Homegeneous distributions and the Mellin transform

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Introduction. We establish some results where analytic function theory is used in connection with distributions and asymptotic expansions. Special attention in the first sections is given to homogeneous distributions in $\mathbf{R}^2$ while the last section exposes a famous result due to Mellin in [Mellin] which has a wide range of applications. A separate section is devoted to the Radon transform which is established directly via Fourier analysis, i.e. here one does not need analytic function theory.

A. Polar distributions

In the (x,y)-plane we can use polar coordinates where $x = r \cdot \cos\theta$ and $y = r \cdot \sin\theta$. If $\phi(x,y)$ belongs to the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions we restrict ϕ to the circle of radius r which after a dilation is identified with unit circle T and obtain the θ-periodic function

$$\theta \mapsto \phi_r(\theta) = \phi(r \cdot \cos\theta, r \cdot \sin\theta) \tag{0.1}$$

Let ν be a distribution on T which via Fourier series expansions is defined as in XX. For each $r > 0$ we can evaluate ν on the $C^\infty(T)$-function ϕ_r which yields a map:

$$r \mapsto \nu(\phi_r) \quad : \, r > 0 \tag{0.2}$$

A.0 Exercise. Show that (2) gives a C^∞-function defined on $\{r > 0\}$. More precisely, verify that the derivative becomes:

$$\frac{d}{dr}(\nu(\phi_r)) = \nu(\cos\theta \cdot \partial_x(\phi)(r \cdot \cos\theta, r \cdot \sin\theta) + \sin\theta \cdot \partial_y(\phi)(r \cdot \cos\theta, r \cdot \sin\theta))$$

More generally, show that for each $m \geq 2$ the derivative of order m becomes:

$$\frac{d^m}{dr^m}(\nu(\phi_r)) = \sum_{j=0}^{j=m} \binom{m}{j} \nu\big(\cos^j\theta \cdot \sin^{m-j}\theta \cdot \partial_x^j \partial_y^{m-j}(\phi)_r\big) \tag{*}$$

Show also that the function in (0.2) decreases as $r \to +\infty$. More precisely, for each positive integer N one has

$$\lim_{r \to \infty} r^N \cdot \nu(\phi_r) = 0 \tag{**}$$

A.1 The function V_λ. Let $\phi \in \mathcal{S}$. Using (**) it follows that if λ is a complex number with $\mathfrak{Re}(\lambda) > -2$, then there exists the absolutely convergent integral

$$V_\lambda(\phi) = \int_0^\infty r^{\lambda+1} \cdot \nu(\phi_r) \cdot dr \tag{1}$$

A.2 Exercise Show that V_λ is an analytic function of λ in the open half-plane $\mathfrak{Re}(\lambda) > -2$. Next, use a partial integration with respect to r and show that:

$$(\lambda+2)V_\lambda(\phi) = -\int_0^\infty r^{\lambda+2}\cdot\frac{d}{dr}[\nu(\phi_r)]\cdot dr$$

Continue this procedure and show that for every $N \geq 1$ one has:

$$(*) \qquad (\lambda+2)\cdots(\lambda+2+N)\cdot V_\lambda(\phi) = (-1)^{N+1}\int_0^\infty r^{\lambda+2+N}\cdot\frac{d^N}{dr^N}[\nu(\phi_r)]\cdot dr$$

From (*) in the exercise above we can conclude that following:

A.3 Proposition. *The function $V_\lambda(\phi)$ extends to a meromorphic function in the whole complex λ-plane with at most simple poles at the integers $-2, -3, \ldots$.*

A.4 Polar distributions. As ϕ varies in $\mathcal{S}$ we obtain a distribution-valued function V_λ. If $\delta > 0$ and $\phi(x, y) \in \mathcal{S}$ is identically zero in the disc $\{x^2 + y^2 < \delta^2\}$, then we only integrate (1) in A.1 when $r \geq \delta$ and we notice that the function

$$\lambda \mapsto \int_\delta^\infty r^{\lambda+1}\cdot\nu(\phi_r)\cdot dr$$

is an entire function of λ whose complex derivative is given by

$$\lambda \mapsto \int_\delta^\infty \log r\cdot r^{\lambda+1}\cdot\nu(\phi_r)\cdot dr$$

