Almost periodic functions.

Introduction. The theory about almost periodic functions is due to Harald Bohr and was developed in the years 1910-1930. The reader may consult the text-book [Bohr] for a detailed account of Bohr's contributions. Here is the baisc defintion:

Defintion. *A continuous function $f(x)$ on the real line is called almost periodic if there to each $\epsilon > 0$ exists some positive number $\ell(\epsilon)$ such that every interval of length $\ell(\epsilon)$ contains a point τ such that the maximum norm*

$$(*) \qquad \max x \in \mathbf{R} |f(x+\tau) - f(x)| \leq \epsilon$$

The class of these functions is denoted by $\mathcal{AP}$.

Trigonometric polynomials. They consist of functions given by finite sums:

$$A(x) = c_1 e^{i\lambda x} + \ldots + c_N e^{i\lambda_N x}$$

where $\lambda_1, \ldots, \lambda_N$ is some N-tuple of real numbers and $c_1, \ldots, c_N$ are complex numbers. Every trigonometric polynomial belongs to $\mathcal{AP}$. Indeed this follows from the following elementary fact:

Proposition. *Let $\lambda_1, \ldots, \lambda_N$ be a finite set of real numbers. To each $\epsilon > 0$ there exists some positive number $\ell(\delta)$ such that every interval on the real line if length $\ell(\delta)$ contains a point τ such that*

$$\frac{\lambda_\nu}{2\pi} \cdot \tau = \delta_\nu + b_\nu \quad : \quad |\delta_\nu| < \epsilon \quad : b_\nu \in \mathbf{Z}$$

Apply this to a trigonometric polynomial $A(x)$. Then we see that

$$A(x+\tau) - A(x) = \sum c_\nu (\, e^{2\pi i \delta_\nu} - 1) \cdot e^{i\lambda_\nu x}$$

and conclude that $A(x) \in \mathcal{AP}$. Conversely one has

Theorem. *Every $f \in \mathcal{SAP}$ can be uniformly approximated by trigonometric polynomials, i.e. there exists a sequence of trigonometric polynomials $A_1, A_2, \ldots$ such that*

$$\lim_{n\to\infty} \max_{x \in \mathbf{R}} |A_n(x) - f(x)| = 0$$

The proof is left as an exercise. The hint is to use approximation for periodic functions via Fourier series.

The spectral function. Let $f(x)$ be almost periodic. From Theorem 1 it follows easily that we get a continuous and bounded function $\Phi(x)$ defined by

$$\Phi(x) = \lim_{w\to\infty} \int_{-w}^{w} f(x+t)\bar{f}(t) \cdot dt$$

Following Bohr we refer to $\Phi(x)$ as the spectral function of f. If x_p and x_q are teo real numbers we notice that

$$\Phi(x_p - x_q) = \lim_{w\to\infty} \int_{-w}^{w} f(x_p+t)\bar{f}(x_q+t) \cdot dt$$

So of $x_1, \dots, x_N$is an N-tuple of real numbers and $a_1, \dots, a_N$ an N-tuple of complex numbers we get

$$(*) \qquad \sum\sum a_p \bar{a}_q \Phi(x_p - x_q) = \lim_{w \to \infty} \int_{-w}^{w} |\sum a_p f(x_p + t)|^2 \cdot dt$$

Hence $\Phi(x)$ is a $\mathcal{B}$-function and Theorem X.1 gives a non-negative measure σ such that

$$\Phi(x) = \int e^{ix\xi} \cdot d\mu(\xi)$$

Following Bohr the non-negative measure μ is called the spectral measure associated to f. A major result about almost periodic functions is:

Theorem. *The spectral measure of an almost periodic function is always discrete. Moreover one has the limit formula:*

$$\lim_{w \to \infty} \frac{1}{2w} \int_{-w}^{w} \left| f(x) - \sum c_\nu \cdot e^{i\lambda_\nu x} \right|^2 \cdot dx = 0$$

where $\mu = \sum c_\nu \cdot \delta_{\lambda_\nu}$.

