0.5.5 Adjoint operators and closed extensions.

Let T be densely defined. But for the moment we do not assune that it is closed. In the dual space X^* we have the family of vectors y for which there exists a constant $C(y)$ such that

(i) $$|y(Tx)| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

It is clear that the set of such y-vectors is a subspace of X^*. Moreover, when (i) holds the density of $\mathcal{D}(T)$ gives a unique vector $T^*(y)$ in X^* such that

(ii) $$y(Tx) = T^*(y)(x) \quad : x \in \mathcal{D}(T)$$

One refers to T^* as the adjoint operator of T whose domain of definition is denoted by $\mathcal{D}(T^*)$.
Exercise. Show that the graph of T^* is closed in $X^* \times X^*$. However, $\mathcal{D}^*(T)$ is in general not a dense subspace of X^*. See § xx for an example.

Closed extensions. There may exist several closed operators S with the property that

$$\Gamma(T) \subset \Gamma(S)$$

When this holds we refer to S as a closed extension of T. Notice that the inclusion above is strict if and only if $\mathcal{D}(S)$ is strictly larger than $\mathcal{D}(T)$.

Exercise. Use the density of $\mathcal{D}(T)$ to show that

$$T^* = S^*$$

hold for every closed extension S of T.

The case when $\mathcal{D}(T^*)$ is dense. Let T be densely defined and assume that its adjoint has a dense domain of definition. In this situation the following holds:

0.5.6 Theorem. *If $\mathcal{D}(T^*)$ is dense then there exists a closed operator $\widehat{T}$ whose graph is the closure of $\Gamma(T)$.*

Proof. Consider the graph $\Gamma(T)$ and let $\{x_n\}$ and $\{\xi_n\}$ be two sequences in $\mathcal{D}(T)$ which both converge to a point $p \in X$ while $T(x_n) \to y_1$ and $T(\xi_n) \to y_2$ hold for some pair y_1, y_2. We must rove that $y_1 = y_2$. To achieve this we take some $x^* \in \mathcal{D}(T^*)$ which gives

$$x^*(y_1) = \lim\, x^*(Tx_n) = \lim\, T^*(x^*)(x_n) = T^*(x^*)(p)$$

In the same way we get $x^*(y_2) = T^*(x^*)(p)$. Now the density of $\mathcal{D}(T^*)$ gives $y_1 = y_2$ which proves that the closure of $\Gamma(T)$ is a graphic subset of $X \times X$ and gives the closed operator $\widehat{T}$ with

$$\Gamma(\widehat{T}) = \overline{\Gamma(T)}$$

The case when X is reflexive. Assume this and let T be a densely defined and closed operator. Suppose in addition that T^* also is densely defined. Now we can construct the adjoint of T^* which is denoted by T^{**}. Since X is reflexive it follows that T^{**} is a closed and densely defined operator on X. If $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$ we have the vector $\widehat{x} \in X^{**}$ and

$$\widehat{x}(T^*(y)) = T^*(y)(x) = y(T(x))$$

From this it is clear that $\widehat{x} \in \mathcal{D}(T^{**})$ and one has the equality

$$T^{**}(\widehat{x}) = T(x)$$

Hence the graph of T is contained in that of T^{**}, i.e. T^{**} is a closed extension of T.

0.5.7 The spectrum of T^*. Let X and T be as above. Then one has the inclusion

(*) $$\rho(T) \subset \rho(T^*)$$

Proof. By translations it suffices to show that if the origin belongs to $\rho(T)$ then it also belongs to $\rho(T^*)$. So now the resolvent $R_T(0)$ exists which means that T is surjective and there is a constant $c > 0$ such that

(i) $$||x|| \leq c^{-1} \cdot ||Tx|| \quad : x \in \mathcal{D}(T)$$

Consider some $y \in \mathcal{D}(T^*)$ of unit norm. Since T is surjective we find $x \in \mathcal{D}(T)$ with $||Tx|| = 1$ and

