16. Entire functions of exponential type

Introduction. The class $\mathcal{E}$ of entire functions of exponential type is defined as follows:

0.1 Definition. *An entire function f belongs to $\mathcal{E}$ if and only if there exists constants A and C such that*

$$(*) \qquad |f(z)| \leq C \cdot e^{A|z|} \quad : \ z \in \mathbf{C}$$

We refer to the literature for studies of of the more extensive class of entire functions with arbitrary finite order, i.e. those f where (*) is replaced by $|z|^\rho$ for some $\rho > 0$. The results in Sections A-B are foremost due to Hadamard and Lindelöf. The class $\mathcal{N}$ appears in Section 3 and was was introduced by Carleman who used it to prove certain approximation theorems related to moment problems. Our main concern deal with Tauberian theorems which is treated in section D whose material is based upon Chapter V in [Paley-Wiener]. Let us present some of the results to be proved in Section D while we refer to Section A and B for more elementary material about the class $\mathcal{E}$. Consider a non-decreasing sequence $\{\lambda_\nu\}$ of positive real numbers such that the series

$$(1) \qquad \sum \lambda_\nu^{-2} < \infty$$

When this holds there exists the entire function given by a product series:

$$H(z) = \prod (1 - \frac{z^2}{\lambda \nu^2})$$

Indeed, this was already proved in III: Chapter. Notice tha $H8z)$ is even and on the positive imaginary axis it is positive. We study the function defined for real $y > 0$:

$$y \mapsto \frac{\log H(iy)}{y} = \frac{1}{y} \sum \log (1 + \frac{y^2}{\lambda \nu^2})$$

At the same time we can consider the intergals

$$J(R) = \int_{-R}^{R} \frac{\log |H(x)| \cdot dx}{x^2}$$

With these notations one has

0.1 Theorem. *The statements*

$$\text{(i)} \qquad \lim_{y \to \infty} \frac{\log H(iy)}{y} = \pi A y$$

and

$$\text{(ii)} \qquad \lim_{R \to \infty} J(R) = -\pi^2 A$$

hold for some $A \geq 0$ are completely equivalent

Example. Consider the case when $\{\lambda \nu\}$ is the set of positive integers. In XX we explain the formula

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$$

Notice that

$$|\sin \pi i y| = \frac{e^{\pi y} - e^{-\pi y}}{2}$$

when $y > 0$. From this it follows that we get the constant A in (i) above. At the same time we proved by redidue calculus in [Residue:XXX] that if $f(z) = \frac{\sin \pi z}{\pi z}$ then

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\log |f(x)| \cdot dx}{x^2} = -\pi^2$$

which is in accordance with (ii) in Theorem 0.1. Of special interest is the case when the limit in (ii) is automatic via an integrability condition, i.e. when we from start has an absolutely convergent integral

$$(*) \qquad \int_{-\infty}^{\infty} \frac{\big|\log|H(x)|\big| \cdot dx}{x^2} < \infty$$

In this case the J-integrals converge and as a consequence there is a limit in (i) for some $A \geq 0$. It turns out that further conclusions can be made. namely, the convergence of (*) implies that the sequence $\{\lambda_\nu\}$ also has a regular growth in the sense that if $N(r)$ is the counting function which for every $r > 0$ counts the number of $\lambda\nu \leq r$, then there exists the limit

$$\lim_{R\to\infty} \frac{N(R)}{R} = A$$

with A determined via Theorem 0.1. We shall prove this in Section D and remark that the integrability condition (*) is related to the study of the Carleman class in Section C.

Remark. The general Tauberian theorems for entire functions of exponential type are foremost due to Wiener. See his pioneering article [Wiener].

A. Growth of entire functions.

here we start the study with arbitrary entire functions $f(z)$. To each such f we associate certain functions which describe the growth and the number of its zeros in discs of radius R centered at the origin. We can always write

$$f(z) = az^m \cdot f_*(z)$$

where f_* is entire and $f_*(0) = 1$. The case when $f(0) = 1$ is therefore not so special and several formulas take a simpler form when this holds.

