A hyperbolic boundary value equation,.

Introduction. Let x, s be coordinates in $\mathbf{R}^2$ and consider the rectangle

$$\square = \{(x,y)\colon\ 0 \leq x \leq \pi\colon\ 0 \leq s \leq s^*\}$$

for some $s^* > 0$. A continuous and real-valued function $g(x,s)$ in $\square$ is x-periodic if

$$g(0,s) = g(\pi,s) \quad : 0 \leq s \leq s^*$$

More generally, if $k \geq 1$ and $g(x,s)$ belongs to $C^k(\square)$ then it is x-periodic if

(i) $$\partial_x^\nu(g(0,s)) = \partial_x^\nu(g(\pi,s) \quad : 0 \leq \nu \leq k$$

In particular we can consider real-valued C^∞-functions on $\square$ for which (i) hold for every $\nu \geq 0$. Let $a(x,s)$ and $b(x,s)$ be a pair real-valued C^∞-functions on $\square$ which are periodic in x. They give the PDE-operator

(*) $$P = \partial_s - a \cdot \partial_x - b$$

A boundary value problem. Let $p \geq 1$ and $f(x)$ is a periodic function on $[0,\pi]$ which is p-times continuously differentiable. Now we seek $F(x,s) \in C^p(\square)$ which is x-periodic and satisfies $P(F) = 0$ in $\square$ and the initial condition

$$F(x,0) = f(x)$$

We are going to prove that this boundary value equation has a unique solution for every $f \in C^p[0,\pi]$. The proof requires several steps and is not finished until § 4. We shall use Hilbert space methods. If $k \geq 2$ there exists the Hilbert space $\mathcal{H}^{(k)}$ which arises via the completetion of $C^k(\square)$ with respect to the sum of L^2-norms of derivatives up to order k of x-periodic C^∞-functions in $\square$. Sobolev's inquality gives

$$\mathcal{H}^{(k)} \subset C^{k-2}(\square) \quad : \ k \geq 2$$

Staying in the interval $\{0 \leq x \leq \pi\}$ we also have the Hilbert space $H^k[0,\pi]$ which is the completion of periodic C^∞-functions $f(x)$. For a fixed $k \geq 2$ we denote by $\mathcal{D}_k(P)$ the set of $f \in H^k[0,\pi]$ such that there exists $F \in \mathcal{H}^{[k)}$ where $P(F) = 0$ and $F(x,0) = f(x)$ on $[0,\pi]$.

In 1§ xx we prove the following Hilbert space version of the boundary problem.

0.1 Theorem. *For each $k \geq 2$ the equality $\mathcal{D}_k(P) = H^k[0,\pi]$ holds and the map $f \to F$ from $H^k[0,\pi]$ to $\mathcal{H}^{(k)}$ is bijective.*

About the proof. The material in § 1 is used to prove that $P\colon \mathcal{D}_k(P) \to H^k[0,\pi]$ is injective. The next step is to show that $\mathcal{D}_k(P)$ is a dense subspace of $H^k[0,\pi]$, and once this has been achieved we can finish the proof rather easily. To prove the density of $\mathcal{D}_k(P)$ we shall consider the the linear operator S_k which for each $f \in \mathcal{D}_k(P)$ associates the function $x \mapsto F(x,s^*)$ on $[0,\pi]$. So here the domain of definition $\mathcal{D}(S_k) = \mathcal{D}(P_k)$. Material from § 1 will be used to prove that S_k is a bounded operator, i.e. there exists a constant C such that

$$||S_k(f)||_k \leq C \cdot ||f||_k \quad : f \in \mathcal{D}(S_k)$$

Armed with this we prove that in § xx that the requested density of $\mathcal{D}_k(P)$ follows from the following:

0.1.1 Proposition. *For each $k \geq 2$ there exists a positive number $\alpha(k)$ such that the range of $E - \alpha \cdot S_k$ contains all periodic C^∞-functions on $[0,\pi]$ when $\alpha < \alpha(k)$.*

0. A periodic equation.

To prove of Proposition 0.1.1 we shall work with doubly periodic functions $g(x,s)$ defined in the rectangle $\{0 \leq x \leq \pi\}\times\{0 \leq s \leq 2\pi\}$. When $k \geq 2$ we get the Hilbert space $\mathcal{H}^{(k)}$ after the completion of doubly periodic C^∞-functions with L^2-norms of derivatives up to order k. This time we are given a differential operator

$$P = \partial_s - a(x,s)\partial_x - b(x,s)$$

where a and b are doubly periodic C^∞-functions. Set

$$\mathcal{D}_k(P) = \{g \in \mathcal{H}^{(k)} : P(g) \in \mathcal{H}^{(k)}\}$$

In the product space $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$ we have the graphic set

$$\gamma_k = \{(g, P(g) : \ g \in C^\infty\}$$

where we always refer to doubly periodic C^∞-functions as above. The closure of γ_k is the graph of a closed and densely defined linear operator on $\mathcal{H}^{(k)}$ denoted by T_k. With these notations the following holds, which apart from its use during the proof of Theorem 0.1 has independent interest:

1.1 Theorem. *There exists a positive number $\lambda(k)$ such that*

$$\lambda \cdot E - T_k : \ \mathcal{D}(T_k) \to \mathcal{H}^{(k)}$$

are surjective fo every $\lambda > \lambda(k)$.