Operational calculus on $L^1(\mathbf{R})$

Let $f(x)$ be in $L^1(\mathbf{R})$ and denote its Fourier transform by $g(\xi)$, i.e.

$$(*) \qquad g(\xi) = \int e^{-ix\xi} f(x) dx$$

We know that $g(\xi)$ is a continuous and complex-valued function. Let $[a, b]$ be a closed interval on the real ξ-line, Put $w = g(\xi)$ which gives the compact subset $g[a, b])$ of the complex w-plane. Let $\Phi(w)$ be an analytic function defined in some open neighborhood of $g[a, b]$. With these notations one has

Theorem. *There exists a function $\phi(x) \in L^1(\mathbf{R})$ whose Fourier transform satisfies*

$$\hat{\phi}(\xi) = \Phi(g(\xi)) \quad : \quad a \leq \xi \leq b$$

Proof. Consider some point $a \leq \xi_* \leq b$. By the analyticity of Φ there exists a series expansion

$$\Phi(w) = \Phi(w_*) + \sum_{\nu=1}^{\infty} c_\nu (w - w^*)^\nu \quad : w^* = g(\xi_*)$$

which is convergent in some open disc centered at w_*. In particular we can find $\delta > 0$ and a constant M such that

$$\text{(i)} \qquad |c_\nu| \leq M \cdot \delta^{-\nu} \quad : \quad \nu = 0, 1, \dots$$

Next, we consider the special function

$$W(\xi) = 1 \quad : \quad |\xi| \leq 1 \quad : W(\xi) = 2 - |\xi| \quad : \quad 1 \leq |\xi| \leq 2$$

Recall from XX that W is the Fourier transform of an L^1-function $P(x)$. Fourier's inversion formula gives:

$$\text{(ii)} \qquad P(x) = \frac{1}{2\pi} \int e^{ix\xi} \cdot W(\xi) d\xi$$

Next, when $|g(\xi) - g(\xi_*)| < \delta$ it follows from (i) that

$$\text{(iii)} \qquad \Phi(g(\xi)) - \Phi(g(\xi_*) = \sum c_\nu (g(\xi) - g(\xi_*))^\nu$$

Let $k > 0$ and put

$$\text{(iii)} \quad \psi_k(\xi) = W(k(\xi - \xi_*)) \cdot \Phi(g(\xi_*)) + \sum c_\nu \cdot \big[W(k(\xi - \xi_*)) \cdot (g(\xi) - g(\xi_*)\big]^\nu$$

Rules for dilation under the Fourier transform and (ii) give

$$\text{(iv)} \qquad \frac{1}{k} \cdot e^{i\xi_* \cdot x} \cdot P(\frac{x}{k}) = \text{inverse Fourier transform of } \ W(k(\xi - \xi_*))$$

More precisely, we have

$$(*) \qquad \frac{1}{k} \cdot e^{i\xi_* \cdot x} \cdot P(\frac{x}{k}) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot W(k(\xi - \xi_*)) \cdot d\xi$$

Define the function $Q_k(x)$ by:

$$\text{(v)} \qquad Q_k(x) = \frac{1}{k} \int e^{i\xi_*(x-y)} \big[P(\frac{x-y}{k} - P(\frac{x}{k})\big] f(y) dy$$

Then (*) and Fourier's inversion formula give:

$$\text{(vi)} \qquad W(k(\xi - \xi_*)) \cdot (g(\xi) - g(\xi_*) = \int e^{-ix\xi} \cdot Q_k(x) dx$$

Next, the triangle inequality applied to the right hand side in (vi) gives:

$$\int |Q_k(x)| \cdot dx \leq \frac{1}{k} \iint |P(\frac{x-y}{k} - P(\frac{x}{k})| \cdot |f(y)| \cdot dx dy =$$

$$(**) \qquad ||f||_1 \cdot \int \big|P(x - \frac{y}{k}) - P(\frac{x}{k})\big| \cdot dx$$

Since $P \in L^1(\mathbf{R})$ the Riemann-Lebesgue theorem gives

$$\lim_{k \to 0} \int \big|P(x - \frac{y}{k}) - P(\frac{x}{k})\big| \cdot dx = 0$$

Together with the inequality (**) we therefore obtain

Sublemma. One has

$$||Q_k||_1 = \lambda(k) \quad : \quad \lim_{k \to 0} \lambda(k) = 0$$

For each $\nu \geq 2$ we construct the ν:th fold convolution of Q_k which we denote by $Q_k(\nu)$. The multiplicative inequality for L^1-norms and the Sublemma give:

$$(***) \qquad ||Q_k(\nu)||_1 \leq \lambda(k)^\nu \quad : \quad \nu = 1, 2, 3, \ldots$$

