Förslag till Sparrerska priset

Johan Alm inledde högskolestudier vid Uppsala Universitet 2004 där han förutom matematik också studerade fysik i fem terminer. Alm antogs till forskarutbildning i matematik vid Stockholms Universitet i augusti 2009 med Sergei Merkulov som huvudhandledare och Torsten Ekedahl som biträdande handledare. Från Alms studier i Uppsala ingick flera rätt avancerade kurser som Modules and Homologcal algebra (2007-12-18) samt Analysis on Manifolds (2009-04-17). Vidare skrev Alm ett imponerade examensarbete som redan i ett tidigt skede vittnar om mycket goda insikter och förmåga att arbeta med avancerad differentialgeometri.

Hösten 2009 inledde Alm ett forskningsprojekt som gavs av hans handledare Merkulov vilket ledde till ett arbete vars resultat vida överträffar vad som normalt kan förväntas hos en nybliven forskarstuderande. Alms presenterade sin licentiatavhandling [A:1] 30 maj 2011. Där införs bland annat två helt nya konfigurationsoperader i § 1.3.14 i [A:1] och Alm härleder egenskaper hos dessa för att efteråt uvidga och skärpa tidigare resultat av bland andra Kontsevich och Duflo. Jag vill gärna tillägga att Alms inledning till licentiatavhandlingen visar prov på en mognad som vore den skriven av en redan eminent matematiker med mångårig erfarenhet. Förutom nya värdefulla resultat i [A:1] visar arbetet prov på bredd och djup hos där metoder från en rad olika ämnesomtåden som differentialgeometri, topologi och algebra ingår på ett vis man ytterst sällan möter hos en ung forskarstuderande.

Artikeln [A:2] är skrivet med Merkulov som medförfattare. Här unyttjas dels resultat från [A:1] samt tidigare arbeten av Merkulov, bland annat om Exotic automorphisms of Shouten algebras and deformation quantization polyvectorfields. Huvudresultatet i [A:2] ger nya och värdefulla inblickar om Poissonstrukturer på mångfalder där de rent matematiska upptäckterna har anknytning till fysik, bland annat kring Feymanngrafer.

Alm har sedan hösten 2012 arbetat med nya problemställningar där man torde kunna förvänta sig nya starka resultat som bland annat kan komma att knyta an till kvantfältsteori där det finns många fundamentala och ännu olösta problem för såväl teoretiska fysiker om matematiker.

Avslutningsvis instämmer jag till fullo med de lovord som Sergei Merkulov framför i sitt rekommendationsbrev om Johan Alm.

Stockholm 21 december 2012

Jan-Erik Björk

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Hörmander presenterade Atiyah och Singer insatser när de fick Abelpriset för deras gemensamma artikel The index of elliptic operators. Han har själv också publicerat en uppsats med anknytning till indexteoremet Se ref 42 i hans böcker om PDE: Elliptische Differentialgleichungen Akademie Verlag Berlin 1971. Beträffande originalbeviset av Atiyah och Singer fordrade den rent analytiska delen av beviset inte allmänna resultat om pseudodifferentialoperatorer. Deras bevis utnyttjade resultat om singulära operatorer, främst via arbeten av Calderon från mitten av 1950-talet. Henri Cartans presentationsföredrag från IMU i Moskava 1966 om Atiyahs Fieldspris tydliggör detta.

Däremot lyfter J.J Kohn i sitt 30-minutersföredrag med titeln Differential Complexes fram ett då aktuellt arbete av Hörmander: Pseudo-Differential operators and non-elliptic boundary value problems (Ann. of Math. 83 (1966). Bland annat skriver Kohn:

In a joint paper (Annals of Math, 1965) with Hugo Rossi we study the problem of extending holomorphic forms from the boundary of manifold. This suggested the investigation of an induced $\bar{\partial}$-operator on real submanifolds of a complex manifold which provide examples of non-elliptic boundary value complexes on compact manifolds with good regularity properties and finite cohomology. Lars Hörmander has studied a much more general class of such problems, and recently Sweeney has extended earlier work about Spencer resolutions where he has applied general facts from Hörmander's cited artiicle.

