordinates in the chart V and (x,y) are local coordinates in U. Then

(i)
$$w|V = g \cdot d\xi + h \cdot d\eta \quad : \quad g, h \in \mathcal{E}(V)$$

In U we get the C^∞-functions $\xi \circ F$ and $\eta \circ F$ whose exterior differentials give smooth 1-forms in U denoted by $F^*(\xi)$ and $F^*(d\eta)$. Now

$$F^*(w) = g \circ F \cdot F^*(d\xi) + h \circ F \cdot F^*(d\eta)$$

3.3.1 Exercise. Verify that the 1-form $F^*(w)$ does not depend upon the chosen local coordinates in the two charts. Conclude that if $w \in \mathcal{E}^1(Y)$ then there exists a smooth 1-form $F^*(w)$ on X obtained in local charts as above. Hence there exists a pull-back map:

$$w \mapsto F^*(w)$$

In the same way one constructs the pull-back of each smooth 2-form on Y. A notable fact is that these pull-back mappings commute with exterior differentials. One has for example

$$d(F^*(w)) = F^*(dw) \quad : \quad w \in \mathcal{E}^1(Y)$$

We leave out the detailed verifications which are exposed in text-books devoted to differential geometry.

3.4. Direct images of currents. The construction of inverse images of smooth forms give by duality F_*-mappings on currents. More precisely, let $\gamma \in \mathfrak{c}^\nu(X)$ for some $0 \leq \nu \leq 2$. The direct image current $F_*(\gamma)$ is defined by

$$F_*(\gamma)(w) = \gamma(F^*(w)) \quad : \quad w \in \mathcal{E}^\nu(Y)$$

By this construction the passage to direct images of currents commute with exterior differentials. For example, let γ be a 1-current on X. Then

$$d_Y(F_*(\gamma)) = F_*(d_X(\gamma))$$

where we for convenience introduced a sub-index to indicate on which manifold one has performed an exterior differential. In particular F_* sends d-closed currents on X to d-closed currents on Y.

3.5. Integration chains and 1-currents. Let c be an oriented C^1-curve on X which means that c is defined via a C^1-mapping $t \to x(t)$ from a closed t-interval $[0, T]$ into X. Each smooth 1-form γ in X can be integrated along c and the map

$$\gamma \mapsto \int_c \gamma$$

yields a 1-current denoted by $\int_c$. More generally we can take a finite sum of such integration currents, i.e. every chain c of class C^1 gives a current $\int_c$ which belongs to $\mathfrak{c}^1(X)$. By definition the exterior differential $d\int_c$ is the 2-current defined by

$$g \mapsto \int_c dg \quad : \quad g \in \mathcal{E}(X)$$

The resulting current is denoted by ∂c. In simplicial topology one refers to ∂c as the boundary of the 1-chain c. Here it has been given an operative definition. If c is an elementary chain defined by a single oriented C^1-curve with end-points a and b, then the 2-current ∂c is a current supported by the end-points a and b. In general ∂c is a sum of Dirac measures assigned at a finite set of points, i.e.

$$d\int_c = \sum k_\nu \cdot \delta(x_\nu)$$

where $\{k_\nu\}$ are integers and $\sum k_\nu = 0$.

3.6. Critical values of a map F. Let $F\colon X \to Y$ be a surjective mapping with finite fibers. One says that F is smooth at a point $p \in X$ if the following hold when a pair of local charts U and V have been chosen as in § XX: The 1-forms $F^*(d\xi)$ and $F^*(d\eta)$ are non-zero and **R**-linearly independent. Next, a point $q \in Y$ non-critical if F is smooth at each $p \in F^{-1}(\{q\})$. The complement is denoted by $\mathrm{Crit}(F)$ and is called the set of the critical values. A welknown fact - called Sard's Lemma - asserts that the set of F-critical values is a (possibly empty) finite subset of Y. Since X and Y both are connected it follows that when $q \in Y \setminus \mathrm{Crit}(F)$ then the number of points in the inverse fiber $F^{-1}(\{q\})$ does not depend upon q. This integer n is called the degree of the mapping F and denoted by $\deg(F)$. Put $X_* = X \setminus F^{-1}(\mathrm{Crit}(F)$. In topology one says that the restricted mapping

$$F\colon X_* \to Y \setminus \mathrm{Crit}(F)$$

is an unbranched covering map of degree n. If $\Omega \subset Y \setminus \mathrm{Crit}(F)$ is open and simply connected one proves that the inverse image $F^{-1}(\Omega)$ consists of pairwise disjoint open subsets $U_1, \ldots, U_n$ and $F\colon U_k \to \Omega$ is a diffeomorphism for each k. One refers to $\{U_k\}$ as the sheets above Ω.

3.7. Direct images of integration currents. Let F be as above and consider a 1-current $\int_c$ on X given by an oriented 1-chain c. Now there exists the direct image current $F_*(\int_c)$. To describe this 1-current on Y we proceed as follows. To begin with c is a sum of small 1-chains $\{c_k\}$ where each c_k is a single C^1-curve and we can also choose c_k so that its parametrized curve $t \to x(t)$ stays in a small subset of X. So $a = x(0)$ and $b = x(T)$ are close to each other and the whole curve c_k stays inside a chart U in X where a is the origin in U and we may also assume that $F(U)$ stays in a chart V around $F(a)$.

Let (ξ, η) be local coordinates in $F(U)$. If $g(\xi, \eta) \in C_0^\infty(F(U))$ we get the differential 1-form $g \cdot d\xi$ whose pull-back can be integrated on the chain c_k. This gives a linear functional

$$g \mapsto \int_{c_k} F^*(g \cdot \xi)$$

A similar linear functional arises when we instead use the 1-form $d\eta$.

3.8 Exercise. Conclude from the above that there exists the direct image current $F_*(\int_{c_k})$ which has a compact support in $F(U)$ and is given by a 1-current of the form

$$\mu_1 \cdot d\xi + \mu_2 \cdot d\eta$$

where μ_1, μ_2 are two Riesz measures with compact support in U. It means that the direct image current has order zero. But unless $F\colon U \to F(U)$ is a diffeomorphism the direct image current is in general not given by integration along a 1-chain. Funally, by additivity one has

$$F_*(\int_c) = \sum F_*(\int_{c_k})$$

3.9. Further results. Let $F\colon X \to Y$ be as above. A special current on X is the 2-current $\square_X$ defined by

$$\omega^2 \mapsto \int_X \omega^2$$

i.e. one integrates 2-forms over X. If ϕ is a bounded Borel function on X, or more generally an L^1-function on x in Lebegue's sense, then there exists the 2-current defined by

$$\omega^2 \mapsto \int_X \phi \cdot \omega^2$$

It is denoted by $\phi \cdot \square_X$.

3.9.1 Exercise. Let $F\colon X \to Y$ where $F(X) = Y$ and F is finite but may have some critical values. Show that there exists a Borel function ϕ on Y such that

$$F_*(\square_X) = \phi \cdot \square_Y$$

and describe the ϕ-function via some sort of Jacobian of F. The case when X is a compact Riemann surface and $Y = \mathbf{P}^1$ while F is a meromorphic function on X is of special interest.

3.10 Cohomology and duality.

Let X be a compact Riemann surface. The constant sheaf $\mathbf{C}_X$ has a soft resolution

$$0 \to \mathbf{C}_X \to \mathcal{E} \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \to 0$$

It follows that the ordinary cohomology groups $\{H^p(X, \mathbf{C})\}$ are equal to the cohomology of the complex

$$0 \to \mathcal{E}(X) \to \mathcal{E}^1(X) \to \mathcal{E}^2(X) \to 0 \tag{3.10.1}$$

Now (3.1.1) is the dual complex. So we have for example that

$$H^2(X, \mathbf{C})^* \simeq \ker_d(\mathfrak{c}^0(X))$$

By 3.2.3 the last space is 1-dimensional and hence $H^2(X, \mathbf{C})$ is also 1-dimensional. Using this the reader can give a proof of Theorem 3.1. Concerning the cohomology group $H^1(X, \mathbf{C})$ its dual space is given by

$$\frac{\ker_d(\mathfrak{c}^1(X))}{d(\mathfrak{c}^0(X)} \tag{3.10.2}$$

Among d-closed 1-currents we have those which arise as integration cycles along closed and oriented C^1-curves γ. A major result due to Riemann asserts that this family is amply enough to generate the quotient space (3.10.2) and the conclusive result is

3.10.3 Theorem. *Let X be a compact Riemann surface which is not biholomorphic with $\mathbf{P}^1$. Then there exists a positive integer g such that $H^1(X, \mathbf{C})$ is a $2g$-dimensional complex vector space. Moreover, there exist closed 1-chains $\gamma_1, \ldots, \gamma_{2g}$ whose images in (3.10.2) is a basis and there exist closed 1-forms $\omega_1, \ldots, \omega_{2g}$ such that*

$$\int_{\gamma_k} \omega_j = \text{Kronecker's delta-function}$$

3.10.4 Remark. Theorem 3.10.3 is discussed further in a separate chapter devoted to compact Riemann surfaces. A notable fact is that the complex analytyic structure leads to a decomposition of d-closed 1-forms which represent the cohomology via (3.10.1).

3.11 Further results.

On a compact Riemann surface X the complex analytic structure gives a decomposition:

$$\mathcal{E}^1(X) = \mathcal{E}^{1,0}(X) \oplus \mathcal{E}^{0,1}(X)$$

The exterior differential on X is decomposed into a sum $d = \partial + \bar{\partial}$ and when $f \in \mathcal{E}(X)$ one gets the 1-forms

$$\partial f \in \mathcal{E}^{1,0}(X) \quad \text{and} \quad \bar{\partial} f \in \mathcal{E}^{0,1}(X)$$

This gives two complexes

$$0 \to \mathcal{E}(X) \xrightarrow{\partial} \mathcal{E}^{1,0}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{1,1}(X) \to 0$$

$$0 \to \mathcal{E}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(X) \xrightarrow{\partial} \mathcal{E}^{1,1}(X) \to 0$$

where we remark that $\mathcal{E}^{1,1}(X) = \mathcal{E}^2(X)$. Since $\bar{\partial}$ is elliptic a current $\gamma \in \mathfrak{c}^0(X)$ is $\bar{\partial}$-closed if it is a holomorphic function, i.e.

$$\text{Ker}_{\bar{\partial}}(\mathfrak{c}^0(X)) = \mathcal{O}(X) = \mathbf{C}$$

where the last equality follows since holomorphic functions on the compact Riemann surface are constant.

