Analytic function theory in one complex variable

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Introduction.

These notes are foremost devoted to analytic function theory of one variable but there also occur material about measures and Fourier analysis. The borderline between complex and real situations is not strict since each profits from the other. The level varies where results of foundational character appear together with more advanced material. The less prepared reader may first study basic material and for experienced readers the chapters provide background to *Special Topics*. Most of the results in these notes are of a relatively old vintage which is no surprise since analytic function theory is a subject which has been developed for more than two centuries. Contributions by Gauss, Cauchy and Abel, followed by Riemann, Schwarz, Weierstrass and Poincaré, put the subject into a level which has inspired later research in analytic function theory. Let us finsh the introduction by some examples. The reader may of course skip this at a first instant and move directly to the heading *An overwiev*. But the subsections below illustrate the flavour of these notes where an ambition is to show how analytic function theory intervenves with other subjects.

The heat equation.

An equation which appears in many situations, including probability theory, where it is related to the Brownian motion, is the PDE-equation

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2}$$

which is satisfied in a domain $B = \{-\infty < x < \infty\} \times \{0 < y < A\}$. The question arises if the vanishing of u on some line $\{y = a\}$ with $0 < a < A$ implies that u is identically zero in B. If there exists a constant k such that

$$\text{(i)} \qquad |u(x,y)| \leq e^{kx^2} \quad : (x,y) \in B$$

the answer is affirmative. In fact, for each $0 < a < A$ one has the integral formula

$$\text{(ii)} \qquad u(x,y) = \frac{1}{2\sqrt{\pi(y-a)}} \int_{-\infty}^{\infty} u(\xi,a) \cdot U(x-\xi, y-a)\, d\xi \quad : a < y < a$$

where $U(x,y)$ is the function which is zero when $y \leq 0$ and

$$U(x,y) = e^{-x^2/y} : \ y > 0$$

If (i) does not hold uniqueness can fail, i.e. without growth conditions there exist solutions to the heat equation in B which vanish identically on a line $\{y = a\}$. A sufficient and "almost necessary" condition for the uniqueness was proved by Erik Holmgren in the article *Sur les solution quasi-analytique d l'équation de chaleur* [Arkiv för matematik. 1924].

Theorem. *Suppose there exists a constant k such that*

$$|u(x,y]| \leq e^{kx^2 \cdot \log(e+|x|)} \quad : \ (x,y) \in B$$

Then, if $u(x,a) = 0$ holds identically in x for some $0 < a < A$, it follows that u is identically zero.

Remark. A complete investigation of the uniqueness problem for the heat equation was carried out by Täcklund in [Täcklund] which shows that Holmgren's sufficiency theorem is quite sharp. In § xx we expose constructions due to Beurling devoted to solutions of the heat equation satisfying

$$\text{(iii)} \qquad u(x,y)| \leq e^{\epsilon(|x|)\cdot x^2\cdot\log(e+|x|)}$$

where $\epsilon(r)$ tends to zero as $r \to +\infty$. Here Holmgren's growth condition holds and hence uniqueness holds. Beurling constructed a kernel U^* which is used to express a solution u satisfying (iii) by an integral formula like (ii) above. Several steps are needed to get U^* where a crucial role is played by the analytic function defined in $\mathbf{C} \setminus (-\infty, -e]$ by

$$\gamma(z) = [\log(z+e)]^z = e^{z\cdot\log(z+e)}$$

Here e is Neper's constant and a single-valued branch of the complex log-function $\log(z+e)$ is found when the real interval $(-\infty, -e]$ has been removed.

The discussion above illustrates that analytic function theory is needed in refined questions from PDE-theory. Readers who feel "an appetite" for the results above will hopefully find these notes interesting, while those who prefer "abstract theories" without involved estimates may just as well refrain from studying these notes.

Runge's theorem and Schwarz' reflection principle.

With $z = x + iy$ we consider a rectangle $\square = \{-a < x < a\} \times \{-b < y < b\}$. It contains the upper part $\square_+ = \{-a < x < a\} \times \{0 < y < b\}$ and the lower part $\square_- = \square \cap \{y < 0\}$. Let $\phi(z)$ be an analytic function in $\square_+$ which extends continuously to its closure. A classic result, known as Runge's theorem, entails that for every $\delta > 0$ there exists a sequence of polynomials $\{P_n(z)$ such that the maximum norms $|P_n - \phi|_{\square_+}$ tend to zero and at the same time the maximum norms of $|P_n|$ taken over the rectangle $\square(-\delta) = \{-a < x < a\} \times \{-b < y < \delta\}$ tend to zero. While such an approximation takes place the maximum norms of $\{P_n\}$ taken over $\square$ cannot remain uniformly bounded when $\delta \to 0$. For if this holds, another classic result, known as Montel's theorem, implies that there exists a sequence of polynomials which converge to an analytic function g in $\square$, where this convergence is uniform over compact subsets of $\square$. Here g is zero in $\square(-\delta)$ which by analyticity entails that g is identically zero in $\square$. At the same time $g = \phi$ in $\square_+$ which contradicts the assumption that ϕ is not identically zero.

To compensate for this one introduces the family $\mathcal{W}$ of real-valued continuous functions $\omega(y)$ which are even and positive $y \neq 0$, and decrease to zero as y approaches zero through positive or negative values. An example is $\omega(y) = y^2$ and another is $\omega(y) = e^{-1/y^2}$ which decreases quite rapidly to zero with y. With $\phi(z)$ given as above and $\omega \in \mathcal{W}$ we say that a weighted Runge approximation holds if there exists a sequence of polynomials $\{P_n(z)\}$ such that

$$(*) \qquad \lim_{n\to\infty} ||\omega(y)(P_n(z) - \phi(z)||_{\square_+} + ||\omega(y)(P_n(z)||_{\square_-} = 0$$

It turns out that such a weighted approximation exists if $\omega(y)$ tends to zero sufficiently fast. The following necesssary and sufficient condition was established by Beurling ín the article *Analytic continuations along a linear boundary: Acta mathematatica 1953*.

A. Theorem *The weighted Runge approximation (*) exists if and only if*

$$(*) \qquad \int_0^b \log\log\frac{1}{\omega(y)}\,dy = +\infty$$

Beurling's result will be proved in § xx from the appendix about distributions. Another fundamental issue is about analytic extensions across a linear boundary. Schwarz' reflection principle in its most wellknown version asserts that if ϕ and ψ is a pair of analytic functions in the upper, resp. the lower rectangle which extend continuously to the real interval $(-a < x < a)$ where $\phi(x) = \psi(x)$, then they are analytic continuations of each other. Relaxed conditions which entail analytic continuation across $(-a, a)$ were established by Carleman in lectures at the Mittag-Leffler Institute in 1935. Here one starts with a pair ϕ, ψ where no growth conditions are imposed as one approaches the real axis, i.e. $y \mapsto |\phi(x_0 + iy)|$ can increase to $+\infty$ with no restriction upon its rate as y descreases to zero while $-a < x_0 < a$ is kept fixed, and similar increasing slice functions of ψ can be present. However, the following sufficient condition holds:

B. Theorem. *Assume that*

$$\lim_{y\to 0}\int_a^b \big|\phi(x+iy) - \psi(x-iy)\big|\,dx = 0$$

Then ϕ and ψ are analytic continuations of each other.

The proof is given in § xx from the appendix about distributions.

Remark. The two results above illustrate the spirit of these notes. The proofs rely upon solutions to the Dirichlet problem and constructions of suitable subharmonic functions.

The Dirichlet problem.

Let D be a bounded and simply connected domain in $\mathbf{C}$. Given a continuous function ϕ on the boundary ∂D one seeks a harmonic function Φ in D whose boundary values agree with ϕ. With no regularity conditions upon ∂D one cannot achive a solution. However, there always exists a unique Φ whose generalised boundary values agree with ϕ. The precise assertion is due to Norbert Wiener and is given in Chapter V. Various applications and refinements of this result will appear in more advanced sections. Let us illustrate this by a result due to Beurling where geometric constructions interfere with analytic estimates. Here is the set-up: Let $\square = \{-a < x < a\} \times \{-b < y < b\}$ be a rectangle in the complex z-plane where $z = x + iy$. Denote by $\mathcal{D}^*$ the family of simply connected subdomains D with the property that if $\zeta = \xi + i\eta \in \partial D$ then the rectangle

$$P_\zeta = \{x + iy \colon |x| < |\xi,\ |y| < \eta|\} \subset D$$

Next, denote by $\mathcal{F}$ the family of even and real-valued functions $\omega(y)$ defined when $0 < |y| < b$ with the property that $y \mapsto \omega(y)$ descreases on $(0, b]$ and there is a constant C such that $\omega(y) + C \geq 0: \ 0 < y \leq b\}$. Moreover,

$$\int_0^b \omega(y)\, dy < \infty$$

The last condition means roughly that ω is not too large when $y \simeq 0$.

Theorem. *For each $\omega \in \mathcal{F}$ there exists a constant c and some $\omega_1 \in \mathcal{F}$ with $\omega_1 \geq \omega$ and a domain $D \in \mathcal{D}^*$ such that the generalized Dirichet problem for D with boundary function*

$$\phi(\xi + i\eta) = e^{\omega_1(\eta)}$$

has a solution Φ satisfying

$$0 < \phi(x + iy) < 2e^{\omega_1(\eta)} + c \quad : x + iy \in P_\zeta : \ \zeta = \xi + i\eta \in \partial D$$

Remark. This result appears in the article *Analytic continuations along a linear boundary* cited before and used to prove Theorem A in the previous paragraph. It goes without saying that the theorem above belongs to one of the harder results in these notes. Here "good familarity" with subharmonic functions is essential to pursue the details in Beurling's proof which is presented in § xx from the appendix about distributions.

The Riemann-Schwarz inequality.

Let σ be a hyperbolic metric on a Riemann surface X, i.e. it has negative curvature which means that in an open chart identified with a subset of the complex z-plane, the σ-distance is given by $e^{u(z)} \cdot |dz|$ where $u(z)$ is a subharmonic function. To each pair of points p, q in X we denote by $\mathcal{C}(p, q)$ the family of curves in x with end-points at p and q.

Theorem. *For every pair γ_1, γ_2 in $\mathcal{C}(p, q)$ and each point $\xi \in \gamma_1$, there exist curves α, β with $\alpha \in \mathcal{C}(p, q)$ while β has end-points at ξ and some point $\eta \in \gamma_2$ such that*

$$(*) \qquad \ell(\alpha)^2 + \ell(\beta)^2 \leq \frac{1}{2}\big(\ell(\gamma_2)^2 + \ell(\gamma_3)^2\big)$$

where $\ell(\gamma)$ denotes the σ-length of a curve in X. Moreover, the inequality (*) is strict with the exception for special rhombic configurations.

We prove (*) in a chapter devoted to Riemann surfaces. The theorem gives the existence of unique geodesic curves on X which join a pair of points. The proof employs the uniformisation theorem for Riemann surfaces which reduces the proof to show the following:

Let D be the unit disc and $0 < \theta < \pi/2$ and consider the unique circle C_θ which contains $e^{i\theta}$ and $e^{-i\theta}$ and intersects the unit circle at right angles at these points. Let α denote the circular subarc of C_θ contained in D. Then the following inequality holds for every subharmonic function u in D:

$$\Big(\int_{-1}^1 e^{u(x)}\, dx\Big)^2 + \Big(\int_\alpha e^{u(z)}\, |dz|\Big)^2 \leq \frac{1}{4} \cdot \Big(\int_0^{2\pi} e^{u(e^{i\phi})}\, d\phi\Big)^2$$

An inequality for Dirichlet integrals.

Let Ω be a bounded open set in $\mathbf{R}^2$ and f a real-valued continuously differentiable function in Ω with partial derivatives f_x and f_y. The Dirichlet integral is defined by

$$(*) \qquad D^2_\Omega(f) = \iint_\Omega (f_x^2(x,y) + f_y^2(x,y))\, dxdy$$

When Ω is the square $\{0 \leq x \leq \pi\} \times \{0 \leq y \leq \pi\}$, $D^2_\Omega(f)$ is given by a Fourier series. More precisely, with

$$a_0 = \frac{1}{\pi^2} \iint_\Omega f(x,y)\, dxdy \quad : \widehat{f}(k,m) = \iint_\Omega f(x,y) \cdot \sin kx \cdot \sin my\, dxdy$$

it follows that

$$D^2_\Omega(f) = \sum_{k=1}^\infty \sum_{m=1}^\infty (k^2 + m^2) \cdot \widehat{f}(k,m)^2$$

At the same time the L^2-integral

$$\iint_\Omega f^2(x,y)\, dxdy = \pi^2 \cdot a_0^2 + \sum_{k=1}^\infty \sum_{m=1}^\infty \widehat{f}(k,m)^2$$

So if the mean value of f over Ω vanishes, i.e. if $a_0 = 0$ then

$$(*) \qquad \iint_\Omega f^2(x,y)\, dxdy \leq D^2_\Omega(f)$$

It turns out that (*) holds when the square is replaced by an arbitrary bounded and simply connected open set U. The proof follows from the existence of a conformal mapping from the square above onto U and is given in (*) where we also describe how (*) can be used to settle the Dirichlet problem for simply connected domains with smooth boundaries.

Carleman-Hadamard functions.

Let Ω be a bounded and connected open set in $\mathbf{C}$. The family of analytic functions in Ω is denoted by $\mathcal{O}(\Omega)$. Each $f \in \mathcal{O}(\Omega)$ is locally represented by convergent power series:

$$(1) \qquad f(z) = f(z_0) + \sum_{n=1}^\infty \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n \quad : z_0 \in \Omega$$

Let $d(z_0, \partial\Omega)$ be the distance of z_0 to the boundary and $D_\Omega(z_0)$ is the open disc of radius $d(z_0, \partial\Omega)$ centered at z_0 The power series in (1) may converge in a larger disc whose radius is denoted by $\rho_f(z_0)$. We exclude the case when f is the restriction to Ω of an analytic function F defined in some large disc containing Ω which means that there exists a constant M such that $\rho_f(z_0) \leq M$ hold for every $z_0 \in \Omega$. Hadamard's formula gives:

$$\frac{1}{\rho_f(z_0)} = \limsup_{n \to \infty} \Big[\frac{|f^{(n)}(z_0)|}{n!} \Big]^{\frac{1}{n}} \tag{2}$$

Regarding a picture one verifies the inequality below for each $z_0 \in \Omega$:

$$|\rho_f(z) - \rho_f(z_0)| \leq |z - z_0| \quad : z \in D_\Omega(z_0)$$

It follows that $\rho_f(z)$ is a continuous and positive function in Ω. The Carleman-Hadamard function is defined by

$$\mathcal{CH}_f(z) = \log \frac{1}{\rho_f(z)}$$

A crucial fact is that this function is subharmonic, i.e its Laplacian taken in the sense of distributions yields a non-negative Riesz measure in Ω. To prove this we take a positive integer N and set

$$\gamma_N(z) = \max_{n \geq N} \frac{1}{n} \cdot \log \frac{|f^{(n)}(z)|}{n!}$$

Since the complex derivatives $\{f^{(n)}(z)\}$ belong to $\mathcal{O}(\Omega)$, $\gamma_N(z)$ is the maximum of subharmonic functions and therefore itself subharmonic. Now $\{\gamma_N(z)\}$ is a non-increasing sequence of functions which by (2) converge pointwise to $\log \frac{1}{\rho_f}$. The subharmonicity of the γ-functions imply that $\{\Delta(\gamma_N)\}$ are non-negative Riesz measures in Ω and then the pointwise convergence implies convergence in the weak distribution sense which implies that $\Delta(\log \frac{1}{\rho_f})$ is a non-negative Riesz measure in Ω and hence $\mathcal{CH}_f$ is subharmonic.

Remark. The subharmonicity of $\mathcal{CH}_f$ was put forward by Carleman in lectures at the Mittag-Leffler Institute in 1935 who used this fact to construct boundary values of analytic functions which led to an extension of Fourier's inversion formula where so called hyperfunctions extend the class of distributions on real intervals. Let us also remark that the Carleman-Hadamard function measures the singularities of f when one approaches the boundary. Namely, the condition that f does not extend analytically across in a neighborhood of a boundary point $p \in \partial\Omega$ means that $\rho_f(z) \to 0$ as $z \to \partial\Omega$, or equivalently that

$$\lim_{z \to p} \mathcal{CH}_f(z) = +\infty \quad : p \in \partial\Omega$$

In addition to this the mass distribution of non-negative Riesz measure $\Delta(\mathcal{CH}_f)$ reflects the boundary behaviour of f in a refined way. A special subharmonic function arises when we take

$$u_\Omega(z) = \log \frac{1}{d(z, \partial\Omega)}$$

The non-negative Riesz measure $\Delta(u_\Omega(z))$ is an invariant attached to the bounded domain Ω which measure proportions in a logarithmic scale and leads us to the next topic.

Logarithmic capacity.

A new era in analytic function theory started with Iversen's Ph.d-thesis *Recherches sur les fonctions meromorphes* from Helsinki University in 1914 and led for example to the theory about value distributions of meromorphic functions where major results are due to R. Nevanlinna. In these notes attention is given to logarithmic capacity. A compact set E of the unit circle has logarithmic capacity equal to zero if the energy integral

$$\iint_{E\times E} \log \frac{1}{|z-\zeta|}\, d\mu(z)\cdot d\mu(\zeta) = +\infty$$

for every probability measure supported by E. *Special Topics* contain sections devoted to thin sets and boundary values of analytic functions. Here is an example where logarithmic capacity intervenes with function theory. Let $f(z)$ be a meromorphic function in the unit disc with no pole at $z=0$. Put

$$T_f(r) = \frac{1}{2\pi}\cdot \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta \quad : \; P_f(r) = \int_0^r \frac{n_\infty(\rho)}{\rho}\, d\rho$$

where $n_\infty(\rho)$ counts the number of poles with their orders in the disc $\{|z| < \rho\}$. One says that the meromorphic function f has unbounded characteristic if

$$\lim_{r\to 1} T_f(r) + P_f(r) = +\infty$$

Next, a complex number α belongs to the essential range set $R(f)$ if there exists a sequence $\{z_n\}$ in D where $|z_n| \to 1$ and $f(z_n) = \alpha$ hold for every n. With these notation the following result was proved by Frostman in [xxx]:

Theorem. *If f has unbounded characteristic then the logarithmic capacity of the set $\mathbf{C} \setminus R(f)$ is zero.*

Results as above motivate why an extensive appendix devoted to measure theory has been included. The next example illustrates how complex analysis in one variable intervenes with measure theoretic considerations.

Borel-Denjoy series.

Let $A_1, A_2, \ldots$ be a sequence of complex numbers which tends to zero and assume that

$$\sum_{k=1}^{\infty} |A_k| \cdot \log \frac{1}{|A_k|} < \infty$$

Let $\{\alpha_\nu\}$ be a sequence of complex numbers where the sole condition is that $\alpha_k \neq \alpha_\nu$ when $\nu \neq k$. An example could be a sequence with rational coordinates arranged in such a way that every point in $\mathbf{C}$ is a cluster point for the α-sequence. Next, let $P(t)$ be a polynomials with complex coefficients. To each $N \geq 1$ we consider the function

$$S_N(t) = \sum_{k=1}^{k=N} \frac{A_k}{P(t) - \alpha_k}$$

As a complex-valued function of the real variable t this partial sum function is defined outside a finite set of t-values where $P(t)$ happens to be α_k for some $k \leq N$. It means that one only has to avoid a denumerable set of t-values in order that the partial sums are defined for every N. But absolute values $|P(t) - \alpha_k|$ can be small quite often and the question arises when there exists a limit

$$\lim_{N\to\infty} S_N(t) = S_*(t)$$

for those t-values where $P(t) \neq \alpha_k$ for all k. It turns out that convergence holds almost everywhere in the sense of Lebesgue. In other words, there exists a subset $\mathcal{N}_P$ of the real t-line whose linear Lebegue measure is zero and the partial sums converge to a limit for all t outside $\mathcal{N}_P$. How does one start the proof of this assertion? An ambition in these notes is to supply the reader with "enough muscles" in analytic function theory so that articles which settle a result as above can be studied with relative ease.

The Γ-function

.

