## Fourier series

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**Introduction.** In section A we recall basic constructions of Fourier series. The major results give solutions to extremal problems such as the theorems in B.8, C.9 and D.4. The final section is more extensive and gives an exposition of Harald Bohr's theory about almost periodic functions on the real line.

### A: Dini's and Fejer's kernels

We consider complex-valued and continuous functions $f(\theta)$ defined in the interval $[0,2\pi]$ which satisfiy $f(0)=f(2\pi)$. To each such function $f$ and every integer $n$ we set

$$\widehat{f}(n)=\frac{1}{2\pi}\cdot\int_0^{2\pi} e^{-in\phi}f(\phi)\cdot d\phi$$

We refer to $\{\widehat{f}(n)\}$ as the Fourier coefficients of $f$. Next, if $N\geq 1$ we set

$$S_N(\theta)=\sum_{n=-N}^{n=N}\hat{f}(n)\cdot e^{in\theta}$$

We refer to $S_N$ as Fourier's partial sum function of degree $N$.

**A.1.The Dini kernel.** If $N\geq 1$ we set

$$D_N(\theta)=\frac{1}{2\pi}\sum_{n=-N}^{n=N}e^{in\theta}$$

**A.2 Proposition.** *One has the formula*

$$D_N(\theta)=\frac{\sin((N+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}$$

*Proof.* We have

$$\sum_{n=-N}^{n=N}e^{in\theta}=e^{-iN\theta}\cdot\frac{1}{2\pi}\sum_{n=0}^{n=2N}e^{in\theta}=e^{-iN\theta}\cdot\frac{e^{i(2N+1)\theta}-1}{e^{i\theta}-1}$$

Proposition A.2 follows after multiplication with $e^{i\theta/2}$ and the two equalities

$$e^{i(N+1/2)\theta}-e^{i(N+1/2)\theta}=2i\cdot\sin((N+1/2)\theta)$$

$$e^{i\theta/2}-e^{-i\theta/2}=2i\cdot\sin(\theta/2)$$

**A.3 Exercise.** Show that

$$S_N(\theta)=\int_0^{2\pi}D_N(\theta-\phi)\cdot f(\phi)\cdot d\phi=\int_0^{2\pi}D_N(\phi)\cdot f(\theta+\phi)\cdot d\phi$$

**A.4 The Fejer kernel.** Given $f$ as above and $N\geq 1$ we set

$$F_N(\theta)=\frac{S_0(\theta)+\ldots+S_N(\theta)}{N+1}\implies$$

$$F_N(\theta) = \int_0^{2\pi} \mathcal{F}_N(\phi)\cdot f(\theta+\phi)\cdot d\phi \quad \text{where} \quad \mathcal{F}_N(\phi) = D_0(\phi)\ldots + D_N(\phi)$$

**A.5 Proposition** *One has the formula*

$$F_N(\theta) = \frac{1}{N+1}\cdot\frac{1-\cos((N+1)\theta)}{2\cdot\sin^2(\frac{\theta}{2})}$$

*Proof.* To each $\nu \geq 0$ we have

$$\sin((\nu+1/2)\theta) = \mathfrak{Im}\big[e^{i(\nu+1/2)\theta)}\big]$$

It follows that $F_N(\theta)$ is equal to the imaginary part of

$$\frac{e^{i\theta/2}}{\sin(\theta/2)}\cdot\sum_{\nu=0}^{\nu=N} e^{i\nu\theta} = \frac{e^{i\theta/2}}{\sin(\theta/2)}\cdot\frac{e^{i(N+1)\theta}-1}{e^{i\theta-1}} = \frac{e^{i(N+1)\theta}-1}{\sin(\theta/2)}\cdot\frac{1}{e^{i\theta/2}-e^{-i\theta/2}} =$$

$$\frac{e^{i(N+1)\theta}-1}{2i\cdot\sin^2(\theta/2)} = i\cdot\frac{1-e^{i(N+1)\theta}}{2\cdot\sin^2(\theta/2)}$$

Finally, the imaginary part of the last term is equal to

$$\frac{1-\cos((N+1)\theta)}{2\cdot\sin^2(\theta/2)}$$

which proves Proposition A.5.

**A.6 A limit formula.** When $f$ is given we set

$$\mathcal{F}_N(\theta) = \int_0^{2\pi} F_N(\phi)\cdot f(\theta+\phi)\cdot d\phi$$

If $a>0$ and $a\leq\theta\leq 2\pi-a$ the sine-function $\sin^2(\theta/2)$ is bounded below, i.e.

$$\sin^2(\theta/2)\geq\sin^2(a/2)$$

So if $M$ is the maximum norm of $|f(\theta)|$ over $[0,2\pi]$ it follows that

$$\int_a^{2\pi-a} F_N(\phi)\cdot f(\theta+\phi)\cdot d\phi \leq$$

$$\frac{M}{(N+1)\cdot\sin^2(a/2)}\int_a^{2\pi-a}(1-\cos(N\phi))\cdot d\phi \leq \frac{2M}{(N+1)\cdot\sin^2(a/2)}$$

**A.7 Exercise.** Given some $\theta_0$ and $a>0$ we set

$$\omega_f(a) = \max_{|\theta-\theta_0|\leq a} |f(\theta)-f(\theta_0)|$$

Show that

$$|\mathcal{F}_N(\theta_0)-f(\theta_0)| \leq \frac{2M}{(N+1)\cdot\sin^2(a/2)} + \omega_f(a) \quad \text{for all} \quad 0<a<\pi$$

Finally, use the *uniform continuity* of the function $f$ over the interval $[0,2\pi]$ to conclude that the sequence $\{\mathcal{F}_N\}$ converges uniformly to $f$ over the interval $[0,2\pi]$.

## B. Legendre polynomials.

If $n\geq 1$ we denote by $\mathcal{P}_n$ the linear space of real-valued polynomials of degree $\leq n$. A bilinear form is defined by

$$\langle q,p\rangle = \int_{-1}^1 q(x)p(x)\cdot dx$$

Since $1,x,\ldots,x^{n-1}$ generate a subspace of co-dimension one in $\mathcal{P}_n$ we get:

**B.1 Proposition.** *There exists a unique* $Q_n(x) = x^n + q_{n-1}x^{n-1} + \ldots + q_0$ *such that*

$$\int_{-1}^{1} x^\nu \cdot Q_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

To find $Q_n(x)$. we consider the polynomial $(1-x^2)^n$ which vanishes up to order $n$ at the end-points 1 and -1. Its the derivative of order $n$ gives a polynomial of degree $n$ and partial integrations show that

$$\int_{-1}^{1} x^\nu \cdot \partial^n((x^2-1)^n)) \cdot dx = 0 \leq \nu \leq n-1$$

The leading coefficient of $x^n$ in $\partial^n((x^2-1)^n))$ becomes

$$c_n = 2n(2n-1)\cdots(n+1)$$

Hence we have

$$Q_n(x) = \frac{1}{c_n} \cdot \partial^n((x^2-1)^n)$$

**B.2 Definition.** *The Legendre polynomial of degree n is given by*

$$P_n(x) = k_n \cdot \partial^n((x^2-1)^n)$$

*where the constant* $k_n$ *is determined so that* $P_n(1) = 1$.