To prove this theorem we shall consider the closed and densely defined operator $\mathcal{T}_k$ on $\mathcal{H}^{(k)}$ where

$$\Gamma(\mathcal{T}_k) = \{(g, P(g) : \ g \in \mathcal{D}_k(P)\}$$

Since doubly periodic C^∞-functions belong to $\mathcal{D}_k(P)$ we have $\Gamma(T_k) \subset \Gamma(\mathcal{T}_k)$, i.e. $\mathcal{T}_k$ is an extension of T_k. Since T_k is densely defined this entails that the adjoint operators T_k^* and $\mathcal{T}_k^*$ are equal. A crucial step in the proof of Theorem 1.1 is the following:

1.2 Theorem. *One has the equality $\mathcal{D}_k(P) = \mathcal{D}(T_k^*)$ and there exists a densely defined self-adjoint operator B_k such that*

$$T_k^* = -\mathcal{T}_k + B_k$$

§ 1. Differential inequalities and energy integrals.

Let $M(s)$ be a non-negative real-valued continuous function on a closed interval $[0, s^*]$. To each $0 \leq s < s^*$ we set

$$d_M^+(s) = \limsup_{\Delta s \to 0} \frac{M(s + \Delta s) - M(s)}{\Delta s}$$

where Δs are positive during the limit.

1.1 Proposition. *Let B be a real number such that $d_M^+(s) \leq B \cdot M(s)$ holds in $[0, s^*)$. Then*

$$M(s) \leq M(0) \cdot e^{Bs} \quad : 0 < s \leq s^*$$

The proof of this result is left as an exercise. The hint is to consider the function $N(s) = M(s)e^{-Bs}$ and show that $d_N^+(s) \leq 0$ for all s. Notice that B is an arbitrary real number, i.e. it may also be < 0. More generally, let $k(s)$ be a non-decreasing continuous function with $k(0) = 0$ and suppose that

$$d_M^+(s) \leq B \cdot M(s) + k(s) \quad : 0 \leq s < s^*$$

Now the reader may verify that

$$M(s) \leq M(0) \cdot e^{Bs} + \int_0^s k(t)\, dt \tag{1.1.1}$$

Next, consider the set $\square = [0, \pi] \times [0, s^*]$ as above. A C^1-function g is periodic with respect to x if g and the partial derivatives $\partial_s(g)$,$\partial_x(g)$ are periodic in x, i.e.

$$g(0, s) = g(\pi, s) \quad : 0 \leq s \leq s^*$$

and similarly for $\partial_x(g)$ and $\partial_s(g)$.

1.2 Theorem. *Let g be a periodic C^1-function which satisfies the PDE-equation*

$$\partial_s(g) = a \cdot \partial_x(g) + b \cdot g \tag{*}$$

in $\square$ where a and b are x-periodic real-valued continuous functions on $\square$. Set

$$M_g(s) = \max_x |g(x, s)| \quad : \ B = \max_{x,s} |b(x, s)|$$

Then one has the inequality

$$M_g(s) \leq M_g(0) \cdot e^{Bs}$$

Proof. Consider some $0 < s < s^*$ and let $\epsilon > 0$. Put

$$m^*(s) = \{x : \ g(x, s) = M_g(s)\}$$

The continuity of g entials that the function $M(s)$ is continuous and the sets $m^*(s)$ are compact. If $x^* \in m^*(s)$ the periodicity of the C^1-function $x \mapsto g(x, s)$ entails that $\partial_x(x^*, s) = 0$ and (*) gives

$$\partial_s(g)(x, s) = b(x, s)g(x, s) \quad : x \in m^*(s)$$

Next, let $\epsilon > 0$. We find an open neighborhood U of $m^*(s)$ such that

$$|\partial_x(g)(x, s)| \leq \epsilon \quad : x \in U$$

Now there exists $\delta > 0$ such that

$$|g(x, s)| \leq M(s) - 2\delta \quad : x \in [0, \pi] \setminus U$$

Continuity gives some $\rho > 0$ such that if $0 < \Delta s < \rho$ then the inequalities below hold:

(i) $\quad |g(x, s + \Delta s)| \leq M(s) - \delta \quad : x \in [0, \pi] \setminus U \quad : \ M(s + \Delta s) > M(s) - \delta$

(ii) $\qquad M(s+\Delta s) \leq M(s)+\epsilon \quad : |\partial_x(g)(x,s+\Delta s)| \leq 2\epsilon \quad : x \in m^*(s)$

If $0<\Delta s<\rho$ we see that (i) gives $x \in m^*(s+\Delta s) \subset U$ and for such x-values Rolle's mean-value theorem and the PDE-equation give

$$M_g(x,s+\Delta s)-g(x,s) = \Delta s \cdot \partial_s(g(x,s+\theta\cdot\Delta s) =$$

(iii) $\qquad \Delta s \cdot \big[a(x,s+\Delta s)\cdot\partial_x(g)(x+\theta\cdot\Delta s) + b(x,s+\Delta s)\cdot g(x,s+\theta\cdot\Delta s)\big]$

Let A be the maximum norm of $|a(x,s)|$ taken over $\square$. Since $|g(x,s)| \leq M(s)$ the triangle inequality and (iii) give

$$M(s+\Delta s) \leq M(s) + \Delta s[\cdot A\cdot 2\epsilon + B\cdot M(s+\theta\cdot\Delta s)]$$

Since the function $s \mapsto M(s)$ is continuous it follows that

$$\limsup_{\Delta s\to 0} \frac{M(s+\Delta s)-M(s)}{\Delta s} \leq A\cdot 2\epsilon + BM(s)$$

Above ϵ can be arbitrary small and hence

$$d^+(s) \leq B\cdot M(s)$$

Then Proposition 1.1 gives (*) in the theorem.

1.3 L^2-inequalities. Let $g(x,s)$ be a C^1-function satisfying (*) in Theorem 1.2. Set

$$J_g(s) = \int_0^\pi g^2(x,s)\,dx$$

Taking the s-derivative we obtain with respect to s and (*) give

$$\frac{dJ_g}{ds} = 2\cdot\int_0^\pi g\cdot\partial_s(g)\,ds = 2\cdot\int_0^\pi (a\partial_x(g)\cdot\partial g + b\cdot g)\,dx$$

The periodicity of g with respect to x gives $\int_0^\pi \partial_x(ag^2)\,dx = 0$. This entails that the right hand side becomes

$$\int_0^\pi (-\partial_x(a)+b)\cdot g^2\,dx$$

So if K is the maximum norm of $-\partial_x(a)+b$ over $\square$ it follows that

$$\frac{dJ_g}{ds}(s) \leq K\cdot J_g(s)$$

Hence Theorem 1.2 gives

(1.3.1) $$\int_0^\pi g^2(x,s)\,dx \leq e^{Ks}\cdot\int_0^\pi g^2(x,0)\,dx \quad : 0<s\leq s^*$$

Integration with respect to s entails that

(1.3.2) $$\iint_\square g^2(x,s)\,dxds \leq \int_0^{s^*} e^{Ks}\,ds\cdot\int_0^\pi g^2(x,0)\,dx$$

Thus, the L^2-integral of $x\to g(x,0)$ majorizes both the area integral and each slice integral when $0<s\leq s^*$.

§ 2. A boundary value equation

Let $a(x,s)$ and $b(x,s)$ be real-valued C^∞-functions on $\square$ which are periodic in x and consider the PDE-operator

$$P = \partial_s - a\cdot\partial_x - b$$

2.1 Theorem. *For every positive integer p and each periodic $f\in C^p[0,\pi]$ there exists a unique periodic $g\in C^p(\square)$ where $P(g)=0$ and $g(x,0)=f(x)$.*

The uniqueness follows from the results in § 1. For if g and h are solutions in Theorem 2.1 then $\phi = g - h$ satisfies $P(\phi) = 0$. Here $\phi(x, 0) = 0$ which gives $\phi = 0$ in $\Box$ via (1.3.2). The proof of existence requires several steps and employs Hilbert space methods. So first we introduce certain Hilbert spaces.

2.2 The space $\mathcal{H}^{(k)}$. To each integer $k \geq 2$ the complex Hilbert space $\mathcal{H}^{(k)}$ is the completion of complex-valued C^k-functions on $\Box$ which are periodic with respect to x. A trivial Sobolev inequality entails that every function in $\mathcal{H}^{(2)}$ is continuous, and more generally

$$\mathcal{H}^{(k)} \subset C^{k-2}(\Box) \quad : k \geq 3$$

and it clear that the first order PDE-operator P maps $\mathcal{H}^{(k+1)}$ into $\mathcal{H}^{(k)}$. Next, on the periodic x-interval $[0, \pi]$ we have the Hilbert spaces $H^k[0, \pi]$ for each $k \geq 2$.