3.11.1 Application. On X one has the exact sheaf sequence

(i) $$0 \to \Omega \to \mathcal{E}^{1,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{1,1} \to 0$$

This is a soft resolution of the sheaf Ω and entails that

(ii) $$H^1(X, \Omega) = \frac{\mathcal{E}^{1,1}(X)}{\bar{\partial}(\mathcal{E}^{1,0}(X))}$$

It follows that

$$\big[H^1(X, \Omega)\big]^* \simeq \ker_{\bar{\partial}}(\mathfrak{c}^0(X))$$

Since $\bar{\partial}$ is elliptic the last space is $\mathcal{O}(X)$ and this is a 1-dimensional complex vector space becuause X is compact and connected. We eonclude that $H^1(X, \Omega)$ is a 1-dimensional complex vector space.

3.12 Principal value distributions. Let $f \in \mathfrak{M}(X)$. It defines a distribution VP(f) as follows: When σ is a smooth 2-form on X we avoid poles of f and consider the limit integrals:

(*) $$\lim_{\epsilon \to 0} \int_{X_f(\epsilon)} f \cdot \sigma$$

where $X_f(\epsilon)$ is the open complement of small removed ϵ-discs centered at the finite set of poles of f. The existence of this limit was explained in §§, i.e. in local coordinates it amounts to show that there exist limits

$$\int_{|z|>\epsilon}, \frac{\phi(x,y)dxdy}{z^k}$$

for every positive integer k and each test-function $\phi(x,y)$ in $\mathbf{C}$. Next, construct the current

$$R_f = \bar{\partial}(\text{VP}(f))$$

It is called the residue current of the meromorphic function f. Since f is holomorphic outside the poles the (0,1)-current R_f is supported by the discrete set of poles. In fact, Stokes Theorem gives

(**) $$R_f(\gamma) = \lim_{\epsilon \to 0} \int_{X_f[\epsilon]} f \cdot \gamma$$

where $X_f[\epsilon]$ consists of disjoint small circles centered at the poles $p_1,\ldots,p_k$ of f.

3.13 The current $\frac{\partial f}{f}$. Let $f\in\mathfrak{M}(X)$. Now $\frac{\partial f}{f}$ is a $(1,0)$-current. Let U be a chart around a zero q of f of some order e so that $f=z^e\phi(z)$ where $\phi\neq 0$ holds in the chart. Then

$$\frac{\partial f}{f}|U=\frac{\partial\phi}{\phi}+e\cdot\frac{dz}{z}$$

If V instead is a chart around a pole of some order e we have $f=z^{-e}\phi(z)$ and obtain

$$\frac{\partial f}{f}|V=\frac{\partial\phi}{\phi}-e\cdot\frac{dz}{z}$$

3.14 Exercise. Show that if $g\in\mathcal{E}(X)$ then

$$\frac{\partial f}{f}(dg)=2\pi i\cdot\big[\sum e_k\cdot g(p_k)-\sum\ e_j\cdot g(q_j)\,\big]$$

where $\{p_k\}$ are the poles and $\{q_k\}$ the zeros of f and $\{e_k\}$ the order of poles, respectively the multiplicities at the zeros of f.

3.15 A special direct images. Let $w\in\Omega(X)$ be an abelian differential. It yields a smooth current and there exists the dirct image current $F_*(\omega)$. Since the passage to direct images commute with differentials we have

$$\bar\partial_y(F_*(\omega))=0$$

Hence the direct image current belongs to $\Omega(Y)$. If $Y=\mathbf{P}^1$ there are no non-zero holomorphic forms so when $f\colon X\to\mathbf{P}^1$ is given via some $f\in\mathfrak{M}(X)$ then $f_*(\omega)=0$. The vanishing of this direct image current means that for every $\beta\in\mathcal{E}^{0,1}(\mathbf{P}^1)$ one has

$$\int_X w\wedge f^*(\beta)=0 \tag{4.4.1}$$

3.16 Divisors and adapated currents.

A divisor D on the compact Riemann surfaces consists of a finite set of pairs (p_ν,k_ν) whrere $\{p_\nu\}$ are points in X and $\{k_\nu\}$ are integers which may be positive or negative. Around each p_ν we choose a small chart Δ_ν with local coordinate z_ν and construct a function

$$f_\nu(z_\nu)=z_\nu^{k_\nu}\cdot\psi_\nu(z_\nu) \tag{3.16.0}$$

By a C^∞-partition of the unity we get a function f in X which is zero-free and C^∞ in $X\setminus\{p_\nu\}$ and in each chart above it has the form (3.16.0). If $k_\nu<0$ for some p_ν the construction of principal value distributions entails that f restricts to a distribution in Δ_ν. The conclusion is that f is a well-defined distribution in X, i.e. a zero-current in X.

3.16. The 1-current $\frac{df}{f}$. Outside the finite set $\{p_\nu\}$ this is a smooth 1-current. If $k_\nu>0$ then df is smooth in the chart Δ_ν and can therefore be multiplied with the principal value distribution defined by f. If $k_\nu<0$ then f restricts to a C^∞-function in δ_ν which can be multiplied with the current df, i.e. here we use that the exterior differential d sends zero-currents to 1-currents.

Exercise. In a chart where $f=z_\nu^{k_\nu}\cdot\psi$ the reader should verfy that

$$\frac{df}{f}=k_\nu\cdot\frac{dz_\nu}{z_\nu}+\frac{d\psi}{\psi} \tag{3.16.1}$$

Decompising this 1-current the part of bi-degree (0,1) becomes

$$\frac{\bar\partial\psi}{\psi}$$

which is smooth since ψ is zero-free. Conclude from the above that there exists a smooth differential form ρ_f if bi-degree $(0,1)$ such that

$$\frac{df}{f}(\alpha^{1,0}) = \int_X \rho_f \wedge \alpha^{1,0}$$

for every test-form of bi-degree $(1,0)$.

3.17 The case when ρ_f is $\bar\partial$-exact.. Suppose that

(i) $$\rho_f = \bar\partial(g)$$

for some C^∞-function g. In the open set $X \setminus \{p_\nu\}$ it is celar that (3.,16.1) entails that

(ii) $$\rho_f = f^{-1} \cdot \bar\partial(f)$$

Together (i-ii) give

$$\bar\partial(e^{-g}f) = 0$$

in $X \setminus \{p_\nu\}$. Hence $\phi = e^{-g} \cdot f$ is holomorphic in the complement of $\{p_\nu\}$ and (3.16.0) entails that it extends to a meromorphic function in X. Moreover, since e^{-g} is zero-free it is clear that the principal divisor $\mathrm{Div}(\phi)$ via (3.16.0) coincides with the given divisor D.

3.18 A general construction. Let D be a divisor as above. Then there exists the familly $\mathfrak{c}^1(X;D)$ of 1-currents γ on X such that

$$\gamma(dg) = \sum k_\nu \cdot g(p_\nu) \quad : g \in C^\infty(X)$$

In the case when the divisor D has degree zero, i.e. $\sum k_\nu = 0$ holds, then we can construct a subfamily using integration chains as in (3.5). For eample, fix a piint p_* in X and let $\{\gamma_\nu\}$ be simple and oriented curves where γ_ν has end-points at p_* and p_ν. Then

$$\int_{\gamma_\nu} dg = g(p_\nu) - g(p_*)$$

Since $\sum k_\nu = 0$ it follows that the 1-current

$$\gamma = \sum k_\nu \cdot \int_{\gamma_\nu}$$

belongs to $\mathfrak{c}^1(X;D)$. Moroever, each single γ_ν can be exrpoessed as a sum of integration currents taken over the all curves parametrised by $t \mapsto \gamma_\nu(t)\colon\ t_j \leq t \leq t_{j+1}$ where $0 = t_0 < t-1 < \ldots < t_N = 1$ is a partiton of $[0,1]$.

3.18.1 Exercise. Use the last partitions above and the fact that closed and smooth 1-forms in a chart are d-exact to to prove that when γ is given as above then there exists f in (3.16.0) such that

$$\gamma(\alpha) = \sum k_\nu \cdot \int_{\gamma_\nu} \alpha = \frac{1}{2\pi i} \cdot \frac{df}{f}(\alpha) = \frac{1}{2\pi i} \cdot \int_X \frac{df}{f} \wedge \alpha$$

for every closed and smooth 1-form α on X.

§ 4. Topological results.

We restrict the attention to topological spaces X which are Hausdorff. One says that the Hausdorff space X has a countable topology if there exista a denumerable family $\mathfrak{U}$ of open subsets $\{U_n\}$ such that every open set in X is the union of sets from $\mathfrak{U}$. Next, the Hausdorff space X is arc-wise connected if there to every pair of points p, q in X exists a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. The following remarkable result is due to Poincaré and Volterra:

4.1 Theorem. *Let X be arc-wise connected and Hausdorff which in addition is locally compact. Suppose there exists a continuous map $f\colon X \to Y$ where Y is a Hausdorff space with countable topology and the fibers $f^{-1}(y)$ are discrete subsets of X for every $y \in Y$. Then X has countable topology.*

Proof. Denote by $\mathfrak{B}$ the family of open sets V in X which satisfy the following:

(i) $$V \text{ has a countable topology}$$

(ii) $$V \text{ is a connected component of } f^{-1}(U) \text{ for some open set } U \text{ in } Y$$

Theorem 4.1 follows if we prove that $\mathfrak{B}$ is a countable family and every open set in X is a union of $\mathfrak{B}$-sets.

Proof that $\mathfrak{B}$ is countable. Let V_0 be a set in $\mathfrak{B}$ and consider an open set U in Y. The connected components of $f^{-1}(U)$ are disjoint by (i) V_0 has countable topology. It follows that at most countably many of the connected components of $f^{-1}(U)$ have a non-empty intersection with V. Next, since every $V \in \mathfrak{B}$ is of the form $f^{-1}(U)$ for an open set in Y and Y has a countable topology we conclude that

(iii) $$V_0 \cap V \neq \emptyset$$

only can hold for countably many V in $\mathfrak{B}$. Next, fix $V_* \in \mathfrak{B}$ and set

$$\mathfrak{B}_0 = \{V\colon V_* \cap V \neq \emptyset\}$$

By (iii) this is a countable set in $\mathfrak{B}$. If $n \geq 1$ we define inductively

$$\mathfrak{B}_n = \{V\colon \exists\, W \in \mathfrak{B}_{n-1} : W \cap V \neq \emptyset\}$$

An induction over n entails that each $\mathfrak{B}_n$ is countable and since X is connected it is clear that their union is equal to $\mathfrak{B}$ which therefore is a countable family.

