A uniqueness theorem for PDE-equations.

Introduction. The development analytic function theory has to a large extent been inspired by boundary value problems for PDE:s. Working in $\mathbf{R}^2$ with the coordinates (x,y) a typical Cauchy problem is to determine a function $u(x,y)$ which is harmonic in some open half-disc

$$D_+(r) = \{x^2 + y^2 < r^2\} \cap \{x > 0\}$$

and satisfies the boundary conditions:

$$(*) \qquad u(0,y) = \psi(y) \quad \text{and} \quad u_x(0,y) = \phi(y)$$

where ϕ and ψ are given in advance. When ϕ and ψ are real-analytic one proves rather easily both existence and uniqueness, i.e. (*) has a unique solution. With less regularity Hadamard gave examples where this fails to hold. In fact, Hadamard proved that a necessary and sufficient condition for the Cauchy problem to be well posed is that the function

$$y \mapsto \phi(y) + \frac{1}{\pi} \int_a^b \operatorname{Log}\Big[\frac{1}{|s-y|}\Big] \cdot \psi(s) \cdot ds$$

is real analytic. More generally we can consider elliptic boundary value problems expressed by a system of first order partial differential equations in the two real variables (x,y). Let $n \geq 2$ and consider two $n \times n$-matrices $\mathcal{A} = \{A_{pq}\}$ and $\mathcal{B} = \{B_{pq}\}$ whose elements are real-valued functions of x and y where the B-functions are continuous and the A-functions of class C^2. The two matrices give a system of first order PDE-equations whose solutions are vector valued functions $(f_1, \ldots, f_n)$ defined in a half-disc

$$D_+(\rho) = \{x^2 + y^2 < \rho^2 \quad : \; x > 0\}$$

where these f-functions satisfy the system:

$$(**) \qquad \frac{\partial f_p}{\partial x} + \sum_{q=1}^{q=n} A_{pq}(x,y) \cdot \frac{\partial f_p}{\partial x} + \sum_{q=1}^{q=n} B_{pq}(x,y) \cdot f_q(x,y) = 0 \quad : \quad 1 \leq p \leq n$$

together with the boundary conditions:

$$f_p(0,y) = 0 \quad \text{for all} \quad 1 \leq p \leq n$$

Above we get eigenvalues of the $\mathcal{A}$-matrix when (x,y)-varies, i.e. the n-tuple of roots $\lambda_1(x,y), \ldots, \lambda_n(x,y)$ which solve

$$(1) \qquad \det\big(\lambda \cdot E_n - \mathcal{A}(x,y)\big) = 0$$

If the λ-roots are non-real we say that (*) above is an elliptic system. Assuming vanishing Cauchy data one expects that a solution f is identically zero. That such a uniqueness holds was proved by Erik Holmgren in an article from 1901 under the assumption that the A-functions and the B-functions are real analytic. The question remained if the uniqueness still holds under less regularity on the coefficient functions. This was settled 30 years later by Holmgren's former Ph.D-student Carleman. In [Carleman] the following is proved:

1. Theorem. *Assume that the λ-roots are all simple and non-real as (x,y) varies in the open half-disc. Then every solution f to (**) with vanishing Cauchy-data is identically zero.*

The proof requires several steps. The methods which occur below are perhaps of greater value than the actual result. In fact, they have inspired more recent work where various Carleman estimates are used to handle boundary value problems in PDE-theory.

0.1 Preliminary constructions.

The system of $2m$-many real equations in (*) is equivalent to a system of m-many equations where one seeks complex-valued functions $g_1,\dots,g_m$ satisfying a system:

$$\frac{\partial g_p}{\partial x}+\sum_{q=1}^{q=m}\lambda_p(x,y)\cdot\frac{\partial g_p}{\partial y}=$$

$$(*)\qquad \sum_{q=1}^{q=m} a_{pq}(x,y)\cdot g_q(x,y)+b_{pq}(x,y)\cdot\bar{g}_q(x,y)=0\ :\ 1\leq p\leq m$$

This easy reduction left to the reader. From now on we study the system (*). So Theorem 1 amounts to prove that if the g-functions satisfy (*) above in a half-disc $D_+(\rho)$ and

$$g_p(0,y)=0\quad:\quad 1\leq p\leq m$$

then there exists some $0<\rho_*\leq\rho$ such that the g-functions are identically zero in $D_+(\rho_*)$.

