VI. Lindelöf functions.

Introduction. For each real number $0 < a \leq 1$ there exists the entire function

$$Ea_(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+na)}$$

which for $a = 1$ gives the exponential function e^z. Growth properties of the E-functions were investigated in a series of articles by Mittag-Leffler between 1900-1904 using integral formulas for the entire function $\frac{1}{\Gamma}$. This inspired Phragmén to study entire functions $f(z)$ such that there are constants C and $0 < a < 1$ with:

$$\log|f(re^{i\theta})| \leq C \cdot (1+|r|)^a) \quad : -\alpha < \theta < \alpha$$

for some $0 < \alpha < \pi/2$ while $f(z)| \leq C$ for all $z \in \mathbf{C} \setminus S$. When this holds we get the entire function

$$g(z) = \int_0^{\infty} f(sz) \cdot e^{-s} \cdot ds$$

If z is outside the sector S it is clear that $|g(z)|$ is bounded by $C \cdot \int_0^\infty e^{-s} ds = C$. When $z = re^{i\theta}$ is in the sector we still get a bound from (1) since $0 < a < 1$ and conclude that the entire function g is bounded and hence a constant. Since the Taylor coefficients of f are recaptured from g it follows that f must be constant. More general results of this kind were obtained in the joint article [PL] by Phragmén and Lindelöf from 1908 and led to what is called the Phragmén-Linedlöf principle. Here we discuss a continuation of [PL] from the article *Remarques sur la croissance de la fonction* $\zeta(s)$ where Lindelöf employed results in collaboration with Phragmén to investigate the growth of Riemann's ζ-function along vertical lines in the strip $0 < \mathfrak{Re}(z) < 1$. This leads to the study of various indicator functions attached to analytic functions. Here is the set-up: Consider a strip domain in the complex s-plane:

$$\Omega = \{s = \sigma + it \quad : t > 0 \quad \text{and} \quad 0 \leq a < \sigma < b\}$$

An analytic function $f(z)$ in Ω is of *finite type* if there exists some integer k, a constant C and some $t_0 > 0$ such that

$$|f(\sigma + it)| \leq C \cdot t^k \quad \text{hold for} \quad t \geq t_0$$

To every such f we define the Lindelöf function

$$(*) \qquad \mu_f(\sigma) = \limsup_{t \to \infty} \frac{\operatorname{Log}|f(\sigma+it)|}{\operatorname{Log} t}$$

Lindelöf and Phragmén proved that μ_f is a continuous and convex function on (a, b). No further restrictions occur on the μ-function because one has:

1. Theorem. *For every convex and continuous function $\mu(\sigma)$ defined in $[a, b]$ there exists an analytic function $f(z)$ without zeros in Ω such that $\mu_f = \mu$.*

2. Exercise. Prove this result using the Γ-function. First, to a pair of real numbers (ρ, α) we set

$$\text{(i)} \qquad f(s) = e^{-\frac{\pi i \cdot \rho s}{2}} \cdot \Gamma(\rho(s-a) + \frac{1}{2})$$

Use properties of the Γ-function to show that f has finite type in Ω and its indicator function becomes a linear function:

$$\mu_f(\sigma) = \rho \cdot (\sigma - a)$$

More generally one gets a function f where μ_f is piecewise linear by:

$$\text{(ii)} \qquad f = \sum_{k=1}^{k=m} c_k e^{-\frac{\pi i \cdot \rho_\nu s}{2}} \Gamma(\rho_k(s - a_k) + \frac{1}{2})$$

where $\{c_k\}$, $\{\rho_k\}$ and $\{a_k\}$ are m-tuples of real numbers. Finally, starting from an arbitrary convex curve we can choose some dense and enumerable set of enveloping tangents to this curve. Then an infinite series of the form above gives an analytic function $f(s)$ such that

$$\sigma \mapsto \mu_f(\sigma)$$

yields an arbitrarily given convex μ-function on (a, b).