(ii) $$|y(Tx)| \geq 1/2$$

Now

(iii) $$y(Tx) = T^*(y)(x)$$

and from (i) we have

(iv) $$||x|| \leq c^{-1} \cdot ||Tx|| = c^{-1}$$

Then (ii) and (iv) entail that

$$||T^*(y)|| \geq c/2$$

This proves that

(v) $$||T^*(y)|| \geq c/2 \cdot ||y|| \quad : y \in \mathcal{D}(T^*)$$

Hence the origin belongs to $\rho(T^*)$ if we prove that T^* has a dense range. If the density fails there exists a non-zero linear functional $\xi \in X^{**}$ such that

$$\xi(T^*(y)) = 0 \quad : y \in \mathcal{D}(T^*)$$

Since X is reflexive we have $\xi = \mathfrak{i}_X(x)$ for some vector x and obtain

$$y(Tx) = 0 \quad : y \in \mathcal{D}(T^*)$$

The density of $\mathcal{D}(T^*)$ gives $Tx = 0$ which contradicts the hypothesis that T is injective and (*) follows.

0.5.8 The case when X is a Hilbert space. In this case we shall prove that when both T and T^* are closed and densely defined, then one has the equality

$$\sigma(T) \subset \sigma(T^*)$$

We refer to § xx for the proof.

0.5.3 The case when resolvent operators are compact. Let T be such that $R_T(\lambda_0)$ is a compact operator for some resolvent value. We assume of course that the Banach space X is not finite dimensional. In §§ we shall learn that the spectrum of a compact operator always contains zero and outside the origin the spectrum is a discrete set with a sole cluster point at the origin. From (0.5.1) it follows that $\sigma(T)$ is a discrete set in $\mathbf{C}$, i.e. its intersection with every disc $\{|\lambda| \leq R\}$ is finite.

0.5.4 Remark. In § xx we shall learn that if S is a compact operator then $S \circ U$ and $U \circ S$ are compact for every bounded operator U. Applying Neumann's equation (*) in (0.3) it follows that if one resolvent operator $R_T(\lambda_0)$ is compact, then all resolvents of T are compact.

0.1 The class $\mathcal{I}(X)$. It consists of bounded linear operators R on X with the property that R is injective and the range $R(X)$ is a dense subspace of X. We do not exclude the possibility that R is surjective. Each such operator R gives a densely defined operator T as follows: If $x \in R(X)$ the injectivity of R gives a unique vector $\xi \in X$ such that $R(\xi) = x$ and we set

(i) $$T(x) = \xi$$

It means that the composed operator $T \circ R = E$, where E is the identity operator on X. Here the domain of defintion for T is equal to the range $R(X)$ and this dense subspace of X is denoted by $\mathcal{D}(T)$. By construction we have

$$R \circ T(x) = x \quad : x \in \mathcal{D}(T)$$

Next, the bounded operator R has a finite operator norm $||R||$ and (i) entails that

$$\text{(ii)} \qquad ||x||| \leq ||R|| \cdot ||T(x)||$$

Thus, with $c = ||R||^{-1}$ one has

$$\text{(iii)} \qquad ||T(x)|| \geq c \cdot ||x||| \quad : x \in \mathcal{D}(T)$$

The graph $\Gamma(T)$. It is the subset of $X \times X$ given by $\{(x, Tx) : x \in \mathcal{D}(T)\}$. The construction of T gives

$$\Gamma(T) = \{(Rx, x) \colon x \in X\}$$

Since R is a bounded inear operator it is clear that the last set is closed in $X \times X$, i.e. $\Gamma(T)$ is closed which means that T is a densely defined and closed linear operator on X. The inequality (iii) shows that T is injective and since

$$T(Rx)) = x \quad : x \in X$$

the range of T is equal to X.

A converse result. Assume that T is a densely defined and closed operator such that (iii) holds and in addition the range of T is dense in X. It turns out that this gives the equality

$$\text{(1)} \qquad T(\mathcal{D}(T)) = X$$

For if $y \in X$ the density of the range gives a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that

$$\text{(2)} \qquad \lim_{n \to \infty} ||T(x_n) - y|| = 0$$

Now

$$||x_n - x_m|| \leq c^{-1} \cdot ||T(x_n) - T(x_m)||$$

and (2) entails that $\{T(x_n)\}$ is a Cauchy sequence. Since X is a Banach space it follows that $\{x_n\}$ converges to a limit vector x. Now $\Gamma(T)$ is closed which implies that (x, y) belongs to the graph, i.e. $x\mathcal{D}(T)$ and $T(x) = y$ which proves (1).

