§ XX. Carleman's work in PDE-theory.

Introduction. Early contributions by Weyl and Carleman laid foundations for much of later work on partial diffferential equations. The sections in this chapter expose material from articles by Carleman during the period 1917 until 1938. The most remarkable result is the uniqueness theorem in § 6 where the methods of the proof has inspired later work. See for example [Hörmander ??]. The boundary value problem for the wave equation in § 7 is nowadays standard. But the strategy to construct a spectral function was revolutionary in 1922 and has after become a basic tool. A merit is that this makes it possbile to achieve various limit properties for the actual solutions by deriving properties of the spectral function. A typical example is Theorem 0.1 in § 7. Another remarkable result appears in § 3 which not only gives asymptotic formulas for eigenvalues but also for values taken by the eigenfunctions. The method to treat a non-linear boundary value problem in § xx also merits attention. It goes without saying that many of the results which are established in dimension 2 or 3 can be extended to higher dimensions. But to avoid technicalities we are content to deal with "classical situations". Let us remark that we have not discussed Carleman's work in kinetic gaz theory which contains far-reaching studies of non-linear equations of the Bolzmann type. For this the reader may consult the monograph [Carleman].

Let us remark that the use ofdistributions is not needed in the subsequent sections since fundamental solutions and other weak type equations are expressed in a direct way via integrals. in § 1 we expose Carelan's construction of fundamental solutions to an elliptic second order differential operor in $\mathbf{R}^3$. I think this gives a very good lesson for beginners in the subject since the result is that one gets a fundamental solution in a unique way via rather straightforward calcuations whose sole analytic tool is to solve integral equations via Neumann series for resolvents. A reward is that one gets a fundamental solution with good regularity properties and the reader may notice that we do not assume that the differential operators is self-adjoint. Of course, many important topics are not overed by the subsequent sections, and we remakr that fgor cetain problems it is better to adopt the methods by Weyl via variational methods and a more extensive use of Hilbert space methods where higher order derivatives occur and various Sobolev inequalities are used. Examples where this becomes crucial occurs for symmetric hyperbolic systems where a quite conclusive theorem was established by Friedrichs in 1946. In § xx we give an account about this which can be read independently of the sections devoted to Carleman's work.

§ 0. Fundamental solutions to second order Elliptic operators.

Introduction. To avoid cumbersome technical points we shall restrict the attention to second order elliptic operators in dimension 2 or 3. But it goes without saying that the subsequent results can be extended after suitable changes of constants which appear in the theorems. Elliptic partial differential operators appear frequently and in order to solve inhomogeneous equations one often employs a fundamental solution. To attain this one seeks fundamental solutions with best possible regularity conditions. For PDE-operators with constant coefficients one can employ Fouriers' inversion fourmula and we refer to Chapter X in vol.2 in Hörmander's text-book series on linear partial differential operators for a detailed account about constructions of fundamental solutions with optimal regularity in the case of constant coefficients. Passing to the case of variable coefficients we expose a construction by Carleman in § 2 which give fundamental solutions to second order elliptic operators in a canonical fashion. They are found by solutions to integral equations. Let us remark that there is no need for distribution theory since we construct fundamental solutions which are locally integrable and in such situations the notion of fundamental solutions were well understood at an early stage after pioneering work by Weyl and Zeilon prior to 1925. In his article Carleman refers to *Grundlösungen*.

Remark. We have restricted the study to $\mathbf{R}^3$ and remark only that similar constructions can be performed when $n \geq 4$ starting from Newton's potential $|x-\xi|^{-n+2}$. Here it would be interesting to clarify the precise estimates when $n \geq 4$ and establish similar inequalities as in the Main Theorem One can also try to extend the constructions in § 1 to elliptic operators of order ≥ 3. In the case of even order $2m$ one can employ the canonical fundamental solutions for elliptic operators of even order with constant coefficents given by F. John. So here replaces Newton's fundamental solutions which appear in § 1 below by those of John and after it is tempting to perform similar constructions as those by Carleman. This appears to be a "profitable research problem" for ph.d-students. Let us also remark that one does not assume that the elliptic operators are symmetric, i.e. both the constructions as well as estimates for the fundamental solutions do not rely upon symmetry conditions.

An asymptotic formula for the spectrum. Let $n=3$ and consider a second order PDE-operator

$$L=\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x)\cdot\frac{\partial^2}{\partial x_p\partial x_q}+\sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p}+a_0(x)$$

The a-functions are real-valued and defined in a neighborhood of the closure of a bounded domain Ω in $\mathbf{R}^3$ with a C^1-boundary. Here one has the symmetry $a_{pq}=a_{qp}$, and $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x\in\Omega$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under these conditions, a result which goes back to work by Neumann and Poincaré, gives a positive constant κ_0 such that if $\kappa\geq\kappa_0$ then the inhomogeneous equation

$$L(u)-\kappa^2\cdot u=f \quad : f\in L^2(\Omega)$$

has a unique solution u which is a C^2-function in Ω and extends to the closure where it is zero on $\partial\Omega$. Moreover, there exists some κ_0 and for each $\kappa\geq\kappa_0$ a Green's function $G(x,y;\kappa)$ such that

$$\text{(i)}\qquad (L-\kappa^2)(\frac{1}{4\pi}\cdot\int_\Omega G(x,y;\kappa)\,f(y)\,dy)=-f(x)\quad : f\in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

$$\text{(ii)}\qquad f\mapsto -\frac{1}{4\pi}\cdot\int_\Omega G(x,y;\kappa)\,f(y)\,dy$$

is Neumann's resolvent to the densely defined operator $L-\kappa^2$ on the Hilbert space $L^2(\Omega)$. Next, one seeks pairs (u_n,λ_n) where u_n are $L2$-functions in Ω which extend to be zero on $\partial\Omega$ and satisfy

$$L(u_n)+\lambda_n\cdot u_n=0$$

It turns out that the set of eigenvalues is discrete and moreover their real parts tend to $+\infty$. They are arranged with non-decreasing absolute values and in § xx we prove that there exist positive constants C and c such that

$$|\mathfrak{Im}(\lambda_n)| \leq C \cdot (\mathfrak{Re}(\lambda_n) + c)$$

hold for every n. Next, the elliptic hypothesis means that the determinant function

$$D(x) = \det(a_{p,q}(x))$$

is positive in Ω. With these notations one has

Theorem. *The following limit formula holds:*

$$(*) \qquad \lim_{n\to\infty} \frac{\mathfrak{Re}(\lambda_n)}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_\Omega \frac{1}{\sqrt{D(x)}}\, dx$$

Remark. The formula above is due to Courant and Weyl when P is symmetric and extended to non-symmetric operators during Carleman 's lectures at Institute Mittag-Leffler in 1935. Weyl and Courant used calculus of variation in the symmetric case while Carleman employed different methods which have the merit that the passage to the non-symmetric case does not cause any trouble. A crucial step during the proof of the theorem above is to construct a fundamental solution $\Phi(x, \xi; \kappa)$ to the PDE-operators $L - \kappa^2$ which done in § 1 while § 2 treats the asymptotic formula above. As pointed out by Carleman the methods in the proof give similar asymptotic formulas in other boundary value problems such as those considered by Neumann where one imposes boundary value conditions on outer normals. As an example we consider an elliptic operator of the form

$$L = \Delta + \sum_{p=1}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + a_0(x)$$

where Δ is the Laplace operator. Given a positive real-valued continuous function $\rho(x)$ on $\partial\Omega$ we obtain the Neumann-Poincaré operator $\mathcal{NP}$ which sends each $u \in C^0(\partial\Omega)$ to

$$\mathcal{NP}(u) = \frac{\partial u^*}{\partial \mathbf{n}_i} - \rho \cdot u$$

Here u^* is the Dirichlet extension of u to Ω which is equal to u on $\partial\Omega$ and satisfies $L(u) = 0$ in Ω, while $\frac{\partial u^*}{\partial \mathbf{n}_i}$ is the inner normal along the boundary. In the special case when $L = \Delta$ this boundary value problem has unique solutions, i.e. for every $f \in C^0(\partial\Omega)$ there exists a unique u such that

$$\mathcal{NP}(u) = f$$

This was proved by Poincaré in 1897 for domains in $\mathbf{R}^3$ whose boundaries are of class C^2 and the extension to domains with a C^1-boundary is also classic. Passing to general operators L as above which are not neccesarily symmetric one encounters spectral problems, i.e. above $\mathcal{NP}$ regared as a linear operator on the Banach space $C^0(\partial\Omega)$ is densely defined and one seeks its spectrum, i.e, compex numbers λ for which there exists a non-zero u such that

$$\mathcal{NP}(u) + \lambda \cdot u = 0$$

I do not know if there exists an analytic formula for these eigenvalues. Notice that a new feauture is that the ρ-function affects the spectrum.

§ 1 The construction of fundamental solutions.

In $\mathbf{R}^3$ with coordinates $x=(x_1,x_2,x_3)$ we consider a second order PDE-operator

$$L=\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x)\cdot\frac{\partial^2}{\partial x_p\partial x_q}+\sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p}+a_0(x)$$

where a-functions are real-valued and one has the symmetry $a_{pq}=a_{qp}$. To ensure existence of a globally defined fundamental solutions we suppose the the following limit formulas hold as $(x,y,z)\to\infty$:

$$\lim a_\nu(x,y,z)=0\colon 0\leq p\leq 3\quad:\ \lim a_{pq}(x,y,z)=\text{Kronecker's delta function}$$

Thus, L approaches the Laplace operator as (x,y,z) tends to infinity. Moreover L is elliptic which means that the eigenvalues of the symmetric matrix with elements $\{a_{pq}(x)\}$ are positive for every x. Recall the notion of fundamental solutions. Consider the adjoint operator:

$$L^*(x,\partial_x)=P-2\cdot(\sum_{p=1}^{p=3}(\sum_{q=1}^{q=3}\frac{\partial a_{pq}}{\partial x_q})\cdot\frac{\partial}{\partial x_p}-\sum_{p=1}^{p=3}\frac{\partial a_p}{\partial x_p}+2\cdot\sum\sum\frac{\partial^2 a_{pq}}{\partial x_p\partial x_q}\tag{0.1}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ,ψ in $\mathbf{R}^3$ with compact support:

$$\int L(\phi)\cdot\psi\,dx=\int\phi\cdot L^*(\psi)\,dx\tag{0.2}$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $L(x,\partial_x)$ if

$$\psi(0)=\int\Phi\cdot L^*(\psi)\,dx\tag{0.3}$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $L-\kappa^2$ and a function $x\mapsto\Phi(x;\kappa)$ is a fundamental solution to $L-\kappa^2$ if

$$\psi(0)=\int\Phi(x:\kappa)\cdot(L^*-\kappa^2)(\psi(x))\,dx\tag{0.4}$$

hold for compactly supported C^2-functions ψ. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x,\xi;\kappa)$ with the property that

$$\psi(\xi)=\int\Phi(x,\xi;\kappa)\cdot(L^*(x,\partial_x)-\kappa^2)(\psi(x))\,dx\tag{*}$$

hold for all $\xi\in\mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x,\xi;\kappa)$ is a function of six variables defined in $\mathbf{R}^3\times\mathbf{R}^3$. With these notations we announce the main result:

Main Theorem. *There exists a constant κ_* such that for every $\kappa\geq\kappa_*$ one can find a fundaemtal solution $\Phi(x,\xi;\kappa)$ which is locally integrable in the 6-dimensional (x,ξ)-space. Moreover, there exist positive constants C and k and for each $0<\gamma\leq 2$ a constant C_γ such that*

$$|\Phi(x,\xi;\kappa)|\leq C\cdot\frac{e^{-k\kappa|x-\xi|}}{|x-\xi|}+\frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for all pairs (x,ξ) in $\mathbf{R}^3$ and every where the constants k and C do not depend upon κ.

1. The construction of $\Phi(x,\xi;\kappa)$.

When L has constant coefficients the construction of fundamental solutions was (at least essentially) given by Newton in his famous text-books from 1666 and goes as follows: Consider a positive and symmetric 3×3-matrix $A=\{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix which gives the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p x_q$$

Put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq} a_p a_q - a_0}$$

where κ is chosen so large that the term under the square-root is > 0. Finally, put

$$\Delta = \det(A)$$

With these notations we get a function:

$$H(x;\kappa) = \frac{1}{4\pi \cdot \sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq} a_p \cdot x_q} \tag{1.1}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ yields a fundamental solution to the PDE-operator $L(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients. Now L has variable coefficients. For each $\xi \in \mathbf{R}^3$ the elements of the inverse matrix to $\{a_{pq}(\xi)$ are denoted by $\{b_{pq}(\xi)\}$. Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi)} \tag{i}$$

Following Newton's construction in (1.1) we put:

$$H(x,\xi;\kappa) = \frac{1}{4\pi} \cdot \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) \cdot x_q} \tag{ii}$$

When ξ is kept fixed this function of x is real analytic outside the origin and $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. We are going to construct a fundamental solution which takes the form

$$\Phi(x,\xi;\kappa) = H(x-\xi,\xi;\kappa) + \int_{\mathbf{R}^3} H(x-y,\xi;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy \tag{iii}$$

where the Ψ-function is the solution to an integral equation which we construct in (1.5). But first we need another construction.

1.3 The function $F(x,\xi;\kappa)$. For every fixed ξ we get the differential operator in the x-space:

$$L_*(x,\partial_x,\xi;\kappa) =$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi)) \frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

Apply L_* to the function $x \mapsto H(x-\xi,\xi;\kappa)$ and put

$$F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)) \tag{1.3.1}$$

1.4 Two estimates. The limit conditions in (0.0) give positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \quad : \; [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|^2} \tag{1.4.1}$$

The verification of (1.4.1) is left as an exercise.

1.5 An integral equation. We seek $\Psi(x,\xi;\kappa)$ which satisfies the equation:

$$\Psi(x,\xi;\kappa)=\int_{\mathbf{R}^3} F(x,y;\kappa)\cdot\Psi(y,\xi;\kappa)\,dy+F(x,\xi;\kappa) \tag{1.5.1}$$

To solve (1.5.1) we construct the Neumann series of F. Thus, starting with $F^{(1)}=F$ we set

$$F^{(k)}(x,\xi;\kappa)=\int_{\mathbf{R}^3} F(x,y;\kappa)\cdot F^{(k-1)}(y,\xi;\kappa)\,dy\quad:\quad k\geq 2 \tag{1.5.2}$$

Then (1.4.1) gives the inequality

$$|F^{(2)}(x,\xi;\kappa)|\leq C_1^2\iiint\frac{e^{-k\kappa|\xi-y|}}{|x-y|^2\cdot|\xi-y|^2}\cdot dy \tag{i}$$

To estimate (i) we notice that the triple integral after the substitution $y-\xi\to u$ becomes

$$C_1^2\iiint\frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2\cdot|u|^2}\cdot du \tag{ii}$$

In (ii) the volume integral can be integrated in polar coordinates and becomes

$$C_1^2\cdot\int_0^\infty\int_{S^2}\frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2}\cdot dwdr \tag{iii}$$

where S^2 is the unit sphere and dw the area measure on S^2. It follows that (iii) becomes

$$2\pi C_1^2\cdot\int_0^\infty\int_0^\pi\frac{e^{-k\kappa r}}{(x-\xi)^2+r^2-2r\cdot|x-\xi|\cdot\sin\theta}\cdot d\theta dr=$$

$$\frac{2\pi C_1^2}{|x-\xi|}\cdot\int_0^\infty e^{-k\kappa|x-\xi|t}\cdot\log|\frac{1+t}{1-t}|\cdot\frac{dt}{t} \tag{iv}$$

where the last equality follows by a straightforward computation.

1.6 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)|\leq\frac{2\pi\cdot C_1^2\cdot C_1^*}{\kappa\cdot|x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

$$|F^{(n)}(x,\xi;\kappa)|\leq\frac{C_1}{|x-\xi|^2}\cdot\Big[\frac{2\pi C_1^2\cdot C_1^*}{\kappa}\Big]^{n-1}\quad\text{for every}\quad n\geq 2 \tag{*}$$

1.6 Conclusion. Choose κ_0^* so large that

$$2\pi C_1^2\cdot C_1^*<\kappa_0^* \tag{1.6.1}$$

Then (*) implies that the Neumann series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $\kappa\geq\kappa_0^*$ and gives the requested solution $\Psi(x,\xi;\kappa)$ in (1.5.1).

1.7 Exercise. We have found Ψ which satisfies the integral equation in § 1.5.1. Next, since the H-function in (ii) from § 1.2 is everywhere positive the integral equation (iii)in § 1.2 has a unique solution $\Phi(x,\xi;\kappa)$. Using Green's formula the reader can check that $\Phi(x,\xi;\kappa)$ yields a fundamental solution of $L(x,\partial_x)-\kappa^2$.

1.8 Some estimates. The constructions above show that the functions

$$x\mapsto\Phi(x,\xi;\kappa)\quad\text{and}\quad x\mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x = \xi$. Consider the difference

$$Q(x, \xi; \kappa) = \Phi(x, \xi; \kappa) - H(x - \xi, \xi; \kappa) \tag{1.8.1}$$

1.8.2 Exercise. Use the previous constructions to show that for every $0 < \gamma \leq 2$ there is a constant C_γ such that

$$\big|Q(x, \xi; \kappa)\big| \leq \frac{C_\gamma}{(\kappa|x - \xi|)^\gamma}$$

hold for every pair (x, ξ) and every $\kappa \geq \kappa_0$. Finally, the reader can apply the inequality for the H-function in (1.4.1) to conclude the results in the Main Theorem.

§ 2. Green's functions.

Let Ω be a bounded domain in $\mathbf{R}^3$, and L an elliptic differential operator as in § xx. Let $\kappa > 0$ and suppose we have found a function $G(x, y; \kappa)$ defined when $(x, y) \in \Omega \times \Omega$ with the property that $G(x, y) = 0$ if $x \in \partial\Omega$ and $y \in \Omega$. Moreover

$$(L(x, \partial_x) - \kappa^2)(G(x, y; \kappa)) = \delta(x - y)$$

With G as kernel we get the integral operator

$$\mathcal{G}(f)(x) = \int_\Omega G(x, y; \kappa)\, f(y)\, dy$$

Then $\mathcal{G}(f)(x) = 0$ on $\partial\Omega$ and the composed operator

$$(L(x, \partial_x) - \kappa^2) \circ \mathcal{G} = E$$

To construct G we use the fundamental solution $\Phi(x, y; \kappa)$ from § xx which satisfies

$$(L(x, \partial_x) - \kappa^2)(\Phi(x, y; \kappa)) = \delta(x - y)$$

With $y \in \Omega$ kept fixed we have the continuous boundary functon

$$x \mapsto \Phi(x, y; \kappa)$$

Solving the Dirchlet problem we find $w(x)$ such that $w(x) = \Phi(x, y; \kappa)$ on the boundary while $(L(x, \partial_x) - \kappa^2)(w) = 0$ holds in Ω. Then we can take

$$G(x, y; \kappa) = \Phi(x, y; \kappa) - w(x)$$

Using the estimates for the Φ-function from § 1 we get estimates for the G-function above. We choose a sufficiently large κ_0 so that $\Phi(x, \xi; \kappa_0)$ is a positive function of (x, ξ). Then the following hold:

2.1 Theorem. *One has*

$$G(x, \xi; \kappa_0) = \frac{1}{\sqrt{\Delta(x)} \cdot \sqrt{\Phi(x, \xi; \kappa_0)}} + R(x, \xi)$$

where the remainder function satisfies the following for all pairs (x, ξ) in Ω:

$$|R(x, \xi)| \leq C \cdot |x - \xi|^{-\frac{1}{4}}$$

and the constant C only depends on the domain Ω and the PDE-operator L.

Remark. Above the negative power of $|x - \xi|$ is a fourth-root which means that the remainder term R is more regular compared to the first term which behaves like $|x - \xi|^{-1}$ on the diagonal $x = \xi$.

2.2 Exercise. Prove Theorem 2.1 If necessary, consult [Carleman: page xx-xx9 for details.

2.3. Almost reality of eigenvalues.

Consider the set of eigenvalues λ for which there exists a function u in Ω which is zero on $\partial\Omega$ while

$$L(u) + \lambda \cdot u = 0$$

holds in Ω.

