II. Null solutions of PDE:s with constant coefficients.

Introduction. We expose material from the article *Null solutions to partial differential operators* [Arkiv för matematik. 1959] by Lars Hörmander. The proof of the Main Theorem below is instructive proof using complex line integrals taken over contours adapted to the complex zeros of the polynomial $P(\zeta)$ of n independent complex variables which corresponds to a PDE-operator $P(D)$ in order to obtain an ample family of null solutions, i.e. functions $u(x)$ for which $P(D)u = 0$. Hörmander employs Puiseux series constructed via embedded curves in the zeros of $P(\zeta)$ to get such u-functions supported by the half-space $\{x_n \geq 0\}$ in the case when the hyperplane $\{x_n = 0\}$ is characterstic to the differential operator $P(D)$. The remaining part of the proof of the Main Theorem is based upon the Paley-Wiener theorem and duality results from general distribution theory. Here one crucial point appears. Namely, thanks to constructions due to Gevrey, there exists test-functions whose higher order derivatives have a good control which entail that their Fourier transforms enjoy certain decay conditions. So the subsequent material offers an instructive mixture of algebra and analysis.

One ingredient in the proof of the Main Theorem employs a density result which goes back to Pusieeux which is a sharp version of the standard Nullstellen Satz. Namely. for every algebraic hypersurface $S = \{P(\zeta) = 0\}$ in the n-dimensional complex ζ-space there exists an ample family of curves of two independent complex contained in S with the property that if $g(\zeta)$ is an entire function which vanishes on all these curves then is is identically zero on S. This entails that

$$g(\zeta) = P(\zeta)h(\zeta)$$

for another entire function h. Moreover, when g is the Fourier-Laplace transform of a distribution μ with compact support, the Paley-Wiener theorem entails that $h = \widehat{\gamma}$ for another distribution whose compact support is contained in the convex hull of μ which is used during the final step in the proof of the Main Theorem.

Before we announce our Main Theorem we need some notations. Let $n \geq 2$ and in $\mathbf{R}^n$ we consider the hyperplane $H = \{x_n = 0\}$. Let $P(D)$ be a differential operator with constant coefficients. Here $D_k = -i \cdot \partial/\partial x_k$ and by Fourier's inversion formula

$$P(D)f(x) = (2\pi)^{-n} \cdot \int e^{i\langle x,\xi\rangle}\, \widehat{f}(\xi)\, d\xi$$

for test-functions $f(x)$. Let m be the order of $P(D)$ which means that

$$P(D) = \sum c_\alpha \cdot D^\alpha$$

where the sum is taken over multi-indices α for which $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq m$. The leading form is defined by

$$P_m(D) = \sum_{|\alpha|=m} c_\alpha \cdot D^\alpha$$

The hyperplane H is characteristic if $P_m(N) = 0$ where $N = (0, \ldots, 1)$, i.e. the term D_n^m does not appear in $P_m(D)$ with a non-zero coefficient. Put

$$H_+ = \{x_n > 0\} \quad \& \quad \mathcal{N}_+ = \{g \in C^\infty(H_+) : \, P(D)(g) = 0$$

Thus, we consider C^∞-functions in the open half-plane H_+ which are null solutions to $P(D)$ in this open half-plane. A smaller space is given by

$$\mathcal{N}_* = \{g \in C^\infty(\mathbf{R}^n) :\ P(D)(g) = 0 \text{ and } \mathrm{Supp}(g) \subset \overline{H_+}\}$$

Denote by $\mathcal{N}_*^\perp$ the family of distributions μ with compact support in H_+ which are zero on $\mathcal{N}_*$.

Main Theorem. *Every distribution μ in $\mathcal{N}_*^\perp$ is zero on $\mathcal{N}_+$*

The proof requires several steps. The crucial step is to construct a sufficiently ample family of functions in $\mathcal{N}_*$ and after prove that this family is a dense subspace of $\mathcal{N}_+$. So we begin with:

1. A construction of null solutions.

Let ξ_0 be a real n-vector such that $P_m(\xi_0) \neq 0$ and ζ_0 some complex n-vector. Let s and t be independent complex variables and set

$$p(s,t) = P(s \cdot N + t\xi_0 + \zeta_0)$$

This gives a polynomial where the term t^m appears since $P_m(\xi_0) \neq 0$. At the same time s^m does not appear because $P_m(N) =$ is assumed. A classic result due to Pusieux from 1852 shows that there exists a positive integer p and a series

$$t(s) = s^{k/p} \cdot \sum_{j=0}^{\infty} c_j \cdot s^{-j/p} \tag{1.1}$$

where $0 \leq k < p$ which converges when $|s|$ is large, i.e. there exists some $M > 0$ such that

$$\sum_{j=0}^{\infty} |c_j| \cdot M^{-j/p} < \infty$$

Moreover,

$$P(s \cdot N + t(s)\xi_0 + \zeta_0) = 0 \quad : \ |s| \geq M \tag{1.2}$$

In the lower half-plane $\mathfrak{Im}(s) < 0$ we choose a single valued branch of $s^{1/p}$ where

$$s = |s| \cdot e^{i\phi} \implies s^{1/p} = |s|^{1/p} \cdot e^{i\phi/p} \quad : \quad -\pi < \phi < 0$$

