Hörmander's L^2-estimate in dimension one

Let Ω be an open set in $\mathbf{C}$. Every real-valued continuous and non-negative function ϕ in Ω gives the Hilbert space $\mathcal{H}_\phi$ whose elements are Lebesgue measurable functions f in Ω such that

$$\int_\Omega |f|^2 \cdot e^{-\phi}\, dxdy < \infty \tag{0.0}$$

and equipped with a norm defined by the square root above. Notice that $\mathcal{H}_\phi$ contain the space of test-functions wtih compact support in Ω as a dense subspace. Let ψ be another continuous and non-negative function which gives the Hilbert space $\mathcal{H}_\psi$ where the norm of an element g is denoted by $||g||_\psi$. The $\bar\partial$-operator sends a test-function f to $\bar\partial(f)$. Consider the equation

$$\bar\partial(f) = w \quad : w \in H_\psi \tag{5.0}$$

where f belongs to $\mathcal{H}_\phi$ with the additional condition that its $\bar\partial$-derivative belongs to $\mathcal{H}_\psi$, i.e this puts a constraint on f. Notice that f is not unique since every holomorphic function in Ω with a finite norm in (0.0) belongs to the $\bar\partial$-kernel. conversely, since $\bar\partial$ is an elliptic differential operator this means the kernel of the densely defined linear operator

$$\bar\partial \colon \mathcal{H}_\phi \to \mathcal{H}_\psi \tag{0.1}$$

consists of holomorphic functions for which (0,0) is finite. We shall find condions on the pair ϕ, ψ such that there exists a constant C where (5.0) has a solution f with

$$||f||_\phi \leq C \cdot ||w||_\psi \quad : \forall\, w \in H_\psi \tag{*}$$

A sufficent condtion to obtain solutions in (*) with a constant C is that the *adjoint* operator in (0.1) has special properties. Denote this adjoint by $\bar\partial^*$. Notice that it is densely defined since it contains the space of test-funtions in Ω. Suppose there exists a positive contant c_0 such that

$$||\bar\partial^*(g)||_\phi \geq c_0 \cdot ||g||_\psi \quad : \quad g \in \mathcal{D}(\bar\partial^*) \tag{0.2}$$

where $\mathcal{D}(\bar\partial^*$ is the domain of defintion for the adjoint operator. Then standard Hilbert space theory gives (*) where onr can take $C = c_0^{-1}$. To ensure that (0.2) we give:

1.1 Definition. *The pair ϕ, ψ satisfies the Hörmander condition if there exists some positive constant c_* such that*

$$\Delta(\psi) - 2 \cdot (\psi_x^2 + \psi_y^2) + \psi_x\phi_x + \psi_y\phi_y \geq 2 \cdot c_* \cdot e^{\psi - \phi} \tag{1.1.1}$$

1.2 Theorem. *If (1.1.1) holds then (*) has solutions with*

$$C \leq \frac{1}{\sqrt{c_*}} \quad : \forall\, w \in \mathcal{H}_\psi$$

Proof. Let w be in the domain of definition for the adjoint operator $\bar\partial^*$. If $f \in C_0^\infty(\Omega)$ one has

$$\langle \bar\partial(f), w\rangle = \int \bar\partial(f) \cdot \bar w \cdot e^{-\psi}\, dxdy = -\int f \cdot \big[\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)\big] \cdot e^{-\psi}\, dxdy \tag{i}$$

where Stokes theorem gives the last equality. Since ψ is real-valued, $\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)$ is equal to the complex conjugate of $\partial(w) - w \cdot \partial(\psi)$. Hence (i) and the construction of adjoint operators give

$$\bar\partial^*(w) = -\big[\partial(w) - w \cdot \partial(\psi)\big] \cdot e^{\phi - \psi} \tag{ii}$$

Taking the squared L^2-norm in $\mathcal{H}_\phi$ we obtain

$$||\bar\partial^*(w)||_\phi^2 = \int |\partial(w) - w \cdot \partial(\psi)|^2 \cdot e^{\phi - 2\psi} =$$

$$\int \big(|\partial(w)|^2 + |w|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi - 2\psi} - 2 \cdot \mathfrak{Re}\Big(\int \partial(w) \cdot \bar w \cdot \bar\partial(\psi) \cdot e^{\phi - 2\psi}\Big) \tag{iii}$$

By partial integration the last integral in (iii) is equal

(iv) $$2 \cdot \mathfrak{Re}\Big(\int w \cdot [\partial(\bar w) \cdot \bar\partial(\psi) + \bar w \cdot \partial\bar\partial(\psi) - 2\bar w \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar w \cdot \bar\partial(\psi) \cdot \partial(\phi)] \cdot e^{\phi - 2\psi} \Big)$$

Next, the Cauchy-Schwarz inequality gives

(v) $$\Big|2 \cdot \int w \cdot \partial(\bar w) \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi} \Big| \leq \int \big(|\partial(w)|^2 + |w|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi-2\psi}$$

Together (iii-v) give

(iv) $$||\bar\partial^*(w)||_\phi^2 \geq 2 \cdot \mathfrak{Re} \int |w|^2 \cdot \big[\, \partial\bar\partial(\psi) - 2 \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar\partial(\psi) \cdot \partial(\phi) \,\big] \cdot e^{\phi-2\psi}$$

Now we recall that

$$\partial\bar\partial(\psi) = \frac{1}{4}\Delta(\psi) \quad \& \quad \bar\partial(\psi) \cdot \partial(\psi) = \frac{1}{4} \cdot (\psi_x^2 + \psi_y^2)$$

It follows that (iv) is equal to

(vi) $$2 \cdot \mathfrak{Re} \int |w|^2 \cdot \frac{1}{4}\big[\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y\,\big] \cdot e^{\phi-2\psi}$$

Hence (1.1.1) gives

(vi) $$|\bar\partial^*(w)||_\phi^2 \geq c_0^2 \cdot \int |w|^2 \cdot e^{\psi-\phi} \cdot e^{\phi-2\psi} = c_* \cdot ||w||_\psi^2$$

This lower bound gives solutions to (*) by general facts about densely defined operators on Hilbert spaces. and Theorem 5.2 follows.

