Compact Riemann surfaces

Contents

Introduction.

Before we present a more detailed account of the contents in this chapter we summarize some of the main results and describe some tools which will be used in the proofs. Geometry, algebra and analysis appear in a fruitful blend. during the study of Riemann surfaces which therefore is an instructive subject since it teaches the power when methods and results from different subjects are combined. The reader may turn directly to the section entitled *Algebraic function fields* and after to the section devoted to Compact Riemann surfaces, while the material in this introduction can be consulted later. But the more experienced reader may prefer to study this expositary material directly. An appendix covers general background including shef theory and the construction of cohomology of sheaves.

Algebraic function fields. In § XX we start from an irreducible polynomial $P(x, y)$ of two variables and construct its associated compact Riemann surface X which entails that the field $\mathfrak{M}(X)$ of meromorphic functions on X is isomorphic to the algebraic function field given by the quotient field of the integral domain $\frac{\mathbf{C}[x,y]}{(P)}$ where (P) denotes the principal ideal generated by P in the unique factorisation domain $\mathbf{C}[x, y]$. Conversely, a result due to Weyl shows that every "abstract" compact and 1-dimensional complex manifold is biholomorphic with an algebraically defined Riemann surface as above. Hence there is a 1-1 correspondence between isomorphism classes of algebraic function fields in one variable and compact Riemann surfaces, identified up to biholomorphic mappings. A result from combinatorical topology will be taken for granted, namely that every oriented and compact 2-dimensional manifold M is homeomorphic to a sphere with a finite number of handles attached. This number is called the genus of X and denoted by $g(X)$. The case $g(X) = 0$ means that X is homeomorphic to the unit sphere which is chosen as the Riemann sphere, or equivalently identified with the projective line $\mathbf{P}^1$. The integer $g(X)$ determines X as a topological space. However, a pair of Riemann surfaces X and Y with equal genus are in general not isomorphic in the sense above. An exception is the case $g = 0$, i.e. every compact Riemann surface X for which $g(X) = 0$ is isomorphic to $\mathbf{P}^1$ in the strong sense which means that $\mathfrak{M}(X)$ is isomorphic to the field of rational functions on one variable over $\mathbf{C}$. When $g \geq 1$ the family of all compact Riemann surfaces X for which $g(X) = g$ turns out to be quite an extensive family. Riemann proved that the space $\mathcal{M}_g$ of all compact Riemann surfaces with presribed genus $g \geq 2$ is parametrized by $3g - 3$ many independent complex variables. We shall not discuss the classification problem in detail since this would require a far more extensive account.

A first example

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Let a be a complex number which differs from zero and 1 and consider the algebraic equation

$$y^2 = x(x-1)(x-a) \tag{i}$$

Here $P = y^2 - x(x-1)(x-a)$ is an irreducible polynomial in $\mathbf{C}[x, y]$ which gives the integral domain $A = \frac{\mathbf{C}[x,y]}{(P)}$ where (P) is the principal ideal generated by P. The quotient field K of A is an example of an algebraic function field. It contains the rational field $\mathbf{C}(x)$ and K is an algebraic exension of degree two where (i) gives the equation when y is adjoined to thid field. A geometric object attched to the algebraic equation (i) arises when we consider the curve S in the 2-dimensional complex (x, y)-space defined by the zeros of $P(x, y)$. Notice that the gradient vector $\nabla(P) = (P'_x, P'_y)$ is non-zero at all points in S which means that S is non-singular, i.e. it appears as a 1-dimensional compl,ex submanifold of $\mathbf{C}^2$. Next, $\mathbf{C}^2$ appears as an open subset of the projective space $\mathbf{P}^2$ whose homogeneuos coordinates are (z, x, y) and points $(x, y) \in \mathbf{C}^2$ are represented by $(1, x, y)$. When $(x, y)| \in S$ with $|x|$ large we see that $|y|^2 \simeq |x|^3$ which entails that the closure of S in the compact metric space $\mathbf{P}^2$ contains one extra point $p^* = (0, 0, 1)$. At this point one has local coordinates (ξ, η) and points in $\mathbf{C}^2$ close to p^* are of the form

$$(1, x, y) = (y^{-1}, xy^{-1}, 1) = (\xi, \eta, 1)$$

The result is that $S \cup \{p^*\}$ is a non-singular projective curve which inherits a complex analytic structure from $\mathbf{P}^2$ and hence becomes a compact complex manifold of dimension one. It is called the associated Riemann surface of the equation (1) and let us denote it by X. Now there exists the field $\mathfrak{M}(X)$ whose elements are meromorphic functions on X. One verifies that it is isomorphic to the previous algebraic function field. Here x and y are both identified with meromorphic functions on X. We shall learn how local charts on the manifold X are adapted to the given equation and conclude that the meromorphic function y has a triple pole at p^* and simple zeros at the points in $(0,0),(1,0),(b,0)$ in S. Similarly we shall learn that x has a double pole at p^* and a double zero at $(0,0)$ and otherwise holomorphic and zero-free. This gives a globally defined holomiorphic 1-form on X given by

$$\omega = \frac{dx}{y}$$

One refer to ω as an abelian differential. Its existence implies that X as a topological space is quite special, i.e. it is homeomorphic to a torus T^2 which arises as the quotient space

$$\frac{\mathbf{C}}{\Gamma}$$

where Γ is a lattice generated by a pair of $\mathbf{R}$-linearly independent vectors in the complex plane.

The moduli problem. The compact Riemann surface was constructed via (1) and is therefore determined by a. So let us denote it by X_a. If b is another complex number which is $\neq \{0,1\}$ we get the compact Riemann surface X_b starting with (1) where b has replaced a. As topological spaces X_a and X_b are homeomorphic but this need not imply that they are biholomorphic as compact complex manifolds. So one is led to analyze for which pairs a,b one has $X_a \simeq X_b$ where $\simeq$ indicates that the complex manifolds are biholomorphic. Here one encounters a crucial problem in the study of Riemann surfaces. More generally one can consider equations of the form

$$y^2 = (x-a_1)\cdots(x-a_k)$$

where $k \geq 4$ and $\{a_\nu\}$ are different complex numbers. We get a non-singular projective curve X by the same procedure as above. Here y has simple zeros at the a-points and a pole of order k at p^*. Here the topological space X need not be a torus. But it is homeomorphic to a compact manifold with g handles attached for some integer $g \geq 2$ which are determined via k as explained in § xx below. Here $X = X(a_1,\dots,a_k)$ depends on the k-tuple and when we consider the family of k-tuples of distinct numbers in $\mathbf{C}^k$ and one encounters a moduli problem which was investigated by Weierstrass after Riemann had introduced the general notion of Reimann surfaces. In these notes we shall not try to expose this in detail since this would require another extensive chapter. Let us only remark that general moduli problems for compact Rieann surfaces is a very rich subject which is an active field of reserach since many delicate problems remain. But the special case $k=3$ will be treated in detail in these notes and may be considered as the first basic lesson about Riemann surfaces.

An open problem.

The subsequent material deals foremost with results which are classic and established more than a century ago. But many problems which at a first sight appear to be quite modest remain open. Let us give an example where one seeks solutions to a certain variational problem. Let a,b,c,d be four distinct complex numbers and consider the algebraic equation

$$y^4 = \frac{1}{(x-a)(x-b)(x-c)(x-d)} \tag{1}$$

Passing to the fourth root we set

$$y^*(x) = \frac{1}{x} \cdot [(1-a/x)(1-b/x)(1-c/x)(1-d/x)]^{-\frac{1}{4}} \tag{2}$$

This yields a single valued analytic function defined in the exterior disc $\{|x| > R$ where R is chosen so that the four numbers a,b,c,d have absolute value $\leq R$. When $|x| \to +\infty$ the function

$y^*(x)$ is $\simeq x^{-1}$. Denote by $\mathcal{J}$ the family of closed Jordan curves which border Jordan domains containing a, b, c, d. Residue calculus gives for each such Jordan curve a unique Riesz measure γ supported by Γ such that

$$y^*(z) = \int_\Gamma \frac{d\gamma(\zeta)}{z-\zeta}$$

From (2) it follows that the mean-value

$$\int_\Gamma d\gamma(\zeta) = 1$$

Hence the total variation of the Riesz measure γ is ≥ 1 and we set

$$\mathcal{E}(\Gamma) = \int_\Gamma |d\gamma(\zeta)| \tag{3}$$

Question. Is it true that

$$\min_{\Gamma \in \mathcal{J}} \mathcal{E}(\Gamma) = 1 \tag{1}$$

Moreover, if (1) holds we ask if there exists a unique compact set Γ_* with the following properties: It consists of a finite union union of simple and real-analytic Jordan arcs $\{\rho_\nu\}$ which in general need not be disjoint, but each pair intersect at most in one point given as a common boundary point of the two arcs. Moreover, there exists a probability measure μ_* supported by Γ_* such that $y^*(z)$ equals the Cauchy transform

$$\int_{\Gamma_*} \frac{d\mu_*(\zeta)}{z-\zeta} \quad : z \in \mathbf{P}^1 \setminus \Gamma_*$$

Remark. If Γ_* exists the construction of μ_* means that the compact Rieman surface associated to the given algebraic equation (1) can be equipped with a special triangulation. See § xx for further comments. The question above is closely related to problems about asymptotic distributions of zeros to eigenpolynomials of ordinary differential equations. The case above where $a < b < c < d$ are real was treated in [B-S]. Here the question was settled and it was proved that Γ_* is the disjoint union of the intervals $[a, b]$ and $[c, d]$. Moreover the measure μ_* was determined in this case. But when a, b, c, d do not belong to a common line the existence and uniqueness of Γ_* remains open. Similar questions as above can of course be posed for a more extensive class of Riemann surfaces attched to algebraic equations.

Another variational problem. It is tempting to try to extend results in analytic function theory on the complex plane to Riemann surfaces. Here is an example: Let X be a compact Riemann surface and σ a metric whose curvatute is everywhere strictly negative and Φ is a positive continuous function on X such that $\log \frac{1}{\Phi}$ is subharmonic. Let D be the unit disc and $f \colon D \to X$ a bijective analytic function whose image $f(D)$ is a Jordan domain in X. If $z \in D$ and dz is small we get the points $f(z)$ and $f(z + dz)$ in X and can measure their σ-distance. Put

$$\rho_f(z) = \lim_{dz \to 0} \frac{\sigma(f(z+dz), f(z))}{|dz|}$$

When one deals with planar domains so that $w = f(z)$ and employ the euclidian metric on the complex w-plane then $\rho_f(z)$ is the absolute value of the complex derivative $f'(z)$. Inspired by Beurling's mapping theorem from § XX one is led to study pairs (Φ, σ) and seek bianalytic mappings f as above such that

$$\lim_{r \to 1} \max_{0 \leq \theta \leq 2\pi} |\Phi(f(re^{i\theta}) - \rho_f(re^{i\theta})| = 0$$

Recall from § xx that the eventual existence of f for a given pair Φ, σ relies upon a sophisticated problem in the calculus of variation. Here we shall not try to discuss this problem in detail but mention that special cases are related to free boundary value problems in connection with the Helmholtz equation. See § xx.

0.1 Non-compact Riemann surfaces.

Let X be a connected and 1-dimensional complex manifold which means that X is covered by some family of charts $\{U_\alpha, z_\alpha\}$ where every U_α is a copy of the open unit disc in the complex z_α-plane and the glueing functions in non-empty intersections $U_\alpha \cap U_\beta$ are biholomorphic mappings. In addition we assume that X is not compact and refer to X as an open Riemann asurface. A theorem due to Rado asserts that every open Riemann surface X has countable topology. This means that X can be covered by denumerable family of charts which can be chosen to be locally finite. In particular the topology on X can be defined by a metric and is also paracompact. Moreover, by solving the Dirichlet problem in subdomains one proves that X can be exhausted by an increasing sequence of relatively compact Runge domains, i.e. relatively compact open subsets Y such that every connected component in the closed complement $X \setminus Y$ is non-compact. Next, denote by $\mathcal{O}$ the sheaf of holomorphic functions on X. A result due to Behnke and Sstein asserts that the cohomology group $H^1(X, \mathcal{O})$ is zero. It is proved in the special section devoted to open Riemann surfaces. Next, let $\mathcal{O}^*$ be the multiplicative sheaf whose sections are zero-free holomorphic functions. In § XX we also prove that $H^1(X, \mathcal{O}^*) = 0$ which means that every holomorphic line bundle is holomorphically trivial. An example of a line bundle is the sheaf Ω of holomorphic 1-forms. Its triviality entails that there exists a global section ω which is everywhere $\neq 0$. This gives a "planar property" of a non-compact Riemann surface since the holomorphic 1-form ω serves like dz for open domains in $\mathbf{C}$.

The vanishing results above have several consequences. To begin with there exists an ample family of globally defined meromorphic functions. More preciely, let $\{p_k\}$ and $\{q_j\}$ be a pair of discrete subsets in X, i.e. every relatively compact set in X contains at most a finite set of p-points or q-points. Then, if $\{m_k\}$ and $\{n_j\}$ is an arbitrary pair of sequences of positive integers there exists $f \in \mathfrak{M}(X)$ with poles of order m_k at p_k for every k while f is holomorphic and zero free in $X \setminus \{p_k\} \cup \{q_j\}$ and has a zero of order n_j at every q_j. We prove this result in § XX. Next, let Ω be the sheaf of holomorphic 1-forms. In local charts every analytic function has a primitive which gives an exact sequence of sheaves

$$0 \to \mathbf{C}_X \to \mathcal{O} \xrightarrow{\partial} \Omega \to 0$$

where $\mathbf{C}_X$ is the sheaf whose sections are locally constant complex-valued functions. The vanishing of $H^1(X, \mathcal{O})$ entails that one has an isomorphism

$$(*) \qquad H^1(X, \mathbf{C}_X) \simeq \frac{\Omega(X)}{\partial(\mathcal{O}(X))}$$

A wellknown fact in topology asserts that $H^1(X, \mathbf{C}_X)$ is equal to the the complex vector space $G(X)$ of group homomorphisms from the fundamental group $\pi_1(X)$ into the additive group of complex numbers. This leads to an interpretation of (*). Namely, choose some $x_* \in X$ and identify $\pi_1(X)$ with homotopy classes of differentiable and closed curves γ which have a common start- and endpoint at x_*. Since a 1-form $\omega \in \Omega(X)$ is closed elementary calculus implies that the line integrals

$$\gamma \mapsto \int_\gamma \omega$$

only depend upon the homotopy class of γ and hence ω determines an element $G(\omega) \in G(X)$. If $G(\omega) = 0$ we get a single-valued holomorphic function f on X defined by

$$f(x) = \int_{x_*}^x \omega$$

where the line integral is taken over some curve from x_* to x and $\omega = \partial(f)$. On the other hand, (*) entails that if a holomorphic 1-form ω is not ∂-exact then the function f above is multi-valued. Its multi-valued behaviour reflects the image $G(\omega)$.

A result on compact Riemann surfaces. Let X be a compact Riemann surfacesd and fix some point $p \in X$ and a local chart Δ centered at p where z is a local coordinate. Now $X \setminus \{p\}$

is an open Riemann surface and hence $H^1(X \setminus \{p\}, \mathcal{O}) = 0$. In the puntured disc $\Delta \setminus \{z=0\}$ we consider an analytic function

(i)
$$\phi(z) = \sum_{\nu=1}^{\infty} c_k z^{-k}$$

where $\{c_k\}$ are such that $\sum |c_k| \cdot \delta^k < \infty$ for some $\delta > 0$. Thus, we have taken a negative Laurent series. The Behnke-Stein theoren entails that $H^1(X \setminus \{p\}, \mathcal{O}) = 0$. From this a trvial sheaf-theoretic result entails that there exists an injective map from the space of all negative Laurent series (i) to the complex vector space $H^1(X, \mathcal{O}) = 0$. As we shall see later on the last cohomology group is finite dimensional. Moreover, a function ϕ is sent into zero in $H^1(X, \mathcal{O}) = 0$ if and only if there exists $\Phi \in \mathcal{O}(X \setminus \{p\}$ such that the restriction of $\Phi - \phi$ to the punctured disc extend to a holomorphic function in Δ. In the special case when the negative Luarent series in (i) is finite it follows that Φ is a globally defined meromorphic function on X with a single pole at p whose negative series expansion is given via (i). In this way one gets a quite ample familly of meromorphic functions on the compact Riemann surface. Precise results will be established later on where ther material about Weiertrass points in § xx is an example.

The vanishing of $H^2(X, \mathbf{Z}_X)$. Let X be an open Riemann surface. Since $H^1(X, \mathcal{O}^*) = 0$ it follows that the cohomology group $H^2(X, \mathbf{Z}_X) = 0$. To see this we regard the exact sequence of sheaves

$$0 \to \mathbf{Z}_X \to \mathcal{O} \overset{exp}{\to} \mathcal{O}^* \to 0$$

where the exponential map sends each $f \in \mathcal{O}$ to $e^{2\pi i f}$. The long exact sequence for sheaf cohomology entails that $H^2(X, \mathbf{Z}_X) = 0$. We remark that this vanishig no longer holds when X is a compact Riemanbn surfce, i.e. the reader should be aware of the different topological properites in the compact versus the non-compact case.

0.2 The uniformisation theorem and Fuchsian groups.

In § XX we prove that if X is a non-compact Riemann surface whose fundamental group $\pi_1(X)$ is zero then X is biholomorphic to the open unit disc D or the complex plane $\mathbf{C}$. If X is a compact Riemann surface it has a fundamental group $\pi_1(X)$ and there exists the universal covering space $\widehat{X}$ which yields another Riemann surface. One shows easily that it is non-compact unless $X = \mathbf{P}^1$. Ignoring this case it follows that $\widehat{X}$ is either D or $\mathbf{C}$. We shall learn that $\widehat{X} \simeq D$ when the genus of X is ≥ 2. Here the holomorphic map $\rho\colon \widehat{X} \to X$ is surjective and locally biholomorphic. Moreover, every fiber $\rho^{-1}(x)$ can be identified with $\pi_1(X)$ which leads to constructions of Fuchsian groups. Namely, fix a point $x_* \in X$ and identify $\pi_1(X)$ with homotopy classes of closed curves at x_*. Now $\widehat{X} = D$ and suppose that x_* is chosen so that $\rho(0) = x_*$, i.e, the origin in D is mapped to x_*. Let $s \mapsto \gamma(s)$ be a closed parametrized curve which starts and ends at x_*. It has a unique lifting $\widehat{\gamma}$ to the open unit disc D where

$$\rho(\widehat{\gamma}(s) = \gamma(s) \quad : 0 \leq s \leq 1$$

and $\widehat{\gamma}(s) = 0$. The end-point $\widehat{\gamma}(1)$ depends only upon the homotopy class of γ and when γ represent different elements $\pi_1(X)$ these end-points change. In this way the set of end-points $\{\widehat{\gamma}(1)\}$ identifies $\pi_1(X)$ with a discrete subset of D. If $a = \widehat{\gamma}(1)$ for some γ one can associate the Möbius transform

$$M_a(z) = \frac{z+a}{1+\bar{a}z}$$

In this way $\pi_1(X)$ is identified with a group of Möbius transforms and $\mathfrak{M}(X)$ with automorphic functions, i.e. meromorphic functions in D which are invariant under the group of Möbius transforms above. In these notes we shall not enter a detailed discussion sbout Fuchsian groups and automorphic functions which is an extensive subject. It has the merit that various compact Riemann surfaces arise via group theoretic constructions.

0.3 Tools for the study of Riemann surfaces.

Sheaf theory is used without hesitation where relevant background is provided in the appendix. Now follows some other basic constructions.

0.3.1 Puiseux expansions. The local ring $\mathbf{C}\{z\}$ of convergent power series is identified with the local ring of germs of analytic functions at the origin. Let w be a new complex variable and consider a polynomial

$$Q(z,w)=w^n+q_{n-1}(z)w^{n-1}+\ldots+q_1(z)w+q_0(z)$$

Here $n\geq 2$ and we assume that this w-polynomial is irreducible in the unique factorisation domain $\mathbf{C}\{z\}[w]$ and that $q_\nu(0)=0$ for every ν. Under these conditions Puiseux proved that with a new complex variable ζ there exists a unique holomorphic function $A(\zeta)$ defined in some disc $\{|\zeta|<r\}$ such that

$$Q(\zeta^n,A(\zeta))=A(\zeta)^n+q_{n-1}(\zeta^n)A(\zeta)^{n-1}+\ldots+q_0(\zeta^n)=0$$

This is proved in § XX and will be used to construct the associated compact Riemann surface to an irreducible polynomial $P(x,y)$.

0.3.2 Differential forms. Analysis on a general Riemann surface X - compact or not - is rather similar to the planar case. The complex analytic structure implies that X is oriented and hence there exist area integrals of test-forms of degree 2 and we often employ its underlying structre as a C^∞-manifold of dimension two. Denote by $\mathcal{E}(X)$ the space of complex-valued C^∞-functions on X and $\mathcal{E}^1(X)$ is the space of differential 1-forms wiith smooth coefficients. There exists the exterior differential map:

$$d\colon \mathcal{E}(X)\to\mathcal{E}^1(X)$$

whose image consists of exact 1-forms. The complex analytic structure on X yields a direct sum decomposition

$$\mathcal{E}^1(X)=\mathcal{E}^{1,0}(X)\oplus\mathcal{E}^{0,1}(X)$$

To be precise, a differential 1-form α is of bi-degree $(1,0)$ if it is expressed as $f\cdot dz$ in every chart (U,z) of X where z is the local analytic coordinate and $f\in C^\infty(U)$. In a chart we also have the conjugate 1-forms $d\bar z$ and a 1-form β is of bi-degree $(0,1)$ when it is expressed as $g\cdot d\bar z$ in every chart of X. We have also the space $\mathcal{E}^2(X)$ of differential 2-forms. In a chart (U,z) each 2-form is expressed as $h\cdot d\bar z\wedge dz$ and we prefer to write $\mathcal{E}^2(X)=\mathcal{E}^{(1,1)}(X)$. Here we have maps:

$$\partial\colon \mathcal{E}^{0,1}(X)\to\mathcal{E}^{(1,1)}(X)\quad:\quad\bar\partial\colon \mathcal{E}^{1,0}(X)\to\mathcal{E}^{(1,1)}(X)$$

0.3.3 Stokes Theorem. Let U be a bounded open subset of a Riemann surface x whose boundary ∂U is some finite union of rectifiable and closed curves. Then

$$\int_{\partial U}\alpha=\iint_U d\alpha$$

for every $\alpha\in\mathcal{E}^1(X)$. If u is a real-valued C^∞-function we get the $(0,1)$-form ∂u and after we apply $\bar\partial$ which gives

$$\int_{\partial U}\partial u=\iint_U\bar\partial\partial(u)$$

If (U,z) is a chart in X we recall from Chapter V that

$$\bar\partial\partial(u)(x+iy)=\frac{i}{4}\cdot\Delta(u)\cdot dx\wedge dy$$

If the Laplacian $\Delta(u)$ is a non-negative function in every chart one says that u is a subharmonic function. The condition is intrinsic because we know from Chapter 5 that subarmonicity is unchanged under local conformal mappings. This gives the class $\mathrm{SH}(X)$ of real-valued C^∞-functions u on X such that

$$\iint_U\bar\partial\partial(u)$$

is equal, to i times a non-negative real number for every bounded open set U. If $\bar{\partial}\partial(u)$ vanishes we say that u is a harmonic function.

0.3.4 The Dirichlet problem. Let U be a bounded open set whose boundsry ∂U is as above Perron's method to solve the Dirichlet problem in the planar case extends verbatim and shows that for every real-valued and continuous function $h \in C^0(\partial U)$ there exists a unique harmonic function H in U whose boundary values are equal to h.

0.3.5 A residue formula Let U be a bounded domain as above and ϕ is an analytic function in U which extends to a continuous and zero-free function to the closure. Then $\frac{\partial\phi}{\phi}$ is a smooth 1-form on ∂U and there exists the line integral

$$(*) \qquad \frac{1}{2\pi i} \cdot \int_{\partial U} \frac{\partial\phi}{\phi}$$

Let $\{q_k\}$ be the finite set of zeros of ϕ in U. Around each q_k we choose a chart D_k and the discs $\{D_k\}$ are pairwise disjoint. Stokes Theorem applies to the open set $U_* = U \setminus \cup\bar{D}_k$. Notice that

$$d(\frac{\partial\phi}{\phi}) = 0$$

holds in U_*. Hence (*) is equal to

$$\frac{1}{2\pi i} \cdot \sum \int_{\partial D_k} \frac{\partial\phi}{\phi}$$

The line integrals over the boundary of the small discs are evaluated exactly as in the planar case and hence (*) is equal to the number of zeros of ϕ in u counted with multiplicities. A special case occurs i the absolute value $|\phi(x)| = 1$ for every $x \in \partial U$. Then, if a is a complex number with $|a| < 1$ we consider the analytic function $\phi(x) - a$ and by the above

$$(*) \qquad \frac{1}{2\pi i} \cdot \int_{\partial U} \frac{\partial\phi}{\phi - a}$$

is an integer. Since (*) depends continuously on a this integer is constant and hence the number of zeros of $\phi - a$ is equal to that if ϕ in U. When ϕ from the start has a simple zero in U it follows that $x \mapsto \phi(x)$ yields a bijective map from U onto the open unit disc D and it is therefore a conformal mapping. In § xx this is used during the proof of the uniformisation theorem.

0.4 Currents.

A distribution on X is a continuous linear functional on the topological space of test-forms of bi-degree 2. An example is when f is an arbitrary C^∞.function on X which yields the distribution defined by

$$\alpha \mapsto \int_X f \cdot \alpha \quad : \alpha \in \mathcal{E}^{(1,1)}(X)$$

In general, if $\{U_\nu\}$ is some locally finite covering of X by charts a distribution γ on X is determined by its restriction to every chart. More precisely, we first find a C^∞-partition of the unity, i.e. a family $\{\phi_\nu \in C0^\infty(U_\nu)\}$ such tht $\sum \phi_n u(x) = 1$ for all $x \in X$. If ν is fixed and α_ν is a smooth 2-form in U_ν we set

$$\gamma_\nu(\alpha_\nu) = \gamma(\phi_\nu \cdot \alpha_\nu)$$

Working with local coordinates in U_ν it means that γ_ν yields an ordinary planar distribution in U_ν expressed by a finite sum of derivatives of Riesz measures as explained in § XX. The distributions $\{\gamma_\nu\}$ recaputre γ since every test-form α on X is equal to $\sum \phi_\nu \cdot \alpha$. An exampe of a non-smooth distribution on X arises when U is an arbitrry open set and the functional is given by

$$\alpha \mapsto \int_U \alpha$$

Another case is when f is a globally defined meromorphic function on X. If f has a pole of some order $n \geq 1$ at a point x_0 we consider a chart $(U, z9$ around x_0 where x_0 is the origin and

$f(z) = z^{-n}$. In this chsrt the riniciapmvalue distribution z^{-n} exists as exaplined in §§ and by partitions of thr unity we get a current on X defined by

$$\int_X f \cdot \alpha$$

where the rule is that during the integration one takes principal values to avoid the discrete set of poles of f. The space of distributions on X is denoted by $\mathfrak{Db}(X)$. Next, we have the topological vector spaces $\mathcal{E}^{0,1}(X)$ and $\mathcal{E}^{1,0}(X)$. We set

$$\mathfrak{c}^{1,0}(X) = \mathcal{E}^{0,1}(X)^* \quad \text{and} \quad \mathfrak{c}^{0,1}(X) = \mathcal{E}^{1,0}(X)^*$$

Here $\mathfrak{c}^{1,0}(X)$ is the space of currents of bi-degree $(1,0)$. An example occurs of $\alpha^{1,0}$ is a smooth form of bi-degree $(1,0)$ which yields the curent defined by

$$\beta^{0,1} \mapsto \int_X \alpha^{1,0} \wedge \beta^{0,1}$$

Hence $\mathcal{E}^{1,0}$ is a subspace of $\mathfrak{c}^{1,0}(X)$ and similarly $\mathcal{E}^{10,1}$ is a subspace of $\mathfrak{c}^{0,1}(X)$. If $\gamma \in \mathfrak{Db}(X)$ there exists the current $\partial\gamma$ defined by

(i) $$\partial(\gamma)(\beta^{0.1}) = -\gamma(\partial(\beta^{0,1})$$

Indeed, this is so because ∂ yields a continuous map from the topological vector space of test-forms of bi-degree $(0,1)$ to those of bi-degree $(1,1)$. For example, let γ be the distribution defined by a $C\infty$-function f. Stokes Theorem entails that

$$0 = \int_X d(f \cdot \beta^{0,1}) = \int_X \partial f \wedge \beta^{0,1}) + \int_X f \cdot \partial(\beta^{0,1})$$

The sign rule in (i) shows that the current $\partial(f)$ is defined by the smooth $(1,0)$-form ∂f. In a similar way we get a differential map

$$\bar{\partial} \colon \mathfrak{Db}(X) \to \mathfrak{c}^{0,1}(X)$$

Finally there exists the space

$$\mathfrak{c}^{1,1}(X) = \mathcal{E}(X)^*$$

Here $\mathcal{E}^{(1,1)}()$ appears as a subspace and we leave it to the reader to detect the mappings:

$$\bar{\partial} \colon \mathfrak{c}^{1,0}(X) \to \mathfrak{c}^{1,1}(X) \quad \text{and} \quad \partial \colon \mathfrak{c}^{0,1}(X) \to \mathfrak{c}^{1,1}(X)$$

Currents will be used both to study compact as well as non-compact Riemann surfaces. A fundamental fact is the elliptic propery of the $\bar{\partial}$-map. Fir example, if μ is a distribution such that $\bar{\partial}(\mu) = 0$ then it is a holomorphic density function. For compact Riemann surfaces we shall use currents in § XX to prove the Hodge decomposition which asserts that if X is a compact Riemann surface then there exists a direct sum decomposition

(*) $$\mathcal{E}^1(X) = d(\mathcal{E}(X)) + \Omega(X) \oplus \bar{\Omega}(X)$$

where $\Omega(X)$ is the space of abelian differentials of the first kind, i.e. globally defined holomorphic 1-forms, and $\bar{\Omega}(X)$ is the conjugate space of differential forms of bidegree $(0,1)$ which are ∂-closed.

Remark. The calculus based upon currents is not only efficient but enable us to give an operative meaning to geometric objects. For example, an oriented and C^1-parametrized curve γ in a Riemann surface X yields a current of bi-degree $(1,0)$ defined by

$$\beta^{0,1} \mapsto \int_\gamma \beta^{0,1}$$

and taking $(1,0)$-forms we get a current of bi-degree $(0,1)$. The proof of Abel's theorem in § E is an example where currents given by such integration chains will be used.

0.4.1 Boundary value distributions. Let X be a Riemann surface and Ω some open and relatively compact subset. Let $f(z)$ be a holomorphic function in Ω, i.e. it belongs to $\mathcal{O}(\Omega)$. If $p \in \partial\Omega$ we choose a local chart (U, z) around p and impose the condition that

$$\iint_{U \cap \Omega} |f(z)|\, dxdy < \infty$$

where $z = x + iy$. It is expressed by sying that if χ_Ω is the characteristic functuon of Ω then $f \cdot \chi_\Omega$ is a locally integrable function on. Under this condition there exists the current

$$\rho = \bar{\partial}(f \cdot \chi_\Omega)$$

The case when $\partial\Omega$ is recitifiable Jordan curve which may be simple or closed is of special interest. Let us give a simple example: First we have the Riemann surface attched to the algebraic equation $y^2 = \frac{1}{x(x-1)}$. Removing the unit interval $[0,1]$ from the complex x-plane we can pick a single valued branch of the root function and find the analytic function

$$f(x) = \frac{1}{\sqrt{x(x-1)}}$$

where the branch is cohsen so that $f(x) \simeq \frac{1}{x}$ when $|x| >> 0$. To pbe precise, additing the point at inifintity one has the prpjective line $\P^1$ and f extends to be analytic at $x = \infty$ where $\zeta = x^{-1}$ is a new coordinate and

$$f(\zeta) = \frac{\zeta}{\sqrt{1-\zeta}}$$

Now we have the simply connectd set $\Omega_* = \mathbf{P}^1 \setminus [0,1]$ and $f \in \mathcal{O}(\Omega_*)$. Here Ω_* corresponds to a sheet Ω of the associated Riemann surface X to the given algebraic equation. More precisely, one first identifies x with a meromorphic function on X and as p varies in X we get points $x(p) \in \mathbf{P}^1$, i.e. x yields a holomorphic map from X to the complex manifold $\mathbf{P}^1$. Now the inverse image $x^{-1}(\Omega_*)$ is a disjonit union of two sheets on which single-valued branches of the root function above are determined. One of these sheets is denoted by Ω where the specific function f is given as above. On X the boundsry $\partial\Omega$ has one realtively open part which is homeomorphic to the open interval $(0,1)$. In addition $\partial\Omega$ contains the two points in X above $\{x = 0\}$ and $\{x = 1\}$. The local constructions of charts in the Riemann surface associated to an algebraic equation will teach us that $\zeta = y^{-1}$ serves as a local coordinate at the point above $\{x = 0\}$ and since $y = \zeta^{-1}$ is locally integrable in a disc centered at $\{\zeta = 0\}$ this entials that when f is lifted to the holomorphic function f^* on Ω then $f^* \cdot \chi_\Omega$ is locally integrable in the sense above. The computation of $\bar{\partial} f^*$ as a current on X can be carried out by firt finding the usual $\bar{\partial}$-derivative of f in the complex x-plane. In this way we are led to calculus in analytic function theory. As explained in §§ the result is that the distribution derivative $\bar{\partial}_x(f)$ is supported by the real interval $\{0 \leq t \leq 1\}$ where $t = \mathfrak{Re}\, x$ and given by the L^1-density

$$\phi(t) = -2i \cdot \frac{1}{\sqrt{t(1-t)}}$$

From this one derives that $f(x)$ is given by a Cauchy transform in $|bfP^1 \setminus [0,1]$, i.e. one has the equation

$$f(x) = \frac{1}{\pi} \cdot \int_0^1 \frac{1}{\sqrt{t(1-t)}} \cdot \frac{dt}{x-t}$$

Remark. The discussion above shows that during the study of Riemann surfaces basic material from analytic function theory intervenes and it is also essential to be familiar with basic facts in distribution theory, especially the construction and prooperties of boundary value distributions.

Historic comments

The theory about algebraic curves has foremost been created by Abel and Riemann. Abel's article *Remarques sur quelques propriétés génerales d'une classe de fonctions trancendantes* from 1826 and Riemann's thesis from 1851 laid the table for later generations. A high point in the theory is Abel's addition theorem which gives a bridge between eliminations in systems of algebraic equations depending upon parameters and evaluations of certain integrals. It is exposed in § XX.

Work by Jacobi and Puiseux.

Abel's pioneering ideas and results were developed further by Jacobi. His main concern was to exhibit inversion formulas which arise via abelian intergrals. The article *Jacobi: Crelle 17* from 18xx studies meromorphic functions of two independent complex variables with four periods which led to inversion formulas on hyperelliptic curves with genus two. This work inspired later studies of integrals which in general depend on several complex variables. Of special interest is the Jacobi residue for a pair of irreducible polynomials in two variables which is used to define the intersection number between a pair of projective curves even when the intersection is not transversal, i.e. the Jacobi residue gives a consistent counting of multiplicities when a pair of algebraic curves do not intersect transversallly. Important contributions are also due to Puiseux. His monograph [§ p-p] from 1854 established the local uniformisation theorem which produces local charts in Riemann surfaces associated to a given algebraic equation and clarified the notion of critical points under maps with finite fibers.

Work by Riemann

Riemann gave the general construction of Riemann surfaces attached to an algebraic equation. He also developed differential calculus on manifolds which extend integral formulas by Cauchy in the complex plane. Recall from § XX that if U is a bounded domain with a boundary of class C^1 in $\mathbf{C}$ and $f = u + iv \in \mathcal{O}(U)$ extends to a C^1-function on the closure, then

$$\int_{\partial U} udv = \iint_U (u_x^2 + u_y^2)\, dxdy \tag{1}$$

Riemann established a similar formula for pairs of analytic functions in domains of a Riemann surface which may have several sheets above the projective x-line and used this to construct a canonical basis for the homology of compact surfaces where a bilinear inequality for holomorphic 1-forms plays a central role. In this way he clarified the equality between the topologically defines genus number $g(X)$ of a compact Riemann surfaces and the dimension of the vector space of globally defined holomorphic 1-forms. He also realised the crucial role of solutions to the Dirichlet problem. Riemann defined a 1-dimensional complex manifold X to be simply connected if Jordan's theorem holds, i.e. for every simple closed curve γ in X the open complement $X \setminus \gamma$ has exactly two connected. Solving the Dirichlet problem Riemann proved that every simply connected Riemann surface is biholomorphic to the projective line $\mathbf{P}^1$, the complex plane or the open unit disc. Riemann's original proof was a "bit shaky" and it was not until 1907 that Koebe and Poincaré gave the complete proof. But the original ideas and essential steps are due to Riemann.

The Riemann-Schwarz inequality. If δ is a metric on a Riemann surface X and (U, z) is a chart, the restriction of δ to U is expressed by a positive function $\lambda(z)$. It means that if γ is a rectifiable Jordan arc in U then its δ-length is given by

$$\delta(\gamma) = \int_\gamma \lambda(z) \cdot |dz|$$

In the chart we can find a real-valued function u such that

$$\lambda(z) = e^{u(z)}$$

One says that δ-metric has negative curvature if the Laplacian of u is strictly positive, i.e. $\log \lambda$ is strictly subharmonic.

Theorem. *Let δ be a metric on X whose curvature is everywhere strictly negative. Then each pair of points a, b on X can be joined by a unique geodesic curve.*

The proof of this result relies upon the Riemann-Schwarz inequality which goes as follows: Denote by $\mathcal{C}(a, b)$ the family of all recitifiable curves γ on X which join a and b. Set

$$\delta_*(a, b) = \min_\gamma \, \delta(\gamma)$$

with the minimum taken over curves in $\mathcal{C}(a, b)$. When γ_1, γ_2 is a pair in $\mathcal{C}(a, b)$ and $p \in \gamma_1$ we set

$$\delta(p, \gamma_2) = \inf_{q \in \gamma_2} \, \delta_*(p, q) \quad \text{and} \quad \text{dist}(\delta; \gamma_1, \gamma_2) = \sup_{p \in \gamma_1} \, \delta(p, \gamma_2)$$

Theorem. *For each pair of points a, b in X and every pair γ_1, γ_2 in $\mathcal{C}(a, b)$ one has the inequality*

$$(*) \qquad \delta_*(a, b)^2 + \text{dist}(\delta; \gamma_1, \gamma_2)^2 \leq \frac{1}{2} \cdot \big(\delta(\gamma_1)^2 + \delta(\gamma_2)^2 \, \big)$$

The proof is given in Chapter VI § XX. We refer to the result above as the Riemann-Schwarz inequality since the proof relies upon results due to them. The general set-up was later put forward by Poincaré who constructed metrics with negative curvature on an extensive class of Riemann surfaces.

The Riemann-Hurwitz formula.. Another important contribution by Riemann is a formula which computes the genus. His original result was later reconsidered by Hurwits so one often adds his name this result. Let us illustrate a specific case where one considers consider Fermat's algebraic equation

$$y^p + x^p = 1$$

Here $p \geq 3$ is a prime number. While x varies in the complex plane and avoids the roots of unity for which $x^p = 1$ one gets a p-tuple of corresponding y-values which means that the Riemann surface X has p many sheets above $\mathbf{C} \setminus \Sigma\}$ where $\Sigma = \{e^{2\pi i k/p} \colon k = 0, \ldots, (p-1)\}$. The glueing of these sheets gives a compact and connected complex manifold. We shall learn that the Riemann-Hurwitz formula entails that the genus of X is equal to $g = 1 + \frac{p(p-3)}{2}$, i.e X is homeomorphic to a sphere with g attached handles. The example illustrates the power of the abstract theory since it is not easy to discover the genus in an intuitive fashion when the prime number p is large. To be precise, with g as above one has to construct $2g$ many closed curves which have $(0, 1)$ as commen start- and end-points and give a basis of the fundamental group of X which is free of rank $2g$.

Triangulations and the Euler-Riemann formula. On a compact Riemann surface X there exist triangulations where A vertex points, B arcs and C open triangles appear. Riemann proved that for every triangulation on X one has the equality

$$(*) \qquad 2g(X) - 2 = B - A - C$$

The reader may check this if $X = S^1$ is the simply connected sphere and construct a triangulation where $A = C = 6$ while $B = 10$ and as a result

$$10 - 6 - 6 = -2$$

Let us remark that it is essential that X is oriented in (*). Starting from a triangle Δ_1 one introduces a positive orientation. Now $\partial \Delta_1$ consists of three arcs which get positive orientations via the rule of the thumb. They determine orientations on the three triangles which share an arc with Δ_1 and after their boundaries determine new positive orientations on arcs. The process continues and since X is assumed to be oriented and connected it gives a *unique* orientation on every arc and every open triangle. Riemann's formula follows when each triangle is idenified with a 2-simplex, each arc with a 1-simplex and vertices with 0-simplexes. To see this one considers the free abelian group of rank C whose basis are the C-many triangles, and similarly the free abelian group of rank B whose basis consists of oriented arcs. Finally we get the free abelian group of vertices. Now one has the complex

$$0 \to \mathbf{Z}^C \xrightarrow{d^0} \mathbf{Z}^B \xrightarrow{d^1} \mathbf{Z}^A \to 0$$

It is easily seen that an element $\sum_{\nu=1}^{\nu=C} k_\nu \cdot \Delta_\nu$ belongs to the d^0-kernel if and only if all the integers k_ν are equal. So the image of d_0 is an abelian group of rank $C-1$. Next, it is easily seen that the image of d^1 is the subgroup of elements $\sum_{\nu=1}^{\nu=A} k_\nu \cdot p_\nu$ for which $\sum k_\nu = 0$. Hence the d^1-kernel has rank $B-A+1$. It follows that the quotient group

$$\frac{\mathrm{Ker}(d^1)}{\mathrm{Im}(d^0)}$$

has rank $B-A+1-C+1 = B_A - C + 2$. This quotient represents the homology group $H_1(X;\mathbf{Z})$ which by the definition of g is a free abelian group of rank $2g$ and Riemann's formula follows.

Remark. The reader may consult Riemann's article *Abelschen Funktionen* from his collected work for the construction of canonical triangulations of a sphere with g handles via $2g$ many closed curves. A detailed account of his constructions supplemented with instructive figures appears also in [Appel-Goursat: chapter V]. Let us recall that the existence of triangulations on differentiable manifolds in arbitrary dimension $n \geq 2$ was given by André Weil in [Weil].

Work by Weierstrass.

In a series of articles starting from the construction of doubly meromorphic functions associated to curves of genus one, Weierstrass studied meromorphic functions of n complex variables which admit $2n$ periods and investigated their relation to abelian integrals on compact Riemann surfaces of higher genus. Articles such as *Allgemeine untersuchungen über $2n$-fach periodische functionen von n veränderlichen* or the inversion formulas in [Crelle: Bd 89] go beyond the material in this chapter. Readers who want study the interplay between analytic function theory and hyperelliptic curves should consult original work by Weierstrass which contains a wealth of results and valuable methods. See also the article *Der Weierstrassche Satz der algebraischen Abgängigkeit von Abelschen Funktionen und seine Verallgemeinerungen* by Thimm in [Weierstrass festschrift 1966].

Work by Poincaré

The article *Mémoire sur les groupes fuchsiens* (Acta math. Vol 1) gave a new perspective on Riemann surfaces, though one should also give credit to Schwarz who constructed special classes of Riemann surfaces using Fuchsian groups via his conformal mapping formulas between polygons and the unit disc. This inspired Poincaré to investigate more general classes than those treated by Schwarz. Poincaré employed Fuchsian groups to construct Riemann surfaces and studied the inverse process where automorphic functions with respect to a group serve as a field of meromorphic functions. It would lead us too far to enter a serious discussion about Poincaré's work. Let us just mention that he generalised Abel's additivity theorem to algebraic surfaces and paved the way towards the proof of the uniformisation theorem in its general form. Important contributions are also due to Picard, foremost his studies of both linear and non-linear systems of differential equations. The interested reader should consult Poincaré's account of the historic development of Fuchisan groups and algebraic functions which appears in his collected work: [Poincaré vol. xxx]. Volume 2 in *Théorie des fonctions algébriques* by Appel and Goursat covers Poincaré's theory where a text covering 500 pages offers a wealth of examples and instructive proofs related to Poincaré's theory. Properties of a compact Riemann surface genus $g \geq 2$ related to its complex analytic structure leads to many delicate results. One can mention Picard's theorem which asserts that on Riemann surface of genus $g \geq 2$ there exists at most $84(g-1)$ many birational transformations and that every birational transformation has at most $2g+1$ many fixed points, where equality only hold for hyperelliptic curves.

Remark. Even though sheaf theory can be used to get simple proofs the constructive nature often becomes less transparent. So it is valuable to become familiar with proofs of a more direct nature. An example is Courant's proof of the uniformisation theorem for Kleinian groups of the third kind. Here the details are considerably harder to follow than those which we are going to present but have the merit that they are constructive when a Riemann surface is defined via a Kleinian group.

Literature on Riemann surfaces

A classic is *Die Idée der Riemanschen Flächen* by Weyl from 1913. Here it is proved that every compact 1-dimensional complex manifold X is defined via an algebraic equation which amounts to show that there exists an ample family of meromorphic functions on X. The key point is to use the elliptic property of the $\bar{\partial}$-operator. Apart from the use of sheaf-theoretic vocabulary the proof in § XX that every compact 1-dimensional complex manifold is algebraic is verbatim to Weyl's original proof. General results about non-compact Riemann surfaces are of a more recent origin. The vanishing of $H^1(X, \mathcal{L})$ when X is a non-compact Riemann surface and $\mathcal{L}$ an arbitrary holomorphic line bundle is due to Behnke and Stein in their joint article *Entwicklungen analytischer funktionen auf Riemannschen Flächen* [Math. Ann. 120 (1948)]. Further results dealing with Runge's approximation theorem on Riemann surfaces were established in the monograph *Reguläre und meromorphe funktionen auf nicht geschlossene Riemannschen Flächen* by Herta Florack [Münster 1848]. These articles have been quite influtential and together with Leray's sheaf theory opened a veritable gateway to complex analysis on several variables presented by Stein in the article *xxxx* from 1951.

