V. Uniqueness theorems for analytic functions.

0. Introduction.

A. A sharp version of the Phragmén-Lindelöf theorem

B. Asymptotic series.

C. A uniqueness theorem for aymptotic series

Introduction.

A sharp version of the Phragmén-Lindelöf theorem is proved in Theorem A.2. It is preceeded by a differential inequality where Carelman's result in Theorem A.1 was inspired by earlier constructions due to Lindelöf. Asymptotic series are studied in section B where earlier work by Borel led Carleman to the general construction in Theorem B.1. The question of uniqueness is expressed via Theorem B.6 and is settled via solutions to a variational problem in Section C.

A. The Phragmén-Lindelöf theorem.

Let $f(z)$ be an entire function. To each $0 \leq \phi \leq 2\pi$ we set

(*) $$\rho_f(\phi) = \max_r |f(re^{i\phi})|$$

The text-book *Le calcus des residues* by Ernst Lindelöf contains examples of entire functions f where $\rho_f(\phi)$ is finite for all ϕ with the exception $\phi = 0$, i.e. only along the positive real axis the ρ-number fails to be bounded. An example is the entire function

$$f(z) = \frac{1}{z^2} \cdot \sum_{\nu=2}^{\infty} \frac{z^\nu}{(\log \nu)^\nu}$$

Here one verifies that that there exists a constant k such that:

(**) $$|f(re^{i\phi})| \leq \exp(e^{\frac{k}{|\phi|\cdot|2\pi-\phi|}})$$

It turns out that the example above is essentially sharp. Namely, assume that the ρ-number in (*) is finite for almost every ϕ. Then the ρ_f-function cannot be too small, unless f is reduced to a constant. Before Theorem A.1 is announced we introduce the non-negative function

(***) $$\omega(\phi) = \log\big[\log^+ \rho_f(\phi)\big]$$

Thus, we have taken a two-fold logarithm which means that $\omega(\phi)$ is considerably smaller compared to the ρ-function.

A.1.Theorem. *For every non-constant entire function $f(z)$ one has*

$$\int_0^{2\pi} \omega(\phi) \cdot d\phi = +\infty$$

Proof. Assume that f is not a constant. Consider the maximum modulus function

$$M(r) = \max_{|z|=r} |f(z)|$$

By the ordinary Liouville theorem the M-function increases to infinity. So we may assume that $M(r) \geq 1$ when $r \geq r_*$ for some r_*. Put

(i) $$v(r) = \operatorname{Log} M(r) \quad : \quad U(z) = \operatorname{Log}|f(z)|$$

Given r we consider the domain

(ii) $$\Omega_r = \{U > \frac{v(r)}{2}\} \cap \{|z| < r\}$$

Next, let ζ_r be some point on the circle $|z| = r$ where $|f(\zeta_r)| = M(r)$ where ζ_r always can be chosen so that there exist arbitrary small δ where $|f(\zeta_{r-\delta})| = M(r-\delta)$ and $\lim_{\delta\to 0} \zeta_{r-\delta} = \zeta_r$. Next, in Ω we get the connected component Ω_* whose boundary contains ζ_r. Put

(iii) $$\gamma = \partial\Omega_* \cap \{|z| = r\}$$

Notice that

(iv) $$U(z) \leq \frac{v(r)}{2} \quad : \quad z \in \partial\Omega_* \cap \{|z| < r\}$$

So if W is the harmonic function in the disc D_r with boundary values 1 on γ and 0 on $\{|z| = 1\} \setminus \gamma$ we have:

(v) $$U(z) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(z) \leq 0 \quad : \quad z \in \partial\Omega_*$$

The maximum principle entails that (v) also holds in Ω_*. Hence there exist arbitrary small $\delta > 0$ such that

(vi) $$v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(\zeta_{r-\delta}) \leq 0$$

Let $2r \cdot \ell$ be the total length of the intervals which belong to γ. By the general inequality from XX we have

(vii) $$W(\zeta_{r-\delta}) \leq \frac{1}{2\pi} \int_{-\ell}^{\ell} \frac{r^2 - (r-\delta)^2}{r^2 - 2r(r-\delta)\cos\theta + (r-\delta)^2} d\theta$$

