The p^*-function.

We shall construct a special harmonic function which is used to get solutions of the Dirichlet problem. Let Ω be an open and connected set in $\mathbf{C}$ and consider the connected components of its closed complement. Let E be such a connected component which is not reduced to a single point. Let us then consider some closed and simple Jordan curve γ contained in Ω. For each point $a \in E$ there exists the winding number $\mathfrak{w}_a(\gamma)$ defined by the integer

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

If b is another point in the connected set E the continuity of the integer-valued winding number implies that

$$(*) \qquad \mathfrak{w}_a(\gamma) = \mathfrak{w}_b(\gamma)$$

This equality yields a single-valued analytic function in Ω given by the difference

$$\log(z-a) - \log(z-b)$$

Taking the exponential we find an analytic function $\psi(z)$ in Ω such that

$$e^{\Psi(z)} = \frac{z-a}{z-b}$$

Since $a \neq b$ we see that $\Psi(z) \neq 0$ for all $z \in \Omega$ and obtain the harmonic function in Ω defined by

$$(*) \qquad p(z) = \mathfrak{Re}\big(\frac{1}{\Psi(z)}\big) = \frac{\mathfrak{Re}(\Psi(z)}{|\Psi(z)|^2}$$

Next, we have

$$\mathfrak{Re}(\Psi(z)) = \text{Log}\,|z-a| - \text{Log}|z-b|$$

and since $\log|z-a| \to -\infty$ as $z \to a$ we see from (*) that

$$(**) \qquad \lim_{z\to a} p(z) = 0$$

Notice also that $\Psi(z)$ exends to a continuous function on $\bar{\Omega} \setminus (a,b)$ and we can choose a small $\delta > 0$ such that

$$\text{(ii)} \qquad \text{Log}|z-a| - \text{Log}\,|z-b| < -1 \quad : \quad |z-a| \leq \delta$$

From the above we can conclude:

7.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the single point a. Then there exists a harmonic function $p^*(z)$ in Ω for which*

$$\lim_{z\to a} p^*(z) = 0$$

and there exists $\delta > 0$ such that

$$0 < r < \delta \implies \max_{\{|z-a|=r\}\cap\Omega} p^*(z) < 0 \quad :$$

1. The Dirichlet Problem.

Introduction. Let Ω be a bounded open set in $\mathbf{C}$. No connectivity assumptions are imposed, i.e. neither Ω or $\partial\Omega$ have to be connected. To each $\phi \in C^0(\partial\Omega)$ we shall construct a harmonic function H_ϕ in Ω by a contruction due to Perron. Denote by $\mathcal{P}(\phi)$ the family of subharmonic functions $u(z)$ in Ω satisfying

$$\limsup_{z\to w} u(z) \leq \phi(w) \quad : \; w \in \partial\Omega \tag{0.1}$$

One verifies that the function defined for $z \in \Omega$ by

$$H_\phi(z) = \max_{u\in\mathcal{P}(\phi)} u(z) \tag{*}$$

is harmonic in Ω. A boundary point a is called Dirichlet regular if

$$\lim_{z\to a} H_\phi(z) = \phi(a) \quad : \; \phi \in C^0(\partial\Omega) \tag{**}$$

If (**) holds for every boundary point then Perron's solution extends to a continuous function on the closure $\bar{\Omega}$ and solves the Dirichlet problem with the prescribed boundary function ϕ. We shall prove that (**) holds under a certain geometric condition.

1.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of a in the closed complement $\mathbf{C} \setminus \Omega$ is not reduced to the singleton set $\{a\}$. Then (**) holds for every ϕ.*

The proof involves several steps. First we consider Perron's solution in the special case when

$$\phi(z) = |z-a|$$

which we denote by $H_a(z)$. Notice that the function

$$z \mapsto |z-a|$$

is subharmonic and hence Perron's function $H_a(z)$ satisfies

$$|z-a| \leq H_a(z) \quad : \quad z \in \Omega \tag{1}$$

Now we shall prove:

1.2 Boulignad's Lemma. *Let $a \in \partial\Omega$ satisfy the condition in Theorem 1.1. Then*

$$\lim_{z\to a} H_a(z) = 0$$

Proof. The assumption upon a and Theorem 0.1 gives a harmonic function $p^*(z)$ in Ω such that

$$\lim_{z\to a} p^*(z) = 0 \quad \text{and} \quad p^*(z) < 0 \quad : z \in \Omega \tag{1}$$

Next, let $\epsilon > 0$. Since $a \in \partial\Omega$ we find $0 < r \leq \epsilon$ such that the circle $|z-a| = r$ has a non-empty intersection Γ with Ω. Put

$$M = \max_{z\in\Omega} |z-a| \tag{2}$$

We can choose a compact subset Γ_* of Γ such that

$$\ell = \text{arc-length } (\Gamma \setminus \Gamma_*) \leq \frac{\epsilon}{M} \tag{3}$$

In the disc $D = \{|z-a| < r\}$ we find the harmonic function $V(z)$ whose boundary values on $|z-a| = r$ are zero outside the open set $\Gamma \setminus \Gamma_*$ while $V = M$ holds on $\Gamma \setminus \Gamma_*$. Next, since Γ_* is a compact subset of Ω and $p_a^* < 0$ in Ω there exists $\delta > 0$ such that

$$p^*(z) \leq -\delta \quad : \quad z \in \Gamma_* \tag{4}$$

Set

$$B(z) = V(z) - \frac{M}{\delta} \cdot p^*(z) \tag{5}$$

This is a harmonic function in $\Omega \cap D$ and the construction of V together with (4) give

$$B(z) \geq M \quad : \quad z \in \Gamma \tag{6}$$

Next, in the the open set $U = \Omega \cap D$ we have the subharmonic function

(7) $$g = H_a - B$$

Since $|z-a| \leq \epsilon$ holds in the closed disc in $\bar{D}$ we have

(8) $$\limsup_{z \to w} H_a(z) \leq \epsilon \quad : \; w \in \bar{D} \cap \partial\Omega$$

and (2) implies that $H_a(z) \leq M$ holds in Ω which gives

(9) $$\limsup_{z \to w} H_a(z) \leq M \quad : \; w \in \Gamma$$

At this stage we use the set-theoretic inclusion

(10) $$\partial(D \cap \Omega) \subset \Gamma \cup (\bar{D} \cap \partial\Omega)$$

Hence (6) together with (7-8) entail that

(10) $$\limsup_{z \to w} H_a(z) - B(z) \leq \epsilon \quad : \; w \in \partial(\Omega \cap D)$$

The maximum principle applied to the subharmonic function $H - B$ in $\Omega \cap D$ and (10) give

(11) $$\limsup_{z \to a} H_a(z) \leq \epsilon + \limsup_{z \to a} B(z) = \epsilon + V(a) + \limsup_{z \to a} p^*(z) = \epsilon + V(a)$$

where (1) gives the last equality. Finally, the mean-value formula for the harmonic function V and (3) entail that

(12) $$V(a) = \ell \cdot M \leq \epsilon$$

Hence the limes superior in the left hand side of (11) is $\leq 2\epsilon$ and since $\epsilon > 0$ was arbitrary small we get $\limsup_{z \to a} H_a(z) \leq 0$ which finishes the proof of Boulignad's lemma.

§ 1.3 Proof of Theorem 1.1.

Let $\phi \in C^0(\partial\Omega)$ with the Perron solution $H_\phi(z)$. If c is a constant it is clear that $H_{\phi - c} = H_\phi - c$. Replacing ϕ by $\phi(z) - \phi(a)$ we may therefore assume that $\phi(a) = 0$ and it remains to show that

(1) $$\lim_{z \to a} H_\phi(z) = 0$$

First we consider the limes superior and show that

(2) $$\limsup_{z \to a} H_\phi(z) \leq 0$$

To get (2) we take some $\epsilon > 0$ and the continuity of ϕ gives $\delta > 0$ such that

$$\phi(z) \leq \epsilon \quad : \quad z \in \partial\Omega \cap D_a(\delta)$$

Put $M^* = \max_{z \in \partial\Omega} |\phi(z)|$ and define the harmonic function in Ω by

$$g^*(z) = \epsilon + \frac{M^*}{\delta} \cdot H_a(z)$$

Since $H_a(z) \geq |z-a|$ we have:

$$\liminf_{z \to b} g^*(z) \geq M^* \quad : \quad b \in \partial\Omega \setminus D_a(\delta)$$

At the same time $g^*(z) \geq \epsilon$ for every $z \in \Omega$ so $g^* \geq \phi$ on the whole boundary and the maximum principle for harmonic functions gives:

$$u \leq g^* \quad : \; u \in \mathcal{P}(\phi)$$

The construction of H_ϕ entails that $H_\phi \leq g^*$ holds in Ω which implies that

$$\limsup_{z \to a} H_\phi(z) \leq \limsup_{z \to a} g^*(z) = \epsilon$$

where the last equality follows from Boulignad's Lemma. Since ϵ can be arbitrary small we get (2). There remains to show that

(3) $$\liminf_{z \to a} H_\phi(z) \geq 0$$

To prove (3) we put

$$g_*(z) = -\epsilon - \frac{M^*}{\delta} \cdot H_a(z)$$

It is clear that

$$\limsup_{z \to \xi} g_*(z) \leq \phi(\xi) \quad \text{for all boundary points } \xi \in \partial\Omega$$

Hence $g_* \in \mathcal{P}(\phi)$ which gives $g_* \leq H_\phi$. So now we have

$$\liminf_{z \to a} H_\phi(z) \geq \liminf_{z \to a} g_*(z) = -\epsilon$$

Since ϵ can be arbitrary small we get (3) and Theorem 1.1 is proved.

1.5 Wiener's solution.

Let Ω be a bounded open set which in general consists of infinitely many connected components. There exists a family $\mathcal{F}$ of nested sequences of strictly increasing relatively compact subsets $\{\Omega_n\}$ such that $\cup\,\Omega_n = \Omega$ and Dirichlet's problem is solvable for each Ω_n. The nested property means that $\bar{\Omega}_{n-1}$ appears as a compact subset of Ω_n for every n. For example, if $N \geq 1$ we consider the family $\mathcal{D}_N$ of dyadic cubes whose sides have length 2^{-N}. We find the finite family $\mathcal{D}_N(\Omega)$ of cubes in this family whose closure stay in Ω. Their closed union is relatively compact in Ω and it is clear that Dirichlet's problem is solvable for the open set given by the union of interior points in this family. With increasing N this is an example of a nested sequence $\mathcal{F}$.

1.6 Theorem. *For each $z \in \Omega$ and every nested sequence $\{\Omega_n\}$, the harmonic measures $\{\mathfrak{m}_z^{\Omega_n}\}$ converge weakly to a unique probability measure $\mathfrak{m}_z^*$ supported by $\partial\Omega$, and for each $f \in C^0(\partial\Omega)$ the function*

$$W_f(z) = \int f(\zeta) \cdot d\mathfrak{m}_z^*(\zeta)$$

is harmonic in Ω.

Proof. Let ϕ be a continuous function ϕ on the closure $\overline{\Omega}$ which is subharmonic in Ω. For every n its restriction to $\partial\Omega_n$ has a harmonic extension Φ_n to Ω_n. Next, let z be a point in Ω and start with some n_* so that $z \in \Omega_{n_*}$. To each $n \geq n_*$ we get the harmonic measure $\mathfrak{m}_z^{\Omega_n}$ and obtain

$$\Phi_n(z) = \int_{\partial\Omega_n} \phi \cdot d\mathfrak{m}_z^{\Omega_n} \tag{1}$$

Since ϕ is subharmonic and $\phi = \Phi_{n+1}$ holds on $\partial\Omega_{n+1}$, it follows that $\phi \leq \Phi_{n+1}$ in Ω_{n+1} This holds in particular on $\partial\Omega_n$ which entails that

$$\Phi_n(z) = \int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \leq \int_{\partial\Omega_n} \Phi_{n+1} \cdot \mathfrak{m}_z^{\Omega_n} = \Phi_{n+1}(z) \tag{2}$$

where the last equality holds since the restriction of Φ_{n+1} to Ω_n is harmonic. Hence (1-2) entail that

$$\int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \leq \int_{\partial\Omega_{n+1}} \phi \cdot \mathfrak{m}_z^{\Omega_{n+1}} \quad \text{hold for every} \quad n \geq n_*$$

So the integrals above give a non-decreasing sequence of real numbers which in addition is bounded above because the maximum norm of ϕ on $\bar{\Omega}$ is finite. Hence there exists the limit

$$L(\phi) = \lim_{n\to\infty} \int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \tag{3}$$

In § XX:B we show that every C^2-function can be expressed as the difference of two subharmonic functions and hence the limit in (3) exists for every $f \in C^2(\partial\Omega)$. Since $C^2(\partial\Omega)$ is dense in $C^0(\partial\Omega)$, Riesz representation Theorem implies that weak-star limits from subsequences of $\{\mathfrak{m}_z^{\Omega_n}\}$ are equal and hence the whole sequence has a weak-star limit $\mathfrak{m}_z^*$. Moreover, the boundaries $\partial\Omega_n$ approach $\partial\Omega$ which entails that $\mathfrak{m}_z^*$ is supported by $\partial\Omega$. At this stage we leave to the reader to show that if $\{U_n\}$ is another nested sequence then the weak star limit of of $\{\mathfrak{m}_z^{U_n}\}$ is equal to the limit measure $\mathfrak{m}_z^*$ constructed via $\{\Omega_n\}$. Finally, the constructions above show that when z varies in Ω then

$$z \mapsto \int_{\partial\Omega} f \cdot \mathfrak{m}_z^*$$

is harmonic in Ω for every continuous boundary function f.

1.7 Regular boundary points. Consider a point $a \in \partial\Omega$. It is called Wiener-regular if

$$\lim_{z\to a} W_f(z) = f(a) \tag{1.7.1}$$

hold for every $f \in C^0(\partial\Omega)$. We shall find a potential-theoretic condition in order that (1.7.1) holds. Without loss of generality we can take a as the origin and for each $n \geq 1$ we consider the closed annulus

$$A_n = \{2^{-n} \leq |z| \leq 2^{-n+1}\}$$

For each fixed n the compact set $\partial\Omega \cap A_n$ has a capacity $e^{-\gamma_n}$ which by § XX means that there exists a probability measure μ_n supported by $\partial\Omega \cap A_n$ such that the logarithmic potential

$$U_n(z) = \frac{1}{\gamma_n} \int \log \frac{1}{|z-\zeta|} \cdot d\mu_n(\zeta) = 1 \quad : z \in \partial\Omega \cap A_n$$

1.8 Theorem. *The boundary point a is regular if and only if*

(*)
$$\sum_{n=1}^{\infty} \frac{n}{\gamma_n} = +\infty$$

Proof. Suppose first that (*) is convergent and then we must prove that (1.7.1) does not hold. To achieve this we take some $0 < \epsilon < 1/3$ which is kept fixed from now on where the assumed convergence in (*) gives an integer N_* such that

(i)
$$\sum_{n=N_*}^{\infty} \frac{n}{\gamma_n} \leq \epsilon$$

Choose $F \in C^0(\partial\Omega)$ with $F(0) = 1$ and $F = 0$ on $\{|z| \geq 2^{-N_*+1} \cap \partial\Omega\}$ while the whole range $F(\bar{\partial}\Omega)$ is contained in $[0,1]$. Now we shall argue by contradiction and suppose that (1.7.1) holds for F. This gives the existence of a small $s > 0$ such that

(ii)
$$W_F(z) \geq 1 - \epsilon \quad \text{holds on} \quad D(s) \cap \Omega$$

where $D(s)$ is the disc of radius s centered at the origin. Next, as $r \to 0$ the discs $D(r)$ shrink to the singleton set $\{0\}$. So by § XX the capacities $e^{-\gamma_r}$ of $D(r) \cap \Omega$. tend to zero with r. Let ν_r be the equilibrium distribution on $D(r) \cap \partial\Omega$ and put

(iii)
$$V_r(z) = \frac{1}{\gamma_r} \cdot \int \log \frac{1}{|z-\zeta|} \cdot d\nu_r(\zeta)$$

So here $V(z) = 1$ on $D(r) \cap \partial\Omega$. Since $\gamma_r \to +\infty$ as $r \to 0$ we can choose $r < s$ so small that

(iv)
$$\max_{|z| \geq s} V_r(z) \leq \epsilon$$

Next, let $M > N_*$ be so large that $2^{-M} \leq r$ and put

$$U_M^*(z) = \sum_{n=N_*}^{n=M} \frac{1}{\gamma_n} \cdot U_n(z)$$

With this choice it is clear that

(v)
$$V(z) + U_M^*(z) \geq 1 \quad \text{holds on} \quad D(2^{-N_*+1}) \cap \partial\Omega$$

Since $0 \leq F \leq 1$ and $F = 0$ on $\partial\Omega \setminus D(2^{-N_*+1})$ we have $F \leq V + U_M^*$ on the whole boundary of Ω. The minimum principle for super-harmonic functions gives the inequality

(vi)
$$V + U_M^* \geq W_F \quad \text{in} \quad \Omega$$

Now (i) and (iv) entail that

(vii)
$$\min_{z \in \Omega \cap D(s)} U_M^*(z) \geq 1 - 2\epsilon$$

At the same time we have $2^{-M} \leq s$ so the construction of U_M^* entails that $U_M^* = 1$ holds on $|z| \cap \partial\Omega$. Together with (vii) we obtain

(viii)
$$\min_{|z|=s} U_M^*(z) \geq 1 - 2\epsilon$$

Now U_M^* restricts to a super-harmonic function on $D(s)$ so the minimum principle for super-harmonic functions gives

$$1-2\epsilon \leq U_M^*(0) = \sum_{n=N_*}^{n=M} \frac{1}{\gamma_n} \int \cdot \log \frac{1}{|\zeta|} \cdot d\mu_n(\zeta) \tag{ix}$$

Finally, $|\zeta| \geq 2^{-n}$ holds on the support of μ_n and hence each integral is $\leq n \cdot \log 2$. So (ix) gives the inequality

$$1-2\epsilon \leq \sum_{n=N_*}^{n=M} \frac{n \cdot \log 2}{\gamma_n} < \log 2 \cdot \epsilon$$

This cannot hold since we have chosen $\epsilon < 1/3$. It means that (ii) cannot hold and hence a is not a regular boundary point.

Proof of sufficiency.

Above we proved that if a is regular then the series (*) in Theorem 1.8 must diverge. There remains to show that if (*) is divergent then a is regular. The proof of this converse is left as an exercise to the reader.

Chapter 5.A Harmonic functions

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Introduction.

Chapter V is divided in two parts, where § B treats subharmonic functions. A logical procedure might have started with subharmonic functions. However, we prefer to first study harmonic functions even though we are obliged to appeal to some facts from V:B. Hopefully the reader will accept this and whenever needed consult results in V:B. A crucial fact is that every positive harmonic function u in the open unit disc is the Poisson extension of a non-negative Riesz measure μ with finite total mass on the unit circle, i.e. with $z = x + iy$ one has

$$u(x,y) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} \cdot d\mu(\theta)$$

Another useful result is that every real-valued continuous function ϕ whose Laplacian in the sense of distribution theory is of order zero, i.e. given by a signed Riesz measure, is the difference of two subharmonic functions. In particular this is true for real-valued C^2-functions. Armed with these results to be proved in V:B we solve the Dirichlet problem in a quite general context where the major results are due to Perron and Wiener. In fact, we shal, learn that there exist several ways to attain solutions of Dirchlet's problem. The less experienced reader may first consult the material in § 3 and § 4 where we study the Poisson kernel in the unit disc and establish the Herglotz formula, while § 4 exhibits maximum principles for harmonic functions which are derived from local mean-value equalities.

Comments on other sections. In § 5 we study harmonic measures and use these to prove results due to Lindelöf concerning limits of bounded analytic functions. In §6 we construct Green's functions and the Neumann problem is treated in § 7. An example related to hydromechanics appears in § 8 and § 9 presents a proof by Carleman which uses differential inequalities in order to estimate harmonic functions. Section 10 treats a result due to Nevanlinna about critical points for level curves of harmonic measure functions. Sections 11-13 contain supplementary facts about harmonic functions. For example, the estimates in § 12 will be used when Lindelöf indicators are studied for analytic functions in strip domains in Special Topics: § XX. Sections 14 -15 contain results whose proofs are more involved, especially in § 15. The results in § 8-15 are not neeeded for the "general theory". However, the proofs are instructive and the theorems quite striking. So they deserve to be studied since they teach lessons about the usefulness of both harmonic and subharmonic functions.

Remark. Many results can be exended to a more general set-up. One can for example replace the Laplace operator by an elliptic second-order PDE-operator with variable coefficients and establish solutions to boundary value problems, both in the sense of Dirichlet and Neumann. So the subsequent material is rather restricted, but has the merit that proofs are quite constructive and the special results which occur for the Laplace operator are used in analytic function theory.

The Dirichlet problem.

Consider a bounded open and connected domain Ω in $\mathbf{C}$. To every $f \in C^0(\partial\Omega)$ one seeks a harmonic function H_f in Ω which extends continuously to $\bar{\Omega}$ and whose boundary value function is f. The maximum principle for harmonic functions entails that H_f is unique if it exists. When Ω is of class $\mathcal{D}(C^1)$, i.e. the boundary is a finite union of pairwise disjoint closed Jordan curves of class C^1, then Dirichlet's problem can be solved for every continuous boundary function. This relies upon properties of Dirchlet integrals. If F is a C^1-function defined in a neighborhood of the closure of Ω the Dirichlet norm is defined by

$$\text{(i)} \qquad D(F) = \sqrt{\iint_\Omega (F_x^2 + F_y^2)\, dxdy}$$

If F and G are two C^1-functions which extend to be of class C^1 on the boundary, then Greens' formula entails that

$$\text{(ii)} \qquad \iint_\Omega [G \cdot \Delta F + (G_x F_x + G_y F_y)]\, dxdy = \int_{\partial\Omega} G \cdot \frac{\partial F}{\partial n}\, ds$$

The case when F is harmonic. Then (ii) gives

$$\text{(iii)} \qquad \iint_\Omega (G_x F_x + G_y F_y)\, dxdy = \int_{\partial\Omega} G \cdot \frac{\partial F}{\partial n}\, ds$$

With $F = G$ we obtain in particular

$$\text{(iv)} \qquad D^2(F) = \int_{\partial\Omega} F \cdot \frac{\partial F}{\partial n}\, ds$$

Next, we notice that

$$\text{(v)} \qquad D^2(F - G) = D^2(F) + D^2(G) - 2 \cdot \iint_\Omega (G_x F_x + G_y F_y)\, dxdy$$

If $G = F$ holds on $\partial\Omega$ we see that (iii-iv) give

$$D^2(F - G) = D^2(G) + D^2(F) - 2 \cdot \int_{\partial\Omega} F \cdot \frac{\partial F}{\partial n}\, ds = D^2(G) - D^2(F)$$

Hence we have established

Dirichlet's equation *Let F be harmonic in Ω. Then*

$$(*) \qquad D^2(G) = D^2(F) + D^2(F - G)$$

hold for every G such that $G = F$ on $\partial\Omega$.

