Sonja Kovalevsky

She was born in Moscow in January 1850 and baptized in a church which is still exists. She grew up at her family estate in Palibino, a small village situated 600 kilometers south of S:t Petersburg. Sonja never attended a regular school but received very good education at home from private teachers, her parents and several uncles. As a teenager she read literature in English, French and German fluently. Sonja was very attached to her seven years older sister Anjuta who inspired her to read poetry and literature. A dramatic even took place at Palibino in 1865 when the parents discovered that Anjuta had published several novels in Epocha, a literary magazine edited by Feodor Dostojevski. After a time Anjuta's correspondence with this famous author was not only accepted but both Anjuta and Sonja could move to S:t Petersburg where they lived with relatives to their mother. Sonja was very impressed by Dostojevski and wrote later about her meetings with him in her book *My Russian childhood* which was published in a Swedish original and became a veritable bestseller in Stockholm at Christmas 1888.

After the arrival to S:t Petersburg Sonja started serious studies in mathematics. She received private lessons by a prominent university professor and already after one year she mastered advanced spherical geometry and basic analysis comparable to contemporary university education in mathematics up to final undergraduate level. Soon after her 18:th birthday she married Vladimir Kovalevsky. After a period where Sonja studied both medicine and mathematics, Sonja and Vladimir travelled together with Julia Lermontova to Heidelberg. Julia became like Sonja a pioneer for women's succes in science. In 1874 she was the first women in history to defend a doctor's thesis in chemistry at Göttingen. Vladimir became one of Charles Darwin's most prominent pupils. His dissertation from the University of Jena in 1874 in palenteology was of a very high standard. Sonja studies in mathematics and theoretical physics at Heidelberg were also succesful and she was of course recognized among the other graduate students at that time. An example is the British mathematician Hill who told about Sonja's talents as a mathematican to the author Mary Evans while he was on leave from Heidelberg to visit in London. Half a year later while Vladimir and Sonja made a travel in England this led to an invitation from Mary Evans so that Sonja could meet her idol in literature whose famous novels she had read in English original since a teenager. For readers less aquainted with literature we mention that Mary Evans wrote under the male pseudo-name George Eliot which most readers have heard about. Sonja's meeting with Mary in 1870 was later resumed by Sonja in a series of articles in Stockholms Dagblad in 1885 shortly after Mary's desease. They tell also about Sonja's own character and except for her her admiration and respect for Mary as an author her articles in the newspaper are still worth reading since they may inspire new generations of readers to meet novels by Mary Evans.

After having solved some hard "test problems" in 1871, Sonja was accepted as a private student by Karl Weierstrass who at that time was the world-leading mathematician. After three years of intensive research in Berlin, Sonja presented a doctor's thesis containing several important articles. The major work, devoted to the theory of partial differential equations was published in Journal für die reine und angewandte mathematik in 1875. It was recognized and appreciated by the whole mathematical community in Europe at that time. After their disserations in Germany, Sonja and Vladimir returned to Russia in the autumn 1874. They were received by a family party in Palibino but both met soon after obstacles since their doctor's degrees from abroad did not help them to get academic positions. They settled in S:t Petersburg where Vladimir started a journal foremost devoted popular articles about science and also politics. For Sonja's the doors were closed. She was not even admitted to teach mathematics on a special gymnasium for girls. So for several years Sonja left research in mathematics. But she followed advancements in science where she was trained in medicine and chemisitry. She followed for example closey the new discoveries by Pasteur in medicine and even applied Pasteur's technology to improve local production of beer at a factory in S:t Petersburg. The "Fermanta Scandal" in Sweden twenty years ago had hardly occurred if more people had read Sonja's articles about fermentation from 1876. One of her heroes was John Ericsson. It may be of interest today that Sonja studied reports by Ericsson in American journals around 1875 at a time when he was engaged in methods to transform *solar energy*. Several of Ericsson's contributions which he carried out during his last twenty years until his disease 1889 have had a lasting impact on methods for production of solar energy.

Sonja's and Vladmir's daughter Sofie was born in 1878 and a year later they moved to Moscow where Vladimir finally got a position at the university. At that time Sonja's high standard as mathematician had been recognised and she got support by several prominent mathematicians such as Chebyshef. She became also member of the Moscow mathematical society and was invited to give lectures at conferences in S:t Petersburg at several occasions between 1878 and 1880. But as a woman she could not get an academic position. After the murder of tsar Alexander I in March 1881, the political climate became hard in Russia. Many friends to Sonja and Vladimir were imprisoned and Vladimir came under severe attack when he was accused - without any justification - for criminal affairs by an oil-prospecting company which he had helped as a scientist. The next two years became a disaster for Vladimir and in March 1883 he committed suicide at his home in Moscow. At that time Sonja lived with her sister Anjuta in Paris while her daughter stayed with Vladimir's elder brother Alexander Kovalesvky. Thanks to her sister Anjuta, Alexander and Julia Lermontova who was Sofie's godmother, Sonja was able to recover during her stay at Alexander's home in Odessa during the summer 1883. At this critical moment, Gösta Mittag Leffler invited her to come to Stockholm and give seminars in mathematics and eventually get a position at the young University in Stockholm which in those days only had 20 employed teachers with chemistry, biology and mathematics as main subjects.

Life in Stockholm. Sonja arrived to Stockholm in November 1883. In those days the city was small and shortly after her arrival she was presented in the newspaper as "The Queen of Science". Her first public seminar took place in central Stockholm at Lundberska Huset situated at the corner of Kungsgatan and Vasagatan. A large audience attended, though only a few probably understood a word while she talked about new methods in calculus of variation in order to settle the Dirichlet problem based upon recent discoveries by Weierstrass. In any case, her talk which was delivered in German was saluted. At Stockholm University she met collegues of high standard, except for Gösta Mittag Leffler the young doctors Phragmén and Bendixson who later both became professors in mathematics. In June 1884 she got a five year appointment as associate professor in higher analysis.

Since "Stockholms Högskola" was not an official University like Uppsala and Lund the financial support was based upon private donations and to some extent also by the City of Stockholm. Sonja could live without problem on her salary which - as a comparison - was five times higher than the best paid fully employed and skilled industrial workers. During the academic year in 1886 Sonja was even "double professor" when she in addition to her position at the university was temporary professor in mechanics at the Royal Institute of Technology until Anders Lindstedt got the permanent professorship.

Socially Sonja got many friends in Stockholm. Among these one should mention Nordenskjöld who at that time was the chairman of the academy of science. He admired Sonja's scientic career and was specially interested in her background from Russia. As an example where Sonja took indirect part in other scientific areas than mathematics, one may mention that she was in a jury to deliver special scholarships for inviting scientists to Stockholm. This led to the visit of Maxim Kovalevsky. His aftername was only by accident the same as Sonja's. During his visit to Sweden Maxim gave invaluable help for future studies in history when he read original handwritten letters in the Russian language by Johan Gabriel Sparwenfeld from his famous visit to Russia between 1684-87. These collections placed at Uppsala University are of considerable historic interest since Sparvenfeld not only knew the Russsin language quite well but also documented meeting with interesting people during his stay in Russia from this remote period. For the history of mathematics one may also mention that Sparwenfeldt had intense contact with Leibniz for several decades when they delivered books and communicated about linguistic questions.

During Sonja's first two years in Stockholm her daughter Sofie stayed with her uncle and cousins in Odessa while Sonja rented a room where Julia Kjellberg had lived. Julia, recognised as the leading person in the women's movement in those days, was married to Georg Wollmar whom Sonja had met in Paris already in 1882. Sonja was very impressed by his political vision and they became close friends. Wollmar was the founder of the social democratic party in Germany and became also a veritable mentor to Hjalmar Branting . His visit to Stockholm in 1885 was a historic landmark for the future evolution of the Social Democratic party in Sweden. Branting assisted by translating Wolmar's speech to the public at "Folkets Hus" into Swedish. Hjalmar Branting was 10 years younger than Sonja and had started his university studies with mathematics and was for some time employed at the institute of astronomy, a subject he had adored since early childhood. So Hjalmar and his wife Anna also became two of Sonja's close friends during her time in Stockholm.