Regarding the distribution-valued function V_λ this means that eventual poles consist of Dirac distributions at the origin. Let us first study if a pole can occur at -2. With $\lambda = -2+\zeta$ we have

$$\text{(i)} \qquad \zeta\cdot V_{-2+\zeta}(\phi) = -\int_0^\infty r^\zeta\cdot\frac{d}{dr}[\nu(\phi_r)]\cdot dr$$

Since $r^\zeta \to 1$ holds for each $r > 0$ as $\zeta \to 0$, the right hand side has the limit

$$\text{(ii)} \qquad \int_0^\infty \frac{d}{dr}[\nu(\phi_r)]\cdot dr = \nu(\phi_0) = \nu(1_T)\cdot\phi(0)$$

where 1_T is the identity function on T on which ν is evaluated. Hence V_λ has a pole at $\lambda = -2$ if and only if $\nu(1_T) \neq 0$ and in this case the polar distribution is $\nu(1_T)$ times the Dirac distribution δ_0.

A.5 Exercise Use the functional equation formula (*) in A.2 to show the following:

A.6 Proposition *For each $N \geq 1$ the polar distribution at $\lambda = -N-2$ is zero if and only if*

$$\nu(\cos^j\theta\cdot\sin^k\theta) = 0 \quad \text{when} \quad j+k = N$$

Thus, no pole occurs if and only if ν vanishes on the $N+1$-dimensional subspace of $C^\infty(T)$ spanned by $\{\cos^j\theta\cdot\sin^{N-j}\theta) \quad : 0 \leq j \leq N\}$. Next, if a pole occurs we have a Laurent series:

$$V_{-N-2+z} = \frac{\gamma_N}{\zeta} + V_{-N-2} + \sum_{n=1}^\infty \rho_j\cdot\zeta^j$$

where γ_N is the polar distribution.

A.7 Exercise, Show that if a pole occurs then γ_N is the Dirac distribution given by:

$$(*) \qquad \gamma_N(\phi) = \frac{1}{N!}\cdot\sum_{j=0}^N \nu((\cos^j\theta\cdot\sin^{N-j}\theta)\cdot\partial_x^j\partial_y^{N-j}(\phi)(0)$$

B. Homogeneous distributions.

A distribution μ defined outside the origin in $\mathbf{R}^2$ is homogeneous of degree λ if

(*) $$\mathcal{E}(\mu) = \lambda \cdot \mu$$

where $\mathcal{E} = x\partial_x + y\partial_y$ is the radial vector field. Denote by $\mathcal{S}^*(\lambda)$ the family of all λ-homogeneous distribution in $\mathbf{R}^2 \setminus \{0\}$.

B.1 Proposition. *$\mathcal{S}^*(\lambda)$ is in a 1-1 correspondence with $\mathfrak{Db}(T)$ when we for every distribution ν on T consider the restriction of V_λ to $\mathbf{R}^2 \setminus \{0\}$.*

B.2 Exercise. Prove this result. The hint is to verify that one has the equality

$$\mathcal{E}(V_\lambda) = \lambda \cdot V_\lambda$$

when one starts from an arbitrary distribution ν on T.

B.3 The space $\mathcal{S}^*[\lambda]$. This the space of tempered distributions on $\mathbf{R}^2$ which are everywhere homogeneous. So a tempered distribution μ belongs to $\mathcal{S}^*[\lambda]$ if and only if

$$\mathcal{E}(\mu) = \lambda \cdot \mu$$

where the equality holds in $\mathcal{S}^*$.