It is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} \cdot e^{-t}\, dt$$

The function is analytic in the half-space $\mathfrak{Re}\, z > 0$ and partial integration gives the functional equation

$$z \cdot \Gamma(z) = \Gamma(z+1)$$

From this it follows that the Γ-function extends to a meromorphic function in the whole complex plane with simple poles at $0, -1, -2, \ldots$. Less obvious is that $\frac{1}{\Gamma(z)}$ is an entire function. Another remarkable result is Euler's integral formula which will be proved in § XX devoted to the Γ-function:

$$(*) \qquad \Gamma(z) = \frac{1}{z} \cdot e^{-\gamma \cdot z + z^2 \int_1^\infty \frac{[t]}{t^2(t+z)}\, dt}$$

Here γ is Euler's constant and $[t]$ the integral part of t which gives

$$z^2 \int_1^\infty \frac{[t]}{t^2(t+z)}\, dt = \sum_{n=1}^\infty z^2 \cdot n \cdot \int_n^{n+1} \frac{dt}{t^2(t+z)} =$$

$$\sum_{n=1}^\infty n \cdot \int_n^{n+1} [\frac{1}{t+z} + \frac{z-t}{t^2}]\, dt = n\big[\log(1 + \frac{1}{n+z}) + \frac{z}{n(n+1)} - \log(1 + \frac{1}{n})\,\big]$$

Euler's clever choice of the constant γ gives the equation (*). An example where (*) is used appear during the construction of kernels which give solutions to the heat equation in the non-analytic case where the usual heat kernel cannot be applied. This illustrates how analytic function theory applies to problems about differential equations.

Critical values for harmonic level curves.

Let D be the unit disc and consider a sequence of real numbers

$$0<\alpha_1<\beta_1<\alpha_2<\ldots<\beta_{m-1}<\alpha_m<\beta_m<2\pi$$

They give rise to m disjoint intervals on the unit circle:

$$J_\nu=\{e^{i\theta}\colon\ \alpha_\nu\leq\theta<\beta_\nu\}\quad :1\leq\nu\leq m$$

Now there exists the harmonic function $\omega(z)$ in the unit disc whose boundary values are one on each J_ν and zero on the closed complement of these J-intervals. Poisson's integral formula gives

$$\omega(z)=\frac{1}{2\pi}\cdot\sum_{\nu=1}^{\nu=m}\int_{\alpha_\nu}^{\beta_\nu}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\theta$$

To each $0<s<1$ we get the level set $L(s)=\{\omega(z)=s\}$ in D. When s is close to one it is easily seen that $L(s)$ is the disjoint union of arcs which stay close to the unit circle and join end-points of the J-intervals. Similarly, if $s\simeq 0$ then $L(s)$ is a disjoint union of arcs which join end-points of the complementary intervals. When $m\geq 2$ one sees from a picture that in between there must occur s-values where the level set $L(s)$ cannot be a union of m disjoint Jordan arcs. It means that self-intersections occur and we shall learn that they are caused by zeros of the gradient vector for ω. Here $\omega(z)$ has a harmonic conjugate $\omega^*(z)$ such that the complex-valued function $\omega(z)+i\cdot\omega^*(z)$ is analytic given by the Herglotz formula:

$$(*)\qquad \omega(z)+i\cdot\omega^*(z)=\frac{1}{2\pi}\cdot\sum_{\nu=1}^{\nu=m}\int_{\alpha_\nu}^{\beta_\nu}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta$$

The Cauchy-Riemann equations show that the gradient vector $\nabla(\omega)=(\omega_x,\omega_y)$ vanishes at a point z if and only if the complex derivative of (*) is zero at z. Taking the derivative with respect to z in (*) we get the analytic function

$$\phi(z)=\frac{1}{2\pi}\cdot\sum_{\nu=1}^{\nu=m}\int_{\alpha_\nu}^{\beta_\nu}\frac{2\cdot e^{i\theta}}{(e^{i\theta}-z)^2}\,d\theta$$

For each ν the θ-derivative of $(e^{i\theta}-z)^{-1}$ is equal to $-ie^{i\theta}\cdot(e^{i\theta}-z)^{-2}$ which gives

$$\phi(z)=\frac{1}{\pi\cdot i}\cdot\Big[\sum_{\nu=1}^{\nu=m}\frac{1}{e^{i\beta_\nu}-z}-\sum_{\nu=1}^{\nu=m}\frac{1}{e^{i\alpha_\nu}-z}\Big]$$

It turns out that the rational function $\phi(z)$ has $m-1$ zeros counted with multiplicity in the unit disc. The proof is very instructive and relies upon the argument principle. Details appear in § XX where we treat a general result due to Nevanlinna which applies to level curves of harmonic measure functions in domains which are not necessarily simply connected.

Absolute values of complex polynomials.

Let $P(z) = (z - \alpha_1) \cdots (z - \alpha_n)$ be a polynomial of degree n with simple zeros which means that $\{\alpha_k\}$ is an n-tuple of disctinct numbers. To every $a > 0$ we get the subset of the complex z-plane defined by the level curve of $|P(z)|$:

$$\gamma_a = \{|P(z)| = a\}$$

We have also the open sets

$$\Omega_a = \{|P(z)| < a\}$$

If a is small it is clear that γ_a is the disjoint union of n many small simple and closed curves around the zeros. If a is very large the term z^n dominates and γ_a is a simple closed curve which stays close to the circle $|z| = a^{\frac{1}{n}}$. Hence some changes must occur as a increases from small to large values. In particular the number of connected components of the Ω-sets change. Under the extra assumption that the derivative $P'(z)$ also has simple zeros, say $\beta_1, \ldots, \beta_{n-1}$ and the absolute values $\{|P(\beta_\nu)|\}$ are all distinct one gets the following topological picture. Arrange the absolute values $\{a_k = |P(\beta_k)|\}$ so that $0 < a_1 < \ldots < a_{n-1}$. In § XX we show that if $a_k < a < a_{k+1}$ for some $1 \leq k \leq n-2$, then γ_k is the union of $n-k$ simple closed curves and hence Ω_k has $n-k$ connected components. If $a > a_{n-1}$ then γ_a is a simple closed curve so that ω_a is a connected a Jordan domain. One can visualize the change of the Ω-domains as a increases on a computer. As a specific example one can take the polynomial $P(z) = z^7 - \delta \cdot z - 1$ where δ is a positive real number. One easily verifies that the conditions above hold for every $\delta > 0$. With δ small, say 10^{-2} the computer illustrates how the domains $\{|P| < a\}$ change and thanks to contemporary programs one watch the changing configurations like a moving film which gives intuitive feeling for a complex analytic situation

Ordinary differential equations.

Let $m \geq 2$ and $p_0(x), \ldots, p_m(x)$ are polynomials which in general may have complex coefficients. With $\partial = \frac{\partial}{\partial x}$ one has the differential operator

$$P(x, \partial) = p_m(x)\partial^m + \ldots + p_1(x)\partial + p_0(x)$$

If the leading polynomial p_m has no real zeros one says that the elliptic case occurs. Then it is wellknown that the homogeneous solutions to P is an m-dimensional vector space formed by real-analytic functions on the x-line. If p_m has real zeros the determination of all distributions μ on the x-line satisfying $P(\mu) = 0$ is not so obvious. Expressing distributions via boundary values of analytic functions we shall prove that the P-kernel on the space $\mathfrak{Db}(\mathbf{R})$ is a vector space of dimension $m + k$, where k is the number of real zeros of p_m counted with multiplicities. Moreover, the P-kernel contains specific subspaces for each real zero a of p_m. Namely, let $p_m(z) = (z - a)^e \cdot q(z)$ for some $e \geq 1$ where $q(a) \neq 0$. In §§ we prove that $\mathrm{Ker}_P \mathfrak{Db}(\mathbf{R})$ contains an e-dimensional subspace of distributions supported by the half-line $[a, +\infty)$ and that P has a fundamental solution γ supported by this half-line, i.e.

$$P(x, \partial)(\gamma) = \delta_a$$

where δ_a is the Dirac distribution at a.

Example. Set $\nabla = (x\partial)^2 = x^2\partial^2 - x\partial$ and consider Bessel's differential operator

$$P(x,\partial) = \nabla^2 + x^2 - 1$$

Here one finds an entire function $f(z)$ such that $P(f) = 0$ whose series expansion at $\{z = 0\}$ starts with $f(z) = z + c_3z^3 + \ldots$. In addition there exists a multi-valued solution

$$\phi(z) = f(z) \cdot \log z + a(z)$$

where $a(z)$ is a meromorphic function with a simple pole at $\{z = 0\}$. Now we get the distribution supported by $\{x \geq 0\}$ defined by

$$\mu = f(x) \cdot (\log x)_+ + a_+(x)$$

where $(\log x)_+$ and $a_+(x)$ are Euler's distribution extension to $\{x \geq 0\}$ of the densities $\log x)$ and $a(x)$ on $\{x > 0\}$ to be defined in § xx where we also verify that μ is a fundamental solution to P.

An overwiew to these notes.

The material in § 0.1-0.10 describes various aspects in analytic function theory and applications to other subjects. The less experienced reader may prefer to study the subsections (0.1-0.10) at a later occasion since rather advanced concepts sometimes appear. The section devoted to work by Beurling and Carleman has been inserted both for its historic reconsiliation, and since many results in these notes are due to Beurling and Carleman.

References.

Much inspiration in these notes, including sources for historic accounts, come from Ludwig Bieberbach's text-books [Bi:1,2] which cover analytic function theory up to 1925. In several sections I borrow proofs from these text-books where my presentation is only slightly different. An example is the Uniformisation Theorem for multiply connected domains in **C** where I have followed Bieberbach's presentation of the elegant proof due to Caratheodory. The text book [Cartan] by Henri Cartan has inspired the way I have organised the chapters devoted to analytic functions. Here the reader may find exercises which give very good illustrations to theoretical results. The book [Ko] by Paul Koosis has also been a valuable source. At several places material from Koosis' original text are presented with minor changes, such as in the proof of Donald Marshall's convexity theorem for inner functions in the unit disc. I have also profited upon material from the two volumes about the logarithmic potential by Koosis which is a good reference for further studies in analytic function theory. Material devoted to Runge's Theorem is inspired by Raghavan Narasimhan's text-book [Na] where the reader also may find a nice introduction to Riemann surfaces. Basic facts about the harmonic measure and the Lindelöf-Pick principle is inspired by R. Nevanlinna's book [xx]. E. Landau's book [xx] has contributed to sections devoted to series. Topics from the book [PE] by Raymond Paley and Norbert Wiener also appear. Apart from the text-books above there also occur some problems taken from [Po-Sz] by Polya and Szegö. An example is a result due to Siegel which gives an upper bound for products of roots to polynomial in any degree expressed by the size of its coefficients. Siegel's proof gives a good lesson since it teaches how complex analysis can be used in an unexpected fashion.

In addition to the references above I have been inspired by the over-all presentation in the text-book [Kr] by Steven G. Krantz. Here the reader will find a number of results about conformal maps of multiply connected domains and their associated automorphism groups together with a study of the Bergman kernel and constructions of various metrics. The reader may also consult the text-book by John Conway, especially part II which contains material of considerable interest. For example a chapter is devoted to De Brange's proof of the Bieberbach conjecture which for more than half a century was an open problem in a analytic function theory. Chapters devoted to harmonic and subharmonic functions also merit attention such the exposition of the Fine Topology in **C** which by definition is the weakest topology making every subharmonic function with values in the extended interval

$-\infty, \infty]$ into continuous functions. Another recommended text-book is [Andersson] which presents elegant proofs of advanced results such as Carleson's Corona theorem.

Finally I must mention the text-books [1-2] by Lars Ahlfors. His masterful presentation in [1] is recommended to everybody who enters studies devoted to complex analysis. Since [1] was a veritable "bible" for me as a graduate student, it is clear that the text by Ahlfors has inspired much material in these notes, foremost in sections devoted to the foundations in analytic function theory. Concerning the more advanced book [Ahl-2] there occur references in these notes to the general Uniformisation Theorem. But I have not tried to pursue this in detail. One reason is that these notes do not treat potential theory. But hopefully these notes may inspire readers to study [Ahl:2]. One must also mention the deep studies of moduli problems in articles such as [Ahl-Beu] where the authors seek invariants to decide when a pair of multiple connected domains are conformally equivalent and from work by Ahlfors one should mention his far reaching studies of quasi-conformal maps. His article [Ah. Acta 1935] is even today an inspiration for contemporary research. The lyric comments when Caratheodory presented Ahlofor's contributions at the IMU-congress in 1936 where he bacame the first mathematician to receive a Fields medal, is a recommended reading even today. Twenty years later Ahlfors discovered an explicit analytic solution to the Beltrami equation and soon after, in collaboration with Lipman Bers, quasiconformal variations were applied to study Kleinian groups. Among more recent topics of advanced nature we mention complex dynamics where Sullivan's theorem about Julia sets is an example. A proof of this result appears in the text-book [Ca-Ga] by L. Carleson and J. Garnett. Finally, we already mentioned the text-book [Ga-Ma] by J. Garnett and Marshall devoted to recent discoveries about harmonic measure and extremal length.

Remarks about the contents.

I have tried to make the notes reasonably self-contained. For example, a detailed proof of *Stokes Theorem* is given together with the formulas of Green and Gauss where attention also is given to the notion of differential forms. With $z = x + iy$ we identify the complex z-plane with $\mathbf{R}^2$. Consider a differential 1-form $W = f \cdot dx + g \cdot dy$ where f and g is a pair real-valued C^1-functions. The 1-form W is closed when

$$(*) \qquad dW = (-f'_y + g'_x) \cdot dx \wedge dy = 0$$

In Chapter II we establish a result expressing the difference of two line integrals of a 1-form W which is not necessarily closed, taken over a pair of curves Γ_1 and Γ_0 with common end-points and linked by a continuous family of similar curves. The formula in Theorem 5.7 from Chapter II is a result in real 2-variable analysis which contains the subsequent study of complex line integrals as a special case when the integrands are complex analytic functions. So the reader should be aware of the fact that ordinary Calculus provide many tools to attain integral formulas in analytic function theory. However, we shall prove results in the complex analytic set-up for its own sake. One reason for doing this extra job is that complex line integrals are nicely expressed by taking complex Riemann sums. Another is the special flavour which arises from complex arguments. A typical case occurs in § 2 from Chapter 4 where the argument principle counts zeros of an analytic function via its complex logarithmic derivative.

An example where calculus in one real variable becomes useful in analytic function theory is a convexity theorem from 1906 by Jensen which goes as follows: Let $\{\alpha_\nu\}$ be a decreasing sequence of positive real numbers with $\alpha_1 \leq 1$ and let $\{p_\nu\}$ another sequence of positive numbers. Assume that there exists some $\delta > 0$ such that the series $\Phi(r) = \sum p_\nu \cdot r^{\alpha_\nu}$ converges when $0 < r < \delta$. Then Jensen proved that $\log \Phi(r)$ is a convex function of $\log r$. This can be used to prove convexity theorems expressed by integrals connected to analytic functions and more generally to subharmonic functions. For example, Jensen's result was used when Hardy in his work *The Mean Value of the Modulus of an Analytic function* from 1915 proved that the function $r \mapsto \int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta$ is a convex function of $\log r$ when $f(z)$ is analytic in some disc centered at the origin. Many early results such as those of Hardy were later superseeded by the systematic use of subharmonic functions and Thorin's convexity theorem which is exposed in § XX.

A general comment.

These notes do not strive to present the theory in its most perfect form and I often try to give a historic perspective, perhaps a bit too affected by my own experience as undergraduate student at Stockholm University in the early 60:s when I was taught function theory by Otto Frostman in a way I appreciated very much. Theorems are often proved in a "pedestrian manner" rather than using the full force of concepts from functional analysis. This is not just a matter of a "historic reconcilation". One reason is that results are better understood if one can find a constructive proof rather than to rely upon the axiom of choice via the Hahn Banach theorem or the existence of maximal ideals which appear in commutative Banach algebras. A typical case is when we start from a function $f(x) \in L^1(\mathbf{R})$ and regard its Fourier transform $\hat{f}(\xi)$. Suppose that $a \leq \xi \leq b$ is an interval on the real ξ-line and let $\Phi(z)$ be an analytic function defined in an open neighborhood of the compact image set $\hat{f}([a, b])$. Then there exists $g(x) \in L^1(\mathbf{R})$ whose Fourier transform $\hat{g}(\xi)$ is equal to $\Phi(\hat{f}(\xi))$ when $a \leq \xi \leq b$. The mere existence of g follows easily by an abstract reasoning. However, one can construct g in a rather explicit manner using the local power series expansions of the analytic Φ-function. We shall give this "pedestrain proof" based upon a lecture by Carleman at Institute of Mittag Leffler in 1935 which has the merit that it is constructive. Of course, this requires more effort as compared to the "abstract proof". But I think it teaches the student more.

The reader will also find repetitions of similar arguments which appear in slightly different contexts. A typical case occurs in studies of various maximal functions. It would be possible to concentrate all this to some few general results in a general context where one regards Riesz measures on metric spaces whose distribution of mass satisfies certain conditions with respect to the given metric. This would lead to short proofs but often with less control of a priori constants. So we often prefer to repeat similar arguments in specific situations. For an account about general maximal functions we refer to the elegant article [Smith] which becomes especially useful when analytic function theory in one complex variable is replaced by studies of harmonic and subharmonic functions in $\mathbf{R}^d$ for $d \geq 3$.

A (nasty ?) remark. Personally I do not understand the contemporary trend in the teaching of mathematics where subjects tend to be isolated from each other into small separate courses devoted to measures, function theory, Fourier series and so on. Not to mention the unreasonable borderline between algebra and analysis. To give an example. In the past Eisenstein's famous theorem from 1852 was demonstrated to most students entering graduate studies. The elegant proof of Heine from 1854 (presented in Chapter XX) combines algebraic facts about irreducible polynomials in two variables with analytic function theory. Recall that Eisenstein's theorem says that if the power series expansion $\sum c_\nu (z - z_0)^\nu$ of a regular local branch of an algebraic function is such that every c_ν is a rational number, then there exists a positive integer k so that $k^\nu \cdot c_\nu \in \mathbf{Z}$ for all ν. My question to contemporary teaching is whether any course in mathematics for beginning graduate students is suited to present Eisenstein's theorem.

Measure theory.

An appendix is devoted to Lebegue theory and Riesz measures. A major result due to Lebesgue asserts that every non-decreasing and continuous function on the real line has a classical derivative outside a null-set. We give a detailed proof which illustrates the high standard in classical analysis and remark that "abstract methods" do not improve Lebegue's original proof. In analytic function theory Lebegue's result was used by Fatou to show that if $f(z)$ is a bounded analytic function in the unit disc, then the radial limit $\lim_{r \to 1} f(re^{i\theta})$ exists for almost all θ. Fatou used the primitive function F of f. Here F has a continuous extension to the closed disc and Fatou's proved that if the function $F(e^{i\theta})$ of the angular variable has an ordinary derivative at a point θ_0, then the radial limit for f exists at this θ-value. Later Friedrich Riesz adopted measure theory in a most succesful manner and proved that there is a 1-1 correspondence between the class of non-negative Riesz measures μ on the unit circle T and the family of positive harmonic functions u in the unit disc. More precisely, every such u is the Poisson extension P_μ of a unique non-negative Riesz measure on T. This is proved in Chapter V and for readers who are less familiar with measure theory the result by Riesz helps to "feel more comfortable" with abstract notions of measures, especially the "hidden family" of singular Riesz measures which carry their whole mass on null sets in the sense of Lebesgue.

Many subtle results in analytic function theory are expressed via measures. An example is a result due to Beurling which goes as follows: Let μ be a non-negative Riesz measure on the unit circle with no atom at $\theta = 0$, i.e. it has no mass at the singleton set $\{z = 1\}$. Let

$$P_\mu(z) = \frac{1}{2\pi} \int \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu(\theta)$$

be Poisson's harmonic extension of μ to the unit disc. To each real number a we put

$$E_a = \{z \in D : \ P_\mu(z) \geq \frac{a}{|1 - z|}\}$$

Beurling proved that the integral

$$\iint_{E_a} \frac{dxdy}{|1 - z|} < \infty$$

At first sight this may appear as an exercise in measure theory. But in § xx we shall learn that the proof requires results based upon harmonic measures and estimates using the theory of extremal lengths. This illustrates that in order to understand measure theory in "concrete situations" it is far from sufficient to digest some abstract notions which interpretate measures as representatives of linear functionals on spaces of continuous functions.

Series.