Proof that $\mathfrak{B}$ is a basis of the topology. Let $\Omega \subset X$ be open and take some $x \in \Omega$. We must prove that there exists $V \in \mathfrak{B}$ such that

$$x \in V \subset \Omega$$

To show this we consider the discrete fiber $f^{-1}(f(x))$. Since X is Hausdorff and locally compact there exists a small open and relatively compact neighborhood W of x such that $W \subset \Omega$ and $\partial W \cap f^{-1}(f(x)) = \emptyset$. Now $f(\partial W)$ is a compact set in Y which does not contain $f(x)$ which gives $U \in \mathfrak{U}$ such that

$$f(x) \in U \quad : f(\partial W) \cap U = \emptyset$$

Let V be the connected component of $f^{-1}(U)$ which contains x. From the above $V \cap \partial W = \emptyset$ and we conclude that

$$x \in V \subset W \subset \Omega$$

which finishes the proof.

4.2 Covering spaces.

Let X be arc-wise connected and Hausdorff. A pair (Y, ρ) is a covering of X if $\rho\colon Y \to X$ is a continuous map and for each point $p \in X$ there exists some open neighborhood U such that

$$\rho^{-1}(U) = \cup V_\alpha$$

where the right hand side is a disjoint union of open sets in Y and $\rho\colon V_\alpha \to U$ are homeomorphisms for each α.

4.2.1 Exercise. Let $\gamma\colon [0,1] \to X$ be a continuous map and let $y_0 \in Y$ be a point such that

$$\rho(y_0) = x_0 = \gamma(0)$$

Show that there exists a unique continuous map $\gamma^*\colon [0,1] \to Y$ such that $y_0 = \gamma^*(0)$ and

(i) $$\rho(\gamma^*(t)) = \gamma(t) : \ 0 \leq t \leq 1$$

which means that there exist unique liftings of curves.

Next, let $\pi_1(X)$ be the fundamental group of X. Since X is arc-wise connected it can be identified with homotoy classes of closed curves γ which have a given $x_0 \in X$ as a common initial and terminal point. Let us also fix a point y_0 in the fiber $\rho^{-1}(x_0)$. To each closed γ-curve as above the exercise gives a unique lifted curve γ^* where (i) entials that

(ii) $$\rho(\gamma(1)) \in \rho^{-1}(x_0)$$

4.2.2 Exercise. Show that the end-point $\gamma(1)$ in (ii) only depends upon the homotopy class of γ and conclude that one has a map

(*) $$\rho_*\colon \pi_1(X) \to \rho^{-1}(x_0)$$

Show also that ρ_* is surjective if Y is connected.

4.2.3 The universal covering property. A covering $\phi\colon Z \to X$ has a universal property if the following holds: For every covering $\rho\colon Y \to X$ and each pair of points $z_0 \in \phi^{-1}(x_0)$ and $y_0 \in \rho^{-1}(x_0)$ there exists a unique fiber preserving continuous map $f\colon Z \to Y$ with $f(z_0) = y_0$ and

(**) $$\rho \circ f = \phi$$

Notice that (**) means that f is fiber preserving.

4.2.4 Exercise. Show that a covering with the universal property is unique in the sense that if (Z_1, ϕ_1) and $Z_2, \phi_2)$ is such a pair then there exists a homeomorphism $g\colon Z_1 \to Z_2$ with

$$\phi_1 = \phi_2 \circ g$$

4.2.5 Existence. Show that an arc-wise connected Hausdorff space X has a universal covering $\widehat{X}$ and that the fundamental group $\pi_1(\widehat{X}) = 0$.

4.2.6 The Galois property. A covering $\rho\colon Y \to X$ where Y is connected has the Galois property if there to each $p \in X$ and every pair of points y_1, y_2 in $\rho^{-1}(p)$ exista a fiber presering homeomorphism $f\colon Y \to Y$ with $f(y_1) = y_2$.

4.2.7 Exercise. Verify that the universal covering $\widehat{X}$ has the Galois property.

4.3 Runge sets on Riemann surfaces.

If K is a compact subset of the Riemann surface X we get the conncted components $\{\Omega_\alpha\}$ of the open complement $X \setminus K$. Since X is connected we notice that

(i) $$\partial\Omega_\alpha \subset K$$

hold for every connected component in $X \setminus K$. Next, choose an open set U which contains K where $\overline{U}$ is compact. This is of course possible since K can be covered by a finite number of charts where each chart has a compact closure in X. If a component Ω_α is such that

(ii) $$\partial U \cap \Omega_\alpha = \emptyset$$

the connectivity of X and (i) entail that

(iii) $$\Omega_\alpha \subset U$$

Next, the Heine-Borel Lemma a finite family of $\Omega_{\alpha_1},\ldots,\Omega_{\alpha_N}$ whose union cover the compact set ∂U. Among these occur a finite family of non-compact sets, say $\Omega_{\alpha_1},\ldots,\Omega_{\alpha_k}$ where $1\leq k\leq N$. Set

(iv) $$\mathfrak{h}(K)=K\cup\bigcup_{*}\Omega_\alpha$$

where the *-marked union is taken over all components for which (iii) hold together with the finite family of relatively compact components which intersect ∂U, or equivalently

$$\mathfrak{h}(K)=X\setminus\Omega_{\alpha_1}\cup\ldots\cup\Omega_{\alpha_k}$$

4.3.1 Exercise. Verify that $\mathfrak{h}(K)$ is compact and that

$$\mathfrak{h}(\mathfrak{h}(K))=\mathfrak{h}(K)$$

4.3.2 Open Runge sets. An open subset Y in X is Runge if

$$K\subset Y\implies\mathfrak{h}(K)\subset Y$$

hold for every compact set K, i.e. the Runge hull of a compact subset of Y stays in Y.

4.3.3 Proposition. *Let K be a compact Runge set in X with a non-empty interior. Let $\Delta_1,\ldots,\Delta_N$ be a finite number of open discs whose union cover ∂K. Then the open set below is Runge:*

$$Y=K\setminus\cup\overline{\Delta}_\nu$$

The easy proof is left as an exercise.

§ 5. Runge's theorem.

Let Y be a relatively compact open set in the non-compact Riemann surface X and $K \subset Y$ is a compact subset such that $K = \mathfrak{h}(K)$. Let $\mathcal{O}(K)$ be the space of germ of analytic functions on K. Each $f \in \mathcal{O}(K)$ has a restriction to K which yields a continuous function $f_* \in C^0(K)$.

5.1 Proposition. *Let Y, K be a pair as above. Then, for each $f \in \mathcal{O}(K)$ there exists a sequence $\{g_n\}$ in $\mathcal{O}(Y^*)$ such that the maximum norms $|g_n - f_*|_K \to 0$.*

Proof. Since Y is relatively compact we have $H^1(Y, \mathcal{O}) = 0$ by Proposition 0.2.4. By Dolbeault's isomorphism it follows that if $\omega \in \mathcal{E}^{0,1}(X)$ then there exists $g \in \mathcal{E}(Y)$ such that

$$\text{(i)} \qquad \omega|Y = \bar{\partial}(g)$$

Next, let μ be a Riesz measure with compact support contained in K such that

$$\text{(ii)} \qquad \int f \cdot d\mu = 0 \quad : f \in \mathcal{O}(Y)$$

For a given ω in (i) the g-function is unique up to $\mathcal{O}(Y)$ and hence (ii) entails that

$$\int g \cdot d\mu$$

only depends on ω. This gives a current S of bi-degree (1,0) acting on smooth (0,1)-forms ω via the rule above. Next, consider an open chart Δ contained in $X \setminus \text{supp}(\mu)$. If $g \in C_0^\infty(X)$ has compact support in Δ we take $\omega^{0,1} = \bar{\partial}(g)$ and since $g = 0$ on the support of μ one has

$$\text{(iii)} \qquad S(\bar{\partial}(g)) = 0$$

Since this hold for every pair (Δ, g) as above we conclude that S is $\bar{\partial}$-closed in $X \setminus \text{supp}(\mu)$ and the elliptic property of $\bar{\partial}$ gives a holomorphic 1-form σ in $X \setminus \text{supp}(\mu)$ such that

$$\text{(iv)} \qquad S(\omega^{0,1}) = \int \omega^{0,1} \wedge \sigma$$

holds when $\omega^{0,1}$ has compact support in $X \setminus \text{supp}(\mu)$.

Consider a point $p \in X \setminus K$. Since $K = \mathfrak{h}(K)$ there exists a non-compact open and connected subset Ω of $X \setminus\ K$ which contains p. Now the current S has a compact support which implies that

$$\text{(v)} \qquad \Omega \setminus \text{Supp}(S) \neq \emptyset$$

Let q be a point in this set and choose a small open chart Δ centered at q which stays in Ω and is disjoint from the support of S. Then

$$\int \omega^{0,1} \wedge \sigma = 0$$

for every test-form $\omega^{0,1}$ with compact support in Δ. Hence the restriction of σ to Δ is zero. Since σ is a holomorphic 1-form and Ω is connected, analyticity gives $\sigma|\Omega = 0$. In particular $\sigma = 0$ holds in a neighborhood of p which proves the inclusion

$$\text{(vi)} \qquad \text{Supp}(S) \subset K$$

Finally, let $f \in \mathcal{O}(K)$ which by defintion means that it is a holomorphic function defined in some open neighborhood U of K. There exist a test-function g with compact support in Y such that $g = f$ holds in a neighborhood of K. Then $\bar{\partial}(g)$ vanishes in a neighborhood of K so (vi) and the construction of S entail that

$$\text{(vii)} \qquad 0 = S(\bar{\partial} g) = \int_K g \cdot d\mu = \int_K f \cdot d\mu$$

Since μ was an arbitrary Riesz meausure for which (ii) holds and we have proved that (ii) implies (vii) for every $f \in \mathcal{O}(K)$, we get Proposition 5.1 from the Hahn-Banach theorem.

Remark. Above we have followed a proof given by Malgrange.