0.2 A useful coordinate transform. The system (*) suggests that we consider first order differential operators of the form

$$(*)\qquad Q=\partial_x+\lambda(x,y)\cdot\partial_y$$

where $\lambda(x,y)$ is a complex valued C^2-function whose imaginary part is >0. The case when the imaginary part instead is everywhere <0 can be handled in a similar way, where complex conjugation means that we arrive to anti-holomorphic functions below. Set

$$\lambda(x,y)=\mu(x,y)+i\cdot\tau(x,y)\quad:\ \tau(x,y)>0$$

Now we look for solutions $h(x,y)$ to the equation $Q(h)=0$. To find this family we first determine a special solution $\xi(x,y)+i\cdot\eta(x,y)$ where ξ and η are real-valued C^2-functions. This means that they satisfy the differential system:

$$\frac{\partial\xi}{\partial x}+\mu_p\cdot\frac{\partial\xi}{\partial y}-\tau_p\cdot\frac{\partial\eta}{\partial y}=0$$
$$\frac{\partial\eta}{\partial x}+\mu_p\cdot\frac{\partial\eta}{\partial y}+\tau_p\cdot\frac{\partial\xi}{\partial y}=0$$

Suppose we have found one solution $\xi + i \cdot \eta$ where the Jacobian defined by

$$\xi_x \eta_y - \xi_y \eta_x$$

is $\neq 0$ at the origin. Then $(x, y) \mapsto (\xi, \eta)$ is a local C^2-diffeomorphism. In the (ξ, η) coordinates we have the usual Cauchy Riemann operator. If $g(\xi + i\eta)$ is a holomorphic function in the complex ζ-space with $\zeta = \xi + i\eta$ it is easily seen the function $g_*(x, y$ defined by

$$g_*(x, y) = g(\xi(x, y) + i\eta(x, y))$$

satisfies $Q(g_*)$.

0.2 Conclusion. *If a non-degenerate solution $\xi + i\eta$ has been found then the homogenous solutions to Q is in a 1-1 correspondence to analytic functions in the ζ-variable.*

Remark. Of course, the effect of the coordinate transformation is that the Q-operator is transported to the Cacuhy-Riemann operator in the complex ζ-space where $\zeta = \xi + i\eta$. Later we employ the (ξ, η)-transformation to construct solutions to an inhomogeneous equation of the form

$$Q(\psi) = (t - \alpha x + 2y\lambda(x, y)) \cdot \psi(t, x, y)$$

where t is a positive parameter and the ψ-functions will have certain specified properties. See (B) below.

A. Proof of Theorem 1: First part

For a pair $\alpha > 0$ and $\ell > 0$ we get the domain:

$$D_\ell(\alpha) = \{x + y^2 - \alpha x^2 < \ell^2\} \cap \{x > 0\} \tag{1}$$

Notice that the boundary

$$\partial D_\ell(\alpha) = \{0\} \times [-\ell, \ell] \cup T_\ell \quad \text{where} \quad x + y^2 - \alpha x^2 = \ell^2 \text{ holds on } T_\ell$$

Above α and ℓ are small so the the g-functions satisfy (*) from (0.1). Next, to each $t > 0$ we define the m-tuple of functions by

$$\phi_p(x, y) = g_p(x, y) \cdot e^{-t(x+y^2-\alpha x^2)} \tag{2}$$

Since the g-functions satisfy the system in (0.1) one verifies easily that the ϕ-functions satisfy the system

$$\frac{\partial \phi_p}{\partial x} + \frac{\partial}{\partial y}(\lambda_p \cdot \phi_p) + t(1 - 2\alpha x + 2y\lambda_p) \cdot \phi_p = H_p(\phi) \tag{3}$$

where

$$H_p(\phi) = \sum_{q=1}^{q=n} a_{pq}(x, y) \cdot \phi_q(x, y) + b_{pq}(x, y) \cdot \bar{\phi}_q(x, y) = 0 \,:\, 1 \leq p \leq m$$

Next, we set

$$\Phi(x,y) = \sum_{p=1}^{p=m} |\phi_p(x,y)| \tag{4}$$

With these notations the crucial step for the proof of Theorem 1 is to establish the following inequality.