Since analytic functions are locally represented by convergent power series the study of series is an imporant issue. In § 2 from Chapter I we start with elementary facts about series and proceed to more involved results. An example is a theorem by Hardy and Littlewood which asserts that if $\{a_n\}$ is a sequence of positive real numbers such that there exists the limit

$$\lim_{x \to 1} (1 - x) \cdot \sum_{n=1}^{\infty} a_n \cdot x^n = A$$

then there exists the mean-value limit

$$\lim_{n \to \infty} \frac{a_1 + \ldots + a_n}{n} = A$$

The remarkable fact is that no other condition than positivity of the a-numbers has been imposed. It is very instructive to follow the single steps in the proof which are put together in an ingenious fashion. One might say that the Hardy-Littlewood theorem illustrates what analysis is all about, i.e. *to put small seemingly unrelated threads together.* The proof of the theorem above is "elementary" in the sense that no functional analysis is used. A case where functional analysis is used appears in *Ikehara's Theorem* which goes as follows: Let μ be a non-negative Riesz measure on $[1, +\infty)$ such that the integrals

$$\int_1^\infty x^{-\delta} \cdot d\mu(x) < \infty \quad \text{for all} \quad \delta > 1$$

Assume also that there exists a continuous function $G(u)$ such that

$$\lim_{\epsilon \to 0} \Big[\int_0^\infty x^{iu-(1+\epsilon)} \cdot d\mu(x) - \frac{A}{\epsilon + iu} \Big] = G(u)$$

holds uniformly over all bounded intervals $-b \leq u \leq b$. Then one has the limit formula:

$$\lim_{x \to \infty} \frac{1}{x} \int_0^x d\mu(t) = A$$

Let us finish this discussion with a result proved by Volterra in 1913. He considered an L^1-function $K(x)$ supported by $[0, +\infty)$ and defined the function

$$\Phi(x) = \sum_{n=1}^\infty (-1)^n \cdot K^n(x) \quad K^n(x) \text{ is the n-fold convolution of } K(x)\,.$$

The question arises when $\Phi(x)$ also belongs to $L^1(\mathbf{R}^+)$. To settle this we consider the Laplace transform

$$\hat{K}(w) = \int_0^\infty e^{-wx} \cdot K(x) \cdot dx\,, \quad \mathfrak{Re}\, w \geq 0\,.$$

Volterra proved that Φ is integrable if and only if

$$\frac{\hat{K}(w)}{1 + \hat{K}(w)} \neq 0 \quad \text{for all} \quad \mathfrak{Re}\, w \geq 0\,.$$

In 1913 the proof was quite involved. Today, with the aid of general results about Banach algebras, Volterra's result is an easy exercise which we explain in the Chapter XX. So this example illustrates that it is profitable to learn abstract theories.

The use of geometry.

It is often profitable to employ geometric considerations. An example is *Poisson's formula* expressing the values taken by a harmonic function in the unit disc from its boundary values. Actually it was Hermann Schwarz, one of the great pioneers in function theory, who established this formula in the context of analytic function theory. His geometric description of the Poisson kernel leads to invariance properties for the spherical, respectively the hyperbolic metric. An example where geometric considerations appear in analytic function theory is a result due to Julia and Caratheodory which goes as follows:

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in the right half-plane $U = \mathfrak{Re}(z) > 0$ whose real part u is a positive function and satisfies:

$$\inf_{x+iy\in U} \frac{u(x+iy)}{x} = 0$$

Then it follows that

$$\lim_{x\to+\infty} \frac{f(x+iy)}{x} = 0 \quad \textit{uniformly every sector } \{y \le N\cdot x\} \quad : N = 1,2,\ldots$$

This is remarkable since no assumption is imposed on the imaginary part of f. As we shall see in XX the proof of the Julia-Caratheodory theorem relies upon a general principle for the spherical measure on domains in $\mathbf{C}$. The result above was used when Julia studied iterated maps which leads to Julia sets and fractals, a subject which has become "fashion" since many phenomena can be illustrated by computers. Deeper results arise when geometric ideas and hard analysis are put together. An example is the notion *Carleson measures* which is essential in the proofs of two famous theorems due to Lennart Carleson: The Corona problem and the interpolation by bounded analytic functions. These results are exposed in a section from *Special Topics* and may be regarded as a veritable "high-light" in these notes.

Hardy spaces.

Problems in complex analysis has initiated many results in measure theory such as in studies of Hardy spaces which appear in many places, foremost in § 2-3 from Special Topics. A result due to C. Fefferman and E. Stein asserts that a measurable function $f(\theta)$ on the unit circle has bounded mean oscillation if and only if the squared length of the gradient vector $\nabla(H_f) = (\partial H_f/\partial x, \partial H_f/\partial y)$ of its harmonic extension $H_f(z)$ to the unit disc satisfies:

$$\frac{1}{h}\cdot \iint_{S_h(\theta)} |z|\cdot \log\frac{1}{|z|}\cdot \big|\nabla(H_f)\big|^2 \cdot dxdy \le C$$

for some constant C and every $0 < h < 1/2$, where $S_h(\theta)$ is the subset of D defined by

$$S_h(\theta) = \{z = re^{i\phi} : 1-h < r < 1 : |\phi-\theta| < h\}$$

This result illustrates how geometric ideas together with analytic estimates are used to attain deep results. Another result which exhibits the distinction between absolutely continuous, respectively singular measures on the unit circle T appears in a closure theorem due to Beurling. Every non-negative measure μ on T gives a bounded analytic function $g(z)$ in the unit disc D defined by

$$g(z) = \exp\big[-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\cdot d\mu(t)\big]$$

The g -function is quite special. It has no zeros in D and $g(0) = e^{-a}$ where $a = \frac{1}{2\pi}\int_0^{2\pi} d\mu(t)$ is a positive constant. At the same time $\log|g|$ is the Poisson integral of $-\mu$ and by a result which goes back to F. Riesz and Fatou this implies that the radial limit

$$\lim_{r\to 1} |g(re^{i\theta})| = 1$$

exists for almost every θ in the sense of Lebesgue. Next, let $H^2(T)$ be the Hardy space whose elements are square integrable functions on T which are boundary values of analytic functions in D. This is a Hilbert spaces where the exponential functions $\{e^{in\theta} : n = 0, 1, 2, \ldots\}$ is an orthonormal basis. One might expect that multiplication with g yields a linear operator on $H^2(T)$ whose range is dense. But this turns out to be false in general. Beurling proved that $g \cdot H^2(T)$ is a closed proper subspace of $H^2(T)$ as soon as the singular part of the measure μ is $\neq 0$. On the other hand, if $\mu = \phi(t) \cdot dt$ is absolutely continouos, i.e when $\phi(t)$ is an L^1-function on T, then $g_\mu \cdot H^2(T)$ is a dense subspace of $H^2(T)$. The proof appears in Section XX and gives an instructive lesson since a mixture of measure theory, functional analysis and analytic function theory appear in the proof.

A final comment.

My personal opinion while entering graduate studies in mathematics is that it is rewarding to pursue specific proofs of a more involved nature, rather than to "swallow a mass of general concepts". My favourite example is due to Carleman. In the article [Ca] it is proved that there exists an absolute constant $\mathcal{C}$ with the following property:

For every pair (f, n) where n is a positive integer and f a non-negative real-valued C^∞- function defined on the closed unit interval $[0, 1]$ whose derivatives up to order n vanish at the two end points, one has the inequality

$$(*) \qquad \sum_{\nu=1}^{\nu=n} \frac{1}{[\beta_\nu]^{\frac{1}{\nu}}} \leq \mathcal{C} \cdot \int_0^1 f(x)dx \quad : \quad \beta_\nu = \sqrt{\int_0^1 [f^{(\nu)}(x)]^2 \cdot dx}$$

The remarkable fact is that $\mathcal{C}$ is independent of n. Carleman's inequality is sharp in the sense that there exists a constant C_* such that one for every $n \geq 2$ can find functions $f_n(x)$ as above so that the opposed inequality (*) holds with C_*. Let us remark that (*) demonstrates that the standard cut-off functions which in many applications are used to keep maximum norms of derivatives small up to order n small are optimal up to a constant. So the theoretical result in (*) plays a role in numerical analysis where one often uses smoothing methods. The ingenious proof of (*) employs certain estimates for harmonic measures applied to the subharmonic Log-function of the absolute value of the Laplace transform of f.

An open problem.

Let me finish by describing an open problem in the spirit of these notes. Let $n \geq 2$ and consider a polynomial

$$f(z) = a_0 + a_1 \cdot x + \ldots + a_n \cdot z^n$$

where the sole assumption is that a_0 and a_n both are $\neq 0$. The coefficients are in general complex numbers. To each $0 < \theta < \pi$ we count the number of zeros in the sector of the complex z-plane whose points have argument in the interval $(-\theta, \theta)$.

Denote this integer by $n_f(\theta)$ where multiple zeros are repeated according to their multiplicities. By the fundamental theorem of algebra f has n complex zeros and if n is large one expects that in average

$$(*) \qquad \frac{n_f(\theta)}{n} \simeq \frac{\theta}{\pi}$$

This "statistical result" is indeed true and goes back to work by Schur. The search for a more precise control of the deviation in (*) stimulated further research. The following result was proved by Erdös and Turan in [Annals of math. vol. 51 (1950)]

Theorem. *There exists a constant C^* which is independent of n such that the following hold for every $0 < \theta < \pi$ and every polynomial f as above:*

$$(*) \qquad \left|\frac{n_f(\theta)}{n} - \frac{\theta}{\pi}\right| \leq C^* \cdot n^{-1/2} \cdot \log \frac{|a_0| + |a_1| + \ldots + |a_n|}{\sqrt{|a_0 \cdot a_n|}}$$

In § XX we give a proof due to Ganelius which shows that one can take

$$C^* = \sqrt{\frac{2\pi}{\mathcal{G}}} \quad : \quad \mathcal{G} = 1 + \sum_{\nu=1}^{\infty} (-1)^\nu \cdot \frac{1}{(2\nu+1)^2} \simeq 0,916\ldots$$

However, this upper bound is not sharp so the question remains if one can determine the best constant C^* above, or at least "as sharp as possible" which eventually only has to work for all sufficiently large n.

A related problem about Fourier series. The major step to estimate C^* from above stems from a result about trigonometric polynomials which goes as follows: Let H and K be positive real numbers. Consider all pairs of finite sequences of real numbers $\{a_k\}$ and $\{b_k\}$ indexed by positive integers k such that the following two inequalities hold for every $0 \leq \phi \leq 2\pi$:

$$\text{(i)} \qquad \sum a_k \cdot \cos k\phi + b_k \cdot \sin k\phi \leq K$$

$$\text{(ii)} \qquad \sum k \cdot (a_k \cdot \sin k\phi - b_k \cdot \cos k\phi) \leq H$$

Set

$$v(\phi) = \sum a_k \cdot \sin k\phi - b_k \cdot \cos k\phi$$

In § XX we expose the following result which is due to T. Ganelius:

B. Theorem. *Put $C_* = 2\pi^{\frac{3}{2}} \cdot \mathcal{G}^{-1/2}$. Then (i-ii) entail that*

$$\left|v(\beta) - v(\alpha)\right| \leq C_* \cdot \sqrt{HK}$$

hold for all pairs $0 \leq \beta < \alpha < 2\pi$.

However, this constant is not sharp so one can ask for a best constant which would improve the choice of C^* in Ganelius' theorem.

Work by Beurling and Carleman.

Many results in these notes are due to Arne Beurling and Torsten Carleman. This reflects my personal taste since I regard them as two of the most prominent mathematicians ever in analysis. Let me first recall some of Carleman's contributions. The appendix devoted to functional analysis treats spectral resolutions for unbounded self-adjoint operators on a Hilbert space which Carleman established in 1923. This result has a wide range of applications. Carleman also introduced a method to handle non-symmetric operators which admit suitable factorisations in order to reduce the spectral analysis to the case of symmetric kernels. We describe such a case in § 11 in the appendix devoted to functional analysis. Among contributions prior to 1920 one can mention Carleman's inequality for the operator norm of resolvents which is a veritable cornerstone in the spectral theory of linear operators. He also used subharmonic functions to extend results of the Phragmén-Lindelöf type. Here is an example. Put $B = \{0 < \mathfrak{Re}\, z < 0\}$ which is an open and simply connected domain in $\mathbf{C}$. Let $f(z)$ be analytic in B and for each $0 < x < 1$ we put

$$m(x) = \max_y |f(x+iy)| \quad \text{and} \quad \phi(x) = \int_{-\infty}^{\infty} |f(x+iy]|^2\, dy$$

We assume that these functions are bounded on all intervals $\delta \leq x \leq 1-\delta$ when $\delta > 0$ are small. Phragmén and Lindelöf proved that the function $\phi(x)$ is convex on $(0,1)$. Carleman proved that $\phi(x)$ satisfies the differential inequality

$$(*) \qquad \phi''(x) \geq \frac{\phi'(x)^2 + m(x)}{\phi(x)}$$

which in particular gives the convexity. In § xx we shall give the proof of (*) which employs the subharmonic function $\log |f|$.

Liouville's theorem. Consider arcs parametrized by injective and continuous maps $s \to \gamma(s) = x(s) + iy(s)$ from the half-open interval $[0,1)$ in $\mathbf{C}$ where $\gamma(0) = 0$ is the origin and $|\gamma(s)| \to +\infty$ as $s \to 1$. Let γ_1, γ_2 be a pair of such arcs whose sole common point is the origin and together give an unbounded domain Ω whose boundary is the union of the two arcs. Suppose that $f(z)$ is analytic in Ω and extends continuously to $\partial\Omega$ with a finite maximum norm

$$m^* = \sup_{z \in \gamma_1 \cup \gamma_2} |f(z)|$$

Since we can find γ-arcs inside Ω which tend to inifinity one cannot appeal to the ordinary maximum principle and conclude that the absolute value $|f(z)|$ stays below m^* for all $z \in \Omega$. However, if the maximum principle fails then the function

$$\omega_f(\theta) = \max_r |f(re^{i\theta})| \quad : re^{i\theta} \in \Omega$$

cannot stay too small. Lindelöf gave examples of non-constant entire functions f where the ω_f-functions is bounded for all θ-angles outside a null-set. By scrutinizing how these Lindelöf-functions increase, Carleman established the following in the article *Sur l'éxtension d'un theoréme de Liouville*: *When the γ-arcs are straight* *half-lines so that* $\Omega = \{\alpha < \arg z < \beta\}$ *is a sector and the maximum principle fails, then one must have a divergent integral*:

$$\int_\alpha^\beta \log\,\log^+ \omega_f(\theta)\; d\theta = +\infty$$

Above a double log-function appears so the the divergence forces the ω_f-function to be very large in the avergage. A proof is given in § XX and in § XX we also expose an improved version due to Beurling where the pair of γ-curves are of a more general type than straight lines.

Carleman's work on PDE-equations.

The student in analysis is confronted with several disciplines where analytic function theory and Fourier analysis is one aspect, and PDE-theory another. As expected the interaction is close and many results in PDE-theory use complex methods and Fourier analysis. Let us describe one of Carleman's results whose proof rely upon methods from all these disciplines.

Let Ω be a bounded open domain in $\mathbf{R}^3$ with a C^1-boundary. We are given a symmetric 3×3-matrix $A(x)$ whose elements $\{a_{pq}(x)\}$ are real-valued continuous functions on the closure $\bar{\Omega}$. Let b_1, b_2, b_3 and c be four other real-valued continuous functions on $\bar{\Omega}$. We get the differential operator L acting on C^2-functions $u(x)$ by

$$L(u) = \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} a_{pq}(x) \frac{\partial^2 u}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} b_p(x) \frac{\partial u}{\partial x_p} + c(x)u$$

Assume that L is elliptic which means that there is a constant $\rho > 0$ such that

$$\min_\xi \,|\, \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} a_{pq}(x) \cdot \xi_p \xi_q \,| \geq \rho$$

where the minimum is taken over all $x \in \bar{\Omega}$ and every real ξ-vector of unit length. Fredholm's classical theory about integral equations implies that the eigenvalues λ for which there exists a non-zero function u such that

$$L(u) + \lambda \cdot u = 0 \quad \text{and} \quad u|\partial\Omega = 0$$

is a discrete set of complex numbers. When L is symmetric these eigenvalues are real and tend to $+\infty$ and results about their asymptotic distribution were established by Weyl and Courant. In the article [Carleman 1936] the asymptotic expansion was extended to cover non-symmetric elliptic operators. The result from [ibid] goes as follows where we for each $x \in \Omega$ introduce the non-zero value of the determinant of $A(x)$ which by the elliptic condition may be taken to be everywhere positive.

Theorem. *Let* $|\lambda_n|$ *be the absolute value of the n:th eigenvalue ordered in a non-decreasing sequence. Then*

$$|\lambda_n| \simeq \Big[\, \frac{1}{6\pi} \int_\Omega \frac{dx}{\det A(x)} \,\Big]^{-\frac{2}{3}} \cdot n^{2/3}$$

Remark. The $\simeq$-sign means that the difference of the terms above are of small ordo $n^{2/3}$ as $n \to +\infty$. Notice that the asymptotic formula is the same as in the symmetric case, i.e. the lower order differentials of L do not affect the asymptotic distribution of eigenvalues. This is no surprise since one can easily show that even if eigenvalues can be complex, they are asymptotically real. More precisely, when the elliptic operator L is given, then there are positive constants c_1, c_2 such that $\mathfrak{Re}\,\lambda \geq -c_1$ and

$$\big|\mathfrak{Im}\,\lambda\,| \leq c_2 \cdot \sqrt{\mathfrak{Re}\,\lambda + c_1}$$

hold for all eigenvalues.

Work by Arne Beurling.

Several sections expose results by Beurling which not only offer striking theorems but have the merit that the proofs are both highly original and instructive. Many of Beurling's methods have led to new areas in analytic function theory, harmonic analysis and potential theory. We have not tried to include his deepest results which for example appear in joint articles with Lars Ahlfors and Paul Malliavin. For a more detailed account we refer to [Beurling:Collected Works 1-2] whose foreword contains a very informative text written by Lars Ahlfors and Lennart Carleson. A feature in Beurling's work is the mixture of geometric constructions and "hard calculus". An example is the notion of extremal distances which Beurling introduced in 1929 and applied in many different situations, foremost to estimate harmonic measures. Below follows a description of some major results in his thesis *Études sur un problème de majoration* form Uppsala in 1933. Personally I find Beurling's proofs in [ibid] utmost intstructive and transparent. So [ibid] constitutes a veritable classic in analytic function theory. The results below speak for themselves, though the less experienced reader may consider the subsequent material as a "bird's view" and whenever necessary consult Chapters for background about concepts which are used without hesitation below.

A. The Milloux problem.

Consider an analytic function $f(z)$ in the unit disc where $|f(z)| < 1$ for all z and suppose there exist some $\delta > 0$ such that

$$\min_\theta\, |f(re^{i\theta})| \leq \delta \quad \forall\ 0 \leq r < 1$$

Then it follows that

$$(*) \qquad \max_\theta\, |f(re^{i\theta})| \leq \delta^{\frac{2}{\pi}\cdot \arcsin\, \frac{1-r}{1+r}} \quad : \ 0 < r < 1$$

Prior to this result M.E Schmidt had proved (*) under the constraint that there exists a Jordan arc from the origin to the unit circle along which $|f|$ is $\leq \delta$ in which case one profits upon a suitable conformal mapping which in a simllar context had been used by Koebe. To prove (*) in the general case Beurling introduced totally new ideas.

Minimum modulus of analytic functions.

Chapter XX in [ibid] studies the minimum modulus of an analytic function. When $f(z)$ is analytic in a disc $\{|z| < R\}$ we set

$$m_f(r) = \max_{|z|=r} |f(z)| \quad : \mu_f(r) = \min_{|z|=r} |f(z) \quad : 0 < r < R$$

The following is proved in [ibid: page 94]:

Theorem. *Let $0 < r_1 < r_2 < R$ and $\alpha > 0$ satisfies $0 < \alpha \leq m_f(r_1)$. Then*

$$(**) \qquad \int_{r_1}^{r_2} \chi(\{\mu_f \leq \alpha\} \cdot \log r\, dr \leq \log 4 + 2 \int_{r_1}^{r_2} \log\log \frac{m_f(r)}{\alpha}\, dr$$

where $\chi(\{\mu_f \leq \alpha\})$ is the characteristic function of the set where $\mu_f(r) \leq \alpha$.

Remark. In the article [xxx] from 1933 Nevanlinna found another proof of (*) above which has the merit that it consolidates why (*) is sharp. Using the constructions by Nevanlinna, Beurling could then demonstrate that the inequality (**) gets sharp when α tends to zero which is the interesting part of (**) above.