Exercise. Let T be densely defined and closed where (iii) holds and $T(\mathcal{D}(T)) = X$. Show that there exists a unique bounded operator $R \in \mathcal{I}(X)$ such that T is the attached operator as in 0.1 above.

0.2 Spectra of densely defined operators.

Let T be a densely defined and closed linear operator. Each complex number λ gives the densely defined operator $\lambda \cdot E - T$. We say that λ is a resolvent value of T if $\lambda \cdot E - T$ is surjective and there exists a positive constant c such that

$$||\lambda \cdot x - T(x)|| \geq c \cdot ||x||$$

The set of resolvent values is denoted by $\rho(T)$. Its closed complement is called the spectrum of T and we put

$$\sigma(T) = \mathbf{C} \setminus \rho(T)$$

Each $\lambda \in \rho(T)$ gives a unique bounded operator $R_T(\lambda) \in \mathcal{I}(X)$ such that

$$(\lambda \cdot E - T) \circ R_T(\lambda)(x) = x$$

Since $\mathcal{D}(T) = \mathcal{D}(\lambda \cdot E - T)$ it follows that the range of $R_T(\lambda)$ is equal to $\mathcal{D}(T)$.

0.2.1 Definition. *Tbe family $\{R_T(\lambda) : \lambda \in \rho(T)\}$ are called Neumann's resolvents of T.*

An example. Let X be the Hilbert space ℓ^2 whose vectors are complex sequences $\{c_1, c_2, \ldots\}$ for which $\sum |c_n|^2 < \infty$. We have the dense subspace ℓ^2_* vectors such that $c_n \neq 0$ only occurs for finitely many integers n. If $\{\xi_n\}$ is an arbitrary sequence of complex numbers there exists the densely defined operator T on ℓ^2 which sends every sequence vector $\{c_n\} \in \ell^2_*$ to the vector $\{\xi_n \cdot c_n\}$. If λ is a complex number the reader may check that (i) holds in (0.0.1) if and only if there exists a constant C such that

$$\text{(v)} \qquad |\lambda - \xi_n| \geq C \quad : n = 1, 2, \ldots$$

Thus, $\lambda \cdot E - T$ has a bounded left inverse if and ony if λ belongs to the open complement of the closure of the set $\{\xi_n\}$ taken in the complex plane. Moreover, if (v) holds then $R_T(\lambda)$ is the bounded linear operator on ℓ^2 which sends $\{c_n\}$ to $\{\frac{1}{\lambda - \xi_n} \cdot c_n\}$. Since every closed subset of $\mathbf{C}$ is equal to the closure of a denumerable set of points our construction shows that the spectrum of a densely defined operator $\sigma(T)$ can be an arbitrary closed set in $\mathbf{C}$.

§ 9. Neumann's resolvent operators

Throughout X denotes a complex Banach space. densely defined linear operator on X isa liner map

$$T\colon \mathcal{D}(T) \to X$$

where $\mathcal{D}(T)$ is a dense subspace of X, called the domain of definition for T. To each such T we associate the graph

$$\Gamma(T) = \{(x, Rx) :\ x \in \mathcal{D}(T)\}$$

So $\Gamma(T)$ is a subspace of the product $X \times X$ and if this graph is a closed subspace we say for brevity that T is closed. From now on T is densely defined and closed. Let E be the identity operator on X. Then each complex number λ gives the operator

$$\lambda \cdot E - T$$

whose domain of definition is equal to $\mathcal{D}(T)$.

Definition. *A complex number λ is called a resolvent value of T if there exists a constant $c > 0$ such that*

$$(*) \qquad ||\lambda \cdot x - Tx|| \geq c \cdot ||x|| \quad : \quad x \in \mathcal{D}(T)$$

and in addition the range of $\lambda \cdot E - T$ is equal to X.

