0.4 Fekete's inequality. The interplay between Fourier series and analytic functions in the unit disc $D = \{|z| < 1\}$ leads to many interesting results. Consider as in (0.2) the sine series

$$\phi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}$$

In D we have the analytic function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z)$$

Notice that

$$f(x) = -\log(1-x)$$

tends to $+|infty$ as $x \to 1$ along the real axis. The Taylor polynomials

$$S_N(z) = \sum_{n=1}^{n=N} \frac{z^n}{n}$$

attain their maximum norms on the closed unit disc when $z = 1$ and here

$$S_N(1) = \sum_{n=1}^{n=N} \frac{1}{n} \simeq \log N$$

At the same time we have seen in (0.2) that $\mathfrak{Im}(f(z))$ is a bounded function in D. Fekete proved that the example above is extremal in the following sense:

Theorem. *There exists an absolute constant C such that if $g(z) = \sum c_n z^n$ is an analytic function in D for which the maximum norm of $\mathfrak{Im}(g(z))$ is ≤ 1, then*

$$\max_\theta |\mathfrak{Re} \sum_{n=0}^{n=N} c_n \cdot e^{in\theta}| \leq C \cdot \log N \quad : N \geq 2$$

Fekete's result will be proved in § XX during a closer study about analytic functions in the unit disc and their associated Fourier series.

A result by Carleman. Let $f(\theta)$ be a 2π-periodic and continuous function. If $\epsilon > 0$ and $N \geq 1$ we denote by $\rho(N; \epsilon)$ the number of integers $0 \leq n \leq N$ for which the maximum norm

$$\max_\theta |S_n(\theta) - f(\theta)| \geq \epsilon$$

With these notations we prove in § x that

$$\text{(i)} \qquad \lim_{N \to \infty} \frac{\rho(N; \epsilon)}{N+1} = 0$$

hold for every $\epsilon > 0$. It means that Gibb's phenomenon form a statistical point of view is exceptional, i.e. "failure of convergence" occurs only for a sparse subsequence of Fourier's partial sums. Actually (i) is a consequence of a more precise result whicgh goes as follows: The continuous function f is uniformly continuos and we set

$$\omega_f(\delta) = \max |f(\theta_1) - f(\theta_2)| \quad : |\theta_1 - \theta_2| \leq \delta$$

0.3 Theorem. *There exists an absolute constant K such that the following hold for every 2π-periodic continuous function f whose maximum norm is ≤ 1*

$$(*) \qquad \frac{\rho(N; \epsilon)}{N} \leq \frac{K}{\epsilon^2} \cdot (\frac{1}{N} + \omega_f(\frac{1}{N})^2)$$

Remark. Since $\frac{1}{N} + \omega_f(\frac{1}{N})^2$ tends to zero as $N \to +\infty$ and (*) holds for each $\epsilon > 0$ we get (i).

Fejer's inequality. Several remarkable inequalities for trigonometric polynomials were established by Fejer in [Fejer] where a central issue is to construct trigonometric polynomials expressed by a sine series which are ≥ 0 on the interval $[0, \pi]$. Consider as an example is the sine-series

$$S_n(\theta) = \sum_{k=1}^{k=n} \frac{\sin k\theta}{k}$$

Here $\{S_n(\theta)\}$ are trigonometric polynomials which are odd functions of θ. They are realted to the analytic function in the unit disc given by the series

$$f(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

Notice that this series represents the analytic function in D given by $\log{(1-z)}$. This complex log-function extends analytically across the unit circle T outside $\{z = 1\}$. If $0 < \theta < \pi$ we notice that

$$\mathfrak{Im}(f(e^{i\theta}) = -\arg(1 - e^{i\theta}) = \frac{\pi - \theta}{2}$$

At the same time $e^{ik\theta} = \sin k\theta$ and therefore

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k} = \frac{\pi - \theta}{2} \quad : 0 < \theta < \pi$$

Moreover, there exist pointwise limits

$$\lim_{n \to \infty} S_n(\theta) = \frac{\pi - \theta}{2} \quad : 0 < \theta < \pi$$

At the same time $S_n(0) = 0$ for every n so one cannot expect that the pointwise convergence for small positive θ holds uniformly. In this connection Fejer proved the following:

0.2 Theorem. *For every $n \geq 1$ one has the inequality*

$$0 < S_n(\theta) \leq 1 + \frac{\pi}{2} \quad : \quad 0 < \theta < \pi$$

The upper bound was proved by in [Fej] and Fejer conjectured that $S_n(\theta)$ stays positive on $(0, \pi)$. This was later confirmed by Jackson in [xx] and Cronwall in [xx]. Here is an occasion to use a computer and plot graphs of the functions $\{S_n(\theta)\}$ to analyze the rate of convergence when $\theta \simeq 0$ and also confirm the inequality in Fjher's theorem numerically ,i.e. plot graphs of $S - N(\theta)$ for large values of n and check the validity of the inequalities above numerically.

Fourier series

Contents

Introduction.

Fourier series were invented by Fourier to solve heat and the wave equations. We expose results in the 1-dimensional case and remark only that one also constructs Fourier series in several variables of functions $f(x_1, \ldots, x_n)$ which are 2π-periodic with respect to each variable in $\mathbf{R}^n$.

Outline of contents.

Section A contains basic material about Fourier series where the kernels of Dini and Fejer are introduced. At the end of § A we construct the Jackson kernel which give approximations of a given periodic function f by trigonometric polynomials where the rate of approximation is controlled by the modulos of continuity of f. Sections B-C are devoted to results about extremal polynomials. A complex version appears in § D where Theorem D.4 relates the transfinite diameter of compact subsets of $\mathbf{C}$ with Tchebysheff polynomials. § F is devoted to a result by Carleman about convergence in the mean of partial Fourier sums. From a statistical point of view this result confirms the convergence of Fourier's partial sums where Theorem F.2 gives an absolute constant K such that for every 2π-periodic and continuous function f whose maximum norm is ≤ 1, the following inequality holds for every positive integer n and each $0 < \delta < \pi$ where $\{s_\nu\}$ are Fourier's partial sums of f:

$$\sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} ||s_\nu - f||^2} \leq \frac{1}{\sqrt{n+1}} \cdot [n^{1+1/2} \cdot \delta \cdot \omega_f(\delta) + 2K\delta^{-1/2} + K]$$

where $\{||s_\nu - f||\}$ denote maximum norms over $[0, 2\pi]$ and ω_f the modulos of continuity. If $\epsilon > 0$ we take $\delta = \frac{1}{\epsilon n}$ for large n, the left hand side is majorised by

$$\frac{\omega_f(1/\epsilon n)}{\epsilon} + 2K\sqrt{\epsilon} + \frac{1}{\sqrt{n+1}} \cdot K$$

Keeping ϵ fixed while n increases this tends to zero which entials that "with high probability" the maximum norms of $|||s_\nu - f||$ are small as ν varies over large integer intervals.

Section § H treats results due to de Vallé Poussin about best approximations by trigonometric polynomials of prescribed degree where one starts with some real-valued and continuous 2π-periodic function f. If $n \geq 1$ we denote by $\mathcal{T}_n$ the $2n+1$)-dimensional real vector space of trigonometric polynomials of degree $\leq n$, i.e. functions of the form

$$P(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\nu=n} a_\nu \cdot \cos \nu x + \sum_{\nu=1}^{\nu=n} b_\nu \cdot \sin \nu x$$

The best approximation of degree n is defined by:

(*) $$\rho_f(n) = \min_{P \in \mathcal{T}_n} ||f - P||$$

Among the results in § H we mention the following lower bound inequality expressed by the Fourier coefficients of f defined by

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} \cdot f(x)\, dx$$

Theorem. *For each $n \geq 1$ one has the inequality*

$$\rho_f(n) \geq |\widehat{f}(n+1)| - \sum_{j=1}^{\infty} |\widehat{f}((n+1)(2j+1))|$$

We remark that this lower bound of the ρ-numbers are of special interest when the Fourier coefficients of f have many gaps.

Carleson's theorem.

