(*)

III.B The Hardy space H^1

0. Introduction.

At several occasions we have met the situation where $F(z) \in \mathcal{O}(D)$ has bounded L^1-norms over circles of radius $r < 1$. The Brothers Riesz theorem in section I shows that if there is a constant M such that

$$\int_0^{2\pi} |F(re^{i\theta})| d\theta \leq M \quad : \quad 0 < r < 1$$

then there exists an L^1-function $F(e^{i\theta})$ on the unit circle and

$$(*) \qquad \lim_{r \to 1} \int_0^{2\pi} |F(re^{i\theta}) - F(e^{i\theta}| \cdot d\theta = 0$$

The class of analytic functions F with boundary function in $L^1(T)$ is denoted by $H^1(T)$ and called the *Hardy space*. It is tempting to start with a real valued L^1-function $u(\theta)$ on the unit circle and apply the Herglotz integral formula which produces both the harmonic extension of u and its conjugate harmonic function by the equation:

$$(**) \qquad g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot u(\theta) d\theta$$

It turns out that g_μ in general does not belong to $H^1(T)$, i.e. the condition that $u \in L^1(T)$ does not imply that $g_\mu \in H^1(T)$. Theorem 0.1 below is due to Zygmund and gives a necessary and sufficient condition for the inclusion $F \in L^1(T)$ when u is non-negative.

0.1 Theorem. *Let $u(\theta)$ be a non-negative L^1-function on T. Then $g_\mu(z) \in H^1(T)$ if and only if*

$$\int_0^{2\pi} u(\theta) \cdot \log^+ |u(\theta)| \cdot d\theta < \infty$$

Remark. That the condition is necessary is proved in section 1. The proof of sufficiency relies upon study of linear operators satisfying weak type estimates where a result due to Kolmogorov is the essential point. To profit upon Kolmogorov's result in section 3 we need a weak-type estimate for the harmonic conjugation functor which is proved in section 2.

0.2 The dual space of $H^1(T)$. On the unit circle the Banach space $C^0(T)$ of continuous complex-valued functions contains the closed subspace $A_*(D)$ which consists of those continuous function $f(e^{i\theta})$ on T which extend to analytic functions in the open disc $|z| < 1$ and vanish at $z = 0$. In Theorem 4.3 we prove that $H^1(T)$ is the dual of the quotient space

$$B = \frac{C^0(T)}{A_*(D)}$$

The proof uses the Brothers Riesz theorem. We shall also consider the subspace $H_0^1(T)$ of those functions in the Hardy space for which $f(0) = 0$. Here we find that

$$(1) \qquad H_0^1(T) \simeq \big[\frac{C^0(T)}{A(D}\big]^*$$

Next, we seek the dual space of $H_0^1(T)$. Using the Brothers Riesz theorem one finds that

$$(2) \qquad H_0^1(T)^* \simeq \frac{L^\infty(T)}{H^\infty(T)}$$

where $H^\infty(T)$ is the space of boundary values of bounded analytic functions in D.

0.3 The dual of $\mathfrak{Re}\, H_0^1(T)$. The real part determine functions in $H_0^1(T)$ which means that we can identify $H_0^1(T)$ with a real subspace of $L^1_{\mathbf{R}}(T)$ whose elements consist of those real-valued and integrable functions $u(\theta)$ for which the the Riesz transform also is integrable. Or equivalently, if we take the harmonic extension H_u then the harmonic conjugate has a boundary function in $L^1_{\mathbf{R}}(T)$ which we denote by u^*. The norm of such a u-function is defined as

$$(*) \qquad ||u|| = ||u||_1 + ||u^*||_1$$

The norm in (3) is not equivalent to the L^1-norm so we cannot conclude that the dual space is reduced to real-valued functions in $L^\infty(T)$. To exhibit elements in the dual space we first consider some real-valued function $F(\theta)$ on T. Let H_F be its harmonic extension to D. For each $0 < r < 1$ we define the linear functional on $\mathfrak{Re}\, H_0^1(T)$ by:

$$(**) \qquad u \mapsto \int_0^{2\pi} H_F(re^{i\theta}) \cdot u(\theta) \cdot d\theta$$

If the limit (*) exists for every u when $r \to 1$ and the absolute value of this limit is $\leq C \cdot ||u||$ for a constant C, then we have produced a continuous linear functional on $\mathfrak{Re}\, H_0^1(T)$. This leads to a description of the dual space which goes as follows. The definition of the norm in (*) and the Hahn-Banach theorem yields for each Λ in the dual space a pair (ϕ, ψ) in $L^\infty(T)$ such that when $f = u + iu^*$ is in $H_0^1(T)$ then

$$\Lambda(u + iu^*) = \int_0^{2\pi} u(\theta) \cdot \phi(\theta) \cdot d\theta + \int_0^{2\pi} u^*(\theta) \cdot \psi(\theta) \cdot d\theta$$

Let ψ^* be the harmonic conjugate of ψ which gives the analytic function $H_\psi + iH_\psi^*$ in D. Since $f = u + iu^*$ vanishes at $z = 0$ we get

$$\int_0^{2\pi} (u + iu^*)(\psi + i\psi^*) \cdot d\theta = 0$$

Regarding the imaginary part it follows that

$$\int_0^{2\pi} u^* \cdot \psi \cdot d\theta = -\int_0^{2\pi} u \cdot \psi^* \cdot d\theta$$

We conclude that Λ is expressed by

$$\Lambda(u) = \lim_{r \to 1} \int_0^{2\pi} H_F(re^{i\theta}) \cdot u(\theta) \cdot d\theta$$

where

$$(***) \qquad F(\theta) = \phi(\theta) - \psi^*(\theta)$$

Above ψ^* is the harmonic conjugate of a bounded ψ-function where an arbitrary $\psi \in L^\infty(T)$ can be chosen. Next, recall from XXX that if $\psi \in L^\infty(T)$ then its conjugate ψ^* belongs to BMO(T). Hence (***) identifies the of $\mathfrak{Re}\, H_0^1(T)$ with a subspace of BMO(T). It turns out that one has equality. More precisely, Theorem 0.4 below which is due to C. Fefferman and E. Stein asserts that F yields such a continuous linear form if and only if F has a bounded mean oscillation in the sense of F. John and L. Nirenberg.

0.4 Theorem. *A real-valued L^1-function F on T yields a continuous linear functional on $H_0^1(T)$ as above if and only if $F \in BMO(T)$. Moreover, there exists an absolute constant C such that*

$$\Big|\int_0^{2\pi} H_F(re^{i\theta}) \cdot u(\theta) \cdot d\theta\Big| \leq C \cdot ||F||_{\text{BMO}} \cdot ||u||_1$$

for all $r < 1$ and $u \in H_0^1(T)$.

We refer to Section 6 for details of the proof which involves several steps where the essential step is to exhibit certain Carleson measures. The space of real-valued functions of bounded mean oscillation is denoted by BMO(T) and studied in Section 5 where Theorem 5.5 is an important result which clarifies many properties of functions in BMO(T).

0.5 The Hardy space on R. It consists of analytic functions $F(z)$ in the upper half-plane for which there exists a constant C such that

$$(*) \qquad \int_{-\infty}^{\infty} |F(x+i\epsilon) \cdot dx \leq C$$

hold for every $\epsilon > 0$. This space is denoted by $H^1(\mathbf{R})$. Let us remark that it differs from $H^1(\mathbf{T})$ even if we employ the conformal map

$$\text{(i)} \qquad w = \frac{z-i}{z+i}$$

onto the unit disc. More precisely, with $F(z)$ given in the upper half-plane we set

$$\text{(ii)} \qquad f(w) = F(\frac{i+iw}{1-w})$$

Then the reader can verify that

$$\text{(iii)} \qquad \lim_{r\to 1} \int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta = \int_{-\infty}^{\infty} \frac{|F(x)|}{1+x^2} \cdot dx$$

where $F(x)$ is the almost everywhere defined limit of F on the real x-line which by (*) identifies $F(x)$ with an element in $H^1(\mathbf{R})$. Since $\frac{1}{1+x^2}$ is bounded it follows that the right hand side is finite in (iii) and hence f belongs to $H^1(\mathbf{T})$. However, the map $F \to f$ is not bijective because the convergence in (iii) need not imply that (*) is finite. In other words, the Hardy space on the real line is more constrained and via $F \mapsto f$ it appears s a proper subspace of $H^1(\mathbf{T})$ and the corresponding norms are not equivalent.

