Carleman's solution to the Bohr-Schrödinger equation

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We expose a proof which was presented by Carleman's at the Scandinavian Congress in mathematics held at Copenhagen 1925. Consider a potential function

$$W(p) = \sum \frac{a_\nu}{|p-\xi_\nu|} + b$$

where $\{\xi_\nu\}$ is a finite set of points in $\mathbf{R}^3$ while $\{a_\nu\}$ and b are positive real numbers. Then

$$T\colon\ \phi \mapsto \Delta(\phi) + W\cdot\phi$$

is a densely defined operator on $L^2(\mathbf{R}^3)$.

Main Theorem. *T is self-adjoint with a discrete spectum and Neumann's resolvent operators $R_T(\lambda)$ defined when $\lambda \in \mathbf{C}\setminus\sigma(T)$ are compact.*

To prove this theorem we shall construct a compact operator $\mathcal{G}$ on $L^2(\mathbf{R}^3$ and also find a positive number κ such that

$$(T-\kappa^2\cdot E)\circ\mathcal{G} = -4\pi\cdot E \tag{0.1}$$

where E is the identity operator on $L^2(\mathbf{R}^3)$. If this has been achieved we seek pairs (ϕ,λ) where $\phi\in L^2(\mathbf{R}^3)$ and λ is a real number such that

$$\Delta(\phi) + (W+\lambda)\phi = 0 \tag{*}$$

To find solutions to (*) we shall construct a Greens function $G(p,q)$ defined in $\mathbf{R}^3\times\mathbf{R}^3$ which satisfies

$$\Delta G + W\cdot G - \kappa^2 G = -4\pi\delta(p-q) \tag{**}$$

where the right hand side is Dirac's delta-distribution with unit mass at $p=q$ and κ a positive number which will be found in § 1. Here we recall that (**) means that if f is a test-function in $\mathbf{R}^3$ then

$$\int G(p,q)\Delta(f)(q)\,dq + (W(p)-\kappa^2)\int G(p,q)f(q)\,dq = -4\pi f(p) \tag{0.1}$$

hold for each p. Moreover, the constructions in § 1 ential that G is symmetric, i.e.

$$G(q,p) = G(p,q) \tag{0.2}$$

and $G(p,q)$ yields a kernel of a compact linear operator $\mathcal{G}$ defined on $L^2(\mathbf{R}^3)$ by

$$\mathcal{G}(g)(p) = \int G(p,x)g(x)\,dx \tag{0.3}$$

Solutions to (*). Suppose that G has been found and let ϕ be an L^2-function which satisfies the integral equation

$$\phi = \frac{\lambda+\kappa^2}{4\pi}\cdot\int G(p,x)\phi(x)\,dx \tag{***}$$

for some real number λ. Then (**) gives

$$\Delta(\phi) = (\kappa^2 - W)\phi - (\lambda+\kappa^2)\phi$$

which entails that ϕ solves (*). Conversely the reader may check that if ϕ solves (*) then it satisfies the integral equation above.

1. The construction of G.

When $\kappa > 0$ we define a function $H(p,q)$ in $\mathbf{R}^3\times\mathbf{R}^3$ by

$$H(p.q) = \frac{e^{-\kappa\cdot|p-q|}}{|p-q|} \tag{1.1}$$

Newton's classical formula gives

$$\Delta(H) = \kappa^2 \cdot H - 4\pi \cdot \delta(p-q) \tag{1.2}$$

Next, introduce the kernel function:

$$\Omega(p,q) = H(p,q) \cdot \sqrt{W(p)} \cdot \sqrt{W(q)} \tag{1.3}$$

It is clear that Ω is everywhere positive and $\Omega(p,q) = \Omega(q,p)$. Close to the diagonal $p = q$ which avoids the points $\{(\xi_\nu, \xi_\nu)\}$ it has the same singularity as H, i.e. like $\frac{1}{|p-q|}$. When both p and q approach the point (ξ_1, ξ_1) we set $p = \xi_1 + x$ and $q = \xi_1 + y$. Then $\Omega(p,q)$ increases like

$$\frac{1}{|x-y| \cdot |x| \cdot |y|} \tag{1.4}$$

1.5 Exercise. Show that the function in $\mathbf{R}^3 \times \mathbf{R}^3$ defined by

$$(x,y) \mapsto \frac{1}{|x-y| \cdot |x| \cdot |y|^{\frac{3}{2}}}$$

is locally integrable in a neighbohood of the origin of $\mathbf{R}^3 \times \mathbf{R}^3$. Conclude that the function

$$q \mapsto \Omega(p,q) \cdot W(q)$$

is locally integrable in the 3-dimensional q-space for each fixed $p \in \mathbf{R}^3$, and that the construction of Ω gives the equation Moreover, the construction of the Ω in (iii) gives the equality below for every p:

$$\int \Omega(p,q)\sqrt{W(q)}\, dq = \sqrt{W(p)} \cdot \int H(p,q) \cdot W(q)\, q \quad : \, p \in \mathbf{R}^3 \tag{1.5.1}$$

1.6 Exercise. Show that if κ sufficiently large then

$$\rho = \max_{p \in \mathbf{R}^3} \int H(p,q) \cdot W(q)\, dq < 4\pi \tag{1.6.1}$$

Now (1.5.1) and (1.6.1) give

$$\int \Omega(p,q)\sqrt{W(q)}\, dq \leq \rho \cdot \sqrt{W(p)} \tag{1.7}$$

for all p. Above $\sqrt{W}$ is a positive function and hence the general result in § xxx implies that the operator on $L^2(\mathbf{R}^3)$ defined by

$$g \mapsto \int \Omega(p,x) g(x)\, dx$$

has norm $\leq \rho$. Dividing by 4π, the strict inequality $\rho < 4\pi$ entials that the linear operator below has norm < 1:

$$\mathcal{S}(g)(p) = \frac{1}{4\pi} \cdot \int \Omega(p,q) g(q)\, dq \tag{1.8}$$

1.9 An integral equation. Since $\mathcal{S}$ has norm < 1, there exists the bounded linear operator

$$\mathcal{L} = (E - S)^{-1} \circ S \tag{1.9.1}$$

where E is the identity operator on $L^2(\mathbf{R}^3)$. Since $\Omega(p,q)$ is a symmetric function it follows that $\mathcal{S}$ is a symmtric operator which entails that $\mathcal{L}$ also is symmetric, and the reader may check that $\mathcal{L}$ is defined by the kernel function which satisfies the integral equation

$$L(p,q) = \frac{1}{4\pi} \cdot \int L(p,x) \cdot \Omega(x,q)\, dx + \Omega(p,q) \tag{1.9.2}$$

So here $\mathcal{L}$ is the bounded operator on $L^2(\mathbf{R}^3)$ defined by

$$\mathcal{L}(g)(p) = \int L(p,x) g(x)\, dx \tag{1.9.3}$$

Now we put

$$G(p,q) = \frac{1}{\sqrt{W(p)}\cdot\sqrt{W(q)}}\cdot L(p,q) \tag{1.10}$$

The positive constant b in (**) gives the inequality below for each pair p,q in $\mathbf{R}^3$:

$$\frac{1}{\sqrt{W(p)}\cdot\sqrt{W(q)}} \leq b^{-1} \tag{1.11}$$

Since $L(p,q)$ is the kernel of the bounded operator $\mathcal{L}$ is, it follows from (1.11) that $G(p,q)$ also is the kernel of a bounded operator denoted by $\mathcal{G}$.

1.12 Exercise. Show from the above that $G(p,q)$ which satisfies the integral equation

$$G(p,q) = H(p,q) + \frac{1}{4\pi}\int H(p,x)W(x)G(x,q)\,dx \tag{1.12.1}$$

and that this equation entails that

$$\Delta(G)(p,q) = \kappa^2\cdot G(p,q) - 4\pi\delta(p-q) - W(p)G(p,q) \tag{1.12.2}$$

which means that G satisfies (**) above. .

2. Spectral values.

Let us first notice that (1.9) and (1.11) imply that if $\mathcal{S}$ is a compact operator so is $\mathcal{G}$.

2.1 Exercise. Show via the explicit construction of $\Omega(p,q)$ in (1.3) and basic measure theory that $\mathcal{S}$ is a compact operator.

The exercise entails that $\mathcal{G}$ is compact. Moreover, since $\Omega(p,q)$ is a positive function it follows that the spectrum of $\mathcal{S}$ is confined to positive eigenvalues and then the general formulas from § xx imply that non-zero points in $\sigma(\mathcal{G})$ are real and positive. Now (*) holds for a real λ if and only if

$$\phi = \frac{\lambda+\kappa^2}{4\pi}\cdot\mathcal{G}(\phi)$$

which means that

$$\frac{4\pi}{\lambda+\kappa^2}\in\sigma(\mathcal{G}) \implies \lambda > 4\pi-\kappa^2$$

2.2 Remark. From the abve we get some information about the true spectrum of the densely defined operator T from (0.x). More precisely it is discrete and real and there exists a smallest eigenvalue λ_* of T. it dpends on the chosden W.function asnd here one needs numerical investigations to determine approximations of λ_*, as well as subsequent eigenvalues and their corresponding eigenfunctions.

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1.5 The case $L = \Delta + c(x, y, z.$. We shall not give all details of the proof of Theorem 0.2.1 but describe a crucial step the proof which is used to prove that L is of Class I when c satisfies the condition in the theorem. Here is the situation. Let B_r be the open ball of radius r centered at the origin and $S^2[r]$ thr unit sphere. The class of functions u which are continuous on the closed ball and whose interior normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ is continuous on the boundary $S^2[r]$ is denoted by $\mathfrak{Neu}(B_r)$.

A Neumann equation. Next, consider a pair a, H where a be a continuous function on $S^2[r]$ and $H(p, q)$ a continuous hermitian function on $S^2[r] \times S^2[r]$, i.e. $H(q, p) = \bar{H}(p, q)$ hold for all pairs of point p, q on the sphere ccc. Finally, c is some real-valued function in $L^2(B_r)$. With these notations the following hold:

1.5.1 Theorem. *For each* $f \in L^2(B_r)$ *and every non-real complex number* λ *there exists a unique* $u \in \mathfrak{Neu}(B_r)$ *which satisfies the two equations:*

(i)
$$\Delta(u) + \lambda \cdot c \cdot u = f \quad \text{holds in} \quad B_r$$

(ii)
$$\partial u/\partial \mathbf{n}(p) + a(p)u(p) + \int_{S^2[r]} H(p, q)u(q)\, dA(q) = 0$$

Moreover, one has the L^2*-estimate*

(iii)
$$\int_{B_r} |u|^2 \cdot dxdydz \leq \Big|\frac{1}{\mathfrak{Im}(\lambda)}\Big| \leq \int_{B_r} |f|^2 \cdot dxdydz$$

Remark. The point is that the L^2-estimate above is independent of the triple a, c, H. The verification that the two equations (i-ii) give (iii) follows easily via Greens formula and is left to the reader. The fact that (i-ii) has a solution is classic and goes back to work by Neumann and Poincaré.

Mathematics by Torsten Carleman

Introduction.

Carleman's collected work covers fifty articles of high standard together with several monographs. He entered university studies in 1911 at Uppsala and five years later he presented his doctors thesis. After the end of World War I he visited Paris frequently where he met many distinguished mathematicians, among those Hadamard, Picard, Borel and Denjoy whose work gave him much inspiration during the years 1919-1924 when many of his most important discoveries were achieved, such as the theory about quasi-analytic functions. He also published some joint papers with Hardy about Fourier series, and other articles from his early period in his career were inspired from previous work by Weyl, Schur, Faber, Gros, Hamburger and the brothers F. and M. Riesz. Of course, from the very start of his studies in mathematics, Carleman read work by the great masters Abel, Riemann, Weierstrass and Poincaré. Several sections in these notes contain results which have emerged from these eminent mathematicians. Among Scandinavian mathematicians one must mention Lindelöf whose contributions in analytic function theory was another source of inspiration for the young Carleman. During his early years at Uppsala he became aquaintd with many current research problems from lectures by the two professors Holmgren and Wiman.

He became professor at Stockholm University in 1924, and from 1927 also director at Institute Mittag-Leffler where he delivered frequent seminars during the period 1928-1938. His text-book (in Swedish) for undergraduate mathematics was published in 1926 and used for several decades. Personally I find it outstanding as a beginner's text for students and during my early education Carleman's presentation of basic material in analysis and algebra served as a "veritable bible".

Carleman was always concerned with the interaction between "pure mathematics" and experimental sciences. After World War I he studied a year at an engineering school in Paris. The article entitled *Sur les équations différentielles de la mecanique d'avion* published in [La Technique Aéronautique, vol. 10 1921) was inspired by Lanchester's pioneering work *Le vol aérien* which played a significant role while airplanes were designed at an early stage. Carleman's article ends with the following conclusion after a very interesting investigation of integral curves to a certain non-linear differential system: *Quelle que soit la vitesse initiale, l'avion, après avoir éxecuté s'il y lieu, un nombre fini des loopings, prend un mouvement qui s'approche indéfinement du régime de descente rectiligne ert uniforme.*

The reader may also consult his lecture held 1944 at the Academy of Science in Sweden entitled *Sur l'action réciproque entre les mathématiques et les sciences expérimentales exactes* which underlines Carleman's concern about applications of mathematics. Let us also mention that Carleman got much inspiration from Ivar Fredholm's work on integral equations. For many decades Fredholm was professor in mathematical physics at Stockholm University until his retirement in 1930 when Oscar Klein became the new professor in theoretical physics.

During the last years in life Carleman suffered from health problems which caused his decease on January 11 1949 at the age of 56 years. A memorial article about his scientific achievement appears in [Acta. Math. 1950] written by Fritz Carlson who was Carleman's collegue at the department of mathematics at Stockholm university for several decades.

Collaboration with Erik Holmgren. Carleman's last major publication *Sur un probléme d'unicité pour les systèmes d'équations aux dérivees partielles à deux variables indépendantes* [Arkiv för matematik 1938] is concerned with a uniqueness theorem for elliptic PDE-systems where the variable coefficients of the PDE-operators are non-analytic. This extended an earlier result by Erik Holmgren from 1901 when the coefficients are real-analytic. Let us recall that Holmgren served as supervisor while Carleman prepared his thesis entitled *Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, presented at Uppsala University in 1916 when he was 23 years old. Later they shared many ideas. An example is Holmgren's uniqueness theorem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t)$ be a solution in a domain $\{-\infty < x < \infty\} \times \{-A < t < A\}$. If u satisfies the growth condition

$$|u(x,t)| \leq e^{kx^2 \cdot \log(|x|+e)}$$

for some constant k where e is Neper's number, then Holmgren proved thst u is determined by its values on a line $\{t = t_0\}$. He employed results by Denjoy and Carleman about quasi-analytic classes to prove the uniqueness, and via constructions outside the class of quasi-analytic functions Täcklund later proved that the result above is quite sharp. Namely, for every $\delta > 0$ there exists a non-zero solutions u such that $u(x,0) = 0$ for all x while

$$|u(x,t)| \leq e^{x^2 \cdot (\log(|x|+e))^{1+\delta}}$$

It appears that Holmgren's result is rarely treated in text-books in spite of its importance related to diffusion which for example appears in stochastic PDE-theory. So in § xx we expose Holmgren's proof from his article published in Arkiv för Matematik och Fysik in 1924. In connection with this we recall a result due to Hadamard which goes as follows: Consider the boundary value problem where one seeks a harmonic function $u(x,y)$ in an open half-disc

$$D_+(r) = \{x^2 + y^2 < r^2\} \cap \{x > 0\}$$

satisfying the boundary conditions:

$$u(0,y) = \psi(y) \quad \text{and} \quad u_x(0,y) = \phi(y)$$

where the functions ϕ and ψ are given in advance. Hadamard proved that a necessary and sufficient condition for this Cauchy problem to be well posed is that the function

$$y \mapsto \phi(y) + \frac{1}{\pi} \int_a^b \log \frac{1}{|s-y|} \cdot \psi(s) \cdot ds$$

is real analytic on the interval (a,b) for all pairs $-r < a < b < r$.

Hadamard's result illustrates how sensible boundary value problems of elliptic type can be and therefore the uniqueness result in § xx is quite remarkable.

Let us now describe some central themes in Carleman's work.

0.1 Analytic functions.

Function theory appears often as a tool in Carleman's work. But he also gave some important contributions to the subject itself. Among these one must mention the fundamental discovery which aserts that if $f(z)$ is an analytic function in the upper half-plane U which extends continuously to the real axis where $|f(z)| \leq e^{A|z|}$ hold for all $z \in U$ and some constant A, then the following implication holds:

$$\int_{-\infty}^{\infty} \log^+ |f(x)| \cdot \frac{dx}{1+x^2} < \infty \implies \int_{-\infty}^{\infty} \log^+ \frac{1}{|f(x)|} \cdot \frac{dx}{1+x^2} < \infty$$

A result which illustrates Carleman's vigour in analysis is his solution to a problem concerned with asymptotic expansions which originally was raised by Poincaré. It goes as follows: Let $\mathcal{A} = \{A_n\}$ be a sequence of positive real numbers. Denote by $H(\mathcal{A})$ the class of analytic functions $f(z)$ in the unit disc for which

$$(*) \qquad \max_{z \in D} \frac{|f(z)|}{|(1-z)^n|} \leq A_n \quad : \quad n = 1, 2, \ldots$$

Theorem. *The necessary and sufficient condition that $H(\mathcal{A}) \neq 0$ is that there exists a constant C such that*

$$\max_{n \geq 1} \int_1^{\infty} \log \Big(\sum_{\nu=1}^{\nu=n} \frac{r^{2\nu}}{A_\nu^2} \Big) \, dr \leq C \quad : \quad n = 1, 2 \ldots$$

This is proved in the book [xxx] on quasi-analytic functions. The proof relies upon a variational problem which is exposed in § xx.

Conformal maps of circular domains. In 1906 Koebe proved a result about conformal mappings between domains bordered by a finite set of circles. Let $p \geq 2$ and denote by $\mathcal{C}^*(p)$ the family of connected bounded domains Ω in $\mathbf{C}$ for which $\partial\Omega$ is the union of p many disjoint circles.

Theorem. *Let $f \colon \Omega \to U$ be a conformal map between two domains in $\mathcal{C}^*(p)$. Then $f(z)$ is a linear function, i.e. $f(z) = Az + B$ for some constants A and B.*

Koebe's proved this via the uniformisation theorem for Riemann surfaces and extensive use of reflections over circular boundaries. Carleman gave a direct proof which we expose in § xx since it gives a good lesson how one computes winding numbers in specific situations. In § x we expose a result whose proof relies heavily upon original work by Weierstrass, but the "final point" which leads to a sharp isometric inequality for minimal surfaces bordered by a fixed curve in $\mathbf{R}^3$ employs analytic function theory.

An application of Hadamard's spectral radius formula. Consider a power series

$$(*) \qquad u(z) = \sum_{n=0}^{\infty} c_n z^n$$

which has some positive and finite radius of convergence r_0. Recall the wellknown formula

$$\frac{1}{r_0} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$$

In his thesis from 1894, Hadamard went much further. Following an original device by Kronecker who determined the class of local Taylor series representing rational functions, Hadamard considered the recursive Hankel determinants of $\{c_n\}$ and found precise criteria in order that (*) extends to a meromorphic function in larger discs. This powerful result was adopted by Carleman. In a letter to Hadamard from 1919, Carleman showed how one gets expansions of resolvent operators which describe the spectrum of compact linear operators arising in the Fredholm-Hibert theory about integral kernels. Since Hadamard's fundamental discovery is rarely treated in text-books we have included an account about this in § xx together with Carleman's application to integral operators.

Hadamard's famous work *Théorie des ondes* was another inspiration for Carleman. In his brief note *Sur les sytèmes linèaires aux dérivées partielles du premier ordre à deux variables*, Carleman considered the system of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = au + bu \quad : \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = cu + du$$

where a, b, c, d are real-valued continuous functions defined in a domain in of $\mathbf{R}^2$. One says that a pair (u, v) is a solution if they are contonuous in D and their first order partial derivatives exists as continous functions in in $D \setminus \gamma$ where γ is a finite union of rectifiable Jordan curves in D and their maximum norms of these first order partial derivatives taken over $D \setminus \gamma$ are finite. Under this condtion we show in § xx that if there exists a point $p \in D \setminus \gamma$ where u and v both vanish of infinite order, then they are identically zero in D. Let us remark that this result can be used to relax certain regularity conditions concerned with uniqueness of solutions to non-linear elliptic equations. which were previously ontained by Hadamard.

0.2 Vibrating planes.

Let us describe a result which illustrates Carleman's concern to provide strict mathematical proofs of expected physical results. Let D be a membrance with constant density of mass m and constant tension $k > 0$. The boundary is fixed by a plane curve C placed in the horizontal (x, y)-plane and the function $u = u(x, y, t)$ is the deviation in the vertical direction while the membrance is in motion. Here t is a time variable and by Hooke's law u satisfies the wave equation

(*)
$$\frac{d^2u}{dt^2} = \frac{k}{m} \cdot \Delta u$$

where the boundary condition is that $u(p, t) = 0$ for each $p \in C$. The time dependent kinetic energy becomes

$$T(t) = \frac{m}{2} \iint_\Omega (\frac{du}{dt})^2 dxdy$$

The potential energy becomes

$$V(t) = \frac{m}{2} \iint_\Omega [(\frac{\partial u}{\partial x})^2 + \frac{\partial u}{\partial x})^2]\, dxdy$$

One solves (*) by a separation of variables. More precisely, there exists an orthonormal sequence of functions $\{\phi_\nu\}$ in $L^2(D)$ and a non-decreasing sequence of positive numbers $\{\lambda_\nu\}$ such that

$$\Delta(\phi_\nu) + \lambda_\nu \cdot \phi_\nu = 0$$

hold in D and each $\{\phi_\nu\}$ i vanishes on the boundary curve C. This is a classic result which goes back to work by Poincaré and Fredholm and is proved in § XX. The solution to (*) can be written in the form

(**)
$$u(x, y, t) = \sum c_\nu(t)\phi_\nu(x, y)$$

where each c-function satisfies the second order differential equation

$$\frac{d^2c_\nu}{dt^2} + \frac{k}{m} \cdot \lambda_\nu \cdot c_\nu(t) = 0$$

The c-functions are determined via expansions in the ϕ-functions of the intial value functions $u(xy, 0)$ and $\frac{du}{dt}(x, y, 0)$ at time $t = 0$. The mean kinetic energy over large time intervals at individual points $p \in D$ are defined by

(1)
$$L(p) = \frac{m}{2} \cdot \lim_{\tau \to \infty} \frac{1}{\tau} \cdot \int_0^\tau (\frac{du}{dt})^2(p) \cdot d\tau$$

Using (**) one has the equality

$$L(p) = k \cdot \sum |c_\nu|^2 \lambda_\nu \phi_\nu(p)^2 \tag{2}$$

We refer to § xx for a detailed verification.

High frequencies. For each positive number w the contribution from high frequencies is defined by:

$$L_w(p) = k \cdot \sum_{\lambda_\nu > w} |c_\nu|^2 \lambda_\nu \cdot \phi_\nu(p)^2$$

Similarly, the mean potential energy from high frequencies is defined by

$$V_w(p) = k \cdot \sum_{\lambda_\nu > w} |c_\nu|^2 \cdot \big[\frac{\partial \phi_\nu}{\partial x}(p)^2 + \frac{\partial \phi_\nu}{\partial y}(p)^2\big]$$

Using limit formulas for the eigenvalues $\{\lambda_\nu\}$ as well as values of the eigenfunctions in D, Carleman proved the following equality for each $p \in D$:

$$\lim_{w \to \infty} \frac{L_w(p)}{V_w(p)} = 1 \tag{1}$$

Moreover, he proved that

$$\lim_{w \to \infty} \frac{L_w(p)}{L_w(q)} = 1 \tag{1}$$

hold for all pairs of point p, q in Ω. Let us remark that the proof relies upon a deep limit theorem for the values taken by the eigenfunctions $\{\phi_n u\}$ and their derivaties in § xx.

0.3 Spectra of compact operators.

Let $k(x,y)$ be a Lebesgue measurable function on $\square = \{0 \leq x \leq 1\} \times \{0 \leq y \leq 1\}$ which in general is complex-valued. We assume that

$$\int_\square |k(x,y)|^2 \, dxdy < \infty$$

This gives the compact Hilbert-Schmidt operator on $L^2[0,1]$ defined by

$$\mathcal{K}(u)(x) = \int_\square k(x,y) u(y) \, dxdy$$

It has a discrete spectrum outside the origin and there exists Fredholm's resolvent function $D(\lambda)$ for which the operator valued function

$$\lambda \mapsto \frac{1}{D(\lambda)} \cdot (E - \lambda \mathcal{K})^{-1}$$

is an entire function of the complex variable λ. For readers who are not familiar with Fredholm's theory we refer to § xx for the construction of $D(\lambda)$. In the article *Sur le genre du dénominateur $D(\lambda)$ de Fredholm* Carleman proved the following conclusive result:

1. Theorem. *$D(\lambda)$ is an entire function of the form*

$$D(\lambda) = e^{a\lambda} \prod (1 - \frac{\lambda}{\lambda_\nu}) \cdot e^{\frac{\lambda}{\lambda_\nu}}$$

where a is some complex constant and

$$\sum \frac{1}{|\lambda_\nu|^2} < \infty$$

The proof relies upon a number of basic facts from analytic function theory due to Poincaré, Lindelöf and Wiman which illustrates the usefulness of complex analysis in operator theory. The result above becomes especially useful when it is combined with the inequality for resolvents of matrices in § xx. To begin with every Hilbert-Schmidt operator on a separable Hilbert space is up to a unitary equivalence given by an integral operator as above. In particular $\mathcal{K}$ is compact asnd hence the spectrum $\sigma(\mathcal{K})$ is discrete outside the origin. Let $\{\mu_n\}$ be the set of spectral values arranged with non-increasing absolute values which implies that $|\mu_n| \to 0$ when $\sigma(\mathcal{K} \setminus \{0\}$ is an infinite set. As usual spectral values are repeated when their eigenspaces have dimension ≥ 2. Theorem 1 entails that the function

$$\phi_{\mathcal{K}}(\mu) = e^{a\lambda} \prod (1 - \frac{\mu_\nu}{\mu}) \cdot e^{\frac{\mu_\nu}{\lambda}}$$

is analytic outside the origin. Next, let $\mathfrak{h}(\mathcal{K})$ denotes the Hilbert-Schmidt norm of $\mathcal{K}$. With these notation we have

2. Theorem. *For every complex number μ outside $\sigma(T\mathcal{K})$ one has the inequality*

$$|\phi_{\mathcal{K}}(\mu)| \cdot ||(\mu \cdot E - \mathcal{K})^{-1}|| \leq |\mu| \cdot e^{\frac{1}{2}(1+\frac{\mathfrak{h}(\mathcal{K})^2}{|\lambda|^2})}$$

This result follows from Therem 1 above together with the matrix-valued version in § xx. The effectiveness of Carleman's inequality is apparent in the study of the completeness properties of eigenfunctions of Hilbert-Schmidt operators and has therefore a wide range of applications. We present results about this in § xx.

0.4 The use of subharmonic functions.

In his text-book *Uniformisation* Rolf Nevanlinna remarks that it was Carleman who first recognized the power of this technique and applied it in many situations. An example appears in his article *Sur les fonctions inverse des fonctions entières d'ordre fini* from 1920 where the crucial result goes as follows: Let Ω be an unbounded comnected domain in the half-plane $\mathfrak{Re}(z) > 0$ where the sets

$$\Omega[\xi] = \Omega \cap \{\mathfrak{Re}(z) = \xi\}$$

are non-empty for each $\xi > 0$ and the sum of the lengths of the intervals in this slice is a finite positive number $h(\xi)$. Let f be an analytic function in Ω which extends to a continuous function on its closure and suppose that $|f(z)| \leq 1$ hold for each z on the boundary. Set

$$M(\xi) = \max_{z \in \Omega[\xi]} |f(z)|$$

With these notations the following is proved in [ibid]:

Theorem. *Either one has $M(\xi) \leq 1$ for all ξ, or else there exists some ξ_0 such that $M(\xi_0) > 1$ and for each $\xi > \xi_0$ it holds that*

$$M(\xi) \geq \log M(\xi) \cdot e^{\frac{4}{\pi} \cdot \int_{\xi_0}^{\xi} h(t)^{-1}\, dt}$$

Let us remark that inequalities as in the theorem above later were extended to cover a more general set-up by Ahlfors and Beurling in their proofs of the Denjoy conjecture which they found independently of each other in 1929.

A Phragmén-Lindelöf result. Carleman used harmonic measures to give simplified proofs of earlier results by Lindelöf and Phragmén. Here is an example whose proof appears in § xx. Let D^* be the punctured disc $\{0 < |z| < 1\}$ and when $0 < \alpha < \pi$ we set

$$D^*(\alpha) = \{z \in D : \ 0 < \arg(z) < \alpha\}$$

Let Ω be a Jordan domain which is contained in $D^*(\alpha)$ whose boundary consists of two simple Jordan curves contained in the closed sector $\overline{D^*(\alpha)}$ and share the origin as an end-point while they have distinct end-points on the unit circle T which are end-points of an interval of T which gives the remaining part of $\partial\Omega$. Let $g(z)$ be analytic in Ω and suppose it extends to a continuos function on $\partial\Omega \setminus \{0\}$ with a maximum norm ≤ 1. Next, if $0 < \delta < 1$ we set $\Omega[\delta] = \Omega \cap \{|z| = \delta\}$ and

$$\rho(\delta) = \max_{z \in \Omega[\delta]} e^{-\delta^{\frac{\pi}{\alpha}}} \cdot |g(z)|$$

Using the subharmonic function $\log|g|$ we prove the following in § xx:

Theorem. *Assume that*

$$\liminf_{\delta\to 0}\ \rho(\delta) < \infty$$

Then it follows that $|g(z)| < 1$ *for all* $z \in \Omega$

A deeper result is the following uniqueness result which appears in Carleman's book *xxx* from 1922: Let $a > 0$ and consider an analytic function $\Phi(z)$ in the half-plane $\mathfrak{Re}(z) > a$ which extends continuously to the closed half-plane and is not identically zero. Suppose there exist two sequences of strictly increasing real numbers $\{\beta_k\}$ and $\{\lambda_k\}$ such that

$$|\Phi(z)| \leq \big(\frac{\beta_k}{|z|}\big)^{\lambda_k}$$

hold for each $k = 1, 2, \ldots$. Then the series

$$\sum \frac{\lambda_{k+1} - \lambda_k}{\beta_k} < \infty$$

The proof appears in § xx and gives an instructive lesson about harmonic majorisation.

Finally the reader may contemplate upon the following result which is proved in § xx using the subharmonicity for the radius of convergence defined via Taylor series of analytic functions defined in open subsets of $\mathbf{C}$. Consider a pair f and g where f is analytic in the rectangle $\square_+ = \{-1 < x < 1\} \times \{0 < y < a\}$ while g is analytic in $\square_- = \{-1 < x < 1\} \times \{-a < y < 0\}$. Then they are analytic continuations of each other over the real interval $-1 < x < 1$ if

$$\lim_{\epsilon\to 0} \int_{-1}^{1} |f(x + i\epsilon) - g(x - i\epsilon)|\, dx = 0$$

Notice that no growth condition is imposed on the functions from the start. This result was established by Carleman's lectures at Mittag Leffler in 1935 and can be regarded as the first major result about hyperfunctions. See § xx for further comments.

0.5 Extensions of Fredholm's theory.

Operator methods were introduced by Carl Neumann in a pioneeering work from 1878 for a special class of ellitptic boundary value problems. Neumann's method was later refined by Poincaré and Fredholm proved existence theorems for a quite extensive class of elliptic boundary value problems via his theory about integral operators. The merit of constructing operators is that one can establish existence theorems without variational methods which therefore leads to a more constructive procedure to attain solutions. For example the standard Dirichlet problem can be proved via operator methods as follows: Consider a bounded domain Ω in $\mathbf{R}^n$ with a C^1-boundary and let dA denote the area measure on $\partial\Omega$. When $n \geq 3$ there exists the bounded linear operator on $C^0(\partial\Omega$ defined by

$$N(f)(p) = \int_{\partial\Omega} \frac{1}{|p - q|^{n-2}} \cdot f(q)\, dA(q)$$

If N has a dense range in $C^0(\partial\Omega)$ the maximum principle for harmonic functions entails that the Dirichet problem is solvable for every continuous boundary function. The proof that N has dense range follows by elementary measure theory and is left as an exercise to the reader. Next, we consider the inhomogeneous Neumann-Poincaré problem. Given a strictly positive continouos function a on $\partial\Omega$ one seeks for each $f \in C^0(\Omega)$ a harmonic function u in Ω which together with its inner normal derivative extends continuously to the boundary where

$$(*) \qquad \frac{\partial u}{\partial \mathbf{n_i}} = a \cdot u + f$$

The maximum principle for harmonic functions and the positivity of a easily entials that the solution u is unique if it exists. To prove existence one introduces the kernel function defined outside the diagonal in $\partial\Omega) \times \partial\Omega$ by:

$$N(p,q) = \frac{\langle p-q, \mathbf{n_i}(q)\rangle}{|p-q|^{n-1}}$$

where $\mathbf{n_i}(q)$ is the inner normal at the boundsry point q.

Next, when $g \in C^0(\partial\Omega)$ we construct the harmonic function U_g in Ω via Newton's potential, i.e.

$$U_g(p) = xxx \cdot \int_{\partial\Omega} \frac{g(q)}{|p-q|^{n-2}}\, dA(q)$$

Now $xx \cdot \frac{1}{|x|^{n-2}}$ is the fundamental solution for the Laplace operator so by Greens' formula

$$\frac{\partial U_g}{\partial \mathbf{n_i}}(p) = g(p) + \int_{\partial\Omega} N(p,q)g(q)\, dA(q)$$

Following the device by Fredholm and Poincaré we introduce the kernel function on the product $\partial\Omega \times \partial\Omega$ defined by

$$K(p,q) = N(p,q) - xxx \cdot a(p) \cdot \frac{1}{|p-q|^{n-2}}$$

From the above we wee that U_g solves (*) if the g-function satisfies the intgral equation

$$\text{(**)} \qquad \int_{\partial\Omega} K(p,q)g(q)\, dA(q) = g(p) + f(p)$$

To see that (**) has a solution one uses the C^1-hypothesis on $\partial\Omega$ which implies that the linear operator on $C^0(\partial\Omega)$ defined by

$$\mathcal{K}(g)(p) = \int_{\partial\Omega} K(p,q)g(q)\, dA(q)$$

is compact. With an arbitrary f we conclude that (**) has a unique solution g if the spectrum of $\sigma(\mathcal{K})$ does not contain 1 which follows from the uniqueness. See § xx for further details amndlet jus also remark that Carelan also analyzed the spectrum of $\mathcal{K}$ via a symmetrization of a general nature which reduces the calcuations to determine spectra of self-adjointopertores and after solve explicit systems of linear equations. See § xx for this result which later has been adopted extensively in operator theory.

Remark. Carleman's thesis treats also boundary value problems where the regularity on $\partial\Omega$ is relaxed. For C^1-domains the kernel $K(p,q)$ above is a bounded function which entails compactness of $\mathcal{K}$. If "corners" are allowed on a sufficiently thin subset of $\partial\Omega$ one has a finite L^1-integral

$$\text{(i)} \qquad \iint_{\partial\Omega\times\partial\Omega} |K(p,q)|\, dA(p)dA(q) < \infty$$

When (i) holds one seeks solutions to the boundary value problem where (*) only is requested at boundary points outside the set of corners. In his thesis, Carleman treated the case when (i) holds and regarded functions f which are integrable with respect to the area measure. Then $\mathcal{K}$ becomes a densely defined operator on a suitable weighted L^1-space. It was analysis of this kind which led Carleman to extend the Fredholm theory to a wider class where singular kernels appear in integral equations.

A non-linear boundary value problem. In the article *Über eine nichtlineare Randwertaufgabe bei der Gleichung* $\Delta u = 0$ (Mathematisches Zeitschrift vol. 9 (1921), Carleman considered the following equation: Let Ω be a bounded domain in $\mathbf{R}^3$ with C^1-boundary and $\mathbf{R}^+$ the non-negative real line where t is the coordinate. Let $F(t,p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$\text{(0.1)} \qquad t \mapsto F(t,p)$$

is strictly increasing for every $p \in \partial\Omega$ and that $F(0,p) \geq 0$. Moreover,

$$\lim_{u \to \infty} F(t,p) = +\infty \tag{0.2}$$

holds uniformly with respect to p. For a given point $q_* \in \Omega$ we seek a function $u(x)$ which is harmonic in $\Omega \setminus \{q_*\}$ and at q_* it is locally $\frac{1}{|x-q_*|}$ plus a harmonic function. Moreover, it is requested that u extends to a continuous function on $\partial\Omega$ and that $u \geq 0$ in $\overline{\Omega}$. Finally, along the boundary the inner normal derivative $\partial u/\partial n$ satisfies the equation

$$\frac{\partial u}{\partial n}(p) = F(u(p), p) \quad : p \in \partial\Omega \tag{*}$$

Let us remark that the case when $F(t,p) = kt^4$ for some positive constant k means that we regard the Stefan-Boltzmann equation whose physical interpretation ensures that (*) has a unique non-negative solution u.

Theorem. *For each F satisfying (0.1-0.2) the boundary value problem has a unique solution u.*

We prove this theorem in § xx. Apart from the conclusive result, the strategy in Carleman's proof deserves attention. The crucial idea is to consider a family of boundary value problems where one for each $0 \leq h \leq 1$ seeks u_h to satisfy

$$\frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p), p) \quad : p \in \partial\Omega \tag{*}$$

and u_h has the same pole as u above. The device to pass via linear systems was put forward by Poincaré in his famous lecture *L'avenir des mathématiques* held in Rome 1908. Let us remark that the methods which prove the theorem above can be adopted to other situations and it goes without saying that extensions to higher dimensions are available and one can also replace the Laplace operator by elliptic operators of positive type.

0.6 Taylor series and quasi-analytic functions.

Let $f(x)$ an infinitely differentiable function defined on the interval $[0,1]$. At $x=0$ we can take the derivatives and set

$$C_\nu = f^{(\nu)}(0)$$

In general the sequence $\{C_\nu\}$ does not determine $f(x)$. The standard example is the C^∞-function defined for $x>0$ by $e^{-1/x}$ and zero on $x \leq 0$. Here $\{C_\nu\}$ is the null sequence and yet the function is no identically zero. Now we seek growth conditions on the derivatives of f over the whole interval $(-1,1)$ such that $\{C_\nu\}$ determines f over this interval. For each non-decreasing sequence of $\mathcal{A} = \{\alpha_\nu\}$ of positive real numbers we denote by $\mathcal{C}_\mathcal{A}$ the class of C^∞-functions on $[0,1]$ where the maximum norms of the derivatives satisfy

$$(*) \qquad \max_{0 \leq x \leq 1} |f^{(\nu)}(x)| \leq k^\nu \cdot \alpha_\nu^\nu \quad : \quad \nu = 0, 1, \ldots$$

for some $k>0$ which may depend upon f. One says that $\mathcal{C}_\mathcal{A}$ is a quasi-analytic class if every $f \in C_\mathcal{A}$ whose Taylor series is identically zero at $x=0$ vanishes identically on $[0,1)$.

1. The Denjoy class. Let $\mathcal{A} = \{\alpha_\nu\}$ be such that the series

$$(**) \qquad \sum \frac{1}{\alpha_\nu} = +\infty$$

In the article [Denjoy 1921), Denjoy proved that (**) entails that $C_\mathcal{A}$ is quasi-analytic.

2. The general case. A conclusive result which gives a necessary and sufficient condition on the sequence $\{\alpha_\nu\}$ in order that $C_\mathcal{A}$ is quasi-analytic was proved by Carleman and goes as follows:

3. Theorem. *The class $C_\mathcal{A}$ is quasi-analytic if and only if*

$$\int_1^\infty \log \left[\sum_{\nu=1}^{\infty} \frac{r^{2\nu}}{a_\nu^{2\nu}} \right] \cdot \frac{dr}{r^2} = +\infty$$

For the proof of this result we refer to § XX in Special Topics.

4. The reconstruction theorem. Since quasi-analytic functions by definition are determined by their Tayor series at a single point there remains the question how to determine $f(x)$ in a given quasi-analytic class $C_\mathcal{A}$ when the sequence of its Taylor coefficients at $x=0$ are given. To achieve this Carleman considered a class of variational problems. Let $n \geq 1$ and for a given sequence of real numbers $\{C_0, \ldots, C_{n-1}\}$ we consider the class of n-times differentiable functions f on $[0,1]$ for which

$$\text{(i)} \qquad f^{(\nu)}(0) = C_\nu \quad : \quad \nu = 0, \ldots, n-1$$

Next, let $\{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ be some $n+1$-tuple of positive numbers and consider the variational problem

$$\text{(ii)} \qquad \min_f J_n(f) = \sum_{\nu=0}^{\nu=n} \gamma_\nu^{-2\nu} \cdot \int_0^1 [f^{(\nu)}(x)]^2 \cdot dx$$

where the competing family consist of n-times differentiable functions on $[0,1]$ satisfying (i) above. The strict convexity of L^2-norms entail that the variational problem has a unique minimizing function f_n which depends linearly upon $C_0, \ldots, C_{n-1}$. In other words, there exists a unique doubly indexed sequence of functions $\{\phi_{p,n}\}$ defined for pairs $0 \leq p \leq n$ such that

$$f_n(x) = \sum_{p=0}^{p=n-1} C_p \cdot \phi_{p,n-1}(x)$$

where the functions $\{\phi_{0,n-1}, \ldots \phi_{n-1,n-1}\}$ depend upon $\gamma_0, \ldots, \gamma_n$ and

$$\phi_{n,p}^{(\nu)}(0) = \text{Kronekcer's delta-function } \delta(\nu, p)$$

5. A specific choice of the γ-sequence. Let $\{\alpha_\nu\}$ be a Denjoy sequence, i.e. (**) above diverges. Set $\gamma_0 = 1$ and

$$\gamma_\nu = \frac{1}{\alpha_\nu} \cdot \sum_{p=1}^{p=\nu} \alpha_p \quad : \quad \nu \geq 1$$

To each $n \geq 1$ we consider the variational problem above using the n-tuple $\gamma_0, \ldots, \gamma_{n-1}$ which yields the extremal function $f_n(x)$. With these notations the following result is proved in Carleman's cited monograph:

6. Theorem. *If $F(x)$ belongs to some $\mathcal{A}$ given by a Denjoy sequence it follows that*

$$\lim_{n \to \infty} f_n(x) = F(x)$$

where the convergence holds uniformly on interval $[0, a]$ for every $a < 1$. Moreover, there exists a doubly indexed sequence $\{a_{\nu,n}\}$ defined for pairs $0 \leq \nu \leq n$ which only depends on the sequence $\{\alpha_\nu\}$ such that every $F(x) \in \mathcal{C}_\mathcal{A}$ then is given as a limit of the form:

$$F(x) = \lim_{n \to \infty} \sum_{\nu=0}^{\nu=n} a_{\nu,n} \cdot \frac{F^{(\nu)}(0)}{\nu\,!} \cdot x^\nu \quad : \quad 0 \leq x < 1$$

7. Remark. Theorem 6 gives a reconstruction for quasi-analytic classes of the Denjoy type. For a general quasi-analytic class $C_\mathcal{A}$ a similar result is proved in [Carleman]. Here the final result is more involved and the doubly indexed a-sequence it is found in a rather implicit manner via solutions to the variational problems which in general depend upon the given sequence $\mathcal{A}$. So up to the present date there remain open problems about effective reconstruction formulas in quasi-analytic classes. The interested reader should consult [Carleman: page xxx] for some specific open questions about this.

8. Quasi-analytic boundary values. Another problem is concerned with boundary values of analytic functions where the set of non-zero Taylor-coefficients is sparse. In general, consider a power series $\sum a_n z^n$ whose radius of convergence equal to one. Assume that there exists an interval ℓ on the unit circle such that the analytic function $f(z)$ defined by the series extends to a continuous function in the closed sector where $\arg(z) \in \ell$. So on ℓ we get a continuous boundary value function $f^*(\theta)$ and suppose that f^* belongs to some quasi-analytic class on this interval. Let f be given by the series

$$f = \sum a_n \cdot z^n$$

Suppose that gaps occur and write the sequence of non-zero coefficients as $\{a_{n_1}, a_{n_2} \ldots\}$ where $k \mapsto n_k$ is a strictly increasing sequence. With these notations the following result is due to Hadamard:

9. Theorem. *Let $f(z)$ be analytic in the open unit disc and assume it has a continuous extension to some open interval on the unit circle where the boundary function $f^*(\theta)$ is real-analytic. Then there exists an integer M such that*

$$n_{k+1} - n_k \leq M \tag{9.1}$$

for all k. In other words, the sequence of non-zero coefficients cannot be too sparse.

Hadamard's result was extended to the quasi-analytic case in [Carleman] where it is proved that if f^* belongs to some quasi-analytic class determined by a sequence $\{\alpha_\nu\}$ then the gaps cannot be too sparse, i.e. after a rather involved analysis one finds that f must be identically zero if the integer function $k \mapsto n_k$ increases too fast. The rate of increase need not be of the type (9.1), but depends upon $\{\alpha_\nu\}$. Up to the present date it appears that no precise descriptions of the growth of $k \mapsto n_k$ which would ensure unicity is known for a general quasi-analytic class, i.e. even in the situation considered by Denjoy. So there remains many basic questions concerned with quasi-analyticity.

Symmetric operators on Hilbert spaces.

Among Carleman's contributions to mathematics one should first of all mention the monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923]. Here the existence of spectral resolutions for unbounded and self-adjoint operators on separable Hilbert space was established. The expositary article *La theorie des equations intégrales singuliérs* [Ann. l'Institut Poincaré Vol. 1 (1931)] from his lectures in Paris during the spring 1930 gives a good introduction to [ibid] and contains several instructive examples. The study in [ibid] was inspired by Hilbert's spectral theorem for bounded self-adjoint operators from his famous book *Integralgleichungen* [xxx: 1904], and put forward by Carleman during his plenary talk at the IMU-congress at Zürich in 1932:

La théorie, crée par Hilbert, des formes quadratiques (ou hermitiennes) à une infinité de variables en connexion avec la théorie des équations intégrales à noyeau symmetrique est certainement la plus importante découverte qui ait été faite dans la théorie des équations intégrales aprés les travaux fondamentaux de Fredholm.

The need for a theory of unbounded operators was put forward by Weyl in 1908 in a famous article devoted to second order differential operator on the real x-line defined by

$$P = \frac{d^2}{dx^2} + q(x)$$

where $q(x)$ is a real-valued and locally square integrable function. Weyl found examples of such operators for which there exists a complex.valued L^2-function $u(x)$ on the real line such that

$$P(u) = i \cdot u$$

where i is the imaginary unit. This appears to contradict the usual spectral theorem for real and symmetric matrices and raised at an early stage many new problems devoted to spectral theory of linear operators. It goes without saying that Weyl's studies also served as a major inspiration for Carleman when he developed a general theory in the cited monograph. Let us give an example from [ibid] where a wave equation is solved via an operator-valued spectral function.

