Analytic function theory and Fourier analysis offer useful tools in many applications to theoretical probability theory. We shall illuminate this by some specific examples and start with the moment problem. In general, let $g(x)$ be a non-negative continous function on $[0, +\infty]$ whose integral

$$\int_0^\infty g(x)\,dx = 1$$

Thus, g is the frequency function of a probability distribution with mass concentrated to $\{x \geq 0\}$. Let us assume that the distribution has finite moments, i.e.

$$\mathfrak{m}_k^2 = \int_0^\infty x^k \cdot g(x)\,dx < \infty \quad : \; k = 1, 2, \ldots$$

where $\mathfrak{m}_k$ is a sequence of positive numbers. The pioneering work by Stieltjes about continued fractions from 1894 gave examples of distinct frequency functions f and g as above with equal moments. Moreover, it was proved by Stieltjes , and independently by Emile Borel, that the family of all possible moment sequences $\{\mathfrak{m}_k\}$ "is as ample as it possibly can be". More precisely, a classic result which goes back to Heine in 1850 shows that a sequence $\{\mathfrak{m}_k\}$ give moments of a frequency function if and only if the quadratic forms

$$\sum_{p=0}^{p=n} \mathfrak{m}_{p+q}^2\, x_p x_q$$

are positive definite for every $n \geq 1$. To obtain such sequences one can Fourier's expansions of real-valued infinitely differentiable functions. Let us take a test-function $f(x)$ on the real x-line with support contained in the closed unit intervak $[0, 1]$. Notice that this implies that f is flat at the end-points $x = 0$ and $x = \pi$, i.e. at these points the derivaties vanish in every order. We define the function of the real variable s by:

$$\Phi(s) = \int_0^1 \cos(st) \cdot f(t)\,dt$$

Parseval's equality applied to f gives

$$\int_0^\infty f(x)^2\,dx = \frac{2}{\pi} \cdot \int_0^\infty \Phi(s)^2\,ds$$

next, define a non-decreasing function $G(x)$ on $x \geq 0$ by

$$G(x) = \frac{2}{\pi} \cdot \int_0^{\sqrt{x}} \Phi(s)^2\,ds$$

With f choen so that the left hand side in (x) us equal to one,it follows that $G(x)$ is a distribnution function of a stochastic variable with mass concenttrated to $\{x \geq 0\}$. The frequerncy functiuon of G becomes

$$g(x) = \frac{1}{\pi \cdot \sqrt{x}} \cdot \Phi(\sqrt{x})^2$$

If $p \geq 1$ we obtain

$$\int_0^\infty x^p \cdot g(x)\,dx = \int_0^\infty s^{2p} \cdot 2s \cdot g(s^2)\,ds = \frac{2}{\pi} \cdot \int_0^\infty s^{2p} \cdot \Phi^2(s)\,ds$$

These moment integrals can be recaptured from the given test-function f. Namely, since f is flat at $x = 0$ and $x = 1$, a partial integration gives the following equality for every $p \geq 1$:

$$(-1)^p \cdot \int_0^1 \cos(st) \cdot f^{(p)}(t)\,dt = \int_0^1 \frac{d^p}{dt^p}(\cos(st)) \cdot f(t)\,dt$$

Applying Paresval's equality to $f^{(p)}$ and cosinie. respecetly sdine-ttenaforms the reasder can deduce that (xx) gives the equlaity

$$\mathfrak{m}_p^2 = \int_0^1 f^{(p)}(t)^2\,dt$$

In particular all the moments of the distribution function G are finite.

Lecture IV: Eigenvalues for the Laplace operator.

Let Ω be a connected bounded domain in the complex z-plane of class $\mathcal{D}(C^1)$. In the Hilbert space $L^2(\Omega)$ we seek functions u satisfying

$$\Delta(u) + \lambda \cdot u = 0 \tag{0.1}$$

for some constnat λ, which in addition extend to continuous functions which are identically zero on the boundary. Stokes theoren gives for every u satisfying (0.1):

$$\iint_\Omega [\,(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]\, dxdy = \lambda \cdot \iint_\Omega u^2\, dxdy \tag{0.2}$$

Hence non-zero solutions in (0.1) can only exist when λ are real and positive. The result below is a special case from the Fredholm-Hilbert theory:

0.3 Theorem. *There exists a non-decreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of positive real numbers and to each λ_n one has a real-valued function ϕ_n where $\phi_n = 0$ on $\partial\Omega$ and*

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0$$

holds in Ω. Here $\{\phi_n\}$ is an orthonormal set in the Hibert space $L^2(\Omega)$ and Green's function satisfies the equation

$$G(p,q) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n}$$

where the right hand side converges when $p \neq q$.