2.3.1 Proposition. *There exist positive constants C_* and c_* such that every eigenvalue λ above satisfies*

$$|\mathfrak{Im}\, \lambda|^2 \leq C_*(\mathfrak{Re}\, \lambda) + c_*)$$

Proof. Let u be an eigenfunction where $L(u) + \lambda \cdot u = 0$. Stokes theorem and the vanishing of $u|\partial\Omega$ give:

$$0 = \int_\Omega \bar{u} \cdot (L + \lambda)(u)\, dx = -\int_\Omega \sum_{p,q} a_{pq}(x) \cdot \frac{\partial u}{\partial x_p} \frac{\partial \bar{u}}{\partial x_q}\, dx + \int_\Omega \bar{u} \cdot (\sum a_p(x) \frac{\partial u}{\partial x_p})\, dx +$$

$$\int_\Omega |u(x)|^2 \cdot b(x)\, dx + \lambda \cdot \int |u(x)|^2\, dx$$

Write $\lambda = \xi + i\eta$. Separating real and imaginary parts we find the two equations:

$$\text{(i)} \qquad \xi \int |u|^2\, dx = \int \sum_{p,q} a_{p,q}(x) \frac{\partial u}{\partial x_p} \cdot \frac{\partial \bar{u}}{\partial x_q}\, dx + \int \big(\frac{1}{2} \cdot \sum \frac{\partial a_p}{\partial x_p} - b\big) \cdot |u|^2\, dx$$

$$\text{(ii)} \qquad \eta \int |u|^2\, dx = \frac{1}{2i} \int \sum a_p \big(u \frac{\partial \bar{u}}{\partial x_p} - \bar{u} \frac{\partial u}{\partial x_p}\big)\, dx$$

Set

$$A = \int |u|^2\, dx \quad : \quad B = \int |\nabla(u)|^2\, dx$$

Since L is elliptic there exists a positive constant k such that

$$\sum_{p,q} a_{p,q}(x) \frac{\partial u}{\partial x_p} > k \cdot |\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants c_1, c_2, c_3 such that

$$\text{(iii)} \qquad A\xi > c_1 B - c_2 B \quad : \quad A|\eta| < c_3 \cdot \sqrt{AB}$$

Here (iii) implies that $\xi > -c_2$ and the reader can also confirm that

$$\text{(iv)} \qquad B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A \cdot c_2 \cdot \sqrt{\frac{\xi + c_2}{c_1}} \quad : \quad |\eta| < c_3 \cdot \sqrt{\frac{\xi + c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 2.3.1.

2.4. Asymptotic formula for eigenvalues

Consider a function f which satisfies

$$\mathcal{G}(f) = -\frac{1}{\lambda} \cdot f$$

for some non-zero complex number λ. With $u = \mathcal{G}(f)$ it follows from (xx) that

$$(L - \kappa^2)(u) = f = -\lambda \cdot u$$

Hence

$$L(u) + (\lambda - \kappa^2)u = 0$$

About the proof of Theorem xx. From the above the asymptotic formula in Theorem xx can be derived from asymptotic properties of eigenvalues to the integral operator $|mathcalG$. Using Theorem 2.1 and the estimates for the fundsemtnal solution Φ in § 1, one can proceed as in the the next section where a Tauberian theorem is employed to finish the proof of Theorem xx. The reader may try to supply details or consult [Carleman: page xx-xx] for details.

§ 3. A study of $\Delta(\phi) + \lambda \cdot \phi$.

Introduction. We expose material from Carleman's article *xxx* whose contents were presented at the Scandinavian Congress in Stockholm 1934. In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact established by G. Neumann and H. Poincaré during the years 1879-1895 gives the following: First there exists the Greens' function

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$ with the property that the operator $\mathcal{G}$ defined on $L^2(\Omega)$ by

$$f \mapsto \mathcal{G}_f(p) = \frac{1}{2\pi} \iint G(p,q) f(q)\, dq$$

satisfies

$$\Delta \circ \mathcal{G}_f = -f \quad : f \in L^2(\Omega)$$

Moreover, $\mathcal{G}$ is a compact operator on the Hilbert space $L^2(\Omega)$ and there exists a sequence $\{f_n\}$ in $L^2(\Omega$ such that $\{\phi_n = \mathcal{G}_{f_n}\}$ is an orthonormal basis in $L^2(\Omega$ and

$$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : n = 1, 2, \ldots$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$. When eigenspaces have dimension ≥ 2, the eigenvalues are repeated by their multiplicity.

Main Theorem. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N \to \infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

The strategy in the proof is to consider the function of a complex variable s defined by

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

and show that it is a meromorphic function in the whole complex s-plane with a simple pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. More precisely we shall prove:

0.1 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem 0.1 gives ther main theorem by a result due to Wiener in the article *Tauberian theorem* [Annals of Math.1932]. Wiener's theorem asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s \to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n \to \infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Exercise. Derive the main theorem from Wiener's result and Theorem 0.1.

About Wiener's result. It is a version of an famous Tauberian theorem proved by Hardy and Littlewood in 1913 which goes as follows:

0.2 The Hardy-Littelwood theorem. *Let $\{a_n\}$ be a sequence of non-negative real numbers such that*

$$(*) \qquad A = \lim_{r \to 1} (1-r) \cdot \sum a_n r^n$$

exists. Then there also exists the limit

$$(**) \qquad A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Notice that no growth condition is imposed on the sequence $\{a_n\}$, i.e. the sole assumption is the existing limit (*). The proof is quite demanding and does not follow by "abstract nonsense" from functional analysis. For the reader's convenience we include details of the proof in a separate appendix since courses devoted to series rarely appear in contemporary education.

§ 1. Proof of Theorem 0.

Let Ω be a bounded and Dirichlet regular domain. For ech fixed point $p \in \Omega$ we get the continuous function on $\partial\Omega$ defined by

$$q \mapsto \log \frac{1}{|p-q|}$$

We find the harmonic function $u_p(q)$ in Ω such that $u_p(q) = \log \frac{1}{|p-q|} \,:\, q \in \partial\Omega$. Green's function is defined for pairs $p \neq q$ in $\Omega \times \Omega$ by

$$(1) \qquad G(p,q) = \log \frac{1}{|p-q|} - u_p(q)$$

Keeping if $p \in \Omega$ fixed, the function $q \mapsto G(p,q)$ extends to the closure of Ω where it vanishes if $q \in \partial\Omega$. If $f \in L^2(\Omega$ we set

$$(2) \qquad \mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot f(q)\, dq$$

where $q = (x,y)$ so that $dq = dxdy$ when the double integral is evaluated. From (1) we see that

$$\iint_{\Omega \times \Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence $\mathcal{G}$ is of the Hilbert-Schmidt type and therefore a compact operator on $L^2(\Omega)$. Next, recall that $\frac{1}{2\pi} \cdot \log \sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator. From this the reader can deduce the following:

1.1 Theorem. *For each $f \in L^2(\Omega)$ the Lapacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

$$(*) \qquad \Delta(\mathcal{G}_f) = -f$$

The equation (*) means that the composed operator $\Delta \circ \mathcal{G}$ is minus the identity on $L^2(\Omega)$. We are led to introduce the linear operator S on $L^2(\Omega)$ defined by Δ, where $\mathcal{D}(S)$ is the range of $\mathcal{G}$. If $g \in C_0^2(\Omega)$, i.e. twice differentiable and with compact support, it follows via Greens' formula that

$$\frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot \Delta(g)(q)\, dq = -g(p)$$

In particular $C_0^2(\Omega) \subset \mathcal{D}(S)$ which implies that S is densely defined and we leave it to the reader to verify that

$$\mathcal{G}(\Delta(f)) = -f \quad : f \in \mathcal{D}(S)$$

Remark. By Carl Neumann's classic construction of resolvent operators from 1879, the result above means that $-\mathcal{G}$ is Neumann's inverse of S. Since $-\mathcal{G}$ is compact it follows by Neumann's formula for spectra that S has a discrete spectrum,and we recall the following wellknown fact which goes back to work by Poincaré:

1.2 Proposition. *There exists an orthonormal basis $\{\phi_n\}$ in $L^2(\Omega)$ where each $\phi_n \in \mathcal{D}(S)$ is an eigenfunction, and a non-decreasing sequence of positive real numbers $\{\lambda_n\}$ such that*

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots \tag{1.2.1}$$

Remark. Above (1.2.1) means that

$$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n$$

This, $\{\lambda_n^{-1}\}$ are eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. As usual eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times.

After these preliminaries we embark upon the proof of Theorem 0.1. First, since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\sum \lambda_n^{-2} < \infty \tag{i}$$

This convergence entails that various constructions below are defined. For each complex number λ outside $\{\lambda_n\}$ we set

$$G(p, q; \lambda) = G(p, q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda - \lambda_n)} \tag{ii}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p, q; \lambda) \cdot f(q)\, dq \tag{iii}$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p, \lambda)$. Set

$$F(p, q, \lambda) = G(p, q; \lambda) - G(p, q)$$

Keeping p fixed we see that (ii) gives

$$\lim_{q \to p} F(p, q, \lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)} \tag{B.1}$$

Set

$$F(p, \lambda) = \lim_{q \to p} F(p, q, \lambda)$$

From (i) and (B.1) it follows that it is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Show via residue calculus that one has the equality below in a half-space $\mathfrak{Re}\, s > 2$:

$$\Phi(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a - i\infty}^{a + i\infty} F(p, \lambda) \cdot \lambda^{-s}\, d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\, \lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p, \lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\Phi(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta}\, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} F(p, -x) \cdot x^{-s}\, dx \tag{E.1}$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^{\infty} F(p, -x) \cdot x^{-s}\, dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p, -x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^{\infty} \frac{e^{-zt}}{\sqrt{t^2 - 1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$\text{(i)} \qquad I_0(z) = \sum_{m=0}^{\infty} \frac{2^{-2m}}{(m!)^2} \cdot z^{2m}$$

$$\text{(ii)} \qquad I_1(z) = \sum_{m=0}^{\infty} \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p, q; \kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p, q; \kappa) = K(\sqrt{\kappa}|p - q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p, q; -\kappa) = K(\sqrt{\kappa} \cdot |p - q|) - H(q; \kappa) \quad : \kappa > 0$$

Next, the construction of $G(p, q)$ gives

$$F(p, -\kappa) = \lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q|] + \lim_{q \to p} [u_p(q) + H(p, q, \kappa)] \tag{G.1}$$

The last term above has the "nice limit" $u_p(p) + H(p, p, \kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q|] = -\frac{1}{2} \cdot \log \kappa + \log 2 - \gamma \tag{G.2}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p, -\kappa) = -\frac{1}{2} \cdot \log \kappa + A + H(p, p; -\kappa)$$

With $x = \kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty x^{-s}\, dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2} \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty \log x \cdot x^{-s}\, dx$$

Notice that the complex derivative

$$\frac{d}{ds} \int_a^\infty x^{-s}\, dx = -\int_a^\infty \log x \cdot x^{-s}\, dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem 0.1 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin\, \pi s}{2\pi^2} \cdot \int_a^\infty H(p,p,\kappa) \cdot \kappa^{-s}\, d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \leq H(p,q;\kappa) \leq \max_{q \in \partial\Omega} K(\kappa|p-q|)$$

With $p \in \Omega$ fixed there is a positive number δ such that $|p-q| \geq \delta : q \in \partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.

Appendix. Theorems by Abel, Tauber, Hardy and Littlewood

Introduction. Consider a power series $f(z) = \sum a_n z^n$ whose radius of convergence is one. If $r < 1$ and $0 \leq \theta \leq 2\pi$ we are sure that the series

$$f(re^{i\theta}) = \sum a_n r^n e^{in\theta}$$

is convergent. In fact, it is even absolutely convergent since the assumption implies that

$$\sum |a_n| \cdot r^n < \infty \quad \text{for all} \quad r < 1$$

Passing to $r = 1$ it is in general not true that the series $\sum a_n e^{in\theta}$ is convergent. An example arises if we consider the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

This leads to the following problem where we without loss of generality can take $\theta = 0$. Consider as above a convergent power series and assume that there exists the limit

$$(*) \qquad \lim_{r \to 1} \sum a_n r^n$$

When can we conclude that the series $\sum a_n$ also is convergent and that one has the equality

$$(**) \qquad \sum a_n = \lim_{r \to 1} \sum a_n r^n$$

The first result in this direction was established by Abel in a work from 1823:

A. Theorem *Let $\{a_n\}$ be a sequence such that $\frac{a_n}{n} \to 0$ as $n \to \infty$ and there exists*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Then $\sum a_n$ is convergent and the sum is A.

An extension of Abel's result was established by Tauber in 1897.

B. Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists the limit*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Set

$$\omega_n = a_1 + 2a_2 + \ldots + na_n \quad : \ n \geq 1$$

If $\lim_{n \to \infty} \omega_n = 0$ it follows that the series $\sum a_n$ is convergent and the sum is A.

C. Results by Hardy and Littlewood. In their joint article *xxx* from 1913 the following extension of Abel's result was proved by Hardy and Littlewood:

C. Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a constant C so that $\frac{a_n}{n} \leq C$ for all $n \geq 1$. Assume also that the power series $\sum a_n z^n$ converges when $|z| < 1$. Then the same conclusion as in Abel's theorem holds.*

Remark. In addition to this they proved a result about positive series from the cited article which has independent interest.

D. Theorem. *Assume that each $a_n \geq 0$ and that there exists the limit:*

$$(*) \qquad A = \lim_{r \to 1} (1 - r) \cdot \sum a_n r^n$$

Then there exists the limit

$$(**) \qquad A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Remark. The proofs of Abel's and Tauber's results are easy while C and D require more effort and rely upon results from calculus in one variable. So before we enter the proofs of the theorems above insert some preliminaries.

1. Results from calculus

Below $g(x)$ is a real-valued function defined on $(0,1)$ and of class C^2 at least.

1.1 Lemma *Assume that there exists a constant $C>0$ such that*

$$g''(x) \leq C(1-x)^{-2} \quad : \, 0<x<1 \quad \text{and} \quad \lim_{x\to 1} g(x)=0$$

Then one has the limit formula:

$$\lim_{x\to 1}\, (1-x)\cdot g'(x)=0$$

1.2 Lemma *Assume that the second order derivative $g''(x)>0$. Then the following implication holds for each $\alpha>0$:*

$$\lim_{x\to 1}\, (1-x)^\alpha \cdot g(x)=1 \implies \lim_{x\to 1}\, (1-x)^{\alpha+1}\cdot g'(x)=\alpha$$

Remark. If $g(x)$ has higher order derivatives which all are >0 on $(0,1)$ we can iterate the conclusion in Lemma 1.2 where we take α to be positive integers. More precisely, by an induction over ν the reader may verify that if

$$\lim_{x\to 1}\, (1-x)\cdot g(x)=1$$

exists and if $\{g^{(\nu)}(x)>0\}$ for all every $\nu\geq 2$ then

(*) $$\lim_{x\to 1}\, (1-x)^{\nu+1}\cdot g^{(\nu)}(x)=\nu\,! \quad : \, \nu\geq 2$$

Next, to each integer $\nu\geq 1$ we denote by $[\nu-\nu^{2/3}]$ the largest integer $\leq\, (\nu-\nu^{2/3})$. Set

$$J_*(\nu)=\sum_{n\leq[\nu-\nu^{2/3}]} n^\nu e^{-\nu} \quad : \quad J^*(\nu)=\sum_{n\geq[\nu+\nu^{2/3}]} n^\nu e^{-\nu}$$

1.3 Lemma *There exists a constant C such that*

$$\frac{J^*(\nu)+J_*(\nu)}{\nu\,!}\leq \delta(\nu) \quad : \quad \delta(\nu)=C\cdot \exp\big(-\frac{1}{2}\cdot \nu^{\frac{1}{3}}\big) \quad : \, \nu=1,2,\ldots$$

Proofs

We prove only Lemma 1.1 which is a bit tricky while the proofs of Lemma 1.2 and 1.3 are left as exercises to the reader. Fix $0<\theta<1$. Let $0<x<1$ and set

$$x_1=x+(1-x)\theta$$

The mean-value theorem in calculus gives

(i) $$g(x_1)-g(x)=\theta(1-x)g'(x)+\frac{\theta^2}{2}(1-x)^2\cdot g''(\xi) \quad \text{for some} \quad x<\xi<x_1$$

By the hypothesis

$$g''(\xi)\leq C(1-\xi)^{-2}\leq C)1-x_1)^{-2}$$

Hence (i) gives

$$(1-x)g'(x)\geq \frac{1}{\theta}(g(x_1)-g(x))-C\cdot \frac{\theta}{2}\frac{(1-x)^2}{1-x_1)^2}=$$

$$\frac{1}{\theta}(g(x_1)-g(x))-\frac{C\cdot\theta}{2(1-\theta)^2}$$

Keeping θ fixed we have by assumption

$$\lim_{x\to 1}\, g(x)=0$$

Notice also that $x\to 1 \implies x_1\to 1$. It follows that

$$\liminf_{x\to 1}\, (1-x)g'(x)\geq -\frac{C\cdot\theta}{2(1-\theta)^2}$$

Above $0 < \theta < 1$ is arbitrary, i.e. we can choose small $\theta > 0$ and hence we have proved that

(*) $$\liminf_{x \to 1} (1-x)g'(x) \geq 0$$

Next we prove the opposed inequality

(**) $$\limsup_{x \to 1} (1-x)g'(x) \leq 0$$

To get (**) we apply the mean value theorem in the form

(ii) $$g(x_1) - g(x) = \theta(1-x)g'(x_1) - \frac{\theta^2}{2}(1-x)^2 \cdot g''(\eta) \quad : \ x < \eta < x_1$$

Since $(1-x_1) = \theta(1-x)(1-\theta)$ we get

(iii) $$(1-x_1)g'(x_1) = \frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + \frac{(1-\theta)\theta}{2} \cdot (1-x)^2 g''(\eta)$$

Now $g''(\eta) \leq C(1-\eta)^{-2} \leq C(1-x_1)^{-2}$ so the right hand side in (iii) is majorized by

$$\frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{(1-\theta)\theta}{2} \cdot (1-x)^2(1-x_1)^2 =$$

(iv) $$\frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{\theta}{2(1-\theta}$$

Keeping θ fixed while $x \to 1$ we obtain:

$$\liminf_{x \to 1} (1-x)g'(x) \leq C \cdot \frac{\theta}{2(1-\theta}$$

Again we can choose arbitrary small θ and hence (**) holds which finishes the proof of Lemma 1.1.

2. Proof of Abel's theorem.

Without loss of generality we can assume that $a_0 = 0$ and set $S_N = a_1 + \ldots + a_N$. Given $0 < r < 1$ we let $f(r) = \sum a_n r^n$. For every positive integer N the triangle inequality gives:

$$\big|S_N - f(r)\big| \leq \sum_{n=1}^{n=N} |a_n|(1-r^n) + \sum_{n \geq N+1} |a_n| r^n$$

Set $\delta(N) = \max_{n \geq N} \frac{|a_n|}{n}$. Since $1 - r^n) = (1-r)(1 + \ldots + r^{n-1} \leq (1-r)n$ the last sum is majorised by

$$(1-r) \cdot \sum_{n=1}^{n=N} n \cdot |a_n| + \delta(N+1) \cdot \sum_{n \geq N+1} \frac{r^n}{n}$$

Next, the obvious inequality $\sum_{n \geq N+1} \frac{r^n}{n} \leq \frac{1}{N+1} \cdot \frac{1}{1-r}$ gives the new majorisation

(1) $$(1-r) \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \frac{\delta(N+1)}{N+1} \cdot \frac{1}{1-r}$$

This hold for all pairs N and r. To each $N \geq 2$ we take $r = 1 - \frac{1}{N}$ and hence the right hand side in (1) is majorised by

$$\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \delta(N+1) \cdot \frac{N}{N+1}$$

Here both terms tend to zero as $N \to \infty$. Indeed, Abel's condition $\frac{a_n}{n} \to 0$ implies that $\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n}$ tends to zero as $N \to \infty$. Hence we have proved the limit formula:

$$\lim_{N\to\infty} \left| s_N - f(1-\frac{1}{N}) \right| = 0 \tag{*}$$

Finally it is clear that (*) gives Abel's result.