Let $h(r-\delta)$ denote the right hand side in (vii) which by (vi) gives us arbitrary small $\delta > 0$ such that

(viii) $$v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot h(r-\delta) \leq 0$$

Rewriting this inequality we obtain

(*) $$\frac{v(r) - v(r-\delta)}{\delta} \geq \frac{v(r)}{2} \cdot \frac{1 - h(r-\delta)}{\delta}$$

Next, from the definition of the h-function one has the limit formula

(ix) $$\lim_{\delta\to 0} \frac{1 - h(r-\delta)}{\delta} = \frac{1}{2\pi} \cdot \frac{\cos\ell}{\sin\ell}$$

Passing to the limit as $\delta \to 0$ in (vii) we get the differential inequality:

(**) $$v'(r) \geq \frac{v(r)}{2\pi r} \cdot \frac{\cos\ell}{\sin\ell}$$

Next, put

$$\log r = s \quad \text{and} \quad \log \frac{v(r)}{2} = g(s)$$

By derivation rules we see that (**) gives

(***) $$\frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos\ell}{\sin\ell}$$

Next, identifying γ with a subset of the periodic interval $0 \leq \phi \leq 2\pi$ it is clear that the definition of the ω-function gives the inclusion

(x) $$\gamma \subset \{\omega(\phi) \geq g(s)\}$$

So if $\lambda(s)$ is the Lebesgue measure of the set $\{\omega(\phi) \geq g(s)\}$ then $\ell \leq \lambda(s)$ and (***) gives

(****) $$\frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos\lambda(s)}{\sin\lambda(s)}$$

Next, the inequality $\sin(t) \geq \frac{2}{\pi} \cdot t$ gives a positive constant k which is independent of s such that the following hold for sufficiently large s, i.e. to ensure that the corresponding r-value satisfies $M(r) \geq 1$:

(xi)
$$\frac{dg}{ds} \geq \frac{k}{\lambda(s)}$$

Hence, starting from some sufficiently large s_0 one has

(xii)
$$\int_{s_0}^{s} \lambda(s) \cdot dg(s) \geq k(s - s_0)$$

This inequality implies in particular that one has a divergent integral:

(xiii)
$$\int_{s_0}^{\infty} \lambda(s) \cdot dg(s) = +\infty$$

Finally, the general equality for distribution functions from XXX gives:

(xiiii)
$$\int_0^{2\pi} \omega(\phi) \cdot d\phi = \int_0^{\infty} \lambda(s) \cdot dg(s)$$

The last integral is $+\infty$ by (xiii) and the requested divergence for the intergal of the ω-function follows.

Remark. At the end of the article [XXX] Carleman points out that the proof above gives a sharp version of the Phragmén- Lindelöf theorem. More precisely one has the following: Let $f(z)$ be analytic in a sector

$$S_\alpha = \{z = re^{i\phi} \quad : \quad -\alpha < \phi < \alpha\}$$

Define $\omega(\phi)$ as above when when $-\alpha < \phi < \phi$. With these notations one has:

A.2. Theorem. *Let f be bounded on the half-lines arg$(z) = \alpha$ and arg$(z) = -\alpha$ and assume also that*

$$\int_{-\alpha}^{\alpha} \omega(\phi) \cdot d\phi < \infty$$

Then $f(z)$ is bounded in the whole sector.

A.3. Exercise. Deduce Theorem A.2 from the preceeding results.

B. Asymptotic series.

Introduction. The notion of asymptotic series was expressed as follows by Poincaré:

Let $f(z)$ be complex-valued function defined in some subset E of $\mathbf{C}$ and z_0 is a boundary point. We say that f has an asymptotic series expansion at z_0 if there exists a sequence of complex numbers $c_0, c_1, \ldots$ such that $\lim_{z \to z_0} f(z) = c_0$ and for each $n \geq 0$ one has:

(*)
$$\lim_{z \to z_0} (z - z_0)^{-n-1} \left[f(z) - (c_0 + c_1 + \ldots + c_n z^n)\right] = c_{n+1}$$

where the limit is taken as z stay in E.

