Fixed point theorems.

A compact topological space S has the fixed point property if every continuous map $T\colon S \to S$ has at least one fixed point. An example is the closed unit ball in $\mathbf{R}^n$ whose fixed-point property is proved in § 5.xx. More generally, consider a locally convex vector space X whose dual space is denoted by X^*. Now one equips X with the weak topology whose open sets are generated by pairs $x^* \in X^*$ and positive numbers δ) of the form:

$$B_\delta(x^*) = \{x \in X : |x^*(x)| \leq \delta\}$$

Denote by $\mathcal{K}(X)$ the family of convex subsets of X which are compact with respect to the X^*-topology.

The Schauder-Tychonoff fixed point theorem. *Each K in $\mathcal{K}(X)$ has the fixed point property.*

The merit of this result is of course that one allows non-linear maps. The next result is due to Kakutani and goes as follows: By a group of linear transformations $\mathbf{G}$ on a real vector space X we mean a family of bijective linear maps $g\colon X \to X$ such that composed maps $g_2 \circ g_1$ again belong to the group as well as the inverse of every g.

Kakutani's theorem. *Let $K \in \mathcal{K}(X)$ be $\mathbf{G}$-invariant, i.e. $g(K) \subset K$ hold for every $g \in \mathbf{G}$. Assume in addition that the family of the restricted $\mathbf{G}$-maps to K is equicontinuous. Then there exists at least some vector $k \in K$ such that $g(k) = k$ for every $g \in \mathbf{G}$.*

Remark. The equicontinuous assumption means that to each pair every (x^*, ϵ) with $x^* \in X^*$ and $\epsilon > 0$, there exists a finite family $x_1^*, \ldots, x_M^*$ and some $\delta > 0$ such that the following hold: If p and q is a pair of points in K such that $p - q$ belongs to $\cap B_\delta(x_\nu^*)$, then

$$g(p) - g(q) \in B_\epsilon(x^*)$$

hold for all $g \in \mathbf{G}$.

Haar measures. Let G be a compact topological group which means that the group is equipped with a Hausdorff topology where the group operations are continuous, i.e the map from $G \times G$ into G which sends a pair of group elements g, h to the product gh is continuous, and the inverse map $g \mapsto g^{-1}$ is bi-continuous. Now there exists the Banach apace $C^0(G)$ of continuous real-valued functions on G. Recall from basic measure theory that the dual space consists of Riesz measures. Denote by $P(G)$ the family of non-negative measures with total mass one, i.e. probability measures on G. If $\phi \in C^0(G)$ and $g \in G$ we get the new continuous function $S_g(\phi)$ defined by

$$S_g(\phi)(h) = \phi(gh) \quad : h \in G$$

Next, if $\mu \in P(G)$ we get the new probability measure $T_g(\mu)$ given by the linear functional

$$\phi \mapsto \int_G S_g(\phi)\, d\mu$$

In this way G is identified with a group of transformations on $P(G)$. Next, $P(G)$ is equipped with the weak-star topology where open neighborhoods of a given $\mu \in P(G)$ consists of finite intersections of sets $\{\gamma \in P(G) : |\gamma(\phi) - \mu(\phi)| < \delta\}$ for pairs $\delta > 0$ and $\phi \in C^0(G)$. The uniform continuity of every $\phi \in C^0(G)$ entails that that the group action on $P(G)$ is equi-continuous on $P(G)$ with respect to the weak-star topology. As explained in § xx, Kakutani's theorem also applies in this situation which yields a fixed point. Hence there is a probability measure μ such that

$$(*) \qquad \int_G \phi(gh)\, d\mu(h) = \int_G \phi(h)\, d\mu(h)$$

hold for every pair $g \in G$ and $\phi \in C^0(G)$. In § xx we show that μ is uniquely determined by (*), i.e. only one probability measure enjoys the invariance above. Moreover, starting with the operators

$$S_g^*(\phi)(h) = \phi(hg) \quad : h \in G$$

one finds a probability measure μ^* such that

$$(**) \qquad \int_G \phi(hg)\, d\mu(h) = \int_G \phi(h)\, d\mu(h)$$

hold for every pair $g \in G$ and $\phi \in C^0(G)$. In § xx we prove that $\mu = \mu^*$ which means that the unique Haar measure is both left and right invariant.

§ 1. Convex sets and their ρ-functions.