Remark. This equality suggests that if $f \in C^0(\partial\Omega)$ then the solution to Dirchlet's problem amounts to settle a variational problem, i.e. find F such that $D(F)$ is minimized while $F = f$ on the boundary. It was pointed out by Weierstrass that a variational problem of this kind is not automatically well posed. After a rather refined analysis Poincaré demonstrated that the variational problem indeed leads to the solution of the Dirichlet problem for domains of class C^2. For domains with less regularity we shall learn that solutions to the Dirchlet problem can be found by a Perron's method. His solution is given in § 5.B after we have become familiar with subharmonic functions.

0.2 Wiener's generalised solution. Examples of "ugly domains" show that the Dirichlet problem in general cannot be solved. However, for an arbitrary bounded open set Ω there exists for each $\phi \in C^0(\partial\Omega)$ a unique harmonic function W_f which is constructed as follows: The domain Ω can be exhausted by a nested sequence of increasing subdomains $\{U_n\}$ such that $\cup U_n = \Omega$, and for each n the boundary ∂U_n is a finite union of closed Jordan curves of class C^2. Moreover,

$$\lim_{n\to\infty} \operatorname{dist}(\partial U_n, \partial\Omega) = 0$$

Next, extending ϕ to a continuous function Φ on $\bar{\Omega}$ we can solve the Dirichlet problem in U_n which gives a harmonic function H_n in U_n whose boundary value function on ∂U_n is the restriction of Φ. In § xx we show that the sequence $\{H_n\}$ converges uniformly over compact subsets to a limit function which is harmonic in Ω. Wiener proved that this limit function does not depend upon the chosen nested U-sequence, nor upon the chosen continuous extension of ϕ. The resulting function is denoted by W_ϕ and called Wiener's generalised solution. The map $\phi \to W_\phi$ is linear and Riesz representation theorem gives for each point $z \in \Omega$ a unique probability measure $\mathfrak{m}_z$ on $\partial\Omega$ such that

$$(*) \qquad W_\phi(z) = \int \phi(\zeta) \cdot d\mathfrak{m}_z(\zeta)$$

holds for every $\phi \in C^0(\partial\Omega)$.

0.3 Regular boundary points. A point $z_0 \in \partial\Omega$ is called regular if

$$\lim_{z\to z_0} W_\phi(z) = \phi(z_0)$$

hold for every continuous boundary function. A criterion for a boundary point to be regular was established by Boulignad in 1923. Namely, a boundary point z_0 is regular if and only if there exists a positive harmonic function V in Ω such that

$$\lim_{z\to z_0} V(z) = 0$$

This condition is rather implicit so one seeks geometric properties to decide if a boundary point is regular or not. In his Phd-thesis from 1933, Beurling established a sufficient regularity condition which goes as follows: Let $z_0 \in \partial\Omega$ and for a given $R > 0$ we consider the circular projection of the closed complement of Ω onto the real interval $0 < r < R$ defined by:

$$E_\Omega(0, R) = \{0 < r < R : \exists\, z \in \{|z - z_0| = r\} \cap \mathbf{C} \setminus \Omega\}$$

Following Beurling one says that z_0 is logarithmically dense (Point frontière de condensation logarithmique) if the integral

$$(1) \qquad \int_{E_\Omega(0,R)} \log r \, dr = +\infty$$

Beurling proved that if (1) is divergent then z_0 is a regular boundary point. The proof relies upon an inequality which has independent interest. Here is the situation considered in [ibid: page 64-66]: Let Ω be an open set - not necessarily connected - and $R > 0$ is such that Ω contains points of absolute value $\geq R$. Consider a harmonic function h in Ω with the following properties:

$$\text{(i)} \qquad \limsup_{z\to z_*} h(z) \leq 0 \quad : z_* \in \partial\Omega \cap \{|z| \leq R\}$$

$$\text{(ii)} \qquad h(z) \leq M \quad : z \in \{|z| = R\} \cap D$$

0.3.1 Theorem. *When (i-ii) hold one has the inequality below for every $0 < r < R$*

$$\max_z h(z) \leq 2M \cdot e^{-\frac{K}{2}}$$

where the maximum in the left hand side is taken over $\Omega \cap \{r < |z| < R\}$ *and*

$$K = \int_{E_\Omega(r,R)} \log r \, dr$$

In this chapter we shall not expose Beurling's result but refer to § xx. Instead we give an account about Wiener's potential-theoretic condition for boundary points to be regular in § 1.6.

0.4 Harmonic measures. Let Ω be a bounded and connected domain. By (*) in (0.2) we find a probability measure $\mathfrak{m}_z$ on $\partial\Omega$ for each $z \in \Omega$ and refer to $\mathfrak{m}_z$ as the harmonic measure of z. Let g be a bounded Borel function defined on $\partial\Omega$. Recall from *Measure Appendix* that we can construct integrals of Borel functions with respect to an arbitrary Riesz measure. If $z \in \Omega$ we put

$$G(z) = \int_{\partial\Omega} g(\zeta) \cdot d\mathfrak{m}_z(\zeta) \quad :$$

Then G is a bounded harmonic function in Ω. In particular we take a Borel set E in $\partial\Omega$ and integrate its characteristic function which gives the harmonic function

$$\omega_E(z) = \int_E d\mathfrak{m}_z(\zeta)$$

One refers to ω_E as the harmonic measure function with respect to the E.

Remark. When z varies in Ω, Harnack's inequality for harmonic functions shows that the family of harmonic measures are absolutely continuous with respect to each other. So with $z_* \in \Omega$ kept fixed every point $z \in \Omega$ gives a function $\rho_z \in L^\infty(\mathfrak{m}_{z_*})$ such that

$$\mathfrak{m}_z = \rho_z \cdot \mathfrak{m}_{z_*}$$

Since $C^0(\partial\Omega)$ is a dense subspace of $L^1(\mathfrak{m}_{z_*})$ every $f \in L^1(\mathfrak{m}_{z_*})$ yields a harmonic function in Ω defined by

$$f^*(z) = \int \rho_z(\zeta) \cdot f(\zeta) \cdot d\mathfrak{m}_{z_*}(\zeta)$$

0.5 Other topics.

In § 2 we construct the harmonic conjugate of a harmonic function. § 3 is devoted to the case when Ω is the unit disc where we confirm the solution to the Dirichlet problem using the Poisson kernel

$$P(r,\theta) = \frac{1}{2\pi} \cdot \frac{1-r^2}{1+r^2-2r\cos\theta}$$

where $0 < r < 1$ while $0 \leq \theta \leq 2\pi$. When $u(\theta)$ is a continuous function on the circle its harmonic extension to D is given by

$$U(r,\phi) = \int_0^{2\pi} P(r,\theta-\phi) \cdot U(\theta) \, d\theta \tag{1}$$

One often rewrites this integral as:

$$U(r,\phi) = \int_0^{2\pi} P(r,\theta) \cdot u(\theta+\phi) \, d\theta \tag{2}$$

0.6 An estimate of derivatives. Suppose that u is Lipschitz continuous, i.e. there is a constant C such that

$$|u(\theta_2) - u(\theta_1)| \leq C \cdot |\theta_1 - \theta_2|$$

Then (2) implies that U restricts to a Lipschitz continuous function of the angular variable ϕ with norm $\leq C$ on every circle of radius $r < 1$. Hence the angular derivatives satisfy

$$|\frac{\partial U}{\partial \phi}(r,\phi)| \leq C \quad : \; 0 < r < 1$$

To study radial derivatives we use the equation

$$\frac{\partial P(r,\theta)}{\partial r}=\frac{1}{2\pi}\cdot\frac{2\cos\theta+2r^2\cos\theta-4r}{(1+r^2-2r\cos\,\theta)^2}$$

With a fixed ϕ_* we consider the function

$$(*)\qquad r\mapsto\frac{\partial U}{\partial r}(r,\phi_*)$$

Now $\theta\mapsto u(\theta+\phi_*)$ has the same Lipschitz norm and we may therefore take $\phi_*=0$ and since the partial r-derivative of U is unchanged when we add a constant we can replace $u(\theta)$ by $u(\theta)-u(0)$, i.e assume that $u(0)=0$. This entails that

$$|u(\theta)|\leq C\cdot|\theta|\quad:\quad-\pi<\theta<\pi$$

Now (*) is majorized in absolute value by the integral

$$\frac{C}{2\pi}\int_{-\pi}^{\pi}\frac{\left|2\cos\theta+2r^2\cos\theta-4r\right|}{(1+r^2-2r\cos\,\theta)^2}\cdot|\theta|\cdot d\theta$$

Exercise. Show that the last integral is majorized by an absolute constant which is independent of $0<r<1$ and conclude the following:

0.6.1 Theorem. *There exists an absolute constant C_* such that*

$$\max_{(r,\phi)\in D}\,|\frac{\partial U}{\partial r}(r,\phi)|+|\frac{\partial U}{\partial\phi}(r,\phi)|\leq C_*\cdot\frac{||u||_{\mathrm{Lip}}}{2\pi}$$

where the last term is the Lipschitz norm of u over T.

Remark. If we remove a small disc centered at the origin the left hand side can be replaced by the length of the gradient vector $\nabla U=(U_x,U_y)$. For example, we find an absolute constant C^* such that

$$\max_{1/2\leq|z|<1}\,||\nabla(U)(z)||\leq C^*\cdot||u||_{\mathrm{Lip}}$$

1. The Dirichlet Problem.

Introduction. Let Ω be a bounded open set in $\mathbf{C}$. No connectivity assumptions are imposed, i.e. neither Ω or $\partial\Omega$ have to be connected. To each $\phi \in C^0(\partial\Omega)$ we shall construct as harmonic function H_ϕ in Ω using a procedure introduced by J. Perron. Denote by $\mathcal{P}(\phi)$ the family of subharmonic functions $u(z)$ in Ω satisfying

$$\limsup_{z \to w} u(z) \leq \phi(w) \quad : w \in \partial\Omega \tag{0.1}$$

In Theorem XX we prove that the function defined by

$$H_\phi(z) = \max_{u \in \mathcal{P}(\phi)} u(z) \quad : z \in \Omega \tag{*}$$

is harmonic in Ω. A boundary point a is called Dirichlet regular if

$$\lim_{z \to a} H_\phi(z) = \phi(a) \quad : \phi \in C^0(\partial\Omega) \tag{**}$$

If (**) holds for every boundary point then Perron's solution extends to a continuous function on the closure $\bar{\Omega}$ and solves the Dirichlet problem with the prescribed boundary function ϕ. It turns out that (**) holds under a fairly mild geometric condition.

1.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of a in the closed complement $\mathbf{C} \setminus \Omega$ is not reduced to the singleton set $\{a\}$. Then (**) holds for every ϕ.*

The idea is to consider Perron's solution when $\phi(z) = |z-a|$. Let us denote it by $H_a(z)$. We remark that outside the origin in $\mathbf{C}$ the Laplacian of the function $|z| = \sqrt{x^2+y^2}$ is equal to $|z|^{-1}$ and hence a positive function. After a translation it follows that $\phi(z) = |z-a|$ is subharmionic in Ω so Perron's construction entails that H_a gives:

$$|z-a| \leq H_a(z) \quad : \quad z \in \Omega \tag{1}$$

The crucial step in the proof of Theorem 1.1 is:

1.2 Boulignad's Lemma. *Let $a \in \partial\Omega$ satisfy the condition in Theorem 1.1. Then*

$$\lim_{z \to a} H_a(z) = 0$$

Proof. The assumption on a and Theorem XX from Chapter 4 gives a harmonic function $p^*(z)$ in Ω which satisfies:

$$\lim_{z \to a} p^*(z) = 0 \quad \text{and} \quad p^*(z) < 0 \quad : z \in \Omega \tag{1}$$

Next, let $\epsilon > 0$. Since $a \in \partial\Omega$ we find $0 < r \leq \epsilon$ such that the circle $|z-a| = r$ has a non-empty intersection Γ with Ω. Put

$$M = \max_{z \in \Omega} |z-a| \tag{2}$$

We can choose a compact subset Γ_* of Γ such that

$$\ell = \text{arc-length } (\Gamma \setminus \Gamma_*) \leq \frac{\epsilon}{M} \tag{3}$$

In the disc $D = \{|z-a| < r\}$ we find the harmonic function $V(z)$ whose boundary values on $|z-a| = r$ are zero outside the open set $\Gamma \setminus \Gamma_*$ while $V = M$ holds on $\Gamma \setminus \Gamma_*$. Next, since Γ_* is a compact subset of Ω and $p_a^* < 0$ in Ω there exists $\delta > 0$ such that

$$p^*(z) \leq -\delta \quad : \quad z \in \Gamma_* \tag{4}$$

Set

$$B(z) = V(z) - \frac{M}{\delta} \cdot p^*(z) \tag{5}$$

This is a harmonic function in $\Omega \cap D$ and the construction of V together with (4) give

$$B(z) \geq M \quad : \quad z \in \Gamma \tag{6}$$

Next, in the the open set $U = \Omega \cap D$ we have the subharmonic function

$$g = H_a - B \tag{7}$$

Since $|z - a| \leq \epsilon$ holds in the closed disc in $\bar{D}$ we have

$$\limsup_{z \to w} H_a(z) \leq \epsilon \quad : \; w \in \bar{D} \cap \partial\Omega \tag{8}$$

From (2) it follows that $H_a(z) \leq M$ holds in Ω which entails that

$$\limsup_{z \to w} H_a(z) \leq M \quad : \; w \in \Gamma \tag{9}$$

At this stage we use the set-theoretic inclusion

$$\partial(D \cap \Omega) \subset \Gamma \cup (\bar{D} \cap \partial\Omega) \tag{10}$$

Hence (6) together with (7-8) entail that

$$\limsup_{z \to w} H_a(z) - B(z) \leq \epsilon \quad : \; w \in \partial(\Omega \cap D) \tag{10}$$

The maximum principle applied to the subharmonic function $H - B$ in $\Omega \cap D$ and (10) give

$$\limsup_{z \to a} H_a(z) \leq \epsilon + \limsup_{z \to a} B(z) = \epsilon + V(a) + \limsup_{z \to a} p^*(z) = \epsilon + V(a) \tag{11}$$

where (1) gives the last equality. Finally, the mean-value formula for the harmonic function V and (3) entail that

$$V(a) = \ell \cdot M \leq \epsilon \tag{12}$$

Hence the limes superior in the left hand side of (11) is $\leq 2\epsilon$ and since $\epsilon > 0$ was arbitrary small we get $\limsup_{z \to a} H_a(z) \leq 0$ which finishes the proof of Boulignad's lemma.

§ 1.3 Proof of Theorem 1.1.

Let $\phi \in C^0(\partial\Omega)$ with the Perron solution $H_\phi(z)$. If c is a constant it is clear that $H_{\phi - c} = H_\phi - c$. Replacing ϕ by $\phi(z) - \phi(a)$ we may therefore assume that $\phi(a) = 0$ and it remains to show that

$$\lim_{z \to a} H_\phi(z) = 0 \tag{1}$$

First we consider the limes superior and show that

$$\limsup_{z \to a} H_\phi(z) \leq 0 \tag{2}$$

To get (2) we take some $\epsilon > 0$ and the continuity of ϕ gives $\delta > 0$ such that

$$\phi(z) \leq \epsilon \quad : \quad z \in \partial\Omega \cap D_a(\delta)$$

Put $M^* = \max_{z \in \partial\Omega} |\phi(z)|$ and define the harmonic function in Ω by

$$g^*(z) = \epsilon + \frac{M^*}{\delta} \cdot H_a(z)$$

Since $H_a(z) \geq |z - a|$ we have:

$$\liminf_{z \to b} g^*(z) \geq M^* \quad : \quad b \in \partial\Omega \setminus D_a(\delta)$$

At the same time $g^*(z) \geq \epsilon$ for every $z \in \Omega$ so $g^* \geq \phi$ on the whole boundary and the maximum principle for harmonic functions gives:

$$u \leq g^* \quad : \; u \in \mathcal{P}(\phi)$$

The construction of H_ϕ entails that $H_\phi \leq g^*$ holds in Ω which implies that

$$\limsup_{z \to a} H_\phi(z) \leq \limsup_{z \to a} g^*(z) = \epsilon$$

where the last equality follows from Boulignad's Lemma. Since ϵ can be arbitrary small we get (2). There remains to show that

$$\liminf_{z \to a} H_\phi(z) \geq 0 \tag{3}$$

To prove (3) we put

$$g_*(z) = -\epsilon - \frac{M^*}{\delta} \cdot H_a(z)$$

It is clear that

$$\limsup_{z\to\xi} g_*(z) \le \phi(\xi) \quad \text{for all boundary points } \xi \in \partial\Omega$$

Hence $g_* \in \mathcal{P}(\phi)$ which gives $g_* \le H_\phi$. So now we have

$$\liminf_{z\to a} H_\phi(z) \ge \liminf_{z\to a} g_*(z) = -\epsilon$$

Since ϵ can be arbitrary small we get (3) and Theorem 1.1 is proved.

1.5 Wiener's solution.

Let Ω be a bounded open set which in general consists of infinitely many connected components. There exists a family $\mathcal{F}$ of nested sequences of strictly increasing relatively compact subsets $\{\Omega_n\}$ such that $\cup\,\Omega_n = \Omega$ and Dirichlet's problem is solvable for each Ω_n. The nested property means that $\bar{\Omega}_{n-1}$ appears as a compact subset of Ω_n for every n. For example, if $N \ge 1$ we consider the family $\mathcal{D}_N$ of dyadic cubes whose sides have length 2^{-N}. We find the finite family $\mathcal{D}_N(\Omega)$ of cubes in this family whose closure stay in Ω. Their closed union is relatively compact in Ω and it is clear that Dirichlet's problem is solvable for the open set given by the union of interior points in this family. With increasing N this is an example of a nested sequence $\mathcal{F}$.

1.6 Theorem. *For each $z \in \Omega$ and every nested sequence $\{\Omega_n\}$, the harmonic measures $\{\mathfrak{m}_z^{\Omega_n}\}$ converge weakly to a unique probability measure $\mathfrak{m}_z^*$ supported by $\partial\Omega$, and for each $f \in C^0(\partial\Omega)$ the function*

$$W_f(z) = \int f(\zeta) \cdot d\mathfrak{m}_z^*(\zeta)$$

is harmonic in Ω.

Proof. Let ϕ be a continuous function ϕ on the closure $\overline{\Omega}$ which is subharmonic in Ω. For every n its restriction to $\partial\Omega_n$ has a harmonic extension Φ_n to Ω_n. Next, let z be a point in Ω and start with some n_* so that $z \in \Omega_{n_*}$. To each $n \ge n_*$ we get the harmonic measure $\mathfrak{m}_z^{\Omega_n}$ and obtain

$$\Phi_n(z) = \int_{\partial\Omega_n} \phi \cdot d\mathfrak{m}_z^{\Omega_n} \tag{1}$$

Since ϕ is subharmonic and $\phi = \Phi_{n+1}$ holds on $\partial\Omega_{n+1}$, it follows that $\phi \le \Phi_{n+1}$ in Ω_{n+1} This holds in particular on $\partial\Omega_n$ which entails that

$$\Phi_n(z) = \int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \le \int_{\partial\Omega_n} \Phi_{n+1} \cdot \mathfrak{m}_z^{\Omega_n} = \Phi_{n+1}(z) \tag{2}$$

where the last equality holds since the restriction of Φ_{n+1} to Ω_n is harmonic. Hence (1-2) entail that

$$\int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \le \int_{\partial\Omega_{n+1}} \phi \cdot \mathfrak{m}_z^{\Omega_{n+1}} \quad \text{hold for every} \quad n \ge n_*$$

So the integrals above give a non-decreasing sequence of real numbers which in addition is bounded above because the maximum norm of ϕ on $\bar{\Omega}$ is finite. Hence there exists the limit

$$L(\phi) = \lim_{n\to\infty} \int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \tag{3}$$

In § XX:B we show that every C^2-function can be expressed as the difference of two subharmonic functions and hence the limit in (3) exists for every $f \in C^2(\partial\Omega)$. Since $C^2(\partial\Omega)$ is dense in $C^0(\partial\Omega)$, Riesz representation Theorem implies that weak-star limits from subsequences of $\{\mathfrak{m}_z^{\Omega_n}\}$ are equal and hence the whole sequence has a weak-star limit $\mathfrak{m}_z^*$. Moreover, the boundaries $\partial\Omega_n$ approach $\partial\Omega$ which entails that $\mathfrak{m}_z^*$ is supported by $\partial\Omega$. At this stage we leave to the reader to show that if

$\{U_n\}$ is another nested sequence then the weak star limit of of $\{\mathfrak{m}_z^{U_n}\}$ is equal to the limit measure $\mathfrak{m}_z^*$ constructed via $\{\Omega_n\}$. Finally, the constructions above show that when z varies in Ω then

$$z \mapsto \int_{\partial\Omega} f \cdot \mathfrak{m}_z^*$$

is harmonic in Ω for every continuous boundary function f.