2.1. Propagation of sound. With (x, y, z) as space variables in $\mathbf{R}^3$ and a time variable t one considers the propagation of sound in the infinite open complement $U = \mathbf{R}^3 \setminus \overline{\Omega}$ of a bounded open subset Ω with a C^1-boundary $\partial\Omega$. It is governed by solutions $u(x, y, z, t)$ defined in $U \times \{t \geq 0\}$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \tag{1}$$

where Δ is the Laplace operator in x, y, z. Moreover, the normal derivative taken along $\partial\Omega$ are zero, i.e.

$$\frac{\partial u}{\partial n}(p, t) = 0 \quad : p \in \partial\Omega \tag{2}$$

hold for each $t \geq 0$. It turns out that there exists a unque u-function satisfying (1-2) and initial conditions

$$u(p, 0) = f_0(p) \quad : \quad \frac{\partial u}{\partial t}(p, 0) = f_1(p) \tag{3}$$

for a pair of functions f_0, f_1 in $L^2(U)$ such that $\Delta(f_1)$ and $\Delta(f_2)$ also belong to $L^2(U)$, and whose normal derivatives along $\partial\Omega$ are identically zero. In [ibid] this result is proved via the construction of an operator-valued spectral function $\Theta(\lambda)$ defined on an interval $[c, +\infty)$ for a positive real number c and the unque solution u is given by

$$u = \int_c^\infty \cos(\sqrt{\lambda} t) \cdot \frac{d\Theta}{d\lambda}(f_0) + \int_c^\infty \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} \cdot \frac{d\Theta}{d\lambda}(f_1) \tag{*}$$

The new feature in Carleman's proof as compared to earlier studies was the construction of Θ which arises via a certain densely defined and self-adjoint operator. More precisely, denote by

$L^2_*(U)$ the family of functions g in U for which both g and $\Delta(g)$ are in $L^2(U)$ and $\frac{\partial g}{\partial n} = 0$ on $\partial\Omega$. Then the following hold:

(i)
$$g = \int_c^\infty \frac{d\Theta}{d\lambda}(g) \quad : g \in L^2_*(U)$$

Moreover, for every interval $[a,b] \subset [c,+\infty)$ and continuous function $\rho(\lambda)$ on $[a,b]$ one has

(ii)
$$\Delta\big(\int_a^b \rho(\lambda)\cdot \frac{d\Theta}{d\lambda}(g)\big) = -\int_a^b \lambda\cdot\rho(\lambda)\cdot\frac{d\Theta}{d\lambda}(g) \quad : g \in L^2_*(U)$$

Since the second order derivative of $t \mapsto \cos(\sqrt{\lambda}t)$ is equal to $-\lambda\cdot\cos(\sqrt{\lambda}t$ and similarly for the sine-function, we see from (i-ii) that (*) yields a solution to the wave equation for each pair f_0, f_1 in $L^2_*(U)$. When $t = 0$ the vanishing of the sine-function at zero gives $u(p,0) = f_0(p)$ and taking the first order t-derivative the reader can confirm the second initial condition in (3).

Remark. In § xx we expose the constructions of the Θ-function. Carleman's original methods from 1923 was later been adopted in work related to boundary value problems and scattering. The reader may consult Chapter XIV in volume 2 of Hörander's text-books in linear partial differential operators which contains a wealth of results dealing with boundary problems as above, based upon the use of spectral functions.

A limit formula. One merit in the constructions from [ibid] is that they confirm that the solution $u(x,t)$ satisfies:

(**)
$$\lim_{t\to\infty} (\frac{\partial u}{\partial x})^2(p,t) + (\frac{\partial u}{\partial y})^2(p,t) + (\frac{\partial u}{\partial z})^2(p,t) = 0$$

with uniform convergence when p stays in a relatively compact subset of U. Namely, the Riemann-Lebesgue theorem and the equation (*) for the u-solution, entail (**) if the spectral Θ-function is absolutely continuous with respect to λ. In § xx we expose Carleman's proof that this holds which relies upon the following result:

Let $\{a \leq s \leq b\}$ be a compact interval on the real s-line and $s \mapsto G_s$ is a function with values in the Hilbert space $L^2_*(U)$ which is continuous in the sense that

$$\lim_{s\to s_0} ||G_s - G_{s_0}||_2 = 0$$

It is called absolutely continuous if there to each $\epsilon > 0$ exists some $\delta > 0$ such that

$$\sum ||G_{s_{\nu+1}} - G_{s_\nu}||_2 < \epsilon$$

hold for every partition $a = s_0 < s_1 < \ldots < s_M = b$ with $\sum (s_{\nu+1} - s_\nu) < \delta$.

Theorem. *Assume that the following holds for every sub-interval $[\alpha,\beta]$ of $[a,b]$:*

$$\Delta(G_\beta - G_\alpha) = -\int_\alpha^\beta s\cdot\frac{dG_s}{ds}$$

Then $s \mapsto G_s$ is absolutely continuous.

§ 0.1 Spectral functions of unbounded hermitian operators

In view of the importance we describe already in this introduction some results from [ibid]. The general situation below covers unbounded self-adjoint operators as a special case. We expose the theory from [ibid] in a general context since this is needed to grasp material about moment problems in § xx. Let us ony remark that spectral resolutions of of unbounded self-adjoint operators is an easy consequence of Hilbert's results for bounded self-adjoint operators and exposed in many text-books. See for example my notes devoted to functional analysis for the details about the extension of Hilbert's theorem to unbounded self-adjoint operators.

Recall that a separable Hilbert space is isomorphic to ℓ^2 whose vectors are sequences of complex numbers $\{x_p\}$ indexed by integers with $\sum |x_p|^2 < \infty$. A doubly indexed sequence $\{c_{pq}\}$ is Hermitian if:

$$c_{q,p} = \bar{c}_{p,q}$$

Impose the condition that each column of this infinite matrix belongs to ℓ^2, i.e.

$$\sum_{q=0}^{\infty} |c_{pq}|^2 < \infty \quad : p = 1, 2, \dots \tag{1}$$

The Cauchy-Schwarz inequality entails that if $x \in \ell^2$ then the series

$$\sum_{q=0}^{\infty} c_{p,q} \cdot x_q \tag{2}$$

converges absolutely for each p and let y_p denote the sum. But (1) does not imply that $\{y_p\}$ belongs to ℓ^2. So we get a subspace $\mathcal{D}$ of ℓ^2 which consists of vectors x such that

$$\sum_{p=0}^{\infty} \big|\sum_{q=0}^{\infty} c_{p,q} \cdot x_q\big|^2 < \infty \tag{3}$$

Notice that (1) implies that $\mathcal{D}$ contains the ℓ^2-vectors $\{x_p\}$ for which only finitely many x_p are non-zero. Hence there exists the densely defined linear operator S sending a vector x to the vector $Sx = y$ where

$$y_p = \sum_{q=0}^{\infty} c_{qp} \cdot x_q \quad : p = 0, 1, 2 \dots$$

By definition the domain of definition of S is the subspace $\mathcal{D}$ of ℓ^2.

Spectral decomposition. Given an infinite matrix as above we get finite hermitian matrices $\{C_N\}$ for each positive integer N whose elements are c_{pq} for pairs $1 \leq p, q \leq N$ and otherwise zero. Each C_N gives bounded linear opertors on the complex Hilbert space ℓ^2 denoted by $\mathcal{C}_N$. The spectral theorem for hermitian matrices gives for each N an N-tuple of pairwise orthogonal vectors $\phi_1^{(N)} \dots, , \phi_1^{(N)}$ such that

$$\mathcal{C}_N(\phi_k^{(N)}) = \mu_k \cdot (\phi_k^{(N)})$$

where the eigenvalues $\{\mu_k\}$ are real. A positive real number λ gives for each N a bounded linear operator $E_N(\lambda)$ defined on vectors $x \in \ell^2$ by

$$E_N(\lambda)(x) = \sum^{\lambda} \langle \phi_k^{(N)}, x \rangle \cdot \phi_k^{(N)}$$

where the summation index indicates that e the sum is restricted to eigenvectors of $\mathcal{C}_N$ for which

$$0 < \mu_k \leq \frac{1}{\lambda} \tag{i}$$

It is clear that $E_N(\lambda)$ is a self-adjoint projection with a finite dimensional range given by the subspace of ℓ^2 whose basis consists of eigenvectors of $\mathcal{C}_N$ for which (i) holds. It is now tempting to perform limits as $N \to infty$. In [ibid: Chapter I] it is shown via basic Hilbert space theory that there exist weakly convergent subsequences of these E-operators. More precisely, there exists a family of operator valued functions $\lambda \mapsto \Theta(\lambda)$ is where each such Θ is a limit from a subsequence of the E-operators, i.e. there exists a sequence $1 \leq N_1 < N_2 < \dots$ such that

$$\Theta(\lambda)(x) = \lim_{\nu \to \infty} E_{N_\nu}(\lambda(x)$$

hold for every ℓ^2-vector x. Here the limit is weak which means that

$$\langle \Theta(\lambda)(x), y \rangle = \lim_{\nu \to \infty} \langle E_{N_\nu}(\lambda(x), y \rangle$$

hold for all pairs x, y in ℓ^2. One refers to every such limit Θ as a spectral operator associated to $\mathcal{C}$. From the constructions above it is clear that the functions

$$\lambda \mapsto \langle E_{N_\nu}(\lambda(x), x\rangle$$

are non-decreasing on the positive λ-line for each vector x. After a passage to the limit it follows that

$$\lambda \mapsto \langle \Theta(\lambda)(x), y\rangle$$

is non-decreasing function which is left continuous, i.e.

$$\lim_{\epsilon\to 0}\langle \Theta(\lambda-\epsilon)(x), y\rangle = \langle \Theta(\lambda)(x), y\rangle$$

hold for each fixed vector x and every $\lambda > 0$, where $\epsilon > 0$ in the limit above. Next, for each N we notice that the opertors $\{E_N(\lambda) : \lambda > 0\}$ commute and

$$E_N(\lambda_2) \circ E_N(\lambda_1) = E_N(\lambda_1)$$

hold for pairs $0 < \lambda_1 < \lambda_2$. Passing to a limit the same hold for each spectral Θ-family. Moreover, if $x \in \ell^2$ then the vector-valued function $\lambda \mapsto \Theta(\lambda)(x)$ has a finite total variation. In fact, for each finite partition $0 < \lambda_1 < \ldots < \lambda_M$ over $(0, +\infty)$ one has the Bessel inequality

$$||\Theta(\lambda_k)(x)|| + \sum_{k=1}^{k=M-1} ||\Theta(\lambda_{k+1}(x) - \Theta(\lambda_k)(x)|| \leq ||x||$$

where we have taken norms in ℓ^2. It follows that if $0 < a < b < \infty$ and $g(\lambda)$ is a continuous function on $[a, b]$, then there exists a bounded linear operator on ℓ^2 defined by

$$x \mapsto \int_a^b g(\lambda) \cdot \frac{d\Theta}{d\lambda}(x)$$

where the ℓ^2-valued integral is taken in the sense of Stieltjes. Thus, a pair (Θ, g) where Θ is a spectral operator on the positive real λ-line and $g \in C^0[a, b]$ gives a bounded linear operator denoted by Θ_g. In [ibid] it s proved that all these operators have range contained in $\mathcal{D}(S)$ and commute with S in the sense that

$$S \circ \Theta_g(x) = \Theta_g \circ S(x) \quad : \, x \in \mathcal{D}(S)$$

In particular we apply this to an interval $[\delta, 1/\delta]$ with $0 < \delta < 1$ where $g(\lambda) = 1/\lambda$. A crucial result from [ibid] asserts the following:

Proposition. *For each vector $x \in \mathcal{D}(S)$ there exists a constant M such that*

$$\int_\delta^{1/\delta} \lambda^{-1} \cdot ||\frac{d\Theta}{d\lambda}(x)|| \leq M$$

for all $\delta > 0$. In other words, the vector-valued integral below is absolutely convergent

$$\int_0^\infty \lambda^{-1} \cdot \frac{d\Theta}{d\lambda}(x)$$

Similar constructions as above can be done when we regard negative eigenvalues of the E-operators. If $\lambda < 0$ we set

$$E_N(\lambda)(x) = \sum^{\lambda} \langle \phi_k^{(N)}, x\rangle \cdot \phi_k^{(N)}$$

where the summation index means that we take the sum over those eigenvectors of $\mathcal{C}_N$ for which

$$\frac{1}{\lambda} \leq \mu_k < 0$$

Notice that if δ is so small that the absolute values of all eigenvalues of $\mathcal{C}_N$ belong to $[\delta, 1/\delta]$, then

$$\mathcal{C}_N(x) = \int_{-1/\delta}^{-\delta} \lambda^{-1} \cdot \frac{dE_N}{d\lambda}(x) + \int_\delta^{1/\delta} \lambda^{-1} \cdot \frac{dE_N}{d\lambda}(x)$$

Next, exactly as above we take subsequences which give weak limits and operator Θ-functions which now are defined on $(-\infty, 0)$. In particular we take subsequences $1 \leq N_1 < N_2 < \ldots$ for which weak limits exist for all real $\lambda \neq 0$. This gives operator-valued spectral Θ-function defined for all real λ. The same finiteness as in Proposition 1 hold when we integrate $1/\lambda$ over intervals $[-1/\delta, -\delta]$ and the following conclusive result was proved in [ibid:Chapter 2]:

Theorem. *For each spectral Θ-function and every $x \in \mathcal{D}(S)$ one has*

$$S(x) = \lim_{\delta \to 0} \int_{-1/\delta}^{-\delta} \lambda^{-1} \cdot \frac{d\Theta}{d\lambda}(x) + \int_{\delta}^{1/\delta} \lambda^{-1} \cdot \frac{d\Theta}{d\lambda}(x)$$

The inhomogenous equation $S(x) = \zeta \cdot x + y$. Let y be a given no-zero vector in ℓ^2 and ζ is a complex number whose imaginary part is > 0. Since each operator $\mathcal{C}_N$ has a real spectrum there exists a unique vector $x_N(\zeta)$ such that

$$\mathcal{C}_N(x_N(\zeta)) = \zeta \cdot x_N(\zeta) + y$$

We leave it to the reader to verify that the reality of $\sigma(\mathcal{C}_N)$ entails that

$$||x_N(\zeta)|| \leq \frac{1}{\mathfrak{Im}(\zeta)} \cdot ||y||$$

Recall that bounded subsets of ℓ^2 are relatively compact with respect to the weak topology. We can therefiore pass to subsequences where $x_{N_k}(\zeta)$ converges weakly to a oimit vector $x_*(\zeta)$ and at the same time $\{N_k\}$ produces a spectral Θ–function. Using Theorem xx it follows that

$$S(x_*(\zeta)) = \zeta \cdot x_*(\zeta) + y$$

So this homogeneous equation has at least one solution. However, it is in general not unique and we shall give examples in § xx in connection with the moment problem where a description of all solution vectors to (xx) is available. In a similar way we get solutions to the equation (xx) for complex numbers ζ with negative imaginary part.

Class I operators.

The densely defined hermitian operator S is of Class I if $iE - S$ and $iE + S$ both are injective. When this holds it is proved in [ibid] that if $\mathfrak{Im}(\zeta) \neq 0$ then the equation

$$S(x) = \zeta \cdot x$$

has no non-zero solution. It means that the inhomgenous equation above has a unique solution $x_*(\zeta)$ for every y and gives an everywhere defined linear operator $R(\zeta)$ where

$$R(\zeta)(y) = x_*(\zeta)$$

A detailed study of Class I operators appears in Chapter III-IV in [ibid]. Let us remark that one also refers to Class I opertors as self-adjoint operators. In fact, this terminology stems from the following result in [ibid]:

Theorem. *The operator S is of class I if and only if*

$$\langle Sx, y \rangle = \langle x, Sy \rangle$$

hold for each pair x, y in $\mathcal{D}(S)$. Moreover, a Case I operator has a unique spectral Θ-function obtained as the unrestricted limit

$$\lim_{N \to \infty} E_N(\lambda) = \Theta(\lambda)$$

The closure property. Let S be of Class I which gives a unique spectral function Θ where $\mathcal{D}(S)$ consists of vctor x for which the integral

$$\int_{-\infty}^{\infty} \lambda^{-1} \cdot \frac{d\Theta}{d\lambda}(x)$$

is absolutely convergent. One says that S has the closure property if

$$x = \lim_{\delta \to 0} \int_{-1/\delta}^{\delta} \frac{d\Theta}{d\lambda}(x) + \int_{\delta}^{1/\delta} \frac{d\Theta}{d\lambda}(x)$$

hold for every vector x. In [ibid] it is proved that this closure property holds if S is injective, i.e. $S(x) \neq 0$ for every non-zero vector in $\mathcal{D}(S)$.

§ 0.2 Applications to quantum mechanics.

Carleman's cited monograph was published before quantum mechanics was born and around 1920 he was concerned with applications to the moment problem of Stieltjes and extension of Fredholm's theory to cases where singular integral operators appear. It was therefore quite exciting when Niels Bohr in a lecture held at the Scandinavian Congress in Copenhagen 1925, talked about the new quantum mechanics and adressed new problems to the mathematical community. Recall that a crucial point in quantum mechanics is the hypothesis on energy levels which correspond to orbits in Bohr's theory of atoms. For this physical background the reader should consult Bohr's plenary talk when he received the Nobel Prize in physics 1923. In the "new-born" quantum mechanics the following second order PDE-equation plays a crucial role:

$$(*)\qquad \Delta\phi + 2m\cdot\big(E-U\big)\big(\frac{2\pi}{h}\big)^2\cdot\phi = 0$$

Here Δ is the Laplace operator in the 3-dimensional (x,y,z)-space, m the mass of a particle and h Planck's constant while $U(x,y,z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Leaving physics aside, the mathematical problem amounts to study second order PDE-operators:

$$(**)\qquad L = \Delta + c(x,y,z) \quad : \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

where $c(x,y,z)$ is a real-valued and Lebesgue measurable function which is locally square integrable in $\mathbf{R}^3$. Above L is defined on the dense subspace of $L^2(\mathbf{R}^3)$ which consists of test-functions. Moreover, Greens' formula entails that it is symmetric, i.e. for each pair f,g in $C_0^\infty(\mathbf{R}^3)$ one has:

$$\iint L(f)\cdot g\,dxdydz = \iint L(g)\cdot f\,dxdydz$$

So the eigenvalue equation (*) boils down to find conditions on the potential function c in order that the operator L is of class I. In physical applications one is foremost concerned with the case when c is a potential function

$$W(p) = \sum \frac{\alpha_\nu}{|p-\xi_\nu|} + \beta$$

where $\{\xi_\nu\}$ is a finite subset of $\mathbf{R}^3$ and $\{\alpha_k\}$ and β are real and positive numbers. For this special c-function results from Carleman's cited monograph easily imply that L is of Class I and the real spectrum of L is a discrete sequence of real numbers whose absolute values tend to $+\infty$. See § xx for the proof. In general one does not know precise conditions on c in order that L is of Class I. The following sufficient condition was presented during in Carleman's lectures at Sorbonne in 1930 and a detailed account appears in the article *Sur la théorie mathématique de l'équation de Schrödinger* [Arkiv för matematik och fysik: 1934]:

Theorem *If there exists if there is a constant M such that*

$$\limsup_{x^2+y^2+z^2\to\infty} c(x,y,z) \leq M$$

Then the operator L is of Class 1.

Remark. Hundreds - or rather thousands - of articles have later exposed the Bohr-Schrödinger equation with various variants where one also regards a time variable in Schrödinger's equation

$$i\cdot\frac{\partial u}{\partial t}(x,y,z,t) = L(u)(x,y,z,t)$$

In all these studies the spectral theorem for unbounded self-adjoint operators plays a significant role. My personal opinion is that Carleman's original proofs superseed most of later articles since they are carried out in a constructive way where the sole technical ingredients involve Green's formula. But it is of course valuable to to give probabalistic interpretations of solutions since they appear naturally in the context of quantum mechanics via the uncertainty principle.

Another Schrödinger equation.. In the article *Théorie relativiste de l'electron et l'interprétation de la mécanique quantique* [xxx 1932.] Schrödinger raised new and unorthodox question leading to mathematical problems of considerable interest. More precisely, consider a Brownian motion which takes place in a bounded region Ω of some euclidian space $\mathbf{R}^d$ for some $d \geq 2$. At time $t = 0$ the densities of particles under observation is given by some non-negative function $f_0(x)$ defined on Ω. Classically the density at a later time $t > 0$ is equal to a function $x \mapsto u(x,t)$ where $u(x,t)$ solves the heat equation

$$\frac{\partial u}{\partial t} = \Delta(u)$$

with boundary conditions

$$(1) \qquad u(x,0) = f_0(x) \quad \text{and} \quad \frac{\partial u}{\partial \mathbf{n}}(x,t) = 0 \quad \text{when} \quad x \in \partial\Omega \quad \text{and} \quad t > 0$$

Schrödinger took into the account the reality of quantum physics which means that in an actual experiment the observed density of particles at a time $t_1 > 0$ does not coincide with $u(x,t_1)$. He posed the problem to find the most probable development during the time interval $[0,t_1)$ which leads to the state at time t_1. He concluded that the the requested density function which substitutes the heat-solution $u(x,t)$ should belong to a non-linear class of functions formed by products

$$(*) \qquad w(x,t) = u_0(x,t) \cdot u_1(x,t)$$

where u_0 is a solution to (1) while $u_1(x,t)$ is a solution to an adjoint equation

$$(2) \qquad \frac{\partial u_1}{\partial t} = -\Delta(u) \quad : \quad \frac{\partial u_1}{\partial \mathbf{n}}(x,t) = 0 \quad \text{on} \quad \partial\Omega$$

defined when $t < t_1$. This leads to a new type of Cauchy problems where one asks if there exists a w-function in (*) satisfying the initial conditions

$$w(x,0) = f_0(x) \quad : \quad w(x,t_1) = f_1(x)$$

when f_0, f_1 are non-negative functions such that

$$\int_\Omega f_0 \cdot dx = \int_\Omega f_1 \cdot dx$$

The solvability of this non-linear boundary value problem was left open by Schrödinger and the search for solutions remains open up to the present date. So far the sole essential contribution is due to Beurling in an article from 1956 whose contents is exposed in my notes about his work. Let us remark that when Ω is a bounded set with a smooth boundary, then one can use the Poisson-Greens function for the classical equation (*) and rewrite Schrödinger's equation to a system of non-linear integral equations. Examples occur already on the product of two copies of the real line where Schrödinger's equations lead to a non-linear equation for measures which goes as follows: Consider the Gaussian density function

$$g(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

Next, consider the family $\mathcal{S}_g^*$ of all non-negative product measures $\gamma_1 \times \gamma_2$ for which

$$(i) \qquad \iint g(x_1 - x_2) \cdot d\gamma_1(x_1) \cdot d\gamma_2(x_2) = 1$$

The product measure gives another product measure

$$\mathcal{T}_g(\gamma_1 \times \gamma_2) = \mu_1 \times \mu_2$$

where

$$\mu_1(E_1) \cdot \mu_2(E_2) = \iint_{E_1 \times E_2} g(x_1 - x_2) \cdot d\gamma_1(x_1) \cdot d\gamma_2(x_2)$$

hold for all pairs of bounded Borel sets. Notice that $\mu_1 \times \mu_2$ becomes a probability measure since (i) above holds. With these notations one has:

0.1 Theorem. *For every product measure $\mu_1 \times \mu_2$ which in addition is a probability measure there exists a unique $\gamma_1 \times \gamma_2$ in S_g^* such that*

$$\mathcal{T}_g(\gamma_1 \times \gamma_2) = \mu_1 \times \mu_2$$

In [Beurling] a more general result is established where the g-function can be replaced by an arbitrary non-negative and bounded function $k(x_1, x_2)$ such that

$$\iint_{\mathbf{R}^2} \log k \cdot dx_1 dx_2 > -\infty$$

At the end of his article Beurling points out that the variational methods which he used to achieve this result cannot be duplicated in higher dimensions. So Schrödinger's original set-up remains as a veritable challenge for the mathematical community.

Carleman's solution to the Bohr-Schrödinger equation

.

We expose a proof baed upon Carleman's article from the congress in Copenhagen 1925. Consider the potential function

$$W(p) = \sum \frac{a_\nu}{|p - \xi_\nu|} + b$$

We seek pairs (ϕ, λ) where $\phi \in L^2(\mathbf{R}^3)$ and λ is a real number such that

$$(*) \qquad \Delta(\phi) + (W + \lambda)\phi = 0$$

To find solutions to (*) we shall construct a Greens function $G(p,q)$ defined in $\mathbf{R}^3 \times \mathbf{R}^3$ which satisfies

$$(**) \qquad \Delta G + W \cdot G - \kappa^2 G = -4\pi\delta(p - q)$$

where the right hand side is Dirac's delta-distribution with unit mass at $p = q$ and κ a positive number which will be chosen in § xx below. To be precise, (**) means that if f is a test-function in $\mathbf{R}^3$ then

$$\int G(p,q)\Delta(f)(q)\, dq + (W(p) - \kappa^2) \int G(p,q) f(q)\, dq = -4\pi f(p)$$

hold for each p. Moreover, we will show that G is symmetric, i.e.

$$G(q,p) = G(p,q)$$

and $G(p,q)$ yields a kernel of a bounded linear operator $\mathcal{G}$ defined on $L^2(\mathbf{R}^3)$ by

$$\mathcal{G}(g)(p) = \int G(p,x) g(x)\, dx$$

Solutions to (*). Suppose that G has been found and let ϕ be an L^2-function which satisfies the integral equation

$$(***) \qquad \phi = \frac{\lambda + \kappa^2}{4\pi} \cdot \int G(p,x)\phi(x)\, dx$$

for some real number λ. Then (**) gives

$$\Delta(\phi) = (\kappa^2 - W)\phi - (\lambda + \kappa^2)\phi$$

which entails that ϕ solves (*). Conversely the reader may check that if ϕ solves (*) then it satisfies the integral equation above.

1. The construction of G.

When $\kappa > 0$ we define a function $H(p,q)$ in $\mathbf{R}^3 \times \mathbf{R}^3$ by

$$H(p.q) = \frac{e^{-|p-q|}}{|p - q|}$$

Newton's classical formula gives

(i) $$\Delta(H) = \kappa^2 \cdot H - 4\pi \cdot \delta(p-q)$$

Suppose we have found $G(p,q)$ which satisfies the integral equation

(ii) $$G(p,q) = H(p,q) + \frac{1}{4\pi}\int H(p,x)W(x)G(x,q)\,dx$$

Then (i) entails that

(iii) $$\Delta(G) = \kappa^2 G - 4\pi\delta(p-q) - W(p)G(p,q)$$

and hence G satisfies (**) above. To show that (ii) has a solution we introduce the kernel function:

(iii) $$\Omega(p,q) = H(p,q) \cdot \sqrt{W(p)} \cdot \sqrt{W(q)}$$

Notice that Ω is everywhere positive and $\Omega(p,q) = \Omega(q,p)$. Close to the diagonal $p = q$ which avoids the points $\{(\xi_\nu, \xi_\nu)\}$ it has the same singularity as H, i.e. $\frac{e^{|p-q|}}{|p-q|}$. When both p and q approach the diagonal point (ξ_1, ξ_1) we set $p = \xi_1 + x$ amd $q = \xi_1 + y$. Then $\Omega(p,q)$ increases like

$$\frac{1}{|x-y| \cdot |x| \cdot |y|}$$

and the reader can check that $(x,y) \mapsto \Omega(\xi_1 + x, \xi_1 + y)$ is locally integrable as a function of (x,y) close to the origin in $\mathbf{R}^3 \times \mathbf{R}^3$. Moreover, the construction of the Ω in (iii) gives the equality below for every p:

(iv) $$\int \Omega(p,x)\sqrt{W(x)}\,dx = \sqrt{W(p)} \cdot \int H(p,x) \cdot W(x)\,dx$$

1.2. Exercise. Show that if κ sufficiently large then

$$\rho = \max_{p \in \mathbf{R}^3} \int H(p,x) \cdot W(x)\,dx < 4\pi$$

This gives

(1.2.1) $$\int \Omega(p,x)\sqrt{W(x)}\,dx \leq \rho \cdot \sqrt{W(p)}$$

for all p. Here $\sqrt{W}$ is a positive function and by the general result in § xx, (1.2.1) implies that the operator on $L^2(\mathbf{R}^3)$ defined by

$$g \mapsto \int \Omega(p,x)g(x)\,dx$$

has norm $\leq \rho$. Dividing by 4π the linear operator below has norm < 1:

(1.2.2) $$S(g)(p) = \frac{1}{4\pi} \cdot \int \Omega(p,x)g(x)\,dx$$

1.3 Another integral equation. Since S has norm < 1, there exists a kernel function $L(p,q)$ which solves the equation

(1.3.1) $$L(p,q) = \frac{1}{4\pi} \cdot \int L(p,x) \cdot \Omega(x,q)\,dx + \Omega(p,q)$$

In fact, L is found by a Neumann series and the symmetry of Ω entails that $L(p,q) = L(q,p)$. Moreover, we get a bounded operator on $L^2(\mathbf{R}^3)$ defined by

$$\mathcal{L}(g)(p) = \int L(p,x)g(x)\,dx$$

To be precise, if E is the identity operator on $L^2(\mathbf{R}^3)$ we have

(1.3.2) $$\mathcal{L} = (E - S)^{-1} \circ S$$

Now we put

$$G(p,q) = \frac{1}{\sqrt{W(p)}\cdot\sqrt{W(q)}}\cdot L(p,q) \tag{1.3.3}$$

The positive constant b in (0.1) gives

$$\frac{1}{\sqrt{W(p)}\cdot\sqrt{W(q)}} \leq b^{-1}$$

for all p and q. Since $\mathcal{L}$ is a bounded linear operator it follows that

$$g \mapsto \int G(p,x)g(x)\,dx \tag{1.3.4}$$

also is a bounded linear operator. At this stage we leave it to the reader to verify that G satisfies the equation (**) which finishes the construction.

1.4 Spectral values. There remains to find those λ for which the integral equation (***) has a solution. This amounts to seek eigenvalues of the bounded operator $\mathcal{G}$. Here the spectrum of $\mathcal{G}$ is a discrete real set the origin and the corresponding eigenspaces at the non-zero real eigenvalues are finite dimensional. However, the determination of spectral λ-values for which (*) have non-zero L^2-solutions is a non-trivis affair. Here numerical solutions are needed which has been studied extensively in numerical analysis.

1.5 The case $L = \Delta + c(x,y,z.$**.** We shall not give all details of the proof of Theorem 0.2.1 but describe a crucial step the proof which is used to prove that L is of Class I when c satisfies the condition in the theorem. Here is the situation. Let B_r be the open ball of radius r centered at the origin and $S^2[r]$ thr unit sphere. The class of functions u which are continuous on the closed ball and whose interior normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ is continuous on the boundary $S^2[r]$ is denoted by $\mathfrak{Neu}(B_r)$.

A Neumann equation. Next, consider a pair a, H where a be a continuous function on $S^2[r]$ and $H(p,q)$ a continuous hermitian function on $S^2[r]\times S^2[r]$, i.e. $H(q,p) = \bar{H}(p,q)$ hold for all pairs of point p,q on the sphere *ccc*. Finally, c is some real-valued function in $L^2(B_r)$. With these notations the following hold:

1.5.1 Theorem. For each $f \in L^2(B_r)$ and every non-real complex number λ there exists a unique $u \in \mathfrak{Neu}(B_r)$ which satisfies the two equations:

$$\Delta(u) + \lambda\cdot c\cdot u = f \quad \text{holds in} \quad B_r \tag{i}$$

$$\partial u/\partial\mathbf{n}(p) + a(p)u(p) + \int_{S^2[r]} H(p,q)u(q)\,dA(q) = 0 \tag{ii}$$

Moreover, one has the L^2-estimate

$$\int_{B_r} |u|^2\cdot dxdydz \leq \Big|\frac{1}{\mathfrak{Im}(\lambda)}\Big| \leq \int_{B_r} |f|^2\cdot dxdydz \tag{iii}$$

Remark. The point is that the L^2-estimate above is independent of the triple a, c, H. The verification that the two equations (i-ii) give (iii) follows easily via Greens formula and is left to the reader. The fact that (i-ii) has a solution is classic and goes back to work by Neumann and Poincaré.

Glimpses from Carleman's work.

Contents.

The subsequent sections present material from some of his articles devoted to analytic function theory and Fourier analysis. Personally I find that few mathematical texts (if any) superseed Carleman's fundamental approach to many problems. Several of his articles appear as veritable "classics" which merit a study up to the present date. The reader is expected to be familiar with basic facts in analytic function theory of one complex variable and some measure theory. Apart from this the proofs in the subsequent sections are self-contained, and for students interested in analysis I personally think that a detailed study of these proofs offer more valuable lessons compared to digesting "general concepts". Before we enter sections § 0-xx we insert a classic result to illustrate the flavour of these notes. Readers who do not appreciate the convergence theorem below will probably not be prepared pursue the studies of the subsequent material where no "abstract concepts" appear while most of the results require a fairly involved analysis.

Power series and arithmetic means.

Consider a power series

$$(*)\qquad f(x)=\sum a_n\cdot x^n$$

which converges when $|x|<1$ and assume also that

$$(**)\qquad \lim_{x\to 1}\sum a_n\cdot x^n=0$$

Euler's example with $a_\nu=(-1)^\nu$ shows that this need not entail that $\sum a_\nu$ converges with a series sum equal to zero. So one needs extra conditions to get this convergence. A sufficient condtion expressed in the lemma below was already known in special cases by Abel, and later demonstrated in detail by Riemann. For each pair of integers p and n_0 we set

$$J(n_0,p)=\int_0^1\frac{\sin 2p\pi(x-1)}{x-1}\cdot\sum_{n=n_0}^\infty a_nx^n\cdot dx$$

The Abel-Riemann Lemma. *The series $\sum a_n$ converges if there to every $\epsilon>0$ exists a pair (p_0,n_0) such that*

$$p\geq p_0\implies |J(n_0,p)|<\epsilon$$

Exercise. Prove this result or consult the literature.

A criteron via Cesaro sums. Given $f(x)$ as in (*) we get for each $k\geq 0$ a unique sequence $\{S_n^{(k)}\colon n=0,1,\ldots\}$ such that

$$(1)\qquad f(x)=\frac{(1-x)^{k+1}}{(k+1)!}\cdot\sum S_n^{(k)}n^k\cdot x^n$$

One has for example

$$S_n^{(1)}=\frac{na_0+(n-1)a_1+\ldots+a_n}{n}$$

while $S_n^{(0)}$ give the arithmetical means. One refers to $\{S_n^{(k)}$ as the Cesaro sequence of order k.

A convergence theorem. *Assume (**) and that there exists some integer $k\geq 1$ such that*

$$\lim_{n\to\infty}S_n^{(k)}=0$$

Then the series $\sum a_n$ converges.

Proof. First the inequalities below hold for all pairs of positive integers p and n:

$$(\text{i})\qquad \Big|\int_0^1\sin(2p\pi x)\cdot x^k(1-x)^n\cdot dx\Big|\leq 2\pi(k+2)\,!\cdot\frac{p}{n^{k+2}}$$

$$(\text{ii})\qquad \Big|\int_0^1\sin 2p\pi x\cdot x^k(1-x)^n\cdot dx\Big|\leq\frac{C(k)}{p\cdot n^k}$$

where the constant $C(k)$ in (ii) only depends upon k. The verification of (i-ii) is left to the reader. Next, let $\epsilon>0$. The hypothesis in the theorem gives some n_0 such that

$$(\text{iii})\qquad n\geq n_0\implies |S_n^{(k)}|<\epsilon$$

Next, from (1) it follows that

$$(\text{iv})\qquad \sum_{n=n_0}^\infty a_nx^n=\frac{(1-x)^{k+1}}{(k+1)!}\cdot\sum_{n=n_0}^\infty S_n^{(k)}n^k\cdot x^n+Q(x)$$

where $Q(x)$ is a polynomial of degree n_0+k at most. Then (iii-iv) and the triangle inequality give

$$(\text{v})\quad |J(n_0,p)|\leq\epsilon\cdot\sum_{n=n_0}^\infty\frac{n^k}{(k+1)!}\cdot\Big|\int_0^1\sin(2p\pi x)\cdot x^k(x-1)^n\cdot dx\Big|+\Big|\int_0^1\frac{\sin 2p\pi(x-1)}{x-1}\cdot Q(x)\,dx\Big|$$

For an arbitrary $p \geq n_0 + 1$ we decompose the sum in the first term above from n_0 up to p and after we take a sum with $n \geq p+1$. Hence this first term is majorized by the sum of the following two expressions:

$$\text{(vi)} \qquad \sum_{n=n_0}^{n=p} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big|$$

$$\text{(vii)} \qquad \sum_{n=p+1}^{\infty} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big|$$

Using (i) above it follows that (vi) is estimated by

$$2\pi \cdot (k+2)! \cdot \frac{C(k)}{p} \cdot (p - n_0) \leq 2\pi \cdot (k+2)! \cdot C(k) = K_1$$

Next, using (ii) it follows that (vii) is estimated by

$$\pi \cdot \frac{k+2}{k+1} \cdot p \cdot \sum_{n=p+1}^{\infty} n^{-2} \leq \pi \cdot \frac{k+2}{k+1} = K_2$$

So with $K = K_1 + K_2$ we obtain from (iv)

$$\text{(viii)} \qquad |J(n_0, p)| \leq 2K \cdot \epsilon + \Big| \int_0^1 \frac{\sin 2p\pi(x-1)}{x-1} \cdot Q(x)\, dx \Big|$$

Here (viii) hold for every $p \geq n_0 + 1$. Since $Q(x)$ is a polynomial of degree $\leq n_0 + k$ it follows from the Riemann-Lebegue Lemma that the last term in (viii) tends to zero when p increases. Hence we can find p_0 such that

$$p \geq p_0 \implies |J(n_0, p)| \leq 2K \cdot \epsilon +$$

Since ϵ can be arbitrary small the condition in the Abel-Riemann Lemma holds and the proof is finished.

§ 0. Abel's inversion formula.

In the article *xxx* from 1823, Niels Henrik Abel established an inversion formula for the potential function $U(x)$ in a conservative field of forces which goes as follows:

Let $U(x)$ be an even function of x with $U(0) = 0$ and $x \to U(x)$ is strictly increasing and convex on $x \geq 0$. A particle of unit mass which moves on the real x-line satisfies Newton's equation

$$\ddot{x}(t) = -U'(x(t))$$

where the initial conditions are $x(0) = 0$ and $\dot{x}(0) = v > 0$. It follows that

$$\frac{\dot{x}(t)^2}{2} + U(x(t)) = \frac{v^2}{2} \implies$$

$$\dot{x}(t) = \sqrt{v^2 - 2U(x(t))} \tag{1}$$

during a time interval $[0, T]$ where $\dot{x}(t) > 0$ when $0 \leq t < T$ and $\dot{x}(T) = 0$. From (1) we get the equation

$$T = \int_0^{x(T)} \frac{dx}{\sqrt{v^2 - 2U(x)}} \quad : \ 2 \cdot U(x(T)) = v^2$$

Abel's inversion formula recaptures U when the function $v \mapsto T(v)$ is known. For suppose that $v \mapsto T(v)$ is given on an interval $0 \leq v \leq v^*$.

In the article *Abelsche Intergalgleichung mot konstanten Integralgrenzen* [Mathematische Zeitschrift 1922], Carleman established inversion formulas in Abel's spirit. An example is as follows: For every fixed real $0 \leq x \leq 1$

$$t \mapsto \log |x - t|$$

is integrable on the unit interval $[0, 1]$ and yields a bounded linear operator on the Banach space $C^0[0, 1]$ sending every $g \in C^0[0, 1]$ to

$$T_g(x) = \int_0^1 \log |x - t| \cdot g(t)\, dt$$

It is not difficult to show that T_g is injective. Moreover there is an inversion formula which at the same time gives a description of its range.

0.1 Theorem. *With $f = T_g$ one has the inversion formula*

$$\sqrt{x(1-x)} \cdot g(x) = \frac{1}{\pi^2} \cdot \int_0^1 \frac{f'(t) \cdot \sqrt{t(1-t)}}{x - t}\, dt + \frac{1}{\pi} \cdot \int_0^1 g(t)\, dt \tag{*}$$

Moreover one has the equation

$$\int_0^1 g(t)\, dt = -\frac{1}{2\pi \cdot \log 2} \cdot \int_0^1 \frac{f(x)}{\sqrt{(1-x)x}}\, dx \tag{**}$$

Remark. This inversion formula shows that a function f in the range of T must satisfy certain regularity properties. The reason is that in (*) the first order derivative $f'(t)$ appears in an integral where we have taken a principal value. Before we enter the proof we recall how one should grasp principal values. In general, let $h(t)$ be a contionuous funvtionon $[0, 1]$. In $\mathbf{C}|setminus[0, 1]$ there exists the analytic function

$$H(z) = \int_0^1 \frac{h(t)}{z - t}\, dt$$

With $\epsilon > 0$ we consider the difference

$$H(x + i\epsilon) - H(x - i\epsilon) = \int_0^1 \frac{h(t)}{z - t}\, dt = -2i\epsilon \int_0^1 \frac{h(t)}{(x - t)^2 + \epsilon^2}\, dt$$

For each $0 < t < 1$ the reader can check that the limit of the right hand side as $\epsilon \to 0$ besomes $-2\pi i \cdot h(t)$. This

$$Acon \lim_{\epsilon \to 0} H(x + i\epsilon) - H(x - i\epsilon) = -2\pi i \cdot h(t)$$

Next, we have

$$\text{(i)} \qquad H(x+i\epsilon)+H(x-i\epsilon)=2\cdot\int_0^1 \frac{x-t}{(x-t)^2+\epsilon^2}\cdot h(t)\,dt$$

When $0<x<1$ it is not always true that the right hand side has a limit as $\epsilon\to 0$. A counterexample os to take $h=0$ when $t\leq x$ while $h(t)=(\log(t-x))^{-1}$ when $t>x$. Moreover, from cacluus one learns that (i) has a limit at x if and only if there exists the limit

$$\lim_{\epsilon\to 0}\int_0^{x-\epsilon}\frac{h(t)}{x-t}\,dt+\int_{x+\epsilon}^1\frac{h(t)}{x-t}\,dt$$

and when the latter limit exists one refers to a principal value and writes

$$\text{PV}\int_0^1\frac{h(t)}{x-t}\,dt$$

Let us remark that this principal value exists for all x when h is a C^1-function on $[0,1]$.

Proof of Theorem 0.1

The complex log-function

$$z\mapsto\log(z-t)$$

is defined when $z\in\mathbf{C}\setminus(-\infty,1]$ for each $0\leq t\leq 1$. The single-valued branches are chosen so that the argument of these log-functions stay in $(-\pi,\pi)$ and $\log x-t$ is real if $x>t$. It follows that

$$\text{(1)} \qquad \lim_{\epsilon\to 0}\log(x+i\epsilon-t)=\log|x-t|+\pi i \quad : x<t$$

where the limit is taken as $\epsilon>0$ decrease to zero. Let $g(t)$ be a continuous function on $[0,1]$ and put

$$\text{(2)} \qquad G(z)=\int_0^1\log(z-t)\cdot g(t)\,dt$$

A. Exercise. Show that (1) gives:

$$\text{(i)} \qquad G(x+i0)=T_g(x)+\pi i\cdot\int_x^1 g(t)\,dt \quad : 0<x<1$$

where $G(x+i0)$ is the limit as $z=x+i\epsilon$ and $\epsilon>0$ decrease to zero. Show in a similar way that

$$\text{(ii)} \qquad G(x-i0)=T_g(x)-\pi i\cdot\int_x^1 g(t)\,dt \quad : 0<x<1$$

Next, outside $[0,1]$ the complex derivative of G becomes

$$G'(z)=\int_0^1\frac{g(t)}{z-t}\,dt$$

Then (i-ii) give

$$\text{(iii)} \qquad G'(x+i0)+G'(x-i0)=2\cdot\frac{T_g(x)}{dx} \quad : \quad G'(x+i0)-G'(x-i0)=-2\pi\cdot g(x)$$

B. The Φ-function. In $\mathbf{C}\setminus[0,1]$ we have the analytic function $h(z)=\sqrt{z(z-1)}$ whose branch is chosen so that it is real and positive when $z=x>1$. It follows that

$$\text{(b.1)} \qquad h(x+i0)=i\cdot\sqrt{x(1-x)} \quad : \quad h(x-i0)=i\cdot\sqrt{x(1-x)} \quad : 0<x<1$$

Consider the analytic function

$$\Phi(z)=\sqrt{z(z-1)}\cdot G'(z)$$

With $f=T_g$ we see that (iii-iv) give the two equations (b.2) $$\Phi(x+i0)+\Phi(x-i0)=2\pi\cdot\sqrt{x(1-x)}\cdot g(x)$$

(b.3) $$\Phi(x+i0)-\Phi(x-i0)=2i\cdot f'(x)\cdot\sqrt{x(1-x)}$$

C. The Ψ-function. Set

(c.1) $$\Psi(z)=\int_0^1\frac{1}{z-t}\cdot f'(t)\cdot\sqrt{t(1-t)}\,dt$$

The equation (b.3) and the general formula from § XX give

(c.2) $$\Phi(x+i0)-\Phi(x-i0)=\Psi(x+i0)-\Psi(x-i0)\quad:\ 0<x<1$$

D. Exercise. Deduce from (c.2) that

$$\Phi(z)=\Psi(z)+\int_0^1 g(t)\,dt$$

where the equality holds when $z\in\mathbf{C}\setminus(-\infty,1]$. Conclude from the above that

(d.1) $$2\pi\cdot\sqrt{x(1-x)}\cdot g(x)=\Psi(x+i0)+\Psi(x-i0)+2\cdot\int_0^1 g(t)\,dt$$

Next, from (c.1) and the general formula in § XX we have

(d.2) $$\Psi(x+i0)+\Psi(x-i0)=\frac{2}{\pi}\cdot\int_0^1\frac{f'(t)\cdot\sqrt{t(1-t)}}{x-t}\,dt$$

where the last integral is taken as a principal value. Together (d.1-2) give (*) in Theorem 0.1.

§ 1. An approximation theorem.

The result below was proved in the article *Sur un théorème de Weierstrass* [Arkiv för mathematik och fysik. vol 20 (1927]:

Theorem. *Let f be a continuous and complex valued function on the real x-line. To each $\epsilon > 0$ there exists an entire function $\phi(z) = \phi(x + iy)$ such that*

$$\max_{x \in \mathbf{R}} |f(x) - \phi(x)| < \epsilon$$

In the cited article Carleman gave an elementary proof using Cauchys integral formula. His constructions can be extended to cover a more general situation which goes as follows. Let K be an unbounded closed null-set in $\mathbf{C}$. If $0 < R < R^*$ we put

$$K[R, R^*] = K \cap \{R \leq |z| \leq R^*\}$$

and if $R > 0$ we put $K_R = K \cap \bar{D}_R$ where $\bar{D}_R = \{|z| \leq R\}$.