Let E be a real vector space. A convex set U which contains the origin is said to be absorbing if there for each vector $x \in E$ exists some real $s > 0$ such that $s \cdot x \in U$. It may occur that the whole line $\mathbf{R}x$ is contained in U, and then we say that x is fully absorbed by U. The convexity of U entails that the set of fully absorbed vectors is a linear subspace of E which we denote by $\mathcal{L}_U$.

1.1 The function ρ_U. Let x be a non-absorbed vector x. Then there exists a positive real number

$$\mu(x) = \max\{s : sx \in U\}$$

If x is absorbed we put $\mu(x) = +\infty$ and for every non-zero vector x we set

$$\rho_U(x) = \frac{1}{\mu(x)}$$

it is clear that if $x \in U$ then $\mu(x) \geq 1$ and hence $\rho_U(x) \leq 1$. Notice that we also have

$$\rho_U(x) = \min\{s : \, x \in s^{-1}U\}$$

1.2 Exercise. Show that the convexity of U entails that ρ_U satisfies the triangle inequlity

$$(1.2.1) \qquad \rho_U(x_1 + x_2) \leq \rho_U(x_1) + \rho_U(x_2)$$

for all pairs of vectors in E. Moreover, check also that $\rho_U(x) = 0$ if and only if x belongs to $\mathcal{L}_U$ and that ρ_U is positively homogeneous, i.e. the equality below holds when a is real and positive:

$$(1.2.2) \qquad \rho_U(ax) = a\rho_U(x) \quad : a > 0$$

1.3 The Hahn-Banach theorem. Keeping U fixed we set $\rho(x) = \rho_U(x)$. An $\mathbf{R}$-linear map λ from E to the 1-dimensional real line is majorised by ρ if

$$(1.3.1) \qquad \lambda(x) \leq \rho(x)$$

hold for every vector x. More generally, let E_0 be a subspace of E and $\lambda_0 \colon E_0 \to \mathbf{R}$ a linear map such that (1.3.1) hold for vectors in E_0. Then there exists a linear map $\lambda \colon E \to \mathbf{R}$ which extends λ_0 and is again majorised by ρ. To prove this we use Zorn's Lemma which gives a maximal extension, i.e. we find a subspace E_* which contains E_0 and a linear map $\lambda_* \colon E_* \to \mathbf{R}$ which is majorised by ρ and extends λ_0. If $E_* \neq E$ we pick a non-zero vector $y \in E \setminus E_*$ and consider the linear space $E_{**} = E_* + \mathbf{R}y$. For each pair of vectors x and ξ in E_* the triangle inequality from (1.2.1) gives

$$\rho(\xi + x) \leq \rho(y + x) + \rho(\xi - y)$$

Next, put

$$\alpha = \min_{x \in E} \rho(x + y) - \lambda(x) \quad : \beta = \max_{\xi \in E} \lambda(\xi) - \rho(\xi - y)$$

It follows that when x and ξ are vectors in E, then

$$\text{(i)} \qquad \alpha - \beta \geq \rho(x + y) - \lambda(x) - \lambda(\xi) + \rho(\xi - y) = \rho(x + y) + \rho(\xi - y) - \lambda(x + \xi)$$

Next, the triangle inequality applied to the vectors $x - y$ and $\xi + y$ gives

$$\text{(ii)} \qquad \rho(x + \xi) = \rho((x + y) + (\xi - y)) \leq \rho(x + y) + \rho(\xi - y)$$

Moreover, since ρ majorises λ on E_* we have

$$\text{(iii)} \qquad \lambda(x + \xi) \leq \rho(x + \xi)$$

it follows from (i-iii) that

(iv) $$\alpha \geq \beta$$

Let us then take a real number a such that $\beta \leq a \leq \alpha$ and extend λ to E_{**} by

$$\lambda(x + sy) = \lambda(x) + sa$$

where $x \in E_*$ and s are real numbers. if $s > 0$ we have

$$\lambda(x + sy) = s\lambda(s^{-1}x + y) = s((\lambda(s^{-1}x + a) \leq s((\lambda(s^{-1}x + \alpha) \leq s\rho(s^{-1}x + y)$$

where the last inequality follos from the defintion of α in (x) applied to the vector $s^{-1}x \in E_*$. Finally, since ρ is positively homogeneous the last term is equal to $\rho(x + sy)$. Hence we have proved the inequality

$$\lambda(x + sy) \leq \rho(x + sy)$$

when $s > 0$. Next, since $a \geq \beta$, the reader can check by a similar method as above that

$$\lambda(x + sy) \leq \rho(x + sy)$$

hold when $s < 0$. This proves that the choice of a above gives an extension of λ to E_{**} which again is majorised by ρ and via Zorn's Lemma we get the Hahn-Banach theorem.