Sections 7 and 9 study $H^1(\mathbf{R})$ and at the end of section 9 we introduce Carleson norms on non-negative Riesz measures in $\mathfrak{Im}(z) > 0$ which will be used for interpolation of bounded analytic functions in Chapter XXX.

1. Zygmund's inequality

Let $u(\theta)$ be a non-negative real-valued function on T such that

$$(*) \qquad \frac{1}{2\pi} \int_0^{2\pi} u(\theta) d\theta = 1$$

Put

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot u(\theta) d\theta$$

We write

$$F = u + iv$$

where u is the harmonic extension of $u(\theta)$ from T to D and v is the harmonic conjugate which by the Herglotz formula is normalised so that $v(0) = 0$ The sufficency part in Zygmund's theorem follows from the general inequality below:

1.1 Theorem. *When $u(\theta)$ is non-negative and (*) holds we have*

$$(*) \qquad \int_0^{2\pi} u(\theta) \cdot \mathrm{Log}^+ |u(\theta)| \cdot d\theta \leq \frac{\pi}{2} \cdot \int_0^{2\pi} |v(\theta)| \cdot d\theta + \int_0^{2\pi} \mathrm{Log}^+ |F(e^{i\theta}]| \cdot d\theta$$

Proof. Since $\mathfrak{Re}(F) > 0$ holds in D we can write

$$\text{(i)} \qquad \log F(z) = \log |F(z)| + i\gamma(z) \quad : \quad -\pi/2 < \gamma(z) < \pi/2$$

Set $G(z) = F(z) \cdot \log F(z)$. Since $F(0) = 1$ we have $G(0) = 0$ and the mean value formula for harmonic functions gives

$$\text{(iii)} \qquad \int_0^{2\pi} u(e^{i\theta}) \cdot \mathrm{Log}\, |F(e^{i\theta})| \cdot d\theta = \int_0^{2\pi} \gamma(e^{i\theta)}) \cdot v(e^{i\theta}) \cdot d\theta$$

By (i) the absolute value of the right hand side is majorised by

$$\text{(iii)} \qquad \frac{\pi}{2} \cdot \int_0^{2\pi} |v(\theta)| \cdot d\theta$$

Now we use the decomposition

$$\log |F(e^{i\theta})| = \mathrm{Log}^+ |F(e^{i\theta})| + \mathrm{Log}^+ \frac{1}{|F(e^{i\theta})|}$$

Then (ii) and (iii) give the inequality

$$\int_0^{2\pi} u(e^{i\theta}) \cdot \mathrm{Log}^+ |F(e^{i\theta})| \cdot d\theta \leq$$

$$\text{(iv)} \qquad \frac{\pi}{2} \cdot \int_0^{2\pi} |v(\theta)| \cdot d\theta + \int_0^{2\pi} u(e^{i\theta}) \cdot \mathrm{Log}^+ \frac{1}{|F(e^{i\theta})|} \cdot d\theta$$

Since $\mathrm{Log}^+ \frac{1}{|F(e^{i\theta})|} \neq 0$ entails that $|F| \leq 1$ and hence also $u \leq 1$, it follows that the last integral above is majorised by

$$\text{(v)} \qquad \int_0^{2\pi} \mathrm{Log}^+ \frac{1}{|F(e^{i\theta})|} \cdot d\theta$$

Next, $\log |F(z)|$ is a harmonic function whose value at $z = 0$ is zero. So the mean-value formula for harmonic functions in D gives the equality:

$$\text{(vi)} \qquad \int_0^{2\pi} \mathrm{Log}^+ \frac{1}{|F(e^{i\theta})|} \cdot d\theta = \int_0^{2\pi} \mathrm{Log}^+ |F(e^{i\theta})| \cdot d\theta$$

Using this and (iv) we get the requested inequality in Theorem 1.1 since we also have the trivial estimate

$$\text{(vii)} \qquad \mathrm{Log}^+ u(e^{i\theta}) \leq \mathrm{Log}^+ |F(e^{i\theta})|$$

2. The weak type estimate.

Let $u(\theta)$ be non-negative and denote by $v(\theta)$ its harmonic conjugate function which is obtained via Herglotz formula.If E is a subset of T we denote its linear Lebesgue measure by $\mathfrak{m}(E)$. With these notations the following weak.type estimate hods:

2.1 Theorem. *For each non-negative u-function on T with mean-value one the following holds:*

$$\mathfrak{m}(\{|v|) > \lambda\} \leq \frac{4\pi}{1+\lambda} \quad : \quad \lambda > 0$$

Proof. For a given $\lambda > 0$ we set

$$\phi(z) = 1 + \frac{F(z) - \lambda}{F(z) + \lambda} \tag{1}$$

where $F(z)$ the analytic function constructed as in section 1. Here $F(0) = u(0) = 1$ which gives:

$$\phi(0) = \frac{2}{\lambda + 1} \tag{2}$$

Next, since $\mathfrak{Re}\, F = u \geq 0$ it follows that

$$|\frac{F(z) - \lambda}{F(z) + \lambda}| \leq 1 \tag{3}$$

Hence (1) gives $\mathfrak{Re}(\phi) \geq 0$ and mean value formula for the harmonic function $\mathfrak{Re}(\phi)$ gives:

$$\frac{4\pi}{1+\lambda} = \int_0^{2\pi} \mathfrak{Re}\, \phi(e^{i\theta}) \cdot d\theta \geq \int_{\mathfrak{Re}\, \phi \geq 1} \mathfrak{Re}\, \phi(e^{i\theta}) \cdot d\theta \geq \mathfrak{m}(\{\mathfrak{Re}\, \phi \geq 1\}) \tag{4}$$

Rewriting the last inequality we get:

$$\mathfrak{m}(\{\mathfrak{Re}\, \phi \geq 1\}) \leq \frac{4\pi}{1+\lambda} \tag{5}$$

Next, the construction of ϕ yields the following equality of sets:

$$\{\mathfrak{Re}\, \phi(e^{i\theta}) \geq 1\} = \{\mathfrak{Re}\, \frac{F(e^{i\theta}) - \lambda}{F(e^{i\theta}) + \lambda} \geq 0\} \tag{6}$$

Finally, with $F(e^{i\theta}) = u(\theta) + iv(\theta)$ one has

$$\mathfrak{Re}\,[\frac{F(e^{i\theta}) - \lambda}{F(e^{i\theta}) + \lambda}] = \frac{u^2 + v^2 - \lambda^2}{(u+\lambda)^2 + v^2}$$

The right hand side is ≥ 0 when $|v| \geq \lambda$ which gives the set-theoretic inclusion:

$$\{|v| > \lambda\} \subset \{\mathfrak{Re}\, \phi \geq 1\}). \tag{7}$$

Then (5) above gives Theorem 2.1.