A.1 Proposition. *Provided that α from the start is sufficiently large there exists some $0 < \ell_* \leq \ell$ and a constant C which is independent of t such that*

$$\iint_{D_{\ell_*}} \Phi(x,y) \cdot dxdy \leq C \cdot \int_{T_{\ell_*}} \sum_{p=1}^{p=n} |\phi_p| \cdot |dy - \lambda_p \cdot dx|$$

How to deduce Theorem 1. Let us show why Proposition A.1 gives Teorem 1. In addition to ℓ_* we fix some $0 < \ell_{**} < \ell_*$. In (2) above we have used the function

$$w(x,y) = e^{-t(x+y^2-\alpha x^2)}$$

It gives

$$\text{(i)} \quad w(x,y) = e^{-t\ell_*^2} \quad : \quad (x,y) \in T_{\ell_*} \quad : \quad w(x,y) \geq e^{-t\ell_{**}^2} \quad : \quad (x,y) \in D_{\ell_{**}}$$

Next, we have $|\phi_p| = |g_p| \cdot w$ for each p. Replacing the left hand side in Proposition A.1 by the area integral over the smaller domain $D_{\ell_{**}}$ we obtain the inequality;

$$\text{(ii)} \quad \iint_{D_{\ell_{**}}} \sum_{p=1}^{p=m} |g_p(x,y)| \cdot dxdy \leq C \cdot e^{t(\ell_{**}^2 - \ell_*^2)} \cdot \int_{T_{\ell_*}} \sum_{p=1}^{p=n} |g_p| \cdot |dy - \lambda_p \cdot dx|$$

Here (ii) holds for every $t > 0$. Passing to the limit as $t \to +\infty$ we have $e^{t(\ell_{**}^2 - \ell_*^2)} \to 0$ and can therefore conclude that

$$\iint_{D_{\ell_{**}}} \sum_{p=1}^{p=m} |g_p(x,y)| \cdot dxdy = 0$$

But this means of course that the g-functions are all zero in $D_{\ell_{**}}$ and Theorem 1 follows.

B. Proof of Proposition A.1

The proof relies upon the construction of certain ψ-functions. More precisely, when $t > 0$ and a point $(x_*, y_*) \in D_\ell$ are given we shall construct an m-tuple of ψ-functions satisfying the following three conditions:

Condition 1. Each ψ_p is defined in the punctured domain $D_\ell \setminus \{(x_*, y_*)\}$ where ψ_p for a given $1 \leq p \leq m$ satisfies the equation

$$\text{(i)} \quad \frac{\partial \psi}{\partial x} + \lambda_p \cdot \frac{\partial \psi}{\partial y} - t(1 - 2\alpha x + 2y\lambda_p)\psi_p = 0$$

Condition 2. For each p the singularity of ψ_p at (x_*, y_*) is such that the line integrals below have a limit:

$$\text{(ii)} \qquad \lim_{\epsilon \to 0} \int_{[z-z_*|=\epsilon} \psi_p \cdot (dx - \lambda_p \cdot dy) = 2\pi$$

Condition 3. There exists a constant K which is independent both of (x_*, y_*) and of t such that

$$\text{(iii)} \qquad |\psi_p(z)| \leq \frac{K}{|z - z_*|}$$

Notice that the ψ-functions depend on the parameter t, i.e. they are found for each t but the constant K in (3) is independent of t.

The deduction of Proposition A.1

Before the ψ-functions are constructed in Section C below we show how they give Proposition A.1. Consider a point $z_* \in D_+(\ell)$. We get the associated ψ-functions from (B.0) at this particular point. Remove a small disc γ_ϵ centered at z_* and consider some fixed $1 \leq p \leq m$. Now ϕ_p satisfies the differential equation (3) from section A and ψ_p satisfies (i) in Condition 1 above. Then Stokes theorem gives:

$$\int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) = \iint_{D_\ell \setminus \gamma_\epsilon} H_p(\phi) \cdot \psi_p \cdot dxdy + \int_{|z-z_*|=\epsilon} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big)$$

Passing to the limit s $\epsilon \to 0$, Condition 2 for ψ_p gives

$$\text{(1)} \qquad \phi_p(x_*, y_*) = \frac{1}{2\pi} \int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) - \frac{1}{2\pi} \cdot \iint_{D_\ell} H_p(\phi) \cdot \psi_p \cdot dxdy$$