The theory about Fourier series containes quite adavanced results. Among these is Carelaon's theorem sabout the almost everywhere convergence of Fourier's partial sums. Let $f(\theta)$ be a complex-valued and continuous function defined on the interval $\{0 \leq \theta \leq 2\pi\}$ which satisfies $f(0) = f(2\pi)$. For every integer n we set

$$\widehat{f}(n) = \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{-in\phi} f(\phi) \cdot d\phi$$

One refers to $\{\widehat{f}(n)\}$ as the Fourier coefficients of f and Fourier's partial sum function of degree N is defined by

$$S_N(\theta) = \sum_{n=-N}^{n=N} \hat{f}(n) \cdot e^{in\theta} \tag{0.0}$$

The question arises if

$$\lim_{N \to \infty} \max_{\theta} |S_N(\theta) - f(\theta)| = 0 \tag{0.1}$$

So (0.1) means that Fourier's partial sums convege uniformyly to f. Examples where (0.1) fails were discovered at an early stage and lead to Gibb's phenomenon. More precisely, there exists contionuous functions f where the uniform not only fails, but for certain θ-values the sequence $\{S_N(\theta)\}$ even fails to converge. A relaxed condition of (0.1) is to ask if the pointwise limit

$$\lim_{N \to \infty} S_N N(\theta) = f(\theta) \tag{0.2}$$

exists for all θ outside a null set in the sense of Lebesgue, i.e. is it true that Fourier's partial sums converge almost everywhere to f. This question was open for more than a half century until the affirmative answer was established by Carleson in 1965. This result constitutes one of the greatest achievements ever in analysis, and the proof goes beyond the level of these notes. The reader may consult Carleson's article [xxx] which includes a remarkable inequality which goes as follows: Let ℓ^2 be the Hilbert space of sequences of complex numbers $c_0, c_1, \ldots$ such that $\sum |c_n|^2 < \infty$. To every such a sequence we introduce trigonometric polynomials

$$S_N(\theta) = \sum_{n=0}^{n=N} c_n e^{in\theta}$$

Define the maximal function by

$$S^*(\theta) = \max_{N \geq 1} |S_N(e^{i\theta})|$$

Carleson's inequality. *There exists a constant C such that the following hold when $\{c_n\} \in \ell^2$:*

$$\int_0^{2\pi} S^*(\theta)^2\, d\theta \leq C \cdot \sum_{n=1}^{\infty} |c_n|^2$$

Bernstein's example.

A remarkable construction was given by S. Bernstein in the article [Comptes Rendus 1914]. Let p be a prime number of the form $4\mu + 1$ where μ is a positive integer. For each integer $n \geq 1$ we have the Legendre symbol $L(n;p)$ which is $+1$ is k has a quadratic remainder modolu p and otherwise $L(n;p) = -1$. Define the trigonometric polynomial

$$\mathcal{B}_p(\theta) = \frac{2}{p^{\frac{3}{2}}} \cdot \sum_{n=1}^{n=p-1} (p-n) \cdot L(n;p) \cdot \cos n\theta$$

Then Bernstein proved that

(i)
$$\max_\theta |\mathcal{B}_p(\theta)| \leq 1$$

At the same time we notice that

(ii)
$$\frac{2}{p^{\frac{3}{2}}} \cdot \sum_{n=1}^{p-1} |(p-n) \cdot L(n;p)| = \frac{p-1}{\sqrt{p}} \simeq \sqrt{p}$$

Bernstein's trigonometric polynomials have extremal properties. For consider an arbitrary cosine series

$$u(\theta) = \sum_{n=1}^{n=N} a_n \cdot \cos\theta$$

Now the L^2-integral

$$\frac{1}{\pi} \int_0^{2\pi} u^2(\theta)\, d\theta = \sum_{n=1}^{n=N} a_n^2$$

If the maximum norm of u is one the L^2 integral is majorized by 2 and the Cauchy-Schwarz inequality gives

$$\sum |a_n| \leq \sqrt{2 \cdot N}$$

Bernstein's example shows that this inequality is essentially sharp. Another notable phenomenon in Bernstein's example is the following: We have

$$\int_0^{2\pi} \mathcal{B}_p^2(\theta)\, d\theta = \frac{4}{p^3} \cdot \pi \cdot \sum_{n=1}^{n=p-1} n^2$$

The right hand side is bounded by an absolute constant C. Hence the maximum norm and the L^2-norm of $\mathcal{B}_p$ are comparable , i.e. there is a fixed constant $0 < c < 1$ such that

$$\frac{c}{\pi} \leq \frac{||\mathcal{B}_p||_\infty}{||\mathcal{B}_p||_2} \leq \frac{1}{\pi c}$$

Remark. Bernstein's construction was based upon arithmetic. Later Salem proved that the Bernstein's example is generic in the sense that by random choice of signs in a given sequence $\{a_k\}$ with prescribed L^2-norm equal to one, the corresponding maximum norms of the partial sums are not so large with "high probabilities". To give an example: Let $N \geq 2$ and consider the family $\mathcal{F}_N$ of cosine series

$$f(\theta) = \frac{1}{\sqrt{N}} \cdot \sum_{n=1}^{n=N} \epsilon_n \cdot \cos n\theta$$

Here$\{\epsilon_n\}$ is random sequence where each ϵ_n is +1 or -1. Notice that the L^2-integrals

$$\int_0^\pi f(\theta)^2\, d\theta = \frac{\pi}{2}$$

Above one has a sample space where each choice of a siugn-sequence $\{\epsilon_n\}$ produces a function in $\mathcal{F}_N$. To be precise, we get 2^N many functions in this family. The evaluation at $\theta = 0$ corerssponds to a Bernoulli trial, i.e,. tossing a coin N times and measure the difference of heads and tails, divided by $\sqrt{N}$. Here the centeral limit theorem applies, i.e by de Moivre's discovery from 1733, the random outcome of the numbers $\{f(0) : F \in \mathcal{F}\}$ is expressed by a discrete random variable whose densities converge to the normal distribution as $N \to \infty$.

A more involved study arises when one regards vakues of the f-functions over the whole interval $[0, \pi]$. Fir example, one can consider the random variabke on the sampåle space above defined by

$$f \mapsto \max_{0 \leq \theta \leq \pi} |f(\theta)|$$

Results about the asymptotic behaviour of the distributions of these random variables as N increases have been obtained by Salem and inspired much later work. The reader may consult [Salem] and [Kahane] for an account about Fourier series with random coefficients. It goes without saying that this leads to a quite involved theory.

A: The kernels of Dini, Fejer and Jackson

Denote by $C^0_{\text{per}}[0,2\pi]$ the family of complex-valued continuous functions $f(\theta)$ on $[0,2\pi]$ which satisfy $f(0)=f(2\pi)$. The Fourier coefficients of such a function f are defined by:

$$\widehat{f}(n)=\frac{1}{2\pi}\cdot\int_0^{2\pi} e^{-in\phi}f(\phi)\cdot d\phi$$

where n are integers. Fourier's partial sum of degree N is defined by

$$S^f_N(\theta)=\sum_{n=-N}^{n=N}\hat{f}(n)\cdot e^{in\theta} \tag{A.0}$$

Exercise. Show that if f is real-valued and N is a positive integer, then the Fourier series takes the form

$$S^f_N(\theta)=\frac{a_0}{2}+\sum_{k=1}^{N} a_k\cdot\cos k\theta+\sum_{k=1}^{N} b_k\cdot\sin k\theta$$

where $a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx$ and when $k\geq 1$:

$$a_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos kx\cdot dx \quad : \quad b_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin kx\cdot dx$$

A.1.The Dini kernel. If $N\geq 0$ we set

$$D_N(\theta)=\frac{1}{2\pi}\sum_{n=-N}^{n=N} e^{in\theta}$$

A.2 Proposition. *One has the formula*

$$D_N(\theta)=\frac{1}{2\pi}\cdot\frac{\sin((N+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}} \tag{*}$$

Proof. We have

$$\sum_{n=-N}^{n=N} e^{in\theta}=e^{-iN\theta}\cdot\sum_{n=0}^{n=2N} e^{in\theta}=e^{-iN\theta}\cdot\frac{e^{i(2N+1)\theta}-1}{e^{i\theta}-1}=$$

$$e^{-iN\theta-i\theta/2}\cdot\frac{e^{i(2N+1)\theta}-1}{2i\cdot\sin\theta/2}=\frac{2i\cdot\sin((N+1/2)\theta)}{2i\cdot\sin\theta/2}$$

and (*) follows after division with $2i$.

A.3 Exercise. Show that the folloeing hold for each $N\geq 0$:

$$S^f_N(\theta)=\int_0^{2\pi} D_N(\theta-\phi)\cdot f(\phi)\cdot d\phi=\int_0^{2\pi} D_N(\phi)\cdot f(\theta+\phi)\cdot d\phi$$

A.4 The Fejer kernel. For each $N\geq 0$ we set

$$\mathcal{F}_N(\theta)=\frac{D_0(\theta)+\ldots+D_N(\theta)}{2\pi(N+1)}$$

A.5 Proposition *One has the formula*

$$\mathcal{F}_N(\theta)=\frac{1}{2\pi(N+1)}\cdot\frac{1-\cos((N+1)\theta)}{2\cdot\sin^2(\frac{\theta}{2})} \tag{**}$$

Proof. To each $\nu \geq 0$ we have $\sin((\nu+1/2)\theta) = \mathfrak{Im}\big[e^{i(\nu+1/2)\theta)}\big]$. Hence $F_N(\theta)$ is the imaginary part of

$$\frac{1}{2\pi(N+1)} \cdot \frac{e^{i\theta/2}}{\sin(\theta/2)} \cdot \sum_{\nu=0}^{\nu=N} e^{i\nu\theta}$$

Next, we have

$$e^{i\theta/2} \cdot \sum_{\nu=0}^{\nu=N} e^{i\nu\theta} = e^{i\theta/2} \cdot \frac{e^{i(N+1)\theta}-1}{e^{i\theta-1}} = \frac{e^{i(N+1)\theta}-1}{2i\cdot\sin(\theta/2)}$$

Since $i^2=-1$ we see that the imaginary part of the last term is equal to

$$\frac{1-\cos((N+1)\theta)}{2\cdot\sin(\frac{\theta}{2})}$$

and then (**) follows.