The set of resilient values is denoted by $\rho(T)$. There exist ugly operators for which $\rho(T) = \emptyset$. We shall ignore this car and assume that T has at least one resolvent value.

The resolvent operator $R_T(\lambda)$. Let $\lambda \in \rho(T)$ be given. Sinc the range of $\lambda \cdot E - T$ is equal to X we find for every $y \in X$ some $x(y) \in \mathcal{D}(T)$ such that

$$\lambda \cdot x(y) - Tx(y) = y$$

and here (*) entails that x is unique and

$$||x|| \leq c^{-1}||y||$$

Hence there exists a bounded liner operator S on X with operator norm $\leq c^{-1}$ such that

$$Sy = x(y)$$

We put $S = R_T(\lambda)$ and it is called Neumannn's resovlent operator of T for at the given resolvent value λ. The construction shows that the range

$$R_T(\lambda)(X) = \mathcal{D}(T)$$

and passing to composed operators we have

$$R \circ T(x) = x :\ x \in \mathcal{D}(T) \ \&\ T \circ R(x) = x :\ x \in X$$

0.3 Neumann's equation.

Let T as above be densely defined and closed where $\rho(T)$ is non-empty. Then one has:

For each pair $\lambda \neq \mu$ in $\rho(T)$ the operators $R_T(\lambda)$ and $R_T(\mu)$ commute and

$$(*) \qquad R_T(\mu)R_T(\lambda) = \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu}$$

Proof. Notice that

$$(\mu \cdot E - T) \cdot \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} =$$

$$\text{(i)} \qquad \frac{E}{\lambda - \mu} - (\mu - \lambda) \cdot \frac{R_T(\lambda)}{\lambda - \mu} - (\lambda \cdot E - T) \cdot \frac{R_T(\lambda)}{\lambda - \mu} = R_T(\lambda)$$

Multiplying to the left by $R_T(\mu)$ gives (*).

0.4 Neumann series.

If $\lambda_0 \in \rho(T)$ we construct the operator valued series

$$S(\zeta) = R_T(\lambda_0) + \sum_{n=1}^{\infty} (-1)^n \cdot \zeta^n \cdot R_T(\lambda_0)^{n+1} \tag{1}$$

where the series converges in the Banach space of bounded linear operators when

$$|\zeta| < \frac{1}{||R_T(\lambda_0)||}$$

Next, one has

$$(\lambda_0 + \zeta - T) \cdot S(\zeta) = (\lambda_0 - T) \cdot S(\zeta) + \zeta S(\zeta) =$$
$$E + \sum_{n=1}^{\infty} (-1)^n \cdot \zeta^n \cdot R_T(\lambda_0)^{n+1} + \zeta S(\zeta)$$

and thanks to the alternating signs the last term is reduced to E. We conclude that

$$S(\zeta) = R_T(\lambda_0 + \zeta)$$

give resolvent operators. This the open disc of radius $\frac{1}{||R_T(\lambda_0)||}$ centered at λ_0 stays in $\rho(T)$. Hence the set of reovlent values for T is open. Moreover, the operator-valued function

$$\lambda \mapsto R_T(\lambda)$$

is an analytic function in $\rho(T)$. For if $\lambda \in \rho(T)$ we can pass to the limit as $\mu \to \lambda$ in Neumann's equation and conclude that the complex derivative exists and is given by

$$\frac{d}{d\lambda}(R_T(\lambda) = -R_T^2(\lambda) \tag{**}$$

Thus, Neumann's resolvent operator satisfies a specific differential equation for every densely defined and closed operator T with a non-empty resolvent set.

0.5 The position of $\sigma(T)$.

Assume that $\rho(T) \neq \emptyset$. For a pair of resolvent values of T we can write Neumann's equation in the form

$$R_T(\lambda)(E + (\lambda - \mu)R_T(\mu)) = R_T(\mu) \tag{1}$$

Keeping μ fixed we conclude that $R_T(\lambda)$ exists if and only if $E + (\lambda - \mu)R_T(\mu))$ is invertible. This gives the set-theoretic equality

$$\sigma(T) = \{\lambda : \frac{1}{\mu - \lambda} \in \sigma(R_T(\mu))\} \tag{0.5.1}$$

Hence one recovers $\sigma(T)$ via the spectrum of any given resolvent operator. Notice that (0.5.1) holds even when the open component of $\sigma(T)$ has several connected components.