6. Proof of the Behnke-Stein theorem.

By § xx we can exhaust the open Riemann surface X by an increasing sequence of Runge domains $\{Y_n\}$ where the closure of Y_n is a compact subset of Y_{n+1} for each n. Consider some $\phi \in \mathcal{E}^{0,1}(X)$. By Theorem 1.3 the restriction of ϕ to Y_n is $\bar{\partial}$-exact for every n. Hence there exists $f_n \in \mathcal{E}(Y_n)$ such that

$$\bar{\partial} f_n = \phi | Y_n \tag{6.1}$$

It follows that the restriction to Y_n of $f_{n+1} - f_n$ is holomorphic and since Y_n is Runge and $\overline{Y}_{n-1}$ is a compact subset, Proposition 5.1 entails that $f_{n+1} - f_n$ can be uniformly approximated on Y_{n-1} by functions from $\mathcal{O}(Y_{n+1})$. Hence there exists $g_{n+1} \in \mathcal{O}(Y_{n+1})$ such that the maximum norm

$$||g_{n+1} + f_{n+1} - f_n||_{Y_{n-1}} < 2^{-n}$$

With $f^*_{n+1} = f_{n+1} + g_{n+1}$ we still have

$$\bar{\partial} f^*_{n+1} = \phi | Y_{n+1}$$

Perfoming these modifications inductively with increasing n we obtain a sequence $\{f^*_n\}$ such that (6.1) hold for every n and

$$||f^*_{n+1} - f^*_n||_{Y_{n-1}} < 2^{-n} \quad : \; n = 1, 2, \ldots \tag{6.2}$$

It follows that $\{f^*_n\}$ converge uniformly on X to a continuous function F which entails that the distributions $\{\bar{\partial} f^*_n\}$ converge to the distribution derivative $\bar{\partial}(F)$. Moreover, since (6.2) hold for every n it is clear that

$$\bar{\partial}(F) = \phi \tag{6.3}$$

holds everywhere in X. Finally, ϕ is a smooth form so the elliptic property of $\bar{\partial}$ implies that F is a C^∞-function. Hence the mapping $\bar{\partial} \colon \mathcal{E}(X) \to \mathcal{E}^{(0,1)}(X)$ is surjective which proves the Behnke-Stein theorem.

§ 7. Further results on non-compact Riemann surfaces.

A locally free sheaf of $\mathcal{O}$-modules with rank one is called a holomorphic line bundle. Let $\mathcal{L}$ be such a sheaf. By definition it has local trivialisations, i.e. we can find a locally finite covering $\mathfrak{U}$ by charts such that the restricted sheaf $\mathcal{L}|U_\alpha \simeq \mathcal{O}$. In each non-empty intersection $U_\alpha \cap U_\beta$ we find a zero free function $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ so that the space $\mathcal{L}(V)$ of sections over an open subset V in X consists of a family $\{\xi_\alpha \in \mathcal{O}(U_\alpha \cap V)\}$ satisfying th glueing conditions

$$\xi_\alpha | U_\alpha \cap U_\beta \cap V = g_{\alpha\beta} \cdot \xi_\beta | U_\alpha \cap U_\beta \cap V \tag{1}$$

Now one can repeat the methods which were used in the previius sections and extend the Behnke-Stein theorem as follows:

7.1 Theorem. *For every holomorphic line bundle $\mathcal{L}$ one has $H^1(X, \mathcal{L}) = 0$.*

Example. Notice that the sheaf Ω of holomorphic 1-forms is a holomorphic line bundle. Another example is the sheaf Θ_X whose sections are holomiorphic vector fields. here we notice that

$$\Omega = Hom_{\mathcal{O}}(\Theta_X, \mathcal{O})$$

is the dual line bundle. So Theorem 7.1 entails that

$$H^1(X, \Omega) = H^1(X, \Theta_X) = 0 \tag{7.2}$$

7.3 Weierstrass' Theorem

Let D be a divisor on X which means that integers $\{k_\nu\}$ are assigned to a discrete sequence of points $\{p_\nu\}$. We shall prove that there exists a globally deefined meromorphic function ψ whose principal divisor is D. To begin with one has:

7.3.1 Exercise. Use a C^∞-partition of the unity to find a function ϕ which is zero-free and C^∞ in the open complement of the discrete set $\{p_\nu\}$, and for each p_ν there is a small open chart Δ_ν centered at p_ν with a local coordinate z_ν where

$$\phi | \Delta_\nu = z_\nu^{k_\nu}$$

Next, we find a locally finite covering $\mathfrak{U} = \{U_\alpha\}$ of charts in X which contains the discs $\{\Delta_\nu\}$ while each U_α which differs from these discs does not contain points in $\{p_\nu\}$. For each U_α which does not contain any point in $\{p_\nu\}$ we take $f_\alpha = 1$ while $f_\alpha = z_\nu^{k_\nu}$ when $U_\alpha = \Delta_\nu$. Then we can write

$$\phi | U_\alpha = f_\alpha \cdot \psi_\alpha \quad : \psi_\alpha \in C^\infty(U_\alpha) \tag{i}$$

Since ϕ is zero-free outside $\{p_\nu\}$ we see that every ψ-function is zero-free. Next, since every chart U_α is simply connected there exist $\{\rho_\alpha \in C^\infty(U_\alpha)\}$ such that

$$\psi_\alpha = e^{\rho_\alpha} \tag{ii}$$

In each non-empty intersection $U_\alpha \cap U_\beta$ it follows that

$$e^{\rho_\alpha - \rho_\beta} = \frac{f_\beta}{f_\alpha} \tag{iii}$$

Each quotient $\frac{f_\beta}{f_\alpha} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and then (iii) entails that the restricted differences

$$\rho_\alpha - \rho_\beta | U_\alpha \cap U_\beta) \in \mathcal{O}(U_\alpha \cap U_\beta)$$

So with $g_{\alpha\beta} = \rho_\alpha - \rho_\beta | U_\alpha \cap U_\beta)$ we get a Cech-cocycle in $H^1(\mathfrak{U}, \mathcal{O})$. The Behnke-Stein theorem entails that it is a δ-coboundary which give $\{g_\alpha \in \mathcal{O}(U_\alpha)\}$ such that

$$\rho_\alpha - \rho_\beta | U_\alpha \cap U_\beta) = g_\alpha - g_\beta \tag{iv}$$

Taking exponentials in (iv) and using (iii) it follows that the following hold in $U_\alpha \cap U_\beta$:

$$e^{g_\alpha} \cdot f_\alpha = e^{g_\beta} \cdot f_\beta \tag{v}$$

This gives a globally defined meromorphic function ψ such that

(vi) $$\psi|U_\alpha = e^{g_\alpha} \cdot f_\alpha$$

Finally, the construction of the f-functions implies that the principal divisor $D(\psi)$ is equal to the given divisor D which finishes the proof of Weierstrass' theorem.

7.3.2 The construction of a non-vanishing holomorphic form. Take a non-zero global section ω in $\Omega(X)$. If (Δ, z) is a chart we have

$$\omega|\Delta = g(z) \cdot dz$$

where $g \in \mathcal{O}(\Delta)$ is non-constant and hence has a discrete set of zeros. The locally defined divisors of these g-functions match each other and yield a globally defined divisor $D = \{(p_\nu; k_\nu)\}$ in X. Notice that D is positive, i.e. the integers $k_\nu > 0$. Weierstrass's theoren gives $\psi \in \mathcal{O}(X)$ with $D(\psi) = D$. Then it is clear that

$$\omega^* = \psi^{-1} \cdot \omega$$

yields a holomorphic 1-form whose associated g-functions in local charts are zero-free.

§ 8. The uniformisation theorem

Consider a non-compact Riemann surface X where every holomorphic 1-form is ∂-exact. By (7.3.2) there exists $\omega \in \Omega(X)$ which is everywhere non-zero. So if Y is an open set in X and $\mu \in \Omega(Y)$ there exists $g \in \mathcal{O}(Y)$ such that

(i) $$\mu = g \cdot \omega_Y$$

where ω_Y is the restriction of Ω to Y. We use this to prove

8.1 Proposition. *If Y is an open Runge set in X one has*

(8.1.1) $$\Omega(Y) = \partial(\mathcal{O}(Y))$$

Proof. Let μ be as in (i). Consider a point $y_0 \in Y$ and a closed curve γ in Y which has y_0 as initial and terminal point. Since Y is Runge, Theorem § XX gives a sequence $\{g_n\}$ in $\mathcal{O}(X)$ and the maximum norms $|g_n - g|_\gamma \to 0$. It follows that

$$\int_\gamma \mu = \lim_{n\to\infty} \int_\gamma g_n \cdot \omega$$

Here γ is a closed curve in X and since each $g_n \cdot \omega$ belong to $\Omega(X)$ the hypothesis on X implies that the integrals in the right hand side above are all zero. Hence the line integrals of μ are zero for all closed curves at y_0 and then § xx shows that it is ∂-exact in Y which finishes the proof.

8.2 Semi-local conformal mappings. Let Y be a relatively compact open subset of X which is Runge and Dirichlet regular. Let $y_0 \in Y$ and by §§ there exists $g \in \mathcal{O}(X)$ with a simple zero at y_0 and zero-free in $X \setminus \{y_0\}$. Now $\mathfrak{Re}(g)$ restricts to a continuous function on ∂Y and we find a harmonic function u in Y whose boundary value function is $\log|g|$. The equality (5.1.1) and the remark in § xx entail that u has a harmonic conjugate v in Y which gives $u + iv \in \mathcal{O}(Y)$. Set

$$f = g \cdot e^{-(u+iv)}$$

On ∂Y it follows that

$$|f| = |g| \cdot e^{-\log|g|} = 1$$

At the same time f has a simple zero at y_0 and is zero-free in $Y \setminus \{y_0\}$. The result in § xx shows that $f\colon Y \to D$ is biholomorphic. Around y_0 we choose a small chart (Δ, z) and get the non-zero complex derivative $f'(0)$ of f at $z = 0$. With

(8.2.1) $$f^* = \frac{1}{f'(0)} \cdot f$$

we have a biholomorphic mapping from Y into the open disc of radius $|f'(0)|$ and the f^*-derivative at $z = 0$ is one.