3. Proof of Tauber's theorem.

We may assume that $a_0 = 0$. Notice that

$$a_n = \frac{\omega_n - \omega_{n-1}}{n} \quad : \, n \geq 1$$

It follows that

$$f(r) = \sum \frac{\omega_n - \omega_{n-1}}{n} \cdot r^n = \sum \omega_n \big(\frac{r^n}{n} - \frac{r^{n+1}}{n+1}\big)$$

Using the equality $\frac{1}{n} = \frac{1}{n+1} = \frac{1}{n(n+1)}$ we can rewrite the right hand side as follows:

$$\sum \omega_n \big(\frac{r^n - r^{n+1}}{n+1} + \frac{r^n}{n(n+1)}\big)$$

Set

$$g_1(r) = \sum \omega_n \cdot \frac{r^n - r^{n+1}}{n+1} = (1-r) \cdot \sum \frac{\omega_n}{n+1} \cdot r^n$$

By the hypothesis $\lim_{n\to\infty} \frac{\omega_n}{n+1} = 0$ and then it is clear that we get

$$\lim_{r\to 1} g_1(r) = 0$$

Since we also have $f(r) \to 0$ as $r \to 1$ we conclude that

$$\lim_{r\to 1} \sum \frac{\omega_n}{n(n+1)} \cdot r^n = 0 \tag{1}$$

Next, with $b_n = \frac{\omega_n}{n(n+1)}$ we have $nb_n = \frac{\omega_n}{n+1} \to 0$. Hence Abel's theorem applies so (1) gives convergent series

$$\sum \frac{\omega_n}{n(n+1)} = 0 \tag{2}$$

If $N \geq 1$ we have the partial sum

$$S_N = \sum_{n=1}^{n=N} \frac{\omega_n}{n(n+1)} = \sum_{n=1}^{n=N}, \omega_n \cdot \big(\frac{1}{n} - \frac{1}{n+1}\big)$$

The last term becomes

$$\sum_{n=1}^{n=N} \frac{1}{n}(\omega_n - \omega_{n-1}) - \frac{\omega_N}{N+1} = \sum_{n=1}^{n=N} a_n - \frac{\omega_N}{N+1}$$

Again, since $\frac{\omega_N}{N+1} \to 0$ as $N \to \infty$ we conclude that the convergent series from (2) implies that the series $\sum a_n$ also is converges and has sum equal to zero. This finishes the proof of Tauber's result.

4. Proof of Theorem D.

Set $f(x) = \sum a_n x^n$ which is defined when $0 < x < 1$. Notice that

$$(1-x)f(x) = \sum s_n x^n \quad \text{where} \quad s_n = a_1 + \ldots + a_n$$

Set $g(x) = \sum s_n x^n$ which is defined when $0 < x < 1$. Since $s_n \geq 0$ for all n all the higher order derivatives

$$g^{(p)}(x) = \sum_{n=p}^{\infty} n(n-1)\cdots(n-p+1)a_n x^{n-p} > 0$$

when $0 < x < 1$. The hypothesis that $\lim_{x\to 1} g(x) = A$ and Lemma 1.1 and the inductive result in the remark after Lemma 1.2 give:

(1) $$\lim_{x\to 1} (1-x)^{\nu+2} \cdot \sum s_n \cdot n^\nu x^n = (\nu+1)! \quad : \nu \geq 1$$

We shall use the substitution $e^{-t} = x$ where $t > 0$. Since $t \simeq 1 - x$ when $x \to 1$ we see that (1) gives

(2) $$\lim_{t\to 0} t^{\nu+2} \cdot \sum s_n \cdot n^\nu e^{-nt} = (\nu+1)! \quad : \nu \geq 1$$

Let us put

$$J_*(\nu,t) = \frac{t^{\nu+2}}{(\nu+1)!} \cdot \sum_{n=1}^{\infty} s_n \cdot n^\nu e^{-nt}$$

So for each fixed ν one has

(3) $$\lim_{t\to 0} J_*(\nu,t) = 1$$

Next, for each pair $\nu \geq 2$ and $0 < t < 1$ we define the integer

(*) $$N = [\frac{\nu - \nu^{2/3}}{t}]$$

Since the sequence $\{s_n\}$ is non-decreasing we get

(i) $$s_N \cdot \sum_{n\geq N} n^\nu e^{-nt} \leq \sum_{n\geq N} s_n \cdot n^\nu e^{-nt} \leq \frac{(\nu+1)! \cdot J_*(\nu,t)}{t^{\nu+2}}$$

Next, the construction of N and Lemma 1.3 give:

(ii) $$\sum_{n\geq N} n^\nu e^{-nt} \geq \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta(\nu))$$

where the δ function is independent of ν and tends to zero as $\nu \to \infty$. Hence (i-ii) give

(iii) $$s_N \leq \frac{(\nu+1)}{t} \cdot \frac{1}{1-\delta(\nu)} \cdot J_*(\nu,t)$$

Next, by the construction of N one has

$$N + 1 \geq \frac{\nu - \nu^{2/3}}{t} = \frac{\nu}{t} \cdot (1 - \nu^{-1/3})$$

It follows that (iii) gives

(iv) $$\frac{s_N}{N+1} \leq \frac{\nu+1}{\nu} \cdot \frac{1}{1-\nu^{-1/3}} \cdot \frac{1}{1-\delta(\nu)} \cdot J_*(\nu,t)$$

Since $\delta(\nu) \to 0$ it follows that for any $\epsilon > 0$ there exists some ν_* such that

(v) $$\frac{\nu_*+1}{\nu_*} \cdot \frac{1}{1-\nu_*^{-1/3}} \cdot \frac{1}{1-\delta(\nu_*)} < 1 + \epsilon$$

Keeping ν_* fixed we now consider pairs t_n, N such that (*) above hold with $\nu = \nu_*$. Notice that

(vi) $$N \to +\infty \implies t_N \to 0$$

It follows from (iv) and (v) that we have:

(vii) $$\frac{s_N}{N+1} < (1+\epsilon) \cdot J_*(\nu_*, t_N) \quad : N \geq 2$$

Now (vi) and the limit in (3) which applies with ν_* while $t_N \to 0$ entail that

$$\lim_{N\to\infty} J(\nu_*, t_N) = 1$$

We have also that $\frac{N}{N+1} \to 1$ and since $\epsilon > 0$ was arbitrary we see that (vii) proves the inequality

$$(1) \qquad \limsup_{N\to\infty} \frac{s_N}{N} \leq 1$$

So Theorem 2 follows if we also prove that

$$(2) \qquad \liminf_{N\to\infty} \frac{s_N}{N} \geq 1$$

The proof of (II) is similar where we now define the integers N by:

$$N = [\frac{\nu + \nu^{2/3}}{t}]$$

Then we have

$$S_N \cdot \sum_{n\leq N} n^\nu e^{-nt} \geq \frac{(\nu+1)! \cdot J_*(\nu,t)}{t^{\nu+2}} - \sum_{n>N} s_n \cdot n^\nu e^{-nt}$$

Here the last term can be estimated above since the Lim.sup inequality (I) gives a constant C such that $s_n \leq Cn$ for all n and then

$$\sum_{n>N} s_n \cdot n^\nu e^{-nt} \leq C \cdot \sum_{n>N} n^{\nu+1} e^{-nt} \leq C \cdot \delta^*(\nu) \cdot \frac{(\nu+1)!}{t^{\nu+2}}$$

where Lemma 1.3 entails that $\delta^*(\nu) \to 0$ as ν increases. At the same time Lemma 1.3 also gives

$$\sum_{n\leq N} n^\nu \cdot e^{-nt} = \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta_*(\nu)$$

where $\delta(\nu_*) \to 0$. At this stage the reader can verify that (2) by similar methods as in the proof of (I).

5. Proof of Theorem C

Set $f(x) = \sum a_n x^n$. Notice that it suffices to prove Theorem C when the limit value

$$\lim_{x\to 1} \sum a_n x^n = 0$$

Next, the assumption that $a_n \leq \frac{c}{n}$ for a constant c gives

$$f''(x) = \sum n(n-1)a_n x^{n-2} \leq c \sum (n-1)x^{n-2} = \frac{c}{1-x)^2}$$

The hypothesis $\lim_{x\to 1} f(x) = 0$ and Lemma xx therefore gives

$$(i) \qquad \lim_{x\to 1} (1-x)f'(x) = 0$$

Next, notice the equality

$$(ii) \qquad \sum_{n=1}^{\infty} \frac{na_n}{c} x^n = \frac{x}{c} \cdot f'(x)$$

At the same time $\sum_{n=1}^\infty x^n = \frac{x}{1-x}$ and hence (i-ii) together give:

$$\lim_{x\to 1} (1-x) \cdot \sum (1 - \frac{na_n}{c}) \cdot x^n = 1$$

Here $1 - \frac{na-n}{c} \geq 0$ so Theorem 2 gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n=N} (1 - \frac{na_n}{c}) = 1$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{n=N} na_n = 0$$

This means precisely that the condition in Tauber's Theorem holds and hence $\sum a_n$ converges and has series sum equal to 0 which finishes the proof of Theorem C.

§ 4. The Neumann-Poincaré boundary value problem

Introduction. Several fundamental results were achieved by Carl Neumann in the article emphxxx from 1877 which in particular solved boundary valued problems where double-layer potentials occur. The results were carried out in dimension 3 which for physical reasons is the most relevant part. Here we expose Neumann's theory in dimension two and give some remarks about the 3-dimensional case in § xx. The crucial steategy is to use Neumann's analytic series expansions which reduces the proof of existence to show that certain poles are absent while meromorphic extensions of Neumann series are constructed. We remark that Neumann's existence results were confined to convex domains where certain majorizations become straighforward since this gives rise to positive integral kernels. The extension of Neumann's results to general domains was achieved by Poincaré in the article *xxx* from 1897 where some ingenoius new methods were introduced to overcome the failure of positivity for the non-symmetric kernel defining the double-layer potential.

The smoothness of boundaries was relaxed in later work. Existence results for planar domains where isolated corner points are allowed were established by Zarmela in 1904. Further studies of Neumann's problem for planar domains with non-regular boundary appear in Carleman's thesis from 1916. A novelty in this work is that solutions to the Neumann's boundary value problem also are exhibited for functions which only are integrable on the boundary. This leads to new phenomena for the spectrum of ïntegral operators, i.e the spectrum is not always confined to discrete subsets of the complex λ-plane. It will take us too far to go into details, especially in the delicate analysis from Part 3 in [ibid]. The interested reader can also consult the expositary article *xxxx* by Holmgen which describes how non-discrete spectral can occur for integral kernels associated to the Neumann problem which in those days was a new phenomenon in operator theory. Let us now expose the methods introduced by Neumann and Poincaré.

Preliminaries. Let $\mathcal{C}$ be a closed Jordan curve of class C^2 whose arc-lengt measure is denoted by σ. If g is a continuous function on $\mathcal{C}$ the logarithmic potential

$$U_g(z) = \frac{1}{\pi} \int_{\mathcal{C}} \log \frac{1}{|z-q|} \cdot g(q)\, d\sigma(q)$$

yields a harmonic function in open the complement of $\mathcal{C}$. Since log $|z|$ is locally integrable in $\mathbf{C}$ and $U_g(z)$ the convolution of this log-function and the compactly supported Riesz measure $g \cdot \sigma$. By elementary measure theory this implies that U_g extends to a continuous function. In particular the pair of harmonic functions in the inner respectively outer component of $\mathcal{C}$ are equal on $\mathcal{C}$. Moreover, the Laplacian of U_g taken in the distribution sense is equal to the measure $g \cdot \sigma$. Now we consider partial derivatives of U and study the inner normal derivative as z approaches points $p \in \mathcal{C}$ from the inside. Let $\mathbf{n}_*$ denote the inner normal derivative along $\mathcal{C}$ which gives the function on $\mathcal{C}$ defined by:

$$p \mapsto \frac{\partial U_g}{\partial \mathbf{n}_*}(p)$$

With $p \neq q$ we have the vector $p - q$. We take its inner product with the unit vector $\mathbf{n}_*(q)$ and set set

$$(*) \qquad K(p,q) = \frac{\langle p-q, \mathbf{n}_*(q)\rangle}{|p-q|^2}$$

Let analyze the behaviour of K close to a point on the diagonal. Working in local coordinates we can take $p = q = (0,0)$ and close the this boundary point the C^2-curve $\mathcal{C}$ is locally defined by a function

$$y = f(x)$$

where $\phi(x)$ is a C^1-function and the (x, y) belong to the bounded Jordan domain when $y > f(x)$. By drawing a figure the reader can verify that

$$\mathbf{n}_*(x, f(x)) \cdot d\sigma = (-f'(x), 1)\dot{d}x$$

So with $p = (t, f(t))$ and $q = (x, f(x))$ we have

$$K(p,q) \cdot d\sigma(q) = \frac{f(t) - f(x) - f'(x)(t-x)}{(t-x)^2 + (f(t) - f(x))^2} \cdot dx$$

By hypothesis f is of class C^2 which implies that the right hand side stays bounded as y and x independently of each other approach zero. This eanble us to construct integrals and Green's formula yields:

0.1 Theorem. *For each $p \in \mathcal{C}$ one has*

$$\frac{\partial U_g}{\partial \mathbf{n}_*}(p) = g(p) + \int_{\mathcal{C}} K(p,q) \cdot g(q)\, d\sigma(q)$$

Exercise. Prove this equality. A hint is by additivity it suffices to take g-functions with supports confined to small sub-intervals of $\mathcal{C}$ and profit upon local coordinates and parametrizations as above for $\mathcal{C}$ close to the support of g.

1. Neumann's boundary value problem.

Let Ω be a bounded domain where $\partial\Omega$ consists of a finite set of closed Jordan curves of class C^2. Let h and f be a pair of real-valued continuous functions on $\partial\Omega$ where h is positive. We seek a function U which is harmonic in Ω and on the boundary satisfies

$$(*) \qquad \frac{\partial U}{\partial \mathbf{n}_*}(p) = h(p)U(p) + f(p)$$

1.1 Theorem. *The boundary value problem above has a unique solution U.*

The uniqueness amounts to show that if V is harmonic in Ω and

$$\frac{\partial V}{\partial \mathbf{n}_*}(p) = h(p)V(p)$$

holds on $\partial\Omega$, then $V = 0$. Since h is positive this follows from § XX: Chapter V.

Proof of existence. For each $g \in C^0(\partial\Omega)$ we construct U_g which by Theorem 0.1 solves (*) if the g satisfies the integral equation

$$(1) \qquad g(p) + \int_{\mathcal{C}} K(p,q) \cdot g(q)\, d\sigma(q) = h(p) \cdot \frac{1}{\pi} \cdot \int_{\partial\Omega} \log \frac{1}{|p-q|} \cdot g(q)\, d\sigma(q) + f(p)$$

With h kept fixed we introduce the kernel

$$K_h(p,q) = h(p) \cdot \frac{1}{\pi} \cdot \log \frac{1}{|p-q|} - K(p.q)$$

and (1) reduces to the equation

$$(2) \qquad g(p) - \int_{\partial\Omega} K_h(p,q) g(q)\, d\sigma(q) = f(p)$$

Define the linear operator on the Banach space $C^0(\partial\Omega)$ defined by

$$(3) \qquad \mathcal{K}_h(g) = \int_{\partial\Omega} K_h(p,q) g(q)\, d\sigma(q) \quad : \quad g \in C^0(\partial\Omega)$$

With this notation a g-function satisfies (2) if

$$(4) \qquad (E - \mathcal{K}_h)(g) = f$$

where E is the identity operator on $C^0(\partial\Omega)$. Next, from the general result in §§ $\mathcal{K}_h$ is a compact linear operator. By another general result from § xx it follows that each $f \in C^0(\partial\Omega)$ yields a meromorphic function of the complex parameter λ given by

$$N_f(\lambda) = f + \sum_{n=1}^{\infty} \lambda^n \cdot \mathcal{K}_h^n(f)$$

If $\delta > 0$ is so small that $||\mathcal{K}_h|| < \delta^{-1}$ it is clear that

$$(E - \lambda\mathcal{K}_h)(N_f(\lambda)) = f \tag{5}$$

By analyticity (5) reamins valied when λ stays outside the poles of $N_f(\lambda)$. In particular, if no pole occurs at $\lambda = 1$ we have the equation

$$(E - \mathcal{K}_h)(N_f(1) = f$$

This means that $g = N_f(1)$ solves (4) and the existence part in Theorem 1.1 follows. So there remains only to show:

The absence of a pole at $\lambda = 1$. If $N_f(\lambda)$ has a pole at $\lambda = 1$ there is a positive integer m such that

$$N_f(\lambda) = \sum_{k=1}^{k=m} \frac{a_k}{(1-\lambda)^k} + b(\lambda)$$

hold when $|\lambda - 1|$ is small where $a_m \neq 0$ in $C^0(\partial\Omega)$ and $b(\lambda)$ is analytic in some disc centered at $\lambda = 1$. It follows that

$$(1-\lambda)^m N_f(\lambda) = a_m + (1-\lambda)\beta(\lambda)$$

where $\beta(\lambda)$ again is an analytic $C^0(\partial\Omega)$-valued function close to 1. Apply $E - \lambda\mathcal{K}_h$ on both sides which gives

$$(1-\lambda)^m(E - \lambda\mathcal{K}_h)(N_f(\lambda)) = (E - \lambda\mathcal{K}_h)(a_m) + (1-\lambda)(E - \lambda\mathcal{K}_h)(\beta(\lambda))$$

By (5) the left hand side is equal to $(1-\lambda)^m f$. So with $\lambda = 1$ we get

$$(E - \mathcal{K}_h)(a_m) = 0 \implies a_m = \mathcal{K}_h(a_m)$$

This contradicts the uniqueness part which already has been proved.

2. The case when $\mathcal{C}$ has corner points.

In the preceeding section we found a unique solution to Neumann's boundary problem where the inner normal derivative of U along $\partial\Omega$ is a continuous function. If corner points appear this will no longer be true. But stated in an appropriate way we can extend Theorem 1.1. Let us analyze the specific case when the boundary curves are piecewise linear, i.e. each closed Jordan curve in $\partial\Omega$ is a simple polygon with a finite number of corner points. Given one of these we begin to study the K-function. Let $\xi_1, \ldots, \xi_N$ be the corner points on $\mathcal{C}$. On the linear interval ℓ_i which joints two succescive corner points ξ_i and ξ_{i+1} we notice that $\mathbf{n}_*$ is constant and it is even true that

$$K(p,q) = 0 \quad : \quad p, q \in \ell_i$$

Indeed, this is obvious for if p and q both belong toℓ_i then the vector $p - q$ is parallell to ℓ_i and hence $\perp$ to the normal of this line. Next, keeping q fixed on the open interval ℓ_i while p varies on $\mathcal{C} \setminus \ell_i$ the behaviour of the function

$$p \mapsto \langle p - q, \mathbf{n}_*(q) \rangle$$

is can be understood via a picture and it is clear that (x) is a continuous function. By a picture the reader should discover the different behaviour in the case when $\mathcal{C}$ is convex or not. For example, in the non-convex case it is in general not true that $\mathcal{C} \setminus \ell_i$ stays in the half-space bordered by the line passing ℓ_i and then (*) can change sign, i.e. take both positive and negative values. In the special case when $\mathcal{C}$ is a convex polygon the reader should confirm that (x) is a positive function of p because we have taken the *inner* normal $\mathbf{n}_*(q)$.