Remark. As usual eigenvalues are repated when the corresponding finite dimensional eigenspace has dimension > 1.

§ 1. Asymptotic formulas.

Theorem 0.1 and Ikehara's theorem lead to asymptotic formulas. The first result is due to Weyl:

1.1 Theorem. *One has the limit formula*

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{\text{area}(\Omega)}$$

Concerning the eigenfunctions the following remarkabel resut was presented by Carleman at the Scandinavian Congress in Copenhagen 1934.

1.2 Theorem. *For every $p \in \Omega$ one has*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \cdot \sum_{k=1}^{k=n} \phi_k(p)^2 = \frac{1}{4\pi}$$

Passing to partial derivatives of the ϕ-functions similar asymptotic formulas hold. The result for first order partial derivatives is:

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \cdot \sum_{k=1}^{k=n} \big(\frac{\partial \phi_k}{\partial x}(p)\big)^2 = \frac{1}{16\pi} \tag{1.2.1}$$

and similarly for the partial y-derivative.

The remarkable faxt is of course thst thr teo limit formulas hold for every point in Ω and the limit does not depend on the domain. The detailed proof is presented in § xx.

4. Vibrating planes.

Let D be a membrance with contant density of mass m and tension $k > 0$. The boundary is fixed by a plane curve C placed in the horizontal (x, y)-plane and the function $u = u(x, y, t)$ is the deviation in the vertical direction while the membrance is in motion. Here t is a time variable and by Hooke's law the y-function satisfies the wave equation

$$(*) \qquad \frac{d^2u}{dt^2} = \frac{k}{m} \cdot \Delta u$$

where the boundary condition is that $u(p, t) = 0$ for each $p \in C$. The time dependent kinetic energy becomes

$$T(t) = \frac{m}{2} \iint_\Omega (\frac{du}{dt})^2 dxdy$$

The potential energy becomes

$$V(t) = \frac{k}{2} \iint_\Omega \left[(\frac{\partial u}{\partial x})^2 + \frac{\partial u}{\partial x})^2 \right] dxdy$$

With $\kappa_\nu = \sqrt{\lambda_\nu}$ the general solution to (*) becomes:

$$(**) \qquad u(p, t) = 2 \cdot \sum_{\nu=1}^{\infty} c_\nu \cos(\kappa_\nu t)\, \phi_\nu(p)$$

where $\{c_\nu\}$ is a sequence of real numbers. Define the mean kinetic energy at individual points $p \in D$ by

$$L(p) = \frac{m}{2} \cdot \lim_{\tau \to \infty} \frac{1}{\tau} \cdot \int_0^\tau (\frac{du}{dt})^2(p) \cdot d\tau$$

Exercise. Show that (**) entails that

$$(***) \qquad L(p) = k \cdot \sum |c_\nu|^2 \lambda_\nu \phi_\nu(p)^2$$

The c-numbers decay in a physically realistic solution so that the series above converges.

High frequencies. For each positive number w the contribution from high frequencies is defined by:

$$L_w(p) = k \cdot \sum_{\lambda_\nu > w} |c_\nu|^2 \lambda_\nu \cdot \phi_\nu(p)^2$$

Similarly, the mean potential energy from high frequencies is defined by

$$V_w(p) = k \cdot \sum_{\lambda_\nu > w} |c_\nu|^2 \cdot \left[\frac{\partial \phi_\nu}{\partial x}(p)^2 + \frac{\partial \phi_\nu}{\partial y}(p)^2 \right]$$