0.5.2 Example. Suppose that $\mu = i$ and that $\sigma(R_T(i))$ is contained in a circle $\{|\lambda + i/2| = 1/2\}$. If $\lambda \in \sigma(T)$ the inclusion (0.0.5.1) gives some $0 \leq \theta \leq 2\pi$ such that

$$\frac{1}{i - \lambda} = -i/2 + 1/2 \cdot e^{i\theta} \implies 1 - i \cdot e^{i\theta} = \lambda(e^{i\theta} - i)$$

The last equation entails that

$$\lambda = \frac{2 \cdot \cos\theta}{|e^{i\theta} - i|^2}$$

and hence λ is real.

0.6 Operational calculus.

Let T be a densely defined and closed operator on a Banach space X. To each pair (γ, f) where γ is a rectifiable Jordan arc contained in $\mathbf{C} \setminus \sigma(T)$ and $f \in C^0(\gamma)$, there exists the bounded linear operator

$$T_{(\gamma,f)} = \int_\gamma f(z) R_T(z)\, dz \tag{0.6.1}$$

The integral is calculated via a Riemann sum where the integrand has values in the Banach space of bounded linear operators on X. More precisely, let $s \mapsto z(s)$ be a parametrisation with respec to arc-length. If L is the arc-length of γ we get Riemann sums

$$\sum_{k=0}^{k=N-1} f(z(s_k)) \cdot (z(s_{k+1}) - z(s_k)) \cdot (s_{k+1} - s_k) \cdot R_T(z(s_k))$$

where $0 = s_0 < s_1 < \ldots s_N = L$ is a partition of $[0, L]$. These Riemann sums converge to a limit when $\{\max (s_{k+1} - s_k\} \to 0$ with respect to the operator norm and give the T-operator in (0.6.1). The triangle inequality entails that

$$T_{(\gamma,f)} \leq L \cdot |f|_\gamma \cdot \max_{z\in\gamma} ||R_T(z)||$$

where $|f|_\gamma$ is the maximum norm of f on γ..

Neumann's equation in (0.3) entails that $R_T(z_1)$ and $R_T(z_2)$ commute for all pairs z_1, z_2 on γ. It follows that if g is another function in $C^0(\gamma)$ then the operators $T_{f,\gamma}$ and $T_{g,\gamma}$ commute. Moreover, for each $f \in C^0(\gamma)$ the reader may verify that the closedeness of T implies that the range of $T_{f,\gamma}$ is contained in $\mathcal{D}(T)$ and one has

$$T_{f,\gamma} \circ T(x) = T \circ T_{f,\gamma}(x) \quad : x \in \mathcal{D}(T)$$

Next, let Ω be an open set of class $\mathcal{D}(C^1)$, i.e. $\partial\Omega$ is a finite union of closed differentiable Jordan curves. When $\partial\Omega \cap \sigma(T) = \emptyset$ we construct line integrals as in (0.6.1) for continuous functions on the boundary. Consider the algebra $\mathcal{A}(\Omega)$ of analytic functions in Ω which extend to be continuous on the closure. Each $f \in \mathcal{A}(\Omega)$ gives the operator

$$T_f = \int_{\partial\Omega} f(z) R_T(z)\, dz \tag{0.6.2}$$

0.6.3 Theorem. *The map $f \mapsto T_f$ is an algebra homomorphism from $\mathcal{A}(\Omega)$ into a commutative algebra of bounded linear operators on X whose image is a commutative algebra of bounded linear operators denoted by $T(\Omega)$.*