2.3 Definition. *For each integer $k \geq 2$ we denote by $\mathcal{D}_k(P)$ the family of all $f(x) \in H^k[0, \pi]$ for which there exists some $F(x, s) \in \mathcal{H}^{(k)}$ such that*

$$(*) \qquad P(F) = 0 \quad : \; F(x, 0) = f(x)$$

The results in § 1 show that F is uniquely determined by (*). Moreover, there exists a constant C_k which only depends upon the C^∞-functions a and b and the given integer k such that

$$(2.3.1) \qquad ||F||_k \leq C_k \cdot ||f||_k$$

where we have taken norms in $\mathcal{H}^{(k)}$ and $H^k[0, \pi]$ respectively. Next, the last inequality in (1.3.2) shows that C_k can be chosen such that

$$(2.3.3) \qquad ||f^*||_k \leq C_k \cdot ||f||_k$$

where $f^*(x) = F(x, s^*)$ belongs to $H^k[0, \pi]$.

2.4 A density principle Above we introduced the space $\mathcal{D}_k(P)$. Now the following hold:

2.4.1 Proposition. *If $\mathcal{D}_k(P)$ is dense in $\mathcal{H}^k[0, \pi]$, then one has the equality*

$$(2.4.1) \qquad \mathcal{D}_k(P) = \mathcal{H}^k[0, \pi]$$

Proof. Suppose that $\mathcal{D}_k(P)$ is dense. So if $f \in \mathcal{H}^k[0, \pi]$ there exists a sequence $\{f_n\}$ in $\mathcal{D}_k(P)$ where $||f_n - g||_k \to 0$. By (2.2.2) we have

$$||F_n - F_m||_k \leq C||f_n - f_m||_k$$

Hence $\{F_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}^{(k)}$ and converges to a limit F. Since each $P(F_n) = 0$ it follows that $P(F) = 0$ and it is clear that the continuous boundary value function $F(x, 0)$ is equal to $f(x)$ which entails that f belongs to $\mathcal{D}_k(P)$.

2.5 The operators S_k. Each $f \in \mathcal{D}_k(P)$ gives the function $f^*(x) = F(x, s^*)$ in $\mathcal{H}^k[0, \pi]$ and set

$$S_k(f) = f^*(x)$$

So the domain of definition of S_k is equal to $\mathcal{D}_k(P)$ and (2.3.3) gives a constant M_k such that

$$||S_k(f)|| \leq M_k \cdot ||f||_k \quad : f \in \mathcal{D}_k(P)$$

where M_k only depends on the integer k and the given PDE-operator P. The next result constitutes a crucial point to attain Theorem 2.1.

2.6 Proposition. *For each $k \geq 2$ there exists some $\alpha(k) < 0$ such that for every $0 < \alpha < \alpha(k)$ the range of the operator $E - \alpha \cdot S_k$ contains all periodic C^∞-functions on $[0, \pi]$.*

2.7 The density of $\mathcal{D}_k(P)$. We prove Proposition 2.6 in § xx and proceed to show that it gives the density of $\mathcal{D}_k(P)$. For if $\mathcal{D}_k(P)$ fails to be dense there exists a non-zero $f_0 \in \mathcal{D}_k(P)$ which is $\perp$ to $\mathcal{D}_k(P)$. In Proposition 2.6 we choose $0 < \alpha \leq \alpha(k)$ so small that

(i) $$\alpha < M_k/2$$

Since periodic C^∞-functions are dense in $\mathcal{H}^k[0,\pi]$, Proposition 2.6 gives a sequence $\{h_n\}$ in $\mathcal{D}_k(P)$ such that

(ii) $$\lim_{n\to\infty} ||h_n - \alpha \cdot S_k(h_n) - f_0||_k \to 0$$

It follows that

(iii) $$\langle f_0, f_0\rangle = 1 = \lim \langle f_0, h_n - \alpha \cdot S_k(h_n)\rangle = -\alpha \cdot \lim \langle f_0, S_k(h_n)\rangle$$

Next, the triangle inequality and (ii) give

(iv) $$||h_n||_k \leq 1 + \alpha \cdot ||(S_k(h_n)|| \leq 1 + 1/2 \cdot ||h_n|| \implies ||h_n||_k \leq 2$$

Finally, by the Cauchy-Schwarz inequality the absolute value in the right hand side of (iii) is majorized by

$$\alpha \cdot M_K \cdot 2 < 1$$

which contradicts (iii). Hence the orthogonal complement of $\mathcal{D}_k(P)$ is zero which proves the requested density.