The text-book *Lectures on Riemann surfaces* by Forster has inspired much of the subsequent material. The reader may also consult the text-books by Gunning which not only cover the basic theory about compact Riemann surfaces but also treat the Jacobi mapping and holomorphic vector bundles over Riemann surfaces. A concise and elegant introduction to Riemann surfaces is also given by Goenot and Narasimhan in [G-N]. At several places we also profit upon material from the classic *Theorie des fonctions algebriques et leurs integrales* by Appel and Goursat. This text-book contain many instructive figures and a wealth of examples and detailed calculations of invariants attached to a Riemann surface and we follow [ibid] while proofs of Abel's resuts are exposed in § XX. Readers interested in applications of Riemann surfaces to geometry should consult the last chapter in [ibid] which describes many elegant applications such as Laguerre's directed curves (courbes de directions). Another classic which treats the interplay between geometry and Riemann surfaces is Klein's book *Theorie des Ikosaheder.* But it would bring us too far to enter this subject where unexpected phenomena during topological constructions appear which are caused by failure of differentiability. Arithmetic problems where one for example seeks rational points on projective curves will not be discussed. This is a highly non-trivial issue which is illustrated by the Fermat curves

$$y^p + x^p = 1$$

where $p \geq 3$ are prime numbers. The *non- existence* of rational points on these curves is equivalent to the Fermat conjecture and was settled by Wiles in 199X. The proof requires methods which go far beyond the material in this chapter. For more advanced text-books devoted to non-compact Riemann surfaces and related moduli probems the reader may consult the text-book [xxx] by Ahlfors and Sario while the book [xxx] by Netto and Sario goes quite far and contains material which is still open for further research. Below we add some comments about some specific topics.

Specific topics.

1. A construction by R. Nevanlinna Following Nevanlinna's plenary talk *Über die Riemannsche Fläche einer analytischen funktion* held at the IMU-congress at Zürich in 1932 we recall a construction which Nevanlinna applied to his theory of value distribution of meromorphic functions on non-compact Riemann surfaces.

Consider a polygon Π with q corner points placed on the unit circle where $q \geq 3$. As illustrated by figure XX each corner point has an angle angles $\frac{\pi}{m_\nu}$. where $m_\nu > 1$ and Π is chosen so that

$$(*) \qquad \sum (1 - \frac{1}{m_\nu}) > 2$$

Next, there exists the conformal mapping f from the unit disc D onto Π which extends to a bi-continuous map from the unit circle T onto the boundary of Π. The corner points on Π correspond to a q-tuple of points $\{p_\nu = e^{i\theta_\nu}\}$ on T and we have q many intervals $\ell_\nu = (e^{i\theta_\nu}, e^{i\theta_{\nu+1}}\}$. Now f maps every such circular interval onto line segment on $\partial\Pi$. If we fix one such interval ℓ_ν then Schwarz' general reflection theorem implies that f extends analytically across ℓ_ν and yields an analytic function $f^*_\nu(z)$ defined in the exterior disc $\{|z| > 1\}$. Next, f^*_ν can be extended analytically across another circular interval ℓ_k which yields an analytic function$f^{**}_{k,\nu}$ in D. The process can be continued and gives a multi-valued analytic function F defined in $\mathbf{C} \setminus \{p_1, \ldots, p_k\}$. Since we have started from a conformal mapping its complex derivative is everywhere $\neq 0$. The full range of F in the sense of Weierstrass is determined via succesive applications of Schwarz' reflections. Suppose for example that we continue f along and arc ℓ_1 where $f(\ell_1)$ is a horisontal line segement of $\partial\Pi$. In this case the analytic continuation f^* is obtained via complex conjugation so the range of f^* on the exterior unit disc is equal to the polygon $\Pi(\ell_1)$ which is obtained by reflecting Π in ℓ_1. The reader is invited to contemplate upon the result after a finite number of reflections or consult Nevanlinna's text-book *Uniformisation* for details and further comments about the whole construction. Let us remark that this can be used to extend Picard's theorem to cases where an analytic functions exclude a finite set of values. See again [ibid]. The constructions above also appear in the theory about automorphic functions and properties of elementary domains. In this context analytic function theory has initiated topics in combinatorical topology which was put forward by Speiser in the influential article *Probleme aus dem gebiet der ganzen tranzendenten Funktionen* from 1929.

2. Algebraic function fields. Every compact Riemann surface arises from an algebraic function field. The article *Theorie der algebraischen functionen einer verändelichen* by Dedekind and Weber from 1882 clarified this result which is used in arithmetical problems and has the merit that it also applies in characteristic p. Recall that Weil has proved the Riemann hypothesis for fields of algebraic functions over finite fields. His book *Sur les courbes algebriques* treats the p-adic case where the geometric and algebraic methods are quite different from those we are going to expose when the groundfield is $\mathbf{C}$. Let us also recall that Weil's influential article *Zur algebraischen Theorie der algebraischen Funktionen* (Çrelle: 179: 1938) gives general constructions of differentials on function fields which can be adapted in every characteristic. So the reader should be aware of the fact that there are many ways to enter studies of Riemann surfaces, and more generally algebraic curves over arbitrary fields. A book which covers both the algebraic constructions and the topology on compact Riemann surfaces is Chevalley's book *Introduction to the theory of algebraic functions in one variable.* The text contains many proofs which are useful to study before entering deeper areas in algebraic geometry.

3. Moduli problems. The classification of compact Riemann surfaces with prescribed genus is another central issue. We shall not try to enter into a detailed account. The interested reader should first of all consult the article *The complex analytic structure of the space of closed Riemann surfaces* by Ahlfors, who like Riemann has given many important contributions to the theory about Riemann surfaces, foremost in the non-compact case. The moduli problem was already considered by Riemann who proved that when $g \geq 2$ then the number of independent parameters which are needed to classify isomorphism classes of Rieman surfaces with genus g is equal to $3g-3$. Riemann gave two proofs which are exposed in Picard's the text-book *Traité d'Analyse* (Part II: page 482]. In the article [Teich] Teichmüller asked for the realisation of a global complex analytic structure for the $3g-3$-dimensional moduli space. The case $g = 2$ is instructive for if a Riemann surface X has genus 2 there exist Weierstrass points on X which means that one can find $f \in \mathfrak{M}(X)$ whose set of poles is confined to a single point where f has a double pole. Hence the holomorphic map from X onto the projective line which sends $p \in X$ into $f(p)$ is 2-sheeted. It follows that $\mathfrak{M}(X)$ is an algebraic field extension of degree two over the field of meromorphic functions generated by f. As explained in § XX this implies that X is the Riemann surface associated to an algebraic equation of the form

$$y^2 = P(x)$$

where $P(x)$ is a polynomial with simple zeros. Moreover, since $g = 2$ the Riemann-Hurwitz formula to be exposed in § xx that $\deg P$ only can be five or six which comes quite close to Riemann's result that the moduli space for curves of genus two is described via $3 \cdot 2 - 3 = 3$ many parameters. See § xx for a further discussion. In connection with the moduli problem a theorem due to Weber in [Crelle: vol. LXXVI] asserts the following: *Let X and Y be a pair of Riemann surfaces with the same genus $g \geq 2$. Then, if $\phi\colon X \to Y$ is a surjective holomorphic map it follows that ϕ is biholomorphic.*. We give the proof in § X based upon constructions of adjoint curves.

4. Relation to algebraic geometry. Results about compact Riemann surfaces have inspired later studies in algebraic geometry. Apart from facts which already have been mentioned and will be covered in § A-H, we recall the Reciprocity Law by the Brill and Noether [Appel-Goursat: page 371-374]. This result was originally announced for compact Riemann surfaces but inspired more general considerations in algebraic geometry where one encounters singular projective curves which are not identfied in a direct way with a compact Riemann surfaces. More precisely, let $N \geq 2$ and consider the N-dimensional projective space $\mathbf{P}^N$ over the complex field. It contains algebraic subsets which are irreducible and of dimension one and called projective curves embedded in $\mathbf{P}^N$. In general such a projective curve S is not regular so one needs to construct the associated Riemann surface. This is achieved by local Puiseux-Riemann charts and gives a unique compact Riemann surface X called the normalisation of S. The map $\rho\colon X \to S$ is in general is not homeomorphic since some singular points in S may be surrounded by more than one Puiseux-Riemann chart. This is explained in § XX. We shall not deal with more extensive material in algebraic geometry and therefore not study embedded algebraic curves in projective spaces except for intersection numbers between irreducible algebraic curves which is treated in § XX.

§ 0. Specific examples.

POSTPONE
The sections below serve to illustrate subsequent material. The less experienced reader may prefer to return to the material below later except for the first head-line:

The Riemann-Hurwitz formula. We shall foremost consider compact Riemann surfaces X which are attached to an algebraic equation $F(y,x)=0$. When this algebraic equation is given there exists a procedure to find g via the duality theorem in § X. The method is to take a meromorphic 1-form w on X and determine its zeros and poles which gives a divisor D on X such that one has an isomorphism of sheaves:

$$\mathcal{O}_D \cdot w \simeq \Omega \tag{*}$$

where Ω is the sheaf of holomorphic 1-forms on X. The divisor has a degree and we shall learn that

$$2g-2=\deg D$$

holds for every meromorphic 1-form w. Using this one can compute the genus for arbitrary given algebraic equations. We shall give many examples to illustrate the calculation of g Consider for example the case when the equation is of the form

$$y^m = Q(x)$$

where $m \geq 2$ and $Q(x)$ is a polynomial of some degree n and $y^m - Q(x)$ is irreducible in the unique factorisation domain $\mathbf{C}[x,y]$. On the associated Riemann surface we have the meromorphic 1-form $\omega = dx$. If the zeros of Q are all simple and the integers m and n are relatively prime we shall learn that the degree of the divisor which appears in (*) satisfies

$$\deg D = n(m-1)-m \implies g = \frac{1}{2}(n-1)(m-1)$$

More involved computations arise when Q has multiple zeros and the integers m and n are not prime. More precisely, suppose that Q has k distinct zeros $\alpha_1, \ldots \alpha_k$ where the zero at α_ν has some order e_ν. So here $e_1+\ldots+e_k=n$. The positive contribution to the divisor D at a zero α_ν is found as follows: Denote by ρ_ν the common divisor of m and e_ν. So $\rho_\nu = 1$ if and only if m and e_ν are relatively prime. Then the positive contribution at α_ν is given by

$$m-\rho_\nu \tag{i}$$

Next, poles of dx occur at points where $x=\infty$. Another computation in local charts shows that if ρ_∞ is the common divisor of n and m then D gets a negative contribution $m+\rho_\infty$. Hence

$$g = 1+\frac{1}{2}\cdot\sum_{\nu=1}^{\nu=k}(m-\rho_\nu)-\frac{1}{2}\cdot(m+\rho_\infty) \tag{ii}$$

Example. Consider the equation where a differs from 0 and 1 and

$$y^4 = x^2(x-1)^3(x-a)^3 \tag{i}$$

So here Q has degree 8 and (*) gives

$$g = 1+\frac{1}{2}\cdot[(4-2)+(4-1)+(4-1)]-\frac{1}{2}\cdot(4+4)=1$$

The low g-number is not so obvious from the start since (1) is an algebraic equation of degree 8. Notice also that the g-number does not change in a continuous manner. For example, if the right hand side is replaced by a monic polynomial $Q(x)$ of degree 8 with simple zeros which be arbitrary close to the corresponding multiple zeros then

$$g = 1+8\cdot 3-4=21$$

In general, if $n \geq 3$ and $Q(x)$ is a polynomial of degree n with simple zeros then the genus of X associated to the equation $y^n = Q(x)$ becomes

$$g = 1+n(n-1)-n=(n-1)^2$$

The subsequent material is inserted to describe the flavour during the studies of Riemann surfaces. The reader may postpone § 0.1-0.8 until later and proceed directly to the headline *Outline of the Contents.*

0.1 Humbert's rosette.

Consider the real (x, y) plane where (r, θ) are the polar coordinates and the curve defined by the equation

$$(*) \qquad r^3 = \cos 3\theta \quad : \quad 0 \le \theta \le 2\pi$$

Drawing a picture the reader can recognize a rosette denoted by $\mathfrak{R}$. When $\pi/6 \le \theta \le \pi/6 + \pi/2$ we see from a figure that $\mathfrak{R}$ contains a simple closed curve which passes the origin and for other θ-intervals we get two other simple curves and the orgin is a common point to these. Hence $\mathfrak{R}$ is a plane curve with a self-intersection at the origin and oriented by θ. With $z = x + iy$ we have $r^3 \cos 3\theta = \mathfrak{Re} z^3$ while $r^6 = (x^2 + y^2)^6$ so (*) can be written in the form

$$(x^2 + y^2)^3 = \mathfrak{Re}(x + iy)^3) = x^3 - 3xy^2$$

Treating x and y as independent complex variable we get the algebraic equation

$$(1) \qquad y^6 + 3x^2y^4 + (3x^4 + 3x)y^2 = x^3 - x^6$$

This is an irreducible equation which gives a compact Riemann surface X whose real points belong to $\mathfrak{R}$. The equation (1) has degree six and looks rather complicated. But it turns out that X has genus one and is therefore homeomorphic with a torus. To show this one considers the arc-length measure $d\sigma$ which in polar coordinates is given by

$$d\sigma^2 = dr^2 + r^2 d\theta^2$$

On $\mathfrak{R}$ we have $r^2 dr = -\sin 3\theta \, d\theta$ and a computation gives

$$d\sigma = \frac{d\theta}{r^2} = \frac{1}{x^2 + y^2} \cdot d \arctan \frac{y}{x} = \frac{xdy - ydx}{(x^2 + y^2)^2}$$

Hence the arc-length measure along $\mathfrak{R}$ has a differential given by a rational function of x and y. Passing to the Riemann surface X the right hand side represents a differential 1-form and in § X we show that is everywhere holomorphic and $\neq 0$ which explains why X has genus one and therefore homeomorphic with a torus. Let us now consider a circle $\mathcal{C}$ defined by an equation

$$(x - \alpha)^2 + (y + \beta)^2 = \alpha^2 + \beta^2$$

where α, β are small positive constants. By a figure the reader can recognize that $\mathcal{C}$ intersects $\mathfrak{R}$ at three points A_1, A_2, A_3. Denote by $\ell(A_K)$ the arc-length of the curve from the origin to A_k taken along the corresponding simple arc on $\mathfrak{R}$. Keeping trace of the orientation in Figure XX it turns out that there exists a contant C which is independent of the chosen circle such that

$$\ell(A_1) + \ell(A_2) - \ell(A_3) = C$$

It is possible to show this after some rather involved calculation but the point is that the existence of C is an immediate consequence of Abel's Theorem where one has used the fact that $d\sigma$ yields a holomorphic 1-form on the torus X, together weith some easy calculations which show that when the equation for $\mathcal{C}$ determines an algebraic curve in the 2-dimensional complex projective (x, y)-space, then this curve has a triple intersection with X at the $(0, 0)$ and six points intersect at infinity whose positions are independent of the chosen circle. In addition we have the three simple intersections at the A-points. This reflects a general fact since the circle is a curve of degree two while $\rfloor$ has degree 6 and therefore the number of intersection points counted with multiplicities is 12. We explain this in § xx after general results about Riemann surfaces have been exposed. The example above was treated by Humbert in an article from 1887. It is instructive since it shows that the abstract theory about compact Riemann surfaces has concrete geometric applications.

0.2 Currents and duality.

On a compact Riemann surface X there exist the space $\mathfrak{c}^{0,1}(X)$ of currents with bidegree $(0, 1)$ which is the topological dual of the Frechet space $\mathcal{E}^{1,0}(X)$ of differential C^∞-forms with bidegree $(1, 0)$. The complex analytic structure gives the $\bar{\partial}$-operator which sends C^∞-functions g to smooth (0,1)-forms $\bar{\partial}(g)$. More generally $\bar{\partial}$ sends a distribution μ on X to the (0,1)-current $\bar{\partial}(u)$. A fundamental fact is that the $\bar{\partial}$-operator is elliptic which means that a distribution μ satisfying

$\bar\partial(\mu) = 0$ is a holomorphic function. This was demonstrated and applied by Hermann Weyl whose book *The concept of a Riemann surface* from 1913 contains the proof that every compact 1-dimensional complex manifold is given by an algebraic function field. As we shall see later on currents give quick proofs of central results about compact Riemann surfaces. For example, the fact that $\bar\partial$-closed $(0,1)$-currents are holomorphic 1-forms easily entails that

$$\Omega(X) \simeq \Big[\frac{\mathfrak{c}^{0,1}(X)}{\bar\partial\mathfrak{c}(X)}\Big]^* \tag{0.1}$$

where the right hand side refers to the dual vector space. At the same time Dolbeault resolutions will teach us that

$$\frac{\mathfrak{c}^{0,1}(X)}{\bar\partial\mathfrak{c}(X)} \simeq H^1(X,\mathcal{O}) \tag{0.2}$$

Examples of (0,1)-currents are integration cycles. Namely, let γ be an oriented and closed Jordan curve in X of class C^1. Every test-form ϕ of bi-degree (1,0) can be integrated on γ. Specializing to holomorphic 1-forms w we get a $\mathbf{C}$-linear functional on $\Omega(X)$ defined by

$$w \mapsto \int_\gamma w$$

One refers to this as period integrals and (0.1) means that the integration current γ is $\bar\partial$-exact if and only if the period integrals above are zero for every holomorphic 1-form w. If these periodic integrals vanish for every $w \in \Omega(X)$ it turns out that the closed Jordan curve γ is homotopic to zero, i.e. it can be retracted to a single point which means that its image in the fundamental group $\pi_1(X)$ vanishes. This illustrates a flavour during the study of Riemann surfaces where geometry and analytic results often intervene with each other.

Next, consider the cohomology group $H^1(X,\Omega)$. In §xx we show that the elliptic property of $\bar\partial$ entails that the dual space

$$\big[H^1(\Omega)\big]^* \simeq \Big[\frac{\mathcal{E}^{1,1}(X)}{\bar\partial\mathcal{E}^{1.0}}\Big]^* \simeq \mathcal{O}(X) = \mathbf{C} \tag{i}$$

Hence $H^1(\Omega)$ is 1-dimensional. Next there exists the following exact sequence of sheaves

$$0 \to \mathbf{C}_X^* \to \mathcal{O}^* \xrightarrow{\partial} \Omega \to 0$$

where $\mathcal{O}^*$ is the multiplicative sheaf of holomorphic functions without zeros. Sheaf theory teaches that this gives an exact sequence of complex vector spaces:

$$0 \to H^0(X;\Omega) \to H^1(X,\mathbf{C}_X^*) \to H^1(X,\mathcal{O}^*) \xrightarrow{\rho} H^1(X;\Omega) \to H^2(X,\mathbf{C}_X^*) \to 0 \tag{ii}$$

In addition to this, a topological result which we prove in § xx shows that

$$H^2(X,\mathbf{C}_X^*) \simeq \frac{H^2(X,\mathbf{C}_X^*)}{H^2(X,\mathbf{Z}_X^*)} = \frac{\mathbf{C}}{\mathbf{Z}} \tag{iii}$$

where $\mathbf{Z}$ is the additive group of integers. Together with (i) this implies that the image under the map

$$\rho\colon H^1(X,\mathcal{O}^*) \to H^1(X;\Omega)$$

from (ii) is reduced to the set of integers. Apart from the "yoga" based on diagram chasings, there remains to understand what is really going on. To grasp the ρ-image one must construct a cohomology class $\xi \in H^1(X,\mathcal{O}^*)$ whose ρ-image corresponds to the integer 1 in the image and therefore is a generator of the abelian group of all integers. This will be done in § xx where we also show that there exists a 1-1 correspondence with isomorphism classes of twisted rings of holomorphic differential operators.

Remark. The calculus with currents clarifies many constructions which otherwise are not so easy to grasp. For example, let X be equipped with a triangulation. Each 2-simplex Δ appears as an open subset of X where the identity function gives a current of degree zero, i.e. one integrates 2-forms on the oriented open subset Δ. Now there exists the (1,0)-current $\partial(1_\Delta$ which is supported by the boundary arcs of Δ and the operative equation becomes:

$$\partial(1_\Delta(\phi^{0,1}) = -\int_\Delta \bar{\partial}(\phi^{0,1}) = -\int_{\partial\Delta} \phi^{0,1}$$

where the last equality follows from Stokes theorem and the line integral is computed using the orientationon $\partial\Delta$ derived via the rule of thumbs. In the same way there exists the current (0,1)-current $\bar{\partial}(1_\Delta)$.

0.3 Branch cuts on Riemann surfaces.

One gets a picture of a Riemann surface X using the algebraic equation which defines it and the projection from X onto the projective line. Consider an equation of the form

$$Q(x)y^n = 1$$

where $n \geq 2$ and $Q(x)$ is a monic polynomial of degree n with simple zeros $c_1, \ldots, c_n$. We shall learn that the meromorphic function y has simple poles at the points $\{c_k, 0\}$ and above $x = \infty$ there occur n points and y has a simple zero at each such point. The map π from X onto the projective x-line is n-sheeted and to find single-valued sheaets one proceeds as follows: In the complex x-plane we can construct compact sets Γ which are connected and given as a finite union of Jordan arics while $\Omega = \mathbf{P}^1 \setminus \Gamma$ is connected and all the zeros $c_1, \cdots, c_n$ belong to Γ. One refers to Γ as a tree and since Ω is simply connected there exists the n-root of $Q(z)$ which gives the analytic root function

$$y_*(x) = Q(x)^{-1/n}$$

whose branch is chosen so that its Taylor series above $x = \infty$ starts with the linear term $\frac{1}{x}$. By results from §§ there exists a unique Riesz measure μ supported by Γ such that

$$y_*(x) = \frac{1}{2\pi i} \int_\Gamma \frac{d\mu(\zeta)}{x - \zeta}$$

Since $y_*(x) \simeq \frac{1}{x}$ at inifinity the mean-value $\int_\Gamma d\mu(\zeta) = 1$. Hence the total variation of μ is at least one which suggests a variational problem. Namely, try to find a tree Γ such that μ is a probability measure.. This corresponds to a special branch cut of the Riemann surface and the qurestion arises if Γ exists and if it is unique. An affirmative answer was given in the article [xxx] by Bergqvist and Rullgård which shows that there exists a unique tree Γ such that μ is a probability measure. The proof relies upon a study of roots of eigenpolynomials to differential equations and will be exposed in § XX.

0.4 The Cauchy transform on Riemann surfaces.

A construction which extends the Cauchy transform of Riesz measures with compact support in the complex plane goes as follows: Let U be an open and connected subset of some compact Riemann surface X and suppose that ∂U is a union of a finite number of rectifiable Jordan arcs. If $\phi \in \mathcal{O}(U)$ has the property that it is locally integrable over sets $D \cap U$ when D are small charts around boundary points, there exists a unique complex valued Riesz measure μ supported by ∂U such that

$$(*) \qquad \iint_U \phi \cdot \partial\bar{\partial}(g) = \int_{\partial U} g \cdot d\mu$$

hold for every $g \in C^\infty(X)$. Above the composed operator $\partial \circ \bar{\partial}$ sends every C^∞-function g on X to a smooth $(1,1)$-form $\partial\bar{\partial}(g)$ is and in the left hand side above its product wuth ϕ has been integrated over U which is okay since the complex analytic structure on X yields a positive orientation on its underlyng real and 2-dimensional manifold. In § XX we study special situations of this construction where ϕ from the start is a meromorphic function in X and discuss variational problems of a similar nature as in the case above.

0.5 Riemann's canonical basis for the singular holomlogy

Riemann surfaces are constructed via a glueing of charts. There are many ways to achieve this and at the same time Riemann surfaces with a prescribed genus g are all homeomorphic to a sphere with g handles attached. So in principle it suffices to perform suitable constructions for each g but for a specific surface X with genus $g \geq 1$ one proceeds as follows: Each oriented and closed Jordan curve γ on X of class C^1 gives a 1-current which sends a test-form ϕ to the line integral along γ:

$$\mathfrak{c}_\gamma(\phi) = \int_\gamma \phi$$

If $\phi = dg$ is exact where $g \in C^\infty(X)$ then the integral is zero since γ is closed. Hence $\mathfrak{c}_\gamma$ gives a linear functional on the Dolbeault quotient of closed modulu exact 1-forms which is a complex vector space of dimension $2g$ and dual to the singular homology over X in degree one. There exists the free abelian group $H_1(X, \mathbf{Z})$ of rank $2g$ whose elements represent group homomorphisms from the fundamental group $\pi_1(X)$ into the additive group of integers. Here one has used the topological fact that $\pi_1(X)$ is a free group with $2g$ many generators. If we choose some point $p \in X$ and identify $\pi_1(X)$ with homotopy classes of closed curves which have p as a common start- and end-point, then there exist $2g$ many closed Jordan curves $\gamma_1, \ldots, \gamma_{2g}$ whose homotopy classes generate $\pi_1(X)$. The Dolbeault isomorphism gives a $2g$-tuple of closed 1-forms $\{\phi_\nu\}$ such that

$$\int_{\gamma_k} \phi_j = \text{Kronecker's delta function}$$

Since images of these 1-forms in the Dolbeault quotient are unique. the integrals

$$\int_X \phi_k \wedge \phi_j \tag{1}$$

yield an anti-symmetric $2g \times 2g$-matrix whose elements only depend on the chosen γ-basis. A simple matrix arises of of this kind is a block matrix where formed by 2×2-matrices where +1 and -1 are the non-diagonal elements. One says that the closed curves $\{\gamma_\nu\}$ give a canonical basis if (1) takes the value +1 when for pairs $\{(j, j+g) \colon 1 \leq j \leq g\}$ and zero for all other pairs $k < j$.

Remark. In the text-book [Appel-Goursat] one refers to such a basis as a *système canoniques des rétrosections* and there it is explained how one constructs such a basis after g handles have been removed from the sphere. When a Riemann surface X is defined by an algebraic equation one has a map from X onto the projective x-line. which can be used to obtain a canonical γ-basis. However, these constructions can be quite involved so the mere existence of a canonical γ-basis via differential calculus is therefore a notable point if one wants to avoid hard initial constructions when a Riemann surface is presented by an algebraic equation. Riemann studied integrals of holomorphic 1-forms and their conjugate forms with respect to a canonical γ-basis. Riemann's *Bilinear Inequality* asserts that if $\{\gamma_\nu\}$ is a canonical homology basis and w an arbitrary holomorphic 1-form with periods

$$\int_{\gamma_j} w = \pi_j \quad : \quad 1 \leq j \leq 2g$$

then

$$\frac{1}{2\pi i} \cdot \sum_{k=1}^{k=g} \left(\pi_{k+g} \cdot \bar{\pi}_k - \pi_k \cdot \bar{\pi}_{k+g} \right) > 0$$

We shall prove this result in § XX.

An example. Consider an irreducible algebraic equation of degree four:

$$P(x, y) = y^4 + q_1(x)y^3 + q_2(x)y^2 + q_3(x)y^3 + x^4 = 0$$

where $\{\deg q_\nu \leq \nu\}$. If the gradient vector $(P'_x, P'_y) \neq (0, 0)$ for all $(x, y) \in \mathbf{C}$ where $P(x, y) = 0$ we shall learn in § xx that the associated Riemann surface has genus 3 and the 3-dimensional vector space of holomorphic 1-forms has a basis given by:

$$\omega_1 = \frac{ax+by+c}{P'_y}\cdot dx \quad : \omega_2 = \frac{x}{P'_y}\cdot dx \quad : \omega_3 = \frac{y}{P'_y}\cdot dx$$

then we have a basis for the complex vector space $\Omega(X)$ and taking their conjugate (0,1)-forms $\{\bar\omega_j\}$ we get the Hermitian matrix with elements

$$\rho_{kj} = \int_X \omega_k \wedge \bar\omega_j$$

From this we can find a triple of holomorphic 1-forms $\{\omega_k^*\}$ such that

$$\int_X \omega_k \wedge \bar\omega_j^* = \text{Kronecker's delta function}$$

Now the material above gives the existence of a canonical basis $\{\gamma_j\}$ for which the ω^*-forms satisfy (xx).

Remark. Notice that this gives an inverse process to attain a canonical basis of homology. It goes without saying that the constructions for a general algebraic equation is not easy to carry out. In any case, Riemann's results give theoretical foundations for topological constructions.

0.5.1 Glueing To grasp a compact Riemann surface one can glue simply connected subsets across arcs. As an illustration we consider an elliptic curve X defined by the equation

$$y^2 = x(x-a)(x+b)$$

where a and b are positive real constants. Remove the interval $[-b,a]$ from the complex x-plane. In § XX we have explained that there exists the square root function

$$y^*(x) = \sqrt{x(x-a)(x+b)}$$

which is holomorphic in the open set $U = \mathbf{C}\setminus[-b,a]$ and takes positive real values when x is real and $> a$. Next, the projection $\pi\colon \mathbf{P}^1$ onto the projective x-line is 2-sheeted an unramified above U. So $\pi^{-1}(U)$ contains two sheets U^* and U_* which are mapped biholomorphically onto U. When $x=\infty$ we shall learn that $\pi^{-1}(\infty)$ is reduced to a single point p^* where the meromorphic function y has a triple pole and x a double pole. So U^* and U_* have a common boundary point at p^*. Their boundaries also share the points $(0,0),(0,a),(0,-b)$. The intervals $(-b,0)$ and $(0,a)$ appear as boundary arcs for the two sheets. While one is crossing $(0,a)$ on the Riemann surface the meromorphic y-function on X is denoted by y^* in U^* moves to the holomorphic function y_* in U_*. Now we construct two simple closed γ-curves on X. Choose first some simple Jordan arc J_1 which joins 0 and a and stays in the upper half-plane of the x-plane. Via π it is lifted to a simple arc J_1^* in U^* with the end-point $(0,a)$ on X. Now we can move in the reverse direction along J_1 but this time its inverse π-image is taken in the sheet U_* until one arrives to the origin. This gives a closed Jordan curve γ_1 in X. In a similar fashion we take a Jordan arc J_1 from the origin to $(0,-b)$ which again moves in $\mathfrak{Im} x) > 0$ and construct a closed Jordan arc γ_2 in X which first moves in U^* until it turns into U_* when it passes $(0,-b)$. Next, on X we shall later on learn that the differential 1-form $\frac{dx}{y}$ which a priori is meromorphic actually is holomorphic on the whole of X gives a basis vector for the 1-dimensional complex vector space $\Omega(X)$ of holomorphic 1-forms.

Exercise. Show that one has the two equations:

(i) $$\int_{\gamma_1} \frac{dx}{y} = \frac{2}{i}\int_0^a \frac{dx}{\sqrt{x(x+a)(x+b)}}$$

(ii) $$\int_{\gamma_2} \frac{dx}{y} = 2\cdot\int_0^b \frac{dx}{\sqrt{x(x+a)(x+b)}}$$

Above one integral is purely imaginary and the other is real which reflects the fact that the two γ-curves is a canonical basis for the 2-dimensional integral homology group on X. The two numbers which appear in the right hand side of (i-ii) are called the periods of the 1-form $\frac{dx}{y}$. Their siginificance will be explained in § xx when we classify elliptic curves, i.e. the family of compact Riemann surfaces with genus one.

0.5.2 The torus The torus $\mathbf{T}^2$ can be identified with the quotient space $\frac{\mathbf{C}}{\Gamma}$ where Γ is the lattice of Gaussian integers. The open unit square $\{0 < x, y < 1\}$ is a fundamental domain and the two line segments $\gamma_1 = \{0 \leq x \leq 1\}$ on the x-axis and $\gamma_2 = \{0 \leq y \leq 1\}$ on the y-axis give simple closed curves on the torus since both $(1,0)$ and $(0,1)$ are identified with the origin. A complex analytic structure is derived from $\mathbf{C}$ and the holomorphic 1-form dz yields an abelian differential on the torus, i.e. a globally defined holomorphic 1-form. With chosen orientations on γ_1 and γ_2 we have

$$\int_{\gamma_1} dz = 1 \quad : \quad \int_{\gamma_2} dz = i$$

These two periods are $\mathbf{R}$-linearly independent and we shall learn that this reflects the topological fact that the two γ-curves generate the singular homology group which is a free abelian group of rank two.

0.6 Automorphic functions.

Another gateway to study of Riemann surfaces was created by Poincaré which employs Fuchsian groups and their associated automorphic functions. Hilbert proved that automorphic functions can be found via solutions to integrals equations. This has an advantage when one seeks approximative solutions numerically. The fundamental equation goes as follows: Identify the unit circle with the 2π-periodic θ-interval. Let

$$\theta \mapsto \Psi(\theta)$$

be an involutive map form the unit circle into itself, i.e. the composed map $\Psi \circ \Psi$ is the identity. Examples of such θ-maps occur when the fundamental substitutions of a Fuchsian group define the involutive Ψ-map The search of automorphic functions associated to the Fuchsian group amounts therefore to find meromorphic functions $f(z)$ defined in $\mathbf{C}$ with a finite number of poles in the disc $\{|z| < 1\}$ which on the unit circle satisfy

$$f(\Psi(\theta)) = f(\theta) \tag{2}$$

To get such functions we start from a rational function $R(z) = \frac{Q(z))}{P(z)}$ where the polynomial Q has degree strictly less than P and the zeros of P belong to $\{|z| < 1\}$. Now one seeks f as a solution to the integral equation

$$f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Big[\frac{1}{e^{i\theta} - z} - \frac{\Psi'(\theta)}{\Psi(\theta) - \Psi(z)}\Big] \cdot f(e^{i\theta}) \cdot e^{i\theta} \cdot d\theta + R(z) + R(\Psi(z)) \tag{*}$$

which to begin with holds when $|z| = 1$ and after f extends to a meromorphi functions with the same poles as R. The kernel function in (*) is singular. But as explained in Carleman's lecture at the IMU-congress in Zürich 1932, see [xx- page 148-150], one can rewrite the integral equation and obtain a kernel where Fredholm's theory applies and in this way obtain automorphic functions associated to a discrete Fuchsian group. The interested reader may consult [ibid:page 149] for explicit integral equations whose solutions produce doubly periodic meromorphic functions with prescribed periods. With today's use of computers it appears that these integral equations can be used to get accurate numerical solutions to extensive families of doubly periodic functions. More could have been said about this approach to study Riemann surfaces. Let us only mention that studies of solutions to differential systems with prescribed monodromy were carried out with great succes in work by Plemelj and readers interested in the theory of fuchsian zeta-functions should consult articles by Uhler.

0.6.1 Fuchsian systems. By elimination one shows that if $y^n + q_1(x)y^{n-1} + \ldots + q_n(x) = 0$ is an irreducible algebraic equation where $\{q_\nu(x)\}$ are rational functions of x, then there exists a unique differential operator

$$Q(x, \partial) = g_{n-1}(x)\partial^{n-1} + \ldots + g_1(x)\partial + g_0(x) \tag{*}$$

where $\{g_\nu(x)\}$ are polynomials without a common factor and the leading polynomial $g_{n-1}(x)$ is monic and every local branch $\alpha(x)$ of a root to the algebraic equation satisfies the homogeneous equation $Q(x,\partial)(\alpha) = 0$. One refers to Q as the associated Fuchsian operator. In § xx we shall construct such Q-operators in some specific examples and discuss how they can be used to investigate the multi-valuedness of the root functions. The operators in (*) belong to the Weyl algebra $A_1(\mathbf{C})$ and are quite special because their generic solution spaces consist of roots to an algebraic equation. A more extensive class of differential operators appears when integrals of algebraic root functions are constructed. To give an example, suppose that the algebraic equation above has a single-valued branch of a root function $\alpha(x)$ defined in a neighborhood of the real interval [0.1] in the complex x-plane. Let ζ be a new complex variable and consider a function

$$F(x,\zeta) = \zeta^m + p_1(\alpha(x)) \cdot \zeta^{m-1} + \ldots p_{m-1}(\alpha(x)) \cdot \zeta + p_m(\alpha(x))$$

where $\{p_\nu(\alpha)\}$ are some polynomials. Suppose that $p_m(\alpha(x)) \neq 0$ when x belongs to the real interval $[0,1]$. Then there exists a small open disc D centered at $\zeta = 0$ and an anlytic function $g(\zeta)$ in this disc defined by

$$g(\zeta) = \int_0^1 \frac{dx}{F(x,\zeta)}$$

It turns out that there exists a differential operator $G(\zeta, \partial_\zeta)$ in the Weyl algebra such that $G(g) = 0$ holds in D and from this one can easily show that g extends to a - in general multi-valued - analytic function in an open complement of a finite subet of the projective ζ-line. Moreover, the differential operator G has a special form, i.e. it is a Fuchsian operator which is used to investigage properties of the multi-valued extension of the locally defined g-function.

0.7 Dynamical systems.

Riemann surfaces and algebraic function fields appear in systems of non-linear differential equations and lead to dynamical considerations. This is an extensive subject which will not be considered here. Let us only remark that Poincare's thesis from 1879 opened a new era in mathematics. An example is the article is *Les fonctions fuchsiennes et l'équation* $\Delta(u) = e^u$ where Poincaré proved that this non-linear second order differential equation has a subharmonic solution with prescribed singularities on every closed Riemann surface attached to an algebraic equation, a work which started potential theoretic analysis on complex manifolds. Let us describe one example of a differential system which appears in the article *xxxx* by Sonja Kovalevsky from 1889. Here one encounters a differential system of six independent functions real-valued functions which depend on the time variable t

$$xxxx$$

In [ibid] it is proved that there exist four integrals to this system, i.e there exist real constants x, x, x such that the following four algebraic equations hold:

$$xxx$$

Starting from this Kovalevsky proved that the solution to the differential system are expressed by θ-functions of the second kind which means that one has constructed specific trancendental functions via integrals of algebraic functions of two variables. To attain this Kovalevsky uses the algebraic structure in the equations to pass into the complex domain where the time variable t takes complex values. The physical siginficance is that the equations above determine the motion for a non-symmetric gyroscope which rotates around a fixed point in $\mathbf{R}^3$ where gravity is the sole external force. A notable fact is that Kovalevsky's gyroscope is the sole non-symmetric example where the Euler equations of a rotating body in Newton's gravitational field can be solved by quadrature. The uniquess was proved by Lyapounov in 1892 and the proof employs geometric constructions which ressemble those which appear for Riemann surfaces.

0.8 Periods of abelian integrals.

Let us finish this introduction by describing a specific result from Abel's work which has a geometric flavour. Consider an equation of the form

$$y^2 = x(x-1)(x-\beta)$$

where the complex number β differs from 0 and 1. This yields a Riemann surface X where $g=1$ and a global holomorphic 1-form is given by

$$\frac{dx}{y}$$

Let γ_1^*, γ_2^* be a canonical basis for the homology which gives the two period numbers

$$w_k = \int_{\gamma_k^*} \frac{dx}{y}$$

Next, consider a complex line ℓ in the 2-dimensional complex (x,y) space defined by some equation

$$\alpha x + \beta y + \gamma = 0$$

When $\beta \neq 0$. we see that $X \cap \ell$ consists of three points whose x-coordinates are roots of the cubic equation

$$x(x-1)(x-\beta) = \frac{1}{\beta^2}(\alpha x + \gamma)^2 \tag{1}$$

One can solve this equation by Cardano's formula. But Abel took another point of view to describe the three intersection points. Suppose the line is chosen so that (1) has three simple roots $\{x_\nu\}$ which determine three points p_1, p_2, p_3 on X where

$$p_\nu = (x_\nu, -\frac{1}{\beta}(\alpha x_\nu + \gamma)$$

Next, starting at the point $p_* = (0,0)$ on X we choose simple curves $\{\gamma_\nu\}$ which join p_* with $\{p_\nu\}$ and integrate ω along each such γ-curve. With these notations the following result is proved in [Abel]:

Theorem. *There exists a constant C which depends on X only such that for every line ℓ as above one has an equation*

$$\sum_{\nu=1}^{\nu=3} \int_{\gamma_\nu} \frac{dx}{dy} = C + m_1 \cdot w_1 + m_2 \cdot m_2 \tag{*}$$

where m_1, m_2 is a pair of integers.

Remark. This equation implies that two intersection points determine the third. Let us for example see why p_3 is determined by p_1 and p_2. Choose the line ℓ^* which passes through p_1 and p_2 and let p_3^* be the third point in $\ell^* \cap XD$. The equation applied to ℓ^* gives a pair of integers m_1', m_2' and an equation

$$\sum_{\nu=1}^{\nu=2} \int_{\gamma_\nu} \frac{dx}{dy} + \int_\gamma \frac{dx}{dy} = C + m_1' + m_2'$$

where γ is a curve from $(0,0)$ to p_3^*. The composed curve $\gamma_3^{-1} \circ \gamma$ joins p_3 with p_3^* and subtracting the equations for the two lines the constant C disappears and we get

$$\int_{p_3}^{p_3^*} \frac{dx}{dy} = (m_1' - m_1)w_1 + (m_2' - m_2)w_2$$

As explained in § xx this implies that $p_3^* = p_3$ which shows that two intersection points determine the third via Abel's addition theorem. A notable fact is the equation (*) in the Theorem not only is necessary, but also gives a sufficient condition in order that three given points on the elliptic curve stay on a line. This could have been established by algebraic computations based upon Cardano's formula but Abel's method has a much wider range and applies to intersection points of the elliptic curves with algebraic curves of arbitrary high degree. The special case above has an obvious conceptual merit and passing to algebraic curves of arbitrary degree $s \geq 3$ Abel proved the following result:

Theorem. *Let $s \geq 3$ and $p_1, \ldots, p_{3s}$ are $3s$ distinct points on the elliptic curve. Then they all belong to an algebraic curve of degree s if and only if there exists a pair of integers m_1, m_2 such that*

$$(*) \qquad \sum_{\nu=1}^{\nu=3s} \int_{\gamma_\nu} \frac{dx}{dy} = C \cdot s + m_1 \cdot w_1 + m_2 \cdot m_2$$

Where $\{\gamma_\nu\}$ are curves on X which join $(0,0)$ to $\{p_\nu\}$ and C the constant of the preceeding theorem.

A result about hyperelliptic curves. Let $g \geq 1$ and consider a Riemann surface X defined by

$$y^2 = p(x)$$

where $p(x)$ is a monic polynomial of degree $2g + 2$ with simple zeros $\{c_\nu\}$. As we shall explain later on X has genus g and a basis for $\Omega(X)$ consists of the 1-forms

$$\frac{x^k dx}{y} \quad : \quad 0 \leq k \leq g - 1$$

Let us now consider polynomials $Q(x)$ of the form

$$Q(x) = x^{g+1} + a_g x^g + \ldots + a_1 x + a_0$$

where $\{a_\nu\}$ is an arbitary $(g+1)$-tuple of complex numbers. As will be explained in § x the number of intersection of the two curves $\{Q = 0\}$ and $\{u^2 - p(x) = 0\}$ when x is finite consists of $2g + 1$ points on X counted with multplicity. Next, let $\{\xi_\nu = (x_\nu, y_\nu)\}$ be some $g + 1$-tuple of points on X where the x-coordinates are finite. Above we dispose $g + 1$ many a-parameters and they can be choosen so that the intersection above contains $\xi_1, \ldots, \xi_{g+1}$ together with another p-tuple of points $\{\xi_{g+2}, \ldots, \xi_{2g+1}\}$ on X. For the generic choice of the first $(g + 1)$-tuple of ξ-points the a-tuple is unique and the last g-tuple depends only upon the assigned $(g + 1)$-tuple. Set

$$\eta_j = \xi^*_{g+1+j} \quad : \quad 1 \leq j \leq g$$

where the *-sign means that when $\xi_{g+1+j} = (x, u)$ in $\mathbf{C}^2$ then we take the *-marked point changes u to $-u$, i.e. the η−-point becomes $(x, -u)$. With these notations Abel proved the following:

Theorem. *There exists a constant C such that the following hold for each $\omega \in \Omega(X)$ and every tuple $(\xi_1, \ldots, \xi_{g+1})$*

$$\sum_{k=1}^{k=g+1} \int_{\xi_*}^{\xi_\nu} \omega = C + \sum_{j=1}^{j=g} \int_{\xi_*}^{\eta_j} \omega$$

Remark. As explained above the η-points are algebraic functions of $\xi_1, \ldots, \xi_{g+1}$ and since we have taken a sum of integrals in the right hand side it is determined by the unordered g-tuple of η-points on X. The theorem constitutes an addition formula since it asserts that the sum of line integrals over a family of $g + 1$ curves is equal to a sum over g curves. When $g = 1$ the theorem above corresponds to an equation which was established by Euler and presumably inspired Abel while he developed the theory of general hyperelliptic equations. The case $g = 2$ can solved rather explicitly and the reader may consult [Appel-Goursat: page 423-425] for detailed formulas expressing Abel's addition theorem for hyper-elliptic curves.

Exercise.

We pose the exercise below in two variables and remark that Abel proved a similar result for any number of variables. By "contemporary terminology" it means that if $T \subset \mathbf{C}^n$ is a hypersurface with k irreducible components defined by irreducible polynomials $P_1, \ldots, P_k$ in $\mathbf{C}[z_1, \ldots, z_n]$, then the integral cohomology of order one over the open set $\mathbf{C}^N \setminus T)$ is a free abelian group generated by the k-tuple of ∂-closed 1-forms $\{\frac{\partial P_\nu}{P_\nu}\}$. Now we give the exercise: Let $R(x,y)$ and $Q(x,y)$ be a pair of rational functions in two variables. Since the polynomial ring $\mathbf{C}[x,y]$ is a unique factorisation domain each rational funtion is the quotient of two polynomials with no common factor. Suppoose that the denomantors of r and W are non-zero at the origin in $\mathbf{C}^2$ and set

$$\Phi(x,y) = \int_0^x R(\zeta,y)\,d\zeta + \int_0^y Q(x,\eta,y)\,d\eta +$$

which is defined when (x,y) stays close to the originö Assume also that the differential 1-form $Rdx+Qdy$ is exact which means that

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial x}$$

Show that there exists a unique pair F,P where P is a polynomial with $P(0,0)=1$ and F a rational function such that

$$\Phi(x,y) = R(x,y) + \log P(x,y)$$

holds in a neighborhood if the origin.