A.6 Fejer sums. For each f and every $N \geq 0$ we set

$$F_N^f(\theta) = \int_0^{2\pi} \mathcal{F}_N(\phi)\cdot f(\theta+\phi)\cdot d\phi$$

A.7 An inequality. If $a>0$ and $a \leq \theta \leq 2\pi - a$ we have the inequality

(i) $$\sin^2(\theta/2) \geq \sin^2(a/2)$$

Let f be given and denote by $M(f)$ the maximum norm of $|f(\theta)|$ over $[0,2\pi]$. Then (i) gives

$$\int_a^{2\pi-a} \mathcal{F}_N(\phi)\cdot f(\theta+\phi)\cdot d\phi \leq$$

(A.7.1) $$\frac{M}{2\pi(N+1)\cdot\sin^2(a/2)}\int_a^{2\pi-a}(1-\cos(N\phi))\cdot d\phi \leq \frac{2M}{(N+1)\cdot\sin^2(a/2)}$$

A.8 Exercise. Given some θ_0 and $0<a<\pi$ we set

$$\omega_f(a) = \max_{|\theta-\theta_0|\leq a} |f(\theta)-f(\theta_0)|$$

Use (A.7.1) to prove that

$$|F_N^f(\theta_0)-f(\theta_0)| \leq \frac{2M}{(N+1)\cdot\sin^2(a/2)} + \omega_f(a)$$

Conclude that the *uniform continuity* of the function f on $[0,2\pi]$ implies that the sequence $\{F_N^f\}$ converges uniformly to f over the interval $[0,2\pi]$.

A.9 Exercise. Use (A.7-8) to show that there exists an absolute constant C such that

(A.9.1) $$||f-\mathcal{F}_n(f)|| \leq C\cdot\omega_f(\frac{1}{n})\cdot\big(1+\log^+\frac{1}{\omega_f(\frac{1}{n})}\big)$$

hold for all continuous 2π-periodic functions f.

A. 10 The Jackson kernel

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One may ask if (A.9.1) can be improved in the sense that there exists a constant C which is independent of both f and of n such that

(*) $$\max_\theta ||f(\theta)-F_n^f(\theta)| \leq C\cdot\omega_f(\frac{1}{n})$$

Examples show that no such uniform bound expressed by a constant C exists. To obtain an inequality such as (*), D. Jackson introduced a new kernel in his thesis *Über die Genauigkeit der*

Annährerung stegiger funktionen durch ganze rationala funktionen from Göttingen in 1911. To each 2π-periodic and continuous function $f(x)$ on the real line and every $n \geq 1$ we set

$$\mathcal{J}_n^f(x) = \frac{3}{2\pi} \cdot \int_{-\infty}^{\infty} f(x + \frac{2t}{n}) \cdot (\frac{\sin t}{t})^4 \cdot dt$$

A.11 Theorem. *The function $\mathcal{J}_n^f(x)$ is a trigonometric polynomial of degree $2n-1$ at most and one has the inequality*

$$\max_x |f(x) - \mathcal{J}_n^f(x)| \leq (1 + \frac{6}{\pi}) \cdot \omega_f(\frac{1}{n})$$

Proof. The variable substitution $t \to nt$ gives

(1) $$\mathcal{J}_n^f(x) = \frac{3}{2\pi n^3} \cdot \int_{-\infty}^{\infty} f(x + 2t) \cdot (\frac{\sin nt}{t})^4 \cdot dt$$

Since $t \mapsto f(x+2t) \cdot sin^4\, nt$ is π-periodic it follows that (1) is equal to

(2) $$\frac{3}{2\pi n^3} \cdot \int_0^{\pi} f(x + 2t) \cdot \sum_{k=-\infty}^{\infty} \frac{\sin^4(nt)}{(k\pi + t)^4} \cdot dt$$

Next, recall from § XX that

$$\frac{1}{\sin^2 z} = \sum_{k=-\infty}^{\infty} \frac{1}{(z + k\pi)^2}$$

Taking a second derivative when $z = t$ is real it follows that

(3) $$\partial_t^2(\frac{1}{\sin^2 t}) = \frac{1}{6} \cdot \sum_{k=-\infty}^{\infty} \frac{1}{(t + k\pi)^4}$$

Hence we obtain

(*) $$\mathcal{J}_n^f(x) = \frac{1}{4\pi n^3} \cdot \int_0^{\pi} f(x + 2t) \cdot \sin^4(nt) \cdot \partial_t^2(\frac{1}{\sin^2 t})\, dt$$

Next, the function

$$\sin^4(nz) \cdot \partial_z^2(\frac{1}{\sin^2 z})$$

is entire and even and the reader may verify that it is a finite sum of entire cosine-functions which implies that the Jackson kernel is expressed by a finite sum of integrals:

(4) $$\mathcal{J}_f^n(x) = \sum_{k=0}^{2n-1} c_k \int_0^{2\pi} f(u) \cdot \cos\, k(x - u))\, du$$

In particular $\mathcal{J}_f^n(x)$ is a trigonometric polynomial of degree $2n-1$ a most. Integration by parts give the equality

(5) $$\int_{-\infty}^{\infty} (\frac{\sin nt}{t})^4\, dt = \frac{1}{6} \int_0^{\pi} \sin^4 t \cdot \partial_t^2(\frac{1}{\sin^2 t})\, dt = \frac{4}{3} \int_0^{\pi} \cos^2 t\, dt = \frac{2\pi}{3}$$

Next, we leave it to the reader to verify the inequality

(6) $$\frac{3}{2\pi} \int_{-\infty}^{\infty} (1 + 2|t|) \cdot (\frac{\sin t}{t})^4 \cdot dt \leq 1 + \frac{6}{\pi}$$

From the above where we use (1) and (*) it follows that

(7) $$\mathcal{J}_n^f(x) - f(x) = \frac{3}{2\pi} \cdot \int_{-\infty}^{\infty} [f(x + \frac{2t}{n}) - f(x)] \cdot (\frac{\sin t}{t})^4 \cdot dt$$

Now

$$|f(x + \frac{2t}{n}) - f(x)| \leq \omega_f(\frac{2t}{n}) \leq (2|t| + 1) \cdot \omega_f(\frac{1}{n})$$

where the last equality follows from Lemma XX. Hence (7) gives

$$\max_x |\mathcal{J}_n^f(x) - f(x)| \leq \omega_f(\frac{1}{n}) \cdot \frac{3}{2\pi} \cdot \int_{-\infty}^{\infty} (2|t|+1) \cdot (\frac{\sin t}{t})^4 \cdot dt$$

Finally, by (6) the last factor is majorized by $1 + \frac{6}{\pi}$ and Jackson's inequality follows.

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A.12 A lower bound for polynomial approximation.

Denote by $\mathcal{T}_n$ the linear space of trigonometric polynomials of degree $\leq n$. For a 2π-periodic and continuous function f we put

$$\rho_f(n) = \min_{T \in \mathcal{T}_n} ||f - T||$$

where $||\cdot||$ denotes the maximum norm over $[0, 2\pi]$. We shall establish a lower bound for the ρ-numbers when certain sign-conditions hold for Fourier coefficients. In general, let f be a periodic function and for each positive integer n we find $T \in \mathcal{T}_n$ such that $||f - T|| = \rho_f(n)$. Since Fejer kernels do not increase maximum norms one has

(i) $$||F_k^f - F_k^T|| \leq \rho_f(n)$$

for every positive integer k. Apply this with $k = n$ and $k = n + p$ where p is another positive integer. If $T \in \mathcal{T}_n$ the equation from Exercise XX gives

(ii) $$T = \frac{(n+p) \cdot \mathcal{F}_{n+p}(T) - n \cdot \mathcal{F}_n(T)}{p}$$

Since (i) hold for n, $n + p$ and $||f - T|| \leq \rho_f(n)$, the triangle inequality gives

(iii) $$||f - \frac{(n+p) \cdot \mathcal{F}_{n+p}(f) - n \cdot \mathcal{F}_n(f)}{p}|| \leq 2 \cdot \frac{n+p}{p} \cdot \rho_f(n)$$

Next, by the formula (§ xx) it follows that (iii) gives

$$||f - \frac{S_n(f) + \cdots S_{n+p-1}(f)}{p}|| \leq 2 \cdot \frac{n+p}{p} \cdot \rho_f(n)$$

In particular we take $p = n$ and get the inequality

(*) $$||f - \frac{S_n(f) + \cdots S_{2n-1}(f)}{n}|| \leq \frac{4}{n} \cdot \rho_f(n)$$

A.12 A special case. Assume that $f(x)$ is an even function on $[-\pi, \pi]$ which gives a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos kx$$

A.12 Proposition *Let f be even as above and assume that $a_k \leq 0$ for every $k \geq 1$. Then the following inequality holds for every $n \geq 1$:*

$$f(0) - \frac{S_n(f)(0) + \cdots S_{2n-1}(f)(0)}{n} \leq -\sum_{k=2n}^{\infty} a_k$$

The easy verification is left to the reader. Taking the maximum norm over $[-\pi, \pi]$ it follows from (*) that

(**) $$\rho_f(n) \geq \frac{n}{4} \cdot \sum_{k=2n}^{\infty} |a_k|$$

holds when the sign conditions on the Fourier coefficients above are satisfied. Notice that (**) means that one has a lower bound for polynomial approximations of f.

