Chapter II: Stokes Theorem

0. Introduction.

The results in § 0:A cover the basic integral formulas which appear in analytic function theory. However, these results require certain regularity assumptions and we give therefore another proof of Stokes Theorem under relaxed regularity conditions in § 2. Stokes theorem in the plane follows from Fermat's descriptions of normals and the arc-length of a plane curve and the fundamental theorem of calculus. More precisely, let $f(x)$ be a positive and continuously differentiable function defined on some closed interval $[a,b]$. Let Ω be the domain bordered by $\{y=0\}$ and the graph of f, i.e.

$$\Omega = \{x,y)\colon 0 < y < f(x)\colon b < x < b\}$$

Let $g(x,y)$ be a C^1-function in Ω which extends to a continuous function on its closure. Set

$$J(x) = \int_0^{f(x)} g(x,y)\,dy \quad : a \leq x \leq b$$

Rules for derivation give

(i) $$J'(x) = \int_0^{f(x)} g_x(x,y)\,dy + g(x,f(x)) \cdot f'(x)$$

where g_x is the partial derivative with respect to x. The fundamental theorem of calculus gives

(ii) $$J(b) - J(a) = \int_a^b J'(x)\,dx$$

The right hand side in (ii) is the sum of two integrals. The first is the double integral

(iii) $$\iint_\Omega g_x(x,y)\,dxdy$$

The second becomes

(iv) $$\int_a^b g(x,f(x)) \cdot f'(x)\,dx$$

The crucial fact is Fermat's formula:

(v) $$-\frac{f'(x)}{\sqrt{1+f'(x)^2}} = \mathbf{n}_x$$

where $\mathbf{n}_x$ is the x-component of the outer normal to the plane curve $\gamma = \{y = f(x)\}$. The reader should check the minus sign in (v) with the aid of a figure. Moreover, Pythagoras' theorem entails that the arc-length along the curve is $\sqrt{1+f'(x)^2} \cdot dx$ and the equations above give

(vi) $$\iint_\Omega g_x(x,y)\,dxdy = J(b) - J(a) + \int_\gamma g \cdot \mathbf{n}_x\,ds$$

Finally $J(b)$ and $J(a)$ can be expressed as line integrals along the pair of vertical lines in the boundary of Ω. Along $\{x = b\}$ the outer normal $\mathbf{n} = e_x$ and $ds = dy$ and along $\{x = a\}$ $\mathbf{n} = -e_x$ and $ds = dy$. The right hand side in (vi) can therefore be written as

$$(*) \qquad \int_{\partial\Omega} g \cdot \mathbf{n}_x \, ds$$

The equality between the line integral (*) and the area integral of g_x is a special case of Stokes theorem. In § xx we establish the general case using partitions of the unity which reduce the proof to the case described above.

The use of differential forms. Line integrals are often expressed via differential 1-forms which lead to alternative expressions of Stokes Theorem in § 3 and § 5. An important result appears in Theorem 10 from § 2 where a pair of functions appear in integral formulas. This will be applied in § 4 where integral formulas related to the Laplace operator occurs. Here is an example which will be used later to study harmonic and more generally subharmonic functions. Let $u(x, y)$ be a C^2-function defined in some open disc of radius R centered at the origin in $\mathbf{R}^2$. Then the following hold for all pairs $0 < s < R$:

$$(1) \qquad u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(s, \theta) \cdot d\theta + \iint_{D(s)} \log\Big(\frac{s}{\sqrt{x^2 + y^2}}\Big) \cdot \Delta u(x, y) \cdot dxdy$$

where $D(s)$ is the disc of radius s centered at the origin. After another integration while (1) is applied as $0 < s < r < R$ we obtain:

$$(2) \qquad u(0) = \frac{1}{\pi r^2} \cdot \iint_{D(r)} u(x, y) \cdot dxdy - \int_0^r K(r, s) \cdot \Big[\int_0^{2\pi} \Delta(u)(s, \theta) \cdot d\theta\Big] \cdot ds$$

where $K_r(s)$ is a kernel function defined for pairs $0 < s \leq r$ by:

$$(3) \qquad K(r, s) = s^3 \cdot \int_1^{\frac{r}{s}} u \cdot \text{Log}(u) \cdot du$$

In (2) the first double integral in the right hand side is the mean-value of u over the disc $D(r)$. The second double integral describes the difference between this mean-value and the value of u at the center of the disc. If the function $\Delta(u)$ is non-negative in the whole disc $D(R)$ we get the mean-value inequality

$$u(0) \leq \frac{1}{\pi r^2} \cdot \iint_{D(r)} u(x, y) \cdot dxdy$$

for every $0 < r < R$. In Chapter V:B we shall learn that this gives the starting point for a study of subharmonic functions.

Polar coordinates. Set

$$x = r \cdot \cos\theta \quad \text{and} \quad y = r \cdot \sin\theta$$

Looking at the first order partial derivatives outside the origin we have

$$\partial_r = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y \quad \text{and} \quad \frac{1}{r} \cdot \partial_\theta = -\sin\theta \cdot \partial_x + \cos\theta \cdot \partial_y$$

It follows that

$$\partial_x = \cos\theta \cdot \partial_r - \frac{1}{r} \cdot \sin\theta \quad \text{and} \quad \partial_y = \sin\theta \cdot \partial_r + \frac{1}{r} \cdot \cos\theta$$

We have the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$ and leave it to the reader to verify that

$$(*) \qquad \Delta = \partial_r^2 + \frac{1}{r} \cdot \partial_r + \frac{1}{r^2} \cdot \partial_\theta^2$$

Next, consider a C^2-function $F(x, y)$ defined in the unit disc $D = \{x^2 + y^2 < 1\}$ with $F(0, 0) = 0$.

0.1 Theorem. *The following equality holds for every* $0 < r < 1$*:*

$$(*)\qquad \int_0^{2\pi} \big[r^2 \cdot (\partial_r(F))^2 - (\partial_\theta(F))^2 \big] \cdot d\theta = 2 \cdot \int_0^r \big[\int_0^{2\pi} s^2 \partial_s(F) \cdot \Delta(F) \cdot d\theta \big] \dot{d}s$$

To prove this via Stokes or Green's theorem would be cumbersome. Instead on employs a Fourier series expansion. Namely, we can write

$$F(r,\theta) = \sum_{k=1}^{\infty} a_k(r) \cdot \cos k\theta + b_k(r) \cdot \sin k\theta$$

where $\{a_k(r)\}$ and $\{b_k(r)\}$ are two sequences of functions which only depend on r.

0.2 Exercise. Use the familiar vanishing results for integrals of sine-and cosine-functions to conclude that it suffices to prove (*) in the special case when $F = a_k(r) \cdot \cos k\theta$ or $F = b_k(r) \cdot \sin k\theta$ for a single integer k. Next, consider the case when

$$F = a(r) \cdot \cos k\theta$$

for some positive integer k. Using the polar formula for Δ the right hand side in (*) becomes

$$\int_0^r \big[\int_0^{2\pi} s^2 \cdot a'(s) \cdot (a''(s) + \frac{1}{s} \cdot a'(s) - \frac{k^2}{s^2} a(s)] \cdot \cos^2 k\theta \cdot d\theta \big] \cdot ds$$

Notice that partial integration gives

$$\int_0^r s^2 \cdot a'(s) \cdot (a''(s) + \frac{1}{s} \cdot a'(s) - \frac{k^2}{s^2} a(s)] \cdot ds = \frac{r^2 a'(r)^2 - k^2 a(r)^2}{2}$$

It follows that the right hand side in (*) is equal to

$$(1)\qquad \big[r^2 a'(r)^2 - k^2 a(r)^2 \big] \cdot \int_0^{2\pi} \cos^2 k\theta \cdot d\theta$$

At this stage the reader can confirm that (1) is equal to the right hand side in (*).

0.3 Remark. The integral formula in Theorem 0.1 will be used to study subharmonic functions, i.e functions f for which $\Delta(F) \geq 0$. For example, if $\partial_r(F) \geq 0$ holds in the disc, then the right hand side in (*) is ≥ 0 which gives the inequality

$$(**)\qquad \int_0^{2\pi} \partial_\theta(F)^2 \cdot d\theta \leq r^2 \cdot \int_0^{2\pi} \partial_r(F)^2 \cdot d\theta \quad \text{for every} \quad 0 < r < 1$$

0:A. Calculus in $\mathbf{R}^2$

A parametrised curve γ in $\mathbf{R}^2$ is defined by a vector-valued function

(*) $$\gamma\colon \quad t \mapsto (x(t), y(t)) \quad : \quad 0 \leq t \leq T$$

The curve is of class C^2 if $x(t)$ and $y(t)$ are both of class C^2. We do not assume that (*) is 1-1 so the parametrized curve can have self-intersections. Line integrals are constructed as follows: Let $u(x, y)$ and $v(x, y)$ be a pair of continuous functions and set

(1) $$\int_\gamma u \cdot dx = \int_0^T u(x(t), y(t)) \cdot x'(t)\, dt$$

(2) $$\int_\gamma v \cdot dy = \int_0^T v(x(t), y(t)) \cdot y'(t)\, dt$$

We refer to (1) as a line integral in the x-direction and (2) as a line integral in the y-direction. The notation $\int_\gamma$ is consolidated by the fact that if $s \mapsto t(s)$ is some strictly increasing C^1-function from an interval $0 \leq s \leq S$ onto $[0, T]$ then the s-parametrisation does not change the integrals since

$$\int_0^T u(x(t), y(t) \cdot x'(t)\, dt = \int_0^S u(x(t(s)), y(t(s)) \cdot \frac{d}{ds}(x(t(s))\, ds$$

where we used the differential equality

$$\frac{d}{ds}(x(t(s)) = x'(t(s)) \cdot t'(s)$$

However, the orientation is essential. One moves from the initial point $\gamma(0) = P$ to the end-point $\gamma(T) = Q$ where an arrow along the curve is used to indicate the direction during the integration. Thus, up to a sign the line integrals depend upon the chosen orientation.

A.1 Homotopy. Consider a family of C^2-curves $\{\gamma_s : 0 \leq s \leq 1\}$ where each single γ_s is parametrised over an interval $[0, T]$ as above and the curves have common end-points $P = (x_*, y_*)$ and $Q = (x^*, y^*)$. So when $t \mapsto (x(t, s), y(t, s))$ is a parametrisation of γ_s, then

(1) $$x(0, s) = x_* \quad \text{and} \quad x(T, s) = x^* \quad \text{for all} \quad 0 \leq s \leq 1$$

and similarly for the y-function. No further assumption is imposed, i.e. the γ curves in the family need not be simple and it may occur that $P = Q$ or that $P \neq Q$. Concerning the functions $x(t, s)$ and $y(t, s)$ we impose the condition that both are of class C^2. In particular the mixed second order derivatives are equal.

(2) $$x''_{ts} = x''_{st} \quad \text{and} \quad y''_{ts} = y''_{st}$$

A.2 Theorem. *Let $u(x, y)$ and $v(x, y)$ be C^1-functions. Then one has the two equalities*

(*) $$\int_{\gamma_1} u \cdot dx - \int_{\gamma_0} u \cdot dx = \iint_\square u'_y \cdot (y'_s \cdot x'_t - y'_t \cdot x'_s)\, dsdt$$

(**) $$\int_{\gamma_1} v \cdot dy - \int_{\gamma_0} v \cdot dy = \iint_\square v'_x \cdot (x'_s \cdot y'_t - x'_t \cdot y'_s)\, dsdt$$

Proof. We prove (*) while (**) is left to the reader since the proof is the same when x and y are interchanged. The fundamental theorem of calculus gives

$$\int_{\gamma_1} u\, dx - \int_{\gamma_0} u\, dx = \int_0^1 \frac{d}{ds}\big(\int_0^T u \cdot x'_t\, dt\big)\, ds =$$

(i) $$\iint_\square \left[u'_x \cdot x'_s x'_t + u'_y \cdot y'_s x'_t + u \cdot x''_{st}\right] dsdt$$

Next, consider for each *fixed* $0 \leq s \leq 1$ the integral

(ii) $$\int_0^T \frac{d}{dt}(u(x(t, s), y(t, s)) \cdot x'_s(t, s)) \cdot dt$$

By (1) the functions $s \mapsto x(0,s)$ and $s \mapsto x(T,s)$ are constant and therefore

(iii) $$x'_s(0,s) = x'_s(T,s) = 0 \quad \text{for each} \quad 0 \leq s \leq 1$$

Hence the integral (ii) is zero for every s. At the same time we can differentiate the function under the integral sign and conclude that

(iv) $$\int_0^T (u'_x \cdot x'_t \cdot x'_s + u'_y \cdot y'_t \cdot x'_s + u \cdot x''_{ts}) \cdot dt = 0$$

Since (iv) holds for every s we get a vanishing double integral

(v) $$\iint_\square (u'_x \cdot x'_t \cdot x'_s + u'_y \cdot y'_t \cdot x'_s + u \cdot x''_{ts}) \cdot dsdt = 0$$

Finally, since $x(t,s)$ is assumed to be a C^2-function the mixed derivatives x''_{ts} and x''_{st} are equal. Subtracting the zero integral in (v) from (i) we conclude that (i) is equal to the double integral

$$\iint_\square [u'_y \cdot (y'_s \cdot x'_t - x'_t \cdot y'_s) \cdot dsdt$$

which gives (*) in Theorem A.2.