Further comments on Abel's work.

Prior to the cited article in the introduction we recall that Abel proved that the general algebraic equations of order five cannot be solved by roots and radicals. The proof uses the symmetric group with 120 elements and Abel also introduced the general notion of field extensions with arbitrary degree of transcendency. These revolutionary algebraic methods appear frequently in Abel's later work. For an expositions of Abel's contributions in mathematics the reader can consult the book *The Abel Legacy* published at the innaguration of the Abel Prize which since 2004 is considered as one the most prestigous prizes in mathematics. In my article from [ibid] comments are given on Abel's proof that the general algebraic equation of degree 5 cannot be solved by roots and radicals where an example is $x^5+x+1=0$. This was proved in 1823 and appears in the article [Abel] which was published shortly after his decease. The demonstration is not via a "single example" but derived from an explicit over-determined system of linear equations which exhibits when a given algebraic equation of degree five cannot be solved by roots and radicals. For equations of higher order the strategy in Abel's proof leads to complicated linear systems of high order since the symmetric group of n many elements increases very fast. Two years after Abel's decease, Galois discovered that group theory leads to criteria fror non-solvability which is more transparent and Galois theory has therefore become a subject in many text-books. However, Abel's original proof has the merit that today's use of computers can be applied to analyze his linear systems and in this way get decide when an specific algebraic equation has solutions by roots and radicals in situations where the its associated Galois group is not easy to handle.

I. Algebraic functions and Riemann surfaces.

In this section we demonstrate how algebraic function fields determine a compact Riemann surface. Before this is done in § 1.3 and § 1.4 we discuss the genus and the degree of meromorphic functions on a compact Riemann surface.

1.1 The genus number.

Denote by $\mathfrak{M}(X)$ the field of meromorphic functions on X on a compact one-dimensionsal complex manifold. A major result is that $\mathfrak{M}(X)$ is a finitely generated field extension over $\mathbf{C}$ whose degree of trancendency is equal to one. To prove this one must establish the existence of an ample family of meromorphic functions on an "abstract" compact Riemann surface. This is done in § 1 using sheaf theory where the crucial result is that the cohomology space $H^1(X,\mathcal{O})$ is a finite dimensional complex vector space when $\mathcal{O}$ is the sheaf of holomorphic functions on X. This gives the integer

$$(*) \qquad g = \dim_{\mathbf{C}} H^1(X,\mathcal{O})$$

It turns out that g is a topological invariant of X. Namely, X is an oriented compact manifold of real dimension two and a wellknown result in topology asserts that X is homeomorphic to a sphere where a finite number of handles attached. Denote this number by $\text{gen}(X)$. With these notations one has the equality:

$$(**) \qquad g = \text{gen}(X)$$

The quickest proof is described in § xx using an induction over g. However, more insight is obtained by a duality result which shows that $H^1(X,\Omega)$ in a canonical way is isomorphic to the dual of the finite dimensional vector space $H^0(X,\Omega_X)$ which consists of globally defined holomorphic 1-forms, or as one says abelian differentials of the first kind on X. The proof is given in § 2. If we admit this duality one derives (**) as follows: Let Ω_X denote the sheaf of holomorphic functions. In local charts every holomorphic function f has a primitive which gives an exact sequence of sheaves:

$$0 \to \mathbf{C}_X \to \mathcal{O}_X \xrightarrow{\partial} \Omega_X \to 0$$

By the maximum principle a globally defined holomorphic function on the compact manifold X is reduced to a constant which means that $H^0(X,\mathcal{O}_X)$ is reduced to a 1-dimensional complex vector space. Hence the construction of long exact sequences in sheaf theory yields an exact sequence of finite dimensional complex vector spaces:

$$(***) \qquad 0 \to H^0(X,\Omega_X) \to H^1(X,\mathbf{C}_X) \to H^1(X,\mathcal{O}_X) \to H^1(X,\Omega_X) \to H^2(X,\mathbf{C}_X) \to 0$$

Since X is an oriented and compact real manifold of dimension two, Poincaré duality shows that $H^2(X,\mathbf{C}_X)$ is 1-dimensional. Moreover, the duality in § XX will show that $H^1(X,\Omega_X)$ is isomorphic to the dual of the 1-dimensional space $H^0(X,\mathcal{O}_X)$ and therefore also 1-dimensional. Counting dimensions in (***) it follows that

$$(1) \qquad \dim(H^1(X,\mathbf{C}_X)) = \dim(H^0(X,\Omega_X)) + \dim(H^1(X,\mathcal{O}_X))$$

Next, by the duality above $\dim(H^0(X,\Omega_X))$ is isomorphic to the dual vector space of $H^1(X,\mathcal{O}_X)$ so (1) and the definition of g gives the equality

$$(2) \qquad \dim(H^1(X,\mathbf{C}_X)) = 2 \cdot g$$

which by general topology gives (**).

Remark. When X and Y are two compact Riemann surfaces with the same genus they are homeomorphic as manifolds. But this does not imply that they are biholomorphic as complex manifolds. An exception is the case $g = 0$ when X always is biholomorphic with the projective line $\mathbf{P}^1$ which is identified with the Riemann sphere where the point at infinity is added to the complex z-plane. Here the field $\mathfrak{M}(\mathbf{P}^1)$ is equal to the field of rational functions of z. If $g = 1$ then X is homeomorphic to a torus $\mathbf{T}^2$ and in § XX we classify the conformal equivalence classes of such compact Riemann surfaces.

A formula for g

The results above enable us to compute g starting from an arbitrary non-constant meromorphic function f on X. Namely, let $p_1,\ldots,p_k$ be the set of poles and denote by $\{\mu_f(p_\nu)\}$ the orders of these poles. Next, the 1-form ∂f is holomorphic outside the poles and let $n_+(\partial f)$ denote the number of zeros of ∂f counted with their multiplicities. With these notations one has the equality below referred to as the Riemann-Hurwitz formula:

$$g = 1 + \frac{n_+(f)}{2} - \frac{1}{2}\cdot\sum_{\nu=1}^{\nu=k}(\mu_f(p_\nu)+1) \tag{1.1}$$

The proof is given in § X. As an illustration, consider the case when $X = \mathbf{P}^1$ is the projective line. Let $p(z)$ be a polynomial of degree $n \geq 1$. It is a meromorphic function on X whose pole at the point at infinity has order $n+1$ and zeros of ∂p correspond to zeros of the derivative p' so that $n_+(p) = n-1$. Now (i) gives $g = 0$ which reflects that $\mathbf{P}^1$ has genus number zero. A more involved case occurs if we consider the algebraic curve defined by the equation

$$y^3 + x^3 = 1 \tag{1.2}$$

where x and y are independent complex variables. In § XX we expain that its closure taken in the 2-dimensional projective space $\mathbf{P}^2$ yields a non-singular projective curve which therefore is a compact Riemann surface denoted by X. Here y is a meromorphic function with three simple poles and at the three points above $x = 0$ where y is a third root of unity, one finds that ∂y has zeros of order 2. From (1.1) we deduce that $g = 1$ which means that X is homeomorphic to a torus. It is not easy to visualize this and the present author is unable to get a picture which explain why X is homeomorphic with a torus. So in the long run it is essential to carry out calculations with the aid of theoretical results. In our example the 1-form

$$\omega = \frac{dy}{x^2}$$

is everywhere holomorphic and $\neq 0$ on X and in § XX we construct a pair of simple closed curves γ_1, γ_2 on X such that the two line integrals

$$\int_{\gamma_\nu} \frac{dy}{x^2}$$

are non-zero and yield a pair of $\mathbf{R}$-linearly independent complex numbers which confirms that X is a torus.

1.2 The degree of a meromorphic function.

With $\mathbf{P}^1$ considered as a compact complex manifold a non-constant meromorphic function f on a compact Riemann surface X yields a holomorphic map from X onto $\mathbf{P}^1$. It turns out that there exists a finite set of points $a_1,\ldots,a_k$ in $\mathbf{P}^1$ such that if $Y = \mathbf{P}^1 \setminus (a_1,\ldots,a_k)$, then f yields a locally biholomorphic map from $f^{-1}(Y)$ onto Y where each inverse fiber consists of n points in X. The integer n called the degree of f and is denoted by $\deg(f)$. This integer is robust in the sense that for every complex number $c \in \mathbf{P}$ the inverse fiber $f^{-1}(c)$ consists of a finite number of points and counting multiplicities this number is equal to n. To see this one employs the meromorphic 1-form ∂f.

1.2.1 The use of Stokes Theorem. On a compact Riemann surface X one can perform line integrals of differential 1-forms in the same way as in open subsets of $\mathbf{C}$. Let $f \in \mathfrak{M}(X)$ and $\{p_k\}$ the finite set of poles. If c is a complex constant we consider $g = f - c$ which has the same poies as f and let $\{q_j\}$ be the zeros of g, i.e. points in X where $f(q_j) = c$. Around the zeros and the poles of g we choose small charts which are biholomorphic to open discs and take their union U. The boundary ∂U consists of a finite set of disjoint closed curves where each is a copy of a circle in a chart. In $X \setminus \bar{U}$ there exists the 1-form

$$\frac{\partial g}{g} = \frac{\partial f}{f-c}$$

Since g is holomorphic in U this 1-form is closed and Stokes theorem gives

(1) $$\int_{\partial U} \frac{\partial f}{f-c} = 0$$

In a chart with local coordinate z around a zero of $f-c$ with multiplicity e we have

$$\frac{\partial f}{f-c} = \frac{f'(z)}{f(z-c)} \cdot dz = e \cdot \frac{dz}{z} + \quad \text{a holomorphic form}$$

Hence the line integral along a small circle centered at $z=0$ has value $2\pi i \cdot e$. Next, consider a chart around a pole of some order e. Here

$$\frac{\partial f}{f-c} = -e \cdot \frac{dz}{z} + \quad \text{a holomorphic form}$$

where the sign is reversed. Now (1) implies that the number n of zeros counted with multiplicity is equal to the number of poles counted with their order and by definition this common integer n is the degree of f. This confirms the robustness of the degree.

1.2.2 Weierstrass points. Let X be a Riemann surface where $g \geq 2$. Then there exists a finite set of points on X denoted by $\mathcal{W}(X)$ and for each $p \in \mathcal{W}(X)$ there exists a non-constant meromorphic function f on X which is holomorphic in $X \setminus \{p\}$ and has a pole of order $\leq g$ at p. This result is due to Weierstrass and the proof is given in § XX. The case $g=2$ is of special interest for then the meromorphic function f has degree two and yields a 2-sheeted covering map from x onto $\mathbf{P}^1$.

1.2.3 An algebraic equation. Let f be meromorphic on X with some degree n. A complex number c is called an f-critical value if $\partial f(z) = 0$ for some $z \in f^{-1}(z)$. The set of critical points is finite and we let σ_f be the union of these points and the point at infinity, i,e. we avoid poles on f when we take the open set $X \setminus f^{-1}(\sigma_f)$. For each $z \in \mathbf{P}^1 \setminus \sigma_f$ the inverse fiber $f^{-1}(z)$ is an unordered n-tuple $x_1(z), \ldots x_n(z)$ of points in X. Next, consider another meromorphic function ϕ on X and with a new variable λ we set

$$\Phi(\lambda) = \prod_{k=1}^{k=n} (\lambda - \phi(x_k(z))$$

When z varies in $\mathbf{P}^1 \setminus \sigma_f$ the coefficients of the λ-polynomial are symmetric polynomials of the n-tuple $\{\phi(x_k(z))\}$ which entails that

$$\phi(\lambda) = \lambda^n + r_{n-1}(z)\lambda^{n-1} + \ldots + r_1(z)\lambda + r_0(z)$$

where $\{r_j(z)\}$ are ordinary rational functions of the single complex variable z. Returning to X it follows that

(1) $$\phi^n + r_{n-1}(f) \cdot \phi^{n-1} + \ldots + r_1(f)\phi + r_0(f)$$

Since such an equation exists for every $\phi \in \mathfrak{M}(X)$ it follows from elementary algebra that the field $\mathfrak{M}(X)$ is an algebraic extension of the subfield $\mathbf{C}(f)$ formed by rational expressions of the meromorphic function f whose dimension as a complex vector space over $\mathbf{C}(f)$ is equal to $\deg(f)$. Here $\mathbf{C}(f)$ can be identified with the standard field $\mathbf{C}(z)$ of rational funtions of a single variable and one says therefore that $\mathfrak{M}(X)$ is an algebraic function field of one variable.

1.3 Algebraic function fields

An algebraic function field K over $\mathbf{C}$ is an abstract field which contains $\mathbf{C}$ as a subfield with degree of trancendency equal to one and generated by a finite number of elements $k_1, \ldots, k_m$. This means that every $k \in K$ can be expressed as quotient of polynomials in $k_1, \ldots, k_m$ with coefficients in $\mathbf{C}$. Elementary algebra teaches that when K is such a field and $\xi \in K \setminus \mathbf{C}$ then there exists $\eta \in K$ such that K is generated by η and the field $\mathbf{C}(\xi)$ whose elements are rational functions in ξ with complex coefficients. Here η satisfies an equation

$$\eta^m + r_{m-1}(\xi)\eta^{m-1} + \ldots + r_1(\xi)\eta + r_0(\xi) = 0$$

where $\{r_j(\xi)\}$ belong to $\mathbf{C}(\xi)$ and each $k \in K$ can be written as

$$k = q_{m-1}(\xi)\eta^{m-1} + \ldots + q_1(\xi\eta + q_0(\xi) = 0$$

where $\{q_j(\xi)\}$ is a unique m-tuple in $\mathbf{C}(\xi)$. Moreover, the polynomial in a new indeterminate variable t given by

$$P(t, \xi) = t^m + r_{m-1}(\xi)t^{m-1} + \ldots + r_1(\xi) + r_0(\xi)$$

is irreducible in the polynomial ring of t over the field $\mathbf{C}(\xi)$. This is be expressed by saying that if $K_* = \mathbf{C}(z)$ is the standard field of rational functions in one variable, then each algebraic function field is isomorphic to a field

$$\frac{K_*[t]}{(P)}$$

where (P) denotes the principal ideal generated by an irreducible polynomial in $K_*[t]$. Here we used that $K_*[t]$ is an euclidian ring which implies that this K_*-algebra is a unique factorisation domain and hence onre has a precise meaning for a polynomial $P[t)$ to be irreducible.

Remark. When the algebraic function field K is given we can choose a trancendental element ξ in many ways. Once ξ is chosen we obtain the finite field extension where $\mathbf{C}(\xi)$ appears as a subalgebra and K regarded as a vector space over this subfield has a finite dimension. More precisely one has the equality

$$\dim_{\mathbf{C}(\xi)}(K) = \deg(\eta)$$

where η is a primitive element whose minimal equation (*) has degree m. Let us also remark that if a trancendental elment ξ is chosen then there is a whole family of primitive η-generators, i.e. for every $\eta \in K$ whose minimal equation (1) has a degree which equals the dimension of K as a vector space over $\mathbf{C}(\xi)$ yields a primitive element in K with respect to the chosen trancendental ξ-element. Finally, two algebraic function fields K_1 and K_2 are isomorphic if there exists a $\mathbf{C}$-algebra isomorphism between them. With this kept in mind one has the following:

1.3.1 Theorem. *There is a 1-1 correspondence between the family of algebraic function fields in one variable and the family of compact Riemann surfaces.*

Above a pair of Riemann surfaces are identified when they are bi-holomorphic. Several steps are needed to prove (1.3.1). First Weyl's theorem shows that $\mathfrak{M}(X)$ is an algebraic function field for every compact Riemann surface. Next, let X_1, X_2 be a pair of biholomorphic Riemann surfaces and let $\rho\colon X_1 \to X_2$ be a biholomorphic mapping. Then $\mathfrak{M}(X_1) \simeq \mathfrak{M}(X_2)$ since ρ gives the algebra isomorphism which sends $f \in \mathfrak{M}(X_2)$ to the meromorphic function on X_1 defined by

$$f^*(x_1) = f(\rho(x_1) \quad : \quad x_1 \in X_1$$

There remains to show show that for every algebraic function field K there exists a compact Riemann surface X such that $K \simeq \mathfrak{M}(X)$. Moreover we must show that if X_1 and X_2 are two Riemann surfaces where the algebraic function fields $\mathfrak{M}(X_1)$ and $\mathfrak{M}(X_2)$ are isomorphic then X_1 and X_2 are biholomorphic. To prove we consider an arbitrary algebraic function field K. A valuation map on K is an injective algebra homomorphism

$$\rho\colon\ K \to \mathbf{C}\{t\}[t^{-1}] \tag{1}$$

where the right hand side is the standard field of germs of meromorphic functions at the origin with t regarded as a complex variable. In addition one requires that ρ is non-degenerated in the sense that there exists some $k \in K$ such that

$$\rho(k) = t + \sum_{\nu=2}^{\infty} c_\nu \cdot t^\nu \tag{2}$$

i.e. this germ is holomorphic and its t-derivative is non-zero when $t = 0$. The ρ-map defines a valuation on K as follows: Each non-zero element k gives the meromorphic germ $\rho(k)$ and we find the unique integer $\rho_*(k)$ such that

$$\rho(k) = t^{\rho_*(k)} \cdot \phi(t)$$

where $\phi(t)$ is a unit in the local ring $\mathbf{C}\{t\}$, i.e its constant term is $\neq 0$. Let $\mathcal{V}(K)$ denote the family of all valuations maps on K. It turns out that $\mathcal{V}(K)$ corresponds to points in a compact Riemann surface X and that $K \simeq \mathfrak{M}(X)$ where each point $x \in X$ yields a valuation since there exists a local chart around X with a coordinate t so that every $f \in \mathfrak{M}$ has a series expansion at x expressed by a an element in $\rho_f(t) \in \mathbf{C}\{t\}[t^{-1}]$. By analyticity the map $f \to \rho_f(t)$ is injective which clarifies the1-1 correspondence between points on X and valuation maps on $\mathfrak{M}(X)$.

So the main burden is to prove that there exists an ample family of valuations on a given algebraic function field K and explain how these valuations fabricate points on a Riemann surface. This is done in § 1.4 below.

1.4 Algebraic curves.

Recall that the polynomial ring $\mathbf{C}[x, y]$ in two variabes is a unique factorisation domain. Let $n \geq 2$ and consider an irreducible polynomial

$$P(x, y) = y^n + q_1(x)y^{n-1} + \ldots + q_{n-1}(x)y + q_n(x) \tag{1}$$

To P corresponds the algebraic function field K whose elements are

$$k = r_0(x) + r_1(x)y + \ldots + r_{n-1}(x)y^{n-1} \quad \text{where} \quad r_0, \ldots, r_{n-1} \in \mathbf{C}(x) \tag{2}$$

Next, we get the algebraic curve S in $\mathbf{C}^2$ defined by $P = 0$. It has a closure in the projective space $\mathbf{P}^2$ whose homogeneous coordinates are $(\zeta_0, \zeta_1, \zeta_2)$. Points in the (x, y)-space are represented by $(1, x, y)$. The hyperplane at infinity is $\{\zeta_0 = 0\}$ and points $(x, y) \in S$ converge to this hyperplane when $|x| \to +\infty$ which yields the closure $\bar{S}$ in the compact manifold $\mathbf{P}^2$. Tbe boundary $\partial S = \bar{S} \setminus S$ is a finite set of at most n points. As an example, suppose that $\deg q_\nu \leq \nu$ hold for the q-polynomials in (1) For each $1 \leq \nu \leq n$ we denote by c_ν the coefficient of x^ν in $q_\nu(x)$. Now

$$P^*(x, y) = y^n + c_1 x y^{n-1} + \ldots + c_{n-1} y x^{n-1} + c_n x^n$$

is a homogeneous polynomial and the fundamental theorem of algebra entails that

$$P^*(x, y) = \prod_{k=1}^{k=m} (y - \beta_k x)^{e_k}$$

where $\{\beta_k\}$ are distinct complex numbers and $e_1 + \ldots + e_m = n$. Then ∂S consists of the m-tuple points $(0, 1, \beta_1), \ldots, (0, 1, \beta_m)$. The case where $\deg(q_\nu) < n$ for each ν is not excluded in which case $P^*(y) = y^n$ and ∂S is reduced to the single point $(0, 1, 0)$.

1.4.1 Regular points on S. Let us restrict the attention to the affine curve S. To analyze this affine curve we first consider the polynomial

$$P'_y(x, y) = ny^{n-1} + (n-1)q_1(x)y^{n-2} + \ldots + q_{n-1}(x)$$

and let us recall the following from Euclides' algebra:

1.4.2 The discriminant polynomial. By assumption P is irreducible in the polynomial ring $K[y]$ in the single variable y where k denotes the field $\mathbf{C}(x)$. It follows by euclidian divisions that there exists a unique pair $A(y), B(y)$ in $K[y]$ such that

$$A(y)\dot{P}(x, y) + B(y) \cdot P'_y(x, y) = 1$$

where the degree of the y-polynomial A is at most $n - 1$ and that of B at most $(n - 2)$. Next, we find a common factor for the denominstors in the K-coefficients of these two y-polynomials whch yields a unique monic polynomial $\delta(x)$ in $\mathbf{C}[x]$ such that one has the equation

$$A_*(x, y)\dot{P}(x, y) + B_*(x, y) \cdot P'_y(x, y) = \delta(x) \tag{1.4.2}$$

where A_* and B_* now are polynomials in x and y which both are primitive with respect to x. For example, one has

$$A_*(x, y) = a_{n-2}(x)y^{n-2} + \ldots a_1(x(y + a_0(x)$$

where the x-polynomials $\{a_k(x)\}$ have no common factor. We refer to $\delta(x)$ as the discriminant polynomial of P.

1.4.3 Root fiunctions. For each fixed n the fundamenntal theorem of algebra yields an n-tuple of roots to the equation $P(y,x)=0$ and we can write

$$P(y,x)=\prod_{k=1}^{k=n}(y-\alpha_k(x))$$

From (1.4.2) we see that the roots are all simle if and only if $\delta(x)\neq 0$. The finite set $\{\delta(x)=0\}$ is called the discriminant locus so for each z in the open and connected set $\mathbf{C}\setminus\delta^{-1}(0)$ we find the unordered n-tuple of simple roots. As explained in § XX they give rise to germs of analytic functions of x and can be extended to multi-valued analytic functions in $\mathbf{C}\setminus\delta^{-1}(0)$. By analyticity each new local bracnh is again a root. For example, start at some point $x_0\in\mathbf{C}\setminus\delta^{-1}(0)$ and pick one root $\alpha_1(x_0)$ which to begin with gives an analytic function $\alpha_1(z)$ in a small open disc centered at z_0. It is now extended in the sense of Weierstrass and the multi-valued function produces a finite set of local branches at z_0. The fact that P from the start is irreducible entails that the local branches under all possible the analytic continuations of α_1 along closed curves in $\mathbf{C}\setminus\delta^{-1}(0)$ which start and finish at z_0, will produce local branches of all roots. The conclusion is that the set

$$S_*=S\setminus\delta^{-1}(0)$$

is connected and the projection $\pi(x,y)=x$ restricts to an n-sheeted covering map from S_* onto $\mathbf{C}\setminus\delta^{-1}(0)$. Moeover, since the root functions are analytic it follos that S_* appears as a 1-dimensional complex submanifold of $\mathbf{C}^2\setminus\delta^{-1}(0)$. Finally, by continuity of roots the closure of S_* taken in $\mathbf{C}^2$ is equal to S.

Remark. The claims made above are classic and exposed in many text-books. We remark only that this was the starting point for Riemann to construct Riemann surfaces and that Weierstrass extended the ressukt above to polynomials in y which may depend upon several x-variables. But in this case the discrimant locus is a hypersurface in a multi-dimensional complex vector space so the properties of the algebraic hypersurface $P^{-1}(0)$ when $P=P(x_1,\ldots,x_n,y)$ is an irreducible polynomial of $n+1$ many variables is more involved and goes far beyond the material in these notes. Let us only mention that the subsequet construction for curves were extended by Zariski to the case of surfaces, i.e. when $n=2$ and for arbitrary $n\geq 3$ a so called desingularisation of the algebraic hypersurface $P^{-1}(0)$ was established by Hironka in the article *xxx* from 1962.

1.4.4 The construction of local charts. let $p=(x_*,y_*)$ be a point in S. Consider the local ring $\mathcal{O}=\mathbf{C}\{x-x_*\}$ of germs of analytic functions in the complex x-variable at x_*. Now $P(x,y)$ is an element in the polynomial ring $\mathcal{O}[y]$ which by elementary algebra is a so called unique factroisation domain. it follows that there exists a unique factorisation

$$P(y,x)=q_*(x,y)\cdot\prod_{k=1}^{k=r}\phi_k(x,y)$$

where $q_*(x_*,y_*)\neq 0$ while $\{\phi_k\}$ are irreducible Weierstrass polynomials in y with coefficients $\mathcal{O}$. It means that each ϕ_k is of the form

$$\phi_k(y,x)=y^{e_k}+\rho_{1,k}(x)\cdot y^{e_k-1}+\ldots+\rho_{e_k,k}(x)$$

where the ρ-functions belong to $\mathcal{O}$ and all vanish at $x=x_*$. So in a neighborhood of $p=(x_*,y_*)$ the curve S is defined by common zeros of the ϕ-functions.

Puiseux charts. Let us fix one ϕ-polynomial say ϕ_1. In a small punctured disc centered at x_* the y-polynomial $\phi_1(x,y)$ has e_1 many simple zeros which occur among roots of $P(x,y)$. As explained in § xx we introduce a new complex variable ζ and find an analytic function $A_1(\zeta)$ in a disc of some radius $r_1>0$ centered at $\zeta=0$ such that

$$q_1(x_*+\zeta^{e_1},y_*+A_1(\zeta)=0\quad:\ |\zeta|<r_1$$

Moreover, since q_1 was irreducible the Taylor series

$$A_1(\zeta) = \sum_{\nu=1}^{\infty} a_\nu \zeta^\nu$$

is such that the principal ideal in $\mathbf{Z}$ generated by those integers for which $a_\nu \neq 0$ does not contain any prime divisor of e_1. This entails that the map

$$\zeta \to (x_* + \zeta^{e-1}, y_* + A_1(\zeta))$$

is bijective, i.e. the open ζ-disc can be identified with a subset of the given affine curve S where $\zeta = 0$ is mapped to $p = (x_*, y_*)$. In this subset of S we can wirte

$$x = x_* + \zeta^{e_1} \quad : \ y = y_* + A_1(\zeta)$$

It means that one can use ζ as a local coordinate and the image of the ζ-disc constitues a chart in the Riemann surface X attached to the curve S. We refer to this as a Puiseux chart. It can be constructed for each ϕ-function which appears in (1.4.4 *).

1.4.5 Riemann's construction. To get a Riemann surface X associated to the curve S one must separate the Puiseux charts above. in orher words, Riemann regarded these charts as disjoint. So if $r \geq 2$ then X contains r *distinct* points above (x_*, y_*). In this way one gets a complex 1-dimensional manifold X and a map $\rho\colon X \to S$ which is bijective except for those points $p = (x_*, y_*)$ in S where more than one irreducible ϕ-function appears in (1.4.3 *).

1.4.6 Example. Consider the irreducible polynomial

$$P(x,y) = y^4 - x^2(x+1)$$

At the point $p = (0,0)$ on the curve $S = P^{-1}(0)$ we get a factorisation

$$P = \phi_1 \cdot \phi_2$$

where we choose a local bracnh of $\sqrt{1+x}$ so that

$$\phi_1(x,y) = y^2 - x \cdot \sqrt{1+x} \quad : \phi_2(x,y) = y^2 + x \cdot \sqrt{1+x}$$

In this way we obtain teo distinct Puiseux-Riemann charts so on X the map $\rho\colon X \to S$ has an inverse fiber above the origin which contains two points. In our special case we notice that y serves as a local coordinate in each of these charts. Passing to the Rieann surfsce it means that y has a simple zero in each of the two Puiseux-Riemann charts. In addition one finds that y has a simple zero above $x = 1$ and after we have constructed charts above $x0\infty$ we shall be abel to conclude that y as a meromorphic funtion on X has three simple zeros and a triple pole at the unique point on X where $x = \infty$.

Remark. The reder shoud be aware of the spearated charts which occur when $r \geq 2$. one tries to "detect" a Riemann surface by drawing pictures the cases when such separated charts appear often cause visual problems. So in the long run it is not only conventient but more or less necessary to perform local calculations which lead to the complete description of Puiseux-Riemann charts above singular points on the affine curve S.

1.4.7 The passage to infinity. There remains to construct charts around points in ∂S. In § 1.4.x we explain how this is reduced to the affine case above where the strategy is to perform a suitable transformation of $[\mathbf{P})^2$. But let is first give some examples to illustrate the calculations. Consider the polynomial

$$P(x,y) = y^3 - x^6 - 1$$

When $|x|$ is large we have $|y| \simeq |x|^2$ which means that the y-coordinate tends faster to infinity than the x-coordinate. So ∂S is reduced to the single point $p^* = (0,0,1)$. In $\mathbf{P}^2$ we have local coorfionstes (ζ, η) around p^* which corresponds to points (ζ, η). When $\zeta \neq 0$ we are outside the hyplerplane at infinity and have

$$x = \frac{\eta}{\zeta} \quad : \ y = \frac{1}{\zeta}$$

So the equation $P = 0$ means that

$$\zeta^{-3} = \frac{\eta^6}{\zeta^6} - 1 \implies \zeta^3 = \eta^6 - \zeta^6$$

The equation $\zeta^3 + \zeta^6 - \eta^6 = 0$ corrsspinds to a union of three zero sets expressed by three functions ϕ_0, ϕ_1, ϕ_2 where

$$\phi_k(\zeta, \eta) = \zeta - b_k(\eta)$$

and $b_k(\eta)$ has a Taylor series of the form

$$b_k(\eta) = e^{2\pi ik/3} \cdot \eta^2 + \text{higher order terms}$$

The conclusion is that there exists three Puiseux-Riemann charts and in each chart we can take η as a local coordinate. Since the the b-functions have zeros of order two it follos that y gets a double pole in each chart. Passing to the compact Riemann surface $\ddot{}X$ itmeans that the meromorphic function y gets three distinct double poles. At thes same time we ntoice that in the finite affine part the y-fuynction has six simple zeros which appear when x solves the equation $x^6 = 1$. So the number of poles counted with multiplicity is equal to the number of zeros as it should be. treating x as a meromorphic function we see that it has simple poles in each of the three Puiseux-Riemann charts arond $p*$. At thes same time se cencounter three simple zeros ehich correspond to the points $(0, j_\nu)$ where $j_\nu^3 = 1$ are 3rd-roots of unity. Let us also consider the 1-form

$$\omega = \frac{dx}{y}$$

Since y has a double pole and x a simple pole in the charts around p^* we see that ω is holomorphic in these charts and also $\neq 0$. Next, in the finite part S we notice that $dx = 0$ can only occur when $y = 0$ an this occurs at six points $\{j_\nu, 0)\}$ where $j_\nu^6 = 1$. At each of these points$y$ is a local coordinste and x has a zero of order three. Hence the 1-form dx has a zero of order teo. We conclude that ω has six simple zeros. From this the Hurwitz-Riemann formula implies that X has genus four.

1.4.8 The curve $y^3 = x(x-1)^2$. Here $|y|^2 \simeq |x|$ and this time ∂S contains three points:

$$p_k^* = (0, 1, e^{2\pi ik/3}) \quad : \ k = 0, 1, 2$$

We leve it to the reder to verify that around each p_k^* we get a single Puiseux-Riemann chart and both x and y have simple poles at these three p^*-points. Next, in the finite part x has a triple zero at $(0, 0)$ where y serves as a local coorfinste. At $(1, 0)$ we have a cusp-like sungularity and here the Pusiuex-Riemann chart has a local coordinate ζ where

$$x = 1 + \zeta^3 \quad : \quad y = A(\zeta)$$

where $A(\zeta)$ has a zero of order two. Let us then consider the 1-form

$$\omega = \frac{dx}{y^2}$$

From the above it is holomorhic and $\neq 0$ at the p^*-points. At $(0, 0)$ x has a triple zero so dx has a double zero and since y has a simple zero we conclude that ω is holomorphic and $\neq 0$ at $(0, 0)$ At $(1, 0)$ the reader may verify that ω has a double pole. The result is that the divisor D for which $\mathcal{O}_D \cdot \omega = \Omega$ has degree -2 and it follows from the Riemann-Hurwitz formula that the genus of X is zero. If we set

$$f = \frac{x-1}{y}$$

then ∞f is a meromorphic function and from the above we see that it is holomophic and $\neq 0$ at the p^*-points while it has a simple pole at $(0, 0)$ and a simple zero at $(1, 0)$. Therefore $\mathfrak{M}(X)$ is reduced to the field $\mathbf{C}(f)$ which means that both x and y can be expressed in this field. That this holds is easily verified. The reader may for example show that

$$x = \frac{1}{1 - f^3}$$

Remark. The example above illustrates that even if one may be "lucky" to discover the existence of a function like f, it is more systematic to proceed with a construction of charts in X and eventually discover f via the positions of poles and zeros of y and $x-1$.

1.6 Intersection numbers.

A pair of different irreducible polynomials $P(x,y)$ and $Q(x,y)$ give the two curves $\{P=0\}$ and $\{Q=0\}$ in $\mathbf{P}^2$. We are going to assign an intersection number which takes into the account eventual multiplicities. A "nice case" occurs if the points of intersection appear in the regular parts of the two curves and the gradient vectors of p and q are linearly independent at every such point. Then one refers to a simple transversal intersection and the intersection number is equal to the number of transversal intersection points. A general procedure to assign an integer to the pair P and Q is the following: Let X be the Riemann surface associated to the projective curve defined by P. Now Q is a meromorphic function on X and we can count its number of zeros with multiplicites which yields an integer denoted by $\mathbf{i}(P;Q)$. Reversing the role and we take the Riemann surface Y associated with $\{Q=0\}$ and count the number of zeros for the meromorphic P-function on Y which gives another integer $\mathbf{i}(Q;P)$. It turns out that one has the equality

$$(*) \qquad \mathbf{i}(P;Q) = \mathbf{i}(P;Q)$$

Keeping P fixed the integer $\mathbf{i}(P;Q)$ is the degree of the meromorphic function Q on X. This entails that the intersection number is unchanged when Q is replaced by $Q-\alpha$ for a complex number as long as $Q-\alpha$ is not reduced to a constant function on X. A similar invariance hold when we repace P by $P-|beta$. Si intersection numbers enjoy a robust property. The proof of (*) uses Jacobi's residue for the pair P and Q which we construct below in a local analytic context, i.e. we shall work in small neighborhoods of the points of intersection between the two projective curves.

1.6.1 Example. Let $P(x,y) = x^2+a-y$ and $Q(x,y) = y^2-b-x^3$ where both where a and b are complex numbers. If S is the closure of $\{P=0\}$ taken in $\mathbf{P}^2$ then ∂S is reduced to the single point $(0,0,1)$ while the projective closure of $\{Q=0\}$ consists of $(0,1,0)$. Hence the curves only intersect in $\mathbf{C}^2$. On the Riemann surface X attached to $\{P=0\}$ we see that the meromorphic Q-function becomes

$$(x^2+a)^2-b-x^3$$

The number of zeros is 4 for all pairs a,b. In the case $a=b=0$ we get $x^3(x-1)$ which means that a zero of order three occurs at $(0,0)$ and geometrically one verifies that the two curves $\{P=0\}$ and $\{Q=0\}$ do not intersect transversally at the origin but have a contact of order three while a transveral intersection occurs when $x=1$, i.e at the point $(1,1-a)$ on X.

$$x^2-a-y=0 \quad : \quad y^2-b-x^3=0$$

This gives

$$(x^2+a)^2 = b+x^3$$

Let us reverse the role and with $b=0$ we consider the Riemann surface Y attached to $\{y^2-x^3\}$. At the origin we get the local coordinate ζ with

$$x=\zeta^2 \quad : \quad y=\zeta^3$$

Here $P=x^2-y=\zeta^4-\zeta^3$ has a zero of order three and at the point $(1,1)$ on Y one verifies that P has a simple zero, i.e. the total number of zeros of the meromorphic function P on Y is equal to four as predicted by Jacobi.

1.6.2 Example. Let $P(x,y) = y^2-x^2-1$ while $Q(x,y) = y^2-2x^2+L(x,y)$ where $L(x,y)$ is some linear polynomial. Here $\{P=0\}$ and we notice that its boundary at infinity is reduced to the points $(0,1,1)$ and $(0,1-1)$ while those of $\{Q=0\}$ are $(0,1,\sqrt{2})$ and $(0,1,-\sqrt{2})$ To compute the number of zeros of Q on the Riemann surface X we can equally well count the number of poles. At the two points $p_1^*=(0,1,1)$ and $p_2^*=(0,1,-1)$ we notice that the meromorphic function Q

has poles of order two and it follows that the number of zeros counted with multiplicities is equal to four.

1.6.3 Example. Let P be as above but this time $Q(x,y) = (y-x)(y-\alpha x) + L(x,y)$ where α differs from one and -1. Again Q has a double pole at $(0,1,-1)$ but at $(0,1,1)$ we must analyze the pole in more detail. With ζ as a local coorniate on X at $(0,1,1)$ we have

$$x = \zeta^{-1} \quad : y = \zeta^{-1} \cdot \sqrt{1+\zeta^2}$$

So with $L(x,y) = ax + by + c$ we get

$$Q = (1 - \sqrt{1+\zeta^2})\zeta^{-1} \cdot \zeta^{-1}(\sqrt{1+\zeta^2} - \alpha) + a\zeta^{-1} + b\zeta^{-1} \cdot \sqrt{1+\zeta^2} + c$$

It follows that the coefficient of ζ^{-1} becomes

(i) $$(1-\alpha) + a + b$$

So if this number is $\neq 0$ then Q has a simple pole at $(0,1,1)$ and if it is zero no pole at all. This, if (i) is $\neq 0$ then the number of zeros is three and if (i)=0 then Q has two zeros on X.

Exercise. Find the equation which determines the three zeros of Q when (i) is $\neq 0$ and analyze under which conditions on the numbers α, a, b, c we get three transversal intersection points. Since the elimination to achieve these equations is rather cumbersome the material above illustrates the efficiency of Jacobi's counting method for the number of intersection points.

1.7 Jacobi's residue

Introduction. The pioneering work by Jacobi essentially covers the results below. Extensions to more than two variables was given by Weil in the article *L'Integrale de Cauchy et les fonctions des plusieurs variables.* Here we follow Weil's methods applied to the case of a pair of polynomials in two variables which form a complete intersection. The material in § A suffices to get the requested facts about Intersection Numbers. In § B we include some extra material which goes beyond our present study of Riemann surfaces. It has been inserted since it illustrates calculus in several complex variables and show the usefulness of currents. The reader may also consult the text-book [Griffiths-Harris] which has inspired the subsequent material.

A. The construction of residues.

Let $P(x,y)$ and $Q(x,y)$ be a pair of polynomials. We do not assume that they are irreducible and they may even have multiple factors. But we suppose that they have no common factor in the unique factorisation domain $\mathbf{C}[x,y]$. This entails that the common zero set $\{P=0\} \cap \{Q=0\}$ is a finite subset of $\mathbf{C}^2$ and one says that the pair (P,Q) is a complete intersection. We shall work close to the origin and choose some $r > 0$ such that

$$\min_{(x,y) \in B(r)} |P(x,y)|^2 + |Q(x,y)|^2 = \rho > 0$$

where $B(r) = \{|x|^2 + |y|^2 \leq r^2\}$ is a closed ball centered at the origin and $\{P=0\} \cap \{Q=0\} \cap B(r)$ is reduced to the origin. With ρ and r kept fixed we consider pairs α, β such that $|\alpha|^2 + |\beta|^2 < \rho$ and in the common zero set $\{P=\alpha\} \cap \{Q=\beta\}$ it is understood that we only pick points (x,y) which belong to ball $B(r)$. Recall from calculus that the real-analytic function $|P|^2$ only has a discrete set of critical values which entails that there exists some $\epsilon^* > 0$ such that the real hypersurfaces $\{|P|^2 = \epsilon\}$ are non-singular for every $0 < \epsilon < \epsilon^*$. One can therefore perform integrals on these. With $x = u + iv$ and $y = \xi + i\eta$ we identify the 2-dimensional complex (x,y)-space with the 4-dimensional real space where (u,v,ξ,η) are coordinates. One can integrate differential 3-forms ψ over the smooth hypersurfaces $\{|P|^2 = \epsilon\}$ which are oriented in a natural way. See §§. If $0 < \delta_* < \delta^*$ we set

$$\square(\epsilon; \delta_*, \delta^*) = \{|P|^2 = \epsilon\} \cap \{\delta_* < |Q|^2 = \delta^*\}$$

The Federer-Stokes Theorem from § xx gives:

(*) $$\iiint_{\square(\epsilon;\delta_*,\delta^*)} d\phi = \iint_{\partial\square(\epsilon;\delta_*,\delta^*)} \phi$$

for every test-form ϕ of degree two. When $g(x,y)$ is a polynomial we apply this starting from the 2-form

$$\phi = \frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}$$

Here ϕ is d-closed since we already have occupied the holomorphic 1-forms dx and dy while the rational function $\frac{g}{P\cdot Q}$ is holomorphic in a neighborhood of $\square(\epsilon;\delta_*,\delta^*)$. Hence (*) gives the equality

$$(**)\qquad \iint_{\sigma(\epsilon;\delta_*)} \frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)} = \iint_{\sigma(\epsilon;\delta^*)} \frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}$$

where

$$\sigma(\epsilon,\delta) = \{|P|^2=\epsilon\}\cap\{|Q|^2=\delta\}\quad : \ \epsilon,\delta>0$$

To be precise this is okay provided that the pair (ϵ,δ) are sufficiently small so that integration only takes place over small compact sets close to the origin in $\mathbf{C}^2$. One refers to $\sigma(\epsilon,\delta)$ as integration chains of degree two. In a similar fashion we can make a variation of ϵ and arrive at the following:

A.1. Proposition. *There exists a pair of positive numbers a,b such that the integrals*

$$\iint_{\sigma(\epsilon;\delta)} \frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}\quad : 0<\epsilon<a:\ 0<\delta<b$$

are independent of ϵ,δ while the 2-chain $\sigma(\epsilon;\delta)$ stays in the ball $B(r)$. The common value is denoted by $\mathfrak{res}_{P,Q}(g)$ and called the Jacobi residue of g with respect to P and Q.

Remark. The residue depends on the ordering of P and Q because we started from the oriented real hypersurface $\{|P|^2=\epsilon\}$ which induces a positive orientation on the σ-chains which determines the sign of the integrals. If the role is changed so that we start with a hypersuface $|Q|^2=\delta$ then

$$\mathfrak{res}_{Q,P}(g) = -\mathfrak{res}_{P,Q}(g)$$

2. A continuity property. Keeping ϵ,δ fixed it is clear that

$$\lim_{(\alpha,\beta)\to(0,0}\iint_{\sigma(\epsilon;\delta)} \frac{g(x,y)\cdot dx\wedge dy}{(P(x,y)-\alpha)\cdot(Q(x,y)-\beta)} = \iint_{\sigma(\epsilon;\delta)} \frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}$$

This entails that

$$(2.1)\qquad \lim_{(\alpha,\beta)\to(0,0}\mathfrak{res}_{P-\alpha,Q-\beta}(g) = \mathfrak{res}_{P,Q}(g)$$

3. The Jacobian. Given the pair P,Q we set

$$\mathcal{J}(x,y) = P'_x\cdot Q'_y - P'_y\cdot Q'_x$$

This is a new polynomial. As explained in § XX $\mathcal{J}$ is not identically zero. The curve $\{\mathcal{J}=0\}$ is a sparse subset of $\mathbf{C}^2$ which entails that there exists many pairs of complex numbers α and β which both are close to zero such that the finite set $\{P=\alpha\}\cap\{Q=\beta\}$ stays outside $\mathcal{J}^{-1}(0)$. When this holds we say that the pair (α,β) is admissable. Notice that the Jacobian is unchanged when the pair P,Q is replaced by $P-\alpha$ and $Q-\beta$.

4. Jacobi's residue formula. Let α,β be a pair of small complex numbers such that $\mathcal{J}\neq 0$ at the common zeros of $P-\alpha$ and $Q-\beta$ which are close to the origin in $\mathbf{C}^2$. Repeated use of Cauchy's residue formula from § xx gives

$$\mathfrak{res}_{P-\alpha,Q-\beta}(g) = (2\pi i)^2\cdot\sum\frac{g(p_k)}{\mathcal{J}(p_k)}$$

where the sum extends over the distinct points in $\{P=\alpha\}\cap\{Q=\beta\}$. which belong to $B(r)$.

A.5. A special case. Above we can take $g = \mathcal{J}$ in which case (2.1) is $-4\pi^2$ times an integer. The continuity in § 2 shows that this integer is constant as α, β varies in the admissable set from § 3. Passing to the limit it follows that

$$\mathfrak{res}_{P,Q}(\mathcal{J}) = K$$

where K is the set of points when $\{P - \alpha\}$ and $\{Q - \beta\}$ have transversal intersections. The absolute value of K is called the intersection number and is denoted by $\mathbf{i}(P, Q)$.