Let L be the maximum over D_ℓ of the coefficient functions of ϕ and $\bar{\phi}$ which appear in $H_p(\phi)$ from (3) i A.0. above. We have also the constant K from Condition 3 for ψ_p. With these constants the triangle inequality gives:

$$\text{(*)} \qquad \big|\phi_p(x_*, y_*)\big| \leq \frac{K}{2\pi} \int_{T_\ell} \frac{|\phi_p| \cdot |dy - \lambda_p \cdot dx|}{z - z_*|} + \frac{LK}{\pi} \cdot \sum_{q=1}^{q=m} \iint_{D_\ell} \frac{|\phi_q|}{|z - z_*|} \cdot dxdy$$

Next, we use the elementary inequality

$$\text{(**)} \qquad \iint_\Omega \frac{dxdy}{\sqrt{x-a)^2 + (y-b)^2}} \leq 2 \cdot \sqrt{\pi} \cdot \sqrt{\text{Area}(\Omega)}$$

where Ω is an arbitrary bounded domain and $(a, b) \in \Omega$ Now we apply (**) with $\Omega = D_\ell$ and set $S = \text{area}(D_\ell)$. Integrating both sides in (*) over D_ℓ for every p and taking the sum we get

$$\iiint_{D_\ell} \Phi \cdot dxdy \leq$$

$$K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |\phi_p| \cdot |dy - \lambda_p \cdot dx| + 2\pi mLK \cdot \sqrt{\frac{S}{\pi}} \iint_{D_\ell} \Phi \cdot dxdy$$

This inequality hold for all small ℓ. Choose ℓ so small that

$$2\pi mLK \cdot \sqrt{\frac{S}{\pi}} \leq \frac{1}{2}$$

Then the inequality above gives

$$(***) \qquad \iiint_{D_\ell} \Phi \cdot dxdy \leq 2 \cdot K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |\phi_p| \cdot |dy - \lambda_p \cdot dx|$$

Finally, consider some relatively compact domain Δ in D_ℓ. Then there exists $0 < \ell_* < \ell$ such that

$$\Delta \subset D_{\ell_*}$$

Now we notice that

$$|\phi_p(z)| \geq e^{-t\ell_*^2} \cdot |u_p(z)| \quad : \quad z \in \Delta \quad : \quad |\phi_p(z)| \geq e^{-t\ell^2} \cdot |u_p(z)| \quad : \quad z \in T_\ell$$

We conclude that

$$(****) \quad e^{-t\ell_*^2} \iiint_\Delta \sum_{p=1}^{p=m} |u_p(z)| \cdot dxdy \leq e^{-t\ell^2} \cdot 2 \cdot K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |u_p| \cdot |dy - \lambda_p \cdot dx|$$

Here (****) hold for every $t > 0$. Passing to the limit as $t \to +\infty$ it follows that

$$\cdot \iiint_\Delta \sum_{p=1}^{p=m} |u_p(z)| \cdot dxdy \leq$$

Since Δ was any relatively compact subset of D_ℓ, we conclude that the u-functions are zero in D_ℓ and Theorem 1 follows.

C. Construction of the ψ-functions.

Notice that t suffices to construct the ψ-functions separately, i.e. we no longer have to bother about a differential system. With a fixed p fixed we have $\lambda_p(x,y) = \mu_p + \tau_p$ and from now on we may drop the index p and explain how to obain ψ-functions satisfying the three conditions from (B) above. We have the first order differential operator

$$(1) \qquad Q = \frac{\partial}{\partial x} + \lambda(x,y) \cdot \frac{\partial}{\partial y}$$

where the assumption is that $\mathfrak{Im}(\lambda) > 0$. This enable us to make a change of variables so that Q is expressed in new real coordinates (ξ, η) by the Cauchy-Riemann operator

$$\frac{\partial}{\partial \xi} + i \cdot \frac{\partial}{\partial \eta} \tag{2}$$

There exist many coordinate transforms $(x, y) \to (\xi, \eta)$ which change Q into (2). This *flexible choice* of coordinate transforms is used to construct the required ψ-functions. Notice also that Condition (2) is of a pointwise character, i.e. it suffices to find a ψ-function for a given point $z_* = x_* + iy_*$. With this in mind the required construction boils down to perform a suitable coordinate transformation adapated to z_*, and after use the existence of a ψ-function which to begin with is expressed in the (ξ, η)-variables where the Q-operator is replaced by the Cauchy-Riemann operator. In this special case the required ψ-function is easy to find, i.e. see the remark in B.0.

