A Non-Linear PDE-equation

Introduction. In the article *Über eine nichtlineare Randwertaufgabe bei der Gleichung* $\Delta u = 0$ (Mathematisches Zeitschrift vol. 9 (1921), Carleman considered the following equation: Let Ω be a bounded domain in $\mathbf{R}^3$ with C^1-boundary and $\mathbf{R}^+$ the non-negative real line where t is the coordinate. Let $F(t,p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$t \mapsto F(t,p) \tag{0.1}$$

is strictly increasing for every $p \in \partial\Omega$ and that $F(0,p) \geq 0$. Moreover,

$$\lim_{u\to\infty} F(t,p) = +\infty \tag{0.2}$$

holds uniformly with respect to p. For a given point $q_* \in \Omega$ one seeks a function $u(x)$ which is harmonic in $\Omega \setminus \{q_*\}$ and at q_* it is locally $\frac{1}{|x-q_*|}$ plus a harmonic function. Moreover, it is requested that u extends to a continuous function on $\partial\Omega$ and that $u \geq 0$ in $\overline{\Omega}$. Finally, along the boundary the inner normal derivative $\partial u/\partial n$ satisfies the equation

$$\frac{\partial u}{\partial n}(p) = F(u(p),p) \quad : p \in \partial\Omega \tag{*}$$

Remark. The case when $F(t,p) = kt^4$ for some positive constant k means that we regard the Stefan-Boltzmann equation whose physical interpretation ensures that (*) has a unique non-negative solution u.

Theorem. *For each F satisfying (0.1-0.2) the boundary value problem has a unique solution u.*

The strategy in the proof is to consider a family of boundary value problems where one for each $0 \leq h \leq 1$ seeks u_h to satisfy

$$\frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p),p) \quad : p \in \partial\Omega \tag{*}$$

and u_h has the same pole as u above. Let us begin with

0.1 The case $h = 0$.

Here we seek u_0 so that

$$\frac{\partial u_0}{\partial n}(p) = u_0 \tag{i}$$

To solve (i) we let $G(p)$ be the Greens' function with a pole at q_* and determine a harmonic function h in Ω such that

$$u_0 = G - h \tag{ii}$$

Since $G(p) = 0$ on $\partial\Omega$, the equation (i) holds if

$$\frac{\partial h}{\partial n}(p) = h(p) + \frac{\partial G}{\partial n}(p) \quad : \ p \in \partial\Omega \tag{iii}$$

This is a classic linear boundary value problem which has a unique solution h. See § xx for further details.

0.2 Properties of u_0. The construction in (ii) entails that u_0 is superharmonic in Ω and therefore attains its minimum on the boundary. Say that

$$u_0(p_*) = \min_{p\in\overline{\Omega}} u_0(p)$$

It follows that $\frac{\partial u_0}{\partial n}(p_*) \geq 0$ and the equation (i) gives

$$u_0(p_*) \geq 0$$

Hnece our unique solution u_0 is non-negative. We can say more. For consider the harmonic function h in (ii) which takes a maxium at some $p^* \in \partial\Omega$. Then $\frac{\partial h}{\partial n}(p^*) \leq 0$ so that (iii) gives

$$h(p^*) + \frac{\partial G}{\partial n}(p^*) \leq 0$$

Hence

$$\max_{p \in \partial\Omega} h(p) \leq -\frac{\partial G}{\partial n}(p^*)$$

which entails that

$$\min_{p \in \partial\Omega} u(p) = -\max_{p \in \partial\Omega} h(p) \geq \frac{\partial G}{\partial n}(p^*) \tag{0.2.1}$$

Here the function

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is continuous and positive on $\partial\Omega$ and if γ_* is the minimum value we conclude that

$$\min_{p \in \partial\Omega} u(p) \geq \gamma_* \tag{0.2.2}$$

Next, let h attain its minimum at some $p_* \in \partial\Omega$ which entails that $\frac{\partial h}{\partial n}(p_*) \geq 0$ and then (iii) gives

$$h(p_*) + \frac{\partial G}{\partial n}(p^*) \geq 0$$

It follows that

$$\max_{p \in \partial\Omega} u_0(p) = \min_{p \in \partial\Omega} h(p) = -h(p_*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^* \tag{0.2.3}$$

where

$$\gamma^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p) \tag{0.2.4}$$

So the unique solution u_0 in (i) satisfies

$$\gamma_* \leq u(p) \leq \gamma^* \quad : p \in \partial\Omega \tag{0.2.5}$$

where the positive constants γ_* and γ^* depend on the point $q_* \in \Omega$ and the given domain Ω.