A.13 The function $f(x) = \sin|x|$ It is obvious that

$$\omega_f(\frac{1}{n}) = \frac{1}{n}$$

Next, the periodic function $f(x($ is even and hence we only get a cosine-series. For each positive integer m we have:

$$a_k = \frac{2}{\pi} \int_0^\pi \sin x \cdot \cos kx \cdot dx$$

To evaluate these integrals we use the trigonometric formula

$$sin\,(k+1)x - \sin(k-1)x = 2\sin x \cdot \cos kx$$

Now the reader can verify that $a_\nu = 0$ when ν is odd while

$$a_{2k} = -\frac{4}{\pi} \cdot \frac{1}{2k^2 - 1}$$

Hence the requested sign conditions hold and (**) entails that

$$\rho_f(n) \geq \frac{n}{\pi} \cdot \sum_{k=n}^\infty \frac{1}{2k^2 - 1}$$

Here the right hand side is $\geq \frac{C}{n}$ for a constant C which is independent of n. So this example shows that the inequality (*) in § A.11 is sharp up to a multiple with a fixed constant.

B. Legendre polynomials.

If $n \geq 1$ we denote by $\mathcal{P}_n$ the linear space of real-valued polynomials of degree $\leq n$. An inner product is defined by is defined by

$$\langle q, p \rangle = \int_{-1}^{1} q(x)p(x) \cdot dx$$

Since $1, x, \ldots, x^{n-1}$ generate a subspace of co-dimension one in $\mathcal{P}_n$ we get:

B.1 Proposition. *There exists a unique $Q_n(x) = x^n + q_{n-1}x^{n-1} + \ldots + q_0$ such that*

$$\int_{-1}^{1} x^\nu \cdot Q_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

To find $Q_n(x)$. we consider the polynomial $(1-x^2)^n$ which vanishes up to order n at the end-points 1 and -1. Its the derivative of order n gives a polynomial of degree n and partial integrations show that

$$\int_{-1}^{1} x^\nu \cdot \partial^n((x^2-1)^n)) \cdot dx = 0 \leq \nu \leq n-1$$

The leading coefficient of x^n in $\partial^n((x^2-1)^n))$ becomes

$$c_n = 2n(2n-1)\cdots(n+1)$$

Hence we have

$$Q_n(x) = \frac{1}{c_n} \cdot \partial^n((x^2-1)^n)$$

B.2 Definition. *The Legendre polynomial of degree n is given by*

$$P_n(x) = k_n \cdot \partial^n((x^2-1)^n)$$

where the constant k_n is determined so that $P_n(1) = 1$.

Since P_n is equal to Q_n up to a constant we still have

$$\int_{-1}^{1} x^\nu \cdot P_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

From this we conclude that

$$\int_{-1}^{1} P_n(x) \cdot P_m x)dx = 0 \quad n \neq m$$

Thus, $\{P_n\}$ is an orthogonal family with respect to the inner product defined above.

B.3 A generating function. Let w be a new variable and set

$$\phi(x, w) = 1 - 2xw + w^2$$

Notice that $\phi \neq 0$ when $-1 \leq x \leq 1$ and $|w| < 1$. Keeping $-1 \leq x \leq 1$ fixed we have the function

$$w \mapsto \frac{1}{\sqrt{1 - 2xw + w^2}}$$

Next, as $|\zeta| < 1$ one has the Newton series

$$\frac{1}{\sqrt{1-\zeta}} = \sum g_n \cdot \zeta^n \quad \text{where} \quad g_n = \frac{3 \cdot 5 \cdots (2n-1)}{2^n}$$

It follows that

$$\frac{1}{\sqrt{1 - 2xw + w^2}} = \sum g_n (2xw - w^2)^2$$

With x kept fixed the series is expanded into w-powers, i.e. set

$$\frac{1}{\sqrt{1 - 2xw + w^2}} = \sum \rho_n(x) \cdot w^n$$

It is easily seen that as x varies then $\rho_n(x)$ is a polynomial of degree n. Moreover, we notice that the coefficient of x^n in $\rho_n(x)$ is equal to

$$g_n \cdot 2^n$$

Next, if $x = 1$ we have

$$\frac{1}{\sqrt{1-2w+w^2}} = \frac{1}{1-w} = \sum w^n$$

From this we conclude that

$$\rho_n(1) = 1 \quad \text{for all} \quad n \geq 0$$

B.4 Theorem. *One has the equality $\rho_n(x) = P_n(x)$ for each n, i.e.*

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum P_n(x) \cdot w^n \quad \text{holds when} \quad -1 \leq x \leq 1 : \, |w| < 1$$

B.5 Exercise. Prove this result.

B.6 The series for $P_n(\cos\theta)$. With x replaed by $\cos\theta$ we notice that

$$1 - 2\cos\theta \cdot w + w^2 = (1 - e^{i\theta}w)(1 - e^{-i\theta}w)$$

It follows that

$$\frac{1}{\sqrt{1-2\mathrm{cos}(\theta)w+w^2}} = \frac{1}{\sqrt{1-1-e^{i\theta}w)}} \cdot \frac{1}{\sqrt{1-e^{-i\theta}w)}}$$

The last product becomes

$$\sum\sum g_m e^{im\theta} w^m \cdot g_\nu e^{-i\nu\theta} w^\nu$$

Collecting w powers the double sum becomes

$$\sum \gamma_n(\theta) \cdot w^n \quad \gamma_n(\theta) = \sum_{m+\nu=n} g_m g_\nu e^{i(m-\nu)\theta}$$

By Theorem B.4 the last sum represents $P_n(\cos(\theta))$. One has for example

$$P_3(\cos(\theta) = 2g_3 \cdot \cos(3\theta) + 2g_2 g_1 \cdot \cos(\theta)$$

where we used that $g_0 = 1$.

B.7 An inequality for $|P(x)|$. Since the g-numbers are ≥ 0 we obtain

$$|P_n(\cos(\theta)| \leq g_n g_0 + g_{n-1}g_1 + \ldots + g_1 g_{n-1} + g_0 g_n = P_n(1) \quad : \; 0 \leq \theta \leq 2\pi$$

Hence we have proved

B.8 Theorem. *For each n one has*

$$|P_n(x)| \leq 1 \quad : \quad -1 \leq x \leq 1$$

Next, we study the values when $x > 1$. Here one has

B.9 Theorem. *For each $x > 1$ one has*

$$1 < P_1(x) < P_2(x) < \ldots$$

Proof. Let us put

$$\psi(x.w) = 1 + \sum_{n=1}^{\infty} [P_n(x) - P_{n-1}(x)] \cdot w^n$$

By Theorem B.4 this is equal to

$$(*) \qquad \frac{1-w}{\sqrt{1-2xw+w^2}}$$

With $x > 1$ we set $x = 1 + \xi$ and notice that

$$1 - 2xw + w^2 = (1-w)^2 - 2\xi w$$

Hence (*) becomes

$$(**)\qquad \frac{1}{\sqrt{1-\frac{2\xi w}{1-w^2}}}=\sum g_n\cdot\frac{(2\xi w)^n}{(1-w^2)^n}=\sum g_n\cdot(2\xi)^n\cdot\frac{w^n}{(1-w^2)^n}$$

Next, for each $n\geq 1$ we notice that the power series of $\frac{w^n}{(1-w^2)^n}$ has positive coefficients. Since $g_n(2\xi)^n>0$ also hold we conclude that (**) is of the form

$$1+\sum_{n=1}^{\infty} q_n(\xi)\cdot w^n \quad \text{where} \quad q_n(\xi)>0$$

Finally, Theorem B.9 follows since

$$P_n(1+\xi)-P_{n-1}(1+\xi)=q_n(\xi)$$

B.10 An L^2-inequality.