Since $P_n$ is equal to $Q_n$ up to a constant we still have

$$\int_{-1}^{1} x^\nu \cdot P_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

From this we conclude that

$$\int_{-1}^{1} x^\nu \cdot P_n(x) \cdot P_m x)dx = 0 \quad n \neq m$$

Thus, $\{P_n\}$ is an orthogonal family with respect to the inner product defined by the integral over $[-1,1]$.

**B.3 A generating function.** Let $w$ be a new variable and set

$$\phi(x,w) = 1 - 2xw + w^2$$

Notice that $\phi \neq 0$ when $-1 \leq x \leq 1$ and $|w/ < 1$. Keeping $-1 \leq x \leq 1$ fixed we have the function

$$w \mapsto \frac{1}{\sqrt{1-2xw+w^2}}$$

Recall that when $|\zeta| < 1$ one has the Newton series

$$\frac{1}{\sqrt{1-\zeta}} = \sum g_n \cdot \zeta^n \quad \text{where} \quad g_n = \frac{3 \cdot 5 \cdots (2n-1)}{2^n}$$

It follows that

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum g_n(2xw - w^2)^2$$

With $x$ kept fixed the series is expanded into $w$-powers, i.e. set

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum \rho_n(x) \cdot w^n$$

It is easily seen that as $x$ varies then $\rho_n(x)$ is a polynomial of degree $n$. Moreover, we notice that the coefficient of $x^n$ in $\rho_n(x)$ is equal to

$$g_n \cdot 2^n$$

Next, if $x = 1$ we have

$$\frac{1}{\sqrt{1-2w+w^2}} = \frac{1}{1-w} = \sum w^n$$

From this we conclude that

$$\rho_n(1) = 1 \quad \text{for all} \quad n \geq 0$$

**B.4 Theorem.** *One has the equality $\rho_n(x) = P_n(x)$ for each $n$, i.e.*

$$\frac{1}{\sqrt{1 - 2xw + w^2}} = \sum P_n(x) \cdot w^n$$

*holds when $-1 \leq x \leq 1$ and $|w| < 1$.*

**B.5 Exercise.** Prove this result.

**B.6 The series for $P_n(\cos\theta)$.** With $x$ replaed by $\cos\theta$ we notice that

$$1 - 2\cos\theta \cdot w + w^2 = (1 - e^{i\theta}w)(1 - e^{-i\theta}w)$$

It follows that

$$\frac{1}{\sqrt{1 - 2\cos(\theta)w + w^2}} = \frac{1}{\sqrt{1 - 1 - e^{i\theta}w)}} \cdot \frac{1}{\sqrt{1 - e^{-i\theta}w)}}$$

The last product becomes

$$\sum\sum g_m e^{im\theta} w^m \cdot g_\nu e^{-i\nu\theta} w^\nu$$

Collecting $w$ powers the double sum becomes

$$\sum \gamma_n(\theta) \cdot w^n \quad \gamma_n(\theta) = \sum_{m+\nu=n} g_m g_\nu e^{i(m-\nu)\theta}$$

By Theorem B.4 the last sum represents $P_n(\cos(\theta))$. One has for example

$$P_3(\cos(\theta) = 2g_3 \cdot \cos(3\theta) + 2g_2 g_1 \cdot \cos(\theta)$$

where we used that $g_0 = 1$.

**B.7 An inequality for $|P(x)|$.** Since the $g$-numbers are $\geq 0$ we obtain

$$|P_n(\cos(\theta)| \leq g_n g_0 + g_{n-1}g_1 + \ldots + g_1 g_{n-1} + g_0 g_n = P_n(1) \quad : \quad 0 \leq \theta \leq 2\pi$$

Hence we have proved

**B.8 Theorem.** *For each $n$ one has*

$$|P_n(x)| \leq 1 \quad : \quad -1 \leq x \leq 1$$

Next, we study the values when $x > 1$. Here one has

**B.9 Theorem.** *For each $x > 1$ one has*

$$1 < P_1(x) < P_2(x) < \ldots$$

*Proof.* Let us put

$$\psi(x.w) = 1 + \sum_{n=1}^{\infty} [P_n(x) - P_{n-1}(x)] \cdot w^n$$

By Theorem B.4 this is equal to

$$(*) \qquad \frac{1 - w}{\sqrt{1 - 2xw + w^2}}$$

With $x > 1$ we set $x = 1 + \xi$ and notice that

$$1 - 2xw + w^2 = (1 - w)^2 - 2\xi w$$

Hence (*) becomes

$$(**) \qquad \frac{1}{\sqrt{1 - \frac{2\xi w}{1 - w^2}}} = \sum g_n \cdot \frac{(2\xi w)^n}{(1 - w^2)^n} = \sum g_n \cdot (2\xi)^n \cdot \frac{w^n}{(1 - w^2)^n}$$

Next, for each $n \geq 1$ we notice that the power series of $\frac{w^n}{(1-w^2)^n}$ has positive coefficients. Since $g_n(2\xi)^n > 0$ also hold we conclude that (**) is of the form

$$1 + \sum_{n=1}^{\infty} q_n(\xi) \cdot w^n \quad \text{where} \quad q_n(\xi) > 0$$