2 Locally convex topologies.

Denote by $\mathcal{C}_E$ the family of convex sets U as in § 1. Let $\mathfrak{U} = \{U_\alpha\}$ be a family in $\mathcal{C}_E$ such that

$$\bigcap \mathcal{L}_{U_\alpha} = \{0\}$$

i.e. the intersction is reduced to the origin. Now there exists a topology on E where a basic for open neighborhoods of the origin consists of sets:

(1) $$\cap \{\rho_{U_{\alpha_i}}(x) < \epsilon\}$$

where $\epsilon > 0$ and $\{\alpha_1, \ldots, \alpha_k\}$ is a finite set of indices defining the U-family. If x_0 is a vector in XS, then a basis for its open neihghborhoods are given by sets of the for $x_0 + U$ where U is a set from (1). In general, a subset Ω in E is open if there to eah $x_0 \in \Omega$ exists some U from (1) such that $x_0 + U \subset \Omega$. it is clear that this gives a topology and (1) entaills that it is separated, i.e. a Huasdorff topology on E. Notice also that eqch set in 82) is convex. One therefore refers to a locally con vex topology on E.

2.1 Remark. The locally convex topology above depends upon the family $\mathfrak{U}$-toplogy. Its topology is not changed if we enlarge the family to consist of all finite intersection of its convex subsets. When this has been done we notice that if $U_1, \ldots, U_n$ is a finite family in $\mathfrak{U}$ then the norm defined by $U = U_1| \cap \ldots \cap U_n$ is stronger than the individual ρ_{U_i}-norms. Hence a fundamental system of neighborhoods consists of single ρ-balls:

$$\Omega = \{\rho_U < \epsilon\} \quad : U \in \mathfrak{U}$$

2.2 The dual space E^*. Let E be equipped with a locally convex $\mathfrak{U}$-topology. As above $\mathfrak{U}$ has been enlarged so that the balls above give a basis for neighborhoods of the origin. A linear functional ϕ on E is $\mathfrak{U}$-continuous if there exists some $U \in \mathfrak{U}$ and a constant C such that

$$|\phi(x)|| \leq C \cdot \rho_U(x)$$

2.3 Closed half-spaces. To each pair $\phi \in E^*$ and a real number a one assigns the closed half-space

$$H = \{x \in X : \phi(x) \leq a\}$$

Notice that $a < 0$ can occur in which case H does not contain the origin.

2.4 The separation theorem. *Each closed convex set K in E is the intersection of closed half-spaces.*

2.5 Exercise. Show (2.4) using the Hahn-Banach theorem.

Next, let K_1 and K_2 be a pair of closed and disjoint convex sets. Then they can be spearated by a hyperplane. More precisley, there exists some $\phi \in E^*$ and a positive number δ such that

$$\max_{x \in K_1} \phi(x) + \delta \leq \min_{x \in K_2} \phi(x) \tag{2.6}$$

Again we leave the proof as an exercise to the reader.

3. Support functions of convex sets.

Let E be a locally convex space as above. Vectors in E are denoted by x, while y denote vectors in E^*. To each closed and convex subset K of E we define a function $\mathcal{H}_K$ on the dual E^* by:

$$\mathcal{H}_K(y) = \sup_{x \in K} y(x)$$

Notice that $\mathcal{H}_K$ take values in $(-\infty, +\infty]$, i.e. it may be $+\infty$ for some vectors $y \in E^*$. For example, let $K = \{\mathbf{R}^+ x_0$ be a half-line. Then $\mathcal{H}_K(y) = +\infty$ when $y(x_0) > 0$ and otherwise zero. So here the range consisists of 0 and $+\infty$. It is clear that

$$\mathcal{H}_K(sy) = s\mathcal{H}_K(y)$$

hold when s is a positive real number, i.e $\mathcal{H}_K$ is positively homogeneous.

3.1 Exercise. Show that the convexity of K entails that

$$\mathcal{H}_K(y_1 + y_2) \leq \mathcal{H}_K(y_1) + \mathcal{H}_K(y_2)$$

for each pair of vectors in E^*.