B.4 Example of distributions in $\mathcal{S}^*[\lambda]$. If ν is a distribution on T we construct the meromorphic function V_λ here we have the equality

(i) $$\mathcal{E}(V_\lambda) = \lambda \cdot V_\lambda \quad : \mathfrak{Re}(\lambda) > -2$$

Let λ_* be a complex number such that V_λ has no pole at λ_*. By analyticity it follows from (i) that the constant term V_{λ_*} satisfies

(ii) $$\mathcal{E}(V_{\lambda_*}) = \lambda_* \cdot V_{\lambda_*}$$

Hence V_{λ_*} belongs to $\mathcal{S}^*[\lambda_*]$. By Proposition A.3 no poles occur when λ_* is outside the set $\{-2, -3, \ldots\}$. This gives the following:

B.5 Proposition. *For each λ_* outside the set $\{-2, -3, \ldots\}$ there exists a map*

$$\nu \mapsto V_{\lambda_*}$$

from $\mathfrak{Db}(T)$ into $\mathcal{S}^*[\lambda_*]$.

B.6 The action $\mathcal{E}$ on Dirac distributions. The complex vector space of all Dirac distributions is a direct sum of the subspaces

(1) $$\text{Dirac}[m] = \oplus\, \mathbf{C} \cdot \partial_x^k \partial_y^j(\delta_0) \quad : j + k = m$$

Next, in the ring $\mathcal{D}$ of differential operators we have the identity

$$\mathcal{E} = \partial_x \cdot x + \partial_y \cdot y - 2$$

Since $x \cdot \delta_0 = y \cdot \delta_0 = 0$ it follows that

$$\mathcal{E}(\delta_0) = -2 \cdot \delta_0$$

In general the reader may verify by an induction over m that

(2) $$\mathcal{E}(\gamma) = -(m+2) \cdot \gamma \quad \text{hold for all} \quad \gamma \in \text{Dirac}[m]$$

B.7 Exercise. Apply Proposition B.1 above and the results about the action by $\mathcal{E}$ on Dirac distributions to show that the map in Proposition B.5 is *bijective*, i.e. it gives linear isomorphism between $\mathfrak{Db}(T)$ and $\mathcal{S}^*[\lambda]$.

B.8 The description of $\mathcal{S}^*[-2-N]$

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Let N be a non-negative integer. Denote by $\mathfrak{Db}(T)[N+2]$ the set of distributions ν on T such that V_λ has no pole at $-2-N$. Proposition A.6 shows that the quotient space

$$\frac{\mathfrak{Db}(T)}{\mathfrak{Db}(T)[N+2]}$$

is a complex vector space whose dimension is $N+1$. Now the following conclusive result holds:

B.9 Theorem. *The map $\nu \to V_{-2-N}$ from $\mathfrak{Db}(T)[N+2]$ into $S^*[-2-N]$ is bijective*

Proof. That the map $\nu| \to V_{-2-N}$ is defined and injective on $\mathfrak{Db}(T)[N+2]$ is clear. There remains to show that the map is surjective. So let $\mu \in S^*[-2-N]$ be given. By Proposition B.1 we find $\nu \in \mathfrak{Db}(T)$ such that $\nu = \mu$ outside the origin. it means that

(1) $$\mu - \nu = \eta$$

for some Dirac distribution η. There remains to see that this implies that ν belongs to $\mathfrak{Db}(T)[N+2]$. To show this we first study V_λ where $\lambda = -N-2+\zeta$ and consider the Laurent series after Proposition A.6. By Exercise B.2 we have

$$\mathcal{E}(\frac{\gamma_N}{\zeta} + V_{-N-2} + \sum \rho_j \zeta^j) = (-N-2+\zeta)V_{-N-2+\zeta} =$$

$$(-N-2)\cdot\frac{\gamma_N}{\zeta} + \gamma_N + (-N-2)V_{-N-2} + (-N-2+\zeta)\cdot\sum \rho_j\zeta^j$$

Identifying the constant term we get

(2) $$\mathcal{E}(V_{-N-2}) = \gamma_N - (N+2)V_{N-2}$$

At the same time $\mu = V_{-N-2} + \eta$ and since $\mu \in S^*[-N-2]$ is assumed, it follows that

(3) $$\mathcal{E}(V_{-N-2}) + (N+2)V_{-N-2} + \mathcal{E}(\eta) + (N+2)\eta = 0$$