Upper bounds for harmonic measures

Let D be a simply connected domain and $\gamma \subset \partial D$ a subarc. If $z \in D$ we denote by $r(z; \partial D)$, resp. $r(z, \gamma)$ the euclidian distance from z to the boundary and to γ respectively. With these notations Theorem III : page 55 in [ibid] asserts that

$$(***) \qquad \mathfrak{m}(D, \gamma; z) \leq \frac{4}{\pi} \cdot \arctan \sqrt{\frac{r(z, \partial D)}{r(z, \gamma)}}$$

where the left hand side is the harmonic measure at z with respect to γ. Examples show that this upper bound is sharp. Next, an inequality, which in a slightly weaker form was established in Beulring's thesis to demonstrate the Denjoy conjecture, was refined later in the article [xx] from 1940 and goes as follows: Let Ω be a Jordan domain and $\omega \subset \partial\Omega$ a sub-arc of its boundary. Let $z_0 \in \Omega$ and suppose there exists a positive harmonic function ψ in Ω satisfying:

$$\psi(z_0) = 0 \quad : A = \iint_\Omega |\nabla(\psi)|^2\, dxdy < \infty$$

Theorem. *Set* $\min_{z \in \omega} |, \psi(z) = L$. *Then one has the inequality*

$$\mathfrak{m}(\Omega; \omega)(z_0) \leq e^{-\pi \cdot \frac{L^2}{A}}$$

Remark. A detailed proof appears in section §§ from Special Topics.

Regular points for the Dirichlet problem.

Sections in [ibid:page 63-69] are devoted to Dirchlet's problem. Let Ω be an open subset of $\mathbf{C}$. If f is a real-valued and continuous function on $\partial\Omega$ it can be extended to some $F \in C^0(\bar{\partial}\Omega)$. Now Ω can be exhausted by an increasing sequence of subdomains $\{\Omega_n\}$ for which the Dirichlet problem is solvable. Such exhaustions were considered by Lebesgue and explicit constructions were given by de Vallé Poussin in 1910. To each n we find the harmonic function H_n in Ω_n which solves the Dirichlet problem with boundary values given by the restriction of F to $\partial\Omega_n$. In Chapter V we prove Wiener's result that the sequence $\{H_n\}$ converges to a unique harmonic function W_f in Ω which is independent of the chosen exhaustion. One refers to W_f as Wiener's generalised solution. A boundary point z_0 is called regular if

$$\lim_{z\to z_0} W_f(z) = f(z_0) \quad : \forall\, f \in C^0(\partial\Omega)$$

A criterion for a boundary point to be regular was established by Boulignad in 1923. Namely, a boundary point z_0 is regular if and only if there exists a positive harmonic function V in Ω such that

$$\lim_{z\to z_0} V(z) = 0$$

This condition is rather implicit so one seeks geometric properties to decide if a boundary point is regular or not. In [ibid] Beurling established a sufficient regularity condition which goes as follows: Let $z_0 \in \partial\Omega$ and for a given $R > 0$ we consider the circular projection of the closed complement of Ω onto the real interval $0 < r < R$ defined by:

$$E_\Omega(0,R) = \{0 < r < R : \exists\, z \in \{|z - z_0| = r\} \cap \mathbf{C} \setminus \Omega\}$$

Following Beurling one says that z_0 is logarithmically dense (Point frontière de condensation logarithmique) if the integral

$$\int_{E_\Omega(0,R)} \log r\, dr = +\infty \tag{1}$$

Beurling proved that if (1) is divergent then z_0 is a regular boundary point. The proof relies upon an inequality which has independent interest. Here is the situation considered in [ibid: page 64-66]: Let Ω be an open set - not necessarily connected and $R > 0$ where Ω is general contains points of absolute value $\geq R$. Consider a harmonic function U in Ω with the following properties:

$$\limsup_{z\to z_*} U(z) \leq 0 \quad : \; z_* \in \partial\Omega \cap \{|z| \leq R\} \tag{i}$$

$$U(z) \leq M \quad : \; z \in \{|z| = R\} \cap D \tag{ii}$$

Theorem. *When (i-ii) hold one has the inequality below for every* $0 < r < R$

$$\max_z U(z) \leq 2M \cdot e^{-\frac{K}{2}}$$

where the maximum in the left hand side is taken over $\Omega \cap \{r < |z| < R\}$ *and*

$$K = \int_{E_\Omega(r,R)} \log r\, dr$$

Extremal length.

Let us first cite Beurling where he gives the following attribute to original work by Poincaré: *Rappelons que dans la théorie des fonctions analytiques, on a introduit des élements géometriques non euclidiennes, invariants par rapports a certaines transformation, et cela surtout pour simplifier la thérie dont il s'agit.* In his lecture at the Scandinavian Congress in Copenhagen 1946, Beurling describes in more detail the usefulness of extremal lengths.

In geometric function theory one often tries to characterize or determine a certain mapping or quantity by an extremal property. The method goes back to Riemann who introduced variational methods in function theory in the form of Dirichlet's principle. When you want to characterize a function by an extremal property , then the class of competing functions is very important. The wider you can make this class the more you can say about the extremal function and the easier it becomes to find good majorants or minorants, as the case may be.

In his thesis Beurling defined an extremal distance between pairs of points z_0, z_1 in simply conected domains Ω with a finite area. The construction goes as follows: First the interior distance is defined by

$$\rho(z_0, z_1; \Omega) = \inf_\gamma \int_\gamma |dz|$$

where the infimimum is taken over rectifiable Jordan arcs which stay in Ω and join the two points. Set

$$\lambda_*(z_0, z_1; \Omega) = \sqrt{\frac{\pi}{\text{Area}(\Omega)}}| \cdot \rho(z_0, z_1, \Omega)$$

One refers to λ_* as the reduced distance. It is not a conformal invariant and to overcome this default Beurling considered the family of *all triples* (Ω^*, z_0^*, z_1^*) which are conformally equivalent to the given triple, i.e. there exists a conformal mapping $f\colon \Omega \to \Omega^*$ such that $f(z_\nu) = z_\nu^*$. The *extremal distance* is now defined by

$$\lambda(z_0, z_1; \Omega) = \sup \lambda_*(z_0^*, z_1^*; \Omega^*)$$

with the supremum taken over all equivalent triples. By this construction λ yields a conformal invariant. Next, consider a triple (z_0, z_1, Ω) is the Green's function $G(z_0, z_1; \Omega)$. Recall that G is a symmetric function of the pair z_0, z_1 and keeping z_1 fixed

$$z \mapsto G(z, z_1; \Omega) - \log \frac{1}{z - z_1|}$$

is a harmonic function Ω which is zero on $\partial\Omega$. In [ibid: Théorème 1: page 29] the following fundamental result is proved:

Theorem. *For each triple* $(z_0, z_1; \Omega)$ *one has the equality*

$$(*) \qquad e^{-2G} + e^{-\lambda^2} = 1$$

About the proof. By conformal invariance it suffices to prove the equality for a triple $(0, a, D)$ where D is the unit disc where the pair of points is the origin and a real point $0 < a < 1$. In this case

$$G = \log \frac{1}{a}$$

Hence the theorem amounts to prove the equality

$$e^{-\lambda^2} = 1 - a^2$$

or equivalently that

$$(*) \qquad \lambda^2 = \log \frac{1}{1-a^2}$$

We will show (*) in § XX. The equation has several applications but at this monent we refrain form discussing more details.

General extremal metrics.

Ten years later Beurling realised the need for a more extensive class of extremal distances which can be applied for domains which are not simply connected. The constructions below appear in the section *Extremal Distance and estimates for Harmonic Mesure* [Collected work. Vol 1. page 361-385].

The numbers $\lambda(E,K;\Omega)(z_0)$. Let Ω be a Jordan domain in the complex z-plane and consider a pair of sets E,K where $E \subset \partial\Omega$ and K is a compact subset of Ω. To a point $z_0 \in \Omega \setminus K$ we introduce the family $\mathcal{J}(E,K;z_0)$ of rectifiable Jordan arcs γ with the following properties: Apart from its end-points γ stays in $\Omega \setminus K$ and passes through z_0. Moreover, the end-points of γ divides the Jordan curve $\partial\Omega$ into a pair of closed intervals ω_1 and ω_2 and the last constraint on γ is that E is contained in one of these ω-intervals. Notice that neither K or E are assumed to be connected. Next, let $\mathcal{A}$ be the family of positive and continuous functions ρ in Ω for which the squared area integral

$$\iint_\Omega \rho^2(x,y)\,dxdy = 1$$

To each such ρ-function we set

$$(*) \qquad L(\rho) = \inf_{\gamma \in \mathcal{J}} \int \rho(z)\,|dz|$$

where we for simplicity put $\mathcal{J} = \mathcal{J}(E,K;z_0)$. The extremal distance between E and z_0 taken in $\Omega \setminus K$ is defined by

$$(**) \qquad \lambda(E,K;\Omega)(z_0) = \sup_{\rho \in \mathcal{A}} L(\rho)$$

These λ-numbers are conformal invariants. More precisely, let (E^*,Ω^*,K^*) be another triple and $f\colon \Omega \setminus K \to \Omega^* \setminus K^*$ is a conformal mapping which extends continuously to the boundary and gives a homeomorphism between the closed Jordan arcs $\partial\Omega$ and $\partial\Omega^*$ where $E^* = f(E)$. Then one has the equality

$$\lambda(E,K;\Omega)(z_0) = \lambda(E^*,K^*;\Omega^*)(f(z_0))$$

Consider a triple $(E;\Omega,K)$ as above and when $z_0 \in \Omega \setminus K$ there exists the harmonic measure $\mathfrak{m}(E;\Omega,K)(z_0)$.

Theorem. *For every triple* $E;\Omega,K)$ *and each* $z_0 \in \Omega \setminus K$ *where* $\lambda(E;\Omega,K)(z_0) \geq 2$ *one has the inequality*

$$\mathfrak{m}(E;\Omega,K)(z_0) \leq 3\pi \cdot e^{-\pi \cdot \lambda(E;\Omega,K)(z_0)}$$

This result is often used to estimate harmonic measures. The point is that in the right hand side we can use ρ-functions which need not maximize the L-functional in (**) and since $L(\rho) \leq \lambda(E;\Omega,K)(z_0)$ one has in particular

$$\mathfrak{m}(E;\Omega,K)(z_0) \leq 3\pi \cdot e^{-\pi \cdot L(\rho)}$$

Examples in § XX demonstrate the usefulness of such majorisations.

The simply connected case. Here Ω is a Jordan domain and $K = \emptyset$. Consider a pair of closed subsets E and F in $\partial\Omega$ which both are a finite unions of closed subintervals. Denote by $\mathcal{J}(E,F)$ the family of Jordan arcs γ with one end-point in E and the other in F while the interior stays in Ω. Set

$$\lambda(E;F:\Omega) = \max_{\rho} \min_{\gamma \in \mathcal{J}} \int_{\gamma} \rho \cdot d|z| \quad : \ \rho \in \mathcal{A}$$

This yields a conformal invariant which can be determined under the extra condition that the sets E and F are *separated* which means that there exists a pair of points p,q in $\partial\Omega \setminus E \cup F$ which divide $\partial\Omega$ in two intervals where E belongs to one and F to the other. Under this condition one has the equality

$$\lambda(E;F:\Omega) = \mathcal{NS}(E;F:\Omega) \tag{1}$$

Here $\mathcal{NS}(E;F:\Omega)$ is the Neumann-Schwarz number associated to the triple $(E,F;\Omega)$ which is found as follows: Let u be the harmonic function in Ω with boundary values $u=1$ on E and zero on F while the normal derivative $\frac{\partial\phi}{\partial n} = 0$ on $\partial\Omega \setminus E \cup F$. If v is the harmonic conjugate then v is increases on E and decreases on F and when v is normalised so that the range $v(E)$ is an interval $[0,h]$ it follows that $f = u + iv$ is a conformal mapping from Ω onto a rectangular slit domain:

$$\{0 < x < 1\} \times \{0 < y < h\} \cup S_\nu$$

where $\{S_\nu\}$ are horizontal intervals directed into the rectangle and with end-points on one of the vertical sides $\{x = 0\}$ or $\{y = 0\}$. See § XX for a figure. The Neumann-Schwarz number is given by

$$\mathcal{NS}(E,F;\Omega) = \frac{1}{\sqrt{h}}$$

In § xx we prove that it is a conformal invariant which reduces the proof of (1) to the case when Ω is a rectangle as above with some removed spikes while E and F are the opposed vertical lines. If ρ^* is the constant function $\frac{1}{\sqrt{h}}$ the fact that horisontal straight lines minimize arc-lengths when we move from E to F imply that

$$L(\rho^*) = \frac{1}{\sqrt{h}} \tag{2}$$

On the other hand, let $\rho \in \mathcal{A}$ be non-constant. Now

$$0 < \iint_{\Omega} (\rho - \frac{1}{\sqrt{h}})^2 dxdy = 2 - \frac{2}{\sqrt{h}} \iint_{\Omega} \rho \, dxdy \implies \iint_{\Omega} \rho \, dxdy < \sqrt{h}$$

It follows that

$$\min_{0 \leq y \leq h} \int_0^1 \rho(x,y) \, dx < \frac{1}{\sqrt{h}}$$

In the left hand side appear competing γ-curves and hence

$$L(\rho) < \frac{1}{\sqrt{h}}$$

This proves that ρ^* maximizes the L-function and the equality (2) entails (1).

Another inequality. Let Ω be a bounded and connected domain whose boundary consists of a finite family of disjoint closed Jordan curves $\{\Gamma_\nu\}$. Consider a pair (z_0, γ) where $z_0 \in \Omega$ and γ is a Jordan arc starting at z_0 and with an end-point p on one boundary curve, say Γ_1. Next, let K be a compact subset of $\Gamma_1 \setminus \{p\}$. For each point $z \in \gamma$ we get the harmonic measure $\mathfrak{m}_K(z)$ which is the value at z of the harmonic function in Ω whose boundary values are one on K and zero in $\partial\Omega \setminus K$. Set

$$h^*(K, \gamma; z_0) = \max_{z \in \gamma} \mathfrak{m}_K(z)$$

The interpretation is that one seeks a point $z \in \gamma$ where the probability for the Browninan motion which starts at z and escapes at some point in K before it has reached points in $\partial\Omega \setminus K$, is as large as possible. An upper bound of h^* is found using the notion of extremal length. Namely, let $\lambda(K; \gamma)$ be the extremal length for the family of curves in Ω with one end-point in K and the other on γ. Then Beurling proved that

$$(*) \qquad h^*(K, \gamma; z_0) \leq 5 \cdot e^{-\pi \cdot \lambda(K;\gamma)}$$

Remark. In random walks under a Brownian motion the inequality above shows that if K is a large obstacle and γ small, then the h^*-function can be majorised if some lower bound for the extremal length can be proved. This is of interest in configurations with "narrow channels" when one seeks paths out to the boundary and illustrates a typical application of extremal lengths.

On quasianalytic series.

Let us finish with a result which illustrates Beurling's vigour in "hard analysis". The theorem below was presented in a lecture at the Scandinvian congress in Helsinki 1938. A closed subset E of $\mathbf{C}$ has positive perimeter if there exists a positive constant c such that

$$\sum \ell(\gamma_\nu) \geq c$$

hold for every denumerable famly of closed and rectifiable Jordan curves $\{\gamma_\nu\}$ which surround E, i.e. E is contained in the union of the associated open Jordan domains. Let $\{\alpha_k\}$ be a sequences of distinct complex numbers and $\{A_k\}$ another sequence sequence such that

$$\text{(i)} \qquad \sum |A_k| < \infty$$

Set

$$(*) \qquad F(z) = \sum \frac{A_k}{z - \alpha_k}$$

To each k we put $r_k = |A_1| + \ldots |A_k|$ and define $\mathcal{E}_f$ by

$$\mathcal{E}_f = \{z \in \mathbf{C} \quad : \sum_{k=2}^{\infty} \frac{A_k}{\sqrt{r_{k-1}}} \cdot \frac{1}{|z - \alpha_k|} < \infty\}$$

It is easily seen that (i) entails that the complement of $\mathcal{E}_f$ is a null set in $\mathbf{C}$. One says that (*) is quasi-analytic if $F(z)$ cannot vanish on a closed subset E of $\mathcal{E}_F$ with positive perimeter. The following was proved in [ibid];

Theorem. *The series (*) is quasi-analytic in the sense above if*

$$\liminf_{n \to \infty} r_n^{\frac{1}{n}} < 1$$

In § XX we expose some steps in the proof based upon some remarkable inequalities for rational functions.

A final comment

Carleman and Beurling studied at Uppsala University, where Carleman entered in 1911 and Beurling in 1924. Both had Erik Holmgren and Anders Wiman as teachers and to this one can add that as young researchers they also met Ernst Lindelöf whose classic text-book *Théorie des Residues* from 1907 was studied at an early stage by them both. The major contents in Beurling's Ph.D-thesis was already written in 1929 and presented on November 4 1933. At that time Carleman had served during five years as director at Institute Mittag-Leffler. In the period 1933-1937 Beurling attended Carleman's lectures at the Mittag-Leffler institute and got considerable inspiration from this. One can mention Carleman's lectures about the generalised Fourier transform in 1935 and refined versions of Ikehara's theorem and other Tauberian results. A few years later Beurling proved Tauberian theorems with remaineder terms. There was never a "competetive situation" during the years when Beurling prepared his application for the chair at Uppsala University replacing Homgren after his retirement in 1937. On the contrary, preserved letters between Carleman and Beurling show that they esteemed each others work. For example, in his thesis Beurling inserted the following comments upon Carleman's new proof from 1933 of the Denjoy conjecture with a sharp bound $2n$ for entire functions of order n: *Plus récemmment M. Carleman en perfectionnant sa méthode initiale est arrivé au resultat* $\leq 2n$ *d'une manière fort élegante.* Beurling's proof from 1929 has like Carleman's proof the merit that it can be adapted to other cases than asymptotic values. An example is the following remarkable result from his thesis:

Theorem *Let $f(z)$ be a bounded analytic function in the half space $\mathfrak{Re}\, z > 0$. Set*

$$\mu(r, f) = \min_{-\pi/2<\theta<\pi/2} |f(re^{i\theta})|$$

Then f converges uniformly to zero in every sector $-\pi/2 + \delta \leq \theta \leq \pi/2 - \delta$ under the condition that

$$\lim_{r\to\infty} \mu(r, f) = 0$$

Carleman's last major article At the end of 1930:s Carleman's health became weak and from this time Beurling took the leading role in Swedish mathematics. His work after 1940 contains a wealth of new results where one may mention his ingenious idea to relate inner functions in the unit disc with closed subspaces of the Hilbert space $H^2(T)$ which are invariant under multiplication with z. While reading articles from their collected work I always get fascinated by the transparency and generality while they announce and prove deep results. I can only hope that the sections in these notes which expose material from their work may inspire readers to continue studies of articles by these eminent mathematicians.

The use of computers.

Today's student is confronted with another world than in the past since computers offer numerical solutions and provide figures which can be illustrated in a mobile way. This is a veritable revolution and one may even ask if traditional text-books in mathematics are out-dated. Hopefully the answer is a compromise, i.e. pure theory should help to create more accurate computer programs and conversely computers contribute to new theoretical discoveries. Apart from the standard programs *Matlab, Mathematica* and *Maple* we recommend the excellent programs delivered by *Comsol*. Here refined numerical mathematics has been developed during the last decade which in particular are suitable for numerical solutions to non-linear PDE:s. The computer offers accurate numerical solutions to both linear and non-linear boundary value problems. An example is the boundary value problem where one seeks a function $u(x, y)$ in the open unit disc which satisfying the non-linear PDE-equation

$$\Delta(u) + u(u - 1)$$

and a boundary condition

$$u(e^{i\theta}) = P(\theta)$$

where P for example is a trigonometric polynomial $A\cos^{\theta} + \sin^2\theta$ for some positive constant $A > 1$. The solution u will be subharmonic in the region $\{u < 1\}$ while it is super-harmonic in $\{u > 1\}$. To find critical points in the open disc where the gradient vector $\nabla(u) = (0, 0)$ one must rely upon the numerial solutions based upon the computer's efficency to solve non-linear PDE-equations by the method of finite elements in fine grids.

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0.1 The isoperimetric inequality.