3.2 Upper semi-continuity. For each fixed vector $x \in E$ the function

$$y \mapsto y(x)$$

is weak-star continuous on E^*. Since the supremum function attached to an arbitrary family of weak-star continuoes functions is upper semi-continuous, it follows that $\mathcal{H}_K$ is upper semi-contiuous.

3.4 Exercise. Let K and K_1 be a pair of closed convex sets such that $\mathcal{H}_K = \mathcal{H}_{K_1}$. Show that this entails that $K = K_1$. The hint is to use the separation theorem.

3.5 The class $\mathcal{S}(E)$. It consids of all all upper semi-continuous functions G on E^* with values in $(-\infty, +\infty]$ which satisfy (x) and (xx). The next result was proved by Hörmander in the article *Sur la fonction d'appui des ensembles convexes dans un espaces localementt convexe* [Arkiv för mat. Vol 3: 1954].

3.6 Theorem. *Each $G \in \mathcal{S}(E)$ is of the form $\mathcal{H}_K$ for a unique closed convex subset K in E.*

Remark. As pointed out by Hörmander in [ibid] this result is closely related to earlier studies by Fenchel in the article *On conjugate convex functions* Canadian Journ. of math. Vol 1 p. 73-77) where Legendre transforms are studied in infinite dimensional topological vector spaces. The novely in Theorem 3.3 is the generality and we remark that various separation theorems in text-books dealing with notions of convexity are easy consequences of Theorem 5.C.2.

Proof of Theorem 3.6 Put $F = E \oplus \mathbf{R}$ which is a new vector space where the 1-dimensional real line is added. It dual space $F^* = E^* \oplus \mathbf{R}$. We are given $G \in \mathcal{S}(E)$ and put

$$G_* = \{(y, \eta) \in E^* \oplus \mathbf{R} \quad : \; G(y) \leq \eta\} \tag{i}$$

Condition in (*) entails that G_* is a convex cone in F^* and the semi-continuous hypothesis on G implies that G_* is closed with respect to the weak-star toplogy on F^*. Next, in F we define the set

$$G_{**} = \{(x, t) \in E \oplus \mathbf{R}^+ : \; y(x) \leq \eta t : \; (y, \eta) \in G_*\} \tag{ii}$$

This gives a set $\widehat{C}$ in F^* which consists of vectors (y, η) such that

$$\max_{(x,t) \in G_{**}} y(x) - \eta t \leq 0$$

It is clear that $G_* \subset \widehat{C}$. Now we prove the equality

$$(*) \qquad G_* = \widehat{C}$$

To get (*) we use Theorem 2.4. Namely, since the two sets in (*) are weak-star closed a strict inequality gives a separating vector $(x_*, t_*) \in E$, i.e. there exists $(y_*, \eta_*) \in \widehat{C}$ and a real number α such that

$$\text{(iv)} \qquad y_*(x_*) - \eta_* t_* > \alpha \quad \text{and} \quad (y, \eta) \in D_K \implies y(x_*) - \eta t_* \leq \alpha$$

Since G_* contains $(0, 0$ we have $\alpha \leq 0$. and since it also is a cone the last implication gives $(x_*, t_*) \in G_{**}$. Now the construction of $\widehat{C}$ in (iii) contradicts the strict inequality in the left hand side of (iv). Hence there cannot exist a separating vector and (*) follows.

Next, in E we consider the convex set

$$K = \{x : (x, 1) \in G_{**}\}$$

Using (*) the reader can check that

$$\mathcal{H}_K(y) = G(y)$$

for all $y \in E^*$ which proves that G has the requested form. The uniqueness of K follows from Exercise 3.4.

3.7 The case of normed spaces. If X is a normed vector space Theorem 3.6 leads to a certain isomorphism of two families. Denote by $\mathcal{K}$ the family of all convex subsets of E which are closed with respect to the norm topology. A topology on $\mathcal{K}$ is defined when we for each $K_0 \in \mathcal{K}$ and $\epsilon > 0$ declare an open neighborhod

$$U_\epsilon(K_0) = \{K \in \mathcal{K} : \operatorname{dist}(K, K_0) < \epsilon\}$$

where the norm defines the distance between K and K_0 in the usual way. Denote by $\mathfrak{H}$ the family of all functions G on E^* which satisfy (*) in 5.B.1 and are continuous with respect to the norm topology on E^*. A subset M of $\mathfrak{H}$ is equi-continuous if there to each $\epsilon > 0$ exists $\delta > 0$ such that

$$||y_2 - y_1|| < \delta \implies ||G(y_2) - G(y_1)|| < \epsilon$$

for every $G \in M$ and all pairs y_1, y_2 in E^*. The topology on $\mathfrak{H}$ is defined by uniform convergence on equi-continuous subsets.

