XI. The Denjoy conjecture

Introduction. Let ρ be a positive integer and $f(z)$ is an entire function such that there exists some $0 < \epsilon < 1/2$ and a constant A_ϵ such that

$$|f(z)| \leq A_\epsilon \cdot e^{|z|^{\rho+\epsilon}} \tag{0.1}$$

hold for every z. Then we say that f has integral order $\leq \rho$. Next, the entire function f has an asymptotic value a if there exists a Jordan curve Γ parametrized by $t \mapsto \gamma(t)$ for $t \geq 0$ such that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and

$$\lim_{t \to +\infty} f(\gamma(t)) = a \tag{0.2}$$

In 1920 Denjoy raised the conjecture that (0.1) implies that the entire function f has at most 2ρ many different asymptotic values. Examples show that this upper bound is sharp. The Denjoy conjecture was proved in 1930 by Ahlfors in [Ahl]. A few years later T. Carleman found an alternative proof based upon a certain differential inequality. Theorem A.3 below has applications beyond the proof of the Denjoy conjecture for estimates of harmonic measures. See [Ga-Marsh].

A. The differential inequality.

Let Ω be a connected open set in $\mathbf{C}$ whose intersection S_x between a vertical line $\{\mathfrak{Re}\, z = x\}$ is a bounded set on the real y-line for every x. When $S_x \neq \emptyset$ it is the disjoint union of open intervals $\{(a_\nu, b_\nu)\}$ and we set

$$\ell(x) = \max_\nu\,(b_\nu - a_\nu) \tag{*}$$

Next, let $u(x, y)$ be a positive harmonic function in Ω which extends to a continuous function on the closure $\bar{\Omega}$ with the boundary values identical to zero. Define the function ϕ by:

$$\phi(x) = \int_{S_x} u^2(x, y) \cdot dy \tag{1}$$

The Federer-Stokes theorem gives the following formula for the derivatives of ϕ:

$$\phi'(x) = 2\int_{S_x} u_x \cdot u(x, y)dy \tag{2}$$

$$\phi''(x) = 2\int_{S_x} u_{xx} \cdot u(x, y)dy + 2\int_{S_x} u_x^2 \cdot dy \tag{3}$$

Since $\Delta(u) = 0$ when $u > 0$ we have

$$2\int_{S_x} u_{xx} \cdot u(x, y)dy = -2\int_{S_x} u_{yy} \cdot u(x, y)dy = 2\int u_y^2 dy \tag{4}$$

The Cauchy-Schwarz inequality applied in (2) gives

$$\phi'(x)^2 \leq 4 \cdot \int_{S_x} u_x^2 \cdot \int_{S_x} u^2(x, y)dy = 4 \cdot \phi(x) \cdot \int_{S_x} u_x^2 dy \tag{5}$$

Hence (4) and (5) give:

$$\phi''(x) \geq 2\int_{S_x} u_y^2(x, y) \cdot dy + \frac{1}{2} \cdot \frac{\phi'^2(x)}{\phi(x)} \tag{6}$$

Next, since $u(x, y) = 0$ at the end-points of all intervals of S_x, *Wirtinger's inequality* and the definition of $\ell(x)$ give:

$$\int_{S_x} u_y^2(x, y) \cdot dy \geq \frac{\pi^2}{\ell(x)^2} \cdot \phi(x) \tag{7}$$

Inserting (7) in (6) we have proved

A.1 Proposition *The ϕ-function satisfies the differential inequality*

$$\phi''(x) \geq \frac{2\pi^2}{\ell(x)^2} \cdot \phi(x) + \frac{\phi'^2(x)}{2\phi(x)}$$

Proof continued. The maximum principle for harmonic functions implies that the $\phi(x) > 0$ when $x > 0$ and hence there exists a ψ-function where $\phi(x) = e^{\psi(x)}$. It follows that

$$\phi' = \psi' e^{\psi} \quad \text{and} \quad \phi'' = \psi'' e^{\psi} + \psi'^2 e^{\psi}$$

Now Proposition A.1 gives

(*)
$$\psi'' + \frac{\psi'^2}{2} \geq \frac{2\pi^2}{\ell(x)^2}$$

A.2 An integral inequality. From (*) we obtain

$$\frac{2\pi}{\ell(x)} \leq \sqrt{\psi'(x)^2 + 2\psi''(x)} \leq \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

Taking the integral we get

(**)
$$2\pi \cdot \int_0^x \frac{dt}{\ell(t)} \leq \psi(x) + \log\, \psi'(x) + O(1) \leq \psi(x) +, \psi'(x) + O(1)$$

where $O(1)$ is a remainder term which is bounded independent of x. Taking the integral once more we obtain:

A.3 Theorem. *The following inequality holds:*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + O(x)$$

where the remainder term $O(x)$ is bounded by Cx for a fixed constant.

