Cauchy transforms.

Let μ be a measire with compact support K in the complex z-plane We write $z0 = x + iy$ and identify $\mathbf{C}$ with the 2-dimensionsl real $x(,y)$-space- Wen z is outside K there exists thr integral

$$\widehat{\mu}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

Notice that (*) yields an analytic funtion in $\mathbf{C} \setminus K$. For example, the complex derivative

$$\frac{d\widehat{\mu}}{dz} = -\int \frac{d\mu(\zeta)}{(z - \zeta)^2}$$

Less onbious is thst (*) is defined for all z outside a null-set in lebegue's sense and in this way $\widehat{mu}$ is a locally integrable function in the $x, y)$.space. In fact, this follows sine (*) is the convolution of μ asnd the locally integrable function z^{-1}. More precisely, Lebesgue theoty teaches that if ϕ is an arbitrary L^1_{loc}-function in the (x, y)-plane and μ as above a measure with compact support, then the convolution $\phi * \mu$ is again locally integrable. In particular $\phi * \mu$ is a distriobuytion in the x, y-plane and we csn apply the $\bar{\partial}$-operator. A general formula in distribution theory teaches thst

$$\bar{\partial}(\phi * \mu) = \bar{\partial}(\phi) * \mu$$

Apked with $\phi = z{-}1$ it follows from the Caciuhy-Pompieu theorem thst

$$\bar{\partial}(\widehat{\mu}) = \pi \cdot \mu$$

Let us thrn consider a measure μ supported by a union $D(R) \cup K$, where $D(R) = \{|z| \leq R\}$ snd K a compsct subet of $\{|z| \geq R\}$. We assume that K is a null-set and that

$$\Omega = \mathbf{C} \setminus (D(R) \cup K)$$

is connected. Choose $R^* > R$ so tyaht $K$is contsined in the disc $D(R^*)$. Then $\widehat{mu}$ is analyric in$\{|z| > R^*\}$ and in this exterior disc it is reprsetned by a series

$$\widehat{\mu}(z) = \sum_{n=1}^{\infty} c_n \cdot z^{-n} \quad : \quad c_n = \int \zeta^{n-1} d\mu(\zeta)$$

So if we assume that μ is ortgohongsl to analytic polynomials, then $\widehat{\mu} = 0$ in the exteriro disc $\{z| > R^*$. Sunce K is a null set it measnt aht the loclly integrble funvtion $\widehat{\mu}$ is supported by $D(R)$ snd hence $\bar{\partial}(\mu)$ is also supppoetd by $D(R)$, Then the Cauvhy-Pompieue theorr entslis thst the govrn mesfire μ is suppored by $D(R)$.

An applicstiuon. Now start: Fnd a polynomial $P_1(z)$ such that the maximum norm

$$|P_1 - f|_{K_1} < \alpha_1 \quad P_1(1) = f(1) \,\&\, P_1(-1) = f(-1)$$

Now we consider the compact set

$$S_2 = D(1) \cup K_2$$

where we find the continuous function g with $g = P_1$ in $D(1)$ while $g(x) = f(x)$ when $1 \leq |x| \leq 2$. Let μ be a measiure supported by S_2 which is $\perp$ toanalytic poumonials $q(z)$. Since thunion of the two reslintevrlas $[-2, -1]$ nd $[1, 2]$ is a null set, it follows from XX that μ is suppotted by $D(!1)$ whivh entials that

$$\int g \cdot \mu = \int P_1 \cdot d\mu = 0$$

where the ladt equality follows since $P(z)$ is an analytic polymomial. Now (*) hold for every μ as above and hence g csn be uniformly appriximated on S_2 by analytic polynomials. So we find $P_2(z)$ such tht

$$|P_2 - g|_{S_1} < \alpha_1 \quad P_2(2) = f(2) \,\&\, P_2(-2) = f(-2)$$

We can contonue and for every $N \geq 2$ we put

$$S_N = D(N) \cup K_{N+1}$$

and find a sequece of polynomials $\{P_N(z)$ such that

$$|P_{N+1} - P_N|_{D(N)} < \alpha_{N+1} \quad \& \quad |P_{N+1} - f| < \alpha_{N+1}$$

Then the series

$$P_1 + (P_2 - P_1) + (P_3 - P_2) + \ldots$$

xonverges uniforly on every cpmaåct disc $D(R)$ and the o it is an entire function $p(z)$. done ....