3.8 Theorem. *If E is a normed vector space the set-theoretic bijective map $K \to \mathcal{H}_K$ is a homeomorphism when $\mathcal{K}$ and $\mathfrak{H}$ are equipped with the described topologies.*

3.9 Exercise. Deduce this result from Theorem 3.6

4. The Krein-Smulian theorem.

Let X be a Banach space and X^* its dual. The weak-star topology on X^* was defined in § 5.4. We have also the bounded weak-star topology desrcibed in § xx from the introduction. So now we have a pair of locally convex spaces X^*_w and X^*_{bw}. Open sets in the weak-star topology are by definition also open in the bounded weak-star topology, i.e. the latter topology contains more open sets and is therefore stronger than the ordinary weak-star topology. Hence there is a natural inclusion of dual spaces:

$$(X^*_w)^* \subset (X^*_{bw})^* \tag{*}$$

The Krein-Smulian theoren assers that eqaulity holds in (*). To prove this we proceed as follows. For each finite subset A of X we put

$$\widehat{A} = \{x^* : \max_{x \in A} |x^*(x)| \leq 1\}$$

Let U be an open set in X^*_{bw} which contains the origin and S^* is the closed unit ball in X^*. The construction of the bounded weak-star topology gives a finite set A_1 in X such that

$$S^* \cap \widehat{A}^0 \subset U \tag{i}$$

Next, let $n \geq 1$ and suppose we have constructed a finite set A_n where

$$nS^* \cap \widehat{A_n} \subset U \tag{ii}$$

To each finite set B of vectors in X with norm $\leq n^{-1}$ we notice that

$$\widehat{A_n \cup B} \subset \widehat{A_n} \tag{iii}$$

Put

$$F(B) = (n+1)S^* \cap \widehat{A_n \cup B} \cap (X^* \setminus U)$$

It is clear that $F(B)$ is weak-star closed for every finite set B as above. If these sets are non-empty for all B, it follows from the weak-star compactness of $(n+1)S^*$ that the whole intersection is non-empty. So we find a vector

$$x^* \in \bigcap_B F(B)$$

Notice that $F(B) \subset \widehat{B}$ for every finite set B as above which means that $|x^*(x)||leq1$ for every vector x in in X of norm $\leq n^{-1}$. Hence the norm

$$||x^*|| \leq n$$

But then (iii) gives the inclusion

$$x^* \in nS^* \widehat{A_n}) \cap (X \setminus U) \tag{iv}$$

This contradicts (ii) and hence we have proved that there exists a finite set B of vectors with norm $\leq n^{-1}$ such that $F(B) = \emptyset$.

From the above it is clear that an induction over n gives a sequence of sets $\{A_n\}$ such that (ii) hold for each n and

$$A_{n+1} = A_n \cup B_n \tag{v}$$

where B_n is a finite set of vectors of norm $\leq n^{-1}$.

Proof of the Krein-Smulian theorem.

Let θ be a linear functional on X^* which is continuous with respect to the bounded weak-star topology. This gives an open neighborhood U in X^*_{bw} such that

$$|\theta(x^*) \leq 1 \quad : x^* \in U \tag{i}$$

To the set U we find a sequence $\{A_n\}$ as above. Let us enumerate the vectors in this sequence of finite sets by $x_1, x_2, \ldots$, i.e. start with the finite string of vectors in A_1, and so on. By the inductive construction of the A-sets we have $||x_n|| \to 0$ as $n \to \infty$. If x^* is a vector in X^* we associate the complex sequence

$$\ell(x^*) = \{x^*(x_n)\}$$

which tends to zero since $||x_n|| \to 0$ as $n \to \infty$. Then

$$x^* \mapsto \ell(x^*)$$

is a linear map from X^* into the Banach space $\mathbf{c}_0$. If

$$\max_n |x^*(x_n)| \leq 1$$

we have by definition $x^* \in A_n^0$ for each n. Choose a positive integer N so that $||x^*|| \leq n$. Thus entails that

