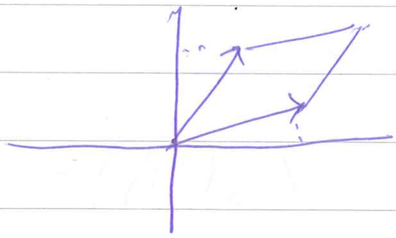
Linear Algebra Sections.

0

- 2×2 matrices
  - arithmetic, identity, ...
  - prop.
  - determinant (from a combinatorial perspect)

Rang



- Cayley-Hamiltonian Thm.
  - Sylvester

done

- Least Square (Graham)

- Turing (triangulation)
  - notes done

- Colman ("∞ equation")
  - notes done $\rightsquigarrow$ Ladder systems

CAYLEY 186.. **Linear Algebra Section**

("Cayley–Hamilton–Sylvester Theorem").

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$\sigma(A)$ = roots of $\Delta(\lambda)$

(n,n)-matrix $\mathbb{C}$

A

$$\begin{pmatrix} * & \rho \\ a & 1 \end{pmatrix}$$

$$\Delta(\lambda) = \det(\lambda E_n - A)$$

$$= \lambda^n + \sum_0^{n-1} c_\nu \lambda^\nu$$

$$\det \begin{pmatrix} \lambda - a_{11} & & \\ & \ddots & \\ & & \lambda - a_{nn} \end{pmatrix}$$

C–H

$$\underline{\Delta(A) = 0 \quad A^n + \sum_0^{n-1} c_\nu \cdot A^\nu = 0}$$

$$R(\lambda) = (\lambda E_n - A)^{-1} \qquad \Delta(\lambda) \neq 0$$

Resolvent of A

$\lambda \mapsto R(\lambda)$ ~~with~~ Rational function

values (n,n)-matrices

$$\Delta(\lambda) \cdot R(\lambda) = C_0 + \lambda C_1 + \quad + \lambda^{n-1} C_{n-1}$$

$$C_0 \quad C_{n-1} \in M_n(\mathbb{C})$$

Cramer (1760)

$$R(\lambda) = \sum_\alpha \left(\sum_{\nu=1}^{\varepsilon_\alpha} \frac{T_{\nu,\alpha}}{(\lambda - \alpha)^\nu} \right), \quad \varepsilon_\alpha \in \mathbb{N}$$

α_m α_2 α_1

(spectral values)

α Run over zeros of $\Delta(\lambda)$

$\{T_{\nu,\alpha}\}_{\nu=1}^{\varepsilon_\alpha}$ Local Cayley matrices of α

"Cayley - Hamiltonian Thm"

Cayley

$$\delta(\lambda) = \prod_{\alpha \in \sigma(A)} (\lambda - \alpha)^{\varepsilon_\alpha}$$

$\delta(\lambda)$ factor of $\Delta(\lambda)$

$\boxed{\text{THM}}$ $\underline{\delta(A) = 0}$!!

ε_α order of pole of $R(\lambda)$ at α!

operational calculus.

$\cancel{\delta(A)}$ $\underline{\text{Fix one } \alpha \in \sigma(A)}$

$$q(\lambda) = q_\alpha(\lambda) \cdot (\lambda - \alpha)^{\varepsilon_\alpha}$$ $\underline{\text{Factorize}}$

$q_\alpha(\alpha) \neq 0$

Now Cauchy 1830

$$q_\alpha(A) = \frac{1}{2\pi i} \int_{|\lambda - \alpha| = \varepsilon} q_\alpha(\lambda) \cdot \left[\sum_{\nu=1}^{\varepsilon_\alpha} \frac{T_{\nu,\alpha}}{(\lambda - \alpha)^\nu} \right] d\lambda$$

$\varepsilon \downarrow 0$

would like To have "spectral Decomposition".

$$E_n = \sum_{\alpha \in \sigma(A)} \mathcal{E}_\alpha$$ $\boxed{\mathcal{E}_\alpha^2 = \mathcal{E}_\alpha}$

$\alpha \neq \beta$ $\mathcal{E}_\alpha \mathcal{E}_\beta = 0$

How to find $\mathcal{E}_\alpha$?s

I_n ~ identity matrix

Resolution of the identity

selvating matrix E_n

Hard Part!!

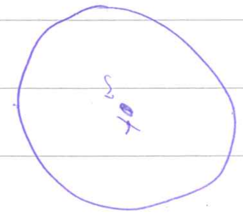
$\alpha \in \sigma(A)$ fixed. ~~Multiplicity~~ $e_\alpha =$ order of Pole for $R(\lambda)$ at α!

<u>Case 1</u> $e_\alpha = 1$ analytic function of λ

(*) $R(\lambda) = \dfrac{T}{\lambda-\alpha} +$ "nice Term"

$|\lambda - \alpha| \ll 1$

α sole pole in



~~Then~~ Claim

$T^2 = T$!!

<u>Reason</u> The function $\lambda \longmapsto R(\lambda)$ is quite special!

<u>First</u> $\lambda \neq \mu$ Distinct in $\mathbb{C} \setminus \sigma(A) \Longrightarrow$

$$R(\lambda) - R(\mu) = (\mu - \lambda)\cdot R(\mu) R(\lambda) \quad !!!$$

Nothing know of T

$$-R(\mu)R(\lambda) = \frac{R(\mu) - R(\lambda)}{\mu - \lambda}$$

Take $\lim_{\mu \to \lambda} \quad -R(\lambda)^2 = R'(\lambda) = \dfrac{dR}{d\lambda}$

Magic
So equation

$$R'(\lambda) = -R(\lambda)^2 \Rightarrow$$

derivative of $(*)$

$$(**) \ -\left(\frac{T}{\lambda-\alpha} + \text{nice}\right)^2 = -\frac{T}{(\lambda-\alpha)^2} + \text{"Nice"}$$

Residue formulas keep

Track of poles!

$\Rightarrow$ Double pole in Left side $(**)$

is

$$-\frac{T^2}{(\lambda-\alpha)^2} \quad \text{Right:} \quad -\frac{T}{(\lambda-\alpha)^2}$$

$$\Rightarrow T = T^2 \quad !!!!$$

~~Eu~~ Swiss, ~~Freanch~~, British "gang", German.

Cramer, 1750; Cauchy 1830; Hamilton-Cayley-Sylvester, $\sim$(1840 - 1870); Frobenius, (1890)

$(*)$ Frobenius has an explicit formulae for $T, E_\alpha, \ldots$

$(*)$ case $e_\alpha > 1$ Extra!!

Camille Jordan (1850)
Decomposition
$e_\alpha > 1$