1.1 Theorem. *Suppose there exists a strictly increasing sequence $\{R_\nu\}$ where $R_\nu \to +\infty$ such that $\mathbf{C} \setminus K_{R_1}$ and the sets*

$$\Omega_\nu = \mathbf{C} \setminus \bar{D}_{R_\nu} \cup K[R_\nu, R_{\nu+1}]$$

are connected for each $\nu \geq 1$. Then every continuous function on K can be uniformly approximated by entire functions.

To prove this result we first establish the following.

1.2 Lemma. *Consider some $\nu \geq 1$ a continuous function ψ on $S = \bar{D}_{R_n} \cup K[R_\nu, R_{\nu+1}]$ where ψ is analytic in the open disc D_{R_ν}. Then ψ can be uniformly approximated on S by polynomials in z.*

Proof. If we have found a sequence of polynomials $\{p_k\}$ which approximate ψ uniformly on $S_* = \{|z| = R_\nu\} \cup K[R_\nu, R_{\nu+1}]$ then this sequence approximates ψ on S. In fact, this follows since ψ is analytic in the disc D_{R_ν} so by the maximum principle for analytic functions in a disc we have

$$||p_k - \psi||_S = ||p_k - \psi||_{S_*}$$

for each k. Next, if uniform approximation on S_* fails there exists a Riesz-measure μ supported by S_* which is $\perp$ to all analytic polynomials while

$$\int \psi \cdot d\mu \neq 0 \tag{1}$$

To see that this cannot occur we consider the Cauchy transform

$$\mathcal{C}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

Since $\int \zeta^n \cdot d\mu(\zeta) = 0$ for every $n \geq 0$ we see that $\mathcal{C}(z) = 0$ in the exterior disc $|z| > R_{\nu+1}$. The connectivity hypothesis implies that $\mathcal{C}(z) = 0$ in the whole open complement of S. Now K was a null set which means that the L^1_{loc}-function $\mathcal{C}(z)$ is zero in the exterior disc $|z| > R_\nu$ and hence its distribution derivative $\bar{\partial}(\mathcal{C}_\nu)$ also vanishes in this exterior disc. At the same time we have the equality

$$\bar{\partial}(C_\nu) = \mu$$

We conclude that the support of μ is confined to the circle $\{|z| = R_\nu\}$. But then (1) cannot hold since the restriction of ψ to this circle by assumption extends to be analytic in the disc D_{R_ν} and therefore can be uniformly approximated by polynomials on the circle.

Proof of Theorem 1.1 Let $\epsilon > 0$ and $\{\alpha_\nu\}$ is a sequence of positive numbers such that $\sum \alpha_\nu < \epsilon$. Consider some $f \in C^0(K)$. Starting with the set K_{R_1} we use the assumption that its complement is connected and using Cauchy transforms as in Lemma 1.2 one shows that the restriction of f to this compact set can be uniformly approximated by polynomials. So we find $P_1(z)$ such that

$$||P_1 - f||_{K_{R_1}} < \alpha_1 \tag{i}$$

From (i) one easily construct a continuous function ψ on $\bar{D}_{R_1} \cup K[R_1, R_2]$ such that $\psi = P_1$ holds in the disc $\bar{D}_{R_1}$ and the maximum norm

$$||\psi - f||_{K[R_1,R_2]} \leq \alpha_1$$

Lemma 1.2 gives a polynomial P_2 such that

$$||P_2 - P_1||_{D_{R_1}} < \alpha_2 \quad \text{and} \quad ||P_2 - f||_{K[R_1,R_2]} \leq \alpha_1 + \alpha_2$$

Repeat the construction where Lemma 1.2 is used as ν increases. This gives a sequence of polynomials $\{P_\nu\}$ such that

$$||P_\nu - P_{\nu-1}||_{D_{R_\nu}} < \alpha_\nu \quad \text{and} \quad ||P_\nu - f||_{K[R_{\nu-1},R_\nu]} < \alpha_1 + \ldots + \alpha_\nu$$

hold for all ν. From this it is easily seen that we obtain an entire function

$$P^*(z) = P_1(z) + \sum_{\nu=1}^{\infty} P_{\nu+1}(z) - P_\nu(z)$$

Finally the reader can check that the inequalities above imply that the maximum norm

$$||P^* - f||_K \leq \alpha_1 + \sum_{\nu=1}^{\infty} \alpha_\nu$$

Since the last sum is $\leq 2\epsilon$ and $\epsilon > 0$ was arbitrary we have proved Theorem 1.1

1.3 Exercise. Use similar methods as above to show that if $f(z)$ is analytic in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$ and has continuous boundary values on the real line, then f can be uniformly approximated by an entire function, i.e. to every $\epsilon > 0$ there exists an entire function $F(z)$ such that

$$\max_{z \in U^+} |F(z) - f(z)| \leq \epsilon$$

2. An inequality for differentiable functions.

A fundamental result was proved by Carleman in the article *Sur un théorème de M. Denjoy* [C.R. Acad. Sci. Paris 1922]

2.1 Theorem. *There exists an absolute constant $\mathcal{C}$ such that the inequality below holds for every pair (f,n), where n is a positive integer and f a non-negative real-valued C^∞- function defined on the closed unit interval $[0,1]$ whose derivatives up to order n vanish at the two end points.*

$$(*)\qquad \sum_{\nu=1}^{\nu=n} \frac{1}{[\beta_\nu]^{\frac{1}{\nu}}} \leq \mathcal{C}\cdot\int_0^1 f(x)dx \quad : \quad \beta_\nu = \sqrt{\int_0^1 [f^{(\nu)}(x)]^2\cdot dx}$$

Remark. The proof below shows that one can take

$$(*)\qquad C \leq 2e\pi\cdot(1+\frac{1}{4\pi^2e^2-1})$$

The best constant $\mathcal{C}_*$ which would give

$$\text{(i)}\qquad \sum_{p=1}^{p=n}\frac{1}{\beta_p(f)} \leq \mathcal{C}^*$$

for all n and every $f\in\mathcal{S}_n$ is not known. Let us also remark that the inequality (*) is sharp in the sense that there exists a constant C_* such that for every $n\geq 2$ there exists a function $f_n(x)$ as above so that the opposed inequality (*) holds with C_*. Hence (*) demonstrates that the standard cut-off functions which are used in many applications to keep maximum norms of derivatives small up to order n small, are optimal up to a constant. So the theoretical result in (*) plays a role in numerical analysis where one often uses smoothing methods. The proof of (*) employs estimates for harmonic measures applied to the subharmonic Log-function of the absolute value of the Laplace transform of f, i.e. via a "detour into the complex domain" which in 1922 appeared as a "revolutionary method".

Proof of Theorem 2.1.

Let $n\geq 1$ and keeping f fixed we put $\beta_p=\beta_p(f)$ to simplify notations. Using partial integrations and the Cauchy-Schwarz inequality one shows that the β-numbers are non-decreasing. i.e.

$$(*)\qquad 1=\beta_0\leq\beta_1\leq\ldots\leq\beta_n$$

Define the complex Laplace transform

$$\Phi(z)=\int_0^1 e^{-zt}f(t)dt$$

Since f by assumption is n-flat at the end-ponts, integration by parts p times gives:

$$\Phi(z)=z^{-p}\int_0^1 e^{-zt}\cdot\partial^p(f^2)(t)dt \quad : \quad 1\leq p\leq n+1$$

where $\partial^p(f^2)$ is the derivative of order p of f^2. We have

$$(1)\qquad \partial^p(f^2)=\sum_{\nu=0}^{\nu=p}\binom{p}{\nu}\cdot f^{(\nu)}\cdot f^{(p-\nu)} \quad : \; 1\leq p\leq n+1$$

Now we study the absolute value of Φ on the vertical line $\mathfrak{Re}(z)=-1$. Since $|e^{t-iyt}|=e^t$ for all y, the triangle inequality gives

$$(2)\qquad |-1+iy|^p\cdot|\Phi(-1+iy)|\leq\sum_{\nu=0}^{\nu=p}\binom{p}{\nu}\cdot\int_0^1 e^t\cdot|f^{(\nu)}(t)|\cdot|f^{(p-\nu)}(t)|\cdot dt$$

Since $e^t\leq e$ on $[0,1]$, the Cauchy-Schwarz inequality and the definition of the β-numbers give:

$$(3)\qquad |-1+iy|^p\cdot|\Phi(-1+iy)|\leq e\cdot\sum_{\nu=0}^{\nu=p}\binom{p}{\nu}\cdot\beta_\nu^\nu\cdot\beta_{p-\nu}^{p-\nu}$$

From (*) it follows that $\beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu} \leq \beta_p^p$ for each ν and since $\sum_{\nu=0}^{\nu=p} \binom{p}{\nu} = 2^p$ we obtain

$$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot 2^p \cdot \beta_p^p \tag{4}$$

Passing to the logarithm we get

$$\log |\Phi(-1+iy)| \leq 1 + p \cdot \log \frac{2\beta_p}{|-1+iy|} \tag{5}$$

Here (5) holds when $1 \leq p \leq n+1$ and the assumption that $\beta_0 = 1$ also gives

$$\log |\Phi(-1+iy)| \leq 1 \tag{6}$$

The ω-function. To each $1 \leq p \leq n+1$ we find a positve number y_p such that

$$|-1+iy_p| = 2e\beta_p$$

Now we define a function $\omega(y)$ where $\omega(y) = 0$ when $y < y_1$ and

$$\omega(y) = p \quad : \quad y_p \leq y < y_{p+1}$$

and finally $\omega(y) = n+1$ when $y \geq y_{n+1}$. Then (5-6) give the inequality

$$\log |\phi(-1+iy)| \leq 1 - \omega(y) \quad : \quad -\infty < y < +\infty \tag{7}$$

A harmonic majorisation. With $1 - \omega(y)$ as boundary function in the half-plane $\mathfrak{Re}(z) > -1$ we construct the harmonic extension $H(z)$ which by Poisson's formula is given by:

$$H(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\omega(y)}{1+y^2} \cdot dy$$

Now $\log |\Phi(z)|$ is subharmonic in this half-plane and hence (7) gives:

$$0 = \log |\Phi(0)| \leq H(0)$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{\omega(y)}{1+y^2} \cdot dy \leq \pi \tag{8}$$

Since $\omega(y) = 0$ when $y \leq y_1$ we see that (8) gives the inequality

$$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy \leq \frac{y_1^2}{1+y_1^2} \cdot \pi \tag{9}$$

The construction of the ω-function gives the equation

$$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy = \frac{1}{y_1} + \ldots + \frac{1}{y_{n+1}} \tag{10}$$

Next, the construction of the y_p-numbers entail that $y_p \leq 2e\beta_p$ so (9-10) give

$$\frac{1}{\beta_1} + \ldots + \frac{1}{\beta_{n+1}} \leq 2e\pi \cdot \frac{1}{1+\frac{1}{y_1^2}} \tag{11}$$

Finally, we have $1 + y_1^2 = 4e^2\beta_1^2$ and recall that Wirtinger's inequality gives $\beta_1 \geq \pi$. Hence

$$\frac{1}{1+\frac{1}{y_1^2}} \leq 1 + \frac{1}{4\pi^2 e^2 - 1} \tag{12}$$

and then (11-12) give the requested inequality in Theorem 2.1.

§ 3. An inequality for inverse Fourier transforms in $L^2(\mathbf{R}^+)$.

By Parsevel's theorem the Fourier transform sends L^2-functions on the real ξ-line to L^2-functions on the x-line. We seek the class of non-negative L^2-functions $\phi(x)$ such that there exists an L^2-function $F(\xi)$ supported by the half-line $\xi \geq 0$ and

$$\phi(x) = \Big| \int_0^\infty e^{ix\xi} \cdot F(\xi) \cdot d\xi \Big| \tag{*}$$

The theorem below was proved in [Carleman] which apart from applications to quasi-analytic functions has several other consequences which are put forward by Paley and Wiener in their text-book [Pa-Wi].

3.1 Theorem. *An L^2-function $\phi(x)$ is of the form (*) if and only if*

$$\int_{-\infty}^\infty \log^+ \Big[\frac{1}{\phi(x)} \Big] \cdot \frac{dx}{1+x^2} < \infty \tag{i}$$

Moreover, when (*) holds and $F(\xi)$ satisfies the weighted mean-value equality

$$\int_0^\infty F(\xi) \cdot e^{-\xi}\, d\xi = 1 \tag{ii}$$

then

$$\int_{-\infty}^\infty \log^+ \Big[\frac{1}{\phi(x)} \Big] \cdot \frac{dx}{1+x^2} \leq \int_{-\infty}^\infty \frac{\phi(x)^2}{1+x^2} \cdot dx \tag{iii}$$

Proof. First we prove the suffiency. Let $\phi(x)$ be a non-negative L^2-function where the integral (i) is finite. The harmonic extension of $\log \phi(x)$ to the upper half-plane is given by:

$$\lambda(x+iy) = \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\log\ \phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0 \tag{1}$$

Let $\mu(z)$ be the conjugate harmonic function of λ and set

$$h(z) = e^{\lambda(z)+i\mu(z)} \tag{2}$$

Fatou's theorem gives for almost every x a limit

$$\lim_{y \to 0} \lambda(x+iy) = \log\ \phi(x) \tag{3}$$

Or, equivalently

$$\lim_{y \to 0} |h(x+iy)| = \phi(x) \tag{4}$$

From (1) and the fact that the geometric mean value of positive numbers cannot exceed their arithmetic mean value, one has

$$\big|h(x+iy)\big| = e^{\lambda(x+iy)} \leq \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0 \tag{5}$$

Then (5) the Schwarz inequality give:

$$\int_{-\infty}^\infty \big|h(x+iy)\big|^2 dx \leq \int_{-\infty}^\infty \big|\phi(x)\big|^2 dx \quad : y > 0 \tag{6}$$

Here $h(z)$ is analytic in the upper half-plane so that (6) and Cauchy's formula entail that if $\xi < 0$, then the integrals

$$J(y) = \int_{-\infty}^\infty h(x+iy) \cdot e^{-ix\xi + y\xi} \cdot dx \quad : \ y > 0 \tag{7}$$

are independent of y. Passing to the limit as $y \to \infty$ and using the uniform upper bounds on the L^2-norms of the functions $h_y(x) \mapsto h(x+iy)$, it follows that $J(y)$ vanishes identically. So the

Fourier transforms of $h_y(x)$ are supported by $\xi \geq 0$ for all $y > 0$. Passing to the limit as $y \to 0$ the same holds for the Fourier transform of $h(x)$. Finally (4) gives

$$\phi(x) = |h(x)| \tag{8}$$

By Parseval's theorem $\widehat{h}(\xi)$ is an L^2-function and hence $\phi(x)$ has the requested form (*).

Necessity. Since F is in L^2 there exists the Plancherel limit

$$\psi(x) = \lim_{N\to\infty} \frac{1}{2\pi} \cdot \int_0^N e^{ix\xi} \cdot F(\xi)\, d\xi \tag{9}$$

and in the upper half plane we get the analytic function

$$\psi(x+iy) = \frac{1}{2\pi} \cdot \int_0^\infty e^{ix\xi - y\xi} \cdot F(\xi)\, d\xi \tag{10}$$

Suppose that $F(\xi)$ satisfies (ii) in the Theorem which gives

$$\psi(i) = 1$$

Consider the conformal map from the upper half-plane into the unit disc where

$$w = \frac{z-i}{z+i}$$

Here $\psi(x)$ corresponds to a function $\Phi(e^{is})$ on the unit circe $|w| = 1$ and:

$$\int_{-\pi}^{\pi} |\Phi(e^{is})|^2\, ds = 2 \cdot \int_{-\infty}^{\infty} \frac{|\phi(x)|^2}{1+x^2}\, dx \tag{11}$$

Similarly let $\Psi(w)$ be the analytic function in $|w| < 1$ which corresponds to $\psi(z)$. From (10-11) it follows that $\Psi(w)$ is the Poisson extension of Φ, i.e.

$$\Psi(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-|w|^2}{|e^{is}-w|^2} \cdot \Phi(e^{is}) \cdot ds \tag{12}$$

If $0 < r < 1$ it follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}| \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\Psi(re^{is}\right|^2 \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{is}|^2 \cdot ds \tag{13}$$

Now (12) gives:

$$\lim_{r\to 1} \Psi(re^{is}) = \Phi(e^{is}) \quad : \text{almost everywhere} \quad 0 \leq s \leq 2\pi \tag{14}$$

Next, since $\psi(i) = 1$ we have $\Psi(0) = 1$ which gives the inequality

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{|\Psi(re^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}| \cdot ds \quad : 0 < r < 1 \tag{15}$$

By (13-15) a passage to the limit as $r \to 1$ gives

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{|\Phi(e^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \left| \Phi(e^{is}\right|^2 \cdot ds \tag{16}$$

Returning to the real x-line the inequality (iii) follows which at the same time finishes the proof of the theorem.

§ 4. The Bergman kernel.

Let Ω be a bounded and simply connected domain in $\mathbf{C}$. If $a \in \Omega$ a famous result due to Stefan Bergman gives the conformal mapping function $f_a \colon \Omega \to D$ via the kernel function of the Hilbert space $H^2(\Omega)$.

A. Bergman's Theorem. *The conformal map f_a is given by*

$$f_a(z) = \sqrt{\frac{\pi}{K(a,a)}} \cdot \int_a^z K(z,a)\, dz$$

Next, the Gram-Schmidt construction gives an orthonormal basis $\{P_n(z)\}$ in $H^2(\Omega)$ where P_n has degree n and

$$\iint_\Omega P_k \cdot \bar{P}_m \cdot dxdy = \text{Kronecker's delta function}$$

From Bergman's result one expects that these polynomials are related to a conformal mapping function. We shall consider the case when Ω is a Jordan domain whose boundary curve Γ is *real-analytic*. Let ϕ be the conformal map from the *exterior domain* $\Omega^* = \Sigma \setminus \bar{\Omega}$ onto the exterior disc $|z| > 1$. Here ϕ is normalised so that it maps the point at infinity into itself. The inverse conformal mapping function ψ is defined in $|z| > 1$ and has a series expansion

$$(*) \qquad \psi(z) = \tau \cdot z + \tau_0 + \sum_{\nu=1}^{\infty} \tau_\nu \cdot \frac{1}{z^\nu}$$

where τ is a positive real number. The assumption that Γ is real-analytic gives some $\rho_1 < 1$ such that ψ extends to a conformal map from the exterior disc $|z| > \rho_1$ onto a domain whose compact complement is contained in Ω.

It turns out that the polynomials $\{P_n\}$ are approximated by functions expressed by ϕ and is complex derivative on $\partial\Omega$. Inspired by Faber's article *Über Tschebyscheffsche Polynome* [Crelle. J. 1920], Carleman proved an asymptotic result in the article *Über die approximation analytischer funktionen durch linearen aggregaten von vorgegebenen potenzen* [Arkiv för matematik och fysik. 1920].

B. Theorem. *There exists a constant C which depends upon Ω only such that to every $n \geq 1$ there is an analytic function $\omega_n(z)$ defined in Ω^* and*

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \cdot \phi'(z) \cdot \phi(z)^n \cdot \big(1 + \omega_n(z)\big) \quad : z \in \partial\Omega$$

where

$$\max_{z \in \partial\Omega} |\omega_n(z)| \leq C \cdot \sqrt{n} \cdot \rho_1^n \quad : \quad n = 1, 2, \ldots$$

Proof.

For each $n \geq 2$ we denote by $\mathcal{M}_n$ the space of monic polynomials of degree n:

$$Q(z) = z^n + b_{n-1} z^{n-1} + \ldots + b_0$$

Put

$$I(Q) = \iint_\Omega |Q(z)|^2\, dxdy$$

and with n kept fixed we set

$$I_*(n) = \min_{Q \in \mathcal{M}_n} I(Q)$$

To each Q we introduce the primitive polynomial

$$\widehat{Q}(z) = \frac{z^{n+1}}{n+1} + \frac{b_{n-1}}{n} z^n + \ldots + b_0 z$$

1. Exercise. Use Green's formula to show that

$$I(Q) = \frac{1}{4}\int_{\partial\Omega} |\partial_n(\widehat{Q})|^2 \, ds$$

where ds is the arc-length measure on $\partial\Omega$ and we have taken the outer normal derivative of $\widehat{Q}$. Next, take the inverse conformal map $\psi(\zeta)$ in (*) and set

$$F(\zeta) = \widehat{Q}(\psi(\zeta))$$

Then F is analytic in the exterior disc $|\zeta| > 1$ and by (*) above, F has a series expansion

$$F(\zeta) = \tau^{n+1}\big[\frac{\zeta^{n+1}}{n+1} + A_n\zeta^n + \ldots + A_1\zeta + A_0 + \sum_{\nu=1}^{\infty} \alpha_\nu \cdot \zeta^{-\nu}\big] \tag{1.1}$$

2. Exercise. Use a variable substitution via ψ to show that

$$I(Q) = \int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2) \, d\theta$$

Show also that the series expansion (1.1) identifies the right hand side with

$$\pi \cdot \tau^{2n+2} \cdot [\frac{1}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 - \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2\,] \tag{2.1}$$

3. An upper bound for $I_*(n)$. In (2.1) the coefficients $A_1, \ldots, A_n$ are determined via Q and the reader may verify that there exists $Q \in \mathcal{M}_n$ such that $A_1 = \ldots = A_n = 0$. It follows that

$$I_*(n) \leq \pi \cdot \tau^{2n+2} \cdot [\frac{1}{n+1} - \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2\,] \leq \pi \cdot \tau^{2n+2} \cdot \frac{1}{n+1} \tag{3.1}$$

4. A lower bound for I_*. The upper bound (3.1) did not use that $\partial\Omega$ is real- analytic, i.e. (3.1) is valid for every Jordan domain whose boundary curve is of class C^1. To get a lower bound we use the constant $\rho - 1 < 1$ from the above and choose $\rho_1 < \rho < 1$. Now ψ maps the exterior disc $|\zeta| > \rho$ conformally to an exterior domain $U^* = \Sigma \setminus \bar{U}$ where U is a relatively compact Jordan domain inside Ω. Choose $Q_n \in \mathcal{M}_n$ so that

$$I(Q_n) = I_*(n)$$

Since $\Omega \setminus \bar{U} \subset \Omega$ we have

$$I_* > \iint_{\Omega\setminus\bar{U}} |Q_n(z)|^2 \, dxdx \tag{4.1}$$

5. Exercise. Show that (4.1) is equal to

$$\int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot d\theta - \int_{|\zeta|=\rho} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot \rho \cdot d\theta =$$

$$\pi \cdot \tau^{2n+2} \cdot [\frac{1-\rho^{2n+2}}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1)\,]$$

and conclude that one has the lower bound

$$I_*(n) \geq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot (1-\rho^{2n+2}) \tag{5.1}$$

Together (4.1) and (5.1) give the inequality

$$\sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1) \,\leq\, \frac{\pi}{n+1} \cdot \rho^{2n+2} \tag{5.2}$$

Since $1-\rho^2 \leq 1-\rho^{2\nu}$ for every $\nu \geq 1$ it follows that

$$\sum_{k=1}^{k=n} k \cdot |A_k|^2 + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \leq \frac{\pi}{(1-\rho^2)\cdot n+1} \cdot \rho^{2n+2} \tag{5.3}$$

6. Conclusion. Recall that $F(\zeta) = \widehat{Q_n}(\psi(\zeta)$. So after a derivation we get

$$F'(\zeta) = \psi'(\zeta) \cdot Q_n(\psi(\zeta))$$

Hence the series expansion of $F(\zeta)$ gives

$$Q_n(\psi(\zeta)) = \frac{\tau^{n+1}}{\psi'(\zeta)} \cdot \big[\zeta^n + \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}\,\big] \tag{6.1}$$

where the equality holds for $|\zeta| > \rho$. Put

$$\omega^*(\zeta) = \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}$$

When $|\zeta| = 1$ the triangle inequality gives

$$|\omega^*(\zeta)| \leq \sum_{k=1}^{k=n} k \cdot |A_k| + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu| \tag{6.2}$$

7. Exercise. Notice that (5.3) holds for every $\rho > \rho_1$ and use this together with suitable Cauchy-Schwarz inequalities to show that (6.1) gives a constant C which is independent of n such that

$$|\omega^*(\zeta)| \leq C \cdot \sqrt{n} \cdot \rho_1^{n+1} \tag{7.1}$$

Final part of the proof. Since ψ is the inverse of ϕ we have

$$\psi'(\phi(z)) \cdot Q_n(\psi(\phi(z) = \frac{Q_n(z)}{\phi'(z)}$$

Define the function on $\partial\Omega$ by

$$\omega_n(z) = \frac{\omega^*(\phi(z))}{\phi'(z)} \tag{i}$$

Then (6.2) gives

$$Q_n(z) = \tau^{n+1} \cdot \phi'(z) \cdot [\phi(z)^n + \omega_n(z)\,] \tag{ii}$$

where Exercise 7 shows that $|\omega_n(z)|$ satisfies the estimate in Theorem 2. Finally, the polynomial Q_n minimized the L^2-norm under the constraint that the leading term is z^n and for this variational problem the upper and the lower bounds in (4.1-5.1) imply that

$$|I_*(n) - \frac{\pi}{n+1} \cdot \tau^{2n+2}| \leq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot \rho^{2n+2}$$

If we normalise Q_n so that its L^2-norm is one gets a polynomial $P_n(z)$ where the factor τ^{n+1} is replaced by $\frac{\sqrt{n+1}}{\sqrt{\pi}}$ which finishes the proof of Theorem B.

§ 5. Fourier series and convergence of arithmetical means

Let $f(x)$ be a real-valued and square integrable function on $(-\pi, \pi)$, i.e.

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$$

We say that f has a determined value $A = f(0)$ at $x = 0$ if the following two conditions hold:

$$\text{(i)} \qquad \lim_{\delta \to 0} \frac{1}{\delta} \cdot \int_0^{\delta} |f(x) + f(-x) - 2A| \, dx = 0$$

$$\text{(ii)} \qquad \int_0^{\delta} |f(x) + f(-x) - 2A|^2 \, dx \leq C \cdot \delta \quad \text{holds for some constant} \quad C$$

Remark. In the same way we can impose this condition at every point $-\pi < x_0 < \pi$. To simplify the subsequent notations we take $x = 0$. If $x = 0$ is a Lebesgue point for f and A the Lebesgue value we have (i). Hence Lebesgue's Theorem entails that (i) holds almost everywhere when $x = 0$ is replaced by other points x_0. We leave it to the reader to show that the second condition also is valid almost everywhere when f is square integrable but in general there appears a null set $\mathcal{N}$ where (ii) fails to hold while $\mathcal{N}$ contains some Lebegue points. Next, expand f in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

and with $x = 0$ we consider the partial sums

$$s_n(0) = \frac{a_0}{2} + a_1 + \ldots + a_n + b_1 + \ldots + b_n$$

The result below is proved in [Carleman] and shows that $\{s_n\}$ are close to the determined value for many n-values.

5.1 Theorem. *Assume that f has a determined value A at $x = 0$. Then the following hold for every positive integer k*

$$(*) \qquad \lim_{n \to \infty} \frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} |s_\nu - A|^k = 0$$

Remark. Recall the famous theorem by Lennart Carleson which asserts that $\{s_n(x)\}$ converge to $f(x)$ almost everywhere for each $f \in L^2$. When pointwise convergence holds the limit formula (*) is obvious. However, it is in general not true that the pointwise convergence exists at *every point* where f has a determined value. So "ugly points" may appear in a null-set where pointwise convergence fails and here Carleman's result offers a substitute.

The case when $f \in \mathbf{BMO}(T)$. If f has bounded mean oscillation it is wellknown that (i-ii) hold at every Lebesgue point of f. So here one has a control for averaged Fourier series of f expressed via its set of Lebesgue points.

The case when f is continuous. Here (i-ii) hold everywhere so the averaged limit formulas hold at every point. We can say more since f is uniformly continuous. Let $\omega_f(\delta)$ be the modulos of continuity function and for each $n \geq 1$, $||s_n - f||$ is the maximum norm of $s_n - f$ over $[0, 2\pi]$. Set

$$\mathcal{D}_n(f) = \sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} ||s_\nu - f||^2}$$

5.2 Theorem. *There exists an absolute constant K such that the following hold for every continuous function f with maximum norm ≤ 1:*

$$\mathcal{D}_n(f) \leq K \cdot \left[\frac{1}{\sqrt{n}} + \omega_f\left(\frac{1}{n}\right)\right]$$

Proof of Theorem 5.1

Set $A = f(0)$ and $s_n = s_n(0)$. Introduce the function:

$$\phi(x) = f(x) + f(-x) - 2A$$

Applying Dini's kernel we have

$$s_n - A = \int_0^\pi \frac{\sin(n+1/2)x}{\sin x/2} \cdot \phi(x) \cdot dx$$

By trigonometric formulas the integral is expressed by three terms for each $0 < \delta < \pi$:

$$\alpha_n = \frac{1}{\pi} \cdot \int_0^\delta \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx$$

$$\beta_n = \frac{1}{\pi} \cdot \int_\delta^\pi \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx$$

$$\gamma_n = \frac{1}{\pi} \cdot \int_0^\pi \cos nx \cdot \phi(x) \cdot dx$$

By Hölder's inequality it suffices to show Theorem F.1 if $k = 2p$ is an even integer. Minkowski's inequality gives

$$(1) \qquad \Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq \Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p}$$

Denote by $o(\delta)$ small ordo and $O(\delta)$ is big ordo. When $\delta \to 0$ we shall establish the following:

$$\text{(i)} \qquad \Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} = n^{1+1/2p)} \cdot o(\delta)$$

$$\text{(ii)} \qquad \Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} \leq K \cdot p \cdot \delta^{-1/2p}$$

$$\text{(iii)} \qquad \Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p} \leq K$$

In (ii-iii) K is an absolute constant which is independent of p, n and δ. Let us first see why (i-iii) give Theorem F.1. Write $o(\delta) = \epsilon(\delta) \cdot \delta$ where $\epsilon(\delta) \to 0$. With these notations (1) gives:

$$(*) \qquad \Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq n^{1+1/2p} \cdot \delta \cdot \epsilon(\delta) + Kp \cdot \delta^{-1/2p} + K$$

Next, let $\rho > 0$ and choose b so large that

$$pKb^{-1/2p} < \rho/3$$

Take $\delta = b/n$ and with n large it follows that $\epsilon(\delta)$ is so small that

$$b \cdot \epsilon(b/n) < \rho/3$$

Then right hand side in (*) is majorized by

$$\frac{2\rho}{3} \cdot n^{1/2p} + K$$

When n is large we also have

$$K \leq \frac{\rho}{3} \cdot n^{1/2p}$$

Hence the left hand side in (*) is majorized by $\rho \cdot n^{1/2p}$ for all sufficiently large n. Since $\rho > 0$ was arbitrary we get Theorem F.1 when the power is raised by $2p$.

Proof of (i-iii)

To obtain (i) we use the triangle inequality which gives the following for every integer $\nu \geq 1$:

(1) $$|a_\nu| \leq \frac{2}{\pi} \cdot \max_{0 \leq x \leq \delta} |\sin \nu x \cdot \cot x/2| \cdot \int_0^\delta |\phi(x)|\, dx = \nu \cdot o(\delta)$$

where the small ordo δ-term comes from the hypothesis expressed by (*) in the introduction. Hence the left hand side in (i) is majorized by

$$\big[\sum_{\nu=1}^{\nu=n} \nu^{2p}\big]^{\frac{1}{2p}} \cdot o(\delta) = n^{1+1/2p} \cdot o(\delta)$$

which was requested to get (i). To prove (iii) we notice that

$$\gamma_0^2 + 2 \cdot \sum_{\nu=1}^\infty \gamma_\nu^2 = \frac{1}{\pi} \int_0^\pi |\phi(x)|^2\, dx$$

Next, we have

$$\sum_{\nu=1}^\infty |\gamma_\nu|^{2p} \leq \big[\sum_{\nu=1}^\infty |\gamma_\nu|^2\big]^{1/2p} \leq K$$

where K exists since ϕ is square-intergable on $[0, \pi]$.

Proof of (ii). Here several steps are required. For each $0 < s < \pi$ we define the function $\phi_s(x)$ by

$$\phi_s(x) = \phi(x) \quad : \quad 0 < x < s$$

and extend it to an odd function, i.e. $\phi_s(-x) = -\phi_s(x)$ while $\phi_s(x) = 0$ when $|x| > s$. This odd function has a sine series

(1) $$\phi_s(x) = \sum_{\nu=1}^\infty c_\nu(s) \cdot \sin x$$

Let us also introduce the functions

(2) $$\rho(s) = \int_0^s |\phi(x)| \cdot dx \quad \text{and} \quad \Theta(s) = \int_0^s |\phi(x)|^2 \cdot dx$$

The crucial step towards the proof of (ii) is the following:

Lemma. *One has the inequality*

$$\sum_{|nu=1}^\infty |c_\nu(s)|^{2p} \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot \rho(s)^{2p-2}$$

Proof. We employ convolutions and define inductively a sequence of functions $\{\phi_{n,s}(x)\}$ where $\phi_{1,s}(x) = \phi_s(x)$ and

$$\phi_{n+1,s}(x) = \frac{1}{\pi} \int_{-\pi}^\pi \phi_{n,s}(y)\phi_s(x+y) \cdot dy$$

Since convolution yield products of the Fourier coefficients and $2p$ is an even integer we have the standard formula:

(1) $$\sum_{\nu=1}^\infty c_n(s)^{2p} = \phi_{2p,s}(0)$$

Next, using the Cauchy-Schwarz inequality the reader may verify that

$$|\phi_{2,s}(x)| \leq \frac{2}{\pi} \cdot \Theta(x)$$

This entails that

$$\phi_{3,s}(x) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |\phi_{2,s}(y)| \cdot |\phi_s(x+y)| \cdot dy \leq \frac{2}{\pi^2} \cdot \Theta(s) \cdot \int_{-\pi}^{\pi} |\phi_s(x+y)| \cdot dy = (\frac{2}{\pi})^2 \cdot \Theta(s) \cdot \rho(s)$$

Proceeding in this way it follows by an induction that

$$\phi_{2p,s}(x) \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot (\rho(s))^{2p-2}$$

This holds in particular when $x = 0$ and then (1) above gives Lemma 1.

A formula for the β-numbers. We have by definition

$$\beta_\nu = \frac{2}{\pi}\int_{\delta}^{\pi} \sin\, \nu x \cdot \frac{1}{2}\cot(\frac{x}{2}) \cdot \phi(x) \cdot dx$$

An integration by parts and the construction of the Fourier coefficients $\{c_\nu(s)\}$ which applies with $s = \delta$ give:

$$(*) \qquad \beta_\nu = -\frac{1}{2} \cdot \cot \delta/2 \cdot c_\nu(\delta) + +\frac{1}{4}\int_{\delta}^{\pi} c_\nu(x) \cdot \mathrm{cosec}^2(\frac{x}{2}) \cdot dx$$

Now we profit upon Minkowski's inequality. Let q be the conjugate of $2p$, i.e $\frac{1}{q} + \frac{1}{2p} = 1$ and choose $\{\xi_\nu\}$ to be the sequence in ℓ^q of unit norm such that

$$|\sum \xi_\nu \cdot \beta_\nu| = ||\beta_\bullet||_{2p}$$

where the last term is the left hand side in (ii). At the same time (*) above and the triangle inequality give

$$||\beta_\bullet||_{2p} \leq -\frac{1}{2} \cdot \cot(\delta/2) \cdot \sum |c_\nu(\delta)| \cdot |\xi_\nu| + \frac{1}{4}\int_{\delta}^{\pi} \mathrm{cosec}^2(\frac{x}{2}) \cdot \sum |c_\nu(x) \cdot \xi_\nu| \cdot dx \leq$$

$$(**) \qquad \frac{1}{2} \cdot \cot(\delta/2) \cdot ||c_\bullet(\delta)||_{2p} + \frac{1}{4}\int_{\delta}^{\pi} \mathrm{cosec}^2(\frac{x}{2}) \cdot ||c_\bullet(x)||_{2p} \cdot dx$$

At this stage we apply Lemma 1 and the assumption which give a constant K such that

$$\Theta(s) \leq K \quad \text{and} \quad \rho(s) \leq K \cdot s$$

The last estimate actually is weaker than the hypothesis but it will be sufficient to get the requested estimate of the ℓ^{2p}-norm in (ii). Lemma 1 gives a constant K_1 such that

$$||c_\bullet(\delta)||_{2p} \leq K_1 \cdot \delta^{1-1/p}$$

At the same time we have a constant K_2 such that

$$\cot(\delta/2) \leq \frac{K_2}{\delta}$$

The product in the first term from (**) is therefore majorized by $K_1K_2 \cdot \delta^{-1/2p}$ as requested in (ii). For the second term we use Lemma 1 which first gives

$$||c_\bullet(x)||_p \leq K \cdot x^{-1/2p}$$

At this stage we leave it to the reader to verify that we get a constant K so that

$$\int_{\delta}^{\pi} x^{-1/2p} \cdot \mathrm{cosec}^2(\frac{x}{2}) \cdot dx \leq K \cdot \delta^{-1/2p}$$

which finishes the proof of (ii).

The case when f is continuous.

Under the normalisation that the L^2-integral of f is ≤ 1 the inequalities (ii-iii) hold for an absolute constant K. In (i) we notice that the construction of ϕ and the definition of ω_f give the estimates

$$|a_\nu| \leq \nu \cdot \delta \cdot \omega_f(\delta)$$

With $p = 2$ this entails that (i) from the proof of Theorem F.1 is majorised by

$$n^{1+1/2} \cdot \delta \cdot \omega_f(\delta)$$

This holds for every $0 \leq x \leq 2\pi$ and from the previous proof we conclude that the following hold for each $n \geq 2$ and every $0 < \delta < \pi$:

(i) $$\mathcal{D}_n(f) \leq \frac{1}{\sqrt{n+1}} \cdot [n^{1+1/2} \cdot \delta \cdot \omega_f(\delta) + 2K\delta^{-1/2} + K]$$

With $n \geq 2$ we take $\delta = n^{-1}$ and see that (i) gives a requested constant in Theorem 5.2.

§ 6. An inequality for resolvents.

Introduction Theorem 6.1 below was proved by Carleman in the article *Sur le genre du denominateur $D(\lambda)$ de Fredholm* from 1917 and used to study integral equations expressed by Hilbert-Schmidt kernels.

The Hilbert-Schmidt norm. It is defined for an $n \times n$-matrix $A = \{a_{ik}\}$ by:

$$\mathfrak{h}(A) = \sqrt{\sum\sum |a_{ik}|^2}$$

where the double sum extends over all pairs $1 \leq i, k \leq n$. Notice that

$$\mathfrak{h}(A)^2 = \sum_{i=1}^{i=n} ||A(e_i)||^2$$

where $e_1, \ldots, e_n$ can be taken as an arbitrary orthogonal basis in $\mathbf{C}^n$. Next, for a linear operator S on $\mathbf{C}^n$ its *operator norm* is defined by

$$||S|| = \max_x ||S(x)|| \quad \text{with the maximum taken over unit vectors.}$$

6.1 Theorem. *Let $\lambda_1, \ldots, \lambda_n$ be the roots of $P_A(\lambda)$ and $\lambda \neq 0$ is outside $\sigma(A)$. Then one has the inequality:*

$$|\prod_{i=1}^{i=n} \big(1 - \frac{\lambda_i}{\lambda}\big) \cdot e^{\lambda_i/\lambda}\big| \cdot ||R_A(\lambda)|| \leq |\lambda| \cdot e^{\frac{1}{2}\big(1+\frac{\mathfrak{h}(A)^2}{|\lambda|^2}\big)}$$

The proof requires some preliminary results. First we need inequality due to Hadamard which goes as follows:

6.2 Hadamard's inequality. *For every matrix A with a non-zero determinant one has the inequality*

$$\big|\det(A)\big| \cdot ||A^{-1}|| \leq \frac{\mathfrak{h}(A)^{n-1}}{(n-1)^{n-1)/2}}$$

Exercise. Prove this result. The hint is to use expansions of certain determinants while one considers $\det(A) \cdot \langle A^{-1}(x), y\rangle$ for all pairs of unit vectors x and y.

6.3 Traceless matrices. Let A be an $n \times n$-matrix. The trace is by definition given by:

(i) $$\mathrm{Tr}(A) = b_{11} + \ldots + b_{nn}$$

Recall that $-\mathrm{Tr}(A)$ is equal to the sum of the roots of $P_A(\lambda)$. In particular the trace of two equivalent matrices are equal. This will be used to prove the following:

6.4 Theorem. *Let A be an $n \times n$-matrix whose trace is zero. Then there exists a unitary matrix U such that the diagonal elements of U^*AU all are zero.*

Proof. Consider first consider the case $n = 2$. By Theorem 4.0.7 it suffices to consider the case when the 2×2-matrix A is upper diagonal and since the trace is zero it has the form

$$A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$$

where a, b is a pair of complex numbers. If $a = 0$ then the two diagonal elements are zero and wee can take $U = E_2$ to be the identity in Lemma 6.5. If $a \neq 0$ we consider a vector $\phi = (1, z)$ in $\mathbf{C}^2$. Then $A(\phi)$ is the vector $(a + bz, -az)$ and hence the inner product becomes:

(i) $$\langle A(\phi), \phi\rangle = a + bz - a|z|^2$$

We can write

$$\frac{b}{a} = re^{i\theta}$$

where $r > 0$ and then (i) is zero if

(ii) $$|z|^2 = 1 + se^{i\theta} \cdot z$$

With $z = se^{-i\theta}$ it amounts to find a positive real number s such that $s^2 = 1 + s$ which clearly exists. Now we get the vector

$$\phi_* = \frac{1}{1+s^2}(1, se^{-i\theta})$$

which has unit length and

(ii) $$\langle A(\phi_*), \phi_* \rangle = 0$$

By 4.0.6 we find another unit vector ψ_* so that ϕ_*, ψ_* is an orthonormal base in $\mathbf{C}^2$ and hence there exists a unitary matrix U such that $U(e_1) = \phi_*$ and $U(e_2) = \psi_*$. If $B = U^*AB$ the vanishing in (ii) gives $b_{11} = 0$. At the same time the trace is unchanged, i.e. $\text{tr}(B) = 0$ holds and hence we also get $b_{22} = 0$. This means that the diagonal elements of U^*AU are both zero as required.

The case $n \geq 3$. For the induction the following is needed:

Sublemma. Let $n \geq 3$ and assume as above that $Tr(A) = 0$. Then there exists some non-zero vector $\phi \in \mathbf{C}^n$ such that

(*) $$\langle A(\phi), \phi \rangle = 0$$

Proof. If (*) does not hold we get the positive number

$$m_* = \min_{\phi} \big| \langle A(\phi), \phi \rangle \big|$$

where the minimum is taken over unit vectors in $\mathbf{C}^n$. The minimum is achieved by some unit vector ϕ_*. Let $\phi_*^\perp$ be its orthonormal complement and E the self-adjoint projection from $\mathbf{C}^n$ onto $\phi_*^\perp$. On the $(n-1)$-dimensional inner product space $\phi_*^\perp$ we get the linear operator $B = EA$, i.e.

(i) $$B(\xi) = E(A(\xi)) \quad : \quad \xi \in \phi_*^\perp$$

If $\psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\phi_*^\perp$ then the n-tuple $\phi_*, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and since the trace of A is zero we get

(ii) $$0 = \langle A(\phi_*), \phi_* \rangle + \sum_{\nu=1}^{\nu=n-1} \langle A(\psi_\nu), \psi_\nu \rangle = m + \sum_{\nu=1}^{\nu=n-1} \langle B(\psi_\nu), \psi_\nu \rangle$$

where we used that $E(\psi_\nu) = \psi_\nu$ for each ν and that E is self-adjoint so that

$$\langle A(\psi_\nu), \psi_\nu \rangle = \langle A(\psi_\nu), E(\psi_\nu) \rangle = \langle E(A(\psi_\nu)), \psi_\nu \rangle = \langle B(\psi_\nu), \psi_\nu \rangle$$

Now (ii) gives

$$\text{Tr}(B) = -m$$

Hence the $(n-1) \times (n-1)$-matrix which represents $B + \frac{m}{n-1} \cdot E$ has trace zero. By an induction over n we find a unit vector $\psi \in \phi_*^\perp$ such that

$$\langle B(\psi_*), \psi_* \rangle = -\frac{m}{n-1}$$

Finally, since E is self-adjoint we have already seen that

$$\langle A(\psi_*), \psi_* \rangle = \langle B(\psi_*), \psi_* \rangle \implies \big| \langle A(\psi_*), \psi_* \rangle \big| = \big| \frac{m}{n-1} \big| = \frac{m_*}{n-1}$$

Since $n \geq 3$ the last number is $< m_*$ which contradicts the minimal choice of m_*. Hence we must have $m_* = 0$ which proves lemma 6.5

Final part of the proof. Let $n \geq 3$. The Sublemma gives unit vector ϕ such that $\langle A(\phi), \phi \rangle = 0$. Consider the hyperplane $\phi^\perp$ and the operator B from the Sublemma which now has trace zero on this $(n-1)$-dimensional space. So by an induction over n there exists an orthonormal basis $\psi_1, \ldots, \psi_{n-1}$ in $\phi^\perp$ such that $\langle B(\psi_\nu), \psi_n u \rangle = 0$ for every ν. Now $\phi, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and if U is the unitary matrix which has this n-tuple as column vectors it follows that the diagonal elements of U^*AU all vanish. This finishes the proof of Theorem 6.4.