$$x^* \in NS^* \cap A_N^0$$

From (ii) during the inductive construction of the A-.sets, the last set is contained in U. Hence $x^* \in U$ which by (i) gives $\theta(x^*)| \leq 1$. We conclude that θ yields a linear functional on on the image space of the ρ-map with norm one at most. The Hahn-Banach theoren gives $\lambda \in \mathbf{c}_0^*$ of norm one at most such that

$$\theta(x^*) = \lambda(\ell(x^*))$$

Next, by a wellknown result due to Banach the dual of $\mathbf{c}_0$ is ℓ^1. Hence there exists a sequence $\{\alpha_n\}$ in ℓ^1 such that

$$\theta(x^*) = \sum \alpha_n \cdot x^*(x_n)$$

In X we find the vector $x = \sum \alpha_n \cdot x_n$ and conclude that $\theta = \widehat{x}$ which proves the Krein-Smulian theorem.

5. Fixed point theorems.

A topological space S has the fixed-point property if every continuous map $f\colon S\to S$ has at least one fixed point.

5.1 Theorem. *The closed unit ball in* $\mathbf{R}^n$ *has the fixed point property for every* $n\geq 2$.

Proof. By Weierstrass approximation theorem every continuous map from B into itself can be approximated unifomly by a C^∞-map. Together with the compactness of B the reader should conclude that it suffices to prove every C^∞-map, $\phi\colon B\to B$ has at least one fixed point. We argue by contradiction, i.e suppose that $\phi(x)\neq x$ for all $x\in B$. Each fixed $x\in B$ gives a quadratic equation of the variable a

$$1=|x+a(x-\phi(x)|^2=|x|^2+2a(1-\langle x,\phi(x)\rangle)+a^2|x-\phi(x)|^2 \tag{i}$$

Exercise 1. Use that $\phi(x)\neq x$, to show that (i) has two simple real roots for each $x\in S$, and if $a(x)$ is the larger then the function $x\mapsto a(x)$ belongs to $C^\infty(B)$. Moreover

$$a(x)=0 \quad : x\in S \tag{E.1}$$

Next, for each real number t we set

$$f(x,t)=x+ta(x)(x-\phi(x))$$

This is a vector-valued function of the $n+1$ variables $t,x_1,\dots,x_n$ where x varies in B. Put

$$g_i(x)=a(x)(x_i-\phi_i(x) \implies f_i)x,t)=x_i+tg_i(x)$$

Taking partial derivatives with respect to x we get

$$\frac{\partial f_i}{\partial x_k}=e_{ik}+t\frac{\partial g_i}{\partial x_k} \tag{ii}$$

where $e_{ii}=1$ and $e_{ik}=0$ if $i\neq k$. Let $D(x;t)$ be the determinant of the $n\times n$-matrix whose elements are the partial derivatives in (ii) and put

$$J(t)=\int_B D(x;t)\,dx \tag{iii}$$

When $t=0$ we notice that the $n\times n$-matrix above is the identy matrix and hence $D(x;0)$ has constant value one so that $J(0)$ is the volume of B. Next, (i) entails that $x\mapsto f(x;1)$ satisfies the functional equation

$$|f(x;1)|^2=1 \implies \sum_{i=1}^{i=n} f_i(x,1)\cdot\partial f_i(x,1)/\partial x_k(x,1)=0$$

for every $1\leq k\leq n$. Since the n-vector $f(x,1)\neq 0$ it means that the columns of the matrix are linearly independent for every $x\in B$ and hence $D(x;1)=0$ holds in B so that $J(1)=0$. The contradiction follows if we show that $t\mapsto J(t)$ is a constant function of t.

2. Exercise. Use Leibniz's rule and that determinants of matrices with two equal columns are zero to conclude that

$$\frac{d}{dt}(D(x;t)=\sum\sum(-1)^{j+k}\cdot\frac{\partial g_i}{\partial x_k} \tag{E.2}$$

where the double sum extends over all pairs $\leq j,k\leq n$. Next, Stokes Theorem gives

$$\int_B\frac{\partial g_i}{\partial x_k}\,dx=\int_S g_i\cdot\mathbf{n}_k\,d\omega \quad : \quad 1\leq i,k\leq n \tag{iv}$$

where ω is the area measure on S. From (E.1) we have $g_i=0$ on S for each i. Hence (E.2) gives

$$\frac{dJ}{dt}=\int_B\frac{d}{dt}(D(x;t)\,dx=0$$

So $t\mapsto J(t)$ is constant which is impossible because $J(0)=1$ and $J(1)=0$ which finishes the proof.