Integralrepresentationer av lösningar till PDE-system med konstanta koefficienter. I Notes till kapitel XV: Analytic function theory and differential equations från vol.2 i Hörmanders PDE-böcker visas att även deriverbara lösningar kan representeras via absoluta konvergenta integraler tagna över exponentiallösningar. Det tekniska problemet som Hörmander löser i detta kapitel handlar om att vissa L^2-estimat med bounds gäller trots att viktfunktionerna inte är överallt plurisubharmoniska. Se Notes s. 360 för en detaljerad kommentar och notera att Hörmander i denna Not till Kapitel XV lyfter fram originalarbeten av Morrey och J.J- Kohn som var de som först använde L^2-olikheter för att studera speciella överbestämda system via Cauchy-Riemann ekvationer i flera komplex variabler.

Hypoelliptiska operatorer. Ett av huvudresultaten i Hörmanders doktorsavhandling handlar om att karakterisera hypoelliptiska operatorer. Under hans år som professor vid SU utvecklade och generaliserade han resultat från avhandlingen där man kan nämna artikeln xxxx (Arkiv för matematik vol. xx 1957).

Lech's theorem on zeros of a polynomial ideal.

We expose a proof of a theorem by Christer Lech from the article [Le]. His result answered a question raised by by Lars Hörmander concerned with over-determined systems of hypoelliptic PDE-equations. Here is the situation. Let $n \geq 3$ and $\mathfrak{p}$ is a prime ideal in $\mathbf{C}[z_1, \ldots, z_n]$ whose locus $\mathfrak{p}^{-1}(0) = V$ yields an irreducible algebraic set. We assume that $\dim(V) = d$ where $d \leq n-2$ and put $p = n - d$. A *Lech polynomial with respect to* V is a polynomial $p(z)$ which is zero on V, and in addition there exists a constant C such that the following inequality hold for each $x \in \mathbf{R}^n$

$$\operatorname{dist}(x, V) \leq C \cdot \operatorname{dist}(x, f^{-1}(0))$$

where dist refers to the euclidian length.

We shall prove that there exists a Lech polynomial with respect to every algebraic set V in $\mathbf{C}^n$. To begin with it suffices to prove this existence when V is irreducible. Namely, in general $V = V_1 \sup \ldots \cup V_m$ where $\{V_j\}$ are irreducible. If $\{f_j\}$ are Lech polynomials asociated to these irreducible sets, then it is trivial to check that the product $f_1 \cdots f_m$ is a Lech polynomial with respect to V.

Let us now consider a prime ideal $\mathfrak{p}$ whose locus is an irreducible algebraic set V where $d = \dim(V) \leq n - 2$ and put $p = n - d$.

To find a polynomial $L(z)$ such that (*) holds we shall use a classic result due to Max Noether. Namely, let V^* be the algwebraic cone of V, i.e. the common zeros of the leading forms of polynomials from $\mathfrak{p}$. It is wellknown that $\dim(V^*) = \dim(V)$ and now we consider elements Π in the Grassmanian $\mathcal{G}(p-1)$ of which conaists of $(p-1)$-dimensional complex subspaces of $\mathbf{C}^n$. Such a subspace Π is said to be V-transversal if

$$\Pi \cap V^* = \{0\}$$

where the right hand side is the origin in $\mathbf{C}^n$. Via algwbraic elimintion one shows easily that for every V-transversal Π, the set-theoretic sum $\pi + V$ is an algebraic hypersurface given by the zeros of an irreducible polynomial $N_\Pi(z)$ whose degree is equal to the muliplicity of the given algewbraic set V. This unifor bound of the degree yields a finite dimensional comålexsubpsace $\mathcal{N}$ of $\mathbf{C}[z]$ generte by these N_π-polynimals as π varies over V-transversal elements in $\mathcal{G}(p-1)$.