1.7 Regular boundary points. Consider a point $a \in \partial\Omega$. It is called Wiener-regular if

$$\lim_{z \to a} W_f(z) = f(a) \tag{1.7.1}$$

hold for every $f \in C^0(\partial\Omega)$. We shall find a potential-theoretic condition in order that (1.7.1) holds. Without loss of generality we can take a as the origin and for each $n \geq 1$ we consider the closed annulus

$$A_n = \{2^{-n} \leq |z| \leq 2^{-n+1}\}$$

For each fixed n the compact set $\partial\Omega \cap A_n$ has a capacity $e^{-\gamma_n}$ which by § XX means that there exists a probability measure μ_n supported by $\partial\Omega \cap A_n$ such that the logarithmic potential

$$U_n(z) = \frac{1}{\gamma_n} \int \log \frac{1}{|z-\zeta|} \cdot d\mu_n(\zeta) = 1 \quad : z \in \partial\Omega \cap A_n$$

1.8 Theorem. *The boundary point a is regular if and only if*

$$\sum_{n=1}^{\infty} \frac{n}{\gamma_n} = +\infty \tag{*}$$

Proof. Suppose first that (*) is convergent and then we must prove that (1.7.1) does not hold. To achieve this we take some $0 < \epsilon < 1/3$ which is kept fixed from now on where the assumed convergence in (*) gives an integer N_* such that

$$\sum_{n=N_*}^{\infty} \frac{n}{\gamma_n} \leq \epsilon \tag{i}$$

Choose $F \in C^0(\partial\Omega)$ with $F(0) = 1$ and $F = 0$ on $\{|z| \geq 2^{-N_*+1} \cap \partial\Omega\}$ while the whole range $F(\bar{\partial}\Omega)$ is contained in $[0,1]$. Now we shall argue by contradiction and suppose that (1.7.1) holds for F. This gives the existence of a small $s > 0$ such that

$$W_F(z) \geq 1 - \epsilon \quad \text{holds on} \quad D(s) \cap \Omega \tag{ii}$$

where $D(s)$ is the disc of radius s centered at the origin. Next, as $r \to 0$ the discs $D(r)$ shrink to the singleton set $\{0\}$. So by § XX the capacities $e^{-\gamma_r}$ of $D(r) \cap \Omega$. tend to zero with r. Let ν_r be the equilibrium distribution on $D(r) \cap \partial\Omega$ and put

$$V_r(z) = \frac{1}{\gamma_r} \cdot \int \log \frac{1}{|z-\zeta|} \cdot d\nu_r(\zeta) \tag{iii}$$

So here $V(z) = 1$ on $D(r) \cap \partial\Omega$. Since $\gamma_r \to +\infty$ as $r \to 0$ we can choose $r < s$ so small that

$$\max_{|z| \geq s} V_r(z) \leq \epsilon \tag{iv}$$

Next, let $M > N_*$ be so large that $2^{-M} \leq r$ and put

$$U_M^*(z) = \sum_{n=N_*}^{n=M} \frac{1}{\gamma_n} \cdot U_n(z)$$

With this choice it is clear that

$$V(z) + U_M^*(z) \geq 1 \quad \text{holds on} \quad D(2^{-N_*+1}) \cap \partial\Omega \tag{v}$$

Since $0 \leq F \leq 1$ and $F = 0$ on $\partial\Omega \setminus D(2^{-N_*+1})$ we have $F \leq V + U_M^*$ on the whole boundary of Ω. The minimum principle for super-harmonic functions gives the inequality

(vi)
$$V + U_M^* \geq W_F \quad \text{in} \quad \Omega$$

Now (i) and (iv) entail that

(vii)
$$\min_{z \in \Omega \cap D(s)} U_M^*(z) \geq 1 - 2\epsilon$$

At the same time we have $2^{-M} \leq s$ so the construction of U_M^* entails that $U_M^* = 1$ holds on $|z| \cap \partial\Omega$. Together with (vii) we obtain

(viii)
$$\min_{|z|=s} U_M^*(z) \geq 1 - 2\epsilon$$

Now U_M^* restricts to a super-harmonic function on $D(s)$ so the minimum principle for super-harmonic functions gives

(ix)
$$1 - 2\epsilon \leq U_M^*(0) = \sum_{n=N_*}^{n=M} \frac{1}{\gamma_n} \int \cdot \log \frac{1}{|\zeta|} \cdot d\mu_n(\zeta)$$

Finally, $|\zeta| \geq 2^{-n}$ holds on the support of μ_n and hence each integral is $\leq n \cdot \log 2$. So (ix) gives the inequality

$$1 - 2\epsilon \leq \sum_{n=N_*}^{n=M} \frac{n \cdot \log 2}{\gamma_n} < \log 2 \cdot \epsilon$$

This cannot hold since we have chosen $\epsilon < 1/3$. It means that (ii) cannot hold and hence a is not a regular boundary point.

Proof of sufficiency.

Above we proved that if a is regular then the series (*) in Theorem 1.8 must diverge. There remains to show that if (*) is divergent then a is regular. The proof of this converse is left as an exercise to the reader.

2. Harmonic conjugates

Let $H(x,y)$ be a harmonic function of class C^2 which means that

$$\Delta(H) = H_{xx} + H_{yy} = 0$$

Put

$$u = H_x \quad v = -H_y$$

Recall from Calculus that the mixed second order derivatives H_{xy} and H_{yx} are equal. Hence $u_y = -v_x$. Next, we have

$$u_x - v_y = H_{xx} + H_{yy} = 0$$

Hence (u,v) is a CR-pair and we can conclude:

2.1 Proposition. *Let H be harmonic in a domain Ω. Then*

$$f(z) = f(x+iy) = H_x - iH_y \in \mathcal{O}(\Omega)$$

2.2 The harmonic conjugate. Suppose that the analytic function $f(z)$ above has a primitive, i.e. there exists an analytic function $F(z) = U + iV$ whose complex derivative is f. We get

$$F'(z) = H_x - iH_y = U_x + iV_x = -iU_y + V_y \tag{1}$$

Identifying real and imaginary parts in (1) give

$$H_x = V_y \quad \text{and} \quad H_y = -V_x \tag{2}$$

We refer to V as the harmonic conjugate of H and notice that it is determined up to a constant. *Conversely*, if H has a harmonic conjugate V we get analytic function $F = H + iV$ where

$$F' = H_x + iV_x = H_x - iH_y = f(z)$$

Hence we have proved

2.3 Theorem *A harmonic function H has a harmonic conjugate if and only if the analytic function $H_x - iH_y$ is the complex derivative of an analytic function.*

2.4 Existence of harmonic conjugates. Consider a domain $\Omega \in \mathcal{D}(C^1)$. Let H be harmonic in Ω which extends to a C^1-function up to the boundary. Put $f(z) = H_x - iH_y$. To find an (eventual) primitive function of f we fix a point $z_0 \in \Omega$ and when $z \in \Omega$ we choose a path γ from z_0 to z and take the complex line integral of f along γ. Put

$$F_\gamma(z) = \int_\gamma f\, d\zeta \tag{i}$$

This gives an analytic function $F(z)$ in Ω provided that the integral above does not depend upon the particular path γ which joins z_0 and z. This independence holds if and only if the integral of f is zero along every *closed* curve inside Ω. Now $\partial\Omega$ consists of k disjoint closed curves $\Gamma_1, \ldots, \Gamma_k$ for some $k \geq 1$. By the *topological fact* from XX the f-intergral is zero along each closed curve in Ω if and only if

$$\int_{\Gamma_i} f(z)\, dz = 0 \quad : \quad 1 \leq i \leq k \tag{ii}$$

So (ii) gives a necessary and sufficient condition in order that f has a primitive analytic function. Let us express (ii) via derivatives of the H-function. We have

$$f dz = (H_x - iH_y)dz = H_x dx + H_y dy + i(H_x dy - H_y dx) \tag{iii}$$

Since Γ_i is closed the observation from § xx in Chapter II gives:

$$\int_{\Gamma_i} H_x\, dx + H_y\, dy = 0 \tag{iv}$$

There remains only the line integrals

$$\int_{\Gamma_i} H_x\, dy - H_y\, dx = \int_{\Gamma_i} \left[H_x \cdot \mathbf{n}_x + H_y \cdot \mathbf{n}_y\right] ds = \int_{\Gamma_i} H_{\mathbf{n}}\, ds \tag{v}$$

Hence the results above give:

2.5 Proposition. *A harmonic function H which extends to a C^1-function on $\bar{\Omega}$ has a harmonic conjugate if and only if*

$$\int_{\Gamma_i} H_{\mathbf{n}}\, ds \quad : \quad 1 \leq i \leq k$$

where $H_{\mathbf{n}}$ is the normal derivative and ds the arc-length measure.

Remark. By Theorem XX from Chapter II the line integral of $H_{\mathbf{n}}$ taken along the whole boundary is always zero, i.e. for every harmonic function H one has

$$\sum_{i=1}^{i=k} \int_{\Gamma_i} H_{\mathbf{n}}\, ds$$

So in order to check the conditions in Proposition 2.5 it suffices to regard $p-1$ many boundary curves.

2.6 The case $k=1$. If Ω only has one boundary curve Γ_1 the remark above shows that every harmonic function has a conjugate. Recall that Ω is a *Jordan domain* when its boundary consists of a single closed curve. Thus, in every Jordan domains there exist harmonic conjugates. If Ω is not a Jordan domain we cannot always find harmonic conjugates.

2.7 Example. Consider an annulus $\Omega = \{r < |z| < R\}$. Here we have the harmonic function $H(z) = \log |z|$. On the boundary curve $|z| = R$ we notice that $H_n = \frac{1}{R}$ and since $ds = Rd\theta$ is the arc-length measure we get

$$\int_{\{|z|=R\}} H_{\mathbf{n}}\, ds = \int_0^{2\pi} d\theta = 2\pi$$

So here there does not exist a harmonic conjugate which reflects the fact that $\log z$ is multi-valued and hence prevents the analytic function $\frac{1}{z}$ to have a primitive in the annulus.

2.8 A special construction. Let $\Omega \in \mathcal{D}(C^1)$ where $\partial\Omega$ consists of p many pairwise disjoint and closed Jordan curves $\Gamma_1, \ldots, \Gamma_p$ and Γ_p is the outer curve. We assume that $p \geq 2$ and consider a harmonic function $H(z)$ which extends to a C^1-function on $\bar{\Omega}$. Put

$$a_j = \int_{\Gamma_j} H_{\mathbf{n}}\, ds \qquad 1 \leq j \leq p-1\,.$$

Suppose there exist integers $m_1, \ldots, m_{p-1}$ such that

$$(*) \qquad a_j = 2\pi \cdot m_j \qquad 1 \leq j \leq p$$

For each $1 \leq j \leq p-1$ we choose a point z_j in the interior of the Jordan domain bordered by Γ_j and set

$$(1) \qquad H^*(z) = H(z) - \sum_{j=1}^{j=p-1} m_j \cdot \log|z - z_j|$$

Then we see that

$$\int_{\Gamma_j} H^*_{\mathbf{n}}\, ds = 0 \qquad 1 \leq j \leq p-1\,.$$

Proposition 2.5 shows that H^* has a harmonic conjugate V^* and there exists the analytic function

$$(2) \qquad f(z) = H^*(z) + iV^*(z)$$

It follows that

$$\left|e^{f(z)}\right| = e^{H^*(z)} = e^{H(z)} \cdot \left|(z-z_1)^{-m_1} \cdots (z - z_{p-1})^{-m_{p-1}}\right|$$

Hence, with $g(z) = (z-z_1)^{m_1} \cdots [z - z_{p-1})^{m_{p-1}} \cdot e^{f(z)}$ we have :

2.9 Theorem. *Let $H(z)$ be a harmonic function in Ω such that the periods $a_1,\ldots,a_{p-1}$ are integer multiples of 2π. Then there exists a zero-free analytic function $g(z)\in\mathcal{O}(\Omega)$ such that*

$$|g(z)| = e^{H(z)}$$

Exercise. By the construction above the g-function extends to a C^1-function on the closure $\bar\Omega$ and the reader should verify that

$$\int_{\Gamma_j} H_{\mathbf{n}}\,ds = \mathfrak{Im}\big[\int_{\Gamma_j}\frac{g'(z)dz}{g(z)-w}\,\big]$$

hold for every boundary curve. Show also the converse to Theorem 2.9. Thus, assume that there exists an analytic function $g(z)$ with no zeros in Ω such that $|g(z)|=e^{H(z)}$ holds and deduce that the period numbers $\{a_j\}$ are integers times 2π.

2.10 The case when H is constant on boundary curves. Let H be as in Theorem 2.8 and suppose also that $H(z)=c_j$ is a constant on every boundary curve of Ω. Then $|g(z)|=e^{c_j}$ is constant on each Γ_j. Let w be a complex number whose absolute value is $\neq e^{c_j}$ for every j. Let $\mathcal{N}(g:w)$ be the number of zeros of $g(z)$ in Ω counted with multiplicity. The formula from Theorem XXX in Chapter 4 gives:

$$\mathcal{N}(g:w) = \mathfrak{Im}\,\big[\frac{1}{2\pi}\sum_{\nu=1}^{p}\int_{\Gamma_\nu}\frac{g'(z)\,dz}{g(z)-w}\big]$$

As explained in XX each single integral

$$\text{(ii)}\qquad \mathfrak{Im}\big[\frac{1}{2\pi}\cdot\int_{\Gamma_\nu}\frac{g'(z)\,dz}{g(z)-w}\big]$$

is equal to the winding number of the image curve $(g(z)-w)\circ\Gamma_\nu$. This individual winding number is zero if $|w|>e^{c_j}$ and if $|w|<e^{c_j}$ the stability of the winding number under a homotopic deformation identifies (ii) with the value when $w=0$. Next, Theorem 2.9 gives the equality

$$g'(z) = g(z)\cdot(H_x - iH_y)$$

So with $w=0$ it follows that (ii) becomes

$$\mathfrak{Im}\big[\frac{1}{2\pi}\cdot\int_{\Gamma_\nu}(H_x-iH_y)\,dz\big] = \frac{1}{2\pi}\cdot\int_{\Gamma_\nu}H_x\,dy + H_y\,dx = \frac{1}{2\pi}\cdot\int_{\Gamma_\nu}H_{\mathbf{n}}\,ds$$

Hence we have proved the following:

2.11 Theorem *Let w be a complex number such that $|w|\neq e^{c_j}$ for every j and denote by $w(*)$ the set of those j:s for which $|w|<e^{c_j}$. Then one has the equality*

$$\mathcal{N}(g:w) = \frac{1}{2\pi}\cdot\sum_{j\in w(*)} a_j$$

2.12 Example. Suppose that $a_p = 2\pi\cdot m$ for some positive integer m while $a_\nu = -2\pi\cdot$ for every $1\le\nu\le p-1$. Suppose that w belongs to the open image set $g(\Omega)$ which means that the sum in Theorem 2.11 is a positive integer. Since $a_j\le 0$ when $j\le p-1$ it follows that the term from a_1 must be >0 and hence $|w|<e^{c-1}$. Thus, $g(\Omega)$ is contained in the disc of radius e^{c_1}. Moreover, there cannot exist $2\le j\le p-1$ with $e^{c_j}>e^{c-1}$ for then we can choose $|w|>e^{c_1}$ while $|w|<e^{c_j}$ and then the right hand side in Theorem 2.11 would be a negative integer.

3. Harmonic functions in a disc.

Above we studied general domains $\Omega \in \mathcal{D}(C^1)$. Here we specialize to the case when $\Omega = D$ is the unit disc centered at the origin. Already in Chapter 1 we encountered the Poisson kernel and described how to solve the Dirichlet problem. Let us resume this and give alternative proofs. If f is analytic in a neighborhood of $\bar{D}$ we have:

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1|} \frac{f(\zeta)\, d\zeta}{\zeta - z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})\, d\theta}{1 - ze^{-i\theta}}$$

Since $z^n f(z)$ is analytic for each non-negative integer n we have

$$\text{(i)} \qquad \int_{|\zeta|=1} \zeta^n f(\zeta)\, d\zeta = i \cdot \int_0^{2\pi} e^{i(n+1)\theta} f(e^{i\theta})\, d\theta = 0$$

Consider the geometric series

$$\frac{\bar{z}e^{i\theta}}{1 - \bar{z}e^{i\theta}} = \sum_{n=0}^{\infty} \bar{z}^{n+1} e^{i(n+1)\theta}$$

Since the series converges absolutely when $|z| < 1$, the vanishing (i) gives

$$(*) \qquad \int_0^{2\pi} \frac{\bar{z}e^{i\theta} \cdot f(e^{i\theta}) \cdot d\theta}{1 - \bar{z}e^{i\theta}} = 0$$

3.1 The Poisson kernel. Let us add the zero term (*) to the Cauchy integral and put

$$P(z, \theta) = \frac{1}{2\pi} \cdot \frac{1}{1 - ze^{-i\theta}} + \frac{\bar{z}e^{i\theta} d\theta}{1 - \bar{z}e^{i\theta}} = \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2}$$

where the last equality follows since $|1 - ze^{-i\theta}|^2 = (1 - ze^{-i\theta})(1 - \bar{z}e^{i\theta})$. We refer to $P(z, \theta)$ as the *Poisson kernel*. The results above give

3.2 Theorem *For each analytic function f in D which extends continuously to $\bar{D}$ one has*

$$f(z) = \int_0^{2\pi} P(z, \theta) f(e^{i\theta})\, d\theta \quad : \quad z \in D$$

Since the Poisson kernel is *real-valued* we can decompose $f = u + iv$ into its real and imaginary parts and obtain

$$u(z) = \int_0^{2\pi} P(z, \theta) u(e^{i\theta})\, d\theta \quad : \quad v(z) = \int_0^{2\pi} P(z, \theta) v(e^{i\theta})\, d\theta$$

3.3 Solution to Dirichlet's Problem Keeping θ fixed we have a function in D defined by $z \mapsto P(z, \theta)$. From the construction we have

$$2\pi \cdot P(z, \theta) = \frac{1}{1 - ze^{-i\theta}} + \frac{\bar{z}e^{i\theta} d\theta}{1 - \bar{z}e^{i\theta}}$$

The first term is an analytic function of z in D and since both the real and imaginary parts of an analytic function are harmonic, this complex valued function satisfies the Laplace equation. The second term is an *anti-analytic function*, i.e. analytic with respect to the conjugate complex variable. These functions also satisfy the Laplace equation and hence the function $z \to P(z, \theta)$ is harmonic in D. Now we solve the Dirichlet problem in D.

3.4 Theorem *Let $h \in C^0(T)$. Put*

$$H(z) = \int_0^{2\pi} P(z,\theta)h(e^{i\theta})\,d\theta$$

Then H solves the Dirichlet problem, i.e. it extends to a continuous function on T where it is equal to h.

Proof. We use that h is uniformly continuous and define its modulus of continuity:

$$\omega(\delta) = \max |h(\theta_1) - h(\theta_2)| \quad : \quad |\theta_1 - \theta_2| \leq \delta$$

For each $0 \leq \alpha \leq 2\pi$ we take a difference and obtain

$$2\pi \cdot [H(re^{i\alpha}) - h(e^{i\alpha})] = \int_0^{2\pi} \frac{(1-r^2)[h(e^{i(\theta+\alpha)} - h(e^{i\alpha})] \cdot d\theta}{1 + r^2 - 2r\cos\theta}$$

By the triangle inequality and the definition of the ω-function, the absolute value of the left hand side is majorized by

$$\int_0^{2\pi} \frac{(1-r^2)\cdot\omega(\theta)}{1+r^2-2r\cos\theta}\,d\theta$$

With $\delta > 0$ given we split the θ-integral in two parts, where we integrate over $-\delta < \theta < \delta$ and in the second part $|\theta| \geq \delta$. Since the ω-function is non-decreasing, i.e. $\omega(\theta) \leq \omega(\delta)$ hold when $|\theta| \leq \delta$ we then majorize the integral above by

$$\omega(\delta) + M \cdot \int_{|\theta|\geq\delta} \frac{(1-r^2)\,d\theta}{1+r^2-2r\cos\theta}$$

where M is the maximum norm of the ω-function. With $r = 1-s$ and $0 < s \leq 1$ the last integral above is majorized by

$$2Ms \cdot \int_{|\theta|\geq\delta} \frac{d\theta}{(2-s)(1-\cos\theta)} \leq 2Ms \cdot \int_{|\theta|\geq\delta} \frac{d\theta}{1-\cos\theta}$$

where the positive contribution of s^2 in the denominator is removed since this only strenghtens the inequality. Next, since $(1-\cos\theta \geq \theta^2/2$ the lat integral is majorised by

$$\int_{|\theta|\geq\delta} \frac{2d\theta}{\theta^2} = 4\frac{\pi-\delta}{\delta} \leq \frac{4\pi}{\delta}$$

Hence (*) is majorized by

$$\omega(\delta) + \frac{8\pi M(1-r)}{\delta}$$

The required limit as $r \to 1$ follows since the ω-function tends to zero, i.e. given ϵ we choose δ so that $\omega(\delta) < \epsilon$ and after we choose $1 - r < \frac{1}{8\pi M}\delta\epsilon$ which majorizes (*) with 2ϵ.

3.5 Remark When the boundary function h has extra continuity, say that it is of class C^1 one can improve the rate of convergence. But we shall not to discuss detailed estimates concerned with the rate of convergence of

$$r \mapsto \max_\theta |\, H(re^{i\theta}) - h(e^{i\theta})\,|$$

as $r \to 1$. One may also consider boundary functions for any L^1-function $h(\theta)$ on the unit circle we can apply the Poisson kernel and obtain a harmonic function $H(z)$ in the unit disc defined by

$$H(z) = \int_0^{2\pi} P(z,\theta)\cdot h(\theta)d\theta$$

The question arises if the H-function has *radial limit values*, i.e. if there exist limits of the form

$$(*) \qquad \lim_{r\to 1} H(re^{i\theta}) = h(\theta)$$

One expects that the radial limit exists for all θ outside a null set, i.e. convergence holds almost everywhere in the sense of Lebesgue. This is indeed true. More precisely (*) holds at every

Lebesgue point of the L^1-function h. We refer to § 3.18 for further details about boundary values of H when we start from a Riesz measure on T.

3.6 Herglotz formula.

Given a real-valued continuos function u on the unit circle we define a function $U(z)$ in D by

$$(*)\qquad U(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\cdot u(e^{i\theta})\cdot d\theta$$

It is clear that $U(z)$ is an analytic function in D since the geometric series of the denominator converges when $|z|<1$. Next, multiply with $e^{-i\theta}-\bar{z}$ which is the complex conjugate of the denominator and notice that:

$$(e^{-i\theta}-\bar{z})(e^{i\theta}-z)=1-|z|^2-2i\cdot\mathfrak{Im}(ze^{-i\theta})$$

It follows that

$$(**)\qquad U(z)=\frac{1}{2\pi}\int_0^{2\pi}[\frac{1-|z|^2}{|e^{i\theta}-z|^2}\cdot u(e^{i\theta})d\theta+\frac{i}{2\pi}\int_0^{2\pi}[\frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta}-z|^2}\cdot u(e^{i\theta})d\theta$$

Hence the real part of U is the Poisson integral of u, i.e. $\mathfrak{Re}(U)$ is the harmonic extension of u. The second integral which yields the harmonic conjugate to $\mathfrak{Re}(U)$. Starting with an analytic function $f(z)=u+iv$ and choosing $h=u|T$ as a boundary function, the construction gives

3.7 Theorem. *Let $f(z)$ be analytic in D with continuous boundary values. Then*

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\cdot\mathfrak{Re}(f(e^{i\theta}))\cdot d\theta+i\cdot\mathfrak{Im}(f(0))\quad:\quad z\in D$$

Hence f is represented by a boundary integral expressed by its real part only, except for the value of its imaginary part at the origin.