A.1 The functions $T_f(R)$ and $m_f(R)$. They are defined for every $R > 0$ by

$$\text{(i)} \qquad T_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \text{Log}^+ \big|f(R(e^{i\theta})\big| \cdot d\theta$$

$$\text{(ii)} \qquad m_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \text{Log}^+ \big[\frac{1}{\big|f(R(e^{i\theta})\big|}\big] \cdot d\theta$$

A.2 The maximum modulus function. It is defined by

$$M_f(R) = \max_{0\leq\theta\leq 2\pi} |f(Re^{i\theta})|$$

A.3 The counting function $N_f(R)$. To each $R > 0$ we count the number of zeros of f in the punctured disc $0 < |z| < R$. This integer is denoted by $N_f(R)$, where multiple zeros are counted according to their multiplicities. Jensen's formula shows that if $f(0) = 1$ then

$$(*) \qquad \int_0^R \frac{N_f(s)}{s} \cdot ds = \frac{1}{2\pi} \cdot \int_0^{2\pi} \text{Log}\big|f(R(e^{i\theta})\big| \cdot d\theta = T_f(R) - m_f(R)$$

Since the left hand side always is ≥ 0 the inequality below holds under the hypothesis that $f(0) = 1$:

$$(**) \qquad m_f(R) \leq T_f(R)$$

Next, since $N_f(R)$ is increasing we get

$$\log 2 \cdot N_f(R) \leq \int_R^{2R} \frac{N_f(s)}{s} \cdot ds \leq T_f(2R) \implies$$

(***) $$N_f(R) \leq \frac{T_f(2R)}{\log 2}$$

A.4 Harnack's inequality. The function $\text{Log}^+|f|$ is subharmonic which implies that whenever $0 < r < R$ then one has

$$\text{Log}^+|f(re^{i\alpha})| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{R+r}{R-r} \cdot \text{Log}^+ \big|f(R(e^{i\theta})\big| \cdot d\theta$$

It follows that

$$M_f(r) \leq \frac{R+r}{R-r} \cdot T_f(R)$$

In particular we can take $R = 2r$ and conclude that

$$M_f(r) \leq 3 \cdot T_f(2r) \quad \text{hold for every} \quad r > 0$$

The last inequality gives:

A.5 Theorem. *An entire function f belongs to $\mathcal{E}$ if and only if there exists a constant A such that*

$$T_f(R) \leq A \cdot R$$

holds for every R.

A.6 A division theorem. Let f and g be in $\mathcal{E}$ and assume that $h = \frac{f}{g}$ is entire. Now

(i) $$\log^+ |h| \leq \log^+ |f| + \log^+ |g|$$

In the case when $g(0) = 1$ we apply (**) in A.3 and conclude that

$$T_h(R) \leq T_f(R) + T_g(R)$$

Hence Theorem A.5 implies that h belongs to $\mathcal{E}$. We leave it to the reader to verify that this conclusion holds in general, i.e. without any assumption on $g(0)$.

A.7 Hadamard products. Let $\{\alpha_\nu\}$ be a sequence of complex numbers arranged so that the absolute values are non-decreasing. The counting function of the sequence is denoted by $N_{\alpha(\bullet)}(R)$. Suppose that the counting function satisfies:

(*) $$N_{\alpha(\bullet)}(R) \leq A \cdot R \quad \text{for all} \quad R \geq 1$$

A.8 Theorem *When (*) holds the infinite product*

$$\prod (1 - \frac{z}{\alpha_\nu}) \cdot e^{\frac{z}{\alpha_\nu}}$$

converges for every z and gives an entire function to be denoted by $H_{\alpha(\bullet)}$ and called the Hadamard product of the α-sequence.