An extension. We do P_1 so that $|P_1 - f|_{K-1} < \alpha - 1$. On $K[1] = K \cap \{z| = 1\}$ we have the contomoud funtion $P_1 - f$ whose maximum norm is small. on this null set the disc algebra interpolates. But how good is sup-norm prdsered. i.e. given g on a null set of thr unit circle. To find G in disc algebra with $G = g$ on the null set and not too large. Thus, are null sets of interpolation . So one regards

$$g \mapsto g_S \in C^0(S)$$

from $A(D)$ into $C^0(S)$. Image is dense by the Brother Riesz's theorem. So we do it for small ϵ This, first a polynomial $P - 1$ with

$$|P - 1 - f[_{K-1} < \epsilon$$

Next, we have $P_1 - f$ on $K[1]$ with a small sup-norm. Find g in disc algebra so that

$$|g + (P_1 - f)|_{K[1]} < \epsilon$$

and not too large g-norm on D. So we can modify ... Better, modify f to $f_1 = f + \rho$ so thst

$$P_1 = f + \rho$$

holds on $K[1]$. At he same time ρ has a small sup-norm on K.

A uniqueness theorem for PDE-equations.

Introduction. We shall work in $\mathbf{R}^2$ with coordinates (x,y). An example of a boundary value problem is to determine a function $u(x,y)$ which is harmonic in some open half-disc

$$D_+(r) = \{x^2+y^2<r^2\} \cap \{x>0\}$$

and satisfies the boundary conditions:

$$u(0,y) = \psi(y) \quad \text{and} \quad u_x(0,y) = \phi(y)$$

where ϕ and ψ are given in advance. When ϕ and ψ are real-analytic one proves easily that (*) has a unique solution. With less regularity Hadamard gave examples where this fails to hold. In fact, Hadamard proved that a necessary and sufficient condition for the Cauchy problem to be well posed is that the function

$$y \mapsto \phi(y) + \frac{1}{\pi}\int_a^b \operatorname{Log}\bigl[\frac{1}{|s-y|}\bigr]\cdot\psi(s)\cdot ds$$

is real analytic. Here we shall focus upon uniqueness of solutions to homogeneous elliptic boundary value problems expressed by a system of first order partial differential equations. Let $n=2m$ be an even positive integer and consider two $n\times n$-matrices $\mathcal{A}=\{A_{pq}\}$ and $\mathcal{B}=\{B_{pq}\}$ whose elements are real-valued functions of x and y where the B-functions are continuous and the A-functions of class C^2. The two matrices give a system of first order PDE-equations whose solutions are vector valued functions $(f_1,\ldots,f_n)$ defined in a half-disc

$$D_+(\rho) = \{x^2+y^2<\rho^2 \quad : \; x>0\}$$

where these f-functions satisfy the system:

$$(*) \qquad \frac{\partial f_p}{\partial x} + \sum_{q=1}^{q=n} A_{pq}(x,y)\cdot\frac{\partial f_p}{\partial y} + \sum_{q=1}^{q=n} B_{pq}(x,y)\cdot f_q(x,y) = 0$$

together with the boundary conditions:

$$(**) \qquad f_p(0,y) = 0 \quad \text{for all} \quad 1\le p\le n$$

We get eigenvalues of the $\mathcal{A}$-matrix when (x,y)-varies, i.e. the n-tuple of roots $\lambda_1(x,y),\ldots,\lambda_n(x,y)$ which solve

$$(1) \qquad \det\bigl(\lambda\cdot E_n - \mathcal{A}(x,y)\bigr) = 0$$

If the λ-roots are non-real we say that (*) is an elliptic system. Assuming vanishing Cauchy data expressed by (**) above one expects that a solution f is identically zero. This uniqueness was proved by Erik Holmgren in the article [Holmgren] from 1901 under the assumption that the A-functions and the B-functions are real analytic. The question remained if the uniqueness still holds under less regularity on the coefficient functions. This was settled 30 years later by Carleman where the following is proved in [Carleman]:

1. Theorem. *Assume that the λ-roots are all simple and non-real as (x,y) varies in the open half-disc. Then every solution f to (**) with vanishing Cauchy-data is identically zero.*

The proof requires several steps and the methods which occur below have inspired more recent work where Carleman estimates are used to handle boundary value problems in PDE-theory.