Finally, Theorem B.9 follows since

$$P_n(1+\xi) - P_{n-1}(1+\xi) = q_n(\xi)$$

**B.10 An $L^2$-inequality.**

Let $n \geq 1$ and denote by $\mathcal{P}_n[1]$ the space of real-valued polynomials $Q(x)$ of degree $\leq n$ for which $\int_{-1}^{1} Q(x)^2 \cdot dx = 1$ and set

$$\rho(n) = \max_{Q \in \mathcal{P}-n[1]} |Q|_\infty$$

where $|Q|_\infty$ is the maximum norm over $[-1,1]$. To find $\rho(n)$ we use the orthonormal basis $\{P_k^*\}$ and write

$$Q(x) = t_0 \cdot P_0^*(x) + \ldots + t_n \cdot P_n^*(x)$$

Since $Q \in \mathcal{P}_n[1]$ we have $t_0^2 + \ldots + t_n^2 = 1$. Recall also that

$$P_\nu^*(x) = \sqrt{\frac{2\nu+1}{2}} \cdot P_\nu(x)$$

Given $-1 \leq x_0 \leq 1$ the Cauchy-Schwarz inequality gives

$$Q(x_0)^2 \leq \sum_{\nu=0}^{\nu=n} \frac{2\nu+1}{2} \cdot |P_\nu(x_0)| \leq \sum_{\nu=0}^{\nu=n} \frac{2\nu+1}{2}$$

where the last inequality follows since the maximum norm of each $P_\nu$ is $\leq 1$. Finally, we notice that

$$\sum_{\nu=0}^{\nu=n} \frac{2\nu+1}{2} = \frac{(1-n)^2}{2}$$

We conclude that

$$|Q(x_0)| \leq \frac{n+1}{\sqrt{2}}$$

**B.11 The case of equality.** To have equality above we take $x_0 = 1$ and

$$t_\nu = \alpha \cdot P_\nu^*(1) \quad : \quad \nu \geq 0$$

## C. The space $\mathcal{T}_n$

Let $n \geq 1$ be a positive integer. A real-valued trigonometric polynomial of degree $\leq n$ is given by

$$g(\theta) = a_0 + a_1 \cos\theta + \ldots + a_n \cos n\theta + b_1 \sin\theta + \ldots b_n \sin n\theta$$

Here $a_0, \ldots, a_n, b_1, \ldots, b_n$ are real numbers. The space of such functions is denoted by $\mathcal{T}_n$ which is a vector space over $\mathbf{R}$ of dimension $2n+1$. We can write

$$\cos kx = \frac{1}{2}[e^{ikx} + e^{-ikx}] \quad \text{and} \quad \sin kx = \frac{1}{2i}[e^{ikx} - e^{-ikx}] \quad : \ k \geq 1$$

It follows that there exist complex numbers $c_0, \ldots, c_{2n}$ such that

$$g(\theta) = e^{-in\theta} \cdot [c_0 + c_1 e^{i\theta} + \ldots + c_{2n} e^{i2n\theta}]$$

**Exercise.** Show that

$$c_\nu + c_{2n-\nu} = 2a_\nu \quad \text{and} \quad c_\nu - c_{2n-\nu} = 2b_\nu \implies$$

$$c_{2n-\nu} = \bar{c}_\nu \quad 0 \leq \nu \leq n$$

**C.1 The polynomial $G(z)$.** Given $g(\theta)$ as above we set

$$G(z) = c_0 + c_1 z + \ldots + c_{2n} z^{2n}$$

Then we see that

$$e^{-in\theta} \cdot G(e^{i\theta}) = g(\theta)$$

**C.2 Exercise.** Set

$$\bar{G}(z) = \bar{c}_0 + c - 1z + \ldots + \bar{c}_{2n} z^{2n}$$

and show that

(*) $$z^{2n} G(1/z) = \bar{G}(z)$$

Use this to show that if $0 \neq z_0$ is a zero of $G(z)$ then $\frac{1}{\bar{z}_0}$ is also a zero of $G(z)$.

**C.3 The case when $g \geq 0$.** Assume that the $g$-function is non-negative. Let

$$0 \leq \theta_1 < \ldots < \theta_\mu < 2\pi$$

be the zeros on the half-open interval $[0, 2\pi)$. Since $g \geq 0$ every such zero has a multiplicity given by an *even* integer. Consider also the polynomial $G(z)$. From Exercise C.2 it follows that $\{e^{i\theta_\nu}\}$ are complex zeros of $G(z)$ with multiplicities given by even integers. Next, if $\zeta$ is a zero where $\zeta \neq 0$ and $|\zeta| \neq 1$, then (*) in C.2 implies that $\frac{1}{\bar{\zeta}}$ also is a zero and hence $G(z)$ has a factorisation

$$G(z) = c_{2n} \cdot \prod_{\nu=1}^{\nu=\mu} (z - e^{i\theta_\nu})^{2k_\nu} \cdot \prod_{j=1}^{j=m} (z - \zeta_j)(z - \frac{1}{\bar{\zeta}_j}) \cdot z^{2r} \quad \text{where} \quad 2\mu + 2m + 2r = 2n$$

Here $0 < |\zeta_j| < 1$ hold for each $j$ and it may occur that multiple zeros appear, i.e. the $\zeta$-roots need not be distinct and the integer $r$ may be zero or positive.

**C.4 The $h$-polynomial**. Let $\delta = \sqrt{|\zeta_1| \cdots |\zeta_m|}$ and put

$$h(z) = c_{2n} \dot{\delta} \cdot \prod_{\nu=1}^{\nu=\mu} (z - e^{i\theta_\nu})^{k_\nu} \cdot \prod_{j=1}^{j=m} (z - \zeta_j) \cdot z^r$$

**C.5 Proposition.** *One has the equality*

$$|h(e^{i\theta})|^2 = g(\theta)$$

*Proof.* With $z = e^{i\theta}$ and $0 < |\zeta| < 1$ one has

$$(e^{i\theta} - \zeta)(e^{i\theta} - \frac{1}{\bar{\zeta}}) = (e^{i\theta} - \zeta) \cdot (\bar{\zeta} - e^{-i\theta}) \cdot e^{i\theta} \cdot \frac{1}{\bar{\zeta}}$$

Passing to absolute values it follows that

$$\left|(e^{i\theta} - \zeta)(e^{i\theta} - \frac{1}{\bar{\zeta}})\right| = \left|e^{i\theta} - \zeta\right|^2 \cdot \frac{1}{|\zeta|}$$

Apply this to every root $\zeta_\nu$ and take the product which gives Proposition C.5.