8.3 A global construction. By § xx we can exhaust X with an increasing sequence of relatively compact open subsets $\{Y_n\}$ which are Runge and Dirichlet regular. Fix a point $p \in Y_0$ and a local chart (Δ, z) as above. For each $n \geq 0$ (8.2.1) gives some $r_n > 0$ and a biholomorphic map

$$f_n \colon Y_n \to D(r_n)$$

where $f_n(p) = 0$ and $f_n'(0) = 1$. The inverse mapping f_n^{-1} is holomorphic in the open disc $D(r_n)$ and we notice that the complex derivative

$$\text{(i)} \qquad (f_n^{-1})'(0) = 1$$

Next, to each $n \geq 0$ we have the composed function

$$h_n = f_{n+1} \circ f_n^{-1}$$

which maps $D(r_n)$ into $D(r_{n+1})$ where $h_n(0) = 0$ and (i) gives $h_n'(0) = 1$. Schwarz' inequality implies that

$$r_n \leq r_{n+1}$$

Set

$$R = \lim_{n \to \infty} r_n$$

We shall construct a biholomorphic map $f^* \colon X \to D(R)$ where the case $R = +\infty$ means that $D(+\infty) = \mathbf{C}$. To find f^* we shall pass to certain subsequences of $\{f_n\}$. Consider first the composed functions

$$g_n = \frac{1}{r_0} \cdot f_n \circ f_0^{-1}(r_0 z)$$

which by the choice of r_0 and the inclusion $Y_0 \subset Y_n$ are holomorphic in the open unit disc D of the complex z-plane. It is clear that we have

$$g_n(0) = 0 \quad : g_n'(0) = 1$$

Moreover, g_n is a biholomorphic map from D onto the open set $r_0^{-1} \cdot f_n(Y_0)$. By the result from § xx $\{g_n\}$ is a normal family in the sense lof Montel. Hence we can extract a subsequence $\{g_{n_k}\}$ which converges uniformly on compact subsets of D.

Exercise. Verify that the convergence of $\{g_{n_k}\}$ above entails that $\{f_{n_k}\}$ converges uniformly over compact subsets of Y_0 to a limit function $f_0^* \in \mathcal{O}(Y)$. Moreover, since each restriction $f_{n_k}|Y_0$ is biholomorphic and the derivatives f_{n_k}' at p all are equal to one, it follows from the result in § xx that $f_0^* \colon Y_0 \to f_0^*(Y_0)$ is biholomorphic.

Next, starting with the sequence

$$g_k = r_1^{-1} \cdot f_{n_k} \circ f_1^{-1}$$

we repeat the arguments above and find a subsequence $\{\phi_{k_j}\}$ which conveges uniformly in the unit disc and returning to the f-sequence, its corresponding subsequence converges uniformly on Y_1 to a limit function f_1^* which yields a biholomorphic map from Y_1 onto $f_1^*(Y_1)$. We continue by an induction over n and obtain biholomorphic maps $\{f_n^* \colon Y_n \to f_n^*(Y_n)\}$ and after the passage to a diagonal subsequence we find a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of x to a limit function f^* which gives the requested biholomorphic map from X onto $D(R)$.

Representations of fundamental groups.

Let X be an open Riemann surface which is not simply connected, i.e. the fundamental group $\pi_1(X) \neq 0$. Recall that it can be identfied with homotopy classes of closred and differentiable curves which start and terminte at a given point $p_* \in X$. Let f be a globally defined and zero-free holomorphic function. Then there exists the holomorphic 1-form $f^{-1}\partial f$ which can be integrated along every closed curve γ above. The monodromy theorem shows that

$$\int_\gamma f^{-1}\partial f \tag{1}$$

only depends on the homotopy class of γ. Moreover, locally $f = e^g$ where g is determined up to an integer multiple of 2π. Covering γ by a finite sequence of charts it follows that (1) is an integer times $2\pi i$. So if $\{\gamma\}$ is the homotopy class we get a map

$$\{\gamma\} \mapsto \frac{1}{2\pi i} \cdot \int_\gamma f^{-1}\partial f \tag{2}$$

from $\pi_1(X)$ into the additive group of integers.

Theorem. *Every group homomorphism from $\pi_1(X)$ into $\mathbf{Z}$ is given by (2) for a zero-free holomorphic function f.*

The proof follows from the Behnke-Stein theorem. To begin with one regards the universal covering space of X which gives a 1-1 correspondence between additive group homeomorphisms of $\pi_1(X)$ and the cohomology group $H^1(X; \mathbf{Z}_X$. Next, one has the exact sheaf sequence

$$0 \to \mathbf{Z}_X \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0$$

Since $H^1(X, \mathcal{O}) = 0$ it follows that

$$H^1(X; \mathbf{Z}_X) \simeq \frac{\mathcal{O}^*(X)}{\exp(\mathcal{O}(X))}$$

The theorem follows from the observation that a zero-free holomorphic function f on X is of the form e^g with $g \in \mathcal{O}(X)$ if and only if the integrals in (ii) are zero for all closed curves γ. Here one direction is clear, i.e. if $f = e^g$ we have

$$f^{-1}\partial f = \partial g$$

and use that line integrals of exact 1-forms taken along a closed curve are zero. Conversely, if (2) is zero for all γ there exists a holomorphic function g on X defined by

$$g(x) = \int_{p_*}^x f^{-1}\partial f$$

where the integral is taken ovar a curve starting at p_* and terminating at x and itis clear that $\partial g = f^{-1}\partial f$ which finishes the proof.

Appendix. Linear operators and spectral theory.

Introduction. We shall expose basic facts about spectra of densely defined linear operators on Banach spaces. Let us remark that the special case of bounded linear operators is incorporated in Neumann's general theory as a special case. The major result is Theorem A.6.3 which extends thr calculus for matrices with complex elements where the spectrum is finite and resolvent operators are expressed via Cayley-Hamilton equations. In addition to the basic material in § A.2-A.7 below we include a result about an inequality for resolvent operators in § A.8. In contrast to the previous material the proof is more involved but has been inserted to give a glimpse of more advanced results in spectral theory. At the same time Carleman's proof offers a very instructive lesson.

The finite dimensional case. Let $T\colon V \to V$ be a linear operator on a finite dimensional complex vector space V of some dimension $N \geq 2$. Let E be the identity operator and with a complex number λ there exists the determinant polynomial

$$P_T(\lambda) = \det(\lambda \cdot E - T)$$

This is a monic polynomial in λ of degree N which we write as

(1) $$\lambda^N + c_{N-1}\lambda^{N-1} + \ldots + c_1\lambda + c_0$$

A fundamental fact in matrix calculus is Hamilton's equation

(*) $$T^N + c_{N-1} \cdot T^{N-1} + \ldots + c_1 \cdot T + c_0 \cdot E = 0$$

The zeros of the determinant polynomial is the spectrum $\sigma)T)$. If μ is outside the spectrum there exists the inverse linear operator

$$R_T(\mu) = (\mu \cdot E - T)^{-1}$$

A crucial point is that

(2) $$\mu \mapsto R_T(\mu)$$

is an analytic matrix-valued function in $\mathbf{C} \setminus \sigma(T)$. The analyticity is expressed by saying that the N^2 many elements of $R_T(\mu)$ are complex-valued analytic functions of μ.

Cayley's equation. If μ is outside the spectrum we have the polynomial

$$\lambda \mapsto \det((\lambda - \mu) \cdot E - T) = P_T(\lambda - \mu)$$

which via (1) is of the form

(iii) $$\lambda^N + c_{N-1}(\mu) \cdot (\lambda^{N-1} + \ldots + c_1(\mu) \cdot \lambda + c_0(\mu)$$

where $\{c_k(\mu)\}$ are polynomials in μ. Here

$$c_0(\mu) = (-1)^N \cdot \det(\mu \cdot E - T) \neq 0$$

and the Hamilton equation gives

(**) $$(\mu \cdot E - T)^N + c_{N-1} \cdot (\mu \cdot E - T)^{N-1} + \ldots + c_1 \cdot (\mu \cdot E - T) + c_0 \cdot E = 0$$

This entails that

(***) $$R_T(\mu) = \frac{(-1)^{N+1}}{c_0(\mu)} \cdot \big[(\mu \cdot E - T)^{N-1} + c_{N-1}(\mu) \cdot (\mu \cdot E - T)^{N-2} + \ldots + c_1(\mu) \cdot E\,\big]$$

One refers to the right hand side as Cayley's equation for the resolvent $R_T(\mu)$ which exibits the function in (2) as a polynomial of degree $N-1$ in T whose coefficients are rational functons of μ with poles confined to $\sigma(T)$.

A.0 Densely defined operators. The Cayley-Hamilton calculus extends verbatim to a quite general case using Neumann's definition of resolvent operators in A.1 below. Let X be a Banach space and $T\colon X \to X$ a linear and densely defined operator whose domain of definition is denoted by $\mathcal{D}(T)$. In general T is unbounded:

$$\max_{x \in \mathcal{D}_*(T)} ||T(x)|| = +\infty$$

where the maximum is taken over unit vectors in $\mathcal{D}(T)$.

A.1 Inverse operators. A densely defined operator T has a bounded inverse if the range $T(\mathcal{D}(T))$ is equal to X and there exists a positive constant c such that

(i) $$||T(x)|| \geq c \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

Since $T(\mathcal{D}(T)) = X$, (i) gives for each $x \in X$ a unique vector $R(x) \in \mathcal{D}(T)$ such that

(ii) $$T \circ R(x) = x$$

Moreover, the inequality (i) gives

(iii) $$||R(x)|| \leq c^{-1} \cdot ||x|| \quad : x \in X$$

and when R is applied to the left on both sides in (ii), it follows that

(iv) $$R \circ T(x) = x \quad : x \in \mathcal{D}(T)$$

A.2 The spectrum $\sigma(T)$. Let E be the identity operator on X. Each complex number λ gives the densely defined operator $\lambda \cdot E - T$. If it fails to be invertible one says that λ is a spectral point of T and denote this set by $\sigma(T)$. If $\lambda \in \mathbf{C} \setminus \sigma(T)$ the inverse to $\lambda \cdot E - T$ is denoted by $R_T(\lambda)$ and called a Neumann resolvent to T. By the construction in (A.1) the range of every Neumann resolvent is equal to $\mathcal{D}(T)$ and one has the equation:

(A.2.1) $$T \circ R_T(\lambda)(x) = R_T(\lambda) \circ T(x) \quad : x \in \mathcal{D}(T)$$

Example. Let X be the Hilbert space ℓ^2 whose vectors are complex sequences $\{c_1, c_2, \ldots\}$ for which $\sum |c_n|^2 < \infty$. We have the dense subspace ℓ^2_* vectors such that $c_n \neq 0$ only occurs for finitely many integers n. If $\{\xi_n\}$ is an arbitrary sequence of complex numbers there exists the densely defined operator T on ℓ^2 which sends every sequence vector $\{c_n\} \in \ell^2_*$ to the vector $\{\xi_n \cdot c_n\}$. If λ is a complex number the reader may check that (i) holds in (A.1) if and only if there exists a constant C such that

(i) $$|\lambda - \xi_n| \geq C \quad : n = 1, 2, \ldots$$

Thus, $\lambda \cdot E - T$ has a bounded inverse if and ony if λ belongs to the open complement of the closure of the set $\{\xi_n\}$ taken in the complex plane. Moreover, if (i) holds then $R_T(\lambda)$ is the bounded linear operator on ℓ^2 which sends $\{c_n\}$ to $\{\frac{1}{\lambda - \xi_n} \cdot c_n\}$. Since every closed subset of $\mathbf{C}$ is equal to the closure of a denumerable set of points our construction shows that the spectrum of a densely defined operator $\sigma(T)$ can be an arbitrary closed set in $\mathbf{C}$. The equation below is due to G. Neumann:

A.3 Neumann's equation. *Assume that $\sigma(T)$ is not the whole complex plane. For each pair $\lambda \neq \mu$ outside $\sigma(T)$ the operators $R_T(\lambda)$ and $R_T(\mu)$ commute and*

(*) $$R_T(\mu) R_T(\lambda) = \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu}$$

Proof. Notice that

$$(\mu \cdot E - T) \cdot \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} =$$

(i) $$\frac{E}{\lambda - \mu} - (\mu - \lambda) \cdot \frac{R_T(\lambda)}{\lambda - \mu} - (\lambda \cdot E - T) \cdot \frac{R_T(\lambda)}{\lambda - \mu} = R_T(\lambda)$$

Multiplying to the left by $R_T(\mu)$ gives (*) which at the same time this shows that the resolvent operators commute.