Together with Propostion 2.4.1 we get the following conclusive result:

2.8 Theorem. *For each $k \geq 2$ and $f(x) \in \mathcal{H}^k[0,\pi]$ there exists a unique function $F(x,s) \in \mathcal{H}^{(k)}$ such that (*) holds in Definition 2.3.*

2.9 Remark. The result above soplves the requested boundsry valued problem in $\mathcal{H}^{(k)}$ -spaces. Using Sobolev inequalities oner easily derives Theorem 2.1.

§ 3. A doubly periodic class of inhomogeneous PDE-equations.

Before Theorem 3.2 is announced we introduce some notations. Put

$$\square = \{0 \leq x \leq \pi\} \times \{0 \leq s \leq 2\pi\}$$

In this section we shall consider doubly periodic functions $g(x,s)$ on $\square$, i.e.

$$g(\pi, s) = g(0,s) \quad : g(x,0) = g(x,2\pi)$$

For each non-negative integer k we denote by $C^k(\square)$ the space of k-times doubly periodic continuously differentiable functions. If $g \in C^k(\square)$ we set

$$||g||^2_{(k)} = \sum_{j,\nu} \int_\square \Big|\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}(x,s)\Big|^2 \, dxds$$

with the double sum extended pairs $j+\nu \leq k$. This gives the complex Hilbert space $\mathcal{H}^{(k)}$ after a completion of $C^k(\square)$ with respect to the norm above. Recall that a Sobolev inequality entails that a function $g \in \mathcal{H}^{(2)}$ is automatically continuous and doubly periodic on the closed square. More generally, if $k \geq 3$ each $g \in \mathcal{H}^{(k)}$ has continuous and doubly periodic derivatives up to order $k-2$. Next, consider a first order PDE-operator

$$P = \partial_s - a(x,s)\partial_x - b(x,s) \tag{3.1}$$

where a and b are real-valued doubly periodic C^∞-functions. It is clear that P maps $\mathcal{H}^{(k)}$ into $\mathcal{H}^{(k+1)}$ for every $k \geq 2$. Keeping $k \geq 2$ fixed we set

$$\mathcal{D}_k(P) = \{g \in \mathcal{H}^{(k)} : \ P(g) \in \mathcal{H}^{(k)}\}$$

Since $C^\infty(\square)$ is dense in $\mathcal{H}^{(k)}$ this yields a densely defined operator

$$P \colon \mathcal{D}_k(P) \to \mathcal{H}^{(k)} \tag{i}$$

In $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$ we get the graph

$$\Gamma_k = \{(g, P(g) \colon \ g \in \mathcal{D}_k(P)\}$$

Since P is a differential operator we know from general results that Γ_k is a closed subspace. Hence there exists a densely defined linear operator and closed operator on $\mathcal{H}^{(k)}$ which we denote by $\mathcal{T}_k$. So here $\mathcal{D}(\mathcal{T}_k) = \mathcal{D}_k$. Set

$$\gamma_k = \{(g, P(g) \colon \ g \in C^\infty(\square)\} \tag{ii}$$

This is a subspace of Γ_k and denote by $\overline{\gamma}_k$ its closure taken in $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$. Since Γ_k is closed we have

$$\overline{\gamma}_k \subset \Gamma_k$$

We get the densely defined linear operator T_k whose graph is $\overline{\gamma}_k$. By this construction $\mathcal{T}_k$ is an extension of T_k which in particular gives the inclusion

$$\mathcal{D}(T_k) \subset \mathcal{D}(\mathcal{T}_k) \tag{iii}$$

Next, let E be the identity operator on $\mathcal{H}^{(k)}$. With these notations we shall prove:

3.2 Theorem. *For each integer $k \geq 2$ there exists a positive real number $\rho(k)$ such that the map*

$$T_k - \lambda \cdot E \colon \mathcal{H}^{(k)} \to \mathcal{H}^{(k)}$$

is bijective for every $\lambda > \rho(k)$.

The proof requires several steps and is not finished until § 3.x. First we shall study the adjoint operator T_k^* and establish the following:

3.3 Proposition. *One has the equality $\mathcal{D}(T_k^*) = \mathcal{D}_k(P)$ and there exists a bounded self-adjoint operator B_k on $\mathcal{H}^{(k)}$ such that*

$$T_k^* = -\mathcal{T}_k + B_k$$

Proof of Proposition 3.3 Keeping $k \geq 2$ fixed we set $\mathcal{H} = \mathcal{H}^{(k)}$. For each pair g, f in $\mathcal{H}$ their inner product is defined by

$$\langle f, g \rangle = \sum \int_\square \frac{\partial^{j+\nu} f}{\partial x^j \partial s^\nu}(x,s) \cdot \overline{\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}}(x,s)\, dxds$$

where the sum is taken when $j + \nu \leq k$. Introduce the differential operator

$$\Gamma = \sum_{j+\nu \leq k} (-1)^{j+\nu} \cdot \partial_x^{2j} \cdot \partial_s^{2\nu}$$