A.3 Application. If the partial derivatives u'_y and v'_x are equal, then the reversed signs for $y'_s x'_t - y'_t x'_s$ which appear in (*) and (**) give the equality

(***) $$\int_{\gamma_1} (u \cdot dx + v \cdot dy) = \int_{\gamma_0} (u \cdot dx + v \cdot dy)$$

A.4 Remark. Given the pair (u,v) one refers to $u \cdot dx + v \cdot dy = 0$ as a differential 1-form. By definition it is closed if and only if

$$u'_y = v'_x$$

So Theorem A.2 shows that the line integral of a closed 1-form is not changed under a homotopy deformation where the end-points are kept fixed.

The case of closed curves. The same proof as in Theorem A.2 shows that if $\{\gamma_s\}$ is a homotopic family of closed curves and $u'_y = v'_x$ then we also have the equality (***) above.

A.5 Stokes Theorem. Consider a map Φ from a rectangle

$$\square = \{0 \leq t \leq T\} \times \{0 \leq s \leq 1\}$$

into $\mathbf{R}^2$. We write $\Phi(t,s) = (x(t,s), y(t,s))$ and assume that $x(t,s)$ and $y(t,s)$ are C^2.-functions. The *Jacobian* of Φ becomes

$$\mathcal{J}_\Phi = y'_t \cdot x'_s - x'_t \cdot y'_s$$

So if $u(x,y)$ is some C^1-function then the double integral

(1) $$\iint_{\Phi(\square)} u'_y \cdot dxdy = \iint_\square u'_y(x(t,s), y(t,s) \cdot [y'_t \cdot x'_s - x'_t \cdot y'_s]\dot{d}sdt$$

By Theorem A.2 the double integral is a difference of two line integrals taken over $s = 0$ and $s = 1$ respectively. Suppose now that Φ is 1-1 so that $\square$ is mapped to a Jordan domain Ω whose boundary $\partial\Omega$ is a simple Jordan curve Γ. Here it may occur that Γ has corner points at the images of the four corner points of $\square$. But in any case the situation is sufficiently regular in order that we get a well defined line integral

$$\int_\Gamma u \cdot dx$$

Inspecting the sign of $y'_s x'_t - y'_t x'_s$ which appears in the Jacobian $\mathcal{J}_\Phi$ respectively in (*) from Theorem A.2 we conclude that one has a minus sign in the equation below:

$$\iint_{\Omega} u'_y \cdot dxdy = -\int_{\partial\Omega} u \cdot dx \tag{2}$$

Here the simple closed Jordan curve $\partial\Omega$ is oriented in the positive direction, i.e. one moves counter-clockwise as this curve encloses Ω. The reader should illustrate this by drawing a figure. In similar fashion one derives the formula

$$\iint_{\Omega} v'_x \cdot dxdy = \int_{\partial\Omega} v \cdot dy \tag{3}$$

A.5 Remark. The results above are in principle all we need to move directly to Chapter 3 and study analytic functions where the Cauchy-Riemann equations are used to ensure that we are in "favourable situations" such as (***) above. But for the reader's convenience we shall repeat certain arguments and give another proof in § 2 which has the merit that regularity conditions can be relaxed.

A result by Schwarz. Above we used the equality of second order mixed derivatives. A sufficient condition for its validity is due to Schwarz where the notable point is that one only requires that one of the mixed derivatives exists as a continuous function. Consider a function $f(x, y)$ of two real variables whose first order partial derivatives f_x and f_y exist as continuous functions. Then one has:

A.6 Theorem. *Assume that the mixed second order derivative f_{xy} exists as a continuos function. Then the mixed derivative f_{yx} exists and is equal to f_{yx}.*

Proof. With y kept fixed and some chosen constant k we consider the function

$$\phi(x) = f(x, y + k) - f(x, y)$$

Taylor expansion up to order two with the usual mean-value taken as second term gives for every h a pair $0 < \theta, \theta' < 1$ such that

$$\phi(x + h) - \phi(x) = h\big(f_x(x + \theta h, y + k) - f_x(x + \theta h, y)\big) = hkf_{xy}(x + \theta h, y + \theta' k) \tag{1}$$

The continuity of f_{xy} entails that

$$f_{xy}(x + \theta h, y + \theta' k) = f_{x,y}(x, y) + \epsilon(h, k)$$

where the ϵ-function tends to zero as $(h, k) \to (0, 0)$. Next, division with k in (1) and a passage to the limit as $k \to 0$ gives

$$f_y(x + h, y) - f_y(x, y) = h(f_{xy}(x + \theta h, y) + \epsilon(h, 0)) \tag{2}$$

Since $\epsilon(h, 0) \to 0$ when $h \to 0$ it follows from (2) that

$$\lim_{h \to 0} \frac{f_y(x + h, y) - f_(x, y)}{h} = f_{xy}(x, y)$$

By definition the left hand side yields the mixed derivative f_{yx} and Schwarz' theorem is proved.

1. Some physical explanations.

Introduction. The material in this section is not necessary in the sequel. It is included to give a perspective upon Stokes Theorem and we also discuss results in dimension 3. The first lessons about line integrals, area integrals and volume integrals go back to Archimedes. The beginner should first of all understand how Archimedes computed the volume of a pyramid Δ. Start in the (x,y)-plane from a bounded and convex domain U bordered by a piecewise linear boundary ∂U with some finite set of corner points $\{p_k = (x_k, y_k)\}$. They are arranged so that a line ℓ_k joints p_k to p_{k+1} when $1 \leq k \leq N-1$ and a line ℓ_N joins p_N with p_1. Next, consider a point $p^* = (x_0, y_0, z_0)$ where $z_0 > 0$. We obtain the pyramid Ω by joining p^* to each corner point. It is clear that $\partial\Omega$ consists of has N many planar pieces plus lines when a pair intersect and a number of corner points. The reader may illustrate this by a figure when U is a square. With the notations above Arkimedes' proved the formula:

$$\text{(*)} \qquad \operatorname{Vol}\Omega = \frac{z_0}{3} \cdot \operatorname{Area}(U)$$

Remark. The proof relies upon the fundamental fact that under dilation expressed by some $s > 0$ areas change with the scale factor s^2 and volumes by s^3. Then (*) follows when we regard portions of Ω where the z-coordinate is restricted to small intervals $\{\frac{k}{N} \leq z \leq \frac{k+1}{N}\}$. Taking a limit as $N \to \infty$ the scale principles give

$$\operatorname{Vol}\Omega = \operatorname{Area}(U) \cdot \int_0^{z_0} (z_0 - z)^2 \cdot dz = \frac{z_0}{3} \cdot \operatorname{Area}(U)$$

A more involved case compared with a pyramid arises when we consider a bounded open set Ω where $\partial\Omega$ consists of N many planar sets $U_1, \ldots, U_N$ and each intersection between two such planar sets is a line segment. Several U-sets may also intersect at corner points on the boundary. Now we have the *area measure* dA_ν on every U_ν and the *outer normal* vector $\mathbf{n}_\nu$ to U_ν, i.e. the unit vector which is $\perp$ to U_ν and points out from Ω. If $\mathbf{n}_\nu(x)$ denotes the x-component of this unit vector one has the formula

$$\text{(***)} \qquad \operatorname{Vol}(\Omega) = \sum_{\nu=1}^{N} \iint_{U_\nu} x \cdot \mathbf{n}_\nu(x) \cdot dA_\nu$$

Above we can replace x by y or z using the components $\mathbf{n}_\nu(y)$ or $\mathbf{n}_\nu(z)$ when we compute area integrals over $\{U_\nu\}$. This volume formula goes back to Archimedes via physical considerations.

The principle of Arkimedes. An experience which every child less than five years has already discovered, is that a body which is gently placed in water eventually comes to a position at rest when there are no waves or streams. The shape of the body can be highly irregular. Imagine a piece of a broken tree with several branches where some of them may be above the waterline in the floating position. The intersection with the free water line $z = 0$ and the boundary of the tree need not even be connected. In § 1.3 we explain how the principle of Arkimedes confirms the validity of (***) even for domains Ω with a highly irregular boundary. The equation (***) is therefore a Law of Nature rather than a "theorem" derived in mathematics.

1.1 Stokes Theorem. Let us describe how Newton and his contemporaries Boyle and Hooke were aware of a *conceptual proof* of Stokes formula in three dimensions. Newton attributes the basic ideas below to Déscartes in his work *Principia* from 1687 when he argues as follows to confirm Stokes theorem:

A bounded connected domain Ω is given in $\mathbf{R}^3$. The boundary may consist of several pairwise disjoint closed surfaces of class C^1 at least, i.e. sufficiently regular in order that we can refer to surface area measure and the outer normal along every component of $\partial\Omega$. Inside Ω a large number of small particles - think of small balls - are moving. They have equal mass and when they impinge with each other the impact is elastic. At a point $p = (x,y,z)$ the "mean neighbor velocity" of balls close to p is a vector valued function $v(p) = (f(p), g(p), h(p))$, i.e. $f(p)$ is the velocity in the x-direction and so on. The interior of Ω is divided into small pairwise disjoint cubes $\{\square_\alpha\}$ with sides parallell to the coordinate axes. The effect of all impacts from balls inside one cube $\square_\alpha$ which hit the boundary of $\square_\alpha$ during a small time interval is a force vector which approximately will be

$$F_\alpha = \rho \cdot (f_x, g_y, h_z) \cdot \mathrm{vol}(\square_\alpha)$$

Here f_x, g_y, h_z are the partial derivatives inside $\square_\alpha$, ρ a constant density of mass and $\mathrm{vol}(\square)_\alpha$ the volume of the square. The boundary of each cube is some hard material so that via the force vectors F_α, each cube "pushes" - or alternatively gets a push - from some of its six many neighbor cubes with which it has a common side. For example, if $f_x > 0$ in a given cube $\square_\alpha$ then $\square_\alpha$ tends to push on the cube next to the right. Along the boundary the impact is expressed by the area integral

$$(*) \qquad \int_{\partial\Omega} \rho \cdot (fn_x + gn_y + hn_z) dA \quad : \quad dA = \text{area measure}$$

where (n_x, n_y, n_z) the outer normal. Since the balls cannot escape the container Ω the *principle of reaction forces* implies that we must have the equality

$$(**) \qquad \sum F_\alpha = \int_{\partial\Omega} \rho \cdot (fn_x + gn_y + hn_z) dA$$

The sum to the left approximates the volume integral of the function $\rho \cdot (f_x + g_z + h_z)$. Dividing out ρ we get the equality

$$(***) \qquad \int\int\int (f_x + g_y + h_z) dxdydz = \int_{\partial\Omega} (fn_x + gn_y + hn_z) dA$$

Remark. In (***) we encounter an arbitrary triple of functions and hence one has three equations:

$$\iiint f_x \cdot dxdydz = \iint_{\partial\Omega} fn_x \cdot dA$$
$$\iiint g_y \cdot dxdydz = \iint_{\partial\Omega} gn_y \cdot dA$$
$$\iiint h_z \cdot dxdydz = \iint_{\partial\Omega} hn_z \cdot dA\,,$$

Personally I find Newton's proof convincing. Advancements in mathematics rely upon ideas as above.