Many problems which from the start are given in a real context are solved via complex analysis. An example is the isoperimetric inequality which asserts that if C is a simple closed curve in $\mathbf{R}^3$ of length L, then the area of the minimal surface bordered by C is majorized by $\frac{1}{4\pi} \cdot L^2$ and equality holds only if C is a planar circle. The proof relies upon a result established by Weierstrass in 1861 which shows that every minimal surface is parametrized via an analytic function defined on a simply connected domain. More precisely, let M be the minimal surface bordered by C and (x, y, z) denote coordinates in $\mathbf{R}^3$. After a change of variables which includes sterographic projection and Riemann's conformal mapping theorem, Weierstrass proved there exists an analytic function $F(u)$ defined in the open unit disc D where u is the complex coordinates such that points (x, y, z) in M are given by the equations

$$x = \mathfrak{Re} \int_0^u (1 - \zeta^2) F(\zeta) d\zeta : \ y = \mathfrak{Re} \int_0^u i(1 + \zeta^2) F(\zeta) d\zeta : z = \mathfrak{Re} \int_0^u 2\zeta F(\zeta) d\zeta$$

From this the sharp isoperimetric inequality boils down to prove the following inequality for every pair of analytic functions f, g in the unit disc:

$$(*) \qquad \iint_D |f(z)| \cdot |g(z)| \, dxdy \leq \frac{1}{4\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta \cdot \int_0^{2\pi} |g(e^{i\theta})| \, d\theta$$

That (*) holds was demonstrated by Carleman in an article from 1921. One reason that Weierstrass himself did not find the sharp isoperimetric inequality might be that the Blaschke-Jensen factorization was not known until 1910. In § XX from Special Topics we shall learn that this factorisation leads to the proof of (*).

Another example of an isoperimetric inequality related to function theory goes as follows: Let $r(z) = \frac{p(z)}{q(z)}$ be a rational function where the polynomials p, q have no common root. No hypothesis is imposed on their degrees. Let $D = \{|z| < 1\}$ be the open unit disc and for each real number t we put

$$D_r(t) = \{z \in D : \ |r(z)| < t\}$$

The boundary of $D_r(t)$ is a union of Jordan arcs and some finite set of corner points. In this family occur subarcs of the unit circle. Denote by $\lambda_r(t)$ the sum of their lengths while $L_r(t)$ is the sum the lengths of the boundary arcs contained in D. With these notations one has the inequality

$$(*) \qquad L_r(t) \geq 2 \sin \frac{\lambda_r(t)}{2}$$

In §§ we shall prove (*) together with other inequalities about rational functions which are due to Beurling. Here geometry, variational inequalities and analytic function theory appear in a most fruitful manner.

0.2 Neumann's boundary value problem.

In the pioneering article *Untersuchungen über das Logarithmische und Newtonsche Potential* from 1877, Gustav Neumann introduced operator-valued meromorphic series to solve the boundary value problem

$$\frac{\partial U}{\partial \mathbf{n}_*}(p) = h(p) \cdot U(p) + f(p) \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^3$, U is harmonic in Ω and (1) hold for points p on the boundary and $\frac{\partial U}{\partial \mathbf{n}_*}$ denotes the inner normal derivative. Above h is assumed to be a positive continuous function on $\partial\Omega$ while f is an arbitrary real-valued continuous function. Neumann proved existence and uniqueness for each such pair (h, f) when Ω is a bounded and strictly convex domains with a C^2-boundary. His work was later resumed by Poincaré whose article [Acta:20: 1897] *La méthode de Neumann et le problème de Dirichlet* where existence and uniqueness was proved under the solve assumption that $\partial\Omega$ is of class C^2, i.e. the convexity assumption is not neeeded. In § xx we shall consider neumann's boundsry value problem in dimension two while the 3-dimensional case is treated in § xx.

The inhomogeneous Laplace equation. We discuss the 2-dimensional case where Ω is a connected and bounded bounded domain in $\mathbf{R}^2$ whose boundary is a finite union of pairwise disjoint and differentiable closed Jordan curves. Under this condition the Diriclhet probelm is sovlable, i.e., for every $h \in C^0(\partial\Omega)$ there exists a unique harmonic function H in Ω whichextends to a continuous function on the closure and $H = h$ on $\partial\Omega$. This will be proved in § xx from Chapter V. In particular we take a point $p \in \Omega$ and find the harmonic function $H_p(q)$ in Ω such that

$$H_p(q) = \log |p - q| \quad : q \in \partial\Omega$$

The Greens' function is defined for pairs (p, q) in in $\Omega \times \Omega$ with $p \neq q$ by

$$G(p, q) = \log \frac{1}{|p - q|} - H_p(q)$$

In § xx from Chapter V we prove that G is symmetric, i.e. $G(p, q) = G(q, p)$. Next one gets a nintegral operator $\mathcal{G}$ defined by

$$\mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p, q) \cdot f(q)\, dq$$

So above $q = (x, y)$ and $dq = dxdy$. For each fixed $p \in \Omega$ the function $q \mapsto G(p, q)$ is integrable over Ω and from this we shall learn that (x) is defined for fun cions $f \in L^2(\Omega)$, i.e. complex-valued square integrable functions in the sense of Lebesgue. Moreover one easily verifies that

$$\iint_{\Omega \times \Omega} |G(p, q)|^2\, dpdq < \infty$$

It means that $\mathcal{G}$ is a Hilbert-Schmidt operator and in particular compact. Next, we can apply the Laplce operator and a crucial resut is that for every $f \in L^2(\Omega)$ it holds that $\Delta(\mathcal{G}_f)$ exists in the distribution sense and is equal to $-f$ in the Hilbert space $L^2(\Omega$, i.e. one has the operator equation

$$\Delta \circ \mathcal{G} = -E$$

where E is the identity operator on $L^2(D)$. In §§ we shall learn that $L^2(\Omega$ has an orthonormal basis $\{\phi_n\}$ where

$$\frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot \phi_n(q)\, dq = \lambda_n^{-1} \cdot \phi_n(p)$$

and $\{\lambda_n\}$ is a non-decreasing sequence of positive real numbers which tend to $+\infty$ as $n \to \infty$. From (x) we see that (xx) means that

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0$$

holds in Ω. Moreover, by the construction of $\mathcal{G}$ one verifies that each ϕ_n extends to a contionuous function on the closure with boundary value equal to zero.

Asymptotic results. The resuts above are due to Neumann and Poincaré and sved as a model in more general studies of integral operatoes by Fredholm and Hilbert. A regined study of the asymptotic behaviour of the eigenvalues $\{\lambda_n\}$ was later established by Weyl who proved the following limit formula for every domain Ω as above:

(*) $$\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{\text{area}(\Omega)}$$

Using some further refinements in Weyl's proof and a general Tuaberian theorem due to Wiener the following result was proved by Carleman:

Theorem. *For each $p \in \Omega$ one has*

$$\lim_{n \to \infty} \frac{1}{n} \cdot \sum_{k=1}^{k=n} \phi_k^2(p) = \frac{1}{\text{area}(\Omega)}$$

—noindent The proolf of this theorem is given in § xx where the crucial point is to study the Dirichlet series below for each $p \in \Omega$:

$$\Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

It is easily seen that $\Phi_p(s)$ is analytic in a half-space $\mathfrak{Re}\, s > b$ for a large b. Less trivial is that there exists an entire function $\Psi_p(s)$ such that

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Hence $\Phi_p(s)$ is meromorphic in the complex s-plane with a simple pole at $s = 1$. The proof that (xx) holds is given in § xx and offers an instructive lesson in analytic functiuon theory where residue caluculs and changes of contioours appear. We remark that this elegant procedure goes back to constructions by Riemann whose original work paved the way towards many discoveries. Concernibngt Theorem xx we also mention that it can be used to analyze asymptotic behaviour of heat-eqautions with related to vibrating membrans. See § xx for an account.

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We the proof in § which uses Ikehara's Theorem. So here "pure real analysis" and Dirichlet series with a complex parameter intervene during the proof. By a further study of estimates for Green's resolvents, Carleman proved that the eigenfunctions and all their higher derivatives also satisfy a limit formula. The result is that for each pair of non-negative integers j, m and every $p \in \Omega$ one has

$$\lim_{n \to \infty} \frac{1}{\lambda_n^{j+m+1}} \cdot \sum_{k=1}^{k=n} \big(\frac{\partial^{j+m}\phi_k}{\partial x^j \partial y^j}\big)^2(p) = \frac{1}{\pi \cdot 2^{2m+2j+2}} \cdot \frac{(2m)! \cdot (2j)!}{m! \cdot j! \cdot (m+j+1)!}$$

So in particular

$$\lim_{n \to \infty} \frac{1}{n} \cdot \sum_{k=1}^{k=n} \phi_k^2(p) = \frac{1}{\text{area}(\Omega)}$$

where a notable point is that the limit is independent of p. Background to the proofs of the results above is covered by material in the chapters where the ordinary Dirichlet problem is solved and Green's functions are constructed in Chapter 5, while material related to harmonic analysis and spectral theory which is needed to get Ikehara's theorem is covered by an appendix devoted to functional analysis.

Poincaré solved the boundary value problem as follows: Given a positive continuous function h on $\partial\Omega$ he associated the kernel function $K_h(p, q)$ defined outside the diagonal of $\partial\Omega \times \partial\Omega$ by

$$K_h(p, q) = \frac{1}{4\pi} \cdot \frac{\langle p - q, \mathbf{n}_*(q) \rangle}{|p - q|^3}$$

where inner product between the vectors $p - q$ and the inner normal $\mathbf{n}_*(q)$ appear in the denominator while the third power of the euclidian distance between points p, q in $\partial\Omega$ appear in the numerator. With a complex parameter λ one gets the integral equatiion

$$u(p) - \lambda \cdot \int_{\partial\Omega} K_h(p, q) u(q) \cdot d\sigma(q) = f(q) \tag{2}$$

where $d\sigma(q)$ denotes the area measure on $\partial\Omega$. It means that for each $f \in C^0(\partial\Omega)$ one seeks $u \in C^0(\partial\Omega)$ such that (2) holds. If u solves (2) when $\lambda = 1$, then Green's formula shows that the solution U to (1) is found via Newton's harmonic extension of u, i.e.

$$U(p) = \int_{\partial\Omega} \frac{1}{p - q|} \cdot u(q)\, d\sigma(q) \quad : \quad p \in \Omega \tag{3}$$

The merit in the approach by Neumann and Poincaré is that these authors realized the importance of studying the operator-valued function

$$\lambda \mapsto E - \lambda \cdot \mathcal{K}_h$$

where $\mathcal{K}_h$ is the integral operator defined by the kernel function $K_h(p, q)$. For domains whose boundary is sufficiently regular $\mathcal{K}_h$ is a bounded linear operator and to solve (2) one considers the resolvent

$$R_h(\lambda) = (E - \lambda \cdot \mathcal{K}_h)^{-1} \tag{*}$$

Poincaré proved that for a domain Ω with C^2-boundary, this operator-valued function extends to a meromorphic function of λ defined in the whole complex plane whose poles are real and simple. He also demonstrated that no pole occurs when $\lambda = 1$ which gives the requested solution u to (2) given by

$$u = R_h(1)(f)$$

Inspired by the Neumann-Poincaré methods to handle boundary value problems, Fredholm and Hilbert studied the eigenvalue problem where one seeks functions u in a domain Ω as above such that

$$\Delta(U) + \lambda \cdot U = 0 \quad \text{and} \quad U|\partial\Omega = 0 \tag{1}$$

In § xx we shall treat this in the planar case where $\partial\Omega$ consists of a finite family of disjoint and closed Jordan curves of class C^1. The first fact is that non-trivial solutions to (1) only occur when λ is real and positive and using general facts about integral equations, Fredholm and Hilbert demonstrated that the eigenvalues for a discrete sequence $\{0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$ where eventual eigenvalues whose eigenspace has dimension > 1 is repated according to this multiplicity. The associated eigenfunctions $\{\phi_k\}$ are real-valued and can be chosen so that they form an orthonormal set in the Hilbert space $L^2(\Omega)$. The search for asymptotic formulas of the eigenvalues became a natural issue and was settled by Weyl who proved the limit formula:

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{\text{area}(\Omega)}$$

We give the proof in § which uses Ikehara's Theorem. So here "pure real analysis" and Dirichlet series with a complex parameter intervene during the proof. By a further study of estimates for Green's resolvents, Carleman proved that the eigenfunctions and all their higher derivatives also satisfy a limit formula. The result is that for each pair of non-negative integers j, m and every $p \in \Omega$ one has

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{j+m+1}} \cdot \sum_{k=1}^{k=n} \Big(\frac{\partial^{j+m}\phi_k}{\partial x^j \partial y^j}\Big)^2(p) = \frac{1}{\pi \cdot 2^{2m+2j+2}} \cdot \frac{(2m)! \cdot (2j)!}{m! \cdot j! \cdot (m+j+1)!}$$

So in particular

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{k=n} \phi_k^2(p) = \frac{1}{\text{area}(\Omega)}$$

where a notable point is that the limit is independent of p. Background to the proofs of the results above is covered by material in the chapters where the ordinary Dirichlet problem is solved and Green's functions are constructed in Chapter 5, while material related to harmonic analysis and spectral theory which is needed to get Ikehara's theorem is covered by an appendix devoted to functional analysis.

0.2.1 Comments by Beurling about the Helmholtz equation.

While referring to applications in physics of analytic function theory and elliptic boundary value problems in $\mathbf{R}^3$ one should be prudent. Here follows an excerpt from a lecture by Beurling in Lund 1951 which points out some problematic features.

Hydromechanics is filled with paradoxes which begin already with the simplest problem; the motion of an ideal incompressible fluid in a force-free field. In that case the velocity vector is the gradient of a harmonic function ϕ whose normal derivative vanishes on fixed walls which limit the fluid's expansion in space. If we bound a certain portion of the fluid by two surfaces A and B which are orthogonal to the stream-lines, we obtain a domain $\mathcal{R}$ bounded by certain fixed wall C. The function ϕ is then a solution to Laplace's differential equation with boundary conditions $\phi = a$ on A; $\phi = b$ on B; $\frac{\partial\phi}{\partial n} = 0$ on C where a and b are constants.

In spite of exact mathematical solutions to the boundary value problem describing a potential flow, it is of limited interest both from a theoretical and practial point of view. The mathematical equations which describe the flow is usually in poor agreement with the empirical facts and give rise to a variety of contradictions, i.e. d'Alemebert's paradox. In 1868 Helmholtz introduced a new model with the objective of fitting the theory better to physical facts. Here the flowing fluid only fills up a certain subdomain $\mathcal{R}_0$ not given in advance, while the remaining set $\mathcal{R} \setminus \mathcal{R}_0$ is filled with fluid at rest, so called "deadwater" or "wakes". By Bernoulli's relations the flowing fluid has constant speed on the boundary F between $\mathcal{R} \setminus \mathcal{R}_0$ and $\mathcal{R}_0$ which mathematically leads to a free boundary value problem.

However, the Helmholtz's flow problem leads to great difficulties and still belongs on the whole to an unexplored region of potential theory. Certain 2-dimensional problems were settled by Weinstein when polygonal boundaries appear where analytic function theory and hodographic methods can be used, and some other 2-dimensional problems have been solved by Leray using rather complicated techniques. Passing to dimension three there is of now the lack of mathematical methods ...

Remark. The reader may consult Birkhoff's text-book *Hydromechanics, a Study in Logic, fact and Similtude* (Princeton University Press 1950) and articles by Weinstein, especially *Non-linear problems in the Theory of Fluid Motion with free Boundaries* [Proc.of first symp. applied math. 1949] for a further account. The article *On measurement of velocity by Pitot tube* by B.J. Andersoon in [Arkiv för mat. 1955] was inspired by Beurling's lecture and contains a specific solution to a 2-dimensional problem with a free boundary where analytic function theory is used to construct theoretical solutions. This theoretical work was carried out together with empirical experiments which showed discrepancy between expermintal data and the theoretical solution but within errors of a few percentages. A Pitot tube is well adapted for a mathematical treatment because it is supposed to have a long range which makes it easy to start with a stationary fluid at a long distance before wakes appear. Passing to dimension three the discrepancy increases rather drastically between experminental data and mathematical solutions.

Beurling's lecture was delivered more than half a century. Thanks to comuters one can nowadays exhibit numerical solutions. But from a pure theoretical point if view little progress has been achieved about equations of Helmholtz type during more recent decades and one may ask if it is worth efforts to find more theoretical results for various linear boundsry value problems. Well, hopefully some n further mathematical discoveries can contribute to develop a more efficient numerical analysis. Non-linearity caused by viscosity lead to more involved equations such as a mixture of a classical Navier-Stokes equation and free boundaries in the sense of Helmholtz and here one is more or less totally dependent upon numerical solutions. This means that a *very important issue* in current mathematical research is the search for improved numerical algorithms. A typical problem is to adapt grids in order that the computer can start calculations to find approximative solutions to PDE-equations. Recall that the major difficulity in Navier-Stokes equations occur in the physically relevant case when the viscosity is small and here one is more or less totally dependent to find numerical solutions since the associated PDE-equations do not fit with any manageable class.

0.2.2 Calculus of variation.

This is a subject which would require an extensive chapter. Optimization problems solved by calculus of variation played a significant role while analysis was developed. Fermat's early studies led to the principle of least action and Huyghen's applied this to many delicate variational problems. Recall for example his fundamental discovery that the Law of Momentum implies that a hanging chain under gravity with its end-points fixed on some veritcal wall, minimizes the potential energy. Dynamical situations led to the solution of the Brachistone problem and extensive classes of variational problems were considered by Euler. Here the Euler equations express necessary conditions for extremals in a variational problem. In classical mechanics the principle of virtual work was introduced by d'Alembert and a conclusive result appears in Lagrange's famous text-book on *Mechanique* from 1770 (?) which gives the equations of motion for every system with a finite degree of freedom. Here the external forces can be arbitrary, i.e. non-conservative cases can occur. The d'Alembert-Lagrange equations have a wide range of applications, foremost in engineering. The crucial role of convexity (or concavity) in order to ensure that a specific problem has a unique minimum (or maximum) was clarified by Legendre. Later physical discoveries, such as the Briot-Savart Law from 1800 led to more involved problems related to mathematical physics. From a geometric point of view one should mention contributions by Gauss who introduced the notion of curvature in differential geometry. A specific case is the Dirichlet problem where we consider the following boundary value problem in dimension two: Let Ω be a connected domain of class $\mathcal{D}(C^1)$ which means that $\partial\Omega$ consists of a finite union of disjoint closed and differentiable Jordan curves. Let $g \in C^0(\partial\Omega)$ be real-valued and $\phi(x,y)$ is a continuous function in $\bar{\Omega}$ which takes positive values in the open interior. One seeks a function u such that $u = g$ on $\partial\Omega$ while

$$\Delta(u) + \phi \cdot u = 0$$

holds in Ω. To solve this one introduces the functional

$$J(u) = \iint_\Omega (u_x^2 + u_y^2)\,dxdy + \iint_\Omega \phi \cdot u^2\,dxdy$$

It is defined on functions $u \in C^0(\bar{\Omega})$ for which $u = g$ on $\partial\Omega$ and its first order partial derivatives exist in Ω so that the first integral above is finite, i.e. u has a finite Dirichlet integral. The J-functional is strictly convex in Legendre's sense. For if $h \in C^1(\Omega)$ vanishes on $\partial\Omega$ then the function of the real variable s defined by

$$s \mapsto J(u + sh)$$

is strictly convex. In fact, this follows because the second order derivative at $s = 0$ is equal to $J(h)$ which is > 0 for every h above which is not identically zero. If u_* minimizes the J-functional it gives a solution to the boundary value problem. The reason is that if h is an arbitrary C^2-function with compact support in Ω then the inequality

$$J(u_* + \Delta \cdot h) \geq J(u_*)$$

for all small δ which may be either positive or negative entails that

$$\iint (u_x h_x + u_y h_y)\,dxdy + \iint \phi \cdot u \cdot h\,dxdy = 0$$

Greens' formula applied to the pair u_*, h and the elliptic property of the Laplace operator imply that u_* is a C^2-function in Ω where it satisfies the requested PDE-equation $\Delta(u_*) + \phi \cdot u_* = 0$.

Remark. Above the competing family of u-functions are of the form $u = G + w$ where G is the ordinary solution to the Dirichlet problem, i.e. harmonic in Ω while $G = g$ on $\partial\Omega$, while w are functions with a finite Dirichlet integral and boundary values zero. After Hilbert's creation of general Hilbert spaces the solution to the variational probelm above was consolidated by Weyl who regarded the space of w-functions as above for which $J(w) < \infty$. Weyl considered the class $C_0^1(\Omega)$ of continuously differentiable w-functions with compact support which gives a dense subspace of a Hilbert space denoted by $W_0^1(\Omega)$ whose functions have first order derivatives in $L^2_{\text{loc}}(\Omega)$ in the sense of distributions, while $J(w) < \infty$ and the inner product of a pair (w, v) is defined by

$$\langle w, v \rangle = \iint (w_x v_x + w_y v_y)\,dxdy + \iint \phi \cdot w \cdot v\,dxdy$$

From this the existence and uniqueness of u_* follows from general facts about strictly convex functionals on Hilbert spaces.