We refer to $\mathcal{N}$ as the Noether space asociated to V and now we announce the major result from [Lech].

Tjoerem xxx

The proof requires several steps.

Consider a finite family $g_1, \ldots, g_m$ in $\mathfrak{p}$ whose set of common zeros is V. To each polynomial $p(z) = \sum c_\alpha z^\alpha$ we can take complex conjugates of the coefficients and get the polynomial $\widehat{p}(z) = \sum \bar{c}_\alpha \cdot z^\alpha$. In particular we get the polynomial

$$g_*(z) = \sum_{j=1}^{j=m} \widehat{g_j}(z) \cdot g_j(z) \tag{0.1}$$

If $x = (x_1, \ldots, x_n)$ is a real point in $\mathbf{C}^n$ we have

$$g_*(x) = \sum |g_j(x)|^2 \implies \mathbf{R}^n \cap V = \mathbf{R}^n \cap g_*^{-1}(0)$$

Thus, $g_* \in \mathfrak{p}$ and it has the same set of real zeros as V. Less obvious is the following:

Lech's Theorem. *There exists a finite set $f_1, \ldots, f_m$ in $\mathfrak{p}$ and a constant C such that the following inequality hold for each $x \in \mathbf{R}^n$:*

$$\operatorname{dist}(x, V) \leq C \cdot \operatorname{dist}(x, f_*^{-1}(0))$$

A polynomial f for which there exists a constant C as above is called a Lech polynomial attached to $\mathfrak{p}$. The construction of a Lech polynomial relies upon material in algebraic geometry which appears in André Weil's book *Foundations of Algebraic Geometry.* But let us first describe the construction of a Lech polynomial using the classic normalization theorem by Max Noether. Given the prime ideal $\mathfrak{p}$ as above we denote by $\mathfrak{p}^*$ the ideal in $\mathbf{C}[z_1, \ldots, z_n]$ generated by the leading forms of polynomials in $\mathfrak{p}$. This yields the algebraic set V^* of common zeros of the homogeneous polynomials in $\mathfrak{p}^*$. It is wellknown that V^* has the same dimension as V. So the codimension of V^* is $p = n - d$. Denote by $\mathcal{G}$ the Grassmanian of $(p-1)$-dimensional subspaces of $\mathbf{C}^n$. A subspace $\Pi \in \mathcal{G}$ is transversal to V if the intersection $V^* \cap \Pi$ is reduced to the origin in $\mathbf{C}^n$. It is wellknown that that for every transversal Π, the set-theoretic sum

$$V + \Pi = \{z + w : \ z \in V \quad \& \quad w \in \Pi$$

is an algebraic hypersurface $P_\Pi^{-1}(0)$ where the degree of P_Π is majorized by the multiplicity $e(V)$ of the given algebraic set. It follows that the family $\{P_\Pi\}$ generates a finite dimensional complex vector space in $\mathbf{C}[z]$ denoted by $\mathcal{N}(\mathfrak{p})$. Let $f_1, \ldots, f_m$ be a basis of $\mathcal{N}(\mathfrak{p})$ and put

$$\text{(i)} \qquad f_*(z) = \sum \widehat{f}_\nu(z) \cdot f_\nu(z)$$

It turns out that f_* is a Lech polynomial attached to $\mathfrak{p}$. The proof relies upon Lech's inequality in § 0 expressed by Theorem 0.1. The proof is then finished using the Zariski-Weil theory about generic points. This is exposed in § 1 and the final part of the proof appears in § 2.

§ 0. Lech's inequality.