5.2 The Hilbert cube $\mathcal{H}_\square$. It is the closed subset of the Hilbert space ℓ^2 which consists of vectors $x = (x_1, x_2, \ldots)$ such that $|x_k| \leq 1/k$ for each k.

5.3 Proposition. *Every closed and convex subset of $\mathcal{C}$ has the fixed point property.*

5.4 Exercise. Deduce this result from Theorem 5.1.

Next, let X be a locally convex vector space and X^* its dual. Denote by $\mathcal{K}(X)$ the family of convex subsets which are compact with respect to the weak topology on X. Let $K \in \mathcal{K}(X)$ and $T\colon K \to K$ a continuous map with respect to the weak topology. For each fixed $f \in X^*$, it follows from our assumptions that the complex-valued function on K defined by

$$p \mapsto f(T(p))$$

is uniformly contnuus with respect to the weak topology. So for each positive integer n there exists a finite set $G_n = (x_1^*, \ldots, x_N^*)$ and some $\delta > 0$ such that the following implication holds for each pair of points p, q in K:

$$\text{(i)} \qquad p - q \in \cap B_\delta(x_\nu^*) \implies |f(T(p)) - f(T(q))| \leq n^{-1}$$

We can attain this for each positive integer n and get a denumerable set

$$G = \cup G_n$$

From (i) it is clear that if p, q is a pair in K and $g(p) = g(q)$ hold for every $g \in G$, then $x^*(T(p)) = x^*(T(q)$. We refer to G as a determining set for the map T. In a similar way we find a denumerable determining set $G^{1)}$ for g_1, By a standard diagonal argument the reader may verify the following:

5.5 Proposition. *There exists a denumerable subset G in X^* which contains f and is self-determining in the sense that it determines each of its vectors as above.*

5.6 An embedding into the Hilbert cube. During the construction of the finite G_m-sets which give (i), we can choose small δ-numbers and take $\{x_\nu^*\}$ such that the maximum values

$$\max_{p \in K} |x_\nu^*(p)|$$

are small. From this observation the reader should confirm that in Proposition 5.5 we can construct the sequence $G = (g_1, g_2, \ldots)$ in such a way that

$$\max_{p \in K} |g_n(p)| \leq n^{-1}$$

hold for every n. Hence each $p \in K$ gives the vector $\xi(p) = (g_1(p), g_2(p), \ldots)$ in the Hilbert cube and now

$$K_* = \{\xi(p) :\ p \in K\}$$

yields a convex subset of $\mathcal{C}$. Since G is self-determininig we have $T(p) = T(q)$ whenever $\xi(p) = \xi(q)$. Hence there exists a map from K_* into itself defined by

$$T_*(\xi(p)) = \xi(T(p) \tag{4.1}$$

5.7 Exercise. Use the compact property of K to show that K_* is closed in the Hibert cube and that T_* is a continuous map with respect to the induced strong norm topology on K_* derived from the complete norm on ℓ^2.

5.8 Consequence. Suppose from the start that we are given a pair of points p_1, p_2 in K and some $f \in X^*$ where $f(p_1) \neq f(p_2)$. From the above f appears in the G-sequence and put

$$K_0 = \{p \in K\colon\ \xi(p) = \xi(p_1)\}$$

Then K_0 is a convex subset of K, and since f appears in the G-sequence it follows that p_2 does not belong to K_0. Moreover, since G is self-determining with respect to T it is clear that

$$T(K_0) \subset K_0$$

Hence we have proved:

5.9 Proposition. *For each pair K and T as above where K is not reduced to a single point, there exists a proper and X^*-closed convex subset K_0 of K such that $T(K_0) \subset K_0$.*

5.10 Proof of the Schauder-Tychonoff theorem.

Let $T\colon K \to K$ be a continuous map where K belongs to $\mathcal{K}(X)$. Consider the family $\mathcal{F}$ of all closed and convex subsets which are T-invariant. It is clear that intersections of such sets enjoy the same property. So we find the minimal set

$$K_* = \bigcap K_0$$

given by the intersection of all sets K_0 in $\mathcal{F}$. If K_* is not reduced to a single point then Proposition 5.9 gives a proper closed subset which again belongs to $\mathcal{F}$. This is contradicts the minimal property. Hence $K_* = \{p\}$ is a singleton set and p gives the requested fixed point for T.