A. Proof of Theorem 1: First part

The system in (*) is equivalent to a system of m-many equations where one seeks complex-valued functions $g_1, \ldots, g_m$ satisfying:

$$\frac{\partial g_p}{\partial x} + \sum_{q=1}^{q=m} \lambda_p(x,y) \cdot \frac{\partial g_p}{\partial y} =$$

$$(**) \qquad \sum_{q=1}^{q=m} a_{pq}(x,y) \cdot g_q(x,y) + b_{pq}(x,y) \cdot \bar{g}_q(x,y) = 0 \, : \, 1 \leq p \leq m$$

Above $\{a_{pq}\}$ and $\{b_{pq}\}$ are complex-valued. The reduction to this complex family of equations is left to the reader and from now on we study the system (**). Theorem 1 amounts to prove that if the g-functions satisfy (**) in a half-disc $D_+(\rho)$ and

$$g_p(0,y) = 0 \quad : \quad 1 \leq p \leq m$$

then there exists some $0 < \rho_* \leq \rho$ such that the g-functions are identically zero in $D_+(\rho_*)$. To attai this we introduce domains as follows: For a pair $\alpha > 0$ and $\ell > 0$ we put

$$(1) \qquad D_\ell(\alpha) = \{x + y^2 - \alpha x^2 < \ell^2\} \cap \{x > 0\}$$

Notice that the boundary

$$\partial D_\ell(\alpha) = \{0\} \times [-\ell, \ell] \cup T_\ell \quad \text{where} \quad x + y^2 - \alpha x^2 = \ell^2 \text{ holds on } T_\ell$$

Above α and ℓ are small so the the g-functions satisfy (**) in $D_\ell(\alpha)$. For each $t > 0$ we define the m-tuple of functions by

$$(2) \qquad \phi_p(x,y) = g_p(x,y) \cdot e^{-t(x+y^2-\alpha x^2)}$$

Since the g-functions satisfy (**) one verifies easily that the ϕ-functions satisfy the system

$$(3) \qquad \frac{\partial \phi_p}{\partial x} + \frac{\partial}{\partial y}\big(\lambda_p \cdot \phi_p\big) + t\big(1 - 2\alpha x + 2y\lambda_p\big) \cdot \phi_p = H_p(\phi)$$

where

$$H_p(\phi) = \sum_{q=1}^{q=n} a_{pq}(x,y) \cdot \phi_q(x,y) + b_{pq}(x,y) \cdot \bar{\phi}_q(x,y) = 0 \, : \, 1 \leq p \leq m$$

Next, we set

$$(4) \qquad \Phi(x,y) = \sum_{p=1}^{p=m} |\phi_p(x,y)|$$

The crucial step in the proof of Theorem 1 is to establish the following inequality.

A.1 Proposition. *Provided that α from the start is sufficiently large there exists some $0 < \ell_* \leq \ell$ and a constant C which is independent of t such that*

$$\iint_{D_{\ell_*}} \Phi(x,y) \cdot dxdy \leq C \cdot \int_{T_{\ell_*}} \sum_{p=1}^{p=n} |\phi_p| \cdot |dy - \lambda_p \cdot dx|$$

How to deduce Theorem 1.