Let $\phi(z_1,\dots,z_n)$ be a polynomial of n variables which is not reduced to a constant, i.e. its degree is >0. If $\alpha\in\mathbf{C}^n\setminus\phi^{-1}(0)$ we put

$$\text{dist}(\alpha,\phi^{-1}(0))=\min_{z\in\phi^{-1}(0)}|z-\alpha|$$

where $|z-\alpha|$ is the euclidian distance. Next, let $\Theta=(\theta_1,\dots,\theta_n)$ be a complex vector of unit length, i.e. $|\theta_1|^2+\ldots+|\theta_n|^2=1$. This gives a polynomial

$$t\mapsto\phi(\alpha+t\cdot\Theta)$$

and a Taylor expansion yields

$$\phi(\alpha+t\cdot\Theta)=\phi(\alpha)+\sum_{k=1}^{k=m}D_k(\alpha;\Theta)\cdot t^k$$

where m is the degree of ϕ and $\{D_k(\alpha;\Theta)\}$ is an m-tuple of complex numbers. For each $1\leq k\leq m$ we set

$$\mathcal{D}_k^*(\phi;\alpha)=\max_{\Theta}|D_k(\alpha;\Theta)|$$

Definition. *The Lech number of ϕ at the point α is defined by*

$$\mathcal{L}(\phi;\alpha)=\max_{1\leq k\leq m}\big[\frac{|\phi(\alpha)|}{\mathcal{D}_k^*(\phi;\alpha)}\big]^{\frac{1}{k}}$$

0.1 Theorem. *For every positive integer m and each polynomial ϕ of degree m the following inequality holds when α is outside the zero-set of ϕ:*

$$(*)\qquad\frac{1}{2}\leq\frac{\text{dist}(\alpha,\phi^{-1}(0))}{\mathcal{L}(\phi;\alpha)}\leq m$$

Proof Replacing ϕ by $c\cdot\phi$ for some constant we may assume that $\mathcal{L}(\phi;\alpha)=1$ which means that

$$(1)\qquad|\phi(\alpha)|=\max_{k,\Theta}D_k(\alpha;\Theta)$$

with the maximum taken over pairs (k,Θ) where $1\leq k\leq m$ and Θ are complex n-vectors of unit length. Every unit vector Θ gives the ζ-polynomial

$$(2)\qquad\phi(\alpha+\Theta\cdot\zeta)=\phi(\alpha)+\sum D_k(\alpha;\Theta)\cdot\zeta^k$$

If the absolute calue $|\zeta|<1/2$ the triangle inequality gives:

$$|\phi(\alpha+\Theta\cdot\zeta)|\geq|\phi(\alpha)|-\sum_{k=1}^{k=m}|\zeta|^k\cdot|D_k(\alpha,\Theta)|$$

Now (1) entails that the right hand side above majorises

$$|\phi(\alpha)|\cdot(1-\sum_{k=1}^{k=m}|\zeta|^k)$$

Since $|\zeta|<1/2$ the last factor above is >0 and hence $\zeta\mapsto\phi(\alpha+\Theta\cdot\eta)$ is zero-free in the disc $\{|\zeta|<1/2$ for every unit vector Θ. Hence $\text{dist}(\alpha,\phi^{-1}(0))\geq 1/2$ which proves the lower bound in (*) from Theorem 0.1. To get the upper bound we choose a pair k_*,Θ_* such that

$$|\phi(\alpha)|=D_{k_*}(\alpha;\Theta_*)$$

and consider the polynomial

$$(3)\qquad g(\zeta)=\zeta^m\cdot\phi(\alpha+\Theta_*\cdot\zeta^{-1})$$

Write

$$g(\zeta)=c_m\zeta^m+\ldots+c_0$$

Now (1) implies that the absolute values of c_m and c_{m-k} are equal. Let $\beta_1,\ldots,\beta_m$ be the zeros of g where eventual multiple zeros are repeated. The symmetric polynomial of order $m-k$ of this m-tuple is a sum of monomials in the roots of degree k and equal to

$$(-1)^{m-k}\cdot\frac{c_{m-k}}{c_m}$$

whose absolute value is 1. If all the zeros have absolute value $\leq 1/m$ the absolute value of the symmetric sum above is majorised by

$$m^{-k}\binom{m}{k}$$

Since this term is <1 we conclude that g must have a zero of absolute value $>1/m$ which by (3) implies that the right hand side in (*) is $\leq m$.