1.2. The principle of Archimedes

Consider a 3-dimensional body K placed in $\mathbf{R}^3$ where (x, y, z) are the coordinates and z is vertical so that the force of gravity is $-g \cdot e_z$. The body has some distribution of mass which need not have a constant density. Imagine a ship where the density is large in the machine room and considerably lighter in the lounge bar. In any case, the body has a specific weight $0 < s < 1$, i.e. if V is the volume then the mass of K is $s \cdot V$. Now K is gently put into the water by a five year old child in the middle of a reasonably large lake at a time when there are no winds or waves. A child predicts correctly that K will float and after a short time even come to rest. The problem is to determine the floating position using the force of gravity and the law of momentum in statics. This was solved by the the genius Archimedes. His studies about floating bodies were reconsidered by Stevin around 1600 after original work by Archimedes' had been rediscovered. Like Galilei in Pisa, Stevin also performed experiments in Amsterdam to show that velocities of falling bodes are independent of their specific weight or total mass and he is regarded as the creator of modern statics. Let $\mathfrak{o}$ be the center of mass in K. Notice that $\mathfrak{o}$ need not be contained in K. A typical example is an oil-platform. Let K_* be the portion of K below the free waterline determined by the equation $z = 0$. In general K_* can be a disconnected set. But connected components of K_* are bounded by a surface below the free waterline and some area domain in the plane $z = 0$ which serves as a "roof" for this component. The reader should illustrate this by a figure. Now there exists the point $\mathfrak{o}^*$ which corresponds to the center of mass which is determined when K_* has a uniform density of mass. With these notations Archimedes stated that when the body has a resting floating position, then the mass M of the body is equal to the mass of water which would fill the portion of K below the free waterline. So with $0 < s < 1$ it means that

$$\text{vol}(K_*) = s \cdot M$$

Moreover, the vector $\mathfrak{o} - \mathfrak{o}^*$ is $\perp$ to the horizontal water line, i.e. parallell to the direction where the force of gravity acts. Finally, these two conditions are both necessary and sufficient for a floating position at rest. That the two conditions are *necessary* seem likely while the *sufficiency* is more subtle. The reason is that there may exist *several floating positions* where the two conditions above hold. For example, let K be a solid cube with $\mathfrak{o}$ placed in the center and constant density of mass which gives some specific weight $0 < s < 1$. The reader should discover that there exist several positions where Arhimedes' conditions hold. Take for example $s = 0.4$ and draw figures to find different solutions where $\mathfrak{o} - \mathfrak{o}^*$ is vertical. This leads to the question of *stability.* Stability conditions were found by Christian Huyghens. But his subtle analysis goes beyond the scope of these notes. Let us only mention that when Ω is a square the floating position where the free waterline is parallel to a pair of sides of Ω, then Huyghens proved that this floating position is stable if and only $0 < s < 2 - \sqrt{3}$ or $\sqrt{3} - 1 < s < 1$. So we have an unstable equilibrium when $2 - \sqrt{3} < s < \sqrt{3} - 1$. To obtain stable floating positions Ω must be tilted. For example when $s = \frac{1}{2}$ one gets a stable position where the sides of the cube have the angle $\pi/4$ to the free waterline.

1.23 Proof of Archimedes' theorem. To begin with we must understand why the body can float at rest. This amounts to determine the *forces of lifting* on the part of K below the waterline. Following Stevin - and the later refinement by Huyghens - the force of lifting is found as follows. From the inside close to a point $p \in \partial K_*$ placed at some distance h below the waterline one makes a small circular hole of radius ϵ. A cylinder of equal radius is pressed a small bit ℓ outside the surface of the submarine. The effect is that a volume of water equal to $\pi\ell\epsilon^2$ is lifted to the free waterline. This requires a work equal to $gh \cdot \pi\ell\epsilon^2$. Then, if P is the force of *pressure* on the submarine close to p the work to push the cylinder is equal to $P \cdot \ell$. At the same time the area removed from the surface of the submarine is $\pi\epsilon^2$. The result is that the infinitesmal lifting force becomes

$$F = gh$$

Next, the force of pressure at p from the outside water on the surface of the submarine is parallell to the normal $\mathfrak{n}$ of ∂K_* where the reader by the aid of a figure realizes that one uses the *inward normal.* Thus, if $\mathfrak{n}$ denotes the *outer normal* to ∂K_* the discussion gives:

1.3 Proposition. *The total lifting force on the floating body is*

$$-g \cdot \int_{\partial K_*} h(p) \cdot \mathfrak{n}(p) d\sigma$$

where $d\sigma$ is the area measure on ∂K_*.

Next, Archimedes' first principle asserts that the vertical component of the vector valued integral above is equal to g times the volume of K_*. Since $z = 0$ on the free waterline, this gives the equality

$$\iint_{K_*} dxdydz = \int_{\partial K_*} z \cdot \mathfrak{n}_z \cdot d\sigma$$

where $h = -z$ above since we stay below the free waterline. As we shall see later on this equation corresponds to Stokes applied to the z-component of the normal vector $\mathfrak{n}$.

1.4 A vanishing result. Since K comes to rest it cannot behave like a moving fish, i.e. the two horisontal components of the total lifting force must be zero. This means that

$$(1) \qquad \int_{K_*} z \cdot \mathfrak{n}_x \cdot d\sigma = \int_{K_*} z \cdot \mathfrak{n}_y \cdot d\sigma = 0$$

Again we shall learn that these two area integrals are zero by Stokes formula. Hence we have consolidated the first principle of Archimedes, and conversely this principle already predicted general integration formulas since the shape of K_* can be arbitrary.

1.5 Proof that $\mathfrak{o}-\mathfrak{o}^*$ is vertical. To show this we analyze the force of momentum. We may assume that coordinates are chosen so that $\mathfrak{o}=(0,b)$ for some b on the y-axis. The *Law of Momentum* gives

$$\mathcal{M}=g\cdot\int_{\partial K_*}(x,y-b)\times(-y\mathfrak{n}(x,y))\cdot ds \tag{2}$$

Here the minus sign for y appears since the force of pressure was found via the distance h from a point below the water line up to $y=0$. In (2) we decompose the vector $\mathbf{n}$ and expanding the vector product it follows that:

$$\mathcal{M}=g\cdot\int_{\partial K_*}-xy\cdot\mathfrak{n}_y(x,y))\cdot ds-\int_{\partial K_*}y(y-b)\cdot\mathfrak{n}_x(x,y)\cdot ds$$

Now Stokes formula entails that

$$\cdot\int_{\partial K_*}y\cdot\mathfrak{n}_x(x,y)\cdot ds=\int_{\partial K_*}y^2\cdot\mathfrak{n}_x(x,y)\cdot ds=0\implies$$

$$\mathcal{M}=-g\cdot\int_{\partial K_*}xy\cdot\mathfrak{n}_y(x,y))\cdot ds \tag{3}$$

By Stokes formula (3) is equal to the area integral

$$-g\int_{K_*}xdxdy \tag{*}$$

This integral must be zero when the body is at rest. This means precisely that the x-component of $\mathfrak{o}^*$ is zero and Archimedes' second assertion follows.

1.5 Curvature and arc-length.

Arc length measure of curves in the (x,y)-plane and the curvature occur frequently in complex analysis. Following Huyghens we explain how to determine curvature by dynamical considerations. Let a plane curve be defined by an equation $y=y(x)$ where $y''(x)>0$ for $x>0$ and $y(0)=y'(0)=0$. Up to translation and rotation this is a general situation. So we have a convex curve which we follow as x increases. To express the curvature Huyghen's considered a particle of unit mass which can slide on the curve, say on the side just above the curve. No gravity occurs, i.e. imagine that a vertical wall is placed along the curve which prevents the particle to leave the curve. At time zero it has velocity v. No friction forces are present which means that the force acting on the particle at every moment is directed along the normal to the plane curve. Let t be the time variable which gives a time dependent function $t\mapsto(x(t),y(x(t))$. At a moment t we denote by $\rho(t)$ the reaction force on the particle, i.e. the force which keeps the particle to move on along the curve. Our assumptions imply that $\rho(t)$ is normal to the curve and directed upwards in the y-direction. Regarding a figure the reader discovers that the components are given by:

$$\rho_x(t)=\rho(t)\cdot\frac{-y'(x(t))}{\sqrt{1+y'(x(t))^2}}\quad:\ \rho_y(t)=\rho(t)\cdot\frac{1}{\sqrt{1+y'(x(t))^2}}$$

Newton's Law that "force=mass times acceleration" gives:

$$\ddot{x}=\rho(t)\frac{-y'(x(t))}{\sqrt{1+y'(x(t))^2}}\quad\text{and}\quad\ddot{y}=\rho(t)\cdot\frac{1}{\sqrt{1+y'(x(t))^2}}$$

Let us now notice that $\dot{y}=y'(x(t))\cdot\dot{x}$. Hence

$$\dot{x}\cdot\ddot{x}=\rho(t)\frac{-\dot{y}}{\sqrt{1+y'(x(t))^2}}=-\ddot{y}\dot{y}$$

It follows that $\dot{x}\ddot{x}+\dot{y}\ddot{y}=0$ which means that $v^2=\dot{x}^2+\dot{y}^2$ is constant. This proves the preservation of kinetic energy which is valid since no other forces than the normal pressure acts on the particle.

1.6 Determination of ρ. To find ρ we start with $\dot{y}=y'(x(t))\cdot\dot{x}$ and taking the time derivative once more it follows that

$$\ddot{y}=y''(x(t))\cdot\dot{x}^2+y'(x(t))\cdot\ddot{x}$$

Inserting the two formulas above for $\ddot{x}$ and $\ddot{y}$ it follows that

$$\sqrt{1+y'(x(t))^2}\cdot\rho(t)=y''(x(t))\cdot\dot{x}^2$$

Now we also have $v^2=\dot{x}^2+\dot{y}^2=\dot{x}^2+(1+y'(x(t))^2\cdot\dot{x}^2$. We conclude that

$$\rho(t)=\frac{y''(x(t))}{[1+y'(x(t))^2]^{\frac{3}{2}}}\cdot v^2$$

This is gives the formula for the *centrifugal force*. The term

$$\mathfrak{c}(x)=\frac{y''(x)}{[1+y'(x)^2]^{\frac{3}{2}}}$$

is the *geometric curvature* of the plane curve. Huyghen's conclusion was that the centrifugal force is the quotient of v^2 with the curvature expressed as above. Having attained this one may give a geometric description of $\mathfrak{c}(x)$. Namely, $\frac{1}{\mathfrak{c}(x)}$ is the radius of a circle placed along the normal to the curve passing through x which has *best contact* with the curve at the point $(x,y(x))$. This geometric description of the curvature could of course have been given from the start. But the dynamical consideration gives a better insight and is extremely important in mechanics. Moreover, Huyghens clarified why the geometric description must be valid by computing the centrifugal force when a particle is constrained to move along a circular wall of some radius R. Namely, in this case the centrifugal force is constant during the motion and given by

$$(*) \qquad \mathcal{C}=\frac{v^2}{R}$$

1.7 Huyghen's proof in the circular case. His proof to obtain th centrifugal force under a circular motion is extremely elegant. To begin with Huyhens regards a particle which moves along a *regular polygon* with N corners inscribed in the circle of radius R with constant velocity v and hits the circle N times during one full turn. The *impact force* each time the particle of unit mass hits the circle is given by:

$$(**) \qquad 2\cdot\sin\frac{\pi}{N}\cdot v$$

The reader should draw a figure and use that the corners of the polygon give rise to N many triangles with angle $2\pi/N$ at the center and discover that the sudden direction of the velocity vector is changed by $2\pi/N$ at every impact. Then (**) follows by decomposing the reaction force and the definition of the sine-function. Next, the total length of the polygon is $N\cdot2\cdot\sin\frac{\pi}{N}$. Hence the time T to perform a full circular turn is

$$T=\frac{N\cdot2\cdot\sin\frac{\pi}{N}}{v}$$

Finally, we have N many instants when impact takes place. So after one circular turn we get

$$F_{\text{imp}}=N\cdot2\cdot\sin\frac{\pi}{N}\cdot v \implies \frac{F_{\text{imp}}}{T}=\frac{v^2}{R}$$

The left hand side expresses the effect of impact while the particle impinges the circle at the corner points and the effect of force per unit time does not depend on N. When $N\to\infty$ we get the "continuous formula" expressed by (*) above.