It is obvious that if f has an asymptotic expansion at z_0 then the sequence $\{c_n\}$ is unique. Constructions of functions which admit asymptotic expansions appear in Emile Borel's thesis *Sur quelques points de la théorie des fonctions* from 1895 and he proved for example that for every sequence of real numbers $\{c_n\}$ there exists a C^∞-function $f(x)$ on the real line whose Taylor expansion at $x = 0$ is given by the sequence, i.e.

$$\frac{f^{(n)}(0)}{n!} = c_n \quad : \quad n = 0, 1, \ldots$$

Following [Car: xx, page 29-31] we prove a complex version of Borel's result where D_+ denotes the open half-disc $\{\mathfrak{Re}(z) > 0 \cap \{|z| < 1\}$.

B.1. Theorem. *To each sequence $\{c_n\}$ of complex numbers there exists a bounded analytic function $F(z)$ in D_+ which has an asymptotic series expansion at $z = 0$ given by $\{c_n\}$.*

Proof. It suffices to prove this when $c_0 = 0$. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $\sum a_\nu < \infty$. Given $\{c_n\}$ we construct a sequence of functions $P_1(z), P_2(z), \ldots$ which are analytic in the half plane $\mathfrak{Re}(z) > 0$ as follows: First

$$\text{(i)} \qquad P_1(z) = c_1 z(1 - \frac{z}{z+\epsilon_1}) \quad : \epsilon_1 = \frac{\alpha_1}{|c_1|} \implies$$

$$\text{(ii)} \qquad |P_1(z)| = |c_1| \cdot \epsilon_1 \cdot \frac{|z|}{|z+\epsilon_1} \leq \alpha_1 \quad : \quad \mathfrak{Re}(z) \geq 0$$

Now $P_1(z)$ has a series expansion at $z = 0$:

$$\text{(ii)} \qquad P_1(z) = \sum_{\nu=1}^{\infty} c_\nu^{(1)} \cdot z^\nu$$

Notice that the series converges in the disc $|z| < \epsilon_1$. Set

$$\text{(iii)} \qquad P_2(z) = \left[c_2 - c_2^{(1)}\right] \cdot z^2 \cdot \left(1 - \frac{z}{z+\epsilon_2}\right) \quad : |c_2 - c_2^{(1)}| \cdot \epsilon_2 \leq a_2$$

With such a careful choice of a small positive ϵ_2 we see that

$$\text{(iii)} \qquad |P_2(z)| \leq a_2 \cdot |z| \quad : \quad \mathfrak{Re}(z) \geq 0$$

Again we obtain a convergent series at $z = 0$:

$$\text{(iv)} \qquad P_2(z) = P_1(z) = \sum_{\nu=2}^{\infty} c_\nu^{(2)} \cdot z^\nu$$

The inductive construction. Let $n \geq 3$ and suppose that $P_1, \ldots, P_{n-1}$ have been constructed where we for each $1 \leq k \leq n-1$ have a series expansion

$$\text{(v)} \qquad P_k(z) = \sum_{\nu=k}^{\infty} c_\nu^{(k)} \cdot z^\nu$$

Then we define

$$P_n(z) = \left[c_n - (c_n^{(1)} + \ldots + c_n^{(n-1)}\right] \cdot z^n \cdot \left(1 - \frac{z}{z+\epsilon_n}\right) \quad : \quad \left| \, c_n - (c_n^{(1)} + \ldots + c_n^{(n-1)} \, \right| \cdot \epsilon_n \leq \alpha_n$$

So we obtain a new series at $z = 0$:

$$\text{(vi)} \qquad P_n(z) = \sum_{\nu=n}^{\infty} c_\nu^{(n)} \cdot z^\nu$$

Staying in the half-disc D_+, the inductive construction gives

$$\max_{z \in D_+} |P_n(z)| \leq \alpha_n \quad : \quad n = 1, 2, \ldots$$

Hence there exists a bounded analytic function in D_+ defined by

$$F(z) = P_1(z) + P_2(z) + \ldots$$

At this stage we leave as an exercise to the reader to verify that

$$\lim_{z \to 0} z^{-n-1} \cdot \left[\, F(z) - (c_1 z + \ldots + c_n z^n)\right] = c_{n+1}$$

B.2. Uniqueness of asymptotic expansions.