Proof Theorem 6.1

Set $B=\lambda^{-1}A$ so that $\sigma(B)=\{\lambda_i/\lambda\}$ and $\text{Tr}(B)=\sum\frac{\lambda_i}{\lambda}$. We also have

$$\mathfrak{h}(B)=\frac{\mathfrak{h}(A)}{|\lambda|}\quad\text{and}\quad \big|\lambda\big|\cdot||R_A(\lambda)||=||(E-B)^{-1}||$$

Hence Theorem 6.1 follows if we prove the inequality

$$(*)\qquad \big|e^{\text{Tr}(B)}\big|\cdot\big|\prod_{i=1}^{i=n}\big[1-\frac{\lambda}{\lambda_i}\big]\cdot||E-B)^{-1}||\leq\exp\big[\frac{1+\mathfrak{h}(B)^2}{2}\big]$$

To prove (*) we choose an arbitrary integer N such that $N>\big|\text{Tr}(B)\big|$ and for each such N we define the linear operator B_N on the $n+N$-dimensional complex space with points denoted by (x,y) with $y\in\mathbf{C}^N$ as follows:

$$(**)\qquad B_N(x,y)=(Bx\,,\,-\frac{\text{Tr}(B)}{N}\cdot y)$$

The eigenvalues of the linear operator $E-B_N$ is the union of the n-tuple $\{1-\frac{\lambda_i}{\lambda}\}$ and the N-tuple of equal eigenvalues given by $1+\frac{\text{Tr}(B)}{N}$. This gives the determinant formula

$$(1)\qquad \det(E-B_N)=\big(1+\frac{\text{Tr}(B)}{N}\big)^N\cdot\prod_{i=1}^{i=n}(1-\frac{\lambda_i}{\lambda})$$

The choice of N implies that (1) is $\neq 0$ so the inverse $(E-B_N)^{-1}$ exists. Moreover, the construction of B_N gives for any pair (x,y) in $\mathbf{C}^{N+n}$:

$$(E-B_N)^{-1}(x,y)=\big(E-B)^{-1}(x),\frac{y}{1+\frac{1}{N}\cdot\text{Tr}(B)}\big)$$

It follows that

$$||(E-B)^{-1}||\leq||(E-B_N)^{-1}||\implies$$

$$(2)\qquad \big|\det(E-B_N)\cdot||(E-B)^{-1}||\leq\big|\det(E-B_N)\big|\cdot||(E-B_N)^{-1}||$$

Hadarmard's inequality estimates the hand side in (2) by:

$$(3)\qquad \frac{\mathfrak{h}(E-B_N)^{N+n-1}}{(N+n-1)^{N+n-1)/2}}$$

Next, the construction of B_N implies that its trace is zero. So by the result in 6.3 we can find an orthonormal basis $\xi_1,\ldots,\xi_{n+N}$ in $\mathbf{C}^{n+N}$ such that

$$\langle B_N(\xi_k),\xi_k\rangle=0\quad:1\leq k\leq n+N$$

Relative to this basis the matrix of $E-B_N$ has 1 along the diagonal and the negative of the elements of B_N elsewhere. It follows that the Hilbert-Schmidt norm satisfies the equality:

$$(4)\qquad \mathfrak{h}(E-B_N)^2=N+n+\mathfrak{h}(B_N)^2=N+n+\mathfrak{h}(B)^2+N^{-1}\cdot|\text{Tr}(B)|^2$$

Hence, (1) and the inequalities from (2-3) give:

$$\big(1+\frac{\text{Tr}(B)}{N}\big)^N\cdot\prod_{i=1}^{i=n}(1-\frac{\lambda_i}{\lambda})\cdot||(E-B)^{-1}||\leq$$

$$\frac{\big(N+n+\mathfrak{h}(B)^2+N^{-1}\cdot|\text{Tr}(B)|^2\big)^{(N+n-1)(2}}{\big(N+n-1\big)^{N+n-1/2}}=\frac{\big(1+\frac{\mathfrak{h}(B)^2}{N+n}+\frac{|\text{Tr}(B)|^2}{N(N+n)}\big)^{(N+n-1)/2}}{(1-\frac{1}{N+n}\big)^{N+n-1/2}}$$

This inequality holds for arbitrary large N. Passing to the limit as $N\to\infty$ the definition of Neper's constant e give

$$\lim_{N\to\infty}\left(1+\frac{\mathrm{Tr}(B)}{N}\right)^N = e^{\mathrm{Tr}(B)}$$

and the reader may also verify that the limit of the last term above is equal to $\exp\left[\frac{1+\mathfrak{h}(B)^2}{2}\right]$ which finishes the proof of (*) above and hence also of Theorem 6.1.

§ 7. The Denjoy conjecture

Introduction. Let ρ be a positive integer and $f(z)$ is an entire function such that there exists some $0<\epsilon<1/2$ and a constant A_ϵ such that

$$|f(z)|\leq A_\epsilon\cdot e^{|z|^{\rho+\epsilon}} \tag{0.1}$$

hold for every z. Then we say that f has integral order $\leq\rho$. Next, the entire function f has an asymptotic value a if there exists a Jordan curve Γ parametrized by $t\mapsto\gamma(t)$ for $t\geq 0$ such that $|\gamma(t)|\to\infty$ as $t\to+\infty$ and

$$\lim_{t\to+\infty} f(\gamma(t))=a \tag{0.2}$$

In 1920 Denjoy raised the conjecture that (0.1) implies that the entire function f has at most 2ρ many different asymptotic values. Examples show that this upper bound is sharp. The Denjoy conjecture was proved in 1930 by Ahlfors in [Ahl]. A few years later T. Carleman found an alternative proof based upon a certain differential inequality. Theorem A.3 below has applications beyond the proof of the Denjoy conjecture for estimates of harmonic measures. See [Ga-Marsh].

A. The differential inequality.

Let Ω be a connected open set in $\mathbf{C}$ whose intersection S_x between a vertical line $\{\mathfrak{Re}\,z=x\}$ is a bounded set on the real y-line for every x. When $S_x\neq\emptyset$ it is the disjoint union of open intervals $\{(a_\nu,b_\nu)\}$ and we set

$$\ell(x)=\max_\nu\,(b_\nu-a_\nu) \tag{*}$$

Next, let $u(x,y)$ be a positive harmonic function in Ω which extends to a continuous function on the closure $\bar\Omega$ with the boundary values identical to zero. Define the function ϕ by:

$$\phi(x)=\int_{S_x} u^2(x,y)\cdot dy \tag{1}$$

The Federer-Stokes theorem gives the following formula for the derivatives of ϕ:

$$\phi'(x)=2\int_{S_x} u_x\cdot u(x,y)dy \tag{2}$$

$$\phi''(x)=2\int_{S_x} u_{xx}\cdot u(x,y)dy+2\int_{S_x} u_x^2\cdot dy \tag{3}$$

Since $\Delta(u)=0$ when $u>0$ we have

$$2\int_{S_x} u_{xx}\cdot u(x,y)dy=-2\int_{S_x} u_{yy}\cdot u(x,y)dy=2\int u_y^2dy \tag{4}$$

The Cauchy-Schwarz inequality applied in (2) gives

$$\phi'(x)^2\leq 4\cdot\int_{S_x} u_x^2\cdot\int_{S_x} u^2(x,y)dy=4\cdot\phi(x)\cdot\int_{S_x} u_x^2dy \tag{5}$$

Hence (4) and (5) give:

$$\phi''(x)\geq 2\int_{S_x} u_y^2(x,y)\cdot dy+\frac{1}{2}\cdot\frac{\phi'^2(x)}{\phi(x)} \tag{6}$$

Next, since $u(x,y)=0$ at the end-points of all intervals of S_x, *Wirtinger's inequality* and the definition of $\ell(x)$ give:

$$\int_{S_x} u_y^2(x,y)\cdot dy\geq\frac{\pi^2}{\ell(x)^2}\cdot\phi(x) \tag{7}$$

Inserting (7) in (6) we have proved

A.1 Proposition *The ϕ-function satisfies the differential inequality*

$$\phi''(x) \geq \frac{2\pi^2}{\ell(x)^2} \cdot \phi(x) + \frac{\phi'^2(x)}{2\phi(x)}$$

Proof continued. The maximum principle for harmonic functions implies that the $\phi(x) > 0$ when $x > 0$ and hence there exists a ψ-function where $\phi(x) = e^{\psi(x)}$. It follows that

$$\phi' = \psi' e^{\psi} \quad \text{and} \quad \phi'' = \psi'' e^{\psi} + \psi'^2 e^{\psi}$$

Now Proposition A.1 gives

$$(*) \qquad \psi'' + \frac{\psi'^2}{2} \geq \frac{2\pi^2}{\ell(x)^2}$$

A.2 An integral inequality. From (*) we obtain

$$\frac{2\pi}{\ell(x)} \leq \sqrt{\psi'(x)^2 + 2\psi''(x)} \leq \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

Taking the integral we get

$$(**) \qquad 2\pi \cdot \int_0^x \frac{dt}{\ell(t)} \leq \psi(x) + \log\, \psi'(x) + O(1) \leq \psi(x) +, \psi'(x) + O(1)$$

where $O(1)$ is a remainder term which is bounded independent of x. Taking the integral once more we obtain:

A.3 Theorem. *The following inequality holds:*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + O(x)$$

where the remainder term $O(x)$ is bounded by Cx for a fixed constant.

B. Solution to the Denjoy conjecture

B.1 Theorem. *Let $f(z)$ be entire of some integral order $\rho \geq 1$. Then f has at most 2ρ many different asymptotic values.*

Proof. Suppose f has n different asymptotic values $a_1, \ldots, a_n$. To each a_ν there exists a Jordan arc Γ_ν as described in the introduction. Since the a-values are different the n-tuple of Γ-arcs are separated from each other when $|z|$ is large. So we can find some R such that the arcs are disjoint in the exterior disc $|z| > R$. We may also consider the tail of each arc, i.e. starting from the last point on Γ_ν which intersects the circle $|z| = R$. So now we have an n-tuple of disjoint Jordan curves in $|z| \geq R$ where each curve intersects $|z| = R$ at some point p_ν and after the curves moves to the point at infinity. See figure. Next, we take one of these curves, say Γ_1. Let D_R^* be the exterior disc $|\zeta| > R$. In the domain $\Omega = \mathbf{C} \setminus \Gamma_1 \cup D_R^*$ we can choose a single-valued branch of $\log \zeta$ and with $z = \log\, \zeta$ the image of Ω is a simply connected domain Ω^* where S_x for each x has length strictly less than 2π The images of the Γ-curves separate Ω^* into n many disjoint connected domains denoted by $D_1, \ldots, D_n$ where each D_ν is bordered by a pair of images of Γ-curves and a portion of the vertical line $x = \log\, R$.

Let $\zeta = \xi + i\eta$ be the complex coordinate in Ω^*. Here we get the analytic function $F(\zeta)$ where

$$F(\log(z)) = f(z)$$

We notice that F may have more growth than f. Indeed, we get

$$(1) \qquad |F(\xi + i\eta)| \leq \exp\bigl(e^{(\rho+\epsilon)\xi}\bigr)$$

With $u = \mathrm{Log}^+ |F|$ it follows that

$$(2) \qquad u(\xi, \eta) \leq e^{(\rho+\epsilon)\xi}$$

Hence the ϕ-function constructed during the proof of Theorem A.3 satisfies

$$\phi(\xi) \leq e^{2(\rho+\epsilon)\xi}$$

It follows that the ψ-function satisfies

$$\psi(\xi) = 2 \cdot (\rho + \epsilon)\xi + O(1) \tag{3}$$

Now we apply Theorem A.3 in each region D_ν where we have a function $\ell_\nu(\xi)$ constructed by (0) in section A. This gives the inequality

$$2\pi \cdot \int_R^\xi \frac{\xi - s}{\ell_\nu(s)} \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1) \quad : \quad 1 \leq \nu \leq n \tag{4}$$

Next, recall the elementary inequality which asserts that if $a_1, \ldots, a_n$ is an arbitrary n-tuple of positive numbers then

$$\sum a_\nu \cdot \sum \frac{1}{a_\nu} \geq n^2 \tag{5}$$

For each s we apply this to the n-tuple $\{\ell_\nu(s)\}$ where we also have

$$\sum \ell_\nu(s) \leq 2\pi$$

So a summation in (4) over $1 \leq \nu \leq n$ gives

$$n \cdot \int_R^\xi (\xi - s) \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1) \tag{6}$$

Another integration gives:

$$n \cdot \frac{\xi^2}{2} \leq (\rho + \epsilon) \cdot \xi^2 + O(\xi) \tag{7}$$

This inequality can only hold for large ξ if $n \leq 2(\rho + \epsilon)$ and since $\epsilon < 1/2$ is assumed it follows that $n \leq 2\rho$ which finishes the proof of the Denjoy conjecture.

§ 8. Approximation by fractional powers

Here is the set-up in the article *Über die approximation analytischer funktionen* by Carleman from 1922. Let $0 < \lambda_1 < \lambda_2 < \ldots$ be a sequence of positive real numbers and Ω is a simply connected domain contained in the right half-space $\mathfrak{Re}(z) > 0$. Notice that the functions $q_\nu(z) = z^{\lambda_\nu}$ are analytic in the half-plane, i.e. with $z = re^{i\theta}$ and $-\pi/2 < \theta < \pi/2$ we have:

$$q_\nu(z) = r^{\lambda_\nu} \cdot e^{i\lambda_\nu \cdot \theta}$$

C.1 Definition. *We say that the sequence $\Lambda = \{\lambda_\nu\}$ is dense for approximation if there for each $f \in \mathcal{O}(\Omega)$ exists a sequence of functions of the form*

$$Q_N(z) = \sum_{\nu=1}^{N} c_\nu(N) \cdot q_\nu(z) \quad : \quad N = 1, 2, \ldots$$

which converges uniformly to f on compact subsets of Ω.

C.2 Theorem. *A sequence Λ is dense if*

$$(*) \qquad \limsup_{R \to \infty} \frac{\sum_R \frac{1}{\lambda_\nu}}{\operatorname{Log} R} > 0$$

where $\sum_R$ means that we take the sum over all $\lambda_\nu < R$.

Remark. Above condition (*) is the same for every simply connected domain Ω. Theorem C.2 gives a *sufficient* condition for an approximation. To get necessary condition one must specify the domain Ω and we shall not try to discuss this more involved problem. The proof of Theorem C.2 requires several steps, the crucial is the uniqueness theorem in C.4 while the proof of Theorem C.2 is postponed until C.5.

C.3 A uniqueness theorem.

Consider a closed Jordan curve Γ of class C^1 which is contained in $\mathfrak{Re}\, z > 0$. When $z = re^{i\theta}$ stays in the right half-plane we get an entire function of the complex variable λ defined by:

$$\lambda \mapsto z^\lambda = r^\lambda \cdot e^{i\theta \cdot \lambda}$$

We conclude that a real-valued and continuous function g on Γ gives an entire function of λ defined by:

$$G(\lambda) = \int_\Gamma g(z) \cdot z^\lambda \cdot |dz|_\Gamma$$

where $|dz|_\Gamma$ is the arc-length on Γ. With these notations one has

C.4 Theorem. *Assume that Λ satisfies the condition in Theorem C.2. Then, if $G(\lambda_\nu) = 0$ for every ν it follows that the g-function is identically zero.*

Proof. If we have shown that the G-function is identically zero then the reader may verify that $g = 0$. There remains to show that if $G(\lambda_\nu) = 0$ for every ν then $G = 0$. To attain this one first shows that there exist constants A, K and $0 < a < \frac{\pi}{2}$ such that:

$$\text{(i)} \qquad |G(\lambda)| \leq K \cdot e^{|\lambda|} \quad \text{and} \quad |G(is)| \leq K \cdot e^{|s| \cdot a} \quad : \; \lambda \in \mathbf{C} \; : \; s \in \mathbf{R}$$

The easy verification of (i) is left to the reader. Next, the first inequality in (i) means that G is an entire function of exponential type one. By assumption $G(\lambda_\nu) = 0$ for every ν. Now we can use Carleman's formula for analytic functions in a half-space from XXX to conclude that $G = 0$. Namely, set

$$\text{(ii)} \qquad U(r, \phi) = \log \left|G(re^{i\phi})\right|$$

Let $\{r_\nu e^{i\phi_\nu}\}$ be the zeros of G in $\mathfrak{Re}(z) > 0$ which by the hypothesis contains the set Λ. By Carleman's formula the following hold for each $R > 1$:

$$\sum_{1 < r_\nu < R} \Big[\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\Big] \cdot \cos \theta_\nu = \frac{1}{\pi R} \cdot \int_{-\pi/2}^{\pi/2} U(R, \phi) \cdot \cos \phi \cdot d\phi +$$

$$\frac{1}{2\pi R}\cdot\int_1^R\Big(\frac{1}{r^2}-\frac{1}{R^2}\Big)\cdot\big[\,U(r,\pi/2)+U(r,-\pi/2)\,\big]\cdot dr+c_*(R)$$

where $c_*(R)\le K$ holds for some constant which is independent of R. Finally, the set Λ satisfies (*) in Theorem C.2 and the sum over zeros in Carleman's formula above majorizes the sum extended over the real λ-numbers from Λ satisfying $1<\lambda_\nu<R$. At this stage we leave it to the reader to verify that the second inequality in (i) above implies that G must be identically zero.

Proof of Theorem C.2

Denote by $\mathcal{O}^*(\Lambda)$ the linear space of analytic functions in the right half-plane given by finite $\mathbf{C}$-linear combinations of the fractional powers $\{z^{\lambda_\nu}\}$. To obtain uniform approximations over relatively compact subsets when Ω is a simply connected domain in $\mathfrak{Re}(z)>0$, it suffices to regard a closed Jordan arc Γ which borders a Jordan domain U where U is a relatively compact subset of Ω. In particular Γ has a positive distance to the imaginary axis and there remains to show that when (*) holds in Theorem C.2, then an arbitrary analytic function $f(z)$ defined in some open neighborhood of $\bar{U}$ can be uniformly approximated by $\mathcal{O}^*(\Lambda)$-functions over a relatively compact subset U_* of U. To achieve this we shall use a trick which reduces the proof of uniform approximation to a problem concerned with L^2-approximation on Γ. To begin with we have

C.5 Lemma. *The uniqueness in Theorem C.4 implies that if V is a real-valued function on Γ then there exists a sequence $\{Q_n\}$ from the family $\mathcal{O}(\Lambda)$ such that*

$$\lim_{n\to\infty}\int_\Gamma\big|Q_n-V|^2\cdot|dz|=0$$

The proof of this result is left as an exercise.

C.6 A tricky construction. Let $f(z)$ be analytic in a neighborhood of the closed Jordan domain $\bar{U}$ bordered by Γ. Define a new analytic function

$$F(z)=\int_{z_*}^{z}\frac{f(\zeta)}{\zeta}\cdot d\zeta \tag{1}$$

where z_* is some point in $\bar{U}$ whose specific choice does not affect the subsequent discussion. We can write $F=V+iW$ where $V=\mathfrak{Re}(F)$. Lemma 6.5 gives a sequence $\{Q_n\}$ which approximates V in the L^2-norm on Γ. Using this L^2-approximation we get

Lemma C.7 *Let U_0 be relatively compact in U. Then there exists a sequence of real numbers $\{\gamma_n\}$ such that*

$$\lim_{n\to\infty}\big|\big|Q_n(z)-i\cdot\gamma_n-F(z)\big|_{U_0}=0$$

Again we leave out the proof as an exercise. Next, taking complex derivatives Lemma C.7 implies that if U_* is even smaller, i.e. taken to be a relatively compact in U_0, then we a get uniform approximation of derivatives:

$$Q_n'(z)\to F'(z)=\frac{f(z)}{z}$$

Well, this means that

$$z\cdot Q_n'\to f(z)$$

holds uniformly in U_*. Next, notice that

$$z\cdot\frac{d}{dz}(z^{\lambda_\nu})=\lambda_\nu\cdot z^{\lambda_\nu}$$

hold for each ν. Hence $\{z\cdot Q_n'(z)\}$ again belong to the $\mathcal{O}(\Lambda)$-family. So we achieve the required uniform approximation of the given f function on U_*. This completes the proof of Theorem C.2.

§ 9. Theorem of Müntz

Introduction. Theorem D.1 below is due to Müntz. See his article *Über den Approximationssatz von Weierstrass* from 1914. The simplified version of the original proof below is given in [Car]. Here is the set up: Let $0 < \lambda_1 < \lambda_2 < \ldots$. To each ν we get the function x^{λ_ν} defined on the real unit interval $0 \leq x \leq 1$. We say that the sequence $\Lambda = \{\lambda_\nu\}$ is L^2-dense if the family $\{x^{\lambda_\nu}\}$ generate a dense linear subspace of the Hilbert space of square integrable functions on $[0, 1]$.

9.1 Theorem. *The necessary and sufficient condition for Λ to be L^2-dense is that $\sum \frac{1}{\lambda_\nu}$ is convergent.*

9.2 Proof of necessity. If Λ is not L^2-dense there exists some $h(x) \in L^2[0, 1]$ which is not identically zero while

(1) $$\int_0^1 h(x) \cdot x^{\lambda_\nu} \cdot dx = 0 \quad : \quad \nu = 1, 2, \ldots$$

Now consider the function

(2) $$\Phi(\lambda) = \int_0^1 h(x) \cdot x^{-i\lambda} \cdot dx$$

It is clear that Φ is analytic in the right half plane $\mathfrak{Im}\,\lambda > 0$. If $\lambda = s + it$ with $t > 0$ we have

$$|x^\lambda| = x^t \leq 1$$

for all $0 \leq x \leq 1$. From this and the Cauchy-Schwarz inequality we see that

(3) $$|\Phi(\lambda)| \leq ||h||_2 \quad : \lambda \in U_+$$

Hence Φ is a bounded analytic function in the upper half-plane. At the same time (1) means that the zero set of Φ contains the sequence $\{\lambda_\nu \cdot i\}$. By the integral formula formula we have seen in XX that this entails that

(*) $$\sum \frac{1}{\lambda_\nu} < \infty$$

which proves the necessity.

Proof of sufficiency. There remains to show that ifwe have the convergence in (*) above then there exists a non-zero h-function i $L^2[0, 1]$ such that (1) above holds. To find h we first construct an analytic function Φ by

(i) $$\Phi(z) = \frac{\prod_{\nu=1}^\infty \left(1 - \frac{z}{\lambda_\nu}\right)}{\prod_{\nu=1}^\infty \left(1 + \frac{z}{\lambda_\nu}\right)} \cdot \frac{1}{(1+z)^2} \quad : \quad \mathfrak{Re}\, z > 0$$

Notice that $\Phi(z)$ is defined in the right half-plane since the series (*) is convergent. When $\mathfrak{Re}(z) \geq 0$ we notice that each quotient

$$\frac{1 - \frac{z}{\lambda_\nu}}{1 + \frac{z}{\lambda_\nu}}$$

has absolute value ≤ 1. It follows that

(ii) $$|\Phi(x + iy)| \leq \frac{1}{1 + x + iy|^2} = \frac{1}{(1 + x)^2 + y^2}$$

In particular the function $y \mapsto \Phi(iy)$ belongs to L^2 on the real y-line. Now we set

(ii) $$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ity} \cdot \Phi(iy) \cdot dy$$

using the inequality (ii) If $t < 0$ we can move the line integral of $e^{tz} \cdot \Phi(z)$ from the imaginary axis to a line $\mathfrak{Re}(z) = a$ for every $a > 0$ and it is clear that

$$\lim_{a \to +\infty} \int_{-\infty}^\infty e^{-at+ity} \cdot \Phi(a + iy) \cdot dy = 0$$

We conclude that $f(t)=0$ when $t<0$. Next, since $y\mapsto \Phi(iy)$ is an L^2-function it follows by Parseval's equality that

$$\int_0^\infty |f(t)|^2\cdot dt<\infty$$

Moreover, for a fixed λ_ν we have

$$\int_0^\infty f(t)e^{-\lambda_\nu t}\cdot dt=\frac{1}{2\pi}\cdot\int_0^\infty [\int_{-\infty}^\infty e^{ity}\cdot\Phi(iy)\cdot dy\,]\cdot e^{-\lambda\nu t}\cdot dt=$$

$$\int_{-\infty}^\infty \frac{1}{iy-\lambda_\nu}\cdot\Phi(iy)\cdot dy$$

where the last equality follows when the repeated integral is reversed. By construction $\Phi(z)$ has a zero at λ_ν and therefore (xx) above remains true with Φ replaced by $\frac{\Phi(z)}{z-\lambda_\nu}$ which entails that

$$\int_0^\infty f(t)\cdot e^{-\lambda_\nu t}\cdot dt=0$$

At this stage we obtain the requested h-function. Namely, since $t\mapsto e^{-t}$ identifies $(0,+\infty)$ with $(0,1)$ we get a function $h(x)$ on $(0,1)$ such that

$$h(e^{-t})=e^{t/2}\cdot f(t)$$

The reader may verify that

$$\int_0^1 |h(x)|^2\cdot dx=\int_0^\infty |f(t)|^2\cdot dt$$

and hence h belongs to $L^2(0,1)$. Moreover, one verifies that the vanishing in (xx) above entails that

$$\int_0^1 h(x)\cdot x^{\lambda_\nu}\cdot dx=0$$

Since this holds for every ν we have proved the sufficiency which therefore finishes the proof of Theorem XX.

§ 10. Ikehara's theorem.

Introduction. The proof below was presented by Carleman during lectures at Institute Mittag-Leffler in 1935. Ikehara's original result appears in Theorem 10.1 while Theorem 10.x gives a slight extension since certain regulasrity properties are relaxed. Let ν be a non-negative Riesz measure supported on $[1, +\infty)$ and assume that

$$\int_1^\infty x^{-1-\delta} \cdot d\nu(x) < \infty \quad \text{for all } \delta > 0$$

This gives an analytic function $f(s)$ of the complex variable s defined in the right half plane $\mathfrak{Re}(s) > 1$ by

$$f(s) = \int_1^\infty x^{-s} \cdot d\nu(x)$$

10.1 Theorem. *Assume that there exists a constant A and a locally integrable function $G(u)$ defined on the real u-line such that*

$$(*) \qquad \lim_{\epsilon \to 0} \int_{-b}^{b} \big| f(1+\epsilon+iu) - \frac{A}{1+\epsilon+iu} - G(u) \big| \cdot du = 0 \quad \text{holds for each } b > 0$$

Then it follows that

$$(**) \qquad \lim_{x \to +\infty} \frac{1}{x} \int_1^x d\nu(t) = A$$

Proof. Define the measure ν^* on the non-negative real ξ-line by

$$(1) \qquad d\nu^*(\xi) = e^{-\xi} \cdot d\nu(e^\xi) - A \cdot d\xi \quad : \ \xi \geq 0$$

If $\eta > 1$ we notice that

$$\int_0^\eta e^{-\eta+\xi} \cdot d\nu^*(\xi) = e^{-\eta} \int_0^\eta d\nu(e^\xi) - A(1-e^{-\eta}) = e^{-\eta} \int_0^{e^\eta} d\nu(t) - A(1-e^{-\eta})$$

hence (**) holds if and only if

$$(2) \qquad \lim_{\eta \to \infty} \int_0^\eta e^{-\eta+\xi} \cdot d\nu^*(\xi) = 0$$

It is also clear that condition (xx) for ν entails that

$$(3) \qquad \int_0^\infty e^{-\delta \cdot \xi} \cdot d\nu^*(\xi) < \infty \quad \text{for all } \delta > 0$$

Moreover, a variable substitution gives

$$(4) \qquad f(s) - \frac{A}{s-1} = \int_0^\infty e^{(1-s)\xi} d\nu^*(\xi)$$

10.2 A reformulation of Ikehara's theorem.

From (1-4) we can restate Ikehara's theorem. Let ν^* be a non-negative measure on $0 \leq \xi < +\infty$ such that

$$(1.1) \qquad \int_0^\infty e^{-\delta \cdot \xi} \cdot d\nu^*(\xi) < \infty \quad \text{for all } \delta > 0$$

Let $A > 0$ be some positive constant and define the measure μ by

$$(1.2) \qquad d\mu(\xi) = d\nu^*(\xi) - A \cdot d\xi$$

Then (1.1) gives the analytic function $g(s)$ defined in $\mathfrak{Re}(s) > 0$ by

$$g(s) = \int_0^\infty e^{-s\cdot\xi} \cdot d\mu(\xi) \tag{1.3}$$

10.2.1. Definition. *We say that the measure μ is of the Ikehara type if there exists a locally integrable function $G(u)$ defined on the real u-line such that*

$$\lim_{\epsilon\to 0} \int_{-b}^{b} \big|g(\epsilon + iu) - G(u)\big| \cdot du = 0 \quad \text{holds for each } b > 0$$

10.2.2 The space $\mathcal{W}$. Let $\mathcal{W}$ be the space of continuous functions $\rho(\xi)$ defined on $\xi \geq 0$ which satisfy:

$$\sum_{n\geq 0} ||\rho||_n < \infty \quad \text{where } ||\rho||_n = \max_{n\leq u\leq n+1} |\rho(u)|$$

The dual space $\mathcal{W}^*$ consists of Riesz measures γ on $[0, +\infty)$ such that

$$\max_{n\geq 0} \int_n^{n+1} |d\gamma(\xi)| < \infty$$

With these notations we shall prove:

10.2.3. Theorem. *Let ν^* be a non-negative meausure on $[0, +\infty)$ and $A \geq 0$ some constant such that the measure $\mu = \nu^* - A \cdot d\xi$ is of Ikehara type. Then $\mu \in \mathcal{W}^*$ and for every function $\rho \in \mathcal{W}$ one has*

$$\lim_{\eta\to+\infty} \int_0^\eta \rho(\eta - \xi) \cdot d\mu(\xi) = 0$$

10.2.4 Exercise. Use the material above to show that Theorem 10.2.3 gives Theorem 10.1 where a hint is to use the function $\rho(s) = e^{-s}$ above.

Proof of Theorem 10.2.3.

Let $b > 0$ and define the function $\omega(u)$ by

$$\omega(u) = 1 - \frac{|u|}{b}\,, \quad -b \leq u \leq b \quad \text{and } \omega(u) = 0 \text{ outside this interval} \tag{i}$$

Set

$$J_b(\epsilon, \eta) = \int_{-b}^{b} e^{i\eta u} \cdot g(\epsilon + iu) \cdot \omega(u) \cdot du \tag{ii}$$

From Definition C.1.1 we have the L^1_{loc}-function $G(u)$ and since $\omega(u)$ is a continuous function on the compact interval $[-b, b]$ we have

$$\lim_{\epsilon\to 0} J_b(\epsilon, \eta) = J_b(0, \eta) = \int_{-b}^{b} e^{i\eta u} \cdot G(u) \cdot \omega(u) \cdot du \tag{iii}$$

With b kept fixed the right hand side is a Fourier transform of an L^1-function. So the Riemann-Lebesgue theorem gives:

$$\lim_{\eta\to+\infty} J_b(0, \eta) = 0 \tag{iv}$$

Moreover, the triangle inequality gives the inequality:

$$|J_b(0, \eta)| \leq \int_{-b}^{b} \big|G(u)\big| \cdot du \tag{v}$$

Some integral formulas. From the above it is clear that

$$J_b(\epsilon,\eta)=\int_0^\infty \Big[\int_{-b}^b e^{i\eta u-i\xi u}\cdot\omega(u)\cdot du\Big]\cdot e^{-\epsilon\cdot\xi}\cdot d\mu(\xi) \tag{1}$$

Next, notice that

$$\int_{-b}^b e^{i\eta u-i\xi u}\cdot\omega(u)\cdot du=2\cdot\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2} \tag{2}$$

Hence we obtain

$$J_b(\epsilon,\eta)=2\cdot\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\mu(\xi) \tag{3}$$

From (iii) above it follows that (3) has a limit as $\epsilon\to 0$ which is equal to the integral in the right hand side in (iii) which is denoted by $J_b(0,\eta)$. Next, it is easily seen that there exists the limit

$$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot Ad\xi=2\pi\cdot A \tag{4}$$

Hence (3-4) imply that there exists the limit

$$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\nu^*(\xi)=J_b(0,\eta)+2\pi\cdot A \tag{5}$$

Next, the measure $\nu^*\geq 0$ and the function $\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\geq 0$ for all ξ. So the existence of a finite limit in (5) entails that there exists the convergent integral

$$\int_0^\infty \frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot d\nu^*(\xi)=J_b(0,\eta)+2\pi\cdot A \tag{6}$$

Proof that $\mu\in\mathcal{W}^*$. Since $A\cdot d\xi$ obviously belongs to $\mathcal{W}^*$ it suffices to show that $\nu^*\in\mathcal{W}^*$. To prove this we consider some integer $n\geq 0$ and with $b=1$ it is clear that (6) gives

$$\Big|\int_n^{n+1}\frac{1-\cos{(\eta-\xi)}}{(\eta-\xi)^2}\cdot d\nu^*(\xi)\Big|\leq |J_1(0,\eta)|+2\pi=\int_{-1}^1|G(u)|\cdot du+2\pi\cdot A$$

Apply this with $\eta=n+1+\pi/2$ and notice that

$$\frac{1-\cos{(n+1+\pi/2-\xi)}}{(n+1+\pi/2-\xi)^2}\geq a\quad\text{for all } n\leq\xi\leq n+1$$

This gives a constant K such that

$$\int_n^{n+1} d\nu^*(\xi)\leq K\quad n=0,1,\ldots$$

Final part of the proof. We have proved that $\mu\in\mathcal{W}^*$. Moreover, from (iv) above and the integral formula (6) we get

$$\lim_{\eta\to+\infty}\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot d\mu(\xi)=0\quad\text{for all } b>0 \tag{*}$$

Next, for each fixed $b>0$ we notice that the function

$$\rho_b(\xi)=2\cdot\frac{1-\cos{(b\xi)}}{b\cdot\xi^2}$$

belongs to $\mathcal{W}$ and its Fourier is $\omega_b(u)$. Here $\omega_b(u)\neq 0$ when $-b<u<b$. So the family of these ω-functions have no common zero on the real u-line. By the Remark in XX this means that the linear subspace of $\mathcal{W}$ generated by the translates of all ρ_b -functions with arbitrary large b is dense in $\mathcal{W}$. Hence (*) above implies that we get a zero limit as $\eta\to+\infty$ for every function $\rho\in\mathcal{W}$. But this is precisely the assertion in Theorem 10.2.3.

§ 11. Fourier-Carleman transforms

Introduction. The Fourier transform can be obtained from a pair of analytic functions defined in the upper - resp. the lower half plane. The idea is that a Fourier transform

$$\widehat{g}(\xi) = \int e^{-ix\xi} \cdot g(x)\, dx$$

becomes a sum when we integrate over $(-\infty, 0)$ respectively $(0, +\infty)$. For each of these we get analytic functions $G_+(\zeta$ and $G_-(\zeta)$ in the upper, resp. the lower half plane of the complex ζ-plane where $\zeta = \xi + i\eta$. After one can take their boundary values. This construction has special interest when the support of the Fourier transform has gaps, i.e. when its complement consists of many open intervals.

3.1 The functions G_+ and G_-. Consider a continuous complex-valued function $g(x)$ defined on the real x-line which is absolutely integrable:

$$\int_{-\infty}^{\infty} |g(x)|\, dx < \infty$$

We obtain analytic functions $G_+(\zeta)$ and $G_-(\zeta)$ defined in the upper, respectively the lower half-plane of the complex ζ-plane where $\zeta = \xi + i\eta$.

$$(*) \qquad G_+(\zeta) = \int_{-\infty}^{0} g(x) e^{-i\zeta x}\, dx \quad : \quad G_-(\zeta) = \int_{0}^{\infty} g(x) e^{-i\zeta x}\, dx$$

With $\zeta = \xi + i\eta$ we have $|e^{-i\zeta x}| = e^{\eta x}$. This number is < 1 when $x < 0$ and $\eta > 0$, and vice versa. We conclude that $G_+(\zeta)$ is analytic in $\mathfrak{Im}(\zeta) > 0$ while $G_-(\zeta)$ is analytic in $\mathfrak{Im}(\zeta) < 0$. Since $|g|$ is integrable we see that G_+ extends continuously to the closed upper half plane where

$$G_+(\xi) = \int_{-\infty}^{0} g(x) e^{-i\xi x} dx$$

Similarly G_- extends to $\mathfrak{Im}(\zeta) \leq 0$ and we have:

$$(**) \qquad G_+(\xi) + G_-(\xi) = \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx = \widehat{g}(\xi)$$

3.2 The case when $\widehat{g}$ has compact support. Then there are two intervals $(-\infty, a)$ and $(b, +\infty)$ and and family of bounded interval $\{(\alpha_\nu, \beta_\nu)\}$ whose union is the open complement of $\text{Supp}(\widehat{g})$. On each such interval $G_+(\xi) + G_-(\xi)$ is identically zero. Hence we get

3.3 Theorem *Put $\Omega = \mathbf{C} \setminus Supp(\widehat{g})$. Then there exists a function $\mathcal{G} \in \mathcal{O}(\Omega)$ such that $\mathcal{G} = G_+$ in the upper half plane and $\mathcal{G} = -G_-$ in the lower half plane.*

Consider some $R > 0$ which is chosen so large that the open disc D_R centered at the origin in the ζ-space contains the compact set $\text{Supp}(\widehat{g})$. For each real x the function $e^{ix\zeta}\mathcal{G}(\zeta)$ is again analytic in Ω. Put

$$\text{(i)} \qquad J_R(x) = \int_{|\zeta|=R} e^{ix\zeta} \cdot \mathcal{G}(\zeta)\, d\zeta$$

Exercise. Show that if $[-R, R]$ contains the support of g then

$$J_R(x) = \int_{-R}^{R} e^{ix\xi} G_+(\xi)\, d\xi + \int_{-R}^{R} e^{ix\xi} G_-(\xi)\, d\xi = \int_{-R}^{R} e^{ix\xi} \cdot \widehat{g}(\xi)\, d\xi$$

The last integral appears in Fourier's inversion formula which gives:

3.4 Theorem *Let $g(x) \in L^1(\mathbf{R})$ be such that $\widehat{g}$ has compact support in the interval $[-R_*, R_*]$. Then*

$$g(x) = \frac{1}{2\pi} \cdot \int_{|\zeta|=R} e^{ix\zeta} \cdot \mathcal{G}(\zeta)\, d\zeta \quad : \quad R > R_*$$

3.5 A more general case. The condition that $\widehat{g}$ has compact support is restrictive since it implies that $g(x)$ extends to an entire function in the complex z-plane. A relaxed condition is that the complement of $\mathrm{Supp}(\widehat{g})$ contains some open intervals both on positive and the negative real ξ-axis. With $\Omega = C \setminus \mathrm{Supp}(\widehat{g})$ we have $\mathcal{G} \in \mathcal{O}(\Omega)$. So if $R > 0$ is a positive number such that R and $-R$ both are outside the support of $\widehat{g}$ we have the equality

$$(*) \qquad \frac{1}{2\pi} \cdot \int_{|\zeta|=R} e^{ix\zeta} \mathcal{G}(\zeta)\, d\zeta = \frac{1}{2\pi} \cdot \int_{-R}^{R} e^{ix\xi} \cdot \widehat{g}(\xi)\, d\xi$$

If $\widehat{g}$ is absolutely integrable Fourier's inversion formula gives

$$g(x) = \lim_{R\to\infty} \frac{1}{2\pi} \cdot \int_{-R}^{R} e^{ix\xi}\, \widehat{g}(\xi)\, d\xi$$

So when $\widehat{g} \in L^1(\mathbf{R})$ and there exists some sequence $\{R_\nu\}$ where R_ν and $-R_\nu$ both are outside $\mathrm{Supp}(\widehat{g})$, then

$$(**) \qquad g(x) = \lim_{\nu\to\infty} \frac{1}{2\pi} \cdot \int_{|\zeta|=R_\nu} e^{ix\zeta} \cdot \mathcal{G}(\zeta)\, d\zeta$$

3.6 Further extensions

Above we assumed that $g(x)$ was absolutely integrable which implies that $\widehat{g}$ is a bounded and continuous function. Suppose now that $g(x)$ is a continuous function such that

$$\int_{-\infty}^{\infty} \frac{|g(x)|}{1 + |x|^N} \cdot x < \infty$$

holds for some positive integer N. We can still define the two analytic functions G_+ and G_-. Consider the behaviour of G_+ as ζ approaches the real ξ-line. We have by definition

$$G_+(\xi + i\eta) = \int_{-\infty}^{0} g(x)^{-i\xi x} e^{\eta x} dx$$

Taking absolute values we get for $\eta > 0$:

$$|G_+(\xi + i\eta)| \leq \int_{-\infty}^{0} \frac{|g(x)|}{(1 + |x|)^N} \cdot (1 + |x|)^N \cdot e^{\eta x} dx$$

Notice that when $\alpha > 0$, then the function

$$t \mapsto (1 + t)^N e^{-\alpha t} : t \geq 0$$

takes its maximum when $1 + t = \frac{N}{\alpha}$ so the maximum value over $[0, +\infty)$ is $\leq \frac{N^N}{\alpha^N}$. Apply this with $\eta > 0$ and $x < 0$ above which gives a constant C such that

$$(*) \qquad |G_+(\xi + i\eta)| \leq \frac{C}{\eta^N} \cdot \int_{-\infty}^{\infty} \frac{|g(x)| \cdot dx}{1 + |x|^N}$$

Hence G_+ has temperate growth as $\eta \to 0$ so its boundary value distribution $\mathfrak{b}(G_+)$ exists. Similarly we find the boundary value distribution $\mathfrak{b}(G_-)$. The Fourier transform of $g(x)$ regarded as a tempered distribution is equal to $\mathfrak{b}(G_+) + \mathfrak{b}(G_-)$. Again, if $\text{Supp}(\widehat{g})$ has gaps we can proceed as in 3.5 and construct the complex line integral

$$J_R(x) = \frac{1}{2\pi} \cdot \int_{|\zeta|=R} e^{ix\zeta} \mathcal{G}(\zeta)\, d\zeta$$

for those values of R such that $-R$ and R are outside the support of $\hat{g}$.

Exercise. Let g be as above and assume that there exists a sequence $\{R_\nu\}$ where $-R_\nu$ and R_ν are outside the support of g. Show that the following hold for every test-function $f(x)$

$$\int g(x) \cdot f(-x)\, dx = \lim_{\nu \to \infty} \frac{1}{2\pi} \cdot \int_{|\zeta|=R_\nu} \mathcal{G}(\zeta) \cdot \widehat{f}(\zeta)\, d\zeta$$

where $\widehat{f}(\zeta)$ is the entire Fourier-Laplace transform of f.

3.7 Use of Fourier's inversion formula.

Consider the following situation: Let $f(x)$ be a function in the Schwartz class and assume that $\widehat{f}(\xi)$ vanishes on some open interval $a < \xi < b$. Set $c = \frac{a+b}{2}$ and $g(x) = e^{ixc} \cdot f(x)$. Then we get

$$\widehat{g}(\xi) = \widehat{f}(\xi + c)$$

Here $\widehat{g}$ is zero on an interval centered at $\xi = 0$ and we may therefore assume from the start that $\widehat{f}$ is zero on some interval $(-A, A)$. Set

$$F_+(x+iy) = \frac{1}{2\pi} \cdot \int_A^\infty e^{(x+iy)i\xi}\, \widehat{f}(\xi)\, d\xi \quad : \quad F_-(x+iy) = -\frac{1}{2\pi} \cdot \int_{-\infty}^{-A} e^{(x+iy)i\xi}\, \widehat{f}(\xi)\, d\xi$$

When $y = 0$ we see that

$$(*) \qquad F_+(x) - F_-(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^\infty e^{ix\xi}\, \widehat{f}(\xi)\, d\xi$$

By Fourier's inversion formula the last integral is equal to $f(x)$ since $\widehat{f} = 0$ on $(-A, A)$. Hence $f(x)$ is represented as a difference of two analytic functions defined in the upper and the lower half-plane respectively where one has the estimates:

$$(1) \qquad \big|F_+(x+iy)\big| \leq \int_A^\infty e^{-y\xi} \cdot |\widehat{f}(\xi)|\, d\xi \leq e^{-Ay} \cdot \int_A^\infty |\widehat{f}(\xi)|\, d\xi$$

$$(2) \qquad |F_-(x+iy)| \leq e^{-A|y|} \cdot \int_{-\infty}^{-A} |\widehat{f}(\xi)|\, d\xi$$

Suppose now that $f(x)$ also is zero on some interval, say $a < x < b$. This means that the two analytic functions $F_+(z)$ and $F_-(z)$ agree on this interval and by the Schwarz reflection principle they are analytic continuations of each other. Hence, we get the following:

3.8 Proposition. *Assume that Supp(f) is a proper subset of* $\mathbf{R}$ *and consider the open complement*

$$U = \cup\, (a_\nu, b_\nu)$$

where $\{(a_\nu, b_\nu)\}$ is a family of disjoint open intervals. Then there exists an analytic function $\mathcal{F}(z)$ defined in the connected set $\mathbf{C} \setminus$ Supp(f) where

$$\mathcal{F}(z) = F_+(z) \quad : z \in U_+ \quad : \quad \mathcal{F}(z) = F_-(z) \quad : z \in U_*$$

We refer to $\mathcal{F}$ as the inverse Fourier-Carleman transform of $\widehat{f}(\xi)$.

3.9 A local estimate. Consider an open interval (a_ν, b_ν) in U. Set

$$r = \frac{b_\nu - a_\nu}{2} \quad : \quad c = \frac{a_\nu + b_\nu}{2}$$

Hence the open disc $D_r(c)$ stays in the open set $\Omega = \mathbf{C} \setminus \mathrm{Supp}(f)$. Next, when $0 < \phi < \pi$ we have

$$\mathcal{F}(c + re^{i\phi}) = \frac{1}{2\pi} \cdot \int_A^\infty e^{(c+r\cos\phi)i\xi - r\sin\phi\cdot\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

Since $\left|e^{(c+r\cos\phi)i\xi}\right| = 1$ the triangle inequality gives

$$|\mathcal{F}(c + re^{i\phi})| \leq \frac{1}{2\pi} \cdot e^{-rA\cdot\sin\phi} \cdot \int_A^\infty \left|\widehat{f}(\xi)\right| \cdot d\xi$$

When $-\pi \leq \phi \leq 0$ we get a similar estimate where we now use that $\mathcal{F} = F_-$. Introducing the L^1-norm of $\widehat{f}$ we conclude

3.10 Proposition. *One has the inequality*

$$\mathcal{F}(c + re^{i\phi}\big| \leq \frac{||\widehat{f}||_1}{2\pi} \cdot e^{-Ar\cdot|\sin\phi|} \quad : \quad 0 \leq \phi \leq 2\pi$$

3.11 The subharmonic function $U = \mathbf{Log}\,|\mathcal{F}|$. Replacing f by $c \cdot f$ with some constant c we assume that $\frac{||\widehat{f}||_1}{2\pi} \leq 1$. Then Proposition 3.10 gives the inequality

$$U(c + re^{i\phi}) \leq -Ar \cdot |\sin\phi| \quad : \quad -\pi \leq \phi \leq \pi$$

Since U is subharmonic we can apply Harnack's inequality from XX and conclude

3.12 Proposition. *One has the inequality*

$$U(x) \leq -\frac{Ar}{2\pi} \cdot \quad : \quad c - \frac{r}{2} \leq x \leq c + \frac{r}{2}$$

3.13 A vanishing theorem. In addition to the inequality in Proposition 3.12 which is valid for every open interval of the x-axis outside the support of f, we also have the estimate from Proposition 3.10. This gives

$$(*) \qquad U(x + iy) \leq -A|y|$$

when $||\widehat{f}||_1 \leq 2\pi$. Now we can apply the general result from XX. Namely, if suppose that there is a sequence of disjoint intervals $\{(a_\nu, b_\nu)$ are outside the support of f. Then

$$(**) \qquad \sum (b_\nu - a_\nu) \cdot \int_{a_\nu}^{b_\nu} \frac{dx}{1 + x^2} < \infty$$

must hold unless f is identically zero. This gives a uniqueness theorem which can be phrased as follows. Let $0 < c_1 < c_2 \ldots$ where each c_ν is the mid-point of an interval (a_ν, b_ν) and these intervals are disjoint. We say that this interval family is thick if

$$\sum \frac{(b_\nu - a_\nu)^2}{c_\nu^2} = +\infty$$

3.14 Theorem. *Let $f(x)$ be a continuous function on the x-line be such its support is disjoint from a thick union of intervals and $\widehat{f}(\xi)$ is integrable. Then $\widehat{f}$ cannot be identically zero on any open subinterval of the ξ-line unless f is identically zero.*

Remark. The proofs above are taken from i [Benedicks] and we refer to [loc.cit] for further gap-theorems which are derived using the Fourier-Carleman transform.