2. Stokes Theorem in $\mathbf{R}^2$

Introduction *Stokes Theorem* is often proved in an "intuitive fashion" where figures illustrate how one divides a domain into simpler so that repeated double integrals can be used. We give a proof without such artificial constructions. In the long run this is essential since the cutting of domains becomes messy when the number of its boundary components increases. This section may appear to be "overkilling" for the beginner. But my opinion is that the subsequent material belongs to the foundation for complex analysis and learning a proof of Stokes Theorem is both important and instructive. In § X from the appendix about measure theory we derive Stokes theorem in every dimension $n \geq 2$. Here we present the results when $n = 2$.

2.A. The case of graphic domains

The FCT = fundamental theorem of calculus - asserts that if $g(x)$ is a function whose derivative exists as a continuous function, then

$$g(x) = g(a) + \int_a^x g'(t)dt$$

Next, consider $\mathbf{R}^2$ where (x, y) are the coordinates. A real valued function $f(x, y)$ is of class C^1 when the two partial derivatives f_x and f_y both exist as continuous functions. Consider a C^1-function $\phi(x)$ which depends on x only and is defined on a closed interval $0 \leq x \leq A$. Assume that $\phi(x) > 0$ which gives the open set

$$\Omega = \{(x, y) \colon\ 0 < x < A \quad 0 < y < \phi(x)\}$$

Now we have the double integrals

$$(*) \qquad \iint_\Omega f_x \cdot dxdy \quad : \quad \iint_\Omega f_y \cdot dxdy$$

We shall express these by certain line integrals. The second double integral is easy to handle since it is a repeated integral:

$$(1) \qquad \iint_\Omega f_y \cdot dxdy = \int_0^A [\int_0^{\phi(x)} f_y(x, y)dy]\, dx = \int_0^A [f(x, \phi(x) - f(x, 0)]dx$$

Later we explain the intrinsic nature of this formula. Let us turn to the double integral in the left hand side of (*). Here we cannot find a repeated integral when horizontal lines $\{x = a\}$ cut the domain Ω so that the sets $\Omega \cap \{x < a\}$ and $\Omega \cap \{x > a\}$ have several connected components. The reader should illustrate this by drawing some figures using an "ugly" ϕ-function. However, we can express the double integral as a sum of line integrals !

1. A clever construction of line integrals. Put

$$J(f) = \iint_\Omega f_x \cdot dxdy$$

Consider the following function ψ of a single variable

$$\psi(x) = \int_0^{\phi(x)} f(x, y)dy \implies$$

$$\psi'(x) = f(x, \phi(x))\phi'(x) + \int_0^{\phi(x)} f_x(x, y)dy$$

The FTC gives

$$\psi(A) - \psi(0) = \int_0^A \psi'(x)dx = \int_0^A f(x, \phi(x)\phi'(x) + \int_0^A [\int_0^{\phi(x)} f_x(x, y)dy\,] \cdot dy$$

The last term is $J(f)$ and hence we get

$$(*) \qquad J(f) = \int_0^{\phi(A)} f(A,y)dy - \int_0^{\phi(0)} f(0,y)dy \ - \int_0^A f(x,\phi(x)\phi'(x)dx$$

In (*) three line integrals appear. The first is taken along the vertical line $x = A$, the second along $x = 0$ in the negative direction and the last along the curve $y = \phi(x)$. Now we explain their geometric meaning.

First, the vertical line $L_+ = \{x = A \quad 0 \leq y \leq \phi(A)\}$ is part of the boundary $\partial\Omega$. On this line the *arc-length measure* is equal to dy and *the outward normal* along L_+ is parallell to the x-axis so its component $n_x = 1$. Hence we can write

$$(1) \qquad \int_0^{\phi(A)} f(A,y)dy = \int_{L_+} fn_x \cdot ds$$

Second, consider $L_- = \{x = 0 \quad 0 \leq y \leq \phi(0)\}$. Again dy is the arc-length measure while the outer normal is directed in the negative x-direction, i.e. $n_x = -1$. Hence:

$$(2) \qquad -\int_0^{\phi(0)} f(0,y)dy = \int_{L_-} fn_x \cdot dy$$

Third. On the curve $\Gamma = \{y = \phi(x)\}$ the arc-length is $ds = \sqrt{1 + \phi'(x)^2} \cdot dx$ and the x-component of the outer normal is

$$n_x = \frac{-\phi'(x)}{\sqrt{1 + \phi'(x)^2}}$$

Here the minus sign becomes clear by inspecting a figure of the curve $y = \phi(x)$ where you discover that $n_x > 0$ if $\phi' < 0$ and vice versa ! Hence we obtain

$$(3) \qquad -\int_0^A f(x,\phi(x)\phi'(x)dx = \int_\Gamma fn_x ds$$

where the first minus sign reflects the sign-rule for n_x along the boundary curve Γ. Putting all this together we have proved the equality

$$(**) \qquad J(f) = \int_\Gamma fn_x ds + \int_{L_+} fn_x ds + \int_{L_-} fn_x ds$$

No "ackward signs" occur in the line integrals above since arc-length measure is always defined and the outer normal along the boundary of an open set is clarified by a picture. There remains the portion of $\partial\Omega$ defined by $I = \{0 \leq x \leq A \quad y = 0\}$. Here the outer normal is in the negative y-direction and hence $n_x = 0$. Therefore we only add a zero term by $\int_I fn_x ds$ and arrive at

2. Theorem. *One has the equality*

$$\iint_\Omega f_x \cdot dxdy = \int_{\partial\Omega} fn_x \cdot ds$$

3. An area formula. Let $\phi(x)$ be a piecewise linear function which is positive when $0 < x < A$ while $\phi(0) = \phi(A) = 0$. So we have corner points

$$p_\nu = (x_\nu, y_\nu) \quad 0 = x_0 < x_1 \ldots < x_N = A$$

while the $y_1, \ldots, y_{N-1}$ is any sequence of positive numbers. Apply Theorem 2 with $f(x,y) = x$. The double integral is the area of Ω, i.e. the domain bounded by the piecewise linear curve Γ and the interval $[0, A]$ on the x-axis. On the line segment ℓ_ν which joins (x_ν, y_ν) with $(x_{\nu+1}, y_{\nu+1})$ the reader may verify by figure and Pythagoras's theorem that along $\ell\nu$ one has:

$$n_x = \frac{y_\nu - y_{\nu+1}}{\sqrt{(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2}} \quad \text{and} \quad ds = \frac{\sqrt{(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2}}{x_{\nu+1} - x_\nu} \cdot dx$$

It follows that

$$\int_{\partial\Omega} xn_x \cdot ds = \sum \frac{y_\nu - y_{\nu+1}}{x_{\nu+1} - x_\nu} \cdot \int_{x_\nu}^{x_{\nu+1}} x\,dx =$$

on each linear piece of Γ. After a summation the line integral becomes

$$(*) \qquad \frac{1}{2} \cdot \sum_{\nu=0}^{\nu=N-1} (x_{\nu+1}^2 - x_\nu^2)(y_\nu - y_{\nu+1}) = \frac{1}{2} \cdot \sum (y_\nu - y_{\nu+1}) \cdot (x_{\nu+1} + x_\nu)$$

Example. Consider the simplest case where just one corner point appears at $(A/2, H)$. So here Ω is a triangle whose area is $AH/2$. The formula gives on the other hand

$$\frac{1}{2} \cdot [-H \cdot A/2 + H \cdot (A + A/2)) = AH/2$$

which confirms (*). The reader should continue to test the area formula (*) in more complicated situations where the picewise linear curve has several negative and positive slopes.

4. Expression of the first double integral Consider the component n_y of the outer normal to Ω. On the piece $I = \{0 \leq x \leq A: \quad y = 0\}$ we see that $n_y = -1$. On the curve $y = \phi(x)$ a picture shows that

$$n_y = \frac{1}{\sqrt{1 + \phi'(x)^2}}$$

At the same time $dx = \frac{1}{\sqrt{1+\phi'(x)^2}} ds$ and hence $n_y ds = dx$ holds on the curve. Finally, the outer normal is in the x-direction on the two vertical lines of $\partial\Omega$. Hence we have proved:

5. Theorem. *One has*

$$\iint_\Omega f_y \cdot dxdy = \int_{\partial\Omega} fn_y \cdot ds$$

Thus, we have a similar formula as in Theorem 2 for the y-coordinate. There remains to extend these two formulas to general domains which even may have several disjoint closed boundary curves. But first we resume our special case a bit further. Consider an open cube $\square$ in $\mathbf{R}^2$ where we after a translation may assume that it is centered at the origin. Let $\psi(x, y)$ be a C^1-function whose gradient is $\neq 0$ at the origin and to make a choice we assume that $\psi_y(0, 0) < 0$. The *implicit function theorem* gives a C^1-function $\phi(x)$ and a positive function $h(x, y)$ such that

$$\psi(x, y) = (\phi(x) - y)h(x, y)$$

Shrinking $\square$ if necessary we may assume that $h > 0$ in the whole of $\square$. Put

$$\Omega = \square \cap \{\psi > 0\} = \square \cap \{y < \phi(x)\}$$

The previous results show that if f is a C^1-function with a compact support in $\square$, then the two FCT-formulas hold for Ω. We refer to Ω as a *graphic domain*. Next, the validity of the FCT-theorem is obviously invariant under a *linear change of coordinates*. Hence we can start with a cube whose sides are not parallell to the coordinate axis and use some direction of the gradient of ψ when the implicit function theorem is used to obtain a graphic domain. With this kept in mind we begin the proof in the general case.

6. The case of domains in $\mathcal{D}(C^1)$.

First we give

7. Definition. *A bounded open and connected subset Ω of $\mathbf{R}^2$ has a C^1-boundary if $\partial\Omega$ is the disjoint union of a finite family of simple and closed curves $\Gamma_1, \ldots, \Gamma_k$ each of which are of class C^1. The family of such domains is denoted by $\mathcal{D}(C^1)$.*

8. Remark about the arc-length. A simple closed curve Γ of class C^1 is the image of a vector-valued function

$$t \mapsto \gamma(t) = (x(t), y(t)) \quad 0 \leq t \leq T$$

which is 1-1 except that $\gamma(0) = \gamma(T)$. Moreover, the functions $x(t)$ and $y(t)$ are both of class C^1 and $\gamma(t)$ is "moving" which means that $\dot{\gamma}(t) \neq 0$ for all t, or equivalently $\dot{x}^2(t) + \dot{y}^2(t) > 0$ for all $0 \leq t \leq T$. The curve Γ can be parametrized in several ways. Among those one has the *parametrization by arc-length.* In this case we use s as parameter and then

$$(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 = 1 \quad : 0 \leq s \leq L$$

where L is the total arc-length of Γ. The arc-length measure along Γ is denoted by ds.

9. The normal to Γ. Let Γ be a C^1-curve. To each point $p \in \Gamma$ we find a unit vector $n(p)$ which is normal to Γ. Given a parametrization by arc-length we have:

$$n = (n_x, n_y) \quad : \quad n_x = \frac{dy}{ds} \quad \text{and} \quad n_y = -\frac{dx}{ds}$$

However, the *sign* of the normal depends on the chosen *orientation* of Γ. For example, let Γ be a circle of radius R centered at the origin. Its arc-lengtth is $2\pi R$ and it has the parametrisation

$$s \mapsto (R\cos(s), R\sin(s)) \quad 0 \leq s \leq 2\pi$$

Given this orientation we see that

$$n_x = \cos(s) \quad n_y = \sin(s)$$

Now we can announce the fundamental result called the FTC in dimension 2.

10. Theorem. *Let $\Omega \in \mathcal{D}(C^1)$. For each function $f(x,y)$ of class C^1 one has*

$$\iint_\Omega f_x \cdot dxdy = \int_{\partial\Omega} f n_x \cdot ds \quad : \quad \iint_\Omega f_y \cdot dxdy = \int_{\partial\Omega} f n_y \cdot ds$$

where n is the outward normal to Ω along each boundary curve.