Let us analyze the limit behaviour of the two functions above when $w \to +\infty$. Let $a(\lambda)$ be a C^2-function defined for$\lambda > 0$ such that $a(\lambda_\nu) = |c_\nu|^2$ for each ν and set

$$P(\lambda) = \sum_{\lambda_\nu \le \lambda} \lambda_\nu \cdot \phi_\nu(p)^2$$

It follows that

$$L_w(p) = k \cdot \int_w^\infty a(\lambda) \cdot dP(\lambda)$$

Exercise. Show first that Theorem 0.1 entails that

$$P(\lambda) \simeq \frac{1}{8\pi} \cdot \lambda^2$$

Suppose now that $a(\lambda)$ is decreasing where

$$\lambda^2 \cdot a(\lambda) \leq K \cdot \int_\lambda^\infty \lambda \cdot a(\lambda)\, d\lambda$$

hold for some constant K. Show that this gives the asymptotic formula:

$$L_w(p) \simeq \frac{k}{4\pi} \int_w^\infty a(\lambda)\, d\lambda$$

The point is that the right hand side is independent of p. So when (xx) holds it follows that

$$\lim_{w\to\infty} \frac{L_w(p)}{L_w(q)} = 1$$

hold for all pairs p, q in Ω.

Exercise Use Theorem 0.1 to deduce a similar asymptotic formula for the V-function and conclude that

$$\lim_{w\to\infty} \frac{L_w(p)}{V_w(q)} = 1$$

hold for each point p in Ω.

When the regularity of $\partial\Omega$ is relaxed, for example if $\partial\Omega$ is a union of planar parts where pairs intersect at lines and "ugly corner points" appear when more than two planar parts meet, then the kernel function K_h is unbounded and may even fail to the square integrable, i.e. it can occur that

$$\iint_{\partial\Omega\times\partial\Omega} |K(p,q)|^2 \, d\sigma(p)d\sigma(q) = +\infty$$

In this situation the analysis becomes more involved and leads to the spectral theory of unbounded linear operators. One can also go further and allow u-solutions to the equation $u = \lambda \cdot \mathcal{K}_h(u)$ which are measurable functions. In other words, the domain of definition for the integral operator $\mathcal{K}_h$ is extended. Then it turns out that the spectrum of $\mathcal{K}_h$ may contain non-discrete parts outside the real line. We treat this case for planar domains in § XX where a specific case occurs if Ω is a bounded open subset of $\mathbf{R}^2$ bordered by a finite family of disjoint piecewise linear Jordan curves, i.e. by polygons. When h is a positive function on $\partial\Omega$ the planar kernel is given by

$$K_h(p,q) = \frac{1}{\pi} \cdot \frac{\langle p-q, \mathbf{n}_*(q)\rangle}{|p-q|^2}$$

Let $\{\alpha_\nu\}$ be the family of interior angles at the corner points from the union of the polygons above. So here $0 < \alpha_\nu < \pi$ for each ν and put:

$$R = \min_\nu \frac{\pi}{\pi - \alpha_\nu}$$

In his thesis *Über das Neumann-Poincarésche Problem für ein gebiet mit Ecken* from 1916, Carleman proved that $\mathcal{K}_h(\lambda)$ extends to a meromorphic function in the open disc $|\lambda| < R$ where a finite set of real and simple poles can occur. But in contrast to the smooth case the continuation beyond this disc is in general quite complicated. More precisely, when the domain of $\mathcal{K}_h$ is extended to measurable functions u with finite logarithmic energy:

$$\iint_{\partial\Omega\times\partial\Omega} \left|\log \frac{1}{|p-q|}\right| \cdot |u(p)| \cdot |u(q)| \, d\sigma(p)d\sigma(q) < \infty$$

there appears in general a non-real spectrum outside the disc of radius R which need not consist of discrete points. We remark that Carleman's study of the Neumann-Poincaré operators for non-smooth domains led to the theory about unbounded self-adjoint operators on Hilbert spaces.

Carleman's book *Sur les équations singuliers à noyeau réel et symmetrique* from 1923 proves the spectral theorem for unbounded operators and constitutes one if his major contributions in mathematics.

We shall prove the results above using material from Carleman's lecture at the Scandinavian Congress in Copenhagen 1934. The major step is to establish properties of the function $\Phi(p,s)$ introduced in § 1.4 below. First we recall the construction of Green resolvents. For every complex number λ outside the discrete set $\{\lambda_n\}$ we find the function which for each fixed $q\in\Lambda$ satisfies

$$\Delta G(p,q;\lambda)+\lambda\cdot G(p,q;\lambda)=0 \quad : \quad p\in\Omega\setminus\{q\}$$

and at $p=q$ it has the same singularity as $G(p,q)$, i.e. $\log\frac{1}{|p-q|}$. Finally, $G(p,q;\lambda)=0$ when $p\in\partial\Omega$. Theorem 0.1 gives the equation:

1.4 Theorem *One has the equality*

$$G(p,q;\lambda)-G(p,q)=\frac{1}{2\pi\lambda}\sum_{n=1}^{\infty}\frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda-\lambda_n)}$$

1.5 The function $\Phi(p,s)$. For each $p\in\Omega$ we define a function of the complex variable s:

$$\Phi(p,s)=\sum_{n=1}^{\infty}\frac{\phi_n^2(p)}{\lambda_n^s}$$

1.6 Theorem *The function $\Phi(p,s)$ extends to a meromorphic in the whole complex s-plane with a simple pole at $s=1$ whose residue is $\frac{1}{4\pi}$ and zeros at $0,-1,-2,\ldots$.*

2. Proof of Theorem 1.6.

We shall need two equations which are proved via residue calculus and left to the reader.

2.1 Lemma *For every pair $0<a<b$ of real numbers and each comlex number s with $\mathfrak{Re}\, s>1$ one has the equations:*

$$b^{-s}=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\lambda}{b(\lambda-b)\lambda^s}\cdot d\lambda=$$

$$\frac{a^{s-1}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i(1-s)\theta}}{b(b-a^{i\theta})}\,d\theta++\frac{\sin\pi s}{\pi}\cdot\int_a^{\infty}\frac{1}{b(b+\lambda)\lambda^s}\,d\lambda$$

2.2 The function $F(p,\lambda)$. Since $G(p,q;\lambda)$ and $G(p,q)$ have the same singularity $\log\frac{1}{|p-q|}$ along the diagonal it follows that

(i) $$G(p,p;\lambda)-G(p,p)=\frac{1}{2\pi\lambda}\sum_{n=1}^{\infty}\frac{\phi_n(p)^2}{\lambda_n(\lambda-\lambda_n)}$$

when $p\in\Omega$ where the right hand side is a meromorphic function of λ with poles confined to the set $\{\lambda_n\}$. Set

$$F(p,\lambda)=G(p,p;\lambda)-G(p,p)$$

Keeping $p\in\Omega$ fixed we apply Lemma 2.1 with $b=\lambda_n$ for every $n\geq 1$ and with a real a such that $0<a<\lambda_1$. A summation over n gives the equation below for each $\mathfrak{Re}\, s>1$:

Lemma 2.3 *One has the equality*

(*) $$\Phi(p,s)=\frac{a^{s-1}}{4\pi^2}\int_{-\pi}^{\pi}e^{i(1-s)\theta}\cdot F(p,ae^{i\theta})\,d\theta+\frac{\sin\pi s}{2\pi^2}\cdot\int_a^{\infty}\frac{F(p,-\lambda)}{\lambda^s}\,d\lambda$$

The first term in (*) is an entire function of s since $0<a<\lambda_1$ is a fixed real number and $F(p,\lambda)$ is analytic in the open disc of radius $|\lambda_1|$ centered at the origin. So $\Phi(p,s)$ extends to a meromorphic function with a simple pole at $s=1$ if the same is true for the function

(2.4) $$F_*(p,s)=\frac{\sin\pi s}{2\pi^2}\cdot\int_a^{\infty}\frac{F(p,-\lambda)}{\lambda^s}\,d\lambda$$

where we in addition should verify that the residue at $s=1$ is $\frac{1}{4\pi}$. To attain this we shall find another integral formula for the F_*-function.

2.5 The functions $H(p,q;\kappa)$. Define the analytic function $K(z)$ in the half-plane $\mathfrak{Re}\, z>0$ by

$$K(z)=\int_1^\infty \frac{e^{-zt}}{\sqrt{t^2-1}}\,dt$$

If p and q is a pair of points on Ω their euclidian distance is denoted by $|p-q|$. To each $\kappa>0$ we get the function defined in $\Omega\times\Omega$ by:

$$K_\kappa(p,q)=K(\kappa|p-q|)$$

Exercise. Verify the limit formula below where γ is the Euler constant:

$$\text{(iii)}\qquad \lim_{q\to p} K_\kappa(p,q)=\log\frac{1}{|p-q|}-\log\kappa+\log 2-\log\gamma$$