**C.6 Application.** Let $g \geq 0$ be as above and assume that the constant coefficient $a_0 = 1$. This means that

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} g(\theta) \cdot d\theta$$

With $h(z) = d_0 + d_1 z + \ldots + d_n z^n$ we get

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} h(e^{i\theta})|^2 \cdot d\theta = |d_0|^2 + \ldots + |d_n|^2$$

Notice that

(i) $$|d_n|^2 = |c_{2n}| \cdot \delta \quad \text{and} \quad |d_0|^2 = |c_{2n} \cdot \delta| \cdot \prod |\zeta_j|^2 = |c_{2n}| \cdot \frac{1}{\delta}$$

From this we see that

(iii) $$|c_{2n}| \cdot (\delta + \frac{1}{\delta}) = |d_0|^2 + d_n|^2 \leq 1$$

Here $0 < \delta < 1$ and therefore $\delta + \frac{1}{\delta} \geq 2$ which together with (iii) gives

$$|c_{2n}| \leq \frac{1}{2}$$

At the same time we recall that

$$c_{2n} = \frac{a_n + ib_n}{2}$$

Hence we have proved the inequality:

(*) $$|a_n + ib_n| \leq 1$$

*Summing up* we have proved the following:

**C.7 Theorem.** *Let $g(\theta)$ be non-negative in $\mathcal{T}_n$ with constant term $a_0 = 1$. Then*

$$|a_n + ib_n| \leq 1$$

**C.8 An application**. Let $n \geq 1$ and consider the space of all monic polynomials

$$P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0$$

where $\{c_\nu\}$ are real- To each such polynomial we can consider the maximum norm over the interval $[-1, 1]$. Then one has

**C.9 Theorem.** *For each $P \in \mathcal{P}_n^*$ one has the inequality*

$$\max_{-1 \leq x \leq 1} |P(x)| \geq 2^{-n+1}$$

*Proof.* Consider some $P \in \mathcal{P}_n^*$ and define the trigonometric polynomial

$$g(\theta) = (\cos\theta)^n + c_{n-1}(\cos\theta)^{n-1} + \ldots + c_0$$

So here $P(\cos\theta) = g(\theta)$ and Theorem C.9 follows if we have proved that

(1) $$2^{-n+1} \geq \max_{0 \leq \theta \leq 2\pi} |g(\theta))|$$

To prove this we set $M = \max_{0 \leq \theta \leq 2\pi} |g(\theta))|$. Next, we can write

$$g(\theta) = a_0 + a_1 \cos\theta \ldots + a_n \cos n\theta$$

Moreover, since

$$(\cos\theta)^n = 2^{-n} \cdot [e^{i\theta} + e^{-\theta}]^n$$

we get

$$a_n = 2^n$$

Now we shall apply Theorem C.8. For this purpose we construct non-negative trigonometric polynomials. First we define

$$g^*(\theta) = \frac{M - g(\theta)}{M - a_0}$$

Then $g^* \geq 0$ and its constant term is 1. We have also

$$g^*(\theta) = 1 - \frac{1}{M - a_0} \cdot \sum_{\nu=1}^{\nu=n} a_\nu \cos\nu\theta$$

Hence Theorem C.7 gives

(1) $$\frac{1}{|M - a_0|} \cdot |a_n| \leq 1 \implies |M - a_0| \geq 2^{-n+1}$$

Next, we have also the function

$$g_*(\theta) = \frac{M + g(\theta)}{M + a_0}$$

In the same way as above we obtain:

(2) $$|M + a_0| \geq 2^{-n+1}$$

Finally, (1) and (2) give

$$M \geq 2^{-n+1}$$

which proves Theorem C.9

## D. Tchebysheff polynomials.

The inequality in Theorem C.9 is sharp. To see this we shall construct a special polynomial $T_n(x)$ of degree $n$. Namely, with $n \geq 1$ we can write

$$\cos n\theta = 2^{n-1} \cdot (\cos\theta)^n + c_{n-1} \cdot (\cos\theta)^{n-1} + \ldots + c_0$$

Set

$$T_n(x) = 2^{n-1}x^n + c_{n-1} \cdot x^{n-1} + \ldots + c_0$$

Hence

$$T_n(\cos\theta) = \cos n\theta$$

We conclude that the polynomial

$$p_n(x) = 2^{-n+1} \cdot T_n(x)$$

belongs to $\mathcal{P}_n^*$ and its maximum norm is $2^{-n+1}$. This proves that the inequality in Theorem 10 is sharp.

**D.1 Zeros of $T_n$**. Set

$$\theta_\nu = \frac{\nu\pi}{n} + \frac{\pi}{2n}$$

It is clear that $\theta_1, \ldots, \theta_n$ are zeros of $\cos n\theta$. Hence the zeros of $T_n(x)$ are:

$$x_\nu = \cos\theta_\nu$$

Notice that

$$-1 < x_n < \ldots < x_1 < 1$$

Since $T_n(x)$ is a polynomial of degree $n$ it follows that $\{x_\nu\}$ give all zeros and we have

$$T_n(x) = 2^{n-1} \cdot \prod (x - x_\nu)$$

**D.2 Exercise**. Show that

$$T_n'(x_\nu) \cdot \sqrt{1 - x_\nu^2} = n$$

hold for every zero of $T_n(x)$.

**D.3 An interpolation formula.** Since $x_1, \ldots, x_n$ are distinct it follows that if $p(x) \in \mathcal{P}_{n-1}$ is a polynomial of degree $\leq n-1$ then

$$p(x) = \sum_{\nu 01}^{\nu=n} p(x_\nu) \cdot \frac{1}{T'(x_\nu)} \cdot \frac{T(x)}{x - x_\nu}$$

By the exercise above we get

$$p(x) = \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} (-1)^{\nu-1} p(x_\nu) \cdot \sqrt{1 - x_\nu^2} \cdot \frac{T(x)}{x - x_\nu}$$

We shall use the interpolation formula above to prove

**D.4 Theorem** *Let $p(x) \in \mathcal{P}_{n-1}$ satisfy*

(1) $$\max_{-1 \leq x \leq 1} \sqrt{1 - x^2} \cdot |p(x)| \leq 1$$

*Then it follows that*

(2) $$\max_{-1 \leq x \leq 1} |p(x)| \leq n$$

*Proof.* First, consider the case when

(*) $$-\cos\frac{\pi}{2n} \leq x \leq \cos\frac{\pi}{2n}$$

Then we have

$$\sqrt{1-x^2} \geq \sqrt{1-\cos^2\frac{\pi}{2n}} = \sin\frac{\pi}{2n}$$

Next, recall the inequality $\sin x \geq \frac{2}{\pi} \cdot x$. It follows that

$$\sqrt{1-x^2} \geq \frac{1}{n}$$

So when (1) holds in the theorem we have

$$|p(x)| = \frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} \cdot |p(x)| \leq \frac{1}{\sqrt{1-x^2}} \leq n$$

Hence the required inequality in Theorem D.4 holds when $x$ satisfies (*) above. Next, suppose that

(**) $$x_1 \leq x \leq 1$$

On this interval $T_n(x) \geq 0$ and from the interpolation formula xx and the triangle inequality we have

$$|p(x) \leq \frac{1}{n}\sum_{\nu=1}^{\nu=n} \sqrt{1-x_\nu^2} \cdot |p(x_\nu)| \cdot \frac{T(x)}{x-x_\nu} \leq \frac{1}{n}\sum_{\nu=1}^{nu=n} \frac{T(x)}{x-x_\nu}$$

Next, the sum

$$\frac{T(x)}{x-x_\nu} = T_n'(x) = n \cdot U_{n-1}(x)$$

So when (**) holds we have

(***) $$|p(x)| \leq |U_{n-1}(x)|$$

By xx the maximum normmof $U_{n-1}$ over $[-1,1]$ is $n$ and hence (***) gives

$$|p(x)| \leq n$$

In the same way one proves htat

$$-1 \leq x \leq x_n \implies |p(x)| \leq n$$

Together with the upper bound in the case (xx) we get Theorem D.4.