So all that remains is to exhibit suitable coordinate transformations which send Q to the $\bar{\partial}$-operator.

C.1 A class of (ξ, η)-functions. Let $V(x, y)$ and $W(x, y)$ be two quadratic forms, i.e. both are homogeneous polynomials of degree two. Given a point (x_*, y_*) and with $z = x + iy$ we try to use a coordinate transformation of the form:

$$\xi(z) = \tau_p(z_*) \cdot (x - x_*) + V(x - x_*, y - y_*) + \gamma_1(z) \cdot |z - z_*|^2$$

$$\eta(z) = (y - y_*) - \mu_p(z_*) \cdot (x - x_*) + W(x - x_*, y - y_*) + \gamma_2(z) \cdot |z - z_*|^2$$

Lemma. *There exists a pair of quadratic forms V and W whose coefficients depend on (x_*, y_*) and a pair of γ-functions which both vanish at (x_*, y_*) up to order one such that ξ and η satisfy the equations above.*

The Schrödinger equation.

We work in $\mathbf{R}^3$ with the coordinates (x,y,z). Let $c(x,y,z)$ be a real-valued function in $L^2_{\text{loc}}(\mathbf{R}^3$. In order that the subsequent formulas can be stated in a precise manner we also assume that $c$is almost everywhere continuous which of course is a rather weak condition and in any case satisfied in applications. Next, let Δ be the Laplace operator and define the operator L by

$$(*) \qquad L(u) = \Delta(u) + c \cdot u$$

Denote by $E_L(\mathbf{R}^3)$ the set of functions u such that both u and $L(u)$ belong to $L^2(\mathbf{R}^3)$. Given a pair (f,λ) where $f \in L^2(\mathbf{R}^3$ and λ is a complex number we seek solutions $u \in E_L(\mathbf{R}^3)$ such that

$$(**) \qquad L(u) + \lambda \cdot u = f$$

The case $\mathfrak{Im}(\lambda \neq 0$. By a classic result about solutions to the Neumann boundary value problem in open balls in $\mathbf{R}^3$ one proves that (1) has at least one solution u whenever λ is not real. The remains to investigate the uniqueness, i.e, when one has the implication

$$(***) \qquad \mathfrak{Im}(\lambda \neq 0 \quad \text{and} \quad L(u) + \lambda \cdot u = 0 \implies u = 0$$

This uniqueness property depends on the c-function. A sufficient condition is the following:

Theorem. *Assume that there exists a constant M and some $r_* > 0$ such that*

$$c(x,y,z) \leq M \quad \text{when} \quad x^2 + y^2 + z^2 \geq r_*^2$$

Then (***) above holds.

The spectral θ-function. When (***) holds it was proved in [Carleman] that classical solutions to the Neumanns boundary value problem in open balls yield a θ-function which enable us to describe solutions to (*) for real λ-values. More precisely, there exists two increasing sequence of positive real numbers $\{\lambda^*(\nu)\}$ and $\lambda_*(\nu)\}$ and two sequence of pairwise orthogonal functions $\{\phi_\nu(p))\}$ and $\{\psi_\nu(p)\}$ in $L^(\mathbf{R})^3$ where all these functions have L^2-norm equal to one such that the following hold. First, set

$$\theta(p.q,\lambda) = \sum_{0<\lambda^*(\nu)\leq\lambda} \phi_\nu(p) \cdot \phi_\nu(q) \quad : \quad \lambda > 0$$

$$\theta(p.q,\lambda) = -\sum_{\lambda\leq\lambda_*(\nu)\leq<0} \psi_\nu(p) \cdot \psi_\nu(q) \quad : \quad \lambda < 0$$

such that the following hold:

$$(1) \qquad v(p) = \lim_{R \to \infty} \sum_{[\lambda_\nu < R} \theta(p,q,\lambda) \cdot v(q) \cdot dq \quad \text{for all} \quad v \in L^2(\mathbf{R}^3)$$

$$(2) \qquad v \in E_L(\mathbf{R})^{\mathbf{3}} \quad \text{if and only if} \quad xxxx$$

$$(3) \quad L(v)(p) = \lim_{R \to \infty} \sum_{[\lambda_\nu < R} \lambda \cdot [\int_{\mathbf{R}^3} \theta(p,q,\lambda) \cdot v(q) \cdot dq] \cdot d\lambda \quad \text{for all} \quad v \in E_L(\mathbf{R}^3)$$

Here the equaliy holds in L^2, i.e, in the sense of a Plancherel's limit.