Let $n\geq 1$ and denote by $\mathcal{P}_n[1]$ the space of real-valued polynomials $Q(x)$ of degree $\leq n$ for which $\int_{-1}^1 Q(x)^2\cdot dx=1$ and set

$$\rho(n)=\max_{Q\in\mathcal{P}-n[1]}|Q|_\infty$$

where $|Q|_\infty$ is the maximum norm over $[-1,1]$. To find $\rho(n)$ we use the orthonormal basis $\{P_k^*\}$ and write

$$Q(x)=t_0\cdot P_0^*(x)+\ldots+t_n\cdot P_n^*(x)$$

Since $Q\in\mathcal{P}_n[1]$ we have $t_0^2+\ldots+t_n^2=1$. Recall also that

$$P_\nu^*(x)=\sqrt{\frac{2\nu+1}{2}}\cdot P_\nu(x)$$

Given $-1\leq x_0\leq 1$ the Cauchy-Schwarz inequality gives

$$Q(x_0)^2\leq\sum_{\nu=0}^{\nu=n}\frac{2\nu+1}{2}\cdot|P_\nu(x_0)|\leq\sum_{\nu=0}^{\nu=n}\frac{2\nu+1}{2}$$

where the last inequality follows since the maximum norm of each P_ν is ≤ 1. Finally, we notice that

$$\sum_{\nu=0}^{\nu=n}\frac{2\nu+1}{2}=\frac{(1-n)^2}{2}$$

We conclude that

$$|Q(x_0)|\leq\frac{n+1}{\sqrt{2}}$$

B.11 The case of equality. To have equality above we take $x_0=1$ and

$$t_\nu=\alpha\cdot P_\nu^*(1)\quad:\quad \nu\geq 0$$

C. The space $\mathcal{T}_n$

Let $n \geq 1$ be a positive integer. A real-valued trigonometric polynomial of degree $\leq n$ is given by

$$g(\theta) = a_0 + a_1 \cos\theta + \ldots + a_n \cos n\theta + b_1 \sin\theta + \ldots b_n \sin n\theta$$

Here $a_0, \ldots, a_n, b_1, \ldots, b_n$ are real numbers. The space of such functions is denoted by $\mathcal{T}_n$ which is a vector space over $\mathbf{R}$ of dimension $2n+1$. We can write

$$\cos kx = \frac{1}{2}[e^{ikx} + e^{-ikx}] \quad \text{and} \quad \sin kx = \frac{1}{2i}[e^{ikx} - e^{-ikx}] \quad : \; k \geq 1$$

It follows that there exist complex numbers $c_0, \ldots, c_{2n}$ such that

$$g(\theta) = e^{-in\theta} \cdot [c_0 + c_1 e^{i\theta} + \ldots + c_{2n} e^{i2n\theta}]$$

Exercise. Show that

$$c_\nu + c_{2n-\nu} = 2a_\nu \quad \text{and} \quad c_\nu - c_{2n-\nu} = 2b_\nu \implies$$

$$c_{2n-\nu} = \bar{c}_\nu \quad 0 \leq \nu \leq n$$

C.1 The polynomial $G(z)$. Given $g(\theta)$ as above we set

$$G(z) = c_0 + c_1 z + \ldots + c_{2n} z^{2n} \implies e^{-in\theta} \cdot G(e^{i\theta}) = g(\theta)$$

C.2 Exercise. Set

$$\bar{G}(z) = \bar{c}_0 + c_1 z + \ldots + \bar{c}_{2n} z^{2n}$$

and show that

(*) $$z^{2n} G(1/z) = \bar{G}(z)$$

Use this to show that if $0 \neq z_0$ is a zero of $G(z)$ then $\frac{1}{\bar{z}_0}$ is also a zero of $G(z)$.

C.3 The case when $g \geq 0$. Assume that the g-function is non-negative. Let

$$0 \leq \theta_1 < \ldots < \theta_\mu < 2\pi$$

be the zeros on the half-open interval $[0, 2\pi)$. Since $g \geq 0$ every such zero has a multiplicity given by an *even* integer. Consider also the polynomial $G(z)$. Exercise C.2 shows that $\{e^{i\theta_\nu}\}$ are complex zeros of $G(z)$ whose multiplicities are even integers. Next, if ζ is a zero where $\zeta \neq 0$ and $|\zeta| \neq 1$, then (*) in C.2 implies that $\frac{1}{\bar{\zeta}}$ also is a zero and hence $G(z)$ has a factorisation

$$G(z) = c_{2n} \cdot \prod_{\nu=1}^{\nu=\mu} (z - e^{i\theta_\nu})^{2k_\nu} \cdot \prod_{j=1}^{j=m} (z - \zeta_j)(z - \frac{1}{\bar{\zeta}_j}) \cdot z^{2r} \quad \text{where} \quad 2\mu + 2m + 2r = 2n$$

Here $0 < |\zeta_j| < 1$ hold for each j and it may occur that multiple zeros appear, i.e. the ζ-roots need not be distinct and the integer r may be zero or positive.

C.4 The h-polynomial. Let $\delta = \sqrt{|\zeta_1| \cdots |\zeta_m|}$ and put

$$h(z) = c_{2n} \dot{\delta} \cdot \prod_{\nu=1}^{\nu=\mu} (z - e^{i\theta_\nu})^{k_\nu} \cdot \prod_{j=1}^{j=m} (z - \zeta_j) \cdot z^r$$

C.5 Proposition. *One has the equality*

$$|h(e^{i\theta})|^2 = g(\theta)$$

Proof. With $z = e^{i\theta}$ and $0 < |\zeta| < 1$ one has

$$(e^{i\theta} - \zeta)(e^{i\theta} - \frac{1}{\bar{\zeta}}) = (e^{i\theta} - \zeta) \cdot (\bar{\zeta} - e^{-i\theta}) \cdot e^{i\theta} \cdot \frac{1}{\bar{\zeta}}$$

Passing to absolute values it follows that

$$\left|(e^{i\theta} - \zeta)(e^{i\theta} - \frac{1}{\bar{\zeta}})\right| = \left|e^{i\theta} - \zeta\right|^2 \cdot \frac{1}{|\zeta|}$$

Apply this to every root ζ_ν and take the product which gives Proposition C.5.

C.6 Application. Let $g \geq 0$ be as above and assume that the constant coefficient $a_0 = 1$. This means that

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} g(\theta) \cdot d\theta$$

With $h(z) = d_0 + d_1 z + \ldots + d_n z^n$ we get

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} h(e^{i\theta})|^2 \cdot d\theta = |d_0|^2 + \ldots + |d_n|^2$$

Notice that

(i) $$|d_n|^2 = |c_{2n}| \cdot \delta \quad \text{and} \quad |d_0|^2 = |c_{2n} \cdot \delta| \cdot \prod |\zeta_j|^2 = |c_{2n}| \cdot \frac{1}{\delta}$$

From this we see that

(iii) $$|c_{2n}| \cdot (\delta + \frac{1}{\delta}) = |d_0|^2 + d_n|^2 \leq 1$$

Here $0 < \delta < 1$ and therefore $\delta + \frac{1}{\delta} \geq 2$ which together with (iii) gives

$$|c_{2n}| \leq \frac{1}{2}$$

At the same time we recall that

(*) $$c_{2n} = \frac{a_n + ib_n}{2} \implies |a_n + ib_n| \leq 1$$

Summing up we have proved the following:

C.7 Theorem. *Let $g(\theta)$ be non-negative in $\mathcal{T}_n$ with constant term $a_0 = 1$. Then*

$$|a_n + ib_n| \leq 1$$

C.8 An application. Let $n \geq 1$ and consider the space of all monic polynomials

$$P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0$$

where $\{c_\nu\}$ are real- To each such polynomial we can consider the maximum norm over the interval $[-1, 1]$. Then one has

C.9 Theorem. *For each $P \in \mathcal{P}_n^*$ one has the inequality*

$$\max_{-1 \leq x \leq 1} |P(x)| \geq 2^{-n+1}$$

Proof. Consider some $P \in \mathcal{P}_n^*$ and define the trigonometric polynomial

$$g(\theta) = (\cos\theta)^n + c_{n-1}(\cos\theta)^{n-1} + \ldots + c_0$$

So here $P(\cos\theta) = g(\theta)$ and Theorem C.9 follows if we have proved that

(1) $$2^{-n+1} \geq \max_{0 \leq \theta \leq 2\pi} |g(\theta))|$$

To prove this we set $M = \max_{0 \leq \theta \leq 2\pi} |g(\theta))|$. Next, we can write

$$g(\theta) = a_0 + a_1 \cos\theta \ldots + a_n \cos n\theta$$

Moreover, since

$$(\cos\theta)^n = 2^{-n} \cdot [e^{i\theta} + e^{-\theta}]^n$$

we get

$$a_n = 2^n$$

Now we shall apply Theorem C.8. For this purpose we construct non-negative trigonometric polynomials. First we define

$$g^*(\theta) = \frac{M - g(\theta)}{M - a_0}$$

Then $g^* \geq 0$ and its constant term is 1. We have also

$$g^*(\theta) = 1 - \frac{1}{M - a_0} \cdot \sum_{\nu=1}^{\nu=n} a_\nu \cos\nu\theta$$

Hence Theorem C.7 gives

(1) $$\frac{1}{|M-a_0|}\cdot|a_n|\leq 1 \implies |M-a_0|\geq 2^{-n+1}$$

Next, we have also the function

$$g_*(\theta)=\frac{M+g(\theta)}{M+a_0}$$

In the same way as above we obtain:

(2) $$|M+a_0|\geq 2^{-n+1}$$

Finally, (1) and (2) give

$$M\geq 2^{-n+1}$$

which proves Theorem C.9

D. Tchebysheff polynomials.

The inequality in Theorem C.9 is sharp. To see this we shall construct a special polynomial $T_n(x)$ of degree n. Namely, with $n \geq 1$ we can write

$$\cos n\theta = 2^{n-1} \cdot (\cos\theta)^n + c_{n-1} \cdot (\cos\theta)^{n-1} + \ldots + c_0$$

Set

$$T_n(x) = 2^{n-1}x^n + c_{n-1} \cdot x^{n-1} + \ldots + c_0$$

Hence

$$T_n(\cos\theta) = \cos n\theta$$

We conclude that the polynomial

$$p_n(x) = 2^{-n+1} \cdot T_n(x)$$

belongs to $\mathcal{P}_n^*$ and its maximum norm is 2^{-n+1} which proves that the inequality in Theorem C.9 is sharp.