1. A relation to harmonic functions.

Let $U(x, y)$ be a bounded harmonic function in the strip domain Ω and V its harmonic conjugate. Set

$$f(s) = \exp\left[(\log(s) - \frac{\pi i}{2})(U(s) + iV(s))\right] \tag{*}$$

It is easily see that $f(z)$ has finite type in Ω. With $s = \sigma + it$ we have

$$|f(\sigma + it)| =$$

$$\exp(\frac{1}{2}\log(\sigma^2 + t^2) \cdot U(\sigma + it) \cdot \exp(-(\frac{\pi}{2} - \arg(\sigma + it)) \cdot V(\sigma + it))$$

It follows that

$$\frac{\log |f(\sigma + it)|}{t} = \frac{\log \sqrt{\sigma^2 + t^2} \cdot U(\sigma + it)]}{\log t} + \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t}$$

Exercise. With σ kept fixed one has

$$\arg(\sigma + it) = \tan\frac{t}{s}$$

which tends to $\pi/2$ as $t \to +\infty$. Next, $V(\sigma + it)$ is for large $t > 0$ up to a constant the primitive of

$$\int_1^t \frac{\partial V}{\partial u}(\sigma + iu) \cdot du$$

Here the partial derivative of V is equal to the partial derivative $\partial U/\partial\sigma(\sigma, u)$ taken along $\mathfrak{Re}\, s = \sigma$. Since U is bounded in the strip domain it follows from Harnack's inequalities that this partial derivative stays bounded when $1 \leq u \leq t$ by a constant which is independent of t. Putting this together the reader can verify that

$$\lim_{t \to +\infty} \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t} = 0 \tag{1}$$

From (1) in the Exercise we obtain the equality

$$\mu_f(\sigma) = \limsup_{t \to \infty} U(\sigma + it) \tag{*}$$

This suggests that we study growth properties of bounded harmonic functions in strip domains.

2. The M and the m-functions.

To a bounded harmonic function U in Ω we associate the maximum and the minimum functions:

$$M(\sigma) = \limsup_{t \to \infty} U(\sigma + it) \quad \text{and} \quad \liminf_{t \to \infty} U(\sigma + it)$$

2.1 Proposition. *$M(\sigma)$ is a convex function while $m(\sigma)$ is concave.*

We prove the convexity of $M(\sigma)$. The concavity of m follows when we replace U by $-U$. Consider a pair α, β with $a < \alpha < \beta < b$. Replacing U b $U + A + \cdot Bx$ for suitable constants A and B we may assume that $M(\alpha) = M(\beta) = 0$ and the requested convexity follows if we can show that

$$M(\sigma) \leq 0 \quad : \quad \alpha < \sigma < \beta$$

To see this we consider rectangles

$$\mathcal{R}[T_*, T^*] = \{\sigma + it \quad \alpha \leq \sigma \leq \beta \quad \text{and} \quad T_* \leq t \leq T^*\}$$

Let $\epsilon > 0$ and start with a large T_* so that

$$t \geq T_* \implies U(\alpha + it) \leq \epsilon$$

and similarly with α replaced by β. Next, we have a constant M such that $|U|_\Omega \leq M$. If $z = \sigma + it$ is an interior point of the rectangle above it follows by harmonic majorisation that

$$U(\sigma + it) \leq \epsilon + M \cdot \mathfrak{m}_z(J_* \cup J^*)$$

where the last term is the harmonic measure at z which evaluates the harmonic function in the rectangle at z with boundary values zero on the two verical lines of the rectangle which it is equal to 1 on the horizontal intervals $J^* = (\alpha, \beta) + iT^*$ and $J_* = (\alpha, \beta) + iT_*$

Exercise. Show (via the aid of figure that with $T^* = 2T_*$ one has

$$\lim_{T_* \to +\infty} \mathfrak{m}_{\sigma + 3iT_*/2}(J_* \cup J^*) = 0$$

where this limit is uniform when $\alpha \leq \sigma \leq \beta$. Since $\epsilon > 0$ is arbitrary in (xx) the reader can now conclude that $M(\sigma) \leq 0$ for every $\sigma \in (\alpha, \beta)$.

A special case. Suppose that we have the equalities

(1) $$m(\alpha) = M(\alpha) \quad \text{and} \quad m(\beta) = M(\beta)$$

using rectangles as above and harmonic majorization the reader can verify that this implies that

$$m(\sigma) = M(\sigma) \quad : \quad \alpha < \sigma < \beta$$

This result is due to Hardy and Littlewood in [H-L].