When Sofie was 8 years old she came to Stockholm to start school. At that time Sonja had consolidated her life in Stockholm and together with her daughter she moved to an appartement with the adress Sturegatan 56. The house is still intact. Sofie learnt quickly Swedish and among her teachers at Whitlockska Skolan was Ellen Key who frequently visited Sonja's and Sofie's apartement. During this period her beloved sister Anjuta died in S:t Petersburg in 1887 which was a severe loss for Sonja. Like her sister, Sonja's health gradually became worse. Weak lungs led to her disease on January 11 in 1891 at the age of 41 years. Many of her friends wrote memory articles about her life and personality. Among these one can mention Elisaveta Levontova who like Sonja had studied mathematics abroad at Zürich with Hermann Schwarz as supervisor. Elisaveta met many obstacles after her return to Russia but after twenty years struggle as a barely supported teacher in elementary classes, she finally became - at the age of fifty years - the first women ever to teach high-school pupils in Russia. She wrote many articles which were published both in Russia and abroad concerned with methods to improve education in mathematics for young people. Her collected work in didactics devoted to the teaching of mathematics is classic and was used extensively in the former Sovjetunion until the end of 1950. Her ideas how one should conduct teaching up to beginning university mathematics are still worth studying.

Sonja Kovalevsky held extensive lecture series during eleven academic terms in Stockholm during the years 1884-1890. The subjects were differential equations, calculus of variation, abelian integrals and mechanics. She was also member of the editorial board for Acta mathematica. She is buried in Stockholm where her grave has been kept under very good protection. The family estate from her childhood at at the village of Palibino has been renovated and is today a museum which bears her name.

THE KOVALEVSKY GYROSCOPE

Introduction These notes give background to readers who would like to study in more detail Sophie Kovalevsky's article *Sur le probleme de la rotatiation d'un corps solide autotur d'un point fixe* which appeared in Acta Mathematica 1889. A work for which she received the *Bordin Prize* on Christmas Eve 1888 in Paris. Let us present the first section of the article in English translation.

The problem of the rotation of a rigid body around a fixed point with gravity force g reduces, as is wellknown, to solve the following system of differential equations:

$$
\begin{aligned}
A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma') \quad &:& \quad \frac{d\gamma}{dt} = r\gamma' - q\gamma'' \\
B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma'') \quad &:& \quad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma \\
C\frac{dr}{dt} &= (A-B)rpq + Mg(x_0\gamma' - y_0\gamma) \quad &:& \quad \frac{d\gamma''}{dt} = q\gamma - p\gamma'
\end{aligned}
$$

The constants $A, B, C, M, g, x_0, y_0, z_0$ denote the following: A, B, C are the positive eigenvalues of the intertia operator of the body, i.e. they correspond to lengths of the principal axis when the body is regarded as an ellipsoid. M is the mass of the body; g the intensity of the gravity force; x_0, y_0, z_0 are the coordinates of the center of mass in a coordinate system given by the principal axis above.

Until now one has only found two cases where the equations of motion can be integrated and thus solved by quadrature:

The case of Poisson (or Euler): $x_0 = y_0 = y_0$.

The case of Lagrange: $A = B$: $x_0 = y_0 = 0$.

In these two cases the solution is found after integration by theta functions. The six quantities $p, q, r, \gamma, \gamma', \gamma''$ are time-dependent functions which extend to complex analytic functions whose singularities are at most poles.

At this stage Kovalevsky ends the introduction of the article by raising a question:

Les integrales des equations differentielles considerées conservent-elles cette propriete dans le cas general.

After this Kovalevsky studies power series solutions to the system above and finds constraints upon their complex Laurent series expansions which leads to the conclusion that the constants A, B, C and the position of the center of mass, must be *quite special* in order to achieve a situation of similar nature as in the two classical cases above.

Remark. Above the following notations are used. First (p, q, r) are the components of Euler's angular velocity expressed in an orthonormal frame formed by principal axis in the body space. Next, the γ-vector represents the position of e_z in the coordinates which are fixed in the body. Finally, in the space $\mathbf{R}^3$ where the body rotates at a fixed point centered at the origin, the force of gravity acts in the negative z-direction.

After five pages of tedious calculations, Kovalevsky presents an example in section 2 from [Kov:1] which leads to a rigid body whose equations of motion can be solved by quadrature for any initial position. More precisely, in the Kovalevsky gyroscope one has:

$$A = B = 2 \quad C = 1 \quad x_0 = 1 \quad y_0 = z_0 = 0$$

The equality $A = B$ means that the body has a plane of symmetry and the center of mass is placed in this plane. But this center of mass does not belong to a principal axis so the gyroscope fails to satisfy the assumption which was treated earlier by Lagrange. In the case above Kovalevsky discovered a *fourth integral* of the differential system above which is expressed by a polynomial of the 4:th degree of the variables $p, q, r, \gamma, \gamma', \gamma''$ and their time-derivatives. Once this is achieved there remained a considerable work to express the solution. Here 15 pages in the article contain calculations where various elliptic functions appear. More precisely, in the specific example Kovalevsky derived a solution expressed by *ultra-elliptic functions* for a class of differential systems whose coefficients are square roots polynomials of any order.

Historic comments. Two years after Kovalevsky's disease, Liapounov proved that her example is *unique*, i.e. it gives the sole example except the classical, where the equations of motion can be solved by quadrature. One may also mention that her former teacher Königsberg in an article published in Acta Mathematica expressed Kovalevsky's gyroscope in Eulerian angles where the exact solution in these "concrete" variables are given by generalised theta-functions. Königsberg's formula together with a computer provide an accurate picture of the motion of the Kovalevsky gyroscope. To avoid possible confusions we remark that there exist examples where the differential system for the motion can be integrated *provided* that special initial conditions are given. But these easy examples were not even at stake in the far more difficult problem where Kovalesvky found an example.

A final Remark. Kovalevsky gave courses in many different areas during her years in Stockholm 1884-1890. The connection between equations derived from the "real world" of mechanics and various complex integrals which for example lead to elliptic functions was often put forward in her lectures. Let us illustrate this by an example which normally belongs to an elementary courses devoted to analytic function theory. Consider Weierstrass' $\mathfrak{p}$-function which is doubly meromorphic with respect to some lattice in $\mathbf{C}$ generated by two $\mathbf{R}$-linearly independent complex vectors ω_1 and ω_2. Set

$$e_1 = \frac{\omega_1}{2}\,, \quad e_2 = \frac{\omega_1}{2}\,, \quad e_3 = \frac{\omega_1 + \omega_2}{2}\,.$$

Now there exists Jacobi's elliptic function

$$\mathfrak{sn}(z) = \frac{\sqrt{e_1 - e_2}}{\sqrt{\mathfrak{p}(z) - e_3}}$$

Jacobi's inversion formula asserts that

$$z = \frac{1}{\sqrt{e_2 - e_3}} \cdot \int_0^{\mathfrak{sn}(z)} \frac{dz}{\sqrt{(1 - z^2)(1 - \chi^2 z^2)}}$$

One can show that this inversion formula is a consequence of conservation laws applied to a two-particle system describing the motion of two mass-points which both perform a periodic motion in a simple pendelum. The addition formula for the $\mathfrak{p}$-function, and more generally for any elliptic function can also be derived via laws of classical mechanics. So students interested in analytic function theory or algebraic geometry devoted to projective curves in characteristic zero, should keep in mind that many basic results results concerning properties of rational functions on such curves have a natural interpretation via classical mechanics and may therefore be considered as a Law of Nature rather than a mathematical discovery. For example, in order to fully appreciate the contents in those sections of Kovalevsky's article devoted to hyper-elliptic functions one should be aware of the interpretation from classical mechanics.

A Lecture in Classical Mechanics.

