19. An isoperimetric inequality.

A planar domain whose boundary curve has prescribed length has a maximal area when it it is a disc. It turns out that discs solve a more extensive class of extremal problems. Consider a function $f(r)$ defined for $r > 0$ which is continuous and increasing with $f(0) = 0$. If p and q are two points in $\mathbf{R}^2$ their euclidian distance is denoted by $|p - q|$. When U is a bounded open domain we set

$$J(U) = \iint_{U \times U} f(|p-q|) \cdot dA_p \cdot dA_q$$

where dA_p and dA_q denote the area measures. Given a positive number $\mathcal{A}$ one seeks to maximize the J-functional in the family of domains with prescribed area $\mathcal{A}$. The J-number is unchanged under a translation or a rotation of a domain and the family of discs is stable under these operations. So the following result makes sense:

1. Theorem. *The J-functional takes its maximum on discs D of radius r with $\pi r^2 = \mathcal{A}$. Moreover, for every domain U with area $\mathcal{A}$ which is not a disc one has a strict inequality*

$$J(U) < J(D)$$

When $f(r)$ is a strictly convex function Theorem 1 was established by Blaschke. For a general f-function which need not be convex the theorem was proved by Carleman in [Car] using the symmetrisation process by W. Gros from his article (Monatshefte math.physik 1917). At the end of [Carleman] it is pointed out that Theorem 1 leads to another property of discs.

2. Theorem. *Let Ω be a domain in the family $\mathcal{D}(C^1)$ and denote by ds the arc-length measure on its boundary. Then, if the function*

$$p \mapsto \int_{\partial\Omega} f(|p-q|) \cdot ds(q)$$

is constant as p varies in $\partial\Omega$ it follows that Ω is a disc.

Remark. It is very likely that Theorem 1 can be extended to any dimension $n \geq 3$ using the fact due to Gros that successive symmetrisations of domains taken in different directions converge to the unit ball in $\mathbf{R}^n$. The reader is invited to supply details using similar methods as in the subsequent proof for the case $n = 2$. The crucial step toward the proof of Theorem 1 relies upon Proposition 2.1 below.

2. A fundamental inequality

Let $M > 0$ be a given constant. On the vertical lines $\{x = 0\}$ and $\{x = M\}$ we consider two subsets G_* and G^* which both consist of a finite union of closed intervals. Let $\{[a_\nu, b_\nu]\}$ be the G_*-intervals taken in the y-coordinates and $\{[c_j, d_j]\}$ are the G^*-intervals. Here $a_\nu < b_\nu < a_{\nu+1}$ holds, and similarly the G^*-intervals are ordered with increasing y-coordinates. The number of intervals of the two sets are arbitrary and need not be the same. Given $f(r)$ as in the Theorem 1 we set

$$I(G_*, G^*) = \sum_\nu \sum_j \int_{a_\nu}^{b_\nu} \int_{c_j}^{d_j} f(|y - y'|) \cdot dy dy'$$

Consider the variational problem where we seek to minimize these I-integrals for pairs all pairs (G_*, G^*) as above under the constraints:

$$\sum (b_\nu - a_\nu) = \ell_* \quad \text{and} \quad \sum (d_j - c_j) = \ell^*$$

That is, the sum of the lengths of the intervals are prescribed on G_* and G^*.

2.1 Proposition. *For every pair (ℓ_*, ℓ^*) the I-integral is minimized when both G_* and G^* consist of a single interval and the mid-points of the two intervals have equal y-coordinate.*

Proof. First we prove the result when both $G_* = (a, b)$ and $G^* = (c, d)$ both are intervals. We must prove that the I-intergal is a minimum when

$$\text{(i)} \qquad \frac{a+b}{2} = \frac{c+d}{2}$$

Suppose that inequality holds. Since the I-integral is symmetric with respect to the pair of intervals, we may assume that

$$\frac{c+d}{2} = s + \frac{a+b}{2} \quad \text{where} \quad s > 0$$

Now $I(G_*, G^*)$ is unchanged when we translate the two intervals, i.e. if we for some number ξ take $(a+\xi, b+\xi)$ and $(c+\xi, d+\xi)$. By such a translation we can assume that $a = -b$ so the mid-point of G_* becomes $y = 0$ and we have:

$$I = \int_{-b}^{b} \int_{c}^{d} f(\sqrt{M^2 + (y - y')^2}) \cdot dy dy'$$

Using the variable substitutions $u = y' - y$ and $v = y' + y$ we see that

$$-b + c \leq u \leq d + b$$

and obtain

$$I = 2b \int_{-b+c}^{d+b} f(\sqrt{M^2 + v^2}) \cdot dv$$

With

$$s = d - \frac{d+c}{2} = \frac{d-c}{2}$$

we can write

$$I = 2b \cdot \int_{w-s}^{w+s} f(\sqrt{M^2 + u^2}) \cdot dv \quad : w = b + \frac{d+c}{2}$$

The last integral is a function of s, i.e. for every $s \geq 0$ we set

$$\Phi(s) = 2b \cdot \int_{w-s}^{w+s} f(\sqrt{M^2 + u^2}) \cdot dv \quad : w = b + \frac{d+c}{2}$$

The derivative of s becomes

$$\Phi'(s) = f(\sqrt{M^2 + (w+s)^2}) - f(\sqrt{M^2 + (w-s)^2})$$

Since $f(r)$ was increasing the derivative is > 0 when $s > 0$. Hence the minimum is achieved when $s = 0$ which means that G_* and G^* have a common mid-point and Proposition 2.1 is proved for the case of an interval pair.