2.1 Local behaviour at a corner point. After a linear change of coordinates we take a corner point ξ_* placed at the origin and one ℓ-line is defined by the equation $\{y=0\}$ to the left of ξ_* where $x<0$ while $y=Ax$ hold to the right for some $A\neq 0$. If $A>0$ it means that the angle α at the corner point is detemined by

$$\alpha=\pi-\text{arctg}(A)$$

If $A<0$ the inner angle is between 0 and $\pi/2$ which the reader should illustrate by a picture. Next, consider a pair of points $p=(-x,0)$ and $q=(t,At)$ where $x,t>0$. So p and q belong to opposite sides of the corner point. To be specific, suppose that $A>0$ which entails that

$$\mathbf{n}_*(q)=\frac{(-A,1)}{\sqrt{1+A^2}}\implies$$

$$K(p,q)\cdot d\sigma(q)=\frac{Ax+t}{(x+t)^2+A^2t^2}$$

When x and t decrease to the origin the order of magnitude is $\frac{1}{x+t}$ so the kernel function is unbounded and the order of magnitude is $\frac{1}{x+t}$. If ℓ_+ denotes the boundsry interval to the right of the origin where q are placed we conclude that

$$\int_{\ell_+}K(p,q)\cdot d\sigma(p)\simeq\int_0^1\frac{dt}{x+t}\simeq\log\frac{1}{x}$$

The last function is integrable with respect to x. This local computation shows that the kernel function K is not too large in the average. In particular

$$\iint_{\mathcal{C}\times\mathcal{C}}|K(p,q)|\cdot d\sigma(p)d\sigma(q)<\infty$$

But the growth of K near corner points prevail a finite L^2-integral, i.e. the reader may verify that

$$\iint_{\mathcal{C}\times\mathcal{C}}|K(p,q)|^2\cdot d\sigma(p)d\sigma(q)=+\infty$$

2.2 The integral operator $\mathcal{K}_h$. Let h be a positive continuous function on $\partial\Omega$. Now we define the kernel function $K_h(p,q)$ exactly as in § xx and obtain the corresponding linear operator

$$g\mapsto\int_{\partial\Omega}K_h(p,q)g(q)d\sigma(q)$$

It has a natural domain of definition. Namely, introduce the space L^1_* which consists of functions on g on $\partial\Omega$ for which

$$(*)\qquad\iint\log\frac{R}{|p-q|}\cdot|g(p)|\cdot d\sigma(q)d\sigma(p)<\infty$$

where $R>0$ is so large that $\frac{R}{|p-q|}>1$ hold for pairs p,q on $\partial\Omega$. Return to the local situation in (xx) and consider a g-function in L^1_*. Locally we encounter an integral of the form

$$\iint_{\square_+}\frac{1}{x+t}\cdot|g(t,At))|\,dt$$

where $0\leq x,t\leq 1$ hold in $\square_+$. In this double integral we first perform integration with respect to x which is finite since the inclusion $g\in L^1_*$ entails that

$$\int_0^1\log\frac{1}{t}\cdot|g(t,At)|\,dt<\infty$$

From the above we obtain the following:

2.3 Theorem. *The kernel function K_h yields a continuous linear operator from L^1_* into $L^1(\partial\Omega)$, i.e. there exists a constant C such that*

$$\int_{\partial\Omega}|\mathcal{K}_h(g)|\cdot d\sigma\leq C\cdot\iint_{\partial\Omega\times\partial\Omega}\log\frac{R}{|p-q|}\cdot|g(p)|\cdot d\sigma(q)d\sigma(p)$$

Armed with Theorem 2.3 we can solve Neumann's boundary value problem for domains whose boundary curves are polygons.

2.4 Theorem. *For each $f \in L^1(\partial\Omega$ there exists a unique harmonic function U in Ω such that*

$$\frac{\partial U}{\partial \mathbf{n}_*}(p) = h(p)U(p) + f(p)$$

holds on $\partial\Omega$. Moreover, $U = U_g$ where $g \in L^1_*$ solves the integral equation

$$g - \mathcal{K}_h(g) = f$$

The uniquenss part. At corner points the inner normal of U has no limit and to establish the uniqueness part we use instead an integral formula:

2.5 Proposition. *For each $g \in L^1_*$ the potential function $U = U_g$ satisfies*

$$\iint_\Omega [(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2]\, dxdy + \int_{\partial\Omega} U \cdot \frac{\partial U}{\partial \mathbf{n}_*}\, d\sigma = 0$$

Exercise. Prove this result.

The requested uniqueness follows. For if $\frac{\partial U}{\partial \mathbf{n}_*} = h \cdot U$ holds on the boundary we get

$$0 = \iint_\Omega [(\frac{\partial U}{\partial x})^2 + (\frac{\partial U}{\partial y})^2]\, dxdy = \int_{\partial\Omega} h \cdot U^2\, d\sigma \implies g = 0$$

2.6 Proof of existence.

It is carried out by the same mehod as in § X. The crucial point is that the kernel function K_h is sufficiently well-behaved in order that every $f \in L^1(\Omega)$ yields a meromorphic function $N_f(\lambda$ where $\mathcal{K}_h$-powers are applied to f exactly as in XX.

Exercise. Supply details which prove that $N_f(\lambda)$ is meromorphic.

Remark. In [Carleman: Part 3] it is proved that the unique solution g to the integral equation is represented in a canonical fashion using a certain orthonormal family of functions with respect to the L^2-function $\log \frac{1}{|p-q|}$ wtih respect to the product measure $\sigma \times \sigma$. Moreover, there exists a representation formula expressed by convergent series for the inhomogenous equation

$$g + \lambda \cdot \mathcal{K}_h(g) = f$$

where poles of $N_f(\lambda)$ are taken into the account.

BRA

§ 5. A Non-Linear PDE-equation

Introduction. In the article *Über eine nichtlineare Randwertaufgabe bei der Gleichung* $\Delta u = 0$ (Mathematisches Zeitschrift vol. 9 (1921), Carleman considered the following equation: Let Ω be a bounded domain in $\mathbf{R}^3$ with C^1-boundary and $\mathbf{R}^+$ the non-negative real line where t is the coordinate. Let $F(t,p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$t \mapsto F(t,p) \tag{0.1}$$

is strictly increasing for every $p \in \partial\Omega$ and that $F(0,p) \geq 0$. Moreover,

$$\lim_{u \to \infty} F(t,p) = +\infty \tag{0.2}$$

holds uniformly with respect to p. For a given point $q_* \in \Omega$ we seek a function $u(x)$ which is harmonic in $\Omega \setminus \{q_*\}$ and at q_* it is locally $\frac{1}{|x-q_*|}$ plus a harmonic function. Moreover, it is requested that u extends to a continuous function on $\partial\Omega$ and that $u \geq 0$ in $\overline{\Omega}$. Finally, along the boundary the inner normal derivative $\partial u/\partial n$ satisfies the equation

$$\frac{\partial u}{\partial n}(p) = F(u(p),p) \quad : p \in \partial\Omega \tag{*}$$

Remark. The case when $F(t,p) = kt^4$ for some positive constant k means that we regard the Stefan-Boltzmann equation whose physical interpretation ensures that (*) has a unique non-negative solution u.

Theorem. *For each F satisfying (0.1-0.2) the boundary value problem has a unique solution u.*

The strategy in the proof is to consider a family of boundary value problems where one for each $0 \leq h \leq 1$ seeks u_h to satisfy

$$\frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p),p) \quad : p \in \partial\Omega \tag{*}$$

and u_h has the same pole as u above. Let us begin with

0.1 The case $h = 0$. Here we seek u_0 so that

$$\frac{\partial u_0}{\partial n}(p) = u_0 \tag{i}$$

If $G(p)$ is the Greens' function with a pole at q_* we seek a harmonic function h in Ω such that

$$u_0 = G - h \tag{ii}$$

Since $G(p) = 0$ on $\partial\Omega$, the equation (i) holds if

$$\frac{\partial h}{\partial n}(p) = h(p) + \frac{\partial G}{\partial n}(p) \quad : \ p \in \partial\Omega \tag{iii}$$

This is a classic linear boundary value problem which has a unique solution h. See § xx for further details.

0.2 Properties of u_0. The construction in (ii) entails that u_0 is superharmonic in Ω and therefore attains its minimum on the boundary. Say that

$$u_0(p_*) = \min_{p \in \overline{\Omega}} u_0(p)$$

It follows that $\frac{\partial u_0}{\partial n}(p_*) \geq 0$ and the equation (i) gives

$$u_0(p_*) \geq 0$$

Hnece our unique solution u_0 is non-negative. We can say more. For consider the harmonic function h in (ii) which takes a maxium at some $p^* \in \partial\Omega$. Then $\frac{\partial h}{\partial n}(p^*) \leq 0$ so that (iii) gives

$$h(p^*) + \frac{\partial G}{\partial n}(p^*) \leq 0$$

Hence

$$\max_{p\in\partial\Omega} h(p) \leq -\frac{\partial G}{\partial n}(p^*)$$

which entails that

$$\min_{p\in\partial\Omega} u(p) = -\max_{p\in\partial\Omega} h(p) \geq \frac{\partial G}{\partial n}(p^*) \tag{0.2.1}$$

Here the function

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is continuous and positive on $\partial\Omega$ and if γ_* is the minimum value we conclude that

$$\min_{p\in\partial\Omega} u(p) \geq \gamma_* \tag{0.2.2}$$

Next, let h attain its minimum at some $p_* \in \partial\Omega$ which entails that $\frac{\partial h}{\partial n}(p_*) \geq 0$ and then (iii) gives

$$h(p_*) + \frac{\partial G}{\partial n}(p^*) \geq 0$$

It follows that

$$\max_{p\in\partial\Omega} u_0(p) = \min_{p\in\partial\Omega} h(p) = -h(p_*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^* \tag{0.2.3}$$

where

$$\gamma^* = \max_{p\in\partial\Omega} \frac{\partial G}{\partial n}(p) \tag{0.2.4}$$

So the unique solution u_0 in (i) satisfies

$$\gamma_* \leq u(p) \leq \gamma^* \quad : p \in \partial\Omega \tag{0.2.5}$$

where the positive constants γ_* and γ^* depend on the point $q_* \in \Omega$ and the given domain Ω.

The homotopy method. To proceed from $h = 0$ to $h = 1$ the idea is to use a "homotopy argument" which can be handled via precise estimates of solutions to Neumann's linear boundary value problem which are presented in § B. Thanks to this and some uniquness properties in § A below, the reduction to the case when F is real-analytic is relatively easy. The ciucial steps during the proof appear in § C where we carry out a "homotoy method" to get solutions in (*) as h increases from zero to one.

A.0. Proof of uniqueness.

Suppose that u_1 and u_2 are two solutions to the equation in the main theorem. Notice that $u_2 - u_1$. is harmonic in Ω. If $u_1 \neq u_2$ we may without loss of generality we may assume that the maximum of $u_2 - u_1$ is > 0. The maximum is attained at some $p_* \in \partial\Omega$ and the strict maximum principle for harmonic functions gives:

$$u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*) \tag{i}$$

for all $x \in \Omega$.With $v = u_2 - u_1$ we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p), p) - F(u_1(p), p)$$

Here (0.1) entails that $\frac{\partial v}{\partial n}(p_*) > 0$ and since we have an inner normal derivative this violates (i) which proves the uniqueness.

A.1 Montonic properties.

Let F_1 and F_2 be two functions which both satisfy (0.1) and (0.2) where

$$F_1(u, p) \leq F_2(u, p)$$

hold for all $(u,p) \in \mathbf{R}^+ \times \partial\Omega$. If u_1, respectively u_2 solve (*) for F_1 and F_2 it follows that $u_2(q) \leq u_1(q)$ for all $q \in \Omega$. To see this we set $v = u_2 - u_1$ which is harmonic in Ω. If $p \in \partial\Omega$ we get

$$\text{(i)} \qquad \frac{\partial v}{\partial n}(p) = F_2(u_2(p),p) - F_1(u_1(p),p) \geq 0$$

Suppose that the maximum of v is > 0 and let the maximum be attained at some point p_*. Since (i) is an inner normal it follows that we must have $0 = \frac{\partial v}{\partial n}(p)$ which would entail that

$$F_2(u_2(p_*)p_*) > F_2(u_1(p_*),p_*) \geq F_1(u_1(p_*),p_*) \implies$$

and this contradicts the strict inequality $u_2(p_*) > u_1(p_*)$ since we have an increasing function in (0.1).

A.2. A bound for the maximum norm. Let G be the Green's function which has a pole at q_* while $G = 0$ on $\partial\Omega$. Then

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is a continuous and positive function on $\partial\Omega$. Set

$$m_* = \min_{p\in\partial\Omega} \frac{\partial G}{\partial n}(p) \quad : m^* = \max_{p\in\partial\Omega} \frac{\partial G}{\partial n}(p)$$

Next, let $0 \leq h \leq 1$ and suppose that u_h is a solution to (*). Put

$$(*) \qquad m(h) = \min_{p\in\partial\Omega} u_h(p) \quad : M(h) = \max_{p\in\partial\Omega} u_h(p)$$

To estimate these numbers we proceed as follows. Choose $p^* \in \partial\Omega$ such that

$$(1) \qquad u_h(p^*) = M(h)$$

Now the function

$$H = u - G - M(h)$$

is harmonic function in Ω and non-negative on the boundary. Hence H is positive in Ω and since $H(p^*) = 0$ we have

$$\frac{\partial H}{\partial n}(p^*) \leq 0 \implies$$

which via the equation (*) give

$$(2) \qquad (1-h)M(h) + h \cdot F(M(h),p^*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^*$$

Next, the hypothesis on F entails that

$$(3) \qquad t \mapsto (1-h)t + h \cdot F(t,p^*)$$

is a strictly increasing function for each fixed $0 \leq h \leq 1$ and the hypothesis (0.2) together with the inequality (2) above, give a positive constant A^* which is independent of h such that

$$(3) \qquad M(h) \leq A^* \quad : 0 \leq h \leq 1$$

Next, let $m(h)$ be the minimum of u_h on $\partial\Omega$ and this time we consider the harmonic function

$$H = u - m(h) - G$$

Here $H \geq 0$ on $\partial\Omega$ and if $u_h(p_*) = m(h)$ we have $H(p_*) = 0$ p_* is a minimum for H. It follows that

$$\frac{\partial H}{\partial n}(p_*) \geq 0 \implies F(u(p_*),p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$\gamma_* = \min_{p\in\partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

$$F(m(h), p^*) \geq \gamma_* \tag{4}$$

Above γ^* is the constant from (xx) and the properties of F give a positive constant A_* such that

$$m(h) \geq A_*$$

Conclusion. Above $0 < A_* < A^*$ are constants which are independent of h. Hence the maxima and the minima of u_h stay in a fixed interval $[A_*, A^*]$ as soon as u_h exists.

B. The linear equation.

Let $f(p)$ and $W(p)$ be a pair of continuous functions on the boundary $\partial\Omega$ where W is positive, i.e. $W(p) > 0$ for every boundary point. Set

$$w_* = \min_p W(p)$$

So by the assumption on W we have $w_* > 0$. The classical Neumann theorem asserts that there exists a unique function U which is harmonic in Ω, extends to a continuous function on the closed domain and its inner normal derivative satisfies:

$$\partial U/\partial n(p) = W(p) \cdot U(p) + f(p) \quad p \in \partial\Omega \tag{1}$$

For the unique solution in (1) some estimates hold. Namely, set

$$M^* = \max_p U(p) \quad \text{and} \quad m_* = \min_p U(p)$$

Since U is harmonic in Ω the maximum and the minimum are both taken on the boundary. If $U(p^*) = M^*$ for some $p^* \in \partial\Omega$ we have $\partial U/\partial n(p^*) \leq 0$ which together with (1) entails that

$$M^* \cdot W(p^*) + f(p^*) \leq 0 \implies M^* \leq \frac{|f|_{\partial\Omega}}{w_*}$$

where $|f|_{\partial\Omega}$ is the maximum norm of f on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial\Omega}}{w_*}$$

Hence the following inequality holds for the the maximum norm $|U|_{\partial\Omega}$:

$$|U|_{\partial\Omega} \leq \frac{|f|_{\partial\Omega}}{w_*} \tag{B.0}$$

Notoice that (B.0) and the equation (1) entails that Suppose that $W \in C^0(\partial\Omega)$ satisfies

$$w_* \leq W(p) \leq w^*$$

for a pair of positive constants. If $|f|_{\partial\Omega}$ is the maximum norm of f it follows from (B.0) that

$$\big|W(p) \cdot U(p) + f(p)\big| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega}$$

Hence the equation (1) gives

$$\max_{p \in \partial\Omega} |\frac{\partial U}{\partial n}(p)| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega} \tag{B.1}$$

B.2 An estimate for first order derivatives. Let $p \in \partial\Omega$ and denote by N the inner normal at p. Since $\partial\Omega$ is of class C^1 a sufficiently small line segment from p along N stays in Ω. So for small positive ℓ we have points $q = p + \ell \cdot N$ in Ω and take the directional derivative of U along N_p. This gives a function

$$\ell \mapsto \partial U/\partial N(p + \ell \cdot N)$$

Since the boundary is C^1 these functions are defined on a fixed interval $0 \leq \ell \leq \ell^*$ for all boundsry points p. A classic result which appears in *Der zweite Randwertaufgabe* gives a constant B such that

$$\big|\partial U/\partial N(p + \ell \cdot N)\big| \leq B \cdot \max_{p \in \partial\Omega} |\frac{\partial U}{\partial n}(p)|$$

hold for all $p \in \partial\Omega$ and $0 \leq \ell \leq \ell^*$.

C. Proof of Theorem when $t \mapsto F(t,p)$ is analytic.

Assume that $t \mapsto F(t,p)$ is a real-analytic function on the positive real axis for each $p \in \partial\Omega$ where local power series converge uniformly with respect to p. In this situation we shall prove the *existence* of a solution u in the Theorem. To attain this we proceed as follows. To each real number $0 \leq h \leq 1$ we seek a solution u_h where

$$\frac{\partial u_h}{\partial n}(p) = h \cdot F(u_h, p) + (1-h) \cdot u_h(p) \tag{1}$$

When $h = 0$ we found the solution u_0 in § xx. Next, suppose that $0 \leq h_0 < 1$ and that we have found the solution u_{h_0} to (1). By the result in § B there exists a pair of positive constants $A_* < A^*$ such that

$$A_* \leq u_{h_0}(p) \leq A^* \tag{*}$$

which are independent of h_0 and of p.

Set $u_0 = u_{h_0}$ and with $h = h_0 + \alpha$ for some small $\alpha > 0$ we shall find u_h by a series

$$u_h = u_{h_0} + \sum_{\nu=1}^{\infty} \alpha^\nu \cdot u_\nu \tag{2}$$

The pole at q_* occurs already in u_0. So $u_1, u_2, \ldots$ is a sequence of harmonic functions in Ω and there remains to find them so that u_h solves (1). We will show that this can be achieved when α is sufficiently small. Keeping h_0 fixed we set

$$u_0 = u_{h_0}$$

The analyticity of F with respect to t gives for every $p \in \partial\Omega$ a series expansion

$$F(u_0(p) + \alpha, p) = F(u_0(p), p) + \sum_{k=1}^{\infty} c_k(p) \cdot \alpha^\nu \tag{3}$$

where $\{c_k(p)\}$ are continuous functions on $\partial\Omega$. Here (*) and the hypothesis on F entail that the radius of convergence has a uniform bound below, i.e. there exists $\rho > 0$ which is independent of $p \in \partial\Omega$ and a constant K such that

$$\sum_{k=1}^{\infty} |c_k(p)| \cdot \rho^k \leq K \tag{4}$$

Now the equation (1) can be solved via a system of equations where the harmonic functions $\{u_\nu\}$ are determined inductively while α-powers are identified. The linear α-term gives the equation

$$\frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1-h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p) \tag{i}$$

For u_2 we find that

$$\frac{\partial u_2}{\partial n} = (1-h_0)u_2 - u_1 + h_0 c_1(p) u_2 + c_1(p) u_1 + c_2(p) u_1^2 \tag{ii}$$

In general we have

$$\frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \ldots, u_{\nu-1}, p) \quad : \; \nu \geq 1 \tag{iii}$$

where $\{R_\nu\}$ are polynomials in the preeceding u-functions whose coefficients are continuous functions obtained from the c-functions. The function $c_1(p)$ is given by

$$c_1(p) = \frac{\partial F}{\partial t}(u_0(p), p)$$

which by the hypothesis on F is a positive continuous function on $\partial\Omega$. It follows that the function

$$\text{(iv)} \qquad W(p) = (1 - h_0) + h_0 \cdot c_1(p)$$

also is positive on $\partial\Omega$ and in the recursion above we have

$$\text{(v)} \qquad \frac{\partial u_\nu}{\partial n} = W(p) \cdot u_\nu(p) + R_\nu(u_0, \ldots, u_{\nu-1}, p) \quad : \nu = 1, 2, \ldots$$

Above we encounter linear equations exactly as in (B.0) where the the f-functions are the R-polynomials. Put

$$w_* = \min_{p \in \partial\Omega} W(p)$$

From § B.XX we get

$$\text{(vi)} \qquad |u_\nu|_{\partial\Omega} \leq w_*^{-1} \cdot |R_\nu(u_0, \ldots, u_{\nu-1}, p)|_{\partial\Omega}$$

Finally, (vi) and a majorising positive series expressing maximum norms imply that if α is sufficiently small then the series (2) converges and gives the requested solution for (1). Moreover, α can be taken *independently* of h_0. Together with the established uniqueness of solutions u_h whenever they exist, it follows that we can move from $h = 0$ until $h = 1$ and arrive at the requested solution in Theorem 1.