Introduction. Dynamical problems in classical mechanics are solved with the aid of the d'Alembert-Lagrange equations. These equations have the merit that they apply to all systems, i.e. even non-conservative. For example, d'Alembert-Lagrange equations are used to study complicated machines where external forces via an engine affect the motion. In addition to particle systems we will derive the Euler-Langrange equations for rigid bodies. Another important feature is the notion of *relative motion* where one employs two systems at the same time. The reason for working in two different systems is that Newton's Law of Forces expressd by the equality $F = m \cdot \mathfrak{a}$ does not hold in a moving frame. We shall illustrate this in a section devoted to *Foucault's pendelum* and the *force of Coriolis* which affects motion in the atmosphere causesd by the rotation of the earth around its polar axis. Let us also recall the *Law of Gravity* which asserts that any pair of bodies with masses m_1, m_2 and distance r attract each other with a force given by

$$(*) \qquad \gamma \cdot \frac{m_1 m_2}{r^2}$$

where γ is a universal constant. A modification of (*) appears in Einstein's work from 1916 on General Relativity Theory. But it is of such a small order that equations derived within classical mechanics via Newton's law is sufficient to determine the motion of satellites up to a very high degree of accuracy. Thus, classical mechanics is still "up-to-date" in most applications of "ordinary engineering and for the motion of satellites which do not circulate too far away from the earth. However, even in Newton's mechanics involved computational problems may appear. An example is influence of gravity from three bodies, the earth, the moon and the sun. In addition to this one must take into the account that the earth itself is not a perfect sphere. This illustrates that very few equations in classical mechanics can be solved analytically. Hence the use of computers for numerical solutions is very important. But to to set up equations of motion is a purely theoretical question. For the student it is therefore instructive to learn how to set up in some simple systems such as rotating rigid bodies.

A first exercise. Here is a typical examination problem in mechanics from Cambridge University from 1880: *Let K be a disc of radius R which has constant density of mass and rotates without friction on the horizontal (x, y)-plane with its center fixed and with angular velocity ω. Suddenly it is impinged at a point p whose distance to the center is r for some $0 < r < R$. After this the body rotates with a new angular velocity θ while p stays fixed..*

The problem is to determine θ and also evaluate loss of kinetic energy during this impact. Readers who cannot solve this should learn the subsequent material about rigid bodies and after it becomes an easy exercise.

One may well imagine that the subsequent material was presented more or less verbatim by Sonja Kovalevsky when she gave lectures in mechanics at Stockholm University and at the Royal Institue Technology during the years 1885-1890. So for the student of today it may be of interest to meet the contents of a typical course from these days when emphasis was to relate notions of mathematics with examples from the real world. Let us also recall that questions concerned with teaching of mathematics was discussed at many university and other high schools in Europe in those days. The prominent mathematicans Felix Klein and Henri Poincaré took active part in these discussions. Their articles from this period are worth reading today. Here is a citation from Poincaré in his article *La Logique et l'Intuition dans la Science Mathematique et l'Enseignement*:

Dans l'ensigenment, il est indispensiable de faire appel à l'intuition pour déveloper certains facultés d l'esprit, utiles au savant et surtout à l'ingenieur ....

The subsequent material is presented in the spirit of Poincaré's view upon how basic mathematics should be taught.

The principle of Archimedes

The principle of Archimedes clarifies floating positions of a body K in liquid such as water. We use the euclidian coordinates x, y, z where (x, y) are horizontal and z vertical. This means that gravity acts by the force $-ge_z$ and we suppose that the equation of the free water line is $z = 0$. Thus, K is at rest in a sea where a portion K_* is below the free waterline. The boundary ∂K_* has two parts. One is the intersection with $z = 0$ and the remaining part is a surface $S = \{\partial K_* \cap \{z < 0\}$. In general one does not assume that S is connected. Let dA denote the infinitesmal area of the surface S. As explained by Figure xx the *lifting force* on the body acts on S. When $p = (x, y, z) \in S$ the lifting force on a small area element dA close to p is equal to

$$-gz \cdot -\mathbf{n} \cdot dA = gz\mathbf{n} \cdot dA$$

where $\mathbf{n}$ is the outer normal to the domain K_* and we have assumed that K floats in water whose specific weight one. Hence the total lifting force on K is the area integral

$$g \cdot \iint_S z\mathbf{n} \cdot dA \tag{*}$$

Since $z = 0$ on the remaining portion of ∂K_* we can identify (*) with the full area integral

$$g \cdot \iint_{K_*} z\mathbf{n} \cdot dA \tag{**}$$

The z-component of the lifting force becomes

$$L_z = g \cdot \iint_{K_*} z\mathbf{n}_z \cdot dA$$

where $\mathbf{n}_z$ is the z-component of the outer normal. Stokes Theorem gives the equality

$$L_z = g \cdot \iiint_{K_*} dxdydz = g \cdot \mathrm{Vol}(K_*)$$

In equilibrium we must have $L_z = g \cdot M$ where M is the total mass of K. This gives the principle of Archimedes expressed by the equality

$$M = \mathrm{Vol}(K_*) \tag{1}$$

Let us also notice that the horizontal components of the lifting force vanish. We have for example

$$L_x = g \cdot \iint K_* \, z\mathbf{n}_x \cdot dA$$

and this area integral is zero by another application of Stokes theorem. Similarly $L_y = 0$. Of course this vanishing is evident when K is at rest i.e. the validity of Stokes Theorem in the present case is a consequence of the experiment which shows that a body attains a floating position at rest in a calm lake.

Force of momentum. There remains to find the center of mass in K_* where we now assume that the density is uniform. Let this center of mass be denoted by $\mathbf{o}_*$. So here

$$\mathbf{o}_{*x} = \iint_{K_*} x \cdot dxdydz$$

with similar expressions for the y and the z-components. At the same time the body K has a center of mass denoted by $\mathbf{o}$. Without loss of generality we may assume that the horizontal coordinates have been chosen so that $\mathbf{o} = (0, 0, a)$ for some real number a. Notice that it may occur that $a < 0$. Now one has

Theorem. The vector $\mathbf{o} - \mathbf{o}_*$ is vertical, i.e.

$$\mathbf{o}_* = (0, 0, a_*)$$

holds for some a_*.

Proof. The lifting forces yield a total momentum on K expressed by the vector

$$\mathcal{M} = g \cdot \iint_{K_*} z \cdot (x, y, z - a) \times \mathbf{n} \cdot dA$$

In equilibrium $\mathcal{M} = 0$. Som let us find when it holds. A vector product appears above and we see that

$$\mathcal{M}_x = g \cdot \iint_{K_*} [zy\mathbf{n}_z - z(z - a)\mathbf{n}_y] \cdot dA$$

By Stokes theorem this area integral becomes

$$g \cdot \iiint_{K_*} ydxdydz$$

So the vanishing of $\mathcal{M}$ means that this volume integral is zero which means that the y-component of $\mathbf{o}_*$ is zero. In the same way the reader should calculate $\mathcal{M}_y$ and conclude that the x-component of $\mathbf{o}_*$ also is zero. This finishes the proof of the theorem.

Exercise. Prove directly that $\mathcal{M}_z = 0$ using Stokes Theorem.

Remark. The proofs above were given by Stevin around 1600 while he re-examined original work by Arkimedes. Stevin- considered as the "founder of modern statics" - formulated in a precise way many basic results in statics. Around 1650 Christian Huyghens established more refined results. For example, he analyzed the *stability* of a floating position. A typical example occurs when K is a solid cube whose density of mass is uniform and the specific weight some number $0 < s < 1$. In the (x, y, z) coordinates we assume that K is the unit cube and with total mass is s. A floating position which obeys the principle of Archimedes occurs when the "bottom side" of the portion of K below the waterline is horizontal. Hughen's showed that this floating position is *unstable* when s is relatively close to 1/2. He also found the critical value of s in order that the floating position is stable. More precisely, set

$$s_* = \sqrt{3} - 1$$

Huyghen's proved that the floating position with a horizontal bottom side is stable when $s > s_*$ or when $s < 1 - s_*$. The proof is instructive, i.e. one regards small pertubations from the given floating position when $0 > s < 1$ is given. Here one computes the sign of a suitable momentum to find if the equilibrium is stable or not. The interested reader may consult the classic text-book on Statics by H. Lamb for details. Let us also mention that the stability of a floating position can be analysed for an arbitrary body. Moreover one can give a "measure of stability" in terms of forces of momentum. Results of this kind have of course a significant practical value when ships and so on are designed.

The principle of virtual work.