A.9 Exercise. Prove this theorem and show also that there exists a constant C which is independent of A such that the Hadamard product satisfies the growth condition:

$$\big|H_{\alpha(\bullet)}(z)\big| \leq C \cdot \exp\big[A \cdot |z| \cdot \log |z|\big] \quad \text{for all} \quad |z| \geq e$$

A.10 Lindelöf's condition. For a sequence $\{\alpha_\nu\}$ we define the Lindelöf function

$$L(R) = \sum_{|\alpha_\nu| < R} \frac{1}{\alpha_\nu}$$

We say that $\{\alpha_\nu\}$ is of Lindelöf type if there there exists a constant L^* such that

(**) $$\big|L(R)\big| \leq L^* \quad \text{hold for all} \quad R.$$

A.11 Theorem. *If the α-sequence satisfies (*) in A.8 and is of the Lindelöf type then there exists a constant C such that the maximum modules function of $H_{\alpha(\bullet)}$ satisfies*

$$M_{H_{\alpha(\bullet)}}(R) \leq C \cdot e^{AR}$$

and hence the Hadamard product belongs to $\mathcal{E}$.

A.12 Exercise. Prove this result. A hint is to study the products

$$\prod_{|\alpha_\nu|<2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}} \quad \text{and} \quad \prod_{|\alpha_\nu|\geq 2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}}$$

separately for every $R \geq 1$. Try also to find an upper bound for C expressed by A and L^*.

A converse result. Le f belong to $\mathcal{L}$. Then it turns out that its set of zeros satisfies (**) in A.10 for a constant L^*. To prove this we shall use:

A.13 An integral formula. With $R > 0$ we put $g(z) = \frac{1}{z} - \frac{\bar{z}}{R^2}$. This is a is harmonic function in $\{0 < |z| > R\}$ and $g = 0$ on $|z| = R$. Apply Green's formula to g and $\text{Log}\,|f|$ on an annulus $\{\epsilon < |z| < R\}$. Let $f(z)$ be an entire function with $f(0) = 1$ and consider a pair $0 < \epsilon < R$ where f has not zeros in $|z| \leq \epsilon$.

A.14 Exercise. Show that

$$(*) \qquad \sum_{|\alpha_\nu|<R} \Big[\frac{1}{\alpha_\nu} - \frac{\bar{\alpha}_\nu}{R^2}\Big] = \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \text{Log}\,|f(Re^{i\theta})| \cdot e^{-i\theta} \cdot d\theta - f'(0)$$

where the sum is taken over zeros of f repeated with multiplicities in the disc $\{|z| < R\}$.

A.15 The case $f \in \mathcal{E}$. Asumme this. From XX we have seen that the counting function $N_f(R)$ is bounded by $C \cdot R$ for some constant C and this implies that the series

$$\sum |\alpha_\nu|^{-2} < \infty$$

To show that the Lindelöf function $L(R)$ is bounded it therefore sufficices to show that the function

$$R \mapsto \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \text{Log}\,|f(Re^{i\theta})| \cdot e^{-i\theta} \cdot d\theta$$

is bounded and this follows from A.XX.

B. The factorisation theorem for $\mathcal{E}$

Consider some $f \in \mathcal{E}$. If f has a zero at the origin we can write

$$f(z) = az^m \cdot f_*(z) \quad \text{where} \quad f_*(0) = 1$$

It is clear that f_* again belongs to $\mathcal{E}$ and in this way we essentially reduce the study of $\mathcal{E}$-functions f to the case when $f(0) = 1$. Above we proved that the set of zeros satisfies Lindelöf's condition and therefore the Hadamard product

$$H_f(z) = \prod (1 - \frac{z}{\alpha_\nu}) \cdot e^{\frac{z}{\alpha_\nu}}$$

taken over all zeros of f outside the origin belongs to $\mathcal{E}$. Now the quotient f/H_f is entire and we shall prove:

B.1 Theorem *Let $f \in \mathcal{E}$ where $f(0) = 1$. Then there exists a complex number b such that*

$$f(z) = e^{bz} \cdot H_f(z)$$

Proof. The division in A.6 shows that the function

$$G = \frac{f}{H_f}$$

is entire and belongs to $\mathcal{E}$. By construction G is zero-free which gives the entire function $g = \log G$ for we have the inequality

$$|g(z)| \leq 1 + \log^+ |G(z)| \leq 1 + C|z|$$

Since $G \in \mathcal{E}$ we see that $|g|$ increases at most like a linear function so by Liouvile's theorem it is a polynomial of degree 1. Since $f(0) = 1$ we have $g(0) = 0$ and hence $g(z) = bz$ for a complex number b and the formula in Theorem B.1 follows.