The case when $M(\sigma) - m(\sigma)$ has a tangential zero. Put $\phi(\sigma) = M(\sigma) - m(\sigma)$ and suppose that this non-negative function in (a, b) has a zero at some $a < sigma_0 < b$ whose graph has a tangent at σ_0. This means that if:

$$h(r) = \max_{-r \leq |\sigma - \sigma_0| \leq r} \phi(\sigma)$$

then

(*) $$\lim_{r \to 0} \frac{h(r)}{r} = 0$$

Under this hypothesis the following result is proved in [Carleman].

2.2 Theorem. *When (*) holds we have*

$$m(\sigma) = M(\sigma) \quad \text{holds for all} \quad a < \sigma < b$$

The subsequent proof from [Carleman] was given at a lecture by Carleman in Copenhagen 1931 which has the merit that a similar reasoning can be applied in dimension ≥ 3. Adding some linear function to U we may assume that $M(\sigma_0) = m(\sigma_0) = 0$ which mans that

(1) $$\limsup_{t \to \infty} U(\sigma_0, t) = 0$$

Next, consider the function

(1) $$\phi \colon t \mapsto \partial U / \partial \sigma(\sigma_0, t)$$

The assumption (*) and the result in XXX gives:

(2) $$\lim_{t \to \infty} \partial U / \partial \sigma(\sigma_0, t) = 0$$

Next, consider some $a < \sigma < b$ and let $\epsilon > 0$. By the result from XX there exist finite tuples of constants $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$ and some N-tuple $\{\tau_\nu\}$ which stays in a $[0, 1]$ such that

(5) $$\left|U(\sigma, t) - \sum a_\nu \cdot U(\sigma_0, t_\nu + t) - \sum b_\nu \cdot \partial U / \partial \sigma(\sigma_0, t_\nu + t)\right| < \epsilon \quad \text{hold for all} \quad t \geq 1$$

Since ϵ is arbitrary it follows from (1-2) that

(5) $$\lim_{t \to \infty} U(\sigma, t) = 0$$

for every $a < \sigma < b$ which obviously gives the requested equality in Theorem 2.2.

2.3. Integral indicator funtions.

Let $f(s)$ be an analytic function of finite order in the strip domain Ω and fix some $t_0 > 0$ which does not affect the subsequent constructions. For a pair (σ, p) where $a < \sigma < b$ and $p > 0$ we associate the set of of positive numbers χ such that the integral

$$(*) \qquad \int_{t_0}^{\infty} \frac{|f(\sigma + it)|^p}{t^{\chi}} \cdot dt < \infty$$

We get a critical smallest non-negative number $\chi_*(\sigma, p)$ such that (*) converges when $\chi > \chi_*(\sigma, p)$. In the case $p = 1$ a result due to Landau asserts that $\chi(\sigma, 1)$ determines the half-plane of the complex z-plane where the function

$$\gamma(z) = \int_{t_0}^{\infty} \frac{f(\sigma + it)}{t^z} \cdot dt$$

is analytic and $\sigma \mapsto \chi(\sigma, 1)$ is a convex function on (a, b). A more general convexity result holds when p also varies.

2.4 Theorem. *Define the ω-function by:*

$$\omega(\sigma, \eta) = \eta \cdot \chi(\sigma, \frac{1}{\eta}) \quad : a < \sigma < b \quad : \eta > 0$$

Then ω is a continuous and convex function of the two variables (σ, η) in the product set $(a, b) \times \mathbf{R}^+$.

2.5 Remark. Theorem 2.4 is proved using Hölder inequalities and factorisations of analytic functions which reduces the proof to the case when f has no zeros. The reader is invited to supply details of the proof or consult [Carleman].

3. Lindelöf estimates in the unit disc.

Let $f(z)$ be analytic in the open unit disc given by a power series

$$f(z) = \sum a_n \cdot z^n$$

We assume that the sequence $\{a_n\}$ has temperate growth, i.e. there exists some integer $N \geq 0$ and a constant K such that

$$|a_n| \leq K \cdot n^N \quad : \quad n = 1, 2, \ldots$$

In addition we assume that the sequence $\{a_n\}$ is not too small in the sense that

$$(*) \qquad \sum_{n=1}^{\infty} |a_n|^2 \cdot n^s = +\infty \quad : \quad \forall\, s > 0$$