3. Kolmogorov's inequality

3.1 Notations. Consider a measure space equipped with a probability measure μ. Let f be a complex-valued and μ-measurable function. For each $\lambda > 0$ we get the μ-measurable set $\{|f| > \lambda\}$ and then

$$\lambda \mapsto \mu(\{|f| > \lambda\})$$

is a decreasing function. Construct the differential function defined for every $\lambda > 0$ by:

$$d\rho_f(\lambda) = \lim_{\delta \to 0} \frac{\mu(\{|f| > \lambda - \delta\}) - \mu(\{|f| > \lambda\})}{\delta} \tag{*}$$

For an arbitrary continuous function $Q(\lambda)$ defined when $\lambda \geq 0$ the formula in XX gives the equality:

$$\int_0^\infty Q(|f|) d\mu = \int_0^\infty Q(\lambda) \cdot d\rho_f(\lambda) \tag{**}$$

Recall also from XX the formula

$$\int_0^\infty \mu[\{|f| > \lambda\}] \cdot d\lambda = \int |f| \cdot d\mu \tag{***}$$

3.2 Operators of Weak type (1,1). Let γ be a probability measure on another sample space and T is some linear map from μ-measurable functions into γ-measurable functions.

3.3 Definition. *We say that T satisfies a weak-type estimate of type (1,1) if there is a constant K such that the inequality below holds for every $\lambda > 0$:*

$$\gamma(\{|Tf| > \lambda\}\,) \leq \frac{K}{\lambda} \cdot \int |f| \cdot d\mu \quad \text{when} \quad f \ \text{is} \, \mu - \text{measurable}$$

We can also regard L^2-spaces. The operator T is L^2-continuous if there exists a constant K_2 such that one has the inequality

$$\int |T(f)|^2 \cdot d\gamma \leq K_2^2 \cdot \int |f|^2 \cdot d\mu$$

Taking square roots it means that the L^2-norm is K_2.

3.4 Theorem. *Let T be a linear operator whose L^2-norm is 1 and with finite weak-type norm K. Then the following holds for each μ-measurable function f:*

$$\int |T(f)| \cdot d\gamma \leq 1 + 4 \cdot \int |f| \cdot d\mu + 2K \cdot \int |f| \cdot \text{Log}^+ |f| \cdot d\mu$$

Proof. When $\lambda > 0$ we decompose f as follows:

(i) $$f = f_* + f^* \quad : f_* = \chi_{\{|f| \leq \lambda\}} \cdot f \quad : \quad f^* = \chi_{\{|f| > \lambda\}} \cdot f$$

For the lower f_*-function we use that T has L^2-norm ≤ 1 and get

(ii) $$\gamma[\{|Tf_*| > \lambda/2\}\,] \leq \frac{4}{\lambda^2} \int_0^\lambda s^2 \cdot d\rho_f(s)$$

For Tf^* we apply the weak-type estimate which gives

(iii) $$\gamma[\{|Tf^*| > \lambda/2\}\,] \leq \frac{2K}{\lambda} \cdot \int_\lambda^\infty s \cdot d\rho_f(s)$$

where we used that $\int_\lambda^\infty s \cdot d\rho_f(s)$ is the L^1-norm of f^*. The set-theoretic inclusion

$$\{|Tf| > \lambda\} \subset \{|Tf_*| > \lambda/2\} \cup \{|Tf^*| > \lambda/2\} \implies$$

(iv) $$\gamma[\{|Tf| > \lambda\}\,] \leq \frac{4}{\lambda^2} \int_0^\lambda s^2 \cdot d\rho_f(s) + \frac{2K}{\lambda} \cdot \int_\lambda^\infty s d\rho_f(s)$$

Next, since γ has total mass one the inequality:

(v) $$\int_0^\infty |Tf| \cdot d\gamma \leq 1 + \int_{\{|Tf|>1\}} |Tf| \cdot d\gamma$$

Now (***) in (3.1) is applied to Tf and the measure γ which gives

$$\int_{\{|Tf|>1\}} |Tf| \cdot d\gamma = \int_1^\infty \gamma[\{|Tf| > \lambda\} \cdot d\lambda$$

By (iv) the last integral in (v) is majorised by:

(vi) $$4 \cdot \int_1^\infty [\frac{1}{\lambda^2} \int_0^\lambda s^2 \cdot d\rho_f(s)] \cdot d\lambda + 2K \int_1^\infty \frac{1}{\lambda} \cdot [\int_\lambda^\infty s d\rho_f(s)] \cdot d\lambda$$

Next, from (XX) one has the equality:

(vii) $$\int_0^\infty [\frac{1}{\lambda^2} \int_0^\lambda s^2 \cdot d\rho_f(s)] \cdot d\lambda = \int |f| \cdot d\mu$$

The left hand side is only smaller if the λ-integration starts at 1. It follows that the first term in (vi) above is majorised by $4 \cdot \int |f| \cdot d\mu$ and together with (v) we conclude that

(viii) $$\int_0^\infty |Tf| d\gamma \leq 1 + 4 \int |f| d\mu + 2K \int_1^\infty [\frac{1}{\lambda} \cdot \int_\lambda^\infty s d\rho_f(s)] \cdot d\lambda$$

Finally,

$$\int_1^\infty [\frac{1}{\lambda}\cdot\int_\lambda^\infty s d\rho_f(s)]\cdot d\lambda = \iint_{1\leq\lambda\leq s}\frac{1}{\lambda}\cdot s\rho_f(s)ds = \int_1^\infty s\cdot \text{Log}\, s\cdot d\rho_f(s)$$

The last integral is equal to $\int f\cdot \text{Log}^+|f|\cdot d\mu$ by the general formula XX. Inserting this in (viii) we get Theorem 3.4.

3.5. Final part of Theorem 0.1

There remains to show that if u is non-negative and if $u\cdot\text{Log}^+ u$ is integrable so is v. To prove this we use $d\mu = d\gamma = \frac{d\theta}{2\pi}$ on the unit circle. Theorem 2.1 which shows that the harmonic conjugation operator $T\colon u\mapsto v$ is of weak-type (1,1) and it is continuous on $L^2(T)$ by Parseval's formula. Hence Kolomogorv's Theorem gives $v\in L^1(T)$ which proves the necessity in Theorem 0.1.

Remark. Notice that Theorem 3.4 applies when we start from any real-valued function $u(\theta)$. So have the following supplement to Theorem 0.1.

3.6 Theorem. *There exists an absolute constant A such that*

$$\int_0^{2\pi}|v(\theta)|\cdot d\theta \leq A\cdot\big[\int_0^{2\pi}|u(\theta)|\cdot d\theta + \int_0^{2\pi}|u(\theta)|\cdot\text{Log}^+|u(\theta)|\cdot d\theta\,\big]$$

4. The Dual space of $H^1(T)$

On the unit circle T we have the Banach space $L^1(T)$ where $H^1(T)$ is a closed subspace. Next, let $C^0(T)$ be the Banach space of continuous functions on T equipped with the maximum norm. It contains the closed subspace $A(D)$ whose functions can be extended as analytic functions in the open disc D. We have also the subspace $A_*(D)$ which consists of the functions in $A(D)$ whose analytic extensions are zero at the origin. As explained in XXX a continuous function f on T belongs to $A_*(D)$ if and only if

(*) $$\int_0^{2\pi} e^{in\theta}\cdot f(e^{i\theta})\cdot d\theta = 0 \quad : \quad n = 0,1,\ldots$$

From (*) it follows that

(**) $$\int_0^{2\pi} g(e^{i\theta})\cdot f(e^{i\theta})\cdot d\theta = 0 \quad : \quad f\in A_*(D) \quad \text{and} \quad g\in H^1(T)$$

Let us now regard the Banach space

$$B = \frac{C^0(T)}{A_*(D}$$

Riesz representation formula identifies the dual space of $C^0(T)$ with Riesz measures on T. Since B is a quotient space its dual space becomes

(i) $$B^* = \{\,\mu\in M(T) \quad : \mu\perp A_*(D\,\}$$

Now a Riesz measure μ is $\perp A_*(D$ if and only if

(ii) $$\int_0^{2\pi} e^{in\theta}\cdot d\mu(\theta) = 0 \quad : \quad n = 1,2\ldots$$

The Brothers Riesz theorem means that (ii) holds if and only if μ is absolutely continuous, i.e. μ is given by some L^1-function f which satisfies:

(iii) $$\int_0^{2\pi} e^{in\theta}\cdot f(e^{i\theta})\cdot d\theta = 0 \quad : \quad n = 1,2\ldots$$

This is precisely the condition that $f\in H^1(T)$. Hence the whole discussion gives:

4.1 Theorem. *The Hardy space $H^1(T)$ is the dual of B.*

4.2 The dual of $H^1(T)$. Recall that $L^\infty(T)$ is the dual space of $L^1(T)$. So by a general formulafrom Appendix: Functional analysis we get:

$$H^1(T)^* = \frac{L^\infty(T)}{H^1(T)^\perp}$$

Next, an L^∞-function f is $\perp H^1(T)$ if and only if

$$\int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad n = 0, 1, 2 \ldots$$

But this means precisely that f is the boundary value of an analytic function in D which vanishes at the origin. Let us identify $H^\infty(D)$ with a subalgebra of $L^\infty(T)$ which is denoted by $H^\infty(T)$. Then we also get the subspace $H_0^\infty(T)$ of those functions which are zero at the origin. Hence we have proved

Theorem 4.3 *The dual space of $H^1(T)$ is equal to the quotient space*

$$\frac{L^\infty(T)}{H_0^\infty(T)}$$

5. BMO

Introduction. Functions of bounded mean oscillation were introduced by F. John and L. Nirenberg in [J-N]. This class of Lebesgue measurable functions can be defined in $\mathbf{R^n}$ for every $n \geq 1$. Here we are content to study the case $n = 1$ and restrict the attention to periodic functions which is adapted to the class BMO on the unit circle. So let $F(x)$ be a locally integrable function on the real x-line which is 2π-periodic, i.e. $F(x + 2\pi) = F(x)$. If $J = (a, b)$ is an interval we get the mean value

$$F_J = \frac{1}{b-a} \cdot \int_a^b F(x)dx$$

To every interval J we put

$$|F|_J^* = \int_J |F(x) - F_J| \cdot dx$$

5.1 Definition. *The function F has a bounded mean oscillation if there exists a constant C such that $|F|_J \leq C$ for all intervals J. When this holds the smallest constant is denoted by $|F|_{BMO}$.*

5.2 The case $n \geq 2$. Even though these notes are devoted to complex analysis in dimension one, we cannot refrain from mentioning a result which illustrates how the class BMO enters in Fourier analysis. Namely, let $F(x)$ be an L^1-function with compact support in $\mathbf{R^n}$. Assume that there exists a constant C such that its Fourier transform $\hat{F}(\xi)$ satisfies the decay condition

$$(*) \qquad |\widehat{F}(\xi) \leq C \cdot (1 + |\xi|)^{-n} \quad : \quad \xi \in \mathbf{R^n}$$

This is not quite enough for $\widehat{F}$ to be integrable. So we cannot expect that (*) implies that $F(x)$ is a bounded function. However, its belongs to BM0 and more precisely one has:

5.3 Theorem. *To each $M > 0$ there exists a constant C_M such that if $F(x)$ has support in the ball $\{|x| \leq M\}$ then*

$$||F||_{\text{BMO}} \leq C_M \cdot \max_\xi \left[1 + |\xi|)^n \cdot |\widehat{F}(\xi)|\right]$$

For the proof we refer to [Björk]. See also [Sjölin] for an improved result that F belongs to BMO under less restrictive conditions expressed by certain L^2-integrals of $\widehat{F}$ over dyadic grids.

5.4 The John-Nirenberg inequality.

Now we turn to the main topic in this section and prove an inequality due to F. John and L. Nirenberg which is presented for the 1-dimensional periodic case. See [J-N] for higher dimensional results.

5.5 Theorem *Let $F(x)$ be a 2π-periodic function on the real x-line which belongs to BMO on T. For every interval J on $\mathbf{R}$ and every positive integer n one has*

$$\mathfrak{m}[\{x \in J : \big|F(x) - F_J\big| \geq 4n \cdot |F|_{\text{BMO}}\}] \leq 2^{-n} \cdot |J|$$

The proof requires several steps. To begin with we make some trivial observations. The BMO-norm of F is unaffected when we add a constant to F and also under a translation, i.e. when we regard $F_a(x) = F(x+a)$ for some real number a. Moreover, the BMO-norm is unchanged under dilations, i.e. when $t > 0$ and $F_t(x) = F(tx)$. Before we enter the proof we need a preliminary result.

5.6 Lemma. *Let F belong to BMO. Let $I \subset J$ be two intervals with the same mid-point. Then*

$$|F_J - F_I| \leq 2 \cdot [\text{Log}_2 \frac{|J|}{|I|} + 1] \cdot |F|_{\text{BMO}}$$

Exercise. Prove this result.

Proof of Theorem 5.5. Replacing F by cF for some positive constant we may assume that its BMO-norm is 1/2. and that $F_J = 0$. Moreover, by the invariance under dilations and translations we may assume that J is the unit interval. Thus, there remains to consider the set

(i) $$E_n = \{x \in [0,1] \, : \, F(x) > 2n\}$$

and show that

(ii) $$\mathfrak{m}(E_n) \leq 2^{-n}$$

Let us begin with the case $n = 1$. For every $x \in E_1$ which is a Lebesgue point for F we find the unique largest dyadic interval $J(x)$ such that

(iii) $$x \in J(x) \subset [0,1] \quad : \quad \frac{1}{\mathfrak{m}(J(x))} \int_{J(x)} F(t)dt > 1$$

Up to measure zero, i.e. ignoring the null set where F fails to have Lebesgue points, we have the inclusion

(iv) $$E_1 \subset \cup_{x \in E_1} J(x)$$

Next, suppose we have a *strict* inclusion $J(x) \subset J(y)$ for a pair of dyadic intervals in this family which means that $\mathfrak{m}(J(y)) > \mathfrak{m}(J(x))$. But this is impossible for then $x \in J(y)$ which contradicts the maximal choice of $J(x))$ as the dyadic interval of largest possible length containing x. Hence the family $\{J(x_\nu)\}$ consists of dyadic intervals which either are equal or disjoint. We can therefore pick a disjoint family where the corresponding x-points are denoted by x_ν^* and obtain the set-theoretic inclusion

(v) $$E_1 \subset \cup \, (J(x_\nu^*)$$

Next, put $\mathcal{E} = \cup J(x_\nu^*)$. Since the mean value of F over each $J(x_\nu^*)$ is ≥ 1 we obtain

$$\mathfrak{m}(\mathcal{E}) \leq \sum \int_{J(x_\nu^*)} F(x)dx = \int_{\mathcal{E}} F(x)dx \leq \int_{\mathcal{E}} |F(x)|dx \leq \int_0^1 |F(x)|dx \leq |F|_{\text{BMO}}$$

where the last inequality follows from the definition of the BMO-norm and the condition that the mean-value of F over the unit interval was zero. Since the BMO-norm of F was 1/2 the inclusion (v) gives:

(*) $$\mathfrak{m}(E_1) \leq \mathfrak{m}(\mathcal{E}) \leq 1/2$$

This proves the case $n = 1$ and we proceed by an induction over n. Let us first regard one of the dyadic intervals $J(x_\nu^*)$ from the family covering E_1. If 2^{-N} is the length of $J(x_\nu^*$ the dyadic exhaustion of $[0,1]$ gives a dyadic interval J' of length 2^{-N+1} which contain $J(x_\nu^*)$. The maximal choice of $J(x_\nu^*)$ gives:

(vi) $$\frac{1}{\mathfrak{m}(J')}\int_{J'} F(t)dt \leq 1$$

Apply Proposition XX to the pair $J(x_\nu^*)$ and J'. Since $|F|_{\text{BMO}} = 1/2$ is assumed and $\text{Log}_2(2) = 0$ we obtain