An application. For each finite family of polynomials $\{\phi_1,\ldots,\phi_k\}$ and every real point a we set

$$\text{dist}(a,\phi_\bullet^{-1}(0))=\min_\nu\ \text{dist}(a,\phi_\nu^{-1}(0))$$

0.2 Lech's Lemma *Let $\mathcal{M}$ be a finite dimensional subspace of $\mathbf{C}[z_1,\ldots,z_n]$ and $f_1,\ldots,f_m$ some basis of $\mathcal{M}$. Then, if $\{g_1,\ldots,g_m\}$ is another basis in $\mathcal{M}$ there exists a constant $c>0$ such that the inequality below holds for every real point a:*

$$\text{dist}(a,f_*^{-1}(0))\geq c\cdot\text{dist}(a,g_\bullet^{-1}(0))$$

where $f_*=\sum\bar{f}_\nu\cdot f_\nu$.

Proof. Applying Lech's inequality to the the g-functions and f_* we can reformulate Lech's Lemma as follows: There exists a constant C which is independent of the real point a such that the following hold: If

$$|g_\nu(\alpha)\geq A^k\cdot D_k(g_\nu)(\alpha) \tag{1}$$

hold for all $k\geq 1$ and all ν and some constant A, then

$$|f_*(\alpha)\geq (CA)^k\cdot D_k(f_*)(\alpha) \tag{2}$$

To get (2) we proceed as follows. First, since $f_1,\ldots,f_m$ is a k-basis in $\mathcal{M}$ there exists a constant C_0 which is independent of α and

$$|g_\nu(\alpha)\leq C_0\cdot\sum_{k=1}^{k=m}|f_k(\alpha)|\quad:\ 1\leq\nu\leq m$$

Conversly, since the g-polynomials also is a basis of $\mathcal{M}$ it is clear that (1) gives a constant $C_1>0$ which again is independent of α such that

$$C_1\cdot\max_\nu|f_\nu(\alpha)|\geq A^k\cdot\sum_{\nu=1}^{\nu=m}D_k(f_\nu;\alpha) \tag{3}$$

Next, since α is real we have

$$f_*(\alpha)=\sum|f_\nu(\alpha)|^2\implies C_1\cdot\sqrt{f_*(\alpha)}\geq A^k\cdot\sum_{\nu=1}^{\nu=m}D_k(f_\nu;\alpha) \tag{4}$$

Put

$$D_k^*(\alpha)=\sum_{\nu=1}^{\nu=N}D_k(f_\nu;\alpha) \tag{5}$$

Notice that when f_ν is replaced by $\bar{f}_\nu$ one has

$$D_k(f_\nu;\alpha)=D_k(\bar{f}_\nu;\alpha)\quad:\ k=1,2,\ldots$$

These Taylor expansions give the inequality

$$D_k(f_*;\alpha)\leq N(k+1)\max_{i+j=k}D_i^*(\alpha)\cdot D_j^*(\alpha) \tag{6}$$

Finally, with $k = i + j$ it is clear that (4) gives

(7) $$A^k \cdot D_i^*(\alpha) \cdot D_j^*(\alpha) \leq C_1^2 |f_*(\alpha)|$$

Then (6-7) give

(8) $$A^k \cdot D_k(f_*; \alpha) \leq N(k+1) \cdot C_1^2 \cdot f_*(\alpha)|$$

Since this hold for each k we get the requested positive constant C in (2) above.

§ 1. Specialisations and linear systems.