B. Solution to the Denjoy conjecture

B.1 Theorem. *Let $f(z)$ be entire of some integral order $\rho \geq 1$. Then f has at most 2ρ many different asymptotic values.*

Proof. Suppose f has n different asymptotic values $a_1, \ldots, a_n$. To each a_ν there exists a Jordan arc Γ_ν as described in the introduction. Since the a-values are different the n-tuple of Γ-arcs are separated from each other when $|z|$ is large. So we can find some R such that the arcs are disjoint in the exterior disc $|z| > R$. We may also consider the tail of each arc, i.e. starting from the last point on Γ_ν which intersects the circle $|z| = R$. So now we have an n-tuple of disjoint Jordan curves in $|z| \geq R$ where each curve intersects $|z| = R$ at some point p_ν and after the curves moves to the point at infinity. See figure. Next, we take one of these curves, say Γ_1. Let D_R^* be the exterior disc $|\zeta| > R$. In the domain $\Omega = \mathbf{C} \setminus \Gamma_1 \cup D_R^*$ we can choose a single-valued branch of $\log \zeta$ and with $z = \log\, \zeta$ the image of Ω is a simply connected domain Ω^* where S_x for each x has length strictly less than 2π The images of the Γ-curves separate Ω^* into n many disjoint connected domains denoted by $D_1, \ldots, D_n$ where each D_ν is bordered by a pair of images of Γ-curves and a portion of the vertical line $x = \log R$.

Let $\zeta = \xi + i\eta$ be the complex coordinate in Ω^*. Here we get the analytic function $F(\zeta)$ where

$$F(\log(z)) = f(z)$$

We notice that F may have more growth than f. Indeed, we get

(1)
$$|F(\xi + i\eta)| \leq \exp\bigl(e^{(\rho+\epsilon)\xi}\bigr)$$

With $u = \text{Log}^+ |F|$ it follows that

$$u(\xi, \eta) \leq e^{(\rho+\epsilon)\xi} \tag{2}$$

Hence the ϕ-function constructed during the proof of Theorem A.3 satisfies

$$\phi(\xi) \leq e^{2(\rho+\epsilon)\xi}$$

It follows that the ψ-function satisfies

$$\psi(\xi) = 2 \cdot (\rho + \epsilon)\xi + O(1) \tag{3}$$

Now we apply Theorem A.3 in each region D_ν where we have a function $\ell_\nu(\xi)$ constructed by (0) in section A. This gives the inequality

$$2\pi \cdot \int_R^\xi \frac{\xi - s}{\ell_\nu(s)} \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1) \quad : \quad 1 \leq \nu \leq n \tag{4}$$

Next, recall the elementary inequality which asserts that if $a_1, \ldots, a_n$ is an arbitrary n-tuple of positive numbers then

$$\sum a_\nu \cdot \sum \frac{1}{a_\nu} \geq n^2 \tag{5}$$

For each s we apply this to the n-tuple $\{\ell_\nu(s)\}$ where we also have

$$\sum \ell_\nu(s) \leq 2\pi$$

So a summation in (4) over $1 \leq \nu \leq n$ gives

$$n \cdot \int_R^\xi (\xi - s) \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1) \tag{6}$$

Another integration gives:

$$n \cdot \frac{\xi^2}{2} \leq (\rho + \epsilon) \cdot \xi^2 + O(\xi) \tag{7}$$

This inequality can only hold for large ξ if $n \leq 2(\rho + \epsilon)$ and since $\epsilon < 1/2$ is assumed it follows that $n \leq 2\rho$ which finishes the proof of the Denjoy conjecture.