Together (2-3) give

(4) $$\mathcal{E}(\eta) + (N+2)\eta = -\gamma_N$$

At this stage we use the results from B.6. From the direct sum decomposition (1) in B.6 we can expand γ and write

$$\eta = \sum \eta_m \quad : \eta_m \in \mathrm{Dirac}[m]$$

Then (2) in B.6 gives

(5) $$\mathcal{E}(\eta) + (N+2)\eta = \sum (N+2-m-2)\cdot\eta_m$$

At the same time (*) in Exercise A.7 gives $\gamma_N \in \mathrm{Dirac}[N]$. and by (4) and (5) we have

(6) $$-\gamma_N = \sum (N+2-m-2)\cdot\eta_m$$

The direct sum decomposition (1) in B.6 entails that $\eta_m = 0$ for each $m \neq N$ and with $m = N$ we do not get any contribution in the right hand side which gives $\gamma_N = 0$ and hence ν belongs to $\mathfrak{Db}(T)[N+2]$ as required.

B.10 Example. Let $\nu = 1_T$ and consider the distribution V_{-1}. It is given by

$$V_{-1}(\phi)=\int_0^\infty \big[\int_0^{2\pi}\phi(r,\theta)d\theta\,\big]\cdot dr=\iint\frac{\phi(x,y)}{\sqrt{x^2+y^2}}\cdot dxdy$$

Hence the L^1-density $\frac{1}{\sqrt{x^2+y^2}}$ has no homogeneous extension. On the other hand, with $\nu=\cos\theta$ we see that the L^1-density $\frac{x}{x^2+y^2}$ is -1-homogeneous. With $z=x+iy$ we have the L^1-density

$$\frac{1}{z}=\frac{x-iy}{x^2+y^2}$$

and conclude that it is homogeneous of degree -1.

B.11 Fourier transforms.

The Fourier transform maps tempered distributions in the (x,y)-space to tempered distribution the (ξ,η)-space. By the laws from XX we see tha the radial field $\mathcal{E}=x\partial_x+y\partial_y$ is sent into the first order differential operator

$$(i\partial_\xi)\cdot i\xi+(i\partial_\eta)\cdot i\eta=-\partial_\xi\cdot\xi-\partial_\eta\cdot\eta=-\xi\partial_\xi-1--\eta\partial_\eta-1=-\mathcal{E}^*-2$$

where $\mathcal{E}^*$ is the Euler field in the (ξ,η)-space. It follows that for every complex number one has the implication

$$(*)\qquad \mu\in S^*[\lambda]\implies\widehat{\mu}\in S^*[-\lambda-2]$$

Moreover, by Fourier's inversion formula we get the opposite implication. Hence the Fourier transform yields a bijective map in (*).

B.12 Example. The Dirac measure δ_0 belongs to $S^*[-2]$. Its Fourier transform is therefore in $S^*[0]$ and this is clear since $\widehat{\delta_0}$ is the constant density times $\frac{1}{2\pi}$. Notice also that the Fourier transform sends $S^*[-1]$ into itself.

B.13 The λ-maps on $\mathfrak{Db}(T)$. Let λ be a complex umber outside the set $\{-2,-3,\ldots\}$. To each $\nu\in\mathfrak{Db}(T)$ we get the distribution V_λ which belongs to $S^*[\lambda)$. It follows that the Fourier transform $\widehat{V_\lambda}$ belongs to $S^*[-\lambda-2]$ and this gives a unique distribution ν^* on T such that

$$(*)\qquad \widehat{V_\lambda}=V^*_{-\lambda-2}$$

Keeping λ fixed this means that we get a bijective map from $\mathfrak{Db}(T)$ to itself defined by

$$\nu\mapsto\nu^*$$

where the rule is that (*) holds. We denote this map by $\mathcal{H}_\lambda$ and refer to this as the λ-map on $\mathfrak{Db}(T)$.