Next, choose a number

$$1 - 1/p < \rho < 1$$

Now $(is)^\rho$ has a single valued branch for which

$$\mathfrak{Re}((is)^\rho) = \cos\frac{\rho\pi}{2} \cdot |s|^\rho \cdot \cos(\rho \cdot (\pi/2 + \phi)) \tag{1.3}$$

So if $\epsilon > 0$ we have

$$|e^{-\epsilon(is)^\rho}| = e^{-\epsilon \cdot \mathfrak{Re}(is)^\rho} = e^{-\epsilon \cdot |s|^\rho \cdot \cos(\rho(\pi/2+\phi))} \tag{1.4}$$

Since $\rho < 1$ we notice that

$$\cos(\rho(\pi/2 + \phi)) \geq \cos \rho\pi/2 = a$$

for all $-\pi < \phi < 0$ where a is positive constant a. It follows that

$$|e^{-\epsilon(is)^\rho}| \leq e^{-a\epsilon \cdot |s|^\rho} \tag{1.5}$$

for all s in the lower half-plane, and also when s is real.

Let M be as in (1.2) and denote by C_* the circle in the lower half-pane which consists of the two real intervals $(-\infty, -M)$ and $(M, +\infty)$ and the lower half-circle where $|s| = M$. For each $x \in \mathbf{R}^n$ and every non-negative integer ν we get the complex line integral

$$(*) \qquad \int_{C_*} e^{i\langle x, sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

This integral is absolutely convergent. Namely, during the integration on the real interval $(-\infty, -M)$ or the real interval $[M, +\infty)$ we see that (1.5) gives estimates the absolute value of the integrand by

$$(1.6) \qquad |s|^{\nu/p} \cdot \cdot |e^{it(s)\langle x,\xi_0\rangle}| \cdot e^{-a\epsilon\cdot|s|^\rho}$$

Next, the Puiseux expansion for $t(s)$ entails that

$$|t(s) \leq A|s|^{1-1/p}$$

hold for some constant A. Since $\rho > 1 - 1/p$ It follows that (1.6) is majorised by

$$(1.6) \qquad |s|^{\nu/p} \cdot e^{A\cdot|\langle x,\xi_0\rangle|\cdot|s|^{1-1/p}} \cdot e^{-a\epsilon\cdot|s|^\rho}$$

Since $\rho > 1 - 1/p$ we conclude that the line integral (*) converges absolutely for each positive integers ν.

Exercise. Show by Cauchy's theorem in analytic function theory that the line integral (*) does not depend on M as soon as it has been chosen so that the Puiseux series definiing $t(s)$ exists. The resulting value of (*) is therefore a function of x and ϵ and gives a function $u_\epsilon(x)$ defined for all x in $\mathbf{R}^n$. Moreover, the reader should check that when $\epsilon > 0$ kept fixed this yields a C^∞-function of x. In particular

$$(**) \quad P(D)(u_\epsilon)(x) = \int_{C_*} P(sN + t(s)\xi_0 + \zeta_0) \cdot e^{i\langle x, sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

Since $P(sN + t(s)\xi_0 + \zeta_0) = 0$ when $|s| \geq M$ we conclude that $P(D)(u_\epsilon) = 0$, i.e. u_ϵ is a null solution.

The inclusion Supp$(u) \subset \overline{H}_+$. In (*) we perform a line integral whose integrand is an analytic function in the lower half-plane. Using Cauchy's theorem the reader can check that for any $M^* > M$ we have

$$(**) \qquad u_\epsilon(x) = \int_{\mathfrak{Im}(s)=-M^*} e^{i\langle x, sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

With $s = t - iM^*$ we have

$$|e^{i\langle x,sN\rangle}| = e^{M^*\langle x,N\rangle}$$

If $\langle x, N\rangle < 0$ this decreases exponentially to zero as $M^* \to +\infty$ and then the reader can check that the limit of (**) as $M^* \to +\infty$ is zero. This proves that the null solution u_ϵ is supported by the half-plane $\overline{H}_+$ and hence belongs to $\mathcal{N}_*$.

§ 2. A study of $\mathcal{N}_*^\perp$.

Consider a test-function ϕ with a compact support in H_+ such that $\phi(\mathcal{N}_*) = 0$. It gives the entire function in the n-dimensional complex ζ-space:

$$\Phi(\zeta) = \int e^{i\langle x,\zeta\rangle}\,\phi(x)\,dx \tag{2.0}$$

Using the convergence of the line integrals in (*) from §§ 1 the reader should verify that Fubini's theorem gives the equation

$$\int u_\epsilon(x)\phi(x)\,dx = \int_{C_*} \Phi(sN + t(s)\xi_0 + \zeta_0)\cdot s^{\nu/p}\cdot e^{-\epsilon(is)^\rho}\,ds \tag{2.1}$$

Since $\phi(\mathcal{N}_*) = 0$ is assumed it follows that the last integral is zero for all non-negative integers ν and each $\epsilon > 0$.