Proof. The theorem asserts that the formula holds for every C^1-function. Following the proof in the introduction by Déscartes and Newton we *decompose* a given C^1-function f and represent it by a sum of C^1-functions where each function of the sum has a small support which enable us to apply the previous result for *graphic domains.* To achieve such a decomposition we first consider some boundary point $p \in \partial\Omega$. It belongs to some boundary curve Γ. Let L be the tangent line to Γ passing through p. Then we construct a small square $\square(p)$ centered at p and with two sides parallel to L while the other are $\perp$ to L. Here Γ is locally defined by an equation $\phi(x,y) = 0$ close to p where ϕ is chosen so that

$$\Omega \cap \square(p) = \{\phi > 0\}$$

Regarding a picture - which the reader should draw - it is clear that $\Omega \cap \square(p)$ is a graphic domain if the sides of $\square(p)$ are sufficiently small. By the *Heine-Borel Lemma* we can cover the compact boundary $\partial\Omega$ by a finite set of such squares, say $\square(p_1), \ldots, \square(p_N)$. Notice that $p_1, \ldots, p_N$ are boundary points chosen from different boundary curves but the cubes are so small that

$$\square(p_\nu) \cap \square(p_j) = \emptyset \quad : \quad p_\nu, p_j \text{ belong to different boundary curves}$$

Next, consider the complementary set

$$K = \Omega \setminus \cup \square(p_\nu)$$

This becomes a compact subset of Ω. By the *Heine-Borel Lemma* it can be covered by a finite set of open cubes $W_1, \ldots, W_m$ chosen so small that each closure $\bar{W}_i$ stays inside Ω.

Partition of the unity. We have the finite family of cubes $\{\square(p_\nu), W_j\}$. By the result in XXX there exist C^1-functions $g_1, \ldots, g_N, h_1, \ldots, h_m$ such that

$$\sum g_\nu + \sum h_i = 1 \quad : \quad \text{Supp}(g_\nu) \subset \square(p_\nu) \quad \text{Supp}(h_i) \subset W_i$$

where the first equality holds in some open neighborhood U of $\bar{\Omega}$.

Final part of the proof. We treat the first formula expressing the double integral of f_x by line integral. Replacing x by y is proved in the same way. Given a C^1-function f it is expressed by a sum:

$$f = \sum fg_\nu + \sum fh_i$$

Since the sum of partial derivatives $\partial_x(\sum g_\nu + \sum h_i)$ and $\partial_y(\sum g_\nu + \sum h_i)$ both vanish we get

$$\iint_\Omega f_x \cdot dxdy = \sum \iint_\Omega (fg_\nu)_x \cdot dxdy + \sum \iint_\Omega (fh_i)_x \cdot dxdy$$

and similarly

$$\int_{\partial\Omega} fn_x \cdot ds = \sum \int_{\partial\Omega} fg_\nu n_x \cdot ds + \sum \int_{\partial\Omega} fh_i n_x \cdot ds$$

Hence it suffices to prove Theorem 10 for each term separately, i.e. for the functions fg_ν or fh_i. Here fg_ν has compact support in $\square(p_\nu)$ and the graphic case applies, i.e. the required formula in Theorem 10 holds in this case. Next, consider fh_i. This function has compact support in Ω so no boundary terms, i.e. no line integral appears. But at the same time XX shows that

$$\iint_\Omega (fh_i)_x \cdot dxdy = 0$$

So these double integrals give no contribution and Theorem 10 is proved.

11. Boundary with corner points

Consider a connected and bounded open set Ω. Its boundary is a compact set. A point $p \in \partial\Omega$ is called regular if there exists a small disc D centered at p and a C^1-function ϕ in D whose gradient vector is $\neq 0$ in D such that

$$\Omega \cap D = \{\phi < 0\} \quad : \quad \phi(p) = 0$$

It is obvious that the set of regular points is an open subset of $\partial\Omega$ to be denoted by $\mathrm{reg}(\partial\Omega)$. We assume that the set is non-empty, i.e. we ignore to consider open sets with a very ugly boundary. On the regular part of $\partial\Omega$ the arc-length measure ds and the outer normal are defined. Next, put

$$\sigma_\Omega = \partial\Omega \setminus \mathrm{reg}(\partial\Omega)$$

So this is a compact set and we shall impose a condition on its size.

12. Federer's conditions. First one requires that the two projected images of σ_Ω on the x-line respectively the y-line are null sets in the sense of Lebesgue. Thus, the condition is that to every $\epsilon > 0$ there exists a finite family of disjoint open intervals $J_\nu = (a_\nu, b_\nu)$ on the x-line such that

$$\sum (b_\nu - a_\nu) < \epsilon \quad : \quad (x, y) \in \sigma_\Omega \implies x \in \cup J_\nu$$

and with a similar condition for the y-interval. The second condition is that the arc-length of the regular part is finite, i.e. that

$$\int_{\mathrm{reg}(\partial\Omega)} ds < \infty$$

13. Theorem *Assume that $\partial\Omega$ satisfies Federer's conditions. Then*

$$\iint_\Omega f_y \, dxdy = \int_{\mathrm{reg}(\partial\Omega)} fn_y \cdot ds$$

and similarly with x replaced by y.

Proof. Let $\epsilon > 0$ and choose intervals J_ν of total length $< \epsilon$ to satisfy Federer's condition for x-coordinates of points in σ_Ω. Choose a C^1-function $g(x)$ where $0 \leq g \leq 1$ and $g = 1$ outside $\cup J_\nu$ while $g(x) = 0$ in a neighbourhood of the compact set

$$\{x \colon \exists y \colon (x, y) \in \sigma_\Omega\}$$

Set

$$h = gf$$

By the choice of g the support of h avoids σ_Ω. Repeating the proof of Theorem 10 for domains with regular boundaries we obtain

$$(1)\qquad \iint_\Omega h_y dxdy = \int_{\text{reg}(\partial\Omega)} hn_y ds$$

Here $h_y = g(x)f_y$ and hence

$$(2)\qquad \iint_\Omega f_y dxdy - \iint_\Omega h_y dxdy = \iint_\Omega (1-g(x))f_y(x,y)dxdy$$

The last double integral is estimated as follows. By the choice of g we have $1-g=0$ outside a union of intervals of length $\leq \epsilon$. So if M is the maximum norm of f_y taken over $\bar{\Omega}$ and L is the maximum of two $y-$-coordinates for points in Ω with the same x-coordinate, then we get

$$(3)\qquad |\iint_\Omega (1-g(x))f_y(x,y)dxdy| \leq ML\epsilon$$

14. Estimate of $\int_{\text{reg}(\partial\Omega)} f(1-g)n_y ds$. Here we need a more delicate argument where the reader - as always when it comes to a more involved proof in analysis - should make suitable pictures to discover the geometry. Let $\delta > 0$ and consider the subset $W(\delta)$ of reg$(\partial\Omega)$ where $|n_y| \geq \delta$. On this set we notice that the arc-length is majorised by $|dx|$:

$$\text{(i)}\qquad |n_y| \geq \delta| \implies ds \leq \frac{1}{\delta}|dx|$$

Moreover, the projection $(x,y) \to x$ restricted to $W(\delta)$ has *discrete fibers*, i.e. it is locally 1-1 as you see by drawing a figure with a small curve passing through any point in $W(\delta)$. However, the set $W(\delta)$ need not be compact so we must perform another reduction. Namely, by hypothesis the total length of reg$(\partial\Omega)$ is finite. So with $\epsilon > 0$ we can find a *compact* set $K \subset$ reg$(\partial\Omega)$ such that

$$\text{(ii)}\qquad \int_{\text{reg}(\partial\Omega)\setminus K} ds < \epsilon$$

Next, restrict the projection map π defined by $(x,y) \to x$ to $W(\delta) \cap K$. Since this set is *compact* and the projection is locally 1-1, *Heine-Borel Lemma* gives an integer M_δ such that the inverse fibers

$$\text{(iii)}\qquad \pi^{-1} \cap W(\delta) \cap K$$

contain at most M_δ points for every x. Notice that M_δ depends on δ but not upon ϵ. Using the above we obtain

$$(*)\qquad \int_{W(\delta)\cap K} (1-g)ds \leq \frac{1}{\delta}\int_{W(\delta)\cap K} (1-g)d|x| \leq \frac{M_\delta}{\delta} \cdot \int (1-g)dx \leq \frac{M_\delta}{\delta} \cdot \epsilon$$

where the last inequality follows from the choice of g in XX above. We have also to other estimates. Let $|f[_K$ be the maximum norm of f over K. On On $K \setminus W(\delta)$ we have $|n_y| \leq \delta$ and hence

$$(**)\qquad |\int_{K\setminus W(\delta)} f(1-g)n_y \cdot ds| \leq \delta \cdot \int_{K\setminus W(\delta)} f(1-g)ds \leq \delta \cdot |f|_K \cdot \int_{\partial\Omega_{\text{reg}}} ds$$

Next, using (iii) above we have

$$(***)\qquad |\int_{[\text{reg}(\partial\Omega)\setminus K]} f(1-g)ds| \leq |f|_{\partial\Omega} \cdot \int_{[\text{reg}(\partial\Omega)\setminus K]} ds \leq |f|_{\partial\Omega} \cdot \epsilon$$

Finally, notice that

$$\text{(v)}\qquad \text{reg}(\partial\Omega) = [W(\delta) \cap K] \cup [\text{reg}(\partial\Omega) \setminus K] \cup [K \setminus W(\delta)]$$

Putting all this together we obtain the inequality

$$\text{(vi)}\qquad |\int_{\text{reg}(\partial\Omega)} f(1-g)n_y ds| \leq \frac{M_\delta \epsilon}{\delta} + A \cdot |f|_K \cdot \delta + |f|_{\partial\Omega} \cdot \epsilon$$

Here (vi) hold for all pairs δ, ϵ). To finish the proof of Theorem 13 we choose an arbitrary small$\delta > 0$ and after ϵ is chosen so small that we first have $\epsilon \leq \delta$ and also

$$\frac{M_\delta \epsilon}{\delta} < \delta$$

Then the left hand side is majorised by

$$(1 + A \cdot |f|_{\partial\Omega} + |f|_{\partial\Omega}) \cdot \delta$$

Since $\delta > 0$ is arbitrary we get Theorem 13.

3. Line integrals via differentials

In § 2 arc-length and the outer normal were used to construct line integrals. One may also introduce the differentials dx and dy. The construction of the outer normal shows that

$$(*) \qquad n_x \cdot ds = dy \quad n_y \cdot ds = -dx$$

Hence one can express Stokes Theorem in the form

$$\iint_\Omega f_x \cdot dxdy = \int_{\partial\Omega} f dy \quad : \quad \iint_\Omega f_y \cdot dxdy = -\int_{\partial\Omega} f dx$$

A Warning. When Stokes Theorem is expressed in this way one must be careful with the orientation. The *rule of thumbs* is used whenever $\partial\Omega$ borders an open set. Personally I prefer to express line integrals by $n_x ds$ or $n_y ds$ since the geometric picture becomes transparent. However, differentials have an advantage when calculus is performed on *manifolds* rather than the euclidian plane $\mathbf{R}^2$, since here arc-length and normal derivatives are not even defined until the manifold has been equipped with a metric. Thus, complex analysis on *Riemann surfaces* employs differential forms.