Around 1740 D'Alembert introduced the notion of *virtual work* to find equations of motion for mechanical systems. His ingenious idea was consolidated by Lagrange whose text-book *Mecanique* from 1770 contains basic theory in classical mechanics where mass and time are absolute. In view of all applications in "daily life", one does not exaggerate by saying that the D'Alembert-Lagrange equations constitute one of the most important discoveries ever in theoretical science. Whether these equations belong to physics or mathematics is irrelevant. Classical mechanics - for which no background in physics is needed - should in any case be common knowledge to any student of mathematics since this subject teaches how to derive equations based upon laws of nature. Apart form their use in classical mechanics they also provide solutions in optimal control theory which is used extensively in other applied areas such as theoretical economics. In fact, the mathematical theory developed by Euler, d'Alembert, Lagrange and Legendre cover all basic results in optimal control theory in problems where the constraints lead to continuous solutions. The exception are cases where so called Bang-Bang solutions appear which are handled via the maximum principle introduced by Pontryagin in 1950. So students interested in optimization theory should become familar with the d'Alembert-Lagrange equations. Examples from classical mechanics not only confirm theoretical results but has also inspired the creation of many disciplines in mathematics. To give an example, Euler's method to derive the dynamical equations of a rigid body which moves in the 3-dimensional euclidian space laid the foundation for "abstract" Linear Algebra. For the beginner it is instructive to regard some concrete situation where one first computes the *momentum* arising from a distribution of mass-points of a rigid body and after discovers the great advantage of working in different coordinate spaces in order derive the equations of motion of a rotating body. To me it is incomprehensible how one can appreciate the spectral theorem for a symmetric 3×3-matrix without any reference to the real world which is offered by he Euler equations for a rotating rigid body. Stokes theorem which relates volume integrals to surface integrals is another example where the statement in mathematics comes from Laws of Nature. For example, the principle of Archimedes implies Stokes Theorem and is can therefore be understood by an observer who regards how a body whose specific weight is < 1 attains a floating position in a lake. Going to more advanced topic such as analytic function theory in one complex variable and Fourier analysis, most of the basic results are explained within the frame of mechanics. Riemann's mapping theorem in complex analysis reflects the existence of an equilibrium density on a closed curve caused by the law of Briot-Savart. That is, a fundamental law discovered experimentally led naturally to the conformal mapping theorem. Concerning potential theory one may recall that Gauss developed many of his later mathematical results during intense studies and calculations of electric and magnetic fields after the physical discoveries by Örstedt and Faraday around 1830. In this connection we cannot refrain form mentioning the a result due to the genius Niels Henrik Abel (1802-1829). In 1823 he proved an inversion formula which implies that any one-dimensional potential field of forces is recovered from the determination of time periods under the oscillating motion of a particle whose initial velocity varies. We expose Abel's inversion formula in a separate section since this fundamental result started the whole theory devoted to inversion formula for elliptic functions and led to the study of abelian integrals and functions.

Example.

The best way to become familiar with the D'Alembert-Lagrange equations is to consider specific problems. Let us describe a case whose solution is not so obvious from the start.

Consider a particle p of unit mass attached to a rigid bar whose mass is neglegable. The bar has length L and p is attached to one of its ends, while the other end is fixed at the origin in the (x, y)-plane. The y-axis is vertical and the bar can swing in the (x, y)-plane without friction while gravity acts in the negative y-direction. Let θ is the angle between the bar and the negative y-axis which means that the position of p expressed by coordinates p_x and p_y becomes

$$p_x = L \cdot \sin(\theta) \quad p_y = -L \cdot \cos(\theta) \qquad 0 < \theta < \pi$$

The kinetic energy of the mass-point becomes $T = \frac{L^2}{2} \cdot \dot{\theta}^2$. Draweing a figure and taking greavity into the account we get the second order differential equation

$$\frac{dT}{dt} L^2 \cdot \ddot{\theta} = -gL \cdot \sin\theta$$

It is solved via initial conditions. A typical case is that $\theta(0) = 0$ while $\dot{\theta}(0) = a > 0$. We leave ot to the reader to analyise the solutions and remark only that the pendelum is an example of a conservative mechanical system where kinetic energy plus potential energy is constant. Now we shall consider a *non-conservative* situation. Namely, imagine that an external force changes the x-coordinate at the point of suspension on the horizontal axis $y = 0$. This external force - caused by a machine - determines a function $t \mapsto x(t)$ which gives

$$\text{(i)} \qquad p_x = L \cdot \sin(\theta) + x(t) \quad p_y = -L \cdot \cos(\theta) \qquad 0 < \theta < \pi$$

The *kinetic energy* of the mass particle p becomes:

$$\text{(ii)} \quad T = \frac{1}{2}[(\dot{x} + L\cos(\theta) \cdot \dot{\theta})^2 + L^2 \sin^2\theta)\dot{\theta}^2] = \frac{1}{2}\left[\dot{x}^2 + L^2 \cdot \dot{\theta}^2 + 2L\dot{x} \cdot \cos(\theta) \cdot \dot{\theta}\right]$$

Let g be the gravity force. The D'Alembert-Lagrange equations will teach us that the function $\theta(t)$ satisfies the differential equation:

$$\text{(iii)} \qquad L^2\ddot{\theta} + L\ddot{x} \cdot \cos(\theta) = -g \cdot L \cdot \sin(\theta)$$

Here $t \mapsto \ddot{x}(t)$ is given via the external force. So (*) is a differential equation which is solved in $\theta(t)$ only for a given time-dependent function $x(t)$. In addition to this one also seeks the time dependent force which moves $x(t)$ in the preassigned fashion. To find this one considers

$$\text{(iv)} \qquad \frac{d}{d\dot{x}}(T) = \ddot{x} + L \cdot \frac{d}{dt}(\cos(\theta) \cdot \dot{\theta})$$

By the principle of virtual work (iv) is equal to the intensity of force $f(t)$ which at a given moment of time is needed to keep the given function $x(t)$.

Example. Consider the case where:

$$x(t) = A \cdot \sin(\omega t) \quad : \quad A, \omega > 0 \tag{*}$$

The differential equation for $\theta(t)$ from (iv) becomes

$$L^2\ddot{\theta} + L \cdot A \cdot \omega^2 \cdot \sin(\omega t) \cdot \dot{\theta} + g \cdot L \cdot \sin(\theta) = 0 \tag{**}$$

This is a second order differential equation for $\theta(t)$ which has a unique solution with the prescribed initial conditions for $\theta(0)$ and $\dot{\theta}(0)$. Today's student in mathematics has an *enormous advantage* compared to previous generations since a numerical solution to (**) can be found with the aid of a computer. Moreover, one can vizulise the whole motion which provide a moving picture of the *time dependent* oscillation of the θ-function. One can also solve a problem numerically with specific values such as $L = A = 1$, $g = 9,8$ and some initial velocity $\dot{\theta}(0)$. It is a good exercise to study the oscillation as ω increases. Before an eventual mathematical attempt is made to describe the asymptotic behaviour of a solution when $t \to \infty$ one can first make some numerical experiments with the computer. Of special interest is the behaviour when A is small but ω rather large. The reader is invited to analyze the behaviour in such situations, for example what happens when the time variable increases.

With given inital conditions $\theta(0) = 0$ and $\dot{\theta}(0) = a > 0$ one may also stufy the work by a machine in order to maintain the motion of $x(t)$ determined by (*) above. If the work appears both to contol motion to the right or to the left it means that dyring a time interval $[0, T]$ one seeks the integral

$$\int_0^T \left|\ddot{x}(t) + L \cdot \frac{d}{dt}(\cos\theta(t) \cdot \dot{\theta}(t)\right| dt$$

Another numerical problem. While Johannes Kepler studied motions of planets he performed very skillful computations. As an example we mention the following equation:

$$w(t) = t \cdot \sin(w(t) + a) \tag{*}$$

Here $a > 0$ is a real constant and t the time variable and one seeks the function $w(t)$. Kepler was able to produce several terms in the series expansion - the complete analytic series solution was later found by Lagrange who a confirmed Kepler's original conjecture that the radius of convergence for the series solution is $\geq R_*$ for a positive number which is independent of a. However, the constant a appears as a parameter and therefore the exact radius of convergence for the series solution depends on a and can only be found numerically. Today's student can perform such calculations which 200 years ago would have given highest possible credit while Lagrange served as the leading professor in mathematics at Sorbonne.