Let us show why Proposition A.1 gives Teorem 1. In addition to ℓ_* we fix some $0 < \ell_{**} < \ell_*$. In (2) above we have used the function

$$w(x,y) = e^{-t(x+y^2-\alpha x^2)} \implies$$

$$\text{(i)} \qquad w(x,y) = e^{-t\ell_*^2} \quad : \quad (x,y) \in T_{\ell_*} \quad : \quad w(x,y) \geq e^{-t\ell_{**}^2} \quad : \quad (x,y) \in D_{\ell_{**}}$$

Next, we have $|\phi_p| = |g_p| \cdot w$ for each p. Replacing the left hand side in Proposition A.1 by the area integral over the smaller domain $D_{\ell_{**}}$ we obtain the inequality;

$$\text{(ii)} \qquad \iint_{D_{\ell_{**}}} \sum_{p=1}^{p=m} |g_p(x,y)| \cdot dxdy \leq C \cdot e^{t(\ell_{**}^2 - \ell_*^2)} \cdot \int_{T_{\ell_*}} \sum_{p=1}^{p=n} |g_p| \cdot |dy - \lambda_p \cdot dx|$$

Here (ii) holds for every $t > 0$. When $t \to +\infty$ we have $e^{t(\ell_{**}^2 - \ell_*^2)} \to 0$ and can therefore conclude that

$$\iint_{D_{\ell_{**}}} \sum_{p=1}^{p=m} |g_p(x,y)| \cdot dxdy = 0$$

This means that the g-functions are all zero in $D_{\ell_{**}}$ and Theorem 1 follows.

B. Proof of Proposition A.1

The proof relies upon the construction of certain ψ-functions. More precisely, when $t > 0$ and a point $(x_*, y_*) \in D_\ell$ are given we shall construct an m-tuple of ψ-functions satisfying the following:

Condition 1. Each ψ_p is defined in the punctured domain $D_\ell \setminus \{(x_*, y_*)\}$ where ψ_p for a given $1 \leq p \leq m$ satisfies the equation

$$\text{(i)} \qquad \frac{\partial \psi}{\partial x} + \lambda_p \cdot \frac{\partial \psi}{\partial y} - t(1 - 2\alpha x + 2y\lambda_p)\psi_p = 0$$

Condition 2. For each p the singularity of ψ_p at (x_*, y_*) is such that the line integrals below have a limit:

$$\text{(ii)} \qquad \lim_{\epsilon \to 0} \int_{[z - z_*| = \epsilon} \psi_p \cdot (dx - \lambda_p \cdot dy) = 2\pi$$

Condition 3. There exists a constant K which is independent both of (x_*, y_*) and of t such that

$$\text{(iii)} \qquad |\psi_p(z)| \leq \frac{K}{|z - z_*|}$$

Notice that the ψ-functions depend on the parameter t, i.e. they are found for each t but the constant K in (3) is independent of t.

The deduction of Proposition A.1

Before the ψ-functions are constructed in Section C we show how they give Proposition A.1. Consider a point $z_* \in D_+(\ell)$. We get the associated ψ-functions from § B at this particular point. Remove a small disc γ_ϵ centered at z_* and consider some fixed $1 \leq p \leq m$. Now ϕ_p satisfies the differential equation (3) from section A and ψ_p satisfies (i) in Condition 1 above. Stokes theorem gives:

$$\int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) = \iint_{D_\ell \setminus \gamma_\epsilon} H_p(\phi) \cdot \psi_p \cdot dxdy + \int_{|z - z_*| = \epsilon} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big)$$

Passing to the limit as $\epsilon \to 0$, Condition 2

$$\text{(1)} \qquad \phi_p(x_*, y_*) = \frac{1}{2\pi} \int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) - \frac{1}{2\pi} \cdot \iint_{D_\ell} H_p(\phi) \cdot \psi_p \cdot dxdy$$

Let L be the maximum over D_ℓ of the coefficient functions of ϕ and $\bar{\phi}$ which appear in $H_p(\phi)$ from (3) i § A. We have also the constant K from Condition 3 for ψ_p. The triangle inequality gives:

$$(*) \qquad \big|\phi_p(x_*, y_*)\big| \leq \frac{K}{2\pi} \int_{T_\ell} \frac{|\phi_p| \cdot |dy - \lambda_p \cdot dx|}{z - z_*|} + \frac{LK}{\pi} \cdot \sum_{q=1}^{q=m} \iint_{D_\ell} \frac{|\phi_q|}{|z - z_*|} \cdot dxdy$$