D.1 Zeros of T_n. Set

$$\theta_\nu = \frac{\nu\pi}{n} + \frac{\pi}{2n}$$

It is clear that $\theta_1, \ldots, \theta_n$ are zeros of $\cos n\theta$. Hence the zeros of $T_n(x)$ are:

$$x_\nu = \cos\theta_\nu$$

Notice that

$$-1 < x_n < \ldots < x_1 < 1$$

Since $T_n(x)$ is a polynomial of degree n it follows that $\{x_\nu\}$ give all zeros and we have

$$T_n(x) = 2^{n-1} \cdot \prod (x - x_\nu)$$

D.2 Exercise. Show that

$$T_n'(x_\nu) \cdot \sqrt{1 - x_\nu^2} = n$$

hold for every zero of $T_n(x)$.

D.3 An interpolation formula. Since $x_1, \ldots, x_n$ are distinct it follows that if $p(x) \in \mathcal{P}_{n-1}$ is a polynomial of degree $\leq n-1$ then

$$p(x) = \sum_{\nu 01}^{\nu=n} p(x_\nu) \cdot \frac{1}{T'(x_\nu)} \cdot \frac{T(x)}{x - x_\nu}$$

By the exercise above we get

$$p(x) = \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} (-1)^{\nu-1} p(x_\nu) \cdot \sqrt{1 - x_\nu^2} \cdot \frac{T(x)}{x - x_\nu}$$

We shall use the interpolation formula above to prove

D.4 Theorem *Let $p(x) \in \mathcal{P}_{n-1}$ satisfy*

(1)
$$\max_{-1 \leq x \leq 1} \sqrt{1 - x^2} \cdot |p(x)| \leq 1$$

Then it follows that

(2)
$$\max_{-1 \leq x \leq 1} |p(x)| \leq n$$

Proof. First, consider the case when

(*)
$$-\cos\frac{\pi}{2n} \leq x \leq \cos\frac{\pi}{2n}$$

Then we have

$$\sqrt{1 - x^2} \geq \sqrt{1 - \cos^2\frac{\pi}{2n}} = \sin\frac{\pi}{2n}$$

Next, recall the inequality $\sin x \geq \frac{2}{\pi} \cdot x$. It follows that

$$\sqrt{1-x^2} \geq \frac{1}{n}$$

So when (1) holds in the theorem we have

$$|p(x)| = \frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} \cdot |p(x)| \leq \frac{1}{\sqrt{1-x^2}} \leq n$$

Hence the required inequality in Theorem D.4 holds when x satisfies (*) above. Next, suppose that

(**) $$x_1 \leq x \leq 1$$

On this interval $T_n(x) \geq 0$ and from the interpolation formula xx and the triangle inequality we have

$$|p(x) \leq \frac{1}{n} \sum_{\nu=1}^{\nu=n} \sqrt{1-x_\nu^2} \cdot |p(x_\nu)| \cdot \frac{T(x)}{x-x_\nu} \leq \frac{1}{n} \sum_{\nu=1}^{nu=n} \frac{T(x)}{x-x_\nu}$$

Next, the sum

$$\frac{T(x)}{x-x_\nu} = T_n'(x) = n \cdot U_{n-1}(x)$$

So when (**) holds we have

(***) $$|p(x)| \leq |U_{n-1}(x)|$$

By xx the maximum norm of U_{n-1} over $[-1,1]$ is n and hence (***) gives

$$|p(x)| \leq n$$

In the same way one proves htat

$$-1 \leq x \leq x_n \implies |p(x)| \leq n$$

Together with the upper bound in the case (xx) we get Theorem D.4.

D.5 Bernstein's inequality.

Let $g(\theta) \in \mathcal{T}_n$. The derivative $g'(\theta)$ is another trigonometric polynomial and we have

Theorem. *For each $g \in \mathcal{T}_n$ one has*

$$\max_{0 \leq \theta \leq 2\pi} |g'(\theta)| \leq n \cdot \max_{0 \leq \theta \leq 2\pi} |g(\theta)|$$

Before we prove this result we establish an inequality for certain trigonometric polynomials. Namely, consider a real-valued sine-polynomial

$$S(\theta) = c_1 \sin(\theta) + \ldots + c_n \sin(n\theta)$$

Now $\theta \mapsto \frac{S(\theta)}{\sin\theta}$ is an even function of θ and therefore one has

$$\frac{S(\theta)}{\sin\theta} = a_0 + a_1 \cos\theta + \ldots + a_{n-1}(\cos\theta)^{-n-1}$$

Consider the polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$$

Then e see that:

$$|p(\cos\theta)| = \frac{|S(\theta)|}{\sqrt{1-\cos^2\theta}}$$

Using this we apply Theorem D.4 to the polynomial $p(x)$ and conclude

D.6 Theorem. *Let $S(\theta) = c_1 sin(\theta) + c_n sin(n\theta)$ be a sine-polynomial as above. Then*

$$\max_{0 \leq \theta \leq 2\pi} \frac{|S(\theta)|}{\sin\theta} \leq n \cdot \max_{0 \leq \theta \leq 2\pi} |S(\theta)|$$

D.7 Proof of Bernstein's theorem. Fix an arbitrary $0 \leq \theta - 0 < 2\pi$. Set

$$S(\theta) = g(\theta_0 + \theta) - g(\theta_0 - \theta)$$

We notice that $S(\theta$is an odd polynomial of θ and $S(0) = 0$, It follows that $S(\theta)$ is a sine-polynomial as above of degree $\leq n$. Notice also that

$$\max_{0 \leq \theta \leq 2\pi} |S(\theta)| \leq 2 \cdot \max_{0 \leq \theta \leq 2\pi} |g(\theta)| \max_{0 \leq \theta \leq 2\pi} |g(\theta)|$$

Theorem D.6 applied to $S(\theta)$ gives

$$\text{(i)} \qquad |\frac{g(\theta_0 + \theta) - g(\theta_0 - \theta)}{\sin \theta}| \leq 2n \cdot \max_{0 \leq \theta \leq 2\pi} |g(\theta)|$$

Next, in the left hand side we can take the limit as $\theta| \to 0$ and notice that

$$2 \cdot g'(\theta_0) = \lim_{\theta \to 0} \frac{g(\theta_0 + \theta) - g(\theta_0 - \theta)}{\sin \theta}$$

Hence (i) gives

$$|g'(\theta_0)| \leq n \cdot \max_{0 \leq \theta \leq 2\pi} |g(\theta)|$$

Finally, since θ_0 was arbitrary we get Bernstein's theorem.

E. Fejers sine series and Gibb's phenomenon.

Several remarkable inequalities for trigonometric polynomials were established by Fejer in [Fejer] where a central issue is to find trigonometric polynomials expressed by a sine series which are ≥ 0 on the interval $[0, \pi]$. Consider as an example is the sine-series

$$S_n(\theta) = \sum_{k=1}^{k=n} \frac{\sin k\theta}{k}$$

E.1 Theorem. *For every $n \geq 1$ one has the inequality*

$$0 < S_n(\theta) \leq 1 + \frac{\pi}{2} \quad : \quad 0 < \theta < \pi$$

The upper bound was proved by in [Fej] and Fejer conjectured that $S_n(\theta)$ stays positive on $(0, \pi)$. This was confirmed in articles by Jackson in [xx] and Cronwall in [xx]. The series (*) has a connection with Gibb's phenomenon and Theorem E.1 can be illustrated by drawing graphs of the S-polynomials where the situation when $\theta = \pi - \delta$ for small positive δ has special interest. Since cos $\pi = -1$ the positivity entails that

$$(*) \qquad \sum_{k=2}^{n} (-1)^k \cdot \frac{\sin k\delta}{k} \geq \sin \delta \quad \text{hold for every} \quad n \geq 2 \quad \text{and small} \quad \delta > 0$$

Exercise. Prove Theorem E.1 or consult the literature. It is also instructive to confirm (*) by numerical experiments with a computer.

E.2 Mehler's integral formula. In XX we introduced the Legendre polynomials. It turns out that

$$(*) \qquad \mathcal{P}_n(x) = \sum_{\nu=1}^{\nu=n} P_\nu(x) > 0 \quad : \quad -1 < x < 1$$

is strictly positive for each $-1 < x < 1$.