We shall use specialisations which appear in the Zariski-Weil theory. The reader is expected to know about the fundamental constructions from their theory which is the starting point for everything dealing with algebraic geometry, i.e. a first and invaluable lesson for beginners is to learn about constructions due to these two meters who have contributed so much in algebra and geometry. Their approach has the merit that various geometric questions are carried over to algebraic calculations which are not easily found in an intuitive fashion. We shall only employ the Zariski-Weil theory in characteristic zero and apply results from Weil's book *Foundations of algebraic geometry* published in 1943. Personally I think the material in this outstanding text-book should be introduced at an early stage to students interested in systems of algebraic equations. I have never understood the point in putting so much emphasis upon trivial sheaf theory and the subsequent yoga about schemes which tend to hide relevant calculations. See Weil's critical comments about these matters in the reprinted version of his text-book from 1962.

The Zariski-Weil linear system.

Let k be a subfield of $\mathbf{C}$ which is finitely generated over Q and $\mathfrak{p}$ a prime ideal in the polynomial ring $k[t_1,\ldots,t_n]$ where $n\geq 2$. Generic specialisations of $\mathfrak{p}$ consist of points

$$\xi=(\xi_1,\ldots,\xi_n)\in\mathbf{C}^n$$

with the property that a polynomial $f(t)$ in $k[t]$ belongs to $\mathfrak{p}$ if and only if $f(\xi)=0$. This means that one simply evaluates f at the point $\xi\in\mathbf{C}^n$. The existence of generic specialisations is not difficult to prove and is explained in Chapter 1 in [Weil]. An invariant of $\mathfrak{p}$ is the degree of trancendency of the field extension $k(\xi)$ over k, i.e. this degree does not depend upon the chosen generic specialisation. Moreover, if d is the dimension of the algebraic set $V=\mathfrak{p}^{-1}(0)$ in $\mathbf{C}^n$ then the degree of transcendency of $k(\xi)$ over k is equal to d. Put $p=n-d$ where we assume that

$$2\leq p\leq n-1$$

Next, introduce n-vectors

$$\bar\tau_k=(\tau_k^1\;\ldots,\tau_k^n)\quad : \; 1\leq k\leq p-1$$

where the doubly indexed family $\{\tau_\nu^i\}$ gives $n(p-1)$ algebraically independent elements over $k(\xi)$. Consider also a $(p-1)$-tuple $\lambda_1,\ldots,\lambda_{p-1}$ where $\{\tau_\nu^i\}$ and $\{\lambda_\nu\}$ together give $(n+1)(p-1)$ algebraically independent elements over k. Now we find n-vectors ζ defined by

$$(1)\qquad \zeta_i=\xi_i+\sum_{k=1}^{k=p-1}\lambda_k\cdot\tau_k^i\quad :\quad 1\leq i\leq n$$

Denote by K the field extension of k generated by the n-tuple $\{\xi_\nu\}$, the $(p-1)$-tuple $\{\lambda_j\}$ and the $n(p-1)$-tuple $\{\tau_\nu^i\}$. Notice that K contains the subfield $k(\zeta,\bar\tau)$ generated by $\zeta_1,\ldots,\zeta_n$ and $\bar\tau=\{\tau_\nu^i\}$. With these notations one has the result below whose detailed proof appears in [Weil:Chapter 1]:

1.2 Theorem. *The field K is algebraic over the subfield $k(\zeta,\bar\tau)$ and the degree of trancendency of the last field over k is equal to $np-1$.*

With n variables $\{z_\nu\}$ and arranging the $n(p-1)$ many trancendental τ-variables as $t=(t_1,\ldots,t_{np-n})$, Theorem 1.2. gives an irreducible polynomial $F(z,t)$ in $k[z,t]$ such that

$$F(\zeta,\bar\tau)=0$$

We can express F in the form:

$$\text{(i)}\qquad F(z,t)=\sum_{\nu=1}^{\nu=m}f_\nu(z)\cdot\phi_\nu(t)$$

where m is some positive integer, and $\{f_\nu(z)\}$ are k-linearly independent in $k[z]$ while $\{\phi_\nu(t)\}$ are k-linearly inependent in the polynomial ring of the $n(p-1)$ many t-variables. The polar decomposition is not unique. However, the vector space over k generated by the f-polynomials in $k[z]$ does not depend upon the chosen polar representation of F. It is denoted by $\mathcal{W}(\mathfrak{p})$ and called the Weil-space attached to the prime ideal $\mathfrak{p}$. We also get the complex vector space

$$\widehat{\mathcal{W}}(\mathfrak{p}) = \mathbf{C} \otimes_k \mathcal{W}(\mathfrak{p})$$