(vii) $$F_{J(x_\nu^*)} = \frac{1}{\mathfrak{m}(J(x_\nu^*))}\int_{J(x_\nu^*)} F(t)dt \leq 2$$

Let $n \geq 2$ and for every ν we set:

(viii) $$E_n(\nu) = E_n \cap J(x_\nu^*)$$

Since $F(x) > 2n$ holds on E_n we get

(ix) $$F(x) - F_{J(x_\nu^*)} > 2(n-1) \quad : \quad x \in E_n(\nu)$$

Hence we have the inclusion

(x) $$E_n(\nu) \subset W_n(\nu) = \{x \in J(x_\nu^*) \quad : F(x) - F_{J(x_\nu^*)} > 2(n-1)\}$$

By a change of scale we can use the interval $J(x_\nu^*)$ instead of the unit interval and by an induction assume that the inequality in Theorem xx holds for $n-1$. It follows that the set in right hand side in (x) is estimated by:

(xi) $$\mathfrak{m}(W_n(\nu)) \leq 2^{-n+1} \cdot \mathfrak{m}(J(x_\nu^*))$$

The set-theoretic inclusion (x) therefore gives

(xii) $$\mathfrak{m}(E_n(\nu)) \leq 2^{-n+1} \cdot \mathfrak{m}(J(x_\nu^*))$$

Finally, since $E_n \subset E_1$ and we already have the inclusion (iv) we obtain

$$\mathfrak{m}(E_n) = \sum \mathfrak{m}(E_n(\nu)) \leq 2^{-n+1} \cdot \sum \mathfrak{m}(J(x_\nu^*)) = 2^{-n+1}\mathfrak{m}(\mathcal{E}) \leq 2^{-n+1} \cdot \frac{1}{2} = 2^{-n}$$

This proves the induction step and Theorem 5.5 follows.

5.7 An L^2-inequality

Let $F \in \text{BMO}(T)$ be given. Given some interval $J \subset T$ and $\lambda > 0$ we set

$$\mathfrak{m}_J(\lambda) = \{\theta \in J : \ |F(\theta) - F_J| > \lambda\}$$

Consider the integral

(*) $$I = \frac{1}{|J|} \cdot \int_0^\infty \lambda \cdot \mathfrak{m}_J(\lambda) \cdot d\lambda$$

Set $A = 4 \cdot ||F||_{\text{BMO}}$. Theorem 5.5. gives

$$I = \frac{1}{|J|} \cdot \sum_{n=0}^{\infty} \int_{nA}^{(n+1)A} \lambda \cdot \mathfrak{m}_J(\lambda) \cdot d\lambda \leq \frac{1}{|J|} \cdot \sum_{n=0}^{\infty} (n+1)A \cdot |J| \cdot \cdot 2^{-n} = C||F||_{\text{BMO}}$$

where $C = 4 \cdot \sum_{n=0}^{\infty} (n+1)\, 2^{-n}$ is an absolute constant. Next, by the general result in XX (*) is equal to

(**) $$\frac{1}{|J|} \cdot \int_J |F(x) - F_J|^2 \cdot dx$$

So by the above (**) is majorized by an absolute constant times the BMO-norm of F.

5.8 BMO and the Garsia norm.

Using the L^2-inequality in (5.7) an elegant description of $\mathrm{BMO}(T)$ was discovered by Garsia which we shall use in Section 6. First we give:

5.9 Definition. *To each real-valued $u \in L^1(T)$ we define a function in D by*

$$\mathcal{G}_u(z) = \frac{1}{8\pi^2} \cdot \iint \frac{(1-|z|^2)^2}{|e^{i\theta} - z|^2 \cdot |e^{i\phi} - z|^2} \cdot [u(\theta) - u(\phi)]^2 \cdot d\theta d\phi$$

If this function is bounded we set

(*) $$\mathcal{G}(u) = \max_{z \in D} \sqrt{\mathcal{G}_u(z)}$$

and say that u has a finite Garsia norm.

Remark. Notice that constant functions have zero-norm. So just as for BMO the $\mathcal{G}$-norm is defined on the quotient of functions modulu constants.

5.10 Exercise. Expanding the square $[u(\theta) - u(\phi)]^2$ the reader can verify that

(*) $$\mathcal{G}_u = H_{u^2} - H_u^2$$

where H_{u^2} is the harmonic extension of u^2. and H_u^2 the square of the harmonic extension H_u.

5.11 Theorem. *An L^1-function u has finite Garsia norm if and only if it belongs to BMO. Moreover, there exists a constant $C \geq 1$ such that*

$$\frac{1}{C} \cdot ||u||_{\mathrm{BMO}} \leq \mathcal{G}(u) \leq C \cdot ||u||_{\mathrm{BMO}}$$

5.12 Exercise. The reader is invited to prove this result using the previous facts about BMO and also straightforward properties of the Poisson kernel. if necessary, consult [Koosis p. xxx-xxx] for details.

5.13 The Garsia norm and Carleson measures. Let f be a real-valued continuous function on T. We get the two harmonic functions H_f and H_{f^2} and recall from (5.10) that

$$\mathcal{G}_f = H_{f^2} - (H_f)^2$$

In XX we introduced the family of Carleson sectors in D and now we prove an important inequality.

5.14 Theorem. *For every Carleson sector S_h with $0 < h < 1/2$ and each $f \in C^0(T)$ one has the inequality*

$$\frac{1}{h} \cdot \iint_{S_h} |z| \cdot \log \frac{1}{|z|} \cdot \left|\nabla(H_f)\right|^2 \cdot dxdy \leq 96 \cdot \mathcal{G}(f)^2$$

Proof. We use the conformal map where $z = \frac{\zeta - i}{\zeta + i}$. If $\phi(z)$ is a function in D we get the function $\phi^*(\zeta)$ in the upper half-plane where

$$\phi(\frac{\zeta - i}{\zeta + i}) = \phi^*(\zeta)$$

One easily verifies that

(i) $$(|z| \cdot \log \frac{1}{|z|})^*(\xi + i\eta) \leq 8 \cdot \eta$$

Set $w(\zeta) = H_f^*(\zeta)$. Then (i) implies that the double integral which appears in the Theorem 5.14 is majorised by

(ii) $$J^* = 8 \cdot \iint_{S_h^*} \eta \cdot |\nabla(w)|^2 \cdot d\xi d\eta$$

where S_h^* is the image of S_h under the conformal map and $|\nabla(w)|^2 = w_\xi^2 + w_\eta^2$. Next, from (*) in Exercise 5.10 we have

$$w^2 = H_{f^2}^* - \mathcal{G}_f^*$$

Since $H_{f^2}^*$ is harmonic we obtain

$$\text{(iii)} \qquad 2 \cdot |\nabla(w)|^2 = \Delta(w^2) = -\Delta(\mathcal{G}_f^*)$$

where the first easy equality follows since w is harmonic. As explained by figure XX the set S_h^* is placed above an interval on the real ξ-line and and since the subsequent estimates are invariant under the center of this interval we therefore may take it as $\xi = 0$. Let us introduce the half-disc

$$D_{2h} = \{|\zeta| < 2h\} \cap \{\eta > 0\}$$

Then a figure shows that

$$\text{(iv)} \qquad S_h^* \subset D_{2h}$$

Next, consider the function $1 - \frac{|\zeta|}{2h}$ and notice that it is $\geq 1/4$ in D_{2h}. Recall from the above that

$$\Delta(\mathcal{G}_f^*) = -2 \cdot |\nabla(w)|^2 \leq 0$$

From the inclusion (iv) and taking the minus signs into the account it follows from (ii) that

$$\text{(v)} \qquad J^* \leq -16 \cdot \iint_{D_{2h}} \eta(1 - \frac{|\zeta|}{2h}) \cdot \Delta(\mathcal{G}_f^*) \cdot d\xi d\eta$$