3.8 Formula for a harmonic conjugate. Start with some real-valued continuous function u on the unit circle T. We find the unique analytic function $f(z)$ in D whose real part is u on T while $f(0)$ is real. Now $\mathfrak{Im}\,f$ is the harmonic conjugate of $\mathfrak{Re}\ f$ and (**) above gives:

$$(***)\qquad \mathfrak{Im}f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta}-z|^2}\cdot u(e^{i\theta})d\theta$$

3.9 The conjugate kernel $\mathcal{C}$. In the product set $D\times T$ we define the function

$$\mathcal{C}(z,\theta)=\frac{1}{2\pi}\cdot\frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta}-z|^2}$$

The $\mathcal{C}$-function is no longer positive and the integrals with respect to θ of its absolute value increase when $|z|\to 1$. Namely, let $z=r=1-s$ with $0<s<1$. Then

$$|\mathcal{C}(z,\theta)|=\frac{1-s}{2\pi}\cdot\frac{|\sin\theta\,|}{[\,s^2+2(1-s)(1-\cos\theta)\,]^2}$$

So when s and θ both are close to zero the order of magnitude of $\mathcal{C}$ becomes

$$(1)\qquad |\mathcal{C}(s,\theta)|\simeq\frac{|\theta|}{(s^2+\theta^2)^2}$$

This explains why the map which sends a boundary function $u(e^{i\theta})$ into the boundary value function $v(e^{i\theta}$ of the harmonic conjugate of the harmonic extension of u is not so well-behaved. However, certain estimates are available. In particular the BMO-norm of the conjugate boundary value function v can be estimated by the maximum norm of u.

3.10 Theorem. *There exists an absolute constant C such that*

$$[v|_{\mathrm{BMO}} \leq C \cdot \max_\theta |u(e^{i\theta})|$$

The proof boils down to an exercise in one variable analysis. Namely, regard the following linear operators:

3.11 Definition. *To each $s > 0$ we define the operator T_s acting on real-valued functions $f(x)$ with support in $[-1, 1]$ by*

$$T_s(f)(y) = \int_{-1}^{1} \frac{x}{x^2 + s^2} \cdot f(x + y) dy$$

In the case when f is an even function we notice that $T_s(f)(0) = 0$. From the definition of functions in BMO and recalling that $\sin\theta \simeq \theta$ for small θ, the reader will have no difficulty to derive Theorem 3.10 from the following result:

3.12 Proposition. *There exists a constant C such that the following holds when f is even and has support in $[-1, 1]$:*

$$|\int \frac{1}{2h} \int_{-h}^{h} T_s(f)(y) dy| \leq |f|_\infty \quad : \forall\ 0 < s, h < 1$$

Proof. By partial integration the term above becomes

$$\frac{1}{2h} \int \operatorname{Log}\left[\frac{(\xi - h)^2 + s^2}{(\xi + h)^2 + s^2}\right] \cdot f(\xi) d\xi$$

Hence there only remains to show that there exists C so that

$$\int_{-1}^{1} |\operatorname{Log}\left[\frac{(\xi - h)^2 + s^2}{(\xi + h)^2 + s^2}\right]| d\xi \leq C \cdot h$$

Obviously C exists if there is a constant C^* such that

(i) $$\int_0^1 \operatorname{Log} |\frac{(\xi + h)^2 + s^2}{\xi^2 + s^2}| d\xi \leq C^* \cdot h$$

Now we use that

$$\frac{\xi + h)^2 + s^2}{\xi^2 + s^2} = 1 + \frac{2h\xi + h^2}{\xi^2 + s^2} \leq 1 + \frac{2h^2 + \xi^2}{\xi^2} = 2 + 2\frac{h^2}{\xi^2}$$

So there remains only to show

(ii) $$\int_0^1 \operatorname{Log}\left[2 + 2\frac{h^2}{\xi^2}\right] \cdot d\xi \leq C^* \cdot h \quad : \ 0 < h < 1$$

With $\xi = hu$ this follows since

(iii) $$\int_0^\infty \operatorname{Log}\left[2 + 2\frac{1}{u^2}\right] \cdot du < \infty$$

In other words, we can take C^* as the value of this convergent integral.

3.13 A duality theorem.

Theorem 3.10 shows that the harmonic conjugation functor expressed via the boundary value function

$$u \mapsto \mathcal{C}(u) \quad : \ u(e^{i\theta}) \in L^\infty(T)$$

is a continuous linear operator from $L^\infty(T)$ into BMO. Next, consider the Hardy space $H^1(T)$ which consists of those $f \in L^1(T)$ which are boundary values of analytic functions in D, or equivalently those integrable functions on the unit circle for for which

$$\int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \; n = 1, 2, \ldots$$

The following result is due to C. Fefferman and E. Stein in [F-S]:

3.14 Theorem *There exists an absolute constant C such that*

$$\left| \int_0^{2\pi} \phi(e^{i\theta}) \cdot f(e^{i\theta}) \cdot d\theta \right| \leq C \cdot |\phi|_{\text{BMO}} \cdot |f|_1$$

hold for all pairs $f \in H^1(T)$ and ϕ in BMO.

For the proof of Theorem 3.14 and an account about Hardy spaces and BMO the reader is referred to text-books by E. Stein. See in particular (St:xxx].

3.15 The Riesz transform

Let $f(\theta) \in L^1(T)$. We get the Fourier coefficients

$$\hat{f}(n) = \int_0^{2\pi} e^{-in\theta} f(\theta) d\theta$$

A dense subspace of $L^1(T)$ consists of trigonometric polynomials, i.e. those functions on T with a finite Fourier series. Denote this subspace by $\mathcal{P}$. On this subspace we define the linear operator $\mathcal{R}$ by

(*) $$\mathcal{R}(e^{i\nu\theta}) = -e^{i\nu\theta} \; : \; \nu \leq -1 \quad : \mathcal{R}(e^{i\nu\theta}) = e^{i\nu\theta} \; : \; \nu \geq 1 \quad : \mathcal{R}(1) = 0$$

This operator is closely related to the harmonic conjugation functor. To see this, let $n \geq 1$ and and put

$$u_n(\theta) = \frac{1}{2}[e^{in\theta} + e^{-in\theta}]$$

Its harmonic extension to D is $\mathfrak{Re}\,(z^n)$ and hence the boundary function of the harmonic conjugate is

$$v(\theta) = \frac{1}{2}[e^{in\theta} - e^{-in\theta}] \implies$$

(**) $$\mathcal{C}(u) = v = \mathcal{R}(u)$$

Taking linear sums we conclude that $\mathcal{R}$ via the associated Fourier series coincides with the harmonic conjugation functor. Passing to functions $f(\theta)$ which belong to $H^\infty(T)$ Theorem 3.10 therefore proves that $\mathcal{R}(f)$ has a bounded means oscillation. In other words, $\mathcal{R}$ gives a continuous linear operator from $H^\infty(T)$ to BMO(T). We shall not continue this discussion since it is a topic in the theory about singular integral operators. See the text-books by E. Stein which presents this theory from many aspects including higher dimensional cases.

3.16 Poisson integrals of Riesz measures.

Let μ be a real Riesz measure on T. In general it is decomposed into
$mu_c + \mu_s$ where μ_c is absolutely continuous and μ_s is singular. We obtain a harmonic function $H_\mu(z)$ in D defined by

$$H_\mu(z) = \frac{1}{2\pi} \int \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu(\theta)$$

Exercise. Show the inequality below for each $r < 1$:

(1) $$\int_0^{2\pi} |H(re^{i\phi}| \cdot d\phi \leq ||\mu|| \quad : \quad ||\mu||| \quad \text{is the total variation of} \quad \mu$$

Next, let $g(\theta)$ be a continuous function on T identified with the 2π-periodic θ-interval. Using the uniform continuity of g the reader may verify the limit formula

$$(2) \qquad \lim_{r\to 1} \int_0^{2\pi} g(\theta)\cdot H(re^{i\theta}) = \int_0^{2\pi} g(\theta)\cdot d\mu(\theta)$$

Since (2) holds for every $g \in C^0(T)$ it means that the absolutely continuous measures $\{H(re^{i\theta})\}$ converge weakly to μ where Riesz' representation theorem identifies the dual space of $C^0(T)$ with Riesz measures on T.

A converse result. Let $H(z)$ be harmonic in D and assume that there is a constant C such that

$$(*) \qquad \int_0^{2\pi} |H(re^{i\theta})|\cdot d\theta \leq C \quad \text{hold for all} \quad r<1$$

3.17 Theorem. *When (*) holds there exists a unique Riesz measure μ on T such that $H = H_\mu$.*

Proof. Notice that (*) means that the family of absolutely continuous measures on T given by the family $\{\mu_r(\theta) = H(re^{i\theta})\}$ is bounded. So by the compactness in the weak topology of measures there exists an increasing sequence $\{r_n\}$ where $r_n \to 1$ and a Riesz measure μ such that

$$(\text{i}) \qquad \lim_{n\to\infty} \int_0^{2\pi} g(\theta)\cdot H(r_ne^{i\theta})d\theta = \int_0^{2\pi} g(\theta)\cdot d\mu(\theta)$$

hold for every $g \in C^0(T)$. There remains to see why this gives $H = H_\mu$ in D. To prove this we fix $z \in D$ and if $r_n > |z|$ we have the Poisson formula

$$(\text{ii}) \qquad H(z) = \int_0^{2\pi} \frac{r_n^2 - |z|^2}{r_ne^{i\theta} - z|^2}\cdot H(r_ne^{i\theta})d\theta$$

Keeping $z \in D$ fixed we set

$$g_n(\theta) = \frac{1}{2\pi}\cdot\frac{r_n^2 - |z|^2}{|r_ne^{i\theta} - z|^2} \quad \text{and} \quad g_*(\theta) = \frac{1}{2\pi}\cdot\frac{1-|z|^2}{|e^{i\theta} - z|^2}$$

It is clear that $g_n \to g_*$ holds uniformly over $0 \leq \theta \leq 2\pi$ as $n \to \infty$. So if $\epsilon > 0$ we can find $n*$ such that $n \geq n_*$ entails

$$\big|H(z) - \int_0^{2\pi} g_*(\theta)\cdot H(r_ne^{i\theta})d\theta\,\big| < \epsilon$$

Now

$$H_\mu(z) = \int_0^{2\pi} g_*(\theta)\cdot d\mu(\theta)$$

and at the same time (i) holds. So when $n \geq n*$ we see that

$$|H(z) - H_\mu(z)| < 2\epsilon$$

Finally, since $\epsilon > 0$ is arbitrary we get $H = H_\mu$ as required.

Remark. Theorem 3.17 gives a 1-1 correspondence between the space of Riesz measures on T and harmonic functions in D satisfying the L^1-condition (*).

3.18 Radial limits. Let μ be a Riesz measure on T which gives the harmonic function H_μ. it turns out that radial limits exist almost everywhere.

3.19 Theorem. *Write $\mu = \mu_c + \mu_s$. Then there exists a null set E on T such that*

$$\lim_{r\to 1} H_\mu(re^{i\theta}) = \mu_c(e^{i\theta})$$

holds when $\theta \in T \setminus E$ and θ is a Lebesgue point of the $L^1(T)$-function $\mu_c(e^{i\theta})$

Exercise. Prove this result. A hint concerning the singular part μ_s is as follows: Since μ_s is singular we know from XX that there exists a null set F on T such that

$$\lim_{\delta\to 0} \frac{1}{\delta}\cdot\int_{\theta-\delta}^{\theta+\delta} |d\mu_s(\theta)| = 0$$

for each $\theta \in T \setminus F$. Using the expression of the Poisson kernel the reader can verify that this gives:

$$\lim_{r\to 1} H_{\mu_s}(re^{i\theta}) = 0 \quad \text{for each} \quad \theta \in T \setminus F$$

3.20 A class of analytic functions. Let μ be a singular measure on T. In D we get the analytic function

$$U_\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot d\mu(\theta)$$

We see that $\mathfrak{Re}\, U_\mu = H_\mu$ and taking the exponential we get the analytic function

$$\omega_\mu(z) = e^{U\mu(z)}$$

Here the absolute value

$$|\,\omega_\mu(z)| = e^{H_\mu(z)}$$

Theorem 3.19 entails that there is a null set F on T such that

(1) $$\lim_{r\to 1} |\omega_\mu(re^{i\theta})| = 1 \quad : \quad \theta \in T \setminus F$$

The question arises if there exist radial limits

$$\lim_{r\to 1} \omega_\mu(re^{i\theta}) = \omega_*(e^{i\theta})$$

outside a null set. This turns out to be true but the proof is more involved and relies upon the Brothers Riesz theorem which is established in Section X from Special Topics and a further study of analytic functions of the form $\omega_\mu(z)$ for singular Riesz measures is treated in XX from Special Topics.

4. The mean value property

Let H be a harmonic function defined in an open domain Ω. Let $z_0 \in \Omega$ and D is a disc centered at z_0 of some radius R with R chosen so that $\bar{D} \subset \Omega$. We already know the mean-value formula:

$$(*) \qquad H(z_0) = \frac{1}{2\pi} \int_0^{2\pi} H(z_0 + Re^{i\theta}) d\theta$$

It turns out that the converse is true. More precisely a real-valued and continuous function in Ω satisfying a local mean-value formula at every point in Ω is harmonic. Since the condition to be harmonic is local it suffices to prove this when Ω is a disc.

4.1 Theorem *Let $\bar{D}$ be a closed disc of some radius R centered at the origin. Let $h \in C^0(\bar{D})$ and assume that to every $z_0 \in D$ there exists some $0 < r \leq dist(z_0, \partial D)$ such that $h(z_0)$ equals its mean value over the circle $|z - z_0| = r$. Then h is harmonic in D.*

Proof. We solve the Dirichlet problem and find a harmonic function H in D which extends to a continuous function on $\bar{D}$ where $H = h$ on the circle $\{|z| = R\}$. Now we claim that $h = H$. Assume the contrary and suppose for example that there exists a point z_0 where $h(z_0) > H(z_0)$. Since a continuos function achieves its maximum on the compact set $\bar{D}$ we get the closed set K where $h - H$ takes its maximum. Here K must be inside the open disc since $h = H0$ on the boundary. Choose a point $z^* \in K$ such that $|z^*| \geq |z|$ for any other point $z \in K$. By hypothesis there exists a circle of some radius $r \leq R - |z^*|$ such that $h(z^*)$ is equal to its mean value over the circle $T = \{|z - z^*| = r\}$. Since the harmonic function H satisfies the mean value condition everywhere, it follows that

$$\text{(i)} \qquad h(z^*) - H(z^*) = \frac{1}{2\pi} \int_0^{2\pi} (h - H)(z^* + re^{i\theta}]) d\theta$$

But this gives a contradicton. For $H - h$ takes its maximum at z^* and by the choice of z^* we can find θ such that $z^* + re^{i\theta}$ is outside the set K and then the integral in (i) cannot be equal to $h(z^*) - H(z^*)$. This proves that $h \leq H$ holds in the whole disc. Since $-h$ also satisfies the local mean value condition we prove that $h \geq H$ in the same way and Theorem 3.1 follows.

4.2 Remark Notice that Theorem 4.1 implies that if h is *not* harmonic then there must exist some point $z_0 \in D$ such that

$$h(z_0) \neq \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta}) d\theta$$

hold for all $0 < r \leq R - |z_0|$. Notice also that this inequality means that $h(z_0)$ is either strictly less or strictly larger than all mean-values above.

4.3 A study of $\log|f|$**.** Let $\Omega \in \mathcal{D}(C^1)$ and f is an analytic function in Ω which extends to a continuos function on $\bar{\Omega}$. We also assume that $f \neq 0$ on $\partial\Omega$ but inside Ω it has some zeros $a_1, \ldots, a_k$ where the multiplicity of each zero can be any positive integer. Outside these zeros $\log|f|$ exists as a harmonic function. Now one has the following inequality:

4.4 Proposition *Let h solve Dirichlet's problem for the boundary function* $\log|f|$. *Then*

$$\log|f(z)| \leq h(z) \quad : \quad z \in \Omega \setminus (a_1, \ldots, a_k)$$

Proof. Let $\epsilon > 0$ be small and let Ω_ϵ be the open set where the closed discs of radius ϵ around the zeros of f are removed from Ω. If m is the order of a zero of f at a_ν we have

$$\log(|f|)(z - a_\nu) = m \cdot \log|z - a_\nu| + \text{ a bounded function}$$

when z is close to a_ν. So on the boundaries of the removed discs the harmonic function $\log|f|$ is negative and $\simeq$ to the negative number $m\cdot\log(\epsilon)$. For a sufficiently small ϵ it therefore follows that

$$\log|f|(z)\leq h(z)\quad:\quad z\in\cup\,\partial D_\epsilon(a_\nu)$$

At the same time equality holds on $\partial\Omega$. Hence the maximum principle for harmonic functions gives the inequality

$$\mathrm{Log}(|f|)(z)\leq h(z)\quad:\quad z\in\Omega\setminus\cup\,\bar{D}_\epsilon(a_\nu)$$

Since we can take any small ϵ Proposition 2.4 follows.

4.5 A specific majorisation. Let $f(z)$ be analytic in the half-plane $\mathfrak{Im}\,z>0$ which extends continuously to the real line $\{\mathfrak{Im}\,z=0\}$. Consider a finite sequence of real numbers $a_0<a_1<\ldots a_N$ where some a_ν may be negative. We get the pairwise disjoint intervals

$$J_0=(-\infty,a_0)\quad J_\nu=(a_\nu,a_{\nu+1})\colon\ 0\leq\nu\leq N-1\quad J_N=(a_n,+\infty)$$

Suppose there are constants $c_0,\ldots,c_N$ such that

$$\mathrm{Log}\,|f(x)|\leq c_\nu\quad:\quad x\in J_\nu\quad:\quad 0\leq\nu\leq N$$

In addition to this we assume that

$$(*)\qquad \lim_{|z|\to\infty}f(z)=0$$

where z stay in the closed upper half-plane during the limit. In this case we can chose a specific harmonic majorant to $\log|f(x)|$. Namely, put

$$H(z)=\sum_{\nu=0}^{\nu=N}c_\nu\cdot H_{J_\nu}(z)$$

where $\{H_{J_\nu}\}$ are the harmonic measure functions of the J-intervals. Then we have

$$\log|f(x)|\leq H(x)$$

for real x.

Exercise. Show that (*) entails the global inequality:

$$\log|f(z)|\leq H(z)\quad:\quad z\in U$$

5. Harmonic measures

Let Ω be a bounded connected and open set where Dirichlets problem can be solved. If $z \in \Omega$ we get a linear form on $C^0(\partial\Omega)$ defined by

$$(1) \qquad \phi \mapsto H_\phi(z)$$

Thus, we evaluate the unique harmonic extension of the boundary function ϕ at the point z. If $\phi \geq 0$ then $H_\phi(z) \geq 0$ and when $\phi = 1$ is the identity we have $H_\phi = 1$. Riesz' Representation Formula gives a unique probability measure on $\partial\Omega$ denoted by $\mathfrak{m}_z$ such that

$$(2) \qquad H_\phi(z) = \int_{\partial\Omega} \phi(\zeta) \cdot d\mathfrak{m}_z(\zeta)$$

When z varies in Ω these measures are absolutely continuous with respect to each other. To see this we fix some point p in Ω. Let q be another point and suppose that $\mathfrak{m}_q$ is not absolutely continuous with respect to $\mathfrak{m}_p$ which gives a closed subset E of $\partial\Omega$ where $\mathfrak{m}_p(E) = 0$ and $\mathfrak{m}_q(E) = a > 0$. Now we can choose a sequence of continuous boundary functions $\{\phi_n\}$ such that $0 \leq \phi_n \leq 1$ while $\phi_n = 1$ on E and

$$(1) \qquad \lim_{n\to\infty} \int_{\partial\Omega} \phi_n \cdot d\mathfrak{m}_p = 0$$

Here $\{H_n = H_{\phi_n}\}$ is a sequence of harmonic functions such that $0 \leq H_n \leq 1$ holds and $H_n(p) \to 0$. The minium principle for harmonic functions entails that $H_n \to 0$ holds on uniformly on every compact subset of Ω. In particular $H_n(q) \to 0$. But this gives a contradiction since the ϕ-functions give $H_n(q) \geq a$ for all n.

Öksendal's theorem. If may occur that the boundary $\partial\omega$ has a positive 2-dimensional Lebesgue measure. But the harmonic measures are always concentrated to null sets with respect to the Lebesgue measure in $\mathbf{R}^2$. Nasmely, one has

5.0 Theorem. *There exists a null-set E in $\partial\Omega$ such that $\mathfrak{m}_p(E) = 1$ for every $p \in \Omega$.*

We prove this result in 5.x below after some results concerning majorisation principles for harmonic measures have been established.

The principle of Gebietserweiterung. The inequality in (*) below is used in many situations. Let Ω and U be two domains where the Dirichlet problem is solvable for both. Suppose that $\Omega \subset U$ and put

$$\partial\Omega = \gamma \cup \gamma$$

where γ is a compact subset of ∂U while Γ is a subset of $\bar{\Omega}$. Consider also a compact subset K of ∂U such that $\bar{\Omega} \cap K = \emptyset$. Let z be a point in Ω which also may be regarded as a point in U. We get the harmonic measure $\mathfrak{m}_z^U(K)$, i.e. the value at z of the harmonic function h in U for which $h = 1$ on K while $h = 0$ on $\partial U \setminus K$. At the same time we have the harmonic measure $\mathfrak{m}_z^\Omega(\Gamma)$ which equals $H(z)$ for the harmonic function in Ω which is one on Γ and zero on $\gamma \setminus \Gamma$. Since $0 \leq h \leq 1$ holds in U it follows that $h \leq H$ on Γ and at the same time both H and h are zero on $\gamma \setminus \Gamma$. Hence $h \leq H$ holds on $\partial\Omega$ so the maximum principle for harmonic functions entails that

$$(*) \qquad \mathfrak{m}_z^U(K) \leq \mathfrak{m}_z(\Gamma)$$

5.1. Conformal invariance. Let Ω and Ω^* be two bounded and connected open sets which both are of class $\mathcal{D}(C^1)$. Let $g\colon \Omega \to \Omega^*$ be a conformal map which extends to the boundary and gives a homeomorphism from $\bar{\Omega}$ to $\bar{\Omega}^*$. As explained in XX the g-function sends a harmonic function H in Ω^* to the harmonic function $H \circ g$ in Ω. Let $\Gamma_1, \ldots, \Gamma_p$ be the closed boundary curves to Ω. Their images $\{g(\Gamma_\nu)\}$ give the boundary curves of Ω^*. Let E be some Borel set in $\partial\Omega$ which gives the harmonic measure function

$$H_E(z) = \int_E d\mathfrak{m}_z(\zeta)$$

Similarly we get the harmonic measure function

$$H^*_{g(E)}(w) = \int_{g(E)} d\mathfrak{m}^*_w(\zeta)$$

where $\mathfrak{m}^*_w$ is the harmonic measure on $\partial\Omega^*$. Now we have the equality

$$H_E(z) = H^*_{g(E)} \circ g(z) \implies$$

$$(*) \qquad \int_E d\mathfrak{m}_z(\zeta) = \int_{g(E)} d\mathfrak{m}^*_{g(z)}(\zeta) \quad \text{holds for all points} \quad z \in \Omega$$

Since g restricts to a homeomorphism from $\partial\Omega$ into $\partial\Omega^*$ this gives a transformation rule for the harmonic measure. This invariance is often used and we shall meet such applications in the chapter about conformal mappings.