§ 12. The generalised Fourier transform.

Introduction. The book *L'Integrale de Fourier et questions qui s'y rattachent* published in 1944 by Institute Mittag-Leffler is based upon Carleman's lectures at the institute in 1935. In the introduction he writes: *C'est avant tous les travaux fondamentaux de M. Wiener et Paley qui ont attiré mon attention.* The book *Fourier transforms on the complex domain* by Raymond Paley and Norbert Wiener was published the year before. We expose material from Chapter II in [Car] which leads a generalised Fourier transform and goes beyond the ordinary Fourier transform for tempered distributions on the real line. This generalised Fourier transform is used when analytic function theory is applied to study singular integral equations with non-temperate solutions. An example is the Wiener-Hopf equation where one seeks eigenfunctions $f(x)$ to the integral equation

$$(*) \qquad f(x) = \int_0^\infty K(x-y)f(y)dy$$

In many physical applications the kernel K has exponential decay and one seeks eigenfunctions f which are allowed to increase exponentially. The major result about solutions (*) appear in Theorem XVI on page 56 in [Pa-Wi] based upon the article [Ho-Wi]. This inspired Carleman to the constructions in § 1 below. Let us remark that the generalised inversion formula in Theorem 6.3 leads to the calculus of hyperfunctions. For comments about the relation between Carleman's original constructions and later studies of hyperfunctions we refer to the article [Kis] by Christer Kiselman.

6.0 A special construction.. Let $f(z)$ be a bounded analytic in the upper half plane $\mathfrak{Im}\, z > 0$ and suppose it extends to a continuous function on the closed half-plane and that the boundary function $f(x)$ is integrable, i.e.

$$\int_{-\infty}^\infty |f(x)|\, dx < \infty$$

To each $0 \leq \theta \leq \pi$ we set

$$(6.0.1) \qquad G_\theta(\zeta) = \int_0^\infty e^{i\zeta re^{i\theta}} \cdot f(re^{i\theta})e^{i\theta}\, dr$$

With $\zeta = se^{i\phi}$ we have

$$|e^{i\zeta re^{i\theta}}| = e^{-sr\sin(\phi+\theta)}$$

Hence (6.0.1) converges if $\sin(\phi+\theta) > 0$, i.e when

$$(ii) \qquad -\theta < \phi < \pi - \theta$$

So $G_\theta(z)$ is analytic in a half-space. In particular G_0 is analytic in $\mathfrak{Im}\, \zeta > 0$ while G_π is analytic in the lower half-plane. Moreover these G-functions are glued as θ-varies. To see this we notice that (0.1) is the complex line integral

$$\int_{\ell_+(\theta)} e^{i\zeta z} \cdot f(z)\, dz$$

where $\ell_+(\theta)$ is the half-line $\{re^{i\theta} : r \geq 0\}$. Hence there exists an analytic function $G^*(z)$ in $\mathbf{C} \setminus [0,+\infty)$ which is equal to $G_\theta(z)$ in every half-space defined via (ii). Next, with $\zeta = \xi + i\eta$ there exists a limit

$$\lim_{\epsilon \to 0} G_0(\xi + i\epsilon) = \int_0^\infty e^{i\xi x} f(x)\, dx$$

Similarly the reader may verify that

$$(iii) \qquad \lim_{\epsilon \to 0} G_\pi(\xi - i\epsilon) = -\int_0^\infty e^{-i\xi r} f(-r)\, dr = -\int_{-\infty}^0 e^{i\xi r} f(r)\, dr$$

Passing to the usual Fourier transform of $f(x)$ we therefore get the equation

$$(iv) \qquad \widehat{f}(-\xi) = G_0(\xi + i0) - G_\pi(\xi - i0)$$

where we have taken boundary values of the analytic functions G_0 and G_π. Now

$$\text{(v)} \qquad G^*(\xi) = G_0(\xi + i0) = G_\pi(\xi - i0) \quad : \xi < 0$$

Hence (iv) entials that $\widehat{f}(-\xi) = 0$ when $\xi < 0$ and reversing signs we conclude that the support of $\widehat{f}$ is contained in the half-line $\{\xi \leq 0\}$. This inclusion has been seen before since the L^1-function $f(x) = f(x + i0)$ is the boundary value of an analytic function in the upper half-plane. Moreover (iv) means that $\widehat{f}$ on $\{\xi < 0\}$ is expressed as the difference of the boundary values of G_0 and G_π taken on the positive real ξ-line. Notoce that this difference is expressing obstructions for the G^*-function to extend across intervals on the positive ξ-line. So in this sense G^* alone determines $\widehat{f}$.

The observations above which were used in work by Plaey and Wiener led Carelan to perform similar cointructions where regularity and growth properties are relaxed.

6.1 Carleman's constructions

Let U^* be the upper half-plane. To each pair of real numbers a, b we denote by $\mathcal{O}_{a,b}(U^*)$ the family of functions $f \in \mathcal{O}(U^*)$ such that for every $0 < \theta_0 < \pi/2$ there exists a constant $A(\theta_0)$ and

$$(*) \qquad |f(re^{i\theta})| \leq A(\theta_0) \cdot \big(r^a + \frac{1}{r^b}\big) \quad : \quad \theta_0 < \theta < \pi - \theta_0$$

Remark. No condition is imposed on the A-function as $\theta_0 \to 0$. In particular $f(z)$ need not have tempered growth as one approaches the real line. In the same way we define the family $\mathcal{O}_{a,b}(U_*)$ of analytic functions defined in the lower half-plane U_* satisfying similar estimates as above.

Example. Let $f(z)$ be the ordinary Fourier-Laplace transform of a tempered distribution μ on the real t-line supported by the half-line $t \leq 0$. Recall that this gives an integer N and a constant C such that

$$|f(x + iy)| \leq C \cdot y^{-N} \quad : \quad y > 0$$

Here we can take $a = b = N$ and $A(\theta) = \frac{C}{\sin(\theta)}$ to get $f(z)$ in $\mathcal{O}_{a,b}(U^*)$.

Let us return to the general case. Consider some $f \in \mathcal{O}_{a,b}(U^*)$. If $b \geq 1$ we choose a positive integer m so that $b < 1 + m$ and when $0 < \theta < \pi$ we consider the half space

$$U^*_\theta = \{z = re^{i\phi} \quad : -\pi - \theta < \phi < -\theta\}$$

The choice of m and (*) give an analytic function in $U^*_\theta(z)$ defined by:

$$\text{(i)} \qquad F_\theta(z) = \frac{i}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-izre^{i\theta}} \cdot r^m \cdot e^{im\theta} \cdot f(re^{i\theta})e^{i\theta} \cdot dr$$

Cauchy's theorem applied to the analytic function $f(z)$ shows that these F_θ-functions are glued together as we rotate the angle θ in the open interval $(0, \pi)$. Notice that

$$\text{(ii)} \qquad \cup_{0<\theta<\pi} U^*(\theta) = \mathbf{C} \setminus [0, +\infty)$$

Hence there exists an analytic function $F^*(z)$ in $\mathbf{C} \setminus [0, +\infty)$ such that

$$\text{(iii)} \qquad F^*\big| U^*_\theta = G_\theta \quad : \quad 0 < \theta < \pi$$

Next, let U_* be the lower half-plane where one defines the family $\mathcal{O}_{a,b}(U_*)$. If $g(z)$ belongs to $\mathcal{O}_{a,b}(U_*)$ we obtain exactly as above analytic functions

$$G_\theta(z) = \frac{i}{\sqrt{2\pi}} \cdot \int_0^\infty e^{izre^{i\theta}} \cdot r^m \cdot e^{im\theta} \cdot g(re^{-i\theta})e^{-i\theta} \cdot dr \quad : \quad 0 < \theta < \pi$$

defined in the half-planes

$$U_*(\theta) = \{z = re^{i\phi} \quad : -\pi + \theta < \phi < \theta\}$$

These G_θ-functions are again glued together and give an analytic function $G_*(z)$ in $\mathbf{C} \setminus (-\infty, 0]$ where which satisfies:

(iii) $$G_*\Big| U_*(\theta) = G_\theta \quad : \quad 0 < \theta < \pi$$

The $\mathcal{S}$-transform. Consider a pair $f \in \mathcal{O}_{a,b}(U^*)$ and $g \in \mathcal{O}_{a,b}(U_*)$. We get the functions F^* and G_* and here $G_* - F^*$ is analytic outside the real axis and can be restricted to both the upper and the lower half-plane. This enable us to give the following:

6.2 Definition. *To every pair f, g as above we set*

$$\mathcal{S}^*(z) = G_*(z) - F^*(z) \quad : \quad \mathfrak{Im}(z) > 0$$

$$\mathcal{S}_*(z) = G_*(z) - F^*(z) \quad : \quad \mathfrak{Im}(z) < 0$$

Remark. The constructions of F^* and G_* entail that $G_* - F^*$ restricts to a function in $\mathcal{O}_{a,b}(U)$ when U is the upper or the lower half-plane. Hence $\mathcal{S}$ is a map from $O_{a,b}(U^*) \times \mathcal{O}_{a,b}(U_*)$ into itself.

6.3 The reflection operator. If $\phi \in \mathcal{O}(U^*)$ we get the analytic function in the lower half-plane defined by

$$T(\phi)(z) = \bar{\phi}(\bar{z})$$

In the same way T sends an analytic function defined in U_* to an analytic function defined in U^*. The composed operator $T \circ \mathcal{S}$ gives a pair of analytic functions defined by

$$(T \circ \mathcal{S})^*(z) = \bar{\mathcal{S}}_*(\bar{z}) \quad : \quad \mathfrak{Im}(z) > 0$$

$$(T \circ \mathcal{S})_*(z) = \bar{\mathcal{S}}^*(\bar{z}) \quad : \quad \mathfrak{Im}(z) < 0$$

With the notations above the result below extends Fourier's inversion formula for tempered distributions. Below $\simeq$ means that two functions differ by a polynomial in z.

6.3 Inversion Theorem. *For each pair (f, g) in $\mathcal{O}_{a,b}(U^*) \times \mathcal{O}_{a,b}(U^*)$ one has*

$$T \circ \mathcal{S} \circ T \circ \mathcal{S}(f) \simeq f \quad : \quad T \circ \mathcal{S} \circ T \circ \mathcal{S}(g) \simeq g$$

where $\simeq$ means that the differences are polynomials in z.

Remark. Theorem 6.3 is the assertion from p. 49 in [Car]. For details of the proof we refer to [loc.cit. p. 50-52]. The proof relies upon results of analytic extensions across a real interval. Since these results have independent interest we proceed to discuss material from from [ibid] and once this has been done we leave it to the reader to discover the proof of Theorem 6.3 or consult Carleman's proof.

6.4 Some analytic extensions.

Let D be the unit disc centered at the origin and set

$$D^* = D \cap \mathfrak{Im}(z) > 0 \quad \text{and} \quad D_* = D \cap \mathfrak{Im}(z) < 0$$

6.5 Theorem. *Let $f^* \in \mathcal{O}(D^*)$ and $f_* \in \mathcal{O}(D_*)$ be such that*

(*) $$\lim_{y \to 0} f^*(x + iy) - f_*(x - iy) = 0$$

holds uniformly with respect to x. Then there exists $F \in \mathcal{O}(D)$ with $F|D^* = f^*$ and $F|D_* = f_*$.

Remark. No special assumptions are imposed on the two functions except for (*). For example, it is from the start not assumed that they have moderate growth as one approaches the real x-line in (*).

Proof. In D^* we get the analytic function

(i) $$G(z) = f^*(z) - \bar{f}_*(\bar{z})$$

Write $G=U+iV$ and notice that (*) gives

(ii) $$\lim_{y\to 0} U(x,y)=0$$

Hence the harmonic function U in D^* converges uniformly to zero on the part of ∂D^* defined by $y=0$. If $\delta>0$ is small we restrict U to the upper half-disc $D^*(\delta)$ of radius $1-\delta$. Now (ii) implies that when G is expressed by the Poisson kernel of $D^*(\delta)$ then the boundary integral is only taken over the upper half-circle. It follows by the analyticity of the kernel function for $D^*(\delta)$ that $G(x,y)$ extends to a real analytic function across the the real interval $-1+\delta<x<1-\delta$. The same holds for the derivatives $\partial G/\partial x$ and $\partial G/\partial y$. The Cauchy Riemann equations show that the complex derivative of $F(z)$ extends analytically across the real interval and the reflection principle by Schwarz finishes the proof.

6.6 Another continuation. We expose another result from [ibid]. See [Car: p. 40: Théorème 3] whose the essential ingredient is a subharmonic property for the radius of convergence of analytic functions. Put

$$\square=\{(x,y):\ -1<x<1\quad\text{and}\quad 0<y<1\}$$

Consider some $F(z)\in\mathcal{O}(\square)$. With a small $\ell>0$ we put

$$D_+(\ell)=\{|\zeta|<\ell\cap\mathfrak{Im}(\zeta)>0\}$$

With $z_0=x_0+iy_0$ where $-1/2<x_0<1/2$ and $0<y_0<1-\ell$. we get an analytic function

(i) $$G_\zeta(z)=F(z+\zeta)-F(z)$$

which is defined in some neighborhood of z_0. It has a series expansion:

$$G_\zeta(z)=F(z+\zeta)-F(z)=\sum P_\nu(\zeta)(z-z_0)^\nu\quad\text{where :}$$

(*) $$P_\nu(\zeta)=\frac{1}{\nu!}\cdot[F^{(\nu)}(z_0+\zeta)-F^{(\nu)}(z_0)]$$

Let $\rho(\zeta)$ be the radius of convergence for the series (*). Hadamard's formula gives:

(**) $$\log\frac{1}{\rho(\zeta)}=\limsup_{\nu\to\infty}\frac{\log|P_\nu(\zeta)|}{\nu}$$

Suppose we knew that

(***) $$\rho(\zeta)\geq y_0\quad:\quad\zeta\in D_+(\ell)$$

Then we can pick $\zeta=\frac{iy_0}{2}$ and conclude that the function

(ii) $$z\mapsto F(z+\frac{iy_0}{2})-F(z)$$

is analytic in the disc $|z-z_0|<y_0$. At the same time the function $z\mapsto F(z+\frac{iy_0}{2})$ is analytic when $\mathfrak{Im}(z)>-\frac{y_0}{2}$ and hence $F(z)$ extends as an analytic function across a small interval on the real x-axis centered at x_0. So if (***) holds for every $-1/2<x_0<1/2$. it follows that $F(z)$ extends analytically across the real interval $-1(2<x<1/2$. There remains to find a condition on F in order that (***) holds. Notice that it suffices to get (***) for sufficiently small y_0 if we seek some analytic extension of F across the real x-line. To obtain (***) Carleman imposed the following:

6.7 Hypothesis on F. *There exists a pair $\ell>0$ and $\delta>0$ such that if ξ is real with if $|\xi|<\ell$ then $z\mapsto G_\xi(z)$ extends to an analytic function in the domain where $|z|<1/2$ and $\mathfrak{Im}(y)>-\delta$.*

It is clear that this hypothesis implies that if y_0 is sufficiently small then there exists a constant k such that

(1) $$|P_\nu(\zeta)\leq k^\nu\quad:\quad\zeta\in D_+(\ell)\quad:\nu=1,2,\ldots$$

Moreover, we see from a figure that the hypothesis also implies that

(2) $$\rho(\zeta)\geq y_0\quad:\quad|\zeta|=\ell\quad:\mathfrak{Im}(\zeta)\geq 0$$

It is also trivial that

$$\text{(3)} \qquad \rho(\zeta) \geq y_0 \quad : |\zeta| = \ell \quad : \mathfrak{Im}(\zeta) = 0$$

Proof that (***) holds

The functions $\zeta \mapsto \log |P_\nu(\zeta)|$ are subharmonic in $D_+(\ell)$ for every ν. So if G is Green's function for $D_+(\ell)$ we have the inequality

$$\text{(i)} \qquad \log |P_\nu(\zeta)| \leq \frac{1}{2\pi} \int_{\partial D_*(\ell)} \frac{\partial G(\zeta, w)}{\partial n_w} \cdot \frac{\log |P_\nu(w)|}{\nu} \cdot |dw|$$

Now (1) above entails that

$$\text{(ii)} \qquad \frac{\log |P_\nu(w)|}{\nu} \leq k \quad : w \in \partial D_+(\ell)$$

At the same time (2-3) and Hadamard's formula give

$$\text{(iii)} \qquad \limsup_{\nu \to \infty} \frac{\log |P_\nu(w)|}{\nu} \leq \log \frac{1}{y_0} \quad : w \in \partial D_+(\ell)$$

Thanks to (ii) we can apply Lebesgue's dominated convergence theorem when we pass to the limes superior in (i) and hence (iii) gives

$$\text{(iv)} \qquad \limsup_{\nu \to \infty} \frac{\log |P_\nu(\zeta)|}{\nu} \leq \log \frac{1}{y_0} \quad : \zeta \in D_+(\ell)$$

Now we apply Hadamard's formula for points in $D_+(\ell)$ and (***) follows.

Remark. The continuation found above can be applied to relax the assumption in Theorem 6.5. For example, there exists an analytic extension for a pair f^*, f_* under the less restrictive condition that

$$\lim_{y \to 0} \int_a^b \big[f^*(x+iy) - f_*(x-iy)\big] \cdot dx = 0$$

This follows when 6.6 is applied to the primitive functions of the pair.

§ 13. The Carleman-Hardy theorem.

Before we announce and prove Theorem 13.x we recall the Dini condition which ensures that converges pointwise in Fourier's inversion formula. Let $f(x)$ be an even and continuous function which is zero outside $[-1,1]$.

13.1 Dini's condition. *It holds at $x=0$ when*

$$(*) \qquad \int_0^1 \frac{|f(x)|}{x}\cdot dx<\infty$$

From now on (*) is assumed. Since f is even we have:

$$\widehat{f}(\xi)=2\cdot\int_0^1 \cos(x\xi)\cdot f(x)\cdot dx$$

With $A>0$ we set

$$(1) \qquad \gamma(A)=\frac{1}{2\pi}\int_{-A}^{A}\widehat{f}(\xi)\cdot d\xi$$

Our aim is to show that Dini's condition implies that

$$(**) \qquad \lim_{A\to\infty}\gamma(A)=f(0)$$

To prove (**) we first evaluate (1) which gives

$$(2) \qquad \gamma(A)=\frac{2}{\pi}\int_0^1 \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx$$

Next, we have the limit formula

$$(3) \qquad \lim_{A\to\infty}\frac{2}{\pi}\int_0^1\frac{\sin(Ax)}{x}\cdot dx=\frac{2}{\pi}\int_0^A\frac{\sin(t)}{t}\cdot dt=1$$

So in order to get $\gamma(A)\to f(0)$ we can replace f by $f(x)-f(0)$, i.e. it suffices to show that $\gamma(A)\to 0$ when $f(0)=0$ is assumed. To obtain this we fix some $0<\delta<1$ and put

$$(4) \qquad \gamma_\delta(A)=\frac{2}{\pi}\int_0^\delta \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx$$

Since $|\sin(Ax)|\leq 1$ the triangle inequality gives

$$(5) \qquad \gamma_\delta(A)\leq\int_0^\delta\frac{|f(x)|}{x}\cdot dx<\infty$$

for all A and every $\delta>0$. Dini's condition implies that the right hand side tends to zero as $\delta\to 0$. Next, we set

$$(6) \qquad \gamma^\delta(A)=\frac{2}{\pi}\int_\delta^1 \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx$$

Here $\frac{f(x)}{x}$ is continuous on $[\delta,1]$ and has therefore a finite modulus of continuity, i.e. we get the function

$$(7) \qquad \omega_\delta(r)=\max_{\delta\leq x_1,x_2\leq 1}\Big|\frac{f(x_1)}{x_1}-\frac{f(x_2)}{x_2}\Big| \quad \text{where} \quad |x_1-x_2|\leq r$$

With these notations one has the inequality:

$$(***) \qquad \gamma^\delta(A)\leq\frac{8\pi}{\pi}\cdot\omega_\delta(\frac{2\pi}{A})$$

The verification is left to the reader as an exercise. We remark only that the extra factor 4 replacing 2 by 8 comes form $4=\int_0^{2\pi}|\sin(t)|\cdot dt$. Hence we have

$$(8) \qquad \gamma(A)\leq\gamma_\delta(A)+\frac{8\pi}{\pi}\cdot\omega_\delta(\frac{2\pi}{A})$$

This holds for all pairs δ and A and now we conclude that Dini's condition indeed gives the limit formula in (**).

Remark. Above $x = 0$. More generally we can impose Dini's condition for f at an arbitrary point a, i.e. for every a we set

$$D_f(a) = \int \frac{|f(x) - f(a)|}{|x - a|} \cdot dx$$

The results above show that whenever $D(a) < \infty$ one has a pointwise limit

$$f(a) = \lim_{A \to \infty} \frac{1}{2\pi} \cdot \int_{-A}^{A} e^{ia\xi} \cdot \widehat{f}(\xi) \cdot d\xi \tag{1}$$

An example when this occurs is when $f(x)$ is Hölder continuous of some order > 0.

13.2 Another criterion for pointwise convergence. Let $u(x)$ be an even L^1-function which is zero outside $[-1, 1]$ and of class C^2 when $x \neq 0$. Moreover, we assume that there exists a constant C such that

$$|u''(x)| \leq \frac{C}{x^2} \quad : \quad x \neq 0 \tag{13.2.1}$$

Since u is an L^1-function we can construct the Fourier series

$$F_u(x) = \sum_{n=0}^{\infty} A_n \cdot \cos nx \quad \text{where} \quad A_n = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\xi) \cdot u(\xi) \cdot d\xi$$

Since u is a C^2-function when $x \neq 0$ the series converges uniformly to u on every interval $[\delta, 1]$ when $0 < \delta < 1$. There remains to analyze the situation at $x = 0$. The following result was proved by Carleman and Hardy in their joint article *xxx.*

13.3 Theorem. *Under the conditions above the series $\sum A_n$ converges if and only if there exists a finite limit*

$$\lim_{x \to 0} u(x) = S_*$$

Moreover, when this holds one has the equality $\sum A_n = S_*$.

The proof of relies upon some inequalities about differentiable functions which we begin to expose.

13.4 Lemma. *Let $\psi(x)$ be a C^1-function defined for $x > 0$ such that*

$$\lim_{x \to 0} \psi(x) = 0 \quad \text{and} \quad |\psi'(x)| \leq \frac{C}{x} \quad : x > 0$$

holds for some constant C. Then it follows that

$$\lim_{x \to 0} \psi'(x) = 0 \tag{*}$$

The proof is left as an exercise to the reader. See also the article *Contributions to the Arithmetic Theory of Series* by Hardy and Littlewood for further limit formulas of higher order derivatives. Next we establish a result due to Landau from the article *Einige ungleichungen für zweimal differentierbare Funktionen.*

13.5 Proposition. *Let $\psi(x)$ be a C^3-function defined on $x > 0$ such that*

$$\lim_{x \to 0} \frac{\psi(x)}{x^2} = 0 \quad \text{and} \quad |\psi'''(x)| \leq \frac{C}{x} \tag{i}$$

hold for some constant C. Then it follows that

$$\lim_{x \to 0} \psi''(x) = 0 \tag{ii}$$

Proof. Let $x > 0$ and set $\xi = \zeta \cdot x$ where $0 < \zeta < 1/2$. Keeping these numbers fixed, Taylor's formula gives

$$\psi(x + \xi) + \psi(x - \xi) - 2\psi(x) = \xi^2 \psi''(x) + \frac{\xi^3}{6} \cdot [\psi'''(x + \theta_1 \xi) - \psi'''(x - \theta_2 \xi)] \quad : \; 0 < \theta_1, \theta_2 < 1$$

The triangle inequality gives

$$|\psi''(x)| \leq$$

(2) $$\frac{1}{\xi^2} \cdot [|\psi(x+\xi)| + |\psi(x-\xi)| + 2|\psi(x)|] + \frac{\xi^3}{6} \cdot [\big(|\psi'''(x+\theta_1\xi)| + |\psi'''(x-\theta_2\xi)|\big)$$

By the second condition in (i) the last term above is majorised by

(3) $$\frac{C}{6} \cdot \xi \cdot (\frac{1}{x+\theta_1\xi} + \frac{1}{x-\theta-2\xi}] \leq \frac{C}{6} \cdot \frac{2\zeta}{1-\zeta}$$

Given $\epsilon > 0$ we can choose ζ so small that (3) is $< \epsilon/2$. Next, keeping ζ fixed the first term in (2) above is majorised by

(4) $$\frac{1}{\zeta^2 x^2} \cdot \big[(1+\zeta)^2 \cdot o(x^2) + (1-\zeta)^2 \cdot o(x^2) + 2 \cdot o(x^2)\big]$$

where the small ordo terms follows from the first condition in (i). Now (ii) in Proposition 13.2 follows from the inequality (2) above.

Proof of Theorem 13.3

Assume first that $\sum A_n$ converges and define the following two functions when $x > 0$:

(i) $$U(x) = \frac{1}{2}A_0 \cdot x^2 + \sum_{n=1}^{\infty} \frac{A_n}{n^2} \cdot (1 - \cos(nx))$$

(ii) $$V(x) = \int_0^x \big[\int_0^y u(s) \cdot ds\,\big] \cdot dy$$

In (i) $U(x)$ is the associated Riemann function of u. Since u is of class C^2 when $x > 0$ it is clear that $U''(x) = V''(x)$ when $x > 0$ and hence $U(x) - V(x)$ is a linear function $C + Dx$ on $(0, +\infty)$. Since u by assumption is an L^1-function we see that $V(x) = o(x)$ and we also have $U(x) = o(x^2)$ by a classical result known as Riemann's Lemma. It follows that $C = D = 0$, i.e. the functions U and V are identical which gives

$$V(x) = o(x^2)$$

Next, notice that

(ii) $$x > 0 \implies V'''(x) = u'(x)$$

By (13.2.1) we have $|u''(x)| \leq \frac{C}{x^2}$ which gives another constant C^* such that $|u'(x)| \leq \frac{C^*}{x}$. Hence (ii) implies that V satisfies the Landau conditions in Proposition 13.5 which gives:

(iii) $$\lim_{x \to 0} V''(x) = 0$$

Finally, since $V''(x) = u(x)$ when $x > 0$ we conclude that $\lim_{x\to 0} u(x) = 0$. This proves one half of Theorem 13.3

The case $\lim_{x\to 0} u(x) = 0$. There remains to show that when this limit condition holds, then the series $\sum A_n$ converges. By assumption $|u''(x)| \leq \frac{C}{x^2}$ which gives a constant C^* such that $|u'(x)| \leq \frac{C^*}{x}$ and Lemma 13.4 gives

(i) $$\lim_{x \to 0} u'(x) = 0$$

Next, we use a result by Lebesgue from his book *Lecons des series trigonmétriquees* which asserts that the series $\sum A_n$ converges if

(ii) $$\lim_{\epsilon \to 0} \int_\epsilon^1 \frac{|u(x+\epsilon) - u(x)|}{x} \cdot dx = 0$$

To get (ii) we use Rolle's theorem and write

(iii) $$u(x+\epsilon) - u(x) = \epsilon \cdot u'(x + \theta \cdot \epsilon)$$

Now it is clear that (i) and (iii) give (ii) which finishes the proof that $\sum A_n$ is convergent.

14. The use of subharmonic majorizations.

Let $\{0 < \beta_1 < \beta_2 < \ldots\}$ be a sequence of positive real numbers. Given $a > 0$ we construct harmonic functions in the right half-plane $\Omega = \{\mathfrak{Re}\, z > a\}$ as follows: Set $q_\nu = a + i\beta_\nu$ and to each $z \in \Omega$ we get the triangle with corner points at the three points $z, eq_\nu, eq_{\nu+1}$ where e is Neper's constant. Denote the angle at z by $\theta_\nu(z)$. Notice that $0 < \theta_\nu(z) < \pi$. As explained in § XX $\theta_\nu(z)$ is a harmonic function with the boundary value π on $\mathfrak{Re}\, z = a$ when $\beta_\nu < y < \beta_{\nu+1}$ while the boundary value is zero outside the closed y-interval $[\beta\nu, \beta_{\nu+1}]$. If we instead consider the points $\{q^*_\nu = a - ie\beta_\nu\}$ we get similar harmonic angle functions $\{\theta^*_\nu\}$ when we regard the angle at z formed by the triangle with corner points at $z - eq_\nu, -eq_{\nu+1}$.

Exercise. Show by euclidian geometry that if $b < 0$ is real and positive then

$$\sin\, \theta_\nu(a+b) = \frac{eb(\beta_{\nu+1} - \beta_\nu)}{\sqrt{(\beta_\nu^2 + b^2)(\beta_{\nu+1}^2 + b^2)}} \tag{1}$$

Use also that $\beta_\nu < \beta_{\nu+1}$ and show that (1) gives

$$\theta_\nu(a+b) > \frac{eb\beta_1^2 \cdot (\beta_{\nu+1} - \beta_\nu)}{\beta_{\nu+1} \cdot \beta_\nu \cdot (e^2\beta_1^2 + b^2)} = C(b, \beta_1) \cdot (\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) \quad : \quad C(b, \beta_1) = \frac{eb\beta_1^2}{e^2\beta_1^2 + b^2}$$

In addition to these θ_ν-functions we get the angle functions $\{\theta^*_n\}$ where we for each $n \geq 2$ consider the harmonic extension to the half-plane whose boundary values are π on $y > \beta_n$ and zero when $y < \beta_n$. This harmonic function is denoted by $\theta^*_n(z)$ and by a figure the reader can verify that

$$\sin\, \theta^*_n(a+b) = \frac{b}{\beta_n^2 + b^2)} \implies \theta^*_n(a+b) > \frac{eb\beta_1^2}{e^2\beta_1^2 + b^2} \cdot \frac{1}{\beta_n} \tag{2}$$

7.1 A class of harmonic functions. Let us also consider a sequence of positive real numbers $\{\lambda_\nu\}$. To each $n \geq 2$ we get the harmonic function $u_n(x,y)$ in Ω defined by

$$u_n(x,y) = \frac{1}{\pi} \cdot \sum_{\nu=1}^{\nu=n-1} \lambda_\nu \cdot (\theta_\nu + \theta^*_\nu) + \lambda_n \cdot (\theta_n + \theta^*_n) \tag{3}$$

If $z = a + b$ is real with $b > 0$ the inequalities in (1-2) give

$$u_n(a+b) \geq \frac{C(b, \beta_1)}{\pi} \cdot [\, \sum_{\nu=1}^{\nu=n-1} \lambda_\nu (\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) + \frac{\lambda_n}{\beta_n}\,] \tag{4}$$

From the above we get:

7.2 Proposition. *Let $\{\lambda_\nu\}$ and $\{\beta_\nu\}$ be such that*

$$\lim_{n\to\infty} [\, \sum_{\nu=1}^{\nu=n-1} \lambda_\nu (\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) = +\infty \tag{*}$$

Then the sequence $\{u_n(a+b)\}$ increases to $+\infty$ for every $b > 0$.

Remark. If the λ-sequence increases a partial summation gives

$$\sum_{\nu=1}^{\nu=n} \frac{\lambda_\nu - \lambda_{\nu-1}}{\beta_\nu} = \sum_{\nu=1}^{\nu=n-1} \lambda_\nu (\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}})$$

where we have put $\lambda_0 = 0$. Hence (*) is equivalent to the divergence of the positive series

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu - \lambda_{\nu-1}}{\beta_\nu} \tag{**}$$

7.3 An application. Let $\{\beta_\nu\}$ and $\{\lambda_\nu\}$ be two strictly increasing sequences of positive real numbers. Consider an analytic function $\Phi(z)$ defined in the half-plane $\mathfrak{Re}\, z > a$ with continuous boundary values on $\mathfrak{Re}\, z = a$ which satisfies the inequalities

$$(*) \qquad |\Phi(z)| \leq \big(\frac{\beta_\nu}{|z|}\big)^{\lambda_\nu} \quad : \quad \nu = 1, 2, \ldots$$

7.4 Exercise. Denote by $u_*(z)$ the harmonic function in the half-plane whose boundary values are one on $-\beta_1 < y < \beta_1$ and otherwise zero. Show that (*) gives the following inequality on $\mathfrak{Re}\, z = a$ for every $n \geq 0$ and $-\infty < y < +\infty$

$$(7.4.1) \qquad \log|\Phi(a+iy)| + u_n(a+iy) \leq \log K \cdot u_*(a+iy) \quad : \quad K = \max_{-\beta_1 \leq y \leq \beta_1} |\Phi(a+iy)|$$

Since u_n and u_* are harmonic functions while $\log|\Phi|$ is subharmonic, the principle of harmonic majorization implies that (7.4.1) holds in Ω. In particular, for every real $b > 0$ we have

$$(7.4.2) \qquad \log|\Phi(a+ib)| + u_n(a+ib) \leq \log K \cdot u_*(a+ib)$$

When Φ is not identically zero we can fix some $b > 0$ where $\Phi(a+ib) \neq 0$ and (7.4.2) entails that the sequence $\{u_n(a+ib)\}$ is bounded. Hence Proposition 7.2 gives

7.5 Theorem. *Assume there exists a non-zero analytic function $\Phi(z)$ in the half-plane $\mathfrak{Re}\, z > a$ such that (*) holds in (7.3). Then*

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu - \lambda_{\nu-1}}{\beta_\nu} < \infty$$

Remark. We can rephrase the result above and get a vanishing principle. Namely, if the positive series in Theorem 7.5 is divergent an analytic function $\Phi(z)$ in the half-plane satisfying (7.3) must be identically zero.

7.6 Example. Let $\{c_n\}$ be a sequence of positive real numbers. Suppose that $\Phi(z)$ is analytic in the half-space and satisfies

$$|\Phi(z)| \leq \frac{c_n}{|z|^n} \quad : \quad n = 1, 2, \ldots$$

Then Φ must vanish identically if the series

$$\sum_{n=1}^{\infty} \frac{1}{c_n^{\frac{1}{n}}} = +\infty$$

15. Convergence under substitution.

Introduction. Let $\{a_k\}$ be a sequence of complex numbers where the additive series $\sum a_k$ is convergent. This gives an analytic function $f(z)$ defined in the open disc by

$$f(z) = \sum a_n \cdot z^n \tag{1}$$

If $0 < b < 1$ we can expand f around b and obtain another series

$$f(b+z) = \sum c_n \cdot z^n \tag{2}$$

From the convergence of $\sum a_k$ one expects that the series

$$\sum c_n \cdot (1-b)^n \tag{3}$$

also is convergent. This is indeed true and was proved by Hardy and Littlewood in (H-L]. A more general result was established in [Carleman] and we are going to expose results from Carleman's article. In general, consider a power series

$$\phi(z) = \sum b_\nu \cdot z^\nu \tag{1}$$

which represents an analytic function D where $|\phi(z)| < 1$ hold when $|z| < 1$. There exists the composed analytic function

$$f(\phi(z)) = \sum_{k=0}^{\infty} c_k \cdot z^k \tag{*}$$

We seek conditions on ϕ in order that the convergence of $\{a_k\}$ entails that the series

$$\sum c_k \quad \text{also converges} \tag{**}$$

First we consider the special case when the b-coefficients are real and non-negative.

8.1. Theorem. *Assume that $\{b_\nu \geq 0\}$ and that $\sum b_\nu = 1$. Then (**) converges and the sum is equal to $\sum a_k$.*

Proof. Since $\{b_\nu\}$ are real and non-negative the Taylor series for ϕ^k also has non-negative real coefficients for every $k \geq 2$. Put

$$\phi^k(z) = \sum B_{k\nu} \cdot z^\nu$$

and for each pair of integers k, p we set

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=p} B_{k\nu}$$

If $k \geq 2$ we notice that

$$B_{kj} = \sum_{\nu=0}^{j} b_{j-\nu} \cdot B_{k-1,\nu}$$

Since $\{b_\nu\}$ are non.-begstive with sum equal to one the reader can easiy verify that the following hold:

$$\lim_{N\to\infty} \Omega_{N,p} = 0 \quad \text{for every} \quad p \tag{i}$$

$$k \mapsto \Omega_{k,p} \quad \text{decreases for every} \quad p \tag{ii}$$

$$\sum_{\nu=0}^{\infty} B_{k\nu} = 1 \quad \text{hold for every} \quad k \tag{iii}$$

Next, the Taylor series of $f(\phi(z))$ is given by

$$\sum a_k \cdot \phi^k(z) = \sum_{\nu=0}^{\infty} [\sum_{k=0}^{\infty} a_k \cdot B_{k\nu}] \cdot z^\nu$$

For each positive integer n^* we set

$$\sigma_p[n^*] = \sum_{\nu=0}^{\nu=p} [\sum_{k=0}^{k=n^*} a_k \cdot B_{k,\nu}] \tag{1}$$

$$\sigma_p(n^*) = \sum_{\nu=0}^{\nu=p} [\sum_{k=n^*+1}^{\infty} a_k \cdot B_{k,\nu}] = \sum_{k=n^*+1}^{\infty} a_k \cdot \Omega_{k,p} \tag{2}$$

Notice that

$$\sigma_p[n^*] + \sigma_p(n^*) = \sum_{k=0}^{k=p} c_k \quad \text{hold for each} \quad p$$

Our aim is to show that the last partial sums converge to 1 as $p \to \infty$. To obtain this we study the σ-terms separately. Introduce the partial sums

$$s_n = \sum_{k=0}^{k=n} a_k$$

By assumption there exists a limit $s_n \to S$ where $S = 1$. This entails that the sequence $\{s_k\}$ is bounded and so is the sequence $\{a_k = s_k - s_{k-1}\}$. By (i) above it follows that the last term in (2) tends to zero when n^* increases. So if $\epsilon > 0$ we find $n*$ such that

$$n \geq n^* \implies |\sigma_p(n)| \leq \epsilon \tag{3}$$

A study of $\sigma_p[n^*]$. Keeping n^* and ϵ fixed we apply (iii) for each $0 \leq k \leq n^*$ and find an integer p^* such that

$$1 - \sum_{\nu=0}^{\nu=p} B_{k,\nu} \leq \frac{\epsilon}{n^* + 1} \quad \text{for all pairs} \quad p \geq p^* : \ 0 \leq k \leq n^*$$

The triangle inequality gives

$$|\sigma_p(n^*) - s_{n^*}| \leq \frac{\epsilon}{n^* + 1} \cdot \sum_{k=0}^{k=n^*} |a_k| \quad \text{for all} \quad p \geq p^* \tag{4}$$

Since $\sum a_k$ converges the sequence $\{a_k\}$ is bounded, i.e. we have a constant M such that $|a_k| \leq M$ for all k. Hence (4) gives

$$|\sigma_p(n^*) - S| \leq \epsilon + \epsilon \cdot M \quad : \quad p \geq p^* \tag{5}$$

Together with (3) this entails that

$$n \geq n^* \implies |\sum_{k=0}^{n^*} |c_k - s| \leq 2\epsilon + M \cdot \epsilon$$

Since we can chose ϵ arbitrary small we conclude that $\sum c_k$ converges and the limit is equal to S which finishes the proof of Theorem 8.1.

8.2 A general case. Now we relax the condition that $\{b_\nu\}$ are real and non-negative but impose extra conditions on ϕ. First we assume that $\phi(z)$ extends to a continuous function on the closed disc, i.e. ϕ belongs to the disc-algebra. Moreover, we assume that $\phi(1) = 1$ while $|\phi(z)| < 1$ for all $z \in \bar{D} \setminus \{1\}$ which means that $z = 1$ is a peak point for ϕ. Consider also the function $\theta \mapsto \phi(e^{i\theta})$ where θ is close to zero. The final condition on ϕ is that there exists some positive real number β and a constant C such that

$$|\phi(e^{i\theta}) - 1 - i\beta| \leq C \cdot \theta^2 \tag{8.2.1}$$

holds in some interval $-\ell \leq \theta \geq \ell$. This implies that for every integer $n \geq 2$ we get another constant C_n so that

(8.2.2) $$|\phi^n(e^{i\theta}) - 1 - in\beta| \leq C_n \cdot \theta^2$$

Hence the following integrals exist for all pairs of integers $p \geq 0$ and $n \geq 1$:

(8.2.3) $$J(n,p) = \int_{-\ell}^{\ell} \frac{\phi(e^{i\theta})^n \cdot (1 - \phi(e^{i\theta})}{e^{ip\theta} \cdot (1 - e^{i\theta})} \cdot d\theta$$

With these notations one has

8.2 Theorem. *Let ϕ satisfy the conditions above. Then, if there exists a constant C such that*

(*) $$\sum_{k=0}^{\infty} |J(k,p)| \leq C \quad \text{for all} \quad p \geq 0$$

it follows that the series (**) from the introduction converges and the sum is equal to $\sum a_k$.

Proof With similar notations as in the previous proof we introduce the Ω-numbers by:

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=k} B_{k\nu}$$

Repeating the proof of Theorem 8.1 the reader may verify that the series $\sum c_k$ converges and has the limit S if the following two conditions hold:

(i) $$\lim_{N \to \infty} \Omega_{N,p} = 0 \quad \text{holds for every} \quad p$$

(ii) $$\sum_{k=0}^{\infty} \left|\Omega_{k+1,p} - \Omega_{k,p}\right| \leq C \quad \text{for a constant} \quad C$$

where C is is independent of p. Here (i) is clear since $\{g_N(z) = \phi^N(z)\}$ converge uniformly to zero in compact subsets of the unit disc and therefore their Taylor coefficients tend to zero with N. To get (ii) we use residue calculus which gives:

(iii) $$\Omega_{k+1,p} - \Omega_{k,p} = \frac{1}{2\pi i} \int_{|z|=1} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz$$

Let ℓ be a small positive number and T_ℓ denotes the portion of the unit circle where $\ell \leq \theta \leq 2\pi - \ell$. Since 1 is a peak -point for ϕ there exists some $\mu < 1$ such that

$$\max_{z \in T_\ell} |\phi(z)| \leq \mu$$

This gives

(iv) $$\frac{1}{2\pi} \cdot |\int_{z \in T_\ell} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz| \leq \mu^k \cdot \frac{2}{|e^{i\ell} - 1}|$$

Since the geometric series $\sum \mu^k$ converges it follows from (iii) and the construction of the J-integrals in (3) that (ii) above holds precisely when

$$\sum_{k=0}^{\infty} |J_\ell(k,p]| \leq C$$

for a constant C which is independent of p. This finishes the the proof of Theorem 8.2.

8.3. Oscillatory integrals. Condition (*) in Theorem 8.2.1 is implicit. We shall therefore find a sufficient condition in order that the J-integrals satisfy (*) which is expressed by local conditions on the ϕ-function close to $z = 1$. To begin with (8.2.1) enable us to write

(i) $$\phi(e^{i\theta}) = e^{i\beta\theta + \rho(\theta)}$$

when $\theta \simeq 0$ and

$$\text{(ii)} \qquad \rho(\theta) = O(\theta^2)$$

The next result gives the requested convergence of the composed series expressed by an additional condition on the ρ-function in (i) above.

8.4. Theorem. *Assume that $\rho(\theta)$ is a C^2-function on some interval $-\ell < \theta < \ell$ and that the second derivative $\rho''(0)$ is real and negative. Then (*) in Theorem 8.2 holds.*

Remark. The proof is left as a (hard) exercise to the reader. If necessary, consult Carleman's article [Car] for a detailed proof.

16. The series $\sum \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}}$

Introduction. We shall prove a result from [Carleman:xx. Note V page 112-115]. Let $\{a_\nu\}$ be a sequence of positive real numbers such that $\sum a_\nu < \infty$ and e denotes Neper's constant.

9.1 Theorem. *Assume that the series $\sum a_\nu$ is convergent and let S be the sum. Then one has the strict inequality*

$$(*) \qquad \sum_{\nu=1}^{\infty} \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} < e \cdot S$$

Remark. The result is sharp in the sense that e cannot be replaced by a smaller constant. To see this we consider a large positive integer N and take the finite series $\{a_\nu = \frac{1}{\nu} \quad : 1 \leq \nu \leq N\}$. Stirling's limit formula gives:

$$\left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} \simeq \frac{e}{\nu} \quad : \nu >> 1$$

Since the harmonic series $\sum \frac{1}{\nu}$ is divergent we conclude that for every $\epsilon > 0$ there exists some large integer N such that $\{a_\nu = \frac{1}{\nu}\}$ gives

$$\sum_{\nu=1}^{\nu=N} \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} > (e - \epsilon) \cdot \sum_{\nu=1}^{\nu=N} \frac{1}{\nu}$$

There remains to prove the strict upper bound (*) when $\sum a_\nu$ is a convergent positive series. To attain this we first establish inequalities for finite series. Given a positive integer m we consider the variational problem

$$(1) \qquad \max_{a_1,\ldots,a_m} \sum_{\nu=1}^{\nu=m} \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} \quad \text{when} \quad a_1 + \ldots + a_m = 1$$

Let $a_1^*, \ldots, a_m^*$ give a maximum and set $a_\nu^* = e^{-x_\nu}$. The Lagrange multiplier theorem gives a number $\lambda^*(m)$ such that if

$$y_\nu = \frac{x_\nu + \ldots + x_m}{\nu}$$

then

$$(2) \qquad \lambda^*(m) \cdot e^{-x_\nu} = \frac{1}{\nu} \cdot e^{-y_\nu} + \ldots + \frac{1}{m} \cdot e^{-y_m} \quad : \quad 1 \leq \nu \leq m$$

A summation over all ν gives

$$\lambda^*(m) = e^{-y_1} + \ldots + e^{-y_m} = \sum_{\nu=1}^{\nu=m} \left[a_1^* \cdots a_\nu^*\right]^{\frac{1}{\nu}}$$

Hence $\lambda^*(m)$ gives the maximum for the variational problem which is no surprise since $\lambda^*(m)$ is Lagrange's multiplier. Now we shall prove the strict inequality

$$(3) \qquad \lambda^*(m) < e$$

We prove (3) by contradiction. To begin with, subtracting the successive equalities in (2) we get the following equations:

$$\lambda^*(m)\cdot[e^{-x_\nu}-e^{-x_{\nu+1}}]=\frac{1}{\nu}\cdot e^{-y_\nu}\quad:\quad 1\le\nu\le m-1 \tag{4}$$

$$m\cdot\lambda^*(m)=e^{x_m-y_m} \tag{5}$$

Next, set

$$\omega_\nu=\nu(1-\frac{a_{\nu+1}}{a_\nu}):\quad 1\le\nu\le m-1 \tag{6}$$

With these notations it is clear that (4) gives

$$\lambda^*(m)\cdot\omega_\nu=e^{x_\nu-y_\nu}\quad:\quad 1\le\nu\le m-1 \tag{7}$$

It is clear that (7) gives:

$$\big(\lambda^*(m)\cdot\omega_\nu\big)^\nu=e^{\nu(x_\nu-y_\nu)}=\frac{a_1\cdots a_{\nu-1}}{a_\nu^{\nu-1}} \tag{8}$$

By an induction over ν which is left to the reader it follows the ω-sequence satisfies the recurrence equations:

$$\omega_\nu^\nu=\frac{1}{\lambda^*(m)}\cdot\big(\frac{\omega_{\nu-1}}{1-\frac{\omega_{\nu-1}}{\nu-1}}\big)^{\nu-1}\quad:\quad 1\le\nu\le m-1 \tag{9}$$

Notice that we also have

$$\omega_1=\frac{1}{\lambda^*(m)} \tag{10}$$

A special construction. With λ as a parameter we define a sequence $\{\mu_\nu(\lambda)\}$ by the recursion formula:

$$\mu_1(\lambda)=\frac{1}{\lambda}\quad\text{and}\quad[\mu_\nu(\lambda)]^\nu=\frac{1}{\lambda}\cdot\big[\frac{\mu_{\nu-1}(\lambda)}{1-\frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1}\quad:\quad\nu\ge 2 \tag{**}$$

From (5) and (9) it is clear that $\lambda=\lambda^*(m)$ gives the equality

$$\mu_m(\lambda^*(m))=m \tag{***}$$

Now we come to the key point during the whole proof.