Remark. Here the equaliy holds in L^2, i.e, in the sense of a Plancherel's limit.

Construction of the ϕ-functions. For each finite r we have the ball B_r and consider the space $E_L(B_r)$ of functions u in B_r such that both u and $L(u)$ also belong to $L^2(B_r)$. By a classical result in the Fredholm theory that exist discrete sequences of real numbers $\{\lambda * (\nu)$ and $\lambda_*(\nu)$ as above and two families of orthonormal functions $\{\phi_\nu^{(r)}\}$ and $\{\psi_\nu^{(r)}\}$ satisfying (xx)

and here a classical result shows that the real eigenvalues to the equation $L(u) + \lambda \cdot u = 0$ xxx

xxx

The proofs of the assertions above rely on a systematic use of Green's formula. To begin with we recall how to express solutions to an inhomogeneous the Laplace equation by an integral formula.

A. The equation $\Delta(u) = \phi$. let D be a domain in $\mathbf{R}^3$ and ϕ a function in $L^2(D)$. Then a function u for which both u and $\Delta(u)$ belong to $L^2(D)$ gives $\Delta(u) = \phi$ if and only if the following hold for every $p \in D$ and every $\rho < \text{dist}(p, \partial D)$:

$$\text{(i)} \quad u(p) = \frac{1}{2\pi\rho^2} \cdot \int_{B_p(\rho)} \frac{1}{|p-q|} \cdot u(q) \cdot dq + \frac{1}{4\pi\rho^2} \cdot \int_{B_p(\rho)} A(p,q) \cdot \phi(q) \cdot dq$$

where we have put

$$\text{(ii)} \qquad A(p,q) = \frac{2}{\rho} - \frac{1}{|p-q|} - \frac{|p-q|}{\rho^2}$$

Exercise. Prove this result.The hint is to apply Green's formula while ϕ is replaced by $\Delta(u)$ in the last integral.

Remark. Let us also recall also that when $\Delta(u)$ is in L^2, then u is automatically a continuous function in D.

The class $\mathfrak{Neu}(B_r)$. Let B_r be the open ball of radius r centered at the origin. The class of functions u which are continuous on the closed ball and whose interior normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ is continuous on the boundary $S^2[r]$ is denoted by $\mathfrak{Neu}(B_r)$.

The Neumann equation. Let $c(x, y, z)$ be a function in $L^2(\mathbf{R}$ and consider also a pair a, H where a be a continuous function on $S^2[r]$ and $H(p, q)$ a continuous hermitian function on $S^2[r] \times S^2[r]$, i.e. $H(q,p) = \bar{H}(p,q)$ hold for all pairs of point p,q on the sphere $S^2[r]$. With these notations the following hold:

Theorem For each $f \in L^2(B_r)$ and every non-real complex number λ there exists a unique $u \in \mathfrak{Neu}(B_r)$ such that u satisfies the two equations:

$$L(u + \lambda \cdot u = f \quad \text{holds in} \quad B_r$$

$$\partial u / \partial \mathbf{n}(p) = xx$$

Moreover, one has the L^2-estimate

$$\int_{B_r} |u|^2 \cdot dxdydz \leq \Big| \frac{1}{\mathfrak{Im}(\lambda)} \Big| \leq \int_{B_r} |f|^2 \cdot dxdydz$$

Classic result: $R > 0$ we have unit ball B_R and unit sphere S_R. Let $\mathfrak{Neu}(R)$ set uf u -functions where $\Delta(u)$ in L^2, continuous on closed ball and limit of interior derivative as a continuous funtion. Given $c \in L^2(B_R)$ define

$$L(u) = \delta(u) + c \cdot u$$

Theorem For each pair (a, H) in (*) there exists a unique $u \in \mathfrak{Neu}(R)$ such that

$$L(u + \lambda \cdot u = f \quad \text{holds in} \quad B_R$$

and u saisfies th boundsry condition

$$\partial u / \partial \mathbf{n}(p) = xx$$

from that L^2-esteimate as well.

Do if for $R = m$ running over positive integers. Catch up sequence with L^2-convergence bounded uniformly. $u_m \to u_*$ weak sense and see tha u_* is a solution to (*) on al over space.

Second point about eventual uniquenss. Class I type. Euquialent condition-