5.2 Harmonic measures in half-discs. Let $R > 1$ and consider the half-disc $D_R^+ = D_R \cap \{y > 0\}$. which is contained in the upper half-plane $\mathfrak{Im}(z) > 0$. Here there exists the harmonic function

$$u(z) = \mathfrak{Im}(\text{Log}(z - R)) - \mathfrak{Im}(\text{Log}(z + R)) = \arg(z - R) - \arg(z + R)$$

As already pointed out in XX we have

$$\lim_{y \to 0} u(x + iy) = \pi \quad : \quad -R < x < R$$

Moreover, by euclidian geometry we have

$$u(Re^{i\theta}) = \frac{\pi}{2} \quad : \ 0 < \theta < \pi$$

Set

$$(*) \qquad u^*(z) = \frac{2}{\pi} \cdot (u(z) - \frac{\pi}{2})$$

Then $u^*(x) = 1$ on the real interval $(-R, R)$ while $u^* = 0$ on the portion of ∂D_R^+ where $|z| = R$. Hence, if $0 < a < R$ then the harmonic measure of the point ai with respect to the boundary piece $(-R, R)$ is equal to $u^*(ai)$.

Exercise. Show that

$$u^*(ai) = \frac{2}{\pi} \cdot \text{arctg}(\frac{R}{a})$$

We have also the harmonic measure with respect to the circular boundary where $|z| = R$ which is given by the function $1 - u^*$. hence, the harmonic measure at ai with respect to this circular portion is equal to

$$1 - \frac{2}{\pi} \cdot \text{arctg}(\frac{R}{a}) = \frac{2}{\pi} \cdot \int_{R/a} \frac{dt}{1 + t^2}$$

The last term is majorised by

$$(*) \qquad \frac{2}{\pi} \cdot \int_{R/a} \frac{dt}{t^2} = \frac{2a}{\pi R}$$

In particular this harmonic measure behaves like $O(\frac{1}{R})$ when $R \to +\infty$ while a stays in a bounded interval.

5.3 Lindelöf's theorem.

Let D^+ be a half-disc where $|z| < R$ and $\mathfrak{Im}\, z > 0$. Let $f(z)$ be a bounded analytic function in D^+ with maximum norm ≤ 1. Consider also a Jordan arc γ which belongs to D^+ except for the endpoint $\gamma(1)$ which is taken as the origin. Suppose that f converges to zero along γ, i.e.

$$\lim_{s \to 1} f(\gamma(s)) = 0$$

Then Lindelöf proved that f has a non-tangential limit at $z = 0$. More precisely, for each $0 < \alpha < \pi/2$ one has

$$(*) \qquad \lim_{r\to 0}\,[\max_{\alpha\leq\theta\leq\pi-\alpha}|f(re^{i\theta})|\,]=0$$

Proof. Consider the subharmonic function $u(z)=\log|f(z)|$. For each large $M>0$ it follows from (1) that there exists $s_0<1$ such that

$$\text{(i)} \qquad u(\gamma(s))\leq -M$$

for all $s_0\leq s<1$. Now we choose $0<R_*<R$ such that the smallest s for which $|\gamma(s)|=R_*$ holds is $\geq s_0$. Denote this s-number with s_* So if D_*^+ is the half-disc where $|z|<R_*$ then

$$\text{(ii)} \qquad u(\gamma(s))\leq -M$$

for all $s_*\leq s<1$ and the half-open Jordan curve $\gamma[s_*,1)$ stays in D_*. It follows from Jordan's theorem that this curve divides D_*^+ into a pair of disjoint Jordan domains Ω_1 and Ω_2. Moreover, as illustrated by a picture the real intervals $[-R_*,0]$ and $[0,R_*]$ appear in the boundary of each Ω-domain. We may assume that

$$[-R_*,0]\subset\partial\Omega_1$$

In addition $\partial\Omega_1$ consists of the Jordan arc $\gamma[s_*,1]$ and an arc of the circle $|z|=R_*$ with end-points at $-R_*$ and $\gamma(s_*)$. Next, given $0<\alpha<\pi$ we consider points $z\in\Omega_1$ such that:

$$\text{(iii)} \qquad z=re^{i\theta}\quad:\quad \alpha\leq\theta\leq\pi-\alpha\quad\text{and}\quad 0<r<R_*/2$$

For each such point z we have the harmonic measure $\mathfrak{m}_z$ on $\partial\Omega_1$. Since $|f|\leq 1$ was assumed we have $u\leq 0$ and since u is subharmonic we get:

$$\text{(iv)} \qquad u(z)\leq -M\cdot\mathfrak{m}_z(\Omega_1\cap\gamma)$$

At this stage we use the principle of *Gebietserweiterung*. Consider the whole half-disc D_*^+. Here $[0,R_*]$ is the portion of its boundary and since the Jordan arc γ separates the half disc where $[0,R_*]\subset\partial\Omega_2$ one has the inequality

$$\mathfrak{m}_z(\Omega_1\cap\gamma)\geq\mathfrak{m}_z^*([0,R_*])$$

where $\mathfrak{m}_z^*$ is the harmonic measure for z in the half disc. It follows from (iv) that

$$u(z)\leq -M\cdot\mathfrak{m}_z^*(([0,R_*])/tagv$$

Finally, since z satisfies (1) it follows that

$$\text{(vi)} \qquad \mathfrak{m}_z^*(([0,R_*])\geq\rho(\alpha)$$

where $\rho(\alpha)$ is a positive number which can be computed via a conformal mapping. See XX for details. So when z is as in (iii) we get the estimate

$$u(z)\leq-\rho(\alpha)\cdot M$$

Here $\rho(\alpha)$ depends on the fixed α-value only while M can be arbitrary large while R_* gets small and this obviously entails that we have the requested limit (*) in the sector $\alpha\leq\arg(z)\leq\pi-\alpha$. Finally, we proved that the limit when z stays in Ω_1. Reversing the role where one instead works with the interval $[-R_*,0]$ we get again the limit in (*) and Lindelöf's theorem is proved.

Remark. The original proof by Lindelöf appears in [Lindelöf]. The idea to employ harmonic measures and estimate the subharmonic function $\log|f(z)|$ was introduced by Carleman. See page xxxx in Nevanlinna's book [Nev] for an account about the principle for harmonic measures under a change of domains.

5.4 Application to Laurent series. Let $f(z)$ be analytic in a punctured disc $0<|z|<1$. If $0<r<1/2$ and $|z|=r$ we find estimates for harmonic measures as follows. To each $0<\epsilon<r$ we have the annulus $\epsilon<|z|<1/2$ and put

$$\omega(z)=\frac{1}{\log\frac{1}{2\epsilon}}\cdot\log\frac{|z|}{\epsilon}$$

This yields a harmonic function and we conclude that

$$\mathfrak{m}_z(T_*) = \frac{1}{\log \frac{1}{2\epsilon}} \cdot \log \frac{2r}{2\epsilon} = 1 + \frac{\log 2r}{\log \frac{1}{2\epsilon}}$$

where T_* is the circle $|z| = 1/2$. It follows that the harmonic measure with respect to the inner circle $T_\epsilon = |z| = \epsilon$ becomes

(1) $$\mathfrak{m}_z(T_\epsilon) = 1 - \mathfrak{m}_z(T_*) = \frac{\log \frac{1}{2r}}{\log \frac{1}{2\epsilon}}$$

Suppose now that $f(z)$ is bounded in the punctured disc with a maximum norm M. At the same time we consider the maximum norm of $|f|$ on the circle $|z| = 1/2$ which we denote by M_*. Since $\log |f|$ is subharmonic it follows from (1) that we have

(2) $$\log |f(z)| \leq \log M_* + M \cdot \frac{\log \frac{1}{2r}}{\log \frac{1}{2\epsilon}}$$

This inequality holds for every $|epsilon > 0$ and since $\frac{1}{2\epsilon} \to +\infty$ we conclude that $\log |f(z)| \leq \log M_*$ which entails that

(3) $$|f(z)| \leq M_*$$

The point is that (3) does not depend upon M. Now we give an application:

5.5 Theorem. *Let f be an analytic function in the punctured disc $0 < |z| < 1$ where f and all is derivatives $\{f^{(n)}\}$ are bounded. Then f extends to an analytic function in the disc $|z| < 1$.*

Proof. Let M be the maximum norm of f in $0 < |z| < 1$. If $n \geq 1$ then Cauchy's inequality entails that

(i) $$|f^{(n)}(z)| \leq n\,! \cdot 2^n \cdot M$$

for each $|z| = 1/2$. Using (3) applied to the analytic function $f^{(n)}$ it follows that (i) holds for all $0 < |z| < 1/2$. Now we consider the series expansion at $z = 1/4$:

(ii) $$f(1/4 + z) = \sum \frac{f^{(n)}(1/4)}{n\,!} \cdot z^n$$

Since (i) hold with $z = 1/4$ we see that the series (ii) has a radius of convergence which is at least $1/2$ which implies that $f(z)$ extends beyond $z = 0$.

5.6 Proof of Öksendal's theorem.

We shall interpretate harmonic measures via the hitting probability for a Browninan motion. To begin with we consider the following situation. Let $\delta > 0$ and start a Brownian motion a point p with $|p| = \delta$. Suppose also that there is a closed subset Γ on a circle $|z| = r$ for some $\delta/2 \leq r < 3\delta/4$ whose linear measure is $(1 - \epsilon) \cdot 2\pi r$ for some $0 < \epsilon < 1$. This set is viewed as an obstacle and we seek the probability that the Brownian motion reaches some point on the circle $|z| = \delta/2$ before it has hit Γ which means that a Brownian path during passages through $|z| = r$ must stay in one of the open intervals on this circle which form the open complement of Γ. With this in mind one has

5.7 Proposition. *There exists an absolute constant C which is independent of both δ and ϵ such that the survival probalility is $\leq C \cdot \epsilon$.*

The result is intuitively clear and for a formal proof we refer to XXX. There remains to see why Propostion 5.7 gives Theorem 5.0. To achieve this we study Lebesgue points on the compact set $\partial\Omega$ where we assume that its 2-dimensional measure is > 0 since otherwise Theorem 5.0 is trivial. With C chosen as above we set

(1) $$\epsilon = \frac{1}{8C}$$

For each positive integer M we let K_M be the subset of $\partial\Omega$ such that if $x \in K_M$ then

$$0 < \delta \leq 2^{-M} \implies \frac{|\partial\Omega \cap B_z(\delta)|_2}{\pi\delta^2} \geq 1 - \epsilon \tag{2}$$

By Lebegue's Theorem we have

$$\lim_{M\to\infty} |\Omega \setminus K_M|_2 = 0 \tag{3}$$

Next, fix a point $p \in \Omega$ and consider the measure$\mathfrak{m}_p$. We must show that this Riesz measure is singular and from (3) it suffices to show that its restriction to K_M is singular for all M. Since the sets $\{K_M\}$ increase with M it suffices to show this when we start from M-values such that the distance of p to $\partial\Omega$ is strictly greater than 2^{-M}. By the general result in MEASURE the restriction $\mathfrak{m}_p|K_M$ is singular if

$$\lim_{N\to\infty} 4^N \cdot \mathfrak{m}_p(B_z(2^{-N})) = 0 \tag{5}$$

for each $z_* \in K_M$. To prove (5) we consider integers $N \geq M+1$ and identify $\mathfrak{m}_p(B_{z_*}(2^{-N}))$ with the probability that a Brownian path starting at p reaches the circle $|z - z_*| = 2^{-N}$ before it has hit $\partial\Omega$. In order that such a path survives it must pass the circles $T_k = |z - z_*| = 2^{-k}$ for every $M \leq k \leq N$. If the path has reached a point $p \in T_k$ the probability that it also reaches T_{k+1} can be estimated above by Proposition 5.7. Namely, for a given k we set

$$\rho_k = \max_r \frac{1}{2\pi r} \cdot \text{Length of} \quad \partial\Omega \cap T_r|$$

where the maximum is taken when $2^{-(k+1)} \leq r \leq 3 \cdot 2^{-k}/4$. It follows that the area:

$$|\Omega \cap B_{z_*}(2^{-k}|_2 \geq (1-\rho_k)\pi \cdot 2^{-2k} \cdot (9/16 - 1/4)$$

On the other hand, since $z_* \in K_M$ we also have

$$|\Omega \cap B_{z_*}(2^{-k})|_2 \leq \epsilon \cdot \pi \cdot 2^{-2k}$$

It follows that

$$1 - \rho_k \leq \frac{16}{5} \cdot \epsilon \implies \rho_k \geq 1 - \frac{16}{5} \cdot \epsilon$$

Hence we find a radius r in the interval above where the length of $\partial\Omega \cap T_r$ is at least $(1 - \frac{16}{5} \cdot \epsilon) \cdot 2\pi r$. This set is an obstacle and Proposition 5.7 together with the general principle in 5.1 imply that the probability to move from p to a point on T_{k+1} without reaching $\partial\Omega$ is majorized by

$$C \cdot \frac{16}{5} \cdot \epsilon = \frac{1}{8}$$

where (1) above gives the last inequality. At this stage we easily get (5). For in order to move from a point p on the circle $|z - z_*| = 2^{-M}$ to a point on $|z - z_*| = 2^{-N}$ we must survive the passage for each k as above, i.e. we must survive a passage $N - M - 1$ many times. By (**) the probability for this survivial is estimated above by

$$\frac{1}{8^{N-M-1}} = 4^{-N} \cdot 8^{M+1} \cdot 2^{-N}$$

It follows that

$$4^N \cdot \mathfrak{m}_p(B_z(2^{-N})) \leq 8^{M+1} \cdot 2^{-N}$$

With M fixed the right hand side tends to zero with N and hence (5) follows which finishes the proof of Öksendal's theorem.

6. Green's functions.

Let Ω be a bounded and connected set of class $\mathcal{D}(C^1)$. Let $z_* \in \Omega$ which gives the continuous boundary function

$$(1) \qquad \zeta \to \text{Log}\,|\zeta - z_*|$$

In Ω we get the unique harmonic function whose boundary function is (1). It depends upon z_* and is denoted by $H(z_*; z)$ where z is the "active variable". Keeping z_* fixed the values of H inside Ω are recaptured by the harmonic measure:

$$(2) \qquad H(z_*; z) = \int_{\partial\Omega} \log\,|\zeta - z_*|\, d\mathfrak{m}_z(\zeta) \quad : \ z \in \Omega$$

Keeping z fixed the measure $\mathfrak{m}_z$ can be weakly approximated by a sequence of discrete measures of the form $\rho = \sum p_k \cdot \delta_{\zeta_k}$ where $\sum p_k = 1$ and $\{\zeta_k\}$ is a finite subset of $\partial\Omega$. Since the functions

$$z_* \mapsto \log|z - z_*|$$

are harmonic in Ω for each $\zeta \in \partial\Omega$ it follows that the functions

$$\zeta_* \mapsto H(z_*; z)$$

are harmonic for every fixed z. It turns out that one has symmetry.

6.1. Theorem. *For each pair of points z_*, z in Ω one has:*

$$H(z_*, z) = H(z, z_*)$$

Proof. By the construction of the H-function we have

$$(1) \qquad H(z_*, \zeta) = \log|\zeta - z_*| \quad : \quad \zeta \in \partial\Omega$$

Next, for each pair of distinct points in Ω we set:

$$(2) \qquad G(z_*; z) = H(z_*; z) - \log\,|z - z_*| \quad : \ z_* \neq z$$

Then (1) entails that

$$(3) \qquad G(z_*; \zeta) = 0 \quad : \ \zeta \in \partial\Omega$$

At the same time the function of z defined by

$$-G(z; z_*) = \log\,|z - z_*| - H(z_*; z)$$

is subharmonic in Ω and tends to $-\infty$ as $z \to z_*$. By (3) it vanishes on $\partial\Omega$ so by the maximum principle for subharmonic functions it is < 0 in $\Omega \setminus z_*$ which means that

$$(4) \qquad G(z_*; z) > 0 \quad : \ z \in \Omega \setminus z_*$$

Next, since $\log\,|z - z_*|$ is symmetric we have:

$$G(z_*; z) - G(z; z_*) = H(z_*; z) - H(z; z_*)$$

Keeping z_* fixed (3) and (4) show that the left hand side is < 0 when $z \in \partial\Omega$. Hence the maximum principle for harmonic functions gives

$$G(z_*; z) - G(z; z_*) \geq 0$$

Reversing the role the reader finds the opposite inequality and Theorem 6.1 follows.

6.2 Remark. The G-function is called the Green's function attached to the domain Ω. Theorem 6.1 implies also that Green's function also enjoys symmetry, i.e.

$$(*) \qquad G(z_*, ; z) = G(z; z_*)$$

6.3 Harmonic measure and normal derivatives of G. It turns out that G recaptures harmonic measures. Keeping z_* fixed we know that $z \to G(z_*; z)$ is positive in Ω and tends to zero as $z \to \partial\Omega$. Now we can apply Green's formula starting with an arbitrary harmonic function u in Ω which extends to a C^1-function on the boundary. More precisely, for a small $\epsilon > 0$ we remove the disc $D_\epsilon = |z - z_*| \leq \epsilon$ and with $\Omega_\epsilon = \Omega \setminus D_\epsilon$ we apply Green's formula to the pair u and $G(z_*, \zeta)$. This gives the equality

$$\int_{\partial\Omega_\epsilon} u(\zeta) \cdot \partial G/\partial \mathbf{n}(\zeta) \cdot ds(\zeta) = \int_{\partial D_\epsilon} G(z_*, \zeta) \cdot \partial u/\partial \mathbf{n}(\zeta) \cdot ds(\zeta)$$

where we used that $G(z_*, \zeta) = 0$ on $\partial\Omega$.

Notice that we have taken outer normals with respect to the domain Ω_ϵ. A passage to the limit when $\epsilon \to 0$ gives the equality

$$(*) \qquad u(z_*) = \frac{1}{2\pi} \cdot \int_{\partial\Omega} u(\zeta) \cdot \partial G/\partial \mathbf{n}_* \cdot ds(\zeta)$$

where we have taken the *inner normal* of $G(z_*, \zeta)$ along $\partial\Omega$. This choice is natural since $G > 0$ in Ω so that the inner normal is non-negative which means that we obtain a representation by a non-negative measure. Hence we get the equality

$$(**) \qquad d\mathfrak{m}_{z_*} = \partial G/\partial \mathbf{n}_* \cdot ds$$

where ds is the arc-length measure on the boundary.

6.5 Example. Let $\Omega = D$ be the unit disc. In this case we have:

$$G(z, \zeta) = \text{Log}\, \frac{|1 - z \cdot \bar{\zeta}|}{|z - \zeta|}$$

With $z_* \in D$ kept fixed and $\zeta = e^{i\theta}$ is on the boundary circle T we see the inner normal derivative becomes

$$\frac{d}{dr}\big[(\text{Log}\, \frac{|re^{i\theta} - z_*|}{|1 - re^{-i\theta} \cdot z_*|}\big]$$

where this derivative is evaluated when $r = 1$. At the same time the arc-length measure is $d\theta$. The reader may now verify that the formula in (**) corresponds to Poisson's formula for the harmonic extension of the boundary function $u(e^{i\theta})$.

6.6 Harmonic measures with respect to arcs. Let Ω be a domain in the class $\mathcal{D}(C^1)$ and γ a simple C^1-curve which appears as a compact subset of Ω. compact subset of Ω. The Dirichlet problem has a solution for the domain $\Omega \setminus \gamma$. Let H be harmonic in $\Omega \setminus \gamma$ with continuous boundary values. Along γ the outer normal derivative of H is taken from two sides while we apply Green's formula. See XXX for a detailed discussion. If $z_0 \in \Omega \setminus \gamma$ we apply Green's formula to $H(\zeta)$ and $G(z_0, \zeta)$ while a small disc centered at z_0 is removed. Since $G = 0$ on $\partial\Omega$ we obtain

$$\int_{\partial\Omega} H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds + \int_\gamma H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds + \int_{\partial D_\epsilon} H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds =$$

$$\int_\gamma G(z_0, \zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds + \int_{\partial D_\epsilon} G(z_0, \zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds$$

Passing to the limit as $\epsilon \to 0$ we get the equality

$$(1) \quad \int_{\partial\Omega} H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds + \int_\gamma H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds - 2\pi \cdot H(z_0) = \int_\gamma G(z_0, \zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds$$

Now $\zeta \mapsto G(z_0, \zeta)$ is harmonic in a neighborhood of γ and hence quite regular which implies that the outer normal derivatives along the two sides of γ cancel each other so the second integral above vanishes and there remains the equality

$$(2) \qquad \int_{\partial\Omega} H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds - 2\pi \cdot H(z_0) = \int_\gamma G(z_0, \zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds$$

In particular we can apply (*) when $H = 0$ on $\partial\Omega$ and identically one on γ. This means that $H(z_0)$ evaluates the harmonic measure at $z_0 =$ with respect to the compact set γ inside the domain Ω. We set $H(z) = w_\gamma(z)$ where one should consider Ω as fixed while γ is a simple curve situated inside Ω. Since $w_\gamma = 0$ on $\partial\Omega$ we arrive at the formula

$$(3) \qquad w_\gamma(z_0) = \frac{1}{2\pi} \cdot \int_\gamma G(z_0,\zeta) \cdot \partial w_\gamma/\partial \mathbf{n}_* \cdot ds$$

where we use the inner normal of w_γ. Denote by σ_γ the non-negative measure defined by the density $\frac{1}{2\pi} \cdot \partial w_\gamma/\partial \mathbf{n}_* \cdot ds$. So we can write

$$(*) \qquad w_\gamma(z_0) = \int_\gamma G(z_0,\zeta) \cdot d\sigma_\gamma(\zeta)$$

Remark. Notice that (*) holds for every point $z_0 \in \Omega \setminus \gamma$ while the non-negative measure σ_γ is fixed.