B. Further results.

Above we defined the integers $\mathfrak{res}_{P,Q}(g)$ where g are polynomials. Keeping P and Q fixed it means that one has a $\mathbf{C}$-linear map

$$g \mapsto \mathfrak{res}_{P,Q}(g)$$

It turns out that it can be described in an algebraic fashion. Namely, let $\mathcal{O}_2 = \mathbf{C}\{x, y\}$ be the local ring of convergent power series in two variables. Thus, the elements are germs of analytic functions in x and y. Now P and Q are elements in $\mathcal{O}_2$ and generate an ideal denoted by (P, Q). Set

$$\mathcal{A} = \frac{\mathcal{O}_2}{(P, Q)}$$

The assumption that the origin is an isolated point in the common seros of P and Q entails that the ideal (P, Q) contains a sufficiently high power of the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_2$. It follows that $\mathcal{A}$ is a local and finite dimensional complex algebra. In commutative algebra one refers to $\mathcal{A}$ as a local artinian ring. If g is a polynomial which belongs to the ideal (P, Q) it follows easily from Jacobi's residue formula in § 4 that $\mathfrak{res}_{P,Q}(g) = 0$. Less obvious is the following:

B.1 Theorem. *A polynomial g belongs to the ideal (P, Q) in $\mathcal{O}_2$ if and only if*

$$\mathfrak{res}_{P,Q}(h \cdot g) = 0 \quad \text{hold for all polynomials } h$$

B.2 Noetherian operators. Theorem B.1 entails that there exists a $\mathbf{C}$-linear form on the finite dimensional vector space $\mathcal{A}$ defined by

$$\bar{g} \mapsto \mathfrak{res}_{P,Q}(g)$$

where $\bar{g}$ is the image in $\mathcal{A}$ of a polynomial g. This linear functional can be expressed by a unique differential operator with constant coefficients. More precisely, we have the polynomial ring $\mathbf{C}[\partial_x, \partial_y]$ of differential operators with constant coefficients where ∂_x and ∂_y are the holomorphic first order operators defining partial derivatives with respect to x and y. Then there exists a unique differential operator $\mathcal{N}(\partial_x \partial_y) \in \mathbf{C}[\partial_x, \partial_y]$ such that

$$\mathfrak{res}_{P,Q}(g) = \mathcal{N}(\partial_x, \partial_y)(g)(0)$$

for every polynomial $g(x, y)$. Thus, in the right hand side one evaluates the polynomial $\mathcal{N}(g)$ at the origin. One refers to $\mathcal{N}$ as the noetherian operator attached to the pair P, Q. It has the special property that $\mathcal{N}(\phi) = 0$ for every ϕ in the ideal (P, Q). To avoid possible confusion we remark that these differential operators were introduced by Max Noether, i.e. not by his famous daughter Emmy whose name is attributed to the notion of noetherian rings as well many other deep results in algebra.

B.2.1 The construction of Noetherian operators. They are obtained via the localised Weyl algebra $A_2(*)$ whose elements are differential operators in whose coefficients are rational functions with no pole at the origin. The crucial result is that if $\mathbf{C}[x, y]$ is identified with zero-order differential operators then the right ideal in $A_2(*)$ generated by P and Q yields a left module

$$\frac{A_2(*)}{A_2(*) \cdot P + A_2(*) \cdot Q}$$

which is isomorphic to m copies of the simple left A_2-module

$$\frac{A_2(*)}{A_2(*) \cdot x + A_2(*) \cdot y}$$

and m is the integer which gives the Jordan-Hölder length of the artinian local ring $\mathcal{A}$. We shall not enter a discussion about this and remark only that the result above belongs to basic material in $\mathcal{D}$-module theory.

B.3 The Gorenstein property. The local algebra $\mathcal{A}$ is special, i.e. it is a local Gorenstein ring which means that the socle defined as those elements $a \in \mathcal{A}$ which are annihilated by the maximal ideal $\mathfrak{m}$ in $\mathcal{O}_2$ is a 1-dimensional complex vector space. This fact is easily proved via a diagram chasing in homological algebra where one uses the assumption that the pair P, Q is a complete intersection. The element in $\mathcal{A}$ which generates the 1-dimensional socle is the image of $\mathcal{J}$. This is a consequence of the following vanishing property of residue integrals:

$$\mathfrak{res}(g \cdot \mathcal{J}) = 0 \quad : \forall\, g \in \mathfrak{m}$$

The reader may notice that this is an immediate consequence of Jacobi's residue formula in § A.4.

B.4 The trace map. Let us introduce two new complex variables w and u. In the 4-dimensional complex (x, y, w, u)-space one has the non-singular analytic surface defined by the equation

$$S = \{w = P(x, y)\} \cap \{u = Q(x, y)\}$$

Let $g(x, y)$ be a polynomial. For every test-form $\psi^{0,2}$ of bi-degree $(0, 2)$ in the (x, y, w, u)-space we set

$$\iint_S g(x, y) \cdot dx \wedge \wedge \psi^{0,2}$$

This gives a current in $\mathbf{C}^4$ denoted by $g \cdot \square_S$ Consider the projection $\pi(x, y, w, u) = (w, u)$. The hypothesis that P and Q is a complete intersection entails that π restricts to a proper map on S and hence there exists a direct image current defined by

$$(1) \qquad \phi^{0,2} \mapsto \iint_S g(x, y) \cdot dx \wedge \wedge \pi^*(\phi^{0,2})$$

where $\pi^*(\phi^{0,2})$ is the pull-back of the test-form $\phi^{0,2}$ in the (w, u)-space. On S one has the equality

$$\pi^*(dw \wedge du) = \mathcal{J} \cdot dx \wedge dy$$

Next, in the (w, u)-space there exists the set of admissable points (w, u) for which $\mathcal{J}(x, y) \neq 0$ for all (x, y) with $P(x, y) = w$ and $Q(x, y) = u$. We notice that this is the same as the image set $\pi(S \cap \mathcal{J}^{-1}(0)$. Here $S \cap \mathcal{J}^{-1}(0)$ is an algebraic hypersurface in S and since π restrict to a proper mapping with finite fibers, it follows that the image set is a hypersurface in the (w, u)-space which we denote by Δ and refer to as a discriminant locus. The direct image current (1) can be described in the open complement of Δ. Namely, π restricts to an unramified covering map from $S \setminus \pi^{-1}(\Delta)$ onto the open complement of Δ in the (w, u)-space,. Let K be the number of points in every fiber which is given by an unordered k-tuple $p_k(w, u) = (x_k(w, u), y_k(w, u)\}$, Here the coordinates $\{x_k(w, u)\}$ and $\{y_k(w, u)\}$ are local branches of multi-valued analytic functions in the open complement of Δ. If $h(u, w)$ is a test-function in the (w, u)-space whose compact support does not intersect Δ and we take $\phi^{0,2} = h(w, u) \cdot dw \wedge du$ then one has the equality

$$\iint_S g(x, y) \cdot dx \wedge \wedge \pi^*(\phi^{0,2}) = \iint_{D^2} \mathfrak{Tr}(\frac{g}{\mathcal{J}})(w, u) \cdot h(w, u) \cdot dw \wedge du$$

where

$$(*) \qquad \mathfrak{Tr}(\frac{g}{\mathcal{J}})(w, u) = \sum_{k=1}^{k=K} \frac{g(p_k(w, u)}{\mathcal{J}(p_k(w, u))}$$

We refer to (*) as a trace function of $\frac{g}{\mathcal{J}}$. Next, recall that the passage to direct image currents commute with differentials. Since $g(x, y)$ is holomorphic the current (1) is $\bar{\partial}$-closed. Indeed, Stokes theorem entails that

$$\iint_S g(x, y) \cdot dx \wedge \wedge dy \wedge \bar{\partial}(\psi^{0,1}) = 0$$

hold for every test-form $\psi^{0,1}$ with compact support in the (x, y, w, u)-space.

B.4.1 Conclusion. Denote by γ the direct image current from (1). By (*) its restriction to the open complement of Δ is the density expressed by the $(2, 0)$-form

$$\mathfrak{Tr}(\frac{g}{\mathcal{J}})(w, u) \cdot dw \wedge du$$

Next, in (*) the trace function is constructed by a sum over fibers which entails that it extends to a meromorphic function in the (w, u)-space with eventual poles confined to Δ. Let G denote this meromorphic function so that

$$G(w, u) = \mathfrak{Tr}(\frac{g}{\mathcal{J}})(w, u) \tag{i}$$

holds when (w, u) are outside Δ. By the above the current γ is $\bar{\partial}$-closed, i.e. the $(2, 0)$-current defined outside Δ by $G \cdot dw \wedge du$ can be extends via γ to a $\bar{\partial}$-closed current in the (w, u)-space. But this can only hold if the meromorphic function G has no poles at all. In fact, this follows from Hartogs' extension result in § XX. Hence we have proved:

B.4.2 Theorem. *For every polynomial $g(x, y)$ the trace function defined by (*) in the complement of Δ extends to a holomorphic function in the (w, u)-space.*

C. The local algebra $\mathbf{C}\{P, Q\}$. Given the pair P, Q there exists a subalgebra of $\mathcal{O}_2$ whose elements are germs of analytic functions which can be expanded into a power series in P and Q. More precisely, denote by $\mathbf{C}[P, Q]$ the set of entire functions of the form

$$\phi_n(x, y) = \sum c_{jk} \cdot P^j \cdot Q^k$$

where $\{c_{jk}\}$ is a finite set of doubly-indexed complex numbers. A germ g belongs to $\mathbf{C}\{P, Q\}$ if and only if there exists a small polydisc D^2 centered at the origin in $\mathbf{C}^2$ such that g is holomorphic in D^2 and there exists a sequence $\{\phi_n\}$ of (P, Q)-polynomials as above which converge uniformly to g in D^2. Concerning the algebra $\mathbf{C}\{P, Q\}$ a wellknown result in several complex variables shows that it is isomorphic to the local ring of convergent power series in two variables, i.e. the polynomials considered as germs in $\mathcal{O}_2$ are analytically independent. Moreover, $\mathcal{O}_2$ is a finitely generated module over its subring $\mathbf{C}\{P, Q\}$.

C.1 The algebraic trace. By the above the quotient field of $\mathcal{O}_2$ is a finite algebraic extension of the quotient field of $\mathbf{C}\{P, Q\}$. If m is the dimension we can choose an m-tuple $\phi_1, \ldots, \phi_m$ in the quotient field of $\mathcal{O}_2$ whose images in $\mathcal{A}$ yields a basis for this finite algebraic extension. Let us then consider a polynomial g. For each $1 \leq \nu \leq m$ we can write

$$g \cdot \phi_\nu = \sum_{j=1}^{j=m} \rho_{\nu,j} \cdot \phi_j$$

where $\{\rho_{\nu,j}\}$ belong to the quotient field of $\mathbf{C}\{P, Q\}$. As explained in §§ the trace defined by

$$\mathfrak{Tr}(g) = \sum \rho_{\nu,\nu}$$

does not depend upon the chosen ϕ-basis.

D. More advanced results.

Above we have treated Jacobi's residues where the denominators are holomorphic functions. More generally, when P and Q are in a complete intersection we can still define integral

$$J_g(\epsilon, \delta) =$$

when $g(x, y)$ is a test-function with compact support close to the origin. Here the integral is no longer independent of the pair (ϵ, δ) which leads to a considerably more involved situation first studied by Miguel Herrera who proved that there exists certain limits provided that one pays attention while ϵ and δ tend to zero. We shall not enter a detailed discussiona about multi-residue calculus but mention that a counter-example was discovered by Passare and Tsikh in [P-T] which shows that the unrestricted limit of the integrals above in general does not exist. In fact, these counter-examples are generic and the reader may consult my article [Björk:Abel Legacy] for further comments about the Passare-Tsikh example. On the positive side there exists a remarkable result due to H. Samuelsson in his Ph.d-thesis at Chalmers University 2005. From an analytic point of view Theorem D.1 below is quite useful. Instead of taking "ugly residues" as above one considers regularisations. Namely, for the given polynomial P there exists a smooth current

$$\frac{\bar{P}}{|P|^2 + \epsilon}$$

One can apply the $\bar{\partial}$-current and get a smooth $(0,1)$-current

$$\rho_P(\epsilon) = \bar{\partial}(\frac{\bar{P}}{|P|^2 + \epsilon})$$

Similarly we construct smooth currents $\rho_\delta(Q)$. With these notations Samuelson proved the following:

D.1 Theorem. *In the space of* $(0,2)$*-currents there exists an unrestricted limit*

$$\gamma_{P,Q} = \lim_{(\epsilon,\delta)\to(0,0)} \rho_P(\epsilon) \wedge \rho_\delta(Q)$$

Moreover, this current is of the Coleff-Herrera type which means that

$$\gamma_{P,Q}(\phi^{2,0}) = 0$$

for every test-form $\phi^{2,0}$ *given as* $[\bar{x} \cdot g_1(x,y) + \bar{y} \cdot g_2(x,y)]\ddot{d} \wedge dy$ *where* g_1, g_2 *is a pair of test-functions.*

1.8 Elliptic function fields.

Once the reader has studied the general theory in the subsequent sections the details which follow below become clear. Let y and z be independent and consider the algebraic equation

$$(*) \qquad y^2 = (x-a)(x-b)(x-c)$$

where a, b, c are distinct complex numbers and all are $\neq 0$. Set

$$P(x,y) = y^2 - (x-a)(x-b)(x-c)$$

This is an irreducible polynomial in $\mathbf{C}[x,y]$ and we get the algebraic function field

$$(**) \qquad K = \frac{\mathbf{C}[x,y]}{(P)}$$

where (P) is the principal ideal generated by P. Let X be the associated compact Riemann surface. As a topological space we will show below that X is homeomorphic to a torus, i.e. the sphere with one attached handle.

The construction of X. Consider the algebraic curve S in $\mathbf{C^2}$ defined by the equation $P(x,y) = 0$. With $p(x) = (x-a)(x-b)(x-c)$ we get the complex gradient vector

$$\nabla(P) = (P_x, P_y) = (p'(x), 2y)$$

The gradient vector is $\neq 0$ if $y \neq 0$ and if $y = 0$ while $p(x) = 0$ we have $p'(x) \neq 0$ since the p-roots are simple. Hence $\nabla(P) \neq 0$ on S which means that the curve S is *non-singular*, i.e. it appears as a 1-dimensional complex submanifold of $\mathbf{C}^2$ and since the polynomial $P(x,y)$ is irreducible this manifold is *connected*. In algebraic geometry one refers to S as a smooth and irreducible affine curve.

A compactification. When x tends to infinity on S we notice that

$$|y|^2 \simeq |x|^3$$

In other words, the absolute value of y increases considerably faster than x. Let us then introduce the 2-dimensional projective space $\mathbf{P^2}$ with coordinates $\zeta_0, \zeta_1, \zeta_2$ and identify $\mathbf{C^2}$ with the portion where $\zeta_0 \neq 0$ so that S is identified with the set

$$(1, x, y) \quad : (x,y) \in S$$

When $(x,y) \in S$ and $|x| >> 0$ we can express the points by

$$(\frac{1}{y}, \frac{x}{y}, 1)$$

Since $|y| >> |x|$ on S we see these points converge to a unique point on in $\mathbf{P}^2$ given by $p^* = (0,0,1)$. In a neighborhood of p^* we have local coordinates w and τ which represent points close to p^* by $(w, \tau, 1)$. Outside the origin in the (w,τ) disc we get

$$y = \frac{1}{w} \quad \text{and} \quad x = \frac{\tau}{w}$$

Thus, a point $p = (x,y)$ with $|x|$ large belongs to S if and only if the corresponding (w,τ)-coordinates of p satisfy:

$$w^{-2} = (\frac{\tau}{w} - a)(\frac{\tau}{w} - b)(\frac{\tau}{w} - c) \implies$$

$$\text{(i)} \qquad w = (\tau - aw)(\tau - bw)(\tau - cw)$$

Here (i) determines w as an analytic function of τ given by a convergent power series of the form:

$$w = \tau^3 + \sum_{\nu \geq 4} c_\nu \tau^4 = b(\tau)$$

Using the local coordinates (τ, w) at p^* we conclude that $X = S \cup p^*$ appears a non-singular submanifold of $\mathbf{P^2}$ and is therefore a compact Riemann surface. We conclude that $\mathfrak{M}(X)$ is isomorphic to the function field K from (**) above.

The genus of X. We have the meromorphic function x on X. At p^* where τ is the local coordinate we have seen that

$$x(\tau) = \frac{\tau}{\zeta} = \frac{\tau}{b(\tau)}$$

Since $b(\tau)$ has a zero of order 3 it follows that x has a double pole at p^*. Next, zeros of x occur when

$$y^2 = -abc$$

Since $abc \neq 0$ we see that x has two simple zeros on X. Hence x is a meromorphic function whose degree is two and its principal divisor is given by:

$$D(x) = 2 \cdot p_* - (q_1 + q_2)$$

where q_1, q_2 are the points on S where $x = 0$ and the y-coordinate is plus or minus $\sqrt{abc}$. Next, consider the meromorphic function y. Here we find three simple zeros at the points $(a,0),(b,0),(c,0)$ and at p^* we have seen that y has a triple pole. Now we consider the differential 1-form

$$\omega = \frac{dx}{y}$$

Zeros of y occur at the points $(a,0),(b,0),(c,0)$ and at each of these y serves as a local coordinate. Close to $(a,0)$ we have for example

$$x = a + g(y)$$

where $g(y)$ has a zero of order two at $y = o$ which entails that dx is of the form $h(y)ydy$ and hence ω is holomorphic close to $(a,0)$ and given by

$$\omega = \alpha(y) \cdot dy \quad \text{where} \quad \alpha(0) \neq 0$$

The same hold at $(b,0)$ and at $(c,0)$. Next, at the point at infinity we have the local coordinate τ and from the above one has

$$x = \tau^{-2}(1 + g(\tau)) \quad \text{and} \quad y(\tau) = \tau^{-3}(1 + h(\tau))$$

where g and h are holomorphic at $\tau = 0$. It follows that

$$\omega = \frac{\tau^3}{(1 + h(\tau)} \cdot (-2\tau^{-3}(1 + g(\tau)) + \tau^{-2}g'(\tau)] \cdot d\tau$$

which shows that ω also is holomorphic at p^*.

Exercise. Conclude from the above that Ω not only is holomorphic but also $\neq 0$ everywhere and that the space $H^0(X, \Omega)$ of globally defined holomorphic 1-forms is a 1-dimensional complex space generated by ω. In § we shall learn that this entails that $g = 1$ and hence X is homeomorphic to a torus.

Remark. Let us specify the three constants a, b, c and take a curve

$$y^2 = x(x-1)(x-d)$$

where d is a real number > 1. Even in this case it is not easy to get a picture of X and in a more naive "discover" that X is a torus. So the calculations which show that $X \simeq T^2$ illustrate the power of the general theory to be exposed in the subsequent sections.

1.5.1 The Jacobi map. Let X be the elliptic curve assocaited to the curve (*) above. Fix some point p_0 on X. For every other point $p \in X$ we can choose a simple curve γ from p_0 to p and construct the line integral of ω along γ. This yields a function

$$(*) \qquad \phi_\gamma(p) = \int_\gamma \omega$$

However, the ϕ-function is not uniquely determined since there exist closoed curves γ along which the line integral is non-zero. So values of the ϕ-function are not well-defined. To overcome this we consider the family of all closed curves at p_0 and to every such curve γ we assign the complex number

$$\{\gamma\} = \int_\gamma \omega \tag{1}$$

Above we have used the monodromy theorem which shows that the right hand side only depends upon the homotopy class of γ in the family of closed curves at p_0. In the left hand side $\{\gamma\}$ denotes this homotopy class which is identfied with an an element in the fundamental group $\pi_1(X)$. The set of all complex numbers which are obtained via (1) as $\{\gamma\}$ varies is a subset of $\mathbf{C}$ which we denote by Γ. In Section XX we shall prove that Γ is a discrete set given as a $\mathbf{Z}$-lattice generated by a pair of $\mathbf{R}$-linearly independent vectors. Now we obtain the torus

$$T^2 = \frac{\mathbf{C}}{\Gamma}$$

From the above the ϕ-integral takes well-defined values in T^2 as p varies in X. Hence we obtain a map from X into T^2. Since $\omega \neq 0$ on the whole of X it follows this map is *locally bi-holomorphic.* But then, since X is compact and T^2 connected it follows from general toplogy that ϕ is surjective and *Abel's theorem* will show that the map also is injective. In particular $\mathfrak{M}(X)$ can be identfied with the field of Γ-periodic meromorphic functions defined in the whole complex z-plane. Above we found the special meromorphic function on x given via x which has one double pole and two simple zeros. This led Weierstrass to perform constructiuons in a reverse way, i.e. starting from a $\mathbf{Z}$-lattice γ the considered the field $\mathcal{P}_\Gamma(\mathbf{C})$ of all meromrphic functions which are Γ-period, or as one says doubly-periodic with resepct to this lattice. In § x we expose how to construct special double periodic functions $\mathfrak{p}$ with a double pole at a lattice point and two simple zeros in a fundamental domain.

1.5.2 A cubic equation. Consider the algebraic equation

$$y^3 + x^3 = 1$$

which gives the algebraic curve S defined by the zero set of $P = x^3+y^3-1$. Here $\nabla(P) = (3x^2, 3y^2)$ is non-zero on S and hence the affine curve S is non-singular. Its closure in $\mathbf{P}^2$ adds three points. One is for example

$$p_1^* = (0, 1, -1)$$

At p_1^* we take local coordinates (ζ, η) representing the points $(\zeta, 1, -1+\eta)$. When $\zeta \neq 0$ they correspond to points

$$(1, \frac{1}{\zeta}, \frac{-1+\eta}{\zeta})$$

in $\mathbf{C}^2$. The defining equation for $\bar{S}$ close to p_1^* becomes:

$$\frac{(-1+\eta)^3}{\zeta^3}) + \frac{1}{\zeta^3} = 1 \implies (-1+\eta)^3 + 1 = \zeta^3$$

This gives $\zeta^3 + -3\eta + 3\eta^2 - \eta^3 = 0$ and hence $\bar{S}$ is locally defined by an equation $\eta = \zeta^3 \cdot g(\zeta)$ where $g(0) \neq 0$. Set $j_1 = e^{2\pi i/3}$. A similar computation as above shows that $\bar{S}$ is non-singular at the points $p_2^* = (0, 1, j_1)$ and $p_3^* = (0, 1, j_1^2)$. Hence $\bar{S}$ is a nonsingular projective curve and is identified with a Riemann surface denoted by X.

Exercise. Show that when y is identified with a meromorphic function then it has simple poles at the three points $\{p_\nu^*\}$ and simple zeros at $(1,0), (e^{2\pi i/3}, 0)$ and $e^{4\pi i/3}, 0)$ Similarly, x has simple poles at $\{p_\nu^*\}$ and a simple zeros at $(0,1), (0, e^{2\pi i/3})$ and $(0, e^{4\pi i/3})$. Let us consider the point $(0,1)$ and with $x = 1+\zeta)$ the local equation for S becomes

$$y^3 + (1+\zeta)^3 = 1 \implies y^3 + 3\zeta + \zeta^2 + \zeta^3 = 0$$

Hence y serves as a local coordinate and $\zeta(y) = y^3 \cdot g(y)$ with $g(0) \neq 0$.

1.5.3 Conclusion. Consider the meromorphic function

$$f = \frac{y}{x-1}$$

From the above f has no poles at the boundary points and $f(p_\nu^*) \neq 0$ for each ν. At the point $(1,0)$ the reader can verify that f has a double pole and it has simple zeros at $(e^{2\pi i/3}, 0)$ and $e^{4\pi i/3}, 0)$. Starting from the rational field $\mathbf{C}(f)$ it follows that $\mathfrak{M}(X)$ is a quadratic extension which shows that X is an elliptic curve. So even if we start from a cubic eqaution it may happen that the resulting curve is elliptic !

1.5.4 An exercise about period integrals. Let $0 < a < b$ be real and consider the curve

$$y^2 = x(x-a)(x-b)$$

We have already seen that its closure in $\mathbf{P}^2$ is non-singular and yields a Riemann surface X with genus $g = 1$. Moreover $\omega = \frac{dx}{y}$ is a holomorphic form without zeros. We grasp this Riemann surface via the 2-sheeted map given by the meromorphic function x. If z is the complex coordinate in $\mathbf{P}^1$ then ω corresponds to the 1-form

$$\frac{dz}{\sqrt{z(z-a)(z-b)}}$$

On X there exists a closed curve starting at $(0,0)$ which first reaches $(a,0)$ along a simple curve on which $\mathfrak{Im}(y) > 0$ and then returns to the (0,0) along a simple curve starting at $(a,0)$ where $\mathfrak{Im}(y) < 0$. The resulting composed curve γ is closed and the reader should verify the equality

$$\text{(i)} \qquad \int_\gamma \omega = (1+i) \cdot \int_0^a \frac{dx}{\sqrt{x(a-x)(b-x)}} \cdot dx$$

Next, start at $(a,0)$ and construct a simple curve reaching $(b,0)$ along which $\mathfrak{Im}(y) > 0$ and return after along a simple curve where $\mathfrak{Im}(y) < 0$. This gives a closed curve γ_1 which starts and ends at $(a,0)$. here the reader should verify the equality

$$\text{(ii)} \qquad \int_{\gamma_1} \omega = i \cdot (1+i) \cdot \int_a^b \frac{dx}{\sqrt{x(x-a)(b-x)}} \cdot dx$$

Using the material about doubly periodic functions in § XX the isomorphism class of the elliptic curve is governed by the quotient of the two real numbers which compute the integrals above. With $0 < a < b$ and $\alpha = a/b$ the variable substitution $x \to as$ gives the equality:

$$\text{(iii)} \qquad \int_0^a \frac{dx}{\sqrt{x(a-x)(b-x)}} \cdot dx = \frac{1}{\sqrt{b}} \cdot \int_0^1 \frac{ds}{\sqrt{s(1-s)(1-\alpha s)}}$$

Similarly we find that

$$\text{(iv)} \qquad \int_a^b \frac{dx}{\sqrt{x(x-a)(b-x)}} \cdot dx = \frac{1}{\sqrt{b}} \cdot \int_1^{b/a} \frac{ds}{\sqrt{s(1-s)(1-\alpha s)}}$$

So while $\alpha = a/b$ varies in $(0,1)$ we encounter the quotients

$$\kappa(\alpha) = \frac{\int_0^1 \frac{ds}{\sqrt{s(1-s)(1-\alpha s)}}}{\int_1^{1/\alpha} \frac{ds}{\sqrt{s(1-s)(1-\alpha s)}}}$$

Exercise. Analyze these quotients numerically as a functions of α. For example, do they take distinct values as $0 < \alpha < 1$ and draw conclusions from this concerning the eventual non-isomorphism of the corresponding elliptic curves $y^2 = x(x-a)(x-b)$ when a/b changes.

1.9 Proof of Theorem § XX

Let u be a continuous and subharmonic function in the unit disc D. Set $\lambda(z) = e^{u(z)}$ which yields a metric δ with non-positive curvature. Consider a pair of points a, b in D and a pair of rectifiable Jordan arcs γ_1, γ_2 in $\mathcal{C}(a, b)$. Suppose for the moment that the intersection of the γ-curves only contains the end-points a and b. Their union gives a closed Jordan curve Γ which borders a Jordan domain Ω. The inequality (*) in Theorem XX follows if there to each point $p \in \gamma_1$ exist a Jordan arc β in Ω which joins p with some $q \in \gamma_2$ and a Jordan arc α in Ω which joins a and b such that

$$(1) \qquad [\int_\alpha \lambda(z)\,|dz|]^2 + [\int_\beta \lambda(z)\,|dz|]^2 \leq \frac{1}{2}\big([\int_{\gamma_1} \lambda(z)\,|dz|]^2 + [\int_{\gamma_2} \lambda(z)\,|dz|]^2\big)$$

To prove (1) we employ a conformal mapping $\psi \colon D \to \Omega$ where D is another unit disc with the complex coordinate w and recall from § XX thatψ extends to a homeomorphism from the closed Jordan domain $\bar{\Omega}$ where it sends γ_1 and γ_2 onto a pair of closed intervals on the unit circle which instersect at two points. Next, there exists the composed function $\lambda \circ \psi$ on D. By assumption $\lambda = e^u$ for some subharmonic function u in Ω and since $u \circ \psi$ is subharmonic in D it follows that $\log \lambda \circ \psi$ is subharmonic in D. Notice that if γ is a recitifable arcs in Ω then

$$\int_\gamma \lambda(z) \cdot |dz| = \int_{\psi(\gamma)} \lambda \circ \psi(w) \cdot \frac{|dw|}{|\psi'(w)|}$$

Set

$$\lambda^*(w) = \frac{1}{|\psi'(w)|} \cdot \lambda \circ \psi(w)$$

Then

$$\log \lambda^* = -\log |\psi'(w)| + \log \lambda \circ \psi$$

Here $\log |\psi'(w)|$ is the real part of the analytiuc function $\log \psi'(w)$ in d and hence harmonic which entails that $\log \lambda^*$ is subharmonic. The proof of (1) is now reduced to the case when Ω is replaced by the unit w-disc where λ is replaced by λ^*. Moreover, the Cauchy-Schearz inequality gives

$$\frac{1}{2}\big([\int_{\gamma_1} \lambda(z)\,|dz|]^2 + [\int_{\gamma_2} \lambda(z)\,|dz|]^2\big) \leq \frac{1}{4} \cdot \big(\int_{\gamma_1 \cup \gamma_2} \lambda(z)\,|dz|]^2\,\big) = \frac{1}{4} \cdot \int_{|w|=1\}} \lambda^*(w)\,|dw|$$

There remains only to find a pair of curves α^* and β^* in the w-disc where α^* joins the two points $\{e^{i\theta_\nu}\}$ on the unit circle where the boundary arcs $\psi(\gamma_1)$ and $\psi(\gamma_2)$ intersect while β^* is a curve which joins some point $p \in \psi(\gamma_1)$ with a point $q \in \psi(\gamma_2)$.

The choice of α^* and β^*. First α^* is the circular arc with end-points at $e^{i\theta_1}$ and $e^{i\theta_2}$ and intersects T at right angles. Next, given a point $p \in \psi(\gamma_1)$ there exists the unique circular arc β^* which intersects both α^* and T at right angles. See figure § XX. Now (1) follows if we have proved the inequality

$$(2) \qquad \big(\int_{\alpha^*} \lambda^*(w) \cdot |dw|\,\big)^2 + \big(\int_{\beta^*} \lambda^*(w) \cdot |dw|\,\big)^2 \leq= \frac{1}{4} \cdot \big(\int_{|w|=1\}} \lambda^*(w)\,|dw|\,\big)^2$$

To prove (2) we use a summetrisation of λ^*. Recall from § xx that there exists a Möbius transformation T_1 on the unit disc which is a reflection of α^*, i.e. it restricts to the identity map on α^* and the composed map T_1^2 is the identity in D. Similarly we find the reflection T_2 of β^*. As explained in § xx one has $T_2 \circ T_1 = T_1 \circ T_1$. Let $S_0(w) = w$ be the identity while $S_1 = T_1$, $S_2 = T_2$ and $S_3 = T_2 \circ T_1$. Set

$$(3) \qquad \lambda^{**}(w) = \frac{1}{4} \cdot \sum_{\nu=0}^{\nu=3} \lambda(S_\nu(w)) \cdot |\frac{S_\nu(w)}{dw}|$$

Now T_1 maps α^* into itself, and similarly T_2 maps β^* into itself while the composed Möbius transformation $S_3 = T_2 \circ T_1$ interchanges the two curves. It follows that the left hand side in (2) is unchanged when replace λ^* is replaced by λ^{**} and the right hand side is also unchanged since the S-transformations map T onto itself. Hence it suffices to prove (2) when λ^* from the start is invariant with respect to the four S-transformations. In this case we solve the Dirichlet problem using the boundary value function $\log\lambda^*$ on T so that

$$\log\lambda^* = u + H$$

where H is harmonic in D while u is subharmonic and zero on T. The maximum principle entails that $u \leq 0$ in D which gives

$$\lambda^* = e^u \cdot e^H \leq e^H \tag{4}$$

Since $u = 0$ the right hand side in (2) is unchagend while the left hand side is majorised when λ^* is replaced by e^H. The S-invariance of λ^* implies that H also is S-invariant and as explained in §§ xx it follows that there exists an analytic function $g(w)$ in D such that

$$e^{H(w)} = |g'(w)|$$

where the map $w \mapsto g(w)$ sends α^* to a real interval $[-A, A]$ and β^* to an imaginary interval $[-iB, iB]$. Hence (4) implies that the left hand side in (2) is majorized by

$$\Big(\int_\alpha |g'(w)|\,|dw|)^2 + \Big(\int_\beta |g'(w)|\,|dw|\,\Big)^2 = 4A^2 + 4B^2 \tag{5}$$

Next, since $\lambda^* = |g'|$ holds on T the right hand side in (2) becomes

$$\frac{1}{4}\cdot\Big(\int_{|w|=1} |g'(w)|\,|dw|\,\Big)^2 \tag{6}$$

The S-symmetry of g entails thst $\int_{|w|=1} |g'(w)|\,|dw|$ is four times the integral taken along a subarc of T which joins consequtive points where α^* and β^* intersect and every suvh integral is the euclidian length of the image curvr under g which by the above joins the real point A with iB. So its euclidian length is $\geq \sqrt{A^2 + B^2}$, i.e. we have used that the shortest distance between a pair of points is a straight line and then applied Pythagoras' theorem. Hence (6) majorizes

$$\frac{1}{4}\cdot\big(4\cdot\sqrt{A^2+B^2}\big)^2 = 4(A^2+B^2) \tag{7}$$

Then (5) and (7) give the requested inequality (2).

Compact Riemann surfaces

We assume that the reader is familiar with the notion of an abstract one-dimensional complex manifold X and how a covering by charts defines the sheaf $\mathcal{O}$ whose sections are holomorphic functions and basic sheaf theory. For the reader's convenience we have included appendices about sheaves and analysis on two-dimensional manifolds. From now on X is a compact Riemann surface.

Notations in § A-H. Keeping X fixed we denote by $\mathcal{O}$ the sheaf of holomorphic functions an X. We have also the sheaf Ω whose sections are holomorphic 1-forms. Next, $\mathcal{E}$ denotes the sheaf of complex-valued C^∞-functions and passing to differential forms we have the two sheaves $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$. The complex structure yields a decomposition

$$\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$$

and similarly the exterior differential d from $\mathcal{E}$ into $\mathcal{E}^1$ is decomposed as:

$$d = \partial + \bar{\partial}$$

Finally there exists the sheaf $\mathcal{E}^{1,1}$ of differential forms of maximal degree and sheaf maps:

$$\partial \colon \mathcal{E}^{0,1} \to \mathcal{E}^{1,1} \quad : \quad \bar{\partial} \colon \mathcal{E}^{1,0} \to \mathcal{E}^{1,1}$$

We shall also employ currents. Denote by $\mathfrak{c}$ is the sheaf of currents whose sections over an open set U in X is the dual over the Frechet space $\mathcal{E}^{1,1}(U)$. It means that $\mathfrak{c}(X)$ is the space of distributions on X regarded as an oriented C^∞-manifold. We have also the sheaves $\mathfrak{c}^{1,0}$ and $\mathfrak{c}^{0,1}$ and passing to global sections one has the inclusions

$$\mathcal{E}(X) \subset \mathfrak{c}(X) \quad : \mathcal{E}^{1,0}(X) \subset \mathfrak{c}^{1,0}(X) \quad : \mathcal{E}^{0,1}(X) \subset \mathfrak{c}^{0,1}(X)$$

The elliptic property of $\bar{\partial}$. If $\gamma \in \mathcal{E}^{1,0}(X)$ is a $\bar{\partial}$-closed the result from § XX entails that it is a holomorphic 1-form, i.e in a chart U where x is a local coordinate there exists $g \in \mathcal{O}(U)$ such that

$$\gamma|U = g \cdot dx$$

Passing to global sections we get the equality

$$(*) \qquad \Omega(X) = \ker_{\bar{\partial}}(\mathcal{E}^{0,1}(X))$$

Divisors. A divisor on X is expressed by a finite sum:

$$D = \sum k_\nu \cdot \delta(x_\nu)$$

Here $\{x_\nu\}$ is a finite set of points in X and $\{k_\nu\}$ are integers. The set $\{x_\nu\}$ is called the support of D and the degree is defined by:

$$\deg(D) = \sum k_\nu$$

The class of divisors whose degree is zero is denoted by $\mathfrak{D}_0$.

Principal divisors. Denote by $\mathcal{M}$ the sheaf of meromorphic functions on X. If f is a section over some open subset V it has an order at each point in V. Namely, if $a \in V$ we set

$$\text{ord}_x(f) = k$$

where $f(z) = z^k \cdot f_0(z)$ holds in some chart U which contains z while $U \subset V$ and f_0 is holomorphic in U with $f_0(a) \neq 0$. So here $k > 0$ if f has a zero of multiplicity k and $k < 0$ when f has a pole of order $-k$. It is obvious that set of points where the order is $\neq 0$ is a discrete subset of V. When f is a gobally defined meromorphic functions the set of its poles and zeros are finite and give the divisor

$$D(f) = -\sum \text{ord}_{p_\nu}(f) \cdot \delta(p_\nu) - \sum \text{ord}_{q_k}(f) \cdot \delta(q_k)$$

where the first sum is taken over the poles and the second over the zeros of f. We refer to $D(f)$ as the principal divisor associated to f. Notice that the first minus sign aboe means that poles of f yield positive integers in the sum which defines the degree of $D(f)$ The result from § 0. XX shows that its degree is zero, i.e. every principal divisor belongs to $\mathfrak{D}_0$.

The sheaves $\mathcal{O}_D$. Let D be a divisor on X. Denote by $\mathcal{O}_D$ the subsheaf of $\mathcal{M}$ whose sections satisfy

$$\text{ord}_a(f) \geq -D(a)$$

for all points a in the open set where f is defined as a meromorphic function.

Example. When $p \in X$ and $n \geq 1$ is a positive integer we get the divisor $D = n \cdot p$. Here $\mathcal{O}_D$ is the sheaf of meromorphic functions whose sections have no poles outside p while the order of the pole at p is at most n.

Summary about the material in § A-H.

1. Divisors. In § A we prove that the cohomology group $H^1(X, \mathcal{O}_D)$ is a finite dimensional complex vector space and in § B the finite dimensionality is extended to the sheaves $\mathcal{O}_D$ where D is an arbitrary divisor. which gives the integer

$$(*) \qquad \chi(\mathcal{O}_D) = \dim_{\mathbf{C}}[H^0(X, \mathcal{O}_D)] - \dim_{\mathbf{C}}[H^1(X, \mathcal{O}_D)]$$

The *Riemann-Roch formula* asserts that

$$(**) \qquad \chi(\mathcal{O}_D) = 1 - g + \deg(D)$$

holds for every divisor D. This is used to construct non-constant meromorphic functions on X. For example, let $p \in X$ and consider the divisor $D = D_{(g+1)\cdot p}$ whose degree is $g + 1$. Then (**) gives

$$\dim_{\mathbf{C}} H^0(X, \mathcal{O}_D) = 1 - g + (g + 1) + \dim_{\mathbf{C}} H^1(X, \mathcal{O}_D) \geq 2$$

This shows that there exists a non-constant meromorphic function f on X which is holomorphic outside p and has a pole at p of of order $\leq g + 1$ at most.

2. The Dolbeault isomorphism. By C^∞-partitions of the unity $\mathcal{E}$ is a soft sheaf and has therefore trivial cohomology. The same holds for the sheaf $\mathcal{E}^{0,1}$ whose sections are differential forms of bidegree $(0, 1)$. Next, in every chart of X the inhomogeneous $\bar{\partial}$ equation has a solution which gives an exact sequence of sheaves:

$$0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \to 0$$

This gives an exact sequence of complex vector spaces

$$0 \to \mathcal{O}(X) \to \mathcal{E}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(X) \to H^1(X, \mathcal{O}) \to 0 \implies$$

$$(2^*) \qquad H^1(X, \mathcal{O}) \simeq \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}(\mathcal{E}(X))}$$

If we consider the sheaf of all differential 1-forms one has the isomorphism

$$(2^{**}) \qquad H^1(X, \mathbf{C}) \simeq \frac{\mathcal{E}^1(X)}{d(\mathcal{E}(X))}$$

The finite dimensionality of this topological cohomology group follows from existence of simplicial decompositions. Next, using (2*) we can exhibit the dual space of $H^1(X, \mathcal{O})$. Namely the dual of the Frechet space $\mathcal{E}^{0,1}(X)$ is $\mathfrak{c}^{(1,0)}$. In a chart of X where z is a local coordinate a current $\gamma \in \mathfrak{c}^{1,0}(X)$ is expressed by

$$\gamma = \mu \cdot dz$$

where μ is a distribution in the chart. Since $H^1(X, \mathcal{O})$ is finite dimensional, it follows from (2*) that $\bar{\partial}(\mathcal{E}(X))$ has finite codimension in $\mathcal{E}^{0,1}(X)$ and is therefore a closed subspace. A current $\gamma \in \mathfrak{c}^{1,0}(X)$ vanishes on the subspace $\bar{\partial}(\mathcal{E}(X))$ if and only if it is $\bar{\partial}$-closed. In a chart where γ is expressed as in (1) this means that the distribution μ satisfies the equation

$$(1) \qquad \bar{\partial}(\mu) = 0$$

Since the first order differential operator $\bar\partial$ is elliptic the distribution μ is a holomorphic density. Hence the $\bar\partial$-closed current γ is a holomorphic 1 -form on X, i.e. a global section of the sheaf Ω. Hence there exists a surjective $\mathbf{C}$ -linear map

$$(*)\qquad \Omega(X)\mapsto \mathrm{Hom}_{\mathbf{C}}(\frac{\mathcal{E}^{0,1}(X)}{\bar\partial(\mathcal{E}(X)}),\mathbf{C})$$

The injectivity in (*). If $\omega\in\Omega(X)$ is not identically zero we can construct the conjugate (0,1)-form $\bar\omega$ and find that

$$\int_X \omega\wedge\bar\omega>0$$

We conclude that (*) also is injective which proves that (*) is a duality isimorphism. In particular we get the equality

$$g=\dim_{\mathbf{C}}(\Omega(X))$$

3. Another duality. Starting from the sheaf Ω we have the exact sequence

$$\Omega\to\mathcal{E}^{1,0}\xrightarrow{\bar\partial}\mathcal{E}^{1,1}\to 0$$

It follows that

$$H^1(X,\Omega)\simeq\frac{\mathcal{E}^{1,1}(X)}{\bar\partial(\mathcal{E}^{1,0}(X))}$$

Hence the dual space of $H^1(X,\Omega)$ can be identified with the space of C^∞-functions f on X such that

$$\int_X f\cdot\bar\partial(\gamma^{1,0})=0$$

for every test-form of bi-degree $(0,1)$. Stokes theorem entails that

$$0=\int_X\bar\partial(f)\,\wedge\gamma^{1,0})\quad\text{for all}\quad\gamma^{1,0}\in\mathcal{E}^{1,0}(X)$$

We conclude that f is holomorphic and hence one has the isomorphism

$$[H^1(X,\Omega)]^*\simeq\mathcal{O}(X)$$

which shows that the cohomology group $H^1(X,\Omega)$ is 1-dimensional.

4. The general duality theorem. The duality in (*) extends to an arbitrary divisor D and goes as follows. We have the sheaf $\mathcal{O}_D$ and introduce the sheaf

$$(1)\qquad \mathcal{E}_D=\mathcal{O}_D\otimes_{\mathcal{O}}\mathcal{E}$$

Similarly we get the sheaves $\mathcal{E}_D^{(0,1)}$ and $\mathcal{E}_D^{(1,0)}$ which all become soft sheaves. Let $-D$ be the opposite divisor and put

$$(2)\qquad \Omega_{-D}=\mathcal{O}_{-D}\otimes_{\mathcal{O}}\mathcal{E}$$

Then every global section $\omega\in\Omega_{-D}(X)$ defines a continuous linear form on $\mathcal{E}_D^{(0,1)}(X)$ by

$$\phi\mapsto\int_X\omega\wedge\phi$$

Exactly as in the case of the trivial divisor one gets the isomorphism

$$(***)\qquad \Omega_{-D}(X)\simeq\mathrm{Hom}_{\mathbf{C}}(\frac{\mathcal{E}_D^{(0,1)}(X)}{\bar\partial(\mathcal{E}_D(X)},\mathbf{C})$$

We refer to (***) as the general duality theorem for compact Riemann surfaces and the proof is given in § XX.

5. A consequence of the duality. To each divisor D there exists the sheaf Ω_D whose Euler characteristic is defined by:

$$(1)\qquad \chi(\Omega_D)=\dim_{\mathbf{C}}[H^0(X,\Omega_D)]-\dim_{\mathbf{C}}[H^1(X,\Omega_D)]$$

Using the scyscraper sheaf associated to a divisor one finds the formula

(2) $$\chi(\Omega_D) = \chi(\Omega) + |D|$$

From (*) and the duality in (C.4) we have in partcular: $\chi(\Omega_X) = 1 - g$ and hence that:

(3) $$\chi(\Omega_D) = 1 - g + |D|$$

6. Abel's theorem. Let $\mathfrak{D}_0$ be the family of divisors on X of degree zero. If f is a globally defined meromorphic function we associate its principal divisor $D(f)$ where we assign positve integers at poles and negative integers at zeros. In § (0.2) we have seen that the degree $|D(f)| = 0$. The question arises when a divisor D of degree zero is principal. Let D be such a divisor whose support consists of a family of points points $q_1, \ldots, q_m$ whose k-numbers are < 0 and another family of points $p_1, \ldots, p_m$ whose k-numbers are > 0. Let $M = \sum^+ k_\nu$ with the sum extended over those k_ν which are > 0. Since D has degree zero we can pick M many simple curves $\gamma_1, \ldots, \gamma_M$ where the end-points of γ_j consists of one p-point and one q-point. Let $\omega_1, \ldots, \omega_g$ be a basis of the complex vector space $H^0(X, \Omega)$. To each $1 \leq j \leq g$ we assign the complex number

$$\rho_j = \sum_{\nu=1}^{\nu=M} \int_{\gamma_\nu} \omega_j$$

Abel's theorem asserts that all these ρ-numbers are zero if and only if the divisor D is principal. This result is proved in § E and leads to interesting conclusions in § H relating the Jacobi map with the Picard group.