C. The Carleman class $\mathcal{N}$

Let $f \in \mathcal{E}$. On the real x-axis we have the non-negative function $\log^+ |f(x)|$. If the integral

$$(*) \qquad \int_{-\infty}^{\infty} \frac{\log^+ |f(x)| \cdot dx}{1 + x^2} < \infty$$

we say that f belongs to the Carleman class denoted by $\mathcal{N}$. To study $\mathcal{N}$ the following integral formula plays an important role.

C.1 Integral formula in a half-plane. Let $g(z)$ be analytic in the half plane $\mathfrak{Im}(z) > 0$. Assume that g extends continuously to the boundary $y = 0$, i.e. to the real x-axis and that $g(0) = 1$. Given a pair $0 < \ell < R$ we consider the domain

$$\Omega_{\ell,R} = \{\ell^2 < x^2 + y^2 < R^2\} \cap \{y > 0\}$$

With $z = re^{i\theta}$ we have the harmonic function

$$v(r, \theta) = (\frac{1}{r} - \frac{r}{R^2}) \sin\theta = \frac{y}{x^2 + y^2} - \frac{y}{R^2}$$

Here $v = 0$ on the upper half circle where $|z| = R$ and $y > 0$ and the outer normal derivative along the x-axis becomes

$$\partial_n(v) = -\partial_y(v) = -\frac{1}{x^2} + \frac{1}{R^2} \quad : \quad x \neq 0$$

Let $\{\alpha_\nu\}$ be the zeros of g counted with multiplicites in the upper half-plane.Then Green's formula gives:

C.2 Proposition. *One has the formula*

$$2\pi \cdot \sum \frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha\nu|^2} - \frac{\mathfrak{Im}\,\alpha_\nu}{R^2} =$$

$$(*) \qquad \int_\ell^R (\frac{1}{R^2} - \frac{1}{x^2}) \cdot \mathrm{Log}\,|g(x) \cdot g(-x)| \cdot dx - \frac{2}{R} \int_0^\pi \sin(\theta) \cdot \mathrm{Log}\,|g(Re^{i\theta})| \cdot d\theta + \chi(\ell)$$

where $\chi(\ell)$ is a contribution from line integrals along the half circle $|z| = \ell$ with $y > 0$.

C.3 Exercise Prove via Green's theorem. Notice that the term $\chi(\ell)$ is independent of R so the formula can be used to study asymptotic behaviour as $R \to +\infty$.

Next, the family of analytic functions $g(z)$ in the upper half-plane is identified with $\mathcal{O}(D)$ using a conformal map, i.e. with a given g we get $g_* \in \mathcal{O}(D)$ where

$$g_*(\frac{z - i}{z + i}) = g(z)$$

holds when $\mathfrak{Im}(z) > 0$. When g extends to a continuous function on the real x-axis we have the equality As explained in XXX this gives the equality

$$(*) \qquad \int_0^{2\pi} \log^+ |g_*(e^{i\theta})| \cdot d\theta = 2 \cdot \int_{-\infty}^{\infty} \frac{\log^+ |g(x)| \cdot dx}{1 + x^2}$$

This means that the last integral is finite if and only if g_* belongs to the Jensen-Nevanlinna class and in XX we proved that this entails that

$$\int_0^{2\pi} \log^+ \frac{1}{|g_*(e^{i\theta})|} \cdot d\theta < \infty$$

In particular we conclude that if an entire function f satisfies (*) above then it follows that

$$(**) \qquad \int_{-\infty}^{\infty} \log^+ \frac{1}{|f(x)|} \cdot \frac{dx}{1+x^2}$$

in other words, (*) entails that the absolute value $|\log|f(x)||$ is integrable with respect to the density $\frac{1}{1+x^2}$. Using (**) we can prove:

C.4 Theorem *Let $f \in \mathcal{N}$. Then*

$$\sum^* \mathfrak{Im}\frac{1}{\alpha_\nu} < \infty$$

where the sum is taken over all zeros of f which belong to the upper half-plane.