Apply Green's formula to the pair $\mathcal{G}_f^*$ and $\rho = \eta(1 - \frac{|\zeta|}{2h})$. Here $\rho = 0$ on the boundary of D_{2h} and it is easily checked that the outer normal $\partial_n(\rho) \leq 0$. At the same time $\mathcal{G}_f^* \geq 0$ and from this it follows that (v) gives:

$$\text{(iv)} \qquad J^* \leq -16 \cdot \iint_{D_{2h}} \Delta(\eta(1 - \frac{|\zeta|}{2h})) \cdot \mathcal{G}_f^* \cdot d\xi d\eta$$

Next, using polar coordinates (r, ϕ) an easy computation gives

$$\Delta(\eta(1 - \frac{|\zeta|}{2h})) = -\frac{3}{2h} \cdot \sin \phi$$

It follows that

$$J^* \leq \frac{24}{h} \cdot \iint_{D_{2h}} \mathcal{G}_f^* \cdot \sin \phi \cdot r dr d\phi$$

Finally, by definition $\mathcal{G}(f)^2$ is the maximum norm of $\mathcal{G}_f$ in D which is $\geq$ the maximum norm of $\mathcal{G}_f^*$ in D_{2h}. So the last integral is majorised by

$$\frac{24\mathcal{G}(f)^2}{h} \cdot \iint_{D_{2h}} \sin \phi \cdot r dr d\phi = 96 \cdot \mathcal{G}(f)^2 \cdot h$$

After a division with h we get Theorem 5.14.

5.15 Remark. Since $|z| \geq 1/2$ holds in sectors S_h with $0 < h < 1/2$ we can remove the factor $|z|$ and hence Theorem 5.14 shows that if $\mathcal{G}_f(z)$ is bounded in D then we obtain a Carleson measure in D defined by

$$(*) \qquad \mu_f = \log \frac{1}{|z|} \cdot \big|\nabla(H_f)\big|^2$$

Moreover, its Carleson norm is estimated via Theorem 5.11 and Theorem 5.24 by an absolute constant times $|F|_{\text{BMO}}$.

6. Proof of Theorem 0.4

By the observations before Theorem 0.4 here remains to prove that if $F \in \text{BMO}(T)$ then (*) holds in Theorem 0.4 for some constant C. To obtain this we need some preliminary results derived via Green's formula.

6.1 Some integral formulas. To simplify notations we set

$$\int_0^{2\pi} g(e^{i\theta})\cdot d\theta = \int_T g\cdot d\theta$$

for integrals over the unit circle. Now follow some results which are left as exercises and proved by Green's formula.

A. Exercise. For every C^2-function W in the closed unit disc with $W(0)=0$ we have

$$\int_T W\cdot d\theta = \iint_D \log\frac{1}{|z|}\cdot\Delta(W)\cdot dxdy \tag{1}$$

Next, if

$$W = |z|\cdot W_1 \tag{2}$$

B. Exercise. Let u, v is a pair of C^2-functions which both are harmonic in the open disc. Show that

$$\Delta(uv) = 2\cdot\langle\nabla(u),\nabla(v)\rangle \tag{i}$$

and use (A) to prove the equality

$$\int_T uv\cdot d\theta = \iint_D \log\frac{1}{|z|}\cdot\langle\nabla(u),\nabla(v)\rangle\cdot dxdy \tag{ii}$$

C. Exercise. Let $f=u+iv$ be analytic in D. Verify that

$$\Delta(|f|) = \frac{1}{|f|}\cdot|\nabla(u)|^2 \tag{i}$$

holds outside the zeros of f. Show also that if

$$f = z\cdot g \tag{ii}$$

where g is zero-free in D then

$$\int_T |f|\cdot d\theta = \iint_D \log\frac{1}{|z|}\cdot\frac{|\nabla(u)|^2}{|f|}\cdot dxdy \tag{iii}$$

Finally, let f be as in (ii) and F a real-valued C^2-function in D. Show that

$$\frac{1}{2}\int_T u\cdot F\cdot d\theta = \iint_D \log\frac{1}{|z|}\cdot\langle\nabla(u),\nabla H_F\rangle\cdot dxdy \tag{iii}$$

6.2 Proof of Theorem 0.4

Let $f\in H_0^1(T)$. Then one finds a Blaschke product B such that

$$f(z) = z\cdot B(z)\cdot g(z)$$

where g is zero-free in D. It follows that

$$2f = z(B+1)\cdot g + z(B-1)\cdot g = f_1+f_2$$

where $||f_\nu||_1\leq 2\cdot||f||_1$ hold for each ν. Using this trick we conclude that it suffices to estasblish Theorem 0.4 for $H^1(T)$-functions of the form $f(z)=z\cdot g(z)$ with a zero-free function g in D. We write $f=u+iv$ and for each real-valued C^2-function $F(\theta)$ on T we have by (iii) from Exercise C:

$$\frac{1}{2}\cdot\int_0^{2\pi} F(\theta)\cdot u(\theta)\cdot d\theta = \iint_D \log\frac{1}{|z|}\cdot\langle\nabla(u),\nabla(H_F)\rangle\cdot dxdy \tag{1}$$

Insert the factor $1=\sqrt{|f|}\cdot\frac{1}{\sqrt{|f|}}$ and apply the Cauchy-Schwarz inequality which estimates the absolute value of (i) by

$$(2) \qquad J = \sqrt{\iint_D \log \frac{1}{|z|} \cdot \frac{|\nabla(u)|^2}{|f(z)|} \cdot dxdy} \cdot \sqrt{\iint_D \log \frac{1}{|z|} \cdot |\nabla(H_F)|^2 \cdot |f(z)| \cdot dxdy}$$

The equality (iii) in Exercise C shows the first factor is equal to $\sqrt{||f||_1}$. In the second factor appears the density function $\log \frac{1}{|z|} \cdot |\nabla(H_F)|^2$ in D.

Finally, by the Remark in (5.15) the Carleson norm of the density $\log \frac{1}{|z|} \cdot |\nabla(H_F)|^2$ is bounded by an absolute constant C times the BMO-norm of F. Together with the result in XXX in Section XXX we get an absolute constant C such that the last factor in (2) above is bounded by

$$(3) \qquad C \cdot |F|_{\text{BMO}} \cdot \sqrt{||f||_1}$$

which finishes the proof of Theorem 0.4.

7. A theorem by Gundy Silver

Introduction. Let $U(x)$ be in $L^1(\mathbf{R})$ and construct its harmonic extension to the upper half plane:

$$U(x+iy) = \frac{1}{\pi} \cdot \int \frac{y}{(x-t)^2 + y^2} \cdot U(t) \cdot dt$$

The harmonic conjugate of $U(x+iy)$ is given by:

$$(0.1) \qquad V(x+iy) = \frac{y}{\pi} \int \frac{U(t) \cdot dt}{(x-t)^2 + y2}$$

Next, to each real x_0 the Fatou sector in the upper half-plane is defined by

$$(0.2) \qquad \{x+iy) \quad \text{such that} \quad |x - x_0| \leq y\}$$

and the maximal function U^* over Fatou sectors is defined on the real x-axis by

$$(0.3) \qquad U^*(x_0) = \max |U(x+iy)| \colon \quad : |x - x_0| \leq y$$

In XXX we proved that if $V \in L^1(R)$ then $U^*(x) \in L^1(\mathbf{R})$ and there exists an absolute constant C_0 such that

$$(*) \qquad ||U^*||_1 \leq \int_{-\infty}^{\infty} \big(|U(x)| + |V(x)|\big) dx$$

A reverse inequality is due to Burkholder, Gundy and Silverstein.

Theorem 7.1. *One has the inequality*

$$\int |V(x)| dx \leq 4 \int U^*(x) dx$$

Remark. Hence U^* belongs to L^1 if and only if the boundary value function $V(x)$ belongs to L^1. The original proof in [BGS] used probabilistic methods. Here we give a proof based upon methods from [Feff-Stein]. Since we shall establish an *a priori estimate*, it suffices to assume that $U(x)$ from the start is a nice function. In particular we may assume that both $U(x+iy)$ and $V(x+iy)$ have rapid decay when $y \to +\infty$ in the upper half-plane. This assumption is used below to ensure that a certain complex line integral is zero.