Hence we encounter the same singularity as for $G(p,q)$ which means that the function

$$\text{(iv)}\qquad g_\kappa(p,q)=K_\kappa(p,q)-G(p,q)$$

is defined in the whole product $\Omega\times\Omega$. Next, for each $\lambda>0$ we set $\kappa=\sqrt{\lambda}$ and define the function $H(p,q;\kappa)$ outside the diagonal in $\Omega\times\Omega$ by:

$$\text{(2.5.1)}\qquad H(p,q;\kappa)=K_\kappa(p,q)-G(p,q;\lambda)$$

By (iii) the functions $K_\kappa(p,q)$ and $G(p,q;\lambda)$ have the same logarithmic singularity along the diagonal which entails that $H(p,q;\kappa)$ is defined in the whole prodct $\Omega\times\Omega$. When $p\in\partial\Omega$ is kept fixed the vanishing of the Greeen's function gives

$$H(p,q;\kappa)=K_\kappa(p,q;\kappa)\quad : q\in\Omega$$

The reader may also verifiy that when $p\in\partial\Omega$ is kepts fixed then the function $q\mapsto H(p,q;\kappa)$ satisfies the equation below in Ω:

$$\Delta H(p,q;\kappa)-\kappa^2\cdot H(p,q;\kappa)=0$$

Exercise. Use the results above to show that there exist constants A and $\alpha>0$ which are independent of κ and of $p\in\Omega$ such that

$$0\le H(p,p;\kappa)\le K(\kappa\cdot \text{dist}(p,\partial\Omega)\quad\text{and}\quad H(p,p;\kappa)\le A\cdot e^{-\alpha\kappa}$$

2.6 Passage to a limit. In (2.5.1) we can pass to the limit as $q\to p$ inside Ω and the construction of F in (2.2) gives:

$$\text{(2.6.1)}\qquad H(p,p;\kappa)=-\log\kappa+\log 2-\log\gamma+g_\kappa(p,p)-F(p,-\lambda)$$

where we recall that $\kappa^2=\lambda$. It follows that

$$\text{(2.6.2)}\quad F_*(p,s)=\frac{\sin\pi s}{2\pi^2}\cdot\Big[-\int_a^\infty\frac{\log\sqrt{\lambda}}{\lambda^s}\,d\lambda+\int_a^\infty\frac{1}{\lambda^s}\,[g_\kappa(p,p)+\log 2-\log\gamma-H(p,p;\kappa)]\,d\lambda$$

A computation gives:

$$\text{(2.6.3)}\qquad -\int_a^\infty\frac{\log\sqrt{\lambda}}{\lambda^s}\,d\lambda=-\frac{1}{2}\cdot\frac{a^{s-1}\log a}{s-1}+\frac{1}{2}\cdot\frac{a^{s-1}}{(s-1)^2}$$

In (2.6.3) we have a double pole at $s=1$ which after multiplication with the sine-function gives a simple pole and the reader can verify that the residue is $\frac{1}{4\pi}$. Finally, using the estimate in Exercise 2.5.2 the reader may verify that the second term in (2.6.2) yields an entire function of s Hence $s\mapsto F_*(p,s)$ is meromorphic with a simple pole at $s=1$ with residue $\frac{1}{4\pi}$ and Theorem 1.6 is proved.

3. Proofs of the asymptotic formulas.

Theorem 1.6 and Ikehara's theorem from § xX entail that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\cdot\sum_{k=1}^{k=n}\phi_k(p)^2=\frac{1}{4\pi} \tag{3.1}$$

This proves Theorem 1.2 is proved and to get Theorem 1.1 we perform an integration over Ω so that

$$\sum_{n=1}^\infty\frac{1}{\lambda_n^s}=\iint_\Omega\Phi(p,s)\,dxdy$$

where we simply have used that each ϕ-function has a squared integral equal to one over Ω. When $\mathfrak{Re}\,s>1$ the equations from the proof of Theorem 1.6 show that after an integration over Ω one has

$$\sum_{n=1}^\infty\frac{1}{\lambda_n^s}=\frac{\text{Area}(\Omega)}{4\pi}\cdot\frac{1}{s-1}+J(s)$$

where $J(s)$ is analytic in $\mathfrak{Re}>1$. By (*) in Lemma 2.3 the J-function is a sum of an entire function and the function

$$s\mapsto\iint_\Omega\Big[\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty\frac{1}{\lambda^s}\,[g_\kappa(p,p)+\log 2-\log\gamma-H(p,p;\kappa)]\,d\lambda\Big]\,dxdy \tag{2.6.2}$$