0.3 Hadamard's extension theorem.

A good example of "an old vintage result " is due to Jaques Hadamard. Text-books are in most cases content to announce the formula for the radius of convergence of a power series $\sum c_n z^n$ by the equality

$$(*) \qquad \frac{1}{\rho} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

This is a minor observation compared to the contents in Hadamard's thesis *Essai sur l'études des fonctions donnés par leur développement de Taylor* from 1893. Let us recall the major result in [ibid] since it has a wide range of applications and is seldom mentioned in contemporary text-books. Consider a power series whose radius of convergence is a positive number ρ. To every pair $p \geq 1$ and $n \geq 0$ we have the Hankel determinants:

$$\mathcal{D}_n^{(p)} = \det \begin{pmatrix} c_n & c_{n+1} & \dots & c_{n+p} \\ c_{n+1} & c_{n+2} & \dots & c_{n+p+1} \\ \dots & \dots & \dots & \dots \\ c_{n+p} & c_{n+p+1} & \dots & c_{n+2p} \end{pmatrix}$$

To each $p \geq 1$ we set

$$\limsup_{n\to\infty} |\mathcal{D}_n^{(p)}|^{\frac{1}{n}} = \delta(p)$$

From (*) it follows easily that

$$(1) \qquad \delta(p) \leq \rho^{-p-1} \quad : \ p \geq 1$$

If strict inequality holds for some positive integer in (1) the following result was proved in [loc.cit] where we put $f(z) = \sum c_n z^n$.

Theorem. *Assume there exists a positive integer p and some $\rho_* > \rho$ such that $\delta(q) = \rho^{-q-1}$ when $0 \leq q \leq p-1$ while*

$$\delta(p) = \rho^{-p} \cdot \rho_*^{-1}$$

Then there exists a polynomial $Q(z)$ of degree p at most such that $Q(z)\cdot f(z)$ extends to an analytic function in the disc $\{|z| < \rho^*\}$.

Hadamard's proof shows also that the coefficients of the Q-polynomial are found via robust limit formulas which arise from linear systems of equations derived from the given sequence $\{c_n\}$.

A result by Jentsch.

The following result is due Jentsch in the article *xxx* from 1917. Let $\{a_k\}$ be a sequence of complex numbers such that

$$(1) \qquad \lim_{k\to\infty} |a_k|^{\frac{1}{k}} = 1$$

To each positive integer n we get the polynomial

$$s_n(z) = a_0 + a_1 z + \ldots + a_n z^n$$

Let $\{\alpha_1^{(n)}, \ldots, \alpha_n^{(n)})$ be its zeros and construct the probability measure

(2) $$\mu_n = \frac{1}{n} \cdot \sum_{k=1}^{k=n} \delta(\alpha_n^{(n)})$$

Jentsch proved that the sequence $\{\mu_n\}$ converges to the Haar measure on the unit circle, i.e. for every continuous function f in $\mathbf{C}$ one has the limit formula

(*) $$\lim_{n \to \infty} \int f \cdot d\mu_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta$$

To prove (*) Jentsch considered the sequece of locally integrable functions

(3) $$\phi_n(z) = \frac{\log |s_n(z)|}{n}$$

In § xx we show that (1) entails that the sequence $\{\phi_n(z)\}$ converges in the sense of distributions to the function $U(z)$ which is zero in $|z| < 1$ and equal to log $|z|$ when $|z| \geq 1$. After this (*) follows when one applies the Laplace operator. So here measure theory intervenes with analytic function theory. The result is quite remarkable. The simplest case occurs when $a_n = 1$ for all n. Here

$$s_n(z) = 1 + z + \ldots + z^n$$

whose zeros are the roots of unity $\{e^{2\pi i\nu/(n+1)} \quad : \nu = 1, \ldots, n\}$. These n-tuples are equi-distributed so after the normlisation with the factor $\frac{1}{n}$ one gets (*). But for other sequences the limit (*) is not evident. For example we can take $\{a_\nu = e^{1 \cdot \theta_\nu}\}$ where the sequence $\{0 \leq \theta_\nu \leq 2\pi\}$ is arbitrary.

0.4 The role of inequalities.

For students aiming to learn "big theories" the search for inequalites may appear to be a modest or even trivial issue compared to studies of axiomatic concepts. But the truth is that more or less all deep results in mathematics rely upon inequalities. An example is the Riemann Hypothesis which is the utmost challenge among open problems in the world of mathematics. A necessary and sufficient condition for its validity goes as follows: For each real number x we denote by $\{x\}$ the largest integer such that $0 \leq x - \{x\} < 1$. To each positive integer M we get M many functions $\{\rho_j(x)\}$ defined on the open unit interval $(0,1)$ by

$$\rho_j(x) = \frac{j}{Mx} - \{\frac{j}{Mx}\} \quad : \quad 1 \leq j \leq M$$

Set

$$\beta(M) = \min_{c_\bullet} \int_0^1 \big(\sum_{j=1}^{M} c_j \cdot \rho_j(x) - 1\big)^2 \cdot dx \tag{1}$$

with the minimum taken over all M-tuples of real numbers such that

$$\sum_{j=1}^{j=M} j \cdot c_j = 0$$

Beurling has shown that the Riemann Hypothesis is true if and only if

$$\lim_{M \to \infty} \beta(M) = 0 \tag{*}$$

When M is fixed (1) is minimized by a unique M-tuple $c_\bullet$ since we are dealing with a positive definite quadratic form and then $\beta(M)$ is recovered from Lagrange's multiplier. The obstacle is that the ρ-functions become irregular as M increases and mathematics has not yet become enough developed to establish bounds of the β-numbers in order to decide whether (*) holds or not.

However we are able to exhibit many inequalities. An example is a mini-max result for polynomials which goes as follows. Let $P(z)$ be a polynomial of some degree $n \geq 2$ where $P(0) = 1$. To each pair $0 < a < b < 1$ we set

$$\omega_P(a,b) = \max_{a \leq r \leq b} \min_\theta |P(re^{i\theta})|$$

Thus, while the radius r varies between a and b we seek some r such that the minimum modulus of P on the circle $|z| = r$ is as large as possible. In his thesis *Études sur in problème de majoration* at Uppsala University from 1933, Beurling found a lower bound hold for these ω-functions. More precisely, the following hold for all pairs $0 < a < b$ and each normalised polynomial P as above with degree n:

$$\omega_P(a,b) \geq \big(\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}\big)^n \tag{*}$$

In § XX we expose the proof which relies upon a study of logarithmic potentials. Let us remark that (*) gets sharp when n increases. A result of a similar nature from [ibid: page 96] goes as follows: Let $R > 0$ and $f(z)$ is analytic in $\{|z| < R\}$. To every $0 < r < R$ we denote by $M(r)$ the maximum of $|f(z)|$ taken on the circle of radius r. Let $\mu > 0$ and consider a pair $0 < r_1 < r_2 < R$ where $M(r_1) > \mu$ is holds and put

$$E = \{r_1 < r < r_2 : M(r) \leq \mu\}$$

The integral of r^{-1} over E measures its logarithmic length. Set

$$\mathcal{L}(r_1, r_2; \mu) = \frac{1}{2} \cdot e^{\int_{r_1}^{r_2} \frac{dr}{r}}$$

With this notation one has the inequality

$$\frac{M(r_2)}{\mu} \geq \left[\frac{M(r_1)}{\mu}\right]^{\mathcal{L}(r_1,r_2;\mu)} \tag{**}$$

0.4.1 Distributions of zeros of polynomials

Let $n \geq 2$ and $\{e^{i\theta_k}\}$ is an n-tuple of points on the unit circle where $-\pi < \theta_k < \pi$, i.e. -1 does not occur. For each $0 < \alpha < \pi$ we denote by $N(\alpha)$ the number of θ-points such that $-\alpha < \theta_k < \alpha$ and set

$$J(\alpha) = \left|\frac{N(\alpha)}{n} - \frac{\alpha}{2\pi}\right|$$

It turns out that the J-function is majorized via a maximum norm of the polynomial $p(z) = \prod (1 - e^{-i\theta_\nu} z)$. Namely, one has the inequality

$$|J(\alpha)| \leq \sqrt{\frac{2\pi}{kn}} \cdot \sqrt{\log |p|_D}$$

where $|p|_D$ is the maximum norm of p on the unit disc and $k \simeq 0,916\ldots$ is Catalani's constant. We prove this in Chapter 5. § 13 under the heading Ganelius' theorem.

0.4.2 A result by Lindwart and Polya.

The result below was proved in their joint article *Über eine Zusammenhang zwischen der Konvergenz von Polynomfolgen und der Verteilering ihrer Wurzeln* [Palermo 1914]. Here is the situation: Let $\{P_n(z)\}$ be a sequence of polynomials which converges uniformly on the closed unit disc $\{|z| \leq 1\}$ to a function which is not identically zero. Suppose also that there exists some $0 < \phi < \pi/2$ and some $a > 0$ such that the polynomials are zero free in the union of the sector $\{-\phi < \arg z < \phi\}$ and the half space $\mathfrak{Re}\, z > a$. Under these conditions the sequence $\{P_n(z)\}$ converges uniformly on compact subsets of $\mathbf{C}$ to an entire function $f(z)$ of order $\rho \leq 2$. The last assertion means that if $\{a_k\}$ are the coefficients in the series expansion of f then

$$\liminf_{k \to \infty} \frac{\log \frac{1}{|a_k|}}{k \cdot \log k} = \frac{1}{\rho}$$

The proof contains several unexpected ingredients and illustrates that analytic function theory contains a wealth of results where individual proofs often require some ingenious idea, i.e. the theory is not "stream-lined". The proof of the Lindwart-Polya theorem is given in § XX.

0.4.3 An inequality by Nevanlinna.

A subject which is not covered in these notes is Nevanlinna's value distribution theory for meromorphic functions. It is treated in the text-book (Nevanlinna). But let us recall one major result which leads to an extension of Picard's theorem. Let $f(z)$ be a meromorphic function in $\mathbf{C}$ and consider Nevanlinna's reduced counting functions, i.e. when a is a complex number then $N_*(r;a)$ which counts the number of points $|z|<r$ where $f(z)=a$ irrespectively of multiplicity. If $a=\infty$ we count poles in the same way irrespectively of the order of the poles. Nevannlina proved that there exists a constant C_f which depends on the given meromorphic function such that the inequality below holds for all $r>0$:

$$(*)\quad \frac{1}{2\pi}\cdot\int_0^{2\pi}\log^+\,|f(re^{i\theta})|\,d\theta\leq N_*(0;r)+N_*(1;r)+N_*(\infty;r)+C_f\cdot\log(r+2)$$

0.4.4 An inequality for convolutions.

Let $k(x,y)$ be a complex-valued continuous function on the square $\{0\leq x,y\leq 1\}$. Symmetry is not assumed, i.e, $k(x,y)\neq k(y,x)$ may occur Let $f_0(x)$ be another continuous function on $[0,1]$ and assume that the maximum norms of k and f both are <1. Inductively we get a sequence $\{f_n\}$ where

$$f_n(x)=\int_0^1 k(x,y)\cdot f_{n-1}(y)\cdot dy\quad:\quad n\geq 1$$

To each $0\leq x\leq 1$ and every pair of positive integers n,p one has the Hankel determinant:

$$\mathcal{D}_n^{(p)}(x)=\det\begin{pmatrix} f_{n+1}(x) & f_{n+2}(x) & \dots & \dots & f_{n+p}(x)\\ f_{n+2}(x) & f_{n+3}(x) & \dots & \dots & f_{n+p+1}(x)\\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \\ f_{n+p}(x) & f_{n+p+1}(x) & \dots & \dots & f_{n+2p-1}(x)\end{pmatrix}$$

With these notation the inequality below will be proved in Chapter I:C : § 10:

0.4.5 Theorem *For every $p\geq 2$ and $0\leq x\leq 1$ one has the inequality*

$$(*)\qquad \big|\,\mathcal{D}_n^{(p)}(x))\,\big|\leq (p\,!)^{-n}\cdot\big(p^{\frac{p}{2}}\big)^n\cdot\frac{p^p}{p\,!}$$

Using (*) and Hadamard's inequality from § 0.3 we also show that if λ is a new complex variable then the power series

$$u_\lambda(x)=\sum_{n=0}^\infty f_n(x)\cdot\lambda^n$$

which to begin with converges if $|\lambda|<1$ extends to a meromorphic function in the whole complex plane whose the poles are related to a certain compact integral operator. Above one regards u_λ as a function of λ with values in the Banach space of continuous functions on the closed unit interval and we shall see many examples where the ordinary family of complex-valued analytic functions is extended to analytic functions with values in Banach spaces.

0.4.6 An inequality for Fourier series.

Let $f(x)$ be a continouos and 2π-periodic function. The Fourier coefficients are defined by

$$\widehat{f}(\nu) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu x} f(x)\, dx$$

For each integer n we get the partial sum

$$s_n(x) = \sum_{\nu=-n}^{\nu=n} \widehat{f}(\nu) \cdot e^{i\nu x}$$

In general $\{s_n\}$ does not converge to f pointwise convergence. Classic examples are due to Gibbs. On the positive side a deep result due to Carleson asserts that

$$\lim_{n\to\infty} s_n(x) = f(x)$$

holds almost everywhere, i.e. for every x outside a null set in Lebegue's sense. An averaged version goes as follows: To each n we set

$$\mathcal{D}_n(f) = \sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} ||s_\nu - f||^2}$$

where $||s_\nu - f||$ is the maximum norm taken over $[0, 2\pi]$. The inequality below shows that Gibb's phenomenon is sparse when when partial sums occur via a random choice of n. For a 2π-periodic and continuous function f we denote its maximum norm by $||f||$ and ω_f is its modulos of continuity. The result below is due to Carleman and will be proved in Chapter I: § XX:

Theorem. *There exists an absolute constant K such that the following hold for every 2π-periodic and continuous function f, every $b \geq 1$ and every positive integer n:*

$$\mathcal{D}_n(f) \leq K \cdot ||f|| \cdot \big[\frac{1}{b} + b \cdot \omega_f(\frac{1}{n})\big]$$

Since $\omega_f(\frac{1}{n})$ tends to zero as n increases we can take b arbitrary large and conclude that

$$\lim_{n\to\infty} \mathcal{D}_n(f) = 0$$

This shows that Gibb's phenomenon in the average only can occur at a sparse set of integers.

0.4.7 Infinite systems of linear equations.

Let $\{a_p\}$ be a sequence of positive real numbers such that $a_p \to +\infty$ and associate the function $A(r)$ which to each $r > 0$ is the number of a-values $< r$. We say that $\{a_p\}$ has a positive asymptotic density if there exists the limit

$$\lim_{r\to\infty} A(r) > 0$$

We do not require that the sequence is monotone, i.e $a_p \leq a_{p+1}$ need not always hold. Next, let $\{b_q\}$ be another sequence of real numbers which also has some positive asymptotic density which in general can be different from that of $\{a_p\}$. Given such a pair we take some $\epsilon > 0$ and consider the homogeneous system of equations:

$$(*) \qquad \sum_{q=1}^{\infty} \frac{x_q}{a_p - b_q - i\epsilon \cdot q} = 0$$

where $\{x_q\}$ is a complex sequence which belongs to the Hilbert space ℓ^2, i.e. $\sum |x_q|^2 < \infty$. In § xx we prove that the homogeneous system only has the trivial null solution. The proof relies upon inequalities for certain fractional sums and a construction of a meromorphic function. In the section entitled *Dagerholm Series* in Special Topics we prove a subtle result where $\epsilon = 0$. In this situation we shall find non-trivial solutions $\{x_q\}$ but they need not be in ℓ^2. The lesson is that many problems about infinite systems of linear equations, as well as many problems in functional analysis, can only be proved by analytic function theory.

0.4.8 Blaschke products and interpolation.

Let $n \geq 2$ and $E = \{\alpha_1, \ldots, \alpha_n\}$ is an n-tuple of distinct points in the open unit disc D. To each n-tuple of complex numbers $w_\bullet = (w_1, \ldots, w_n)$ one seeks to minimize the maximum norm

$$|f|_D = \max_{z\in D} |f(z)|$$

in the class of functions which solve the interpolation problem $f(\alpha_k) = w_k$ for every k. A result due to Nevanlinna and Pick shows that there exists a unique interpolating function f with minimal norm. Moreover,

$$f(z) = \rho(w_\bullet) \cdot e^{i\theta} \cdot \prod_{\nu=1}^{\nu=n-1} \frac{z - \beta_\nu}{1 - \bar{\beta}_\nu \cdot z}$$

Here $\rho(w_\bullet) > 0$ while $0 \leq \theta \leq 2\pi$ and $\{\beta_\nu\}$ is an $(n-1)$-tuple in D which is not necessarily distinct. We prove this result in Special Topics § XX. Keeping E fixed we introduce the interpolation constant

$$\mathfrak{int}(E) = \max \rho(w_\bullet)$$

where the mximum is taken over n-tuples $w_\bullet$ such that $|w_k| \leq 1$ for every k. Next, denote by $\mathcal{B}_{n-1}$ the class of functions of the form

$$e^{i\theta} \cdot \prod_{\nu=1}^{\nu=n-1} \frac{z - \beta_\nu}{1 - \bar{\beta}_\nu \cdot z}$$

where $0 \leq \theta < 2\pi$ and $\{\beta - \nu\}$ is an $(n-1)$-tuple of points in D which need not be distinct. Put

$$\tau(E) = \min_{\phi} |\phi|_E$$

where the minimum is taken over all $\phi \in \mathcal{B}_{n-1}$ and $|\phi|_E$ is the maximum norm on E. The Nevanlinna-Pick theorem easilly entails that one has the equality

$$\mathtt{int}(E) = \frac{1}{\tau(E)}$$

Less obvious is the follwing result due to Beurling:

Theorem. *Up to multiplication with $e^{i\theta}$ for some complex number, the minimax problem above has a unique solution ϕ_*. Moreover*

$$|\phi_*(\alpha_k)| = \tau(E) \quad : 1 \leq k \leq n$$

A consequence of this result is that when the set E is given then the "worst scenario" consists of n-vectors $w^*_\bullet$ such that $\rho(w^*_\bullet) = \mathtt{int}(E)$ and Beurling's result shows that $|w^*_k| = 1$ hold for every k. Moreover, this n-tuple of points on the unit circle is unique up to a common rotation. We shall prove this in Special Topics §§ which illustrate a typical complex extremal problem where the given set E determines the $(n-1)$-tuple of zeros of the unique ϕ-function in Beurling's minimax-theorem as well as the n-tuple $w^*_\bullet$. A notable fact which also is due to Beurling shows that the extremals ϕ_* and $w^*_\bullet$ can be found by a variational problem using a suitable Hilbert spaces. More precisely, consider the Hardy space $H^2(D)$ of analytic functions $F(z)$ in the unit disc equipped with the norm

$$||F||_2 = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^2 \, d\theta}$$

Given $E = \{\alpha_k\}$ we have the Blaschke product

$$B_E(z) = \prod_{k=1}^{k=n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k \cdot z}$$

With these notations one considers the variational problem

$$\max_F \sum_{k=1}^{k=n} \frac{|F(\alpha_k)|^2}{|B'_E(\alpha_k)|}$$

with the maximum taken over F of unit norm in $H^2(D)$. Beurling has proved that this variational problem has a unique solution F normalised so that $F(0)$ is real and positive. Moreover, the maximum above is equal to $\mathtt{int}(E)$ and $F(z)$ is a zero-free analytic function in D where the β-zeros of ϕ_* are determined by the equation

$$\prod_{\nu=1}^{\nu=n-1} (\bar{\zeta} - \bar{\beta}_\nu)(1 - \bar{\beta}_\nu \cdot \zeta) = \prod_{k=1}^{k=n} |\zeta - \alpha_k|^2 \cdot \frac{\zeta \cdot F^2(\zeta)}{F^2(0)} \quad : \ |\zeta| = 1$$

0.5 Why one should study complex analysis

Apart from its intrinsic beauty which contains residue theory and conformal mappings, complex analytic methods find applications many areas of mathematics. Let us remark that complex analytic functions are special. For example, an entire function $f(z)$ is uniquely determined by its values in a dense subset of an arbitrary small open interval on the real line. On the other hand the class of entire functions is quite ample. In section XX from Special Topics we prove a result due to Weierstrass which asserts that if $g(x)$ is an arbitrary complex valued function on the real line and $\epsilon > 0$, then there exists an entire function $f(z)$ such that $|f(x) - g(x)| < \epsilon$ hold for all real x. Notice that this is far more precise than the trivial result that approximation holds on compact subsets of the real line. So just as in the case of Hadamard's radius formula this result from 1861 is a good example of "old vintage".