Proof. Since $f \in \mathcal{E}$ there exists a constant C such that $N_f(R) \leq C \cdot R$. If $R \geq 1$ it follows that

$$\big|R^{-2} \sum \bar{\alpha}_\nu| \leq R^{-2} \cdot R \cdot N_f(R) \leq C$$

where the sum is taken over zeros in $\Omega_{\ell,R}$ Next, since $\mathfrak{Im}\,\alpha_\nu > 0$ in this open set it follows that

$$\frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha_\nu|^2} > 0$$

for every zero in the upper half-plane. In particular this holds for the zeros in $\Omega_{\ell,R}$ and passing to the limit as $R \to \infty$ it suffices to establish an upper bound in the right hand side of Proposition C.2 with $g = f$. The integral taken over the half-circle where $|z| = R$ is uniformly bounded with respect to R since $f \in \mathcal{E}$ and we have the inequality XX from A.XX. For the integral on the x-axis we therefore only need an upper bound. Since $R^{-2} - x^{-2} \leq 0$ during the integration it suffices to find a constant C such that

$$\int_\ell^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \cdot \log^+ \frac{1}{|f(x) \cdot f(-x)|} \cdot dx \leq C \quad \text{hold for all} \quad R \geq 1$$

The reader may verify that such a constant C since (**) above holds.

C.5 A limit for the counting function. Using the Tauberian theorem which is proved in Section D one has the following:

C.6 Theorem *For each $f \in \mathcal{N}$ there exists the limit:*

$$\lim_{R\to\infty} \frac{N_f(R)}{R}$$

C.7 Remark. To prove this we first notice that if $f \in \mathcal{N}$ then the product $f(z) \cdot f(-z)$ also belongs to $\mathcal{N}$ and for this even function the counting function is twice that of f. So it suffices to prove Theorem XX when f is even. We may also assume that $f(0) = 1$ and since $f \in \mathcal{E}$ it is given by a Hadamard product

$$(1) \qquad f(z) = \prod^* \left(1 - \frac{z^2}{\alpha_\nu^2}\right)$$

where $\prod^*$ indicates the we take the product of zeros whose real part is > 0 and if they are purely imaginary they are of the form $b \cdot i$ with $b > 0$. We can replace the zeros by their absolute values and construct

$$(2) \qquad f_*(z) = \prod^* \left(1 - \frac{z^2}{|\alpha_\nu|^2}\right)$$

If x is real we see that

(3) $$|f_*[x)| \leq |f(x)|$$

We conclude that if f belongs to $\mathcal{N}$ so does f_*. At the same time their counting functions of zeros are equal. This reduces the proof of Theorem XX to the special case when f is even and the zeros are real. In the next section we study entire and even functions in $\mathcal{E}$ whose zeros are real and via a general Tauberian theorem deduce Theorem C.6 above.

D. A Tauberian Theorem

To every non-decreasing and discrete sequence of positive real numbers $\{0 < t_1 \leq t_2 \leq \ldots\}$ we associate the even sequence where we include $\{-t_\nu\}$. Assume that $\mathcal{N}_\Lambda(R) \leq C \cdot R$ for some constant. We get the entire function

$$f(z) = \prod (1 - \frac{z^2}{t_\nu^2})$$

which by the results in Section A belongs to $\mathcal{E}$. If $R > 0$ we set:

(*) $$J_1(R) = \frac{\log f(iR)}{R} \quad \text{and} \quad J_2(R) = \int_{-R}^{R} \frac{\operatorname{Log}|f(x)|}{x^2} \cdot dx$$

