A spectral theorem for continuous functions.

On the real ξ-line we denote by C_* the set of bounded and uniformly continuous functions $\phi(\xi)$. So there exists a constant m such that $|\phi(\xi) \leq m$ for al ξ and the uniform continuity means that if we put

$$\omega_\phi(\delta) = \max_\xi |\phi(\xi + \delta) - \phi(|xi)|$$

then $\omega_\phi(|delta) \to 0$ as $\delta \to 0$. When $\phi \in C_*$ we get the linear space $\mathcal{T}_\phi$ which consist of functions of the form

(1) $$\sum c_k \cdot \phi(\xi + \tau_k)$$

where $\{\tau_k\}$ is a finite set of real numbers and $\{c_k\}$ a finite tuple of complex numbers.

Following [Beurling] a function a function $f(x)$ in the class C_* belongs to the *tight closure* of $\mathcal{T}_\phi$ if there exists a sequence $\{f_k\}$ in $\mathcal{T}_\phi$ such that $f_k(\xi) \to f(\xi)$ holds uniformly on every bounded interval $-A \leq \xi \leq A$ and moreover the maximum norms taken over the whole ξ-line satisfy

(2) $$||f_|| \leq ||f||$$

The spectrum of ϕ. Given $\phi \in C_*$ its spectrum is defined as the set of real numbers λ such that the exponential function $e^{i\xi\lambda}$ belongs to the tight closure of $\mathcal{T}_\phi$. The major result in [Beurling] asserts that the spectrum always in $\neq \emptyset$ unless ϕ is identically zero.

Proof that $\sigma(\phi) \neq \emptyset$

First we take some even test-function $H(x)$ on the real x-line where $H(0) = \frac{1}{2\pi}$. With $\zeta = \xi + i\eta$ we get the entire function

(i) $$h(\zeta) = \int e^{i\zeta x} \cdot H(x) \cdot dx$$

Since H is a test-function we know from XX that the functions

$$\xi \mapsto h(\xi + i\beta)$$

are rapidly decreasing for every β, i.e. they decrease faster than any negative power of $|\xi|$. Moreover, Fourier's inversion formula gives in particular

$$\frac{1}{2\pi} = H(0) = \frac{1}{2\pi} \cdot \int h(\xi) \cdot d\xi$$

Hence we have

(ii) $$\int h(\xi) \cdot d\xi = 1$$

Next, if $H(x)$ is supported by an interval $[-c, c]$ the result in XXX gives a constant C such that

(iii) $$|h(\xi + i\eta)| \leq C \cdot \frac{e^{c|\eta|}}{1 + \xi^2 + \eta^2}$$

for all $\xi + i\eta$

Next, for each complex number $\alpha + i\beta$ we get a function of ξ defined by

(iv) $$\xi \mapsto= \int_{-\infty}^{\infty} \phi(\xi - s) \cdot h(s + \alpha + i\beta) \cdot ds$$

Exercise. Show that the function in (iv) belongs to the tight closure of $\mathcal{T}_\phi$. The hint is to use (iii) and the uniform continuity of ϕ.

Next, consider the function

$$\psi(\xi + i\eta) = \int_{-\infty}^{\infty} (\xi + i\eta - s) \cdot \phi(s) \cdot ds$$

It is easily seen that $\psi(\zeta)$ is an entire function and using (iii) there is a constant C such that

$$|\psi(\xi + i\eta)| \leq C \cdot e^{c|\eta|}$$

Remark. When ψ is restricted to the real $|xi$-line it is the convolution of h and ϕ. Now $\phi(\xi)$ defines a temperate density function and is therefore equal to the Fourier transform of a temperate distribution μ on the real x-line. From the calculus with tempered distributions in XXX this means that $\psi(\xi)$ up to a constant is the Fourier transform of the distribution $H(x) \cdot \mu$ which has compact support on the x-line. This clarifies why $\psi(\zeta)$ is an entire Fourier-Laplace transform and also the growth condition xx above.

At this stage we announce the following:

Proposition. *There exists a real number a and a sequence of complex numbers $\{alpha_n + i\beta_n\}$ such that the functions*

$$f_n(\xi) = e^{i\alpha\xi} \cdot \frac{\psi(\xi + \alpha_n + i\beta_n)}{|\psi(\alpha_n + i\beta_n)|}$$

converge uniformly to 1 on the real ξ-axis and at the same time the maximum norms $||f_n||$ converge to one. which implies that $e^{ia\xi}$ belongs to the tight closure of $\mathcal{T}_\phi$.

Proof. Introduce the function

$$\mu_f(\eta) = \max_\xi \log |\psi(\xi + i\eta)|$$

By (XX) the functions $\xi \to |\psi(\xi + i\eta)|$ are bounded in every interval $-/rho \leq y \leq \rho$ so the μ-function is defined on the whole η-line. By the result in XX the μ-function is convex and hence the derivative $\mu'(\eta)$ is non-decreasing and the inequality (xx) entails that

$$\mu'(\eta) \leq c$$

hold for every η. As a consequence there exist the two limits

$$a = \lim_{\eta \to +\infty} \mu(\eta) \quad : \quad b = \lim_{\eta \to -\infty} \mu(\eta)$$

where $b \leq a$. Using these limits we will show that a sequence $\{f_n\}$ exists where we can take a to be the first term in (xx) above.

PROOF rather easy now as exposed on page 65 in Collected.

Measures. $\{\mu_n\}$ converge to limit μ meaning pointwise convergence of Fourier weak type ... Many ways to reach μ. Question when ϕ good in that sense. To entail convergence. Meaning $\mu_n \to 0$ in weak sens implies that integrals to zero. True iff ϕ uniformly inverse transofrm.