6.7 Beurling's projection theorem.

The formula (*) in (6.6) has several applications. We shall give one which is due to Beurling. Let Ω be the unit disc D and consider an arc γ inside D which does not contain the origin. Keeping γ fixed we set $\sigma = \sigma_\gamma$. Now we have the circular projection

$$(1) \qquad z \mapsto |z|$$

which sends γ onto some interval $[a,b]$ on the positive x-axis with $0 < a \leq b < 1$. The exceptional equality $a = b$ holds if γ happens an arc on the circle $|z| = a$. As illustrated by a figure the map (1) is in general not bijective. However, there always exists the push-forward measure denoted by σ^* which is supported by $[a,b]$ and

$$(2) \qquad \int_a^b \phi(s) \cdot d\sigma^*(s) = \int_\gamma \phi(|\zeta|) \cdot d\sigma(\zeta)$$

holds for every $\phi \in C^0[a,b]$. We consider only the case $a < b$ and assume also the measure σ^* is absolutely continuous.

Exercise. Show that σ^* is absolutely continuous if the arc γ is real-analytic.

Next, consider the function

$$(3) \qquad V(z) = \int_a^b G(z,s) \cdot d\sigma^*(s)$$

It is clear that $V(z)$ is a harmonic function in $D \setminus \gamma$ and $V = 0$ on the unit circle T. Since $z \mapsto g(z,\zeta)$ are super-harmonic for every ζ and $\sigma^* \geq 0$, it follows that $V(z)$ is super-harmonic in the whole disc D.

6.7.1 Proposition. *For each $a \leq s_* \leq b$ one has*

$$\liminf_{z \to s_*} V(z) \geq 1$$

where the $\liminf$ *is taken as $z \in D \setminus [a,b]$ approach s_*.*

Proof. We have $s_* = |z_*|$ for some $z_* \in \gamma$ where $z_* = s_* \cdot e^{i\theta}$ for some θ. If $\liminf$ is taken along some sequence $\{z_k$ we put $z_k^* = |z_k| \cdot e^{i\theta}$ which implies that $z_k^* \to z_*$. Since $w_\gamma = 1$ on γ it follows that

$$(i) \qquad \lim \int G(z_k^*,\zeta) \cdot d\sigma(\zeta) = 1$$

Since $\sigma \geq 0$ the inequality for G in (xx) entails that the integrals above are majorized by

$$(ii) \qquad \int G(|z_k^*|,|\zeta|) \cdot d\sigma(\zeta) = \int G(|z_k|,s) \cdot d\sigma^*(s)$$

where the last equality used (2) above and the equalities $|z_k^*| = z_k|$ At the same time we have

$$V(z_k) = \int_a^b G(z_k, s) \cdot d\sigma^*(s) \tag{iii}$$

Now we have the limit formulas

$$\lim z_k = \lim |z_k| = s_*$$

We have assumed that $d\sigma^*(s)$ is an absolutely continuous measure Moreover, it is clear the functions $s \mapsto G(z, s)$ are L^1-functions on $[a, b]$ for every $z \in D$ which in addition satisfy

$$\lim \int_a^b |G(z, s) - G(z', s)| \cdot d\sigma^*(s) = 0$$

when $|z| - z'| \to 0$. Since $|z_k - |z_k| \to 0$ we have

$$\lim V(z_k) = \lim V(|z_k|)$$

Finally, the majorization from (ii) and the limit formula (i) give Proposition 6.7.1.

6.7.2 An estimate for harmonic measures. The inequality in Proposition 6.7.1 implies that the V-function majorizes the harmonic measure function with respect to the interval $[a, b]$. Hence we get the inequality

$$w_{[a,b]}(-[z|) \leq V(-|z|) = \int_a^b G(-|z|, s) \cdot d\sigma^*(s) \tag{i}$$

for each $z \in D$. By (2) the right hand side is equal to

$$\int_\gamma G(-|z|, |\zeta|) \cdot d\sigma(\zeta) \tag{ii}$$

Now the inequality from (xxx) shows that (ii) is majorized by

$$\int_\gamma G(z, \zeta) \cdot d\sigma(\zeta) = w_\gamma(z) \tag{iii}$$

Hence we arrive at the following

6.7.3 Theorem. *For every $z \in D$ one has the inequality*

$$w_{[a,b]}(-|z|) \leq w_\gamma(z)$$

where $[a, b]$ is the circular projection of γ.

Remark. The result above gives the starting point fora far-reaching study concerned with estimates of harmonic measures. We refer to Section 9 in chapter III in [Garnett-Marshall] for further results of this nature and mention that [ibid] also contains several instructive exercises and references to the literature.

7. The Neumann problem.

Let Ω be a domain in the class $\mathcal{D}(C^1)$ bordered by p many closed Jordan curves $\Gamma_1, \ldots, \Gamma_p$ where $p \geq 2$. Given a continuous function f on $\partial\Omega$ we seek a harmonic function v in Ω whose outer normal derivative satisfies

$$(*) \qquad \partial v/\partial \mathbf{n}(p) = f(p) \quad : \quad p \in \partial\Omega$$

Since

$$\int_{\partial\Omega} \partial v/\partial \mathbf{n} \cdot ds = 0$$

hold for every harmonic function a necessary condition in order that (*) can be solved is that

$$(**) \qquad \int_{\partial\Omega} f \cdot ds = 0$$

We are going to prove that (*) has a unique solution when f satisfies (**). Before we enter the proof we establish a mean-value inequality which is due to Schwarz. Namely, let v be harmonic in Ω and suppose it extends to a C^1-function on the boundary. Let γ be an arc of one boundary curve, say Γ_1, and suppose that

$$\text{(i)} \qquad \partial v/\partial \mathbf{n}(p) = 0 \text{ for all } p \in \gamma$$

If a and b are the two end-points of γ we can construct a closed Jordan curve J where γ appears as an arc and $J \setminus \gamma$ is a simple Jordan curve contained in Ω. As explained by a figure we may construct J so that a and b are two corner points and otherwise J is of class C^1. With this given one has the following result:

7.1 Schwarz integral formula. *Assume that (i) above holds. Then, for every point $p \in \gamma$ there exists a probability measure μ_p supported by $J \setminus \gamma$ such that*

$$v(p) = \int_{J\setminus\gamma} v(z) \cdot d\mu_p(z)$$

Proof. Riemann's mapping theorem gives a conformal map ϕ from the Jordan domain bordered by J onto the upper half-disc D^+ where $|w| < 1$ and $\mathfrak{Im}(w) > 0\}$, such that $\phi(\gamma)$ is the real interval $(-1, 1)$ while $\phi(J \setminus \gamma)$ is the half circle. Then $V = v \circ \phi^{-1}$ is harmonic in D^+ and with the complex coordinate $\zeta = \xi + i\eta$ in D^+ we see that (i) means that

$$\text{(ii)} \qquad \frac{\partial V}{\partial \eta}(\xi, 0) = 0 \quad : \quad -1 < \xi < 1\,.$$

Let U be the harmonic conjugate of V and consider the analytic function

$$F(\zeta) = U + iV$$

The Cauchy-Riemann equations and (ii) give $\frac{\partial U}{\partial \xi}(\xi, 0) = 0$. Hence U is a constant on this real interval and since we can choose this conjugate harmonic function up to a constant we may assume that $U(\xi, 0) = 0$. Hence Schwarz' reflection principle can be applied, i.e. we find an analytic function $\hat{F}(\zeta)$ defined in the unit disc $|\zeta| < 1$ with

$$\hat{F}(\zeta) = \bar{F}(\bar{\zeta}) \quad : \quad \mathfrak{Im}(\zeta) < 0$$

If $-1 < \xi < 1$ we represent $\hat{F}(\xi)$ by the Poisson integral over $|\zeta| = 1$ and since ξ is real we get

$$\hat{F}(\xi) = 2 \cdot \int_0^\pi P(\xi, e^{i\theta}) \cdot F(e^{i\theta}) \cdot d\theta$$

Taking the real part we can write

$$V(\xi) = 2 \cdot \int_0^\pi P(\xi, e^{i\theta}) \cdot V(e^{i\theta}) \cdot d\theta$$

Here $2\cdot\int_0^\pi P(\xi,e^{i\theta})d\theta = 1$. So $V(\xi)$ is a convex sum of values of V taken on the half-circle $\phi(J\setminus\gamma)$. Returning to the v-function this means that there exists a probability measure μ_ξ supported by $J\setminus\gamma$ such that

$$v(\phi^{-1}(\xi)) = \int_{J\setminus\gamma} v(z)\cdot d\mu_\xi(z)$$

Since $\phi^{-1}(\xi))$ corresponds to any preassigned point on γ we have proved Schwarz' integral formula.

7.2 A uniqueness result. Let v be harmonic in Ω and suppose that

$$\frac{\partial v}{\partial \mathbf{n}}(p) = 0 \text{ holds for every } p\in\partial\Omega$$

Then Schwarz' integral formula and the maximum principle for harmonic function implies that v is a constant. In fact, let M be the maximum of v over $\partial\Omega$ and choose $p\in\partial\Omega$ where $v(p)=M$. If v is not a constant we have seen in XX that $v(q)<M$ for every $q\in\Omega$. This strict inequality means that we cannot obtain a measure μ_p in the integral formula above. On the other hand we can find such a non-trivial formula when $\frac{\partial v}{\partial \mathbf{n}}$ is identically zero on $\partial\Omega$. Hence v must be reduced to a constant. This proves the uniqueness part for a solution to the equation (*). Their remains to prove the existence. Before this is done we establish another result of independent interest.

7.3 A non-singular matrix for harmonic measure functions. To each boundary Γ_i we have the harmonic function H_i in Ω with boundary value one on Γ_i while $H_i=0$ on the remaining boundary curves. Let Γ_p be the outer curve and consider the functions $H_1,\ldots,H_{p-1}$. We get the $(p-1)\times(p-1)$-matrix A with elements

(i)
$$a_{j,\nu} = \int_{\Gamma_\nu} \frac{\partial H_j}{\partial \mathbf{n}}\cdot ds \quad : 1\le j,\nu\le p-1$$

Proposition. *7.4 The matrix A is non-singular.*

Proof. Assume the contrary. This gives some $p-1$-tuple $c_1,\ldots,c_{p-1}$ which are not all zero such that

(ii)
$$c_1\cdot a_{1,\nu}+\ldots+c_{p-1}\cdot a_{p-1,\nu}=0 \quad : \quad 1\le\nu\le p-1$$

Set $V=c_1H_1+\ldots+c_{p-1}H_{p-1}$. Then (i) and (ii) give

(iii)
$$\int_{\Gamma_\nu}\frac{\partial V}{\partial \mathbf{n}}\cdot ds = 0 \quad : \quad 1\le\nu\le p-1$$

From (xx) it follows that V has a harmonic conjugate U and we get the analytic function $F=U+iV$. Next, the *tangential* derivative of a harmonic measure function H_j is identically zero on every boundary curve. By the Cauchy- Riemann equations it follows that the normal derivative of U is identically zero on every boundary curve and then the uniqueness result in 7,2 above implies that U is a constant. But then V would also be reduced to a constant which obviously cannot occur when some $c_k\neq 0$. Hence the A-matrix is non-singular.

7.5 A consequence. Now we consider the whole p-tuple $H_1,\ldots,H_p$ and the $p\times p$-matrix B with elements

(i)
$$b_{j\nu} = \int_{\Gamma_\nu}\frac{\partial H_j}{\partial \mathbf{n}}\cdot ds \quad : 1\le j,\nu\le p$$

By xx we have $b_{1\nu}+\ldots+b_{n\nu}=0$ for every ν and therefore the B-matrix must be singular, From Proposition xx its rank is $p-1$. More precisely, the column vectors when $1\le\nu\le p-1$ are linearly independent. Hence we can find complex numbers $\rho_1,\ldots,\rho_{p-1}$ such that

(ii) $$\int_{\Gamma_\nu} \frac{\partial H_p}{\partial \mathbf{n}} \cdot ds = \sum_{j=1}^{p-1} \rho_j \cdot \int_{\Gamma_\nu} \frac{\partial H_j}{\partial \mathbf{n}} \cdot ds \quad : \; 1 \leq \nu \leq p$$

7.6 Solution to the Neuman problem.

Since the equation (*) is linear it suffices by additivity to show that it has a solution when the function f is identically zero on $(p-1)$ many boundary curves and $\neq 0$ on one of them. Consider the case when $f = 0$ on the inner curves. Notice that (**) entails that

(1) $$\int_{\gamma_p} f \cdot ds = 0$$

where ds is the arc-length measure along γ_p. Fix a point z_* on γ_p and we move in the positive direction of this oriented curve to construct a primitive function with respect to s, i.e. we obtain a continuous function F on γ_p such that $F(z_*) = 0$ and its derivative with respect to s is equal to f on γ_p. .Next, we solve the Dirichlet problem and find G where $\Delta(G) = 0$ in Ω while $G = F$ on γ_p and $G = 0$ on the inner boundary curves. By 7.4 we can find constants $a_1, \ldots, a_{p-1}$ such that the function $g = G - (a_1H_1 + \ldots + a_{p-1}H_{p-1})$ has a harmonic conjugate which we denote by v. Since $H_1, \ldots, H_{p-1}$ are constant functions on the boundary curves, the tangential g-derivative is equal to that of G along each boundary curve. From the construction of G this means that the tangential g-derivatives are zero on all inner curves and equal to f on the outer curve. Finally, if v is the conjugate harmonic function to g it follows that v is a solution to (*).

8. An example from Hydro mechanics.

Introduction. Consider a stationary motion of an ideal fluid in the plane which means that the velocity vector (u, v) at each point (x, y) is independent of the time. We also assume that the fluid is incompressible which means that whenever Ω is some domain, then

$$\int_{\partial\Omega} \big(u \cdot \mathbf{n}_x + v \cdot \mathbf{n}_y\big) \cdot ds = 0$$

Stokes theorem implies that the area integral

$$\iint_\Omega (u_x + v_y) \cdot dxdy = 0$$

Since this holds for every small domain the velocity vector satisfies the equation

(*) $$u_x + v_y = 0$$

This is the continuity equation for an incompressible fluid. Next, one defines the *rotation* by

(i) $$\rho = \frac{1}{2}(v_x - u_y)$$

Assume that (*) holds in the whole plane which gives a function ψ such that

(ii) $$\frac{\partial\psi}{\partial x} = -v \quad \text{and} \quad \frac{\partial\psi}{\partial y} = u\,.$$

From (i-ii) we get

(iii) $$\Delta(\psi) + 2\rho = 0$$

An example. Consider the case when

$$\lim_{x^2+y^2\to+\infty} (u, v) = (0, 0)$$

and there exists a bounded Jordan domain Ω such that

$$\rho(x, y) = A = \text{a constant when} \quad (x, y) \in \Omega \quad \text{and } \rho = 0 \text{ in } \mathbf{C} \setminus \Omega$$

Then the ψ-function satisfies:

(1) $$\Delta(\psi) = 0 \text{ outside } \Omega \quad \text{and} \quad \Delta(\psi)(x, y) = -2A \,:\, (x, y) \in \Omega$$

(2) $$\lim_{x^2+y^2\to\infty} \Big[\Big(\frac{\partial\psi}{\partial x}\Big)^2 + \Big(\frac{\partial\psi}{\partial y}\Big)^2\Big] = 0$$

The Riesz representation formula for subharmonic functions gives:

8.1 Proposition. *The ψ-function is given by*

$$\psi(x, y) = \frac{A}{\pi} \iint_\Omega \log \frac{1}{|z - \zeta|} \cdot dxdy + C$$

where C is some constant.

Above we found a formula for the ψ-function. It turns out the domain Ω cannot be arbitrary. Namely, one has:

8.2 Theorem. *Under the assumptions above Ω is a disc.*

Proof. Since the motion is stationary it follows from the general physical principle by Helmholtz that the boundary curve $\partial\Omega$ must be a streamline. By (ii) above this means that the ψ-function is a constant C_0 on $\partial\Omega$, i.e. we have

(**) $$\frac{A}{\pi} \iint_\Omega \log \frac{1}{|z - \zeta|} \cdot d\xi d\eta = C_0 \quad \text{for all } z \in \partial\Omega$$

But then the general result below shows that Ω is a disc.

8.3 An isoperimetric result. Let $f(r)$ be a positive function defined on $r > 0$ which is strictly decreasing. Next, for an arbitrary Jordan domain Ω with a C^1-boundary we define the function

$$(*) \qquad F_\Omega(z) = \iint_\Omega f(|z-\zeta|) \cdot d\xi d\eta$$

With these notations Theorem 8.2 is a special case of the result below applied to the special function

$$f(r) = \operatorname{Log} \frac{1}{r}$$

8.5 Theorem. *If Ω is a Jordan domain such that $F_\Omega(z)$ is a constant function on $\partial\Omega$ then Ω is a disc.*

Remark. We refer to XXX for the proof of this isoperimetric result. Notice that we can express Theorem 8.2 in physical terms as follows:

8.6 Theorem. *Every stationary motion of an incompressible fluid with constant rotational density and a simply connected (Querschnitt) is circular.*

9. A differential inequality

In this section we expose a result from Carleman's article *Sur les fonctions inverses des fonctions entières d'ordre fini* which was published in 1920 and gave the first example how the principle of harmonic majorisation can be applied in a fruitful manner. Here is the situation: A connected domain Ω is given. It is contained in $\mathfrak{Re}(z) > 0$ and for every $\xi > 0$ we assume that the set

(1) $$\ell(\xi) = \{y \quad : \ (\xi, y) \in \Omega\}$$

is bounded. Hence $\ell(\xi)$ is some union of open intervals. We assume that that $\ell(\xi) \neq \emptyset$ for all ξ and the sum of the lengths of the intervals yields a positive function $h(\xi)$. Next, let $f(z)$ be an analytic function in Ω with maximum norm $|f(z)|_\Omega \leq 1$. For every $\xi > 0$ we set

(2) $$M(\xi) = \max_{z \in \ell(\xi)} |f(z)|$$

We shall assume that $M(\xi)$ tends to $+\infty$ as ξ increases. In particular there exists some ξ_* such that $M(\xi_*) > 1$. The maximum principle f shows that $\xi \mapsto M(\xi)$ is non-decreasing when $\xi \geq \xi_*$.

9.1 Theorem. *One has the inequality*

$$\log\, M(\xi) \geq \log\, M(\xi_*) \cdot \exp\Big[\frac{4}{\pi} \int_{\xi_*}^{\xi} \frac{dt}{h(t)}\Big] \quad \text{for all} \quad \xi > \xi_*$$

Proof. Let $\xi > \xi_*$ and choose $p \in \ell(\xi)$ where $M(\xi) = |f(p)|$ and put

(i) $$U(x + iy) = \log |f(x + iy)|$$

Choose some $\xi^* > \xi$ and let D be the connected subdomain of the set

$$\Omega \cap \{0 < \mathfrak{Re}\, z < \xi^*\}$$

which contains p. In D we have the Green's function G with a pole at p while $G(z)$ is harmonic in $D \setminus \{p\}$ and zero on ∂D. Next, in ∂D we have the portion

$$L(\xi_*) = \partial D \cap \ell(\xi_*)$$

With these notations we apply the general formula in XX to G and U which by § XX gives the inequality

(*) $$2\pi \cdot U(p) \leq \int_{L(\xi_*)} U \cdot \frac{\partial G}{\partial \mathbf{n}} \cdot ds \leq \log\, M(\xi_*) \cdot \int_{L(\xi_*)} \frac{\partial G}{\partial \mathbf{n}} \cdot ds$$

where we have used that $U \leq 0$ on $\partial D \setminus L(\xi)$. Now we use the principle of harmonic majorisation from § XX. Namely, consider the half-plane $\mathfrak{Re}\, z < \xi_*$ and let G^* be the Green's function for this half-plane with a pole at p. The majorisation principle from § XX gives:

$$\frac{\partial G}{\partial \mathbf{n}} \leq \frac{\partial G^*}{\partial \mathbf{n}} \quad \text{on} \quad L(\xi_*)$$

Next, with $p = (\xi, y_0)$ the explicit formula for G^* gives

(1) $$\int_{L(\xi_*)} \frac{\partial G^*}{\partial \mathbf{n}} \cdot ds = \int_{L(\xi_*)} \frac{\xi_* - \xi}{(\xi_* - \xi)^2 + (y - y_0)^2} \cdot dy$$

Now (1) is $\leq$ than the integral taken over $\ell(\xi_*)$ and since $h(\xi_*)$ is the sum of the lengths of these intervals it is clear that (1) is majorised by

(2) $$\frac{2}{\pi} \cdot \text{arctg}\, \frac{h(\xi_*)}{2(\xi_* - \xi)}$$

Hence we have proved the inequality

(3) $$\log\, M(\xi) \leq \frac{2}{\pi} \cdot \text{arctg}\, \frac{h(\xi_*)}{2(\xi_* - \xi)} \cdot \log\, M(\xi) \quad \text{for all} \quad \xi > \xi_*$$

Apply (3) when $\xi = \xi_* + \delta$ and $\delta \to 0$ which gives:

$$\frac{2}{\pi} \cdot \operatorname{arctg} \frac{h(\xi_*)}{2\delta} = 1 - \frac{4\delta}{\pi \cdot h(\xi)} + O(\delta^2)$$

Passing to the limit as $\delta \to 0$ we get the differential inequality

$$\frac{d}{d\xi} \big[\log \, M(\xi) \big] \geq \frac{4}{\pi \cdot h(\xi)}$$

Finally an integration gives the inequality in Theorem 9.1.