A.4 The position of $\sigma(T)$**.** Assume that $\mathbf{C} \setminus \sigma(T)$ is non-empty. We can write (*) in the form

(1) $$R_T(\lambda)(E + (\lambda - \mu) R_T(\mu)) = R_T(\mu)$$

Keeping μ fixed we conclude that $R_T(\lambda)$ exists if and only if $E+(\lambda-\mu)R_T(\mu))$ is invertible which implies that

$$\text{(A.4.1)} \qquad \sigma(T)=\{\lambda : \frac{1}{\mu-\lambda}\in\sigma(R_T(\mu))\}$$

Hence one recovers $\sigma(T)$ via the spectrum of any given resolvent operator. Notice that (A.4.1) holds even when the open component of $\sigma(T)$ has several connected components.

A.4.2 Example. Suppose that $\mu=i$ and that $\sigma(R_T(i))$ is contained in a circle $\{|\lambda+i/2|=1/2\}$. If $\lambda\in\sigma(T)$ the inclusion (A.4.1) gives some $0\leq\theta\leq 2\pi$ such that

$$\frac{1}{i-\lambda}=-i/2+1/2\cdot e^{i\theta}\implies 1-i\cdot e^{i\theta}=\lambda(e^{i\theta}-i)\implies$$

$$\lambda=\frac{2\cdot\cos\theta}{|e^{i\theta}-i|^2}\in\mathbf{R}$$

A.4.3 Neumann series. Let λ_0 be outside $|sigma(T)$ and construct the operator valued series

$$\text{(1)} \qquad S(\zeta)=R_T(\lambda_0)+\sum_{n=1}^{\infty}(-1)^n\cdot\zeta^n\cdot R_T(\lambda_0)^{n+1}$$

It is clear that (1) converges in the Banach space of bounded linear operators when

$$\text{(2)} \qquad |\zeta|<\frac{1}{||R_T(\lambda_0)||}$$

Moreover, the series expansion (1) gives

$$\text{(23)} \qquad (\lambda_0+\zeta-T)\cdot S(\zeta)=(\lambda_0-T)\cdot S(\zeta)+\zeta S(\zeta)=E$$

Hence $S(\zeta)=R_T(\lambda_0+\zeta)$ and the locally defined series in (1) entail the complement of $\sigma(T)$ is open where $\lambda\mapsto R_T(\lambda)$ is an analytic operator-valued function. Finally (*) in (A.3) and a passage to the limit as $\mu\to\lambda$ shows that this analytic function satisfies the differential equation

$$\text{(**)} \qquad \frac{d}{d\lambda}(R_T(\lambda)=-R_T^2(\lambda)$$

A.5. Operators with closed graph.

Let T be a densely defined operator. In the product $X\times X$ we get the graph:

$$\text{(A.5.1)} \qquad \Gamma(T)=\{(x,Tx)\quad : x\in\mathcal{D}(T)\}$$

If $\Gamma(T)$ is a closed subspace of $X\times X$ we say that T is closed.

A.5.1 Exercise. Let T be densely defined and assume that $\sigma(T)$ is not the whole complex plane. Show that T is automatically closed.

In the study of spectra we shall foremost restrict the attention to closed operators. Assume that T is densely defined and closed. Let λ be a complex number such that (i) holds in (A.1) for some constant c and the range of $\lambda\cdot E-T$ is dense.

A.5.2 Exercise. Show that the closedness of T implies that the range of $\lambda\cdot E-T$ is equal to X so that $R_T(\lambda)$ exists. A hint is that if $y\in X$, then the density gives a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $\xi_n=\lambda\cdot x_n-T(x_n)\to y$. In particular $\{\xi_n\}$ is a Cauchy- sequence By (i) in (A.1) we have

$$||x_n-x_m||\leq c^{-1}\cdot||\xi_n-\xi_m||$$

Hence $\{x_n\}$ is a Cauchy sequence and since X is a Banach space there exists $x\in S$ such that $x_n\to x$. But then $\{(x_n,T(x_n))=(x_n,\lambda\cdot x_n-\xi_n)$ converges to $(x,\lambda\cdot x-y)$ and since T is closed it follows that $(x,\lambda\cdot x-y)\in\Gamma(T)$ which gives the requested surjectivity since

$$T(x)=\lambda\cdot x-y\implies y=(\lambda\cdot E-T)(x)$$

A.5.3 Adjoints. Let T be densely defined but not necessarily closed. In the dual space X^* we get the subspace of vectors y for which there exists a constant $C(y)$ such that

(i) $$|y(Tx)| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

When (i) holds the density of $\mathcal{D}(T)$ gives a unique vector $T^*(y)$ in X^* such that

(ii) $$y(Tx) = T^*(y)(x) \quad : x \in \mathcal{D}(T)$$

One refers to T^* as the adjoint operator of T whose domain of definition is denoted by $\mathcal{D}(T^*)$.

Exercise. Show that T^* has a closed graph.

A.5.4 Closed extensions. Let T be densely defined but not closed. The question arises when the closure of $\Gamma(T)$ is the graph of a linear operator $\widehat{T}$ and then we refer to $\widehat{T}$ as a closed extension of T. A sufficient condition for the existence of a close extension goes as follows:

A.5.5 Theorem. *If $\mathcal{D}(T^*)$ is dense in X^* then T has a closed extension.*

Proof. Consider the graph $\Gamma(T)$ and let $\{x_n\}$ and $\{\xi_n\}$ be two sequences in $\mathcal{D}(T)$ which both converge to a point $p \in X$ while $T(x_n) \to y_1$ and $T(\xi_n) \to y_2$ hold for some pair y_1, y_2. We must rove that $y_1 = y_2$. To achieve this we take some $x^* \in \mathcal{D}(T^*)$ which gives

$$x^*(y_1) = \lim\, x^*(Tx_n) = \lim\, T^*(x^*)(x_n) = T^*(x^*)(p)$$

In the same way we get $x^*(y_2) = T^*(x^*)(p)$. Now the density of $\mathcal{D}(T^*)$ gives $y_1 = y_2$ which proves that the closure of $\Gamma(T)$ is a graphic subset of $X \times X$ and gives the closed operator $\widehat{T}$ with

$$\Gamma(\widehat{T}) = \overline{\Gamma(T)}$$

A.5.6 Exercise. Let T be closed and densely defined. There may exist several closed operators S with the property that

$$\Gamma(T) \subset \Gamma(S)$$

Show the equaity

$$T^* = S^*$$

for every closed extension S.

A.5.7 The case when X is reflexive. Recall that a Banach space is reflexive if the bi-dual $X^{**} = X$. Suppose this holds and let T be a linear operator where both $\mathcal{D}(T)$ and $\mathcal{D}(T^*)$ are dense. If $x \in \mathcal{D}(T)$ it yields $x^{**}|inX^{**}$ and by defintion

$$x^{**}(T^*(y)) = y(Tx) \quad : y \in \mathcal{D}(T^*)$$

It follows that $x^{**} \in \mathcal{D}(T^{**}$ and we conclude that T^{**} extends T in the sense that $\Gamma(T) \subset \Gamma(T^{**})$ and from the above T^{**} yields a closed extension of T,.

A.6 Operational calculus.

Let T be densely defined and closed. To each pair (γ, f), where γ is a rectifiable Jordan arc contained in $\mathbf{C} \setminus \sigma(T)$ and $f \in C^0(\gamma)$, there exists the bounded linear operator

(A.6.1) $$T_{(\gamma,f)} = \int_\gamma f(z) R_T(z)\, dz$$

The integrand has values in the Banach space of bounded linear operators on X and (A.6.1) is calculated by Riemann sums. Next, Neumann's equation (A.3) entails that $R_T(z_1)$ and $R_T(z_2)$ commute for all pairs z_1, z_2 on γ. From this it is clear that if g is another function in $C^0(\gamma)$, then the operators $T_{f,\gamma}$ and $T_{g,\gamma}$ commute. Moreover, for each $f \in C^0(\gamma)$ the reader may verify that the closedeness of T implies that the range of $T_{f,\gamma}$ is contained in $\mathcal{D}(T)$ and

$$T_{f,\gamma} \circ T(x) = T \circ T_{f,\gamma}(x) \quad : x \in \mathcal{D}(T)$$

Next, let Ω be an open set of class $\mathcal{D}(C^1)$, i.e. $\partial\Omega$ is a finite union of closed differentiable Jordan curves. When $\partial\Omega \cap \sigma(T) = \emptyset$ we construct the line integrals (A.6.1) for continuous functions on the boundary. Consider the algebra $\mathcal{A}(\Omega)$ of analytic functions in Ω which extend to be continuous on the closure. Each $f \in \mathcal{A}(\Omega)$ gives the operator

$$T_f = \int_{\partial\Omega} f(z) R_T(z)\, dz \tag{A.6.2}$$

A.6.3 Theorem. *The map $f \mapsto T_f$ is an algebra homomorphism from $\mathcal{A}(\Omega)$ into a commutative algebra of bounded linear operators on X whose image is a commutative algebra of bounded linear operators denoted by $T(\Omega)$.*

Proof. Let f, g be a pair in $\mathcal{A}(\Omega)$. We must show the equality

$$T_{gf} = T_f \circ T_g \tag{*}$$

To attain this we choose a slightly smaller open set $\Omega_* \subset \Omega$ which again is of class $\mathcal{D}(C^1)$ and each of it bounding Jordan curve is close to one boundary curve in $\partial\Omega$ and $\Omega \setminus \Omega_*$ does not intersect $\sigma(T)$. By Cauchy's theorem we can shift the integration to $\partial\Omega_*$ and get

$$T_g = \int_{\partial\Omega_*} g(z) R_T(z_*)\, dz_* \tag{i}$$

where we use z_* to indicate that integration takes place along $\partial\Omega_*$. Now

$$T_f \circ T_g = \iint_{\partial\Omega_* \times \partial\Omega} f(z) g(z_*) R_T(z) \circ R_T(z_*)\, dz_* dz \tag{ii}$$

Neumann's equation (*) from (A.3) entails that the right hand side in (ii) becomes

$$\iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z_*)}{z - z_*}\, dz_* dz + \iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z) g(z_*) R_T(z)}{z - z_*}\, dz_* dz = A + B \tag{iii}$$

Here A is evaluated by first integrating with respect to z and Cauchy's theorem gives

$$f(z_*) = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega} \frac{f(z)}{z - z_*} \quad : z_* \in \partial\Omega_*\, dz$$

It follows that

$$A = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega_* \times \partial\Omega} f(z_*) g(z_*) R_T(z_*)\, dz_* = T_{fg}$$

Next, B is evaluated when we first integrate with respect to z_*. Here

$$\iint_{\partial\Omega} \frac{g(z_*)}{z - z_*} \quad : z \in \partial\Omega$$

which entails that $B = 0$ and the theorem follows.