A. The cohomology group $H^1(X,\mathcal{O})$

Let X be a compact Riemann surface and $\mathfrak{U}=\{U_\nu\}$ is a finite covering of charts which gives the Cech complex

$$0\to C^0(\mathfrak{U},\mathcal{O})\xrightarrow{\delta} C^1(\mathfrak{U},\mathcal{O})\xrightarrow{\delta} C^2(\mathfrak{U},\mathcal{O})\to\dots$$

where the arrows are the usual co-boundary maps. Above

$$C^0(\mathfrak{U},\mathcal{O})=\oplus\,\mathcal{O}(U_\nu)\quad\text{and}\quad C^1(\mathfrak{U},\mathcal{O})=\oplus\,\mathcal{O}(U_\nu\cap U_j)$$

A 1-cochain is an alternating family $\{f_{\nu,j}\in\mathcal{O}(U_\nu\cap U_j)\}$, i.e. $f_{\nu,j}=-f_{j,\nu}$ hold for all pairs $\nu,j,$. The δ-map from $C^0(\mathfrak{U},\mathcal{O})$ into $C^1(\mathfrak{U},\mathcal{O})$ is defined by

$$\delta(\oplus f_j)=\{f_{\nu,j}=f_\nu-f_j\}$$

Next, we have 1-cocycles

$$Z^1(\mathfrak{U},\mathcal{O})=\{f_{\nu,j}\in\mathcal{O}(U_\nu\cap U_j)\quad\text{such that}\quad f_{\nu j}+f_{jk}+f_{k\nu}=0\}$$

where the last equality holds whenever the open set $U_\nu\cap U_j\cap U_k\neq\emptyset$. The Cech cohomology group with respect to the covering $\mathfrak{U}$ is defined by:

(*) $$H^1(\mathfrak{U};\mathcal{O})=\frac{Z^1(\mathfrak{U},\mathcal{O})}{\delta(C^0(\mathfrak{U},\mathcal{O}))}$$

A.1 Dolbeault formulas. Recall from § XX that the inhomogenous $\bar\partial$-equation is solvable in every open subdomain of $\mathbf{C}$. If $\mathfrak{U}$ is a finite covering by charts every intersection in this family is biholomorphic with a domain in $\mathbf{C}$. It follows that for each $p\geq 0$ there is an exact sequence:

(i) $$0\to C^p(\mathfrak{U},\mathcal{O})\to C^p(\mathfrak{U},\mathcal{E})\xrightarrow{\bar\partial} C^p(\mathfrak{U},\mathcal{E}^{0,1})\to 0$$

Recall from § xx that C^∞-partitions of the soft sheaves $\mathcal{E}$ and $\mathcal{E}^{0,1}$ entail that

(ii) $$H^p(\mathfrak{U};E)=H^p(\mathfrak{U};E^{0,1})\quad:\quad p\geq 1$$

Then (ii) and the general result in § xx give an exact sequence:

(iii) $$H^0(\mathfrak{U};\mathcal{O})\to H^0(\mathfrak{U};\mathcal{E})\to H^0(\mathfrak{U};E^{0,1})\to H^1(\mathfrak{U};\mathcal{O})\to 0$$

Moreover, the long exact sequence from § xx and the vanishing of higher cohomology in (ii) give:

(iv) $$H^p(\mathfrak{U};O)=0)\quad:\quad p\geq 2$$

Next, by definition

$$H^0(\mathfrak{U};E^{0,1})=Z^0(\mathfrak{U};E^{0,1})=\mathcal{E}^{0,1}(X)$$

Similarly $H^0(\mathfrak{U};\mathcal{E})=\mathcal{E}(X)$ and from (iii) we get:

A.2 Theorem. *For every finite covering by charts one has an isomorphism:*

(*) $$H^1(\mathfrak{U};O)\simeq\frac{\mathcal{E}^{0,1}(X)}{\bar\partial(\mathcal{E}(X))}$$

A.3 The finite dimensionality of $H^1(X,\mathcal{O})$. Every finite chart covering $\mathfrak{U}$ is acyclic with respect to the sheaf $\mathcal{O}$. Hence Leray's theorem in §§ implies that the cohomology group $H^1(X;\mathcal{O})$ is equal to the Cech cohomlogy $H^1(\mathfrak{U};O)$. To see that the complex vector space $H^1(X;\mathcal{O})$ has finite dimension we can argue as follows: Let $\mathfrak{U}_*$ be another covering of charts which is refinement of $\mathfrak{U}$ as explained in § xx. This gives a map

(1.3.i) $$H^1(\mathfrak{U};\mathcal{O})\to H^1(\mathfrak{U}_*;\mathcal{O})$$

Recall Montel's theorem which asserts that when V is a relatively compact subset of a domain U in $\mathbf{C}$ then the restriction map $\mathcal{O}(U)\to\mathcal{O}(V)$ is compact. Since the map (1.3.i) is an isomrphism it follows from the general result from [functional analysis in xx] applied to the Frechet spaces defined by sections of $\mathcal{O}$ over open sets) that the common cohomology groups above have finite dimension.

B. The Riemann-Roch theorem

Let D be a positive divisor, i.e. the integers $\{k_\nu\}$ in (1.2.0) are positive. Now $\mathcal{O}$ is a subsheaf of $\mathcal{O}_D$ which gives an exact sequence of sheaves:

$$0 \to \mathcal{O} \to \mathcal{O}_D \to \mathcal{S}_D \to 0$$

where $\mathcal{S}_D$ is the scyscraper sheaf of D. For the scyscraper sheaf $\mathcal{S}_D$ one has $H^1(X, \mathcal{S}_D) = 0$ which gives an exact sequence of complex vector spaces:

(i) $$0 \to H^0(X, \mathcal{O}) \to 0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{S}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}_D) \to 0$$

Recall from § xx that the higher cohomology groups $H^p(X, \mathcal{O})$ are zero if $p \geq 2$. Set

$$\chi(\mathcal{O}_D) = \dim_{\mathbf{C}} H^0(X, \mathcal{O}_D) - \dim_{\mathbf{C}} H^1(X, \mathcal{O}_D)$$

The exact sequence (i) gives the equality

(ii) $$\chi(\mathcal{O}_D) = \chi(\mathcal{O}) + \deg(D) = 1 - g + \deg(D)$$

Above D was a positive divisor. In general a divisor D is the difference of two positive divisors, i.e. $D = D_1 - D_2$. Here one has the inclusion

$$\mathcal{O}_{-D_2} \subset \mathcal{O}_D$$

and exactly as above we have:

$$\chi(\mathcal{O}_D) = \chi(\mathcal{O}_{-D_2}) + \deg(D_1)$$

The inclusion $\mathcal{O}_{-D_2} \subset \mathcal{O}$ gives

$$\chi(\mathcal{O}) = \chi(\mathcal{O}_{-D_2}) + \deg(D_2)$$

Adding this two equations we obtain

B.1 Theorem. *For every divisor D one has the equality:*

(*) $$\chi(\mathcal{O}_D) = \chi(\mathcal{O}) + \deg(D_1) - \deg(D_2) = 1 - g + \deg(D)$$

B.2 Existence of meromorphic functions. Let $p \in X$ and consider the divisor $D = (g+1) \cdot p$. So here $\mathcal{O}_D$ is the sheaf of meromorphic functions with a pole of order $\leq g+1$ at p and otherwise it is holomorphic. The Riemann-Roch formula gives

(i) $$\dim_{\mathbf{C}} (H^0(X, \mathcal{O}_D)) = \dim_{\mathbf{C}} (H^1(X, \mathcal{O}_D)) + 1 - g + (g+1) \geq 2$$

This proves that $H^0(X, \mathcal{O}_D)$ contains a non-constant meromorphic function. Hence there exists a non-constant meromorphic function f with a pole at p of order $g+1$ at most and otherwise f is holomorphic.

C. The duality theorem

Denote by Ω the sheaf of holomorphic 1-forms. A section ω defined in some open set V is given as follows: If $U \subset V$ is a chart where z is the local coordinate then

(1) $$\omega | U = f(z) \cdot dz \quad \text{where} \quad f \in \mathcal{O}(U)$$

Let $\omega \in \Omega(X)$ be a global section. For every globally defined differential form $\sigma \in \mathcal{E}^{0,1}(X)$ we can integrate the 2-form $\omega \wedge \sigma$ over X and get the map:

(2) $$\sigma \to \int_X \omega \wedge \sigma$$

If σ is $\bar{\partial}$-exact, i.e. if $\sigma = \bar{\partial}(g)$ for some C^∞-function we have:

$$d(g\omega) = \bar{\partial}(g\omega) = \bar{\partial}(g) \wedge \omega = -\omega \wedge \sigma$$

By Stokes Theorem the integral

(3) $$\int_X d(g\omega) = 0$$

We conclude that ω yields a linear functional on the quotient space

(4) $$\frac{\mathcal{E}^{0,1}(X)}{\bar\partial(C^\infty(X))}$$

which by § XX is isomorphic to $H^1(X,\mathcal{O})$. Hence we have found a linear map:

(5) $$L\colon\ \Omega(X)\mapsto H^1(X,\mathcal{O})^*$$

C.1 Theorem. *The L-map is an isomorphism.*

Proof. Recall that $\mathfrak{c}^{1,0}(X)$ is the dual of $\mathcal{E}^{0,1}(X)$ and when a $(1,0)$-current γ is zero on the subspace $\bar\partial(C^\infty(X))$ it is $\bar\partial$-closed which by the elliptic property means that γ is a holomorphic 1-form. This shows that L is surjective. To see that L is injective we take some non-zero $\omega\in\Omega(X)$ and construct the conjugate $(0,1)$-form $\bar\omega$ and by the result in § xx one has

$$\int_X \omega\wedge\bar\omega = i\cdot A$$

where A is a positive real number. This shows that $L(\omega)$ is not identically zero and the proof is finished.

C.2 A more general duality theorem. Let D be a divisor and define the sheaf:

$$\mathcal{E}_D=\mathcal{O}_D\otimes_{\mathcal{O}}\mathcal{E}$$

Similarly we get the sheaf $\mathcal{E}_D^{0,1}$ and passing to the opposed sheaf $-D$ we get the two sheaves

$$\mathcal{E}_{-D}\quad\text{and}\quad\mathcal{E}_{-D}^{1,0}$$

If $\phi\in\mathcal{E}_{-D}^{1,0}(X)$ and $\psi\in\mathcal{E}_D^{0,1}(X)$ the reader may check that there exists an integral

(i) $$\int_X\phi\wedge\psi$$

The Dolbeault isomorphism gives:

(ii) $$H^1(X;\mathcal{O}_D)\simeq\frac{\mathcal{E}_D^{0,1}(X)}{\mathcal{E}_D^(X))}$$

Next, we have the sheaf

$$\Omega_{-D}=\mathcal{O}_{-D}\otimes_{\mathcal{O}}\Omega$$

Now (i-ii) gives a canonical map

(iii) $$L_D\colon\Omega_{-D}(X)\to H^1(X;\mathcal{O}_D)^*$$

Exercise. Repeat the proof from the case when $D=0$ to show that L_D is an isomorphism.

C.3 Another duality. We have the exact sheaf sequence

$$0\to\Omega\to\mathcal{E}^{1,0}\overset{\bar\partial}{\to}\mathcal{E}^{1,1}\to 0$$

This gives

(i) $$H^1(X;\Omega)\simeq\frac{\mathcal{E}^{1,1(X)}}{\bar\partial(\mathcal{E}^{1,0})}$$

The dual space $\mathcal{E}^{1,1}(X)^*=C^\infty(X)$ and a C^∞-function f vanishes on the subspace $\bar\partial(\mathcal{E}^{1,0})$ if and only if $\bar\partial(f)=0$,i.e. if $f\in\mathcal{O}(X)$. Hence we have

(ii) $$H^1(X;\Omega)^*\simeq\mathcal{O}(X)=\mathbf{C}$$

it follows that

(*) $$\chi(\Omega)=g-1$$

D. The Hodge decomposition.

The duality from § C implies that

$$\dim_{\mathbf{C}} \Omega(X) = \dim_{\mathbf{C}} \frac{\mathcal{E}^{0,1}(X)}{\bar\partial(\mathcal{E}(X))} \tag{1}$$

Denote by $\bar\Omega(X)$ be the space of ∂-closed $(0,1)$-forms, i.e they arise as conjugates of holomorphic 1-forms. The map

$$\omega \to \bar\omega$$

is a **C**-linear isomorphism and from this the reader can conclude that we also have the equality

$$\dim_{\mathbf{C}} \bar\Omega(X) = \dim_{\mathbf{C}} \frac{\mathcal{E}^{1,0}(X)}{\partial(\mathcal{E}(X))} \tag{2}$$

D.1 Theorem. *One has the two direct sum decompositions*

$$\mathcal{E}^{1,0}(X) = \partial(\mathcal{E}(X)) \oplus \Omega(X) \quad : \quad \mathcal{E}^{0,1}(X) = \bar\partial(\mathcal{E}(X)) \oplus \bar\Omega(X)$$

Exercise. Prove this result. The hint is to count dimensions via (1-2) above.

Next, $\Omega(X)$ and $\bar\Omega(X)$ both appear as subspaces of $\mathcal{E}^1(X)$ and here we get another decomposition:

D.2 Theorem. *One has*

$$\mathcal{E}^1(X) = d(\mathcal{E}(X)) \oplus \Omega(X) \oplus \bar\Omega(X) \tag{*}$$

Proof. Since $\mathcal{E}^1(X) = \mathcal{E}^{0,1}(X) \oplus \mathcal{E}^{0,1}(X)$ it is clear that the right hand side is equal to $\mathcal{E}^1(X)$. There remains to show that one has a direct sum, i.e. suppose that

$$dg + \omega_1 + \bar\omega_2 = 0$$

where $g \in \mathcal{E}$ and ω_1, ω_2 is a pair of holomorphic 1-forms. Write $dg = \partial g + \bar\partial g$ which gives

$$\partial(g) = \omega_1 \implies \bar\partial(\partial(g)) = 0$$

As explained in §§ x this means that g is a harmonic function and by the maximum principle reduced to a constant so that $dg = 0$. It follows that $\omega_1 = \omega_2 = 0$ and the requested direct sum decomposition follows.

D.3 A basis for $H^1(X, \mathbf{C})$. This cohomology group is obtained via the ordinary d-complex, i.e. we have the exact sequence of sheaves:

$$0 \to \mathbf{C}_X \to \mathcal{E} \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \to 0$$

Using C^∞-partitions of the unity it follows that:

$$H^1(X, \mathbf{C}) = \frac{\mathrm{Ker}_d(\mathcal{E}^1(X))}{d(\mathcal{E}(X)} \tag{1}$$

Theorem D.2 entails that

$$H^1(X, \mathbf{C}) \simeq \Omega(X) \oplus \bar\Omega(X)$$

So if $\omega_1, \ldots, \omega_g$ is a basis for the g-dimensional vector space $\Omega(X)$ then $H^1(X, \mathbf{C})$ is $2g$-dimensional where a basis is given by this g-tuple of holomorphic forms and the corresponding g-tuple $\{\bar\omega_\nu\}$.

E. The Riemann-Hurwitz formula.

Let ω be a non-zero global section in $\Omega(X)$. At each point p we have a local chart U with coordinate z and write $\omega|U = g(z)\cdot dz$ where the holomorphic function g may have a zero of some order ≥ 1 at p. Such zeros can occur at a finite set of points and gives a positive divisor D_ω which assigns positive integers at those $p \in X$ where ω has a zero of some order $k \geq 1$. It is clear that one has an equality of sheaves:

$$\Omega = \mathcal{O}_{D_\omega} \cdot \omega$$

The left hand side is isomorphic to the sheaf $\mathcal{O}_{D_\omega}$ so the Riemann-Roch theorem gives

$$g - 1 = \chi(\Omega) = 1 - g + \text{deg}D_\omega) \tag{1}$$

Hence we have proved

E.1 Theorem. *For every non-zero holomorphic 1-form the number of zeros counted with multiplicity is equal to* $2g-2$.

E.2 The case of meromorphic forms. If Ω is a meromorphic 1-form we also encounter poles and get an associated divisor D_ω whose negative part comes from poles counted by their order and from the above one has

$$2g - 2 = \deg(D_\omega) \tag{*}$$

This applies in particular when f is a meromorphic function and $\omega = \partial f$. The divisor $D_{\partial f}$ has a negative part caused by poles of f. Notice that if f has a pole of order k at some point $p \in X$ then

$$\partial f = d(z^{-k}g) = -kz^{-k-1} \cdot dz + z^{-k} \cdot \partial g$$

which entails that ∂f has a pole of order $k+1$. It follows that if $\{p_\nu\}$ are the poles of f with multiplicities $\{k_\nu\}$ then the negtive part of $D_{\partial f}$ has degree

$$\sum (k_\nu + 1)$$

E.3 Zeros of ∂f. If ζ is a local coordinate at a point q where f has no pole write

$$f(\zeta) = f(q) + a\zeta^k + \text{higher order terms} \implies df = ak\zeta^{k-1} + \text{higher order terms}$$

If $k \geq 2$ we assign the positive one-point divisor $(k-1)\delta(q)$. Taking the sum over all these one-point divisors we get the positive part of the divisor $D_{\partial f}$.

E.4 Example. Suppose that the meromorphic function f has degree $n \geq 2$. Zeros of df outside the poles correspond to critical points under the holomorphic map $f\colon X \to \mathbf{P}^1$. Their images are called critical values of the map. If α is a critical value the inverse fiber $f^{-1}(\alpha)$ consists of strictly less than n many points, say $q_1, \ldots, q_m$ where $df(q_j) = 0$ for at least one j and when this holds we refer to q_j as a branch point under the mapping $f\colon X \to \mathbf{P}^1$. In this way we get a finite set of branch points $q_1, \ldots, q_m$ and let $\rho(q_j)$ be the order of the zero of df at each q_j. Hence

$$\sum (\rho(q_j) - 1) \cdot \delta(q_j)$$

gives positive part of the divisor $D_{\partial f}$. In the case when all poles of f are simpke each pole contributes with degree 2 and as a result one has the degree formua

$$2g - 2 = \deg(D_{\partial f} = \sum (\rho(q_j) - 1) - 2n \tag{*}$$

E.5 Calculations of some g-numbers Consider an irreducible equation of the form

$$y^m - Q(x) = 0$$

where $Q(x)$ is a polynomial with simple zeros of some degree n and $m \geq 2$. Now

$$my^{m-1}dy = Q'(x)dx$$

It follows that dy has simple zeros when $Q'(x) = 0$ and counting multiplicities and recalling that one has an m-sheeted covering from the Riemann surface X onto the projective x-line the positive part of tthe divisor D_{dy} has degree

$$m(n-1)$$

Next, above $x = \infty$ the meromorphic function y has poles. If the integers m and n are relatively prime wr recall from § § that there is a unique point above $x = \infty$ and a local coordinate t where $y = t^{-n}$. So the degree of the negative part of D_{dy} becomes $-(n+1)$. From this we get the genus formula:

$$g = 1 + \frac{(m(n-1)}{2} - \frac{n+1}{2} \tag{1}$$

When n and m have a common factor we write $n = kn_*$ and $m = km_*$ where n_* and m_* are relatively prime. In this case we get k many charts above $x = \infty$ where y has a pole of order n_* in each chart. The result is that

$$g = 1 + \frac{(m(n-1)}{2} - k \cdot \frac{n_* + 1}{2} \tag{2}$$

For example, consider the equation

$$y^4 = x^2 + x + 1$$

Here (xx) gives $g = 1 + 2 - 2 \cdot 2 = 1$ so X is a torus.

Exercise. Verify that in the example above the 1-form $\frac{dx}{y}$ is holomorphic and $\neq 0$ on X which clarifies why $g = 1$.

F. Weierstrass' points

Let X be a Riemann surface of some genus $g \geq 2$ Given a point $p \in X$ we ask if there exists a meromorphic function f on X which is holomorphic in $X \setminus \{p\}$ and the order of the pole at p is $\leq g$. If such a meromorphic function exists we say that p is a Weierstrass point and denote this set by $\mathcal{W}(X)$. We shall prove that this is a non-empty subset of X when $g \geq 2$. To achieve this we take a basis $\omega_1, \ldots, \omega_g$ in the complex vector space $\Omega(X)$.

F.1. Wronskian determinants. Let $p \in X$ and choose a chart U_0 around p where p is the origin and z the local coordinate. We get the g-tuple of holomorphic functions $\{f_\nu\}$ in U_0 defined by

$$\omega_\nu | U_0 = f_\nu(z) \cdot dz$$

In the chart we take higher order derivatives with respect $\partial = \partial/\partial z$ and get the following $m \times m$-matrix with elements in $\mathcal{O}(U_0)$:

$$\phi_{m\nu}(z) = \frac{\partial^{m-1}(f_\nu)(z)}{(m-1)!} \quad 1 \leq m, \nu \leq g$$

Set

(i) $$W(z) = \det(\phi_{m\nu}(z))$$

Since $f_1, \ldots, f_g$ are $\mathbf{C}$-linearly independent in the chart U_0, the general result in § XX shows that $W(z)$ is not identically zero. Next, let $\mathfrak{U} = \{U_i\}$ be a finite covering of X by charts. In each chart U_i we choose a local coordinate z_i and construct the Wronskian $W_i \in \mathcal{O}(U_i)$. Consider a pair of charts where $U_i \cap U_k \neq \emptyset$. In the interesction we have $z_k = z_k(z_i)$ and get the non-vanishing holomorphic function

(ii) $$\frac{dz_k}{dz_i}$$

The transition formula from § XX gives:

F.2. Lemma. *In each non-empty intersection $U_i \cap U_k$ we have*

$$W_k = (\frac{dz_k}{dz_i})^N \cdot W_i \quad : \quad N = \frac{g(g+1)}{2}$$

This gives a well defined finite subset Σ in X such that for each chart U_i one has

(*) $$\Sigma \cap U_i = W_i^{-1}(0)$$

F.3. Theorem. *Every $p \in \Sigma$ is a Weierstrass point.*

Proof. Let (U, z) be a chart around p and for each $1 \leq \nu \leq g$ the restriction $\omega_\nu | U$ is of the form $f_\nu(z) dz$ where $\{f_\nu\}$ in $\mathcal{O}(U)$. Every f_ν has a Taylor expansion

$$f_\nu(z) = a_{0,\nu} + a_{1,\nu} z + \ldots + a_{g-1,\nu} z^{g-1} + \quad \text{higher order terms}$$

The hypotesis that the local Wronskian is zero at p implies that the determinant of the $g \times g$-matrix formed by the doubly-indexed a-coefficients is zero. Hence there exists a non-zero g-tuple $(c_1, \ldots, c_g)$ such that

(1) $$\sum_{j=0}^{j=g-1} a_{j,\nu} \cdot c_j = 0 \quad : \quad 1 \leq \nu \leq g$$

Next, in U_0 we consider the meromorphic function

(2) $$h(z) = \frac{c_1}{z} + \ldots + \frac{c_g}{z^g}$$

It follows from (1) that

(3) $$\mathfrak{res}(\omega_\nu(h)) = 0 \quad : \quad 1 \leq \nu \leq g$$

Since $\omega_1, \ldots, \omega_g$ is a basis for $\Omega(X)$ the general result in § XX gives a globally defined meromorphic function H which is holomorphic in $X \setminus \{p\}$ whose polar expansion at p is h. In particular H has a pole of order $\leq g$ at p and since no other poles occur we have $p \in \mathcal{W}(X)$.

The divisor $\mathfrak{w}$. The functions $\frac{dz_k}{dz_i}$ from (ii) are $\neq 0$ in each pair of overlapping charts. Hence there exists a non-negative divisor $\mathfrak{w}$ on X where one for each chart U_i has:

$$\mathfrak{w}(x) = \mathrm{ord}_x(W_i) \quad x \in U_i$$

F.4. Proposition. *One has the equality*

$$|\mathfrak{w}| = g(g+1)(g-1)$$

Proof. Let $\omega \in \Omega(X)$ be a non-zero holomorphic 1-form. In each chart we can write

$$\omega | U_i = h_i(z) \cdot dz_i$$

where the holomorphic h-functions satisfy

$$h_k = \frac{dz_i}{dz_k} \cdot h_i \quad \text{ in } U_k \cap U_i$$

It follows that

$$h_k^{-N} = (\frac{dz_k}{dz_i})^N \cdot h_i^{-N}$$

Hence $\{h_k^{-N}\}$ satisfies the same transition formulas as the W-functions which gives a globally defined meromorphic function F on X such that

$$F | U_i = W_i \cdot h_i^{-N} \tag{1}$$

Next, the holomorphic 1-form ω has its divisor D_ω whose degree becomes:

$$|D_\omega| = 2(g-2) \tag{2}$$

At the same time the principal divisor of F has degree zero and then (1-2) entail that

$$|\mathfrak{w}| = N|D_\omega| = 2N(g-2) = g(g+1)(g-1)$$

where the last equality follows from Lemma F.2.

F.5 Conclusion. With $g \geq 2$ the number of Weierstrass points is at least $g(g+1)(g-1)$.

G. Abel's theorem.

In § X we have shown that if $f \in \mathcal{M}(X)$ then its principal divisor $D(f)$ has degree zero. Hence we get a subset of $\mathcal{D}_0$ given by

$$\mathcal{P}_0 = \{D(f) \quad : f \in \mathcal{M}(X)\}$$

We shall find a necessary and sufficient condition in order that a divisor $D \in \mathcal{D}_0$ belongs to $\mathcal{P}_0$. To attain this we shall represent divisors in $\mathcal{D}_0$ by chains. Denote by $\mathcal{C}$ the family of C^1-curves in X:

$$\text{(i)} \qquad t \mapsto x(t) \quad : 0 \leq t \leq 1$$

from the closed unit interval into X. The curve in (i) has two end-points p, q in X and if g is any C^∞-function on X we have

$$\text{(ii)} \qquad g(q) - g(p) = \int_0^1 g(x(t)) \cdot \dot{x}(t)\, dt$$

Recall from calculus that (ii) does not depend upon the special parametrization of the curve so we prefer to write (ii) as $\int_c dg$ and c is called an elementary 1-chain whose boundary $\partial(c)$ consists of the two points q and p reflected by the equations

$$\text{(ii)} \qquad g(q) - g(p) = \int_c dg \quad : \quad g \in \mathcal{E}(X)$$

More generally, consider a finite family of such curves $\{c_k\}$. The sum yields a 1-chain $\mathbf{c}$ where

$$\text{(iii)} \qquad \int_{\mathbf{c}} dg = \sum \int_{c_k} dg = \sum g(q_k) - \sum g(p_k)$$

where $\{(p_k, q_k)\}$ are the end-points of the individual c-curves in the chain $\mathbf{c}$. Since the manifold X is connected it is clear that if N is a positive integer and $\{p_k\}$ and $\{q_k\}$ an arbitrary pair of N-tuples of points in X, then there exists a 1-chain $\mathbf{c}$ which consists of N curves $c_1, \ldots, c_N$ where c_k is a curve from p_k to q_k so that (iii) holds. Every divisor D in $\mathcal{D}_0$ is represented by a pair of such N-tuples $\{q_k\}$ and $\{p_k\}$ and we say that a 1-chain $\mathbf{c}$ represents D if

$$(*) \qquad \int_{\mathbf{c}} dg = \sum g(q_k) - \sum g(p_k) \quad : g \in \mathcal{E}(X)$$

G.1 Theorem. *A divisor D in $\mathcal{D}_0$ belongs to $\mathcal{P}_0$ if and only it can be represented by an oriented 1-chain $\mathbf{c}$ such that*

$$(*) \qquad \int_c \omega = 0 \quad \text{for all } \omega \in \Omega(X)$$

First we carry out some general constructions. If c is an elementary and oriented 1-chain then an arbitrary 1-form γ can be integrated along c and in this way c is identified with a current of degree one. More generally, a chain expressed by a sum of elementary chains yields 1-current. Moreover, when an elementary 1-chain c is defined by a map $t \to x(t)$ as in (i) above we can take a partition of the interval $[0,1]$ and express c as a sum of elementary chains where $t \to x(t)$ in each small partition is such that the range stays in a chart of X. Hence every 1-chain c is a finite sum of elementary 1-chains $\{c_\alpha\}$ where each single c_α is contained - or rather supported as a current - by a chart in U_α in x which in local coordinates can be taken as an open disc.

The currents $\frac{d\Phi}{\Phi}$. Let U be a chart which in local coordinates is an open disc centered at $z = 0$. In this chart there exists a function

$$\phi(z) = \frac{z}{z - z_0} \cdot a(z)$$

where $z_0 \neq 0$ in the disc while $a \in C^\infty(X)$ is zero-free and chosen so that ϕ is identicallly one outside a compact subset of U. Now there exists the current of degree one in X which has a compact support contained in U and in the chart it is given by

(1) $$\frac{d\phi}{\phi} = \frac{da}{a} + \frac{dz}{z} - \frac{dz}{z - z_0}$$

Exercise. Above we have a 1-current which can be decomposed into a sum of currents of bi-degree $(0, 1$ and $(1, 0)$. Show that the component of bi-degree $(0, 1)$ is the smooth current

$$a^{-1} \cdot \bar{\partial} a$$

Apply also Cauchy's residue formula in the chart and show that if $g \in \mathcal{E}(X)$ then

$$\int_X \frac{d\phi}{\phi} \wedge dg = 2\pi i(g(q) - g(p))$$

where $p \in X$ corresponds to $z = 0$ and q to z_0 in the chart. The reader should make a figure to check that the signs are okay because Stokes Theorem is applied in exterior domains in the complement of small punctured discs at $z = 0$ and $z = z_0$.

Next, let c be a 1-chain and perform a decomposition so that $c = \sum c_j$ where every c_j is an elementary chain with compact support in a chart U_j. We can carry out the construction above to each j and obtain the current

$$\rho = \sum \frac{d\phi_j}{\phi_j}$$

Next, let γ be a closed 1-form in X. Its restriction to a chart U_j is exact and if K_j is a compact subset of U_J whose interior contains the union of the support of the elementary 1-chain and the support of the current ρ_j, then we can find $g \in \mathcal{E}(X)$ such that

$$\gamma = dg + \gamma^*$$

where the support of the 1-form γ^* is disjoint from K_j. It follows that

(1) $$\rho_j(\gamma) = \rho_j(dg) = \int_c dg = \int_c \gamma$$

where the last equality follows since the support of c does not intersect the support of γ^*. Next, with $c = \sum c_j$ we associate the current

$$\rho = \frac{1}{2\pi i} \cdot \sum \frac{d\phi_j}{\phi_j}$$

A summation over j gives the equation below for every closed 1-form γ:

(*) $$\int_c \gamma = \rho(\gamma)$$

Proof of sufficiency

Suppose the divisor D is represented by a 1-chain c for which (*) holds in Theorem G.1. Construct the current ρ as above. Each $\omega \in \Omega(X)$ is closed and hence (*) above entails that

$$0 = \rho(\omega) = 0$$

Since ω is a test-form of bi-degree $(1, 0)$ we apply the $(0, 1)$-part of ρ which by (xx) is a smooth current given by

$$\rho^{0,1} = \frac{1}{2\pi i} \cdot \sum a_j^{-1} \cdot \bar{\partial} a_j$$

Since $\rho^{0,1}$ applied to every $\omega \in \omega(X)$ gives zero, the duality result in § XX implies that it is $\bar{\partial}$-exact. Hence there exists $G \in C^\infty(X)$ such that

$$\bar{\partial}(G) = \rho^{0,1}$$

Outside the the union of the finite set of pairs $\{p_j, q_j\}$ there exists the function

$$\Psi = e^{-G} \cdot \prod \phi_j$$

Here (xx) and (xx) give $\bar{\partial}(\Psi) = 0$ outside the finite union above and by the local constructions of the ϕ-functions in the charts we see that Ψ has at most simple poles at $\{p_j\}$ and yields a globally defined meromorphic function on X. Next, we also have the equation

$$\frac{d\Psi}{\Psi} = \frac{d\Phi}{\Phi} - dG$$

If $g \in \mathcal{E}(X)$ it follows that

$$\int_X \frac{d\Psi}{\Psi} \wedge dg = \int_X \frac{dF}{F} \wedge dg = \int_c dg$$

Since the 1-chain c represents the given divisor D it follows that the principal divisor $D(\Psi) = D$ which proves the sufficiency in Theorem G.1.

G.2 Proof of necessity

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We must prove that when $f \in \mathfrak{M}(X)$ and $D = \text{div}(f)$ then (*) holds in Theorem G.1. To begin with f yields a holomorphic map from X onto $\mathbf{P}^1$. So if γ is a smooth 1-form on $\mathbf{P}^1$ there exists the pull-back $f^*(\gamma) \in \mathcal{E}^1(X)$. If $\omega \in \Omega(X)$ we therefore obtain a 1-current $f_*(\omega)$ on $\mathbf{P}^1$ defined by

(1)
$$\langle f_*(\omega), \gamma \rangle = \int_X \omega \wedge f^*(\gamma) \quad : \quad \gamma \in \mathcal{E}^1(\mathbf{P}^1)$$

Let us now consider some function $g \in C^\infty(\mathbf{P}^1)$. Since $f \colon X \to \mathbf{P}^1$ is a holomorphic mapping it follows that $g \circ f \in C^\infty(X)$ and since ω is a holomorphic 1-form Stokes Theorem gives

$$\int_X \omega \wedge \bar{\partial}(g \circ f) = 0$$

Recall from § XX that f_* commutes with differentials which entails that the $(1,1)$-current $\bar{\partial}(f_*(\omega))$ on $\mathbf{P}^1$ satisfies

$$\langle \bar{\partial}(f_*(\omega)), g \rangle = \langle f_*(\omega), \bar{\partial}(g) \rangle = \int_X \omega \wedge \bar{\partial}(g \circ f) = 0$$

Since this vanishing hold for every $g \in C^\infty(\mathbf{P}^1$ it follows that $\bar{\partial}(f_*(\omega)) = 0$, i.e. the $(1,0)$-current $f_*(\omega)$ is $\bar{\partial}$-closed. Now $\bar{\partial}$ is elliptic and therefore $f_*(\omega)$ belongs to $\Omega(\mathbf{P}^1)$. But on the projective line there do not exist non-zero holomorphic 1-forms by the result in § XX. We conclude that $f_*(\omega) = 0$, i.e. we have proved the following:

G.3 Proposition. *For each pair $f \in \mathfrak{M}(X)$ and $\omega \in \Omega(X)$ the direct image current $f_*(\omega)$ is zero.*

There remains to see why Proposition G.3 enable us to construct a 1-chain $\mathbf{c} \in \text{rep}(D(f))$ such that (*) holds in Theorem G.1. Let us first consider the case when the meromorphic function f has some degree n with n simple zeros $p_1, \ldots, p_n$ and n simple poles $q_1, \ldots, q_n$. On $\mathbf{P}^1$ we have the finite set $\xi_1, \ldots, \xi_k$ which yield critical values of the mapping $f \colon X \to \mathbf{P}^1$. Let Σ denote the set $\xi_1, \ldots, \xi_k$ and put

$$X_* = X \setminus \cup f^{-1}(\Sigma)$$

Now $f \colon X_* \to \mathbf{P}^1 \setminus \Sigma$ is an n-sheeted unbranched covering map which in addition is locally biholomorphic. Let c_* be a simple closed curve which joins the origin in $\mathbf{P}^1$ with the point at infinity and avoids the set Σ. Then $f^{-1}(c_*)$ consists of n pairwise disjoint simple curves $c_1, \ldots, c_n$ where c_k joins some zero p_k with a pole indexed to be q_k. Hence the 1-chain

$$\mathbf{c} = c_1 + \ldots + c_n$$

represents $D(f)$. Finally, since $f\colon X_* \to \mathbf{P}1$ is locally biholomorphic the general result in § XX gives:

$$\int_{\mathbf{c}} \omega = \langle f_*(\omega), c_* \rangle$$

The right hand side is zero by Proposition G.3 and since this hold for every $\Omega \in \omega(X)$ the representing 1-chain $\mathbf{c}$ satisfies (*) in Theorem G.1 and the necessity is proved.

H. Jacobi integrals.

Let X be a Riemann surface with genus $g \geq 1$. The manifold X has a fundamental group $\pi_1(X)$ whose elements can be identified with homotopy classes of closed curves of class C^1 at any given point in X. Fix $p \in X$ and for each closed C^1-curve γ which has p as commen start- and end-point its homotopy class in this family of closed curves in denoted by $\{\gamma\}$. If $\omega \in \omega(X)$ we get integrals

$$\int_\gamma \omega$$

Since ω is a closed 1-form the general result in § XX shows that this integral only depends upon $\{\gamma\}$. Hence there exists a complex-valued function on $\pi_1(X)$ defined by

$$\text{(i)} \qquad \{\gamma\} \mapsto \int_\gamma \omega$$

If γ_1, γ_2 are two closed curves at p the homotopy class of the composed curve $\gamma_2 \circ \gamma_1$ is the product $\{\gamma_2\} \cdot \{\gamma_2\}$ in $\pi_1(X)$ and we have the additive formula:

$$\int_{\gamma_2 \circ \gamma_1} \omega = \int_{\gamma_2} \omega + \int_{\gamma_1} \omega$$

Hence (i) is a group homomorphism from $\pi_1(X)$ into the additive group of complex numbers.

Next, let $\omega_1, \ldots, \omega_g$ be a basis in $\Omega(X)$. Then we obtain an additive map from $\pi_1(X)$ into the g-dimensional complex vector space defined by

$$(*) \qquad \{\gamma\} \mapsto (\int_\gamma \omega_1, \ldots \int_\gamma \omega_g)$$

Its image yields an additive subgroup of $\mathbf{C}^g$ denoted by Γ and called the period group associated to the given basis $\omega_1, \ldots, \omega_g$ in $\Omega(X)$.

Exercise. Show that Γ only depends on $\omega_1, \ldots, \omega_g$. More precisely, let $q \in X$ be another point where $\pi_1(X)$ is represented by closed curves which start and end at q. Then the resulting Γ-group is the same, i.e. the chosen base point $p \in X$ does not affect the position of Γ.

An additive subgroup G in $\mathbf{C}^g$ is called a $\mathbf{Z}$-lattice if it is a free abelian group of rank $2g$ whose basis vectors are $\mathbf{R}$-linearly independent vectors in $\mathbf{C}^g$ which can be identified with a real vector space of dimension $2g$.

H.1 Theorem. *The period group Γ is a lattice in $\mathbf{C}^g$.*

The proof requires some preliminaries. To begin with, with a chosen base point p we study C^1-curves which in general are non-closed starting at p and with varying end-points in X. If $\omega \in \Omega(X)$ and c_x denotes a curve from p to a point $x \in X$ we get the line integral

$$\int_{c_x} \omega$$

If c_x and d_x are two such curves the composed curve $\gamma = d_x^{-1} \circ c_x$ is closed which entails that the difference

$$\int_{d_x} \omega - \int_{c_x} \omega = \int_\gamma \omega$$

We conclude that there exists a mapping $\mathcal{J}_p$ from X into the quotient group $\frac{\mathbf{C}^g}{\Gamma}$ defined by

$$\mathcal{J}_p(x) = \sum_{k=1}^{k=g} \overline{\int_{c_x} \omega_1, \ldots \int_{c_x} \omega_g)}$$

where c_x is an arbitrary C^1-curve c_x from p to x. and we have taken the image of the complex g-vector in the quotient group above. We refer to $\mathcal{J}_p$ as a Jacobi map at the base point p.

Choose g different base points $p_1,\ldots,p_g$ and and for every g-tuple of points $x_1,\ldots,x_g$ in X we put

$$F(x_1,\ldots,x_g)=\sum_{j=1}^{j=g}\mathcal{J}_{p_j}(x_j)$$

Hence F is a map from the g-fold product X^g into $\frac{\mathbf{C^g}}{\Gamma}$.

H.2 A local study of F. Around each p_j we choose a chart U_j where z_j is a local coordinate so when $(z_1,\ldots,z_g)$ is close to the origin in the g-dimensional z-space we the $\mathbf{C}^g$-valued function

$$\Phi(z)=F(p_1+z_1,\ldots,p_1+z_g)$$

where $\Phi=(\phi_1,\ldots\phi_g)$ and

$$\phi_k(z_1,\ldots,z_g)=\sum_{j=1}^{j=k}\int_0^{z_j}\omega_k$$

At $z=0$ we get the $g\times g$-matrix with elements

$$\rho_{jk}=\frac{\partial\phi_k}{\partial z_j}(0)$$

To find the ρ-numbers we express the 1-forms $\{\omega_k\}$ in each chart, i.e. write

$$\omega_k|U_j=f_{kj}(z)\cdot dz_j$$

Then it is clear that

$$\frac{\partial\phi_k}{\partial z_j}(0)=f_{kj}(0)$$

hold for all pairs j,k. Hence the Jacobian of Φ at $z=0$ is the $g\times g$-matrix with elements $\{f_{kj}(0)\}$.

H.3 A favourable choice of $p_1,\ldots,p_g$. The matrix $\{f_{kj}(0)\}$ depends on the chosen g-tuple $p_1,\ldots,p_g$. We will prove that for a suitable choice of this g-tuple the matrix is non-singular. To see this we notice that if the matrix is singular there exists a non-zero g-tuple of complex numbers $\alpha_1,\ldots,\alpha_g$ such that

(i) $$\sum\alpha_k\cdot f_{kj}(0)=0\quad:\quad 1\leq j\leq g$$

Now we consider the holomorphic 1-form

(ii) $$\omega=\alpha_1\omega_1+\ldots+\alpha_g\omega_g$$

The vanishing in (i) shows that ω has a zero at every p_j. The next result shows that this cannot occur for suitable g-tuples of p-points.

H.4 Lemma. *There exists a g-tuple $p_1,\ldots,p_g$ in X with the property that a holomorphic 1-form ω which is zero at each p_j must be identically zero.*

Proof. To each $p\in X$ we set

$$H_p=\{\omega\in\Omega(X(:\ \omega(p)=0\}$$

Each H_p is either $\Omega(X)$ or a hyperplane and at the same time

$$\bigcap_{p\in X}H_p=\{0\}$$

Since the vector space $\Omega(X)$ is g-dimensional it follows by elementary geometry that there exists some g-tuple $p_1,\ldots,p_g$ such that

$$H_{p_1}\cap\ldots\cap H_{p_g}=\{0\}$$

Clearly every such g-tuple does the job in Lemma H.4.

H.5 Conclusion. We can always find some g-tuple $p_1,\ldots,p_g$ such that the Jacobian of the Φ-function above is $\neq 0$ at $z=0$. This entails that if the charts $U_1,\ldots,U_g$ are taken as small discs then F yields a bijective map from the polydisc $U_1\times\ldots U_g$ onto an open neighborhood V of the origin in $\mathbf{C}^g$. In fact, it is even a biholomorphic mapping. Using this local result one has:

H.6 Proposition. *The intersection* $\Gamma \cap V$ *is reduced to the origin in* $\mathbf{C}^g$.

Proof. Via Abel EASY ...

H.7 The Picard group. Every $f \in \mathfrak{M}(X)$ yields a principal divisor $D(f)$ which belongs to $\mathcal{D}_0$. These principal divisors give an additive subgroup denoted by $\mathcal{P}_0$ and we obtain the quotient group

$$(*) \qquad \operatorname{Pic}(X) = \frac{\mathcal{D}_0}{\mathcal{P}_0}$$

If $D \in \mathcal{D}_0$ we choose a representing 1-chain c and assign the point in $\frac{\mathbf{C}^g}{\Gamma}$ given by

$$\text{(i)} \qquad \overline{(\int_c \omega_1, \ldots, \int_c \omega_g)}$$

Its image in $\frac{\mathbf{C}^g}{\Gamma}$ is unique since two different one-chains c and d for which $D = \partial(c) = \partial(d)$ yield the a closed chain $c - d$. Moreover, Abel's theorem shows that $D = D(f)$ for a meromorphic function f if and only if is a principal divisor if and only if (i) is zero in the quotient group. From this one concludes the following:

H.8 Theorem. *By (i) above we have an injective mapping from Pic*(X) *into* $\frac{\mathbf{C}^g}{\Gamma}$.

H.9 Remark. Since Γ is a lattice in $\mathbf{C}^g$ the complex analytic structure on the affine space induces a complex analytic structure on the quotient which therefore becomes a compact complex analytic manifold of dimension g. One refers to $\frac{\mathbf{C}^g}{\Gamma}$ as a g-dimensional complex torus. One can say more about the map in Theorem H.8:

H.10 Theorem. *The map in Theorem H.8 is also surjective and hence one has a set-theoretic bijection*

$$\operatorname{Pic}(X) \simeq \frac{\mathbf{C}^g}{\Gamma}$$

Proof EASY by previous locally conformal mapping...

H.11 Additive representations of $\pi_1(X)$ Suppose that $\psi\colon \pi_1(X) \to \mathbf{C}$ is a group homomorphism where $\mathbf{C}$ the additive group of complex numbers and hence an abelian group. If g is the genus of S topology teaches that the fundamental group is free of rank $2g$ and generated by $2g$ many simply closed curves $\gamma 1, \ldots, \gamma_{2g}$ which issue from some chosen point $p \in X$ and surround the g many handles in a suitable fashion An arbitary additive representation is therefore determined by a $2g$-tuple of complex numbers $\{c_\nu\}$. If it is induced via integrals of some $w \in \Omega(X)$ one has

$$\int_{\gamma_\nu} w = c_\nu$$

Here $1 \leq \nu \leq 2g$ and since $\Omega(X)$ is g-dimensional it is in general not possibe to find w, i.e. only special $2g$-tuples $\{c_\nu\}$ give the existence of w.

I. Non-compact Riemann surfaces

Introduction. Sheaves and currents will be used in the subsequent proofs. The crucial result in this chapter are the following vanishing theorems.

0.1 Theorem. *For every non-compact Remann surface one has $H^1(X, \mathcal{O}) = 0$.*

Next, let $\mathcal{O}^*$ be the multiplicative sheaf whose sections are zero-free holomorphic functions.

0.2 Theorem. *The cohomology group $H^1(X, \mathcal{O}^*) = 0$.*

A locally free sheaf of $\mathcal{O}$-modules of rank one is called a holomorphic line bundle As explained in § xx the family of holomorphic line bundles correspond to elements in $H^1(X, \mathcal{O}^*) = 0$. So Theorem 0.2 means that every holomorphic line bundle $\mathcal{L}$ is holomorphically trivial and as a sheaf therefore isomorphic to $\mathcal{O}$ which by Theorem 0.1 entails that

(*) $$H^1(X; \mathcal{L}) = 0$$

However, to prove Theorem 0.2 we shall need (*) and we shall later on explain how (*) can be established in a similar way as in the special case $\mathcal{L} = \mathcal{O}$.