Abel's inversion formula

Linear motion in a potential field. Before we expose the general equations by d'Alembert and Lagrange we shall consider a single particle systems and derive Abel's inversion formula. Here is the situation. A particle of unit mass can move on the non-negative x-axis. Let t be the time variable. Under a motion we get the time-depdendent function $x(t)$ and the velocity $\dot{x} = \frac{dx}{dt}$. The second order derivative $\ddot{x}$ is acceleration. A strictly increasing function $U(x)$ with $U(0)$ is also given. It is a potential function where $-U'(x)$ is the force acting at any point $x > 0$. Notice that it is negative. By Newton's law we have

$$\text{(i)} \qquad \ddot{x} = -U'(x)$$

It follows that:

$$\text{(ii)} \qquad \frac{d}{dt}[\frac{\dot{x}^2}{2}] + U(x(t)] = \dot{x}(\ddot{x} + U'(x(t)) = 0$$

Hence $\frac{\dot{x}^2}{2} + U(x(t))$ stays constant under the motion which reflects the energy principle, i.e. the sum of kinetic and potential energy is constant. Suppose the particle at time zero is at $x = 0$ and gets an initial velocity $\dot{x}(0) = v$ caused by a sudden impact. By (ii) the function $\dot{x}(t)$ decreases and we assume that $U(x)$ increases so fast that the particle eventually comes to rest at some time T_* where the position $x(T_*) > 0$. After the force caused by the potential function will push the particle back to $x = 0$. By symmetry we get the equality:

$$\text{(iii)} \qquad x(T_* + t) = x(T_* - t) \quad T_* \leq t \leq 2T_* \,.$$

At time $T_{**} = 2T_*$ the particle arrives at $x = 0$, but this time with velocity $-v$ in the negative x-direction. In many physical applications one does not know $U(x)$. But one may perform experiments, i.e. give the particle an initial velocity v and with the aid of clock measure the return time T_{**}. Notice that we do not assume that one can follows the particle during its travel on the x-axis. For example, one may imagine it is invisible except at $x = 0$. So by an experiment one cannot measure $x^*(T_*)$, nor make any intermediate observations during the motion.

The question arises if it is still possible to determine the U-functions by measuring T_{**} for many initial velocities. It turns out that there exists an *inversion formula* which determines U. Before we announce the result we introduce some notations. Set

$$w = \frac{v^2}{2} \quad \text{where} \;\; v = \dot{x}(0)\,.$$

Next, by succesive experiments the observer placed at $x = 0$ with a clock is able to determine the function

$$w \mapsto T_{**}(w)$$

and in this way also calculate the function defined by:

$$\text{(iv)} \qquad \mathcal{J}(A) = \int_0^A \frac{T_{**}(w) \cdot dw}{\sqrt{A - w}} \quad : A > 0$$

Next, since $U(x)$ is strictly increasing it suffices to determine the inverse function U^{-1} which we denote by $\xi(x)$, So here

(v) $$U(\xi(x)) = x \quad : x > 0$$

With these notations one has

Abel's inversion formula. For every $A > 0$ one has

$$\xi(A) = \frac{\mathcal{J}(A)}{\sqrt{2} \cdot \pi}$$

Proof. During the time interval $0 \leq t \leq T_*$ while the particle moves with positive velocity we have by (ii)

(1) $$\dot{x} = \sqrt{2} \cdot \sqrt{w - U(x)}$$

Now $dx = \dot{x} \cdot dt$ so $dt = \frac{dx}{\dot{x}}$ and an integration gives

(2) $$T_* = \frac{1}{\sqrt{2}} \cdot \int_0^{x(T_*)} \frac{dx}{\sqrt{w - U(x)}}$$

Now $T_* = T_*(w)$ is a function of w and so $x(T_*(w)) = x(w)$. Next, we have the ξ-function and a change of variables in (2) gives

(3) $$T_*(w) = \frac{1}{\sqrt{2}} \cdot \int_0^{\xi(w)} \frac{d\xi}{\sqrt{(w - \xi)}}$$

Next, since $T_{**}(w) = 2T_*(w)$ the definition of $\mathcal{J}(w)$ in (iv) gives:

$$\mathcal{J}(A) = \sqrt{2} \cdot \int_0^A [\int_0^{\xi(w)} \frac{d\xi}{\sqrt{(w - \xi)}}] \cdot \frac{dw}{\sqrt{A - w}}$$

Interchanging the order of integration we get:

(4) $$\sqrt{2} \cdot \int_0^A [\int_\xi^A \frac{dw}{\sqrt{A - w} \cdot \sqrt{w - \xi}}] \cdot d\xi$$

Now we can finish the proof using the following

Sublemma. One has

$$\int_\xi^A \frac{dw}{\sqrt{A - w} \cdot \sqrt{w - \xi}} = \pi \quad : \quad 0 < \xi < A$$

Proof With the substitution $w \to u + \xi$ the integral becomes

(i) $$\int_0^{A-\xi} \frac{du}{\sqrt{A - \xi - u} \cdot \sqrt{u}}$$

With $u = (A - \xi)s$ it follows that (i) becomes

(ii) $$\int_0^1 \frac{ds}{\sqrt{1 - s} \cdot \sqrt{s}}$$

and the reader verifies that the value is π.

Next, the Sublemma and (4) give $\mathcal{J}(A) = \sqrt{2} \cdot \pi \cdot \xi(A)$. Then division with $\sqrt{2}\pi$ gives Abel's inversion formula.

The harmonic oscillator

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It arises when $U(x) = kx^2$ for some $k > 0$. In this case $\xi(A)$ is a constant times $\sqrt{A}$. Abel's inversion formula shows that this holds if and only if the T_{**}-function is independent of w. The fact that T_{**} is independent of w when $U(x) = k \cdot x^2$ was of course known long before. It was for example wellknown to R. Descartes around 1640 and a few years later Christian Huyghens used the isochronic property to construct reliable clocks which apart from daily life use, gave scientists a new powerful tool. For example, thanks to accurate time measure the Danish mathematician and astronomer Ole Brömer gave a reasonable estimate for the speed of light in 1676. His result was a bit slower than today, but his assertion that light travels with a speed exceeding 240 000 kilometers per second was a veritable achievement at that time. The interested reader may consult text-books in physics which explain how Brömer performed the measure by studying the moon Ion which moves around Jupiter. His method to approximate the speed of light used device which later was used to construct so called winding numbers of plane curves and led to residue calculus of complex analytic functions.

Inversion formulas with fixed intervals. Here follows a more recent example of an inversion forumla in the spirit of Abel. In his work *Abelsche Integralgleichung mit konstanten Integrationsgrenzen* from 1922 Carleman studied inversion formulas for integrals of the form

$$(*) \qquad \int_0^1 \frac{1}{|x-y|^\alpha} \cdot \phi(y)dy = f(x) \quad : \quad 0 < \alpha < 1$$

Here $f(x)$ is supposed to be a known function defined on the unit interval $0 \leq x \leq 1$ and one seeks a formula for the ϕ-function. In contrast to Abel's inversion formula the inverse formula which determines ϕ requires a more involved proof where complex analysis is used. Another inversion formula arises when we regard the equation

$$(**) \qquad \int_0^1 \operatorname{Log}|x-y| \cdot \psi(y)dy = f(x)$$

To see an example from the world of mathematics we recall Carleman's inversion formula for the ψ-function in (**):

Theorem. One has the formula:

$$\psi(x) =$$

$$\frac{1}{\pi^2} \cdot \frac{1}{\sqrt{x(1-x)}} \cdot \int_0^1 \frac{f'(s) \cdot \sqrt{s(1-s)}}{s-x} \cdot ds - \frac{1}{2\pi^2 \cdot \operatorname{Log} 2 \cdot \sqrt{x(1-x)}} \cdot \int_0^1 \frac{f(s)}{\sqrt{s(1-s)}} \cdot ds$$

The proof uses residue calculus and is of course beyond the scope of these lectures.

1. The D'Alembert-Lagrange equations

We derive the equations which govern the motion of particle systems in classical mechanics. The restriction to a finite particle system is not essential since they can approximate any continuously distribution of mass. The resulting system of differential equations apply to mechanical system which are exposed by arbitrary and in general time dependent outer forces. The special conservative case occurs when the outer forces are determined by a potential function which is independent of the time variable t.

1.1 Particle system It consists of a system $\mathcal{P}$ with N mass-points $p_1, \ldots, p_N$. Each p_ν has a positive mass m_ν and a *configuration* of $\mathcal{P}$ is specified by the N-vector $(p_1, \ldots, p_N)$ in a $3N$-dimensional space. Now constraints may occur which means that not all configurations can appear. An example occurs when the mass-points belong to a *rigid body* where the distance between any pair of mass-points is the same under all configurations. This led Lagrange to the introduce the following:

1.2 Degree of freedom *We say that $\mathcal{P}$ has a k-dimensional degree of freedom if there exists some subset of $\mathbf{R^k}$ where we $(\xi_1, \ldots, \xi_k)$ are coordinates which determine every possible configuration of the particle system, expressed by an N-tuple of vector-valued functions*

$$\xi \mapsto p_\nu(\xi_1, \ldots, \xi_k) \tag{1.2}$$

Example If K is a rigid plane body which can move freely on the 2-dimensional (x, y)-plane, then its degree of freedom is 3 where the (ξ_1, ξ_2) give coordinates for the mass-point of K and ξ_3 measures the angular rotation. If the plane rigid body is allowed to move freely in the 3-dimensional (x, y, z)-space then its degree of freedom increases and is equal to six.

