Bochner's moment theorem

In probability theory the distribution of a stochastic variable is expressed by a probability measure μ on the real line where we take t as the coordinate. The characteristic function is by definition the Fourier transform of μ and we set

$$(*) \qquad f(x) = \int e^{-ixt} \cdot d\mu(t)$$

Since μ has total mass one we get $f(0) = 1$. Moreover, let $x_1, \ldots, x_N$ be some N-tuple of real numbers and $\alpha_1, \ldots, \alpha_N$ some N-tuple of complex numbers. Then

$$(**) \qquad \sum\sum f(x_p - x_q)\alpha_p \cdot \bar{\alpha}_q = \int |\sum \alpha_p \cdot e^{-ix_p \cdot t}|^2 d\mu(t)$$

Since $\mu \geq 0$ the right hand side is ≥ 0. It turns out that this inequality characterizes the family of bounded continuous functions $f(x)$ which are Fourier transforms of non-negative measures. To make it precise we give

Definition. *Denote by $\mathcal{B}$ the class of continuous functions $f(x)$ on the real x-line such that*

$$\sum\sum f(x_p - x_q)\alpha_p \cdot \bar{\alpha}_q \geq 0 \quad : \quad f(0) = 1$$

where the inequality holds for all pairs of N-tuples $x_\bullet$ and $\alpha_\bullet$ as above.

Remark. Given $x > 0$ we take $N = 2$ with $x_1 = 0$ and $x_2 = x$ and $\alpha_1 = 1$ while $\alpha - 2 = e^{i\theta}$. Then Bochner's condition gives

$$\text{(i)} \qquad 2 \cdot f(0) + e^{i\theta} f(x) + e^{-it\theta} f(-x) \geq 0$$

With $\theta = \pi/2$ it follows that $f(-x) = \bar{f}(x)$ and then the inequality (i) gives

$$\text{(ii)} \qquad |f(x)| \leq f(0) = 1$$

So functions in $\mathcal{B}$ are automatically bounded. Now we announce Bochner's result.

1. Theorem. *For each $f \in \mathcal{B}$ there exists a unique non-negative measure μ such that*

$$f(x) = \int e^{-ixt} d\mu(t)$$

Remark. Theorem 1 appears in the book [Boch] *Vorlesungen über Fouriersche Integrale* from 1932. The essential ingredient in the proof is a representation formula for positive harmonic functions in the upper half-plane. Prior to Bochner's result the periodic version of Theorem 1 was established by G. Herglotz who proved the following in [Herg] from 1911:

2. Theorem. *Let $\{m_n : -\infty < n < \infty\}$ be a sequence of complex numbers. In order that there exists a non-negative Riesz measure μ on the interval $[0, 2\pi]$ such that*

$$m_n = \int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta)$$

it is necessary and sufficient that

$$\sum_{\nu=-N}^{\nu=N} \sum_{j=-N}^{j=N} m_{\nu-j} \cdot \alpha_\nu \cdot \bar{\alpha}_j \geq 0$$

holds for and finite sequence of complex numbers $\alpha_{-N} \ldots, \alpha_N$.

To prove Bochner's theorem we shall need the following result:

3. Proposition. *For each pair of real numbers ξ, η with $\eta > 0$ there exists a function $\phi(x)$ in $L^1(\mathbf{R})$ such that*

$$e^{-i\xi x - \eta|x|} = \int_{-\infty}^{\infty} \phi(x+y) \cdot \bar{\phi}(y) \cdot dy \quad : \quad -\infty < x < \infty$$

Proof Set

(ii) $$\phi(x)=\sqrt{\frac{1}{2\pi}}\cdot\int e^{itx}\cdot\sqrt{\frac{1}{2\pi}}\cdot\sqrt{\frac{\eta}{\eta^2+(\xi+t)^2}}\cdot dt$$

The reader can verify that $\phi(x)\in L^1(\mathbf{R})$ and that the equality in Proposition 3 holds.