3.0 Transformation laws. Let (ξ, η) be the coordinate functions in another copy of $\mathbf{R^2}$. Let Ω be a domain in the (x, y)-plane and consider a bijective map

$$(1) \qquad Q\colon\ (x,y) \to (\phi(x,y), \psi(x,y)) \quad : \xi = \phi(x,y) \quad \eta = \psi(x,y)$$

defined in some neighborhood of $\bar{\Omega}$. Here ϕ and ψ are C^1-functions and we get the domain $Q(\Omega)$ in the (ξ, η)-space. Now we have

$$(2) \qquad d\xi = \phi_x \cdot dx + \phi_y \cdot dy \quad : \ d\eta = \psi_x \cdot dx + \psi_y \cdot dy \quad :$$

The *Jacobian* of the Q-map is defined by the equation

$$(3) \qquad \mathcal{J} = \phi_x \cdot \psi_y - \phi_x \cdot \psi_y$$

The Q-map preserves orientation when $\mathcal{J} > 0$ and from now on this is assumed. The Jacobian changes area which is expressed by

$$(3) \qquad d\xi d\eta = J \cdot dxdy$$

Now we take some C^1-function $g(\xi, \eta)$ defined in the $(\xi, \eta$-space. In the (x, y)-space we get the function

$$(4) \qquad g_*(x,y) = g(\phi(x,y), \psi(x,y))$$

Let us study the effect of the transformation when Stokes formula is applied. Put $\Omega^* = Q(\Omega)$. We have the obvious equality:

$$(5) \qquad \int_{\partial\Omega^*} g d\xi = \int_{\partial\Omega} g_* \cdot (\phi_x dx + \phi_y dy)$$

Stokes formula applied to the right hand side gives

$$(6) \qquad \iint_\Omega -(g_*\phi_x)_y + (g_*\phi_y)_x \cdot dxdy$$

Here we get a cancellation since the mixed derivatives ϕ_{xy} and ϕ_{yx} are equal. Hence (6) becomes:

$$(7) \qquad \iint_\Omega \left[-(g_*)_y \cdot \phi_x + (g_*)_x \cdot \phi_y\right] \cdot dxdy$$

If Stokes formula is applied to the left hand side in (5) we get the area integral

$$(8) \qquad \iint_{\Omega^*} -g_\eta \cdot d\xi d\eta$$

3.1 Theorem. *The integrals (7) and (8) are equal.*

Proof. The equality follows using transformation rules for partial derivatives. Namely, from (4) we get

(9) $$(g_*)_x = \phi_x \cdot g_\xi + \psi_x \cdot g_\eta \quad : (g_*)_y = \phi_y \cdot g_\xi + \psi_y \cdot g_\eta$$

Here we can solve out g_η and find

(10) $$(\phi_x\psi_y - \phi_y\psi_x) \cdot g_\eta = (g_*)_y \cdot \phi_x - (g_*)_x \cdot \phi_y$$

In (10) we discover the Jacobian as a factor for g_η. Hence the rule for area transformation in (3) and the two minus signs in (7) and (8) show that (7)=(8).

3.2 The pull-back of differential forms. The efficient way to analyze transforms which can be extended to maps between manifolds goes as follows: Let

(1) $$Q\colon\ (\xi,\eta) \mapsto (x,y) \quad : x = \phi(\xi,\eta) \quad y = \psi(\xi,\eta)$$

The differential 1-forms dx and dy have inverse images defined by

(2) $$(dx)^* = \phi_\xi \cdot d\xi + \phi_\eta \cdot d\eta \quad : \quad (dy)^* = \psi_\xi \cdot d\xi + \psi_\eta \cdot d\eta$$

More generally, to each pair of C^1-functions $A(x,y), B(x,y)$ we get the 1-form $\alpha = A(x,y)\dot{d}x + B(x,y) \cdot dy$. Its pull-back is

(3) $$\alpha^* = A^*(\xi,\eta) \cdot (dx)^* + B^*(\xi,\eta) \cdot (dy)^*$$

where $A^*(\xi,\eta) = A(\phi(x,y),\psi(x,y))$ and similarly for B^*.

3.3 Exterior differentials. If $\alpha = A \cdot dx + B \cdot dy$ is a 1-form its exterior differential is defined as

(4) $$d\alpha = -A_y \cdot dx \wedge dy + B_x \cdot \wedge dy$$

The minus sign in front of A_y is compatible with Stokes formula is expressed in (3.0),i.e. we get

(5) $$\iint_\Omega d\alpha = \int_\Omega \alpha$$

The formula in (5) summarizes the whole content of Stokes formula. It becomes especaily useful because of the following fundamental fact.

3.4 Theorem. *The pull-back of differential forms commutes with exterior differential.*

3.5 Remark. Theorem 3.4 asserts that if we start from α and get α^* then the 2-form $d(\alpha^*)$ in the $\xi,\eta)$-space is equal to the pull-back of the 2-form $d\alpha$. The reader should verify this or consult some text-book in calculus. Passing to Stokes formula the result is that if α is a 1-form in the (x,y)-space then one has equality for the area integrals

(6) $$\iint_\Omega d\alpha = \iint_{\Omega^*} d\alpha^*$$

3.6 Currents. Above we recalled classic notions which extend to manifolds in dimension 2 and using some calculations with multi-linear forms one introduces differential forms of higher degree to manifolds in any dimension. However, the classic approach has a drawback since an equality like (6) assumes that one has a 1-1 map from the (ξ,η)-plane into the (x,y)-plane. The modern procedure is to use distribution theory. For example, consider a map

$$Q\colon\ (\xi,\eta) \mapsto (\phi(x,y),\psi(x,y))$$

where ϕ and ψ are C^∞-functions. Here the map Q need not be 1-1. Let γ be a Jordan arc in the (ξ,η)-space. For example, an interval on some circle or a line segment. The image set $Q(\gamma)$ can be a curve with self-intersections and so on. A typical case is that for points $p \in Q(\gamma)$ the inverse set $Q^{-1}(p) \cap \gamma$ is a finite set of points on γ but the number may change as p moves in $Q(\gamma)$. So one should regard the image of γ under the Q-map as a *current* acting as a linear form on 1-forms in the (x, y)-space by the rule

$$\alpha \mapsto \int_\gamma \alpha^* \tag{1}$$

The point is that the pull-back α^* is defined even if Q is not 1-1, i.e. it is given by

$$A^*(\xi, \eta) \cdot [\phi_x^* \cdot d\xi + \phi_x^* \cdot d\eta] + B^*(\xi, \eta) \cdot [\psi_x^* \cdot d\xi + \psi_x^* \cdot d\eta] \tag{2}$$

The current defined by (1) is denoted by $Q_*(\gamma)$ and called the direct image of the integration current defined by γ in the (ξ, η)-space. Here it is essential that we have given an orientation on γ its direct image current is constructed. The current $Q_*(\gamma)$ has distribution coefficients when we specialize the 1-form α. That is, there exists a map

$$A(x, y) \in C^\infty(\mathbf{R^2}) \mapsto \int_\gamma A^*(\xi, \eta) \cdot (dx)^*$$

3.7 Stokes formula in higher dimension For readers who already are a bit familiar with differential geometry, the FCT in any dimension goes as follows: Let X be an oriented manifold of some dimension $n \geq 2$ and of class C^2 at least. Let V be a locally closed and oriented submanifold of some dimension $1 \leq k \leq n-1$. Assume that $\bar{V}$ is compact and that the boundary ∂V satisfies Federer's condition, i.e. it contains an open part $\text{reg}(\partial V)$ which is an oriented $k-1$-dimensional manifold whose $(k-1)$-dimensional volume is finite, and the $k-1$-dimensional Haussdorff measure of $\partial V \setminus \text{reg}(\partial V)$ is zero. Then the following hold for every differential $(k-1)$-form α of class C^1 defined in some open neighborhood of $\bar{V}$:

$$\int_V d\alpha = \int_{\text{reg}(\partial V)} \alpha$$

An example Let $n \geq 3$ and $1 \leq k \leq n-1$. Suppose that $P_1(x), \ldots, P_k(x)$ is a k-tuple of real valued polynomials of n variables such that the set where the $k \times k$-matrix whose elements are

$$\partial P_i / \partial x_\nu(x) \quad : \quad 1 \leq i, \nu \leq k$$

is invertible in some non-empty open set U of $\mathbf{R^n}$, i.e. the polynomial defined by the determinant of this matrix is not identically zero. Then we obtain a locally closed submanifold of $\mathbf{R^n}$ defined by

$$W = \{x\colon\ P_1(x) = \ldots = P_k(x) = 0\} \cap U$$

Next, let $Q_(x), \ldots, Q_m(x)$ be some m-tuple of polynomials and put

$$\mathfrak{Q} = \{x\colon\ Q_\nu(x) < 0 : 1 \leq \nu \leq m\}$$

Assume also that the open set $\mathfrak{Q}$ is bounded in $\mathbf{R^n}$, i.e. contained in some open ball with sufficiently large radius centered at the origin. Then the general Stokes Theorem holds for the locally closed k-dimensional submanifold $V = W \cap \mathfrak{Q}$. The proof that Federer's conditions hold follows from a result about *semi-algebraic sets* due to Tarski and Seidenberg. The reader may consult the appendix in Hörmander's text-book [Hö:1] or his more recent text-book series [Hö] for an account about semi-algebraic sets which verify Federer's conditions. We remark that the resulting boundary integrals may become quite involved since no extra conditions are imposed upon the Q-functions so the boundary ∂V may have "corners" which in the case $n \geq 3$ of course are hard to vizualise.

4. Green's formula and Dirichlet's problem

Let $\Omega \in \mathcal{D}(C^1)$ and f is a function of class C^2 which means that f_x and f_y are of class C^1. Stokes Theorem applied to f_x and f_y give

$$\iint_\Omega f_{xx} dxdy = \int_{\partial\Omega} f_x n_x ds \quad : \quad \iint_\Omega f_{yy} dxdy = \int_{\partial\Omega} f_y n_y ds$$

Adding the two equalities we obtain

$$\iint_\Omega (f_{xx} + f_{yy}) dxdy = \int_{\partial\Omega} (f_x n_x + f_y n_y) ds$$

Here $f_x n_x + f_y n_y$ is the *directional derivative* of f along the outer normal n which we denote by f_n. Next, $f_{xx} + f_{yy}$ is the *Laplacian* of f and is denoted by $\Delta(f)$. Hence we have proved

4.1 Theorem *Let $\Omega \in \mathcal{D}(C^1)$. For each f of class C^2 we have*

$$\iint_\Omega \Delta(f)\, dxdy = \int_{\partial\Omega} f_n\, ds$$

4.2 Remark. Stokes Theorem applied to the functions f_x and f_y gives the two formulas

$$\iint_\Omega f_{yx} dxdy = \int_{\partial\Omega} f_x n_y ds \quad : \quad \iint_\Omega f_{xy} dxdy = \int_{\partial\Omega} f_y n_x ds$$

When f is of class C^2, the mixed derivatives f_{yx} and f_{xy} are equal. Hence we obtain the following equality for line integrals

$$\int_{\partial\Omega} f_x n_y ds = \int_{\partial\Omega} f_y n_x ds$$

Notice that this equality is obvious using differentials to express the line integrals, i.e. since $dx = -n_y ds$ and $dy = n_x ds$ the equality above is expressed by

$$\int_{\partial\Omega} f_x dx + f_y dy = 0$$

The vanishing of this line integral follows trivally since each boundary curve of $\partial\Omega$ is closed. However, the two equalities above have a non-trivial consequence. Namely, let f be of class C^2 and consider its *gradient vector*

$$\nabla(f) = (f_x, f_y) = f_x \cdot e_x + f_y \cdot e_y$$

where e_x and e_y are the euclidian basis vectors in $\mathbf{R^2}$. Subtracting the two equalities above we get

$$\int_{\partial\Omega} (f_x n_y - f_y n_x) ds = 0$$

Here $f_x n_y - f_x n_x$ is equal to the *vector product* $\nabla \times n$. Hence we have proved

4.3 Theorem *Let $\Omega \in \mathcal{D}(C^1)$. For each f of class C^2 we have*

$$\int_{\partial\Omega} \nabla(f) \times n \cdot ds = 0$$

4.4 Mean value integrals Consider the case when Ω is an open disc. Without loss of generality we may assume its center is at the origin and let R be the radius. Denote the disc by D_R. Here ∂D_R is parametrized by

$$\theta \mapsto R(\cos(\theta), \sin(\theta)) \quad : \quad 0 \leq \theta \leq 2\pi$$

Let f be a C^2-function defined in some open neighbourhood of the closed disc $\bar{D}_R$. Put

$$M_f(R) = \frac{1}{2\pi} \int_0^{2\pi} f(R \cdot \cos\theta, R \cdot \sin\theta)\, d\theta$$