From this it is clear that the requested continuity of J up to $s=1$ follows if the integrals

$$\iint_\Omega\Big[\int_a^\infty\frac{1}{\lambda}\,H(p,p;\kappa)]\,d\lambda\Big]\,dxdy<\infty \tag{i}$$

and similarly with $H(p,p;\kappa)$ replaced by $g_\kappa(p,p)$. To prove (i) we use the estimate in (2.5.2) which shows that (i) is majorised by

$$A\cdot\iint_\Omega\Big[\int_a^\infty\frac{1}{\lambda}e^{-\alpha\cdot\sqrt{\lambda}\cdot\text{dist}(p,\partial\Omega)}\,d\lambda\Big]\,dxdy$$

Put $\ell(p)=\text{dist}(p,\partial\Omega)$ and consider for a fixed $p\in\Omega$:

$$\int_a^\infty\frac{1}{\lambda}e^{-\alpha\cdot\sqrt{\lambda}\cdot\ell(p)}\,d\lambda$$

Above a and α are fixed positive constants and the reader may verify that this gives another pair of constants b,c which are independent of p such that (i) is majorized by

$$c\cdot[\log^+\frac{b}{\ell(p)}]^2$$

This function of p is integrable over Ω and in the same way the reader can check the convergence when $H(p,p,\kappa)$ is replaced by $g_\kappa(p,p)$.

3.2. The limit formula (1.3.) The proof of (1.3) uses similar methods as above but some extra technicalities appear because estimates for partial derivatives of the Green's function are needed. The reader may consult [Carleman: page 38-40] for the details which gives (1.3) or try to carry out the proof. In [ibid] a general limit formula for higher order mixed partial derivatives of the ϕ-functions is proved. The result is:

3.3. Theorem. *For every pair of non-negative integers j,m and each $p\in\Omega$ one has the limit formula*

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{j+m+1}}\cdot\sum_{k=1}^{k=n}\Big(\frac{\partial^{j+m}\phi_k}{\partial x^j\partial y^j}\Big)^2(p)=\frac{1}{\pi\cdot 2^{2m+2j+2}}\cdot\frac{(2m)!\cdot(2j)!}{m!\cdot j!\cdot(m+j+1)!}$$

4. Vibrating planes.

Let D be a membrance with contant density of mass m and tension $k > 0$. The boundary is fixed by a plane curve C placed in the horizontal (x, y)-plane and the function $u = u(x, y, t)$ is the deviation in the vertical direction while the membrance is in motion. Here t is a time variable and by Hooke's law the y-function satisfies the wave equation

$$(*) \qquad \frac{d^2u}{dt^2} = \frac{k}{m} \cdot \Delta u$$

where the boundary condition is that $u(p, t) = 0$ for each $p \in C$. The time dependent kinetic energy becomes

$$T(t) = \frac{m}{2} \iint_\Omega (\frac{du}{dt})^2 dxdy$$

The potential energy becomes

$$V(t) = \frac{k}{2} \iint_\Omega [(\frac{\partial u}{\partial x})^2 + \frac{\partial u}{\partial x})^2] \, dxdy$$

With $\kappa_\nu = \sqrt{\lambda_\nu}$ the general solution to (*) becomes:

$$(**) \qquad u(p, t) = 2 \cdot \sum_{\nu=1}^{\infty} c_\nu \cos(\kappa_\nu t) \, \phi_\nu(p)$$

where $\{c_\nu\}$ is a sequence of real numbers. Define the mean kinetic energy at individual points $p \in D$ by

$$L(p) = \frac{m}{2} \cdot \lim_{\tau \to \infty} \frac{1}{\tau} \cdot \int_0^\tau (\frac{du}{dt})^2(p) \cdot d\tau$$

Exercise. Show that (**) entails that

$$(***) \qquad L(p) = k \cdot \sum |c_\nu|^2 \lambda_\nu \phi_\nu(p)^2$$

The c-numbers decay in a physically realistic solution so that the series above converges.