We can choose k so large that $\lambda(k) < \delta$. it follow from (xx) that

$$G(x) = \sum_{\nu=1}^{\infty} c_\nu \cdot Q_k(\nu)(x)$$

converges in the Banach space $L^1(\mathbf{R})$. Hence we obtain the $L^1(\mathbf{R})$-function defined by

$$G^*(x) = \frac{1}{k} \cdot \Phi(g(\xi_*)) \cdot e^{i\xi_* \cdot x} \cdot P\big[\frac{x}{k}) + G(x)$$

From the previous constructions it is clear that the Fourier transform of $G^*(x)$ is equal to the function $\psi_k(\xi)$ from (iii). Finally, from the construction of the W-function and the original series expansion of Φ we have the equality

$$(****) \qquad \psi_k(\xi) = \Phi(g(\xi)) \quad : \quad |\xi - \xi_*| \leq \frac{1}{k}$$

Remark. Above the integer k was chosen to ensure that $\lambda(k) < \delta$. In the Sublemma we used the inequality (**) where the P-function does not depend on f or on Φ. So we conclude that when we restrict the attention to L^1-functions f of norm ≤ 1, then it suffices to choose k so large that

$$\int \big|P(x - \frac{y}{k}) - P(\frac{x}{k})\big| \cdot dx < \delta$$

Final part of the proof. By the remark above we find L^1-functions whose Fourier transforms agree with $\Phi(g(\xi))$ on small intervals around every point $a \leq \xi_* \leq b$. By the Heine-Borel Lemma and a C^∞-partition of the unit we can finish the proof of Theorem xx. To be precise, we use that if $h(\xi)$ is a test-functionon the real ξ-line then it is the Fourier transform of some L^1-function.

Bochner's moment theorem

In probability theory the frequency of a stochastic variable is expressed by a probability measure μ on the real line where we take t as the coordinate. The characteristic function is by definition the Fourier transform of μ and we set

$$(*) \qquad f(x) = \int e^{-ixt} \cdot d\mu(t)$$

Let $x_1, \ldots, x_N$ be some N-tuple of real numbers and $\alpha_1, \ldots, \alpha_N$ some N-tuple of complex numbers. Then

$$(**) \qquad \sum \sum f(x_p - x_q)\alpha_p \cdot \bar{\alpha}_q = \int \big|\sum \alpha_p \cdot e^{-ix_p \cdot t}\big|^2 d\mu(t)$$

Since $\mu \geq 0$ it follows that the right hand side is ≥ 0. It turns out that this inequality characterizes the family of bounded continuous functions $f(x)$ which are Fourier transforms of non-negative measures. First we introduce a class of functions:

The class $\mathcal{B}$. It consists of continuous functions $f(x)$ on the real x-line such that $f(0) = 1$ and

$$\sum\sum f(x_p - x_q)\alpha_p \cdot \bar{\alpha}_q \geq 0 \quad : \quad f(0) = 1$$

hold for all N-tuples $x_\bullet$ and $\alpha_\bullet$ as above.

Remark. Given $x > 0$ we take $N = 2$ with $x_1 = 0$ and $x_2 = x$ and $\alpha_1 = 1$ while $\alpha - 2 = e^{i\theta}$. Then Bochner's condition gives

(i) $$2 \cdot f(0) + e^{i\theta} f(x) + e^{-it\theta} f(-x) \geq 0$$

With $\theta = \pi/2$ it follows that $f(-x) = \bar{f}(x)$ and then the inequality (i) gives

(ii) $$|f(x)| \leq f(0) = 1$$

So functions in $\mathcal{B}$ are automatically bounded. Theorem 1 is due to Bochner and was for example presented in his book [Boch] *Vorlesungen über Fouriersche Integrale* from 1932.