Lemma *If $\lambda\ge e$ then the $\mu(\lambda)$ -sequence satisfies*

$$\mu_\nu(\lambda)<\frac{\nu}{\nu+1}\quad:\quad\nu=1,2,\dots$$

Proof. We use an induction over ν. With $\lambda\ge e$ we have $\frac{1}{\lambda}<\frac{1}{2}$ so the case $\nu=1$ is okay. If $\nu\ge 1$ and the lemma holds for $\nu-1$, then the recursion formula (**) and the hypothesis $\lambda\ge e$ give:

$$[\mu_\nu(\lambda)]^\nu=\frac{1}{\lambda}\cdot\big[\frac{\mu_{\nu-1}(\lambda)}{1-\frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1}<\frac{1}{e}\cdot\big[\frac{\frac{\nu-1}{\nu}}{1-\frac{\nu-1}{\nu(\nu-1)}}\big]^{\nu-1}$$

Notice that the last factor is 1 and hence:

$$[\mu_\nu(\lambda)]^\nu<e<\big(1+\frac{1}{\nu}\big)^{-\nu}$$

where the last inequality follows from the wellknown limit of Neper's constant. Taking the ν:th root we get $\mu_\nu(\lambda)<\frac{\nu}{\nu+1}$ which finishes the induction.

Conclusion. If $\lambda^*(m)\ge e$ then the lemma above and the equality (***) would entail that

$$m=\mu(\lambda^*(m))<\frac{m}{m+1}$$

This is impossible when m is a positive integer and hence we must have proved the strict inequality $\lambda^*(m)<e$.

The strict inequality for an infinite series. It remains to prove that the strict inequality holds for a convergent series with an infinite number of terms. So assume that we have an equality

(i) $$\sum_{\nu=1}^{\infty}\big[a_1\cdots a_\nu\big]^{\frac{1}{\nu}}=e\cdot\sum_{\nu=1}^{\infty}a_\nu$$

Put as as above

(ii) $$\omega_n=n(1-\frac{a_{n+1}}{a_n})$$

Since we already know that the left hand side is at least equal to the right hand side one can apply Lagrange multipliers and we leave it to the reader to verify that the equality (i) gives the recursion formulas

(iii) $$\omega_n^n=\frac{1}{e}\cdot\big[\frac{\omega_{n-1}}{1-\frac{\omega_{n-1}}{n-1}}\big]^{n-1}$$

Repeating the proof of the Lemma above it follows that

(iv) $$\omega_n<\frac{n}{n+1}\implies\frac{a_{n+1}}{a_n}>\frac{n}{n+1}$$

where (ii) gives the implication. So with $N\geq 2$ one has:

$$\frac{a_{N+1}}{a_1}>\frac{1\cdots N}{1\cdots N(N+1)}=\frac{1}{N+1}$$

Now $a_1>0$ and since the harmonic series $\sum\frac{1}{N}$ is divergent it would follows that $\sum a_n$ is divergent. This contradiction shows that a strict inequality must hold in Theorem 9.1.

16. Asymptotic series.

0. Introduction. A sharp version of the Phragmén-Lindelöf theorem is proved in Theorem A.1. Asymptotic series are studied in section B where earlier work by Poincaré and Borel led Carleman to the general construction in Theorem B.1. The question of uniqueness is expressed via Theorem B.5 whose proof relies upon a variational problem in Section C.

A. The Phragmén-Lindelöf theorem.

Let $f(z)$ be an entire function. To each $0 \leq \phi \leq 2\pi$ we set

(*) $$\rho_f(\phi) = \max_r |f(re^{i\phi})|$$

The text-book *Le calcus des residues* by Ernst Lindelöf contains examples of entire functions f where $\rho_f(\phi)$ is finite for all ϕ with the exception $\phi = 0$, i.e. only along the positive real axis the ρ-number fails to be bounded. An example is the entire function

$$f(z) = \frac{1}{z^2} \cdot \sum_{\nu=2}^{\infty} \frac{z^\nu}{(\log \nu)^\nu}$$

Here one verifies that that there exists a constant k such that:

(**) $$|f(re^{i\phi})| \leq \exp(e^{\frac{k}{|\phi|\cdot|2\pi-\phi|}})$$

It turns out that the example above is essentially sharp. Namely, assume that the ρ-number in (*) is finite for almost every ϕ. Then the ρ_f-function cannot be too small, unless f is reduced to a constant. Before Theorem A.1 is announced we introduce the non-negative function

(***) $$\omega(\phi) = \log^+ \bigl[\log^+ \rho_f(\phi)\bigr]$$

Since we have taken a two-fold logarithm $\omega(\phi)$ is considerably smaller compared to the ρ-function.

A.1.Theorem. *For every non-constant entire function $f(z)$ one has*

$$\int_0^{2\pi} \omega(\phi) \cdot d\phi = +\infty$$

Proof. Assume that f is not a constant. Consider the maximum modulus function

$$M(r) = \max_{|z|=r} |f(z)|$$

By the ordinary Liouville theorem the M-function increases to infinity. So we may assume that $M(r) \geq 1$ when $r \geq r_*$ for some r_*. Put

(i) $$v(r) = \log M(r) \quad : \quad U(z) = \log |f(z)|$$

Given $r \geq r_*$ we consider the domain

(ii) $$\Omega_r = \{U > \frac{v(r)}{2}\} \cap \{|z| < r\}$$

Next, let ζ_r be some point on the circle $|z| = r$ where $|f(\zeta_r)| = M(r)$ where ζ_r always can be chosen so that there exist arbitrary small δ where $|f(\zeta_{r-\delta})| = M(r-\delta)$ and $\lim_{\delta\to 0} \zeta_{r-\delta} = \zeta_r$. Next, in Ω we get the connected component Ω_* whose boundary contains ζ_r. Put

(iii) $$\gamma = \partial\Omega_* \cap \{|z| = r\}$$

Notice that

(iv) $$U(z) \leq \frac{v(r)}{2} \quad : \quad z \in \partial\Omega_* \cap \{|z| < r\}$$

So if W is the harmonic function in the disc D_r with boundary values 1 on γ and 0 on $\{|z| = 1\} \setminus \gamma$ we have:

(v) $$U(z) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(z) \leq 0 \quad : \quad z \in \partial\Omega_*$$

The maximum principle entails that (v) also holds in Ω_*. Hence there exist arbitrary small $\delta > 0$ such that

(vi) $$v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(\zeta_{r-\delta}) \leq 0$$

Let $2r \cdot \ell$ be the total length of the intervals which belong to γ. By the general inequality from XX we have

(vii) $$W(\zeta_{r-\delta}) \leq \frac{1}{2\pi} \int_{-\ell}^{\ell} \frac{r^2 - (r-\delta)^2}{r^2 - 2r(r-\delta)\cos\theta + (r-\delta)^2} d\theta$$

Let $h(r-\delta)$ denote the right hand side in (vii) which by (vi) gives us arbitrary small $\delta > 0$ such that

(viii) $$v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot h(r-\delta) \leq 0$$

Rewriting this inequality we obtain

(*) $$\frac{v(r) - v(r-\delta)}{\delta} \geq \frac{v(r)}{2} \cdot \frac{1 - h(r-\delta)}{\delta}$$

Next, from the definition of the h-function one has the limit formula

(ix) $$\lim_{\delta \to 0} \frac{1 - h(r-\delta)}{\delta} = \frac{1}{2\pi} \cdot \frac{\cos\ell}{\sin\ell}$$

Passing to the limit as $\delta \to 0$ in (vii) we get the differential inequality:

(**) $$v'(r) \geq \frac{v(r)}{2\pi r} \cdot \frac{\cos\ell}{\sin\ell}$$

Next, put

$$\log r = s \quad \text{and} \quad \log \frac{v(r)}{2} = g(s)$$

By derivation rules we see that (**) gives

(***) $$\frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos\ell}{\sin\ell}$$

Next, identifying γ with a subset of the periodic interval $0 \leq \phi \leq 2\pi$ it is clear that the definition of the ω-function gives the inclusion

(x) $$\gamma \subset \{\omega(\phi) \geq g(s)\}$$

So if $\lambda(s)$ is the Lebesgue measure of the set $\{\omega(\phi) \geq g(s)\}$ then $\ell \leq \lambda(s)$ and (***) gives

(****) $$\frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos\lambda(s)}{\sin\lambda(s)}$$

Next, the inequality $\sin(t) \geq \frac{2}{\pi} \cdot t$ gives a positive constant k which is independent of s such that the following hold for sufficiently large s, i.e. to ensure that the corresponding r-value satisfies $M(r) \geq 1$:

(xi) $$\frac{dg}{ds} \geq \frac{k}{\lambda(s)}$$

Hence, starting from some sufficiently large s_0 one has

(xii) $$\int_{s_0}^{s} \lambda(s) \cdot dg(s) \geq k(s - s_0)$$

This inequality implies in particular that one has a divergent integral:

(xiii) $$\int_{s_0}^{\infty} \lambda(s) \cdot dg(s) = +\infty$$

Finally, the general equality for distribution functions from XXX gives:

$$\text{(xiiii)} \qquad \int_0^{2\pi} \omega(\phi)\cdot d\phi = \int_0^\infty \lambda(s)\cdot dg(s)$$

The last integral is $+\infty$ by (xiii) and the requested divergence for the intergal of the ω-function follows.

Remark. At the end of the article [XXX] Carleman points out that the proof above gives a sharp version of the Phragmén- Lindelöf theorem. More precisely one has the following: Let $f(z)$ be analytic in a sector

$$S_\alpha = \{z = re^{i\phi} \quad : \quad -\alpha < \phi < \alpha\}$$

Define $\omega(\phi)$ as above when when $-\alpha < \phi < \phi$. With these notations one has:

A.2. Theorem. *Let f be bounded on the half-lines $arg(z) = \alpha$ and $arg(z) = -\alpha$ and assume also that*

$$\int_{-\alpha}^{\alpha} \omega(\phi)\cdot d\phi < \infty$$

Then $f(z)$ is bounded in the whole sector.

A.3. Exercise. Deduce Theorem A.2 from the preceeding results.

B. Asymptotic series.

Introduction. The notion of asymptotic series was expressed as follows by Poincaré:

Let $f(z)$ be complex-valued function defined in some subset E of $\mathbf{C}$ and z_0 is a boundary point. We say that f has an asymptotic series expansion at z_0 if there exists a sequence of complex numbers $c_0, c_1, \ldots$ such that $\lim_{z\to z_0} f(z) = c_0$ and for each $n \geq 0$ one has:

$$(*) \qquad \lim_{z\to z_0} (z - z_0)^{-n-1}\left[f(z) - (c_0 + c_1 + \ldots + c_n z^n)\right] = c_{n+1}$$

where the limit is taken as z stay in E.

It is obvious that if f has an asymptotic expansion at z_0 then the sequence $\{c_n\}$ is unique. Constructions of functions which admit asymptotic expansions appear in Emile Borel's thesis *Sur quelques points de la théorie des fonctions* from 1895 and he proved for example that for every sequence of real numbers $\{c_n\}$ there exists a C^∞-function $f(x)$ on the real line whose Taylor expansion at $x = 0$ is given by the sequence, i.e.

$$\frac{f^{(n)}(0)}{n!} = c_n \quad : \quad n = 0, 1, \ldots$$

Following [Car: xx, page 29-31] we prove a complex version of Borel's result where D_+ denotes the open half-disc $\{\mathfrak{Re}(z) > 0 \cap \{|z| < 1\}$.

B.1. Theorem. *To each sequence $\{c_n\}$ of complex numbers there exists a bounded analytic function $F(z)$ in D_+ which has an asymptotic series expansion at $z = 0$ given by $\{c_n\}$.*

Proof. It suffices to prove this when $c_0 = 0$. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $\sum a_\nu < \infty$. Given $\{c_n\}$ we construct a sequence of functions $P_1(z), P_2(z), \ldots$ which are analytic in the half plane $\mathfrak{Re}(z) > 0$ as follows: First

$$\text{(i)} \qquad P_1(z) = c_1 z(1 - \frac{z}{z + \epsilon_1}) \quad : \epsilon_1 = \frac{\alpha_1}{|c_1|} \implies$$

$$\text{(ii)} \qquad |P_1(z)| = |c_1|\cdot\epsilon_1\cdot\frac{|z|}{|z + \epsilon_1} \leq \alpha_1 \quad : \quad \mathfrak{Re}(z) \geq 0$$

Now $P_1(z)$ has a series expansion at $z = 0$:

$$\text{(ii)} \qquad P_1(z) = \sum_{\nu=1}^\infty c_\nu^{(1)}\cdot z^\nu$$

Notice that the series converges in the disc $|z| < \epsilon_1$. Set

(iii) $$P_2(z) = \big[c_2 - c_2^{(1)}\big] \cdot z^2 \cdot \big(1 - \frac{z}{z+\epsilon_2}\big) \quad : \ |c_2 - c_2^{(1)}| \cdot \epsilon_2 \leq a_2$$

With such a careful choice of a small positive ϵ_2 we see that

(iii) $$|P_2(z)| \leq a_2 \cdot |z| \quad : \quad \mathfrak{Re}(z) \geq 0$$

Again we obtain a convergent series at $z = 0$:

(iv) $$P_2(z) = P_1(z) = \sum_{\nu=2}^{\infty} c_\nu^{(2)} \cdot z^\nu$$

The inductive construction. Let $n \geq 3$ and suppose that $P_1, \ldots, P_{n-1}$ have been constructed where we for each $1 \leq k \leq n-1$ have a series expansion

(v) $$P_k(z) = \sum_{\nu=k}^{\infty} c_\nu^{(k)} \cdot z^\nu$$

Then we define

$$P_n(z) = \big[c_n - (c_n^{(1)} + \ldots + c_n^{(n-1)}\big] \cdot z^n \cdot \big(1 - \frac{z}{z+\epsilon_n}\big) \quad : \quad \big| \, c_n - (c_n^{(1)} + \ldots + c_n^{(n-1)} \, \big| \cdot \epsilon_n \leq \alpha_n$$

So we obtain a new series at $z = 0$:

(vi) $$P_n(z) = \sum_{\nu=n}^{\infty} c_\nu^{(n)} \cdot z^\nu$$

Staying in the half-disc D_+, the inductive construction gives

$$\max_{z \in D_+} |P_n(z)| \leq \alpha_n \quad : \quad n = 1, 2, \ldots$$

Hence there exists a bounded analytic function in D_+ defined by

$$F(z) = P_1(z) + P_2(z) + \ldots$$

At this stage we leave as an exercise to the reader to verify that

$$\lim_{z \to 0} z^{-n-1} \cdot \big[\, F(z) - (c_1 z + \ldots + c_n z^n)\big] \, = c_{n+1}$$

B.2. Uniqueness of asymptotic expansions.

There exist functions whose asymptotic series is identically zero. Here is an example:

$$f(z) = e^{-\frac{1}{z^2}}$$

If $z = re^{i\theta}$ with $-\pi/8 \leq \theta \leq \pi/8$ we see that

$$|f(re^{i\theta})| = \exp\big(-\frac{\cos 2\theta}{r^2}\big) \leq \exp\big(-\frac{1}{\sqrt{2} \cdot r^2}\big)$$

It follows that the asymptotic series at $z = 0$ is identically zero. Via a conformal map from the half-disc D_+ to the unit circle we are led to the following problem: Let $f(z)$ be analytic in the open unit disc D. Suppose that

(*) $$\lim_{z \to 1} \frac{f(z)}{(1-z)^n} = 0 \quad : \quad n = 1, 2, \ldots$$

We seek growth conditions on f in order that (*) implies that f is identically zero. An answer to this uniqueness problem was proved by Carleman in [Car]. Namely. consider a sequence of real positive numbers $A_1, A_2, \ldots$. To each $n \geq 1$ we put

(**) $$I_n = \exp\Big(\frac{1}{\pi} \int_1^\infty \log\big[\sum_{\nu=1}^{\nu=n} \frac{r^{2\nu}}{A_\nu^2}\big] \cdot dr\Big)$$

B.3. Definition. *Denote by $\mathfrak{B}$ the set of all sequences $\{A_n\}$ such that $\{I_n\}$ is bounded, i.e. there exists some K such that*

$$I_n \leq K \quad : \quad n = 1, 2 \ldots$$

In [Car: page 7-52] the following existence result is proved:

B.4. Theorem. *To each sequence $\{A_n\} \in \mathfrak{B}$ there exists an analytic function $f(z)$ in D which is not identically zero and satisfies:*

$$\frac{|f(z)|}{|1-z|^n} \leq A_n \quad : \quad n = 1, 2, \ldots \tag{1}$$

while (*) holds.

A converse result. In [loc.cit] appears the converse to the result which ensures uniqueness of the asymoptotic expansion at $z = 1$.

B.5. Theorem. *Let $\{A_n\}$ be a sequence of positive numbers such that there exists an analytic function $f(z)$ in D which is not reduced to a constant and satisfies (*) and (1) in Theorem B.4. Then $\{A_n\} \in \mathfrak{B}$.*

Remark. The results above show that if $\{A_n\}$ is a sequence for which $\{I_n\}$ is unbounded then the asymptotic expansion at $z = 1$ is unique for every analytic function $f(z)$ satisfying (1) in Theorem B.4. The proofs of the two results above rely upon a varational problem which is presented below while the deduction after of the two cited results above are left to the reader who may find details in [Carleman].

C. A variational problem.

Let $n \geq 1$ and $a_0, a_1, \ldots, a_n$ some n-tuple of positive real numbers. Denote by $\mathcal{O}(*)$ the family of analytic functions $f(z)$ in the unit disc which extend to continuos functions on the closed disc and in addition $f(0) = 1$. Put

$$I(f) = \frac{1}{2\pi} \cdot \sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \int_0^{2\pi} \frac{|f(e^{i\theta})|^2}{|e^{i\theta} - 1|^{2\nu}} \cdot d\theta \quad : \quad I_* = \min_{f \in \mathcal{O}(*)} I(f) \tag{0.1}$$

Above we have a variational problem. We shall prove that there exists a unique function in $\mathcal{O}(*)$ which minimizes the functional. To prove this we shall rewrite the variational problem. With n fixed we have the rational function

$$\Omega(z) = \sum_{\nu=0}^{\nu=n} a_\nu^2 \big[(1-z)(1-\frac{1}{z})\big]^{n-\nu} \tag{0.2}$$

It is clear that $\Omega(z)$ has a pole of order n at $z = 0$ and can be written as

$$Q(z) = z^{-n} \cdot (-1)^n \cdot a_0^2 \cdot \Omega^*(z)$$

where $\Omega^*(z)$ is a polynomial of degree $2n$ and $\Omega^*(1) = 1$. Notice that $(1-e^{i\theta})(1-e^{-i\theta}) = |e^{i\theta} - 1|^2$ hold for every θ. This gives

$$\Omega(e^{i\theta}) = a_0^2 + \sum_{\nu=1}^{\nu=n} a_\nu^2 \cdot |e^{i\theta} - 1|^{2n-2\nu} \tag{0.3}$$

In particular $\Omega(z)$ takes real and positive values on the unit circle.

Exercise. Show that (0.3) implies that the polynomial $\Omega^*(z)$ has n-zeros $\rho_1, \ldots, \rho_n$ in the open disc while $1/\rho_1, \ldots, 1/\rho_n$ are the zeros in the exterior disc $\{|z| > 1\}$, and hence

$$\Omega(z) = (-1)^n \cdot a_0^2 \cdot z^{-n} \cdot \prod (z - \rho_\nu) \cdot \prod (z - \frac{1}{\rho_\nu}) \tag{0.4}$$

Next, (0.3) gives the equality

$$I(f) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |e^{i\theta} - 1|^{2n} \cdot |f(e^{i\theta})|^2 \, d\theta \tag{0.5}$$

We use (0.5) to prove

C.1 Theorem. *The variational problem has a unique solution minimizing function f_* given by*:

$$f_*(z) = \frac{(1-z)^n}{\prod 1 - \rho_\nu \cdot z)} \tag{i}$$

Moreover,

$$I(f_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log} [\sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \frac{1}{(2 \cdot \sin \frac{\theta}{2})^{2\nu}}] \cdot d\theta \tag{ii}$$

Proof The choice if f_* and (0.5) give

$$I(f_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot \prod |1 - \rho_\nu e^{i\theta}|^{-2} d\theta \tag{i}$$

It is clear that f_* is minimizing if

$$I(f_*) < I(f_* + h)$$

for every analytic function h in D such that $h(0) = 0$. To prove this we notice that (0.5) gives

$$I(f_* + h) = I(f_*) + I(h) + \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |e^{i\theta} - 1|^{-2n} \cdot \mathfrak{Re}(f_*(e^{i\theta}) \cdot \bar{h}(e^{i\theta}) \, d\theta \tag{0.5}$$

Since $I(h) > 0$ when h is not identically zero we get (xx) it the last integral above is zero. To orove this we use that $\Omega(e^{i\theta})$ is real and hence the requested vanishing follows if

$$\int_0^{2\pi} \Omega(e^{i\theta}) \cdot |e^{i\theta} - 1|^{-2n} \cdot f_*(e^{i\theta}) \cdot \bar{h}(e^{i\theta}) \, d\theta = 0$$

$$F(z) = \frac{f(z)}{(1-z)^n} \tag{i}$$

Then (0.5) gives the equality:

$$I(f) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |F(e^{i\theta})|^2 \cdot d\theta \tag{iii}$$

Hence the variational problem is equivalent to seek the minimum of

$$\min_{f \in \mathcal{O}(*)} \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |F(e^{i\theta})|^2 \cdot d\theta \tag{iv}$$

Now f_* is a unique minimizing function in Theorem C.1 if we have proved the strict inequality

$$I(f_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |F_*(e^{i\theta})|^2 \cdot d\theta < \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |F_*(e^{i\theta}) + H(e^{i\theta})|^2 \cdot d\theta \tag{v}$$

for every analytic function $h(z)$ in D such that $h(0) = 0$. Put

$$I(h) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |H(e^{i\theta})|^2 \cdot d\theta \tag{vi}$$

Then the right hand side in (v) becomes

$$I(f_*) + I(h) + \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot [\bar{F}_*(e^{i\theta}) \cdot h(e^{i\theta}) + F_*(e^{i\theta}) \cdot \bar{h}(e^{i\theta})] \, d\theta \tag{vii}$$

Since $I(h) > 0$ whenever $h \neq 0$ the requested strict inequality in (v) follows if we show that the last integral in (vii) is zero. By (iii) the function $\Omega(e^{i\theta})$ is constant so the requested vanishing follows if

(viii) $$\mathfrak{Re}\,\frac{1}{2\pi}\cdot\int_0^{2\pi}\bar{F}_*(e^{i\theta})\cdot h(e^{i\theta})d\theta = 0$$

To prove (viii) we notice that the construction of f_* gives $F_*(z) = \frac{1}{\prod 1-\rho_\nu\cdot z)}$. Put

(ix) $$k(z) = \frac{1}{\prod (z-\bar{\rho}_\nu)} \implies k(z) = \bar{F}_*(z) \quad : |z| = 1$$

Hence

(x) $$\frac{1}{2\pi}\cdot\int_0^{2\pi}\bar{F}_*(e^{i\theta})\cdot h(e^{i\theta})d\theta = \int_0^{2\pi} k(e^{i\theta})\cdot h(e^{i\theta})d\theta$$

The integral in (x) vanishes since $h(0) = 0$ and $k(z)h(z)$ is analytic in D. In particular the real part is zero which gives (viii) and finishes the proof that f_* is minimizing.

The equality (C.x.2)

§ 17. Representations of rotation invariant harmonic functions.

The results in this section appear in Carleman's article *Applications de la théorie des fonctions analytiques à la resolutions de certaines équations fonctionelles* [Acad. Italia Volta 1940]. Let (x,t,s) be the real coordinates in $\mathbf{R}^3$. Let $u(x,t,s)$ be real-valued and harmonic. We shall consider the subclass of harmonic functions which are invariant under rotations of the (t,s)-coordinates. More precisely, suppose that u is defined in in an open set Ω given as the product of an interval $(-A,A)$ on the x-axis and a disc $\{t^2+s^2<R^2\}$. The invariance means that the function

$$\phi \mapsto u(x, r\cos\phi.r\sin\phi)$$

is constant for each x and $0<r<R$. Let $\square$ be the rectangle in the z-plane defined by

$$\{z=x+iy: \ -A<x<A, : \ -R<y<R\}$$

Let $f(z)$ be analytic in $\square$ assume that $f(x)$ is real-valued when $-A<x<A$. Then we have

17.1 Theorem. *Define u in Ω by*

$$u(x,t,s)=\frac{1}{\pi}\int_0^\pi f(x+i\sqrt{t^2+s^2}\cdot\cos\phi)\,d\phi$$

Then u is harmonic and rotation invariant in Ω

Proof. The existence of complex derivative of f give

(i) $$\partial_x^2(u)=\frac{1}{\pi}\int_0^\pi f''(x+i\sqrt{t^2+s^2}\cdot\cos\phi)\,d\phi$$

$$\partial_t(u)=\frac{1}{\pi}\int_0^\pi i\cos\phi\cdot\frac{t}{\sqrt{t^2+s^2}}\cdot f'(x+i\sqrt{t^2+s^2}\cdot\cos\phi)\,d\phi \implies$$

(ii) $$\partial_t^2(u)=\frac{1}{\pi}\int_0^\pi [-\cos^2\phi\cdot\frac{t^2}{t^2+s^2}+i\cos\phi\cdot\frac{s^2}{(t^2+s^2)^{3/2}}]\cdot f''(x+i\sqrt{t^2+s^2}\cdot\cos\phi)\,d\phi$$

A similar expression is found for $\partial_s^2(u)$ and adding the result we find that

(iii) $$\Delta(u)=\frac{1}{\pi}\int_0^\pi \big(\sin^2\phi\cdot f''+i\cos\phi\cdot\frac{1}{\sqrt{t^2+s^2}}\cdot f'\big)\,d\phi$$

where $1-\cos^2\phi=\sin^2\phi$ was used. Next, notice that

(iv) $$\partial_\phi(f'(x+i\sqrt{t^2+s^2}\cos\phi)=-i\sqrt{t^2+s^2}\cdot\sin\phi\cdot f''(x+\sqrt{t^2+s^2}\cos\phi)$$

By partial integration of the first term in (iii), the reader can check that (iv) entails that $\Delta(u)=0$. Next, the integral equation in the theorem shows that u is rotation invariant and

$$u(x,0,0)=f(x)$$

Since $f(x)$ was real-valued the reader can confirm that u is real-valued in Ω and the proof is finished.

17.2 A converse result. Let u be a rotation invariant harmonic function. Then there exists an analytic function $f(z)$ in $\square$ such that u is represented as in Theorem 17.1 To prove this we use that harmonic functions are real-analytic and define f on the real x-interval $(-A,A)$ by

$$f(x)=u(x,0,0)$$

At this stage we leave as an exercise to the reader to confirm that f extends to an analytic funcituon in $\square$ and that u is represented via f as in the theorem. A hint is to apply the general result below.

17.3 Integrals expression solutions to elliptic equations. Let $a(x, y), b(x, y), c(x, y)$ be real-valued and real-analytic functions in a rectangle

$$\square = \{z = x + iy : \ -A < x < A : \ -B < y < B\}$$

Denote by $\mathcal{S}$ the class of real-valued functions $u(x, y)$ which solve the elliptic equation

$$\Delta(u) + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c = 0$$

in $\square$.

17.4 Theorem. *There exists a function $V(x, y, \zeta)$ where ζ is a new complex variable such that V is complex analytic with respect to ζ, and the general solution u is of the form*

$$u(x, y) = A \cdot V(x, y, 0) + \mathfrak{Re} \int_0^{x+iy} V(x, y, \zeta) \cdot f(\zeta)\, d\zeta$$

where A is a real constant and f an arbitray continous complex analytic function.

17.5 Remark. A special case of Theorem 17.4 appears when we regard solutions to the elliptic equation

$$\Delta(u) - k^2 \cdot u = 0$$

where $k > 0$ is a constant. Here the general u-solution defined in some disc centered at the origin takes the form

$$u(x, y) = \mathfrak{Re} \int_0^z J_0(ik\sqrt{|z|^2 - \bar{z} \cdot \zeta}) \cdot f(\zeta)\, d\zeta$$

where $z = x + iy$ and J_0 the usual Bessel function.

§ 18. Conformal maps of circular domains.

Introduction. In 1906 Koebe proved a result about conformal mappings between domains bordered by a finite set of circles. Let $p \geq 2$ and denote by $\mathcal{C}^*(p)$ the family of connected bounded domains Ω in $\mathbf{C}$ for which $\partial\Omega$ is the union of p many disjoint circles.

Theorem. *Let $f\colon \Omega \to U$ be a conformal map between two domains in $\mathcal{C}^*(p)$. Then $f(z)$ is a linear function, i.e. $f(z) = Az + B$ for some constants A and B.*

Koebe's original proof used reflections over the boundaries and results related to the uniformisation theorem. A more direct proof was given by Carleman in [Car] which we expose below. It teaches how to compute certain winding numbers in specific situations.

§ 19. On spectra of compact operators.

Let $\mathcal{H}$ be a Hilbert space. In §§ we proved that if T is a compact operator on $\mathcal{H}$, then there exists the compact self-adjoint operator $\sqrt{T^*T}$. Denote by $\{\mu_n(T)\}$ its discrete spectrum which arranged so that sequence $\{\mu_n(T)\}$ is non-increasing, where eventual multiplicities are counted as usual.

1.1 Definition. *For each $p > 0$ we denote by $\mathcal{C}_p$ the class of compact operators on $\mathcal{H}$ such that*

$$\Big(\sum_{n=1}^{\infty} \mu_n(T)^p\Big)^{\frac{1}{p}} < \infty$$

Next, denote by $\mathrm{sp}(T)$ the set of all vectors $x \in \mathcal{H}$ for which there exists some non-zero complex number λ and some integer $n \geq 1$ such that

$$(*) \qquad (\lambda E - T)^n(x) = 0$$

Next, for every positive integer N we have operator T^N and get the image space $T^N(\mathcal{H})$. We shall find sufficient conditions in order that

$$(**) \qquad T^N\mathcal{H} \subset \mathrm{sp}(T)$$

holds for some positive integer N. To achieve this we shall use the resolvent operator

$$R(\lambda) = (\lambda \cdot E - T)^{-1}$$

which is defined outside the discrete spectrum of T. Let γ be a simple Jordan arc which has the origin as one end-point while $\gamma^* = \gamma \setminus \{0\}$ stays outside the spectrum of T. Now $R(\lambda)$ exists for every $\lambda \in \gamma^*$ and we can compute operator norms which leads to:

1.2 Definition. *A Jordan arc γ as above is called T-escaping of order N if there exists a constant C such that*

$$||R(\lambda)|| \leq C \cdot |\lambda|^{-N} \quad : \quad \lambda \in \gamma_*$$

Next, let $\gamma_1, \ldots, \gamma_s$ be a finite family of Jordan arics as above whose intersections with a small punctured disc $D^*(\delta) = \{0 < |z| < \delta\}$ gives a disjoint famlily of curves $\{\gamma_\nu^*\}$. Then $D^*(\delta)$ is decomposed into s many pairwise disjoint Jordan domains, each of which is bordered by a pair of γ^*-curves. Let $\rho > 0$ be some positive number. We impose the geometric condition that every Jordan domain above is contained in a sector where $\arg(z)$ stays in an interval of length $< \rho$ as z varies in the Jordan domain. Denote by $\mathcal{J}(\rho)$ the class of all finite families of Jordan curves for which these sector conditions hold.

1.3 Theorem. *Let T be a compact operator of class C_p for some $p > 0$ and suppose there exists a family $\{\gamma_\nu\}$ which belongs to $\mathcal{J}(\pi/p)$ where each γ_ν is T-escaping order N. Then one has the inclusion*

$$T^N(\mathcal{H}) \subset \mathrm{sp}(T)$$

Remark. This result is announced and proved in [Dunford-Schwartz: Theorem XI.9.29 on page 1115]. The proof is bseed upon several results. The first step which is fairly straightforward reduces the proof to the case when the compact operator T is of Hilbert-Schmidt type. We remark only that for this reduction one uses the fact that T belongs to C_p for some $p > 0$. The major step is to extend Carleman's inequality for matrices in § XX from Chapter XX to the case of Hilbert-Schmidt operators on Hilbert spaces. After this the proof is finished by standard applications of the Phragmén-Lindelöf inequalities.

Exampe and a question. Let $\mathcal{H} = L^2[0,1]$ and consider the operator T defined by

$$Tu(x) = \int_0^1 \frac{k(x,y)u(y)}{|x-y|^\alpha}\, dy$$

where $1/2 < \alpha < 1$ and $k(x,y)$ is a real-valued continuous function on the closed unit square. Then T is compact and using the results from § X one finds a p-number which depend upon α such that T belongs to C_p. It would be interesting to investigate how one can produce integers N to get the inclusion in Theorem 1.3. In such an investigations one can try specified k-functions and in partocular regard the symmetric case when $k(x,y) = k(y,x)$.

§ 20. Distribution of eigenvalues for a class of singular operators.

Let $f(x,y)$ be a continuous function on the unit square $0 \leq x,y \leq 1$ which is symmetric, i.e. $f(x,y) = f(y,x)$. When $0 < \alpha < 1$ we set

$$k(x,y) = \frac{f(x,y)}{|x-y|^\alpha}$$

and get the linear operator

$$K_\alpha(u)(x) = \int_0^1 k(x,y)u(y)\,dy$$

By the result in § xx K_α is a compact operator on the Hilbert space $L^2[0,1]$ In fact, by the result in § xx it is even a compact operator and since k is symmetric the eigenvalues are real and non-zero. Let $\{\lambda_n^+\}$ be the positive eigenvalues arranged in a non-decreasing order. Similarly $\{\lambda_n^-\}$ is the set of negative eigenvalues where $\{-\lambda_n^-\}$ is non-decreasing. To the set of eigenvalues corrspond eigenfunctions $\{\phi_n^+\}$ and $\phi_n^-\}$ where

$$K_\alpha(\phi_n^+) = \lambda_n^+ \cdot \phi_n^+$$

1. Theorem. *If $f(x,x) > 0$ hold on some open interval $x_0 < x < x_1$ it follows that*

$$\sum_{n=1}^\infty \big(\frac{1}{\lambda_n^+}\big)^{\frac{1}{1-\alpha}} = +\infty$$

During the proof we use the following notation for real-valued functions u in $L^2[0,1]$:

$$\langle K_\alpha u, u\rangle = \iint f(x,y)u(x)u(y)\,dxdy$$

and for a pair of real-valued L^2-functions u,v we set

$$\langle u,v\rangle = \int_0^1 u(x)v(x)\,dx$$

We shall need the following result from §§:

2. Proposition. *For each $u \in L^2[0,1]$ we have the equality*

(*) $$\langle Ku,u\rangle = \sum \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2 + \sum \frac{1}{\lambda_n^-} \cdot \langle u,\phi_n^-\rangle^2$$

Proof of Theorem 1. Let m be a positive integer. Since $\{\lambda_n^-\}$ are negative (*) gives:

$$\langle Ku,u\rangle \leq \sum_{n=1}^m \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2 + \sum_{n=m+1}^\infty \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2$$

Since $\{\lambda_n^+\}$ is non-decreasing the last sum above is majorized by

$$\frac{1}{\lambda_m^+} \cdot \sum_{n=m+1}^\infty \langle u,\phi_n^+\rangle^2 \leq \frac{1}{\lambda_{m+1}^+} \cdot \langle u,u\rangle$$

where the last inequality follows from Bessel's inequality since the eigenfunctions $\{\phi_n^+\}$ form an orthonormal family. Hence the following inequality holds for every positive integer m:

(i) $$\langle Ku,u\rangle \leq \sum_{n=1}^m \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2 + \frac{\langle u,u\rangle}{\lambda_{m+1}^+}$$

Let $\psi_1,\ldots,\psi_m$ be some orthonormal m-tuple in $L^2[0,1]$. We can apply (i) to each ψ-function and a summation over $1 \leq k \leq m$ gives:

(ii) $$\sum_{k=1}^{k=m} \langle K\psi_k, \psi_k\rangle \leq \sum_{k=1}^{k=m}\sum_{n=1}^{n=m} \frac{1}{\lambda_n^+}\cdot \langle \psi_k, \phi_n^+\rangle^2 + \frac{m}{\lambda_{m+1}^+}$$

Another application of Bessel's inequality gives for each $1 \leq n \leq m$:

$$\sum_{k=1}^{k=m} \langle \psi_k, \phi_n^+\rangle^2 \leq \langle \phi_n^+, \phi_n^+\rangle = 1$$

Hence (ii) entails that

(iii) $$\sum_{k=1}^{k=m} \langle K\psi_k, \psi_k\rangle \leq \sum_{n=1}^{n=m} \frac{1}{\lambda_n^+} + \frac{m}{\lambda_{m+1}^+}$$

A choice of ψ-functions. By assumption we find an interval $[x_0, x_1]$ where $f(x,x) > 0$. Set $d = x_1 - x_0$ and when m is a positive integer we define $\psi_1, \ldots, \psi_m$ where

$$\psi_k(x) = \sqrt{\frac{m}{d}} \quad \text{when} \quad x_0 + (k-1)\frac{d}{m} < x < x_0 + k\frac{d}{m}$$

while $\psi_k = 0$ outside the intervals above. The continuity of f gives some large integer m_* and a positive constant δ such that if $m \geq m_*$ then $f(x,y) \geq \delta$ on each small square where $\psi_k(y)\cdot\psi_k(x) \neq 0$. Denote this small square by $\square_k$ which gives the inequality below for each $1 \leq k \leq m$:

$$\langle K\psi_k, \psi_k\rangle \geq \delta \cdot \frac{m}{d} \cdot \iint_{\square_k} \frac{dxdy}{|x-y|^\alpha}$$

An easy calculation shows that the double integral over $\square_k$ becomes

$$\frac{2}{(1-\alpha)(2-\alpha)} \cdot \big(\frac{d}{m}\big)^{2-\alpha}$$

So with $A = \delta \cdot \frac{2}{(1-\alpha)(2-\alpha)}$ one has the inequality

(iv) $$\sum_{k=1}^{k=m} \langle K\psi_k, \psi_k\rangle \geq A \cdot \big(\frac{d}{m}\big)^{1-\alpha} \cdot m = Ad^{1-\alpha}\cdot m^\alpha$$

By construction $\psi_1, \ldots, \psi_m$ is an orthonormal family and hence (iii) holds which together with (iv) gives the inequality below for every $m \geq m_*$:

(v) $$Ad^{1-\alpha}\cdot m^\alpha \leq \sum_{n=1}^{n=m} \frac{1}{\lambda_n^+} + \frac{m}{\lambda_{m+1}^+}$$

At this stage we shall argue by a contradiction, i.e. we prove that (v) prevents that the positive series in Theorem 1 converges. Namely, suppose that

$$\sum \big(\frac{1}{\lambda_n^+}\big)^{\frac{1}{1-\alpha}} < \infty$$

Since the terms in this positive series decrease with n it follows that

$$\lim_{m\to\infty} m \cdot \big(\frac{1}{\lambda_m^+}\big)^{\frac{1}{1-\alpha}} = 0$$

So if $\epsilon > 0$ we can find $m^* \geq m_*$ such that

$$m \cdot \big(\frac{1}{\lambda_m^+}\big)^{\frac{1}{1-\alpha}} < \epsilon \implies \frac{1}{\lambda_m^+} < \big(\frac{\epsilon}{m}\big)^{1-\alpha}$$

Hence (v) gives the following when $m > m^*$:

$$Ad^{1-\alpha} \cdot m^{\alpha} \leq \sum_{n=1}^{n=m^*} \frac{1}{\lambda_n^+} + \sum_{\nu=m^*+1}^{n=m} (\frac{\epsilon}{\nu})^{1-\alpha} + m \cdot (\frac{\epsilon}{m+1})^{1-\alpha}$$

The middle sum above is majorized by

$$\epsilon^{1-\alpha} \cdot \int_{m^*}^{m} \frac{dx}{x^{1-\alpha}} = \frac{\epsilon^{1-\alpha}}{\alpha} \cdot m^{\alpha}$$

At the same time we notice that the last term is $\leq \epsilon^{1-\alpha} \cdot m^{\alpha}$ and after a division with m^{α} we obtain

$$Ad^{1-\alpha} \leq m^{-\alpha} \cdot \sum_{n=1}^{n=m^*} \frac{1}{\lambda_n^+} + \epsilon^{1-\alpha}(\frac{1}{\alpha} + 1)$$

Above A and d are fixed positive constants while we can choose arbitary large m and arbitrary small ϵ. This gives a contradiction and Theorem 1 is proved.

§ 21. An entire spectral function.

Introduction. Theorem 1 below is due to Carleman in the article *Sur le genre du dénominateur $D(\lambda)$ de Fredholm*. The proof uses some basic results about entire functions due to Poincaré, Lindelöf and Wiman and offers an instructive lesson in analytic function theory. Let $k(x, y)$ be a continuous function on the unit square $\{0 \leq x, y \leq 1\}$. We do not assume that k is symmetric, i.e. $k(x, y) \neq k(y, x)$ can hold. To each n-tuple of points $\{s_\nu\}$ on $[0, 1]$ we assign the determinant function

$$K(s_1, \ldots, s_n) = \det \begin{pmatrix} k(s_1, s_1) & \cdots & k(s_1, s_n) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ k(s_n, s_1) & \cdots & k(s_n, s_n) \end{pmatrix}$$

Set

$$c_n = \int_{\square_n} K(s_1, \ldots, s_n) \cdot ds_1 \cdots ds_n$$

where the integral is taken over the n-dimensional unit cube.

1. Theorem. *Put*

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cdot c_n \cdot \lambda^n$$

Then D is an entire function of the form

$$D(\lambda) = e^{a\lambda} \prod (1 - \frac{\lambda}{\lambda_\nu}) \cdot e^{\frac{\lambda_\nu}{\lambda}}$$

where a is some complex constant and

$$\sum \frac{1}{|\lambda_\nu|^2} < \infty$$

Remark. Prior to Carleman's result in Theorem 1, Schur proved that $D(\lambda)$ is an entire function of the form

$$D(\lambda) = e^{a\lambda + b\lambda^2} \prod (1 - \frac{\lambda}{\lambda_\nu}) \cdot e^{\frac{\lambda_\nu}{\lambda}}$$

for some second constant b. The novelty in [Carleman] is that $b = 0$ always holds. Above we assumed that k is a continuous kernel. This condition was later relaxed in the article [§ xx. 1919] which gives Theorem 1 when $k(x, y)$ is a kernel of the Hilbert-Schmidt type, i.e. it suffices to assume that

$$\iint |k(x, y)|^2 \, dxdy < \infty$$

Proof of Theorem 1

First we approximate k by polynomials. If $\epsilon > 0$ we find a polynomial $P(x,y)$ such that the maximum norm of $k-P$ over the unit square is $< \epsilon$. Write

$$k(x,y) = P(x,y) + B(x,y)$$

So now $|B(x,y)| < \epsilon$ for all $0 \leq x, y \leq 1$. To each pair $0 \leq p \leq n$ we set

$$B_p(s_1,\ldots,s_n) = \det \begin{pmatrix} P(s_1,s_1) & \cdots & k(s_1,s_n) \\ \cdots & \cdots & \cdots \\ P(s_p,s_1) & \cdot & P(s_p,s_n) \\ B(s_{p+1},s_1) & \cdots & B(s_{p+1},s_n) \\ \cdots & \cdots & \cdots \\ B(s_n,s_1) & \cdots & B(s_n,s_n) \end{pmatrix}$$

It is easily seen that

(i) $$c_n = \sum_{p=0}^{p=n} \binom{n}{p} \cdot \int_{\square_n} B_p(s_1,\ldots,s_n) \cdot ds_1 \cdots ds_n$$

Next, let N be the degree of the polynomial $P(x,y)$. The reader can verify that the first p row vectors in the matrix which defines $B_p(s_1,\ldots,s_n)$ are linearly independent as soon as $p > N$ which therefore gives $B_p = 0$. So for every $n \geq N$ one has the equality

(ii) $$c_n = \sum_{p=0}^{p=N} \binom{n}{p} \cdot \int_{\square_n} B_p(s_1,\ldots,s_n) \cdot ds_1 \cdots ds_n$$

Next, if M is the maximum norm of $k(x,y)$, and $\epsilon < M$ the maximum norm of P is $\leq 2M$. Hadamard's determinant inequality in § xx gives

(iii) $$|B_p(s_1,\ldots,s_n)| \leq (2M)^p \epsilon^{n-p} \cdot n^{\frac{n}{2}}$$

Next, when $n \geq p$ and $0 \leq p \leq n$ we set

$$c_n(p) = \binom{n}{p} \cdot \int_{\square_n} B_p(s_1,\ldots,s_n) \cdot ds_1 \cdots ds_n$$

Then (iii) gives:

(iv) $$|c_n(p)| \leq \binom{n}{p} \cdot (2M)^p \epsilon^{n-p} \cdot n^{\frac{n}{2}}$$

Next, recall that $\binom{n}{p} \leq \frac{n^p}{p!}$ and hence (iv) gives

(v) $$|c_n(p)| \leq \frac{(2M)^p}{p!} \cdot \epsilon^{n-p} \cdot n^{\frac{n}{2}+p}$$

At this stage we return to the D-function. To each $0 \leq p \leq N$ we set

$$D_p(\lambda) = \sum_{n=p}^{\infty} \frac{-1)^n}{n!} \cdot c_n(p)\lambda^n$$

Subemma. *For each p we have*

(*) $$\lim_{|\lambda| \to +\infty} e^{-4\epsilon|\lambda|^2} \cdot D_p(\lambda) = 0$$

Exercise. Prove (*). The hint is to use (v) above and Lindelöf's asymptotic formula from § xx.