As R varies we can take the derivative and rules of differentiation yield the following equality for each $0 < r < R$:

$$\frac{d}{dr}(M_f(r)) = \frac{1}{2\pi}\int_0^{2\pi} [f_x(r\cos\theta, r\sin\theta)\cos(\theta) + f_y(r\cos\theta, r\sin\theta)\sin(\theta)]\, d\theta$$

Since $rd\theta = ds$ and $n = (\cos\theta, \sin\theta)$ we can express the equation by

$$\frac{d}{dr}(M_f(r)) = \frac{1}{2\pi r}\int_{\partial D_r} f_n ds$$

Hence Theorem 4.1 gives:

4.5 Theorem *For each* $0 < r < R$ *one has*

$$\int_{D_r} \Delta(f) dxdy = 2\pi r \cdot \frac{d}{dr}(M_f(r))$$

In particular, suppose that f is a *harmonic function* which means that it satisfies the Laplace equation, i.e. $\Delta(f) = 0$. Then the left hand side is zero above and hence the mean-value function $M_f(r)$ is constant. By continuity at the origin we see that

$$\lim_{\epsilon \to 0} M_f(\epsilon) = f(0,0)$$

Hence we get

4.6 Theorem *Let* f *be a harmonic function in disc* D_R. *Then*

$$f(0,0) = M_f(r) \quad : \quad 0 < r < R$$

Staying with harmonic functions we also notice that Theorem 4.1 gives

4.7 Theorem. *Let* $\Omega \in \mathcal{D}(C^1)$. *For each* C^1*-function* h *which is harmonic in* Ω *one has*

$$\int_{\partial\Omega} h_n ds = 0$$

4.8 Formulas with two functions. Let $\Omega \in \mathcal{D}(C^1)$ and f, g is a pair of C^2-functions. Applying Theorem 2.4 to $f_x g$ and $f_y g$ gives after a summation

$$\iint_\Omega [\Delta(f) \cdot g + f_x g_x + f_y g_y]\, dxdy = \int_{\partial\Omega} f_n \cdot g \cdot ds$$

Here $f_x g_x + f_y g_y$ is the *inner product* of the gradient vectors of f and g. Since this term is symmetric for the pair, we obtain the following when the same formula a above is applied with f and g interchanged:

4.9 Theorem *For each pair of* C^2*-functions* f, g *one has:*

$$\iint_\Omega \big[\Delta(f) \cdot g - f \cdot \Delta(g)\big]\, dxdy = \int_{\partial\Omega} \big[f_n \cdot g - f g_n\big] \cdot ds$$

4.10 Application. Let $\Omega = D_R$ be a disc centered at the origin. Let f be harmonic in D_R. Given a point $(a,b) \in D_R$ we define the g-function

$$g(x,y) = \text{Log}(\sqrt{(x-a)^2 + (y-b)^2})$$

An easy computation which is left to the reader shows that g is a harmonic function in $\mathbf{R^2}$ outsidee the point (a,b). With $\epsilon > 0$ small we remove the open disc $D_\epsilon(a,b)$ and apply Green's formula to the domain $\Omega_\epsilon = D_R \setminus \bar{D}_\epsilon(a,b)$. Since both f and g are harmonic in Ω_ϵ we obtain:

$$\int_{\partial\Omega_\epsilon} f_n g\, ds = \int_{\partial\Omega_\epsilon} f g_n\, ds$$

This equality yields an interesting formula when $\epsilon \to 0$. First, $\partial\Omega_\epsilon = \partial\Omega \cup \partial D_\epsilon$. For the line integrals over D_ϵ the following two limit formulas hold:

(*) $$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon} f_n g\,ds = 0 \quad : \quad \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} f g_n\,ds = -2\pi f(a,b)$$

where the last minus sign appears since ∂D_ϵ is a boundary component of Ω_ϵ so that the outward normal points into the disc D_ϵ.

4.11 Exercise. Prove the last limit formula in (*). The hint is: We may assume that (a,b) is the origin and since $\frac{d}{dr}\text{Log}\,r = \frac{1}{r}$ we find that the outer normal $g_n = -\frac{1}{\epsilon}$. At the same time $ds = \epsilon d\theta$ and now the reader verifies the second limit formula.

Using these two limit formulas we obtain

(**) $$2\pi f(a,b) = \int_{\partial D_R} f g_n\,ds - \int_{\partial D_R} f_n g\,ds$$

Hence the value of the harmonic function f at (a,b) can be expressed by sum of two line integrals on ∂D_R where f and f_n appear. Later on we shall find *Poisson's formula* where the value of f is expressed by a line integral where only f appears.

4.12 The Dirichlet problem

The formula 4.8 with two functions was the starting point when Dirichlet around 1840 posed the following problem:

$$\text{Given } h \in C^0(\partial\Omega) \text{ find } f \in C^0(\bar{\Omega}) \quad : \quad f|\partial\Omega = h \quad \Delta(f)|\Omega = 0$$

To solve this Dirichlet considered the following *variational problem*

$$V(f) = \iint_\Omega (f_x^2 + f_y^2)\cdot dxdy \quad : \quad f|\partial\Omega = h$$

Now one seeks

$$V_* = \min_f V(f)$$

If $V(f) = V_*$ one has:

(i) $$\lim_{\epsilon\to 0}\frac{V(f+\epsilon g) - V(f)}{\epsilon} = 0 \quad : \quad \forall\, g \text{ such that } g|\partial\Omega = 0$$

Now we notice that

$$V(f+\epsilon g) - V(f) = 2\epsilon\iint (f_x g_x + f_y g_y)dxdy + \epsilon^2\iint (g_x^2 + g_y^2)dxdy =$$

$$-2\epsilon\iint \Delta(f) g\,dxdy + \epsilon^2\iint (g_x^2 + g_y^2)dxdy$$

Passing to the limit when $\epsilon \to 0$ we conclude that if f minimizes V then

(ii) $$\iint \Delta(f) g\,dxdy = 0$$

Since this hold for all g vanishing on $\partial\Omega$ it follows that $\Delta(f) = 0$ in Ω. Hence f solves Dirichlet's problem.

4.13 An obstacle. Dirichlet's solution is correct but his proof became "shaky" when Weierstrass discovered that there exist variational problems of a similar nature as above which *fail* to have an extremal solution. In 1923 O. Perron gave a rigorous proof using subharmonic functions which answers the question when Dirichlet's problem has a solution for every continuous boundary function.

4.14 Theorem. *Let Ω be a bounded open set such that for every point $a \in \partial\Omega$ the connected component of the set $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the singleton set $\{a\}$. Then each $h \in C^0(\partial\Omega)$ has a unique harmonic extension to $\bar{\Omega}$.*

Remark. We prove this in Chapter V. Notice that Theorem 4.14 applies when Ω is of class $\mathcal{D}(C^1)$.

4.15 Probabilistic solution. Theorem 4.14 can be proved by *probabilistic considerations*. Let us describe this under the assumption that $\partial\Omega$ is "nice". Pick some arc γ from the boundary, i.e. γ is a simple closed curve contained in one of the closed boundary curves Γ. Now one seeks a harmonic function f in Ω such that its boundary value is equal to 1 at interior points of γ and zero on $\partial\Omega \setminus \gamma$. So the boundary values of f are determined except at the the two end-points of the closed C^1-curve γ.

4.16 The harmonic measure. Let $p \in \Omega$. Starting from p we consider the *Brownian motion*, i.e. perform a 2-dimensional random walk which may be approximated by regarding small consequtive steps of length δ which moves the particle with probability 1/4 in each direction - i.e. changing x or y by + or -δ. With probability one the discrete random walk eventually crosses the boundary $\partial\Omega$. Some of these cross γ and this gives a number $0 < \pi_\gamma(p) < 1$ which is the probability for a random walk to cross γ. To be precise, one gets this number in the limit when $\delta \to 0$. Next, we use the fact that a function is harmonic if and only if it satisfies a local *mean-value condition*, i.e. a function f is harmonic in Ω if and only if its value at point $q \in \Omega$ is equal to its mean-value over small discs centered at q. From this it follows easily that the function

$$p \mapsto \pi_\gamma(p)$$

is harmonic in Ω and yields the solution to Dirichlet's problem with boundary values as above. Finally, since this is achieved for boundary arc γ one can deduce Theorem 4.14 by approximating a continuous function h on $\partial\Omega$ with functions which are piecewise constant

Remark The probability expressed by $\pi_\gamma(p)$ plays an important role later on since it is equal to the *harmonic measure* defined as the value at p of the harmonic function in Ω with boundary value zero on $\partial\Omega \setminus \gamma$ and equal to one on γ. This probabilistic interpretation of the harmonic measure gives also an intuitive feeling for the harmonic measure.

A notable point is that the probabilistic solution makes it possible to use *Monte Carlo simulations* in order to obtain good approximative solutions to Dirichlet's problem which in general have no "analytic solutions". For example, presence of several boundary components of $\partial\Omega$ does not in principle cause any problem when a Monte Carlo simulation is used. A drawback is that Monte Carlo simulations tend to be rather time consuming since one must repeat random walks several times over small grids. In an impressive *Examensarbete by Oskar Sandberg* (2003) at the Mathematics Department in Stockholm, a quite rapid Monte Carlo simulation was developed . Here one takes larger random steps in each simulation using the *mean-value property* for harmonic functions. The interested reader may consult Sandberg's work for further details where numerical solutions to the Dirichlet problem in dimension ≥ 3 also are obtained by Monte Carlo simulations. See also the section about the material about the *Brownian motion* in Chapter XX for further comments.

4.18 The Dirichlet problem in a half-space.

Consider the half space in $\mathbf{R}^2$ defined by $U = \{(x,y) \colon y > 0\}$. So here ∂U is the x-axis. We construct harmonic a class of harmonic functions in U by the following procedure:

4.19 Definition *To each pair of real numbers $a < b$ we let $H_{a,b}(x,y)$ be the function in U whose value at (x,y) is the angle at this point in the triangle with corners at $(a,0),(b,0),(x,y)$.*

The reader should draw a figure which explains why $0 < H_{a,b}(x,y) < \pi$ for all $(x,y) \in U$. Moreover one has the limit formulas

$$\lim_{y \to 0} H_{a,b}(x,y) = \pi \quad a < x < b \quad : \quad \lim_{y \to 0} H_{a,b}(x,y) = \pi \quad x < a \quad b < x$$

Finally we also have

$$\lim_{x^2+y^2 \to \infty} H_{a,b}(x,y) = 0$$

Less obvious is that H is a *harmonic* function in U. So let us give:

Proof Recall that the sum of the angles of a triangle is π. Given the poins $a < b$ a figure shows that the angle $H_{a,b}(x,y)$ is equal to $\beta - \alpha$ where α is the angle between the vector from a to (x,y) and the positive x-axis, and similarly β is the angle between the vector from b to (x,y) and the positive x-axis. Since the sum of harmonic functions is again harmonic, i suffices to show that the functions $\alpha(x,y)$ and $\beta(x,y)$ are harmonic. Now it is clear - again by a figure - that

$$\text{tg}(\alpha) = \frac{y}{x-a} \implies \alpha(x,y) = \text{arctg}\big(\frac{y}{x-a}\big)$$

When $y > 0$ and $x = a$ the α-function is $\pi/2$. Outside this vertical line we can take irs derivatives. Using the wellknown formula for the derivative of the arctg-function the reader may verify that $\Delta(\alpha) = 0$.