### D.5 Berstein's inequality.

Let $g(\theta) \in \mathcal{T}_n$. The derivative $g'(\theta)$ is another trigonometric polynomial and we have

**Theorem.** *For each $g \in \mathcal{T}_n$ one has*

$$\max_{0\leq\theta\leq 2\pi} |g'(\theta)| \leq n \cdot \max_{0\leq\theta\leq 2\pi} |g(\theta)|$$

Before we prove this result we establish an inequality for certain trigonometric polynomials. Namely, consider a real-valued sine-polynomial

$$S(\theta) = c_1\sin(\theta) + \ldots + c_n\sin(n\theta)$$

Now $\theta \mapsto \frac{S(\theta)}{\sin\theta}$ is an even function of $\theta$ and therefore one has

$$\frac{S(\theta)}{\sin\theta} = a_0 + a_1\cos\theta + \ldots + a_{n-1}(\cos\theta)^{-n-1}$$

Consider the polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$$

Then e see that:

$$|p(\cos\theta)| = \frac{|S(\theta)|}{\sqrt{1-\cos^2\theta}}$$

Using this we apply Theorem D.4 to the polynomial $p(x)$ and conclude

**D.6 Theorem.** *Let $S(\theta) = c_1 sin(\theta) + c_n sin(n\theta)$ be a sine-polynomial as above. Then*

$$\max_{0\leq\theta\leq2\pi} \frac{|S(\theta)|}{\sin\theta} \leq n \cdot \max_{0\leq\theta\leq2\pi} |S(\theta)|$$

**D.7 Proof of Bernstein's theorem.** Fix an arbitrary $0 \leq \theta - 0 < 2\pi$. Set

$$S(\theta) = g(\theta_0 + \theta) - g(\theta_0 - \theta)$$

We notice that $S(\theta$is an odd polynomial of $\theta$ and $S(0) = 0$, It follows that $S(\theta)$ is a sine-polynomial as above of degree $\leq n$. Notice also that

$$\max_{0\leq\theta\leq2\pi} |S(\theta)| \leq 2 \cdot \max_{0\leq\theta\leq2\pi} |g(\theta)| \max_{0\leq\theta\leq2\pi} |g(\theta)|$$

Theorem D.6 applied to $S(\theta)$ gives

(i) $$|\frac{g(\theta_0 + \theta) - g(\theta_0 - \theta)}{\sin\theta}| \leq 2n \cdot \max_{0\leq\theta\leq2\pi} |g(\theta)|$$

Next, in the left hand side we can take the limit as $\theta| \to 0$ and notice that

$$2 \cdot g'(\theta_0) = \lim_{\theta\to0} \frac{g(\theta_0 + \theta) - g(\theta_0 - \theta)}{\sin\theta}$$

Hence (i) gives

$$|g'(\theta_0)| \leq n \cdot \max_{0\leq\theta\leq2\pi} |g(\theta)|$$

Finally, since $\theta_0$ was arbitrary we get Bernstein's theorem.

**Almost periodic functions.**

The theory about almost periodic functions on the real line was created and developed by Harald Bohr. The original proofs were rather cumbersome and alternative methods and simplifications of proofs which also led to more precise approximations were found by Weyl and Bochner. One should also mention contributions by Besivcovich whose articles [Bes:1+2] completed Bohr's initial studies of almost periodicity for bounded complex analytic functions in strip domains. The interested reader can consult Bohr's plenary talk at the IMU-congress 1950 for a survey about the development of the theory about almost periodic functions where contributions due to xxxx in [Bel ??] are pointed out since his work brings the theory about almost periodic functions closer to problems related to analytic number theory. In addition to this we refer to work by J. Favard about almost periodic harmonic functions and for applications to the study of differential and difference equations an overall reference is the collected work by Bochner and articles by Neugebauer.

We shall restrict the study to almost periodic functions on the real line but remark that the theory extends to a a general set-up where one starts from an arbitrary discrete abelian group $G$ which yields a compact dual group $\widehat{G}$ whose elements are maps $\chi$ from $G$ into the unit circle satisfying

$$(*) \qquad \chi(g_1 + g_2) = \chi(g_1) \cdot \chi(g_2) \quad \text{for all pairs} \quad g_1, g_2 \in G$$

One refers to such maps as characters and when $G$ is equipped with the discrete topology then (*) is the sole assumption, i.e. no continuity property is involved. Keeping $g \in G$ fixed we get the exponential function on $\widehat{G}$ defined by

$$E_g(\chi) = \chi(g) \quad : \quad \chi \in \widehat{G}$$

Finite linear combinations of such $E$-functions form a linear space and using the maximum norm for functions on $\widehat{G}$ the closure yields a Banach space of functions on $\widehat{G}$ denoted by $\mathcal{F}(\widehat{G})$. One equips $\widehat{G}$ with the weakest topology such that every function in $\mathcal{F}(\widehat{G})$ is continuous. Tychonof's Theorem from general topology shows that $\widehat{G}$ becomes a compact topological space equipped with the structure of an abelian group since the product of two $\chi$-functions on $G$ again is a character function. We shall not enter a detailed discussion about this general construction which leads to harmonic analysis on locally compact abelian groups. The interested reader should consult the excellent text-book by Rudin in [Rudin] devoted to Fourier analysis on locally compact abelian groups.

Now we turn to the special study on the real line which means that $G$ as a group is the real $x$-line line equipped with the discrete topology. If $\xi$ is a real number we get the character defined by

$$\chi_\xi(x) = e^{ix\xi}$$

In this way the dual group $\widehat{\mathbf{R}}_{\text{dis}}$ contains a copy of the real $\xi$-line. But the whole compact dual group contains more characters whose existence rely upon the axiom of choice. So apart from the evident characters $\{\chi_\xi\}$ which identify the real $\xi$-line with a subset of $\widehat{\mathbf{R}}_{\text{dis}}$, there exist more characters which no longer are constructible in an elementary fashion. One refers to $\widehat{\mathbf{R}}_{\text{dis}}$ as the Bohr group over the real line. The induced topology on the embedded $\xi$-line is much weaker than the ordinary topology. For example, a fundamental system of relatively open neighborhoods of $x = 0$ consists of sets of the form

$$U_\epsilon(\xi_1, \ldots, \xi_m) = \bigcap_{\nu=1}^{\nu=m} \{e^{i\xi_\nu \cdot x} - 1| < \epsilon\}$$

where $\epsilon > 0$ and $\xi_1, \ldots, \xi_m$ is some $m$-tuple of real numbers. Notice that every such set is unbounded.