1.3 Velocity and acceleration Consider a particle system with k degrees of freedom. A time-dependent motion of $\mathcal{P}$ is therefore determined by k-many functions

$$\xi_j(t) \quad : \quad 1 \leq j \leq k$$

The motion of each particle p_ν becomes

$$t \mapsto p_\nu(t) = p_\nu(\xi_1(t), \ldots, \xi_k(t)) \tag{i}$$

This yields N many vector valued functions of the ξ-variables. In $\mathbf{R}^3$ the position of a single mass point p_ν is expressed as a 3-dimensional vector (x_ν, y_ν, z_ν). So for a fixed ν we get three functions $t \mapsto x_\nu(\xi_1(t), \ldots, \xi_k(t))$ and similarly for the y and z-coordinates. We assume that the functions $t \mapsto \xi_j(t)$ are at least twice differentiable. This gives rise to velocity and acceleration vectors of each single particle. First we get the velocity vectors:

$$\dot{p}_\nu(t) = \sum_{j=1}^{j=k} \partial p_\nu / \partial \xi_j \cdot \dot{\xi}_j(t) \quad 1 \leq \nu \leq N \tag{ii}$$

Taking another time derivative we obtain the acceleration vectors:

$$\text{(iii)} \quad \ddot{p}_\nu(t) = \sum_{j=1}^{j=k} \partial p_\nu / \partial \xi_j \cdot \ddot{\xi}_j(t) + \sum_{j=1}^{j=k} \sum_{j=1}^{j=k} \partial^2 p_\nu / \partial \xi_j \partial \xi_i \cdot \dot{\xi}_j(t) \dot{\xi}_i(t) \quad 1 \leq \nu \leq N$$

1.4 Kinetic energy. The kinetic energy of the particle system is at every time moment given by

$$T = \frac{1}{2} \sum_{\nu=1}^{\nu=N} m_\nu \cdot |\dot{p}_\nu|^2$$

We can express T using time derivatives of the ξ-functions. Namely, for each fixed ν we have a symmetric $k \times k$-matrix A^ν with elements

$$A_{ij}^\nu = \langle \partial p_\nu / \partial \xi_i, \partial p_\nu / \partial \xi_j \rangle$$

Then we obtain

$$T = \frac{1}{2} \sum_{\nu=1}^{\nu=N} m_\nu \cdot \sum \sum A_{ij}^\nu(\xi) \cdot \dot{\xi}_i(t) \dot{\xi}_j(t)$$

Hence the kinetic energy is a quadratic form in the time derivatives $\dot{\xi}_1, \ldots, \dot{\xi}_k$ whose coefficients are

$$a_{ij}(\xi) = \frac{1}{2} \sum_{\nu=1}^{\nu=N} m_\nu \cdot A_{ij}^\nu(\xi)$$

1.5 The Lagrangean functions $\mathcal{L}_i$. Treating $(\xi_1, \ldots, \xi_k, \dot{\xi}_1, \ldots, \dot{\xi}_k)$ as formally independent variables we construct the partial derivatives

$$\partial T / \partial \dot{\xi}_i \quad : \quad \partial T / \partial / \xi_i \quad 1 \leq i \leq k$$

With $i = 1$ we have for example

$$\partial T / \partial \dot{\xi}_1 = \sum_{\nu=1}^{\nu=N} \sum_{j=1}^{j=k} m_\nu \langle \partial p_\nu / \partial \xi_1, \partial p_\nu / \partial \xi_j \rangle \cdot \dot{\xi}_j$$

Now we take its time derivative and obtain the triple sum:

$$\frac{d}{dt} \left[\partial T / \partial \dot{\xi}_1 \right] = A + B + C \quad \text{where}$$

$$A = \sum_{\nu=1}^{\nu=N} \sum_{i=1}^{i=k} \sum_{j=1}^{j=k} m_\nu \langle \partial p_\nu / \partial \xi_1, \partial^2 p_\nu / \partial \xi_i \partial \xi_j \rangle \cdot \dot{\xi}_i \dot{\xi}_j$$

$$B = \sum_{\nu=1}^{\nu=N} \sum_{j=1}^{j=k} m_\nu \langle \partial p_\nu / \partial \xi_1, \partial p_\nu / \partial \xi_j \rangle \cdot \ddot{\xi}_j$$

$$C = \sum_{\nu=1}^{\nu=N} \sum_{i=1}^{i=k} \sum_{j=1}^{j=k} m_\nu \langle \partial^2 p_\nu / \partial \xi_1 \partial \xi_i, \partial p_\nu / \partial \xi_j \rangle \cdot \dot{\xi}_i \dot{\xi}_j$$

The expressions for the acceleration vectors $\ddot{p}_\nu$ gives:

$$A + B = \sum_{\nu=1}^{\nu=N} m_\nu \langle \partial p_\nu / \partial \xi_1, \ddot{p}_\nu \rangle$$

We also notice the equality

$$C = \partial T/\partial \xi_1$$

Hence we arrive at the following

1.6 Proposition *Put* $\mathcal{L}_1 = \frac{d}{dt}(\partial T/\partial \dot{\xi}_1) - \partial T/\partial \xi_1$. *Then we have the equality*

$$\mathcal{L}_1 = \sum_{\nu=1}^{\nu=N} m_\nu \langle \partial \mathbf{p}_\nu / \partial \xi_{\mathbf{1}}, \ddot{\mathbf{p}}_\nu \rangle$$

Remark Of course, a similar conclusion holds for each ξ- variable. Thus, to every $1 \leq i \leq k$ we have

$$\mathcal{L}_i = \frac{d}{dt}(\partial T/\partial \dot{\xi}_i) - \partial T/\partial \xi_i = \sum_{\nu=1}^{\nu=N} m_\nu \langle \partial p_\nu / \partial \xi_i, \ddot{p}_\nu \rangle$$

1.7 The Lagrange-D'Alembert equations The particle system is in motion. At a given time t we suppose that the configuration is instantly changed from its position expressed by the k-vector $\xi(t)$ to a new position $(\xi_1(t)+\delta\xi_1, \xi_2(t), \ldots, \xi_k(t))$ where $\delta\xi_1$ is small. This sudden change of the configuration requires an infinitesmal force. To begin with, each particle p_ν is moved into

$$p_\nu + \partial p_\nu / \partial \xi_1 \cdot \delta\xi_1 + \text{ small ordo of } \Delta\xi_1$$

By Newton's Law the force acting on p_ν at time t is equal to $m_\nu \cdot \ddot{p}_\nu$. After a summation over all ν the *infinitesmal work* for the $\delta\xi_1$-displacement of the configuration at time t becomes:

$$\delta W_1 = \sum_{\nu=1}^{\nu=N} m_\nu \cdot \langle \partial p_\nu / \partial \xi_1, \ddot{p}_\nu \rangle \cdot \delta\xi_1 + \text{small ordo of } \delta\xi_1$$

Passing to the limit as $\delta\xi_1 \to 0$ and using Proposition 1.6 we get:

$$\lim_{\Delta\xi_1 \to 0} \frac{\delta W_1}{\delta \xi_1} = \mathcal{L}_1$$

Following D'Alembert and Lagrange we refer to $\lim_{\delta\xi_1 \to 0} \frac{\delta W_1}{\delta \xi_1}$ as the *virtual work intensity* for an infinitesmal displacement of ξ_1 and denote it by $\mathcal{A}_1$. Of course, we obtain a similar result after infiniesmal displacement of any other ξ_i. Thus, we have a k-tuple $\mathcal{A}_1, \ldots, \mathcal{A}_k$ which are time-dependent functions under the motion of the particle system and for each $1 \leq i \leq k$ we have the equality

(*) $$\mathcal{A}_i = \mathcal{L}_i$$

These k equations constitute the D'Alembert-Lagrange equations which are used to find the equations of motion and also determine the outer forces acting on the system. Thus, every dynamical problem in classical mechanics given by a system with a finite degree of freedom is solved by these equations.