D.1 Theorem. *There exists a limit*

$$\lim_{R\to\infty} \frac{N_f(R)}{R} = 2A$$

if and only if at least one of the J-functions has a limit as $R \to \infty$. Moreover, when this holds one has the equalities:

$$\lim_{R\to\infty} J_1(R) = \frac{\pi \cdot A}{2} \quad \text{and} \quad \lim_{R\to\infty} J_2(R) = -\frac{\pi^2 \cdot A}{2}$$

To prove this we introduce the following:

D.2 The W-functions. On the positive real t-line we define the following functions:

(1) $$W_0(t) = \frac{1}{t} \quad : \ t \geq 1 \quad \text{and} \quad W_0(t) = 0 \quad \text{when} \ t < 1$$

(2) $$W_1(t) = \frac{\operatorname{Log}(1 + t^2)}{t}$$

(3) $$W_2(t) = \int_0^t \frac{\operatorname{Log}|1 - x^2|}{x^2} \cdot dx$$

Next, the real sequence $\Lambda = \{t_\nu\}$ gives a discrete measure on the positive real axis where one assigns a unit point mass at every t_ν. If repetitions occur, i.e. if some t-numbers are equal we add these unit point-masses. Let ρ denote the resulting discrete measure. The constructions of the J-functions obviously give:

(*) $$\frac{\mathcal{N}_\Lambda(R)}{R} = 2 \cdot \int_0^\infty W_0(R/t) \cdot \frac{d\rho(t)}{t}$$

(**) $$J_k(R) = \int_0^\infty W_k\big(\frac{R}{t}\big) \cdot \frac{d\rho(t)}{t} \quad : \ k = 1, 2$$

D.3 Exercise. Show that under the assumption that the function $\frac{\mathcal{N}_\Lambda(R)}{R}$ is bounded, it follows the three $\mathcal{W}$-functions belong to the $\mathcal{BW}$-algebra defined by the measure ρ as explained in XXX.

D.4 Fourier transforms. Recall that on $\{t > 0\}$ we have the Haar measure $\frac{dt}{t}$. We leave it to the reader to verify that all the W-functions above belong to $L^1(\mathbf{R}^+)$, i.e.

(i) $$\int_0^\infty \big|W_k(t)\big| \cdot \frac{dt}{t} < \infty \quad : k = 0, 1, 2$$

The Fourier transforms are defined by

(ii) $$\widehat{W_k}(s) = \int_0^\infty W_k(t) \cdot t^{-(is+1)} \cdot dt$$

We shall prefer to use the functions with reversed sign on s, i.e. set

(iii) $$\mathcal{F}\, W_k(s) = \int_0^\infty W_k(t) \cdot t^{is-1} \cdot dt$$

D.5 Proposition *One has the formulas*

(i) $$\mathcal{F}\, W_0(s) = \frac{1}{1 - is}$$

$$\mathcal{F}\, W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1 - is) \cdot (1 + e^{-\pi s})}$$

(iii) $$\mathcal{F}\, W_2(s) = \frac{2\pi}{(1 - is) \cdot (e^{\pi s/2} + e^{-\pi s/2})}$$

Proof. The equation (i) is easily verified. To prove (ii) we notice that a partial integration gives

$$\mathcal{F}\, W_1(s) = \frac{1}{is - 1} \cdot \int_0^\infty \frac{2 \cdot t^{is} \cdot dt}{1 + t^2}$$

To compute this integral we employ residue calculus where we consider the function

$$\phi(z) = \frac{z^{is}}{1 + z^2}$$

We perform line integrals over large half-circles where $z = Re^{i\theta}$ and $0 \leq \theta \leq \pi$. A reside occurs at $z = i$. Notice also that if $t > 0$ then

$$(-t)^{is} = t^{is} \cdot e^{-\pi s}$$

which gives

$$\mathcal{F}W_1(s) \frac{1}{1 - is) \cdot (1 + e^{-\pi s})} \cdot \lim_{R \to \infty} \int_{-R}^{R} \phi(t) \cdot dt$$

Here ϕ has a simple pole at $z = 1$ so by residue calculus the last integral becomes

$$-2\pi i \cdot (i)^{is} \cdot \frac{1}{2i} = -\pi \cdot e^{-\pi s/2}$$

Taking the minus sign into the account we conclude that

$$\mathcal{F}W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1 - is) \cdot (1 + e^{-\pi s})}$$

For (iii) a partial integration gives

$$\mathcal{F}\, W_2(s) = -\frac{1}{is} \cdot \int_0^\infty \log|1 - t^2| \cdot t^{is-2} \cdot dt$$

Here we computed the right hand side in [Residue Calculus] which gives the requested formula (iii).