5.11 Proof of Kakutani's theorem.

With the notations from the introduction we are given a group $\mathbf{G}$ where each element g preserves the convex set K in $\mathcal{K}(X)$. Zorn's lemma gives a minimal closed and convex subset K_* of K which again is invariant under the group. Kakutani's theorem follows if K_* is a singleton set. To prove that this indeed is the case we argue by contradiction. For K_* is not a singleton set then

$$K_* - K_* = \{p - q\colon p, q \in K_*\}$$

contains points outside the origin, and we find a convex open neighborhood V of the origin such that

(i) $$(K_* - K_*) \setminus \overline{V} \neq \emptyset$$

Since $\mathbf{G}$ is equicontinuous on K and hence also on K_* there exists an open convex neighborhood U of the origin such that whenever k_1, k_2 is a pair in K_* such that $k_1 - k_2 \in U$, then the orbit

(ii) $$\mathbf{G}(k_1 - k_2) \subset V$$

Set

$$U^* = \text{convex hull of } \mathbf{G}(U)$$

Since the $\mathbf{G}$-maps are linear, the set U^* is invariant and continuity entails the equality

(iii) $$\mathbf{G}\overline{U^*}) = \overline{U^*}$$

We find the unique positive number δ such that the following hold for every $\epsilon > 0$:

(iv) $$K_* - K_* \subset (1+\epsilon) \cdot U^* \quad : \ (K_* - K_*) \setminus (1-\epsilon) \cdot \delta \cdot \overline{U^*} \neq \emptyset$$

Next, $\{k + \frac{\delta}{2} \cdot U\colon\ k \in K_*\}$ is an open covering of the compact set K_*. Hence Heine-Borel's Lemma gives a finite set $k_1, \ldots, k_n$ in K_* such that

(v) $$K_* \subset \ \cup (k_\nu + \frac{\delta}{2} \cdot U)$$

Put

(vi) $$K_{**} = K_* \cap \bigcup_{k \in K_*} (k + (1 - 1/4n)\delta \cdot \overline{U})$$

Since $\overline{U}$ is $\mathbf{G}$-invariant and the intersection above is taken over all k in the invariant set K_*, we see that K_{**} is a closed convex and $\mathbf{G}$-invariant set. The requested contradiction follows if we prove that K_{**} is non-empty and is strictly contained in K_*. To get the strict inclusion follows we take some $0 < \epsilon < 1/4n$. Then (iv) gives a pair k_1, k_2 in K_* such that $k_1 - k_2$ does not belong to $(1-\epsilon)\delta \cdot \overline{U^*}$. At the same time the inclusion $k_1 \in K_{**}$ entails that

(v) $$k_1 \in (k_2 + (1 - 1/4n)\delta \cdot \overline{U}) \implies k_1 - k_2 \in (1 - 1/4n)\delta \cdot \overline{U}$$

which cannot hold since $1 - 1/4n < 1 - \epsilon$. The proof of Kakutani's theorem is therefore finished if we have shown that

(vi) $$p \in K_{**}$$

To see this we take an arbitrary $k \in K_*$. From (v) we find some $1 \leq i \leq n$ such that

(vii) $$k_i - k \in \frac{\delta}{2} \cdot U$$

Without loss of generality we can assume that $i = 1$ and get a vector $u \in U$ such that

(viii) $$k_1 = k + \frac{\delta}{2} \cdot u$$

It follows that

(ix) $$p = \frac{k_1 + \ldots + k_n}{n} = k + \frac{\delta}{2n} \cdot u + \sum_{i=2}^{i=n} \frac{1}{n}(k_i - k)$$

Next, for each $\epsilon > 0$ the left hand inclusion in (iv) and the convexity of U give

(x) $$\sum_{i=2}^{i=n} \frac{1}{n}(k_i - k) \subset \frac{n-1}{n}(1+\epsilon) \cdot \delta \cdot U$$

It follows that

(xi) $$\frac{\delta}{2n} \cdot u + \sum_{i=2}^{i=n} \frac{1}{n}(k_i - k) \in (\frac{n-1}{n}(1+\epsilon)\delta + \frac{\delta}{2n}) \cdot U$$

Above we can choose ϵ so small that

$$\frac{n-1}{n}(1+\epsilon) + \frac{1}{2n}) < 1 - 1/4n$$

and then we see that

$$p \in k + (1 - 1/4n)\delta \cdot \overline{U}$$

Since $k \in K_*$ was arbitrary the requested inclusion $p \in K_{**}$ follows.