Next, we use the elementary inequality

$$(**)\qquad \iint_\Omega \frac{dxdy}{\sqrt{x-a)^2+(y-b)^2}} \leq 2\cdot\sqrt{\pi}\cdot\sqrt{\text{Area}(\Omega)}$$

where Ω is an arbitrary bounded domain and $(a,b)\in\Omega$. Apply (**) with $\Omega = D_\ell$ and set $S = \text{area}(D_\ell)$. Integrating both sides in (*) over D_ℓ for every p and taking the sum we get

$$\iiint_{D_\ell} \Phi\cdot dxdy \leq$$

$$K\cdot\sqrt{\frac{S}{\pi}}\cdot\int_{T_\ell}\sum_{p=1}^{p=m}|\phi_p|\cdot|dy-\lambda_p\cdot dx| + 2\pi mLK\cdot\sqrt{\frac{S}{\pi}}\iint_{D_\ell}\Phi\cdot dxdy$$

This inequality hold for all small ℓ. Choose ℓ so small that

$$2\pi mLK\cdot\sqrt{\frac{S}{\pi}} \leq \frac{1}{2}$$

Then the inequality above gives

$$(***)\qquad \iiint_{D_\ell}\Phi\cdot dxdy \leq 2\cdot K\cdot\sqrt{\frac{S}{\pi}}\cdot\int_{T_\ell}\sum_{p=1}^{p=m}|\phi_p|\cdot|dy-\lambda_p\cdot dx|$$

Finally, consider some relatively compact domain Δ in D_ℓ. Then there exists $0<\ell_*<\ell$ such that

$$\Delta\subset D_{\ell_*}$$

Now we notice that

$$|\phi_p(z)|\geq e^{-t\ell_*^2}\cdot|u_p(z)| \quad : \quad z\in\Delta \quad : \quad |\phi_p(z)|\geq e^{-t\ell^2}\cdot|u_p(z)| \quad : \quad z\in T_\ell$$

We conclude that

$$(****)\qquad e^{-t\ell_*^2}\iiint_\Delta\sum_{p=1}^{p=m}|u_p(z)|\cdot dxdy \leq e^{-t\ell^2}\cdot 2\cdot K\cdot\sqrt{\frac{S}{\pi}}\cdot\int_{T_\ell}\sum_{p=1}^{p=m}|u_p|\cdot|dy-\lambda_p\cdot dx|$$

Here (****) hold for every $t>0$. Passing to the limit as $t\to+\infty$ it follows that

$$\cdot\iiint_\Delta\sum_{p=1}^{p=m}|u_p(z)|\cdot dxdy \leq$$

Since Δ was any relatively compact subset of D_ℓ, we conclude that the u-functions are zero in D_ℓ and Theorem 1 follows.

C. Construction of the ψ-functions.

Before we embark upon specific constructions we investigate the whole family of solutions to a first order differential operators of the form

$$(*)\qquad Q = \partial_x + \lambda(x,y)\cdot\partial_y$$

where $\lambda(x,y)$ is a complex valued C^2-function whose imaginary part is >0. Set

$$\lambda(x,y) = \mu(x,y) + i\cdot\tau(x,y) \quad : \tau(x,y)>0$$

Now we look for solutions $h(x,y)$ to the equation $Q(h)=0$. With $h(x,y)=\xi(x,y)+i\cdot\eta(x,y)$ where ξ and η are real-valued C^2-functions this gives the differential system:

$$\frac{\partial\xi}{\partial x} + \mu_p\cdot\frac{\partial\xi}{\partial y} - \tau_p\cdot\frac{\partial\eta}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial x} + \mu_p \cdot \frac{\partial \eta}{\partial y} + \tau_p \cdot \frac{\partial \xi}{\partial y} = 0$$