Resolvents of linear operators. An example where analytic functions play a crucial role occurs in spectral theory for linear operators. Consider a Banach space X and $T\colon X \to X$ is a linear operator whose domain of definition is a dense subspace $\mathcal{D}(T)$ of X. But T is unbounded which means that

$$\max_{x \in \mathcal{D}_*(T)} ||Tx|| = +\infty$$

where $\mathcal{D}_*(T)$ is the intersection of $\mathcal{D}(T)$ and the unit ball of vectors in X with norm ≤ 1. In addition we assume that T is closed which means that the graph

$$\Gamma(T) = \{(x, Tx) \quad : x \in \mathcal{D}(T)\}$$

is a closed subset of $X \times X$. Let T be such a linear operaor and denoted by E the identity operator on X.

Definition. *A resolvent value of T is a complex number λ such that there range of $\lambda \cdot E - T$ is a dense subspace of X and there exists a positive constant c such that*

(i) $$||\lambda \cdot x - Tx|| \geq ||x|| \quad : x \in \mathcal{D}(T)$$

Since T is closed it easily follows from (i) that the dense range $(\lambda \cdot E - T)(X)$ is equal to X and there exists a unique bounded linear operator $R(\lambda; T)$ on X which is injective and has range equal to $\mathcal{D}(T)$ where

$$R(\lambda; T) \circ (\lambda \cdot E - T)(x) = x \quad : x \in \mathcal{D}(T)$$

and the composed operator $(\lambda \cdot E - T) \circ R = E$. One refers to $R(\lambda; T)$ as the resolvent operator of T. The construction of such resolvents appeared for the first time in Neumann's article cited in § xx. The crucial fact is that if λ is a resolvent value of T then the open disc of radius $||R(\lambda; T)||$ centered at λ give resolvent values and one has the Neumann series

$$R(\lambda + \zeta; T) = R(\lambda; T) + \sum_{n=1}^{\infty} (-1)^n \cdot \zeta^n \cdot R(\lambda; T)^n \quad : |\zeta| < ||R(\lambda; T)||$$

So if T has at least one resolvent value the set of all resolvent values is an open subset of $\mathbf{C}$ whose closed complement is denoted by $\sigma(T)$ and called the spectrum of T. A neat desription of $\sigma(T)$ is available using the Neumann equation which asserts that if λ, μ is an arbitrary pair of resolvent values then

$$R(\lambda;T) - R(\mu;T) = \frac{R(\lambda;T)R(\mu;T)}{\mu - \lambda}$$

In the appendix devoted to Functional analysis we shall establish the Neumann equation and use it to show that if μ is an arbitrary resolvent value of T, then

$$\text{(ii)} \qquad \sigma(T) = \{\mu - \frac{1}{s} : \ s \in \sigma(R(\mu;T)) \setminus \{0\}\}$$

A special case occurs when one of the resolvents $R(\mu;T)$ is a compact operator. Neumann's equation entails that all resolvent operators are compact operators and (ii) implies that $\sigma(T)$ is a discrete subset of $\mathbf{C}$. Situations where the resolvent operators are compact occur in Fredholm's studies of integral equations and was put into a general context by Hilbert in his work *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* from 1904 which paved the way to the the theory about spectral resolutions of linear operators. When X is a Hilbert space and T a densely defined but unbounded self-adjoint operator conclusive results were established by Carleman in his book *xxxx* from 1923. The appendix devoted to Funtional Analysis contains proofs of Carleman's spectral resolution for unbounded self-adjoint operators. Here complex methods play a crucial role and at the same time the result has a wide range of applications to situations which at a first glance seem to be of a "purely real character". An example where Carleman's theory about unbounded self-adjoint operators applies is the product formula for an arbitrary densely defined and closed linear operator T on a complex Hilbert space. It asserts that there is a unique pair P, A where A is a densely defined self-adjoint operator and P a partial isometry such that

$$T = PA$$

One can regard this as the ultimate polar representation of complex objects. The proof is given in the appendix devoted to functional analysis.

Applications to probability theory. A theorem due to Harald Cramér asserts that if $\chi_1, \ldots, \chi_N$ is a family of independent random variables whose sum variable $\chi_1 + \ldots + \chi_N$ is normally distributed, then each single χ_ν has a normal distribution. This is a useful fact for applications in statistics. Using the existence of complex log-functions of entire functions without zeros the proof boils down to show that an entire function which only increases like $|z|^2$ is a polynomial of degre ≤ 2. Staying with probability theorem we also recall a result due to Kakutani and Yosida where one regards transition probability functions defining a Markov process. These authors proved that under a rather modest condition there exists a finite partition of the given probability space which consists of a dissipative part and a finite set of ergodic kernels where the process stays once it has arrived to such kernel. The main steps in the proof use analytic function theory and operators with complex eigenvalues. The proof of the Kakutani-Yosida theorem is exposed in § XXX,

Linear algebra and Thorin's convexity theorem. Let A be an $n \times n$-matrix with real elements $\{a_{k\nu}\}$. To get zeros of the characteristic polynomial $P_A(\lambda) = \det(\lambda \cdot E_n - A)$ one must include those which are complex. Denote by $\rho(A)$ the maximum of the absolute value of these zeros. Now $R_A(\lambda) = (\lambda \cdot E_n - A)^{-1}$ is an analytic matrix-valued function defined outside the spectrum $\sigma(A)$. From this it follows that $\rho(A)$ is equal to the spectral radius computed via an an *arbitrary norm* on $\mathbf{R}^n$. This invariance property is for example used to study a matrix A whose elements are positive real numbers where the spectral radius formula gives a quick proof of a theorem due to Perron and Frobenius which asserts that if A is a matrix with real and positive elements then there exists a unique a real eigenvector $\mathbf{x}$ where each $x_k > 0$ such that $\sum x_k = 1$ and $A(\mathbf{x}) = \rho(A) \cdot \mathbf{x}$. This result is a cornerstone in linear programming where one seeks solutions which are constrained to convex sets. Another example where complex methods are used appears in Thorin's convexity theorem. Let $A = \{c_{\nu k}\}$ be an arbitrary $n \times n$-matrix with complex elements. To each pair of real numbers $0 < a, b < 1$ we set

$$M(a,b) = \max_{x,y} |\sum\sum c_{\nu k} \cdot x_\nu \cdot y_k| \quad : \sum |x_\nu|^{1/a} = \sum |y_k|^{1/b} = 1$$

Via Hadamard's maximum principle for analytic functions defined in strip domains, Thorin proved that the function

$$(a,b) \mapsto \log M(a,b)$$

is convex in the square $0 < a, b < 1$. The fundamental theorem of algebra is of course a major motivation why complex numbers are so useful. The variation of the argument for a complex-valued function which is $\neq 0$ along some curve is a fundamental issue. Here is a result which can serve as a first challenge for the reader and will be explained in Chapter 4. Consider a polynomial

$$P(z) = z^{2n+1} + a_{2n}z^{2n} + \ldots a_1 z + a_0$$

with real coefficients and assume that the function

$$P_*(iy) = \sum_{k=0}^{n} a_{2k} \cdot (iy)^{2k}$$

is > 0 for all real y. Then, if n is even it follows that $P(z)$ has n complex zeros counted with multiplicity in the right half-plane $\mathfrak{Re}(z) > 0$, while the number of zeros is $n+1$ if n is odd. For example, if $n = 5$ and

$$P(z) = z^5 + Az^3 + Bz + 1$$

where A, B is any pair of real numbers then P has two roots in $\mathfrak{Re} z > 0$. We shall learn that this follows since the real parti of $P(iy)$ is identically one on the imaginary axis which implies that the variation of $\arg(Piy)$ increases from $-\pi/2$ when $y = -\infty$ to $\pi/2$ when $y = +\infty$ while the argument changes by 5π when we follows $\arg P(z)$ along a half-circle $\{z = Re^{i\theta} : -\pi/2 < \theta < \pi/2\}$ and then a general result from Chapter 4 gives the assertion about the number of zeros in $\mathfrak{Re}\ z > 0$.

0.6 Jensen's formula and the Jensen-Nevanlinna class

One can say that modern analytic function theory started with a discovery by the Danish telephone engineer Jensen. In 1899 he proved that if $f(z)$ is analytic in a disc $|z| \leq R$ where $f(0) \neq 0$, then

$$(*) \qquad \frac{1}{2\pi} \cdot \int_0^{2\pi} \log|f(Re^{i\theta})| \cdot d\theta = \log|f(Re^{i\theta})| + \log \frac{R^n}{|a_1| \cdots |a_n|}$$

where $\{a_k\}$ are the zeros of f in the disc $\{|z| > R$ repeated with their multiplicitites. Today's student learns this theorem with relative ease, i.e. one uses a factorisation to get rid of the zeros by a product of Möbius functions and after the mean value formula for a harmonic functions is applied. But it took almost a century until familarity with complex analysis reached a stage when (*) was announced. Jensen's formula is a cornerstone in analytic function theory and our historic comment illustrates that today's student has to learn a lot of material in order to advance in contemporary mathematics. Another essential result is Herglotz' integral formula which exhibits an analytic function $f(z)$ in the unit disc by its real part. Namely, if $h(\theta)$ is a continuous function on the unit circle there exists an analytic function in the unit disc defined by

$$(**) \qquad \mathcal{H}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot h(\theta) \cdot d\theta$$

One refers to (**) as the Herglotz' extension of h. It is related to the given h-function since the real part $\mathfrak{Re}\,\mathcal{F}$ is expressed by Poisson's formula

$$\mathfrak{Re}\,\mathcal{H}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot h(\theta) \cdot d\theta$$

This means that $\mathfrak{Re}\mathcal{H}_u$ is the harmonic extension of h to the unit disc. Starting from (*) and (**) zero-free functions were analyzed by Jensen and Nevanlinna. Here one takes exponentials and define

$$F_h(z) = e^{\mathcal{H}(z)}$$

This is a zero-free analytic function in D whose absolute value is given by:

$$|F_h(z)| = e^{\mathfrak{Re}\mathcal{H}_u(z)}$$

On T we get the equality

$$\log\,|F_h(e^{i\theta})| = h(\theta)$$

and at the origin we find that

$$\log F_h(0) = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \cdot d\theta$$

A more extensive class of analytic functions in the unit disc are found using Herglotz's extensions of signed Borel measure μ on the unit circle. Here we get

$$(***) \qquad \mathcal{H}_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot d\mu(\theta)$$

and as above we get a zero-free analytic function $F_\mu(z) = e^{\mathcal{H}_\mu(z)}$. Next there exist Blaschke products where Jensen's integral formula gives a precise condition in order that they converge and yield analytic functions in the unit disc with prescribed zeros. In §§ we construct the Jensen-Nevanlinna class of analytic functions f in the unit disc which are charcterised by the property that there exists a constant C so that

$$(***) \qquad \int_0^{2\pi} \log^+ |f(re^{i\theta})\, d\theta \leq C \quad : \ 0 < r < 1$$

A major result due to Jensen and Nevanlinna asserts that when (***) holds and $f(0) = 1$ and $\{\alpha_\nu\}$ are the zeros of f with multiple zeros repeated according to their multiplicities, then there exists a unique Riesz measure μ on T such that

$$f(z) = B(z) \cdot e^{\mathcal{H}_\mu(z)} \quad : \ B(z) = \prod \frac{z - \alpha_\nu}{1 - \bar{\alpha}_\nu \cdot z}$$

0.6.1 A special construction. Let $0 < \delta < 1$ and define the function $\phi(\theta)$ on T by:

$$\phi(\theta) = \log(\delta^2 - \theta^2) \quad : \ -\delta \leq \theta \leq \delta$$

while $\phi = 0$ outside this interval on T. Now we get the analytic function $\mathcal{H}_\phi(z)$ and if K is a positive integer we put

$$\Phi(z) = e^{K \cdot \mathcal{H}_\phi(z)}$$

On the unit circle we have

$$|\Phi(e^{i\theta})| = (\delta^2 - \theta^2)^K \quad : \ -\delta \leq \theta \leq \delta$$

while $|\Phi(e^{i\theta})| = 1$ outside this interval. Next, let μ be a real-valued and non-negative Riesz measure on the unit circle supported by $T \setminus [-\delta, \delta]$ and construct the zero-free analytic function $F_\mu(z)$ as above. Set

$$f(z) = F(z) \cdot \Phi(z)$$

Notice that $|F(e^{is})| = 1$ holds on interval $(-\delta, \delta)$. Let us then consider the complex derivative $f'(z)$. Here an upper bound exists for radial limits as we approach points in $(-\delta, \delta)$. More precisely one has:

0.6.2 Proposition. *There exists an absolute constant C which dpends on K only such that*

$$\limsup_{r \to 1} |f'(re^{i\theta})| \leq C \cdot (\delta^2 - \theta^2)^{K-2} \cdot ||\mu|| \quad : \ -\delta < \theta < \delta$$

The proof employs Herglotz formula which gives the equation below for every z in the open unit disc:

$$f'(z) = \frac{1}{\pi} \cdot \Big[\int_{-\delta}^{\delta} \frac{\phi(s) \cdot e^{is}}{(e^{is} - z)^2}\, ds + \int \frac{d\mu(s) \cdot e^{is}}{(e^{is} - z)^2}\, ds\Big] \cdot f(z)$$

Starting from this we will show Proposition 0.6.2 in § xx from the section devoted to the disc algebra in Special Topics. where we expose results due to Carleson about sets of uniquneness for analytic functions. Apart for a number of conclusive results Carleson's constructions provide some concrete aspects in measure theory where more refined questions require that one goes beyond the notion of null-sets in Lebegue's sense.

0.6.4 Logarithmic potentials.

A central issue in these notes are logarithmic potentials. A closed subset E of the unit circle T has logarithmic capacity zero if there exists a probability measure μ on E such that

$$\lim_{r\to 1}\int_E \log\frac{1}{|e^{i\theta}-r\cdot e^{i\phi}|}\cdot d\mu(\theta)=+\infty \quad \text{for all points} \quad e^{i\phi}\in E$$

Conversely, E has positive capacity if there exists a probability measure μ on E such that the limit of integrals above are bounded by a fixed constant for all $e^{i\phi}\in E$ which amounts to say that the energy integral

$$(1)\qquad J(\mu)=\int_E\int_E \log\frac{1}{|e^{i\theta}-e^{i\phi}|}\cdot d\mu(\theta)\cdot d\mu(\phi)<\infty$$

This energy integral is related to Dirichlet integrals. Namely, starting from μ we obtain a harmonic function in the unit disc defined by

$$(2)\qquad U_\mu(re^{i\phi})=\int_E \log\frac{1}{|e^{i\theta}-r\cdot e^{i\phi}|}\cdot d\mu(\theta)$$

Here U_μ is the real part of the analytic function

$$(3)\quad F_\mu(z)=\int_E \log\frac{1}{1-e^{-i\theta}z}\cdot d\mu(\theta)=\sum_{n=1}^\infty \frac{\widehat{\mu}(n)}{n}\cdot z^n \quad : \quad \widehat{\mu}(n)=\int_E e^{-in\theta}\,d\mu(\theta)$$

Next, to every analytic function $f(z)$ in the unit disc one associates the Dirichlet integral

$$\mathcal{D}(f)=\frac{1}{\pi}\iint_D |f'(z)|^2\,dxdy$$

If $f(z)=f(0)+\sum a_nz^n$ the double integral becomes

$$(4)\qquad \sum_{n=1}^\infty\sum_{m=1}^\infty n\cdot a_n\cdot m\cdot \bar{a}_m\cdot\frac{1}{\pi}\int_0^1\int_0^{2\pi} r^{n+m-1}\cdot e^{i(n-m)\theta}\,d\theta\,dr$$

Since $\int_0^{2\pi} e^{ik\theta}\,d\theta=0$ for every non-zero integer it follows that (4) is equal to

$$\sum_{n=1}^\infty n\cdot |a_n|^2$$

So f has a finite Dirichlet integral if and only if the sequence $\{\sqrt{n}\cdot a_n\}$ belongs to the Hilbert space ℓ^2_+ of complex sequences $\{c_1,c_2,\ldots\}$ for which $\sum |c_n|^2<\infty$. Analytic functions with a finite Dirichlet integral have special interest because $\mathcal{D}(f)$ measures the area of the image domain $f(D)$. When $f=F_\mu$ with F_μ as in (3) one has

$$\mathcal{D}(F_\mu)=\sum\frac{1}{n}\cdot|\widehat{\mu}(n)|^2$$

Next, when (1) is finite one has the limit formula

$$J(\mu)=\lim_{r\to 1}\int_E\int_E \log\frac{1}{|1-r\cdot e^{i\phi-\theta}|}\cdot d\mu(\theta)\cdot d\mu(\phi)<\infty$$

Expanding the log-function one gets the equality

$$J(\mu) = \mathcal{D}(F_\mu) \tag{5}$$

Beurling used the formulas above in the article *Ensembles exceptionnels* from 1940 to prove that when f is an analytic function in the unit disc with a finite Dirichlet integral, then the radial limits

$$\lim_{r \to 1} f(re^{i\theta}) = f^*(\theta)$$

exists for all θ outside a set whose logarithmic capacity is zero. Beurling's theorem is established in § XX from Special Topics.

0.6.5 Carleson's extremal functions.

The results above suggest the study of pairs (f, E) where E is a closed set with positive logarithmic capacity while $\mathcal{D}(f)$ is finite and the boundary value function f^* is zero on E. Denote this family by $\mathcal{D}_E$. In the article *Sets of uniqueness for functions regular in the unit disc* [Acta 1952] Carleson studied the situation where $\mathcal{D}_E$ contains non-constant functions while E has positive logarithmic capacity. Let $\mathcal{D}_E^*$ be the subfamily of functions $f \in \mathcal{D}_E$ where $f(0) = 0$ and consider the variational problem

$$\inf_{f \in \mathcal{D}_E^*} D(f)$$

Carleson proved that there exists a unique $f \in \mathcal{D}_E^*$ with minimal Dirchlet integral with the property that the complex derivative $f'(z)$ extends to an analytic function in $\mathbf{C} \setminus E$. This result is established in Special Topics § XX. The reader who can pursue the details of Carleson's proof with the aid of material from earlier chapters has attained good familarity with analytic function theory.

Remark. We inserted some details at this early stage in order to illustrate what these notes are striving at. The proofs are not "sophisticated" but contain often a sequence of seemingly unrelated threads which constitute the innovative part where more delicate results often rely upon innovative constructions which are not predicted in advance. A brilliant example is Carleson's construction from § XX of a compact set on the unit interval with positive logarithmic capacity while it has vanishing Hausdorff measures for all h-functions satisfying

$$\int_0^1 \frac{h(r)}{r}\, dr < \infty$$

The main lesson is therefore to study details of separate proofs and in this way become sufficiently familiar with methods to be able to make new discoveries.

0..7 Riemann's ζ-function.

These notes are foremost directed to students who are familiar with undergraduate calculus and want to encounter more advanced topics in analysis. While entering such studies it is good to have a goal. I cannot imagine anything more appealing than to learn about Riemann's zeta-function defined by the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \quad \mathfrak{Re}\, s > 1\,.$$

The first result is due to Euler and asserts that $\zeta(s)$ extends to a meromorphic function defined in the whole complex s-plane with a simple pole at $s = 1$ and the entire function $(1-s)\zeta(s)$ has simple zeros at all even negative integers while the remaining zeros belong to the strip domain

$$0 < \mathfrak{Re}\, s < 1$$

In a work which dates back to 1859, Bernhard Riemann conjectured that the zeros of the zeta-function belong to the line $\mathfrak{Re}\, s = \frac{1}{2}$. Numerical calculations confirm the hypothesis up to a very high degree. So it is not likely that one may discover an eventual zero outside the critical line by mere guessing. But there remains to *prove* the hypothesis. The study of the zeta-function has contributed to the development in analytic function theory. An example is the asymptotic formula for the zeros in the critical strip $0 < \mathfrak{Re}(s) < 1$ which was already announced by Riemann. It asserts that there exists a constant C_0 such that if $\mathcal{N}(T)$ is the number of zeros counted with multiplicities in the rectangle where $-T < \mathfrak{Im}\, s < T$ and $0 < \mathfrak{Re}\, s < 1$, then

$$(*) \quad \mathcal{N}(T) = \frac{1}{\pi} \cdot T \cdot \operatorname{Log} T - \frac{1 + \operatorname{Log} 2\pi}{\pi} \cdot T + \rho(T) \cdot \operatorname{Log} T \quad \text{where} \quad |\rho(T)| \leq C_0$$

where the constant C_0 is indepenent of T. We prove (*) in the chapter devoted to Riemann-s ζ-function whose content should be more than sufficient to convince that analytic function theory in one complex variable is as active as ever within the curriculum of mathematics. Anybody who tries to attack the Riemann Hypothesis must be well aquainted with function theory and harmonic analysis. This explains why these notes in addition to analytic function theory contains extensive material about the Fourier transform. Let us illustrate this interaction by the following formula which at first sight appears to "almost settle" the Riemann hypothesis: Hardy proved the equality below when $\mathfrak{Re}\, s > 1$:

$$(**) \qquad \zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \cdot \int_0^{\infty} \omega(x) \cdot x^{\frac{s}{2}-1} \cdot dx \quad : \quad \omega(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$$

where $\Gamma(s)$ is the Gamma-function. The constraint $\mathfrak{Re}\, s > 1$ is needed to ensure that the integral in the right hand side is defined since one has the asymptotic formula

$$\omega(x) \simeq \frac{1}{\sqrt{x}} \quad x \to 0$$

A major fact to be proved in these notes is that $\frac{1}{\Gamma(s)}$ is an entire function of the complex variable s. So (**) can be used to investigate zeros of the zeta-function after its meromorphic continuation is constructed via the Dirichlet integral of the ω-function. The equation (**) was used by Hardy in [Ha] to demonstrate that a "considerable amount" of zeros belong to the critical line $\mathfrak{Re}\, s = 1/2$.