Proof. Let f, g be a pair in $\mathcal{A}(\Omega)$. To show that $T_{gf} = T_f \circ T_g$ we consider a slightly smaller open set Ω_* which again is of class $\mathcal{D}(C^1)$ and each of it bounding Jordan curve is close to one boundary curve in $\partial\Omega$ and $\Omega \setminus \Omega_*$ does not intersect $\sigma(T)$. By Cauchy's theorem we can shift the integration to $\partial\Omega_*$ and get

$$T_g = \int_{\partial\Omega_*} g(z) R_T(z_*)\, dz_* \tag{i}$$

where we use z_* to indicate that integration takes place along $\partial\Omega_*$. Now

$$T_f \circ T_g = \iint_{\partial\Omega_* \times \partial\Omega} f(z) g(z_*) R_T(z) \circ R_T(z_*)\, dz_* dz \tag{ii}$$

Neumann's equation (*) from (0.0.3) entails that the right hand side in (ii) becomes

$$\iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z)g(z_*)R_T(z_*)}{z - z_*}\, dz_* dz + \iint_{\partial\Omega_* \times \partial\Omega} \frac{f(z)g(z_*)R_T(z)}{z - z_*}\, dz_* dz = A + B \tag{iii}$$

Here A is evaluated by first integrating with respect to z and Cauchy's theorem gives

$$f(z_*) = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega} \frac{f(z)}{z - z_*} \quad : z_* \in \partial\Omega_* \, dz$$

It follows that

$$A = \frac{1}{2\pi i} \cdot \iint_{\partial\Omega_* \times \partial\Omega} f(z_*) g(z_*) R_T(z_*) \, dz_* = T_{fg}$$

Next, B is evaluated when we first integrate with respect to z_*. Here

$$\iint_{\partial\Omega} \frac{g(z_*)}{z - z_*} \quad : z \in \partial\Omega$$

which entails that $B = 0$ and the theorem follows.

0.7 Spectral gap sets.

Let K be a compact subset of $\sigma(T)$ such that $\sigma(T) \setminus K$ is a closed set in $\mathbf{C}$. This implies that if V is an open neighborhood of K, then there exists a relatively compact subdomain $U \in \mathcal{D}(C^1)$ which contains K as a compact subset. To every such domain Ω we can apply Theorem 0.0.6.3. If $U_* \subset U$ for a pair of such domains we can restrict functions in $\mathcal{A}(U)$ to U_* which yields an algebra homomorphism

$$\mathcal{T}(U) \to \mathcal{T}(U_*)$$

Next, denote by $\mathcal{O}(K)$ the algebra of germs of analytic functions on K. So each $f \in \mathcal{O}(K)$ comes from some analytic function in a domain U as above. The resulting operator $T_U(f)$ depends on the germ f only. In fact, this follows because if $f \in \mathcal{A}(U)$ and $U_* \subset U$ is a similar $\mathcal{D}(C^1)$-domain which again contains K, then Cauchy's vanishing theorem from § xxx is applied to $f(z)R_T(z)$ in $U \setminus \bar{U}_*$ and entails that

$$\int_{\partial U_*} f(z) R_T(z) \, dz = \int_{\partial U} f(z) R_T(z) \, dz$$

Hence there exists an algebra homorphism from $\mathcal{O}(K)$ into bounded linear operators on X whose image is denoted by $\mathcal{T}(K)$. The identity in $\mathcal{T}_K$ is denoted by E_K and called the spectral projection operator attached to the compact set K in $\sigma(T)$. By this construction one has

$$E_K = \frac{1}{2\pi i} \cdot \int_{\partial U} z \cdot R_T(z) \, dz$$

for every open domain U around K as above.

0.0.6.4 The operator T_K. When K is a compact spectral gap set of T we set

$$T_K = TE_K$$

This bounded linear operator is given by

$$\frac{1}{2\pi i} \cdot \int_{\partial U} z \cdot R_T(z) \, dz$$

where U is a domain as above containing K.

0.0.6.4.1 *Identify T_K with a densely defined operator on the space $E_K(X)$. Then one has the equality*

$$\sigma(T_K) = K$$

Proof. If λ_0 is outside K we can choose U so that λ_0 is outside $\bar{U}$ and construct the operator

$$S = \frac{1}{2\pi i} \cdot \int_{\partial U} \frac{1}{\lambda_0 - z} \cdot R_T(z) \, dz$$

The operational calculus gives

$$S(\lambda_0 E_K - T) = E_K$$

here E_K is the identity operator on $E_K(X)$ which shows that $\sigma(T_K) \subset K$.