C. The family $\int|P|^{2\lambda}$

Let $P(x,y)$ be a polynomial of the two real variables. In general it may have complex coefficients and no special assumption is imposed on its zero set. When $\mathfrak{Re}(\lambda)>0$ it is clear that we obtain a tempered distribution defined by

$$(*)\qquad \phi\mapsto\int|P(x,y)|^{2\lambda}\cdot\phi(x,y)\cdot dxdy$$

Moreover, we see that this gives a distribution-valued holomorphic function in the right half-plane. In the case when P has real coefficients we can consider a connected component Ω of the set $\{P>0\}$ and construct

$$(**)\qquad \phi\mapsto\int_\Omega P(x,y)^{2\lambda}\cdot\phi(x,y)\cdot dxdy$$

It turns out that both (*) and (**) extend to meromorphic distribution valued functions in the whole complex λ-plane and there exists a finite set of positive rational numbers $\{0 < q_\nu\}$ such that the poles are contained in the set

$$\cup_{k=1}^m \mathcal{A}_k = \{-q_k - n \quad : n = 0, 1, 2 \ldots\}$$

Remark. Special cases of this result appeared in work by Marcel Riesz around 1935 who used meromorphic extensions to construct fundamental solutions to PDE-equations. More extensive classes appear in the text-book [G-S] by Gelfand and Shilov. We remark that these authors also treated cases where P depends upon more than two variables. But the class of polynomials was quite restricted. In the impressive work [Nilsson] it was proved the general fact that distribution valued functions as above extend to meromorphic functions to the whole complex λ-plane with poles confined to an arithmetic progression as above. We remark that Nils Nilsson established this result for polynomials in any number of variables. Even though it was not stated explicitly in [Nilsson] who also analyzed multi-valued penomenta caused by non-vanishing monodromy, the meromorphic extension above follows from [Nilsson] when one takes a Mellin transform. Here we are content to treat the case of two variables and the subsequent proof is in the spirit of Nilsson's work which relies upon some very ingenious applications from the classic theory about algebraic functions of two variables.

The functional equation. Using algebraic properties of the Weyl algebra of differential operators with polynomial coefficients, Joseph Bernstein gave a simple proof of the mere existence of the meromorphic extension in [Bernstein]. Of course, just like in Nilsson's case the existence was established in any number of variables. The new point in Bernstein's work is that the meromorphic extension can be achieved by a functional equation. Namely, with P given the required meromorphic extension in (*) follows from the existence of a non-zero polynomial $b(\lambda)$ in $\mathbf{C}[\lambda]$ such that

$$b(\lambda) \cdot \int |P(x,y)|^{2\lambda} \cdot \phi(x,y) \cdot dxdy =$$

$$\sum \lambda^k \cdot \int P(x,y) \cdot |P(x,y)|^{2\lambda} \cdot Q_k(\phi)(x,y) \cdot dxdy$$

hold for every ϕ in $\mathcal{S}(\mathbf{R}^2)$ and $\{Q_k\}$ is a finite set of differential operators indexed by non-negative integers which belong to the Weyl algebra $A_2(\mathbf{C})$. Above one can choose $b(\lambda)$ of smallest possible degree and it is then referred to as the *Bernstein-Sato polynomial* of P. We remark that the tribute to M. Sato stems from his early discoveries of many functional equations of this kind from [Sato]. The fact that the roots of the b-function always are strictly negative rational numbers was established by Masaki Kashiwarin in the article [Kashiwara] from 1975.

The Radon transform

In the article [Radon] from 1917 Johann Radon established an inversion formula which recaptures a test-function $f(x,y)$ in $\mathbf{R}^2$ via integrals over affine lines in the (x,y)-plane. This family is parametrized by pairs (p,α), where $p \in \mathbf{R}$ and $0 \leq \alpha < \pi$ give the line $\ell(p,\alpha)$:

$$t \mapsto \big(p \cdot \cos\alpha - t \cdot \sin\alpha \,,\, p \cdot \sin\alpha + t \cdot \cos\alpha\big)$$

The Radon transform of f is a function of the pairs (α, p) defined by:

$$(*) \qquad R_\alpha(p) = \int_{\ell(p,\alpha)} f \cdot dt = \int_{-\infty}^{\infty} f\big(p \cdot \cos\alpha - t \cdot \sin\alpha \,,\, p \cdot \sin\alpha + t \cdot \cos\alpha\big) \cdot dt$$