D.6 Evaluations at $s = 0$ From (i-iiii) we find that

$$\mathcal{F}W_2(0) = \frac{\pi}{2} \quad \text{and} \quad \mathcal{F}W_2(0) = -\frac{\pi^2}{2}$$

Since we also have $\mathcal{F}_1(0) = 1$ we apply the general Tauberian Theorem in XX and can read off the results in Theorem D.1.

D.8 Proof of Theorem D.1

The formulas for the Fourier transforms in Proposition D.5 show that each of them is $\neq 0$ on the whole real s-line. Hence we can apply the general result in XX to the discrete measure ρ since the $\mathcal{W}$-functions belong to the $\mathcal{BW}$-algebra from XXX. This implies that if one of the three limits in Theorem D.3 above exists, so do the other. To get the relation between the limit values we only have to evaluate the Fourier transform at $s = 0$. From Proposition D.5 we see that

$$(**) \qquad \mathcal{F}W_0(0) = 1 \quad : \quad F\mathcal{W}_1(0) = 1 \quad \text{and} \quad F\mathcal{W}_2(0) = \pi$$

This gives the formulas in Theorem D.3 by the general result for $\mathcal{BW}$-algebras in XXX.

Application to measures with compact support.

Le μ be a Riesz measure on the real t-line with compact support in an interval $[-a, a]$ where we assume that both end-points belong to the support. The measure is in general complex-valued. Now we get the entire function

$$f(z) = \int_{-a}^{a} e^{-izt} \cdot d\mu(t)$$

Here f restricts to a bounded function on the real x-axis with maximum norm $\leq ||\mu||$. Hence f belongs to $\mathcal{N}$ which means that Theorem D.x holds.

Theorem. *One has the equality*

$$\lim_{R \to \infty} \frac{N_f(R)}{R} = \frac{a}{\pi}$$

At the same time we notice that if $z = x + iy$ then we have

$$|e^{-izt}| = e^{yt} \leq e^{a|y|}$$

for all $-a \leq t \leq a$. From this it follows that

$$|f(x+iy)| \leq ||\mu|| \cdot e^{a|y|}$$

E. Tauberian theorems with a remainder term

Results which i addition to Theorem D.1 also contain remainder terms were established by Beurling in 1936. An example of such a theorem with remainder term goes as follows: Let

$$f(z) = \prod (1 - \frac{z^2}{t_\nu^2})$$

be an even and entire function of exponential type with real zeros as in section D. Then one has:

E.1 Theorem. *Let $A > 0$ and $0 < a < 1$ and assume that there exists a constant C_0 such that*

$$\Big| -\frac{1}{\pi^2} \cdot \int_0^R \frac{\log |f(x)|}{x^2} \cdot dx - A \Big| \leq C_0 \cdot R^{-a}$$

hold for all $R \geq 1$. Then there is another constant C such that

$$\Big| N_f(R) - R \Big| \leq C_1 \cdot R^{1-a/2}$$

Beurling's original manuscript which proved Theorem E.1 as well as other results dealing with remainder terms has remained unpublished. But it was resumed with details of proofs in a Master's Thesis at Stockholm University by F. Gülkan in 1994. As remarked by Beurling in his article [Beurling] proofs of results with remainders require the full force from the theory of Fourier integrals in addition to more direct use of analytic functions of exponential type. The interested reader should also consult articles by Beurlings former Ph.d student S. Lyttkens which prove various Tauberian theorems with remainder terms. See also work by T. Ganelius for closely related material.