2.2 Another vanishing integral. In the upper half-plane $\mathfrak{Im}(s) > 0$ we can choose single-valued branches of $s^{1/p}$ and $(-is)^\rho$, where the last branch is chosen so that the value is $a^\rho > 0$ when $s = ai$ for $a > 0$. Now we construct the contour C^* given by the real intervals $(\infty, -M)$ and $(M, +\infty)$ together with the upper half circle of radius M, which for each non-negative integer ν gives the function

$$v_\epsilon(x) = \int_{C^*} e^{i\langle x, sN + t(s)\xi_0 + \zeta_0\rangle}\cdot s^{\nu/p}\cdot e^{-\epsilon(-is)^\rho}\,ds \tag{*}$$

Exactly as in § 1 one verifies that this gives a C^∞-function of x supported by the right half space $\{x_n \leq 0\}$. Since ϕ has compact support in H_+ it follows that

$$0 = \int v_\epsilon(x)\phi(x)\,dx = \int_{C^*} \Phi(sN + t(s)\xi_0 + \zeta_0)\cdot s^{\nu/p}\cdot e^{-\epsilon(-is)^\rho}\,ds \tag{2.2.1}$$

2.3 The limit as $\epsilon \to 0$. In (2.2.1) we have vanshing integrals for each $\epsilon > 0$. If the test-function $\phi(x)$ belongs to a suitable Gevrey class with more regularity than an arbitrary test-function, then the entire function $\Phi(\zeta)$ enjoys a decay condition which enable us to pass to the limit as $\epsilon \to 0$ in (2.2.1). To find a sufficient decay condition we set $\zeta = \xi + i\eta$, and with M kept fixed we study the function

$$s \mapsto \Phi(sN + t(s)\xi_0 + \zeta_0)$$

We already know that there is a constant C such that $|t(s)| \leq C|s|^{1-1/p}$ when $|s| \geq M$. Since ξ_0 and ζ_0 are fixed this gives a constant C_1 such that

$$|\mathfrak{Im}(sN + t(s)\xi_0 + \zeta_0)| \leq C_1(1+|s|)^{1-1/p} \tag{2.3.1}$$

At the same time we have the unit vector N and get a positive constant C_2 such that

$$|\mathfrak{Re}(sN + t(s)\xi_0 + \zeta_0)| \geq C_1(1+|s|) \tag{2.3.2}$$

when $|s|$ is large. Suppose now that the test-function ϕ has been chosen so that

$$|\Phi(\xi + i\eta) \leq C\cdot e^{A|\eta|) - B|\xi|^b} \tag{2.3.3}$$

hold for some constants C, A, B, a where $b < 1$. From (2.3.1-2.3.2) this gives with other positive constants

$$|\Phi(sN + t(s)\xi_0 + \zeta_0) \leq C_1 e^{A_1|s|^{1-1/p} - B_1|s|^b} \tag{2.3.4}$$

With ρ chosen as in § 1 where the equality (1.3) is used, it follows that as sson as

$$a > \rho$$

then we get absolutely convergent integrals

$$\int_{|s|\geq M} |\Phi(sN + t(s)\xi_0 + \zeta_0) \cdot |s|^w| \, ds < \infty$$

for every positive integer w. This enable us to pass to the limit in (2.2) and conclude that

$$\int_{C^*} \Phi(sN + t(s)\xi_0 + \zeta_0) \cdot s^{\nu/p} \, ds = 0 \tag{2.3.5}$$

for every non-negative integer ν. In the same fashion we find vanishing integrals with C^* replaced by C_*. The vanishing of these integrals for all $\nu \geq 0$ entails by the classic result due to Puiseux that that $\frac{\Phi}{P}$ is an entire function. Then a division theorem with bounds due to Lindelöf, together with the Paley-Wiener theorem imply that the entire quotient

$$\frac{\Phi}{P} = \Psi \tag{i}$$

where Ψ is given as in (2.0) for some test-function ψ supported by the convex hull of the support of ϕ. Moreover, (i) entails that

$$P(-D)(\psi) = \phi$$

and then it is obvious that ϕ annihilates $\mathcal{N}_+$. Hence we have proved the implication in the Main Theorem for distributions which are defined by test-functions ϕ whose associated entire Φ-function satisfies (2.3.3) with some a such that

$$1 - 1/p < a < 1$$

But this finishes the proof of the Main Theorem. Namely, fix a as above and put

$$\delta = 1/a$$

Now $\delta > 1$ which by a classic construction due to Gevrey enable us to construct an ample family of test-functions ϕ for which (2.3.3) hold and at the same time this family is weak-star dense in the space of distributions with compact support in H_+ which gives the Main Theorem. For details about this density the reader can consult Hörmander's article or his text-book [Hö:xx] if necessary. See also the article [Bj] by Göran Björck which offers a very detailed study of distributions arising from Gevrey classes.