Thus, for a given α we take the mean value of f over an affine line which is $\perp$ to the vector $(\cos\alpha, \sin\alpha)$ and whose distance to the origin is $|p|$. We give an inversion formula which recaptures

f from the Radon transform. To achieve this we construct the partial Fourier transform of R with respect to p, i.e. set

$$\widehat{R}_\alpha(\tau) = \int e^{-i\tau p} \cdot R_\alpha(p) \cdot dp \tag{1}$$

Consider the linear map $(p,\tau) \mapsto (x,y)$ where

$$x = p \cdot \cos\alpha - t \cdot \sin\alpha \quad \text{and} \quad y = p \cdot \sin\alpha + t \cdot \cos\alpha \implies$$

$$p = \cos(\alpha) \cdot x + \sin(\alpha) \cdot y \tag{2}$$

Since $\cos^2\alpha + \sin^2\alpha = 1$ this substitution gives $dpdt = dxdy$. So (2) gives

$$\widehat{R}_\alpha(\tau) = \int e^{-i\tau(x\cdot\cos\alpha + y\cdot\sin\alpha)} \cdot f(x,y) \cdot dxdy = \widehat{f}(\tau\cdot\cos\alpha\,,\,\tau\cdot\sin\alpha) \tag{3}$$

Next, Fourier's inversion formula applied to f gives:

$$f(x,y) = \frac{1}{(2\pi)^2} \cdot \int e^{i(x\xi + y\eta)} \cdot \widehat{f}(\xi,\eta) \cdot d\xi d\eta$$

Now we use the substitution $(\tau,\alpha) \mapsto (\xi,\eta)$ where

$$\xi = \cos(\alpha)\cdot\tau \quad \text{and} \quad \eta = \sin(\alpha)\cdot\tau$$

Here $d\xi d\eta = |\tau| \cdot d\tau d\alpha$ and then (3) gives the equality

$$f(x,y) = \frac{1}{(2\pi)^2} \int_0^\pi \big[\int_{-\infty}^\infty e^{i\tau(x\cdot\cos\alpha + y\cdot\sin\alpha)} \cdot \widehat{R}_\alpha(\tau) \cdot |\tau| \cdot d\tau\big] \cdot d\alpha \tag{*}$$

To get an inversion formula where the partial Fourier transform $\widehat{R}_f(\alpha,\tau)$ does not appear we apply the Fouriers inversion formula in dimension one. Namely, for each $A > 0$ we set

$$K_A(u) = \frac{1}{2\pi} \int_{-A}^A e^{i\tau u} \cdot |\tau| \cdot d\tau \tag{4}$$

This function admits a alternative description since we have

$$K_A(u) = \frac{1}{\pi} \int_0^A \tau \cdot \cos(\tau u) \cdot d\tau = \frac{1}{\pi} \cdot \frac{d}{du}\big(\int_0^A \sin(\tau u) \cdot d\tau\big) =$$

$$\frac{1}{\pi} \cdot \frac{d}{du}\big(\frac{1-\cos(Au)}{u}\big) = \frac{1}{\pi} \cdot [A \cdot \frac{\sin Au}{u} - \frac{1-\cos(Au)}{u^2}] \tag{5}$$

Next, by the convolution formula for Fourier transforms the right hand side in (*) becomes

$$\lim_{A\to\infty} \frac{1}{2\pi} \int_0^\pi \big[\int_{-\infty}^\infty R_\alpha(x\cdot\cos\alpha + y\cdot\sin\alpha - u) \cdot K_A(u) du\big] \cdot d\alpha \tag{6}$$

Put

$$\phi_\alpha(x,y,u) = R_\alpha(x\cdot\cos\alpha + y\cdot\sin\alpha - u) \tag{7}$$

After the substitution $u \to \frac{s}{A}$ and applying (5) the limit in (*) becomes

$$\lim_{A\to\infty} \int_0^\pi \big[\int_{-\infty}^\infty \phi_\alpha(x,y,\frac{s}{A}) \cdot \big(A\cdot\frac{\sin s}{s} - A\cdot\frac{1-\cos s}{s^2}\big) \cdot ds\big] \cdot d\alpha \tag{**}$$

Remark. For further material about the Radon transform which includes inversion formulas in every dimension $n \geq 2$ we refer to Helgason's text-book [Helgason].