Remark. The reader may consult page 106 in [Carleman] where the existence of a uniform constant $\alpha > 0$ for which the series (2) converge for every h is demonstrated by an explicit majorant series.

§ 6. A uniqueness theorem for an elliptic boundary value problem

Introduction. We shall work in $\mathbf{R}^2$ with coordinates (x, y). Let $n = 2m$ be an even positive integer and consider two $n \times n$-matrices $\mathcal{A} = \{A_{pq}\}$ and $\mathcal{B} = \{B_{pq}\}$ whose elements are real-valued functions of x and y where the B-functions are continuous and the A-functions of class C^2. Eigenvalues of the $\mathcal{A}$-matrix when (x, y)-varies give an n-tuple of roots $\lambda_1(x, y), \ldots, \lambda_n(x, y)$ which solve

$$\det\big(\lambda \cdot E_n - \mathcal{A}(x, y)\big) = 0 \tag{1}$$

Next, we have a system of first order PDE-equations whose solutions are vector valued functions $(f_1, \ldots, f_n)$ defined in a half-disc

$$D_+(\rho) = \{x^2 + y^2 < \rho^2 \quad : \ x > 0\}$$

where the f-functions satisfy the system:

$$\frac{\partial f_p}{\partial x} + \sum_{q=1}^{q=n} A_{pq}(x, y) \cdot \frac{\partial f_p}{\partial y} + \sum_{q=1}^{q=n} B_{pq}(x, y) \cdot f_q(x, y) = 0 \tag{*}$$

together with the boundary conditions:

$$f_p(0, y) = 0 \quad \text{for all} \quad 1 \le p \le n \tag{**}$$

If the λ-roots are non-real in (1) we say that (*) is an elliptic system. When this holds one exocets that the vanishing Cauchy data in (**) entails that the solution f is identically zero. This uniqueness was proved by Erik Holmgren in the article [Holmgren] under the assumption that the A-functions and the B-functions are real analytic. The question remained if the uniqueness still holds under less regularity on the coefficient functions. An affirmative answer was proved by Carleman in the article *xxx*.

1. Theorem. *Assume that the λ-roots are all simple and non-real as (x, y) varies in the open half-disc. Then every solution f to (**) with vanishing Cauchy-data is identically zero.*

The proof requires several steps and the methods which occur below have inspired more recent work where Carleman estimates are used to handle boundary value problems in PDE-theory.

A. First part of the proof

The system in (*) is equivalent to a system of m-many equations where one seeks complex-valued functions $g_1, \ldots, g_m$ satisfying:

$$\frac{\partial g_p}{\partial x} + \sum_{q=1}^{q=m} \lambda_p(x, y) \cdot \frac{\partial g_p}{\partial y} =$$

$$\sum_{q=1}^{q=m} a_{pq}(x, y) \cdot g_q(x, y) + b_{pq}(x, y) \cdot \bar{g}_q(x, y) = 0 \ : \ 1 \le p \le m \tag{**}$$

Above $\{a_{pq}\}$ and $\{b_{pq}\}$ are complex-valued,and by the elliptic hyptheis the complex-valued λ-functions can be chosen so that their imaginary parts are positive functions of (x, y). The reduction of the originalsystem to to this complex family of equations is left to the reader. From now on we study the system (**) and Theorem 1 amounts to prove that if the g-functions satisfy (**) in a half-disc $D_+(\rho)$ and

$$g_p(0, y) = 0 \quad : \quad 1 \le p \le m$$

then there exists some $0 < \rho_* \le \rho$ such that the g-functions are identically zero in $D_+(\rho_*)$. To attain this we introduce domains as follows: For a pair $\alpha > 0$ and $\ell > 0$ we put

$$D_\ell(\alpha) = \{x + y^2 - \alpha x^2 < \ell^2\} \cap \{x > 0\} \tag{1}$$

Notice that the boundary

$$\partial D_\ell(\alpha) = \{0\} \times [-\ell, \ell] \cup \{x + y^2 - \alpha x^2 = \ell^2\}$$

Above α and ℓ are small so the the g-functions satisfy (**) in $D_\ell(\alpha)$. For each $t > 0$ we define the m-tuple of functions by

$$\phi_p(x,y) = g_p(x,y) \cdot e^{-t(x+y^2-\alpha x^2)} \tag{2}$$

Since the g-functions satisfy (**) one verifies easily that the ϕ-functions satisfy the system

$$\frac{\partial \phi_p}{\partial x} + \frac{\partial}{\partial y}\big(\lambda_p \cdot \phi_p\big) + t(1 - 2\alpha x + 2y\lambda_p\big)\cdot\phi_p = H_p(\phi) \tag{3}$$

where

$$H_p(\phi) = \sum_{q=1}^{q=n} a_{pq}(x,y) \cdot \phi_q(x,y) + b_{pq}(x,y) \cdot \bar{\phi}_q(x,y) = 0 \ : \ 1 \leq p \leq m$$

Next, we set

$$\Phi(x,y) = \sum_{p=1}^{p=m} |\phi_p(x,y)| \tag{4}$$

The crucial step in the proof of Theorem 1 is to establish the following inequality.

A.1 Proposition. *Provided that α from the start is sufficiently large there exists some $0 < \ell_* \leq \ell$ and a constant C which is independent of t such that*

$$\iint_{D_{\ell_*}} \Phi(x,y) \cdot dxdy \leq C \cdot \int_{T_{\ell_*}} \sum_{p=1}^{p=n} |\phi_p| \cdot |dy - \lambda_p \cdot dx|$$

How to deduce Theorem 1. Let us show why Proposition A.1 gives Teorem 1. In addition to ℓ_* we fix some $0 < \ell_{**} < \ell_*$. In (2) above we have used the function

$$w(x,y) = e^{-t(x+y^2-\alpha x^2)} \implies$$

$$\text{(i)} \qquad w(x,y) = e^{-t\ell_*^2} \quad : \quad \{x + y^2 - \alpha x^2 = \ell_*^2\} \quad : \quad w(x,y) \geq e^{-t\ell_{**}^2} \quad : \quad (x,y) \in D_{\ell_{**}}$$

Next, we have $|\phi_p| = |g_p| \cdot w$ for each p. Replacing the left hand side in Proposition A.1 by the area integral over the smaller domain $D_{\ell_{**}}$ we obtain the inequality;

$$\text{(ii)} \qquad \iint_{D_{\ell_{**}}} \sum_{p=1}^{p=m} |g_p(x,y)| \cdot dxdy \leq C \cdot e^{t(\ell_{**}^2 - \ell_*^2)} \cdot \int_{T_{\ell_*}} \sum_{p=1}^{p=n} |g_p| \cdot |dy - \lambda_p \cdot dx|$$

Here (ii) holds for every $t > 0$. When $t \to +\infty$ we have $e^{t(\ell_{**}^2 - \ell_*^2)} \to 0$ and conclude that

$$\iint_{D_{\ell_{**}}} \sum_{p=1}^{p=m} |g_p(x,y)| \cdot dxdy = 0$$

This means that the g-functions are all zero in $D_{\ell_{**}}$ and Theorem 1 follows.

B. Proof of Proposition A.1

The proof relies upon the construction of certain ψ-functions. More precisely, when $t > 0$ and a point $(x_*, y_*) \in D_\ell$ are given we shall construct an m-tuple of ψ-functions satisfying the following:

Condition 1. Each ψ_p is defined in the punctured domain $D_\ell \setminus \{(x_*, y_*)\}$ where ψ_p for a given $1 \leq p \leq m$ satisfies the equation

$$\text{(i)} \qquad \frac{\partial \psi}{\partial x} + \lambda_p \cdot \frac{\partial \psi}{\partial y} - t(1 - 2\alpha x + 2y\lambda_p)\psi_p = 0$$

Condition 2. For each p the singularity of ψ_p at (x_*, y_*) is such that the line integrals below have a limit:

$$\text{(ii)} \qquad \lim_{\epsilon \to 0} \int_{[z - z_*| = \epsilon} \psi_p \cdot (dx - \lambda_p \cdot dy) = 2\pi$$

Condition 3. There exists a constant K which is independent both of (x_*, y_*) and of t such that

$$\text{(iii)} \qquad |\psi_p(z)| \leq \frac{K}{|z - z_*|}$$

Notice that the ψ-functions depend on the parameter t, i.e. they are found for each t but the constant K in (3) is independent of t.

The deduction of Proposition A.1

Before the ψ-functions are constructed in Section C we show how they give Proposition A.1. Consider a point $z_* \in D_+(\ell)$. We get the associated ψ-functions from § B at this particular point. Remove a small disc γ_ϵ centered at z_* and consider some fixed $1 \leq p \leq m$. Now ϕ_p satisfies the differential equation (3) from section A and ψ_p satisfies (i) in Condition 1 above. Stokes theorem gives:

$$\int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) = \iint_{D_\ell \setminus \gamma_\epsilon} H_p(\phi) \cdot \psi_p \cdot dxdy + \int_{|z - z_*| = \epsilon} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big)$$

Passing to the limit as $\epsilon \to 0$, Condition 2 gives

$$\text{(1)} \qquad \phi_p(x_*, y_*) = \frac{1}{2\pi} \int_{T_\ell} \phi_p \cdot \psi_p \cdot \big(dy - \lambda_p \cdot dx\big) - \frac{1}{2\pi} \cdot \iint_{D_\ell} H_p(\phi) \cdot \psi_p \cdot dxdy$$

Let L be the maximum over D_ℓ of the coefficient functions of ϕ and $\bar{\phi}$ which appear in $H_p(\phi)$ from (3) i § A. We have also the constant K from Condition 3 for ψ_p. The triangle inequality gives:

$$\text{(*)} \qquad \big|\phi_p(x_*, y_*)\big| \leq \frac{K}{2\pi} \int_{T_\ell} \frac{|\phi_p| \cdot |dy - \lambda_p \cdot dx|}{z - z_*|} + \frac{LK}{\pi} \cdot \sum_{q=1}^{q=m} \iint_{D_\ell} \frac{|\phi_q|}{|z - z_*|} \cdot dxdy$$

Next, we use the elementary inequality

$$\text{(**)} \qquad \iint_\Omega \frac{dxdy}{\sqrt{x - a)^2 + (y - b)^2}} \leq 2 \cdot \sqrt{\pi} \cdot \sqrt{\text{Area}(\Omega)}$$

where Ω is an arbitrary bounded domain and $(a, b) \in \Omega$. Apply (**) with $\Omega = D_\ell$ and set $S = \text{area}(D_\ell)$. Integrating both sides in (*) over D_ℓ for every p and taking the sum we get

$$\iiint_{D_\ell} \Phi \cdot dxdy \leq$$

$$K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |\phi_p| \cdot |dy - \lambda_p \cdot dx| + 2\pi m L K \cdot \sqrt{\frac{S}{\pi}} \iint_{D_\ell} \Phi \cdot dxdy$$

This inequality hold for all small ℓ. Choose ℓ so small that

$$2\pi m L K \cdot \sqrt{\frac{S}{\pi}} \leq \frac{1}{2}$$

Then the inequality above gives

$$\text{(***)} \qquad \iiint_{D_\ell} \Phi \cdot dxdy \leq 2 \cdot K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |\phi_p| \cdot |dy - \lambda_p \cdot dx|$$

Finally, consider some relatively compact domain Δ in D_ℓ. Then there exists $0 < \ell_* < \ell$ such that

$$\Delta \subset D_{\ell_*}$$

Now we notice that

$$|\phi_p(z)| \geq e^{-t\ell_*^2} \cdot |u_p(z)| \quad : \quad z \in \Delta \quad : \quad |\phi_p(z)| \geq e^{-t\ell^2} \cdot |u_p(z)| \quad : \quad z \in T_\ell$$

We conclude that

$$(****) \qquad e^{-t\ell_*^2} \iiint_\Delta \sum_{p=1}^{p=m} |u_p(z)| \cdot dxdy \leq e^{-t\ell^2} \cdot 2 \cdot K \cdot \sqrt{\frac{S}{\pi}} \cdot \int_{T_\ell} \sum_{p=1}^{p=m} |u_p| \cdot |dy - \lambda_p \cdot dx|$$

Here (****) hold for every $t > 0$. Passing to the limit as $t \to +\infty$ it follows that

$$\cdot \iiint_\Delta \sum_{p=1}^{p=m} |u_p(z)| \cdot dxdy \leq$$

Since Δ was any relatively compact subset of D_ℓ, we conclude that the u-functions are zero in D_ℓ and Theorem 1 follows.

C. Construction of the ψ-functions.

Before we embark upon specific constructions we investigate the whole family of solutions to a first order differential operators of the form

$$(*) \qquad Q = \partial_x + \lambda(x, y) \cdot \partial_y$$

where $\lambda(x, y)$ is a complex valued C^2-function whose imaginary part is > 0. Set

$$\lambda(x, y) = \mu(x, y) + i \cdot \tau(x, y) \quad : \tau(x, y) > 0$$

Now we look for solutions $h(x, y)$ to the equation $Q(h) = 0$. With $h(x, y) = \xi(x, y) + i \cdot \eta(x, y)$ where ξ and η are real-valued C^2-functions this gives the differential system:

$$\frac{\partial \xi}{\partial x} + \mu_p \cdot \frac{\partial \xi}{\partial y} - \tau_p \cdot \frac{\partial \eta}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial x} + \mu_p \cdot \frac{\partial \eta}{\partial y} + \tau_p \cdot \frac{\partial \xi}{\partial y} = 0$$

Suppose we have found one solution $h = \xi + i \cdot \eta$ where the Jacobian $\xi_x \eta_y - \xi_y \eta_x$ is $\neq 0$ at the origin. Then $(x, y) \mapsto (\xi, \eta)$ is a local C^2-diffeomorphism. With $\zeta = \xi + i\eta$ we have the usual Cauchy-Riemann operator.

$$\frac{1}{2}(\frac{\partial}{\partial \xi} + i \cdot \frac{\partial}{\partial \eta})$$

Let $g(\xi + i\eta)$ be a holomorphic function in the complex ζ-space with $\zeta = \xi + i\eta$ and put

$$g_*(x, y) = g(\xi(x, y) + i\eta(x, y))$$

Then one easily verifies that $Q(g_*) = 0$ and conversely, every solution to this equation is expressed by a $g - *$function derived from an analytic function in the complex ζspace.satisfies $Q(g_*)$.

Conclusion. *If a non-degenerate solution $h = \xi + i\eta$ has been found then the homogenous solutions to Q is in a 1-1 correspondence to analytic functions in the ζ-variable.*

Remark. The effect of a coordinate transformation as above is that the Q-operator is transported to the Cauchy-Riemann operator in the complex ζ-space where $\zeta = \xi + i\eta$. Later we employ such (ξ, η)-transformations to construct solutions to an inhomogeneous equation of the form

$$Q(\psi) = (t - \alpha x + 2y\lambda(x, y)) \cdot \psi(t, x, y)$$

where t is a positive parameter and the ψ-functions will have certain specified properties. Notice that it suffices to construct the ψ-functions separately, i.e. we no longer have to bother about a differential system. With a fixed p fixed $\lambda_p(x, y) = \mu_p + \tau_p$ and from now on we may drop the index p and explain how to obain ψ-functions satisfying the three conditions from § B. So we consider the first order differential operator

$$(1) \qquad Q = \frac{\partial}{\partial x} + (\mu(x, y) + i\tau(x, y)) \cdot \frac{\partial}{\partial y}$$

where $\tau(x,y) > 0$.

C.1 A class of (ξ,η)-functions. Let $V(x,y)$ and $W(x,y)$ be two quadratic forms, i.e. both are homogeneous polynomials of degree two. Given a point (x_*,y_*) and with $z = x+iy$ we seek a coordinate transformation $(x,y) \mapsto (\xi,\eta)$ of the form:

$$\xi(z) = \tau_p(z_*) \cdot (x - x_*) + V(x - x_*, y - y_*) + \gamma_1(z) \cdot |z - z_*|^2$$

$$\eta(z) = (y - y_*) - \mu_p(z_*) \cdot (x - x_*) + W(x - x_*, y - y_*) + \gamma_2(z) \cdot |z - z_*|^2$$

Lemma. *There exists a pair of quadratic forms V and W whose coefficients depend on (x_*,y_*) and a pair of γ-functions which both vanish at (x_*,y_*) up to order one such that the complex-valued function $\xi + i\eta$ solves the homogeneous equation $Q(\xi + i\eta) = 0$.*

A solution above gives a change of variables so that Q is expressed in new real coordinates (ξ,η) by the operator

$$\frac{\partial}{\partial \xi} + i \cdot \frac{\partial}{\partial \eta} \tag{2}$$

There exist many coordinate transforms $(x,y) \to (\xi,\eta)$ which change Q into (2). This *flexible choice* of coordinate transforms is used to construct the required ψ-functions. Notice that Condition (2) in § B is of a pointwise character, i.e. it suffices to find a ψ-function for a given point $z_* = x_* + iy_*$. With this in mind the required construction in § B boils down to perform a suitable coordinate transformation adapated to z_*, and after use the existence of a ψ-function which to begin with is expressed in the (ξ,η)-variables where the Q-operator is replaced by the Cauchy-Riemann operator. In this special case the required ψ-function is easy to find, i.e. see the remark in § B.0. So all that remains is to exhibit suitable coordinate transformations which send Q to the $\bar{\partial}$-operator. We leave it to the reader to carry out such coordinate transformations. If necessary, consult Carleman's article where a very detailed construction appears.

§ 7. Propagation of sound

With (x, y, z) as space variables in $\mathbf{R}^3$ and a time variable t, the propagation of sound in the infinite open complement $U = \mathbf{R}^3 \setminus \overline{\Omega}$ of a bounded open subset Ω is governed by solutions $u(x, y, z, t)$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \tag{1}$$

where Δ is the Laplace operator in x, y, z. So here (1) holds when $p = (x, y, z) \in U$ and $t \geq 0$. We assume that $\partial\Omega$ is of class C^1, i.e. given as a union of pairwise disjoint closed surfaces of class C^1 along which normal vectors are defined. We seek solutions such that $p \mapsto u(p, t)$ belong to $L^2(U)$ for each t, and the outer normal derivatives taken along $\partial\Omega$ are zero, i.e. for every t

$$\frac{\partial u}{\partial n}(p, t) = 0 \quad : p \in \partial\Omega \tag{2}$$

Initial conditions are expressed by a pair of C^2-functions $f_1 x, y, z)$ and $f_2(x, y, z)$ defined in U such that f_1, f_2 together with $\Delta(f_1$ and $\Delta(f_2)$ belong to $L^2(U)$, and their outer normal derivatives along $\partial\Omega$ are zero. So here $u(p, 0) = f_0(p)$ and $\frac{\partial u}{\partial t}(p, 0) = f_1(p)$ hold for each $p \in U$.