Proof of Theorem 1. Let $f\in\mathcal{B}$ be given and put

(1) $$\Phi(\xi,\eta)=\int_{-\infty}^{\infty}e^{-i\xi x-\eta|x|}\cdot f(x)\cdot dx\quad:\ \xi\in\mathbf{R}\quad:\eta>0$$

Proposition 3 gives:

$$\Phi(\xi,\eta)=\iint\phi(x+y)\cdot\bar{\phi}(y)\cdot f(x)\cdot dxdy=$$

$$\iint\phi(x)\cdot\bar{\phi}(y)\cdot f(x-y)\cdot dxdy$$

Since both f and ϕ belong to L^1 we can approximate the last double integral by Riemann sums which are of the form

$$\sum f(x_p-x_q)\cdot\alpha_p\bar{\alpha}_q$$

The hypotehsis that $f\in\mathcal{B}$ therefore implies that the Φ-function is ≥ 0. Next, for each fixed x we consider the function

(*) $$(\xi,\eta)\mapsto e^{-i\xi x-\eta|x|}$$

Since $i^2=-1$ we see that this function is harmonic. Approximating the integral (1) by Riemann sums we conclude that $\Phi(\xi,\eta)$ is a harmonic function in the upper half-plane $\eta>0$. Since $|f(x)|\leq 1$ for all x and $|e^{-ix\xi}|=1$ the the triangle inequality gives

(i) $$|\Phi(\xi,\eta)|\leq\int_{-\infty}^{\infty}e^{-\eta|x|}\cdot dx=\frac{2}{\eta}$$

Now Φ is harmonic and ≥ 0 in the upper half-plane. Hence the inequality (i) and the general result in § XX gives a non-negative measure μ of finite total mass such that

(2) $$\Phi(\xi,\eta)=\frac{1}{\pi}\cdot\int_{-\infty}^{\infty}\frac{\eta}{\eta^2+(\xi-t)^2}\cdot d\mu(t)$$

With $\eta>0$ kept fixed we notice that (1) means that the function $\xi\mapsto\Phi(\xi,\eta)$ is the Fourier transform of $e^{-\eta|x|}f(x)$. Hence (2) and Fourier's inversion formula yield:

$$e^{-\eta|x|}f(x)=\frac{1}{2\pi^2}\cdot\int_{-\infty}^{\infty}e^{ix\xi}\cdot\Big[\int_{-\infty}^{\infty}\frac{\eta}{\eta^2+(\xi-t)^2}\cdot d\mu(t)\Big]\cdot d\xi=$$

(iii) $$\frac{1}{2\pi^2}\cdot\int\Big[\int_{-\infty}^{\infty}\frac{\eta\cdot e^{ix\xi}}{\eta^2+(\xi-t)^2}\Big]\,d\mu(t)\quad:\quad\eta>0$$

Next, we have the limit formulas

(iv) $$\frac{1}{\pi}\cdot\lim_{\eta\to 0}\int_{-\infty}^{\infty}e^{ix\xi}\cdot\frac{\eta}{\eta^2+(\xi-t)^2}\cdot d\xi=e^{ixt}\quad:\ -\infty<t<\infty$$

(v) $$\lim_{\epsilon\to 0}e^{-\epsilon|x|}\cdot f(x)\to f(x)$$

So after the passage to the limit as $\eta\to 0$ we get the requested formula:

(iv) $$f(x)=\frac{1}{2\pi}\cdot\int e^{ixt}\cdot d\mu(t)$$

Operational calculus on $L^1(\mathbf{R})$

Let $f(x)$ be in $L^1(\mathbf{R})$ and denote its Fourier transform by $g(\xi)$, i.e.

$$(*) \qquad g(\xi) = \int e^{-ix\xi} f(x)dx$$

Let $[a,b]$ be a closed interval on the real ξ-line. Write $w = g(\xi)$ which gives the compact subset $g[a,b])$ of the complex w-plane. Let $\Phi(w)$ be an analytic function defined in some open neighborhood of $g[a,b]$. With these notations one has

Theorem. *There exists a function $\phi(x) \in L^1(\mathbf{R})$ whose Fourier transform satisfies*

$$\hat{\phi}(\xi) = \Phi(g(\xi)) \quad : \quad a \leq \xi \leq b$$

Proof. Consider a point $a \leq \xi_* \leq b$ and put $w_* = g(\xi_*)$. The analyticity of Φ gives a series expansion

$$(*) \qquad \Phi(w) = \Phi(w_*) + \sum_{\nu=1}^{\infty} c_\nu (w - w_*)^\nu$$

which is convergent in some open disc centered at w_*. Hence there exist $\delta > 0$ and a constant M such that

$$\text{(i)} \qquad |c_\nu| \leq M \cdot \delta^{-\nu} \quad : \quad \nu = 0, 1, \ldots$$

Next, consider the function

$$W(\xi) = 1 \quad : \quad |\xi| \leq 1 \quad : W(\xi) = 2 - |\xi| \quad : \quad 1 \leq |\xi| \leq 2$$