The homotopy method. To proceed from $h = 0$ to $h = 1$ the idea is to use a "homotopy argument" which can be pursued thanks to precise estimates of solutions to Neumann's linear boundary value problem which are presented in § B. Thanks to this and some uniquness properties in § A below, the reduction to the case when F is real-analytic is easy, while the crucial steps during the proof appear in § C where a "homotoy procedure" gives solutions in (*) as h increases from zero to one.

A.0. Proof of uniqueness.

Suppose that u_1 and u_2 are two solutions to the equation in the main theorem. Notice that $u_2 - u_1$. is harmonic in Ω. If $u_1 \neq u_2$ we may without loss of generality we may assume that the maximum of $u_2 - u_1$ is > 0. The maximum is attained at some $p_* \in \partial\Omega$ and the strict maximum principle for harmonic functions gives:

$$u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*) \tag{i}$$

for all $x \in \Omega$.With $v = u_2 - u_1$ we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p), p) - F(u_1(p), p)$$

Here (0.1) entails that $\frac{\partial v}{\partial n}(p_*) > 0$ and since we have an inner normal derivative this violates (i) which proves the uniqueness.

A.1 Montonic properties.

Let F_1 and F_2 be two functions which both satisfy (0.1) and (0.2) where

$$F_1(u,p) \leq F_2(u,p)$$

hold for all $(u,p) \in \mathbf{R}^+ \times \partial\Omega$. If u_1, respectively u_2 solve (*) for F_1 and F_2 it follows that $u_2(q) \leq u_1(q)$ for all $q \in \Omega$. To see this we set $v = u_2 - u_1$ which is harmonic in Ω. If $p \in \partial\Omega$ we get

$$\text{(i)} \qquad \frac{\partial v}{\partial n}(p) = F_2(u_2(p),p) - F_1(u_1(p),p) \geq 0$$

Suppose that the maximum of v is > 0 and let the maximum be attained at some point p_*. Since (i) is an inner normal it follows that we must have $0 = \frac{\partial v}{\partial n}(p)$ which would entail that

$$F_2(u_2(p_*)p_*) > F_2(u_1(p_*),p_*) \geq F_1(u_1(p_*),p_*) \implies$$

and this contradicts the strict inequality $u_2(p_*) > u_1(p_*)$ since we have an increasing function in (0.1).

A.2. A bound for the maximum norm. Let G be the Green's function which has a pole at q_* while $G = 0$ on $\partial\Omega$. Then

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is a continuous and positive function on $\partial\Omega$. Set

$$m_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p) \quad : m^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

Next, let $0 \leq h \leq 1$ and suppose that u_h is a solution to (*). Put

$$(*) \qquad m(h) = \min_{p \in \partial\Omega} u_h(p) \quad : M(h) = \max_{p \in \partial\Omega} u_h(p)$$

To estimate these numbers we proceed as follows. Choose $p^* \in \partial\Omega$ such that

$$(1) \qquad u_h(p^*) = M(h)$$

Now the function

$$H = u - G - M(h)$$

is harmonic function in Ω and non-negative on the boundary. Hence H is positive in Ω and since $H(p^*) = 0$ we have

$$\frac{\partial H}{\partial n}(p^*) \leq 0 \implies$$

which via the equation (*) give

$$(2) \qquad (1-h)M(h) + h \cdot F(M(h),p^*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^*$$