A.7 Spectral gap sets.

Let K be a compact subset of $\sigma(T)$ such that $\sigma(T) \setminus K$ is a closed set in $\mathbf{C}$. This implies that if V is an open neighborhood of K, then there exists a relatively compact subdomain $U \in \mathcal{D}(C^1)$ which contains K as a compact subset while $\partial U \cap \sigma(T) = \emptyset$. To every such domain U we can apply Theorem A.6.3. If $U_* \subset U$ for a pair of such domains we can restrict functions in $\mathcal{A}(U)$ to U_*. This yields an algebra homomorphism

$$\mathcal{T}(U) \to \mathcal{T}(U_*)$$

Next, denote by $\mathcal{O}(K)$ the algebra of germs of analytic functions on K. So each $f \in \mathcal{O}(K)$ comes from some analytic function in a domain U as above. The resulting operator $T_U(f)$ depends on the germ f only. In fact, this follows because if $f \in \mathcal{A}(U)$ and $U_* \subset U$ is a similar $\mathcal{D}(C^1)$-domain which again contains K, then Cauchy's vanishing theorem in analytic function theory applies to $f(z)R_T(z)$ in $U \setminus \bar{U}_*$ and entails that

$$\int_{\partial U_*} f(z) R_T(z)\, dz = \int_{\partial U} f(z) R_T(z)\, dz$$

Hence there exists an algebra homorphism from $\mathcal{O}(K)$ into a commutative algebra of bounded linear operators on X denoted by $\mathcal{T}(K)$. The identity in $\mathcal{T}(K)$ is denoted by E_K and called the spectral projection operator attached to the compact set K in $\sigma(T)$. By this construction one has

$$E_K = \frac{1}{2\pi i} \cdot \int_{\partial U} R_T(z)\, dz$$

for every open domain U surrounding K as above.

A.7.1 The operator T_K. When K is a compact spectral gap set of T we set

$$T_K = TE_K$$

This is a bounded linear operator given by

$$\frac{1}{2\pi i} \cdot \int_{\partial U} z \cdot R_T(z)\, dz$$

where U is a domain as above which contains K.

A.7.2 Theorem. *Identify T_K with a linear operator on the subspace $E_K(X)$. Then one has the equality*

$$\sigma(T_K) = K$$

Proof. If λ_0 is outside K we can choose U so that λ_0 is outside $\bar{U}$ and construct the operator

$$S = \frac{1}{2\pi i} \cdot \int_{\partial U} \frac{1}{\lambda_0 - z} \cdot R_T(z)\, dz$$

The operational calculus gives

$$S(\lambda_0 E_K - T) = E_K$$

here E_K is the identity operator on $E_K(X)$ which shows that $\sigma(T_K) \subset K$.

A.7.3 Discrete spectra. Consider a spectral set reduced to a singleton set $\{\lambda_0\}$, i.e. λ_0 is an isolated point in $\sigma(T)$. The associated spectral projection is denoted by $E_T(\lambda_0)$ and given by

$$E_T(\lambda_0) = \frac{1}{2\pi i} \cdot \int_{|\lambda - \lambda_0| = \epsilon} R_T(\lambda)\, d\lambda$$

for all sufficiently small ϵ. Now $R_T(\lambda)$ is an analytic function defined in some punctured disc $\{0 < \lambda - \lambda_0| < \delta\}$ with a Laurent expansion

$$R_T(\lambda) = \sum_{-\infty}^{\infty} (\lambda - \lambda_0)^k \cdot B_k$$

where $\{B_k\}$ are bounded linear operators obtained by residue formulas:

$$B_k = \frac{1}{2\pi i} \cdot \int_{|\lambda| = \epsilon} \frac{R_T(\lambda)}{(\lambda - \lambda_0)^{k+1}}\, d\lambda \quad : \epsilon < \delta$$

Exercise. Show that $R_T(\lambda)$ is meromorphic, i.e. $B_k = 0$ hold when $k << 0$, if and only if there exists a constant C and some integer $M \geq 0$ such that the operator norms satisfy

$$||R_T(\lambda)|| \leq C \cdot |\lambda - \lambda_0)^{-M}$$

Suppose now that R_T has a pole of some order $M \geq 1$ at λ_0 which gives a series expansion

(i) $$R_T(\lambda) = \sum_{1}^{M} \frac{B_{-k}}{(\lambda - \lambda_0)^k} + \sum_{0}^{\infty} (\lambda - \lambda_0)^k \cdot B_k$$

Residue calculus gives

(ii) $$B_{-1} = E_T(\lambda_0)$$

The case of a simple pole. Suppose that $M = 1$. Then it is clear that Operational calculus gives

$$\text{(iii)} \qquad \lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{|\lambda - \lambda_0| = \epsilon} (\lambda - \lambda_0) R(\lambda)\, d\lambda = 0$$

This vanishing and operational calculus entail that

$$(\lambda_0 E - T) E_T(\lambda_0) = 0$$

which means that the ragne of the projection operator $E_T(\lambda_0)$ is equal to the kernel of $\lambda_0 \cdot E - T$, i.e. the set of eigenvectors x for which

$$Tx = \lambda_0 \cdot x$$

The case $M \geq 2$. To begin with residue calculus identifies (iii) with B_2 and at the same time operational calculus which aftera reversed sign gives

$$(\lambda_0 E - T) E_T(\lambda_0) = -B_2$$

Since $E_T(\lambda_0)$ is a projection which commutes with T it follows that

$$E_T(\lambda_0) \cdot B_2 = B_2 \cdot E_T(\lambda_0) = B_2$$

Exercise. Show that if $M \geq 3$ then

$$(\lambda_0 E - T)^k \cdot E_T(\lambda_0) = (-1)^k \cdot B_{k+1} \quad : 2 \leq k \leq M$$

Consider also the subspaces

$$\mathcal{N}_k(\lambda_0) = \{x \colon (\lambda_0 E - T)^k (x) = 0\} \quad : 1 \leq k \leq M$$

and show that they are non-decreasing and for every $k > M$ one has

$$\mathcal{N}_M(\lambda_0) = \{x \colon (\lambda_0 E - T)^k (x) = 0\}$$

A.7.4 The case when $E_T(\lambda_0)$ has a finite dimensional range. Here the operator $T(\lambda_0) = TE_T(\lambda_0)$ acts on this finite dimensional vector space which entails that the nullspaces $\{\mathcal{N}_k(\lambda_0)\}$ above are finite dimensional and if M is the order of the pole one has

$$(T(\lambda_0) - \lambda_0)^M = 0$$

So λ_0 is the sole eigenvalue of $T(\lambda_0)$. Moreover, the finite dimensional range of $E_T(\lambda_0)$ has dimension equal to that of $\mathcal{N}_M(\lambda_0)$.

Exercise. Recall l that finite dimensional subspaces appear as direct sum compoinetns. So if $E_T(\lambda_0)$ is finite dimensional there exists a direct sum decomposition

$$X = E_T(\lambda_0)(X) \oplus E - E_T(\lambda_0)$$

where $V = E - E_T(\lambda_0)$ is a closed subspace of X. Show that V is T-invariant and that there exists some $c > 0$ such that

$$||\lambda_0 x - Tx|| \geq ||x|| \quad x \in V \cap \mathcal{D}(T)$$

A.8 Carleman's inequality.

In many applications one is interested to estimate the operator norm of resolvent operators. In this connection a useful result was proved by Carleman in the article *Zur theorie der linearen Integralgleuchungen*[Mathematische Zeitschrift. 1921]. Let us remark that the finite dimensional case in the theorem below easily extends to analogue results for operators on infinite-dimensional Hilbert spaces. See Chapter XI in [Dunford-Schwartz] for an account and applications of Carleman's inequality in Theorem A.8.1 below. First we recall:

The Hilbert-Schmidt norm. Let n be a positive integer and $A = \{a_{ik}\}$ some $n \times n$-matrix whose elements are complex numbers. Set

$$||A|| = \sqrt{\sum\sum |a_{ik}|^2}$$

where the double sum extends over all pairs $1 \leq i, k \leq n$. Next, for a linear operator S on $\mathbf{C}^n$ its *operator norm* is defined by

$$\text{Norm}[S] = \max_x ||S(x)|| \quad \text{with the maximum taken over unit vectors.}$$

Identifying a matrix A with a linear operator on the Hermitian vector space $\mathbf{C}^n$, it is clear that

$$\text{Norm}[A] \leq ||A||$$

Examples show that the inequality in general is strict. Let $\lambda_1 \ldots, \lambda_n$ be the unordered n-tuple of roots of $\det(\lambda \cdot E_n - A)$ with eventual multiple zeros repeated. The union of these roots give the spectrum $\sigma(A)$.