The Mellin transform

In many situations one encounters a function $J(\epsilon)$ which is defined for $\epsilon > 0$ and has an asymptotic expansion as $\epsilon \to 0$ by fractional powers which means that there exists a strictly increasing sequence of real numbers $0 < q_1 < q_2 \ldots$with $q_N \to +\infty$ and constants $c_1, c_2, \ldots$ such that for every N there exists some $\delta > 0$ which in general depends upon N and:

$$(*) \qquad \lim_{\epsilon \to 0} \frac{J(\epsilon) - (c_1 e^{q_1} + \ldots + c_N e^{q_N})}{\epsilon^{q_N + \delta}} = 0$$

It is clear that the constants $\{c_k\}$ are uniquely determined by J and the q-numbers if (*) holds. We are only concerned with the behavior of J for small ϵ and may therefore assume that $J(\epsilon) = 0$ when $\epsilon > 1$.

Mellin's asymptotic formula. Assume that $J(\epsilon)$ be some bounded and continuous function on $[0, 1]$ and the integral

$$(1) \qquad \int_0^1 |J(\epsilon)| \cdot \frac{d\epsilon}{\epsilon} < \infty$$

We also assume that $J(\epsilon) = 0$ when $\epsilon \geq 1$. When $\mathfrak{Re}(\lambda) > 0$ we set

$$(2) \qquad M(\lambda) = \lambda \cdot \int_0^1 J(\epsilon) \cdot \epsilon^{\lambda - 1} \cdot d\epsilon$$

It is clear that $M(\lambda)$ is an analytic function in the right half-plane $\mathfrak{Re}(\lambda) > 0$ which by (1) extends to a continuous function on the closed half-plane. Moreover (*) implies that $M(\lambda)$ extends to a meromorphic function in the whole complex λ-plane whose poles are contained in the set $\{-q_k\}$. In fact, this follows easily via (*) since

$$\lambda \int_0^1 \epsilon^q \cdot \epsilon^{\lambda - 1} \epsilon = \frac{1}{\lambda + q} \quad \text{for every} \quad q > 0$$

Consider M along the imaginary axis where we have

$$(3) \qquad M(is) = is \int_0^1 J(\epsilon) \cdot \epsilon^{is} \cdot \frac{d\epsilon}{\epsilon}$$

Apart from the factor i this is the Fourier transform of J on the multiplicative line $\mathbf{R}^+$. So by XXX one has the inversion formula

$$(4) \qquad J(\epsilon) = \lim_{R \to \infty} J_R(\epsilon) = \int_R^R \epsilon^{-is} \cdot \frac{M(is)}{is} \cdot ds$$

Mellin found a reverse procedure where one from the star only assumes that (1) holds and to achieve an asymptotic expansion (*) one imposes conditions upon the M-function. More precisely, *assume* that $M(\lambda)$ extends to a meromorphic function with simple vpoles confined to a set $\{-q_k\}$ of strictly negative real numbers. With $\lambda = t + is$ we consider line integrals over rectangles

$$(5) \qquad \square_{R,A} = \{-A < t < 0\} \cap \{-R < s < R\}$$

where one for an arbitrary positive integer N choose A so that

$$(6) \qquad q_N < A < q_{N+1}$$

With $\epsilon > 0$ kept fixed we have the analytic function in $\square_{R,A}$ defined by

$$(6) \qquad \lambda \mapsto \epsilon^{-\lambda} \cdot \frac{M(\lambda)}{\lambda}$$