5.3 Remark. The full strength of L^2-estimate appears in dimension $n \geq 2$ where one works with *plurisubharmonic functions* and impose the condition that Ω is a strictly pesudo-convex set in $\mathbf{C}^n$ and solve inhomogeneous $\bar\partial$-equations for differential forms of bi-degree (p,q). In addition to Hörmander's original article [Hörmander] we refer to his text-book [Hörmander] and Chapter XX in [Hömander XX-Vol 2] where bounds for $\bar\partial$-equations are established with certain relaxed assumptions which are used to settle the fundamental principle for over-determined systems of PDE-equations in the smooth case.

The case $n = 2$

The special case below may help the reader to pursue details from Hörmander's work, where I personally recomend his original article from 1962 in Acta matematica. Take n=2 and let D^2 be the 2-dimensional polydisc in $\mathbf{C}^2$ with coordinates $z = (z_1, z_2)$. Here $\bar\partial_1$ and $\bar\partial_2$ are pairwise commuting operators. Let $\phi(z)$ be a real-valued function in D^2 which is at least of class C^2. We get the Hilbert space $\mathcal{H}$ of locally square integraböe functions with finite norm:

$$||a||_\phi = \sqrt{\int_{D^2} |a(z)|^2 \cdot e^{-\phi(z)}\, d\lambda(z)}$$

where $d\lambda(z)$ is the 4-dimensional Lebesgue meaure. Now we consider the densely defined linear operator T from $\mathcal{H}$ into $\mathcal{H} \oplus \mathcal{H}$ defined by

$$T(a) = \bar\partial_1(a) \oplus \bar\partial_2(a)$$

A. Exercise. Let T^* be the adjoint of T which sends a pair $(f,g) \in \mathcal{H} \oplus \mathcal{H}$ to $\mathcal{H}$. Show that

(A) $$T^*(f,g) = -(\partial_1(f) - f \cdot \partial_1(\phi) + \partial_1(g) - g \cdot \partial_1(\phi)]$$

B. Exercise. Put

$$\delta_1(f) = \partial_1(f) - f \cdot \partial_1(\phi) \quad : \quad \delta_2(g) = \partial_2(g) - g \cdot \partial_2(\phi)$$

Use (A) to show that

(B.1) $$||T^*(f,g)||^2 = ||\delta_1(f)||^2 + ||\delta_2(g)||^2 + 2 \cdot \mathfrak{Re} \int \delta_1(f) \cdot \overline{\delta_2(g)} \cdot e^{-\phi}\, d\lambda$$

Next, use Stokes theorem to show that

$$\int \delta_1(f) \cdot \overline{\delta_2(g)} \cdot e^{-\phi}\, d\lambda = -\int f \cdot \overline{\partial_1(\delta_2(g))} \cdot e^{-\phi}\, d\lambda \tag{B.2}$$

C. Exercise. Put

$$H(z) = \frac{\partial^2 \phi}{\partial z_1 \bar{\partial} z_2} \tag{C.0}$$

and by multiplication one identifies H with a zero-order differential operator. Show the following equality in the ring of differential operators in $\mathbf{C}^2$:

$$\partial_1 \circ \delta_2 = \delta_2 \circ \partial_1 - H \tag{C.1}$$

Conclude that (B.2) becomes

$$\int f \cdot \bar{g} \cdot \bar{H} \cdot e^{-\phi}\, d\lambda - \int f \cdot \overline{\delta_2 \partial_1((g))} \cdot e^{-\phi}\, d\lambda \tag{C.2}$$

D. The case $\bar{\partial}_1(g) = \bar{\partial}_2(f)$. Use the above to show that this equality gives

$$-2 \cdot \mathfrak{Re} \int \bar{f} \cdot \delta_2(\bar{\partial}_2((f)) \cdot e^{-\phi}\, d\lambda = -2 \cdot \mathfrak{Re} \int |\partial_2(f)|^2 \cdot e^{-\phi}\, d\lambda \tag{D.1}$$

E. Conclusion. Show that (A), (B.1-2) and (D.1) and the equality $\bar{\partial}_1(g) = \bar{\partial}_1(f)$ give:

$$||T^*(f,g)||^2 = ||\delta_1(f)||^2 + ||\delta_2(g)||^2 + 2 \cdot \mathfrak{Re} \int f \cdot \bar{g} \cdot H(z) \cdot e^{-\phi}\, d\lambda + ||\partial_2(f)||^2 \tag{E.1}$$

Above the last term is always ≥ 0. To ensure that there exists a constant c_0 such that

$$||f||^2 + ||g||^2 \leq c_0^2 \cdot ||T^*(f,g)||^2 \tag{E.2}$$

one must impose suitable conditions upon ϕ Above the mixed derivative which defines the H-function appears while norms $||\delta_1(f)||^2$ and $||\delta_2(g)||^2$ can be estimated as in the case $n = 1$ applied with $\phi = \psi$. The reader is invited to contemplate upon conditions which give a constant c_0 in (E.2) where Hörmander's work can be consulted. further dertails. Above we treated a special case since we used the same weight function ϕ and not a pair as in the case $n = 1$.