Comments to Bernstein-Sato polynomials, central characters and stratification.

The authors deal with the algebraic case but let me formulate the set-up in the analytic context. Let X be some n-dimensional complex manifold and $f \in \mathcal{O}(X)$. Set $T = f^{-1}(0)$ where f in general can have multiple factors, i.e. the principal ideal (f) can be strictly larger than the defining ideal $\mathcal{I}_T$. Shrinking X to a suitable neighbrhood of T we assume that the 1-form $df \neq 0$ outside T. Next, let $\mathcal{D}_X$ be the sheaf of holomorphic differential operators on X. Let s be a new parameter which yields the left $\mathcal{D}_X$-module

$$\mathfrak{N} = \mathcal{O}_X[f^{-1}, s] \cdot f^s$$

where sections in $\mathcal{O}_X[f^{-1}, s]$ are s-polynomials with coefficients in the sheaf $\mathcal{O}_X[*T]$ of meromorphic functions with poles contained in T. The left $\mathcal{D}_X$-module structure on $\mathfrak{N}$ is determined via the rule

$$\delta(f^s) = \delta(f)^{-1} s \cdot f^s \quad : \delta \in \mathfrak{g}_X$$

where $\mathfrak{g}_X$ is the sheaf of holomorphic vector fields regarded as first order holomorphic differential operators. Following Kashiwara's work in his article on b-functiones (Inventiones 1975) one introduces the $\mathcal{D}_X$-submodule

$$\mathcal{N} = \mathcal{D}_X[s] \cdot f^s$$

Multiplication with s yields a left $\mathcal{D}_X$-linerar map on $\mathcal{N}$. A crucial result from [Ka:1] is that $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module which is subholonomic, i.e. its singular spectrum $\mathrm{SS}(\mathcal{N})$ has dimension $d_X + 1$. Let us remark that this follows easily since f belongs to the integral closure of the $\mathcal{O}_X$-ideal generated by its first order derivatives. Next, $\mathcal{N}$ contains the $\mathcal{D}_X$-submodule

$$\mathcal{N}_* = \mathcal{D}_X[s] \cdot f^{s+1}$$

Since s is a parameter the $\mathcal{D}_X$-modules $\mathcal{N}$ and $\mathcal{N}_*$ are isomorphic. From this it follows that the quotient module

$$\mathcal{M} = \frac{\mathcal{N}}{\mathcal{N}_*}$$

is holonomic. Multiplication with s is a $\mathcal{D}_X$-linear map on $\mathcal{M}$ and holonomicity entails that it has locally minimal polynomials. Working locally around a point $p \in X$ where X is taken as a polydisc centered at p we find a unique minimal polynomial $b(s)$ such that $b(s)\mathcal{M} = 0$ holds in X. In [Ka:1] desingularisation was used to prove that the roots of the b-function are strictly negative rational numbers where eventual multiple zeros can occur. If

$$b(s) = \prod (s + q_\nu)$$

where eventual multipe zeros are repeated we can consider subproducts

$$\beta(s) = \prod (s + q_j)$$

where $\{q_j\}$ is a proper subfamily of the b-roots. Now there exists the analytic set

$$V_\beta = \mathrm{Supp}(\beta(s)\mathcal{M})) \tag{*}$$

From this collection of analytic subsets of X one gets a stratification where the minimal stalkwise defined b-function for f^s is constant over each stratum. This is the topic treated in the abstract by the authors. Of course, it has been considered by many authors and certainly deserves attention since roots of the b-functions are closely related to delicate topological properties of the hypersurface T and also to asymptotic expansion of current-valued functions such as

$$s \mapsto \int_{f=s}$$

studied in great detail by Barlet. Whether it is really possible to exhibit explicit stratifications for a given f is not so clear to me, i.e. even with the aid of computer algebra it appears to be a hard problem which only can be sttled in special cases. The authors give however a nice example for a polynomial in 3 variabes. A special case which was studied at an early stage and was suggested by Frederic Pham in 1978 arises when we start with f and assume that there exists a holomorphic vector field ρ such that

$$\rho(f) = f$$

If we know f and ρ the stratification where the minmal b-function is constant can be expressed as follows: In $\mathcal{D}_X$ where X is the n-dimensional polydisc with coordinstes $x_1, \ldots, x_n$ we have the n-tuple of derivation operators $\{\partial_i\}$. Let $\mathcal{L}$ be the left ideal generated by the n-tuple of first order differential operators

$$(**) \qquad f \cdot \partial_i - \partial_i(f)\rho \quad : 1 \leq i \leq n$$

Then (*) corresond to the inclusions

$$\beta(\rho) \in \mathcal{L} + \mathcal{D}_X f$$

where the last term is the left principal ideal generated by f.

Remark. Above we recalled Kasiwara's studies of b-functions for a single holomorphic function. The interested reader should also consult his plenary talk about b-functions from the IMU-cogress at Helsinki in 1978 for futher interesting comments about b-functions. More generally one associates b-functions to pairs $(\mathcal{M}, f)$ where $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module and f is a non-zero divsor on $\mathcal{M}$, i.e. $\mathcal{M}$ does not contain non-vanishing section supported by the hypersurface T. My book *Analytic $\mathcal{D}$-modules* expose the major results in this situation where similar strafications as for a single holomorphic function are achieved.

Concluding remarks As already said the proposed topics by the authors certainly deserve attention. But their abstract does not make it clear to what extent "computational progess" has been achived to illustrate and eventually also consolidate the general theory which was established a long time ago, foremost by Kashiwara., and in the algebraic case one should also give tribute to Bernsteins's work. In this connection I also think it woud be interesting to study the specific cases by Sato where b-functions appear in a quite explicit way in his work on prehomogenous spaces from 1960. For readers more familiar with analysis one should mention that the issue in the abstract is related to natural and important problems concerning meromorphic extensions of distribution-valued functions. With f as above there exists for every positive integer m the $\mathfrak{Db}(X)$-valued function

$$(*) \qquad \mu_{m\lambda}(\phi) = \int_X f^{-m} \cdot |f|^{2\lambda} \cdot \phi$$

where ϕ are test-forms on X of maximal bi-degree (n, n). The local existence of b-functions imply that (*) extends to a meromorphi function in the whole complex λ-plane whose poles are confined to a finite union of arithmetic progressions of the form $\{-q - \nu \quad : \nu \geq 0\}$ where q are positive rational numbers abnd this inclusion of poles hold for every $m \geq 0$ above. Suppose for example thaat $0 < q < 1$ is a rational number and poles coccur at $-q - \nu$ fora non.empty set of non-negative integers ν. The resulting polar distributions which appear in Laurent expansions of $\mu_{m\lambda}$ at such poles are of interest in many applications to PDE-theory. Recall here a theorem due to Barlet which asserts that poles of $\mu_{m,\lambda}$ give an *effective contribution* to roots of the b-function. So if one for example is content to analyse stratifications where the b-function is constant up to multiple zeros and shifts by integers, the support of such distributions would serve as a stratification. This remark is given to illustrate the great complexity - but also the interest - of the topics proposed by the abstract.

As an *overall evaluation* I suggest that the abstract should be ranked as "weakly accepted" though I would add that it may be regarded to be rather close to the higher evaluation "accept".

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