§ 3. A doubly periodic class of inhomogeneous PDE-equations.

Put

$$\square = \{0 \le x \le \pi\} \times \{0 \le s \le 2\pi\}$$

We shall consider complex-valued and doubly periodic functions $g(x, s)$ on $\square$, i.e.

$$(*) \qquad g(\pi, s) = g(0, s) \quad : g(x, 0) = g(x, 2\pi)$$

For each non-negative integer k we denote by $C^k(\square)$ the space of k-times doubly periodic continuously differentiable functions. It means that (*) hold for all mixed derivatives $\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}$ when $\nu + j \le k$. If $g \in C^k(\square)$ we set

$$||g||^2_{(k)} = \sum_{j,\nu} \int_\square \Big| \frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}(x, s)\Big|^2 \, dxds$$

with the double sum extended pairs $j + \nu \le k$. The completion of $C^k(\square)$ with respect to this norm gives a complex Hilbert space $\mathcal{H}^{(k)}$. Sobolev's inequality entails that a function $g \in \mathcal{H}^{(2)}$ is automatically continuous and doubly periodic on the closed square. More generally, if $k \ge 3$ each $g \in \mathcal{H}^{(k)}$ has continuous and doubly periodic derivatives up to order $k - 2$. Next, consider a first order PDE-operator

$$(1.1) \qquad P = \partial_s - a(x, s)\partial_x - b(x, s)$$

where a and b are real-valued doubly periodic C^∞-functions. It is clear that P maps $\mathcal{H}^{(k)}$ into $\mathcal{H}^{(k+1)}$ for every $k \ge 2$. Keeping $k \ge 2$ fixed we set

$$(1.2) \qquad \mathcal{D}_k(P) = \{g \in \mathcal{H}^{(k)} : \ P(g) \in \mathcal{H}^{(k)}\}$$

Since $C^\infty(\square)$ is dense in $\mathcal{H}^{(k)}$ this yields a densely defined operator

$$(1.3) \qquad \mathcal{T}_k \colon \mathcal{D}_k(P) \to \mathcal{H}^{(k)}$$

Thus, $\mathcal{T}_k$ is a notation for the restriction of P to $\mathcal{D}_k(P)$. So by definition

$$\mathcal{D}(\mathcal{T}_k) = \mathcal{D}_k(P)$$

Next, in $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$ we get the graph

$$\Gamma(\mathcal{T}_k) = \{(g, P(g) \colon \ g \in \mathcal{D}_k(P)\}$$

Since P is a differential operator we know from general results that this is a closed subspace. Set

$$(1.4) \qquad \gamma_k = \{(g, P(g) \colon \ g \in C^\infty(\square)\}$$

This is a subspace of $\Gamma(\mathcal{T}_k)$ and let $\overline{\gamma}_k$ be its closure taken in $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$. Since $\Gamma(\mathcal{T}_k)$ is closed we have

$$\overline{\gamma}_k \subset \Gamma(\mathcal{T}_k)$$

Let T_k be the densely defined linear operator whose graph is $\overline{\gamma}_k$. The construction gives the inclusion

$$(1.5) \qquad \mathcal{D}(T_k) \subset \mathcal{D}(\mathcal{T}_k)$$

Remark. Examples show that this inclusion in general can be strict.

From now on we assume that $k \ge 2$. Keeping k fixed we let E be the identity operator on $\mathcal{H}^{(k)}$. With these notations we shall prove:

A. Theorem. *For each integer $k \ge 2$ there exists a positive real number $\rho(k)$ such that the densely defined orator $T_k - \lambda \cdot E)$ s invertible in the sense of Neumann for every real $\lambda > \rho(k)$.*

Remark. This means that if $\lambda > \rho(k)$ then $T_k - \lambda \cdot E$ has a dense range and there exists a constant $c > 0$ such that

$$||T_k(g) - \lambda \cdot g||_{(k)} \ge c \cdot ||g||_{(k)} \quad : \quad g \in \mathcal{D}(T_k)$$

The proof requires several steps and is not finished until § 3.x. First we shall study the adjoint operator T_k^* and establish the following:

A.1 Proposition. *One has the equality $\mathcal{D}(T_k^*) = \mathcal{D}_k(P)$ and there exists a bounded self-adjoint operator B_k on $\mathcal{H}^{(k)}$ such that*

$$T_k^* = -\mathcal{T}_k + B_k$$

Proof of Proposition A.1

Keeping $k \geq 2$ fixed we set $\mathcal{H} = \mathcal{H}^{(k)}$. For each pair g, f in $\mathcal{H}$ their inner product is defined by

$$\langle f, g \rangle = \sum \int_\Box \frac{\partial^{j+\nu} f}{\partial x^j \partial s^\nu}(x,s) \cdot \overline{\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}}(x,s)\, dxds$$

where the sum is taken when $j + \nu \leq k$. Introduce the differential operator

$$\Gamma = \sum_{j+\nu \leq k} (-1)^{j+\nu} \cdot \partial_x^{2j} \cdot \partial_s^{2\nu}$$