Next, we notice that (ii) gives an equation

$$D(\lambda) = q(\lambda) + \sum_{p=0}^{p=N} D_p(\lambda) \tag{vi}$$

where $q(\lambda)$ is a polyonomial of degree $N-1$ at most. This entails that the entire function $D(\lambda)$ also satisfies (*) above. Let $\{\lambda_\nu\}$ be the zeros of $D(\lambda)$. Then (*) and a classic result due to Poincaré gives the entire function

$$F(\lambda) = \prod (1 - \frac{\lambda}{\lambda_\nu}) \cdot e^{\frac{\lambda}{\lambda_\nu}} \quad \text{where}$$

$$\lim_{|\lambda| \to +\infty} e^{-\delta|\lambda|^2} \cdot F(\lambda) = 0 \quad \text{for all} \quad \delta > 0 \tag{**}$$

To profit upon (**) we use a device introduced by Lindelöf. Let $\omega = e^{2\pi i/5}$ which gives the entire function

$$G(\zeta) = F(\zeta^5)F(\omega\zeta^5)\ldots F(\omega^4\zeta^5) \tag{vii}$$

From (**) we see that the entire function G has order $< 1/2$. Then a result due to Wiman in § xx gives an increasing sequence $\{R_k\}$ which tends to $+\infty$ such that

$$\min_\theta |G(R_k e^{i\theta})| \geq 1 \tag{viii}$$

hold for every k. Taking λ-circles with $r_k^5 = R_k$ and using Poincaré's limit from (**) we see that (vii) and (viii) entail that for every $\delta > 0$ there exist some k_* such that

$$k \geq k_* \implies \max_\theta \frac{1}{|F(r_k e^{i\theta})}| \leq e^{\delta r_k^2} \tag{ix}$$

Finally, we have the zero-free entire function

$$H(\lambda) = \frac{D(\lambda)}{F(\lambda)}$$

In (ix) we can take $\delta = \epsilon$ which gives

$$\limsup_{k \to +\infty} e^{-5\epsilon \cdot r_k^2} \cdot \max_\theta |H(r_k e^{i\theta})| = 0$$

Now Liouville's theorem entails that the entire function $\log H(z)$ must be a linear polynomial and since $D(0) = 1$ we conclude that

$$D(\lambda) = e^{a\lambda} \cdot F(\lambda)$$

for a constant a which finishes the proof of Theorem 1.

§ 22. Two problems in the calculus of variation.

We shall consider two problems with a geometric content which are classic in the sense that they were alreadt treated by Steiner and Weierstrass at an early stage.

§ 0.1 An isoperimetric inequality

Recall that a planar domain whose boundary curve has prescribed length has a maximal area when it it is a disc. It turns out that discs solve a more extensive class of extremal problems. Consider a function $f(r)$ defined for $r > 0$ which is continuous and increasing with $f(0) = 0$. If p and q are two points in $\mathbf{R}^2$ their euclidian distance is denoted by $|p - q|$. When U is a bounded open domain we set

$$J(U) = \iint_{U \times U} f(|p - q|) \cdot dA_p \cdot dA_q$$

where dA_p and dA_q denote area measures. Given a positive number $\mathcal{A}$ one seeks to maximize the J-functional in the family of domains with prescribed area $\mathcal{A}$. The J-number is unchanged under a translation or a rotation of a domain and the family of discs is stable under these operations. So the following result makes sense:

1.Theorem. *The J-functional takes its maximum on discs D of radius r with $\pi r^2 = \mathcal{A}$. Moreover, for every domain U with area $\mathcal{A}$ which is not a disc one has a strict inequality*

$$J(U) < J(D)$$

When $f(r)$ is a strictly convex function Theorem 1 was established by Blaschke. For a general f-function which need not be convex the theorem was proved by Carleman in [Car] using the symmetrisation process by W. Gros from the article (Monatshefte math.physik 1917). In § xx we explain why Theorem 1 leads to another property of discs.

2.Theorem. *Let Ω be a domain in the family $\mathcal{D}(C^1)$ and denote by ds the arc-length measure on its boundary. Then, if the function*

$$p \mapsto \int_{\partial\Omega} f(|p - q|) \cdot ds(q)$$

is constant as p varies in $\partial\Omega$ it follows that Ω is a disc.

Remark. Theorem 1 can be extended to any dimension $n \geq 3$ using successive symmetrisations of domains taken in different directions converge to the unit ball in $\mathbf{R}^n$. Here discs are replaced by $n-1$-dimensional spheres.

2. A variational inequality

We first establish some inequalties which will be used in § 3 to finish the proof of Theorem 1. Let $M > 0$ and on the vertical lines $\{x = 0\}$ and $\{x = M\}$ we consider two subsets G_* and G^* which both consist of a finite union of closed intervals. Let $\{[a_\nu, b_\nu]\}$ be the G_*-intervals taken in the y-coordinates and $\{[c_j, d_j]\}$ are the G^*-intervals. Here $a_\nu < b_\nu < a_{\nu+1}$ holds, and similarly the G^*-intervals are ordered with increasing y-coordinates. The number of intervals of the two sets are arbitrary and need not be the same. Given $f(r)$ as in the Theorem 1 we set

$$I(G_*, G^*) = \sum_\nu \sum_j \int_{a_\nu}^{b_\nu} \int_{c_j}^{d_j} f(|y - y'|) \cdot dy dy'$$

Consider the variational problem where we seek to minimize these I-integrals for pairs (G_*, G^*) as above under the constraints:

$$\sum (b_\nu - a_\nu) = \ell_* \quad \text{and} \quad \sum (d_j - c_j) = \ell^*$$

That is, the sum of the lengths of the intervals are prescribed on G_* and G^*.

2.1 Proposition. *For every pair (ℓ_*, ℓ^*) the I-integral is minimized when both G_* and G^* consist of a single interval and the mid-points of the two intervals have equal y-coordinate.*

Proof. First we prove the result when both $G_* = (a, b)$ and $G^* = (c, d)$ both are intervals. We must prove that the I-integral is a minimum when

(i) $$\frac{a+b}{2} = \frac{c+d}{2}$$

Suppose that inequality holds. Since the I-integral is symmetric with respect to the pair of intervals, we may assume that

$$\frac{c+d}{2} = s + \frac{a+b}{2} \quad \text{where} \quad s > 0$$

Now $I(G_*, G^*)$ is unchanged when we translate the two intervals, i.e. if we for some number ξ take $(a + \xi, b + \xi)$ and $(c + \xi, d + \xi)$. By such a translation we can assume that $a = -b$ so the mid-point of G_* becomes $y = 0$ and we have:

$$I = \int_{-b}^{b} \int_{c}^{d} f(\sqrt{M^2 + (y - y')^2}) \cdot dy dy'$$

Using the variable substitutions $u = y' - y$ and $v = y' + y$ we see that

$$-b + c \leq u \leq d + b$$

and obtain

$$I = 2b \int_{-b+c}^{d+b} f(\sqrt{M^2 + v^2}) \cdot dv$$

With

$$s = d - \frac{d+c}{2} = \frac{d-c}{2}$$

we can write

$$I = 2b \cdot \int_{w-s}^{w+s} f(\sqrt{M^2 + u^2}) \cdot dv \quad : w = b + \frac{d+c}{2}$$

The last integral is a function of s, i.e. for every $s \geq 0$ we set

$$\Phi(s) = 2b \cdot \int_{w-s}^{w+s} f(\sqrt{M^2 + u^2}) \cdot dv \quad : w = b + \frac{d+c}{2}$$

The derivative of s becomes

$$\Phi'(s) = f(\sqrt{M^2 + (w+s)^2}) - f(\sqrt{M^2 + (w-s)^2})$$

Since $f(r)$ was increasing the derivative is > 0 when $s > 0$. Hence the minimum is achieved when $s = 0$ which means that G_* and G^* have a common mid-point and Proposition 2.1 is proved for the case of an interval pair.

The general case. If $G_* = \{(a_\nu, b_\nu)\}$ and $G^* = \{(c_k, d_k)\}$ we make an induction over the total number of intervals which appear in the two families. Let

$$\xi^* = \frac{c^* + d^*}{2}$$

be the largest mid-point from the G^*-family which means that k is maximal, In the G_*-family we also get the largest mid-point is

$$\eta^* = \frac{a^* + b^*}{2}$$

If $\xi^* > \eta^*$ the previous case shows that the double sum representing I decreases as long as when the interval (c^*, d^*) is lowered. In this process two cases can occur: First, suppose that the lowered

(c^*, d^*)-interval hits (c_{k-1}, d_{k-1}) before the mid-point equality appears. To be precise, this occurs if

$$c^* - d_{k-1} < \xi^* - \eta^*$$

In this case we replace G^* by a union of intervals where the number of intervals therefore has decreased by one and we lower (c^*, d^*) until $\xi^* = \eta^*$. After this we lower the two top-intervals at the same time until one of them hits the second largest G-interval and in this way the total number of intervals is decreased while the double sum for I is not enlarged. This gives the requested induction step and the proof of Proposition 2.1 is finished.

3. Proof of Theorem 1.

Consider a domain U defined by

$$g_1(x) \leq y \leq g_2(x) \quad : \quad a \leq x \leq b \tag{1}$$

where $g_1(a) = g_2(a)$ and $g_1(b) = g_2(b)$. To U we associate the symmetric domain U^* defined by

$$-\frac{1}{2}\big[g_2(x) - g_1(x)\big] \leq y \leq \frac{1}{2}\big[g_2(x) - g_1(x)\big] \quad : \quad a \leq x \leq b \tag{2}$$

Notice that U and U^* have the same area. Set

$$J = \iint_{U \times U} f\big(\sqrt{(x-x')^2 + (y-y')^2}\big) \cdot dxdx'dydy'$$

$$J^* = \iint_{U^* \times U^*} f\big(\sqrt{(x-x')^2 + (y-y')^2}\big) \cdot dxdx'dydy'$$

Lemma 3.1. *One has the inequality $J \leq J^*$.*

Proof. Set $h(x) = \frac{1}{2}[g_2(x) - g_1(x)]$ and introduce the function

$$H^*(x, x') = \int_{y=-h(x)}^{h(x)} \int_{y'=-h(x')}^{h(x')} \rho\big(\sqrt{(x-x')^2 + (y-y')^2}\big) \cdot dydy'$$

We have also the function

$$H(x, x') = \int_{y=g_1(x)}^{g_2(x)} \int_{y'=g_1(x')}^{g_2(x')} f\,\big(\sqrt{(x-x')^2 + (y-y')^2}\big) \cdot dydy'$$

It is clear that

$$J = \int_a^b \int_a^b H(x, x')dxdx' \quad \text{and} \quad J^* = \int_a^b \int_a^b H^*(x, x')dxdx'$$

Lemma 3.1 follows if we have proved the inequality

$$H(x, x') \leq H^*(x, x') \tag{*}$$

for all pairs x, x' in $[a, b]$. But this follows via Fubini's theorem when Proposition 2.1 applied in the special case where G_* and G^* both consist of a single interval.

3.2 Variation of convex sets. Let $\mathcal{A}$ be the prescribed area in Theorem 1 and consider a convex domain U whose area is $\mathcal{A}$. By elementary geometry we see that after a translation and a rotation the convex domain U can be represented as in(1) above. We construct U_* as above and notice that it is a new convex domain. Moreover, Proposition 2.1 gives $J(U_*) \leq J(U)$. In the next step we perform a symmetrisation of U_* along some other line which cuts U_* to get a new domain U_{**} where we now have $J(U_{**}) \leq J(U_*) \leq J(U)$. Finally we use the geometric result due to Steiner for convex domains which asserts that when symmetrisations as above are repeated infinitely often while the angles of the directions to the x-axis change with some irrational multiple of 2π, then the resulting sequence of convex domains converge to a disc. This proves that the J-functional on a disc is $\leq J(U)$ for every convex domain.

3.3 The non-convex case Here we use the symmetrisation process by Gros. Let U be a domain. Its symmetrisation in the x-direction is defined as follows: To every x we get the open set

$$\ell_U(x) = \{y : (x,y) \in U\} \tag{1}$$

Let $\{(a_\nu, b_\nu)\}$ be the disjoint intervals of $\ell_U(x)$ and put

$$d(x) = \frac{1}{2} \cdot \sum (b_\nu - a_\nu)$$

We get the domain U^* which is symmetric with respect to the x-axis where

$$\ell_{U^*}(x) = (-d(x), d(x))$$

Notice that U and U^* have equal area. Proposition 2.1 applies and gives the inequality

$$J(U) \leq J(U^*) \tag{2}$$

Now Theorem 1 follows when we start from a non-convex domain U. Namely, by the result proved in [Gros], it holds that after inifinitely many symmetrizations as above using different directions, the sequence of U-sets converge to a disc.

§ 0.2 On minimal surfaces.

We shall consider an isoperimetric problem with a fixed boundary curve. More involved situations arise when the minimal surfaces are bordered by closed Jordan curves which are free to move on prescribed manifolds. This leads to problems by Plateau and Douglas and for an account about this general case we refer to Courant's article *The existence of minimal surfaces of given toplogical structure under prescribed boundary conditions.* (Acta. Math. Vol 72 1940]) where the reader also finds an extensive references to relevant literature.

From now on we discuss the restricted problem when a boundary curve is fixed in $\mathbf{R}^3$ with coordinates (x, y, z). Consider a rectifiable closed Jordan curve C and denote by $\mathcal{S}(C)$ the family of surfaces which are bordered by C. A surface M in this family is minimal if it has smallest possible area. To find such a minimal surface corresponds to a problem in the calculus of variation and was studied by Weierstrass in a series of articles starting from *Untersuchungen über die Flächen deren mittlere Krümmung überall gleich null ist* from 1866. A revised version written by Weierstrass himself appears in volume I of his collected work. He proved that if M is a minimal surface in $\mathcal{S}(C)$ then its mean curvature vanishes identically. Moreover, M has no singular points and is simply connected. More precisely, the exists a homeomorphic parametrization of M above an open disc in the complex u-plane which can be achieved via complex analytic functions, or as expressed by Weierstrass in the introduction to [Wei]:

Ich habe mich mit der Theorie die Flächen, deren mittlere Krümmung überall gleich null ist, besonders auf dem grunde eingehender beschäftigt, weil sie, wie ich zeigen werde, auf das Innigste mit der Theorie der analytischen funktionen einer komplexen Argumentz zusammenhhängt. Or shortly phrased: *The theory about minimal surfaces is closely linked to the theory of analytic functions in one complex variable.*

The isoperimetric inequality. Using Weierstrass' description of minimal surfaces the following was proved by Carleman in the article *Zur Theorie der Minimalflächen* in 1920:

Theorem. *For every rectifiable simple closed curve C the area A of the minimal surface in $\mathcal{S}(C)$ satisfies the inequality*

$$A \leq \frac{\ell(C)^2}{4\pi}$$

where $\ell(C)$ is the arc-length of C.

Remark. For historic reasons one may wonder why this result was not already discovered by Weierstrass. The reason might be that certain facts in analytic function theory was not yet enough developed. Carleaman's proof relies upon the Jensen-Blascke factorisation of analytic functions which was not know prior to 1900. Another obstacle was the discovery by Hermann Schwarz that the minimal surface in the family $\mathcal{S}(C)$ is not determined by vanishing mean curvarute alone. See Volume II, page 264 and 151-167 in the collected work of Hermann Schwarz for this "ugly phenomenon" which was one reason why Weierstrass paid much attention to existence problems in the calculus of variation. As remarked by Carleman at the end of his article, an alternative (and simpler) proof was given by Blaschke after the publication of [Carleman]. However, this proof is restricted to a special class of minimal surfaces where the "ugly phenomena" do not occur so here we rely upon Weierstrass' original parametristions which lead to a proof the the theorem above.

The case when C is piecewise linear. Suppose that the boundary curve consists of n many line segements $L_1,\ldots,L_n$. Following Weierstrass it means that one regards the problem: *Es soll ein einfach zusamenhängenden Minimalflächenstück M analytish bestimmt werden, dessen vorgeschiebenen begrenzung C aus n geradlinigen strecken bestecht, welche eine einfache, geschlossene, nicht verknotete Linie bilden.*

In [Weierstrass] appears a far reaching study of this problem. The main result shows that M is determined via a pair of analytic functions $G(u)$ and $H(u)$ defined in the lower half-plane $\mathfrak{Im}\, u < 0$ for which the three functions defined by

$$\phi_1(u) = \det \begin{pmatrix} G(u) & H'(u) \\ G'(u) & H'(u) \end{pmatrix}$$

$$\phi_2(u) = \det \begin{pmatrix} G(u) & H'(u) \\ G''(u) & H''(u) \end{pmatrix}$$

$$\phi_3(u) = \det \begin{pmatrix} G'(u) & H'(u) \\ G''(u) & H''(u) \end{pmatrix}$$

become rational functions of u. Moreover, [ibid] exhibits second order differential equations of the Fuchsian type satisfied by the rational ϕ-functions and the position of their poles are described in terms of the geometric configuration of C. It would lead us too far to enter the material in [Weierstrass] so its rich contents is left to the interested reader for further studies.

The planar case. If C is a simple closed curve in the complex z-plane the isoperimetric inequality follows easily via analytic function theory. Namely, let M be the Jordan domain bordered by C. By Riemann's theorem there exists a conformal mapping $\phi\colon D \to M$ and we have

$$\ell(C) = \int_0^{2\pi} |\phi'(e^{i\theta})|\, d\theta \quad : \quad \text{area}(M) = \iint_D |\phi'(z)|^2\, dxdy$$

Hence the isoperimetric inequality for planar domains boils down to show that

$$\text{(i)} \qquad 4\pi \cdot \iint_D |\phi'(z)|^2\, dxdy \leq \big(\int_0^{2\pi} |\phi'(e^{i\theta})|\, d\theta\big)^2$$

To prove (i) we use that the derivative $\phi'(z)$ is zero-free and hence it has a suingle-valued square root $f = \sqrt{\phi}$. We have a series expansion

$$f(z) = a_1 z + a_2 z^2 + \ldots$$

The right hand side in (i) becomes

$$4\pi^2 \cdot \big(\int_0^{2\pi} |\sum a_\nu e^{i\nu\theta}|^2\, d\theta\big)^2 = 4\pi^2 \cdot (\sum |a_\nu|^2)^2$$

The left hand side becomes

$$4\pi \cdot \iint_D (\sum |a_\nu z^\nu|)^4\, dxdy$$

Set

$$b_m = \sum_{\nu=1}^{\nu=m} a_\nu \cdot a_{m-\nu} \quad : \; m \geq 2$$

Then (xx) becomes

$$4\pi \cdot \iint_D \sum |b_m z^m|)^2 \, dxdy = 4\pi \sum |b_m|^2 \cdot 2\pi \cdot \iint_D r^{2m+1} \, dr = 8\pi^2 \cdot \sum \frac{|b_m|^2}{2m+2}$$

Hence (i) follows if

(ii) $$\sum_{m=2}^{\infty} \frac{|b_m|^2}{m+1} \leq \sum_{\nu=1}^{\infty} |a_\nu|^2)^2$$

At this stage we leave it to the reader to verify the planar isoperimetric inequality in Theorem 1 and that equality holds if and only $\phi(z)$ is such that the complex derivative takes the form

$$\phi'(z) = \frac{a}{(1-qz)^2}$$

for a pair of constants a, b. This means that ϕ is is a Möbius transform and hence C must be a circle, i.e. equality in Theorem 1 for a planar curve holds if and only if C borders a disc,

B. Proof of Theorem 1.

The crucial step in the proof relies upon the following result which is due to Weierstrass:

B.1 Proposition. *Let M be a minimal surface in $\mathcal{S}(C)$. Then there exists an analytic function $F(u)$ in the open unit disc such that points $(x, y, z) \in M$ are given by the equations:*

$$x = \mathfrak{Re} \int (1-u^2) F(u) \, du \quad : \quad y = \mathfrak{Re} \int i(1+u^2) F(u) \, du \quad : \quad z = \mathfrak{Re} \int 2F(u) \, du$$

The proof of this result occupies five pages in [Weierstrass]. We remark that he employed Riemann's mapping theoren for simply connected domains during the proof. Let us indicate some details. To begin with Weierstrass proved that there exists a planar domain Σ with real coordinates (p, q) and a diffeomorphism between M and Σ which is conformal, i.e. M is defined by the equations

(i) $$x = x(p,q) \quad : y = y(p,q) \quad : z = z(p,q)$$

where the vectors $(\frac{\partial x}{\partial p}, (\frac{\partial y}{\partial p}, (\frac{\partial x}{\partial p},)$ and $(\frac{\partial x}{\partial q}, (\frac{\partial y}{\partial q}, (\frac{\partial z}{\partial q})$ are pairwise orthoginal unit vectors. Moreover, when M is minimal the mean curvature of M vanishes which means that the three functions in (1) are harmonic, i.e.

(ii) $$\Delta(x) = \frac{\partial^2 x}{\partial p^2} + \frac{\partial^2 x}{\partial q^2} = 0$$

and similarly for y and z. Next, the harmonic functions above are real parts of analytic functions which yields a triple f, g, h in $\mathcal{O}(\Sigma)$ such that

$$x = \mathfrak{Re} \, f(u)$$

The orthogonality of the vectors $\mathbf{v}$ and $\mathbf{w}$ above entails via the Cauchy Riemann equations that

$$(f'(u))^2 + (g'(u))^2 + (h'(u))^2 = 0$$

Starting from this, Weierstrass used sterographic projections and Riemann's conformal mapping theorem to construct an analytic function $F(u)$ which gives the equations in Proposition B.1. Admitting Weierstrass' result the following hold:

B.2 Propostion. *When the minimal surface M is parametrized as in Proposition B.1 one has the equations*

$$\text{area}(M) = \iint_D (1+|u|^2)^2 \cdot |F(u)|^2 \, d\xi d\eta \quad : \quad \ell(C) = 2 \cdot \int_0^{2\pi} |F(e^{i\theta})| \, d\theta$$

Proof. With $u = \alpha + i\beta$ this amounts to show that

(i) $$dx^2 + dy^2 + dz^2 = (1 + |u|^2)|F(u)|^2 \cdot (d\alpha^2 + d\beta^2)$$

To prove (i) we consider some point $u \in D$. Set $F(u) = |F(u)| \cdot e^{i\theta}$ and $u = se^{i\alpha}$. With $du = d\alpha$ real we have

$$dx = \mathfrak{Re}(1 - u^2)F(u)) \cdot d\alpha = |F(u)| \cdot \big(\cos\theta - |u|^2 \cos\theta \cdot \cos 2\alpha - |u|^2 \sin\theta \cdot \sin 2\alpha\big) \cdot d\alpha$$

Trigonometric formulas give

(i) $$(dx)^2 = |F(u)| \cdot \big[\cos^2\theta + |u|^4 \cos^2(2\alpha - \theta) - 2|u|^2 \cos\theta \cdot \cos(2\alpha - \theta)\,\big] \cdot (d\alpha)^2$$

(ii) $$(dy)^2 = |F(u)|^2 \cdot \big[\sin^2\theta + |u|^4 \sin^2(2\alpha - \theta) + 2|u|^2 \sin\theta \cdot \sin(2\alpha - \theta)\big] \cdot d\alpha$$

(iii) $$(dz)^2 = 4|F(u)|^2 \cdot |u|^2 \big(\cos^2(\theta - \alpha)\big] \cdot (d\alpha)^2$$

Adding (i-ii) we get

$$(dx)^2 + (dy)^2 = |F(u)|^2 \cdot \big[1 + |u|^4 - 2 \cdot |u|^2 \cos(2\theta - 2\alpha)\big] \cdot (d\alpha)^2$$

Finally, the trigonometric formula

$$4\cos^2\phi = 2 - 2\cos 2\phi$$

and (iii) entail that

(iv) $$(dx)^2 + (dy)^2 + (dz)^2 = |F(u)|^2 \cdot (1 + |u|^2)^2 \cdot (d\alpha)^2$$

The same infinitesmal equality as in (iv) is proved when $u = id\beta$ for some small real β and then we can read off Proposition B.2.

Final part of the proof

Put $f_1(u) = F(u)u^2$ and $f_2(u) = F(u)$. Proposition B.2 gives

(i) $$\text{area}(M) = \iint_D \big(\,|f_1(u)|^2 + |f_2(u)|^2\,\big)\, d\xi d\eta + 2 \cdot \iint_D |f_1(u)| \cdot |f_2(u)|\, d\xi d\eta$$

Since $|f_1| = |f_2| = |F|$ holds on the unit circle we also get

(ii) $$\ell(C)^2 = \big[\int_0^{2\pi} |f_1(e^{i\theta})|\, d\theta\big]^2 + \big[\int_0^{2\pi} |f_2(e^{i\theta})|\, d\theta\big]^2 + 2 \cdot \int_0^{2\pi} |f_1(e^{i\theta})|\, d\theta \cdot \int_0^{2\pi} |f_2(e^{i\theta})|\, d\theta$$

Using (i-ii) Carleman derived the isoperimetric inequality from the following:

B.3 Lemma. *For each pair of analytic functions g, h in the unit disc one has*

$$\iint_D \big[g(u)| \cdot |h(u)|\, d\xi d\eta \leq \frac{1}{4\pi} \cdot \int_0^{2\pi} |g(e^{i\theta})|\, d\theta \cdot \int_0^{2\pi} |h(e^{i\theta})|\, d\theta$$

Let us first notice that Lemma B.3 applied to the pairs $g = h = f_1$, $g = h = f_2$ and the pair $g = f_1$ and $h = f_2$ together with (i-ii) give Theorem 1. So there remains to prove Lemma B.3. We can write

$$g = B_1 \cdot g^* \quad : \quad h = B_2 \cdot h^*$$

where B_1, B_2 are Blaschke products and the analytic functions g^* and h^* are zero free in the unit disc. Since $|B_1| = |B_2| = 1$ hold on the unit circle it suffices to prove Lemma B.2 for the pair g^*, h^*, i.e. we may assume that both g and h are zero-free. Then they posses square roots so we can find analytic functions G, H in the unit disc where

$$g = G^2 \quad : \quad h = H^2$$

Consider the Taylor series

$$G(z) = \sum a_k u^k \quad : \quad H(z) = \sum b_k u^k$$

Now $GH = \sum c_k u^k$ where

(i)
$$c_k = a_0 b_k + \ldots + a_k b_0$$

Using polar coordintes to perform double integrals it follows that

$$\iint_D |G^2(u)| \cdot |H^2(u)| \, d\xi d\eta = \pi \cdot \sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1}$$

At the same time one has

$$\int_0^{2\pi} |G^2(e^{i\theta})| \, d\theta = 2\pi \cdot \sum_{k=0}^{\infty} |a_k|^2$$

with a similar formula for the integral of H^2. Hence Lemma B.2 follows if we have proved the inequality

(ii)
$$\sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1} \leq \sum_{k=0}^{\infty} |a_k|^2 \cdot \sum_{k=0}^{\infty} |b_k|^2$$

To get (ii) we use (i) which for every k gives:

$$|c_k|^2 \leq (|a_0||b_k| + \ldots + |a_k||b_0|)^2 \leq (k+1) \cdot (|a_0|^2 |b_k|^2 + \ldots + |a_k|^2 |b_0|^2)$$

Finally, a summation over k entails (ii) and Lemma B.3 is proved.

§ 23 Lindelöf functions.

Introduction. For each real number $0 < a \leq 1$ there exists the entire function

$$Ea_(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+na)}$$

Growth properties of the E-functions were investigated in a series of articles by Mittag-Leffler between 1900-1904 using integral formulas for the entire function $\frac{1}{\Gamma(z)}$. This inspired Phragmén to study entire functions $f(z)$ such that there are constants C and $0 < a < 1$ with:

$$\log|f(re^{i\theta})| \leq C \cdot (1+|r|)^a) \quad : -\alpha < \theta < \alpha \tag{1}$$

for some $0 < \alpha < \pi/2$ while $f(z)| \leq C$ for all $z \in \mathbf{C} \setminus S$. When this holds we get the entire function

$$g(z) = \int_0^{\infty} f(sz) \cdot e^{-s} \cdot ds$$

If z is outside the sector S it is clear that $|g(z)|$ is bounded by $C \cdot \int_0^{\infty} e^{-s} ds = C$. When $z = re^{i\theta}$ is in the sector we still get a bound from (1) since $0 < a < 1$ and conclude that the entire function g is bounded and hence a constant. Since the Taylor coefficients of f are recaptured from g it follows that f must be constant. More general results of this kind were obtained in the joint article [PL] by Phragmén and Lindelöf from 1908 and led to the Phragmén-Lindelöf principle. A continuation of [PL] appears in Lindelöf's article *Remarques sur la croissance de la fonction* $\zeta(s)$ (Bull. des scienes mathématiques 1908] devoted to the growth of Riemann's ζ-function along vertical lines in the strip $0 < \mathfrak{Re}(z) < 1$. This leads to the study of various indicator functions attached to analytic functions and we shall expose material from Care,an's article *Sur la fonction* $\mu(\sigma)$ *de M. Lindelöf* which was published in 1930 and attributed toi A. Wiman,. Here is the set-up: Consider a strip domain in the complex s-plane:

$$\Omega = \{s = \sigma + it \quad : t > 0 \quad \text{and} \quad 0 \leq a < \sigma < b\}$$

An analytic function $f(z)$ in Ω is of *finite type* if there exists some integer k, a constant C and some $t_0 > 0$ such that

$$|f(\sigma + it)| \leq C \cdot t^k \quad \text{hold for} \quad t \geq t_0$$

To every such f we define the Lindelöf function

$$\mu_f(\sigma) = \limsup_{t \to \infty} \frac{\operatorname{Log}|f(\sigma + it)|}{\operatorname{Log} t} \tag{*}$$

Lindelöf and Phragmén proved that μ_f is a continuous and convex function on (a, b). No further restrictions occur on the μ-function because one has:

1. Theorem. *For every convex and continuous function $\mu(\sigma)$ defined in $[a, b]$ there exists an analytic function $f(z)$ without zeros in Ω such that $\mu_f = \mu$.*

2. Exercise. Prove this result using the Γ-function. First, to a pair of real numbers (ρ, α) we set

$$f(s) = e^{-\frac{\pi i \cdot \rho s}{2}} \cdot \Gamma(\rho(s-a) + \frac{1}{2}) \tag{i}$$

Use properties of the Γ-function to show that f has finite type in Ω and its indicator function becomes a linear function:

$$\mu_f(\sigma) = \rho \cdot (\sigma - a)$$

More generally one gets a function f where μ_f is piecewise linear by:

$$f = \sum_{k=1}^{k=m} c_k e^{-\frac{\pi i \cdot \rho_\nu s}{2}} \Gamma(\rho_k(s - a_k) + \frac{1}{2}) \tag{ii}$$

where $\{c_k\}$, $\{\rho_k\}$ and $\{a_k\}$ are m-tuples of real numbers. Finally, starting from an arbitrary convex curve we can choose some dense and enumerable set of enveloping tangents to this curve. Then an infinite series of the form above gives an analytic function $f(s)$ such that

$$\sigma \mapsto \mu_f(\sigma)$$

yields an arbitrarily given convex μ-function on (a, b).

1. A construction of harmonic functions.

Let $U(x, y)$ be a bounded harmonic function in the strip domain Ω and V its harmonic conjugate. Set

$$(*) \qquad f(s) = \exp\big[(\log(s) - \frac{\pi i}{2})(U(s) + iV(s))\big]$$

It is easily see that $f(z)$ has finite type in Ω. With $s = \sigma + it$ we have

$$|f(\sigma + it)| =$$

$$\exp(\frac{1}{2}\log(\sigma^2 + t^2) \cdot U(\sigma + it) \cdot \exp(-(\frac{\pi}{2} - \arg(\sigma + it)) \cdot V(\sigma + it))$$

It follows that

$$\frac{\log|f(\sigma + it)|}{t} = \frac{\log\sqrt{\sigma^2 + t^2} \cdot U(\sigma + it)]}{\log t} + \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t}$$

1.1 Exercise. With σ kept fixed one has

$$\arg(\sigma + it) = \tan\frac{t}{s}$$

which tends to $\pi/2$ as $t \to +\infty$. Next, $V(\sigma + it)$ is for large $t > 0$ up to a constant the primitive of

$$\int_1^t \frac{\partial V}{\partial u}(\sigma + iu) \cdot du$$

Here the partial derivative of V is equal to the partial derivative $\partial U/\partial\sigma(\sigma, u)$ taken along $\mathfrak{Re}\, s = \sigma$. Since U is bounded in the strip domain it follows from Harnack's inequalities that this partial derivative stays bounded when $1 \leq u \leq t$ by a constant which is independent of t. Putting this together the reader can verify that

$$(1.2) \qquad \lim_{t \to +\infty} \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t} = 0$$

From (1.2) we obtain the equality

$$(*) \qquad \mu_f(\sigma) = \limsup_{t \to \infty} U(\sigma + it)$$

This suggests a further study of growth properties of bounded harmonic functions in strip domains.

2. The M and the m-functions.

To a bounded harmonic function U in Ω we associate the maximum and the minimum functions:

$$M(\sigma) = \limsup_{t \to \infty} U(\sigma + it) \quad \text{and} \quad \liminf_{t \to \infty} U(\sigma + it)$$

2.1 Proposition. *$M(\sigma)$ is a convex function while $m(\sigma)$ is concave.*

We prove the convexity of $M(\sigma)$. The concavity of m follows when we replace U by $-U$. Consider a pair α, β with $a < \alpha < \beta < b$. Replacing U by $U + A + \cdot Bx$ for suitable constants A and B we may assume that $M(\alpha) = M(\beta) = 0$ and the requested convexity follows if we can show that

$$M(\sigma) \leq 0 \quad : \quad \alpha < \sigma < \beta$$

To see this we consider rectangles

$$\mathcal{R}[T_*, T^*] = \{\sigma + it \quad \alpha \leq \sigma \leq \beta \quad \text{and} \quad T_* \leq t \leq T^*\}$$

Let $\epsilon > 0$ and start with a large T_* so that

(i) $$t \geq T_* \implies U(\alpha + it) \leq \epsilon$$

and similarly with α replaced by β. Next, we have a constant M such that $|U|_\Omega \leq M$. If $z = \sigma + it$ is an interior point of the rectangle above it follows by harmonic majorisation that

$$U(\sigma + it) \leq \epsilon + M \cdot \mathfrak{m}_z(J_* \cup J^*)$$

where the last term is the harmonic measure at z which evaluates the harmonic function in the rectangle at z with boundary values zero on the two verical lines of the rectangle which it is equal to 1 on the horizontal intervals $J^* = (\alpha, \beta) + iT^*$ and $J_* = (\alpha, \beta) + iT_*$

Exercise. Show (via the aid of figure that with $T^* = 2T_*$ one has

$$\lim_{T_* \to +\infty} \mathfrak{m}_{\sigma + 3iT_*/2}(J_* \cup J^*) = 0$$

where this limit is uniform when $\alpha \leq \sigma \leq \beta$. Since $\epsilon > 0$ is arbitrary in (i) the reader can conclude that $M(\sigma) \leq 0$ for every $\sigma \in (\alpha, \beta)$.

A special case. Suppose that we have the equalities

(1) $$m(\alpha) = M(\alpha) \quad \text{and} \quad m(\beta) = M(\beta)$$

Using rectangles as above and harmonic majorization the reader can verify that this implies that

$$m(\sigma) = M(\sigma) \quad : \quad \alpha < \sigma < \beta$$

Let us remark that this result was originally proved by Hardy and Littlewood in [H-L].

The case when $M(\sigma) - m(\sigma)$ has a tangential zero. Put $\phi(\sigma) = M(\sigma) - m(\sigma)$ and suppose that this non-negative function in (a, b) has a zero at some $a < \sigma_0 < b$ whose graph has a tangent at σ_0. This means that if:

$$h(r) = \max_{-r \leq |\sigma - \sigma_0| \leq r} \phi(\sigma)$$

then

(*) $$\lim_{r \to 0} \frac{h(r)}{r} = 0$$

Under this hypothesis the following result is proved in [Carleman].

2.2 Theorem. *When (*) holds we have*

$$m(\sigma) = M(\sigma) \quad : a < \sigma < b$$

The subsequent proof from [Carleman] was given at a lecture by Carleman in Copenhagen 1931 which has the merit that a similar reasoning can be applied in dimension ≥ 3. Adding some linear function to U we may assume that $M(\sigma_0) = m(\sigma_0) = 0$ which means that

(1) $$\limsup_{t \to \infty} U(\sigma_0, t) = 0$$

Next, consider the function

(1) $$\phi \colon t \mapsto \partial U / \partial \sigma(\sigma_0, t)$$

The assumption (*) and the result in XXX gives:

(2) $$\lim_{t \to \infty} \partial U / \partial \sigma(\sigma_0, t) = 0$$

Next, consider some $a < \sigma < b$ and let $\epsilon > 0$. By the result from XX there exist finite tuples of constants $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$ and some N-tuple $\{\tau_\nu\}$ which stays in a $[0, 1]$ such that

(5) $$\left| U(\sigma, t) - \sum a_\nu \cdot U(\sigma_0, t_\nu + t) - \sum b_\nu \cdot \partial U / \partial \sigma(\sigma_0, t_\nu + t) \right| < \epsilon \quad \text{hold for all} \quad t \geq 1$$

Since ϵ is arbitrary it follows from (1-2) that

(5) $$\lim_{t \to \infty} U(\sigma, t) = 0$$

for every $a < \sigma < b$ which obviously gives the requested equality in Theorem 2.2.

2.3. Integral indicator funtions.

Let $f(s)$ be an analytic function of finite order in the strip domain Ω and fix some $t_0 > 0$ which does not affect the subsequent constructions. For a pair (σ, p) where $a < \sigma < b$ and $p > 0$ we associate the set of of positive numbers χ such that the integral

$$(*) \qquad \int_{t_0}^{\infty} \frac{|f(\sigma + it)|^p}{t^\chi} \cdot dt < \infty$$

We get a critical smallest non-negative number $\chi_*(\sigma, p)$ such that (*) converges when $\chi > \chi_*(\sigma, p)$. In the case $p = 1$ a result due to Landau asserts that $\chi(\sigma, 1)$ determines the half-plane of the complex z-plane where the function

$$\gamma(z) = \int_{t_0}^{\infty} \frac{f(\sigma + it)}{t^z} \cdot dt$$

is analytic and $\sigma \mapsto \chi(\sigma, 1)$ is a convex function on (a, b). A more general convexity result holds when p also varies.

2.4 Theorem. *Define the ω-function by*:

$$\omega(\sigma, \eta) = \eta \cdot \chi(\sigma, \frac{1}{\eta}) \quad : a < \sigma < b \quad : \eta > 0$$

Then ω is a continuous and convex function of the two variables (σ, η) in the product set $(a, b) \times \mathbf{R}^+$.

2.5 Remark. Theorem 2.4 is proved using Hölder inequalities and factorisations of analytic functions which reduces the proof to the case when f has no zeros. The reader is invited to supply details of the proof or consult [Carleman].

3. Lindelöf estimates in the unit disc.

Let $f(z)$ be analytic in the open unit disc given by a power series

$$f(z) = \sum a_n \cdot z^n$$

We assume that the sequence $\{a_n\}$ has temperate growth, i.e. there exists some integer $N \geq 0$ and a constant K such that

$$|a_n| \leq K \cdot n^N \quad : \quad n = 1, 2, \ldots$$

In addition we assume that the sequence $\{a_n\}$ is not too small in the sense that

$$(*) \qquad \sum_{n=1}^{\infty} |a_n|^2 \cdot n^s = +\infty \quad : \quad \forall\, s > 0$$

Now there exists the smallest number $s_* \geq 0$ such that the Dirichlet series

$$\sum_{n=1}^{\infty} |a_n|^2 \cdot \frac{1}{n^s} < \infty, \quad \text{for all } s > s_*$$

To each $0 \leq \theta \leq 2\pi$ we set

$$(1) \qquad \chi(\theta) = \min_s \int_0^1 \left|f(re^{i\theta})\right| \cdot (1 - r)^{s-1} \cdot dr < \infty$$

$$(2) \qquad \mu(\theta) = \text{Lim.sup}_{r \to 1} \frac{\text{Log}\,|f(re^{i\theta})|}{\text{Log}\,\frac{1}{1-r}}$$

We shall study the two functions χ and μ. The first result is left as an exercise.

3.1. Theorem. *The inequality*

$$\chi(\theta) \leq \frac{s^*}{2}$$

holds almost everywhere, i.e. for all $0 \leq \theta \leq 2\pi$ *outside a null set on* $[0, 2\pi]$.

Hint. Use the formula

$$\frac{1}{2\pi} \cdot \int_{=}^{2\pi} |f(re^{i\theta})|^2 \cdot d\theta = \sum |a_n|^2$$

For the μ-function a corresponding result holds:

3.2. Theorem. *The inequality below holds almost everywhere.*

$$\mu(\theta) \leq \frac{s^*}{2}$$

Proof. Let $\epsilon > 0$ and introduce the function

$$\Phi(z) = \sum a_n \cdot \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{s^*}{2}+\epsilon)} \cdot z^n = \sum c_n \cdot z^n$$

It is clear that the construction of s^* entails

$$\sum |c_n|^2 < \infty$$

Next, set $\Phi_0 = \Phi$ and define inductively the sequence $\Phi_0, \Phi_1, \ldots$ by

$$\Phi_\nu(z) = z^{\nu-1} \cdot \frac{d}{dz}\big[z^\nu \cdot \Phi_{\nu-1}(z)\big) \quad : \quad \nu = 1, 2, \ldots$$

3.3 Exercise. Show that for almost every $0 \leq \theta \leq 2\pi$ there exists a constant $K = K(\theta)$ such that

$$|\Phi_\nu(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{1-r)^\nu} \quad : 0 < r < 1$$

Next, with s^* and ϵ given we define the integers ν and ρ:

$$\nu = \big[\,\frac{s^*}{2} + \epsilon\big] + 1 \quad : \quad \rho = \frac{s^*}{2} + \epsilon - \big[\,\frac{s^*}{2} + \epsilon\big]$$

where the bracket term is the usual notation for the smallest integer $\geq \frac{s^*}{2} + 1$.

Exercise Show that with ν and ρ chosen as above one has

$$\Phi_\nu(z) = \sum a_n \cdot \frac{\Gamma(n+1+\nu)}{\Gamma(n+1+\rho-1)} \cdot z^n$$

and use this to show the inversion formula

$$(*) \qquad f(z) = \frac{1}{z^\nu \cdot \Gamma(1-\rho)} \cdot \int_0^z (z-\zeta)^{-\rho}\dot{\zeta}^{\nu+\rho-1} \cdot \Phi_\nu(\zeta) \cdot d\zeta$$

3.4 Exercise. Deduce from the above that for almost every θ there exists a constant $K(\theta)$ such that

$$(**) \qquad |f(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{(1-r)^{\nu+\rho-1}}$$

Conclusion. From (**) and the construction of ν and ρ the reader can confirm Theorem 3. 2.

3.5 Example. Consider the function

$$f(z) = \sum_{n=1}^{\infty} z^{n^2}$$

Show that $s^* = \frac{1}{2}$ holds in this case. Hence Theorem B.2 shows that for each $\epsilon > 0$ one has

(E) $$\max_r (1-r)^{\frac{1}{4}+\epsilon} \cdot |f(re^{i\theta}| < \infty$$

for almost every θ.

3.6 Exercise. Use the inequality above to show the following: For a complex number $x + iy$ with $y > 0$ we set

$$q = e^{\pi i x - \pi y}$$

Define the function

$$\Theta(x+iy) = 1 + q + q^2 + \ldots$$

Show that when $\epsilon > 0$ then there exists a constant $K = K(\epsilon, x)$ for almost all x such that

$$y^{\frac{1}{4}+\epsilon} \cdot |\theta(x+iy)| \leq K \quad : \quad y > 0$$

The equation $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = af - bf$

Let s and b be complex-valued continous functions defined in a disc of radius R centered at the origin in the complex z-plane where $z = x + iy$. Let f satisfy the equation above where we from the start assume that f is continuous and the equality holds in the sense of distributions. Assume also that

$$\lim_{z\to 0} z^{-n} f(z) = 0$$

hold for every positive integer n. We leave it to the reader to verify that this entails that the function $f_n(z) = f(z)/z^n$ satisfies the differential equation

$$\frac{\partial f_n}{\partial x} + i\frac{\partial f_n}{\partial y} = af_n + b\cdot \frac{\bar{z}^n}{z^n}\cdot \bar{f}(z)$$

Multiplying thie equation with $\frac{1}{z-\zeta}$, Cauchy's formula gives the equation

$$f_n(\zeta) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f_n(z)}{z-\zeta}\,dz - \frac{1}{2\pi}\iint_{|z|\leq r} \big[\,af_n + b\cdot \frac{\bar{z}^n}{z^n}\cdot \bar{f}_n(z)\big]\,dxdy$$

for each $0 < r < R$. Recall that

$$\frac{1}{2\pi}\iint_{|z|\leq r} \frac{d\xi d\eta}{|z-\zeta|}|d\xi d\eta \;\leq 2r$$

hold when $|z| \leq r$. Let M majorise the maximum norms of a and b over the disc of radius r. By the triangle inequality we conclude that an integration over the ζ-disc of radius r gives

$$\iint_{|z|\leq r} |f_n(z)]\,dxdy \leq 2r\cdot \int_{|z|=r} |f_n(z)|\,|dz| + 4Mr\iint_{|z|\leq r} |f_n(z)|\,dxdy$$

With $r < 1/4M$ we obtain

(*)
$$\iint_{|z|\leq r} |\frac{f_n(z)}{z^n}|\,dxdy \leq \frac{2r}{1-4Mr}\cdot\frac{1}{r^n}\iint_{|z|\leq r} |f(z)|\,dxdy$$

This inequality holds for each positive integer n. If $f(z_0) \neq 0$ for some $|z_0| < r$ it is clear that there exists some $0 < \rho < r$ and a positive constant k such that the left hand side is $\geq k/\rho^n$ for every n. Thus contradicts (*) when n is large and hence f must vanish identically in the disc of radius r.

Appendix: Entire functions of exponential type

.

The class $\mathcal{E}$ of entire functions of exponential type is defined as follows:

A.0 Definition. *An entire function f belongs to $\mathcal{E}$ if and only if there exists constants A and C such that*

$$(*) \qquad |f(z)| \leq C \cdot e^{A|z|} \quad : \ z \in \mathbf{C}$$

The results in Sections A-B are due to Hadamard and Lindelöf. The class $\mathcal{N}$ which appears in Section 3 was introduced by Carleman who used it to prove certain approximation theorems related to moment problems. The main results deal with Tauberian theorems which is treated in section D and based upon Chapter V in [Paley-Wiener]. Let us present some of the results to be proved in § D while we refer to § A-B for more elementary material about the class $\mathcal{E}$. Consider a non-decreasing sequence $\{\lambda_\nu\}$ of positive real numbers such that the series

$$(1) \qquad \sum \lambda_\nu^{-2} < \infty$$

When this holds there exists the entire function given by a product series:

$$H(z) = \prod (1 - \frac{z^2}{\lambda_\nu^2})$$

Notice that $H(z)$ is positive on the imaginary axis. We get the function defined for real $y > 0$:

$$y \mapsto \frac{\log H(iy)}{y} = \frac{1}{y} \sum \log (1 + \frac{y^2}{\lambda_\nu^2})$$

At the same time we consider the intergals

$$J(R) = \int_{-R}^{R} \frac{\log |H(x)| \cdot dx}{x^2}$$

0.1 Theorem. *The existence of a constant A such that*

$$\text{(i)} \qquad \lim_{y \to \infty} \frac{\log H(iy)}{y} = \pi A$$

and

$$\text{(ii)} \qquad \lim_{R \to \infty} J(R) = -\pi^2 A$$

are completely equivalent

Remark. Of special interest is the case when the limit in (ii) is automatic via an integrability condition, i.e. when

$$(*) \qquad \int_{-\infty}^{\infty} \frac{\big|\log |H(x)|\,\big| \cdot dx}{x^2} < \infty$$

In this case the J-integrals converge and as a consequence there is a limit in (i) for some $A \geq 0$. It turns out that further conclusions can be made. Namely, the convergence of (*) implies that the sequence $\{\lambda_\nu\}$ has regular growth in the sense that if $N(r)$ is the counting function which for every $r > 0$ counts the number of $\lambda\nu \leq r$, then there exists the limit

$$\lim_{R \to \infty} \frac{N(R)}{R} = A$$

with A determined via Theorem 0.1. We shall prove this in § D and remark that the integrability condition (*) is related to the study of the Carleman class in § C.