Proof of Theorem 7.1

Given $\lambda > 0$ we put

$$J_\lambda = \{x \colon U^*(x) > \lambda\}$$

The closed complement $\mathbf{R} \setminus J_\lambda$ is denoted by E. Let $\{(a_\nu, b_\nu)\}$ be the disjoint intervals of J_λ. Construct the piecewise linear Γ-curve which stays on the real x-line on E while it follows the two sides of the triangle T_ν standing on (a_ν, b_ν) for each ν. So the corner point of T_ν in the upper half-plane is:

$$p_\nu = \frac{1}{2}(a_\nu + b_\nu)(1+i)$$

Set $\partial T = \Gamma \setminus E$ and notice that the construction of Fatou sectors gives

(1) $$U^*(x) \leq \lambda \quad : \quad x \in T$$

In $\mathfrak{Im}(z) > 0$ we have the analytic function $G(z) = (U + iV)^2$. By hypothesis U $y \mapsto G(x+iy)$ decreases quite rapidly which gives a vanishing complex line integral:

$$\int_\Gamma G(z)dz = 0$$

Now Γ is the union of E and the union of of the broken lines which give the two sides of the T_ν-triangles. Let ∂T denote the union of these broken lines. Since the complex differential $dz = dx + idy$ the real part of the complex line integral is zero which gives

(2) $$\int_E (U^2 - V^2) \cdot dx + \int_{\partial T} (U^2 - V^2) \cdot dx - 2 \cdot \int_{\partial T} U \cdot V dy$$

On the sides of the T-triangles the slope is plus or minus $\pi/4$ and hence $|dy| = |dx|$ where $|dx| = dx$ is positive. Hence the he inequality $2ab \leq a^2 + b^2$ for any pair of non-negative numbers gives:

(3) $$2 \cdot \big| \int_{\partial T} UV dy \,\big| \leq \int_{\partial T} U^2 \cdot dx + \int_{\partial T} V^2 \cdot dx$$

Since (2) is zero we see that (3) and the triangle inequality give:

(4) $$\int_E V^2 \cdot dx \leq \int_E U^2 \cdot dx + 2 \cdot \int_{\partial T} U^2 \cdot dx$$

Next, put

$$V_\lambda^+ = \{x : |V(x)| > \lambda\}$$

Then (4) gives:

(5) $$\mathfrak{m}(V_\lambda^+ \cap E) \leq \frac{1}{\lambda^2} \cdot \int_E V^2 \cdot dx \leq \frac{1}{\lambda^2} \cdot \int_E U^2 \cdot dx + \frac{2}{\lambda^2} \cdot \int_{\partial T} U \cdot dx$$

Next, Since the integral $\int_{T_\nu} dx = (b_\nu - a_\nu)$ for each ν and (1) holds we have

(6) $$\frac{2}{\lambda^2} \cdot \int_{\partial T} U^2 \cdot dx \leq 2 \cdot \sum (b_\nu - a_\nu) = 2 \cdot \mathfrak{m}(J_\lambda)$$

Using the set-theoretic inclusion $V_\lambda^+ \subset (V_\lambda^+ \cap E_\lambda) \cup J_\lambda$ it follows after adding $\mathfrak{m}(J_\lambda)$ on both sides in (5):

(6) $$\mathfrak{m}(V_\lambda^+) \leq 3 \cdot \mathfrak{m}(J_\lambda) + \frac{1}{\lambda^2} \cdot \int_E U^2 \cdot dx$$

Finally, $U \leq U^*$ holds on E and since E is the complement of J_λ we have $E = \{x : U^*(x) \leq \lambda\}$. Now we apply general integral formulas which after integration over $\lambda \geq 0$ gives

$$\int |V(x) \cdot dx = 3 \cdot \int U^*(x) \cdot dx + \int_0^\infty \frac{1}{\lambda^2} \big[\int_{(U^* \leq \lambda} (U^*)^2 \cdot dx \big] \cdot d\lambda$$

By the integral formula from XX the last term is equal to $\int U^*(x) \cdot dx$ and Theorem 7.1 follows.

8. The Hardy space on R

Consider an analytic function $F(z)$ in the upper half-plane whose boundary value function $F(x)$ on the real line is integrable. This class of analytic functions in $\mathfrak{Im}\, z > 0$ is denoted by $H^1(\mathbf{R})$. To each such F we introduce the non-tangential maximal function

(*) $$F^*(x) = \max_{z \in \mathcal{F}(x)} |F(z)|$$

where $\mathcal{F}(x)$ is the Fatou sector of points $z = \xi + i\eta$ for which $|\xi - x| \leq \eta$. With these notations one has

8.1 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty}^{\infty} |F^*(x)| \cdot dx \leq C \cdot \int_{-\infty}^{\infty} |F(x)| \cdot dx$$

To prove this we shall first study harmonic functions and reduce the proof of Theorem 8.1 to a certain L^2-inequality. To begin with, let $u(x)$ is a real-valued function on the x-axis such that the integral

$$\int_{-\infty}^{\infty} \frac{|u(x)|}{1+x^2} \cdot dx < \infty$$

The harmonic extension to the upper half-plane becomes:

$$U(x+iy) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} \cdot u(t) \cdot dt$$

The non-tangential maximal function is defined by:

(*) $$U^*(x) = \max_{z \in \mathcal{F}(x)} |U(z)|$$

When $u(x)$ belongs to $L^2(\mathbf{R})$ it turns out that one there is an L^2-inequality.

8.2 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty}^{\infty} (U^*(x))^2 \cdot dx \leq \int_{-\infty}^{\infty} u^2(x) \cdot dx$$

for every L^2-function on the x-axis.

In 8.X below we show how Theorem 8.2 gives Theorem 8.1. The proof of Theorem 8.2 relies upon a point-wise estimate of U via the Hardy-Littlewood maximal function of u. Let us first consider a function $u(x)$ supported by $x \geq 0$ such that the function

$$t \mapsto \frac{1}{t} \int_0^t |u(x)| \cdot dx$$

is bounded on $(0, +\infty)$. Let $u^M(0)$ denote this supremum over t. Then one has

8.3 Proposition. *For each $z = x + iy$ in the upper half-plane one has*

$$|U(x+iy)| \leq (1 + \frac{|x|}{2y}) \cdot u^M(0)$$

Proof. Since the absolute values $|U(x+iy)|$ increase when u is replaced by $|u|$ we may assume that $u \geq 0$ from the start. Put

$$\Phi(t) = \int_0^t u(x) \cdot dx$$

which yields a primitive of u and a partial integration gives

$$U(x+iy) = \lim_{A \to \infty} \frac{1}{\pi} \cdot \Big| \frac{y}{(x-t)^2 + y^2} \cdot \Phi(t) \Big|_0^A + \lim_{A \to \infty} \frac{2}{\pi} \cdot \int_0^A \frac{y(t-x)}{((x-t)^2 + y^2)^2} \cdot \Phi(t) \cdot dt$$

With (x, y) kept fixed the finiteness of $u^M(0)$ entails that $t^{-2} \cdot \Phi(t)$ tends to zero with A and there remains

$$U(x+iy) = \frac{2}{\pi} \cdot \int_0^{\infty} \frac{y(t-x)}{((x-t)^2 + y^2)^2} \cdot \Phi(t) \cdot dt$$

Now $\Phi(t) \leq u^M(0) \cdot t$ gives the inequality

$$U(x+iy)=\frac{2u^M(0)}{\pi}\cdot\int_0^\infty\frac{y(t-x)\cdot t}{((x-t)^2+y^2)^2}\cdot dt$$

To estimate the integrand we notice that it is equal to

$$\frac{y}{((x-t)^2+y^2)}+\frac{y(t-x)x}{((x-t)^2+y^2)^2}$$

The Cauchy-Schwarz inequality gives

$$|\frac{2y(t-x)x}{((x-t)^2+y^2)^2}|\leq\frac{|x|}{(x-t)^2+y^2}$$

It follows that

$$|U(x+iy)|\leq\frac{2u^M(0)}{\pi}\cdot\int_0^\infty\frac{y}{(x-t)^2+y^2)}+\frac{u^M(0)\cdot|x|}{\pi}\cdot\int_0^\infty\frac{1}{(x-t)^2+y^2}\cdot dt$$

The last sum of integrals is obviously majorised by $u^M(0)(1+\frac{|x|}{2y})$ and Proposition XX is proved.