Now there exists the smallest number $s_* \geq 0$ such that the Dirichlet series

$$\sum_{n=1}^{\infty} |a_n|^2 \cdot \frac{1}{n^s} < \infty, \quad \text{for all } s > s_*$$

To each $0 \leq \theta \leq 2\pi$ we set

$$(1) \qquad \chi(\theta) = \min_s \int_0^1 \left| f(re^{i\theta}) \right| \cdot (1 - r)^{s-1} \cdot dr < \infty$$

$$(2) \qquad \mu(\theta) = \text{Lim.sup}_{r \to 1} \frac{\text{Log}\,|f(re^{i\theta})|}{\text{Log}\,\frac{1}{1-r}}$$

We shall study the two functions χ and μ. The first result is left as an exercise.

3.1. Theorem. *The inequality*

$$\chi(\theta) \leq \frac{s^*}{2}$$

holds almost everywhere, i.e. for all $0 \leq \theta \leq 2\pi$ *outside a null set on* $[0, 2\pi]$.

Hint. Use the formula

$$\frac{1}{2\pi} \cdot \int_{=}^{2\pi} |f(re^{i\theta})|^2 \cdot d\theta = \sum |a_n|^2$$

For the μ-function a corresponding result holds:

3.2. Theorem. *The inequality below holds almost everywhere.*

$$\mu(\theta) \leq \frac{s^*}{2}$$

Proof. Let $\epsilon > 0$ and introduce the function

$$\Phi(z) = \sum a_n \cdot \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{s^*}{2}+\epsilon)} \cdot z^n = \sum c_n \cdot z^n$$

It is clear that the construction of s^* entails

$$\sum |c_n|^2 < \infty$$

Next, set $\Phi_0 = \Phi$ and define inductively the sequence $\Phi_0, \Phi_1, \ldots$ by

$$\Phi_\nu(z) = z^{\nu-1} \cdot \frac{d}{dz}\big[z^\nu \cdot \Phi_{\nu-1}(z)\big) \quad : \quad \nu = 1, 2, \ldots$$

3.3 Exercise. Show that for almost every $0 \leq \theta \leq 2\pi$ there exists a constant $K = K(\theta)$ such that

$$|\Phi_\nu(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{1-r)^\nu} \quad : 0 < r < 1$$

Next, with s^* and ϵ given we define the integers ν and ρ:

$$\nu = \big[\frac{s^*}{2} + \epsilon\big] + 1 \quad : \quad \rho = \frac{s^*}{2} + \epsilon - \big[\frac{s^*}{2} + \epsilon\big]$$

where the bracket term is the usual notation for the smallest integer $\geq \frac{s^*}{2} + 1$.

Exercise Show that with ν and ρ chosen as above one has

$$\Phi_\nu(z) = \sum a_n \cdot \frac{\Gamma(n+1+\nu)}{\Gamma(n+1+\rho-1)} \cdot z^n$$

and use this to show the inversion formula

(*) $$f(z) = \frac{1}{z^\nu \cdot \Gamma(1-\rho)} \cdot \int_0^z (z-\zeta)^{-\rho}\dot{\zeta}^{\nu+\rho-1} \cdot \Phi_\nu(\zeta) \cdot d\zeta$$

3.4 Exercise. Deduce from the above that for almost every θ there exists a constant $K(\theta)$ such that

(**) $$|f(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{(1-r)^{\nu+\rho-1}}$$

Conclusion. From (**) and the construction of ν and ρ the reader can confirm Theorem 3. 2.

3.5 Example. Consider the function

$$f(z) = \sum_{n=1}^{\infty} z^{n^2}$$

Show that $s^* = \frac{1}{2}$ holds in this case. Hence Theorem B.2 shows that for each $\epsilon > 0$ one has

(E) $$\max_r \, (1-r)^{\frac{1}{4}+\epsilon} \cdot |f(re^{i\theta}| < \infty$$

for almost every θ.

3.6 Exercise. Use the inequality above to show the following: For a complex number $x + iy$ with $y > 0$ we set

$$q = e^{\pi i x - \pi y}$$

Define the function

$$\Theta(x+iy) = 1 + q + q^2 + \ldots$$

Show that when $\epsilon > 0$ then there exists a constant $K = K(\epsilon, x)$ for almost all x such that

$$y^{\frac{1}{4}+\epsilon} \cdot |\theta(x+iy)| \leq K \quad : \quad y > 0$$