In the sequel we shall not discuss the Bohr group since the constructions and results will be expressed via ordinary calculus, and the subsequent proofs have the merit that no appeal to the Axiom of Choice is needed.

Now we present the major results in Bohr's theory. Denote by $C_*(\mathbf{R})$ the set of bounded and uniformly continuous functions $f(x)$ on the real $x$-line. Recall that uniform continuity means that the non-increasing function defined by

$$\omega_f(\delta) = \max_{0<s\leq\delta} [\max_x |f(x+\delta - f(x)|\,]$$

tends to zero as $\delta \to 0$. Following Bohr we give:

**0.1 Definition.** *A function $f(x)$ is almost periodic if it belongs to $C_*(\mathbf{R})$ and to each $\epsilon > 0$ there exists some $\ell > 0$ such that every interval $(a, a+\ell)$ contains a point $\tau_a$ where the maximum norm*

$$\max_x |f(x+\tau_a) - f(x)| < \epsilon$$

*The class of these functions is denoted by $\mathcal{AP}$.*

**Remark.** Apart from the condition that $\lim \omega_f(\delta) = 0$ as $\delta \to 0$, almost periodicity means that translates of $f$ satisfy an addition condition. To each $\epsilon > 0$ we put

$$E_f(\epsilon) = \{\tau \;:\; \max_x |f(x+\tau) - f(x)| \leq \epsilon\}$$

A point $\tau$ in this set is called a translation number of size $\leq \epsilon$ with respect to $f$. The continuity of $f$ shows that every $E_f(\epsilon)$ is a closed set and almost periodicity means that for every $\epsilon > 0$ there exists $\ell_f(\epsilon) > 0$ such that every open interval of length $\ell_f(\epsilon)$ contains at least one point from $E_f(\epsilon)$. One says that $E_f(\epsilon)$ is a relatively dense subset of $\mathbf{R}$. Notice that the function $\epsilon \mapsto \ell_f(\epsilon)$ is non-decreasing.

**Exercise.** Show that the sum and the product of two almost periodic functions is almost periodic. Show also that if $\{f_n\}$ is a sequence in $\mathcal{AP}$ which converges uniformly to a limit function $f$ then $f$ is almost periodic. Hence $\mathcal{AP}$ appears a a closed subalgebra of $C_*(\mathbf{R})$.

**Exponential polynomials.** If $\tau$ is a real number it is clear that the function $e^{i\tau x}$ is almost periodic, and we consider the algebra of functions of the form

$$p(x) = \sum c_k \cdot e^{i\tau_k \cdot x}$$

where $\{c_k\}$ is a finite set of complex numbers and $\{\tau_k\}$ a finite set of real numbers. Denote this algebra by $\mathcal{TP}$. A first major result in Bohr's theory is the following density result:

**0.2 Theorem.** *$\mathcal{AP}$ is a closed subspace of $C_*(\mathbf{R})$ where $\mathcal{TP}$ appears as a dense subspace.*

The non-trivial part is the density which is proved in XXX. Now we introduce the spectrum of an almost periodic function whose construction relies upon the following:

**0.3 Proposition.** *Let $f \in \mathcal{AP}$. Then there exists a limit*

$$M_f(\lambda) = \lim_{T\to\infty} \frac{1}{T} \cdot \int_0^T e^{-i\lambda x} \cdot f(x) \cdot dx \quad \text{for every real number} \quad \lambda.$$

*Proof.* Since $e^{-i\lambda x} \cdot f(x)$ are almost periodic for every $\lambda$ it suffices to prove the result when $\lambda = 0$. Let $\epsilon > 0$ and pick some $\tau > 0$ in the set $E_f(\epsilon)$. If $T$ is a large number we find the positive integer $N$ such that $N\tau \leq T < (N+1)\tau$. Now

$$\text{(i)} \qquad \int_0^T f \cdot dx = \sum_{k=0}^{k=N-1} \int_{k\tau}^{(k+1)\tau} f \cdot dx + \int_{N\tau}^T f \cdot dx$$

For each $k$ we have

$$\int_{k\tau}^{(k+1)\tau} f \cdot dx = \int_0^\tau f(x+k\tau) \cdot dx$$

At the same time

$$|\int_0^\tau f(x+k\tau) \cdot dx - \int_0^\tau f(x) \cdot dx\,| \leq \epsilon \cdot \tau$$

The triangle inequality therefore gives

$$\Big|\frac{1}{T}\int_0^T f\cdot dx-\frac{N}{T}\cdot\int_0^\tau f(x)\cdot dx\,\Big|\leq\epsilon\cdot\frac{N\cdot\tau}{T}+\frac{1}{T}\cdot\Big|\int_{N\tau}^T f\cdot dx\,\Big|$$

**Bessel's inequality.** It turns out that $M_f(\lambda)\neq 0$. holds in set which is at most denumerable. To prove this we consider the product $f\cdot\bar{f}=|f|^2$ which again is almost periodic and this gives the existence of the number:

$$(*)\qquad ||f||^2=\lim_{T\to\infty}\frac{1}{T}\cdot\int_0^T|f(x)|^2\cdot dx$$

We refer to $||f||^2$ as the squared mean of $f$.

**0.4 Proposition.** *For every finite set* $\{\lambda,\ldots,\lambda_m\}$ *one has the inequality*

$$\sum_{\nu=1}^{\nu=m}|M_f(\lambda_k)|^2\leq||f||^2$$

*Proof.* Put $a_k=M_f(\lambda_k)$. Now

$$|f(x)|^2-\sum a_k\cdot e^{i\lambda_k x}|^2=|f(x)|^2+\sum|a_k|^2-\sum\bar{a}_k\cdot f\dot{e}^{-i\lambda_k x}-\sum a_k\cdot\bar{f}\dot{e}^{i\lambda_k x}$$

Passing to mean values over $[0,T]$ while $T\to\infty$ the reader may verify that

$$||f||^2=\sum|a_k|^2+\lim_{T\to\infty}\frac{1}{T}\cdot\int_0^T|f(x)|^2-\sum a_k\cdot e^{i\lambda_k x}|^2\cdot dt$$

which gives Bessel's inequality.