The general case. If $G_* = \{(a_\nu, b_\nu)\}$ and $G^* = \{(c_k, d_k)\}$ we make an induction over the total number of intervals which appear in the two families. Let

$$\xi^* = \frac{c^* + d^*}{2}$$

be the largest mid-point from the G^*-family which means that k is maximal, In the G_*-family we also get the largest mid-point is

$$\eta^* = \frac{a^* + b^*}{2}$$

If $\xi^* > \eta^*$ the previous case shows that the double sum representing I decreases as long as when the interval (c^*, d^*) is lowered. In this process two cases can occur: Namely the lowered (c^*, d^*)-interval hits (c_{k-1}, d_{k-1}) before the mid-point equality appears. To be precise, this occurs if

$$c^* - d_{k-1} < \xi^* - \eta^*$$

In this case we replace G^* by a union of intervals where the number of intervals therefore has decreased by one. If (ii) does not hold we lower (c^*, d^*) until $\xi^* = \eta^*$. After this we lower the two top-intervals at the same time until one of them hits the second largest G-interval and in this way the total number of intervals is decreased while the double sum for I is not enlarged. This gives the requested induction step and the proof of Proposition 2.1 is finished.

3. Proof of Theorem 1 in the convex case

Consider a domain U defined by

(1) $$g_1(x) \leq y \leq g_2(x) \quad : \quad a \leq x \leq b$$

where $g_1(a) = g_2(a)$ and $g_1(b) = g_2(b)$. To U we associate the symmetric domain U^* defined by

(2) $$-\frac{1}{2}\big[g_2(x) - g_1(x)\big] \leq y \leq \frac{1}{2}\big[g_2(x) - g_1(x)\big] \quad : \quad a \leq x \leq b$$

Notice that U and U^* have the same area. Set

$$J = \iint_{U \times U} f\big(\sqrt{(x-x')^2 + (y-y')^2}\big) \cdot dxdx'dydy'$$

$$J^* = \iint_{U^* \times U^*} f\big(\sqrt{(x-x')^2 + (y-y')^2}\big) \cdot dxdx'dydy'$$

Lemma 3.1. *One has the inequality $J \leq J^*$.*

Proof. Set $h(x) = \frac{1}{2}[g_2(x) - g_1(x)]$ and introduce the function

$$H^*(x, x') = \int_{y=-h(x)}^{h(x)} \int_{y'=-h(x')}^{h(x')} \rho\big(\sqrt{(x-x')^2 + (y-y')^2}\big) \cdot dydy'$$

We have also the function

$$H(x, x') = \int_{y=g_1(x)}^{g_2(x)} \int_{y'=g_1(x')}^{g_2(x')} f\left(\sqrt{(x-x')^2 + (y-y')^2}\right) \cdot dydy'$$

It is clear that

$$J = \int_a^b \int_a^b H(x,x')dxdx' \quad \text{and} \quad J^* = \int_a^b \int_a^b H^*(x,x')dxdx'$$

Lemma 3.1 follows if we have proved the inequality

$$(*) \qquad H(x,x') \leq H^*(x,x')$$

for all pairs x, x' in $[a,b]$. But this follows via Fubini's theorem when Proposition 2.1 applied in the special case where G_* and G^* both consist of a single interval.

3.2 Variation with convex sets. Let $\mathcal{A}$ be the prescribed area in Theorem 1 and consider a convex domain U whose area is $\mathcal{A}$. By elementary geometry we see that after a translation and a rotation the convex domain U can be represented as in(1) from § 3 above. Then we construct U_* which is a new convex domain and by Proposition 2.1 gives $J(U_*) \leq J(U)$. In the next step we perform a symmetrisation of U_* along some other line which cuts U_* to get a new domain U_{**} where we now have $J(U_{**}) \leq J(U_*) \leq J(U)$. Finally we use the geometric result due to Steiner for convex domains which asserts that when symmetrisations as above are repeated infinitely often while the angle to the angle of the directions to the x-axis change with some irrational multiple of 2π, then the resulting sequence of convex domains converge to a disc. This proves that the J-functional on a disc is $\leq J(U)$ for every convex domain.

4. The non-convex case

Here we use the symmetrisation process by Gros. Let U be a domain. Its symmetrisation in the x-direction is defined as follows: To every x we get the open set

$$(1) \qquad \ell_U(x) = \{y : (x,y) \in U\}$$

Let $\{(a_\nu, b_\nu)\}$ be the disjoint intervals of $\ell_U(x)$ and put

$$d(x) = \frac{1}{2} \cdot \sum (b_\nu - a_\nu)$$

We get the domain U^* which is symmetric with respect to the x-axis where

$$\ell_{U^*}(x) = (-d(x), d(x))$$

Notice that U and U^* have equal area. Proposition 2.1 applies and gives the inequality

$$(2) \qquad J(U) \leq J(U^*)$$

Now Theorem 1 follows when we start from a non-convex domain U. Namely, by the result proved in [Gros], it holds that after inifinitely many symmetrizations as above using different directions, the sequence of U-sets converge to a disc.