Recall from the example in § XX that W is the Fourier transform of an L^1-function $P(x)$. Fourier's inversion formula gives:

$$\text{(ii)} \qquad P(x) = \frac{1}{2\pi} \int e^{ix\xi} \cdot W(\xi) d\xi$$

Next, when $|g(\xi) - g(\xi_*)| < \delta$ it follows from (*) that

$$\text{(iii)} \qquad \Phi(g(\xi)) - \Phi(g(\xi_*)) = \sum c_\nu (g(\xi) - g(\xi_*))^\nu$$

Let $k > 0$ and put

$$\text{(iii)} \qquad \psi_k(\xi) = W(k(\xi - \xi_*)) \cdot \Phi(g(\xi_*)) + \sum c_\nu \cdot \big[W(k(\xi - \xi_*)) \cdot (g(\xi) - g(\xi_*) \big]^\nu$$

Rules for dilation under the Fourier transform and (ii) give

$$\text{(iv)} \qquad \frac{1}{k} \cdot e^{i\xi_* \cdot x} \cdot P(\frac{x}{k}) = \text{inverse Fourier transform of } \ W(k(\xi - \xi_*))$$

More precisely, we have

$$(**) \qquad \frac{1}{k} \cdot e^{i\xi_* \cdot x} \cdot P(\frac{x}{k}) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot W(k(\xi - \xi_*)) \cdot d\xi$$

Define the function $Q_k(x)$ by:

$$\text{(v)} \qquad Q_k(x) = \frac{1}{k} \int e^{i\xi_*(x-y)} \big[P(\frac{x-y}{k} - P(\frac{x}{k}) \big] f(y) dy$$

Then (**) and Fourier's inversion formula give:

$$\text{(vi)} \qquad W(k(\xi - \xi_*)) \cdot (g(\xi) - g(\xi_*) = \int e^{-ix\xi} \cdot Q_k(x) dx$$

Next, the triangle inequality applied to the right hand side in (vi) gives:

$$(***) \quad \int |Q_k(x)| \cdot dx \leq \frac{1}{k} \iint |P(\frac{x-y}{k} - P(\frac{x}{k})| \cdot |f(y)| \cdot dxdy = ||f||_1 \cdot \int \big| P(x - \frac{y}{k}) - P(\frac{x}{k}) \big| \cdot dx$$

Since $P \in L^1(\mathbf{R})$ the Riemann-Lebesgue theorem gives

$$\lim_{k\to 0} \int \left|P(x-\frac{y}{k})-P(\frac{x}{k})\right|\cdot dx = 0$$

Together with the inequality (***) we therefore obtain

(****) $$||Q_k||_1 = \lambda(k) \quad \text{where} \quad \lim_{k\to 0} \lambda(k) = 0$$

Next, for each $\nu \geq 2$ we construct the ν:th fold convolution of Q_k which we denote by $Q_k^{(\nu)}$. The multiplicative inequality for L^1-norms and (****) give:

(*****) $$||Q_k^{(\nu)}||_1 \leq \lambda(k)^\nu \quad : \quad \nu = 1,2,3,\dots$$

Choose k so large that $\lambda(k) < \delta$. Then (*****) entails that

(vii) $$G(x) = \sum_{\nu=1}^{\infty} c_\nu \cdot Q_k^{(\nu)}(x)$$

converges in the Banach space $L^1(\mathbf{R})$. Hence we obtain the $L^1(\mathbf{R})$-function defined by

(viii) $$G^*(x) = \frac{1}{k}\cdot \Phi(g(\xi_*))\cdot e^{i\xi_*\cdot x}\cdot P\big(\frac{x}{k}\big)$$

From the constructions above it is clear that the Fourier transform of $G^*(x)$ is equal to the function $\psi_k(\xi)$ in (iii). Moreover, the construction of the W-function and the series expansion of Φ in (*) give the equality

(ix) $$\psi_k(\xi) = \Phi(g(\xi)) \quad : \quad |\xi - \xi_*| \leq \frac{1}{k}$$

Final part of the proof. By (ix) we find L^1-functions whose Fourier transforms agrees with $\Phi(g(\xi))$ on small intervals around every point $a \leq \xi_* \leq b$. By the Heine-Borel Lemma and a C^∞-partition of the unit we finish the proof of Theorem 1. To be precise, use that if $h(\xi)$ is a test-functionon the real ξ-line then it is the Fourier transform of some L^1-function.