**0.5 Bohr's spectral set $\sigma(f)$.** Bessel's inequality shows that $M_f(\lambda)\neq 0$ for at most a denumerable set. The set of all such $\lambda$ is called Bohr's spectrum of $f$ and is denoted by $\sigma(f)$. Every denumerable set $\{\lambda_k\}$ can appear as a Bohr spectrum. For consider some $\ell^1$-sequence of non-zero complex numbers $\{c_k\}$, i.e. $\sum|c_k|<\infty$. We get the almost periodic function

$$\phi(x)=\sum c_k\cdot e^{i\lambda_k x}$$

**Exercise.** Show that $\sigma(\phi)=\{\lambda_k\}$. The hint is that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\alpha x}\cdot dx=0\quad\text{hold for every}\quad\alpha\neq 0$$

**0.7 Parseval's equality.** *For every* $f\in\mathcal{AP}$ *one has*

$$||f||^2=\sum_{\lambda_k\in\sigma(f)}|M_f(\lambda_k)|^2$$

The proof will be given in XX and relies upon the following result which is proved in XX below.

**0.8 Theorem** *The Bohr spectrum is non-empty for every* $f\in\mathcal{AP}$ *which is not identically zero.*

**0.9 Bochner-Fejer kernels.** Parseval's equality leads not only to a proof of Theorem 0.2 but gives a procedure to approximate every $f$ in $\mathcal{AP}$. Namely, let $\{\lambda_k\}=\sigma(f)$ and if $\epsilon>0$ we choose $m$ so large that

$$\sum_{k=1}^{k=m}|M_f(\lambda_k)|^2>||f||^2-\epsilon$$

The finite set $\lambda_1, \ldots, \lambda_m$ generates a free abelian group where we choose some basis $\beta_1, \ldots \beta_p$. So each $\lambda_k$ is an integer combination of the $\beta$-numbers. For each $p$-tuple of integers $w_1, \ldots, w_p$ we assign the exponential function

$$E_{w_1,\ldots,w_p}(x) = e^{i(\beta_1 \cdot w_k 1 + \ldots + \beta_k \cdot w_k)\cdot x}$$

Next, for each positive integer $N$ we set

$$\mathcal{B}_N(x) = \sum \prod \left(1 - \frac{|w_k|}{N}\right) \cdot M_f(w_1\beta_1 + \ldots + w_p\beta_p) \cdot E_{w_1,\ldots,w_p}(x)$$

where $\Sigma$ extends over all $w$-tuples for which

$$-N \leq w_k \leq N \quad : \quad 1 \leq k \leq p$$

We refer to $\mathcal{B}_N$ as a Bochner-Fejer sum associated to $f$. Each $\mathcal{B}_N$ belongs to $\mathcal{TP}$ and its finite spectrum is confined to numbers of the form $w_1\beta_1 + \ldots + w_p\beta_p$ where $M_f \neq 0$. In particular $\sigma(\mathcal{B}_N)$ is a subset of $\sigma(f)$. In XX we prove that these Bochner-Fejer kernels can be used to approximate $f$ uniformly by functions in $\mathcal{TP}$. A special case occurs when the abelian group generated by the spectrum of $f$ is finitely generated. Then we construct $\{\mathcal{B}_N\}$ as above starting from a finite set $w_1, \ldots, w_k$ such that every point in $\sigma(f)$ is an integer combination of this $m$-tuple. In XXX we prove that

(*) $$\lim_{N\to\infty} \max_x |\mathcal{B}_N(x) - f(x)| = 0$$

Notice that this result is similar to the uniform approximation by Fejer sums of continuous and periodic functions on $[0, 2\pi]$.

**0.10 Diophantic considerations.** Let $\{f_k\}$ be a sequence of periodic functions, where the periodic of $f_k$ is some number $\tau_k$. Suppose that this sequence converges uniformly to an almost periodic function $f$ with a spectrum $\sigma(f)$.

**0.11 Theorem.** *For each non-zero $\lambda \in \sigma(f)$ it follows that $\tau_k$ is a rational multiple of $\lambda$ for all sufficiently large $k$. As a consequence every pair of points $\lambda_1, \lambda_2$ in $\sigma(f)$ must be $Q$-linearly dependent.*

**0.12 Remark.** Theorem 0.11 shows that only a restricted class in $\mathcal{AP}$-can be approximated uniformly by periodic functions. Theorem o.10 gives the necessary condition that the the numbers in $\sigma(f)$ are rationally dependent. Conversely, if there exists some $\lambda_* \neq 0$ such that every $\lambda \in \sigma(f)$ is a rational multiple of $\lambda_*$ then we will show in XX that $f$ can be uniformly approximated by periodic functions.

**0.13 Arithmetical properties of translation numbers.** In the remark after definition 0.1 we introduced the sets $E_f(\epsilon)$. We can consider the subset of integers, i.e. put

$$\mathbf{Z}_f(\epsilon) = E_f(\epsilon) \cap \mathbf{Z}$$

It turns out hat this set is nonand even relatively dense, i.e. there exists some $\ell > 0$ such that every interval of length $\ell$ contains at least one integer from $\mathbf{Z}_f(\epsilon)$. Keeping $\epsilon > 0$ fixed we impose an "almost periodic condition" which we begin to describe. Consider an interval $(a, a + \ell)$ which gives us a finite set of integers

(1) $$\mathbf{Z}_f(\epsilon) \cap (a, a + \ell)$$

Consider also some $0 < \rho < \epsilon$ which gives the set $\mathbf{Z}_f(\rho)$. We say that an integer $n$ which belongs to (1) survives up to order $\rho$ if one has the inclusion

(2) $$\{n\} + \mathbf{Z}_f(\rho) \subset \mathbf{Z}_f(\epsilon)$$

**0.14 Definition.** *The set $\mathbf{Z}_f(\epsilon)$ is said to be almost periodic if there for each $\eta > 0$ exists a pair $(\rho, \ell_*)$ for which the following hold: Whenever $\ell > \ell_*$ and $a$ is a real number, it follows that the* *number of points in $\mathbf{Z}_f(\epsilon) \cap (a, a+\ell)$ which survive up to order $\rho$ is $\geq$ then $1-\eta$) times the number of integers in $\mathbf{Z}_f(\epsilon) \cap (a, a+\ell)$.*

**0.15 Theorem.** *The set of all $0 < \epsilon < 1$ for which $\mathbf{Z}_f(\epsilon)$ is almost periodic has Lebesgue measure one, i.e. almost periodicity holds for all $\epsilon$ except for a possible empty null-set.*

The proof is given in XXX below.