1.8 The principle of least action For motion in a conservative field of forces the D'Alembert-Lagrange equations correspond to the principle of least action. This terminology comes from Euler's equations in calculus of variation. Let us explain this in more detail. Suppose there exists a function $U(\xi_1, \ldots, \xi_k)$ such that virtual work under any displacement of ξ at any time t are expressed by

$$\mathcal{A}_i = -\partial U(\xi)/\partial \xi_i \quad 1 \leq i \leq k$$

So here the equations of motion are

$$\frac{d}{dt}(\partial T/\partial \dot{\xi}_i) = \partial T/\partial \xi_i - \partial U(\xi)/\partial \xi_i \quad 1 \leq i \leq k$$

Now we put $H(\xi, \dot{\xi}) = T(\xi, \dot{\xi}) - U(\xi)$. Then the equations above are the Euler's equations from the Calculus of Variation for the H-function. So if the system moves from one configuration at time zero, expressed by the k-vector $\xi(0) = p$ to a position at later time expressed by a given k-vector $\xi(T) = q$, then the *actual motion* of the particle system during the time interval $[0, T]$ minimizes the variational integral

$$\int_0^T H(\xi(t), \dot{\xi}(t))dt$$

among all paths in the ξ-space from p to q. However, most problems in engineering deal with situations which are not governed in this simple fashion.

1.9 Two examples. Consider a rolling wheel on a rough horizontal plane Π. The wheel is a circle of some radius R and to say that the plane on which it rolls is rough means that friction is strong enough to prevent sliding. But a force is needed to prevent sliding. This force acts at any time moment t at the point of the wheel which at time t has contact with Π. The point of contact varies with t and the force which prevents sliding varies also both in strengthand direction while the wheel is rolling. Of course, we consider the non-trivial situation when the wheel does not roll in upward position. When the wheel moves it leaves a track on the ground which as a first approximation is close to a sine curve when the wheel is almost in upward position. At the same time the plane of the wheel changes, i.e it oscillates around the vertical direction which is $\perp$ to Π.

In 1899 the equations of motion for a rolling wheel were derived by Paul Appel and Königsberg. It is an example of a *holonomic system* where the difficulty for deriving the equations of motions comes from the fact that the friction forces which prevent the wheel from sliding are indirectly determined by its actual motion. The procedure to derive the equation in the equations for a rolling wheel are presented in an *Examensarbete* written by by A. Sellerholm at the mathematics department of Stockholm university from 2004. Here the reader not only will find how one derives the equation of motion but also an implemented MatLab program which is used to analyze the motion numerically. An example treated in detail by Sellerholm is the following: Suppose the wheel starts rolling in a non-upward position with a velocity which is strictly smaller than the critical speed which is necessary for stability. Then one gives the wheel a small tilt and now it will eventually fall down. But of one needs a a computer to describe the whole motion and the time it takes until the wheel comes to rest on Π. Another and perhaps not so well known feature in classical mechanics is Poisson's equations for sudden forces, i.e. when an impact takes place between two bodies which collide at a certain time moment.

Here friction may cause loss of energy and the Poisson equations which determine the situation after an impact rely upon dynamical equations which occur during the small time interval while the impact takes place. The interested reader may consult the *Examensarbete by Ninos Poli* which in addition to a theoretical survey of Poisson's equations contains quite exensive MatLab computations which are used to determine the impact between an american football and the ground. Here the football is an ellipsoid and when it hits the ground it may have a spinning velocity and the point of the ellipsoid which hits the ground has a general position. For such an impact the equations of Poisson can only be solved numerically. So this is an example where one must rely upon a computer in order to describe the motion of the football after the impact. Of course, in examples of this kind several physical parameters occur, i.e. apart from the known shape of the fotball, the elasticity constant for the impact between the fooball and the grass may vary as well as the friction between the football and the grass which affects a non-central impact with sliding velocity. So it is essential to use computers in order to solve Poisson's equations when these parameters vary if one wants to obtain a more or less realistic solution to this specific "bouncing problem".

1.10 The Coriolis' force

The French geometer Coriolis proved in 1830 an extension of the D'Alembert-Lagrange equations when a body moves in a rotating space. His mathematical result led the engineer Foucault to construct a pendelum in order to demonstrate the rotation of the earth around its polar axis in 1854. The force of Coriolis is acting all the time in the atmosphere where its influence is far from negligable. To find this force we assume that the earth is a sphere of radius R, i.e. $R \simeq 637 \cdot 10^4$ meter. A particle p of unit mass moves on the sphere. Let θ be the angle which measures latitude, i.e. the angle betwen p and the equator plane. So at Stockholm $\theta \simeq \pi/3$. The earth rotates with constant angular velocity ω around the polar axis joining the North-pole with the South-pole and makes a turn under 24 hours. Hence the angular velocity measured in meter/second becomes:

$$\omega \simeq \frac{2\pi}{24 \cdot 3600}$$

Suppose the particle at an instant moves along a meridian with angular velocity $\dot{\theta}$ on the northern hemi-sphere, i.e. it strives towards the North-pole. Its velocity v measured in meter/second becomes $R \cdot \dot{\theta}$. In this situation one has

Theorem *The Coriolis force acting on a particle of unit mass is directed to the* east *and its magnitude in Newton meter is given by*

$$\mathcal{C} = 2\sin(\theta) \cdot \omega \cdot v$$

Remark. Since ω is quite small the magnitude of $\mathcal{C}$ is relatively small. For example, let $v = 100$ meter/second. At Stockholm one has have

$$\mathcal{C} \simeq \frac{2\pi\sqrt{3}}{24 \cdot 36}$$

Proof of Coriolis' formula. Imagine that the particle moves on the earth without constraint. Then it has two degrees of freedom where the position is measured by the two angular variables θ and ψ. Here ψ rotates around the polar axis. Thus, ψ stays constant along a meridian. The kinetic energy becomes

$$T = \frac{R^2}{2}[\dot{\theta}^2 + \cos^2(\theta)\dot{\psi}^2]$$

It follows that

$$\mathcal{L}_\psi = \frac{d}{dt}(\cos^2(\theta)\dot{\psi}) = R^2[\cos^2(\theta)\ddot{\psi} - 2\dot{\psi}\sin(\theta)\cos(\theta)\dot{\theta}]$$

In order to determine $\mathcal{C}$ we *constrain* the motion of the particle so that it only can move along a meridian. Imagine that it moves inside a small pipeline which is placed along a meridian. At the latitude θ a small displacement $\delta\psi$ gives a length equal to $R \cdot \cos(\theta)\delta\psi$. Under the constrained motion $\dot{\psi} = \omega$ is constant. Hence $\ddot{\psi} = 0$ and the D'Alembert-Lagrange gives the equality

$$R \cdot \cos(\theta) \cdot \mathcal{C} = -2R^2 \cdot \omega\sin(\theta)\cos(\theta)\dot{\theta}$$

After division with $R \cdot \cos(\theta)$ and identifying v with $R\dot{\theta}$ we get the formula of Coriolis:

$$\mathcal{C} = -2R \cdot \omega \cdot \sin(\theta)$$

Remark Observe the *minus sign* which gives the correct direction of the force vector $\mathcal{C}$ when we recall how the earth rotates - i.e. at Stockholm the sunset is in the eastern direction. The reader should contemplate upon daily experience and the wellknown result that the derivate of $\cos(\theta)$ is *minus* $\sin(\theta)$ in order to digest the result by Coriolis. If the particle instead moves on the northern hemisphere on a meridian towards the equator , then the force of Coriolis changes sign.

The θ-velocity Under the constrained motion the second equation gives

$$\ddot{\theta} = -\omega^2\cos(\theta)\sin(\theta)$$

In fact, this holds since $\mathcal{L}_\theta = 0$ when we assume that the particle moves freely inside the pipeline. The magnitude of ω^2 is small, so the effect upon the acceleration $\ddot{\theta}$ is small. But notice that if v is the "true" velocity one has $R\ddot{\theta} = \ddot{v}$. So with $a = \cos(\theta)\sin(\theta)$ which at Stockholm is $\simeq \frac{\sqrt{3}}{4}$ one has

$$\ddot{v} = -a\frac{\omega^2}{R} \simeq a\frac{637}{36^2 \cdot 24^2}$$

The last factor is $\simeq 10^{-4}$. So if the particle starts off in the pipeline with initial velocity 10 meter/second it has lost almost the whole velocity after a bit more than 50 hours when we move on latitudes relatively close to Stockholm. The reader may contemplate upon this and reflect upon what happens if the pipe line for example is directed along a meridian starting at some city in Europe on the same latitude as Paris and reaching Stockholm. The whole discussion shows that the total effect of Coriolis' forces is quite complex.