Exercise. Prove (*) using Theorem E.1 and Mehler's integral formula

$$(*) \qquad \mathcal{P}_n(\cos \theta) = \frac{2}{\pi} \cdot \int_0^\pi \frac{\sin(n + \frac{1}{2})\phi \cdot d\phi}{\sqrt{2 \cos \theta - 2 \cos \phi}}$$

F. Convergence of arithmetical means

Let $f(x)$ be a real-valued and square integrable function on $(-\pi, \pi)$, i.e.

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$$

We say that f has a determined value $A = f(0)$ at $x = 0$ if the following two conditions hold:

(i) $$\lim_{\delta \to 0} \frac{1}{\delta} \cdot \int_0^{\delta} |f(x) + f(-x) - 2A| \, dx = 0$$

(ii) $$\int_0^{\delta} |f(x) + f(-x) - 2A|^2 \, dx \leq C \cdot \delta \quad \text{holds for some constant} \quad C$$

Remark. In the same way we can impose this condition at every point $-\pi < x_0 < \pi$. To simplify the subsequent notations we take $x = 0$. If $x = 0$ is a Lebesgue point for f and A the Lebesgue value we have (i). Hence Lebesgue's Theorem entails that (i) holds almost everywhere when $x = 0$ is replaced by other points x_0. We leave it to the reader to show that the second condition also is valid almost everywhere when f is square integrable but in general there appears a null set $\mathcal{N}$ where (ii) fails to hold while $\mathcal{N}$ contains some Lebegue points. Next, expand f in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

and with $x = 0$ we consider the partial sums

$$s_n(0) = \frac{a_0}{2} + a_1 + \ldots + a_n + b_1 + \ldots + b_n$$

The result below is proved in [Carleman] and shows that $\{s_n\}$ are close to the determined value for many n-values.

F.1 Theorem. *Assume that f has a determined value A at $x = 0$. Then the following hold for every positive integer k*

(*) $$\lim_{n \to \infty} \frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} |s_\nu - A|^k = 0$$

Remark. Carleson's theorem asserts that $\{s_n(x)\}$ converge to $f(x)$ almost everywhere when $f \in L^2$. When a pointwise convergence holds the limit formula (*) is obvious. However, it is in general not true that the pointwise convergence exists at *every point* where f has a determined value. So "ugly points" may appear in a null-set where pointwise convergence fails and here Carleman's result offers a substitute.

The case when $f \in$ BMO(T). If f has bounded mean oscillation the results from § XX in Special Topics show that the conditions (i-ii) hold at every Lebesgue point of f. So here one has a control for averaged Fourier series of f expressed via its set of Lebesgue points.

The case when f is continuous. Here (i-ii) hold everywhere so the averaged limit formulas hold at every point. We can say more since f is uniformly continuous. Let $\omega_f(\delta)$ be the modulos of continuity function and for each $n \geq 1$, $||s_n - f||$ is the maximum norm of $s_n - f$ over $[0, 2\pi]$. Set

$$\mathcal{D}_n(f) = \sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} ||s_\nu - f||^2}$$

F.2 Theorem. *There exists an absolute constant K such that the following hold for every continuous function f with maximum norm ≤ 1:*

$$\mathcal{D}_n(f) \leq K \cdot \left[\frac{1}{\sqrt{n}} + \omega_f\left(\frac{1}{n}\right)\right]$$

Proof of Theorem F.1

Set $A = f(0)$ and $s_n = s_n(0)$. Introduce the function:

$$\phi(x) = f(x) + f(-x) - 2A$$

Applying Dini's kernel we have

$$s_n - A = \int_0^\pi \frac{\sin(n+1/2)x}{\sin x/2} \cdot \phi(x) \cdot dx$$

By trigonometric formulas the integral is expressed by three terms for each $0 < \delta < \pi$:

$$\alpha_n = \frac{1}{\pi} \cdot \int_0^\delta \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx$$

$$\beta_n = \frac{1}{\pi} \cdot \int_\delta^\pi \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx$$

$$\gamma_n = \frac{1}{\pi} \cdot \int_0^\pi \cos nx \cdot \phi(x) \cdot dx$$

By Hölder's inequality it suffices to show Theorem F.1 if $k = 2p$ is an even integer. Minkowski's inequality gives

$$(1) \qquad \Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq \Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p}$$

Denote by $o(\delta)$ small ordo and $O(\delta)$ is big ordo. When $\delta \to 0$ we shall establish the following:

$$\text{(i)} \qquad \Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} = n^{1+1/2p)} \cdot o(\delta)$$

$$\text{(ii)} \qquad \Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} \leq K \cdot p \cdot \delta^{-1/2p}$$

$$\text{(iii)} \qquad \Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p} \leq K$$

In (ii-iii) K is an absolute constant which is independent of p, n and δ. Let us first see why (i-iii) give Theorem F.1. Write $o(\delta) = \epsilon(\delta) \cdot \delta$ where $\epsilon(\delta) \to 0$. With these notations (1) gives:

$$(*) \qquad \Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq n^{1+1/2p} \cdot \delta \cdot \epsilon(\delta) + Kp \cdot \delta^{-1/2p} + K$$

Next, let $\rho > 0$ and choose b so large that

$$pKb^{-1/2p} < \rho/3$$

Take $\delta = b/n$ and with n large it follows that $\epsilon(\delta)$ is so small that

$$b \cdot \epsilon(b/n) < \rho/3$$

Then right hand side in (*) is majorized by

$$\frac{2\rho}{3} \cdot n^{1/2p} + K$$

When n is large we also have

$$K \leq \frac{\rho}{3} \cdot n^{1/2p}$$

Hence the left hand side in (*) is majorized by $\rho \cdot n^{1/2p}$ for all sufficiently large n. Since $\rho > 0$ was arbitrary we get Theorem F.1 when the power is raised by $2p$.

Proof of (i-iii)

To obtain (i) we use the triangle inequality which gives the following for every integer $\nu \geq 1$:

$$|a_\nu| \leq \frac{2}{\pi} \cdot \max_{0 \leq x \leq \delta} |\sin \nu x \cdot \cot x/2| \cdot \int_0^\delta |\phi(x)|\, dx = \nu \cdot o(\delta) \tag{1}$$

where the small ordo δ-term comes from the hypothesis expressed by (*) in the introduction. Hence the left hand side in (i) is majorized by

$$[\sum_{\nu=1}^{\nu=n} \nu^{2p}]^{\frac{1}{2p}} \cdot o(\delta) = n^{1+1/2p} \cdot o(\delta)$$

which was requested to get (i). To prove (iii) we notice that

$$\gamma_0^2 + 2 \cdot \sum_{\nu=1}^{\infty} \gamma_\nu^2 = \frac{1}{\pi} \int_0^\pi |\phi(x)|^2\, dx$$

Next, we have

$$\sum_{\nu=1}^{\infty} |\gamma_\nu|^{2p} \leq [\sum_{\nu=1}^{\infty} |\gamma_\nu|^2]^{1/2p} \leq K$$

where K exists since ϕ is square-intergable on $[0, \pi]$.

Proof of (ii). Here several steps are required. For each $0 < s < \pi$ we define the function $\phi_s(x)$ by

$$\phi_s(x) = \phi(x) \quad : \quad 0 < x < s$$

and extend it to an odd function, i.e. $\phi_s(-x) = -\phi_s(x)$ while $\phi_s(x) = 0$ when $|x| > s$. This odd function has a sine series

$$\phi_s(x) = \sum_{\nu=1}^{\infty} c_\nu(s) \cdot \sin x \tag{1}$$

Let us also introduce the functions

$$\rho(s) = \int_0^s |\phi(x)| \cdot dx \quad \text{and} \quad \Theta(s) = \int_0^s |\phi(x)|^2 \cdot dx \tag{2}$$

The crucial step towards the proof of (ii) is the following:

Lemma. *One has the inequality*

$$\sum_{|nu=1}^{\infty} |c_\nu(s)|^{2p} \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot \rho(s)^{2p-2}$$

Proof. We employ convolutions and define inductively a sequence of functions $\{\phi_{n,s}(x)\}$ where $\phi_{1,s}(x) = \phi_s(x)$ and

$$\phi_{n+1,s}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_{n,s}(y)\phi_s(x+y) \cdot dy$$

Since convolution yield products of the Fourier coefficients and $2p$ is an even integer we have the standard formula:

$$\sum_{\nu=1}^{\infty} c_n(s)^{2p} = \phi_{2p,s}(0) \tag{1}$$

Next, using the Cauchy-Schwarz inequality the reader may verify that

$$|\phi_{2,s}(x)| \leq \frac{2}{\pi} \cdot \Theta(x)$$

This entails that

$$\phi_{3,s}(x) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |\phi_{2,s}(y)| \cdot |\phi_s(x+y)| \cdot dy \leq \frac{2}{\pi^2} \cdot \Theta(s) \cdot \int_{-\pi}^{\pi} |\phi_s(x+y)| \cdot dy = (\frac{2}{\pi})^2 \cdot \Theta(s) \cdot \rho(s)$$

Proceeding in this way it follows by an induction that

$$\phi_{2p,s}(x) \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot (\rho(s))^{2p-2}$$

This holds in particular when $x = 0$ and then (1) above gives Lemma 1.