1. Theorem. *For each $f \in \mathcal{B}$ there exists a unique non-negative measure μ such that*

$$f(x) = \int e^{-ixt} d\mu(t)$$

The subsequent proof of Theorem 1 uses a representation formula for positive harmonic functions in the upper half-plane which in its turn is a special case of more general representations of harmonic functions in half-planes due to the Brothers Nevanlinna in the article [Nev-Nev] from 1920. Let us also remark that the periodic version preceeded Theorem 1 and is due to G. Herglotz who proved the following in his article [Herg] from 1911:

2. Theorem. *Let $\{m_n : -\infty < n < \infty\}$ be a sequence of complex numbers. In order that there exists a non-negative Riesz measure μ on the interval $[0, 2\pi]$ such that*

$$m_n = \int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta)$$

it is necessary and sufficient that

$$\sum_{\nu=-N}^{\nu=N} \sum_{j=-N}^{j=N} m_{\nu-j} \cdot \alpha_\nu \cdot \bar{\alpha}_j \geq 0$$

holds for and finite sequence of complex numbers $\alpha_{-N} \ldots, \alpha_N$.

To prove Theorem 1 we shall need the following result:

3. Proposition. *For each pair of real numbers ξ, η with $\eta > 0$ there exists a function $\phi(x) \in L^1(\mathbf{R})$ such that*

$$e^{-i\xi x - \eta|x|} = \int_{-\infty}^{\infty} \phi(x+y) \cdot \bar{\phi}(y) \cdot dy \quad : \ -\infty < x < \infty$$

Proof Define the function

(i) $$g(t) = \sqrt{\frac{1}{2\pi}} \cdot \sqrt{\frac{\eta}{\eta^2 + (\xi + t)^2}}$$

Put

(ii) $$\phi(x) = \sqrt{\frac{1}{2\pi}} \cdot \int e^{itx} \cdot g(t) \cdot dt$$

We leave as an exercise to the reader to verify that $\phi(x) \in L^1(\mathbf{R})$ and that the equality in Proposition 3 holds.

Proof of Theorem 1.

Consider the function

(1) $$\Phi(\xi, \eta) = \int_{-\infty}^{\infty} e^{-i\xi x - \eta|x|} \cdot f(x) \cdot dx \quad : \xi \in \mathbf{R} \quad : \eta > 0$$

Proposition 3 gives:

$$\Phi(\xi, \eta) = \iint \phi(x+y) \cdot \bar{\phi}(y) \cdot f(x) \cdot dxdy =$$

$$\iint \phi(x) \cdot \bar{\phi}(y) \cdot f(x-y) \cdot dxdy$$

Since both f and ϕ belong to L^1 we can approximate the last double integral by Riemann sums which take the form

$$\sum f(x_p - x_q) \cdot \alpha_p \bar{\alpha}_q$$

Since $f \in \mathcal{B}$ it follows that the Φ-function is ≥ 0. Next, for each fixed x we consider the function

(*) $$(\xi, \eta) \mapsto e^{-i\xi x - \eta|x|}$$

Since $i^2 = -1$ we see that this function is harmonic. Approximating the integral (1) by Riemann sums we conclude that $\Phi(\xi, \eta)$ is a harmonic function in the upper half-plane $\eta > 0$. Since $|f(x)| \leq 1$ for all x and $|e^{-ix\xi}| = 1$ the the triangle inequality gives

$$|\Phi(\xi, \eta)| \leq \int_{-\infty}^{\infty} e^{-\eta|x|} \cdot dx = \frac{2}{\eta} \tag{2}$$

Now Φ is harmonic and ≥ 0 in the upper half-plane. Hence the inequality (i) and the general result in XX gives a non-negative measure μ of finite total mass such that

$$\Phi(\xi, \eta) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\eta}{\eta^2 + (\xi - t)^2} \cdot d\mu(t) \tag{3}$$

With $\eta > 0$ kept fixed we notice that (2) means that the function $\xi \mapsto \Phi(\xi, \eta)$ is the Fourier transform of $e^{-\eta|x|} f(x)$. Hence (3) and Fourier's inversion formula yield:

$$e^{-\eta|x|} f(x) = \frac{1}{2\pi^2} \cdot \int_{-\infty}^{\infty} e^{ix\xi} \cdot \big[\int_{-\infty}^{\infty} \frac{\eta}{\eta^2 + (\xi - t)^2} \cdot d\mu(t) \big] \cdot d\xi =$$

$$\frac{1}{2\pi^2} \cdot \int \big[\int_{-\infty}^{\infty} \frac{\eta \cdot e^{ix\xi}}{\eta^2 + (\xi - t)^2} \big] \, d\mu(t) \quad : \quad \eta > 0 \tag{4}$$