Next, the hypothesis on F entails that

$$(3) \qquad t \mapsto (1-h)t + h \cdot F(t,p^*)$$

is a strictly increasing function for each fixed $0 \leq h \leq 1$ and the hypothesis (0.2) together with the inequality (2) above, give a positive constant A^* which is independent of h such that

$$(3) \qquad M(h) \leq A^* \quad : 0 \leq h \leq 1$$

Next, let $m(h)$ be the minimum of u_h on $\partial\Omega$ and this time we consider the harmonic function

$$H = u - m(h) - G$$

Here $H \geq 0$ on $\partial\Omega$ and if $u_h(p_*) = m(h)$ we have $H(p_*) = 0$ p_* is a minimum for H. It follows that

$$\frac{\partial H}{\partial n}(p_*) \geq 0 \implies F(u(p_*),p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$\gamma_* = \min_{p\in\partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

$$F(m(h), p^*) \geq \gamma_* \tag{4}$$

Above γ^* is the constant from (xx) and the properties of F give a positive constant A_* such that

$$m(h) \geq A_*$$

Conclusion. Above $0 < A_* < A^*$ are constants which are independent of h. Hence the maxima and the minima of u_h stay in a fixed interval $[A_*, A^*]$ as soon as u_h exists.

B. The linear equation.

Let $f(p)$ and $W(p)$ be a pair of continuous functions on the boundary $\partial\Omega$ where W is positive, i.e. $W(p) > 0$ for every boundary point. Set

$$w_* = \min_p W(p)$$

So by the assumption on W we have $w_* > 0$. The classical Neumann theorem asserts that there exists a unique function U which is harmonic in Ω, extends to a continuous function on the closed domain and its inner normal derivative satisfies:

$$\partial U/\partial n(p) = W(p) \cdot U(p) + f(p) \quad p \in \partial\Omega \tag{1}$$

For the unique solution in (1) some estimates hold. Namely, set

$$M^* = \max_p U(p) \quad \text{and} \quad m_* = \min_p U(p)$$

Since U is harmonic in Ω the maximum and the minimum are both taken on the boundary. If $U(p^*) = M^*$ for some $p^* \in \partial\Omega$ we have $\partial U/\partial n(p^*) \leq 0$ which together with (1) entails that

$$M^* \cdot W(p^*) + f(p^*) \leq 0 \implies M^* \leq \frac{|f|_{\partial\Omega}}{w_*}$$

where $|f|_{\partial\Omega}$ is the maximum norm of f on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial\Omega}}{w_*}$$

Hence the following inequality holds for the the maximum norm $|U|_{\partial\Omega}$:

$$|U|_{\partial\Omega} \leq \frac{|f|_{\partial\Omega}}{w_*} \tag{B.0}$$

Notoice that (B.0) and the equation (1) entails that Suppose that $W \in C^0(\partial\Omega)$ satisfies

$$w_* \leq W(p) \leq w^*$$

for a pair of positive constants. If $|f|_{\partial\Omega}$ is the maximum norm of f it follows from (B.0) that

$$\big|W(p) \cdot U(p) + f(p)\big| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega}$$

Hence the equation (1) gives

$$\max_{p\in\partial\Omega} |\frac{\partial U}{\partial n}(p)| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega} \tag{B.1}$$

B.2 An estimate for first order derivatives. Let $p \in \partial\Omega$ and denote by N the inner normal at p. Since $\partial\Omega$ is of class C^1 a sufficiently small line segment from p along N stays in Ω. So for small positive ℓ we have points $q = p + \ell \cdot N$ in Ω and take the directional derivative of U along N_p. This gives a function

$$\ell \mapsto \partial U/\partial N(p + \ell \cdot N)$$

Since the boundary is C^1 these functions are defined on a fixed interval $0 \leq \ell \leq \ell^*$ for all boundsry points p. A classic result which appears in *Der zweite Randwertaufgabe* gives a constant B such that

$$\big|\partial U/\partial N(p+\ell \cdot N)\big| \leq B \cdot \max_{p\in\partial\Omega} |\frac{\partial U}{\partial n}(p)|$$

hold for all $p \in \partial\Omega$ and $0 \leq \ell \leq \ell^*$.