8.4 General cae. If no constraint is imposed on the support of u it is written as u_1+u_2 where u_1 is supported by $x\leq 0$ and u_2 b $x\geq 0$. Here we consider the maximal function

$$u^M(0)=\max_t\frac{1}{2t}\int_{-t}^t|u(x)|\cdot dx$$

Exactly as above the reader may verify that

$$\text{(i)}\qquad |U(x,y)|\leq u^M(0)(2+\frac{|x|}{y})$$

In the Fatou sector at $x=0$ we have $x\leq|y|$ and hence (i) gives

$$U^*(0)\leq\leq 3\cdot u^M(0)$$

After a translation with respect to x a similar inequality holds. More precisely, put

$$u^M(x)=\max_t\frac{1}{2t}\int_{-t}^t|u(x+s)|\cdot ds$$

for every x, Then we have

$$U^*(x)\leq 3\dot{u}^M(x)$$

Now we apply the Hardy-Littlewood inequality from XX for the L^2-case and obtain the conclusive result:

8.5 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty>}^\infty U^*(x)^2\cdot dx\leq C\cdot\int_{-\infty>}^\infty u^2(x)\cdot dx$$

for every L^2-function u on the real line.

8.6 Proof of Theorem 8.1 We use a factorisation via Blaschke products which enable us to write

$$F(z)=B(z)\cdot g^2(z)$$

where $g(z)$ is a zero-free analytic function in the upper half-plane. Since $|B(z)|\leq 1$ holds in $\mathfrak{Im}(z)>0$ we have trivially

$$F^*(x)\leq g^*(x)^2$$

On the real axis we have $|F(x)|=|g(x)|^2$ almost everywhere so the L^1-norm of F is equal to the L^2-norm of g. Next, with $g=U+iV$ we have a pair of harmonic functions and since $|g|^2=U^2+V^2$

we can apply Theorem 8.5 to each of these harmonic functions and at this stage we leave it to the reader to confirm the assertion in Theorem 8.1

8.7 Carleson measures

Let $F(z)$ be in the Hardy space $H^1(\mathbf{R})$. If $\lambda > 0$ we put

$$J_\lambda = \{F^*(x) > \lambda\}$$

We assume that the set is non-empty and hence this open set is a union of disjoint intervals $\{(a_k, b_k)\}$. To each interval we construct the triangle T_k with corners at the points a_k, b_k and $p_k = \frac{1}{2}(a_k + b_k) + \frac{i}{2}(b_k - a_k)$. Put

$$\Omega = \cup T_k$$

Exercise. Use the construction of Fatou sectors and the definition of F^* to show that

$$\{|F(z)| > \lambda\} \subset \Omega \tag{1}$$

Let us now consider a non-negative Riesz measure μ in the upper half-plane. For the moment we assume that μ has compact support and that $F(z)$ extends to a continuos function on the closed upper half-plane This is to ensure that various integrals exists but does not affect the final a priori inequality in Theorem X below. General formulas for distribution functions give:

$$\int |F(z)| \cdot d\mu(z) = \int_0^\infty \lambda \cdot \mu(\{|F(z)| > \lambda\}) \cdot d\lambda \tag{2}$$

To profit upon (1) we impose a certain norm on μ. To each x and every h we construct the triangle $T_x(a)$ standing on the interval $(x - a/2, x + a/2]$.

8.8 Definition. *The Carleson norm of μ is defined as smallest constant C such that*

$$\mu(T_x(a) \leq C \cdot a$$

hold for all pairs $x \in \mathbf{R}$ and $a > 0$ and is denoted by $\mathfrak{car}(\mu)$.

8.9 Application. Given μ with its Carleson norm the inclusion (1) gives

$$\mu(\{|F(z)| > \lambda\}) \leq \sum \mu(T_k) \leq \mathfrak{car}(\mu) \cdot \sum (b_k - a_k) \tag{i}$$

The last sum is the Lebesgue measure of $\{F^* > \lambda\}$ and hence the right hand side in (i) is estimated above by

$$\mathfrak{car}(\mu) \cdot \int_0^\infty \lambda \cdot \mathfrak{m}(\mu(\{F^* > \lambda\}) \cdot d\lambda = \mathfrak{car}(\mu) \cdot \int_{-\infty}^\infty F^*(x) \cdot dx \tag{ii}$$

Together with Theorem 8.1 we arrive at the conclusive result:

8.10 Theorem. *There exists an absolute constant C such that*

$$\int |F(z)| \cdot d\mu(z) \leq C \cdot \mathfrak{car}(\mu) \cdot \int_{-\infty}^\infty |F(x)| \cdot dx$$

hold for each $F \in H^1(\mathbf{R})$ and every non-negative Riesz measure μ in the upper half-plane.

9. BMO and radial norms of measures

Theorem 0.4 together with the preceeding description of the dual space of $\mathfrak{Re}\, H_0^1(T)$ implies that every BMO-function F can be written as a sum

$$F = \phi + v^* \tag{i}$$

where ϕ is bounded and v^* is the harmonic conjugate of a bounded function. However, this decomposition is not unique. A *constructive* procedure to find a pair u, v in for a given BMO-function F was given by P. Jones in [Jones]. See also the article [Carleson] from 1976.

9.1 Radial norms on measures. Let D be the unit disc. An L^1-function $u(z)$ in D is radially bounded if there exists a constant C such that

$$(*) \qquad \frac{1}{\pi}\cdot\iint_{S_h}|u(z)|\cdot dxdy \leq C\cdot h$$

for each sector

$$S_h=\{z\colon \theta-h/2<\arg z\theta+h/2\} \quad : \quad h>0$$

The smallest C for which (*) holds is denoted by $|u|^*$. Notice that $|u|^*$ in general is strictly larger than the L^1-norm over D which occurs when we take $h=\pi$ above. If u satisfies (*) we define a function P_u on the unit circle by

$$P_u(\theta)=\frac{1}{\pi}\cdot\iint_D\frac{1-|z|^2}{|e^{i\theta}-z|^2}\cdot u(z)\cdot dxdxy$$

With these notations Fefferman proved:

9.2 Theorem *There exists an absolute constant C such that*

$$|P_u|_{\mathrm{BMO}}\leq C|u|^*$$

Thus, $u\mapsto P_u$ sends radially bounded $L^1(D)$-functions to BMO(T). The proof of Theorem 8.1 relies upon Theorem 0.4 and the following observation:

9.3 Exercise. Show that when u is radially bounded and $H(z)$ is a harmonic function in D with continuous boundary values on T then

$$\iint_D H(z)\cdot u(z)\cdot dxdy=\int_0^{2\pi}H(e^{i\theta})\cdot P_u(\theta)\cdot d\theta$$

The following result is also due to Fefferman:

9.4 Theorem. *Let $F(\theta)\in BMO(T)$. Then there exists a radially bounded $L^1(D)$-function u and some $s(\theta)\in H^\infty(T)$ such that*

$$F(\theta)=s(\theta)+P_u(\theta)$$

For detailed proofs of the results above we refer to Chapter XX in [Koosis].