Partial integration gives

$$\text{(i)} \qquad \langle f, g \rangle = \int_\square f \cdot \Gamma(\bar{g})\, dxds = \int_\square \Gamma(f) \cdot \bar{g}\, dxds \quad : \ f, g \in C^\infty$$

Now we consider the operator $P = \partial_s - a \cdot \partial_x - b$ and get

$$\text{(ii)} \qquad \langle P(f), g \rangle = \int_\square P(f) \cdot \Gamma(\bar{g})\, dxds$$

Partial integration identifies (ii) with

$$\text{(iii)} \qquad -\int_\square f \cdot \big(\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma(\bar{g})\, dxds$$

1.1 Exercise. In (iii) appears the composed differential operator

$$\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma$$

Show that in the ring of differential operators with C^∞-coefficients this differential operator can be written in the form

$$\Gamma \circ (\partial_s - a \cdot \partial_x - b) + Q(x, s, \partial_x, \partial_s)$$

where Q is a differential of order $\leq 2k$ with coefficients in $C^\infty(\square)$. Conclude from the above that

$$\text{(1.1.1)} \qquad \langle Pf, g \rangle = -\langle f, Pg \rangle + \int_\square f \cdot Q(\bar{g})\, dxds$$

1.2 Exercise. With Q as above we have a bilinear form which sends a pair f, g in $C^\infty(\square)$ to

$$\text{(1.2.1)} \qquad \int_\square f \cdot Q(\bar{g})\, dxds$$

Use partial integration and the Cauchy-Schwarz inequelity to show that there exists a conatant C which depends on Q only such that the absolute value of (1.2.1) is majorized by $C_Q \cdot ||f||_k \cdot ||g||_k$. Conclude that there exists a bounded linear operator B_k on $\mathcal{H}$ such that

$$\text{(1.2.2)} \qquad \langle f, B_k(g) \rangle = \int_\square f \cdot Q(\bar{g})\, dxds$$

1.3 Proof that B_k is self-adjoint From the above we have

$$\text{(1.3.1)} \qquad \langle Pf, g \rangle = -\langle f, Pg \rangle + \langle f, B_k(g) \rangle$$

Keeping f in $C^\infty(\square)$ we notice that $\langle f, B_k(g) \rangle$ is defined for every $g \in \mathcal{H}$. From this the reader can check that (1.3.1) remains valid when g belongs to $\mathcal{D}(\mathcal{T}_k)$ which means that

$$\text{(1.3.2)} \qquad \langle Pf, g \rangle = -\langle f, \mathcal{T}_k g \rangle + \langle f, B_k(g) \rangle \quad : f \in C^\infty(\square)$$

Moreover, when both f and g belong to $C^\infty(\square)$ we can reverse their positions in (*) which gives

$$\text{(1.3.3)} \qquad \langle Pg, f \rangle = -\langle g, Pf \rangle + \langle g, B_k(f) \rangle$$

Since a and b are real-valued it is clear that

(1.3.4) $$\langle Pg, f\rangle = -\langle f, Pg\rangle$$

It follows that

(1.3.5) $$\langle f, B_k(g) = \langle g, B_k(f) \quad : f, g \in C^\infty(\square)$$

Since this hold for all pairs of C^∞-functions and B_k is a bounded linear operator on $\mathcal{H}$ the density of $C^\infty(\square)$ entails that B_k is a bounded self-adjoint operator on $\mathcal{H}$.

1.4 The equality $\mathcal{D}(T_k^*) = \mathcal{D}_k(P)$. The density of $C^\infty(\square)$ in $\mathcal{H}$ entails that a function $g \in \mathcal{H}$ belongs to $\mathcal{D}(T_k^*)$ if and only if there exists a constant C such that

(1.4.1) $$|\langle Pf, g\rangle| \leq C \cdot ||f|| \quad : f \in C^\infty(\square)$$

Since B_k is a bounded operator, (1.3.2) gives the inclusion

(1.4.2) $$\mathcal{D}_k(P) \subset \mathcal{D}(T_k^*)$$

To prove the opposite inclusion we use that the Γ-operator is elliptic. If $g \in \mathcal{D}(T_k^*)$ we have from (i) in § 1.1:

$$\langle Pf, g\rangle = \langle f, T_k^* g\rangle = \int \Gamma(f) \cdot \overline{T_k^*(g)}\, dxds \quad : f \in C^\infty(\square)$$