Beurling's criterion. A necessary and sufficient condition for the validity of the Riemann hypothesis was presented by Beurling in the article *A closure problem related to the Riemann zeta-function* and goes as follows: Denote by $\rho(x)$ the numerically smallest remainder term modulo 1, i.e. $\rho(x) = x$ if $0 < x < 1$ and $\rho(x+1) = \rho(x)$ for all real x. Given some $N \geq 2$ and an N-tuple of real numbers $0 < \theta_\nu < 1$ and real constants $c_1, \ldots, c_N$ such that $\sum c_\nu \cdot \theta_\nu = 0$ we define a function $f(x)$ on the unit interval $(0,1)$ by

$$f(x) = \sum_{\nu=1}^{\nu=N} c_\nu \cdot \rho(\frac{\theta_\nu}{x})$$

Theorem. *The Riemann hypothesis is valid if and only if for each $\epsilon > 0$ there exists some N and a pair of N-tuples $\{\theta_\nu\}$ and $\{c_\nu\}$ as above such that*

$$\int_0^1 (1 - f(x))^2 \cdot dx < \epsilon$$

The sufficieny, i.e. that the existence of f-functions which approximate the identity function in the L^2-norm gives the Riemann Hypothesis is easy to prove, while the proof of necessity relies upon an ingenious blend of functional analysis and analytic function theory. Personally I find Beurling's proof exceedingly instructive and hope that this will be shared by the reader. For details we refer to section xx in Chapter VIII.

Alan Turing and the ζ-function. The creation of computers is foremost due to Turing and von Neumann. Less wellknown is perhaps that Turing also was fascinated by the zeta-function. His article *A method for the calculation of the zeta-function* from 1943 already suggested his ambition to build an analog computer specifically intended for calculating values of $\zeta(s)$. A decade later his article *Some calculations of the Riemann zeta-function* gave the first extensive calculations of zeros of $\zeta(s)$ by a computer. This historic reconcilation illustrates how problems in mathematics has stimulated advancement in applied areas of science. The reader may consult the article by Dennis Hejhal in [Hej] for a further account about Turing's work related to the ζ-function and up-to-date references about the zeta-function.

0.8 Kepler's equation and Lagrange's problem

During intense calculations based upon astronomic data in the beginning of 1600, Kepler encountered the equation

$$(*)\qquad \zeta_a(z) = a + z \cdot \sin(\zeta_a(z))$$

where $a > 0$ is a positive constant which may vary. At that time analytic function theory did not exist and Kepler could only manage to derive a part of the series expansion of the ζ_a-function. In 1760 Lagrange found the whole series expansion of $\zeta_a(z)$ and proved that it is analytic in a disc centered at $z = 0$ whose radius of convergence ρ is independent of a. However, no exact formula for ρ is known, i.e. it can only be found numerically. Personally I find historic examples of this kind both instructive and exciting since they illustrate how "heroic efforts" have preceded a "more developed theory". In these notes the reader will learn how basic analytic function theory can be applied to obtain Lagrange's result about Kepler's $\zeta_a(z)$-functions.

A problem by Lagrange. Let us give another early example where complex methods appear naturally. In a work from 1782 Lagrange studied the motion of the planets in the solar system which by Newton's Law of gravity amounts to solve the N-body problem with $N \geq 3$. As a first approximation for "la longitude du périhélie de l'orbite d'une planete au temps t" one needs to determine the argument of a complex-valued trigonometric polynomial

$$F(t) = a_1 e^{i\lambda_1 t} + \ldots + a_N e^{i\lambda_N t}$$

where $\lambda_1, \ldots, \lambda_N$ are distinct real numbers and $a_1, \ldots, a_N$ some N-tuple of complex numbers. Assume that $F(t) \neq 0$ for all $t \geq 0$. Then there exists a continuous argument function $\psi(t)$ of F such that:

$$F(t) = |F(t)| \cdot e^{i\psi(t)} \quad : t \geq 0\,.$$

When $|a_N| > |a_1| + \ldots + |a_{N-1}|$ it is easily seen that

$$(*)\qquad \lim_{T \to \infty} \frac{\psi(T)}{T} = \lambda_N\,.$$

Lagrange posed the question if there exists a limit in the general case when no single a-coefficient has a dominating absolute value. This problem with its source in astronomy has led to interesting results in analytic function theory. In 1908 it was proved by Bohl that (*) always has a limit when $N = 3$. For $N \geq 4$ quite general results were found by Weyl and later work has studied the existence of (*) for the more extensive class of almost periodic analytic functions. Let us also mention that the study of (*) is closely related to find zeros of analytic functions in vertical strip domains. More precisely, given $F(t)$ as above it is the restriction of an entire function $F(\zeta)$ to the imaginary axis in the complex ζ-plane where $\zeta = \sigma + it$. For each real σ it turns out that there exists the limit

$$\phi(\sigma) = \mathrm{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \log |F(\sigma + it)| \cdot dt\,.$$

A result due to B. Jessen and M. Tornehave in their joint article [Je-Torn] from 1945 shows that the ϕ-function is continuous and convex with respect to σ and has an ordinary derivative outside a set which contains at most denumerable many points. Moreover, assuming that ϕ has a derivative at $\sigma = 0$ one has the equality

$$\mathrm{Lim}_{T\to\infty} \frac{\psi(T)}{T} = \phi'(0)\,.$$

Finally the derivatives of the ϕ-function determine the asymptotic number of zeros of $F(z)$. More precisely, for a given pair of real numbers $a < b$ we denote by $N_{a,b}(T)$ the number of zeros of $F(\zeta)$ in the domain $\{\zeta = \sigma + it \;:\; a < \sigma < b \text{ and } 0 < t < T\}$. Then

$$\mathrm{Lim}_{T\to\infty} \frac{N(T)}{T} = \frac{\phi'(b) - \phi'(a)}{2\pi}$$

hold when ϕ has a derivative at a and b. Results of this kind illustrate the rich interplay between analytic function theory and other areas of mathematics.

Conformal mappings.

In the 3-dimensional space $\mathbf{R}^3$ with coordinates (x, y, z) we consider a surface S defined by an equation

$$z = g(x, y)$$

where g is a real-valued function which is real-analytic as a function of the two real variables (x, y). One may imagine a portion of such a surface which is defined when (x, y) stays in a small square $\square$ centered at the origin in $\mathbf{R}^2$. Given an arbitrary point $p = (x, y, g(x, y))$ on S we consider a pair of curves γ and ρ passing through p. They intersect at p with some angle denoted by $\langle\gamma, \rho\rangle$. Next, let (ξ, η) be another set of coordinates in $\mathbf{R}^2$ and consider a map

$$\phi\colon\; (x, y, g(x, y)) \mapsto (\xi(x, y), \eta(x, y))$$

Working locally we assume that the map is a diffeomorphism and preserves orientation which means that the Jacobian $\xi_x \cdot \eta_y - \xi_y \cdot \eta_x > 0$. Now we can consider the ρ-images of a pair of curves γ, ρ as above and in the (ξ, η)-plane we regard the angle between these two plane curves. If the angle is preserved for every pair of curves passing through arbitrary points on S one says that ϕ is an *conformal* representation of S onto a plane. The first example of such a conformal map goes back to the astronomer Ptolemy who considered the case when S is a portion of a sphere and found the answer by the *stereographic projection* which has become a useful in analytic function theory. Recall also that Mercator constructed another conformal map in 1568 which has been adopted for the construction of sea-maps. In 1779 Lagrange classified all locally defined conformal maps from a portion of the sphere onto the plane and in 1821 Gauss described the whole family of locally defined conformal maps from an arbitrary surface S onto a planar domain. A turning point which led to the contemporary analytic function theory was discovered by Riemann in his thesis from 1851. He introduced ideas which have been used in later investigations of conformal representation and showed that the problem itself is of fundamental importance for analytic function theory. A major result is that every bounded and simply connected domain in $\mathbf{C}$ is conformal with the open unit disc D. This is called *Riemann's Mapping Theorem.* The first rigorous proof was given in 1865 by Hermann Schwartz. More generally there is a mapping theorem for an arbitrary connected domain Ω in $\mathbf{C}$. For topological reasons we cannot find a conformal map from Ω onto D. But there exist locally conformal maps from Ω onto D expressed by *multi-valued* analytic functions on Ω which are used to obtain a holomorphic function $g(z)$ defined in D such that the complex derivative $g'(z) \neq 0$ for all points and the image $g(D) = \Omega$. This identifies the fundamental group of Ω with a group of Möbius transformations on the unit disc D. Starting from this fact, Poincaré began to investigate general groups of Möbius transformations. He considered to begin with a discontinuous group $\mathcal{F}$ of Möbius transforms on D without a fixed point. For such a group there exists a normal domain U inside D characterized by the property that every orbit under $\mathcal{F}$ intersects U in exactly one point. This result, proved in his article *Théorie des groupes fuchsien* was published in the first volume of Acta Mathematica in 1882. All this may at first glance appear as "classic old stuff". But the truth is that the visions by Riemann and Poincaré continue to serve as an inspiration in contemporary mathematics. To this one should also say that analytic function theory in one variable already had reached a quite advanced level more than a century ago. For example, a result which goes beyond the material in these notes is the study of the differential equation $\Delta(u) = e^u$ where one seeks a solution u defined on a closed Riemann surface with prescribed singularities. This second order differential equation was completeley solved by Poincaré in his article [Poinc] *Les fonctions fuchsiennes et l'équation* $\Delta(u) = e^u$ from 1898.

0.9 The Dagerholm series and comments on more recent theories.

Of course, many results in these notes are of a more recent origin. An example is an extension of the Riemann Mapping Theorem by Beurling from an article published in 1953. A result which illustrates that more recent theorems also are concerned with "concrete problems" was published in 1968 by Karl Dagerholm who was Beurling's first Ph.d student from Uppsala University in 1938, and thirty years later Dagerholm was finally able to prove the following result which had remained as an open question from his thesis:

Theorem. *Up to multiplication with a real number there exists a unique sequence* $\{x_q\}$ *of real numbers which is not identically zero and solves the infinite system of equations*

$$\sum_{q \neq p} \frac{x_q}{p-q} = 0 \quad : \quad p = 1, 2, \ldots \quad \text{where } \sum_{q=1}^{\infty} \frac{x_q}{q} \ \text{ is convergent.}$$

As we shall see the proof this result employs many ingredients from analytic function theory.

The Hayman-Wu constant. To illustrate that analytic function theory in one complex variable contains many active research areas we mention the following open problem: Let $\phi\colon D \to \Omega$ be a conformal map from the unit disc onto a simply connected domain Ω. Now $\mathfrak{Im}(\phi)$ is a harmonic function in D whose gradient is everywhere $\neq 0$. As we shall explain later on this implies that the zero set $\{\mathfrak{Im}(\phi) = 0\}$ is a union of disjoint open Jordan arcs $\{\gamma_k\}$ where each single arc has two end-points on the unit circle ∂D. Denote by $\ell(\gamma_k)$ the arc-length of each individual Jordan arc. In 1981 it was proved by Hayman and Wu that there exists an absolute constant C such that

$$(*) \qquad \sum \ell(\gamma_k) \leq C$$

The remarkable fact is that C is independent of the conformal map. To find the best constant in (*) is an open problem. So far one knows that C is strictly smaller than 4π [Rohde 2002] and at least π^2 [Öyma 1993].

Remark. The text-book [GM] *Harmonic measure* by John B. Garnett and Donald E. Marshall has been an inspiration for these notes. Actually I started to write these notes to provide background for the reader who wants to study more advanced topics in [GM]. But as in most projects of this sort my original material eventually expanded and cover subjects in analytic function theory which are not within the main streamline in [GM]. In any case, this text-book is highly recommended. For example, Chapter V offers a brilliant account of results due to Lars Ahlfors and contains also an introduction to the theory of extremal distances which was created by Arne Beurling during the years 1943-44 and presented in his article [Beurling] from 1946. An example of a result which goes beyond the scope of these notes appears in [GM; Chapter V : Theorem 2.1] which gives a quite sharp lower bound for harmonic measures. See also Chapter XX in R. Nevanlinna's text-book [Nev] where the Carleman-Milloux problem is studied via certain solutions which were found independently by Nevanlinna and Beurling in 1933. In these notes we are content to establish results which provide upper bounds for harmonic measures in various situations.

0.10 Mathematical physics.

Analytic function theory of one complex variable is often used in mathematical physics. An example is the *Biot-Savart Law* which was discovered in 1800 and inspired the development of analytic function theory. In 1879 the electric engineer Robin posed a quite general problem which led to the study of thin sets and various potential theoretic questions. The special case occurs when Γ is a closed Jordan curve in the complex z-plane. To determine a density function which gives electric equilibrium on Γ amounts to find a positive function μ on Γ such that the logarithmic potential

$$(*) \qquad \int_\Gamma \log \frac{1}{|q-p|} \cdot \mu(p) \cdot ds(p) \quad : \quad q \in \Gamma$$

where $ds(p)$ is the arc-length along Γ and one requirs that (*) is constant as q varies on Γ. Riemann demonstrated the existence of such a density by his conformal mapping theorem. More precisely, the equilibrium density is unique and equal to a constant times $\frac{1}{|f'(z)|}$ where $f'(z)$ is the complex derivative of the conformal map from the *exterior domain* bordered by Γ to the exterior disc $|w| > 1$ in the complex w-plane. We prove this result in the chapter devoted to conformal mappings. This example is instructive since it shows that one should be aware of the point at infinity. This was the reason why Riemann introduced the Riemann sphere in his fundamental work [Rie:xx] from 1857. Let us also remark that analytic function theory does not "cover everything" as the following example shows. A classical result due to Helmholtz about stationary fluid motion in the plane asserts that if Ω is a Jordan domain and the function

$$\text{(i)} \qquad z \mapsto \iint_\Omega \log \frac{1}{|z-\zeta|} \cdot d\xi d\eta$$

is constant on $\partial\Omega$, then Ω must be a disc. This fact has a natural physical explanation. Less obvious is that that Ω also must be a disc if (i) holds when the log-potentiao is replaced by an integrand of the form $f(|z-\zeta|)$, where $f(r)$ is an arbitrary strictly decreasing function defined on $r \geq 0$. Here the proof uses calculus of variation and the symmetrization process due to W. Gross. See § XX in special topics where the asserted result is proved.

The use of majorant series. The idea to use majorant series is due to Cauchy and is often used to prove existence results for non-linrear equations. Let us give an example where Cauchy's method applies. Let Ω be a bounded open domain in $\mathbf{R}^3$ whose boundary $\partial\Omega$ is of class C^1. Let $p = (x, y, z)$ denote points in $\mathbf{R}^3$. The equation which obeys the Stefan-Bolzmann law for heat conduction of Black Bodies is to find a function u which is harmonic in Ω outside a point p_* (the source of heat) where $u(p)$ is locally of the form $\frac{1}{|p-p_*|}$ plus a harmonic function. On the boundary its interior normal derivative satisfies:

$$(*) \qquad \partial u/\partial \mathbf{n} = k \cdot u^4 \quad : \quad k > 0$$

Green's formula shows that the solution to (*) is unique if it exists. The proof of existence is achieved by regarding intermediate equations where $u_h(p)$ for every $0 < h < 1$ is a solution with the pole at p_* as above and

$$(**) \qquad \partial u_h/\partial \mathbf{n} = k \cdot ((1-h)u_h + h \cdot u_h^4)$$

Using the robust properties of solutions to Neumann's linear boundary value problem one shows that if $0 \leq h_0 < 1$ is given and the unique solution u_{h_0} has been found, then (**) is solved for h-values in some interval $h_0 < h < h_0 + \delta$ via a series expansion $u_h = u_{h_0} + \sum_{\nu=1}^{\infty} (h-h_0)^\nu \cdot w_\nu$. Here $\{w_\nu\}$ are functions which are found inductively by solving a system of *linear* boundary value problems. In this way the main burden to obtain a solution to the non-linear Stefan-Boltzmann equation is to prove that various series admit a positive radius of convergence. In XX we expose Carleman's methods from [Car] to solve the more general non-linear boundary value problem where one seeks u as above in Ω while

$$(***) \qquad \partial u/\partial \mathbf{n}(p) = F(u(p), p) \quad : p \in \partial\Omega$$

Here the sole condition is that $F(u, p)$ is a on-negative continuous function defined on $\mathbf{R}^+ \times \partial\Omega$, and for each fixed $p \in \partial\Omega$ the map $u \mapsto F(u, p)$ is increasing and tends to $+\infty$ with u.

Another boundary value problem. Let Ω be a bounded domain in the real (x, y)-plane of class $\mathcal{D}(C^1)$, i.e. the boundary is a finite union of disjoint and closed differentiable Jordan curves. Denote by $\Delta = \partial_x^2 + \partial_y^2$ the Laplace operator. One seeks functions $\phi(x, y)$ which satisfy

$$\Delta(\phi) + \lambda \cdot \phi = 0$$

in Ω for some constant λ where $\phi = 0$ on the boundary $\partial\Omega$. This equation has solutions for a discrete sequence of eigenvalues $\{\lambda_n\}$ where $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and the corresponding eigenfunctions $\{\phi_n\}$ are normalised so that the integrals

$$\iint_\Omega \phi_n(x, y)^2 \, dxdy = 1$$

Moreover, these ϕ-functions can be chosen to be pairwise orthogonal when we take area integrals. This is proved by classical methods using Fredholm determinants and hence only calculus in real variables is used, while analytic function theory is needed to study asymptotic properties. Tohe crucuial result gies as follows where s is a complex variable and p a variable point in Ω:

Theorem. *The series*

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\phi_n^2(p)}{\lambda_n^s}$$

is a meromorphic function of s defined in the whole complex s-plane which is independent of the point $p \in \Omega$. Moreover, ϕ has a simple pole at $s = 1$ whose residue is $\frac{1}{4\pi}$ and it has zeros at all non-positive integers.

Using Tauberian theorems which also are derived via analytic function theory one gets the following asymptotic formula:

Theorem. *Let S be the area of Ω. Then*

$$\lim_{n \to \infty} \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} \phi_\nu^2(p) = \frac{1}{S}$$

hold for each $p \in \Omega$.

Remark. Results such as thoses above illustrate that one attains "muscles" via studies in analytic function theory in order to establish asymptotic results for solutions to PDE-equations.

Quantum mechanics. Analytic function theory as well as extensive use of Fourier integral operators occur also in quantum mechanics while one investigates special equations. For example, confluent hypergeometric functions appear as solutions in a Coulomb field and to continue the study of scattering in a Coulomb field where the quantum-mechanical collision problem can be solved exactly, it is from a physical point of view important to analyze the asymptotic behaviour of confluent hypergeometric functions. Here asymptotic expansions are derived via complex line integrals where attention must be given to the choice of local branches of certain multi-valued analytic functions. Examples of this kind illustrate that classical topics in analytic function theory remain "up-to date" since they provide useful tools in quantum mechancs. It would lead us to far to even try to give a glimpse of the physical theories. For the mathematically oriented reader we recommend the outstanding text-books by L.D. Landau and his former student E.M Lifschitz. The third edition of *Non-relativistic quantum mechanics* is especially recommended.