1. Exercise. Show that Cauchy's residue formula gives the equality:

$$2\pi i\cdot J_R(\epsilon)=2\pi i\cdot\sum_{k=1}^{k=N} q_k^{-1}\cdot\mathfrak{res}(M(\lambda):q_k)\cdot\epsilon^{q_k}+I_1(R,A)+I_2(R,A) \tag{i}$$

where

$$I_1(R)=\int_0^{-A}\big[\epsilon^{-t-iR}\cdot\frac{M(t+iR)}{t+iR}-[\epsilon^{-t+iR}\cdot\frac{M(t-iR}{t-iR}\big]\cdot dt \tag{ii}$$

$$I_2(R)=-\int_{-R}^{R}\epsilon^{A-is}\cdot\frac{M(-A+is)}{-A+is}\cdot ds \tag{iii}$$

2. The passage to the limit. The triangle inequality gives

$$|I_2(R)|\leq\epsilon^A\cdot\int_{-R}^{R}\big|\frac{M(-A+is)}{-A+iRs}\big|\cdot ds\leq\epsilon^A\cdot\frac{1}{R}\cdot\int_{-R}^{R}\big|M(-A+is)\big|\cdot ds \tag{3.1}$$

Above we have $A>q_N$ and hence ϵ^A gives an admissable error for an asymptotic expansion up to order N. So from (i) in the exercise we conclude the following:

3. Theorem. *Assume that we for every positive integer N can find $q_N<A<q_{N+1}$ such that the following two limit formulas hold:*

$$\lim_{R\to\infty}\frac{1}{R}\cdot\int_{-R}^{R}\big|M(-A+is)\big|\cdot ds=0 \tag{i}$$

$$\lim_{R\to\infty}\int_0^{-A}\big[\epsilon^{-t-iR}\cdot\frac{M(t+iR)}{t+iR}-[\epsilon^{-t+iR}\cdot\frac{M(t-iR}{t-iR}\big]\cdot dt=0 \tag{ii}$$

Then the function $J(\epsilon)$ has an asymptotic expansion (*) where the c-numbers are given by

$$c_k=\mathfrak{res}(M:q_k)\quad:\quad k=1,2,\ldots$$

4. The case of multiple roots. Keeping the conditions (i-ii) in Theorem 3 we can relax the hypothesis that the poles of $M(\lambda)$ are simple and obtain an asymptotic expansion where the terms $\{c_k\epsilon^{q_k}\}$ in the expansion (*) are replaced by finite sums of the form

$$\sum_{\nu=0}^{m_k-1} c_{k,\nu}\cdot(\log\epsilon)^\nu\cdot\epsilon^{q_k} \tag{1}$$

where m_k is the multiplicity of the pole of $M(\lambda)$ at q_k. More precisely, Cauchy's residue formula holds in (i) when the terms $2\pi i\cdot q_k^{-1}\mathfrak{res}(M(\lambda):q_k)$ are replaced by

$$2\pi i\cdot\mathfrak{res}(\epsilon^{-\lambda}M(\lambda):q_k) \tag{2}$$

For a given k we set $\lambda=-q_k+\zeta$ and then the residue above can be calculated using the expansion

$$\epsilon^{q_k+\zeta}=\epsilon^{q_k}\cdot\big[1+\sum_{\nu=1}^{\infty}(\log\epsilon)^\nu\cdot\zeta^\nu\big]$$

For example, assume that $M(\lambda)$ has a double pole at $-q_k$ with a local Laurent expansion

$$M(-q_k+\zeta)=\frac{c_k}{\zeta^2}+a_0+a_1\zeta+\ldots$$

In this case the residue in (2) becomes

$$\mathfrak{res}(\epsilon^{-\lambda}M(\lambda):q_k)=c_k\cdot\epsilon^{q_k}\cdot\log\epsilon \tag{3}$$

Final remark. To appreciate Mellin's result one should consider various examples where the point is that other kind of methods to begin with prove that the M-function has a "nice" meromorphic extension as above. Quite extensive classes of situations where this applies are derived via ⌈-module methods when the M-function satisfies certain functional equations. See the chapter on $\mathcal{D}$-module theory for further comments and examples. The reader may also consult the elegant article [Barlet-Maire] where Mellin's result is extended to give complex expansions, i.e here a J-function is defined in a punctured complex disc and one seeks asymptotic expansions when a complex variable ζ tends to zero instead of taking limits as in (*) over positive real ϵ.