Similarly

$$\langle f, B_k(g)\rangle = \int \Gamma(f) \cdot \overline{B_k(g)}\, dxds$$

Treating $\mathcal{T}_k(g)$ as a distribution the equation (1.3.2) entails that the elliptic operator Γ annihilates $T_k^*(g) - \mathcal{T}_k(g) + B_k(g)$. Since both $T_k^*(g)$ and $B_k(g)$ belong to $\mathcal{H}$ this implies by the general result in § xx that $\mathcal{T}_k(g)$ belongs to $\mathcal{H}$ which proves the requested equality (1.4) and at the same time the operator equation

(1.4.3) $$T_k^* = -\mathcal{T}_k(g) + B_k$$

3.4 An inequality.

Let $f \in C^\infty(\square)$ and λ is a positive real number. Then

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 =$$

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f)||^2 + \lambda^2 \cdot ||f||^2 - \lambda\big(\langle \mathcal{T}_k(f) - \frac{1}{2}B_k(f), f\rangle + \langle f, \mathcal{T}_k(f) - \frac{1}{2}B_k(f)\rangle\big)$$

The last term is λ times

(i) $$\langle \mathcal{T}_k(f), f\rangle + \langle f, \mathcal{T}_k(f)\rangle - \langle f, B_k f\rangle$$

where we used that B_k is symmetric. Now $T_k = \mathcal{T}_k$ holds on $C^\infty(\square)$ and the definition of adjoint operators give

(ii) $$\langle \mathcal{T}_k(f), f\rangle = \langle f, T_k^*\rangle$$

Then (1.4.3) implies that (i) is zero and hence we have proved

(iii) $$||T_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 = \lambda^2 \cdot ||f||^2 + ||T_k(f) - \frac{1}{2}B_k(f)||^2 \geq \lambda^2 \cdot ||f||^2$$

From (iii) and the triangle inequality for norms we obtain

(iv) $$||T_k(f) - \lambda \cdot f|| \geq \lambda \cdot ||f|| - \frac{1}{2}||B_k(f)||$$

Now B_k has a finite operator norm and if $\lambda \geq ||B_k||$ we see that

(v) $$||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f||$$

Finally, since $C^\infty(\square)$ is dense in $\mathcal{D}(T_k)$ it is clear that (v) gives

3.41 Proposition. *One has the inequality*

$$||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f|| \quad : f \in \mathcal{D}(T_k) \tag{3.4.1}$$

§ *3.5. Proof of Theorem 3.2*

Suppose we have found some $\lambda^* \geq \frac{1}{2} \cdot ||B||$ such that $T_k - \lambda$ has a dense range in $\mathcal{H}$ for every $\lambda \geq \lambda^*$. If this is so we fix $\lambda \geq \lambda^*$ and take some $g \in \mathcal{H}$. The hypothesis gives a sequence $\{f_n \in \mathcal{D}(T_k)\}$ such that

$$\lim_{n\to\infty} ||T_k(f_n) - \lambda \cdot f_n - g|| = 0$$

In particular $\{T_k(f_n) - \lambda \cdot f_n\}$ is a Cauchy sequence in $\mathcal{H}$ and (1.5.x) implies that $\{f_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$ and hence converges to a limit f_*. Since the operator T_k is closed we conclude that $f_* \in \mathcal{D}(T_k)$ and we get the equality

$$T_k(f_*) - \lambda \cdot f_* = g$$

Since $g \in \mathcal{H}$ was arbitrary we have proved Theorem 3.2.

3.5.1 Density of the range. There remains to find λ^* as above. By the construction of adjoint operators, the range of $T_k - \lambda \cdot E$ fails to be dense if and ony if $T_k^* - \lambda$ has a non-zero kernel. So assume that

$$T_k^*(f) - \lambda \cdot f = 0$$

for some $f \in \mathcal{D}(T_k^*)$ which is not identically zero. Notice that T_k sends real-valued functions into real-valued functions. So above we can assume that f is real-valued and normalised so that

$$\int_\square f^2(x,s)\, dxds = 1 \tag{i}$$

From (i) and Proposition 3.3 we have

$$\mathcal{T}_k(f) + \lambda \cdot f - B(f) = 0 \tag{ii}$$

Let us consider the function

$$V(s) = \int_0^\pi f^2(x,s)\, dx$$

Since $k \geq 2$ is assumed we recall that the $\mathcal{H}$-function f is of class C^1 at least. The s-derivative of $V(s)$ becomes:

$$\frac{1}{2} \cdot V'(s) = \int_0^\pi f \cdot \frac{\partial f}{\partial s}\, dx \tag{iii}$$

By (ii) we have

$$\frac{\partial f}{\partial s} - a(x)\frac{\partial f}{\partial x} - b \cdot f = B(f) - \lambda \cdot f$$

Hence the right hand side in (iii) becomes

$$-\lambda \cdot V(s) + \int_0^\pi f(x,s) \cdot B(f)(x,s)\, dx + \int_0^\pi a(x,s) \cdot f(x,s) \cdot \frac{\partial f}{\partial x}(x,s)\, dx$$