Returning to the polynomial $F(z,t)$ of the np-many variables (z,t) the constructions above give the following. When the t-vector is freezed to a point $t^* \in \mathbf{C}^{np-n}$ then the λ-polynomial

$$\lambda \mapsto F(\xi + \lambda \cdot t^*, t^*) \tag{1.3}$$

is identically zero, where $\lambda \cdot t^*$ is the complex n-vector with components

$$(\lambda \cdot t^*)_i = \lambda_1 \cdot t_i + \lambda_2 \cdot t_{i+n} + \ldots + \lambda_{p-1} \cdot t_{i+n(p-2)}$$

With t^* given while λ varies over complex $(p-1)$-tuples we get a subspace denoted by $\mathcal{L}(t^*)$ in $\mathbf{C}^n$ which consists of points $z = (z_1, \ldots, z_n)$ of the form

$$z = (\lambda \cdot t^*) \quad : \quad \lambda \in \mathbf{C}^{p-1}$$

With these notations the vanishing of the λ-polynomial in (i) and the fact that ξ was a generic point, entails that the polynomial

$$z \mapsto F(z, t^*)$$

is zero on the set-theoretic sum

$$V + \mathcal{L}(t^*)$$

In other words,

$$F(z + \lambda \cdot t^*) = 0 \quad : \quad z \in V \;\&\; \lambda \in \mathbf{C}^{p-1} \tag{1.4}$$

Expressing F by (i) we have by (1.4)

$$\sum_{\nu=1}^{\nu=m} \phi_\nu(t^*) \cdot f_\nu(z + \lambda \cdot t^*) = 0 \quad : \quad z \in V \;\&\; \lambda \in \mathbf{C}^{p-1} \tag{1.5}$$

In particular we can take $\lambda = 0$ and hence

$$\sum_{\nu=1}^{\nu=m} \phi_\nu(t^*) \cdot f_\nu(z) = 0 \quad : \quad z \in V$$

Above t^* is arbitrary in $\mathbf{C}^{(n-1)p}$ and since the ϕ-polynomials constructed via the polarization in (i) are linearly independent ver k his entails that $f_\nu(\xi) = 0$ for every ν, i.e. the f-polynomials belong to $\mathfrak{p}$. We are going to show that the polynomial $f_*(z)$ constructed in § 0 yields the requested Lech polynomial in the Main Theorem.

§ 2. Proof of the Main Theorem

From § 1 we have the m-tuple $f_1, \ldots, f_m$ in $k[z]$ and construct the polynomial f_* as in (0.1). If it fails to be a Lech polynomial there exists a real sequence $\{a_\mu\}$ in $\mathbf{R}^n$ which stays outside V while

$$(2.0) \qquad \lim_{\mu\to\infty} \frac{\text{dist}(\alpha_\mu, f_*^{-1}(0))}{\text{dist}(\alpha_\mu, V)} = 0$$

We shall prove that this leads to a contradiction. First. passing to a subsequence if necessary we may assume that there exist the limits:

$$(2.1) \qquad d_\nu = \lim_{\mu\to\infty} \frac{f_\nu(\alpha_\mu)}{\sqrt{f_*(\alpha_\mu)}}$$

where we notice that the limit is taken over complex m-vectors of length one. So above