Partial integration gives

(i) $$\langle f, g \rangle = \int_\Box f \cdot \Gamma(\bar{g})\, dxds = \int_\Box \Gamma(f) \cdot \bar{g}\, dxds \quad : \ f, g \in C^\infty$$

Now we consider the operator $P = \partial_s - a \cdot \partial_x - b$ and get

(ii) $$\langle P(f), g \rangle = \int_\Box P(f) \cdot \Gamma(\bar{g})\, dxds$$

Partial integration identifies (ii) with

(iii) $$-\int_\Box f \cdot \big(\partial_s - \partial_x(a) - a \cdot \partial_x - b\big) \circ \Gamma(\bar{g})\, dxds$$

In (iii) appears the composed differential operator

$$(\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma$$

In the ring of differential operators with C^∞-coefficients this differential operator can be written in the form

$$\Gamma \circ P + Q(x, s, \partial_x, \partial_s)$$

where Q is a differential of order $\leq 2k$ with coefficients in $C^\infty(\Box)$. Now (ii-iii) give

(iv) $$\langle Pf, g \rangle = -\langle f, Pg \rangle + \int_\Box f \cdot Q(\bar{g})\, dxds$$

The operator B_k. With Q as above we have a bilinear form which sends a pair f, g in $C^\infty(\Box)$ to

(v) $$\int_\Box f \cdot Q(\bar{g})\, dxds$$

Partial integration and the Cauchy-Schwarz inequality give a constant C which depends on Q only such that

$$\Big| \int_\Box f \cdot Q(\bar{g})\, dxds \Big| \leq C_Q \cdot ||f||_k \cdot ||g||_k$$

It follows that there exists a bounded linear operator B_k on $\mathcal{H}$ such that

(vi) $$\langle f, B_k(g) \rangle = \int_\Box f \cdot Q(\bar{g})\, dxds$$

Lemma. *The operator B_k is self-adjoint*

Proof. From the above we have

$$(1)\qquad \langle Pf,g\rangle=-\langle f,Pg\rangle+\langle f,B_k(g)\rangle$$

for each pair f,g in $C^\infty(\square)$. Thus

$$\langle Pf,g\rangle+\langle f,Pg\rangle=\langle f,B_k(g)\rangle$$

where the left hand side is symmetric in f and g. It follows that

$$\langle f,Pg\rangle=\langle g,B_k(f)\rangle$$

hold for a pair in $C^\infty(\Delta)$. Since B_k is bounded and $C^\infty(\Delta)$ is dense in $\mathcal{H}$ we conclude that B_k is self-adjoint.

Armed with the operator B_k we proceed to the proof of Proposition A.1. First, by the construction of adjoint operators and the density of $C^\infty(\square)$ in $\mathcal{H}$, it follows that a function $g\in\mathcal{H}$ belongs to $\mathcal{D}(T_k^*)$ if and only if there exists a constant C such that

$$\text{(i)}\qquad |\langle Pf,g\rangle|\leq C\cdot ||f|| \quad : f\in C^\infty(\square)$$

Moreover, since B_k is a bounded operator, the equation (1) from the proof of Lemma XX gives the inclusion

$$\text{(ii)}\qquad \mathcal{D}_k(P)\subset\mathcal{D}(T_k^*)$$

To prove the opposite inclusion we shall the Γ-operator. If $g\in\mathcal{D}(T_k^*)$ we have from (i) in § 1.1:

$$\langle Pf,g\rangle=\langle f,T_k^*g\rangle=\int\Gamma(f)\cdot\overline{T_k^*(g)}\,dxds\quad : f\in C^\infty(\square)$$

Similarly

$$\langle f,B_k(g)\rangle=\int\Gamma(f)\cdot\overline{B_k(g)}\,dxds$$

Treating $\mathcal{T}_k(g)$ as a distribution the equation (1) from the proof of Lemma § xx entails that

$$\Gamma(T_k^*(g)-\mathcal{T}_k(g)+B_k(g))=0$$

Now the PDE-opertor Γ is elliptic and since both $T_k^*(g)$ and $B_k(g)$ belong to $\mathcal{H}$ this implies by the general result in § § XX that $\mathcal{T}_k(g)$ belongs to $\mathcal{H}$ which proves that equality holds in (ii) above, and at the same time we get the requested operator equation

$$T_k^*=-\mathcal{T}_k(g)+B_k$$

which finishes the proof of Proposition A.1.