0.3 Some cohomology groups. The sheaf Ω is a locally free sheaf of $\mathcal{O}$-modules of rank one and there is the exact sequence of sheaves:

$$0 \to \mathbf{C}_X \to \mathcal{O} \xrightarrow{\partial} \Omega \to 0$$

Passing the the long exact sequence of cohomology the vanishing in (*) entails that

$$H^2(X, \mathbf{C}_X) = 0$$

From this it follows trivially - see § xx - that if the complex field $\mathbf{C}$ is replaced by the field of rational numbers then

(i) $$H^2(X, Q_X) = 0$$

Next, the exponential map which sends a holomorphic function f to the zero-free function $e^{2\pi i f}$ gives an exact sequence of sheaves:

$$0 \to \mathbf{Z}_X \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

Passing to the long exact sequence of cohomology we get

$$H^1(X, \mathcal{O}^*) \simeq H^2(X, \mathbf{Z}_X$$

So Theorem 0.2 anounts to show that the last cohomology group is zero. We have already (i) so Theroem 0.2 is "almost" derived from (*). However, extra arguments will be needed to get Theorem 0.2 which we begin to explain. Let $\mathcal{L}$ ibea holomorphic line bundle. By definton it is trivial, i.e. isomorphic to $\mathcal{O}$ if and only if it has a global generating section. Suppose to begin with that one has found a non-constant global section $\xi \in \mathcal{L}$. The zeros of ξ is some (possibly empty) discrete set in xX. Let $\{q_\nu\}$ be these zeros and ξ vanishes of some order k_ν at q_ν. Suppose we have found a meromorphic function g in X with poles of order k_ν at every q_ν while g is a zero-free holomorphic function outside this discrete set. Then $g \cdot \xi \in \mathcal{L}(X)$ is a global section which gives the requested trivialisation so that $\mathcal{L} \simeq \mathcal{O}$. So in order to get Theorem 0.2 we need to prove the existence of globally defined meromorphic functions with prescribed poles on discrete subsets of X. The whole discussion has shown that Theorem 0.2 follows if we in addition to Theorem 01 and its extension in (*) have established the following:

0.4 Theorem. *Every divisor D on X is principal, i.e. equal to $D(f)$ for some $f \in \mathfrak{M}(X)$.*

0.5 Runge's theorem. Recall from § xx that an open subset Y in X is a Runge domain if every connectd component of the closed complemnet $X \setminus y$ is non-compact. The following result will be used to prove Theorem 0.1 and has also independent interest:

0.6 Theorem. *Let $Y \subset X$ be a Runge domain. Then the restriction map $\mathcal{O}(X) \to \mathcal{O}(Y)$ has a dense image.*

The results above constitute the major facts in this section. To prove Theorem 0.1 we first establish seeemingly weaker results and after finish the proofs of the the other results.

§ 1. A finiteness result.

Consider an connected open subset Y which is relatively compact in X. The boundary ∂Y is compact but in general it need not be connected. It can for example consist of a finite number of pairwise disjoint and closed Jordan curves together with some Jordan arcs and some isolated points, like an annulus where a finite set of points in the open annulus have been removed. If ∂Y is "nice", for example piecewise analytic then tthe cohomology group $H^1(Y, \mathbf{C})$ is a finite dimensional complex vector space. Recall also from § xx that one has the isomorphism

$$\frac{\mathcal{E}^1(Y)}{d(\mathcal{E}(Y))} \simeq H^1(Y, \mathbf{C})$$

and as explained in § xx there exists an injective map

$$\frac{\mathcal{E}^{0,1}Y)}{\bar{\partial}(\mathcal{E}(Y))} \to H^1(Y, \mathbf{C})$$

Moreover, Dolbeault's Theorem identifies right hand side with $H^1(Y, \mathcal{O})$ and hence this cohomology group is finite dimensional for every Y as above with a nice compact boundary. The finite-dimensionality enable us to construct non-constant meromorphic functions on Y. Namely, if $p \in Y$ is fixed we take a positive integer N and the sheaf $\mathcal{O}_D$ where D is the divisor $N \cdot p$ which means that one allows a pole of order $\leq N$ at p. Exactly as in the compact case from § xx one has an exact sequence

$$0 \to \mathcal{O}(Y) \to \mathcal{O}_D(Y) \to \mathbf{C}^N \to H^1(Y, \mathcal{O}) \to H^1(Y, \mathcal{O}_D) \to 0$$

If N is strictly larger than the dimension of $H^1(Y, \mathcal{O})$ we get a non-zero section of $\mathcal{O}_D$, i.e. a meromorphic function in Y with a pole of order $\leq N$ at p and otherwise holomorphic.

§ 2. A vanishing lemma.

Consider a pair $Y \subset Y_1$ where Y is relatively compact in Y_1. As explained in § xx one has the restriction map

$$\mathcal{L} \colon H^1(Y_1, \mathcal{O}) \to H^1(Y, \mathcal{O}) \tag{2.1}$$

2.2 Proposition. *If $H^1(Y; \mathcal{O})$ is finite dimensional the $\mathcal{L}$-image in (2.1) is zero*

Proof. Choose $\xi_1, \ldots, \xi_K$ in $H^1(Y_1, \mathcal{O})$ such that $\{\mathcal{L}(\xi_k)\}$ gives a basis for the $\mathcal{L}$-image in the finite dimensional vector space $H^1(Y; \mathcal{O})$. Next, we can find a relatively compact set Y_2 in X where $Y_1 << Y_2$ and ∂Y_2 is "nice" so that $H^1(Y_2, \mathcal{O})$ is finite dimensional. Fix a point $p \in Y_2 \setminus Y_1$ and from § 1 we find a meromorphic function f in Y_2 which restricts to a holomorphic funtion in Y_1 which is not constant on the connected components of this set. Now we get the vectors $\{f \cdot \xi_k\}$ in $H^1(Y_1, \mathcal{O})$ and take their $\mathcal{L}$-images. Since we have chosen a basis above there is an $N \times n$-matrix B with complex elements such that

$$\mathcal{L}(f \cdot \xi_k) = \sum_{j=1}^{j=N} b_{kj} \cdot \mathcal{L}(\xi_j) \tag{i}$$

Let us now consider the holomorphic function in Y_1 defined by

$$F = \det(f \cdot E_N - B)$$

Then (i) and Cramer's rule entail that

$$\mathcal{L}(F \cdot \xi_k) = 0 \quad : 1 \leq k \leq N \tag{ii}$$

Above F is meromorphic in Y_2 with no poles in the relatively compact subset Y_1. We can therefore cover Y_1 with a finite number of charts $\mathfrak{U} = \{U_\alpha\}$ such that F has no zeros in non-empty intersections $U_\alpha \cap U_\beta$ when $\alpha \neq \beta$. Let us then take some $\eta \in H^1(Y_1, \mathcal{O})$ which by Lerays theorem is represented by a cocycle $\{\phi_{\alpha,\beta}\}$ in $Z^1(\mathfrak{U}, \mathcal{O})$. Since $F \neq 0$ on each non-empty intersection we get the cocycle $\{\psi_{\alpha,\beta} = F^{-1} \cdot \phi_{\alpha,\beta}\}$ which represents a vector in $H^1(Y_1, \mathcal{O})$ denoted by η^*. From these constructions we get

$$\mathcal{L}(\eta) = \mathcal{L}(F \cdot \eta^*)$$

By (ii) the last term is zero and since $\eta \in H^1(Y_1, \mathcal{O})$ was arbitrary Proposition 2.2 is proved.

2.3 A $\bar{\partial}$-equation. Consider a pair $Y << Y_1$ as above. Dolbeault's theorem and Proposition 2.2 imply that every $\phi \in \mathcal{E}^{0,1}(Y_1)$ restricts to a $\bar{\partial}$-exact form on Y, i.e. there exsits $g \in \mathcal{E}(Y)$ such that

$$\bar{\partial} g = \phi | Y \tag{2.3.1}$$

Now we can establish a density result:

2.4 Proposition. *Let Y be a relatively compact Runge domain in X. Then, for every Y_1 sich that $Y < Y_1$ and $H^1(Y_1, \mathcal{O})$ is finite dimensional it follows that the restriction map from $\mathcal{O}(Y_1)$ to $\mathcal{O}(Y)$ has a dense image.*

Proof. By the Hahn-Banach Theorem it suffices to show that for every (1,1)-current γ with some compact support K in Y one has the implication

$$\gamma(f|Y) = 0 \quad \forall\ f \in \mathcal{O}(Y_1) \implies \gamma|\,\mathcal{O}(Y) = 0 \tag{i}$$

Consider a current γ as above which vanishes on each restricition of $f \in \mathcal{O}(Y_1)$ to Y and let k be its compact support in Y. Next, choose an open set Y_2 where $Y_1 << Y_2$ and (2.3.1) holds for this pair. So when $\phi \in \mathcal{E}^{0,1}(X)$ there exists $g \in \mathcal{E}(Y_1)$ such that

$$\bar{\partial} g = \phi | Y_1$$

Above g is unique up to a function in $\mathcal{O}(Y_1)$ and since γ vanishes on $\mathcal{O}(Y_1)|Y$ the evaluation $\gamma(g)$ only depends upon ϕ. Hence there exists a current S on X defined by the process above, i.e.

$$S(\phi) = \gamma(g|Y) \quad : \bar{\partial} g = \phi | Y_1$$

If $h \in C_0^\infty(X \setminus K)$ we get the $(0,1)$-form $\bar{\partial}(h)$ and by the constructions above

$$S(\bar{\partial} h) = \gamma(h|Y) = 0$$

where the last equality follows since γ is supported by K. This means that the current S is $\bar{\partial}$-closed in $X \setminus K$ and since $\bar{\partial}$ is elliptic the restricttion of S to $X \setminus K$ is expressed by a holomorphic 1-form $\sigma \in \Omega(X \setminus K)$. So far we have not used the assumption that Y is Runge. To profit upon this we consider the compact hull $\mathfrak{h}(K)$ from § XX. If $y \in Y \setminus \mathfrak{H}(K)$ the connected component V in $X \setminus K$ which contains y is non-compact. Since the current S has compact support it follows that

$$V \setminus \bigl(\mathrm{Supp}(S) \cup K\bigr) \neq \emptyset \tag{ii}$$

Now σ respresents S outside K and therefore it must vanishes in the open set (ii). By analyticity σ vanishes in the whole connected set V and hence in a neighborhood of y. Since y was arbitrary we conclude that σ vanishes identically in $Y \setminus \mathfrak{h}(K)$. Let us now consider some $f \in \mathcal{O}(Y)$ and take $g \in C_0^\infty(Y)$ where $g = f$ in an open neighborhood of $\mathfrak{h}(K)$. It follows that

$$S(\bar{\partial}(g)) = \gamma(g|Y) = \gamma(f)$$

where the last equality holds since K is the support of γ. Finally, since f is holomorphic we have $\bar{\partial} g = 0$ in a neighborhood of $\mathfrak{H}(k)$ and since σ is supported by $\mathfrak{h}(K)$ it follows that $S(\bar{\partial}(g)) = 0$. Hence $\gamma(f) = 0$ for every $f \in \mathcal{O}(Y)$ which proves the requested implication (i) above and hence the density in Proposition 2.4.

Remark. The proof above is attributed to Malgrange and we followed the presentation in [Forster: page xx-xx].

2.5 The case of locally free $\mathcal{O}$-modules. If $\mathcal{L}$ is a locally free sheaf of $\mathcal{O}$-modules with rank one similar results as above hold. Theproofs are more or less verbatim. More precisely we first have some locally finite covering $\mathfrak{U} = \{U_\alpha\}$ by simply connected charts and $\mathcal{L}|U_\alpha$ are trvial for every α, i.e. there exists a section $\rho_\alpha \in \mathcal{L}(U_\alpha)$ such that the restricted sheaf $\mathcal{L}|U_\alpha$ is the free $\mathcal{O}_{U_\alpha}$-module generated by ρ_α. Shrinking $\mathfrak{U}$ if necessary we may assume that the transistion functions which give

$$\rho_\beta = \phi_{\beta\alpha} \cdot \rho_\alpha$$

in each non-empty intersection $U_\beta \cap U_\alpha$ yiled bounded holomrphic functions, i.e. the maximum norms

$$||\phi_{\beta\alpha}||_{U_\beta \cap U_\alpha} < \infty$$

When this holds it is meaningful to speak about convergence for $\mathcal{L}$-valued sections. More precisely, let $Y << Y_1$ and $\{\rho_\nu \in (Y_1)\}$ be given. If $\rho_* \in \mathcal{L}(Y)$ and K is a compact subset of Y we leave it to the reader to clarify the condition that $\{\rho_\nu|K$ converges in a uniform way to $\rho|K$, i.e. u Thus, in Propostion 22and 2.4 we can replace $\mathcal{O}$ by any $\mathcal{L}$. Concerning (2,3,1) we ntocie that there exists the sheaf

$$\mathcal{E}_\mathcal{L} = \mathcal{O}_\mathcal{L} \otimes_\mathcal{O} \mathcal{E}$$

and a well defined map

$$\bar\partial \colon \mathcal{E}_\mathcal{L} \to \mathcal{E}_\mathcal{L}^{(0,1)}$$

So now (2,3,1) means that if $\phi \in \mathcal{E}_\mathcal{L}^{(0,1)}(Y_1)$ then there exists $g \in \mathcal{E}_\mathcal{L}(Y)$ such that

$$\bar\partial(g) = \phi|Y$$

2.5 Proof of Theorem 0.1

By § XX we exhaust X by an increasing sequence of Runge domains $\{Y_n\}$ where $Y_n << Y_{n+1}$ and $\{H^1(Y_n, \mathcal{O})\}$ are finite dimensional. Consider some $\phi \in \mathcal{E}^{0,1}(X)$. Now (2.3.1) applies to each piar $Y_n << Y_{n+1}$ and since ϕ restricts to $\mathcal{E}^{(0,1)}(Y_{n+1}$ there exists $f_n \in \mathcal{E}(Y_n)$ such that

$$\bar\partial f_n = \phi|Y_n$$

It follows that the restriction to Y_n of $f_{n+1} - f_n$ is holomorphic. Now $Y_{n-1} << Y_n$ and Proposition 2.4 entails that this restricted holomorphic function can be uniformly approximated on Y_{n-1} by functions from $\mathcal{O}(Y_{n+1}$. In particular we find $g_{n+1} \in \mathcal{O}(Y_{n+1})$ such that the maximum norm

$$||f_{n+1} + g_{n+1} - f_n||_{Y_{n-1}} < 2^{-n}$$

With $f^*_{n+1} = f_{n+1} + g_{n+1}$ we still have

$$\bar\partial f^*_{n+1} = \phi|Y_{n+1}$$

Pwerforibng these modfifications inductively we get anew sequence $\{f^*_n\}$ where (xx) hold for every n while

$$||f^*_{n+1} - f^*_n||_{Y_{n-1}} < 2^{-n} \quad : \; n = 1, 2, \ldots$$

It follows that $\{f^*_n\}$ converge uniformly on X to a continuous function F. This entials that the distributuons
$bar\partial f^*_n\}$ converge to the distribution derivative $\bar\partial(F)$ and by (xx) it is clear that the following hold on the whole of X:

$$\bar\partial(F) = \phi$$

Now ϕ is a smooth form and the elliptic property of $\bar\partial$ implies that F is a C^∞-function. Hence the mapping

$$\bar\partial \colon \mathcal{E}(X) \to \mathcal{E}^{(0,1)}(X)$$

is surjective. Dolbeaualt's theorem gives $H^1(X, \mathcal{O}) = 0$ and Theorem 0.1 is proved.

2.7 Exercise. Use the remarks from § 25.5 and repeat the methods above to show that

$$H^1(X, \mathcal{L}) = 0$$

for every locally free sheaf of $\mathcal{O}$-modules with rank one.

Sheaf theory.

Introduction. Sheaf theory was created by Leray during the years 1943-45. A historic account about sheaves and their cohomology is presented by Houzel in [xxx]. A merit of sheaves is that triangulations on topological spaces is not needed to define cohomology. But specific triangulations are of course be helpful to calculate cohomology of specific topological spaces. The fact that cohomology groups are defined for arbitrary sheaves has many advantages. One reason is that the family of all sheaves on a given topological space is ample enough to produce suitable resolutions of a given sheaf by sheaves whose cohomology are easy to find. For example, on a compact Riemann surface one starts with the sheaf $\mathcal{O}_X$ whose first cohomology group $H^1(X, \mathcal{O}_X)$ is not found directly. But there exists the $\bar\partial$-resolution of $\mathcal{O}(X)$ where the local solvability of inhomogenous $\bar\partial$-equation gives an exact sequence of sheaves

$$\mathcal{O}_X \to \mathcal{E}_X \stackrel{\bar\partial}{\to} \mathcal{E}_X^{0,1} \to 0$$

In § xx we explain why the existence of C^∞-partitions of the unity implies that cohomology groups in positive degree of $\mathcal{E}_X$ and $\mathcal{E}_X^{0,1}$ are zero. Together with a general long exact sequence for sheaf cohomology one arrives at the Dolbeault isomorphism expressed by the equality

$$H^1(X, \mathcal{O}(X) \simeq \frac{\mathcal{E}_X^{0,1}(X)}{\bar\partial(\mathcal{E}_X^{0,1}(X))}$$

We shall expose material which go further than what is needed for compact Riemann surfaces. But for those results which will be used to study compact Riemann surfaces detailed proofs are given. In the long run it is essential to become familiar with general constructions in sheaf theory, and foremost with cohomology which can be defined not only for a single sheaf but even for a complex of sheaves where one encounters hypercohomology. To determine the cohomology of complexes of sheaves one employs spectral sequences which is another major contribution due to Leray in the famous article *L'anneau spectral et anneau filtré d'homologie d'un espace localement compact* from 1950. The construction of spectral sequences of filtered complexes is presented in § XX.

Remark. For students in complex analysis spectral sequences and their use should belong to common knowledge just as Stokes theorem. It suffices to spend a week-end to learn the rather trivial constructions of the derived category associated to an arbitrary abelian category. This "yoga" is applied to construct cohomology for complexes of sheaves and at the same time standard homological functors over associative rings. After this one considers derived functors on ringed spaces where sheaf cohomology and homological functors appear at the same time. Recall that if $\mathcal{A}$ is an abelian category with a finite homological dimension if there exists an integer μ such that each object in $\mathcal{A}$ has an injective resolution of length μ at most. When this holds the derived category $D^b(\mathcal{A})$ whose objects are bounded complexes is equivalent to the triangulated category $K^b(i(\mathcal{A}))$ whose objects are bounded complexes of injectice objects and morphisms between them are identified up to homotopy. If X is a manifold of some real dimension n and $\mathcal{A}$ the abelian category of sheaves whose sections are complex vector spaces, then $i(\mathcal{A})$ consists of flabby sheaves and one verifies easily that its homological dimension is $\dim(X) + 1$. If U is an open subset of such a manifold X and $\mathcal{F}^\bullet$ is a bounded sheaf complex one takes an injective resolution $\mathcal{J}^\bullet$ and gets the complex $\mathcal{J}^\bullet(U)$. In this complex of vector spaces the cohomolgy groups are intrinsically defined, i.e. independent of the chosen injective resolution of $\mathcal{F}$ and one writes

$$H^k(U; \mathcal{F}^\bullet) = H^k(\mathcal{J}^\bullet(U))$$

In addition there exists Leray's spectral sequence whose second table are doubly indexed vector spaces

$$E_2^{p,q} = H^p(U; h^q(\mathcal{F}^\bullet))$$

where $\{h^q(\mathcal{F}^\bullet)\}$ are the cohomology sheaves of $\mathcal{F}^\bullet$. See also § 0.1 for some further comments in the case of sheaves.

The genus of Riemann surfaces. The equality between the dimension of $H^1(X, \mathcal{O}_X)$ and the topological genus is easily proved by sheaves. Let us for example consider a compact Riemann surface X which is homeomorphic to a sphere with two handles. We may assume that they are placed close to the northern respectively the southern pole. Choose a pair of open sets U_1, U_2 whose union cover X while $U_1 \cap U_2$ is an annulus around the equator circle. Here the handle at the northern pole is contained in U_1, and the handle at the south pole in U_2, Since $U_1 \cap U_2$ is biholomorphic with a planar domain we have $H^1(U_1 \cap U_2, \mathcal{O}_X) = 0$. At the same time the cohomology over U_1 and U_2 are equal to that over the torus which is known to be of dimension one. Hence $H^1(U_\nu, \mathcal{O}_X)$ are 1-dimensional for $\nu = 1, 2$. In § xx we explain why $H^1(X, \mathcal{O}_X)$ is a direct sum of these cohomology groups and hence of dimension two which gives the requested equality. By an induction over g the equality is proved by similar elementary sheaf theoretic arguments.

0.1 Flabby sheaves and categorical cohomology. A sheaf $\mathcal{J}$ on a topological space is flabby if the restriction maps $\mathcal{J}(X) \to \mathcal{J}(U)$ are surjective for every open subset U. Denote by $M(\mathbf{C}_X)$ the abelian category of sheaves whose sections are complex vector spaces. It turns out that injective objects in this category consist of flabby sheaves and to compute cohomology of a sheaf $\mathcal{F}$ one can therefore use a flabby resolution

$$0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \dots$$

Passing to global sections one gets a complex

$$0 \to \mathcal{J}^0(X) \to \mathcal{J}^1(X) \to \dots$$

By a general result about injective resolutions in an arbitrary abelian category one shows that the cohomology of this complex does not depend upon the chosen flabby resolutionof $\mathcal{F}$ and in this way one gets intrinsically defined cohomology groups

$$(*) \qquad H^p(X; \mathcal{F}) = H^p(J^\bullet(X))$$

A fundamental fact is that these cohomology groups are equal to the Cech cohomoogy groups when X is paracompact, i.e. when every open covering has a locally finite subcovering. This rather weak condition holds in most applications so one can study cohomology via flabby resolutions which has the merit that many long exact sequences are easier to discover. One can also go further and study ringed spaces where homological algebra intervense with sheaf cohomology. This systematic approach was put forward by Grothendieck in the article *Sur quelques points d'algèbre homologiques* from 1957 which has been very influential. We shall not give a full account about sheaf theory. So we refrain from discussing direct and inverse image sheaves and their higher direct images as well as local cohomology. Excellent material which covers most of the essential results in advanced sheaf theory appear in the text-book *Sheaves on Manifolds* by Kashiwara and Schapira.

0.2 Mittag-Leffler's theorem. In [ibid] Grothendieck proved a result with wide a range of applications. It is called the Mittag-Leffler theorem since the essential ingredient in the proof is to succesively modify sections by a similar procedure as in Mittag-Leffler's constructions of meromorphic functions with prescribed poles in $\mathbf{C}$. For simplicity we announce the result below for a single sheaf but mention that it is valid for complexes of sheaves where single cohomology groups are replaced by derived cohomology groups. Here is the situation: Let X be a topological space which is Hausdorff and paracompact. A family of open set $\{\Omega_t$ indexed by real numbers $0 \leq t \leq 1$ is called a Mittag-Leffler family if it is increasing, i.e $t < s \implies \Omega_t \subset \Omega_s$ and satisfies:

$$\bigcup_{t<s} \Omega_t = \Omega_s \quad : \quad 0 < s \leq 1 \quad \text{and} \quad \bigcup_{t>s} \Omega_t = \bar{\Omega}_s \quad : \quad 0 \leq s < 1$$

Consider a sheaf $\mathcal{F}$. The inclusion $\Omega_0 \subset \Omega_1$ gives for each $p \geq 1$ a restriction map

$$(*) \qquad H^p(\Omega_1, \mathcal{F}) \to H^p(\Omega_0, \mathcal{F})$$

In many applications one wants to decide when these maps are isomorphisms. The following sufficient condition was proved in [ibid]:

Theorem. *The restriction maps in (*) are isomorphisms if the following hold for each $p \geq 1$:*

(i) $$\lim_{U \to \{x_0\}} H^p(U \cap \Omega_t) = 0 \quad 0 < t < 1 : \; x_0 \in \partial\Omega_t$$

(ii) $$\mathcal{H}^0_{X \setminus \Omega_t}(\mathcal{F}(x_0) = 0 \quad : \quad x_0 \in \partial\Omega_t$$

Above condition (ii) means that if ξ is a non-zero element in the stalk $\mathcal{F}(x_0)$, then its support cannot be contained in the closed complement of Ω_t.

0.3 Multiplicative sheaves. There occur also sheaves whose sections over open set no longer are complex vector spaces. If X is compact Riemann surface one has the sheaf $\mathfrak{M}^*$ whose sections are meromorphic functions which are not identically zero and for such a pair f, g one can take the product fg. With this rule the sections $\mathfrak{M}^*(U)$ over an open set is an abelian group whose identity is the constant function which is everywhere one. There is also the subsheaf $\mathcal{O}^*$ whose sections are holomorphic functions with no zeros. The determination of the local cohomology group $H^1(X, \mathcal{O}^*)$ is given in § XX. The exponential map $f \to e^{2\pi i f}$ yields a surjective sheaf morphism from $\mathcal{O}$ onto $\mathcal{O}^*$ whose kernel consists of functions which are locally constant and integer-valued. This sheaf is denoted by $\mathbf{Z}_X$ where each stalk is a copy of the abelian group of integers. The exact sequence

(*) $$0 \to \mathbf{Z}_X \to \mathcal{O} \to \mathcal{O}^* \to 0$$

gives a long exact sequence of cohomology. Since $\mathcal{O}(X) = \mathbf{C}$ and $\mathcal{O}^*(X) = \mathbf{C}^*$ one gets an exact sequence

$$0 \to H^1(X, \mathbf{Z}_X) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbf{Z}_X) \to 0$$

In § xx we show that $H^2(X, \mathbf{Z}_X)$ is reduced to the group of integers which means that to each cohomology class $\xi \in H^1(X, \mathcal{O}^*)$ one assigns an integer denoted by $\mathrm{ch}(\xi)$ and called the Chern class of ξ. In (*) we have also the inclusion $H^1(X, \mathbf{Z}_X) \subset H^1(X, \mathcal{O})$. Topology teaches that $H^1(X, \mathbf{Z}_X)$ is a free abelian group of rank $2g$ where a basis is given by $2g$ many cycles $\gamma_1, \ldots, \gamma_{2g}$ as described in § XX. Recall that the dual of $H^1(X, \mathcal{O})$ is $\Omega(X)$. So every $w \in \Omega(X)$ restricts to an additive map

$$w \colon H^1(X, \mathbf{Z}_X) \to \mathbf{C}$$

It turns out that this map is given by period integrals:

$$w(\gamma_k) = \int_{\gamma_k} w \quad : \quad 1 \leq k \leq 2g$$

where $\{\gamma_k\}$ is a a canonical basis for the integer homology in Riemann's sense. If Γ denotes the image of $H^1(X, \mathbf{Z}_X)$ in $H^1(X, \mathcal{O})$ we get the g-dimensional torus

$$T^g = \frac{H^1(X, \mathcal{O})}{H^1(X, \mathbf{Z}_X)}$$

By the above there remains an exact sequence

(**) $$0 \to T^g \to H^1(X, \mathcal{O}^*) \to \mathbf{Z}$$

At the same time we recall the Jacobi-Picard theorem from § xx which asserts that

(***) $$T^q \simeq \frac{\mathcal{D}_0}{\mathcal{P}_0}$$

In § we shall how (**) can be used to recover (***) in an elegant fashion. The basic point is that $H^1(X, \mathcal{O}^*)$ can be identified with isomorphism classes of holomorphic line bundles over X which correspond to locally free sheaves of $\mathcal{O}$-modules of rank one. If $\mathcal{L}$ is such a sheaf we prove in § xx via results from § x that its meromorphic extension has global sections and that every such global sections assigns a unique divisor $D(\mathcal{L})$ on X. Its the degree is $\neq 0$ precisely when the corresponding cohomology class $\{\mathcal{L}\} \in H^1(X, \mathcal{O})$ is sent to a non-zero integer in (**) above. The class of holomorphic line bundle $\mathcal{L}$ for which $D(\mathcal{L}) = 0$ is therefore identified with the torus T^q which gives another realisation of the Jacobi-Picard torus.

1. Leray's spectral sequence.

Introduction. A complex of abelian groups consists of an indexed family $\{A_k; d^k\}$ where k run over the set of integers and each A^k is an abelian group. Moreover, to each k there is an additive map

$$d^k \colon A^k \to A^{k+1}$$

where the composed maps are zero, i.e. $d^{k+1} \circ d^k = 0$. This gives a complex

$$(*) \qquad \ldots A^{k-1} \overset{d^{k-1}}{\to} \ldots A^k \overset{d^k}{\to} \ldots A^{k+1} \to \ldots$$

It is denoted by $A^\bullet$ and for each integer k there exists the cohomology group

$$H^k(A^\bullet) = \frac{\ker(d^k)}{\operatorname{im}(d^{k-1})}$$

If all these cohomology groups are zero one says that (*) is an exact complex.

Homotopy. Let $\mathcal{C}(A^\bullet)$ be a complex. Supoose we have a family of additive maps

$$\rho^k \colon \ A^k \mapsto A^{k-1}$$

which satisfy

$$d^{k-1} \circ \rho^k - \rho^{k+1} \circ d^k = 1_{A^k}$$

for every integer k where 1_{A^k} is the identity map on A^k. Then the complex is exact. For if k is an integer and $\xi \in \ker(d^k)$ one has:

$$\xi = d^{k-1} \circ \rho^k(\xi) - \rho^{k+1} \circ d^k(\xi) = d^{k-1}(\eta) \quad : \quad \eta = \rho^k(\xi)$$

Hence the cohomology class of ξ is zero. The existence of ρ-maps yields a sufficient condition for exactness only but we remark that there exist exact complexes where such a family of ρ-maps does not exist.

Example. Let $\mathcal{O} = \mathbf{C}\{z\}$ be the local ring of germ of holomprihic functions at the origin. We consider the abelian catrgory of $\mathcal{O}$-modues andnow one has an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}[z^{-1}] \to \mathcal{P} \to 0$$

where $\mathcal{P}$ is thr module of polar parts of germs of holomorphic functions. Now the reader can verify that there cannot exist non-zero $\mathcal{O}$-linear maps from $\mathcal{P} \to \mathcal{O}[t^{-1}]$ or from $\mathcal{O}[z^{-1}]$ into $\mathcal{O}$.

Next, let $A^\bullet$ and $B^\bullet$ be complexes. A morphism from $A^\bullet$ into $B^\bullet$ consists of a family of additive maps $\phi^k \colon \ A^k \to B^k$ such that

$$d_B^k \circ \phi^k = \phi^{k+1} \circ d_A^k \quad \text{hold for all integers} \quad k$$

1. Exercise. Show that if $\phi^\bullet$ is a morphism then there exist additive maps

$$H^k(\phi^\bullet) \colon H^k(A^\bullet) \to H^k(A^\bullet)$$

for all k. One refers to $\{H^k(\phi^\bullet)\}$ as the associated cohomology maps.

Homotopy of morphisms. Let $\phi^\bullet$ and $\psi^\bullet$ be a pair of morphisms. They are homotopic if there exists a family of additive ρ-maps $\rho^k \colon A^k \to B^{k-1}$ such that

$$\psi^k - \phi^k = d_B^{k-1} \circ \rho^k - \rho^{k+1} \circ d_A^k$$

2. Exercise. Show that if $\phi^\bullet$ and $\psi^\bullet$ are homotopic then $H^k(\phi^\bullet) = H^k(\psi^\bullet)$ for all k.

A long exact sequence. Let $\{A_\nu^\bullet\}$ be three complexes and consider a pair of morphisms

$$\phi_\nu^\bullet \colon \ A_\nu^\bullet \to A_{\nu+1}^\bullet \quad : \quad \nu = 1, 2$$

For each integer k we get a complex

$$\text{(i)} \qquad 0 \to A_1^k \overset{\phi_1^k}{\to} A_2^k \overset{\phi_1^k}{\to} A_3^k \to 0$$

If (i) are eact for every integer k we say that the pair $\phi_1^\bullet, \phi_2^\bullet$ yield an exact sequence of the three given complexes. Now there exist the cohomology maps:

$$H^k(\phi_1^\bullet)\colon H^k(A_1^\bullet) \to H^k(A_2^\bullet) \quad \text{and} \quad H^k(\phi_2^\bullet)\colon H^k(A_2^\bullet) \to H^k(A_3^\bullet)$$

The connecting morphisms on cohomology. Consider an exact sequence of complexes. Let $\xi \in \ker(d_3^k)$ and denote its cohomoogy class in $H^k(A_3^\bullet)$ by $\{\xi\}$. The exactness gives $\eta \in A_2^k$ such that

(i) $$\phi_2^k(\eta) = \xi \implies$$

(ii) $$0 = d_3^k(\xi) = d_3^k \circ \phi_2^k(\eta) = \phi_2^{k+1} \cdot d_2^k(\eta)$$

Hence $d_k^2(\eta)$ belongs to the kernel of ϕ_2^{k+1} and by exactness we find $\gamma \in A_1^{k+1}$ such that

(iii) $$\phi_1^{k+1}(\gamma) = d_k^2(\eta)$$

Notice that (iii) entail that

(iii) $$\phi_1^{k+2} \circ d_1^{k+1}(\gamma) = d_2^{k+1} \circ \phi_1^{k+1}(\gamma) = d_2^{k+1} \circ d_k^2(\eta) = 0$$

The last equality holds since $d_2^{k+1} \circ d_2^k = 0$. Hence $d_1^{k+1}(\eta) \in \ker(\phi_1^{k+1})$. Exactness entails that ϕ_1^{k+1} is injective and hence $d_1^{k+1}(\eta) = 0$ which means that η yields a cohomology class $\{\eta\} \in H^{k+1}(A_1^\bullet)$.

3. Exercise. Show that the cohomology class $\{\eta\}$ only depends upon $\{\xi\}$ and gives an additive map

$$\rho^k\colon H^k(A_3^\bullet) \to H^{k+1}(A_1^\bullet)$$

Show also that these ρ maps yield a long exact sequence

(*) $$\dots H^k(A_1^\bullet) \overset{H^k(\phi_1^\bullet)}{\to} H^k(A_2^\bullet) \overset{H^k(\phi_2^\bullet)}{\to} H^k(A_3^\bullet) \overset{\rho^k}{\to} H^{k+1}(A_1^\bullet) \overset{H^{k+1}(\phi_1^\bullet)}{\to} \dots$$

One refers to (*) as the long exact sequence of cohomology.

Double complexes.

Let $A^{k\nu}$ be a doubly indexed family of abelian groups where k, ν are non-negative integers. For each pair we are also given additive mappings:

$$\partial^{k,\nu}\colon A^{k\nu} \to A^{k,\nu+1} \quad : \quad \delta^{k,\nu}\colon A^{k\nu} \to A^{k+1,\nu}$$

where the composed maps below are zero:

(i) $$d^{k,\nu+1} \circ \partial^{k,\nu} = \partial^{k+1,\nu} \circ \delta^{k,\nu} = 0$$

It means that each single row and each single column is a complex. In addition the double complex is commutative which means that

(ii) $$\partial^{k+1,\nu} \circ \delta^{k,\nu} = \delta^{k,\nu+1} \circ \partial^{k,\nu}$$

For each $m \geq 0$ we set $\Delta^0 = A^{0,0}$ and

$$\Delta^m = A^{0,m} \oplus A^{1.m-1} \oplus \cdots \oplus A^{m,0} : \, m \geq 1$$

Now there exist maps $\{d^m\}$ from Δ^m into Δ^{m+1} defined for each pair j,k with $j + k = m$ by:

$$d^m(a_{k,j}) = \partial(a_{k,j}) + (-1)^j \cdot \delta(a_{k,j})$$

From (i-ii) the composed maps $d^{m+1} \circ d^m = 0$ and refer to $\{\delta^\bullet; d^\bullet\}$ as the diagonal complex. Now we have the diagonal cohomology groups $\{H^m(\delta^\bullet)\}$. Each cohomology group can be equipped with two filtrations, the *First* and the *Second.* Both consist of a non-decreasing sequence of subspaces with at most m strict inclusions.

The First filtration. To each m we set

$$\mathcal{F}_\nu(\Delta^m) = A^{m,0} \oplus \ldots \oplus A^{\nu,m-\nu} \quad : \quad 0 \leq \nu \leq m$$

Then d^m maps $F_\nu(\delta^m)$ into $F_{\nu+1}(\Delta^{m+1})$ for all pairs ν, m and in this way the $\Delta^\bullet$-complex is equipped with a filtration where the differentials increase the degree with one unit for every m. This yields a filtration on the diagonal cohomology groups as follows: Put

$$Z_\nu^m = \mathcal{F}_\nu(\Delta^m) \cap \mathrm{Ker}(d^m)$$

Then $H^m(\Delta^\bullet)$ is filtered by the increasing sequence

$$\mathcal{F}_\nu(H^m(\Delta^\bullet)) = \frac{Z_\nu^m + \mathrm{Im}(d^{m-1})}{\mathrm{Im}(d^{m-1})} \simeq \frac{Z_\nu^m}{Z_\nu^m \cap \mathrm{Im}(d^{m-1})}$$

where the last equality comes via Noether's isomorphism. So for each m we have a non-decreasing sequence

$$(*) \qquad 0 \subset \mathcal{F}_0^{(1)}(H^m(\Delta^\bullet)) \subset \mathcal{F}_1^{(1)}(H^m(\Delta^\bullet)) \subset \ldots \subset \mathcal{F}_m^{(1)}(H^m(\delta^\bullet))$$

To simplify notations we set

$$\mathcal{F}^\nu((H^m(\Delta^\bullet)) = \mathcal{F}_\nu(m)$$

and associate the graded cohomology group

$$\mathrm{gr}(H^m(\Delta^\bullet) = \oplus \frac{\mathcal{F}_\nu(m)}{\mathcal{F}_{\nu-1}(m)}$$

The spectral sequence. To each m we have the graded group

$$\mathrm{gr}(\Delta^m) = \oplus \frac{\mathcal{F}_\nu(\Delta^m)}{\mathcal{F}_{\nu-1}(\Delta^m)}$$

The d-differentials are homogenoeous of degree $+1$ and we notice that the δ-maps preserve the first filtration. Hence the ∂-maps alone determine the cohomology of the associated graded complex which by the construction of $\{d^m\}$ means that we for each fixed m consider the ∂-cohomology of the complex

$$0 \to A^{m,0} \xrightarrow{\partial} A^{m,1} \xrightarrow{\partial} \ldots$$

Let $H^\nu(A^{m,\bullet})$ denote the cohomology of degree ν of this complex and define the graded group

$$\mathcal{E}_1^m = H^0(A^{m,\bullet}) \oplus \ldots H^m(A^{0,\bullet})$$

Next, using Noether's isomorphism one shows that there exist graded maps

$$\rho_2^m \colon \mathcal{E}_1^m \to \mathcal{E}_1^{m+1}$$

which are homogenous of degree -1 which means that

$$\rho_1^m \colon H^\nu(A^{m-\nu,\bullet}) \to H^{\nu-2}(A^{m+1-\nu,\bullet}) \quad \text{hold for all pairs} \quad \nu, m$$

Moreover, these ρ-maps appear in a complex

$$\ldots \to \mathcal{E}_1^{m-1} \xrightarrow{\rho_1^{m-1}} \mathcal{E}_1^m \xrightarrow{\rho_1^m} \mathcal{E}_1^{m+1} \xrightarrow{\rho_1^{m+1}} \ldots$$

As a result the cohomology yield graded groups $\{\mathcal{E}_2^m\}$ where

$$\mathcal{E}_2^m = \frac{\mathrm{Ker}(\rho_1^m)}{\mathrm{Im}(\rho_1^{m-1})}$$

This construction can be continued, i.e. there exists for each $r \geq 2$ a family of graded groups $\{\mathcal{E}_r^m\}$ which arise as cohomology groups via graded ρ-differentials on a complex formed by $\{\mathcal{E}_{r-1}^m\}$. After a finite number of steps the construction terminante and for each integer m one has

$$(*) \qquad \mathcal{E}_m^m \simeq \mathrm{gr}(H^m(\Delta^\bullet))$$

This is expressed by saying that the spectral sequence abuts to the graded cohomology.

Remark. Many text-books expose the construction above and are often supplied by diagrams which help to visulize the higher order differentials. The Second filtration is constructed in a similar fashion and the details are left to the reader who may consult the literature for details. A good reference is the text-book *xxx* by Bott.

A general construction.

The construction above is a special case of the spectral sequence associated to a general filtered complex. Let $\{F^\nu\}$ be a family of abelian groups indexed by integers where ν can be negative and infinitely many $F^\nu \neq 0$. Suppose each F^ν is equipped with a filtration $\{F^\nu(j)\}$ where $F^\nu(j) \subset F^\nu(j+1)$ and every $F^\nu(j)$ is an additive subgroup. Next, suppose there are additive maps

$$d^\nu \colon F^\nu \to F^{\nu+1}$$

where each d^ν is filter preserving and $d^{\nu+1} \circ d^\nu = 0$. The cohomology of $F^\bullet$ is filtered by

$$H^\nu(j) = \frac{\mathrm{Ker}(d^\nu) \cap F^\nu(j) + \mathrm{Im}(d^{\nu-1})}{\mathrm{Im}(d^{\nu-1})}$$

For each fixed ν we obtain the graded group

$$\mathrm{gr}(H^\nu) = \oplus \frac{H^\nu(j)}{H^\nu(j-1)} \tag{1}$$

Now we construct a sequence of graded complexes whose graded cohomology groups approximate the graded cohomology (1). To obtain this we set for each triple j,k,ν with $k \geq 0$:

$$Z_j^\nu(k) = \{u \in F^\nu(j) : \; d^\nu(u) \in F^{\nu+1}(j-k)\}$$
$$B_j^\nu(k) = F^\nu(j) \cap \, d^{\nu-1}(F^{\nu-1}(j+k-1)$$

Notice that $Z_j^\nu(0) = F^\nu(j)$ and that $\{Z_j^\nu(k)\}$ decrease with k for each fixed pair j,ν. The B-groups increase with k and since $d^\nu \circ d^{\nu-1} = 0$ we have the inclusions

$$B_j^\nu(k) \subset Z_j^\nu(k)$$

It follows that there exist graded groups

$$E_k^\nu = \oplus_j \, E_k(j) \quad : \quad E_k(j) = \frac{Z_j^\nu(k) + F^\nu(j-1)}{B_j^\nu(k) + F^\nu(j-1)}$$

Exercise. Noether's isomorphism gives

$$E_k(j) = \frac{Z_j^\nu(k)}{B_j^\nu(k) + Z_j^\nu(k \cap \, F^\nu(j-1)} \tag{i}$$

The construction of the Z-groups entail that

$$d^\nu(Z_j^\nu(k)) \subset Z_{j-k}^{\nu+1}(k) \quad : d(Z_j^\nu(k \cap \, F^\nu(j-1)) \subset B_{j-\nu}^{\nu+1}(k) \tag{ii}$$

Show that (i-ii) give additive maps

$$E_k^\nu(j) \mapsto E_k^{\nu+1}(j-k)$$

and taking the direct sum over j we obtain a graded additive map

$$\delta_k^\nu \colon E_k^\nu \to E_k^{\nu+1}$$

which is homogeneous of degre $-k$ where the constructions above entail that

$$\delta_k^{\nu+1} \circ \delta_k^\nu = 0$$

So for each fixed k, $E_k^\bullet$ is a complex of graded groups whose differentials are of degree $-k$.

Finally, show that Noether's isomorphism gives the following:

Proposition. *For each k the cohomology of $E_k^\bullet$ are isomorphic to the graded groups in $E_{k+1}^\bullet$, i.e*

$$H^\nu(E_k^\bullet) \simeq E_{k+1}^\nu \quad : \nu, k \geq 0$$

Convergence of the spectral sequence. The constructions of the Z-groups show that when k increases then $Z_j(k)$ gets close to the d^ν-kernel on $F^\nu(j)$ and similarly $B_j(k)$ increase to

$$B_j(\infty) = F^\nu(j) \cap \mathrm{Im}(d^{\nu-1})$$

So one expects that the graded groups E_k^ν approximate the graded cohomology groups in (1). A sufficient condition for this goes as follows:

Theorem. *Let w be a non-negative integer such that*

$$F^\nu(j) \cap \mathrm{Im}(d^{\nu-1}) \subset d^{\nu-1}(F^{\nu-1}(j+w))$$

hold for all pairs j, ν. Then it follows that

$$E_w^\bullet \simeq \mathrm{gr}(H^\bullet)$$

Remark. The isomorphism above holds in the category of graded groups, i.e for each ν one has the isomorphism

$$E_w^\nu \simeq \mathrm{gr}(H^\nu)$$

The proof of Theorem xx is left as an exercise. When it holds one says that the spectral sequence abuts to the graded cohomology. Notice that the conditions in Theorem xx are satisfied if we start from a complex $F^\bullet$ for which there exist an integer j^* such that

$$F_\nu(j) \neq 0 \implies -j^* \leq j \leq j^*$$

In other words, when the filtrations on the given complex $F^\bullet$ are uniformly bounded the spectral sequence abuts to the graded cohomology. This applies in particular to the first, respectively the second filtration of a bounded double complex.

Sheaf theory.

Let X be a topological space. No special assumptions are made. For example, X need not be a Haussdorff space. We restrict the study to sheaves whose sections are complex vector spaces. Such a sheaf $\mathcal{F}$ on X assigns to every open subset U a complex vector space $\mathcal{F}(U)$ and to each pair $V \subset U$ a $\mathbf{C}$-linear map

(i) $$\rho_{V,U} \colon \mathcal{F}(U) \to \mathcal{F}(V)$$

where the ρ-maps satisfy

(ii) $$\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$$

for every triple $W \subset V \subset U$. Finally the following two axioms hold: First, let U be open and $\{V_\alpha\}$ a family of open subsets of U such that $\cup V_\alpha = U$. Then the map

$$\mathcal{F}(U) \mapsto \prod \mathcal{F}(V_\alpha)$$

is injective. It means that if a section $\xi \in \mathcal{F}(U)$ under the restriction maps is zero in V_α for every α, then ξ is the zero vector in $\mathcal{F}(U)$. Finally the the glueing axiom holds. It means that if $\{V_\alpha\}$ as above is covering of U and $\{\xi_\alpha \in \mathcal{F}(V_\alpha)\}$ is a family of sections such that

$$\rho_{V_\beta \cap V_\alpha, V_\alpha}(\xi_\alpha) = \rho_{V_\beta \cap V_\alpha, V_\beta}(\xi_\beta)$$

hold for all pairs of V-sets with an non-empty intersection, then there exists a section $\xi \in \mathcal{F}(U)$ such that

$$\rho_{V_\alpha,U}(\xi) = \xi_\alpha$$

for every α. Notice that ξ is unique by the first axiom.