A formula for the β-numbers. We have by definition

$$\beta_\nu = \frac{2}{\pi}\int_\delta^\pi \sin\,\nu x \cdot \frac{1}{2}\cot(\frac{x}{2}) \cdot \phi(x) \cdot dx$$

An integration by parts and the construction of the Fourier coefficients $\{c_\nu(s)\}$ which applies with $s = \delta$ give:

$$(*) \qquad \beta_\nu = -\frac{1}{2} \cdot \cot\delta/2 \cdot c_\nu(\delta) + +\frac{1}{4}\int_\delta^\pi c_\nu(x) \cdot \operatorname{cosec}^2(\frac{x}{2}) \cdot dx$$

Now we profit upon Minkowski's inequality. Let q be the conjugate of $2p$, i.e $\frac{1}{q} + \frac{1}{2p} = 1$ and choose $\{\xi_\nu\}$ to be the sequence in ℓ^q of unit norm such that

$$|\sum \xi_\nu \cdot \beta_\nu| = ||\beta_\bullet||_{2p}$$

where the last term is the left hand side in (ii). At the same time (*) above and the triangle inequality give

$$||\beta_\bullet||_{2p} \leq -\frac{1}{2} \cdot \cot(\delta/2) \cdot \sum |c_\nu(\delta)| \cdot |\xi_\nu| + \frac{1}{4}\int_\delta^\pi \operatorname{cosec}^2(\frac{x}{2}) \cdot \sum |c_\nu(x) \cdot \xi_\nu| \cdot dx \leq$$

$$(**) \qquad \frac{1}{2} \cdot \cot(\delta/2) \cdot ||c_\bullet(\delta)||_{2p} + \frac{1}{4}\int_\delta^\pi \operatorname{cosec}^2(\frac{x}{2}) \cdot ||c_\bullet(x)||_{2p} \cdot dx$$

At this stage we apply Lemma 1 and the assumption which give a constant K such that

$$\Theta(s) \leq K \quad \text{and} \quad \rho(s) \leq K \cdot s$$

The last estimate actually is weaker than the hypothesis but it will be sufficient to get the requested estimate of the ℓ^{2p}-norm in (ii). Lemma 1 gives a constant K_1 such that

$$||c_\bullet(\delta)||_{2p} \leq K_1 \cdot \delta^{1-1/p}$$

At the same time we have a constant K_2 such that

$$\cot(\delta/2) \leq \frac{K_2}{\delta}$$

The product in the first term from (**) is therefore majorized by $K_1K_2 \cdot \delta^{-1/2p}$ as requested in (ii). For the second term we use Lemma 1 which first gives

$$||c_\bullet(x)||_p \leq K \cdot x^{-1/2p}$$

At this stage we leave it to the reader to verify that we get a constant K so that

$$\int_\delta^\pi x^{-1/2p} \cdot \operatorname{cosec}^2(\frac{x}{2}) \cdot dx \leq K \cdot \delta^{-1/2p}$$

which finishes the proof of (ii).

The case when f is continuous.

Under the normalisation that the L^2-integral of f is ≤ 1 the inequalities (ii-iii) hold for an absolute constant K. In (i) we notice that the construction of ϕ and the definition of ω_f give the estimates

$$|a_\nu| \leq \nu \cdot \delta \cdot \omega_f(\delta)$$

With $p = 2$ this entails that (i) from the proof of Theorem F.1 is majorised by

$$n^{1+1/2} \cdot \delta \cdot \omega_f(\delta)$$

This holds for every $0 \leq x \leq 2\pi$ and from the previous proof we conclude that the following hold for each $n \geq 2$ and every $0 < \delta < \pi$:

(i) $$\mathcal{D}_n(f) \leq \frac{1}{\sqrt{n+1}} \cdot [n^{1+1/2} \cdot \delta \cdot \omega_f(\delta) + 2K\delta^{-1/2} + K]$$

With $n \geq 2$ we take $\delta = n^{-1}$ and see that (i) gives a requested constnt in Theorem F.2.

G. Best approximation by trigonometric polynomials.

The results below are due to de Vallé Poussin and we follow Chapter VIII in his text-book [V-P]. Consider the $2n+2$-tuple

$$x_j = \frac{2\pi j}{(2n+2)} \quad : \quad 1 \leq j \leq 2n+2$$

Let $P(x)$ be a trigonometric polynomial in $\mathcal{T}_n$:

$$P(x) = \sum_{\nu=-n}^{\nu=n} a_\nu \cdot e^{i\nu x}$$

Let f be a 2π-periodic and continuous function and put

$$\rho_P(f) = \max_j |P(x_j) - f(x_j)|$$

Assume that $\rho_P(f) > 0$ which gives a unique $(2n+2)$-tuple of complex numbers $\{u_j\}$ where every u_j has absolute value ≤ 1 and

$$(1) \qquad f(x_j) = \rho_P(f) \cdot u_j + \sum_{\nu=-n}^{\nu=n} a_\nu \cdot e^{i\nu x_j} \quad : \quad 1 \leq j \leq 2n+2$$

G.1 Proposition. *One has the equality*

$$(*) \qquad \rho_P(f) = \Big|\frac{f(x_1) - f(x_2) + \ldots + f(x_{2n+1} - f(x_{2n})}{u_1 + u_2 + \ldots + u_{2n+1} + u_{2n+2}}\Big|$$

Proof. Consider the $(2n+2) \times (2n+1)$-matrix

$$(i) \qquad \begin{pmatrix} e^{-inx_1} & \ldots & e^{inx_1} \\ e^{-inx_2} & \ldots & e^{inx_2} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ e^{-inx_{2n+2}} & \ldots & e^{inx_{2n+2}} \end{pmatrix}$$

To each $1 \leq k \leq 2n+2$ we denote by $\mathcal{A}_k$ the $(2n+1) \times (2n+1)$-matrix which arises when the k:th row is deleted. Using van der Monde formulas the reader can verify that

$$(ii) \qquad A_k = \det(\mathcal{A}_k) = \prod_{1 \leq i \leq j \leq 2n+2}^{(k)} \sin \frac{x_j - x_i}{2}$$

where (k) above the product sign indicates that i and j both are $\neq 0$ in the product. We leave it to the reader to show that there exists a positive constant A_* such that

$$(iii) \qquad \det A_k = A_* \quad : \quad 1 \leq k \leq 2n+2$$

Now (1) is a system of linear equations where $\rho_*(P), a_{-n}, \ldots, a_n$ are the indeterminate variables. By Cramér's rule we can solve out $\rho_*(P)$ via the $2n+2$-matrix and (iii) gives

$$(iv) \qquad \rho_P(f) = \frac{A_1 f(x_1) - A_2 f(x_2) + \ldots + A_{2n+1} f(x_{2n+1} - A_{2n} f(x_{2n})}{A_1 u_1 + A_2 u_2 + \ldots + A_{2n+1} u_{2n+1} + A_{2n} u_{2n}}$$

Together with (iii) the requested equation (*) in Proposition G.1 follows.

G.2 Conclusion. To find

$$\rho_n(f) = \min_{P \in \mathcal{T}_n} \rho_P(f)$$

we should choose P so that all the u-numbers are +1 or -1. This determines the a-numbers in the system (1), i.e. we find a unique polynomial $P_* \in \mathcal{T}_n$ for which

$$\rho_n(f) = \max_{1 \leq j \leq 2n+2} |P_*(x_j) - f(x_j]|$$

Moreover, the deviation numbers $|P_*(x_j) - f(x_j)|$ are all equal to

$$(**) \qquad \frac{\big|f(x_1) - f(x_2) + \ldots + f(x_{2n+1} - f(x_{2n})\big|}{2n+2}$$

G.3 Exercise. Let f be given with a Fourier series expansion:

$$f(x) = \frac{1}{2} \cdot a_0 + \cdot \sum_{k=1}^{\infty} (\alpha_k \cdot \cos kx + \beta_k \cdot \sin kx)$$

Use the formula (ii) from the proof above to show that

$$(***) \qquad \frac{1}{2n+2} \cdot \sum_{\nu=1}^{\nu=2n+2} (-1)^\nu \cdot f(x_\nu) = \sum_{j=0}^{\infty} a_{(2j+1)(n+1)}$$

Together (**) and (***) give:

G.4 Theorem. *For each integer n and every 2π-periodic function $f(\theta)$ one has the equality*

$$\rho_n(f) = \big|a_{n+1} + a_{3(n+1)} + a_{5(n+1)} + \ldots\big|$$

G.5 Remark. Since the maximum norm taken over the whole interval $[0, 2\pi]$ majorizes the maximum norm over the $2n+2$-tuple above, we get the inequality:

$$\min_{P \in \mathcal{T}_n} ||f - P|| \geq \big|a_{n+1} + a_{3(n+1)} + a_{5(n+1)} + \ldots\big|$$

where $||f - P||$ is the maximum norm over $[0, 2\pi]$. This gives the result announced in § 0.X from the introduction.