By partial integration the last term is equal to

$$-\frac{1}{2}\int_0^\pi \partial_x(a)(x,s) \cdot f^2(x,s)\, dx \tag{iv}$$

Set

$$M = \frac{1}{2} \cdot \max_{(x,s)\in\square} |\partial_x(a)(x,s)|$$

From the above we get the inequality

$$\text{(v)} \qquad \frac{1}{2}\cdot V'(s) \leq (M-\lambda)\cdot V(s) + \int_0^\pi f(x,s)\cdot B(f)(x,s)\,dx$$

Set

$$\Phi(s) = \int_0^\pi |f(x,s)|\cdot|B(f)(x,s)|\,dx$$

Since the L^2-norm of f is one the Cauchy-Schwarz inequality gives

$$\int_{-\pi}^{\pi} \Phi(s)\,ds \leq \sqrt{\int_\Box |B(f)(x,s)|^2\,dxds} \leq ||B(f)||$$

where the last equality follows since the squared integral of $B(f)$ is majorized by its squared norm in $\mathcal{H}$. When $\lambda > M$ it follows from (v) that

$$\text{(vi)} \qquad (\lambda - M)\cdot V(s) + \frac{1}{2}\cdot V'(s) \leq \Phi(s)$$

Next, since f is double periodic we have $V(-\pi) = V(\pi)$ so after an integration (vi) gives

$$\text{(vii)} \qquad (\lambda - M)\cdot \int_\pi^\pi V(s)\,ds = \int_{-\pi}^{\pi} \Phi(s)\,ds \leq ||B(f)||$$

Finally, thr normalisation (i) gives $\int_\pi^\pi V(s)\,ds = 1$ and then (vii) cannot hold if

$$\lambda > M + ||B(f)||$$

Remark. Set

$$\tau = \min_f ||B(f)||$$

with the minimum taken over funtions $f \in \mathcal{D}(T_0^*)$ whose L^2-integral is normalised by (i) above. The proof has shown that the kernel of $T_0^* - \lambda$ is zero for all $\lambda > M + \tau$.

A special solution.

Let $f(x)$ be a periodic C^∞-function on $[0,\pi]$. Put

$$Q = a(x,s)\cdot\frac{\partial}{\partial x} + b(x,s)$$

Let $\eta(s)$ be a C^∞-function of s and m some positive integer If $\lambda > 0$ is a real number. we set

$$\text{(i)} \qquad g_\lambda(x,s) = \eta(s)\cdot f + \eta(s)\cdot\sum_{j=1}^{j=m}\frac{(s-\pi)^j}{j!}\cdot(Q-\lambda)^j(f) \quad : 0\le s\le\pi$$

We choose η to be a real-valued C^∞-function such that $\eta(s) = 0$ when $s \le 1/4$ and -1 if $s \ge 1/2$. Hence $g_\lambda(x,s) = 0$ in (i) when $0 \le s \le 1/4$ and we extend the function to $[-\pi \le s \le \pi$ where $g_\lambda(x,-s) = g_\lambda(x,s)$ if $0 \le s \le \pi$. So now g_λ is π-periodic with respect to s and vanishes when $|s| \le 1/4$.

Exercise. If $1/2 \le s \le \pi$ we have $\eta(s) = 1$. Use (i) to show that

$$(P+\lambda)(g_\lambda) = \frac{\partial g_\lambda}{\partial s} - (Q-\lambda)(g_\lambda) = \frac{(s-\pi)^m}{m!}\cdot(Q-\lambda)^{m+1}(f)$$

hold when $1/2 \le s \le \pi$. At the same time $g_\lambda(s) = 0$ when $0 \le s \le 1/4$. So $(P+\lambda)(g)$ is a function whose derivatives with respect to s vasnish up to order m at $s = 0$ and $s = \pi$ and is therefore doubly periodic of class C^m in $\Box$. Now Theorem 2.2 applies. For a given $k \ge 2$ we choose a sufficently large m and find $h(x,s)$ so that

$$P(h) + \lambda\cdot h = (P+\lambda)(g_\lambda)(x,s)$$

where h is s-periodic, i.e.

$$h(x,0) = h(x,\pi)$$

Notice also that $g_\lambda(x,0) = 0$ while $g_\lambda(x,\pi) = f(x)$. Set

$$g_*(x) = h - g_\lambda$$

Then $P(g_*) + \lambda\cdot g_* = 0$ and

$$g_*(x,0) - g_*(x,\pi) = f(x)$$

Above we started with the C^∞-function. Given $k \ge 2$ we can take m sufficiently large during the constructions above so that g_* belongs to $\mathcal{H}^{(k)}(\Box)$.