Now we use the limit formula:

$$\frac{1}{\pi} \cdot \lim_{\eta \to 0} \int_{-\infty}^{\infty} e^{ix\xi} \cdot \frac{\eta}{\eta^2 + (\xi - t)^2} \cdot d\xi = e^{ixt} \quad : \quad -\infty < t < \infty$$

At the same time we have

$$\lim_{\epsilon \to 0} e^{-\epsilon|x|} \cdot f(x) \to f(x)$$

So after the passage to the limit as $\eta \to 0$ we get:

$$f(x) = \frac{1}{2\pi} \cdot \int e^{ixt} \cdot d\mu(t) \tag{5}$$

If we wish to use e^{-ixt} we just have to replace μ by μ^* where $d\mu^*(t) = d\mu(-t)$ and Theorem 1 follows.

A uniqueness for a moment problem.

Introduction. The first general study of moment problems is due to Stieltjes who asked for a non-negative Riesz measure μ defined on $x \geq 0$ such that the moments

$$(*) \qquad \int_0^\infty t^n \cdot d\mu(t) = c_n \quad : \quad n = 0, 1, \ldots$$

Remark. In 1880 the notion of general measure was not Conditions for the existence of solutions to (*) as well as uniqueness were proved by Stieltjes using continued fraction expansions corresponding to the series

$$\sum \frac{(-1)^\nu \cdot c_\nu}{t^{\nu+1}}$$

We shall not discuss this any further but refer to original work by Stieltjes. Instead we study the moment problem where integration takes place over the whole real line, i.e. this time we regard a non-negative measure μ with support on the real line and set

$$(**) \qquad c_\nu = \int_{-\infty}^\infty t^\nu \cdot d\mu(t) \quad : \quad \nu = 0, 1, \ldots$$

This moment problem was studied by Hamburger in (Math.Annalen 81-82). See also R. Nevanlinna's article *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjesche Momentproblem.* A conclusive uniqueness result was presented by T. Carleman at the Scandianavian Congress of mathematics at Helsinki in 1922. Before we announce this uniqueness theorem we remark that in order that (*) has a meaning we only regard non-negative measures μ on the real t-axis for which

$$\int_{-\infty}^\infty t^{2n} \cdot d\mu(t) < \infty \quad : \quad n = 0, 1, \ldots$$

This means that μ does not carry much mass when $|x|$ is large. In particular the function

$$F(z) = \int \frac{d\mu(t)}{t - z}$$

exists as analytic function of the complex variable z outside the real axis, i.e. $F(z)$ is analytic in both the upper and the lower half-plane. Moreover, $F(z)$ determines μ, i.e. if F is identically zero then $\mu = 0$. This is used to settle uniquness for the moment equation (**) where the result from [Car: p. 80] goes as follows:

Theorem. *Let $\{c_\nu\}$ be a sequence of real numbers such that the series*

$$\sum_{n=0}^\infty \frac{1}{\left|c_{2n}\right|^{\frac{1}{2n}}} = +\infty$$

Then (*) has at most one solution.

Remark. PROOF via cauchy integralsgiven on page 80 is elegant ...
A case of non-uniqueness. Consider a C^∞- function $f(x)$ which vanishes with all its derivatives at $x = 0$ and $x = 1$. Put

$$m_p^2 = \int_0^1 |f^{(p)}(x)|^2 \cdot dx$$

Then there exist several measures μ such that

$$\int_0^\infty x^p \cdot d\mu(x) = m_p^2$$

EXPLICIT constructions appear in Carleman on page 85.

Another condition for uniqueness

Let μ be as above, i.e. denote the class of

$$\int t^{2p} \cdot d\mu(t) < \infty$$

hold for every $p \geq 1$.

Then we can use the Gram-Schmidt procedure and construct a sequence of polynomials $\{P_\nu\}$ which are orthogonal, i.e.

$$(*) \qquad \int P_n(t) \cdot \bar{P}_m(t) \cdot d\mu(t) = \text{Kronecker's delta function}$$

where each $P_n(t)$ is monic and of degree n. So to the given measure μ belongs the series:

$$(**) \qquad \mathcal{P}_\mu^*(z) = \sum_{n=0}^\infty |P_n(z)|^2 \quad : z = t + is$$

A sufficient condition for the uniqueness is given as follows:

Theorem. *Let $\mu \in \mathcal{R}$ be such that (**) is divergent for every $s \neq 0$. Then μ is determined by the sequence $\{c_j\}$.*