High frequencies. For each positive number w the contribution from high frequencies is defined by:

$$L_w(p) = k \cdot \sum_{\lambda_\nu > w} |c_\nu|^2 \lambda_\nu \cdot \phi_\nu(p)^2$$

Similarly, the mean potential energy from high frequencies is defined by

$$V_w(p) = k \cdot \sum_{\lambda_\nu > w} |c_\nu|^2 \cdot \big[\frac{\partial \phi_\nu}{\partial x}(p)^2 + \frac{\partial \phi_\nu}{\partial y}(p)^2\big]$$

Let us analyze the limit behaviour of the two functions above when $w \to +\infty$. Let $a(\lambda)$ be a C^2-function defined for$\lambda > 0$ such that $a(\lambda_\nu) = |c_\nu|^2$ for each ν and set

$$P(\lambda) = \sum_{\lambda_\nu \leq \lambda} \lambda_\nu \cdot \phi_\nu(p)^2$$

It follows that

$$L_w(p) = k \cdot \int_w^\infty a(\lambda) \cdot dP(\lambda)$$

Exercise. Show first that Theorem 0.1 entails that

$$P(\lambda) \simeq \frac{1}{8\pi} \cdot \lambda^2$$

Suppose now that $a(\lambda)$ is decreasing where

$$\lambda^2 \cdot a(\lambda) \leq K \cdot \int_\lambda^\infty \lambda \cdot a(\lambda)\, d\lambda$$

hold for some constant K. Show that this gives the asymptotic formula:

$$L_w(p) \simeq \frac{k}{4\pi} \int_w^\infty a(\lambda)\, d\lambda$$

The point is that the right hand side is independent of p. So when (xx) holds it follows that

$$\lim_{w\to\infty} \frac{L_w(p)}{L_w(q)} = 1$$

hold for all pairs p, q in Ω.

Exercise Use Theorem 0.1 to deduce a similar asymptotic formula for the V-function and conclude that

$$\lim_{w\to\infty} \frac{L_w(p)}{V_w(q)} = 1$$

hold for each point p in Ω.

When the regularity of $\partial\Omega$ is relaxed, for example if $\partial\Omega$ is a union of planar parts where pairs intersect at lines and "ugly corner points" appear when more than two planar parts meet, then the kernel function K_h is unbounded and may even fail to the square integrable, i.e. it can occur that

$$\iint_{\partial\Omega\times\partial\Omega} |K(p,q)|^2\, d\sigma(p) d\sigma(q) = +\infty$$

In this situation the analysis becomes more involved and leads to the spectral theory of unbounded linear operators. One can also go further and allow u-solutions to the equation $u = \lambda \cdot \mathcal{K}_h(u)$ which are measurable functions. In other words, the domain of definition for the integral operator $\mathcal{K}_h$ is extended. Then it turns out that the spectrum of $\mathcal{K}_h$ may contain non-discrete parts outside the real line. We treat this case for planar domains in § XX where a specific case occurs if Ω is a bounded open subset of $\mathbf{R}^2$ bordered by a finite family of disjoint piecewise linear Jordan curves, i.e. by polygons. When h is a positive function on $\partial\Omega$ the planar kernel is given by

$$K_h(p,q) = \frac{1}{\pi} \cdot \frac{\langle p-q, \mathbf{n}_*(q)\rangle}{|p-q|^2}$$

Let $\{\alpha_\nu\}$ be the family of interior angles at the corner points from the union of the polygons above. So here $0 < \alpha_\nu < \pi$ for each ν and put:

$$R = \min_\nu \frac{\pi}{\pi - \alpha_\nu}$$

In his thesis *Über das Neumann-Poincarésche Problem für ein gebiet mit Ecken* from 1916, Carleman proved that $\mathcal{K}_h(\lambda)$ extends to a meromorphic function in the open disc $|\lambda| < R$ where a finite set of real and simple poles can occur. But in contrast to the smooth case the continuation beyond this disc is in general quite complicated. More precisely, when the domain of $\mathcal{K}_h$ is extended to measurable functions u with finite logarithmic energy:

$$\iint_{\partial\Omega\times\partial\Omega} \left|\log \frac{1}{|p-q|}\right| \cdot |u(p)| \cdot |u(q)|\, d\sigma(p) d\sigma(q) < \infty$$

there appears in general a non-real spectrum outside the disc of radius R which need not consist of discrete points. We remark that Carleman's study of the Neumann-Poincaré operators for non-smooth domains led to the theory about unbounded self-adjoint operators on Hilbert spaces. Carleman's book *Sur les équations singuliers à noyeau réel et symmetrique* from 1923 proves the spectral theorem for unbounded operators and constitutes one if his major contributions in mathematics.