Growth of entire functions.

Each entire function $f(z)$ can be written in the form

$$f(z) = az^m \cdot f_*(z)$$

where f_* is entire and $f_*(0) = 1$. The case when $f(0) = 1$ is therefore not so special and several formulas below take a simpler form when this holds.

A.1 The functions $T_f(R)$ and $m_f(R)$. They are defined for every $R > 0$ by

(i)
$$T_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \log^+ \bigl|f(R(e^{i\theta})\bigr|\, d\theta$$

(ii)
$$m_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \log^+ \frac{1}{\bigl|f(R(e^{i\theta})\bigr|}\, d\theta$$

A.2 The maximum modulus function. It is defined by

$$M_f(R) = \max_{0 \leq \theta \leq 2\pi} |f(Re^{i\theta})|$$

A.3 The counting function $N_f(R)$. To each $R > 0$ we count the number of zeros of f in the punctured disc $0 < |z| < R$. This integer is denoted by $N_f(R)$, where multiple zeros are counted according to their multiplicities. Jensen's formula shows that if $f(0) = 1$ then

(A.3.1)
$$\int_0^R \frac{N_f(s)}{s} \cdot ds = \frac{1}{2\pi} \cdot \int_0^{2\pi} \log \bigl|f(R(e^{i\theta})\bigr| \cdot d\theta = T_f(R) - m_f(R)$$

The left hand side is always ≥ 0. So if $f(0) = 1$ one has

(A.3.2)
$$m_f(R) \leq T_f(R)$$

Moreover, since $m_f(R) \geq 0$ we have the inequality

(A.3.3)
$$\int_0^R \frac{N_f(s)}{s} \cdot ds \leq T_f(R)$$

Finally, since $N_f(R)$ is increasing we

(A.3.3)
$$\log 2 \cdot N_f(R) \leq \int_R^{2R} \frac{N_f(s)}{s} \cdot ds \leq T_f(2R) \implies N_f(R) \leq \frac{T_f(2R)}{\log 2}$$

A.4 Harnack's inequality. Since the function $\log^+ |f|$ is subharmonic one has

(A.4.1)
$$\log^+ |f(re^{i\alpha})| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{R+r}{R-r} \cdot \log^+ \bigl|f(R(e^{i\theta})\bigr| \cdot d\theta \quad : 0 < r < R$$

It follows that

(A.4.2)
$$M_f(r) \leq \frac{R+r}{R-r} \cdot T_f(R)$$

With $R = 2r$ we conclude that

(A.4.3)
$$M_f(r) \leq 3 \cdot T_f(2r) \quad : r > 0$$

The last inequality gives:

A.5 Theorem. *An entire function f belongs to $\mathcal{E}$ if and only if there exists a constant A such that the following holds for every R*

$$T_f(R) \leq A \cdot R$$

A.6 A division theorem. Let f and g be in $\mathcal{E}$ and assume that $h = \frac{f}{g}$ is entire. Now

(i) $$\log^+ |h| \leq \log^+ |f| + \log^+ |g|$$

In the case when $g(0) = 1$ we apply (**) in A.3 and conclude that

$$T_h(R) \leq T_f(R) + T_g(R)$$

Hence Theorem A.5 implies that h belongs to $\mathcal{E}$. We leave it to the reader to verify that this conclusion holds in general, i.e. without any assumption on $g(0)$.

A.7 Hadamard products. Let $\{\alpha_\nu\}$ be a sequence of complex numbers arranged so that the absolute values are non-decreasing. The counting function of the sequence is denoted by $N_{\alpha(\bullet)}(R)$. Suppose that the counting function satisfies:

(*) $$N_{\alpha(\bullet)}(R) \leq A \cdot R \quad \text{for all} \quad R \geq 1$$

A.8 Theorem *When (*) holds the infinite product*

$$\prod (1 - \frac{z}{\alpha_\nu}) \cdot e^{\frac{z}{\alpha_\nu}}$$

converges for every z and gives an entire function to be denoted by $H_{\alpha(\bullet)}$ and called the Hadamard product of the α-sequence.

A.9 Exercise. Prove this theorem and show that there exists a constant C which is independent of A such that (*) entails that the Hadamard product satisfies the growth condition:

$$\big|H_{\alpha(\bullet)}(z)\big| \leq C \cdot e^{A \cdot |z| \cdot \log\, |z|} \quad \text{for all} \quad |z| \geq e$$

A.10 Lindelöf's condition. For a sequence $\{\alpha_\nu\}$ we define the Lindelöf function

$$L(R) = \sum_{|\alpha_\nu|<R} \frac{1}{\alpha_\nu}$$

We say that $\{\alpha_\nu\}$ is of Lindelöf type if there there exists a constant L^* such that

(A.10.1) $$\big|L(R)\big| \leq L^* \quad \text{hold for all} \quad R.$$

A.11 Theorem. *If α-sequence is of the Lindelöf type and satisfies (*) in (A.7), then there exists a constant C such that the maximum modulos function of $H_{\alpha(\bullet)}$ satisfies*

$$M_{H_{\alpha(\bullet)}}(R) \leq C \cdot e^{AR}$$

and hence the entire function $H_{\alpha(\bullet)}(z)$ belongs to $\mathcal{E}$.

A.12 Exercise. Prove this result. A hint is to study the products

$$\prod_{|\alpha_\nu|<2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}} \quad \text{and} \quad \prod_{|\alpha_\nu|\geq 2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}}$$

separately for every $R \geq 1$. Try also to find an upper bound for C expressed by A and L^*.

A converse result.

A.13 Theorem. *For each $f \in \mathcal{E}$ the set of zeros $\{\alpha_\nu\}$ is of the Lindelöf type.*

Proof. With $R > 0$ we put

(i) $$g(z) = \frac{1}{z} - \frac{\bar{z}}{R^2}$$

This is a is harmonic function in $\{0 < |z| > R\}$ and $g = 0$ on $|z| = R$. Let $f(z)$ be an entire function with $f(0) = 1$ and consider a pair $0 < \epsilon < R$ where f has not zeros in $|z| \leq \epsilon$. Green's formula applied to g and $\log|f|$ in the annulus $\{\epsilon < |z| < R\}$ gives:

(ii) $$\sum_{|\alpha_\nu|<R} [\frac{1}{\alpha_\nu} - \frac{\bar{\alpha}_\nu}{R^2}] = \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \log|f(Re^{i\theta})| \cdot e^{-i\theta} \cdot d\theta - f'(0)$$

where the sum is taken over zeros of f repeated with multiplicities in the disc $\{|z| < R\}$. Next, the triangle inequality gives

(iii) $$|\sum_{|\alpha_\nu|<R} \frac{\bar{\alpha}_\nu}{R^2}| \leq \sum_{|\alpha_\nu|<R} \leq \frac{|\alpha_\nu|}{R^2}| \leq R^{-2} \cdot \int_0^R s \cdot dN(s) \leq \frac{N(R)}{R}$$

We conclude that

(iii) $$L(R) \leq \frac{N(R)}{R} + \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \log^+ |f(Re^{i\theta})|\, d\theta + |f'(0)|$$

From (A.3.3) the last sum is majorized by

(iv) $$\frac{1}{R} \cdot [\frac{T_f(2R)}{\log 2} + 2 \cdot T_f(R)] + |f'(0)|$$

In the case $f \in \mathcal{E}$ the sum in (iv) is bounded which proves Theorem A.13.

B. The factorisation theorem for $\mathcal{E}$

Consider some $f \in \mathcal{E}$. If f has a zero at the origin we can write

$$f(z) = az^m \cdot f_*(z) \quad \text{where} \quad f_*(0) = 1$$

It is clear that f_* again belongs to $\mathcal{E}$ which essentially reduces the study of $\mathcal{E}$-functions f to the case when $f(0) = 1$. Above we proved that the set of zeros satisfies Lindelöf's condition and therefore the Hadamard product

$$H_f(z) = \prod (1 - \frac{z}{\alpha_\nu}) \cdot e^{\frac{z}{\alpha_\nu}}$$

taken over all zeros of f outside the origin belongs to $\mathcal{E}$. Now the quotient f/H_f is entire and we shall prove:

B.1 Theorem *Let $f \in \mathcal{E}$ where $f(0) = 1$. Then there exists a complex number b such that*

$$f(z) = e^{bz} \cdot H_f(z)$$

Proof. The division in A.6 shows that the function $G = \frac{f}{H_f}$ is entire and belongs to $\mathcal{E}$. Here G is zero-free which gives the entire function $g = \log G$ and one has the inequality

(i) $$|g(z)| \leq 1 + \log^+ |G(z)| \leq 1 + C|z|$$

Since $G \in \mathcal{E}$, (i) implies that $|g|$ increases at most like a linear function so by Liouville's theorem it is a polynomial of degree 1. Since $f(0) = 1$ we have $g(0) = 0$ and hence $g(z) = bz$ for a complex number b and Theorem B.1 follows.

C. The Carleman class $\mathcal{N}$

Let $f \in \mathcal{E}$. On the real x-axis we have the non-negative function $\log^+ |f(x)|$. If the integral

(*) $$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)| \cdot dx}{1 + x^2} < \infty$$

we say that f belongs to the Carleman class denoted by $\mathcal{N}$. To study this class the following integral formula plays a crucial role.

C.1 Integral formula in a half-plane. Let $g(z)$ be analytic in the half plane $\mathfrak{Im}(z) > 0$ which extends continuously to the boundary $y = 0$ and assume also that $g(0) = 1$. For each pair $0 < \ell < R$ we consider the domain

$$\Omega_{\ell,R} = \{\ell^2 < x^2 + y^2 < R^2\} \cap \{y > 0\}$$

With $z = re^{i\theta}$ we have the harmonic function

$$\text{(C.1.1)} \qquad v(r,\theta) = (\frac{1}{r} - \frac{r}{R^2}) \sin\theta = \frac{y}{x^2+y^2} - \frac{y}{R^2}$$

Here $v = 0$ on the upper half circle where $|z| = R$ and $y > 0$ and the outer normal derivative along the x-axis becomes

$$\text{(C.1.2)} \qquad \partial_n(v) = -\partial_y(v) = -\frac{1}{x^2} + \frac{1}{R^2} \quad : \quad x \neq 0$$

Let $\{\alpha_\nu\}$ be the zeros of g counted with multiplicites in the upper half-plane.

C.2 Proposition. *One has the equation*

$$2\pi \cdot \sum \frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha_\nu|^2} - \frac{\mathfrak{Im}\,\alpha_\nu}{R^2} =$$

$$\text{(C.2.1)} \qquad \int_\ell^R (\frac{1}{R^2} - \frac{1}{x^2}) \cdot \log|g(x) \cdot g(-x)|\, dx - \frac{2}{R}\int_0^\pi \sin(\theta) \cdot \log|g(Re^{i\theta})|\, d\theta + \chi(\ell)$$

where $\chi(\ell)$ is a contribution from line integrals along the half circle $|z| = \ell$ with $y > 0$.

C.3 Exercise Prove (C.3.1) by Green's theorem. Above the term $\chi(\ell)$ is independent of R so (*) can be used to study the asymptotic behaviour as $R \to +\infty$.

C.4 Relation to the Jensen-Nevanlinna class. The family of analytic functions $g(z)$ in the upper half-plane is identified with $\mathcal{O}(D)$ via a conformal map, i.e. every g gives $g_* \in \mathcal{O}(D)$ where

$$\text{(C.4.1)} \qquad g_*(\frac{z-i}{z+i}) = g(z)$$

holds when $\mathfrak{Im}(z) > 0$. If g extends to a continuous function on the real x-axis the reader can verify the equality

$$\text{(C.4.2)} \qquad \int_0^{2\pi} \log^+|g_*(e^{i\theta})|\, d\theta = 2 \cdot \int_{-\infty}^{\infty} \frac{\log^+|g(x)|}{1+x^2}\, dx$$

Hence the last integral is finite if and only if g_* belongs to the Jensen-Nevanlinna class in the unit disc. In §§ XX we proved that this entails that

$$\text{(C.4.3)} \qquad \int_0^{2\pi} \log^+ \frac{1}{|g_*(e^{i\theta})|}\, d\theta < \infty$$

In particular we consider an entire function f which satisfies (*) above. Then (C.4.3) implies that

$$\text{(C.4.4)} \qquad \int_{-\infty}^{\infty} \log^+ \frac{1}{|f(x)|} \cdot \frac{dx}{1+x^2}$$

Since the absolute value of $\log|f(x)|$ is equal to the sum

$$\log^+ \frac{1}{|f(x)|} + \log^+|f(x)|$$

we conclude that (*) entails that the absolute value of $\log|f(x)|$ is integrable with respect to the density $\frac{1}{1+x^2}$. Using this we can prove:

C.5 Theorem *For each $f \in \mathcal{N}$ one has*

$$\sum{}^{*}\, \mathfrak{Im}\frac{1}{\alpha_\nu} < \infty$$

where the sum is taken over all zeros of f which belong to the upper half-plane.

Proof. Since $f \in \mathcal{E}$, we have sen in (iii) during the proof of Theorem A.13 that there exists a constant C which is independent of R such that

(i) $$\big|R^{-2}\sum|\alpha_\nu|\leq C$$

where the sum is taken over zeros in $\Omega_{\ell,R}$. Next, we have

(ii) $$\sum\mathfrak{Im}\frac{1}{\alpha_\nu}=-\sum\frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha_\nu|^2}$$

where the sum again is taken over zeros in $\Omega_{\ell,R}$. Now (C.2.1) gives Theorem V.5 follows after a passage to the limit as $R\to+\infty$ when we apply (C.2.1) with $g=f$ More precisely, it suffices to find a constant C such that

it suffices to establish an upper bound in the right hand side of Proposition C.2 with $g=f$. The integral taken over the half-circle where $|z|=R$ is uniformly bounded with respect to R since $f\in\mathcal{E}$ and we have the inequality XX from A.XX. There remains to establish an upper bound for the integral on the x-axis in (C.2.1). Since $R^{-2}-x^{-2}\leq 0$ during the integration it suffices to find a constant C such that

(iii) $$\int_\ell^R\Big(\frac{1}{x^2}-\frac{1}{R^2}\Big)\cdot\log^+\frac{1}{|f(x)\cdot f(-x)|}\cdot dx\leq C\quad:\ R\geq 1$$

The reader may verify that such a constant C exists from the result in (C.4).

D. A Tauberian Theorem

Let Λ be a non-decreasing and discrete sequence of positive real numbers $\{t_\nu\}$ whose counting function satisfies $\mathcal{N}_\Lambda(R)\leq C\cdot R$ for some constant. We get the entire function

$$f(z)=\prod(1-\frac{z^2}{t_\nu^2})$$

which by the results in § A belongs to $\mathcal{E}$. If $R>0$ we set:

(*) $$J_1(R)=\frac{\log f(iR)}{R}\quad\text{and}\quad J_2(R)=\int_{-R}^R\frac{\text{Log}\,|f(x)|}{x^2}\cdot dx$$

D.1 Theorem. *There exists a limit*

$$\lim_{R\to\infty}\frac{N_f(R)}{R}=2A$$

if and only if at least one of the J-functions has a limit as $R\to\infty$. Moreover, when this holds one has the equalities:

$$\lim_{R\to\infty}J_1(R)=\frac{\pi\cdot A}{2}\quad\text{and}\quad\lim_{R\to\infty}J_2(R)=-\frac{\pi^2\cdot A}{2}$$

The proof requires seveeral steps. First we introduce the following:

D.2 The W-functions. On the positive real t-line we define the following functions:

(1) $$W_0(t)=\frac{1}{t}\quad:\ t\geq 1\quad\text{and}\quad W_0(t)=0\quad\text{when}\ t<1$$

(2) $$W_1(t)=\frac{\log(1+t^2)}{t}$$

(3) $$W_2(t)=\int_0^t\frac{\log\,|1-x^2|}{x^2}\cdot dx$$

Next, the real sequence $\Lambda=\{t_\nu\}$ gives a discrete measure on the positive real axis where one assigns a unit point mass at every t_ν. If repetitions occur, i.e. if some t-numbers are equal we add these unit point-masses. Let ρ denote the resulting discrete measure. The constructions of the J-functions obviously give:

$$J_k(R) = \int_0^\infty W_k\big(\frac{R}{t}\big) \cdot \frac{d\rho(t)}{t} \quad : \ k = 1,2 \tag{4}$$

Moreover, the reader can verify that

$$\frac{\mathcal{N}_\Lambda(R)}{R} = 2 \cdot \int_0^\infty W_0(R/t) \cdot \frac{d\rho(t)}{t} \tag{5}$$

D.3 Exercise. Show that under the assumption that the function $\frac{\mathcal{N}_\Lambda(R)}{R}$ is bounded, it follows the three $\mathcal{W}$-functions belong to the $\mathcal{BW}$-algebra defined by the measure ρ as explained in § XXX.

D.4 Fourier transforms. Recall that on $\{t > 0\}$ we have the Haar measure $\frac{dt}{t}$. We leave it to the reader to verify that all the W-functions above belong to $L^1(\mathbf{R}^+)$, i.e.

$$\int_0^\infty \big|W_k(t)\big| \cdot \frac{dt}{t} < \infty \quad : k = 0,1,2 \tag{i}$$

The Fourier transforms are defined by

$$\widehat{W}_k(s) = \int_0^\infty W_k(t) \cdot t^{-(is+1)} \cdot dt \tag{ii}$$

We shall prefer to use the functions with reversed sign on s, i.e. set

$$\mathcal{F}\, W_k(s) = \int_0^\infty W_k(t) \cdot t^{is-1} \cdot dt \tag{iii}$$

D.5 Proposition *One has the formulas*

$$\mathcal{F}\, W_0(s) = \frac{1}{1 - is} \tag{i}$$

$$\mathcal{F}\, W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1 - is) \cdot (1 + e^{-\pi s})}$$

$$\mathcal{F}\, W_2(s) = \frac{1}{is} \cdot \big[\frac{i\pi}{1 - is} + \frac{2\pi}{(i + s) \cdot (e^{\pi s/2} + e^{-\pi s/2})}\big] \tag{iii}$$

Proof. Equation (i) is easily verified and left to the reader. To prove (ii) we use a partial integration which gives

$$\mathcal{F}\, W_1(s) = \frac{1}{is - 1} \cdot \int_0^\infty \frac{2 \cdot t^{is} \cdot dt}{1 + t^2}$$

To compute this integral we employ residue calculus where we shall use the function

$$\phi(z) = \frac{z^{is}}{1 + z^2}$$

We perform line integrals over large half-circles where $z = Re^{i\theta}$ and $0 \leq \theta \leq \pi$. A residue occurs at $z = i$. Notice also that if $t > 0$ then

$$(-t)^{is} = t^{is} \cdot e^{-\pi s}$$

This gives

$$\mathcal{F}W_1(s) = \frac{1}{1 - is) \cdot (1 + e^{-\pi s})} \cdot \lim_{R \to \infty} \int_{-R}^R \phi(t) \cdot dt$$

Here ϕ has a simple pole at $z = 1$ so by residue calculus the last integral becomes

$$-2\pi i \cdot (i)^{is} \cdot \frac{1}{2i} = -\pi \cdot e^{-\pi s/2}$$

Taking the minus sign into the account we conclude that

$$\mathcal{F}W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1-is)\cdot(1+e^{-\pi s})}$$

For (iii) a partial integration gives

$$\mathcal{F}\,W_2(s) = -\frac{1}{is}\cdot\int_0^\infty \log|1-t^2|\cdot t^{is-2}\cdot dt$$

Here we computed the right hand side in [Residue Calculus] which gives (iii).

D.6 Proof of Theorem D.1

The formulas for the Fourier transforms in Proposition D.5 show that each of them is $\neq 0$ on the real s-line. Hence we can apply the general result in XX to the discrete measure ρ since the $\mathcal{W}$-functions belong to the Beurling-Wiener algebra whose defintion and properties are treated in my notes on the mathematics by Beurling. This implies that if one of the three limits in Theorem D.1 exists, so do the other. To get the relation between the limit values we only have to evaluate the Fourier transform at $s=0$. From Proposition D.5 we see that

$$\mathcal{F}W_0(0) = 1 \quad : \quad F\mathcal{W}_1(0) = \frac{\pi}{2}$$

Finally, (iii) in (D.4) and a computation which is left to the reader gives

(**) $$F\mathcal{W}_2(0) = -\frac{\pi^2}{2}$$

This gives the formulas in Theorem D.3 by the general result for $\mathcal{BW}$-algebras in XXX.

D.7 An application. Using Theorem D.1 we can prove the following:

D.8 Theorem *For each $f \in \mathcal{N}$ there exists the limit:*

$$\lim_{R\to\infty}\frac{N_f(R)}{R}$$

Proof. The product $f(z)\cdot f(-z)$ also belongs to $\mathcal{N}$ and for this even function the counting function is twice that of f. So it suffices to prove Theorem D.8 when f is even. We may also assume that $f(0)=1$ and since $f\in\mathcal{E}$ it is given by a Hadamard product

(1) $$f(z) = \prod^{*}(1-\frac{z^2}{\alpha_\nu^2})$$

where $\prod^*$ indicates the we take the product of zeros whose real part is >0 and if they are purely imaginary they are of the form $b\cdot i$ with $b>0$. We can replace the zeros by their absolute values and construct

(2) $$f_*(z) = \prod^{*}(1-\frac{z^2}{|\alpha_\nu|^2})$$

If x is real we see that

(3) $$|f_*[x)| \leq |f(x)|$$

We conclude that if f belongs to $\mathcal{N}$ so does f_*. At the same time their counting functions of zeros are equal. This reduces the proof to the special case when f is even and the zeros are real and at this stage it is clear that Theorem D.1 gives existence of the limit in Theorem D.8.

E. Application to measures with compact support.

Le μ be a Riesz measure on the real t-line with compact support in an interval $[-a,a]$ where we assume that both end-points belong to the support. The measure is in general complex-valued. Now we get the entire function

$$f(z)=\int_{-a}^{a}e^{-izt}\cdot d\mu(t)$$

Here f restricts to a bounded function on the real x-axis with maximum norm $\leq ||\mu||$. Hence f belongs to $\mathcal{N}$ which means that Theorem D.8 holds and the reader may now verify the following:

E.1 Theorem. *One has the equality*

$$\lim_{R\to\infty}\frac{N_f(R)}{R}=\frac{a}{\pi}$$

F. Tauberian theorems with a remainder term

An extension of Theorem D.1 which contains remainder terms were established by Beurling in 1936. An example of Beurling's results goes as follows: Let

$$f(z)=\prod(1-\frac{z^2}{t_\nu^2})$$

be an even and entire function of exponential type with real zeros as in section D.

F.1 Theorem. *Let $A>0$ and $0<a<1$ and assume that there exists a constant C_0 such that*

$$\Big|-\frac{1}{\pi^2}\cdot\int_0^R\frac{\log|f(x)|}{x^2}\cdot dx-A\Big|\leq C_0\cdot R^{-a}$$

hold for all $R\geq 1$. Then there is another constant C such that

$$\Big|\,N_f(R)-R\Big|\leq C_1\cdot R^{1-a/2}$$

Remark. Beurling's original manuscript which contains a proof of Theorem F.1 as well as other results dealing with remainder terms has remained unpublished. It was resumed with details of proofs in a Master's Thesis at Stockholm University by F. Gülkan in 1994. The interested reader should also consult articles by Beurlings former Ph.d student S. Lyttkens which prove various Tauberian theorems with remainder terms.

7. Hadamard's radius theorem.

The thesis *Essais sur l'études des fonctions donnés par leur dévelopment d Taylor* from 1894 by Hadamard contains many interesting results. Here we expose material from Section 2 in [ibid]. Consider a power series

$$f(z) = \sum c_n z^n \tag{*}$$

whose radius is a positive number ρ. So f is analytic in the open disc $\{|z| < \rho\}$ and has at least one singular point on the circle $\{|z| = \rho\}$. Hadamard found a condition in order that these singularities consists of a finite set of poles only so that f extends to be meromorphic in some disc $\{|z| < \rho_*\}$ with $\rho_* > \rho$. The condition is expressed via properties of the Hankel determinants. Let us recall their definition. Let $\{c_0, c_1, \ldots\}$ be a sequence of complex numbers. For each integer $p \geq 0$ and every $n \geq 0$ we obtain the $(p+1) \times (p+1)$-matrix:

$$\mathcal{C}_n^{(p)} = \begin{pmatrix} c_n & c_{n+1} & \ldots & c_{n+p} \\ c_{n+1} & c_{n+2} & \ldots & c_{n+p} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+p} & c_{n+p+1} & \ldots & c_{n+2p} \end{pmatrix}$$

Let $\mathcal{D}_n^{(p)}$ denote the determinant. One refers to $\{\mathcal{D}_n^{(p)}\}$ as the recursive Hankel determinants. Kronecker proved that the series (*) represents a rational function in the complex z-åplane whose poles are outside the origin if and only if there exists some positvie integer p such that $\mathcal{D}_n^{(p)} = 0$ hold for all n. We shall prove this in § xx. Hadamard established a remarkable extensionmof Kronecker's result which goes as follows. WEe are given the series in (*) and assume that it has a finite postive radius of convergence which by a wellknown formula satisfies the equation

$$\frac{1}{\rho} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

This entails that for every $\epsilon > 0$ there exists a constant C_ϵ such that

$$|c_n| \leq C \cdot (\rho - \epsilon)^{-n} \quad \text{hold for every} \quad n$$

It follows trivially that

$$|\mathcal{D}_n^{(p)}| \leq (p+1)! \cdot C^{p+1} (\rho - \epsilon)^{-(p+1)n}$$

Passing to limes superior where high n:th roots are taken we conclude that:

$$\delta(p) = \limsup_{n\to\infty} \left[\mathcal{D}_n^{(p)}\right]^{\frac{1}{n}} \leq \rho^{-(p+1)} \tag{1}$$

Suppose there exists some $p \geq 1$ where a strict inequality occurs:

$$\delta(p) < \rho^{-(p+1)} \tag{2}$$

Let p be the smallest integer for which (2) which gives a number $\rho_* > \rho$ such that

$$\delta(p) = \rho_*^{-1} \cdot \rho^{-p} \tag{3}$$

Hadamard's Theorem. *With p and ρ_* as in (3), it follows that $f(z)$ extends to a meromorphic function in the disc of radius ρ_* where the number of poles counted with multiplicity is at most p.*

The proof requires several steps. We shall first expose some general formulas about determinants while the proof of Hadamard's theorem starts in § xx. But let us first describe an application of Hadamard's theorem. Namely, if

$$\lim_{p\to\infty} \delta(p) = 0 \tag{4}$$

it follows that (*) extends to a meromprohic function inthe whole complex plane. Let us now consider a complex-valued and continuous function $k(x,y)$ defined on the unit square $\{0 \leq x, y \leq 1\}$. We do not assume that k is symmetric, i.e, in general $k(x,y) \neq k(y,x)$. Let $f(x)$ be another continuous function on $[0,1]$. Assume that the maximum norms of k and f both are < 1. By induction over n starting with $f_0(x) = f(x)$ we get a sequence $\{f_n\}$ where

$$f_n(x) = \int_0^1 k(x,y) \cdot f_{n-1}(y) \cdot dy \quad : \quad n \geq 1$$

The hypothesis entails that each f_n has maximum norm < 1 and hence there exists a power series:

$$u_\lambda(x) = \sum_{n=0}^{\infty} f_n(x) \cdot \lambda^n$$

which converges for every $|\lambda| < 1$ and yields a continuous function $u_\lambda(x)$ on $[0,1]$.

0.1 Theorem. *The function $\lambda \mapsto u_\lambda(x)$ with values in the Banach space $B = C^0[0,1]$ extends to a meromorphic B-valued function in the whole λ-plane.*

To prove this we consider the recursive Hankel determinants for each $0 \leq x \leq 1$:

$$\mathcal{D}_n^{(p)}(x) = \det \begin{pmatrix} f_{n+1}(x) & f_{n+2}(x) & \ldots & \ldots & f_{n+p}(x) \\ f_{n+2}(x) & f_{n+3}(x) & \ldots & \ldots & f_{n+p+1}(x) \\ \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ f_{n+p}(x) & f_{n+p+1}(x) & \ldots & \ldots & f_{n+2p-1}(x) \end{pmatrix}$$

With these notations the following inequality holds:

Proposition 0.2 *For every $p \geq 2$ and $0 \leq x \leq 1$ one has*

$$\big|\, \mathcal{D}_n^{(p)}(x)) \,\big| \leq (p\,!)^{-n} \cdot \big(p^{\frac{p}{2}}\big)^n \cdot \frac{p^p}{p\,!} \tag{0.2.1}$$

This result is due to Carleman and exposed in § xx. The inequality (0.2.1) entails that

$$\limsup_{n \to \infty} \big|\mathcal{D}_n^{(p)}(x))\,\big|^{1/n} \leq \frac{p^{p/2}}{p\,!}$$

Next, Stirling's formula gives:

$$\lim_{p \to \infty} \big[\frac{p^{1/2}}{p\,!}\,\big]^{-1/p} = 0$$

Hence the special case of Hadamard's theorem gives Theorem 0.1

A. The Sylvester-Franke theorem.

Let A be some $n \times n$-matrix with elements $\{a_{ik}\}$. Put

$$b_{rs} = a_{11}a_{rs} - a_{r1}a_{1s} \quad : \quad 2 \le r, s \le n$$

These b-numbers give an $(n-1) \times (n-1)$-matrix where b_{22} is put in position $(1,1)$ and so on. The matrix is denoted by $\mathcal{S}^1(A)$ and called the first order Sylvester matrix. If $a_{11} \neq 0$ one has the equality

(A.1.1) $$a_{11}^{n-2} \cdot \det(A) = \det(\mathcal{S}^1(A))$$

Exercise. Prove this result or consult a text-book which apart from "soft abstract notions" does not ignore to treat determinants.

A.1.2 Sylvester's equation. For every $1 \le h \le n-1$ one constructs the $(n-h \times (n-h)$-matrix whose elements are

$$b_{rs} = \det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1h} & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2h} & a_{2s} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{h1} & a_{h2} & \dots & a_{hh} & a_{hs} \\ a_{r1} & a_{r2} & \dots & a_{rh} & a_{rs} \end{pmatrix} \quad : \quad h+1 \le r, s \le n$$

With these notation one has the Sylvester equation:

(*) $$\det \begin{pmatrix} b_{h+1,h+1} & b_{h+1,h+2} & \dots & b_{h+1,n} \\ b_{h+2,h+1} & b_{h+2,h+2} & \dots & b_{h+2,n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ b_{n,h+1} & b_{n,h+2} & \dots & b_{n,n} \end{pmatrix} = \Big[\det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1h} \\ a_{21} & a_{22} & \dots & a_{2h} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{h1} & a_{h2} & \dots & a_{hh} \end{pmatrix} \Big]^{n-h-1} \cdot \det(A)$$

For a proof of (*) we refer to original work by Sylvester or [Kovalevski: page xx-xx] which offers several different proofs of (*).

A.1.3 The Sylvester-Franke theorem. Let $n \ge 2$ and $A = \{a_{ik}\}$ an $n \times n$-matrix. Let $m < n$ and consider the family of minors of size m, i.e. one picks m columns and m rows which give an $m \times m$-matrix whose determinant is called a minor of size m of the given matrix A. The total number of such minors is equal to

$$N^2 \quad \text{where} \quad N = \binom{n}{m}$$

We have N many strictly increasing sequences $1 \le \gamma_1 < \dots \gamma_m \le n$ where a γ-sequence corresponds to preserved columns when a minor is constructed. Similarly we have N strictly increasing sequences which correspond to preserved rows. With this in mind we get for each pair $1 \le r, s \le N$ a minor $\mathfrak{M}_{rs}$ where the enumerated r:th γ-sequence preserve columns and similarly s corresponds to the enumerated sequence of rows. Now we obtain the $N \times N$-matrix

$$\mathcal{A}_m = \begin{pmatrix} \mathfrak{M}_{11} & \mathfrak{M}12 & \dots & \mathfrak{M}_{1N} \\ \mathfrak{M}_{21} & \mathfrak{M}22 & \dots & \mathfrak{M}_{2N} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \mathfrak{M}_{N1} & \mathfrak{M}22 & \dots & \mathfrak{M}_{NN} \end{pmatrix}$$

We refer to $\mathcal{A}_m$ as the Franke-Sylvester matrix of order m. They are defined for each $1 \le m \le n-1$.

A.1.4 Theorem. *For every* $1 \leq m < n$ *one has the equality*

$$\mathcal{A}_m = \det(A)^{\binom{n-1}{m-1}}$$

Example. Consider the diagonal 3×3-matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

With $m = 2$ we have 9 minors of size 2 and the reader can recognize that when they are arranged so that we begin to remove the first column, respectively the first row, then the resulting $\mathfrak{M}$-matrix becomes

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Its determinant is $4 = 2^2$ which is in accordance with the general formula since $n = 3$ and $m = 2$ give $\binom{n-1}{m-1} = 2$. For the proof of Theorem 0.A.1 the reader can consult [Kovalevski: page102-105].

§ B. Hankel determinants.

Let $\{c_0, c_1, \ldots\}$ be a sequence of complex numbers. For each integer $p \geq 0$ and every $n \geq 0$ we obtain the $(p+1) \times (p+1)$-matrix:

$$\mathcal{C}_n^{(p)} = \begin{pmatrix} c_n & c_{n+1} & \ldots & c_{n+p} \\ c_{n+1} & c_{n+2} & \ldots & c_{n+p} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+p} & c_{n+p+1} & \ldots & c_{n+2p} \end{pmatrix}$$

Let $\mathcal{D}_n^{(p)}$ denote the determinant. One refers to $\{\mathcal{D}_n^{(p)}\}$ as the recursive Hankel determinants. They are used to establish various properties of the given c-sequence. To begin with we define the rank r^* of $\{c_n\}$ as follows: To every non-negative integer n one has the infinite vector

$$\xi_n = (c_n, c_{n+1}, \ldots)$$

We say that $\{c_n\}$ has finite rank if there exists a number r^* such that r^* many ξ-vectors are linearly independent and the rest are linear combinations of these.

B.1 Rational series expansions. The sequence $\{c_n\}$ gives the formal power series

$$f(x) = \sum_{\nu=0}^{\infty} c_\nu x^\nu \tag{B.1.1}$$

If $n \geq 1$ we set

$$\phi_n(x) = x^{-n} \cdot (f(x) - \sum_{\nu=0}^{n-1} c_\nu x^\nu) = \sum_{\nu=0}^{\infty} c_{n+\nu} x^\nu$$

It is clear that $\{c_\nu\}$ has finite rank if and only if the sequence $\{\phi_\nu(x)\}$ generates a finite dimensional complex subspace of the vector space $\mathbf{C}[[x]]$ whose elements are formal power series. If this dimension is finite we find a positive integer p and a non-zero $(p+1)$-tuple $(a_0, \ldots, a_p)$ of complex numbers such that the power series

$$a_0 \cdot \phi_0(x) + \ldots + a_p \cdot \phi_p(x) = 0$$

Multiplying this equation with x^p it follows that

$$(a_p + a_{p-1}x + \ldots + a_o x^p) \cdot f(x) = q(x)$$

where $q(x)$ is a polynomial. Hence the finite rank entails that the power series (B.1.1) represents a rational function.

Exercise. Conversely, assume that

$$\sum c_\nu x^\nu = \frac{q(x)}{g(x)}$$

for some pair of polynomials. Show that $\{c_n\}$ has finite rank. The next result is also left as an exercise to the reader.

B.2 Proposition. *A sequence $\{c_n\}$ has a finite rank if and only if there exists an integer p such that*

$$\mathcal{D}_0^{(p)} \neq 0 \quad \text{and} \quad D_0^{(q)} = 0 \quad : \quad q > p \tag{4}$$

Moreover, one has the equality p is equal to the rank of $\{c_n\}$.

B.3 A specific example. Suppose that the degree of q is strictly less than that of g in the Exercise above and that the rational function $\frac{q}{g}$ is expressed by a sum of simple fractions:

$$\sum c_\nu x^\nu = \sum_{k=1}^{k=p} \frac{d_k}{1 - \alpha_k x}$$

where $\alpha_1, \ldots, \alpha_p$ are distinct and every $d_k \neq 0$. Then we see that

$$c_n = \sum_{k=1}^{k=p} d_k \cdot \alpha_k^n \quad \text{where we have put} \quad \alpha_k^0 = 1 \quad \text{so that} \quad c_0 = \sum d_k$$

B.4 The reduced rank. Assume that $\{c_n\}$ has a finite rank r^*. To each $k \geq 0$ we denote by r_k the dimension of the vector space generated by $\xi_k, \xi_{k+1}, \ldots$. It is clear that $\{r_k\}$ decrease and we find a non-negative integer r_* such that $r_k = r_*$ for large k and refer to r_* as the reduced rank. By the construction $r_* \leq r^*$. The relation between r^* and r_* is related to the representation

$$f(x) = \frac{q(x)}{g(x)}$$

where q and g are polynomials without common factor. We shall not pursue this discussion any further but refer to the literature. See in particular the exercises in [Polya-Szegö : Chapter VII:problems 17-34].

B.5 Hankel's formula for Laurent series. Consider a rational function of the form

$$R(z) = \frac{q(z)}{z^p - [a_1 z^{p-1} + \ldots + a_{p-1} z + a_p]}$$

where the polynomial q has degree $\leq p - 1$. At ∞ we have a Laurent series expansion

$$R(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \ldots$$

Consider the $p \times p$-matrix

$$A = \begin{pmatrix} 0 & 0 & & \ldots & 0 & a_p \\ 1 & 0 & 0 & \ldots & 0 & a_{p-1} \\ 0 & 1 & 0 & \ldots & \ldots & a_{p-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & a_1 \end{pmatrix}$$

B.5.1 Theorem. *Let $\mathcal{D}_n^{(p)}$ be the Hankel determinants of $\{c_n\}$. Then the following hold for every $n \geq 1$:*

$$\mathcal{D}_n^{(p)} = \mathcal{D}_0^{(p)} \cdot \big[\det(A)\big)^n$$

Exercise. Prove this result.

B.6 Kronecker's identity. For all pairs of positive integers p and n one has the equality:

$$\mathcal{D}_n^{(p+1)} \cdot \mathcal{D}_{n+2}^{(p-2)} = \mathcal{D}_n^{(p+1)} \mathcal{D}_{n+2}^{(p-1)} - \big[\mathcal{D}_{n+1}^{(p)}\big]^2 \tag{B.6.1}$$

Proof. The equality (B.6.1) is s special case of a determinant formula for symmetric matrices which is due to Sylvester. Namely, let $N \geq 2$ and consider a symmetric matrix

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1N} \\ s_{21} & s_{22} & \dots & s_{2N} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_{N1} & a_{N2} & \dots & s_{NN} \end{pmatrix}$$

Now we consider the $(N-1) \times (N-1)$-matrices

$$S_1 = \begin{pmatrix} s_{22} & s_{23} & \dots & s_{2N} \\ s_{32} & s_{33} & \dots & s_{3N} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_{N2} & s_{N3} & \dots & s_{NN} \end{pmatrix} \quad : \quad S_2 = \begin{pmatrix} s_{12} & s_{13} & \dots & s_{1N} \\ s_{22} & s_{23} & \dots & s_{2N} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_{N-1,2} & s_{N-1,3} & \dots & s_{N-1,N} \end{pmatrix}$$

$$S_3 = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1,N-1} \\ s_{21} & s_{22} & \dots & s_{2,N-1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_{N-1,1} & s_{N-1,2} & \dots & s_{N-1,N-1} \end{pmatrix}$$

We have also the $(N-2) \times (N-2)$-matrix when extremal rows and columns are removed:

$$S_* = \begin{pmatrix} s_{22} & s_{23} & \dots & s_{2,N-1} \\ s_{32} & s_{33} & \dots & s_{3,N-1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_{2,N-1} & s_{3,N-1} & \dots & s_{N-1,N-1} \end{pmatrix}$$

B.7 Sylvester's identity. *One has the equation*

$$\det(S) \cdot \det(S_*) = \det S_1) \cdot \det S_3 - \big(\det S_2\big)^2$$

Exercise. Prove this result and deduce Kronecker's equation.

C. Proof of Hadamard's theorem.

We are given the smallest integer p in Hadamard's theorem. A first step in the proof is:

C.1 Lemma. *When p as above is minimal one has the unrestricted limit formula:*

$$\lim_{n\to\infty} \big[\mathcal{D}_n^{(p-1)}\big]^{\frac{1}{n}} = \rho^{-p} \tag{*}$$

TO BE GIVEN: Exercise power series+ Sylvesters equation.

Lemma 7.2 entails that if n is large $\{\mathcal{D}_n^{(p-1)}\}$ are $\neq 0$. Hence there exists some n_* such that every $n \geq n_*$ we find a unique p-vector $(A_n^{(1)}, \dots, A_n^{(p)})$ which solves the inhomogeneous system

$$\sum_{k=0}^{k=p-1} c_{n+k+j} \cdot A_n^{(p-k)} = -c_{n+p+j} \quad : \quad 0 \leq j \leq p-1$$

Or expressed in matrix notation:

$$(*)\qquad \begin{pmatrix} c_n & c_{n+1} & \cdots & c_{n+p-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+p} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+p-1} & c_{n+p} & \cdots & c_{n+2p-2} \end{pmatrix} \begin{pmatrix} A_n^{(p)} \\ \cdots \\ \cdots \\ \cdots \\ A_n^{(1)} \end{pmatrix} = - \begin{pmatrix} c_{n+p} \\ \cdots \\ \cdots \\ \cdots \\ c_{n+2p-1} \end{pmatrix}$$

C.2 Exercise. Put

$$H_n = c_{n+2p} + A_n^{(1)} \cdot c_{n+2p-1} + \ldots + A_n^{[(p)} \cdot c_{n+p}$$

Show that the evaluation of $\mathcal{D}_n^{(p)}$ via an expansion of the last column gives the equality:

$$\text{(C.2.1)}\qquad H_n = \frac{\mathcal{D}_n^{(p)}}{\mathcal{D}_n^{(p-1)}}$$

Next, the limit formula (3) above Theorem 7.1 together with Lemma C.1 give for every $\epsilon > 0$ a constant C_ϵ such that the following hold for all sufficiently large n:

$$\text{(i)}\qquad |H_n| \leq C_\epsilon \cdot \big(\frac{\rho + \epsilon}{\rho_* - \epsilon}\big)^n$$

Next, put

$$\text{(ii)}\qquad \delta_n^k = A_{n+1}^{(k)} - A_n^{(k)} \quad : \quad 1 \leq k \leq p$$

From the equation in (*) applied with n and $n+1$ it is clear that the δ-numbers satisfy the system

$$\sum_{k=0}^{k=p-1} c_{n+j+k+1} \cdot \delta_n^{(p-k)} = 0 \quad : \quad 0 \leq j \leq p-2$$

$$\text{(iii)}\qquad \sum_{k=0}^{k=p-1} c_{n+p+k} \cdot \delta_n^{(p-k)} = -(c_{n+2p} + A_n^{(1)} \cdot c_{n+2p-1} + \ldots + A_n^{[(p)} \cdot c_{n+p})$$

here the δ-numbers in the linear system (iii) are found via Cramer's rule. The minors of degree $p-1$ in the Hankel matrices $\mathcal{C}_{n+1}^{(p-1)}$ have elements from the given c-sequence and every such minor has an absolute value majorized by

$$C \cdot (\rho - \epsilon)^{-(p-1)n}$$

where C is a constant which is independent of n. We conclude that the δ-numbers satisfy

$$\text{(iv)}\qquad |\delta_n^{(k)}| \leq |\mathcal{D}_n^{(p-1)}|^{-1} \cdot C \cdot (\rho - \epsilon)^{-(p-1)n} \cdot |H_n|$$

Next, the unrestricted limit in Lemma C.1 give upper bounds for $|\mathcal{D}_n^{(p-1)}|^{-1}$ so that (C.2.1) and (iv) give:

C.3 Lemma *To each $\epsilon > 0$ there is a constant C_ϵ such that*

$$|\delta_n^{(k)}| \leq C_\epsilon \cdot \big(\frac{\rho + \epsilon}{\rho_* - \epsilon}\big)^n \quad : \quad 1 \leq k \leq p$$

C.4 The polynomial $Q(z)$. Lemma C.3 and (ii) entail that the sequence $\{A_n^{(k)} : n = 1, 2, \ldots\}$ converges for every k. Set

$$A_*^{(k)} = \lim_{n \to \infty} A_n^{(k)}$$

Notice that Lemma C.3 after summations of geometric series gives a constant C_1 such that

$$\text{(C.4.1)}\qquad |A_*^{(k)} - A_n^{(k)}| \leq C_1 \cdot \big(\frac{\rho + \epsilon}{\rho_* - \epsilon}\big)^n$$

hold for every $1 \leq k \leq p$ and every n.

Now we consider the sequence

$$b_n = c_{n+p} + A_*^{(1)} \cdot c_{n+p-1} + \ldots A_*^{(p)} \cdot c_n \tag{C.4.2}$$

Equation (*) applied to $j = 0$ gives

$$b_n = (A_*^{(1)} - A_n^{(1)}) \cdot c_{n+p-1} + \ldots + (A_*^{(p)} - A_n^{(p)}) \cdot c_n \tag{C.4.3}$$

Next, we have already seen that $|c_n| \leq C \cdot (\rho - \epsilon)^{-n}$ hold for some constant C which together with (C.4.1) gives:

C.5 Lemma. *For every $\epsilon > 0$ there exists a constant C such that*

$$|b_n| \leq C \cdot \big(\frac{1+\epsilon}{\rho_*}\big)^n$$

Finally, consider the polynomial

$$Q(z) = 1 + A_*^{(1)} \cdot z + \ldots A_*^{(p)} \cdot z^p \tag{C.6}$$

Set $g(z) = Q(z)f(z)$ which has a power series $\sum d_\nu z^\nu$ where

$$d_{n+p} = c_n \cdot A_*^{(p)} + \ldots c_{n+p-1} A_*^{(1)} + c_{n+p} = b_n$$

Above p is fixed so Lemma 6.5 and the standard spectral radius formula show that $g(z)$ is analytic in the disc $|z| < \rho_*$. This proves that f extends and the poles are contained in the zeros of the polynomial Q which occur in $\rho \leq |z| < \rho_*$.