10. On harmonic measure functions.

We expose material from [§ 5: Chapter 1] in Nevanlinna's text-book [Nev]. Let Ω be a domain in $\mathcal{D}(C^1)$ with boundary curves $\Gamma_1, \ldots, \Gamma_p$ where Γ_p is taken as the outer boundary curve. On each Γ_ν we consider a function ϕ_ν which takes the value $+1$ on a finite union of intervals and zero on the complementary intervals. So unless ϕ happens to be constant on a single curve Γ_ν we get jumps at common boundary points of intervals where ϕ_ν is constant. The number of such points is an even integer denoted by $2 \cdot k_\nu$ where $k_\nu = 0$ means that ϕ_ν is identically 1 or 0 on Γ_ν. Solving Dirichlet's problem we get a harmonic function $\omega(z)$ whose boundary function is ϕ. At jump points the boundary value is discontinuous. But in any case ω is defined in Ω where it takes values in the open interval $(0,1)$ when we assume that the range of the ϕ-function contains both 1 and 0. In fact, the strict inequality

$$0 < \omega(z) < 1 \quad : z \in \Omega$$

follows from the maximum principle for harmonic functions. For every $0 < \lambda < 1$ the level curve $\{\omega = \lambda\}$ is denoted by $S(\lambda)$. Since $\partial\Omega$ is of class C^1 it follows that $S(\lambda)$ contains all jump points, i.e. if $z_* \in \Gamma_\nu$ is a jump point for some ν then $S(\lambda)$ contains an arc which has z_* as end-point. This is proved using a locally defined conformal mapping and a straighforward local analysis which takes place when Ω is an open half-disc bordered by a real segment $-r \leq x \leq r$ and the upper half-circle $|z| = r$ where $y \geq 0$. When $\phi(x) = 0$ for $-r < x < 0$ and $\phi(x) = 1$ when $0 < x < r$ we describe the level curves $S(\lambda)$ in XX below which proves the claim above.

Let us introduce the critical points for the ω-function defined by:

$$\mathcal{C}_\omega = \{z = x + iy \quad : \nabla(\omega)(x,y) = 0\}$$

where $\nabla(\omega) = (\omega_x, \omega_y)$ is the gradient vector. Consider the analytic function

$$g(z) = \omega_x - i \cdot \omega_y$$

Then $\mathcal{C}_\Omega$ is equal to the zeros of g in Ω.

10.1 Theorem. *The number of zeros of g counted with multiplicities is equal to*

$$k_1 + \ldots + k_p + p - 2$$

10.1 Examples. Consider the when Ω is the unit disc D so that $p = 1$. So here $T = \Gamma_1$ is the sole boundar curve. Let $k_1 = 1$ which means that $\phi = 1$ on some interval (θ_0, θ_1) and 0 on the complementary interval. Theorem 10.1 asserts that g has no zeros and hence each curve $S(\lambda)$ is smooth. In fact, as shown by a figure $S(\lambda)$ will be a simple curve with end-points at ${}^{i\theta_0}$ and $e^{i\theta_1}$. Next, consider the case $k_1 = 2$ which means that ϕ is 1 on two disjoint intervals. Suppose they are $\{-a < \theta < a\}$ and $\pi - b < \theta < \pi + b\}$ for some pair $0 < a, b < \pi/2$. In this case there exists a unique $0 < \lambda_* < 1$ where the level curve $S(\lambda_*)$ fails to be smooth. For symmetric reason the branch point of $S(\lambda_*)$ belongs to the real axis, i.e. at a point $-1 < x_* < 1$ where two pieces of $S(\lambda_*)$ intersect at a right angle by the general result about level curves for harmonic functions in XX. As explained in the introduction to these notes we have

$$g(z) = \frac{1}{e^{ia} - z} + \frac{1}{e^{i\pi + b} - z} - \frac{1}{e^{-ia} - z} - \frac{1}{e^{i\pi - b} - z}$$

With $B(z) = (e^{ia} - z)(e^{-ia} - z)(e^{ib} - z)(e^{-ib} - z)$ a computation shows that the rational function $g(z)$ is of the form

$$g(z) = \frac{A(z)}{B(z)}$$

where $A(z)$ is a polynomial of second degree. Nevanlinna's result is confirmed if one employs a computer to find the two zeros of a, where one is in D and the other belongs to the exterior disc.

More involved cases. Keeping $\Omega = D$ while k_1 increases more involved computations occur. A computer is needed for plots while Nevanlinna's theorem indicates what will occur during such numerical investigations.

The case $p = 2$**.** Let Ω be an annulus $\{1 < |z| < R\}$ for some $R > 1$. Let $\phi = 0$ on $\{|z| = 1\}$, and on the outer circle we take some interval $-a < \theta < a$ where $\phi(Re^{i\theta}) = +1$ while $\phi = 0$ on the remaining part of $\{|z| = R\}$. Here $k_1 = 1$ and $p = 2$ so g has one simple zero which corresponds to a unique λ_* where $S(\lambda_*$ fails to be smooth. The reader is again invited to use a computer and to determine the zero of g and plot the critical level curve $S(\lambda_*)$.

Proof of Theorem 10.1

Along the outer Jordan curve Γ_p we let $s \mapsto z(s)$ be the parametrization with respect to the arc-length. The function

$$s \mapsto \omega(z(s))$$

is constant on each of the intervals between eventual jumps. If J is such an interval we have

$$0 = \frac{d\omega(z(s))}{ds} = \omega_x((z(s)) \cdot \frac{dx}{ds} + \omega_y((z(s)) \cdot \frac{dy}{ds}$$

With $\dot{z}(s) = \frac{dx}{ds} + i \cdot \frac{dy}{ds}$ this means that

$$\mathfrak{Re}\big(\omega_x - i\omega_y) \cdot \dot{z}(s)\big) = 0$$

This entails that the argument function

$$s \mapsto \arg g(z(s)) + \arg z(s)$$

is constant on the interval J. So after integration over J we have the equality for the variation of the arguments

$$\text{(i)} \qquad \int_J \arg g \cdot ds = \int_J \arg z \cdot ds$$

If $k_p = 0$ it follows that

$$\int_{\Gamma_p} \arg g \cdot ds - \int_{\Gamma_p} \arg z \cdot ds = -2\pi$$

where the last equality follows since Γ_p is the outer Jordan curve. If $k_p = m \geq 1$ we consider the m many intervals between jumps. Let p_ν and q_ν be the end-points of the ν:th interval. With a small $\epsilon > 0$ one constructs circular arcs of radius ϵ centered at the $2m$ many jumps as illustrated with Figure XX. Denote these by $\mathcal{C}(P_\nu)$ and $\mathcal{C}(q_\nu)$. With ϵ small we apply (i) to the J-intervals and get

$$\text{(1)} \qquad \int_{\Gamma_p} \arg g \cdot ds \simeq -2\pi + \sum_{\nu=1}^{\nu=m} \int_{\mathcal{C}(p_\nu)} \arg g \cdot ds + \sum_{\nu=1}^{\nu=m} \int_{\mathcal{C}(q_\nu)} \arg g \cdot ds$$

If $p \geq 2$ a similar reasoning applies to each inner curve. Here one has

$$\int_{\Gamma_j} \arg z \cdot ds = -2\pi \quad : \ j = 1, \ldots (p-1)$$

This entails that if $1 \leq k \leq p-1$ then

$$\text{(2)} \qquad \int_{\Gamma_j} \arg g \cdot ds \simeq 2\pi + \sum_{\nu=1}^{\nu=k_j} \int_{\mathcal{C}(p_\nu^j)} \arg g \cdot ds + \sum_{\nu=1}^{\nu=k_j} \int_{\mathcal{C}(q_\nu^j)} \arg g \cdot ds$$

where $\{p_\nu^j, q_\nu^j\}$ are thr jump points when $k_j \geq 1$.

Next, by the argument principle the number of zeros of g counted with multiplicities in Ω is equal to

$$\text{(3)} \qquad \frac{1}{2\pi} \cdot \sum_{j=1}^{j=p} \int_{\Gamma_j} \arg g \cdot ds$$

Adding (1) and (2) we conclude that Theorem 10.1 follows if the following limit formula holds at every jump point ξ which belong to one of thr bordering curves whether it is the outer curve Γ_p or an inner curve:

$$(*) \qquad \lim_{\epsilon \to 0} \int_{\mathcal{C}(\xi)} \arg g \cdot ds = \pi$$

Proof of (*)

By local conformal mappings of small arcs centered at ξ on the Γ-curve which contains this jump point the reader may verify that the proof of (*) boils dow to the study the infinitemal variation of arguments when we take the upper half-plane $\mathfrak{Im}\, z > 0$ and consider the harmonic function $\omega(z)$ which takes the value $+1$ on the negative real x-axis and zero on $x \geq 0$, or vice versa. In the case when $\omega(x) = 1$ for $x < 0$ while $\omega(x) = 0$ when $x > 0$ we have

$$\omega(x) = \frac{1}{\pi i} \cdot \mathfrak{Im}\, \log z$$

It follows that

$$g(z) = \frac{1}{\pi i} \cdot \frac{1}{z}$$

Now z moves on the half-circle of radius ϵ from $-\epsilon$ to ϵ. Here

$$g(-\epsilon) = -\frac{1}{\pi i \cdot \epsilon} \implies \arg\, g(-\epsilon) = \frac{\pi}{2}$$

When $z = \epsilon \cdot i$ we see that

$$g(\epsilon \cdot i) = -\frac{1}{\pi \epsilon} \implies \arg g(\epsilon \cdot i) = \pi$$

Finally the reader can check that

$$\arg g(\epsilon) = \frac{3\pi}{2}$$

So the function

$$\theta \mapsto \arg\, g(\epsilon \cdot e^{i\theta})$$

increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ and hence the jump of the argument is π. Let us instead consider the function

$$g_*(z) = -\frac{1}{\pi i} \cdot \frac{1}{z}$$

Ths time the reader can verify that the functon

$$\theta \mapsto \arg\, g_*(\epsilon \cdot e^{i\theta})$$

increases from $-\frac{\pi}{2}$ to $\frac{pi}{2}$ so again the argument increases by π. This gives (*) and Nevanlinna's theorem is proved.

11. Some estimates for harmonic functions.

Let u be harmonic in the unit disc D and suppose that

$$(*) \qquad \iint_D \sqrt{|u(x,y)|}\cdot dxdy = 1$$

We seek estimates of the maximum modulus function

$$M(r) = \max_\theta |u(r,\theta)|$$

where $re^{i\theta} = x + iy$ are the polar coordinates. Put

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{|u(r,\theta)|}\cdot d\theta$$

If $0 < r < 1$ we represent values of u on $|z| = r^2$ by a Poisson integral over $|z| = r$ and find that

$$M(r^2) \leq \frac{1+r}{1-r}\cdot\frac{1}{2\pi}\int\int_0^{2\pi} |u(r,\theta)|\cdot d\theta$$

The right hand side is estimated above by $\frac{2}{1-r}\cdot\sqrt{M(r)}\cdot I(r)$ which gives the inequality

$$M(r^2) \leq \frac{2}{1-r}\cdot\sqrt{M(r)}\cdot I(r) \implies$$

$$\text{(i)} \qquad \log\, M(r^2) \leq \log\,\frac{2}{1-r} + \frac{1}{2}\cdot\log M(r) + \log I(r)$$

Put

$$\mathcal{M}(A) = \int_A^1 \log\, M(s)\cdot\frac{ds}{s}$$

The substitution $s = r^2$ gives

$$\mathcal{M}(A) = 2\cdot\int_{\sqrt{A}}^1 \log\, M(r^2)\cdot\frac{dr}{r}$$

Using (i) we obtain

$$\text{(ii)} \qquad \mathcal{M}(A) \leq 2\cdot\int_{\sqrt{A}}^1 \log\,\frac{2}{1-r}\cdot\frac{dr}{r} + \int_{\sqrt{A})}^1 \log M(r)\cdot dr + 2\cdot\int_{\sqrt{A}}^1 \log I(r)\cdot\frac{dr}{r}$$

Put

$$C(A) = 2\cdot\int_{\sqrt{A}}^1 \log\,\frac{2}{1-r}\cdot\frac{dr}{r}$$

Then (ii) gives

$$\int_A^{\sqrt{A}} \log\, M(s)\cdot\frac{ds}{s} \leq C(A) + 2\cdot\int_{\sqrt{A}}^1 \log I(r)\cdot\frac{dr}{r}$$

Since $M(r)$ is increasing the left hand side majorizes

$$\log M(A)\cdot\int_A^{\sqrt{A}}\frac{ds}{s} = \log M(A)\cdot\frac{1}{2}\cdot\log\,\frac{1}{A}$$

So for each $1/2 \leq A < 1$ we have

$$\text{(iii)} \qquad \log M(A)\cdot\frac{1}{2}\cdot\log\,\frac{1}{A} \leq C(A) + 2\cdot\int_{\sqrt{A}}^1 \log I(r)\cdot\frac{dr}{r}$$

The case $A \geq 1/2$. Then $r \geq \sqrt{A} \geq \sqrt{1/2}$ during the integration and the last integral in (iii) is estimated by

$$2\sqrt{2}\cdot\int_{\sqrt{A}}^1 \log I(r)\cdot dr$$

Let us also write $\sqrt{A} = 1 - s$ where we will study the situation as $s \to 0$.

Exercise. Show that there exists an absolute constant such that when $A \geq 1/2$ then

$$C(A) \leq C \cdot s$$

Use also that

$$s \mapsto \log \frac{1}{(1-s)^2}$$

starts a Taylor expansion with $2s$ and conclude that there is an absolute constant C such that

$$(*) \qquad \log M((1-s)^2) \leq C\big[1 + s^{-1} \cdot \int_{1-s}^1 \log I(r) \cdot dr\,\big]$$

Next, since the Log-function is concave we get the inequality

$$\frac{1}{s} \cdot \int_{1-s}^1 \log I(r) \cdot dr \leq \log\big[\,\frac{1}{s} \int_{1-s}^1 I(r) \cdot dr\,\big]$$

and the normalisation (*) gives

$$1 = 2\pi \cdot \iint_0^1 I(r) \cdot rdr$$

It follows that the integrals

$$\Phi(s) = \int_{1-s}^1 I(r) \cdot dr$$

converge to zero as $s \to 0$ and with $r = 1 - s$ we obtain

$$(**) \qquad \log M(r^2) \leq C[1 + \frac{\Phi(1-r)}{1-r}]$$

This estimate is probably not very sharp so we are led to the following:

11.1 Question. What kind of upper bounds can hold for $M(r)$ as $r \to 1$.

11.2 Interior estimates. Ignoring the eventual growth of $M(r)$ as $r \to 1$ we fix $r = 1/2$ and get the inequality

$$(1) \qquad M(1/4) \leq C[1 + 2\Phi(1/2)]$$

for an absolute constant C. Moreover, the normalisation

$$1 = 2\pi \cdot \int_0^1 I(r) \cdot r \cdot dr \implies$$

$$\Phi(1/2) \leq 2 \cdot \int_{1/2}^1 I(r) \cdot rdr \leq \frac{1}{\pi}$$

Hence (1) gives another absolute constant C^* such that

$$M(1/2) \leq C^*$$

In particular we can estimate $u(0)$ and the discussion above leads to

11.3 Theorem. *In the family of harmonic functions u where (*) holds there is an absolute constant C such that*

$$|u(0)| \leq C$$

12. A density result for harmonic functions.

Consider the interval $[-a,a]$ on the real x-line for some $0<a<1$ which in general is small. Let $H(x,y)$ be harmonic in D and H^* is the harmonic conjugate which gives the analytic function $f=h+iH^*$ whose complex derivative becomes $f'=H_x-iH_y$. If both H and the partial derivative H_y are identically zero on $[-a,a]$ it follows that $f'=0$ on this interval. So by analyticity f is a constant and it follows that $H=0$ in D. Hence, the pair of restricted functions H and H_y to $[-a,a]$ determine H in D. We use this to establish some density theorems. To each $-a\leq x\leq a$ we have the Poisson kernel

$$P_x(\theta)=\frac{1}{2\pi}\cdot\frac{1-x^2}{1+x^2-2x\cos\theta}$$

We also introduce a partial y-derivative and define

$$\partial_y(P_x)(\theta)=\frac{1}{2\pi}\cdot\frac{(1-x^2)\cdot\cos\theta}{(1+x^2-2x\cos\theta)^2}$$

It means that whenever $h(\theta)$ is a continuous function on the unit circle and $H(z)$ its harmonic extension to D then

$$\partial_y(H)(x,0)=\int_0^{2\pi}\partial_y(P_x)(\theta)\cdot h(\theta)\cdot d\theta$$

12.1 Proposition. *For each $0<a<1$ the linear space of functions on T generated by $\{P_x\}$ and $\partial_y(P_x)\}$ as $-a\leq x\leq a$ is dense in $L^2(T)$.*

Proof. If $h\in L^2$ is $\perp$ to this space it follows from the above that its harmonic extension is identically zero and since $h(\theta)$ is almost everywhere equal to the radial limit of its harmonic extension it follows that $h=0$. This proves the requested density.

Let us now consider a point $z\in D$ which is outside the interval. Now $\theta\mapsto P_z(\theta)$ is an L^2-function. So to each $\epsilon>0$ there exist finite subsets $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ in $[-a,a]$ such that an $\mathbf{R}$-linear combination

$$\sum c_j\cdot P_{\alpha_j}(\theta)++\sum d_k\cdot\partial_y(P_{\beta_k}(\theta) \tag{1}$$

has distance $\leq\epsilon$ in the L^2-norm to P_z. Applying Cauchy-Schwarz inequality it follows that for every harmonic function H in the unit disc whose boundary function $h(\theta)$ belongs to $L^2(T)$ one has

$$\big|H(p)-\sum c_j\cdot H(\alpha_j)-\sum d_k\cdot\partial_y(H)(\beta_k)\big|\leq\sqrt{\epsilon}\cdot||h||_2 \tag{*}$$

Remark. The family of $L^2(T)$-functions used for the density in Proposition 12.1 is not orthogonal and it appears to be cumbersome to exhibit an orthonormal family in an efficient manner. So above we have just proved a density theorem while no specific control is given to the constants $\{c_\nu\}$ or $\{d_k\}$ which in addition depend on z when the ϵ-approximation in (*) is attained. But a merit is that (*) hold for all harmonic functions H with $h\in L^2(T)$. This can for example be used to study growth properties of harmonic functions in strip domains. See the section *Lindelöf functions* in Special Topics XXX for such an application.

12.2 On inner normal derivatives. Consider the domain

$$\Omega_A=\{0<x<1\}\cap\{x>A|y|\}$$

where $A>0$. If A is small this domain is almost a half-disc while it shrinks as A increases. See figure XX. Let $u(x,y)$ be harmonic in Ω_A with continuous boundary values where $u(0,0)=0$ and $u\geq 0$ in $\bar{\Omega}$. This entails that $u>0$ in Ω and in particular we get the positive number $u(1/2)$. Now we estimate $u(x,0)$ from below as $x\to 0$.

Exercise. Show that there exists a positive number A_* which depends on A only such that

$$u(x,0)\geq A_*\cdot u(1/2,0)\cdot x$$

hold for all $0 < x < 1/2$ and every harmonic function u as above. A hint is to employ the conformal map F from Ω_A onto the half-disc D_+ in the right half-plane where

$$F(z) = z^{\frac{1}{A}}$$

Then we can pass to the half-disc and seek B_* such that

$$U(x^{\frac{1}{A}}, 0) \geq B_* \cdot U(2^{-\frac{1}{A}}, 0) \cdot x^{\frac{1}{A}}$$

for non-negative harmonic functions in the half-disc which is achieved using a suitable version of Harnack's inequality which is left to the reader to analyze in detail.

An application. Let Ω be a connected domain where the origin is a boundary point which contain Ω_A for some $A > 0$. Prove that there again exists a constant A_{**} which depends on A only such that (*) above holds for every harmonic function u in Ω which is > 0 in Ω while $u(0,0) = 0$. This entails in particular that

$$\liminf_{x \to 0} \frac{u(x)}{x} > 0$$

Remark. A special case occurs when $\partial\Omega$ is of class C^1 and working locally close to the boundary point $(0,0)$ the domain is defined by an equation $\{x > \rho(y)\}$ where $\rho(y)$ is a real-valued C^1-function with $\rho(0) = 0$. In that case we get (xx) above and this strict inequality for C^1-domains is sometimes referred to as Hopf's lemma who established similar inequalities in higher dimensions. See also [Krantz-Green: page 440-444] which describes how Hopf's Lemma is used to establish certain smoothness results for conformal mappings.

13. Representations of rotation invariant harmonic functions.

Let (x,t,s) be the real coordinates in $\mathbf{R}^3$. A real-valued C^2-function $u(x,t,s)$ is harmonic if it satisfies the Laplace equation. We shall consider the subclass of harmonic functions which are invariant under rotations of the (t,s)-coordinates. More precisely, suppose that u is defined in in an open set U given as the product of an interval $(-A,A)$ on the x-axis and a disc $\{t^2+s^2<R^2\}$. The invariance means that the function

$$\phi \mapsto u(x, r\cos\phi . r\sin\,\phi)$$

is constant for each x and $0<r<R$. Let $\square$ be the rectangle in the z-plane defined by

$$\{z=x+iy:\ -A<x<A,:\ -R<y<R\}$$

Let $f(z)$ be analytic in $\square$ assume that $f(x)$ is real-valued when $-A<x<A$. Then we have

13.1 Theorem. *Define u in U by*

$$u(x,t,s)=\frac{1}{\pi}\int_0^\pi f(x+i\sqrt{t^2+s^2}\cdot\cos\,\phi)\,d\phi$$

Then u is harmonic and rotation invariant in U

Proof. The existence of complex derivative of f give

(i) $$\partial_x^2(u)=\frac{1}{\pi}\int_0^\pi f''(x+i\sqrt{t^2+s^2}\cdot\cos\,\phi)\,d\phi$$

$$\partial_t(u)=\frac{1}{\pi}\int_0^\pi i\cos\,\phi\cdot\frac{t}{\sqrt{t^2+s^2}}\cdot f'(x+i\sqrt{t^2+s^2}\cdot\cos\,\phi)\,d\phi\implies$$

(ii) $$\partial_t^2(u)=\frac{1}{\pi}\int_0^\pi [-\cos^2\,\phi\cdot\frac{t^2}{t^2+s^2}+i\cos\,\phi\cdot\frac{s^2}{(t^2+s^2)^{3/2}}]\cdot f''(x+i\sqrt{t^2+s^2}\cdot\cos\,\phi)\,d\phi$$

A similar expression is found for $\partial_s^2(u)$ and adding the result we find that

(iii) $$\Delta(u)=\frac{1}{\pi}\int_0^\pi \big(\sin^2\phi\cdot f''+i\cos\phi\cdot\frac{1}{\sqrt{t^2+s^2}}\cdot f'\,\big)\,d\phi$$

where $1-\cos^2\phi=\sin^2\phi$ was used. Next, notice that

(iv) $$\partial_\phi(f'(x+i\sqrt{t^2+s^2}\cos\phi)=-i\sqrt{t^2+s^2}\cdot\sin\phi\cdot f''(x+\sqrt{t^2+s^2}\cos\phi)$$

BY partial integration of the first term in (iii), the reader can check that (iv) entails that $\Delta(u)=0$. Next, the integral equation in the theorem shows that u is rotation invariant and

$$u(x,0,0)=f(x)$$

Since $f(x)$ was real-valued the reader can confirm that u is real-valued in U and the proof is finished.