C. Proof of Theorem when $t \mapsto F(t,p)$ is analytic.

Assume that $t \mapsto F(t,p)$ is a real-analytic function on the positive real axis for each $p \in \partial\Omega$ where local power series converge uniformly with respect to p. In this situation we shall prove the *existence* of a solution u in the Theorem. To attain this we proceed as follows. To each real number $0 \leq h \leq 1$ we seek a solution u_h where

$$\frac{\partial u_h}{\partial n}(p) = h \cdot F(u_h, p) + (1-h) \cdot u_h(p) \tag{1}$$

When $h = 0$ we found the solution u_0 in § xx. Next, suppose that $0 \leq h_0 < 1$ and that we have found the solution u_{h_0} to (1). By the result in § B there exists a pair of positive constants $A_* < A^*$ such that

$$A_* \leq u_{h_0}(p) \leq A^* \tag{*}$$

which are independent of h_0 and of p.

Set $u_0 = u_{h_0}$ and with $h = h_0 + \alpha$ for some small $\alpha > 0$ we shall find u_h by a series

$$u_h = u_{h_0} + \sum_{\nu=1}^{\infty} \alpha^\nu \cdot u_\nu \tag{2}$$

The pole at q_* occurs already in u_0. So $u_1, u_2, \ldots$ is a sequence of harmonic functions in Ω and there remains to find them so that u_h solves (1). We will show that this can be achieved when α is sufficiently small. Keeping h_0 fixed we set

$$u_0 = u_{h_0}$$

The analyticity of F with respect to t gives for every $p \in \partial\Omega$ a series expansion

$$F(u_0(p) + \alpha, p) = F(u_0(p), p) + \sum_{k=1}^{\infty} c_k(p) \cdot \alpha^\nu \tag{3}$$

where $\{c_k(p)\}$ are continuous functions on $\partial\Omega$. Here (*) and the hypothesis on F entail that the radius of convergence has a uniform bound below, i.e. there exists $\rho > 0$ which is independent of $p \in \partial\Omega$ and a constant K such that

$$\sum_{k=1}^{\infty} |c_k(p)| \cdot \rho^k \leq K \tag{4}$$

Now the equation (1) can be solved via a system of equations where the harmonic functions $\{u_\nu\}$ are determined inductively while α-powers are identified. The linear α-term gives the equation

$$\frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1-h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p) \tag{i}$$

For u_2 we find that

$$\frac{\partial u_2}{\partial n} = (1-h_0)u_2 - u_1 + h_0 c_1(p) u_2 + c_1(p) u_1 + c_2(p) u_1^2 \tag{ii}$$

In general we have

$$\frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \ldots, u_{\nu-1}, p) \quad : \nu \geq 1 \tag{iii}$$

where $\{R_\nu\}$ are polynomials in the preeceding u-functions whose coefficients are continuous functions obtained from the c-functions. The function $c_1(p)$ is given by

$$c_1(p) = \frac{\partial F}{\partial t}(u_0(p), p)$$

which by the hypothesis on F is a positive continuous function on $\partial\Omega$. It follows that the function

$$\text{(iv)} \qquad W(p) = (1 - h_0) + h_0 \cdot c_1(p)$$

also is positive on $\partial\Omega$ and in the recursion above we have

$$\text{(v)} \qquad \frac{\partial u_\nu}{\partial n} = W(p) \cdot u_\nu(p) + R_\nu(u_0, \ldots, u_{\nu-1}, p) \quad : \nu = 1, 2, \ldots$$

Above we encounter linear equations exactly as in (B.0) where the the f-functions are the R-polynomials. Put

$$w_* = \min_{p \in \partial\Omega} W(p)$$

From § B.XX we get

$$\text{(vi)} \qquad |u_\nu|_{\partial\Omega} \leq w_*^{-1} \cdot |R_\nu(u_0, \ldots, u_{\nu-1}, p)|_{\partial\Omega}$$

Finally, (vi) and a majorising positive series expressing maximum norms imply that if α is sufficiently small then the series (2) converges and gives the requested solution for (1). Moreover, α can be taken *independently* of h_0. Together with the established uniqueness of solutions u_h whenever they exist, it follows that we can move from $h = 0$ until $h = 1$ and arrive at the requested solution in Theorem 1.

Remark. The reader may consult page 106 in [Carleman] where the existence of a uniform constant $\alpha > 0$ for which the series (2) converge for every h is demonstrated by an explicit majorant series.