There exist functions whose asymptotic series is identically zero. Here is an example:

$$F(z) = e^{-\frac{1}{z^2}}$$

If $z = re^{i\theta}$ with $-\pi/8 \leq \theta \leq \pi/8$ we see that

$$|F(re^{i\theta})| = \exp\,(-\frac{\cos 2\theta}{r^2}) \leq \exp(-\frac{1}{\sqrt{2}\cdot r^2})$$

It follows that the asymptotic series at $z = 0$ is identically zero. Via a conformal map from the half-disc D_+ above to the unit circle we are led to the following problem: Let $F(z)$ be analytic in the open unit disc D. Suppose that

$$(*) \qquad \lim_{z \to 1} \frac{F(z)}{(1-z)^n} = 0 \quad : \quad n = 1, 2, \ldots$$

We seek growth conditions on F in order that (*) implies that F is identically zero. A complete answer to this uniqueness problem was proved by Carleman in [Car]. First we exhibit a class of functions whose asymptotic series vanish identically. Namely. consider a sequence of real positive numbers $A_1, A_2, \ldots$. To each $n \geq 1$ we put

$$(**) \qquad I_n = \exp\big(\,\frac{1}{\pi}\int_1^\infty \text{Log}\,[\,\sum_{\nu=1}^{\nu=n}\frac{r^{2\nu}}{A_\nu 2}\,]\cdot dr\,\big)$$

B.3. Definition. *Denote by $\mathfrak{B}$ the set of all sequences $\{A_n\}$ such that the associated sequence $\{I_n\}$ is bounded, i.e. there exists some K such that*

$$I_n \leq K \quad : \quad n = 1, 2 \ldots$$

In [Car: page 7-52] the following existence result is proved:

B.4. Theorem. *To each sequence $\{A_n\} \in \mathfrak{B}$ there exists an analytic function $f(z)$ in D which is not identically zero and satisfies:*

$$(***) \qquad \frac{|f(z)|}{|(1-z)^n|} \leq A_n \quad : \quad n = 1, 2, \ldots$$

B.5. A converse result. In [loc.cit] appears also proof of the converse to the result above.

B.6. Theorem. *Let $\{A_n\}$ be a sequence of positive numbers such that there exists an analytic function $f(z)$ in D which is not reduced to a constant and satisfies (***) in Theorem 4. Then $\{A_n\} \in \mathfrak{B}$.*

The proof of the two results above rely upon a varational problem which is presented below while the deduction after of the two cited results above are left to the reader who may find details in [Carleman].

C. A variational problem.

Let $n \geq 1$ and $a_0, a_1, \ldots, a_n$ some n-tuple of non-negative real numbers where $a_0 > 0$ is assumed. Let $\mathcal{O}(*)$ denote the family of analytic functions $f(z)$ in the unit disc satisfying $f(0) = 1$. Put

$$I(f) = \frac{1}{2\pi}\cdot\sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \int_0^{2\pi} \frac{|f(e^{i\theta})|^2}{|e^{i\theta}-1|^{2\nu}}\cdot d\theta \quad : \quad I_* = \min_{f \in \mathcal{O}(*)} I(f)$$

Remark. Above we have a variational problem. It turns out that there exists a unique function $f_*(z)$ which yields a minimum. To find f_* we shall use the rational function:

$$\Omega(z) = \sum_{\nu=0}^{\nu=n} a_\nu^2 \big[(1-z)(1-\frac{1}{z})\big]^{n-\nu}$$

Notice that

$$\text{(i)} \qquad \Omega(e^{i\theta}) = a_0^2 + \sum_{\nu=1}^{\nu=n} a_\nu^2 \cdot |e^{i\theta} - 1|^{2n-2\nu}$$