4.20 Remark See also XX where we give an alternative proof that H is harmonic using complex valued Log-functions. The $H_{a,b}$-functions can be used to solve the Dirichlet problem since the two limit formulas after Definition 4.19 settle the case when the boundary function is the characteristic function of an interval. To proceed further we need to study the H-functions when $b - a \to 0$. Given $(x,y) \in U$ and some a on the x-axis we consider for a small positive Δ the triangle with corners at $(x,y), (a,0), (a+\Delta,0)$. Let α be the angle at (x,y) which therefore gets small with Δ. By wellknown results about the cosine- and the sine-function - especially that $\frac{\sin\alpha}{\alpha} \to 1$ as $\alpha \to 0$, the reader can easily verify that:

(*)
$$\lim_{\Delta \to 0} \frac{\alpha}{\Delta} = \frac{y}{(x-a)^2 + y^2}$$

Armed this result we solve the Dirichlet problem when $f(x)$ is a continuous function which vanishes outside a bounded interval $[-A, A]$. To keep variables distinct we use ξ as the coordinate on the x-axis. Let $\{\xi_\nu\}$ be a finite and strictly increasing sequence where the differences $\xi_{\nu+1} - \xi_\nu$ are small. Here $\xi_0 = -A$ and $\xi_N = A$ for some $A > 0$. Define the function

(i)
$$G_N(x,y) = \sum f(\xi_\nu) \cdot H_{\xi_\nu, \xi_\nu+1}(x,y)$$

Since f is continuous the limit formulas for the H-functions above show that

(ii)
$$\lim_{y \to 0} G_N(a,y) \simeq \pi \cdot f(a)$$

for every real a where this approximative equality becomes more and more accurate as the maximum of the differences $\xi_{\nu+1} - \xi_\nu$ tends to zero. At the same time the limit formulas for the H-functions imply that the G_N-fuction is approximated by the *Riemann sum*

$$\sum f(\xi_\nu) \cdot (\xi_{\nu+1} - \xi_\nu) \cdot \frac{y}{(x-\xi_\nu)^2 + y^2}$$

Next, by the construction of the Riemann integral of f over $[-A, A]$ we also have

(iii)
$$G_N(x,y) \simeq \int_{-A}^{A} \frac{y}{(x-\xi)^2 + y^2} f(\xi) d\xi$$

Passing to a limit where we take refined ξ-partitions, the sequence $\{G_N(x,y)$ converges to a limit function $G(x,y)$ which is harmonic in the upper half-plane and

$$\lim_{y \to 0} G(a,y) = \pi \cdot f(a) \quad : \quad -A < a < A$$

If a is outside the interval $[-A, A]$ we see that $\lim_{y \to 0} G(a,y) = 0$. Hence we have arrived at Poisson's solution:

4.21 Theorem *Let $f(\xi)$ be a continuous function on the real x-axis which is zero outside a bounded interval. Then the function*

$$G(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2+y^2}\dot{f}(\xi)d\xi$$

is harmonic in the upper half plane and $G(\xi,0)=f(\xi)$ holds on the boundary.

5. Exact versus closed 1-forms

In this section a domain Ω in $\mathcal{D}(C^1)$ is given. It is connected and $\partial\Omega$ has p many boundary curves for some $p \geq 1$. The functions and other objects below are defined in Ω and when integrals are taken over the boundary it is assumed that the functions have been extended to the closure of Ω in order that boundary integrals are defined. A differential 1-form is given by:

(1) $$W = f(x,y)\cdot dx + g(x,y)\cdot dy$$

Here f and g are supposed to be of class C^1 at least. The 1-form is closed if

(2) $$f'_y = g'_x$$

Suppose there exists a C^2-function $U(x,y)$ such that

(3) $$U'_x = f \quad \text{and} \quad U'_y = g$$

Since the mixed second order derivatives U''_{xy} and U''_{yx} are equal we see that (3) gives (2). When U exists we say that the 1-form W is exact and U is called the potential function of W.

Remark. In mechanics one refers to a 1-form W as a field of forces, i.e. to every point one assigns the force vector $F=(f,g)$. If (3) holds we have the equality $\nabla(U)=F$ and one says that F is a potential field where U is its potential function. Notice that U is determined up to a constant.

Let Γ be C^1-curve with end-points $A=(x_0,y_0)$ and $B=(x_1,y_1)$. For every 1-form W we get the line integral

(i) $$\int_\Gamma W = \int_0^T \big[f(x(t),y(t))\cdot\dot{x} + g(x(t),y(t))\cdot\dot{y}\big]\cdot dt$$

If W is exact with a potential function U we notice that

$$\frac{d}{dt}(U(x(t),y(t)) = f(x(t),y(t))\cdot\dot{x} + g(x(t),y(t))\cdot\dot{y}$$

Hence (i) is equal to $U(B)-U(A)$. In other words, when W is exact then the line integral along a curve Γ only depends upon the two end-points and it is expressed by the difference $U(B)-U(A)$.

5.1 Exercise. Let W be a closed 1-form and assume that the line integral along every curve Γ only depends on the end-points. Show that W is exact.

5.2 Non-exact 1-forms. The standard example of a closed but non-exact 1-form occurs when Ω is an annulus $r^2 < x^2+y^2 < R^2$ and we take

$$W = \frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y2}$$

5.3 Starshaped domains. We say that Ω is star-shaped with respect to the origin if the line from 0 to every point $p \in \Omega$ is contained in Ω. In this case every closed 1-form is exact. To see this we define the function $U(x,y)$ in Ω by

$$U(x,y) = \int_0^1 \big[x\cdot f(tx,ty) + y\cdot g(tx,ty)\big]\cdot dt$$

Now we get

(1) $$U'_x = \int_0^1 f(tx,ty) + \int_0^1 \big[tx\cdot f'_x(tx,ty) + ty\cdot g'_x(tx,ty)\big]\cdot dt$$

Next, $g'_x = f'_y$ is assumed and we notice that

$$\frac{d}{dt}(f(tx,ty)) = x \cdot f'_x(tx,ty) + y \cdot f'_y(tx,ty)$$

It follows that the last integral in (1) becomes

$$\int_0^1 t \cdot \frac{d}{dt}(f(tx,ty)) \cdot dt = f(x,y) - \int_0^1 f(tx,ty)) \cdot dt$$

The last integral is cancelled via (1) and hence $U'_x = f$. In the same way one shows that $U'_y = g$ and hence the 1-form is exact.

5.4 The deformation theorem. Let A and B be two given points in Ω. Consider a family of curves $\{\Gamma_s\}$ where each Γ_s has A and B as endpoints and we are given a vector valued function

(*) $$\rho\colon\ (s,t) \mapsto (x(s,t), y(s,t) \quad : 0 \leq s \leq 1 \quad \text{and} \quad 0 \leq t \leq T$$

Here $t \mapsto (x(s,t), y(s,t))$ is the parametrization of Γ_s. With these notations we have

5.5 Theorem. *For every closed 1-form W the function below is a constant*

(1) $$s \mapsto \int_{\Gamma_s} W$$

Remark. We have essentially proved this in the introduction, i.e. see Theorem A.X. But we give another proof below which involves less calculations and at the same time illustrates the efficiency when the calculus of differential forms is used.

Proof. To begin with, we can restrict s to some interval $[0, s_*]$ for every $0 < s_* \leq 1$ and therefore it suffices to prove that (1) takes the same value of Γ_1 and Γ_0. To achieve this we first consider an arbitrary C^1-function f and the 1-form $f \cdot dx$. We shall now find an expression of the difference

(3) $$\int_{\Gamma_1} f \cdot dx - \int_{\Gamma_0} f \cdot dx$$

For this purpose we consider the ρ-map in (*) and construct the inverse function

$$f^*(s,t) = f(x(s,t), y(s,t))$$

We have also the inverse 1-form

$$\rho^*(dx) = x'_s \cdot ds + x'_t \cdot dt$$

Put $\square = \{(s,t) \quad : \{0 \leq s \leq 1\} \cap \{0 \leq t \leq T\}\}$. Since the end-points of the curves $\{\Gamma_s\}$ are equal it follows that (3) is equal to the boundary integral

(4) $$\int_{\partial\square} f^* \cdot \rho^*(dx)$$

Stokes theorem applied to $\square$ implies that (4) is equal to

(5) $$\iint_\square df^* \cdot \rho^*(dx) = \iint_\square \rho^*(f \cdot dx) = \iint_\square \rho^*(f'_y \cdot dy \wedge \cdot dx)$$

Let us remark that the last integral becomes

(6) $$\iint_\square (f'_y)^* \cdot (y'_s \cdot x'_t - y'_t \cdot x'_s) \cdot ds \wedge dt$$

If we instead start with a 1-form $g \cdot dy$ then the same calculation gives

(7) $$\int_{\Gamma_1} g \cdot dy - \int_{\Gamma_0} g \cdot dy = \iint_\square \rho^*(g'_x \cdot dx \wedge \cdot dy)$$

Finally, using the equality $dy \wedge dx = dx \wedge dy$ we obtain the following equality which is valid for an *arbitrary* 1-form $W = f \cdot dx + g \cdot dy$ which need not be closed:

$$(**) \qquad \int_{\Gamma_1} W - \int_{\Gamma_0} W = \iint_{\square} \rho^*((g'_x - f'_y) \cdot dx \wedge \cdot dy)$$

In the special case when $f'_y = g'_x$ the last integral is zero and hence $\int_{\Gamma_1} W = \int_{\Gamma_0} W$ holds, i.e. Theorem 5.3 is a special case of (**).

5.6 When is a 1-form exact. If $p = 1$, i.e. if Ω is a Jordan domain then topology learns that every closed curve in Ω can be deformed to a point. Hence Theorem 5.3 implies that if W is a closed 1-form then its line integral over every closed curve is zero. It follows from Exercise 5.1 that W is exact. If $p > 1$ topology learns that there exist $p - 1$ many closed curves $\Gamma_1, \ldots, \Gamma_{p-1}$ which give a basis for the homology of the multiple connected domain. If W is a closed 1-form one assigns the period numbers

$$(*) \qquad \int_{\Gamma_k} W \quad : \; 1 \leq k \leq p - 1$$

Now W is exact if and only if all the period numbers are zero. Suppose $W_1, \ldots, W_{p-1}$ is some $(p-1)$-tuple of closed 1-forms such that the $(p-1) \times (p-1)$-matrix with elements

$$a_{ik} = \int_{\Gamma_k} W_i$$

is invertible. Then, for every closed 1-form W we can find $c_1, \ldots, c_{p-1}$ such that the periods of $W - (c_1 W_1 + \ldots + c_{p-1} W_{p-1})$ vanish and hence there exists a potential function U such that

$$W = c_1 W_1 + \ldots + c_{p-1} W_{p-1} + dU$$

6. An integral formula for the Laplace operator.

Consider a C^2-function $f(x,y)$ defined in the open disc $D(R)$ of radius R centered at the origin. Let $0 < r < R$ and choose also a small $\epsilon > 0$ and consider the annulus $\Omega = \{\epsilon^2 < x^2 + y^2 < r^2\}$. In Ω we have the function

$$g_r(x,y) = \text{Log}(\frac{r}{\sqrt{x^2+y^2}}) \tag{1}$$

We have already seen that g is harmonic in Ω. The boundary $\partial\Omega$ consists of the circle $x^2+y^2 = \epsilon^2$ and the circe of radius r and we leave it to the reader to verify that Theorem 4.9 applied to f and g give the formula

$$\iint_\Omega \Delta(f)(x,y) \cdot g_r(x,y) \cdot dxdy = \int_0^{2\pi} f(r,\theta) \cdot d\theta - \int_0^{2\pi} f(\epsilon,\theta) \cdot d\theta \tag{2}$$

Keeping r fixed while $\epsilon \to 0$ the continuity of f at the origin gives:

$$f(0) = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(r,\theta) \cdot d\theta - \frac{1}{2\pi} \cdot \iint_{D(r)} \Delta(f)(x,y) \cdot g(x,y) \cdot dxdy \tag{3}$$

With $0 < r_* < R$ we can apply (3) to every $0 < r < r^*$ and after an integration using polar coordinates we obtain:

$$f(0) = \frac{1}{\pi \cdot r_*^2} \cdot \iint_{D(r_*)} f(x,y) \cdot dxdy - \frac{1}{2\pi} \cdot \int_0^{r_*} [\iint_{D(r)} \Delta(f)(x,y) \cdot g_r(x,y) \cdot dxdy\big] \cdot dr$$

The last double integral is evaluated via polar coordinates and the result is the formula from the introduction, i.e. the reader may verify that the double integral becomes

$$\int_0^{r_*} K(r^*,s) \cdot \big[\int_0^{2\pi} \Delta(f)(s,\theta) \cdot d\theta\big] \cdot ds \tag{i}$$

where we for each pair $0 < s \leq r^*$ have:

$$K(r^*,s) = s^3 \cdot \int_1^{\frac{r}{s}} u \cdot \text{Log}(u) \cdot du \tag{ii}$$