1. Sheaf maps The family of sheaves on X whose sections are complex vector spaces is denoted by $\mathcal{S}(\mathbf{C}_X)$. If $\mathcal{F}$ and $\mathcal{G}$ is a pair of sheaves a morphism - or a sheaf map - from $\mathcal{F}$ into $\mathcal{G}$ consists of a family of $\mathbf{C}$-linear maps

$$\gamma_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$$

where U run over all open sets and the γ-maps which commute with the restriction maps of the two sheaves.

2. Stalks of sheaves. Let $\mathcal{F}$ be a sheaf on X. To each $x_0 \in X$ the vector space $\mathcal{F}(x_0)$ is inductive limit of the vector spaces $\{\mathcal{F}(U)\}$ where U run over open neighborhoods of x_0. It means that for every open set U which contains x_0 one has a $\mathbf{C}$-linear map

$$\gamma_U(x_0) \colon \mathcal{F}(U) \to \mathcal{F}(x_0)$$

A vector $\xi_U \in \mathcal{F}(U)$ belongs to the γ_U-kernel if and only if there exists some smaller open set $V \subset U$ with $x_0 \in V$ and $\rho_{V,U}(\xi_U) = 0$.

Finally, every vector $\xi \in \mathcal{F}(x_0)$ is the γ-image for some open neighborhood of x_0. Let us remark that it suffices to use small open neighborhoods of x_0 and choosing a chart around x_0 given by some open disc D in the complex z-plane we can use the decreasing sequence of open discs $\{U_n = \{|z| < 1/n\}$ and then $\mathcal{F}(x_=)$ is identified with the inductive limit of the denumerable family of vector spaces $\{\mathcal{F}(U_n)\}$.

3. Induced stalk maps. Consider a pair of sheaves $\mathcal{F}, \mathcal{G}$ and a sheaf map

$$\phi \colon \mathcal{F} \to \mathcal{G}$$

So for every open set U one has $\mathbf{C}$-linear map α_U and these maps commute with the ρ-maps on the two given sheaves.

3.1 Exercise. Explain how one for each $x_0 \in X$ constructs a $\mathbf{C}$-linear map

$$\phi(x_0) \colon \mathcal{F}(x_0) \to \mathcal{G}(x_0)$$

It is called the induced stalk-map at x_0 of the sheaf map.

Next, to each open set U we have the linear map ϕ_U whose kernel is a complex subspace of $\mathcal{F}(U)$. We can restrict the ρ-maps which define $\mathcal{F}$ to these subspaces and the reader may verify that we obtain a sheaf via the data:

$$U \mapsto \ker \phi_U$$

The resulting sheaf is denoted by $ker(\phi)$.

3.2 Exercise. Prove that for every $x_0 \in X$ there exists a natural **C**-linear map from the stalk at x_0 of the sheaf $\ker \phi$ to the kernel $\phi(x_0)$. Hence the following isomorphism of complex vector spaces holds:

$$(*) \qquad ker(\phi)(x_0) = \ker(\phi(x_0))$$

3.3 Image sheaves. Given ϕ we get image spaces and the resulting map

$$U \mapsto \mathrm{Im}(\phi_U)$$

The ρ-maps on $\mathcal{G}$ induce ρ-maps

$$\rho_{V,U} \colon \mathrm{Im}(\phi_U) \to \mathrm{Im}(\phi_V)$$

for each pair $V \subset U$ and they satisfy (ii) from 0.1. Since the vector spaces $\mathrm{Im}(\phi_U)\}$ appear as subspaces of vector spaces defining the sheaf $\mathcal{G}$, the first sheaf axiom for $\mathcal{G}$ is inherited. There remains to study the glueing condition. So now U is an open set and $\{V_\alpha\}$ some open covering of U. We are given a family $\{\xi_\alpha \in \mathcal{F}(V_\alpha)\}$ such that

$$\phi(\xi_\alpha)|V_\alpha \cap V_\beta = \phi(\xi_\beta)|V_\alpha \cap V_\beta$$

hold for all pairs α, β. It means that there exist sections $\{\eta_{\alpha,\beta} \in ker(\phi)(V_\alpha \cap V_\beta)\}$ such that the restricted difference

$$\xi_\alpha - \xi_\beta|V_\alpha \cap V_\beta = \eta_{\alpha,\beta}$$

At this stage we encounter an "ugly obstruction". Namely, we seek $\xi_U \in \mathcal{F}(U)$ such that

$$\phi(\xi_U)|V_\alpha = \phi(\xi_\alpha)$$

hold for each α. This cannot be achieved in general so we have not found a sheaf above. To compensate for this we enlarge space of sections over open sets. This, when U is open we denote by $Im\,\phi)(U)$ the vector space whose elements are of the form $\{\phi(\xi_\alpha)\}$ for some open covering $\{V_\alpha\}$ of U. If $\{W_\beta\}$ is another open covering for which we assign a vector $\{\phi(\xi^*_\beta) : \xi^*_\alpha \in \mathcal{F}(U \cap W_\beta)\}$, we declare that the two vectors are equal if all the restrictions

$$\phi(\xi_\alpha) - \phi(\xi^*_\beta \,\big|V_\alpha \cap W_\beta = 0$$

In this way we obtain a sheaf denoted by $Im\,\phi$ and called the sheaf-theoretic image of ϕ.

3.4 Quotient sheaves. SAY Defintion ....

3.5 Remark. The results above can be summarized by saying that $\mathcal{S}(X)$ is an abelian catergory. A short exact sequence in this category isd given by

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

where $\mathcal{F}_0$ is a subsheaf of $\mathcal{F}_1$ and $\mathcal{F}_2$ the quotient sheaf $\frac{\mathcal{F}_1}{\mathcal{F}_2}$.

3.6 Cohomology.

Let $\mathfrak{U}$ be an open covering which means that one has an index set A and for each $\alpha \in A$ an open set U_α. The sole condition is that their union is equal to X. Next, let $\mathcal{F}$ be a sheaf on X. A zero-cochain with values in $\mathcal{F}$ consists of a family

$$\{\xi_\alpha \in \mathcal{F}(U_\alpha) : \alpha \in A\}$$

The family of zero-cochains is denoted by $C^0(\mathfrak{U}, \mathcal{F})$ which as a vector space becomes the direct product:

$$C^0(\mathfrak{U}, \mathcal{F}) = \prod_{\alpha \in A} \mathcal{F}(U_\alpha) \tag{1}$$

Next, we introduce the vector space of alternating 1-cochains given by

$$C^1(\mathfrak{U}, \mathcal{F}) = \prod^{*} \mathcal{F}(U_\alpha \cap U_\beta)$$

where (*) indicates the alternating rule which means that

$$\xi_{\beta,\alpha} = -\xi_{\alpha,\beta} \quad \text{for each pair} \quad \alpha, \beta$$

hold for every index pair. In particular $\xi_{\alpha,\alpha} = 0$ for every α.

3.6.1. The coboundary map δ^0. Let $\{\xi_\alpha\}$ in $C^0((\mathfrak{U}, \mathcal{F})$. For each pair of open sets U_α, U_β we can restrict ξ_α and ξ_β to the intersection $U_\alpha \cap U_\beta$ and obtain the alternating 1-chain represented by the sections

$$\xi_{\alpha,\beta} = \xi_\alpha - \xi_\beta | U_\alpha \cap U_\beta$$

This 1-cochain is denoted by $\delta^0(\{\xi_\alpha\})$ and called the coboundary image of the 0-cochain. This yields a $\mathbf{C}$-linear map

$$\delta^0 \colon C^0(\mathfrak{U}, \mathcal{F}) \to C^1(\mathfrak{U}, \mathcal{F})$$

The vector space $C^2(\mathfrak{U}, \mathcal{F})$**.** This tine we consider intersection of triples in $\mathfrak{U}$ and define the vector space

$$C^2(\mathfrak{U}, \mathcal{F}) = \prod^{*} \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma)$$

where (*) again indicates that the sections satify the alternating rule. It means that whenever a triple of $\mathfrak{U}$-sets have a non-empty intersection then:

$$\xi_{\alpha,\beta,\gamma} = -\xi_{\gamma,\alpha,\beta} = \xi_{\beta,\gamma,\alpha}$$

Now there exists a cobuondary map

$$\delta^1 \colon C^1(\mathfrak{U}, \mathcal{F}) \to C^2(\mathfrak{U}, \mathcal{F})$$

Namely, let $\{\xi_{\alpha,\beta}\}$ be a 1-cochain. If $U_\alpha \cap U\beta \cap U_\gamma \neq \emptyset$ we take restrictions of sections and put

$$\xi_{\alpha,\beta,\gamma} = \xi_{\alpha,\beta} - \xi_{\alpha,\gamma} + \xi_{\beta,\gamma}$$

Exercise. Show that (xx) satsifies the alternating rules As a result we obtain a $\mathbf{C}$-linear map

$$\delta^1 \colon C^1(\mathfrak{U}, \mathcal{F}) \to C^2(\mathfrak{U}, \mathcal{F})$$

The case $p \geq 3$**.** In general we define the vector space of alternating p-cochains by

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod^{*} \mathcal{F}(U_{\alpha_0} \cap \ldots \cap U_{\alpha_p})$$

So here $p+1$-fold intersections of open sets occur and the alternating conditions means that for every permutation $\sigma(0), \ldots, \sigma(p)$ of $0, 1, \ldots, p$ one has:

$$\xi_{\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(p)}} = \mathrm{sign}(\sigma) \cdot \xi_{\alpha_0, \ldots, \alpha_p)}$$

where $\mathrm{sign}(\sigma)$ is the signature of the permutation.

Exercise. Show that to every p there exists a map

$$\delta^p\colon\ C^p(\mathfrak{U},\mathcal{F}) \to C^{p+1}(\mathfrak{U},\mathcal{F})$$

and verify that the composed maps

$$\delta^{+1}\circ\delta^p = 0$$

for all p.

3.7 Cech cohomology groups.

From the above we obtain the complex $C^\bullet(\mathfrak{U},\mathcal{F})$. To each $p\geq 0$ we get the cohomology group

$$H^p(\mathfrak{U},\mathcal{F}) = \frac{\mathrm{Ker}(\delta^p)}{\mathrm{Im}(\delta^{p-1})}$$

3.7.1 Subcoverings. Let $\mathfrak{U}$ and $\mathfrak{V}$ be a pair of open coverings. We write $\mathfrak{V}\leq\mathfrak{U}$ if there exists an index map τ which such that

$$V_\beta\subset U_{\tau(\beta)}$$

for all β in the index set B for $\mathfrak{V}$ while $\{\tau(\beta)\}$ belong to the index set A for $\mathfrak{U}$. When this hold we can restrict sections which for each p yields a map

$$\tau^p\colon C^p(\mathfrak{U},\mathcal{F}) \to C^p(\mathfrak{V},\mathcal{F})$$

For example, consider some vector $\xi\in C^1(\mathfrak{U},\mathcal{F})$. To each pair β_1,β_2 in B we get the section over $V_{\beta_1}\cap V_{\beta_2}$ defined by

$$\eta_{\beta_1,\beta_2} = \xi_{\tau(\beta_1),\tau(\beta_2)}|V_{\beta_1}\cap V_{\beta_2}$$

The construction of τ^p for $p\geq 2$ is similar way and we leave it to the reader to check that $\{\tau^p\}$ commute with the coboundary differentials. Hence $\tau^\bullet=\{\tau^p\}$ is a morphism from the complex $C^\bullet(\mathfrak{U},\mathcal{F})$ into $C^\bullet(\mathfrak{V},\mathcal{F})$ which give induced maps on cohomology

$$H^p(\tau)\colon\ H^p(\mathfrak{U},\mathcal{F}) \to H^p(\mathfrak{V},\mathcal{F})$$

3.7.2 Theorem. *The cohomology maps $\{H^p(\tau)\}$ are intrinsic in the sense that they do not depend upon the chosen index-map from A to B.*

Proof when $p=1$. Let u$\mu\colon A\to B$ be another index map and we must show that $H^1(\tau)=H^1(\mu)$. We orefer to use latin letters for the sets in $\mathfrak{V}$, i.e. now we have open sets $\{V_j\}$. Let $\{\xi_{\alpha,\beta}\}$ be a 1-cocycle in $C^1(\mathfrak{U},\mathcal{F})$. The index map τ gives the cochain

$$g_{jk} = \xi_{\tau(j),\tau(k)}$$

Similarly μ gives the cochain

$$g^*_{jk} = \xi_{\mu(j),\mu(k)}$$

For each V_k the open sets $U_{\tau(k)}$ and $U_{\mu(k)}$ contain V_k which enable us to define

$$h_k = \xi_{\tau(k),\mu(k)}$$

$\{\xi_{\alpha,\beta}\}$ Now we use that $\{\xi_{\alpha,\beta}\}$ is a 1-cycycle which applied to the triple $\tau(j),\tau(k),\mu(j)$ in A entails that

$$\xi_{\tau(j),\tau(k)} = \xi_{\tau(j),\mu(j)} + \xi_{\tau(k),\mu(j)}$$

From this the reader may confirm the equality:

$$g_{jk}-g^*_{j,k} = \big(\xi_{\tau(j),\mu(j)}+\xi_{\tau(k),\mu(j)}\big) - \big(\xi_{\mu(j),\mu(k)}+\xi_{\tau(k),\mu(j)}\big) = h_j - h_k$$

Since $\{h_j\in\mathcal{F}(V_j)\}$ this gives the requiested equality between the cohomology classes of g and g^*.

Exercise. Use a similar device as above to prove Theorem XX for every $p\geq 2$.

3.7.3. An injective map.

Let τ be an index map as above. With $p=1$ we have the map

$$H^1(\tau)\colon H^1(\mathfrak{U},\mathcal{F}) \to H^1(\mathfrak{V},\mathcal{F})$$

It turns out that this map always is injective.

Proof. GIVE IT

3.7.4 A surjective map. Now we impose the condition that

(*) $$H^1(U_\alpha \cap \mathfrak{V},\mathcal{F}) = 0$$

holds for every open set in $\mathfrak{U}$. Under this condition we prove that $H^1(\tau)$ is surjective. Let $\{\eta_{j,k}\}$ be a cocycle in $C^1(\mathfrak{V},\mathcal{F})$. For each U_α in $\mathfrak{U}$ we can consider restrictions to the open sets $U_\alpha \cap V_k \cap V_j$. In this way the η-cocylce yields a co-cycle in $C^1(U \cap \mathfrak{V},\mathcal{F})$ which by assumption is a coboundary which means that there exist sections

$$\gamma_k^\alpha \in \mathcal{F}(U_\alpha \cap V_k)$$

such that

(i) $$\eta_{k,j}|U_\alpha \cap V_k \cap V_j = \gamma_k^\alpha - \gamma_j^\alpha$$

Next, consider some pair α_1,α_2 in A. For every $m \in B$ we have the section

$$\gamma_m^{\alpha_1} - \gamma_m^{\alpha_2} \in \mathcal{F}(U_{\alpha_1} \cap U_{\alpha_2} \cap V_m)$$

and (i) gives

$$\gamma_k^{\alpha_1} - \gamma_k^{\alpha_2} = \gamma_j^{\alpha_1} - \gamma_j^{\alpha_2}$$

Since this hold for all pairs β_1,β_2. the glueing condition is satisifed while $\mathfrak{V}$ induces an open covering of $U_{\alpha_1} \cap U_{\alpha_2}$. Hence the sheaf axiom gives a section

$$\xi_{\alpha_1,\alpha_2} \in \mathcal{F}(U_{\alpha_1} \cap U_{\alpha_2})$$

such that

$$\xi_{\alpha_1,\alpha_2}|U_{\alpha_1} \cap U_{\alpha_2} \cap V_m = \gamma_m^{\alpha_1} - \gamma_m^{\alpha_2} \quad \text{hold for all} \quad m \in B$$

It is clear that this entails that these ξ-sections gives a cocycle in $C^1(\mathfrak{U},\mathcal{F})$ and hence it has an image in $H^1(\mathfrak{U},\mathcal{F})$. Its image under $H^1(\tau)$ is represented by

$$\eta^*_{k,j} = \xi_{\tau(k),\tau(j)} = \gamma_k^{\tau(k} - \gamma_j^{\tau(j)}$$

Now $U_{\tau(j)}$ and $U_{\tau(k)}$ both contain $V_k \cap V_j$ and therefore (*) above together with (**) imply that the equality below in $\mathcal{F}(V_k \cap V_j)$:

$$\eta^*_{k,j} = \eta_{k,j}$$

which finshes the proof that $H^1(\tau)$ is surjective

3.8 Leray's Theorem.

Let $\mathcal{F}$ be a sheaf on X. To each open covering $\mathfrak{U}$ we have constructed the Cech cohomlogy groups. Passing to subcoverings one can employ the intrinsically maps in Theorem 3.7.2 which after a passage to the limit give canonical maps for every p:

$$\rho^p_{\mathfrak{U}}\colon H^p(\mathfrak{U},\mathcal{F}) \to H^p(X,\mathcal{F})$$

Following Leray we shall exhibit a condition that a ρ-map is an isomorphism in a given degree p. We say that $\mathcal{F}$ satisfies Leray's p-condition if the following hold:

(i) $$H^{q-k}(U_{\alpha_1} \cap \ldots U\alpha_k) = 0 \quad : \ q+k=p \quad \text{and} \quad 1 \leq q \leq p$$

(ii) $$H^{\nu}(U_{\alpha_1} \cap \ldots U\alpha_k\}) = 0 \quad : \quad q+\nu = p-2 \quad \text{and} \quad 1 \leq q \leq p-2$$

3.8.1 Theorem. *If Leray's p-condition holds for a given p then $\rho^p_{\mathfrak{U}}$ is an isomorphism.*

3.8.2 Example. When $p=1$ Leray's condition means that $H^1(U_\alpha,\mathcal{F})=0$ for each U_α which is a useful criterion for $\rho^p_{\mathfrak{U}}$ to be an isomorphism.

To prove of Theorem 3.8.1 we take a flabby resolution $\mathcal{J}^\bullet$ and identify the cohomology groups of $\mathcal{F}$ with those in the complex $\mathcal{J}^\bullet$ as explained in § xx. Next, when $\mathfrak{U}$ is an open covering we obtain a double complex where

$$A^{p,\nu} = C^p(\mathfrak{U},\mathcal{J}^\nu)$$

Let us first consider the δ-maps which means that with ν fixed we regard the cochain complex $C^\bullet(\mathfrak{U},\mathcal{J}^\nu)$. Since $\mathcal{J}^\nu$is flabby it is exact in every degree except zero where the kernel becomees $J^\nu(X)$. It means that the spectral sequence of the Second Filtration degenerates and as explained in § xx this entials that the diagonal cohomology of the double complex are equal to those of the complex $\mathcal{J}^\bullet(X)$ which by defintion are the cohomology groups $\{H^p(X,\mathcal{F})\}$.

Next we consider the First Filtration where we start with ∂-maps. This time the ∂-cohomology gives the double complex:

$$E_1^{k,\nu} = \oplus\, H^\nu(U_{\alpha_0} \cap \ldots U_{\alpha_k},\mathcal{F})$$

where the direct sum is taken over stictly increasing sequences $\alpha_0 < \ldots < \alpha_k$ in the totally ordered and finite index-set A. If $p \geq 1$ and Leray's p-condition holds only one E_1-term remains on the diagonal $k+\nu=p$, i.e.

$$E_1^{p,0} = C^p(\mathfrak{U},\mathcal{F})$$

Next, passing to the E_2-term leary's condition entails tht nocoboundsry terms appear in position $(p,0)$ where we also have the δ-map. So by inspecting the diagram (*) it follows that:

$$H^p(\Delta^\bullet) \simeq E_2^{p,0} \simeq H^p(\mathfrak{U},\mathcal{F})$$

Since we already have seen that $H^p(\Delta^\bullet) \simeq H^p(X,\mathcal{F})$ Leray's theorem follows.

3.8.3 Example. Let us consider the case $p=1$ but this time we do no assume that $H^1(U\alpha,\mathcal{F})$ are all zero. When we take the Second Filtration we encounter a table where

$$E_2^{1.0} = H^1(\mathfrak{U},\mathcal{F})$$

But we may have also a contribution from $E_2^{0,1}$. In general the Second Filtration induces a fikltrationon $H^1(\delta^\bullet)$ which entails that there is an exact seuence

(*) $$0 \to H^1(\mathfrak{U},\mathcal{F}) \to H^1(X,\mathcal{F}) \to E_2^{0,1}$$

where the klast arrow in general is not surjective. To analyze $E_2^{0,1}$ we recall that

$$E_1^{0,1} = \oplus\, H^1(U_\nu\mathcal{F})$$

Let us now assume that

$$H^1(U_k \cap U_\nu, \mathcal{F}) = 0 \quad \text{for all pairs} \quad k \leq \nu$$

Then the δ-map on $E_1^{0,1}$ is identically zero so that

$$E_2^{0,1} = \oplus H^1(U_\nu \mathcal{F})$$

Next, the spectral sequence gives a canonical apping

$$E_2^{0,1} \to H^2(\mathfrak{U}, \mathcal{F})$$

Let $\mathcal{K}$ be the kernel of this map which gives

$$E_3^{0,1} = \mathcal{K}$$

Nothing more happens in this position since the next differential has bi-degree $(-2, 3)$ and no entries in the double complex occur when the first index is negative. So in general we have an exact sequence

$$(*) \qquad 0 \to H^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \mathcal{K}$$

where $\mathcal{K}$ is the kernel under the canonical map

$$\oplus H^1(U_\nu, \mathcal{F}) \to H^2(\mathfrak{U}, \mathcal{F})$$

Remark The computations above illustrate the "yoga" which appears while one investigates the specrtral sequences of a double complex. A specific case occurs if X is a compact Riemann surface and $\mathcal{F} = \mathcal{O}$. We can consider finite coverings $\mathfrak{U}$ where some of the Cech cohomology groups $H^1(U_\nu, \mathcal{O}) \neq 0$ and when this occurs the exact sequence above shows that the CFech cohomology $H^1(\mathfrak{U}, \mathcal{O})$ in general does not recover the cohomology. Take as an example the case when X has genus 2 and let us imagine that $H^1(\mathfrak{U}, \mathcal{F})$ is 1-dimensional. while $H^1(U_\nu, \mathcal{O}) \neq 0$ for every ν and that $m \geq 2$ is the number of sets in the covering. Now $\mathcal{K}$ has dimension one so the kernel of the map (*) is $m - 1$-dimensional. The reader is invited to find examples where a situation as above occurs. with $m \geq 3$.

Analysis on compact Riemann surfaces.

Each compact Riemann surface X has an underlying real-analytic structure which is an oriented compact real manifold of dimension two denoted by $X_{\mathbf{R}}$. For a while we ignore the complex analytic structure. There exists the sheaf $\mathcal{E}$ whose sections are complex-valued C^∞-functions and the sheaf $\mathcal{E}^1$ of differential 1-forms with C^∞-coefficients. For brevity we refer to smooth 1-forms. Finally there is the sheaf $\mathcal{E}^2$ of smooth 2-forms. Since X is oriented and compact smooth 2-forms σ can be integrated over X and we write

$$\int_X \sigma$$

If α is a smooth 1-form its exterior differential $d\alpha$ belongs to $\mathcal{E}^2(X)$ and Stokes Theorem gives

$$\int_X d\alpha = 0$$

It means that the integral of an exact 2-form over X is zero. A non-trivial fact is the converse whose proof is given in § xx below.

1.Theorem. *A smooth 2-form is exact if and only if its integral over X is zero.*

Currents. A current of degree zero is a continuous linear form on the Frechet space $\mathcal{E}^2(X)$ and we set

$$\mathfrak{c}^0(X) = \mathcal{E}^2(X)^*$$

Let us remark that $\mathfrak{c}^0(X)$ corresponds to the space of distributions on X, i.e.

$$\mathfrak{c}^0(X) = \mathfrak{Db}(X)$$

The dual $\mathcal{E}^1(X)^*$ is denoted by $\mathfrak{c}^1(X)$ whose elements are called 1-currents. Finally, the dual of $\mathcal{E}(X)$ is denoted by $\mathfrak{c}^2(X)$ and its elements are called 2-currents.

Example. A function $f \in \mathcal{E}(X)$ defines a zero-current via the linear form

$$\sigma \mapsto \int_X f \cdot \sigma \quad : \quad \sigma \in \mathcal{E}^2(X)$$

In this way $\mathcal{E}(X)$ appears as a subspace of $\mathfrak{c}^0(X)$ and such currents are called smooth. In the same way we get an inclusion $\mathcal{E}^1(X) \subset \mathfrak{c}^1(X)$ where $\phi \in \mathcal{E}^1(X)$ yields a continuous linear functional on $\mathcal{E}^1(X)$ defined by

$$w \mapsto \int_X \phi \wedge w \quad : \quad w \in \mathcal{E}^1(X)$$

Exterior differential maps on smooth forms extend to currents. If $w \in \mathcal{E}^1(X)$ is a smooth 1-current then dw is the 2-current defined by

$$f \mapsto \int_X f \cdot dw \quad : \quad f \in \mathcal{E}(X)$$

Stokes theorem gives:

$$0 = \int_X d(fw) = \int_X f \cdot dw + \int_X df \wedge w$$

Since $df \wedge w = -w \wedge df$ we get

$$\int_X f \cdot dw = \int_X w \wedge df$$

This leads to the construction of exterior differentials of 1-currents. Namely, if $\gamma \in \mathfrak{c}^1(X)$ then the 2-current $d\gamma$ is defined by

$$f \mapsto \gamma(df) \quad : \quad f \in \mathcal{E}(X)$$

By the above it follows that if $w \in \mathcal{E}^1(X)$ is regarded as a smooth 1-current then its current differential dw is the smooth 2-current expressed by the differential 2-form dw. Thus, the exterior differentials on smooth forms extend to currents and one has a complex

$$0 \to \mathfrak{c}^0(X) \xrightarrow{d} \mathfrak{c}^1(X) \xrightarrow{d} \mathfrak{c}^2(X) \to 0$$

2. d-closed zero currents. A current γ of order zero is d-closed if and only if

$$d\gamma(w) = \gamma(dw) = 0 \quad : \quad w \in \mathcal{E}^1(X) \tag{2.1}$$

To grasp this condition we consider a chart U in X taken as an open disc in the (x,y)-plane. If $f(x,y) \in C_0^\infty(U)$ we have the 1-form $f \cdot dx$ and get $d(fdx) = f'_y \cdot dy \wedge dx$. Hence (2.i) means that

$$\gamma(f'_y \cdot dy \wedge dx) = 0 \quad : \quad f \in C_0^\infty(U) \tag{2.2}$$

The restriction of γ to U is a distribution in the sense of § X and (i) means that the distribution derivative $\frac{\partial\gamma}{\partial y}$ vanishes in U. Similarly, if we start with a 1-form gdx where $g \in C_0^\infty(U)$ we find that $\frac{\partial\gamma}{\partial x} = 0$ in U. It follows that the distribution γ restricts to a constant density function in U. The conclusion is that the d-kernel on $\mathfrak{c}^0(X)$ is reduced to constant functions, i.e.

$$\ker_d(\mathfrak{c}^0(X)) = \mathbf{C} \tag{2.3}$$

3. Inverse and direct images under mappings. Let X, Y be a pair of oriented compact two-dimensional manifolds and $F \colon X \to Y$ a real-analytic map. Assume that F is surjective, i.e. $F(X) = Y$ and that the mapping is finite which means that there exists an integer n such that the inverse finber $F^{-1}(\{y\}$ contains at most n points in X for every $y \in Y$. Now we construct inverse images of smooth forms on Y. First, if $g \in \mathcal{E}(Y)$ the composed function $g \circ F$ belongs to $\mathcal{E}(X)$. Next, let w be a smooth 1-form on Y. Using local coordinates one verifies that there exists a pull-back denoted by $F^*(w)$ which yields a smooth 1-form on X. More precisely, let $p \in X$ be a given point and U is some chart around p while V is a chart around $F(p)$. Let (ξ, η) be local coordinates in the chart V and (x, y) are local coordinates in U. Then

$$w|V = g \cdot d\xi + h \cdot d\eta \quad : \quad g, h \in \mathcal{E}(V) \tag{i}$$

In U we get the C^∞-functions $\xi \circ F$ and $\eta \circ F$ whose exterior differentials give smooth 1-forms in U denoted by $F^*(\xi)$ and $F^*(d\eta)$. Now

$$F^*(w) = g \circ F \cdot F^*(d\xi) + h \circ F \cdot F^*(d\eta)$$

Exercise. Verify that the 1-form $F^*(w)$ does not depend upon the chosen local coordinates in the two charts. Conclude that if $w \in \mathcal{E}^1(Y)$ then there exists a smooth 1-form $F^*(w)$ on X obtained in local charts as above. Hence there exists a pull-back map:

$$w \mapsto F^*(w)$$

In the same way one constructs the pull-back of each smooth 2-form on Y. A notable fact is that these pull-back mappings commute with exterior differentials. One has for example

$$d(F^*(w)) = F^*(dw) \quad : \quad w \in \mathcal{E}^1(Y)$$

We leave out the detailed verifications which are exposed in text-books devoted to differential geometry.

4. Direct images of currents. Tbe construction of the F^*-mappings of smooth forms give by duality F_*-mappings on currents. More precisely, let $\gamma \in \mathfrak{c}^\nu(X)$ for some $0 \le \nu \le 2$. The direct image current $F_*(\gamma)$ is defined by

$$F_*(\gamma)(w) = \gamma(F^*(w)) \quad : \quad w \in \mathcal{E}^\nu(Y)$$

By this construction the passage to direct images of currents commute with exterior differentials. For example, let γ be a 1-current on X. Then

$$d_Y(F_*(\gamma)) = F_*(d_X(\gamma))$$

where we for convenience introduced a sub-index to indicate on which manifold one has performed an exterior differential. In particular F_* sends d-closed currents on X to d-closed currents on Y.

5. Integration chains and 1-currents. Let c be an oriented C^1-curve on X which means that c is defined via a C^1-mapping $t \to x(t)$ from a closed t-interval $[0, T]$ into X. Each smooth 1-form γ in X can be integrated along c and the map

$$\gamma \mapsto \int_c \gamma$$

yields a 1-current denoted by $\int_c$. More generally we can take a finite sum of such integration currents, i.e. every chain c of class C^1 gives a current $\int_c$ which belongs to $\mathfrak{c}^1(X)$. By definition the exterior differential $d \int_c$ is the 2-current defined by

$$g \mapsto \int_c dg \quad : \quad g \in \mathcal{E}(X)$$

The resulting current is denoted by ∂c. In simplicial topology one refers to ∂c as the boundary of the 1-chain c. Here it has been given an operative definition. If c is an elementary chain defined by a single oriented C^1-curve with end-points a and b, then the 2-current ∂c is a current supported by the end-points a and b. In general ∂c is a sum of Dirac measures assigned at a finite set of points, i.e.

$$d \int_c = \sum k_\nu \cdot \delta(x_\nu)$$

where $\{k_\nu\}$ are integers and $\sum k_\nu = 0$.

6. Critical values of a map F. Let $F\colon X \to Y$ be a surjective mapping with finite fibers. One says that F is smooth at a point $p \in X$ if the following hold when a pair of local charts U and V have been chosen as in § XX: The 1-forms $F^*(d\xi)$ and $F^*(d\eta)$ are non-zero and **R**-linearly independent. Next, a point $q \in Y$ non-critical if F is smooth at each $p \in F^{-1}(\{q\})$. The complement is denoted by $\text{Crit}(F)$ and is called the set of the critical values. A welknown fact - called Sard's Lemma - asserts that the set of F-critical values is a (possibly empty) finite subset of Y. Since X and Y both are connected it follows that when $q \in Y \setminus \text{Crit}(F)$ then the number of points in the inverse fiber $F^{-1}(\{q\})$ does not depend upon q. This integer n is called the degree of the mapping F and denoted by $\deg(F)$. Put $X_* = X \setminus F^{-1}(\text{Crit}(F)$. In topology one says that the restricted mapping

$$F\colon X_* \to Y \setminus \text{Crit}(F)$$

is an unbranched covering map of degree n. If $\Omega \subset Y \setminus \text{Crit}(F)$ is open and simply connected one proves that the inverse image $F^{-1}(\Omega)$ consists of pairwise disjoint open subsets $U_1, \ldots, U_n$ and $F\colon U_k \to \Omega$ is a diffeomorphism for each k. One refers to $\{U_k\}$ as the sheets above Ω.

7. Direct images of integration currents. Let F be as above and consider a 1-current $\int_c$ on X given by an oriented 1-chain c. Now there exists the direct image current $F_*(\int_c)$. To describe this 1-current on Y we proceed as follows. To begin with c is a sum of small 1-chains $\{c_k\}$ where each c_k is a single C^1-curve and we can also choose c_k so that its parametrized curve $t \to x(t)$ stays in a small subset of X. So $a = x(0)$ and $b = x(T)$ are close to each other and the whole curve c_k stays inside a chart U in X where a is the origin in U and we may also assume that $F(U)$ stays in a chart V around $F(a)$.

Let (ξ, η) be local coordinates in $F(U)$. If $g(\xi, \eta) \in C_0^\infty(F(U))$ we get the differential 1-form $g \cdot d\xi$ whose pull-back can be integrated on the chain c_k. This gives a linear functional

$$g \mapsto \int_{c_k} F^*(g \cdot \xi)$$

A similar linear functional arises when we instead use the 1-form $d\eta$.

Exercise. Conclude from the above that there exists the direct image current $F_*(\int_{c_k})$ which has a compact support in $F(U)$ and is given by a 1-current of the form

$$\mu_1 \cdot d\xi + \mu_2 \cdot d\eta$$

where μ_1, μ_2 are two Riesz measures with compact support in U. It means that the direct image current has order zero. But unless $F\colon U \to F(U)$ is a diffeomorphism the direct image current is in general not given by integration along a 1-chain. Funally, by additivity one has

$$F_*(\int_c) = \sum F_*(\int_{c_k})$$

9. Further results. Let $F\colon X \to Y$ be as above. A special current on X is the 2-current $\Box_X$ defined by

$$\omega^2 \mapsto \int_X \omega^2$$

i.e. one integrates 2-forms over X. If ϕ is a bounded Borel function on X, or more generally an L^1-function on x in Lebegue's sense, then there exists the 2-current defined by

$$\omega^2 \mapsto \int_X \phi \cdot \omega^2$$

It is denoted by $\phi \cdot \Box_X$.

9.1 Exercise. Let $F\colon X \to Y$ where $F(X) = Y$ and F is finite but may have some critical values. Show that there exists a Borel function ϕ on Y such that

$$F_*(\Box_X) = \phi \cdot \Box_Y$$

and describe the ϕ-function via some sort of Jacobian of F. The case when X is a compact Riemann surface and $Y = \mathbf{P}^1$ while F is a meromorphic function on X is of special interest.

II. The case when X is a Riemann surfaces.

On a compact Riemann surface X the complex analytic structure gives a decomposition:

$$\mathcal{E}^1(X) = \mathcal{E}^{1,0}(X) \oplus \mathcal{E}^{0,1}(X)$$

The exterior differential on X is decomposed into a sum $d = \partial + \bar{\partial}$ and when $f \in \mathcal{E}(X)$ one gets the 1-forms

$$\partial f \in \mathcal{E}^{1,0}(X) \quad \text{and} \quad \bar{\partial} f \in \mathcal{E}^{0,1}(X)$$

This gives two complexes

$$0 \to \mathcal{E}(X) \xrightarrow{\partial} \mathcal{E}^{1,0}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{1,1}(X) \to 0$$

$$0 \to \mathcal{E}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(X) \xrightarrow{\partial} \mathcal{E}^{1,1}(X) \to 0$$

where we remark that $\mathcal{E}^{1,1}(X) = \mathcal{E}^2(X)$. Since $\bar{\partial}$ is elliptic a current $\gamma \in \mathfrak{c}^0(X)$ is $\bar{\partial}$-closed if it is a holomorphic function, i.e.

$$\mathrm{Ker}_{\bar{\partial}}(\mathfrak{c}^0(X)) = \mathcal{O}(X) = \mathbf{C}$$

where the last equality follows since holomorphic functions on the compact Riemann surface are constant.

2.1 Principal value distributions. Let $f \in \mathfrak{M}(X)$. It defines a distribution $\mathrm{VP}(f)$ as follows: When σ is a smooth 2-form on X we avoid poles of f and consider the limit integrals:

(*) $$\lim_{\epsilon \to 0} \int_{X_f(\epsilon)} f \cdot \sigma$$

where $X_f(\epsilon)$ is the open complement of small removed ϵ-discs centered at the finite set of poles of f. The existence of this limit was explained in §§, i.e. in local coordinates it amounts to show that there exist limits

$$\int_{|z|>\epsilon}, \frac{\phi(x,y)dxdy}{z^k}$$

for every positive integer k and each test-function $\phi(x,y)$ in $\mathbf{C}$. Next, construct the current

$$R_f = \bar{\partial}(\mathrm{VP}(f))$$

It is called the residue current of the meromorphic function f. Since f is holomorphic outside the poles the (0,1)-current R_f is supported by the discrete set of poles. In fact, Stokes Theorem gives

(**) $$R_f(\gamma) = \lim_{\epsilon \to 0} \int_{X_f[\epsilon]} f \cdot \gamma$$

where $X_f[\epsilon]$ consists of disjoint small circles centered at the poles $p_1, \ldots, p_k$ of f.

2.2 The current $\frac{\partial f}{f}$. Let $f \in \mathfrak{M}(X)$. Now $\frac{\partial f}{f}$ is a $(1,0)$-current. Let U be a chart around a zero q of f of some order e so that $f = z^e \phi(z)$ where $\phi \neq 0$ holds in the chart. Then

$$\frac{\partial f}{f}|U = \frac{\partial \phi}{\phi} + e \cdot \frac{dz}{z}$$

If V instead is a chart around a pole of some order e we have $f = z^{-e}\phi(z)$ and obtain

$$\frac{\partial f}{f}|V = \frac{\partial \phi}{\phi} - e \cdot \frac{dz}{z}$$

2.3 Exercise. Show that if $g \in \mathcal{E}(X)$ then

$$\frac{\partial f}{f}(dg) = 2\pi i \cdot \left[\sum e_k \cdot g(p_k) - \sum \; e_j \cdot g(q_j)\right]$$

where $\{p_k\}$ are the poles and $\{q_k\}$ the zeros of f and $\{e_k\}$ the order of poles, respectively the multiplicities at the zeros of f.

2.4 A special direct images. Let $w \in \Omega(X)$ be an abelian differential. It yields a smooth current and there exists the dirct image current $F_*(\omega)$. Since the passage to direct images commute with differentials we have

$$\bar{\partial}_y(F_*(\omega)) = 0$$

Hence the direct image current belongs to $\Omega(Y)$. If $Y = \mathbf{P}^1$ there are no non-zero holomorphic forms so when $f\colon X \to \mathbf{P}^1$ is given via some $f \in \mathfrak{M}(X)$ then $f_*(\omega) = 0$. The vanishing of this direct image current means that for every $\beta \in \mathcal{E}^{0,1}(\mathbf{P}^1)$ one has

$$\int_X w \wedge f^*(\beta) = 0 \tag{2.4.1}$$

2.4.2 Exercise. Verify (2.4.1) directly. The hint is to use partitions of the unity and reduce the proof to the case when the test-form β has a small support. The sole local difficulty occurs when the support contains some critical value under the mapping f.

III. Holomorphic line bundles.

Let X be a compact Riemann surface. If U is an open set we denote by $\mathcal{O}^*(U)$ the family of holomorphic functions with no zeros in U. For each pair f,g in $\mathcal{O}^*(U)$ the product fg is also holomorphic without zeros. In this way $\mathcal{O}^*(U)$ is an abelian group whose zero-element corresponds to the constant function in U which is everywhere one. One verifies easily that

$$U \mapsto \mathcal{O}^*(U)$$

satisfies the sheaf axioms and the resulting sheaf is denoted by $\mathcal{O}^*$. Next, if $g \in \mathcal{O}(U)$ the exponential function $e^{2\pi i g}$ yields a section in $\mathcal{O}^*(U)$ which is the zero-element in this multiplicative abelian group if and only if g is a locally constant and integer-valued function on U. In simply connected charts the exist logarithms of $\mathcal{O}^*$-valued sections which gives an exact sheaf complex:

$$0 \to \mathbf{Z}_X \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}^* \to 0$$

where $\mathbf{Z}_X$ is the abelian sheaf of locally constant integer-valued funtions. Hence there is a long exact sequence of cohomology where we notice that $\mathcal{O}^*(X)$ is reduced to the multiplicative group of non-zero complex numbers and the kernel of the exponential map on $\mathbf{C}$ is the additive group of integers . This gives an exact sequence

(1) $$0 \to H^1(\mathbf{Z}_X) \to\to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \to H^2(\mathbf{Z}_X) \to 0$$

where the last arrow comes from $H^2(X,\mathcal{O}) = 0$. A wellknown result in topology asserts that $H^2(\mathbf{Z}_X)$ is reduced to the group of integers. So the last arrow in the exact squence above gives an additive map

$$H^1(X,\mathcal{O}^*) \to \mathbf{Z}$$

To investigate this map we use Cech cohomology to describe $H^1(X,\mathcal{O}^*)$. Let $\mathfrak{U}$ be a finite covering by charts. As explained in § xx this gives:

$$H^1(\mathfrak{U},\mathcal{O}) \simeq H^1(X,\mathcal{O})$$

Next, for the sheaf $\mathbf{Z}_X$ topology teaches that

$$H^\nu(\mathfrak{U},\mathbf{Z}_X) \simeq H^\nu(X,\mathbf{Z}_X) \quad : \quad 1 \le \nu \le 2$$

hold when non-empty intersections of pairs or triples in $\mathfrak{U}$ are simply connected. So when $\mathfrak{U}$ is a simply connected covering in the sense of § XX then (1) gives the exact sequence

(2) $$0 \to H^1(\mathfrak{U},\mathbf{Z}_X) \to H^1\mathfrak{U},(X,\mathcal{O}) \to H^1(\mathfrak{U},\mathcal{O}^*) \to H^2(\mathfrak{U},\mathbf{Z}_X) \to 0$$

Moreover one has the isomorphism

(3) $$H^1(\mathfrak{U},X,\mathcal{O}^*) \simeq H^1(X,\mathcal{O}^*)$$

3.1 Conclusion. The last isomorphism entails that each cohomology class in $H^1(X,\mathcal{O}^*)$ is represented by a Cech cocycle $\{f_{\beta\alpha}\}$ in $C^1(\mathfrak{U},\mathcal{O}^*)$ where the multiplicative cocycle condition means that for every triple $U_\alpha, U_\beta, U_\gamma$ with non-empty intersection it holds that

(3.1.i) $$f_{\gamma\alpha} = f_{\gamma,\beta} \cdot f_{\beta,\alpha}$$

3.2 Associated locally free $\mathcal{O}$-modules. Let $\xi = \{f_{\beta\alpha}\}$ be an element in $C^1(\mathfrak{U},\mathcal{O}^*)$. Then there exists a sheaf $\mathcal{F}$ whose sections are given as follows: If V is an open set one has

$$\mathcal{F}(V) = \prod^{*} \mathcal{O}(U_\alpha \cap V)$$

where (*) indicates that a family $\{\phi_\alpha \in \mathcal{O}(V \cap U_\alpha)\}$ yields an element in the (*)-marked product if and only if

$$\phi_\beta(x) = f_{\beta,\alpha}(x) \cdot \phi_\alpha(x) \quad : \quad x \in V \cap U_\beta \cap U_\alpha$$

If $V = U_{\alpha_*}$ for a given index we see that $\mathcal{F}(U_{\alpha_*})$ is a copy of $\mathcal{O}_{U_{\alpha_*}}$. For if $g_* \in \mathcal{O}(V)$ with $V \subset U_{\alpha_*}$ we get an $\mathcal{F}$-valued section over V where

$$g_\alpha = f_{\alpha,\alpha_*} \cdot g_*$$

Hence $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}$-modules with rank one. Such sheaves are also called holomorphic vector bundles.

3.3 Exercise. Show that conversely each locally free sheaf of $\mathcal{O}$-modules with rank one comes from some $\xi \in H^1(X, \mathcal{O}^*)$. Show also that if $\xi \neq \eta$ in $H^1(X, \mathcal{O}^*)$ then their associated holomorphic vector bundles are non-isomorphic.

3.4 Cohomology of holomorphic vector bundles. Given a Cech cocycle $\xi = \{f_{\beta,\alpha}\}$ in $H^1(\mathfrak{U}, \mathcal{O}^*)$ we denote the associated holomorphic vector bundle with $\mathcal{O}_\xi$. We can also introduce the C^∞-bundle $\mathcal{E}_\xi$ whose sections are C^∞-functions satisfying the glueing conditions in 3.2 and similarly the sheaf of such sections in $\mathcal{E}^{0,1}$. Since the transition functions $f_{\beta,\alpha}$ are holomorphic we get an exact sheaf complex

$$\mathcal{O}_\xi \to \mathcal{E}_\xi \xrightarrow{\bar\partial} \mathcal{E}_\xi^{0,1} \to 0$$

It follows that

$$H^1(X, \mathcal{O}_\xi) \simeq \frac{\mathcal{E}_\xi^{0,1}(X)}{\bar\partial(\mathcal{E}_\xi(X))}$$

Exercise. Prove that the cohomlogy groups $H^0(X, \mathcal{O}_\xi)$ and $H^1(X, \mathcal{O}_\xi)$ are finite dimensional complex vectorspaces. The hint is to pass to subcoverings and use Leray's Theorem.