§ 2. An inequality.

Let $f\in C^\infty(\square)$ and λ is a positive real number. Then

$$||\mathcal{T}_k(f)-\frac{1}{2}B_k(f)-\lambda\cdot f||^2=$$

$$||\mathcal{T}_k(f)-\frac{1}{2}B_k(f)||^2+\lambda^2\cdot||f||^2-\lambda\big(\langle\mathcal{T}_k(f)-\frac{1}{2}B_k(f),f\rangle+\langle f,\mathcal{T}_k(f)-\frac{1}{2}B_k(f)\rangle\big)$$

The last term is λ times

$$\text{(i)}\qquad \langle\mathcal{T}_k(f),f\rangle+\langle f,\mathcal{T}_k(f)\rangle-\langle f,B_kf\rangle$$

where we used that B_k is symmetric. Now $T_k=\mathcal{T}_k$ holds on $C^\infty(\square)$ and the definition of adjoint operators give

$$\text{(ii)}\qquad \langle\mathcal{T}_k(f),f\rangle=\langle f,T_k^*\rangle$$

Then (1) in Lemma § xx implies that (i) is zero and hence we have proved

$$\text{(iii)} \quad ||T_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 = \lambda^2 \cdot ||f||^2 + ||T_k(f) - \frac{1}{2}B_k(f)||^2 \geq \lambda^2 \cdot ||f||^2$$

From (iii) and the triangle inequality for norms we obtain

$$\text{(iv)} \qquad ||T_k(f) - \lambda \cdot f|| \geq \lambda \cdot ||f|| - \frac{1}{2}||B_k(f)||$$

Now B_k has a finite operator norm and if $\lambda \geq ||B_k||$ we see that

$$\text{(v)} \qquad ||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f||$$

Finally, since $C^\infty(\Box)$ is dense in $\mathcal{D}(T_k)$ it is clear that (v) gives

2.1 Proposition. *One has the inequality*

$$\text{(2.1.1)} \qquad ||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f|| \quad : f \in \mathcal{D}(T_k)$$

§ 3. Proof of Theorem A

Suppose we have found some $\lambda^* \geq \frac{1}{2} \cdot ||B||$ such that $T_k - \lambda$ has a dense range in $\mathcal{H}$ for every $\lambda \geq \lambda^*$. If this is so we fix $\lambda \geq \lambda^*$ and take some $g \in \mathcal{H}$. The hypothesis gives a sequence $\{f_n \in \mathcal{D}(T_k)\}$ such that

$$\lim_{n \to \infty} ||T_k(f_n) - \lambda \cdot f_n - g|| = 0$$

In particular $\{T_k(f_n) - \lambda \cdot f_n\}$ is a Cauchy sequence in $\mathcal{H}$ and (2.1.1) implies that $\{f_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$ and hence converges to a limit f_*. Since the operator T_k is closed we conclude that $f_* \in \mathcal{D}(T_k)$ and we get the equality

$$T_k(f_*) - \lambda \cdot f_* = g$$

Since $g \in \mathcal{H}$ was arbitrary Theorem A follows.

3.1 Density of the range. There remains to find λ^* as above. By the construction of adjoint operators, the range of $T_k - \lambda \cdot E$ fails to be dense if and ony if $T_k^* - \lambda$ has a non-zero kernel. Set

$$M = \frac{1}{2} \cdot \max_{(x,s) \in \Box} |\partial_x(a)(x,s)|$$

We have also the operator norm $||B_k||$ and with these two positive constants we prove:

3.2 Proposition *For each $\lambda > M + ||B_k||$ it follows that $T_k^*(f) - \lambda \cdot E$ is injective.*

Proof. Let $\lambda > M + ||B_k||$ and suppose that

$$(*) \qquad T_k^*(f) - \lambda \cdot f = 0$$

for some $f \in \mathcal{D}(T_k^*)$ which is not identically zero. Since T_k^* sends real-valued functions into real-valued functions we can assume that f is real-valued and normalised so that

$$\text{(i)} \qquad \int_\Box f^2(x,s)\, dxds = 1$$

From Proposition A.xx we get

$$\text{(ii)} \qquad \mathcal{T}_k(f) + \lambda \cdot f - B_k(f) = 0$$

Let us now consider the function

$$V(s) = \int_0^\pi f^2(x,s)\,dx$$

The s-derivative of $V(s)$ becomes:

$$\text{(iii)} \qquad \frac{1}{2}\cdot V'(s) = \int_0^\pi f\cdot\frac{\partial f}{\partial s}\,dx$$

By (ii) we have

$$\frac{\partial f}{\partial s} - a(x)\frac{\partial f}{\partial x} - b\cdot f = B_k(f) - \lambda\cdot f$$