Following materail in [carleman] we shall prove the existence and uniqueness using a spectral function θ attached to a Class I operator A constructed via solutions to the Dirichlet problem where no time variable appears. A merit of this proof is that it enable us to confirm the physically expected result expressed by:

$$\lim_{t \to \infty} (\frac{\partial u}{\partial x})^2(p, t) + (\frac{\partial u}{\partial y})^2(p, t) + (\frac{\partial u}{\partial z})^2(p, t) = 0 \tag{*}$$

with uniform convergence when p stays in a relatively compact subset of U. More precisely, Carleman proved (*) by proving that the spectral function of A is *absolutely continuous* with respect to the λ-parameter. Since this result has independent interest we announce it here in the introduction while the proof is given in § xx.

Let $\{a \leq s \leq b\}$ be a compact interval on the real s-line and $s \mapsto G_s$ is a function with values in the Hilbert space $L^2(U)$ which is continuous in the sense that

$$\lim_{s \to s_0} ||G_s - G_{s_0}||_2 = 0$$

hold for each s_0, where we introduced the L^2-norms. The function has a finite total variation if there exists a constant M such that

$$\sum ||G_{s_{\nu+1}} - G_{s_\nu}||_2 \leq M$$

hold for every partition $a = s_0 < s_1 < \ldots < s_M = b$. When this holds one construct Stieltjes integrals and for every subinterval $[\alpha, \beta]$ there exists the L^2-function in U

$$\Phi_{[\alpha,\beta]} = \int_\alpha^\beta s \cdot \frac{dG_s}{ds}$$

Impose the extra conditions that the normal deriviatives $\frac{\partial G_s}{\partial n}$ vanish on $\partial\Omega$ for every $a \leq s \leq b$ and the following differential equation holds for every sub-interval $[\alpha, \beta]$ of $[a, b]$:

$$\Delta(G_\beta - G_\alpha) + \Phi_{[\alpha,\beta]} = 0$$

0.1 Theorem. *The equations above imply that $s \mapsto G_s$ is absolutely continuous which means that whenever $\{\ell_1, \ldots, \ell_M\}$ a finite family of disjoint intervals in $[a, b]$ where the sum of their lengths is $< \delta$, then the sum of the total variations over these intervals is bounded by $\rho(\delta)$ where ρ is a function of δ which tends to zero as $\delta \to 0$.*

§ 1. Preliminary results.

Let U be as above and suppose that u is a real-valued C^2-function in this domain such that both u and $\Delta(u)$ belong to $L^2(/U)$. In addition we assume that the normal derivative of u along ∂U exists as a continuous function. Set

$$D^2(u) = \frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2 + \frac{\partial u}{\partial z}^2$$

We shall prove that

$$\text{(i)} \qquad \int_U D^2(u)\, dxdydz < \infty$$

To prove (ii) we consider large R such that Ω is contained in the ball $B(R)$ of radius R centered at the origin. Set $U(R) = U \cap B(R)$. Then Greens' formula gives the equation

$$\text{(i)} \qquad \int_{U(R)} u\Delta(u) + D^2(u) + \int_{\partial U} u\frac{\partial u}{\partial n}\, dA = \int_{\partial B(R)} u\frac{\partial u}{\partial n}\, dA$$

The Cauchy-Schwarz inequality and the hypothesis that u and $\Delta(u)$ belong to $L^2(U)$ imply that the first term above is bounded as a function of R. We conclude that (i) holds if there exists a sequence $\{r_k\}$ which tends to $+\infty$ such that

$$\text{(ii)} \qquad \lim \int_{\partial B(r_k)} u\frac{\partial u}{\partial n}\, dA = 0$$

To prove (i) we introduce the function

$$\psi(R) = \int_{U(R)} u^2$$

It follows that

$$\psi'(R) = \int_{\partial B(R)} u^2\, dA$$

and the reader can verify that the second order derivative becomes

$$\psi''(R) = \frac{2}{R}\psi'(R) + 2\int_{\partial B(R)} u\frac{\partial u}{\partial n}\, dA$$

Hence

$$\int_{\partial B(R)} u\frac{\partial u}{\partial n}\, dA = \frac{\psi''(R)}{2)} - \frac{\psi'(R)}{R}$$

Next, since u is in $L^2(U)$ the increasing function $\psi(R)$ is bounded and converges to a finite limit as $R \to \infty$ which entails that the positive function $\psi'(R)$ cannot stay $\geq a$ for a positive constant a. Now two possible cases can occur. The first is that $R \mapsto \psi'(R)$ decreases in a monotone fashion to zero. Then Rolle's mean-value theorem implies that there exists a sequence $\{r_k\}$ such that $\psi''(r_k) \to 0$ and (ii) follows.

The second case is that there exists a sequence $\{r_k\}$ where $\psi'(r_k)$ takes a local minimum which gives $\psi''(r_k) = 0$ and since the quotients $\frac{\psi'(r_k)}{r_k}$ also tend to zero we get (i) in this case also.

1.1 Exercise. Let u be a complex-valued function where u and $\delta(u)$ both belong to $L^2(U)$ and the normal derivative along ∂U is zero. Show that the result above gives the equation

$$\int \Delta(u)\bar{u} + \int |D(u)|^2 = 0$$

where we have put

$$|D(u)|^2 |\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2 + |\frac{\partial u}{\partial z}|^2$$

1.2 Application. Suppose that u is a function u in $L^2(U)$ such that

$$\Delta(u)+\lambda u=0$$

hold for some complex number λ and the normal derivative is zero along ∂U. The exercise gives the equality

$$-\lambda\cdot\int_U |u|^2+\int_U D(u)^2=0$$

This entails that λ must be real and positive. The next result requires more effort to prove.

1.3 Theorem. *Let u satisfy the equation in the exercise for some positive real number λ. Then u is identically sero.*

Proof. We shall work with polar coordinates, i.e. employ the Euler's angular variables ϕ and θ where

$$0<\theta \quad : 0<\phi<2\pi$$

The wellknown expression of Δ in the variables r,θ,ϕ shows that the equation (*) corresponds to

(i) $$xxxxx+xxxx=0 page 132$$

Let $n\geq 1$ and $Y_n(\theta,\phi)$ some spherical function of degree n with a normalised L^2-integral equal to one. For each r where $B(r)$ contains $\overline{\Omega}$ we set

(ii) $$Z(r)=\int_0^{2\pi}\int_0^{\pi} Y_n(\theta,\phi)\cdot f(r,\theta,\phi)\,\sin(\theta)\,d\theta d\phi$$

The Cauchy-Schwarz inequality gives

$$Z(r)^2\leq \int_0^{2\pi}\int_0^{\pi} Y_n^2\cdot\sin(\theta)\,d\theta d\phi\cdot\int_0^{2\pi}\int_0^{\pi} f^2(r,\theta,\phi)\cdot\sin(\theta)\,d\theta d\phi$$

Since the L^2-integral if Y_n is normalised the last product is reduced to

$$J(r)=\int_0^{2\pi}\int_0^{\pi} f^2(r,\theta,\phi)\cdot\sin(\theta)\,d\theta d\phi$$

Now

$$\int_{r_*}^{\infty} r^2\cdot J(r)\,dr$$

is equal to the finite L^2-integral of f in the exterior domain taken outside a ball $B(r_*)$. From (ii) we conclude that

(iii) $$\int_{r_*}^{\infty} r^2\cdot Z(r)^2\,dr<\infty$$

A differential equation. Recall that a spherical function of degree n satisfies

(iv) $$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\cdot\frac{\partial Y_n}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_n}{\partial\phi^2}+(n+1)n\cdot Y_n=0$$

Exercise. Show via suitable partial integrations that (i) and (iv) imply that $Z(r)$ satisfies the differential equation

(v) $$\frac{1}{r^2}\cdot\frac{d}{dr}(r^2\cdot\frac{dZ}{dr})+(\lambda-\frac{n(n+1)}{r^2})Z=0$$

The second order differential equation has two linearly independent solutions of the form

$$Z_1=\cos(\sqrt{\lambda}r)\cdot[\frac{1}{r}+\frac{a_2}{r^2}+\ldots]$$

$$Z_2=\sin(\sqrt{\lambda}r)\cdot[\frac{1}{r}+\frac{b_2}{r^2}+\ldots]$$

It follows that there exist a pair of constants c_1, c_2 such that

$$\text{(vi)} \qquad Z = c_1 Z_1 + c_2 Z_2 = \frac{c_1 \cdot \cos(\sqrt{\lambda} r) + c_2 \cdot \sin(\sqrt{\lambda} r)}{r} + \frac{B(r)}{r^2}$$

where $r \mapsto B(r)$ stays bounded as $r \to +\infty$.

Exercise. Show that the finite integral in (iii) and (vi) give $c_1 = c_2 = 0$ and hence $Z(r)$ identically zero for large r. Since this hold for all spherical functions we conclude that f is identically zero outside the closed ball $B(r_*)$. Finally, by assumption U is connected and the elliptic equation (*) implies that f is a real-analytic function in U, So the vanishing outside a large ball entails that f is identically zero in U which finishes the proof of Theorem 1.3.

§ 2. Proof of Theorem 0.1

Using similar methods as in the proof of Theorem xxx one reduces the proof to study functions $g(r;\mu)$ where $r \mapsto g(r;\mu)$ is a C^2-function and square integrable on the interval $[r_*, +\infty)$ for a given $r_* > 0$ while μ as above varies in $[a, b]$. Moreover one has

$$\max_{\mu} \int_{r_*}^{\infty} g(r;\mu)\, dr < \infty$$

Next, for each sub-interval $\ell = [\alpha, \beta]$ we set

$$\delta_\ell(g(r,\mu) = g(r,\beta) - g(r,\alpha)$$

With these notations we say that $g(r;\mu)$ is absolutely continuous with respect to μ if there to each $\epsilon > 0$ exists $\delta > 0$ such that

$$\sum \int_{r_*} |\delta_{\ell_\nu}((g(r,\mu)\big|^2 \cdot r^2\, dr < \epsilon$$

for every finite family of sub-intervals $\{\ell_\nu\}$ when the sum of their lengths is $< \delta$.

2.1 Theorem. *Assume in addition to the above that the equation below holds for each sub-interval ℓ*

$$\frac{1}{r^2}\frac{d}{dr}\big[\, r^2 \frac{d}{dr}(\delta_\ell(g(r,\mu)\big] - \frac{n(n+1)}{r^2} \cdot \delta_\ell(g(r,\mu) + \int_\ell \mu \cdot \frac{d}{d\mu}(g(r,\mu) = 0$$

Then $g(r,\mu)$ is absolutely continuous with respect to μ.

§ 3. Solution of the wave equation.

The result in § 1.2 imples that the symmetric and densely defined operator on $L^2(U)$ expressed by the Green's function for the Laplace operator in U gives a spectral function $\Theta(p,q;\lambda)$ defined for $\lambda \geq 0$ with the property that when f is a function in U such that both f and $\Delta(f)$ are square integrable and the normal derivative is zero on the boundary, then

$$f = \int_0^\infty \frac{d\Theta}{d\lambda}(f)$$

where the integral is absolutely convergent a s explained in § xx. Moreover, one has

$$\Delta(f) = \int_0^\infty \lambda \cdot \frac{d\Theta}{d\lambda}(f)$$

The solution $u(x,t)$ from the introduction with the prescribed initial conditions when $t = 0$ is therefore given by

$$u = \int_0^\infty \cos \sqrt{t\lambda} \cdot \frac{d\Theta}{d\lambda}(f_0) + \int_0^\infty \frac{\sin \sqrt{t\lambda}}{\sqrt{\lambda}} \cdot \frac{d\Theta}{d\lambda}(f_1)$$

This entails for example that

$$\frac{\partial u}{\partial x} = \int_0^\infty \cos \sqrt{t\lambda} \cdot \frac{d\Theta}{d\lambda}(\frac{\partial f_0}{\partial x}) + \int_0^\infty \frac{\sin \sqrt{t\lambda}}{\sqrt{\lambda}} \cdot \frac{d\Theta}{d\lambda}(\frac{\partial f_1}{\partial x})$$

Next, Theorem 0.1 entails that the spectral function is absolutely contoinuous with respect to λ, and then the Riemann-Lebesgue Theorem applies and proves the limit formula (*) from the introduction.

A hyperbolic boundary value equation,.

Introduction. Let x, s be coordinates in $\mathbf{R}^2$ and consider the rectangle

$$\square = \{(x,y) : \ 0 \leq x \leq \pi : \ 0 \leq s \leq s^*\}$$

for some $s^* > 0$. A continuous and real-valued function $g(x,s)$ in $\square$ is x-periodic if

$$g(0,s) = g(\pi, s) \quad : 0 \leq s \leq s^*$$

More generally, if $k \geq 1$ and $g(x,s)$ belongs to $C^k(\square)$ then it is x-periodic if

(i)
$$\partial_x^\nu(g(0,s)) = \partial_x^\nu(g(\pi,s) \quad : 0 \leq \nu \leq k$$

In particular we can consider real-valued C^∞-functions on $\square$ for which (i) hold for every $\nu \geq 0$. Let $a(x,s)$ and $b(x,s)$ be a pair real-valued C^∞-functions on $\square$ which are periodic in x. They give the PDE-operator

(*)
$$P = \partial_s - a \cdot \partial_x - b$$

0.1 A boundary value problem. Let $p \geq 1$ and $f(x)$ is a periodic function on $[0, \pi]$ which is p-times continuously differentiable. Now we seek $F(x,s) \in C^p(\square)$ which is x-periodic and satisfies $P(F) = 0$ in $\square$ and the initial condition

$$F(x,0) = f(x)$$

We are going to prove that this boundary value equation has a unique solution. The proof requires several steps and is not finished until § 4. We shall use Hilbert space methods. If $k \geq 2$ there exists the Hilbert space $\mathcal{H}^{(k)}$ which arises via the completetion of $C^k(\square)$ with respect to the sum of L^2-norms of derivatives up to order k of x-periodic C^∞-functions in $\square$. Sobolev's inquality gives

$$\mathcal{H}^{(k)} \subset C^{k-2}(\square) \quad : \ k \geq 2$$

Staying in the interval $\{0 \leq x \leq \pi\}$ we also have the Hilbert space $H^k[0,\pi]$ which is the completion of periodic C^∞-functions $f(x)$. For a fixed $k \geq 2$ we denote by $\mathcal{D}_k(P)$ the set of $f \in H^k[0,\pi]$ such that there exists $F \in \mathcal{H}^{[k)}$ where $P(F) = 0$ and $F(x,0) = f(x)$ on $[0,\pi]$.

In 1§ xx we prove the following Hilbert space version of the boundary problem.

0.2 Theorem. *For each $k \geq 2$ the equality $\mathcal{D}_k(P) = H^k[0,\pi]$ holds and the map $f \to F$ from $H^k[0,\pi]$ to $\mathcal{H}^{(k)}$ is bijective.*

About the proof. The material in § 1 is used to show that if $\mathcal{D}_k(P)$ is a dense subspace of $H^k[0,\pi]$, then equality holds. To prove the density we consider the the linear operator S_k which for each $f \in \mathcal{D}_k(P)$ associates the periodic function $x \mapsto F(x, s^*)$ on $[0,\pi]$. So here the domain of definition $\mathcal{D}(S_k) = \mathcal{D}(P_k)$. Again material in § 1 is used to prove that S_k is a bounded operator, i.e. there exists a constant C such that

$$||S_k(f)||_k \leq C \cdot ||f||_k \quad : f \in \mathcal{D}(S_k)$$

In § xx we prove that the density of $\mathcal{D}_k(P)$ follows from the following:

0.2.1 Proposition. *For each $k \geq 2$ there exists a positive number $\alpha(k)$ such that the range of $E - \alpha \cdot S_k$ contains all periodic C^∞-functions on $[0,\pi]$ when $\alpha < \alpha(k)$.*

0.3 A periodic equation. To prove of Proposition 0.2.1 we consider doubly periodic functions $g(x,s)$ defined in $\{0 \leq x \leq \pi\} \times \{0 \leq s \leq 2\pi\}$. When $k \geq 2$ we get the Hilbert space $\mathcal{H}^{(k)}$ after the completion of doubly periodic C^∞-functions with L^2-norms of derivatives up to order k. This time we consider a differential operator

$$P = \partial_s - a(x,s)\partial_x - b(x,s)$$

where a and b are doubly periodic C^∞-functions. Set

$$\mathcal{D}_k(P) = \{g \in \mathcal{H}^{(k)} : P(g) \in \mathcal{H}^{(k)}\}$$

In the product space $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$ we have the graphic set

$$\gamma_k = \{(g, P(g) : \ g \in C^\infty\}$$

where we always refer to doubly periodic C^∞-functions as above. The closure of γ_k is the graph of a closed and densely defined linear operator on $\mathcal{H}^{(k)}$ denoted by T_k. With these notations the following holds, which apart from its use during the proof of Theorem 0.2 has independent interest:

0.3.1 Theorem. *There exists a positive number $\lambda(k)$ such that*

$$\lambda \cdot E - T_k : \ \mathcal{D}(T_k) \to \mathcal{H}^{(k)}$$

are surjective fo every $\lambda > \lambda(k)$.

To prove this theorem we shall consider the closed and densely defined operator $\mathcal{T}_k$ on $\mathcal{H}^{(k)}$ where

$$\Gamma(\mathcal{T}_k) = \{(g, P(g) : \ g \in \mathcal{D}_k(P)\}$$

Since doubly periodic C^∞-functions belong to $\mathcal{D}_k(P)$ we have $\Gamma(T_k) \subset \Gamma(\mathcal{T}_k)$, i.e. $\mathcal{T}_k$ is an extension of T_k. Since T_k is densely defined this entails that the adjoint operators T_k^* and $\mathcal{T}_k^*$ are equal. A crucial step in the proof of Theorem 0.3.1 is the following:

0.3.2 Theorem. *One has the equality $\mathcal{D}(P) = \mathcal{D}(T_k^*)$ and there exists a desnely defined self-adjoint operator B_k such that*

$$T_k^* = -\mathcal{T}_k + B_k$$

Remark. As described above general facts about densely defined and closed linear operators on Hilbert spaces play an essential role. The subsequent proofs therefore offer an instructive lesson hos they are applied to "concrete" PDE-equations. In § xx we expose a result in higher dimension for symmetric hyperbolic systems where the proofs are verbatim the same, except for additional technical details.

§ 1. Differential inequalities and energy integrals.

Let $M(s)$ be a non-negative real-valued continuous function on a closed interval $[0, s^*]$. To each $0 \leq s < s^*$ we set

$$d_M^+(s) = \limsup_{\Delta s \to 0} \frac{M(s + \Delta s) - M(s)}{\Delta s}$$

where Δs are positive during the limit.

1.1 Proposition. *Let B be a real number such that $d_M^+(s) \leq B \cdot M(s)$ holds in $[0, s^*)$. Then*

$$M(s) \leq M(0) \cdot e^{Bs} \quad : 0 < s \leq s^*$$

The proof of this result is left as an exercise. The hint is to consider the function $N(s) = M(s)e^{-Bs}$ and show that $d_N^+(s) \leq 0$ for all s. Notice that B is an arbitrary real number, i.e. it may also be < 0. More generally, let $k(s)$ be a non-decreasing continuous function with $k(0) = 0$ and suppose that

$$d_M^+(s) \leq B \cdot M(s) + k(s) \quad : 0 \leq s < s^*$$

Now the reader may verify that

$$M(s) \leq M(0) \cdot e^{Bs} + \int_0^s k(t)\, dt \tag{1.1.1}$$

Next, consider the set $\square = [0, \pi] \times [0, s^*]$ as above. A C^1-function g is periodic with respect to x if g and the partial derivatives $\partial_s(g)$,$\partial_x(g)$ are periodic in x, i.e.

$$g(0, s) = g(\pi, s) \quad : 0 \leq s \leq s^*$$

and similarly for $\partial_x(g)$ and $\partial_s(g)$.