3.5 Meromorphic sheaves. We can consider meromorphic sections and introduce the sheaf

$$\mathfrak{M}_\xi = \mathfrak{M} \otimes_{\mathcal{O}} \mathcal{O}_\xi$$

If D is a divisor we get the sheaf

$$\mathcal{O}_D \otimes_{\mathcal{O}} \mathcal{O}_\xi \tag{1}$$

It is denoted by $\mathcal{O}_\xi[D]$. Since $\mathcal{O}_D$ is a subsheaf of $\mathfrak{M}$ it follows that $\mathcal{O}_\xi[D]$ is a subsheaf of $\mathfrak{M}_\xi$. Next, if D is a positive divisor we get the exact sheaf sequence

$$0 \to \mathcal{O}_\xi \to \mathcal{O}_\xi[D] \to \mathcal{S}_D \to 0$$

where $\mathcal{S}_D$ is the scysraper sheaf of the divisor. Ecxactly as in § Xx we get the equality below for the Euler characteristics:

$$\chi(\mathcal{O}_\xi[D]) = \chi(\mathcal{O}_\xi) + |D|$$

So if $|D|$ is sufficiently large then $H^0(X, \mathcal{O}_\xi[D])$ has a positive dimension which gives a non-zero global section in $\mathcal{O}_\xi[D]$ and hence also in the larger sheaf $\mathfrak{M}_\xi$.

3.6 The divisor of meromorphic sections. Keeping ξ fixed we consider a global section ϕ in $\mathfrak{M}_\xi$. It means that there is a family $\{g_\alpha \in \mathfrak{M}(U_\alpha)\}$ such that

$$g_\beta = f_{\beta,\alpha} \cdot g_\alpha$$

in every non-empty intersection $U_\alpha \cap U_\beta$. Since the f-functions are zero-free the divisors of the g-functions are matching and produce a divisor on X denoted by $D(g_\bullet)$. Let $\{h_\alpha\}$ be another global meromorphic section. Then there exists a meromorphic function ψ on X such that

$$\psi | U_\alpha = \frac{h_\alpha}{g_\alpha}$$

It follows that

$$D(h_\bullet) = D(g_\bullet) + D(h)$$

So the images in the quotient t space $\frac{\mathcal{D}}{\mathcal{P}_0}$ are equal and we have proved

3.7 Proposition. *The holomorphic line bundle defined by ξ yields a unique element in $\frac{\mathcal{D}}{\mathcal{P}_0}$ which we denote by $\mathfrak{D}(\xi)$.*

The degree of $\mathfrak{D}(\xi)$ can be non-zero. See §§ for a specific example. To find those ξ for which the degree is zero so that $\mathfrak{D}(\xi)$ belongs to the Jacobi-Picard torus $\frac{\mathcal{D}_0}{\mathcal{P}_0}$ we employ the exact sequence of sheaves. from (1). Recall from XX that

$$\frac{H^1(\mathcal{O})}{H^1(X, \mathbf{Z}_X)}$$

is equal to the Jacobi-Picard torus $\text{Pic}(X)$ and (1) and (1) gives the exact sequence

$$0 \to \text{Pic}(X) \to H^1(X, \mathcal{O}^*) \xrightarrow{\rho} \mathbf{Z} \to 0$$

3.8 Proposition. *For each $\xi \in H^1(X, \mathcal{O}^*)$ one has the equality*

$$\rho(\xi) = |\mathfrak{D}(\xi)|$$

Hence the degree of $\mathfrak{D}(\xi)$ is zero if and only if ξ belongs to the image of $H^1(X, \mathcal{O})$ from the sequence on (1).

Exercise. Prove this result. The hint is that when ξ is represented by $\{f_{\beta,\alpha}\}$ then we can pick bracnhes of the log-functions and since we have a muyltilicative co-cycle it follows that we get a Cech cocycle in $Z^2(\mathfrak{U}, \mathbf{Z}_X)$ where a triple intersection has the integer

$$\log f_{\gamma,\beta} + \log f_{\beta,\alpha} - \log f_{\gamma,\alpha}$$

This cocycle represents an integer in $H^2(X, \mathbf{Z}_X))$ and the reader should verify that it is equal to $\rho(\xi)$ and after this show that if the cocycle (i) is δ-exact in the Cech complex of $\mathbf{Z}_X$ then ξ belongs to the $H^1(X, \mathcal{O})$-image.

4. Differential operators.

On a given Ruemann surface X we have the sheaf of ring $\mathcal{D}$ whose sections are holomorphic differential operators which is filtered by $\{\mathcal{D}(m)\}$. See § xx for further details. Next, denote by $\mathfrak{g}_X$ the sheaf of holomorphic vector fields which yields the subsheaf of first order differential operators in $\mathcal{D}$. Recall from § XX that $\mathcal{D}(0)$ is a copy of $\mathcal{O}$ and the associated graded sheaf

$$\mathrm{gr}(\mathcal{D}) = \mathcal{D}(0) \oplus \frac{\mathcal{D}(1)}{\mathcal{D}(0)} \oplus \frac{\mathcal{D}(2)}{\mathcal{D}(1)} \oplus \ldots$$

is a commutative graded sheaf of rings which in local charts is isomorphic to the polynomial sheaf in one variable over $\mathcal{O}$.

4.1 Twisted rings of differential operators. Consider a sheaf $\mathcal{A}$ of rings which is filtered where $\mathcal{A}(0) = \mathcal{O}_X$ and the associated graded ring

$$\mathrm{gr}(\mathcal{A}) = \mathcal{A}(0) \oplus \frac{\mathcal{A}(1)}{\mathcal{A}(0)} \oplus \frac{\mathcal{A}(2)}{\mathcal{A}(1)} \oplus \ldots$$

is locally isomorphic to is isomorphic to the polynomial sheaf in one variable over $\mathcal{O}$.

Definition. When $\mathcal{A}$ is as above itis called a twised ring of differential operators if there exists an $\mathcal{O}X$-linear sheaf map ρ which gives an exact sequence

$$(*) \qquad 0 \to \mathcal{A}_0 \to \frac{\mathcal{A}_1}{\mathcal{A}_0} \xrightarrow{\rho} \mathfrak{g}_X \to 0$$

Since $\mathfrak{g}_X$ is a locally free $\mathcal{O}_X$-moduke it follows that (*) is locally split exact. So when $\{U_\alpha\}$ is a covering by charts there exists for each α some $\mathcal{O}_X$-linear map

$$\gamma_\alpha \colon \mathfrak{g}_X \to \mathcal{A}_1$$

defined in U_α where $\rho \circ \gamma_\alpha$ is the identity on the restriction of $\mathfrak{g}_X$ to U_α. So if $\delta \in \mathfrak{g}(U_\alpha \cap U_\beta)$ we obtain

$$(1) \qquad \gamma_\alpha(\delta) - \gamma_\beta(\delta) = f_{\alpha,\beta}$$

where $\{f_{\alpha,\beta}\}$. The assignment of these f-functions give an $\mathcal{O}_X$-linear map from $\mathfrak{g}_X|U_\alpha \cap U_\beta$. Hence there is a holomorphic 1-form $w_{\alpha,\beta} \in \Omega(U_\alpha \cap U_\beta)$ such that

$$w_{\alpha,\beta}(\delta) = \gamma_\alpha(\delta) - \gamma_\beta(\delta)$$

It is clear that $\{w_{\alpha,\beta}\} \in Z^1(\mathfrak{U}, \Omega)$ so this cocycle gives a cohomoogy class in $H^1(X, \Omega)$. Hence every twisted ring of differential operators produces an element in $H^1(X, \Omega)$.

A converse construction. Let $\{w_{\alpha,\beta}\} \in Z^1(\mathfrak{U}, \Omega)$ be given. On $\mathcal{D}$ we construct an isomorphism on the restriction to $U_\alpha \cap U_\beta$ which is the identity on $\mathcal{D}(0)$ while

$$\delta \mapsto \delta - w_{\beta,\alpha}(\delta$$

when $\delta \in \mathfrak{g}_X|U_\alpha \cap U_\beta$.

Exercise. Show that these isomorphisms yield a twisted ring of differential operators whose isomorphism class in the category XX above only depends upon the cohomology class of the cocycle. Conclude that there exists a 1-1 correspondence with elements in $H^1(X, \Omega)$ and isomorphismclasses in XX.

An exact sequence. If $f \in \mathcal{O}^*$ we obtain the holomorphic 1-form $\frac{\partial f}{f}$. The reader can verify that thisyields an exact sequence of sheaves

$$0 \to \mathbf{C}_X^* \to \mathcal{O}^* \xrightarrow{d\log} \Omega \to 0$$

Passing to cohomology there is a long exact sequence:

$$0 \to H^0(X, \Omega) \to H^1(\mathbf{C}_X^*) \xrightarrow{\rho} H^1(X : \mathcal{O}^*) \to H^1(X, \Omega) \to H^1(\mathbf{C}_X^*) \to 0$$

Recall from § XX that $H^1(X,\Omega)$ is one dimensional and as explained in § xx one has

$$H^1(\mathbf{C}_X^*) \simeq \frac{\mathbf{C}}{\mathbf{Z}}$$

We conclude from (x)nthat the image of the ρ-map above is reduced to the additive group of integers. So this image is a sparse subset of the complex vector space $H^1(X,\Omega)$ and there is a family of twisted rings of differential operators indexed by integers which arise via the ρ-image.

The construction of $\mathcal{A}_*$. We shall find the special tweisted ring $\mathcal{A}_*$ which belongs to the ρ-image and corresponds to the integer 1.

Cauchy measures on Riemann surfaces.

Let U be an open and connected subset of X whose boundary ∂U has the following properties: It is a finite union of rectifiable Jordan arcs $\gamma_1, \ldots, \gamma_N$. Each γ-arc is a bijective image of a function from [0,1] into X with bounded variation whose interior part is the image of $(,1)$ is denoted by γ^* and we add the condition that interior arcs are disjoint, i.e. $\gamma_j^* \cap \gamma_k^* = \emptyset$ when $j \neq k$. Next, let $k \geq 2$. and $p \in \partial U$ is a common boundary point of k many γ-curves, say $\gamma_1, \ldots, \gamma_k$. In a small chart centered at p which we can take as a disc where p is the origin and $\gamma_\nu \cap D$ is an arc with one end-point at the origin and the other on the circle ∂D. The complement in D f these k curves consists of k disjoint open sectors $W_1, \ldots, W_k$ which is illustrated by a figure. For each W-sector t holds that $U \cap W_j = W_j$ or that this intersection is empty.

Next, let $\phi \in \mathcal{U}$ and assume that ϕ is locally integrable at every boundary point, i.e. if χ_U is the characteristic function of U we assume that $\chi_U \cdot \phi$ belongs to $L^1(X)$. This L^1-function is considered as a distribution on X and applying the $\bar{\partial}$-operator we get the $(0,1)$-current $\bar{\partial}(\chi_U \cdot \phi)$ which is supported by ∂U. Under these conditions on U and ϕ, the local constructions of boundary values of analytic functions from § XX entail this current has order zero and we can conclude:

Theorem. *With U and ϕ as above there exists a unique Riesz measure μ supported by ∂U such that the following hold for every $g \in C^\infty(X)$:*

$$\int_U \phi \cdot \bar{\partial}\partial\, g = \int g \cdot d\mu$$

Example. Let X be the Riemann surface associated to an equation

$$y^n + q_1(x)y^{n-1} + \ldots + q_n(x) = 0$$

We have the map $\pi \colon X \to \mathbf{P}^1$ where x is the coordinate on the projective line. Above the point at infinity we suppose that there is an unbranched point p_* which means that $\zeta = \frac{1}{x}$ serves as a local coordinare at p_*. Moreover, we suppose that the meromorphic function y has a simple zero at p_* where the Taylor expansion is $y(x) = \frac{a}{x} + \sum_{j=2}^\infty c_j x^{-j}$ with a real and positive. Denote by $\mathcal{F}$ the family of open and connected domains U in X which contain p_* and whose boundary consists of a finite union of Jordan arcs. in addition π restricts to a biholomorphic map from U onto an open domain in $\mathbf{P}^1$ and the meromorphic function y in U has no poles.

Tbe L^1-condition. Suppose that the order of poles of the rational function $q_\nu(x)$ at points $x \in \mathbf{C}$ is $\leq \nu$ for each $1 \leq \nu \leq n-1$. Then we leave it to the reader to check that $y \in L^1(U)$ for each domain in the family $\mathcal{F}$ Hence each $U \in \mathcal{F}$ gives a Riesz- measure μ_U supported by ∂U for which (*) holds in Theorem XX. Put $U_* = \pi(U)$ and when $g \in C^\infty(X)$ we get the funtion g_* in U_* where $g_* = g \circ \pi^{-1}$. Similarly, we get the holomorphic function y_* in U_* where $y_* = y \circ \pi^{-1}$. The complex variable x is written as $t + is$ and with these notations we get

$$\int_U y \cdot \bar{\partial}\partial g = \int_{U_*} y_*(x) \cdot \bar{\partial}_x(g_* \cdot dx)$$

If $\mu_* = \pi_*(\mu)$ is the direct image measure it follows that when $h(x)$ is a C^2-function in U_* with continuous boundary values then

$$\int_{U_*} y_*(x) \cdot \bar{\partial}_x(h \cdot dx) = \int h \cdot d\mu_*$$

The construction above is foremost of interest when U is so large that $\pi(\partial U)$ is a compact subset of the complex x-plane without interior points. Next, (xx) means that the Laplacian of y_* taken in the dsistribution sense is the Riesz measure $4 \cdot \mu_*$ and from § xx we have

$$y_*(x) = \frac{1}{2\pi i} \int \frac{d\mu_*(\zeta)}{x - \zeta}$$

Conjecture Given a local branch for y as above, we ask if $\mathcal{F}$ contains a domain U such that $\mathbf{P}^1 \setminus \pi(U)$ has no interior points and the measure μ_U is real and positive.

Example. Consider the Riemann surface X associated to the equation

$$y^2 = \frac{1}{x(x-1)(x-2)(x-a)}$$

where $a > 2$ is real. Above $x = \infty$ we have the point p where the local series of $y = \frac{1}{x} + c_2 x^{-2} + \ldots$. Put

$$U_* = \mathbf{P}^1 \setminus [0,1] \cup [2,a]$$

Then there exists $U \in \mathcal{F}$ with $\pi(U) = U_*$ and here one finds that the Ruesz easure μ_U is real and positive. The verification is left as an exercise to the reader where material from § X in [Distributions] can be used. On the Riemann surface X we remark that ∂U is the union of two pairwse disjoint and closed Jordan curves.

Some specific calculations.

Consider the equation

$$y^2 + y = \frac{1}{x}$$

Here we can solve out x in y, i.e

$$x = \frac{1}{y+y^2}$$

So the Riemann surface X is $\mathbf{P}^1$. But we have also the 2-sheeted map $\pi\colon X \to \mathbf{P}^1$ where $\mathbf{P}^1$ is the projective x-space. When $x \neq 0$ is stays in the complex plane we write

$$(y+1/2)^2 = \frac{1}{x} + 1/4$$

Hence we get a pair of simple roots except when $x=-4$ where a double root appears and y takes the value $-1/2$. When $x=\infty$ $\zeta = \frac{1}{x}$ is a local coordinate and the equation $(y+1/2)^2 = 1/4+\zeta$ yields a pair of roots functions

$$\alpha_1(\zeta) = \sqrt{1/4+\zeta} - 1/4 \quad : \quad \alpha_2(\zeta) = -\sqrt{1/4+\zeta} - 1/4$$

Let us now remove the line interval $[-4,0]$ from the complex x-plane. Then $U_* = \mathbf{P}^1 \setminus [-4,0]$ is simply connected and homeomorphic to the open unit disc. In U_* there exist single-valued branches of the root functions. In particular we find the holomorphic function $\alpha^*(x)$ in where $\alpha^*(\infty)=0$. When x is close to the origin and stays outside the removed interval we see that

$$|\alpha^*(x)| \simeq |x|^{-\frac{1}{2}}$$

Hence α^* is a locally integrable function close to the origin and regarding α^* as a distribution we get the distribution derivative

$$\frac{\bar\partial}{\bar\partial \bar z}(\alpha^*) \tag{1}$$

which is supported by $[-4,0]$. To find this distribution we consider the behaviour of α^* above and below the open real interval $(-4,0)$. If $0<t<4$ and $\epsilon>0$ is small we have

$$\alpha_1(-t+i\epsilon) = i\cdot\sqrt{\frac{4-t-4i\epsilon t}{4t}} - 1/4 \quad : \quad \alpha_2(-t-i\epsilon) = -i\cdot\sqrt{\frac{4-t+4i\epsilon t}{4t}} - 1/4$$

It follows that

$$\lim_{\epsilon\to 0} \alpha_1(-t+i\epsilon) - \alpha_1(-t-i\epsilon) = 2i\cdot\sqrt{\frac{4-t}{4t}}$$

By the results in § XX this entails that the distribution derivative in (1) is equal to the density on $[-4,0]$ expressed by

$$t \mapsto 2i\cdot\sqrt{\frac{4+t}{-4t}} = \rho(t) \quad : \quad -4<t<0$$

Now the general formula for Cauchy transforms in § xx gives

$$\alpha^*(x) = \frac{1}{2\pi i}\int_{-4}^{0} \frac{1}{x-t}\cdot\rho(t)\,dt \tag{xx}$$

Above the ρ-function is purely imaginary and division with i gives:

$$\alpha^*(x) = \frac{1}{2\pi}\int_{-4}^{0} \frac{1}{x-t}\cdot\sqrt{\frac{4+t}{-4t}}\,dt$$

Hence $\alpha^*(x)$ is the Cauchy transform of a positive measure supported by $[-4,0]$.

Remark. After the variable substitution $u=-t$ we have

$$\frac{1}{2\pi}\cdot\int_{-4}^{0}\sqrt{\frac{4+t}{-4t}}\,dt = \frac{1}{2\pi}\cdot\int_{0}^{4}\sqrt{\frac{4-u}{u}}\,du = \frac{4}{2\pi}\cdot\int_{0}^{1}\sqrt{\frac{1-s}{s}}\,ds = 1$$

where the reader can check the last equality.

The curve $y^3 = \frac{1}{x(x-1)(x-i)}$**.** Let X be the associated Remann surface and π the projection from X onto the projective x-line. Above 0,1 and i the fiber is reduced to a single point and at of these points the meromorphic function y has a simple pole and above $x = \infty$ we encounter three simple zeros. Consider a tree Γ in the complex x-plane which consists of three Jordan arcs $\gamma_0, \gamma_1, \gamma_3$ which have one common boundary pount $p \in \mathbf{C} \setminus \{0, 1, i\}$ and 0,1 and i appear as end-points of this triple of Jordan arcs. See figure XX: Now $U_* = \mathbf{P}^1 \setminus \Gamma$ is simply connected and $\pi^{-1}(U_*)$ consists of three sheets U_1, U_2, U_3. The sheet U_1 is chosen so that the meromorphic function y has the local Taylor series $\frac{1}{x}$ plus higher order terms when $x = \infty$. The L^1–function $y|U_1$ produces a Riesz measure and we take its image under π which gives a Riesz meassure μ supported by Γ such that the chosen single valued branch

$$y^*(x) = \int_\Gamma \frac{d\mu(\zeta)}{x - \zeta}$$

The expansion of the chosen branch at $x = \infty$ entails that the mean-value integral

$$\int_\Gamma d\mu(\zeta) = 1$$

In particular the total variation of μ is always ≥ 1. This total variation is minimized if by a tree Γ for which μ is real and positive. So one is led to a variational problem. In the article [Bergquist-Rullgård] it is proved that there exists a unique Γ-tree such that the associated μ-measure is real and positive. Moreover, the branch point p in Γ where the three γ-curves meet each other is contained in the open triangle with with corner points at 0,1 and i.

Examples and Exercise.

Binomial curves. This is the class of Riemann surfaces defined by an equation of the form

$$y^m = R(x)$$

where $R(x)$ is a rational function and $m \geq 2$ Notice that when $x \in \mathbf{C}$ and R neither has a zero or a pole at x, theny takes m values which differ by roots if the unity. If R has a zero or a pole it is locally like z^q for some integer q and when m and q are prime we get a chart where $y = t^q$ for the chosen local coordinte.

1. Example. Suppose that $R = \frac{P}{Q}$ where P and Q are polynomials with equal degree k and each has simple roots. Now the reader can varify that y has simple zeros at the zeros of P and simple poles at zero of Q. So the degree of the map from X onto the projective x-line is k. Above $x = \infty$ we have k distinct points, i.e. here π is unramified. Hence the sole ramification points on the projective x-line occur at zeros of P or of Q.

Exercise. Show that the meromorphic function x has zeros of order m at the $(a_k, 0)$ and also at $(b_k, 0)$. So the posotove psrt of the principal divisor $D(x)$ has degree $2mq$. An interesting conclusion is that when $x = \infty$ then m points are above where y is a local coordiante and now x should have a plle of order $2q$ at all these points.

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Finally, at ∞ the function x has a pole of order 2 so here dx has a pole of order 3. Starting from this, Abel considered curves defined by

$$u = \frac{R(x)}{Q(x)} \tag{1}$$

where R and Q are polynomials of degree m with no common zero and m is an arbitrary positive integer. The points of intersection between X and such a curve have x-coordinates which satisfy

$$Q^2(x)p(x) - R^2(x) = 0$$

This is a polynomial of degree $2m + 2g + 1$ whose zeros via (1) determines the u-coordinates. So the intersections of the two curves consists of $2m+2g+1$ points $\{\xi_\nu\}$ in X. here eventual multiple points may appear and then we just repeat some of the ξ-points. Let us for brevity write

$$M = 2g + 2m + 1$$

Next, let $q(x)$ be a polynomial where $\deg q \leq g - 1$. Following Abel we set

$$J = \sum_{\nu=1}^{\nu=M} \int_{p_*}^{\xi_\nu} \frac{\phi(x)dx}{y}$$

where p_* is some fixed point on X and curves joining p with the ξ-points are chosen for each ν. By the construction the ξ-points depend upon the chosen pair P, Q. A first remarkable fact from [Abel] is that the sum of integrals in (*) alwaysis zero ! The proof is given in § XX and constitutes a veritable high point in the theory of Riemann surfaces. The vanishing was then used by Abel to prove a fundamental result which for the present hyperelliptic curve goes as follows: Assume for simplicity that the polynomials P and Q are monic of some dgree m. But lower order terms can be arbitrary. Following Abel we consider $2m$ independent variables $(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m1})$ and write

$$R(x) = x^m + \sum_{k=0}^{m-1} a_k x^k \quad : \quad Q(x) = x^m + \sum_{k=0}^{m-1} b_k x^k$$

We consider the case when $2m \geq g + 1$ and let m_* be some integer such that

$$g + 1 \leq m_* \leq 2m$$

Now the ξ-points above depend both on x and

hyperelliptic curves it gives Abel's addition theorem which Abel himself considered as an extension addition formulas for trigonometric functions. Concerning the genus number of a Riemann surface, where Abel of course referred to algebraic functions to interpretate the genus number, he was led count points of intersection between a given curve and whoe families of curves with some arbitrary assigned degree.

Specific examples. To digest general and often rather abstract results one should first consider some specific Riemann surfaces. As a first example we consider the algebraic equation

$$y^2 = x(x-1)$$

When the absolute value $|x| \to +\infty$ we notice that $\frac{y}{x}$ converges to +1 or -1. From this we shall learn that the equation yields a projective and non-singular curve S which is embedded in $\mathbf{P}^2$ and gives the associated compact Riemann surface. On S the meromorphic function y has two simple poles and simple zeros at the points $(0,0)$ and $(1,0)$. The meromorphic function x has also two simple poles and a zero of multiplicity two at $(0,0)$. We shall later on learn that the assertions above are immediate consequences of the construction of charts in S. Next, consider the meromorphic function $\phi = \frac{y}{x}$. From the above it has a sole simple pole at $(0,0)$ and also a simple zero at $(1,0)$. The conclusion is that S is isomorphic to the projective ϕ-line and every meromorphic function on S is a rational function of ϕ. Of course, this can be seen directly since

$$\phi^2 = \frac{y^2}{x^2} = \frac{x(x-1)}{x^2} = \frac{(x-1)}{x} \implies x = \frac{1}{1-\phi^2}$$

And the reader can also confirm that $y = \frac{\phi}{1-\phi^2}$. But the example is instructive since it was not obvious from the start that the quotient $\frac{y}{x}$ can be used to identify S with a projective line. A more involved case appears in § 0.X where we take the curve

$$y^4 = x(x-1)^3$$

It turns out that it also is a projective line where $\phi = \frac{y}{x-1}$ is used. Riemann surfaces with a more complicated topology arise via algebraic equations of the form

$$\text{(i)} \qquad y^m = q(x)$$

where $m \geq 2$ and $q(x)$ is a polynomial with simple zeros of some degree $n \geq 3$ and $y^m - q(x)$ is irreducible in the unique factorization domain $\mathbf{C}[x,y]$ The genus of the associated Riemann surface is calculated via the Hurwitz-Riemann formula which is presented in § XX. If m and n are relatively prime the genus is given by:

$$1 + \frac{m(n-1)}{2} - \frac{n+1}{2} = \frac{(m-1)(n-1)}{2}$$

Notice that the last term is an integer since the hypothesis means that not both n and m are even integers. The formula shows that the genus g increases quite rapidly. For example, if $m = 4$ and $n = 5$ we find that $g = 6$ which means that the Riemann surface is homeomorphic with a sphere with 6 handles attached. The special case in (i) above with $m = 2$ and $n = 3$ lead to elliptic Riemann surfaces which will be treated in § XX.

0.1 Abelian differentials. The notions which appear below will be explained in detail later on. Let $P(x,y) = 0$ be an irreducible algebraic equation which defines a Riemann surface X of some genus $g \geq 1$ and m is the degree of P. Consider also a globally defined holomorphic 1-form ω on X and fix some point $p_* \in X$. Given X and ω one constructs functions which in general depend upon several complex variables. Following Abel the procedure is as follows. Let $N \geq 2$ be a positive integer and $x, y, a_1, \ldots, a_k$ are $k+2$ independent complex variables. Consider a polynomial of the form

$$Q(x, y; a_\bullet) = q_N(x,y) + \sum g_{k,j}(a_1, \ldots a_m) \cdot x^j y^k$$

where the last sum extends over pairs $j.k$ with $j + k \leq N - 1$ while $q_N(x,y)$ is a polynomial of degree N and the doubly-indexed g-functions belong to the polynomial ring $\mathbf{C}[a_1, \ldots, a_k]$. For each $a_\bullet \in \mathbf{C}^k$ the intersection between X and $Q(x, y; a_\bullet)$ consists of mN points on X denoted by $\{\xi_j(a_\bullet) : 1 \leq j \leq mN\}$. Here eventual multiple points are repeated according to their multiplicities. Notice that this mN-tuple of points on X is unordered. Continuity gives for each point $a_\bullet$

in $\mathbf{C}^k$ an unordered mN-tuple of arcs $\{\gamma_j(a_\bullet)\}$ where each $\gamma_j(a_\bullet)$ joins p_* with $\xi_j(a_\bullet)$ and in this way we get a function defined on $\mathbf{C}^k$ by:

$$J(a_\bullet) = \sum_{j=1}^{j=nM} \int_{\gamma_j(a_\bullet)} \omega \tag{1}$$

Abel's first result is that J is a constant function in $\mathbf{C}^k$. More generally we take a meromorphic 1-form ω on X. Then a major result in [loc.cit] is that the function

$$J(a_\bullet) = r(a_\bullet) + \log p(a_\bullet)$$

where r is a rational function of $a_\bullet$ and p a polynomial. This means that the J-function in general is multivalued and must be restricted to the complement of an algebraic hypersurface in $\mathbf{C}^k$ which is natrual since one must avoid poles of ω when we take intersections between X and the varying Q-curves parametrized by $a_\bullet$. From the results above Abel deduced an addition theorem which we describe locally as follows: Given a point $p \in X$ we stay in a small chart Δ around x which can be identified with the open unit disc in a complex plane. Let μ be an arbitrary integer which is strictly greater than the genus g. Let ω be a holomorphic 1-form on x. For each μ-tuple $x_1, \ldots, x_\mu$ of points in Δ we choose simple curves $\{\gamma_j\}$ which stay in Δ with end-points at p_* and $\{x_j\}$. Then we get a complex-valued function defined in the μ-fold product of Δ by

$$J(x_1, \ldots, x_\mu) = \sum_{j=1}^{j=\mu} \int_{\gamma_j} \omega$$

Let z be the local coordinte in Δ which identifies Δ with the open unit disc $D = \{|z| < 1\}$ and here p corresponds to the origin. $x_\mu = p + z_\nu$ where $\{x_\mu\}$ is a μ-tuple of points in D so the J-function above is a function in the polydisc D^μ. It is special because we have taken a sum so that J is invariant under permutations of $z_1, \ldots, z_\mu$, i.e. the J-function is symmetric. Abel's addition theorem expressed locally asserts that in the situation above there exists a g-tuple of analytic functions $\{\phi_k(z_1, \ldots, z_\mu)\}$ such that

$$J(x_1, \ldots, x_\mu) = \sum_{j=1}^{j=g} \int_0^{\phi_k(z_\bullet)} \omega$$

in other words, the symmetric J-function which a priori depends upon μ many independent variables can be expressed by a sum of g integrals which means that if $\{w_k = \phi_k(z_\bullet)\}$ then J only depends on g many independent variables. Abel expressed this by saying that the J-function above has a degree of trancendency given by the genus of X. Let us remark that the local equation above extends when the J-function is constructed on the symmetric product of μ copies of X and where the ϕ-functions extend to rational functions on X^μ. A precise description of this global result requires conceots from analytic function theory in several variables and in this chapter we shall be content to expose consequences of Abel's addition theorem in specific cases.

About Abel's proof. The results above constitute in the present author's opinion the high point for the material in this chapter. As expected the proofs require a considerable amount of preparations. Abel employed differentials in a similar fashion as Lagrange when he derived the equations of motion in an arbitrary particle system with a finite degree of freedom which obeys Newton's Law of Forces while external time dependent outer forces can be arbitrary. Recall that the d'Alembert-Lagrange equations are valid in non-conservative systems and can therefore be applied in many applied situations. The equations constitute a cornerstone in optimization theory and are for example used in optimal control theory as long as the solutions are not of Bang-bang character. A modified version of Lagrange's proof was given by Pontryagin in 1949 adpated to problems where control variables are restricted to bounded sets. It is well documented that Abel as a young student studied mechanics. So it is no surprise that he got much inspiration from from Lagrange's text-book in mechanics from 1780. Let us also recall that one of Abel's first publications was devoted to mechanics where an inversion formula was established which makes it possible to recover a potential field in an oscillating system after experiments have determined the periods as a function of varying amplitudes. Concerning Abel's results above we mention that extensions to higher dimensions were achieved by Nils Nilsson in the impressive article [Nils] from 1965 which has led to the notion of Nilsson class functions in several complex variables.

Jacobi's residue. An elegatn analytic formua to obtain intersection numbners was given by Jacobi based upon residue calculus. Let P and Q be a pair of irreducible polynomials and supose that their projective curves do not intersect at infinity. If (x_*, y_*) is a point in the intersection of $\{P = 0\}$ and $\{Q = 0\}$ a local intersection number is defined by

$$\mathbf{i}(P; Q; (x_0, y_0)) = -\frac{1}{4\pi^2} \cdot \lim_{(\epsilon,\delta)\to(0,0)} \iint \frac{\partial P}{P} \wedge \frac{\partial Q}{Q}$$

where the integration takes place over the 2-dimensional surface defined by

$$\{|P| = \epsilon\} \cap \{|Q| = \delta\}$$

Following Jacobi this yields an integer which deends upon the ordering of P and Q and is called the local intersection number of the two curves at (x_0, y_0). It turns out that this construction gives intersection numbers which enjoy continuity. Let us first compute Jacobi's number for the pair $P = x^2 - y$ and $Q = y^2 - x^3$. Using Stokes Theorem as explained in § xx the limit in (*) can be performed while ϵ tends much faster than δ to zero. During such a limit we set $y = x^2 + \epsilon \cdot e^{i\theta}$ to get points on $|P| = \epsilon$. Jacobis integral becomes

$$-\frac{1}{4\pi^2} \cdot \int_0^{2\pi} [\int_{|Q(x,x^2+\epsilon\cdot e^{i\theta}|=\delta} \frac{\partial Q}{Q}] \cdot i \cdot d\theta$$

We notice that

$$\partial P \wedge \partial Q = 2xdx \wedge 2ydy - dy \wedge -3x^2dx = (4xy - 3x^2) \cdot dx \wedge dy$$

Thus the inner integral is taken over the curve in the x-plane defined by

$$|(x^2 + \epsilon e^{i\theta})^2 - x^3| = \delta$$

and the limit when $\epsilon << \delta$ becomes

$$-\frac{1}{4\pi^2} \cdot 2\pi i \cdot \int_{|x^3-x^4|=\delta} \frac{dx^4 - dx^3}{x^3 - x^4}$$

Notice that the curve $|x^3 - x^4| = \delta$ is approximately the circle when $\delta$is small. Moreover

$$\frac{dx^4 - dx^3}{x^3 - x^4} = \frac{(4x - 3)dx}{x(1 - x)}$$

By Cauchy's residue formula this nmber is equal to $2\pi i\dot{3}$ and hence Jacobi's limit becomes

$$-\frac{1}{4\pi^2} \cdot 2\pi i \cdot 2\pi i \cdot 3 = 3$$

Thus, we have found the previous intersection number.

A general case.

To P we have associated a Riemann surface X where x and y become meromorphic functions. Hence $Q(x, y)$, being a polynomial of the two meromorphic functions x and y on X, belongs to $\mathfrak{M}(S)$. This meromorphic function has some degree N whose zeros correspond to points in the intersection between S and the curve $\{Q = 0\}$ in the projective space $\mathbf{P}^2$.

Proposition. *The degree of the meromorphic Q-function on S is equal to the product* $\deg P \cdot \deg Q$ *of the given polynomials.*

How can we prove this result ? A first observation is that the assertion is invariant under a linear change of coordinates in the (x, y)-space which reduces the proof to the case when $P(x, y)$ is of the form

$$y^n + p_1(x)y^{n-1} + \ldots + p_n(x)$$

where $\deg p_k \leq k$ hold for the q-polynomials. Let us assume that we do not encounter singular points above $x = \infty$. A s explained in § XX it means that the leading n-homogeneous polynomial in (1) is of the form $\prod (y - \alpha_k x)$ where $\alpha_1, \ldots, \alpha_n$ are distinct complex numbers. The degree of the meromorphic function Q can be found by regarding its poles on X which only can occur at points above $x = \infty$. Supoose that we also have:

$$Q(x, y) = y^m + q_1(x) y^{n-1} + \ldots + q_m(x)$$

where $\deg q_k \leq k$ for $1 \leq k \leq m$. On X we have the ponits

$$p_k = (0, 1, \alpha_k) \quad : \quad 1 \leq k \leq n$$

At each of these points the meromorphic functions y and x have simple poles. Now we can find a condition in order that the meromorphic function Q has a pole of order m at each p_k. Namely, the leading m-homogeneous polynomial Q^* of Q is of the form

$$Q^*(x, y) = \prod (y - \beta_\nu \cdot x)$$

If each $\beta - \nu$ differes from the α-values it is clear that Q has a pole of order m at every p_k and conclude that the degree of the meromorphic function Q is equal to nm which confirms Proposition 1.6.1. There reamins to decribe the zeros of Q. In contrast to the poles there may occur multiple zeros, i.e. the set-theoretic intersection between the projective curves $\{P = 0\}$ and $\{Q = 0\}$ may consist of fewer than mn many points. So the intersection number mn takes into the account multiple points when S and $\{Q = 0\}$ intersect at points (x, y) in $\mathbf{C}^2$.

To P one associates the gradient vector

$$\nabla(P) = (P'_x, P'_y)$$

A point $s = (x_*, y_*) \in S$ is called regular if $\nabla(P)(s) \neq 0$. If $P'_y(s) \neq 0$ there exists a local factorisation:

$$P(x, y) = (y - a(x)) \cdot Q(y, x)$$

where $a(x)$ is a holomorphic function of x defined in some disc centered at x_* and $a(x_*) = y_*$. This means geometrically that s is surrounded by a chart defined by a copy of a disc $\{|x - x_*| < \delta$ where the points on S are $(x, a(x)$ and gives a chart on S where one says that $x - x_*$ gives a local coordinate in the chart. Regarding S as a subset of $\mathbf{P}^2$ this means that S appears as locally a 1-dimensional complex submanifold in an open neighbrhood of $(1, x_*, y_*)$. If we instead have $P'_y(x_*, y_*)$ we find in a similar way that y is a local coordinate. This time one has a local factorisation

$$P(x, y) = (x - b(y)) \cdot R(x, y)$$

where $b(y)$ is holomorphic in disc centered at y_*. Put

$$\text{reg}(S) = \{(x, y) \in S : \nabla(P)(x, y) \neq (0, 0)\}$$

A first fact is:

1.4.2 Proposition. *reg*(S) *is a connected and dense open subset of* $\bar{S}$.

The density is clear while the connectivity is not so obvious from the start. But it follows from the hypothesis that the polynomial P is irreducible. See § xx below for details. It follows that reg(S) is a connected complex manifold and the question arises if the complex analytic structure can be extended to $\bar{S}$. This turns out to be "almost true". More precisely one has

1.4.3 Theorem. *There exists a unique compact Riemann surface X where reg(S) can be identfied with a dense open subset of X and we have a surjective map $\rho\colon X \to \bar{S}$ where the inverse fiber of each $p \in S \setminus$ reg(S) is a finite set.*

This result is known as the normalisation theorem for projective curves. It appears that no precise source of this result is known, but was most likely understood by Abel since his work [Abel] contains the requested constructions of complex analytic structures for elliptic curves. The subsequent proof shows that the ρ-map often is bijective which means that the topological spaces X and $\bar{S}$ are the same.

Proof of Theorem 1.4.3.

Let $s_* = (x_*, y_*) \in S$ where $\nabla(P)(s_*) = 0$. With $z = x - x_*$ and $\eta = y - y_*$ we set

$$p(z, \eta) = P(x_* + z, y_* + \eta)$$

It follows that

$$p(z, \eta) = \eta^n + r_1(z)\eta^{n-1} + \ldots + r_{n-1}(z)\eta + r_n(z)$$

where $\{r_j(z)\}$ are polynomials in z. Since $p(0,0) = P(x_*, y_*) = 0$ we have $r_n(0) = 0$ and we notice that $P'_y(x_*, y_*) = 0$ also gives $r_{n-1}(0) = 0$. Hence there exists an integer $2 \leq k \leq n$ such that

$$p(0, \eta) = \eta^k \cdot p_*(\eta) \quad : \quad p_*(0) \neq 0$$

As explained in § xx where we even treated η-polynomials with coefficients in $\mathbf{C}\{z\}$ there exists a factoriasation

$$p_*(z, \eta) = \big(\eta^k + \rho_1(z)\eta^{k-1} + \ldots + \rho_k(z)\big)Q(z, \eta)$$

where $Q(0,0) \neq 0$ and $\rho_j(0) = 0$ for every $1 \leq j \leq k$. Set

$$w(y, z) = \eta^k + \rho_1(z)\eta^{k-1} + \ldots + \rho_k(z)$$

By the above $\rho_\nu(0) = 0$ for every ν which means that w is a Weierstrass polynomial. Zeros of w close to the origin correspond to points on s close to s_*. From § xx it follows that there exists some $\delta > 0$ such that if $0 < |z| < \delta$ then the roots of the η-polynomial above are simple and distinct which by §§ means that w has no multiple factors and is therefore a product

$$w(z, y) = w_1(z, y) \cdots w_r(z, y)$$

where $\{w_j\}$ are irreducible Weierstrass polynomials. Next, the local construction in § xx shows that every irreducible factor w_j produces an analytic disc. More precisely, for a fixed $1 \leq j \leq r$ we let e denote the η-degree of w_j. When $e \geq 2$ the material from §§ gives an analytic function $A(\zeta)$ defined of a new complex variable ζ in a disc $\{|\zeta| < \delta\}$ such that

$$w_j(\zeta^e, A(\zeta)) = 0 \quad : \quad 0 \leq \nu \leq e - 1$$

Moreover, in the Taylor expansion of A there does not exist an integer $e_* \geq 2$ which is a factor of e such that $A(\zeta) = A_1(\zeta^{e_*})$ for some analytic function A_1. Thus implies that the map from the punctured disc $\dot{D} = \{0 < |\zeta| < \delta\}$ which sends ζ to $(\zeta^e, A(\zeta)$ is injective for a sufficiently small δ and the image is a relatively open set in $\text{reg}(S)$. The origin $\zeta = 0$ is mapped to s_* and the local geometric picture is as follows: There exists an open neighborhood U of s_* in S such that

$$U \setminus \{s_*\} = U_1 \cup \ldots, \cup U_r$$

and to each U_j we have a biholomorphic mapping from a punctured ζ-disc which sends $\zeta \neq 0$ to $(\zeta^{e_j}, A_j(\zeta)$ where e_j is the degree of the j:th Weierstrass polynomial w_j in the factorisation of w above. The sets $\{U_j\}$ are disjoint open subsets $\text{reg}(S)$ and have s_* as a common boundary point.

Remark. A favourable case occurs when w from the start is irreducible. Then we get an open neighborhood U of s_* where the map

$$\zeta \to (\zeta^e, A(\zeta))$$

yields a bijective mapping between the open ζ-disc and U. Moreover, the map is holomorphic in the sense that if for every rational function

$$k(x, y) = q_0(x) + q_1(x)y + \ldots + q_{n-1}(x)y^{n-1}$$

where $\{q_j(x)\}$ belong to $\mathbf{C}[x]$, it follows that $k \circ \rho = k^*(\zeta)$ are meromorphic functions in the ζ-disc. So when the local Weierstrass polynomials w are irreducible for all $s_* \in S \setminus \text{reg}(S)$ we have prodiced local charts at all points in S which give a an atlas of a complex manifold X_0 which as a point set coincides with S.

If w has several irreducible factors we get r many charts where the origin in each chart has to be speatted from the others. So here one is obliged to consider r disjoint ζ-discs $D_1, \ldots, D_r$ where the punctured discs are mapped into disjoint sets $U_1, \ldots, U_r$ as above while the origin in each D_j

is mapped into s_*. In the Riemann surface X_0 we therefore find r distinct points above s_*, i.e. the map $X_0 \to S$ is no longer bijective. But it is bijective outside the singular points and every rational function $k(x,y)$ has a pull-back to a meromorphic function on X_0.

Passage to infinity. Above we studied α-roots while x stays in $\mathbf{C}$. Passing to ∞ we replace x by $\zeta = \frac{1}{x}$ and corresponding α-roots are solutions to

$$P^*(\zeta, y) = y^n + q_{n-1}(\zeta^{-1})y^{n-1} + \ldots + q_0(\zeta^{-1})$$

Again the local uniformisation theorem is used to construct a local t-coordinates if multiple roots occur and in this way a finite set of points is added to X_0 which yields a compact Riemann surface X where we have a surjective map $\rho\colon X \to \bar{S}$ which is only fails to be bijective at a finite (possibly empty) set of points.

1.5 Examples.

Consider the curve S defined by the equation $y^2 = x^2 + 1$. With $P = y^2 - x^2 - 1$ we have $\nabla(P)(x,y) = (-2x, 2y) \neq 0$ on S. The projective closure contains two points $p_1 = (0,1,1)$ and $p_2 = (0,1,-1)$. Close to p_1 we have local coordinates (ζ, η) where

$$(\zeta, 1, 1+\eta) \simeq (1, \frac{1}{\zeta}, \frac{1+\eta}{\zeta}) \quad : \quad \zeta \neq 0$$

The equation for $\bar{S}$ close to p_1 becomes

$$\frac{(1+\eta)^2}{\zeta^2} = \frac{1}{\zeta^2} + 1 \implies$$

$$2\eta + \eta^2 = -\zeta^2$$

Hence ζ yields a local coordinate and p_1 is a non-singular point. In the same way one verifies that p_2 is a regular point. Hence $\bar{S}$ is a smooth projective curve and can therefore be identified with a compact Riemann surface.

The curve $y^4 = x(x-1)^3$. At $(0,0)$ we can take y as a local coordinate on S and then x has a zero of order 4 at this point. At the point $p = (1,0)$ S is locally a cusp-like curve where a local coordinate t on the associated Riemann surface X is found such that:

$$x = 1 + t^4 \quad \text{and} \quad y = (1+t^4)^{\frac{1}{3}} \cdot t^3$$

So at p the meromorphic function y has a zero of order three while it has a simple zero at $(0,0)$ since here y itself serves as a local coordinate. Next, as $|x| \to +\infty$ we notice that

$$y^4 = x^4(1 - x^{-1})3 \simeq x^4$$

Gometrically it means that the curve S regarded as a subset of the 2-.dimensional complex (x,y)-space has four asymptotic lines of the form

$$\ell_\nu = \{y = e^{2\pi i\nu/4}x : \, 0 \leq \nu \leq 3\}$$

From this it follows that above $x = \infty$ there occur four points in X and at each of these the meromorphic functions x and y both have simple poles.

From the above the reader can recognize that the degree of the meromorphic function y is four. Similarly, x has a multiple zero of order four at $p = (0,0)$ and four simple poles. So it has also degree four. Next, consider the meromorphic function

$$\phi = \frac{y}{x-1}$$

Then ϕ has a simple zero at $(0,0)$ and at $(1,0)$ we recall that $x - 1 = t^4$ while y has a zero of order 3 and hence ϕ has a simple pole here. At the boundary points at infinity we see that ϕ has no poles and is $\neq 0$. The conclusion is that the map defined by ϕ from X onto $\mathbf{P}^1$ is *biholomorphic* which means that $X \simeq \mathbf{P}^1$ and the associated algebraic function field K is therefore isomorphic to the standard field of rational functions in one variable. The theory therefore predicts that

$$K = \mathbf{C}(\phi)$$

which means that both x and y can be expressed as rational functions of ϕ. Of course, this can be seen directly, i.e. the reader may verify that

$$x = \frac{\phi^4}{\phi^4 - 1} \quad : \quad y = \frac{2\phi^4 - \phi^5}{\phi^4 - 1}$$

Remark. The example is instructive since it teaches how to find the ϕ-function in a systematic way since a mere guessing starting from the algebraic equation (*) is not obvious.