$$|d_1|^2 + \ldots + |d_m|^2 = 1$$

From § 1 we have the polynomial $F(z,t)$ with its polar decomposition and put

$$\text{(i)} \qquad \phi_*(t) = \sum d_\nu \cdot \phi_\nu(t)$$

Here $\phi_*(t)$ is not identically zero since $\{\phi_\nu(t)\}$ are linearly independent and the d-vectors. More precisely, there exists an m-tuple of freezed vectors $t^*[k]$ such that the determinant of the $m\times m$-matrix with elements $\{\, \phi_\nu(t^*[k])\}$ is non-zero, and in addition

$$\text{(ii)} \qquad \phi_*(t^*(k[) \neq 0 \quad : 1 \leq k \leq m$$

We get the polynomials

$$\text{(iii)} \qquad g_k(z) = F(z, t^*[k]) \quad : 1 \leq k \leq m$$

Since

$$\det(\{\, \phi_\nu(t^*[k])\} \neq 0$$

It follows that the g-polynomials is a basis in the Weil space from § xx. At this stage we shall use Lech's Lemma from (0.2). Namely, when (2.0) holds we find some $1 \leq k \leq m$ such that

$$(*) \qquad \liminf_{\mu\to\infty} \frac{\text{dist}(\alpha_\mu, g_k^{-1}(0)}{\text{dist}(\alpha_\mu, V)} = 0$$

Passing to a subsequence of the α-sequence we may assume that (*) holds as an unrestricted limit as $\mu \to \infty$ and there remains only to prove that this gives a contradiction to get Main Theorem. To achieve this we consider the subspace $\mathcal{L}(t^*[k]$ and recall, from § 1 that $g_k(z)$ is zero on

$$V + \mathcal{L}(t^*[k])$$

Hence

$$(2.2) \qquad \text{dist}(\alpha_\mu, g_k^{-1}(0)) \geq \min_{z,\zeta} ||\alpha_\mu - z - \zeta||$$

where the minimum is taken over pairs $z \in V$ and $\zeta \in \mathcal{L}$. Since $\text{dist}(\alpha_\mu, V)) \leq |\alpha_\mu - z|$ for every $z \in V$, the quotient in (*) therefore majorizes

$$(2.3) \qquad \min_{z,\zeta} \frac{||\alpha_\mu - z - \zeta||}{||\alpha_\mu - z||}$$

Since $\mathcal{L}$ is a subspace the minimum is the same as

$$(2.4) \qquad \min_{z,\zeta} ||\frac{\alpha_\mu - z}{||\alpha_\mu - z||} - \zeta||$$

Above we measure distances of unit vectors $\frac{\alpha_\mu - z}{||\alpha_\mu - z||}$ to the subspace $\mathcal{L}$ and by the unrestricted limit in (*) it follows that (2.4) tends to zero when $\mu \to \infty$. Given $\epsilon > A$ we can therefore find subspace $\mathcal{L}(\tau_\mu)$ in the Grassmanian $\mathcal{G}$ with distance ϵ from $\mathcal{L}(t^*[k]$ for large μ, and points $z_\mu \in V$ such that

$$\frac{\alpha_\mu - z}{||\alpha_\mu - z_\mu||} \in \mathcal{L}(\tau_\mu) \implies \alpha_\mu \in V + \mathcal{L}(\tau_\mu)$$

The last inclusion gives

$$0 = F(\alpha_\mu, \tau_\mu) = \sum f_\nu(\alpha_\mu) \cdot \phi_\nu(\tau_\mu) \tag{2.5}$$

We can divide the last sum by $\sqrt{f_*(\alpha_\mu)}$. So with

$$d_\nu(\mu) = \frac{f_\nu(\alpha_\mu)}{\sqrt{f_*(\alpha_\mu)}}$$

it follows that

$$\sum d_\nu(\mu) \cdot \phi_\nu(\tau_\mu) = 0 \tag{2.6}$$

Here

$$d_\nu(\mu) \to d_\nu \quad \& \quad \tau_\mu \to t^*[k]$$

as $\mu \to \infty$. By continuity this implies that (2.6) contradicts there non-vanishing expression from (Ii) which finishes our proof of the Main Theorem.