1.2 Theorem. *Let g be a periodic C^1-function which satisfies the PDE-equation*

$$\partial_s(g) = a \cdot \partial_x(g) + b \cdot g \tag{*}$$

in $\square$ where a and b are x-periodic real-valued continuous functions on $\square$. Set

$$M_g(s) = \max_x |g(x, s)| \quad : B = \max_{x,s} |b(x, s)|$$

Then one has the inequality

$$M_g(s) \leq M_g(0) \cdot e^{Bs}$$

Proof. Consider some $0 < s < s^*$ and let $\epsilon > 0$. Put

$$m^*(s) = \{x : g(x, s) = M_g(s)\}$$

The continuity of g entials that the function $M(s)$ is continuous and the sets $m^*(s)$ are compact. If $x^* \in m^*(s)$ the periodicity of the C^1-function $x \mapsto g(x, s)$ entails that $\partial_x(x^*, s) = 0$ and (*) gives

$$\partial_s(g)(x, s) = b(x, s)g(x, s) \quad : x \in m^*(s)$$

Next, let $\epsilon > 0$. We find an open neighborhood U of $m^*(s)$ such that

$$|\partial_x(g)(x, s)| \leq \epsilon \quad : x \in U$$

Now there exists $\delta > 0$ such that

$$|g(x, s)| \leq M(s) - 2\delta \quad : x \in [0, \pi] \setminus U$$

Continuity gives some $\rho > 0$ such that if $0 < \Delta s < \rho$ then the inequalities below hold:

$$|g(x, s + \Delta s)| \leq M(s) - \delta \quad : x \in [0, \pi] \setminus U \quad : M(s + \Delta s) > M(s) - \delta \tag{i}$$

$$M(s + \Delta s) \leq M(s) + \epsilon \quad : |\partial_x(g)(x, s + \Delta s)| \leq 2\epsilon \quad : x \in m^*(s) \tag{ii}$$

If $0 < \Delta s < \rho$ we see that (i) gives $x \in m^*(s+\Delta s) \subset U$ and for such x-values Rolle's mean-value theorem and the PDE-equation give

$$M_g(x, s+\Delta s) - g(x,s) = \Delta s \cdot \partial_s(g(x, s+\theta \cdot \Delta s) =$$

$$\text{(iii)} \qquad \Delta s \cdot \big[a(x, s+\Delta s) \cdot \partial_x(g)(x+\theta \cdot \Delta s) + b(x, s+\Delta s) \cdot g(x, s+\theta \cdot \Delta s)\big]$$

Let A be the maximum norm of $|a(x,s)|$ taken over $\Box$. Since $|g(x,s)| \leq M(s)$ the triangle inequality and (iii) give

$$M(s+\Delta s) \leq M(s) + \Delta s[\cdot A \cdot 2\epsilon + B \cdot M(s+\theta \cdot \Delta s)]$$

Since the function $s \mapsto M(s)$ is continuous it follows that

$$\limsup_{\Delta s \to 0} \frac{M(s+\Delta s) - M(s)}{\Delta s} \leq A \cdot 2\epsilon + BM(s)$$

Above ϵ can be arbitrary small and hence

$$d^+(s) \leq B \cdot M(s)$$

Then Proposition 1.1 gives (*) in the theorem.

1.3 L^2-inequalities. Let $g(x,s)$ be a C^1-function satisfying (*) in Theorem 1.2. Set

$$J_g(s) = \int_0^\pi g^2(x,s)\, dx$$

Taking the s-derivative we obtain with respect to s and (*) give

$$\frac{dJ_g}{ds} = 2 \cdot \int_0^\pi g \cdot \partial_s(g)\, ds = 2 \cdot \int_0^\pi (a\partial_x(g) \cdot \partial g + b \cdot g)\, dx$$

The periodicity of g with respect to x gives $\int_0^\pi \partial_x(ag^2)\, dx = 0$. This entails that the right hand side becomes

$$\int_0^\pi (-\partial_x(a) + b) \cdot g^2\, dx$$

So if K is the maximum norm of $-\partial_x(a) + b$ over $\Box$ it follows that

$$\frac{dJ_g}{ds}(s) \leq K \cdot J_g(s)$$

Hence Theorem 1.2 gives

$$\text{(1.3.1)} \qquad \int_0^\pi g^2(x,s)\, dx \leq e^{Ks} \cdot \int_0^\pi g^2(x,0)\, dx \quad : 0 < s \leq s^*$$

Integration with respect to s entails that

$$\text{(1.3.2)} \qquad \iint_\Box g^2(x,s)\, dxds \leq \int_0^{s^*} e^{Ks}\, ds \cdot \int_0^\pi g^2(x,0)\, dx$$

Thus, the L^2-integral of $x \to g(x,0)$ majorizes both the area integral and each slice integral when $0 < s \leq s^*$.

§ 2. A boundary value equation

Let $a(x,s)$ and $b(x,s)$ be real-valued C^∞-functions on $\Box$ which are periodic in x and consider the PDE-operator

$$P = \partial_s - a \cdot \partial_x - b$$

2.1 Theorem. *For every positive integer p and each periodic $f \in C^p[0,\pi]$ there exists a unique periodic $g \in C^p(\Box)$ where $P(g) = 0$ and $g(x,0) = f(x)$.*

The uniqueness follows from the results in § 1. For if g and h are solutions in Theorem 2.1 then $\phi = g - h$ satisfies $P(\phi) = 0$. Here $\phi(x,0) = 0$ which gives $\phi = 0$ in $\Box$ via (1.3.2). The proof of existence requires several steps and employs Hilbert space methods. So first we introduce certain Hilbert spaces.

2.2 The space $\mathcal{H}^{(k)}$. To each integer $k \geq 2$ the complex Hilbert space $\mathcal{H}^{(k)}$ is the completion of complex-valued C^k-functions on $\square$ which are periodic with respect to x. A trivial Sobolev inequality entails that every function in $\mathcal{H}^{(2)}$ is continuous, and more generally

$$\mathcal{H}^{(k)} \subset C^{k-2}(\square) \quad : k \geq 3$$

and it clear that the first order PDE-operator P maps $\mathcal{H}^{(k+1)}$ into $\mathcal{H}^{(k)}$. Next, on the periodic x-interval $[0, \pi]$ we have the Hilbert spaces $H^k[0, \pi]$ for each $k \geq 2$.

2.3 Definition. *For each integer $k \geq 2$ we denote by $\mathcal{D}_k(P)$ the family of all $f(x) \in H^k[0, \pi]$ for which there exists some $F(x, s) \in \mathcal{H}^{(k)}$ such that*

$$(*) \qquad P(F) = 0 \quad : F(x, 0) = f(x)$$

The results in § 1 show that F is uniquely determined by (*). Moreover, there exists a constant C_k which only depends upon the C^∞-functions a and b and the given integer k such that

$$(2.3.1) \qquad ||F||_k \leq C_k \cdot ||f||_k$$

where we have taken norms in $\mathcal{H}^{(k)}$ and $H^k[0, \pi]$ respectively. Next, the last inequality in (1.3.2) shows that C_k can be chosen such that

$$(2.3.3) \qquad ||f^*||_k \leq C_k \cdot ||f||_k$$

where $f^*(x) = F(x, s^*)$ belongs to $H^k[0, \pi]$.

2.4 A density principle Above we introduced the space $\mathcal{D}_k(P)$. Now the following hold:

2.4.1 Proposition. *If $\mathcal{D}_k(P)$ is dense in $\mathcal{H}^k[0, \pi]$, then one has the equality*

$$(2.4.1) \qquad \mathcal{D}_k(P) = \mathcal{H}^k[0, \pi]$$

Proof. Suppose that $\mathcal{D}_k(P)$ is dense. So if $f \in \mathcal{H}^k[0, \pi]$ there exists a sequence $\{f_n\}$ in $\mathcal{D}_k(P)$ where $||f_n - g||_k \to 0$. By (2.2.2) we have

$$||F_n - F_m||_k \leq C||f_n - f_m||_k$$

Hence $\{F_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}^{(k)}$ and converges to a limit F. Since each $P(F_n) = 0$ it follows that $P(F) = 0$ and it is clear that the continuous boundary value function $F(x, 0)$ is equal to $f(x)$ which entails that f belongs to $\mathcal{D}_k(P)$.

2.5 The operators S_k. Each $f \in \mathcal{D}_k(P)$ gives the function $f^*(x) = F(x, s^*)$ in $\mathcal{H}^k[0, \pi]$ and set

$$S_k(f) = f^*(x)$$

So the domain of definition of S_k is equal to $\mathcal{D}_k(P)$ and (2.3.3) gives a constant M_k such that

$$||S_k(f)|| \leq M_k \cdot ||f||_k \quad : f \in \mathcal{D}_k(P)$$

where M_k only depends on the integer k and the given PDE-operator P. The next result constitutes a crucial point to attain Theorem 2.1.

2.6 Proposition. *For each $k \geq 2$ there exists some $\alpha(k) < 0$ such that for every $0 < \alpha < \alpha(k)$ the range of the operator $E - \alpha \cdot S_k$ contains all periodic C^∞-functions on $[0, \pi]$.*

2.7 The density of $\mathcal{D}_k(P)$. We prove Proposition 2.6 in § xx and proceed to show that it gives the density of $\mathcal{D}_k(P)$. For if $\mathcal{D}_k(P)$ fails to be dense there exists a non-zero $f_0 \in \mathcal{D}_k(P)$ which is $\perp$ to $\mathcal{D}_k(P)$. In Proposition 2.6 we choose $0 < \alpha \leq \alpha(k)$ so small that

$$\text{(i)} \qquad \alpha < M_k/2$$

Since periodic C^∞-functions are dense in $\mathcal{H}^k[0, \pi]$, Proposition 2.6 gives a sequence $\{h_n\}$ in $\mathcal{D}_k(P)$ such that

$$\text{(ii)} \qquad \lim_{n \to \infty} ||h_n - \alpha \cdot S_k(h_n) - f_0||_k \to 0$$

It follows that

(iii) $$\langle f_0, f_0\rangle = 1 = \lim \langle f_0, h_n - \alpha \cdot S_k(h_n)\rangle = -\alpha \cdot \lim \langle f_0, S_k(h_n)\rangle$$

Next, the triangle inequality and (ii) give

(iv) $$||h_n||_k \leq 1 + \alpha \cdot ||(S_k(h_n)|| \leq 1 + 1/2 \cdot ||h_n|| \implies ||h_n||_k \leq 2$$

Finally, by the Cauchy-Schwarz inequality the absolute value in the right hand side of (iii) is majorized by

$$\alpha \cdot M_K \cdot 2 < 1$$

which contradicts (iii). Hence the orthogonal complement of $\mathcal{D}_k(P)$ is zero which proves the requested density.

Together with Propostion 2.4.1 we get the following conclusive result:

2.8 Theorem. *For each $k \geq 2$ and $f(x) \in \mathcal{H}^k[0, \pi]$ there exists a unique function $F(x, s) \in \mathcal{H}^{(k)}$ such that (*) holds in Definition 2.3.*

2.9 Remark. The result above soplves the requested boundsry valued problem in $\mathcal{H}^{(k)}$ -spaces. Using Sobolev inequalities oner easily derives Theorem 2.1.

§ 3. A doubly periodic class of inhomogeneous PDE-equations.

Before Theorem 3.2 is announced we introduce some notations. Put

$$\square = \{0 \leq x \leq \pi\} \times \{0 \leq s \leq 2\pi\}$$

In this section we shall consider doubly periodic functions $g(x,s)$ on $\square$, i.e.

$$g(\pi, s) = g(0, s) \quad : g(x, 0) = g(x, 2\pi)$$

For each non-negative integer k we denote by $C^k(\square)$ the space of k-times doubly periodic continuously differentiable functions. If $g \in C^k(\square)$ we set

$$||g||^2_{(k)} = \sum_{j,\nu} \int_\square \Big|\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}(x,s)\Big|^2 \, dxds$$

with the double sum extended pairs $j + \nu \leq k$. This gives the complex Hilbert space $\mathcal{H}^{(k)}$ after a completion of $C^k(\square)$ with respect to the norm above. Recall that a Sobolev inequality entails that a function $g \in \mathcal{H}^{(2)}$ is automatically continuous and doubly periodic on the closed square. More generally, if $k \geq 3$ each $g \in \mathcal{H}^{(k)}$ has continuous and doubly periodic derivatives up to order $k - 2$. Next, consider a first order PDE-operator

$$P = \partial_s - a(x,s)\partial_x - b(x,s) \tag{3.1}$$

where a and b are real-valued doubly periodic C^∞-functions. It is clear that P maps $\mathcal{H}^{(k)}$ into $\mathcal{H}^{(k+1)}$ for every $k \geq 2$. Keeping $k \geq 2$ fixed we set

$$\mathcal{D}_k(P) = \{g \in \mathcal{H}^{(k)} : \ P(g) \in \mathcal{H}^{(k)}\}$$

Since $C^\infty(\square)$ is dense in $\mathcal{H}^{(k)}$ this yields a densely defined operator

$$P \colon \mathcal{D}_k(P) \to \mathcal{H}^{(k)} \tag{i}$$

In $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$ we get the graph

$$\Gamma_k = \{(g, P(g) \colon \ g \in \mathcal{D}_k(P)\}$$

Since P is a differential operator we know from general results that Γ_k is a closed subspace. Hence there exists a densely defined linear operator and closed operator on $\mathcal{H}^{(k)}$ which we denote by $\mathcal{T}_k$. So here $\mathcal{D}(\mathcal{T}_k) = \mathcal{D}_k$. Set

$$\gamma_k = \{(g, P(g) \colon \ g \in C^\infty(\square)\} \tag{ii}$$

This is a subspace of Γ_k and denote by $\overline{\gamma}_k$ its closure taken in $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$. Since Γ_k is closed we have

$$\overline{\gamma}_k \subset \Gamma_k$$

We get the densely defined linear operator T_k whose graph is $\overline{\gamma}_k$. By this construction $\mathcal{T}_k$ is an extension of T_k which in particular gives the inclusion

$$\mathcal{D}(T_k) \subset \mathcal{D}(\mathcal{T}_k) \tag{iii}$$

Next, let E be the identity operator on $\mathcal{H}^{(k)}$. With these notations we shall prove:

3.2 Theorem. *For each integer $k \geq 2$ there exists a positive real number $\rho(k)$ such that the map*

$$T_k - \lambda \cdot E \colon \mathcal{H}^{(k)} \to \mathcal{H}^{(k)}$$

is bijective for every $\lambda > \rho(k)$.

The proof requires several steps and is not finished until § 3.x. First we shall study the adjoint operator T_k^* and establish the following:

3.3 Proposition. *One has the equality $\mathcal{D}(T_k^*) = \mathcal{D}_k(P)$ and there exists a bounded self-adjoint operator B_k on $\mathcal{H}^{(k)}$ such that*

$$T_k^* = -\mathcal{T}_k + B_k$$

Proof of Proposition 3.3 Keeping $k \geq 2$ fixed we set $\mathcal{H} = \mathcal{H}^{(k)}$. For each pair g, f in $\mathcal{H}$ their inner product is defined by

$$\langle f, g\rangle = \sum \int_\square \frac{\partial^{j+\nu} f}{\partial x^j \partial s^\nu}(x,s) \cdot \overline{\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}}(x,s)\, dxds$$

where the sum is taken when $j + \nu \leq k$. Introduce the differential operator

$$\Gamma = \sum_{j+\nu \leq k} (-1)^{j+\nu} \cdot \partial_x^{2j} \cdot \partial_s^{2\nu}$$

Partial integration gives

$$\text{(i)} \qquad \langle f, g\rangle = \int_\square f \cdot \Gamma(\bar{g})\, dxds = \int_\square \Gamma(f) \cdot \bar{g}\, dxds \quad : \ f, g \in C^\infty$$

Now we consider the operator $P = \partial_s - a \cdot \partial_x - b$ and get

$$\text{(ii)} \qquad \langle P(f), g\rangle = \int_\square P(f) \cdot \Gamma(\bar{g})\, dxds$$

Partial integration identifies (ii) with

$$\text{(iii)} \qquad -\int_\square f \cdot \big(\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma(\bar{g})\, dxds$$

1.1 Exercise. In (iii) appears the composed differential operator

$$\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma$$

Show that in the ring of differential operators with C^∞-coefficients this differential operator can be written in the form

$$\Gamma \circ (\partial_s - a \cdot \partial_x - b) + Q(x, s, \partial_x, \partial_s)$$

where Q is a differential of order $\leq 2k$ with coefficients in $C^\infty(\square)$. Conclude from the above that

$$\text{(1.1.1)} \qquad \langle Pf, g\rangle = -\langle f, Pg\rangle + \int_\square f \cdot Q(\bar{g})\, dxds$$

1.2 Exercise. With Q as above we have a bilinear form which sends a pair f, g in $C^\infty(\square)$ to

$$\text{(1.2.1)} \qquad \int_\square f \cdot Q(\bar{g})\, dxds$$

Use partial integration and the Cauchy-Schwarz inequelity to show that there exists a conatant C which depends on Q only such that the absolute value of (1.2.1) is majorized by $C_Q \cdot ||f||_k \cdot ||g||_k$. Conclude that there exists a bounded linear operator B_k on $\mathcal{H}$ such that

$$\text{(1.2.2)} \qquad \langle f, B_k(g)\rangle = \int_\square f \cdot Q(\bar{g})\, dxds$$

1.3 Proof that B_k is self-adjoint From the above we have

$$\text{(1.3.1)} \qquad \langle Pf, g\rangle = -\langle f, Pg\rangle + \langle f, B_k(g)\rangle$$

Keeping f in $C^\infty(\square)$ we notice that $\langle f, B_k(g)\rangle$ is defined for every $g \in \mathcal{H}$. From this the reader can check that (1.3.1) remains valid when g belongs to $\mathcal{D}(\mathcal{T}_k)$ which means that

$$\text{(1.3.2)} \qquad \langle Pf, g\rangle = -\langle f, \mathcal{T}_k g\rangle + \langle f, B_k(g)\rangle \quad : f \in C^\infty(\square)$$

Moreover, when both f and g belong to $C^\infty(\square)$ we can reverse their positions in (*) which gives

$$\text{(1.3.3)} \qquad \langle Pg, f\rangle = -\langle g, Pf\rangle + \langle g, B_k(f)\rangle$$

Since a and b are real-valued it is clear that

$$\text{(1.3.4)} \qquad \langle Pg, f\rangle = -\langle f, Pg\rangle$$

It follows that

$$\text{(1.3.5)} \qquad \langle f, B_k(g) = \langle g, B_k(f) \quad : f, g \in C^\infty(\square)$$

Since this hold for all pairs of C^∞-functions and B_k is a bounded linear operator on $\mathcal{H}$ the density of $C^\infty(\Box)$ entails that B_k is a bounded self-adjoint operator on $\mathcal{H}$.