In particular Ω is real and positive on the unit circle and by symmetry it has n zeros $\rho_1, \ldots, \rho_n$ in the unit disc and $\frac{1}{\rho_1}, \ldots, \frac{1}{\rho_n}$ are the zeros in the exterior disc. Thus

$$(*) \qquad \Omega(z) = z^{-n} \cdot \frac{(-1)^n}{a_0^2} \cdot p_n(z) \cdot \prod (z - \frac{1}{\rho_\nu}) \quad : p_n(z) = (z - \rho_1) \cdots (z - \rho_n)$$

Next, let us put

$$\text{(ii)} \qquad \phi(z) = \frac{f(z)}{(1-z)^n}$$

From (i) we see that

$$\text{(iii)} \qquad I(f) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |\phi(e^{i\theta})|^2 \cdot d\theta$$

We will use the last expression to prove

C.1 Theorem. *The variational problem has a unique solution where the minimum I_* is achieved by the function*

$$f_*(x) = (1-z)^n \cdot \frac{1}{\prod 1 - \rho_\nu \cdot z)}$$

Moreover,

$$I_* = \frac{1}{2\pi} \cdot \int_0^{2\pi} \text{Log}\, [\sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \frac{1}{(2 \cdot \sin\frac{\theta}{2})^{2\nu}}] \cdot d\theta$$

Proof By (iii) the variational problem is equivalent to seek the minimum of

$$(1) \qquad I(\phi) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |\phi(e^{i\theta})|^2 \cdot d\theta \quad : \phi(0) = 1$$

With f_* as in the theorem we get the ϕ-function

$$(2) \qquad \phi_*(z) = \frac{1}{\prod 1 - \rho_\nu \cdot z)}$$

Now f_* is a unique minimizing function if we have proved the strict inequality

$$(3) \qquad I(\phi_* + h) < I(\phi_*)$$

for every analytic function $h(z)$ in D with $h(0) = 0$. To show this we notice that (1) can be replaced by a complex line integral over $|z| = 1$ which gives

$$I(\phi_* + h) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \Omega(z) \cdot |\phi(z) + h(z)|^2 \cdot \frac{dz}{z} =$$

$$(4) \qquad I(\phi_* + I(h) + \frac{1}{2\pi i} \cdot \int_{|z|=1} \Omega(z) \cdot [\bar{\phi}_*(z) \cdot h(z) + \phi_*(z) \cdot \bar{h}(z)] \cdot \frac{dz}{z}$$

Since $\Omega = \bar{\Omega}$ holds on $|z| = 1$ where we also have $\bar{z} = z^{-1}$, it follows from (*) that $|z| = 1$ entails

$$\Omega(z) = z^n \cdot \frac{(-1)^n}{a_0^2} \cdot \prod \left(\frac{1}{z} - \frac{1}{\rho_\nu}\right) \cdot \prod \left(\frac{1}{z} - \rho_\nu\right) =$$

(ii) $$\frac{(-1)^n}{a_0^2} \cdot \prod (1 - z\rho_\nu) \cdot \prod \left(\frac{\rho_\nu}{z} - 1\right) \cdot \frac{1}{\rho_1 \cdots \rho_n}$$

At the same time (2) gives

(ii) $$\bar{\phi}_*(z) = \left(\prod \left(1 - \frac{\rho_\nu}{z}\right)\right)^{-1}$$

Hence

$$\Omega(z) \cdot \bar{\phi}_*(z) = \frac{1}{\rho_1 \cdot \rho_n \cdot a_0^2} \cdot \prod (1 - \rho_\nu \cdot z)$$

Since $h(0) = 0$ it follows that

$$\int_{|z|=1} \Omega(z) \cdot \bar{\phi}_*(z) \cdot h(z) \cdot \frac{dz}{z} = 0$$

In the same way we get

$$\int_{|z|=1} \Omega(z) \cdot \phi_*(z) \cdot \bar{h}(z) \cdot \frac{dz}{z} = 0$$

Hence the last integral in (4) is zero which shows that

$$I(\phi_* + h) = I(\phi_* + I(h)$$

and since $I(h) > 0$ we get the strict inequality in (3).