Suppose we have found one solution $h = \xi + i \cdot \eta$ where the Jacobian $\xi_x \eta_y - \xi_y \eta_x$ is $\neq 0$ at the origin. Then $(x, y) \mapsto (\xi, \eta)$ is a local C^2-diffeomorphism. With $\zeta = \xi + i\eta$ we have the usual Cauchy-Riemann operator.

$$\frac{1}{2}(\frac{\partial}{\partial \xi} + i \cdot \frac{\partial}{\partial \eta})$$

Let $g(\xi + i\eta)$ be a holomorphic function in the complex ζ-space with $\zeta = \xi + i\eta$ and put

$$g_*(x, y) = g(\xi(x, y) + i\eta(x, y))$$

Then one easily verifies that $Q(g_*) = 0$ and conversely, every solution to this equation is expressed by a $g - *$function derived from an analytic function in the complex ζspace.satisfies $Q(g_*)$.

Conclusion. *If a non-degenerate solution $h = \xi + i\eta$ has been found then the homogenous solutions to Q is in a 1-1 correspondence to analytic functions in the ζ-variable.*

Remark. The effect of a coordinate transformation as above is that the Q-operator is transported to the Cauchy-Riemann operator in the complex ζ-space where $\zeta = \xi + i\eta$. Later we employ such (ξ, η)-transformations to construct solutions to an inhomogeneous equation of the form

$$Q(\psi) = (t - \alpha x + 2y\lambda(x, y)) \cdot \psi(t, x, y)$$

where t is a positive parameter and the ψ-functions will have certain specified properties. Notice that it suffices to construct the ψ-functions separately, i.e. we no longer have to bother about a differential system. With a fixed p fixed $\lambda_p(x, y) = \mu_p + \tau_p$ and from now on we may drop the index p and explain how to obain ψ-functions satisfying the three conditions from § B. So we consider the first order differential operator

$$Q = \frac{\partial}{\partial x} + (\mu(x, y) + i\tau(x, y)) \cdot \frac{\partial}{\partial y} \tag{1}$$

where $\tau(x, y) > 0$.

C.1 A class of (ξ, η)-functions. Let $V(x, y)$ and $W(x, y)$ be two quadratic forms, i.e. both are homogeneous polynomials of degree two. Given a point (x_*, y_*) and with $z = x + iy$ we seek a coordinate transformation $(x, y) \mapsto (\xi, \eta)$ of the form:

$$\xi(z) = \tau_p(z_*) \cdot (x - x_*) + V(x - x_*, y - y_*) + \gamma_1(z) \cdot |z - z_*|^2$$

$$\eta(z) = (y - y_*) - \mu_p(z_*) \cdot (x - x_*) + W(x - x_*, y - y_*) + \gamma_2(z) \cdot |z - z_*|^2$$

Lemma. *There exists a pair of quadratic forms V and W whose coefficients depend on (x_*, y_*) and a pair of γ-functions which both vanish at (x_*, y_*) up to order one such that the complex-valued function $\xi + i\eta$ solves the homogeneous equation $Q(\xi + i\eta) = 0$.*

A solution above gives a change of variables so that Q is expressed in new real coordinates (ξ, η) by the operator

$$\frac{\partial}{\partial \xi} + i \cdot \frac{\partial}{\partial \eta} \tag{2}$$

There exist many coordinate transforms $(x, y) \to (\xi, \eta)$ which change Q into (2). This *flexible choice* of coordinate transforms is used to construct the required ψ-functions. Notice that Condition (2) in § B is of a pointwise character, i.e. it suffices to find a ψ-function for a given point $z_* = x_* + iy_*$. With this in mind the required construction in § B boils down to perform a suitable coordinate transformation adapated to z_*, and after use the existence of a ψ-function which to begin with is expressed in the (ξ, η)-variables where the Q-operator is replaced by the Cauchy-Riemann operator. In this special case the required ψ-function is easy to find, i.e. see the remark in § B.0. So all that remains is to exhibit suitable coordinate transformations which send

Q to the $\bar{\partial}$-operator. We leave it to the reader to carry out such coordinate transformations. If necessary, consult Carleman's article where a very detailed construction appears.