A.8.1 Theorem. *For each non-zero λ outside $\sigma(A)$ one has the inequality:*

$$|\prod_{i=1}^{i=n}(1 - \frac{\lambda_i}{\lambda})e^{\lambda_i/\lambda}| \cdot \text{Norm}\big[R_A(\lambda)\big] \leq |\lambda| \cdot \exp(\frac{1}{2} + \frac{||A||^2}{2 \cdot |\lambda|^2})$$

The proof requires some preliminary results. First we recall a wellknown inequality due to Hadamard which goes as follows:

1. Hadamard's inequality. *For every matrix A with a non-zero determinant one has the inequality*

$$\big|\det(A)\big| \cdot \text{Norm}(A^{-1}) \leq \frac{||A||^{n-1}}{(n-1)^{n-1)/2}}$$

2. Traceless matrices. The trace of the $n \times n$-matrix A is given by

(i) $$\text{Tr}(A) = b_{11} + \ldots + b_{nn}$$

Recall that $\text{Tr}(A)$ is equal to the sum of the roots of the characterstic polynomial $\det(\lambda \cdot E - n - A)$. In particular the trace of two equivalent matrices are equal. Carleman used this to prove the rewsult below where we follow a simplified proof which was found later by Schur.

3. Lemma. *Let A be an $n \times n$-matrix whose trace is zero. Then there exists a unitary matrix U such that the diagonal elements of U^*AU all are zero.*

Proof. Consider first consider the case $n = 2$. Since A can be transformed to an upper tringular matrix via a unitary transformation in $\mathbf{C}^2$, it suffices to consider the case when the traceless 2×2-matrix A has the form

$$A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$$

where a, b is a pair of complex numbers. If $a = 0$ the diagonal elements are zero and we can take $U = E_2$ to be the identity. If $a \neq 0$ we consider a vector $\phi = (1, z)$ in $\mathbf{C}^2$. Then $A(\phi)$ is the vector $(a + bz, -az)$ and the inner product becomes:

(i) $$\langle A(\phi), \phi \rangle = a + bz - a|z|^2$$

We can write

$$\frac{b}{a} = re^{i\theta}$$

where $r > 0$ and then (i) is zero if

(ii) $$|z|^2 = 1 + se^{i\theta} \cdot z$$

With $z = se^{-i\theta}$ it amounts to find a positive real number s such that $s^2 = 1 + s$ which clearly exists. Now we get the vector

$$\phi_* = \frac{1}{1 + s^2}(1, se^{-i\theta})$$

which has unit length and

(ii) $$\langle A(\phi_*), \phi_* \rangle = 0$$

Now we can find another unit vector ψ_* so that ϕ_*, ψ_* is an orthonormal base in $\mathbf{C}^2$ and hence there exists a unitary matrix U such that $U(e_1) = \phi_*$ and $U(e_2) = \psi_*$. If $B = U^*AB$ the vanishing in (ii) gives $b_{11} = 0$. At the same time the trace is unchanged, i.e. $\mathrm{tr}(B) = 0$ holds and hence we also get $b_{22} = 0$. This means that the diagonal elements of U^*AU are both zero as required.

The case $n \geq 3$. For the induction the following is needed:

Sublemma. For each $n \geq 3$ there exists some non-zero vector $\phi \in \mathbf{C}^n$ such that

(*) $$\langle A(\phi), \phi \rangle = 0$$

Proof. If (*) does not hold we get the positive number

$$m_* = \min_{\phi} \big|\langle A(\phi), \phi \rangle\big|$$

where the minimum is taken over unit vectors in $\mathbf{C}^n$. The minimum is achieved by some unit vector ϕ_*. Let $\phi_*^\perp$ be its orthonormal complement and E is the self-adjoint projection from $\mathbf{C}^n$ onto $\phi_*^\perp$. On the $(n-1)$-dimensional inner product space $\phi_*^\perp$ we get the linear operator $B = EA$, i.e.

(i) $$B(\xi) = E(A(\xi)) \quad : \quad \xi \in \phi_*^\perp$$

If $\psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\phi_*^\perp$ then the n-tuple $\phi_*, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and since the trace of A is zero we get

(ii) $$0 = \langle A(\phi_*), \phi_* \rangle + \sum_{\nu=1}^{\nu=n-1} \langle A(\psi_\nu), \psi_\nu \rangle = m + \sum_{\nu=1}^{\nu=n-1} \langle B(\psi_\nu), \psi_\nu \rangle$$

where we used that $E(\psi_\nu) = \psi_\nu$ for each ν and that E is self-adjoint so that

$$\langle A(\psi_\nu), \psi_\nu \rangle = \langle A(\psi_\nu), E(\psi_\nu) \rangle = \langle E(A(\psi_\nu)), \psi_\nu \rangle = \langle B(\psi_\nu), \psi_\nu \rangle$$

Now (ii) gives

$$\mathrm{Tr}(B) = -m$$

Hence the $(n-1) \times (n-1)$-matrix which represents $B + \frac{m}{n-1} \cdot E$ has trace zero. By an induction over n we find a unit vector $\psi \in \phi_*^\perp$ such that

$$\langle B(\psi_*), \psi_* \rangle = -\frac{m}{n-1}$$

Finally, since E is self-adjoint we have already seen that

$$\langle A(\psi_*), \psi_* \rangle = \langle B(\psi_*), \psi_* \rangle \implies \big|\langle A(\psi_*), \psi_* \rangle\big| = \big|\frac{m}{n-1}\big| = \frac{m_*}{n-1}$$

Since $n \geq 3$ the last number is $< m_*$ which contradicts the minimal choice of m_*. Hence we must have $m_* = 0$ which proves lemma 6.5

Final part of the proof. Let $n \geq 3$. The Sublemma gives unit vector ϕ such that $\langle A(\phi), \phi \rangle = 0$. Consider the hyperplane $\phi^\perp$ and the operator B from the Sublemma which now has trace zero on this $(n-1)$-dimensional space. So by an induction over n there exists an orthonormal basis $\psi_1, \ldots, \psi_{n-1}$ in $\phi^\perp$ such that $\langle B(\psi_\nu), \psi_n u \rangle = 0$ for every ν. Now $\phi, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and if U is the unitary matrix which has this n-tuple as column vectors it follows that the diagonal elements of U^*AU all vanish. This finishes the proof of Lemma 3.

Proof Theorem A.8.1

.

Set $B = \lambda^{-1}A$ so that $\sigma(B) = \{\lambda_i/\lambda\}$ and $\mathrm{Tr}(B) = \sum \frac{\lambda_i}{\lambda}$. We also have

$$||B||^2 = \frac{||A||^2||}{|\lambda|^2} \quad \text{and} \quad \big|\lambda\big| \cdot \mathrm{Norm}[R_A(\lambda)] = \mathrm{Norm}[(E-B)^{-1}]$$

Hence Theorem A.8.1 follows if we prove the inequality

$$(*) \qquad \big|e^{\mathrm{Tr}(B)}\big| \cdot \big|\prod_{i=1}^{i=n}\big[1-\frac{\lambda}{\lambda_i}\big] \cdot \mathrm{Norm}\big[E-B)^{-1}\big] \leq \exp\big[\frac{1+||B||^2}{2}\big]$$

To prove (*) we choose an arbitrary integer N such that $N > \big|\mathrm{Tr}(B)\big|$ and for each such N we define the linear operator B_N on the $n+N$-dimensional complex space with points denoted by (x,y) with $y \in \mathbf{C}^N$ as follows:

$$(**) \qquad B_N(x,y) = (Bx\,,\, -\frac{\mathrm{Tr}(B)}{N} \cdot y)$$

The eigenvalues of the linear operator $E-B_N$ is the union of the n-tuple $\{1-\frac{\lambda_i}{\lambda}\}$ and the N-tuple of equal eigenvalues given by $1+\frac{\mathrm{Tr}(B)}{N}$. This gives the determinant formula

$$(1) \qquad \det(E-B_N) = \big(1+\frac{\mathrm{Tr}(B)}{N}\big)^N \cdot \prod_{i=1}^{i=n}(1-\frac{\lambda_i}{\lambda})$$

The choice of N implies that (1) is $\neq 0$ so the inverse $(E-B_N)^{-1}$ exists. Moreover, the construction of B_N gives for any pair (x,y) in $\mathbf{C}^{N+n}$:

$$(E-B_N)^{-1}(x,y) = \big(E-B)^{-1}(x), \frac{y}{1+\frac{1}{N}\cdot\mathrm{Tr}(B)}\big)$$

It follows that

$$\mathrm{Norm}\big[(E-B)^{-1})\big] \leq \mathrm{Norm}\big[(E-B_N)^{-1}\big] \implies$$

$$(2) \qquad \big|\det(E-B_N)\big| \cdot \mathrm{Norm}\big[(E-B)^{-1})\big] \leq \big|\det(E-B_N)\big| \cdot \mathrm{Norm}\big[(E-B_N)^{-1}\big]$$

Hadarmard's inequality estimates the hand side in (2) by:

$$(3) \qquad \frac{||E-B_N||^{N+n-1}}{(N+n-1)^{N+n-1)/2}}$$

Next, the construction of B_N implies that its trace is zero. So by Lemma 3 we can find an orthonormal basis $\xi_1,\ldots,\xi_{n+N}$ in $\mathbf{C}^{n+N}$ such that

$$\langle B_N(\xi_k),\xi_k\rangle = 0 \quad : 1 \leq k \leq n+N$$

Relative to this basis the matrix of $E-B_N$ has 1 along the diagonal and the negative of the elements of B_N elsewhere. It follows that the Hilbert-Schmidt norm satisfies the equality:

$$(4) \qquad ||E-B_N||^2 = N+n+||B_N||^2 = N+n+||B||^2 + N^{-1}\cdot|\mathrm{Tr}(B)|^2$$

Hence, (1) and the inequalities from (2-3) give:

$$\big(1+\frac{\mathrm{Tr}(B)}{N}\big)^N \cdot \prod_{i=1}^{i=n}(1-\frac{\lambda_i}{\lambda}) \cdot \mathrm{Norm}\big[(E-B)^{-1}\big] \leq$$

$$\frac{\big(N+n+||B||^2+N^{-1}\cdot|\mathrm{Tr}(B)|^2\big)^{(N+n-1)(2}}{\big(N+n-1\big)^{N+n-1/2}} = \frac{\big(1+\frac{||B||^2}{N+n}+\frac{|\mathrm{Tr}(B)|^2}{N(N+n)}\big)^{(N+n-1)/2}}{(1-\frac{1}{N+n}\big)^{N+n-1/2}}$$

This inequality holds for arbitrary large N. Passing to the limit as $N \to \infty$ the definition of Neper's constant e give

$$\lim_{N\to\infty}\big(1+\frac{\mathrm{Tr}(B)}{N}\big)^N = e^{\mathrm{Tr}(B)}$$

and the reader may also verify that the limit of the last term above is equal to $\exp\big[\frac{1+||B||^2}{2}\big]$ which finishes the proof of (*) above and hence also of Theorem A.8.1.