Hence the right hand side in (iii) becomes

$$\text{(iv)} \quad -\lambda\cdot V(s) + \int_0^\pi f(x,s)\cdot B_k(f)(x,s)\,dx + \int_0^\pi a(x,s)\cdot f(x,s)\cdot\frac{\partial f}{\partial x}(x,s)\,dx$$

By partial integration the last term is equal to

$$\text{(v)} \qquad -\frac{1}{2}\int_0^\pi \partial_x(a)(x,s)\cdot f^2(x,s)\,dx$$

and with M as in (3.1.1) it is majorized by $M\cdot V(s)$. From the above we get the inequality

$$\text{(vi)} \qquad \frac{1}{2}\cdot V'(s) \le (M-\lambda)\cdot V(s) + \int_0^\pi f(x,s)\cdot B(f)(x,s)\,dx$$

Set

$$\Phi(s) = \int_0^\pi |f(x,s)|\cdot|B(f)(x,s)|\,dx$$

Since the L^2-norm of f is one the Cauchy-Schwarz inequality gives

$$\int_{-\pi}^\pi \Phi(s)\,ds \le \sqrt{\int_\square |B(f)(x,s)|^2\,dxds} \le ||B||$$

where the last equality follows since the squared integral of $B(f)$ is majorized by its squared norm in $\mathcal{H}$ and $||f|| = 1$. So when $\lambda > M$ it follows from (v) that

$$\text{(vi)} \qquad (\lambda - M)\cdot V(s) + \frac{1}{2}\cdot V'(s) \le \Phi(s)$$

Next, since f is double periodic we have $V(-\pi) = V(\pi)$ so after an integration (vi) gives

$$\text{(vii)} \qquad (\lambda - M)\cdot\int_0^\pi V(s)\,ds = \int_{-\pi}^\pi \Phi(s)\,ds \le ||B||$$

Finally, by the normalisation (i) we have $\int_0^\pi V(s)\,ds = 1$ so (vii) entails that

$$\text{(viii)} \qquad \lambda \le M + ||B||$$

This contradicts the initial inequality and finishes the proof of Proposition 3.2.

A special solution.

Let $f(x)$ be a periodic C^∞-function on $[0,\pi]$. Put

$$Q = a(x,s)\cdot\frac{\partial}{\partial x} + b(x,s)$$

Let $\eta(s)$ be a C^∞-function of s and m some positive integer If $\lambda > 0$ is a real number. we set

$$\text{(i)} \qquad g_\lambda(x,s) = \eta(s)\cdot f + \eta(s)\cdot\sum_{j=1}^{j=m}\frac{(s-\pi)^j}{j!}\cdot(Q-\lambda)^j(f) \quad : 0\leq s\leq\pi$$

We choose η to be a real-valued C^∞-function such that $\eta(s) = 0$ when $s \leq 1/4$ and -1 if $s \geq 1/2$. Hence $g_\lambda(x,s) = 0$ in (i) when $0 \leq s \leq 1/4$ and we extend the function to $[-\pi \leq s \leq \pi$ where $g_\lambda(x,-s) = g_\lambda(x,s)$ if $0 \leq s \leq \pi$. So now g_λ is π-periodic with respect to s and vanishes when $|s| \leq 1/4$.

Exercise. If $1/2 \leq s \leq \pi$ we have $\eta(s) = 1$. Use (i) to show that

$$(P+\lambda)(g_\lambda) = \frac{\partial g_\lambda}{\partial s} - (Q-\lambda)(g_\lambda) = \frac{(s-\pi)^m}{m!}\cdot(Q-\lambda)^{m+1}(f)$$

hold when $1/2 \leq s \leq \pi$. At the same time $g_\lambda(s) = 0$ when $0 \leq s \leq 1/4$. So $(P+\lambda)(g)$ is a function whose derivatives with respect to s vasnish up to order m at $s = 0$ and $s = \pi$ and is therefore doubly periodic of class C^m in $\square$. Now Theorem 2.2 applies. For a given $k \geq 2$ we choose a sufficently large m and find $h(x,s)$ so that

$$P(h) + \lambda\cdot h = (P+\lambda)(g_\lambda)(x,s)$$

where h is s-periodic, i.e.

$$h(x,0) = h(x,\pi)$$

Notice also that $g_\lambda(x,0) = 0$ while $g_\lambda(x,\pi) = f(x)$. Set

$$g_*(x) = h - g_\lambda$$

Then $P(g_*) + \lambda\cdot g_* = 0$ and

$$g_*(x,0) - g_*(x,\pi) = f(x)$$

Above we started with the C^∞-function. Given $k \geq 2$ we can take m sufficiently large during the constructions above so that g_* belongs to $\mathcal{H}^{(k)}(\square)$.