0.0.6.5 Discrete spectra. Consider a spectral set reduced to a singleton set $\{\lambda_0\}$, i.e. λ_0 is an isolated point in $\sigma(T)$. The associated spectral projection is denoted by $E_T(\lambda_0)$ and expressed

$$E_T(\lambda_0) = \frac{1}{2\pi i} \cdot \int_{|\lambda-\lambda_0|=\epsilon} R(\lambda)\, d\lambda$$

for all sufficiently small ϵ. Now $R_T(\lambda)$ is an analytic function defined in some punctured disc $\{0 < \lambda - \lambda_0| < \delta\}$ with a Laurent expansion

$$R_T(\lambda) = \sum_{-\infty}^{\infty} (\lambda - \lambda_0)^k \cdot B_k$$

where $\{B_k\}$ are bounded linear operators obtained by residue formulas:

$$B_k = \frac{1}{2\pi i} \cdot \int_{|\lambda|=\epsilon} \frac{R_T(\lambda)}{(\lambda - \lambda_0)^{k+1}}\, d\lambda \quad : \epsilon < \delta$$

Exercise. Show that $R_T(\lambda)$ is meromorphic, i.e. $B_k = 0$ hold when $k << 0$, if and only if there exists a constant C and some integer $M \geq 0$ such that the operator norms satisfy

$$||R_T(\lambda)|| \leq C \cdot |\lambda - \lambda_0)^{-M}$$

Suppose now that R_T has a pole of some order $M \geq 1$ which gives an expansion

$$R_T(\lambda) = \sum_{1}^{M} \frac{B_{-k}}{(\lambda - \lambda_0)^k} + \sum_{0}^{\infty} (\lambda - \lambda_0)^k \cdot B_k$$

Here $B_{-1} = E_T(\lambda_0)$ and if $M \geq 2$ the negative indexed operators satisfy

$$B_{-k} = B_{-k} E_T(\lambda_0) \quad 2 \leq k \leq M$$

In the case of a simple pole, i.e. when $M = 1$ the operational calculus gives

$$(\lambda_0 E - T) E_T(\lambda_0) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{|\lambda-\lambda_0|=\epsilon} (\lambda_0 - \lambda) R(\lambda)\, d\lambda = 0$$

which implies hat the range of the projection operator $E_T(\lambda_0)$ is equal to the kernel of $\lambda_0 \cdot E - T$.

0.0.6.6 The case $M \geq 2$. Now one has a a non-decreasing family of subspaces

$$N_k(\lambda_0) = \{x \colon (\lambda_0 E - T)^k(x) = 0\} \quad : 1 \leq k \leq M$$

Let us analyzie the special case when the range of $E_T(\lambda_0)$ has finite dimension. Here the operator $T(\lambda_0) = T E_T(\lambda_0)$ acts on this finite dimensional vector space and the B-matrices with negative indices can be expressed as in linear algebra via a Jordan decomposition of $T(\lambda_0)$. More precisely Jordan blocks of size > 1 may occur which occurs of the smallest positive integer m such that

$$(\lambda_0 E - T)^m(x) = \quad : x \in E_T(\lambda_0)(X)$$

is strictly larger than one. Moreover, $E - E_T(\lambda_0)$ is a projection operator and one has a direct sum decompostion

$$X = E_T(\lambda_0)(X) \oplus E - E_T(\lambda_0)$$

Here $V = E - E_T(\lambda_0)$ is a closed subspace of X which is invariant under T and there exists some $c > 0$ such that

$$||\lambda_0 - Tx|| \geq ||x|| \quad x \in V \cap \mathcal{D}(T)$$

Remark. In applications it is often an important issue to decide when $E_T(\lambda_0)$ has a finite dimensional range for an isolated point in $\sigma(T)$. The Kakutani-Yosida theorem in § 11.9 is an example where this finite dimensionality will be established for certain operators T.