### Proof of Theorem 0.8

Let $f$ be almost periodic and assume that $\sigma(f) = \emptyset$. To prove that $f$ is identically zero the crucial step is the following:

**1.1 Lemma.** *Assume that $\sigma(f) = \emptyset$. Then, for each $\epsilon > 0$ there exists $T_\epsilon$ such that*

$$T \geq T_\epsilon \implies \frac{1}{T} \cdot \Big| \int_0^T e^{i\lambda x} f(x) \cdot dx \Big| \leq \epsilon \quad \text{for all} \quad \lambda$$

We prove this technical result in XX below and first show why Lemma 1.1 gives Theorem 0.8. The idea is to employ certain periodic functions. Consider some $T > 0$ and let $F_T(x) = f(x)$ on $0 \leq x \leq T$ and after $F$ is extended to a $T$-periodic function. The function $F_T(x)$ on the bounded interval $(0, T)$ has an ordinary Fourier series

$$F_T(x) = \sum A_k(T) \cdot e^{2\pi i k x/T}$$

where we have

$$A_k(T) = \frac{1}{T} \int_0^T e^{2\pi i k x/T} \cdot f(x) \cdot dx$$

for every integer $k$. Parseval's equality for ordinary Fourier series gives:

(1) $$\sum |A_k(T)|^2 = \frac{1}{T} \int_0^T |f(x)|^2 \cdot dx \leq |f|_\infty^2$$

Next, introduce the following pair of $T$-periodic functions:

(2) $$G_T(x) = \sum |A_k(T)|^2 \cdot e^{2\pi i k x/T} \quad \text{and} \quad H_T(x) = \sum |A_k(T)|^4 \cdot e^{2\pi i k x/T}$$

By the general formula for periodic functions in XX we have

$$G_T(x) = \int_0^T H_T(x+t) \cdot \bar{H}_T(t) \cdot dt$$

At this stage we apply Lemma 1.1 which gives:

(3) $$\lim_{T \to \infty} \max_k A_T(k) = 0$$

Together with the inequality (1) for the sum of squares of $|A_k(T)|$ it follows that

(4) $$\lim_{T \to \infty} \sum |A_T(k)|^4 \to 0$$

This implies that the $T$-periodic function $H_T(x)$ tends uniformly to zero on $[0, T]$ and (2) entails that

$$\lim_{T \to \infty} G_T(0) = 0$$

Next, the equality $F = f$ on $[0, T]$ and the $T$-periodicity of $F$ gives for each $0 < x < T$:

(5) $$G_T(x) = \frac{1}{T_n} \int_0^{T-x} f(x+t) \cdot \bar{f}(t) \cdot dt + \int_{T-x}^T f(x+t-T) \cdot dt$$

Since $f$ is almost periodic we can find an increasing sequence $\{T_n\}$ where $T_n \to +\infty$ where

$$\max_x |f(x - T_n) - f(x)| \leq \frac{1}{n} \tag{6}$$

hold for every $n$. With $0 < x < T_n$ we get from (5):

$$G_{T_n}(x) = \frac{1}{T_n} \int_0^T f(x+t) \cdot \bar{f}(t) \cdot dt + + \frac{1}{T_n} \int_{T_n - x}^{T_n} [f(x+t-T_n) - f(x+t)] \cdot \bar{f}(t) \cdot dt$$

By (6) the absolute value of the last term is majorized by $\frac{|f|_\infty}{n}$ which gives:

$$|G_{T_n}(x) - \frac{1}{T_n} \int_0^T f(x+t) \cdot \bar{f}(t) \cdot dt| \leq \frac{|f|_\infty}{n}$$

Apply this with $x = 0$ and a passage to the limit gives:

$$\lim_{T_n \to \infty} \frac{1}{T_n} \int_0^{T_n} f(t) \cdot \bar{f}(t) \cdot dt = 0$$

By the observation from (XX) this limit formula implies that $f$ is identically zero which finishes the proof that Lemma 1.1 gives of Theorem 0.8

*B. Proof of Lemma 1.1.*

EASY.. via uniform continuity estimates...

**C. Proof of Parseval's formula.**

Let $\{\lambda_k\}$ be the Bohr spectrum of $f$. By Bessel's inequality we know that the series formed by $\{|M_f(\lambda_k)|^2\}$ converges which gives us the almost periodic function

$$\phi(x) = \sum |M_f(\lambda_k)|^2 \cdot e^{\imath \lambda_k \cdot x}$$

At the same time the result in (xx) gives us the almost periodic function

$$g(x) = \lim_{T \to \infty} \frac{1}{T} \cdot \int_0^T f(x+t) \cdot \bar{f}(t) \cdot dt$$

Moreover, $\sigma(g) = \sigma(f)$ and

$$M_g(\lambda_k) = |M_f(\lambda_k)|^2$$

hold for each $k$. The same hold for the Bohr spectrum of $\phi$ and it follows that the Bohr spectrum of the difference $\phi - g$ is empty and hence $\phi = g$ by Theorem 0.8. In particular we get

$$\sum |M_f(\lambda_k)|^2 = \phi(0) = g(0) = ||f||^2$$

which proves Parseval's equality for $f$.

**D. The Bochner-Fejer approximation.**

Let $f \in \mathcal{A}$ be given and consider the $\mathcal{B}$-functions defined as in (xx). They all arise via Fejer kernels and using the formula from the periodic case one has

**Proposition.** *For every $\mathcal{B}$-function constructed via (*) wiuth an arbitrary trary $N$ and an initial finite family from$\sigma(f)$ one has the inequality below for every real number $\tau$:*

$$\max_x |\mathcal{B}_N(x+\tau) - \mathcal{B}_N(x)| \leq \max_x |f(x+\tau) - f(x)|$$

**Exercise.** Prove this result using the explicit formulas for Fejer kernels.

Let us now see how one can approximate the given function $f$. let us consider the case when the abelian group generated by $\sigma(f)$ is finitely generated. So now we have some $p$-tuple $\beta_1, \ldots, \beta_p$ and each $w \in \sigma(f)$ is an integer combination of the $\beta$-tuple. If

$$w = m_1\beta_1 + \ldots + m_p\beta_p$$

then we have the equality

$$M_{\mathcal{B}_N}(w) = \prod (1 - \frac{|m_\nu|}{N}) \cdot M_f(w)$$

From this and the $L^2$-convergence of $\{M_f(w)\}$ taken over $\sigma(f)$, it follows that

$$\lim_{N\to\infty} ||f - \mathcal{B}_N||^2 = 0$$

There remains to see that the $L^2$-convergence entails uniform convergence in the maximum norm, i.e. that

$$\max_x |f(x) - \mathcal{B}_N(x)| = 0$$

Proof is EASY via (xx). Call it the Arzela-Bohr Lemma via equi-continuity and uniform $\ell(\epsilon)$ choice in the almost periodicity condition.