1.4 The equality $\mathcal{D}(T_k^*) = \mathcal{D}_k(P)$. The density of $C^\infty(\Box)$ in $\mathcal{H}$ entails that a function $g \in \mathcal{H}$ belongs to $\mathcal{D}(T_k^*)$ if and only if there exists a constant C such that

$$|\langle Pf, g\rangle| \leq C \cdot ||f|| \quad : f \in C^\infty(\Box) \tag{1.4.1}$$

Since B_k is a bounded operator, (1.3.2) gives the inclusion

$$\mathcal{D}_k(P) \subset \mathcal{D}(T_k^*) \tag{1.4.2}$$

To prove the opposite inclusion we use that the Γ-operator is elliptic. If $g \in \mathcal{D}(T_k^*)$ we have from (i) in § 1.1:

$$\langle Pf, g\rangle = \langle f, T_k^* g\rangle = \int \Gamma(f) \cdot \overline{T_k^*(g)}\, dxds \quad : f \in C^\infty(\Box)$$

Similarly

$$\langle f, B_k(g)\rangle = \int \Gamma(f) \cdot \overline{B_k(g)}\, dxds$$

Treating $\mathcal{T}_k(g)$ as a distribution the equation (1.3.2) entails that the elliptic operator Γ annihilates $T_k^*(g) - \mathcal{T}_k(g) + B_k(g)$. Since both $T_k^*(g)$ and $B_k(g)$ belong to $\mathcal{H}$ this implies by the general result in § xx that $\mathcal{T}_k(g)$ belongs to $\mathcal{H}$ which proves the requested equality (1.4) and at the same time the operator equation

$$T_k^* = -\mathcal{T}_k(g) + B_k \tag{1.4.3}$$

3.4 An inequality.

Let $f \in C^\infty(\Box)$ and λ is a positive real number. Then

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 =$$

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f)||^2 + \lambda^2 \cdot ||f||^2 - \lambda\big(\langle \mathcal{T}_k(f) - \frac{1}{2}B_k(f), f\rangle + \langle f, \mathcal{T}_k(f) - \frac{1}{2}B_k(f)\rangle\big)$$

The last term is λ times

$$\langle \mathcal{T}_k(f), f\rangle + \langle f, \mathcal{T}_k(f)\rangle - \langle f, B_k f\rangle \tag{i}$$

where we used that B_k is symmetric. Now $T_k = \mathcal{T}_k$ holds on $C^\infty(\Box)$ and the definition of adjoint operators give

$$\langle \mathcal{T}_k(f), f\rangle = \langle f, T_k^*\rangle \tag{ii}$$

Then (1.4.3) implies that (i) is zero and hence we have proved

$$||T_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 = \lambda^2 \cdot ||f||^2 + ||T_k(f) - \frac{1}{2}B_k(f)||^2 \geq \lambda^2 \cdot ||f||^2 \tag{iii}$$

From (iii) and the triangle inequality for norms we obtain

$$||T_k(f) - \lambda \cdot f|| \geq \lambda \cdot ||f|| - \frac{1}{2}||B_k(f)|| \tag{iv}$$

Now B_k has a finite operator norm and if $\lambda \geq ||B_k||$ we see that

$$||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f|| \tag{v}$$

Finally, since $C^\infty(\Box)$ is dense in $\mathcal{D}(T_k)$ it is clear that (v) gives

3.41 Proposition. *One has the inequality*

$$||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f|| \quad : f \in \mathcal{D}(T_k) \tag{3.4.1}$$

§ 3.5. Proof of Theorem 3.2

Suppose we have found some $\lambda^* \geq \frac{1}{2} \cdot ||B||$ such that $T_k - \lambda$ has a dense range in $\mathcal{H}$ for every $\lambda \geq \lambda^*$. If this is so we fix $\lambda \geq \lambda^*$ and take some $g \in \mathcal{H}$. The hypothesis gives a sequence $\{f_n \in \mathcal{D}(T_k)\}$ such that

$$\lim_{n\to\infty} ||T_k(f_n) - \lambda \cdot f_n - g|| = 0$$

In particular $\{T_k(f_n) - \lambda \cdot f_n\}$ is a Cauchy sequence in $\mathcal{H}$ and (1.5.x) implies that $\{f_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$ and hence converges to a limit f_*. Since the operator T_k is closed we conclude that $f_* \in \mathcal{D}(T_k)$ and we get the equality

$$T_k(f_*) - \lambda \cdot f_* = g$$

Since $g \in \mathcal{H}$ was arbitrary we have proved Theorem 3.2.

3.5.1 Density of the range. There remains to find λ^* as above. By the construction of adjoint operators, the range of $T_k - \lambda \cdot E$ fails to be dense if and ony if $T_k^* - \lambda$ has a non-zero kernel. So assume that

$$T_k^*(f) - \lambda \cdot f = 0$$

for some $f \in \mathcal{D}(T_k^*)$ which is not identically zero. Notice that T_k sends real-valued functions into real-valued functions. So above we can assume that f is real-valued and normalised so that

(i) $$\int_\square f^2(x,s)\,dxds = 1$$

From (i) and Proposition 3.3 we have

(ii) $$\mathcal{T}_k(f) + \lambda \cdot f - B(f) = 0$$

Let us consider the function

$$V(s) = \int_0^\pi f^2(x,s)\,dx$$

Since $k \geq 2$ is assumed we recall that the $\mathcal{H}$-function f is of class C^1 at least. The s-derivative of $V(s)$ becomes:

(iii) $$\frac{1}{2} \cdot V'(s) = \int_0^\pi f \cdot \frac{\partial f}{\partial s}\,dx$$

By (ii) we have

$$\frac{\partial f}{\partial s} - a(x)\frac{\partial f}{\partial x} - b \cdot f = B(f) - \lambda \cdot f$$

Hence the right hand side in (iii) becomes

$$-\lambda \cdot V(s) + \int_0^\pi f(x,s) \cdot B(f)(x,s)\,dx + \int_0^\pi a(x,s) \cdot f(x,s) \cdot \frac{\partial f}{\partial x}(x,s)\,dx$$

By partial integration the last term is equal to

(iv) $$-\frac{1}{2}\int_0^\pi \partial_x(a)(x,s) \cdot f^2(x,s)\,dx$$

Set

$$M = \frac{1}{2} \cdot \max_{(x,s)\in\square} |\partial_x(a)(x,s)|$$

From the above we get the inequality

(v) $$\frac{1}{2} \cdot V'(s) \leq (M - \lambda) \cdot V(s) + \int_0^\pi f(x,s) \cdot B(f)(x,s)\,dx$$

Set

$$\Phi(s) = \int_0^\pi |f(x,s)| \cdot |B(f)(x,s)|\,dx$$

Since the L^2-norm of f is one the Cauchy-Schwarz inequality gives

$$\int_{-\pi}^{\pi} \Phi(s)\,ds \leq \sqrt{\int_{\square} |B(f)(x,s)|^2\,dxds} \leq ||B(f)||$$

where the last equality follows since the squared integral of $B(f)$ is majorized by its squared norm in $\mathcal{H}$. When $\lambda > M$ it follows from (v) that

(vi)
$$(\lambda - M)\cdot V(s) + \frac{1}{2}\cdot V'(s) \leq \Phi(s)$$

Next, since f is double periodic we have $V(-\pi) = V(\pi)$ so after an integration (vi) gives

(vii)
$$(\lambda - M)\cdot \int_{\pi}^{\pi} V(s)\,ds = \int_{-\pi}^{\pi} \Phi(s)\,ds \leq ||B(f)||$$

Finally, thr normalisation (i) gives $\int_{\pi}^{\pi} V(s)\,ds = 1$ and then (vii) cannot hold if

$$\lambda > M + ||B(f)||$$

Remark. Set

$$\tau = \min_f\, ||B(f)||$$

with the minimum taken over funtions $f \in \mathcal{D}(T_0^*)$ whose L^2-integral is normalised by (i) above. The proof has shown that the kernel of $T_0^* - \lambda$ is zero for all $\lambda > M + \tau$.

A special solution.

Let $f(x)$ be a periodic C^∞-function on $[0,\pi]$. Put

$$Q = a(x,s)\cdot\frac{\partial}{\partial x} + b(x,s)$$

Let $\eta(s)$ be a C^∞-function of s and m some positive integer If $\lambda > 0$ is a real number. we set

$$\text{(i)} \qquad g_\lambda(x,s) = \eta(s)\cdot f + \eta(s)\cdot\sum_{j=1}^{j=m}\frac{(s-\pi)^j}{j!}\cdot(Q-\lambda)^j(f) \quad : 0\leq s\leq\pi$$

We choose η to be a real-valued C^∞-function such that $\eta(s)=0$ when $s\leq 1/4$ and -1 if $s\geq 1/2$. Hence $g_\lambda(x,s)=0$ in (i) when $0\leq s\leq 1/4$ and we extend the function to $[-\pi\leq s\leq\pi$ where $g_\lambda(x,-s)=g_\lambda(x,s)$ if $0\leq s\leq\pi$. So now g_λ is π-periodic with respect to s and vanishes when $|s|\leq 1/4$.

Exercise. If $1/2\leq s\leq\pi$ we have $\eta(s)=1$. Use (i) to show that

$$(P+\lambda)(g_\lambda) = \frac{\partial g_\lambda}{\partial s} - (Q-\lambda)(g_\lambda) = \frac{(s-\pi)^m}{m!}\cdot(Q-\lambda)^{m+1}(f)$$

hold when $1/2\leq s\leq\pi$. At the same time $g_\lambda(s)=0$ when $0\leq s\leq 1/4$. So $(P+\lambda)(g)$ is a function whose derivatives with respect to s vasnish up to order m at $s=0$ and $s=\pi$ and is therefore doubly periodic of class C^m in $\square$. Now Theorem 2.2 applies. For a given $k\geq 2$ we choose a sufficently large m and find $h(x,s)$ so that

$$P(h)+\lambda\cdot h = (P+\lambda)(g_\lambda)(x,s)$$

where h is s-periodic, i.e.

$$h(x,0)=h(x,\pi)$$

Notice also that $g_\lambda(x,0)=0$ while $g_\lambda(x,\pi)=f(x)$. Set

$$g_*(x) = h - g_\lambda$$

Then $P(g_*)+\lambda\cdot g_*=0$ and

$$g_*(x,0)-g_*(x,\pi)=f(x)$$

Above we started with the C^∞-function. Given $k\geq 2$ we can take m sufficiently large during the constructions above so that g_* belongs to $\mathcal{H}^{(k)}(\square)$.

The Schrödinger equation.

We work in $\mathbf{R}^3$ with the coordinates (x,y,z). Let $c(x,y,z)$ be a real-valued function in $L^2_{\text{loc}}(\mathbf{R}^3$. In order that the subsequent formulas can be stated in a precise manner we also assume that $c$is almost everywhere continuous which of course is a rather weak condition and in any case satisfied in applications. Next, let Δ be the Laplace operator and define the operator L by

$$(*) \qquad L(u) = \Delta(u) + c \cdot u$$

Denote by $E_L(\mathbf{R}^3)$ the set of functions u such that both u and $L(u)$ belong to $L^2(\mathbf{R}^3)$. Given a pair (f,λ) where $f \in L^2(\mathbf{R}^3$ and λ is a complex number we seek solutions $u \in E_L(\mathbf{R}^3)$ such that

$$(**) \qquad L(u) + \lambda \cdot u = f$$

The case $\mathfrak{Im}(\lambda \neq 0$. By a classic result about solutions to the Neumann boundary value problem in open balls in $\mathbf{R}^3$ one proves that (1) has at least one solution u whenever λ is not real. The remains to investigate the uniqueness, i.e, when one has the implication

$$(***) \qquad \mathfrak{Im}(\lambda \neq 0 \quad \text{and} \quad L(u) + \lambda \cdot u = 0 \implies u = 0$$

This uniqueness property depends on the c-function. A sufficient condition is the following:

Theorem. *Assume that there exists a constant M and some $r_* > 0$ such that*

$$c(x,y,z) \leq M \quad \text{when} \quad x^2 + y^2 + z^2 \geq r_*^2$$

Then (***) above holds.

The spectral θ-function. When (***) holds it was proved in [Carleman] that classical solutions to the Neumanns boundary value problem in open balls yield a θ-function which enable us to describe solutions to (*) for real λ-values. More precisely, there exists two increasing sequence of positive real numbers $\{\lambda^*(\nu)\}$ and $\lambda_*(\nu)\}$ and two sequence of pairwise orthogonal functions $\{\phi_\nu(p))\}$ and $\{\psi_\nu(p)\}$ in $L^(\mathbf{R})^3$ where all these functions have L^2-norm equal to one such that the following hold. First, set

$$\theta(p.q,\lambda) = \sum_{0<\lambda^*(\nu)\leq\lambda} \phi_\nu(p) \cdot \phi_\nu(q) \quad : \quad \lambda > 0$$

$$\theta(p.q,\lambda) = -\sum_{\lambda\leq\lambda_*(\nu)\leq<0} \psi_\nu(p) \cdot \psi_\nu(q) \quad : \quad \lambda < 0$$

such that the following hold:

$$(1) \qquad v(p) = \lim_{R\to\infty} \sum_{[\lambda_\nu < R} \theta(p,q,\lambda) \cdot v(q) \cdot dq \quad \text{for all} \quad v \in L^2(\mathbf{R}^3)$$

$$(2) \qquad v \in E_L(\mathbf{R})^{\mathbf{3}} \quad \text{if and only if} \quad xxxx$$

$$(3) \qquad L(v)(p) = \lim_{R\to\infty} \sum_{[\lambda_\nu < R} \lambda \cdot [\int_{\mathbf{R}^3} \theta(p,q,\lambda) \cdot v(q) \cdot dq] \cdot d\lambda \quad \text{for all} \quad v \in E_L(\mathbf{R}^3)$$

Here the equaliy holds in L^2, i.e, in the sense of a Plancherel's limit.

Remark. Here the equaliy holds in L^2, i.e, in the sense of a Plancherel's limit.

Construction of the ϕ-functions. For each finite r we have the ball B_r and consider the space $E_L(B_r)$ of functions u in B_r such that both u and $L(u)$ also belong to $L^2(B_r)$. By a classical result in the Fredholm theory that exist discrete sequences of real numbers $\{\lambda * (\nu)$ and $\lambda_*(\nu)$ as above and two families of orthonormal functions $\{\phi_\nu^{(r)}\}$ and $\{\psi_\nu^{(r)}\}$ satisfying (xx)

and here a classical result shows that the real eigenvalues to the equation $L(u) + \lambda \cdot u = 0$ xxx

$$xxx$$

The proofs of the assertions above rely on a systematic use of Green's formula. To begin with we recall how to express solutions to an inhomogeneous the Laplace equation by an integral formula.

A. The equation $\Delta(u) = \phi$. let D be a domain in $\mathbf{R}^3$ and ϕ a function in $L^2(D)$. Then a function u for which both u and $\Delta(u)$ belong to $L^2(D)$ gives $\Delta(u) = \phi$ if and only if the following hold for every $p \in D$ and every $\rho < \text{dist}(p, \partial D)$:

$$\text{(i)} \qquad u(p) = \frac{1}{2\pi\rho^2} \cdot \int_{B_p(\rho)} \frac{1}{|p-q|} \cdot u(q) \cdot dq + \frac{1}{4\pi\rho^2} \cdot \int_{B_p(\rho)} A(p,q) \cdot \phi(q) \cdot dq$$

where we have put

$$\text{(ii)} \qquad A(p,q) = \frac{2}{\rho} - \frac{1}{|p-q|} - \frac{|p-q|}{\rho^2}$$

Exercise. Prove this result.The hint is to apply Green's formula while ϕ is replaced by $\Delta(u)$ in the last integral.

Remark. Let us also recall also that when $\Delta(u)$ is in L^2, then u is automatically a continuous function in D.

The class $\mathfrak{Neu}(B_r)$. Let B_r be the open ball of radius r centered at the origin. The class of functions u which are continuous on the closed ball and whose interior normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ is continuous on the boundary $S^2[r]$ is denoted by $\mathfrak{Neu}(B_r)$.

The Neumann equation. Let $c(x, y, z)$ be a function in $L^2(\mathbf{R}$ and consider also a pair a, H where a be a continuous function on $S^2[r]$ and $H(p, q)$ a continuous hermitian function on $S^2[r] \times S^2[r]$, i.e. $H(q, p) = \bar{H}(p, q)$ hold for all pairs of point p, q on the sphere $S^2[r]$. With these notations the following hold:

Theorem For each $f \in L^2(B_r)$ and every non-real complex number λ there exists a unique $u \in \mathfrak{Neu}(B_r)$ such that u satisfies the two equations:

$$L(u + \lambda \cdot u = f \quad \text{holds in} \quad B_r$$

$$\partial u / \partial \mathbf{n}(p) = xx$$

Moreover, one has the L^2-estimate

$$\int_{B_r} |u|^2 \cdot dxdydz \leq \Big| \frac{1}{\mathfrak{Im}(\lambda)} \Big| \leq \int_{B_r} |f|^2 \cdot dxdydz$$

Classic result: $R > 0$ we have unit ball B_R and unit sphere S_R. Let $\mathfrak{Neu}(R)$ set uf u -functions where $\Delta(u)$ in L^2, continuous on closed ball and limit of interior derivative as a continuous funtion. Given $c \in L^2(B_R)$ define

$$L(u) = \delta(u) + c \cdot u$$

Theorem For each pair (a, H) in (*) there exists a unique $u \in \mathfrak{Neu}(R)$ such that

$$L(u + \lambda \cdot u = f \quad \text{holds in} \quad B_R$$

and u saisfies th boundsry condition

$$\partial u / \partial \mathbf{n}(p) = xx$$

from that L^2-esteimate as well.

Do if for $R = m$ running over positive integers. Catch up sequence with L^2-convergence bounded uniformly. $u_m \to u_*$ weak sense and see tha u_* is a solution to (*) on al over space.

Second point about eventual uniquenss. Class I type. Euquialent condition-

In an article from 1920 Carleman constructed an "ugly example" of a doubly indexed sequence $\{c_{pq}\}$ of real numbers satisfying (1) and the symmetry condition $c_{pq} = c_{qp}$, and yet there exists a non-zero complex vector $\{x_p = a_p + ib_p\}$ in ℓ^2 such that

$$S(x) = ix \tag{4}$$

This should be compared with the finite dimensional case where the spectral theorem due to Cauchy and Weierstrass asserts that if A is a real and symmetric $N \times N$-matrix for some positive integer N, then there exists an orthogonal $N \times N$-matrix U such that UAU^* is a diagonal matrix with real elements. Carleman extended this finite dimensional result to infinite Hermitian matrices for which the densely defined linear operator S has no eigenvectors with eigenvalue i or $-i$. More precisely, one says that the densely defined operator S is of Class I if the equations

$$S(z) = iz \quad : S(\zeta) = -i\zeta \tag{*}$$

do not have non-zero solutions with complex vectors z or ζ in ℓ^2. The major result in the cited monograph is as follows:

0.0.1 Theorem. *Each densely Hermitian operator S of Class I has a unique adapted resolution of the identity.*

0.0.2 Resolutions of the identity. In order to digest Theorem 0.0.1 we recall the notion of spectral resolutions. To begin with, a resolution of the identity on ℓ^2 consists of a family $\{E(\lambda)\}$ of self-adjoint projections, indexed by real numbers λ which satisfies (A-C) below.

A. Each $E(\lambda)$ is an orthogonal projection from ℓ^2 onto the range $E(\lambda)(\ell^2)$ and these operators commute pairwise, i.e.

$$E(\lambda) \cdot E(\mu) = E(\mu) \cdot E(\lambda) \tag{i}$$

hold for pairs of real numbers.

B. To each pair of real numbers $a < b$ we set

$$E_{a,b} = E(b) - E(a)$$

Then

$$E_{a,b} \cdot E_{c,d} = 0 \tag{iii}$$

for each pair of disjoint interval $[a, b]$ and $[b, c]$.

C. For each $x \in \ell^2$ the real-valued function

$$\lambda \mapsto \langle E(\lambda)(x), x \rangle \tag{c}$$

is a non-decreasing and right continuous function.

0.0.3 S-adapted resolutions. Let S be a densely defined and hermitian linear operator on ℓ^2. A spectral resolution $\{E(\lambda)\}$ of the identity is S-adapted if the following three conditions hold:

A.1 For each interval bounded $[a,b]$ the range of $E_{a,b}$ from (B) above is contained in $\mathcal{D}(S)$ and

$$E_{a,b}(Sx) = S \circ E_{a,b}(x) \quad : x \in \mathcal{D}(S)$$

B.1 By (C) each $x \in \ell^2$ gives the non-decreasing function $\lambda \mapsto \langle E(\lambda)(x), x\rangle$ on the real line. Together with the right continuity in (c) there exist Stieltjes' integrals

$$\int_a^b \lambda \cdot \langle dE(\lambda)(x), x\rangle$$

for each bounded interval. Carleman's second condition for $\{E(\lambda)\}$ to be S-adapted is that a vector x belongs to $\mathcal{D}(S)$ if and only if

$$\int_{-\infty}^{\infty} |\lambda| \cdot \langle dE(\lambda)(x), x\rangle < \infty \tag{b.1}$$

C.1 The last condition is that

$$\langle Sx, y\rangle = \int_{-\infty}^{\infty} \lambda \cdot \langle dE(\lambda)(x), y\rangle \quad : x, y \in \mathcal{D}(S) \tag{c.1}$$

where (b.1) and the Cauchy-Schwarz inequality entail that the Stieltjes' integral in (c.1) is absolutely convergent.