A converse result. Let u be a rotation invariant harmonic function. Then there exists an analytic function $f(z)$ in $\square$ such that u is represented as in the theorem. To prove this we use that harmonic functions are real-analytic and define f on the real x-interval $(-A,A)$ by

$$f(x)=u(x,0,0)$$

At this stage we leave as an exercise to the reader to confirm that f extends to an analytic funcituon in $\square$ and that u is represented via f as in the theorem. A hint is to apply the general result below.

13.2 Integrals expression solutions to elliptic equations. Let $a(x,y), b(x,y), c(x,y)$ be real-valued and real-analytic functions in a rectangle

$$\square = \{z = x + iy : \ -A < x < A : \ -B < y < B\}$$

Denote by $\mathcal{S}$ the class of real-valued functions $u(x,y)$ which solve the elliptic equation

$$\Delta(u) + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c = 0$$

in $\square$.

13.3 Theorem. *There exists a function $V(x,y,\zeta)$ where ζ is a new complex variable such that V is complex analytic with respect to ζ, and the general solution u is of the form*

$$u(x,y) = A \cdot V(x,y,0) + \mathfrak{Re} \int_0^{x+iy} V(x,y,\zeta) \cdot f(\zeta)\, d\zeta$$

where A is a real constant and f an arbitray continous complex analytic function.

13.4 Remark. The results above appear in Carleman's article *Applications de la théorie des fonctions analytiques à la resolutions de certaines équations fonctionelles* [Acad. Italia Volta 1940]. A special case of Theorem 13.3 appears whgen we regard solutions to the elliptic equation

$$\Delta(u) - k^2 \cdot u = 0$$

where $k > 0$ is a constant. Here the general u-solution defined in some disc centered at the origin takes the form

$$u(x,y) = \mathfrak{Re} \int_0^z J_0(ik\sqrt{|z|^2 - \bar{z} \cdot \zeta}) \cdot f(\zeta)\, d\zeta$$

where $z = x + iy$ and J_0 the usual Bessel function.

14. Ganelius' inequality for conjugate harmonic functions.

Introduction. We prove a result due to Ganelius from the article *Sequences of analytic functions and their zeros* Arkiv för matematik. Vol 3 (1953). Here is the situation: Let u be a harmonic funtion defined in some open neighborhood of the closed unit disc where $u(0) = 0$. Its harmonic conjugate v is normalised so that $v(0)$. The constants H, K are defined by:

$$(*) \qquad \max_\theta u(e^{i\theta}) = H \quad : \quad \max_\theta \frac{\partial v}{\partial \theta}(e^{i\theta}) = K$$

where the maximum is taken over $0 \leq \theta \leq 2\pi$.

14.1 Theorem. *There exists an absolute constant C such that*

$$\max_{z \in D} |v(z)| \leq C \cdot \sqrt{HK}$$

Proof. For a small $\epsilon > 0$ and a positive number λ we set

$$G(z) = \log \left|\frac{1+z^\lambda}{1-z^\lambda}\right|$$

where G is defined in the intersection of the unit disc D and the sector $S = \{z : |\arg z| < \frac{\pi}{2\lambda}\}$. If $z = r \cdot e^{i\pi/2\lambda}$ we see that $G(z) = \log \frac{/1+ir}{1-ir}| = 0$ and the reader may verify that the normal derviative $\frac{\partial G}{\partial r} = 0$ if $z = e^{i\theta}$ when $0 < |\theta| < \frac{\pi}{2\lambda}$.

1. Exercise. Apply Green's formula to domains $S \cap \{|z-1| > \epsilon\}$ and show that the passage to the limit when $\epsilon \to 0$ gives the equation below for each $0 \leq \phi \leq 2\pi$:

$$2\lambda \cdot \int_0^1 \frac{u(re^{i(\phi - \frac{\pi}{2\lambda})} + u(re^{i(\phi + \frac{\pi}{2\lambda}})}{1+r^2} r^{\lambda-1}\, dr - \pi u(e^{i\phi}) =$$

$$(1) \qquad \int_{\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log \frac{\cos(\lambda\theta/2)}{\sin(\lambda\theta/2)} \cdot \frac{\partial u}{\partial r}(re^{i(\phi+\theta})\, d\theta$$

Next, the restriction of the harmonic conjugate v to the unit circle is written as:

$$v^*(\theta) = v(e^{i\theta})$$

The Cauchy-Riemann equations give

$$\frac{\partial u}{\partial r}(re^{i(\phi+\theta}) = \frac{\partial v^*}{\partial \theta}(\phi + \theta)$$

Choose some $0 < \xi < \pi$ and integrate (1) with respect to ϕ over $(-\xi, \xi)$. The equality

$$(2) \qquad \frac{\partial v^*}{\partial \theta}(\phi + \theta) = \frac{\partial v^*}{\partial \phi}(\phi + \theta)$$

implies that the integral of the last term in (1) becomes

$$(3) \qquad \int_{\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log \frac{\cos(\lambda\theta/2)}{\sin(\lambda\theta/2)} \cdot \left[v^*(\xi + \theta) - v^*(-\xi + \theta)\right] d\theta$$

2. Exercise. Recall that u is harmonic with $u(0) = 0$ and by (*) we have $u^*(\theta) \leq H$ for all θ. Show that

$$\int_a^b u(\theta)\, d\theta \geq -2\pi H$$

hold for all intervals $(a, b)($ on the unit circle and deduce that the integral of the first two terms in (1) taken over an interval $-\xi < \phi < \xi)$ is bounded below by $-4\pi H$ for each $0 < \xi \leq \pi$.

From the above it follows that

$$\int_{\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} \log\frac{\cos(\lambda\theta/2)}{\sin(\lambda\theta/2)}\cdot\big[v^*(\xi+\theta)-v^*(-\xi+\theta)\big]\,d\theta\geq -4\pi H \tag{4}$$

At this stage we use the upper bound for the θ-derivative of v^*. To simplify notations we set

$$J_\lambda(\theta)=\log\frac{\cos(\lambda\theta/2)}{\sin(\lambda\theta/2)} \tag{5}$$

3. Exercise. Use the ineqality $\frac{dv^*}{d\theta}\leq K$ for the θ-periodic v^*-function to show that

$$v^*(\xi+\theta)-v^*(-\xi+\theta)\leq v^*(\xi-\frac{\pi}{2\lambda})-v^*(-\xi+\frac{\pi}{2\lambda})+K\cdot\frac{\pi}{\lambda}\quad:\quad -\frac{\pi}{2\lambda}<\theta<\frac{\pi}{2\lambda} \tag{6}$$

Since $J_\lambda(\theta)\geq 0$ for all θ it follows from (4-5) that we have the inequality

$$v^*(\xi-\frac{\pi}{2\lambda})-v^*(-\xi+\frac{\pi}{2\lambda})+K\cdot\frac{\pi}{\lambda}\geq -\frac{1}{I(\lambda)}\cdot 4\pi H\quad:\quad I(\lambda)=\int_{\frac{\pi}{2\lambda}}^{\frac{\pi}{2\lambda}} J_\lambda\theta)\,d\theta \tag{7}$$

4. Exercise. Show that

$$I(\lambda)=\frac{4}{\lambda}\int_1^\infty\frac{\log t}{1+t^2}\,dt=\frac{4}{\lambda}\cdot k\quad : k=1+\sum_{m=1}^\infty(-1)^m\cdot\frac{1}{(m+1)^2}\simeq 0,916\dots$$

5. The choice of λ. The inequality (7) hold for all ξ and and we choose λ to minimize

$$K\cdot\frac{\pi}{\lambda}+\frac{4}{\lambda}\cdot k\cdot H$$

It means that we take $\lambda=\sqrt{\frac{Kk}{\pi H}}$ and (7) gives the inequality

$$v^*(\xi-\frac{\pi}{2\lambda})-v^*(-\xi+\frac{\pi}{2\lambda})\geq -2\pi\cdot\sqrt{\frac{\pi}{k}HK} \tag{8}$$

Since (8) hold for all ξ and v^* is 2π-periodic we obtain

$$v^*(2\kappa+\xi)-v^*(\xi)\geq -2\pi\cdot\sqrt{\frac{\pi}{k}HK}\quad\forall\ 0\leq\xi,\kappa\leq 2\pi \tag{9}$$

Finally, since $v(0)=0$ and v is harmonic the boundary value function v^* has at least one zero ξ_0 and we can choose κ so that

$$v^*(2\kappa+\xi_0)=\min_\theta V^*(\theta)$$

Then (9) implies that the minimum of v^* is $\geq -2\pi\cdot\sqrt{\frac{\pi}{k}HK}$. In the same way the reader can verify that the maximum of v^* cannot exceed $2\pi\cdot\sqrt{\frac{\pi}{k}HK}$. Hence (*) in Theorem 14.1 holds with

$$C=2\pi\cdot\sqrt{\frac{\pi}{k}} \tag{10}$$

where k is the constant from Exercise 4.

14.2 Applications to zeros of polynomials.

Let $n\geq 1$ and consider a polynomial

$$p(z)=\prod(1-ze^{-i\theta_\nu})$$

where $\{\theta_\nu\}$ is some n-tuple in the periodic interval $[0,2\pi]$. Let (α,β) be an interval on the unit circle. Denote by $N(\alpha,\beta)$ the number of θ_ν satisfying

$$\alpha<\theta_\nu<\beta$$

and $|p|_D$ is the maximun norm of p over the unit disc.

14.3 Theorem. *With the same absolute constant as in Theorem 14.1 one has*

$$[\frac{N(\alpha,\beta)}{n}-\frac{\beta-\alpha}{2\pi}|\leq C\cdot\sqrt{n\cdot\log|p|_D}$$

Proof. Consider the harmonic function $u(z)$ in the unit disc defined by

$$u(z)=\frac{1}{\pi}\log\,|p(z)|=\frac{1}{\pi}\cdot\sum\log\,|1-ze^{-i\theta_\nu}|$$

It follows that the conjugate harmonic function becomes

$$v(z)=\frac{1}{\pi}\cdot\sum\arg\,(1-z\cdot e^{-i\theta_\nu})$$

From this the reader can verify the inequality

$$\frac{\partial v}{\partial\theta}(re^{i\theta})\leq\frac{n}{2\pi}\quad:\;0<r<1$$

From the above Theorem 14.1 gives

$$\max_\theta\,v(\theta)\leq C\sqrt{\frac{n}{2\pi}\cdot\log\,|p|_D}=\sqrt{\frac{2\pi}{k}}\cdot\sqrt{n\cdot\log\,|p|_D}$$

where k is Catalani's constant from Exercise 4.

NOW DONE by generalformula !

14.4 Schur's inequality.

A polynomial $P(z)$ of degree n with constant term $a_0\neq 0$ can be written as

$$P(z)=a_0\cdot\prod\,(1-\frac{z}{\alpha_k})$$

which entails that the absolute value of coefficient a_n of z^n becomes

$$|a_n|=\frac{|a_0|}{\prod\,|\alpha_k|}\tag{1}$$

Put $\alpha_k=\rho_k\cdot e^{i\theta_k}$ and set

$$p(z)=\prod\,(1-e^{-i\theta_k}z)$$

14.5 Theorem. *One has the inequality*

$$|p|_D\leq\frac{|P|_D}{\sqrt{|a_0a_n|}}$$

Proof. For each k one we set $|\alpha_k|=r_k$ and notice that

$$r_k\cdot|1-\frac{e^{i\theta}}{\alpha_k}|^2=r_k+\frac{1}{r_k}-2\cos(\theta-\theta_k)\geq 2-2\cos(\theta-\theta_k)=|1-e^{i(\theta-\theta_k)}|^2$$

This holds for each θ we choose θ where $|p|$ takes its maximum and conclude that

$$|p|_D^2\leq\frac{\prod r_k}{|a_0|^2}\cdot|P|_D^2$$

Taking the square root and using (1) above we get Theorem 14.5

14.6 Carlson's formula. Studies of zeros of polynomials, and more generally of analytic functions, in sectors were treated by F. Carlson in the article *Sur quelques suites de polynomes*. We present a specific formula by Carlson. Let $P(z)$ be a polynomial of some degree n where $P(x)\neq 0$

for every real $x \leq 0$. The argument of each root α_ν of P is taken in the interval $(-\pi, \pi)$. If $0 < \phi < \pi$ we set

$$J(\phi) = \sum (\phi - |\arg \alpha_\nu)|$$

where the sum is taken over roots of P with their arguments in $(-\phi, \phi)$.

14.6.1 Theorem. *For each ϕ one has the equality*

$$J(\phi) = \frac{n\phi^2}{2\pi} + \frac{1}{2\pi} \int_0^\infty \frac{\log |P(re^{i\phi})| \cdot \log |P(re^{-i\phi})|}{|\log P(r)|^2} \, \frac{dr}{r}$$

Proof By a factorisation of P the proof is reduced to the case $n = 1$ with $P(z) = 1 - \frac{1}{\alpha}$ for some non-zero complex number α and here the verification is left as an exercise to the reader.

The formula above is special and yet instructive. Let us finally mention that one can study distributions of zeros for Dirichlet series and exponential polynomials. This requires a further analysis and will not be exposed here. The reader may consult the cited article by Ganelis for results about complex zeros of exponential polynomials.

15. Beurling's minimum principle for positive harmonic functions

Introduction. We expose a result from the article [Beurling:Ann.sci, Fennica]. Let Ω be a simply connected domain. Let $g(z,\zeta)$ be the Green function for Ω. If $p \in \partial\Omega$ a Martin function with respect to p is a positive harmonic function ϕ in Ω such that

$$\lim_{z\to q} \phi(z) = 0 \quad : \forall\, q \in \partial\Omega \setminus \{p\}$$

The class of these Martin functions is denoted by $\mathcal{M}(p)$. Following Beurling we give

15.0 Definition. *A sequence $\mathcal{S} = \{z_n\}$ in Ω which convergs to the boundary point p is an equivalence sequence at p if the following implication hold for every positive harmonic function u in D, every real $\lambda > 0$ and each $\phi \in \mathcal{M}(p)$*

$$u - \lambda\cdot\phi\big|S \geq 0 \implies u \geq \lambda\cdot\phi \quad \text{holds in the whole domain } \Omega$$

15.1 Theorem. *A sequence $\mathcal{S} = \{z_n\}$ which converges to p is an equivalence at p if and only if it constains a subsequence $\{\xi_k = z_{n_k}\}$ such that*

$$\sup_{k\neq j} g(\xi_j,\xi_k) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} g(z,\xi_k)\cdot\phi(\xi_k) = +\infty \quad : \ z \in \Omega$$

Remark. The necessity in Theorem 15.1 proved rather easily. See § xx below. From now on we treat the non-trivial part, i.e.we prove that the two conditions in Theorem 15.1 entail that $\mathcal{S}$ is an equivalence at p. By Riemann's mapping theorem it suffices to prove this sufficiency when Ω is the unit disc D. Since every positive harmonic function u in D is the Poisson extension of a non-negative Riesz measure on T, it follows that every $\phi \in \mathcal{M}(p)$ is the Poisson extension of the unit point mass at p times a positive constant λ, i.e. of the form

$$\phi_\lambda(z) = \frac{\lambda}{2\pi}\cdot\frac{1-|z|^2}{|1-z|^2}$$

where the mean-value property gives $\phi_\lambda(0) = \lambda$. Moreover, if u is a positive harmonic function given by the Poisson extension P_μ of some non-negative Riesz measure μ in D. then the inequality $P_\mu \geq \phi_\lambda$ holds everywhere in D if and only if the measure μ has an atom at $z = 1$ of mass $\geq \lambda$. Up to scaling it suffices to consider the case $\lambda = 1$ and hence $\mathcal{S}$ is an equivalence sequence if and only if the following hold for every non-negative Riesz measure on T:

$$(*) \qquad P_\mu(z_n) \geq \phi_1(z_n) \quad : n = 1,2,\ldots \implies \mu(\{1\}) \geq 1$$

To prove (*) we prefer to work in the upper half-plane where p is taken as ∞. Here we get Martin function $\phi(x,y) = y$ and Green's function is

$$g(\zeta,z) = \log\,\frac{|z-\bar\zeta|}{|z-\zeta)}$$

When $z = re^{i\theta}$ where r is large and $0 < \theta < \pi$ one has

$$\frac{|re^{i\theta}+i|^2}{|re^{i\theta}-i|^2} = \frac{r^2+1+2r\sin\theta}{r^2+1-2r\sin\theta}$$

It follows that

$$g(i, re^{i\theta}) = \frac{2\sin\,\theta}{r} + O(r^{-2})$$

in polar coordinates we write $\{\zeta_k = r_k e^{i\theta_k}\}$ where $\{0 < \theta_k = \arg\xi_k < \pi\}$ and from the above it is clear that the two conditions in Theorem 15.1 are equivalent to

$$(**) \qquad \inf_{j\neq k} \frac{|\xi_j-\xi_k|}{|\xi_j-\bar\xi_k|} \quad : \quad \sum_{k=1}^{\infty} \sin^2\,\theta_k = +\infty$$

At this stage we need a result which has independent interest. In the final part of the proof of Theorem 15.2 below we derive the sufficiency in Theorem 15.1.

15.2 Theorem. *For each positive harmonic function $u(x,y)$ in U_+ we put*

$$E_u = \{x+iy \in U_+ \quad : u(x,y) \geq y\}$$

Then one has the implication

$$U_+ \setminus E_u \neq \emptyset \implies \int_{E_u} \frac{dxdy}{1+x^2+y^2} < \infty \tag{15.2}$$

Proof. The impliction (15.2) is obvious when $u(x,y) = ay$ for some positive real number a, i.e. regard the cases $a < 1$ and $a \geq 1$ separately. From now on we assume that $u(x,y)$ is not a linear function of y and the Harnack-Hopf inequality from § XX gives the strict inequality

$$\frac{\partial u}{\partial y}(x,y) < \frac{u(x,y)}{y} \quad : (x,y) \in U_+ \tag{1}$$

Set $u_* = y - u$ which entails that

$$\{u_* > 0\} = U_+ \setminus E_u \tag{2}$$

3. A simply connected domain. Assume that the set $U_+ \setminus E_u$ is non-emtpy and let Ω be a connected component of this open set. Replacing u by the harmonic function $u(x+a,y)$ for a suitable real a we may assume that Ω contains a point on the imaginary axis and after a scaling that $i \in \Omega$. Next, (1) gives

$$\frac{\partial u_*}{\partial y} = 1 - \frac{\partial u}{\partial y} > 1 - \frac{u}{y} \tag{4}$$

Starting at a point $(x,y) \in \Omega$ it follows that $s \mapsto u_*(x, y+is)$ increases strictly on $\{s \geq 0\}$ and hence the open set Ω is placed above a graph, i.e. there exists an interval (a,b) on the real x-axis and a non-negative continuous function $g(x)$ on (a,b) such that

$$\Omega = \{(x,y) : a < x < b \ : \ y > g(x)\} \tag{5}$$

In (5) it may occur that the interval (a,b) is unbounded and may even be the whole x-axis in which case Ω is the sole component of $U_+ \setminus E_u$. In any case, since $i \in \Omega$ one has $a < 0 < b$ and Ω contains the set $\{ai\colon \ a \geq 1\}$. Next, for each $r > 2$ we put

$$C_r = U_+ \cap \{|z+i| < r\} \tag{6}$$

Notice that C_r contains the point $(r-1)i$ which also belongs to Ω. Denote by γ_r be the largest open interval on the circle $\{|z+i| = r$ which contains $(r-1)i$ and at the same time is contained in $\partial\Omega$. With the aid of a figure it is clear that the set $\Omega_r = \Omega \cap C_r$ is simply connected where γ_r is a part of ∂D_r and $u_* = 0$ on $\partial D_r \setminus \gamma_r$.

The use of harmonic measure. Let $h_r(z)$ be the harmonic function in Ω_r which is 1 on γ_r and zero on the rest of the boundary. By the above and the maximum principle for harmonic functions we have the inequality

$$u_*(i) \leq h_r(i) \cdot \max_{z \in \partial D_r} u_*(z) \leq r \tag{7}$$

where the last inequality follows since $u_*(x,y) \leq y \leq r$. To estimate the function $h_r(z)$ with respect to r and z we use the harmonic function

$$\Psi(z) = \log\left|\frac{z+i}{2i}\right| \implies |\nabla(\Psi)|^2 = \Psi_x^2 + \Psi_y^2 = \frac{1}{|z+i|^2} \tag{8}$$

Set

$$m(r) = \frac{1}{\pi} \iint_{E_u \cap C_r} |\nabla(\Psi)|^2 \, dxdy = \frac{1}{\pi} \iint_{E_u \cap C_r} \frac{1}{|z+i|^2}| \, dxdy \tag{9}$$

Notice that

$$
(10) \qquad \frac{1}{\pi}\iint_{C_r} \frac{1}{|z+i|^2}|\, dxdy < \log r
$$

Together (9-10) give the inequality

$$
(11) \qquad \frac{1}{\pi}\iint_{\Omega_r} |\nabla(\Psi)|^2 \, dxdy < \log r - m(r)
$$

Next, we notice that

$$
(12) \qquad \Psi(z) = \log r - \log 2 \quad \text{on} \quad \gamma_r \quad \text{and} \quad \Psi(i) = 0
$$

Now we can apply the general inequality from § XX in [Beurling-Special Topics] and conclude that (11-12) give the inequality

$$
(13) \qquad h_r(i) \leq e^{-\frac{\pi L^2}{A}} \quad \text{where} \quad L = \log r - \log 2
$$

Next, the reader may verify that there exists a constant C which is independent of $r \geq 2$ such that

$$
(14) \qquad \frac{\pi L^2}{A} \geq \frac{\lg r - \log 2)^2}{\log r - \log m(r)} \geq \log r - 2\log 2 + m(r) \cdot \big[1 - \frac{C}{\log r}\big]
$$

Then (13-14) gives a constant C_1 such that

$$
(15) \qquad h_r(i) \leq e^{-\frac{\pi L^2}{A}} \leq C_1 \cdot \frac{1}{r} \cdot e^{-m(r)} \quad : \; r \geq 2
$$

Then (7) and (15) give

$$
(*) \qquad u_*(i) \leq C_1 \cdot \cdot e^{-m(r)}
$$

This inequality hold for all $r > 2$ and since $u_*(i) > 0$ there must exist a finite limit

$$
(**) \qquad \lim_{r\to\infty} m(r) = m^*
$$

The requested implication in Theorem 15.2 follows since (9) this means that

$$
\iint_{E_u} \frac{1}{|z+i|^2} \, dxdy = \lim_{r\to\infty} m(r)
$$