The drag force on a pendelum

Let us illustrate the d'Alembert-Lagrang equations by an example where the force is non-conservative. At the same time this shows that one cannot expect analytic solutions for the equation of motion even in quite simple situations. In the example below one encounters as a certain trigonometric integral which only can be expressed via one of Legendre's generalised elliptic integrals. In fact, the solution below is non-standard because we do not obtain a symmetric oscillatory motion.

The model. A point of unit mass is attached to one end-point of a rigid bar B with unit length. The mass of the bar itself is ignored, i.e. taken to be zero. The other end-point q is constrained to move on the horisontal x-axis. But B can swing freely in a vertical plane like a plane pendelum where the force of gravity acts on p. Let y be the coordinate so that $-g \cdot e_y$ is the force of gravity. This one-particle system has two degrees of freedom, i.e. the x-coordinate for q and the angle α between B and the y-axis. The position of p in the (x, y)-coordinates becomes

$$p = (x - \sin(\alpha), -\cos(\alpha)) \tag{i}$$

Here the angle α is taken so that $\alpha > 0$ yields a decreasing x-coordinate for p. The reader may illustrate this by a figure. Under motion the kinetic energy becomes:

$$T = \frac{1}{2}\big[\dot{x}^2 + \dot{\alpha}^2 - 2 \cdot x \cdot \dot{\alpha} \cdot \cos(\alpha)\big] \tag{ii}$$

At time $t = 0$ we assume that $\alpha = 0$ and $v = \dot{\alpha}(0) > 0$. Thus, at $t = 0$ the pendelum starts to swing to the left which means that the x-coordinate of p decreases. Next, by an outer force the point q is moved on the x-axis with a constant acceleration $\rho > 0$. So here

$$\dot{x} = \rho t \quad : \quad \ddot{x} = \rho \tag{ii}$$

The d'Alembert-Lagrange equations become:

$$\ddot{\alpha} - \rho \cdot \cos(\alpha) = -g \cdot \sin(\alpha) \tag{iv}$$

$$\rho \frac{d}{dt}(\dot{\alpha} \cdot \cos(\alpha)) = F(t) \tag{v}$$

where $F(t)$ is the drag force which is needed to keep $\ddot{x}$ constant. We reduce as usual (iv) to a first order equation, i.e. multiply both sides with $\dot{\alpha}$ and integrate. This gives

$$\frac{\dot{\alpha}^2}{2} - \rho \cdot \sin(\alpha) = g \cdot (\cos(\alpha) - 1) + \frac{v^2}{2} \tag{vi}$$

Let us assume that neither v nor ρ are too large. Then we get the first moment of time t^* when $\dot{\alpha}(t^*) = 0$ and here $\alpha^* = \alpha(t^*)$ is a positive number between 0 and $\pi/2$. From (*) we have the two equations

$$g(1 - \cos(\alpha^*)) = \frac{v^2}{2} + \rho \cdot \sin(\alpha^*) \tag{*}$$

$$(**) \qquad t^* = \frac{1}{\sqrt{2}} \cdot \int_0^{\alpha^*} \frac{d\alpha}{\sqrt{v^2 + \rho \cdot \sin(\alpha) + 2g(\cos(\alpha) - 1)}}$$

The solution expressed by (*) and (**) is expressed by an integral of the Legendre type. It cannot be solved analytically. But of course one can obtain numerical solutions.

The work integral. During the time interval $[0, t^*]$ work is required to keep $\ddot{x}$ constant. From (v) we get

$$A = \int_0^{t^*} F(t) \cdot dt = \rho \cdot t^* + v$$

Remark. Notice the plus sign for v. This reflects the fact that in the degenerate case $v = 0$ the pendelum stays at rest in its vertical position since the ODE-equation (vi) with initial conditions $\alpha(0) = \dot{\alpha}(0) =$ yields the zero solution. Thus, it is only when $v > 0$ that the pendelum starts to swing and gives rise to a drag force. Let us analyze the solution when $v \simeq 0$, i.e. v is small and positive. In this case Taylor expansions of the sine- and the cosine functions give:

$$(1) \qquad \alpha^* \simeq= \sqrt{v^2 + \frac{\rho^2}{g^2}} - \frac{\rho^2}{g^2} \simeq \frac{gv^2}{2\rho}$$

For t^* we get the approximative formula

$$(2) \qquad t^* = \frac{1}{\sqrt{2}} \cdot \int_0^{\frac{gv^2}{2\rho}} \frac{d\alpha}{\sqrt{v^2 + \rho \cdot \alpha - g \cdot \alpha^2}}$$

By the variable substitution $\alpha = v^2\xi$ we see that $v \simeq 0$ gives the approximative formula

$$(3) \qquad t^* \simeq \frac{1}{\sqrt{2}} \cdot v \cdot \int_0^{\frac{g}{2\rho}} \frac{d\xi}{\sqrt{1 + \rho \cdot \xi}} = \frac{1}{\sqrt{2}} \cdot v \cdot \frac{1}{\rho} \cdot \int_0^{\frac{g}{2}} \frac{ds}{\sqrt{1+s}}$$

Conclusion. Put

$$C_0 = \frac{1}{\sqrt{2}} \cdot \int_0^{\frac{g}{2}} \frac{ds}{\sqrt{1+s}}$$

Then we get

$$(*) \qquad A \simeq (C_0 + 1)v$$

By the previous calculations this equality holds up to order v^2. Notice that A is independent of ρ. On the other hand the acceleration rate ρ affects the time t^* when p has reached its maximal swing to the left, i.e. by the equation (3) above t^* decreases approximately linearly with $\frac{1}{\rho}$. A warning should be inserted, i.e. the approximative solutions above are valid when $v \simeq 0$ while ρ is not too small. For example, in the opposed extreme case when $\rho = 0$ and $v \simeq 0$ we have the wellknown approximative formula

$$t^* \simeq \frac{\pi}{2\sqrt{g}} \cdot v$$

The simple pendelum.

The solution for the motion of the simple pendelum is interesting in the study of elliptic integrals since Jacobi's $\mathfrak{sn}$-function is determined by the osciallation. Namely, consider a simple pendelum which consists of a mass-point p. It moves in a vertical plane and its motion is described by the angle θ to the vertical z-axis. The sole external force is gravity. Let ℓ be the length of the rigid bar from p to the fixed point of suspension. The kinetic energy of this one-particle system becomes

$$T = \frac{\ell^2 \cdot \dot{\theta}^2}{2}$$

The force of gravity is $-g \cdot \ell \cdot \sin\theta$ and we get the differential equation

$$(*) \qquad \ddot{\theta} = -\frac{g}{\ell} \cdot \sin\theta$$

From this we obtain an expression of Jacobi's $\mathfrak{sn}$-function whic appears in theory of elliptic functions, i.e. it is used to recover the $\mathfrak{p}$-function of Weierstrass. Regarding the two-particle system of two suspended pendela one derives from the preservation laws the addition formula for elliptic functions Here we shall not discuss these facts from analytic function theory in detail but only point out that thanks to basic equations in classical mechanics one recovers the main results about elliptic functions and integrals without using residue calculus.

The spherical pendelum.

Here a mass point p can swing in $\mathbf{R}^3$. It is attached to a fixed point of suspension by a rigid bar of unit length. Let (x, y) be the coordinates in a horisontal plane and the point of suspension is placed at $(0, 0, 1)$. Now we have a one-particle system with two degrees of freedom. Introducing the angles θ and ϕ the position of p is given by equations

$$x = \sin\theta \cdot \cos\phi \quad : \quad y = \sin\theta \cdot \sin\phi \quad : \quad z = 1 - \cos\theta$$

The kinetic energy becomes

$$T = \frac{\sin^2\theta}{2} \cdot \dot{\phi}^2 + \frac{\dot{\theta}^2}{2}$$

The D'Alembert-Lagrange equations give a constant C such that

$$\sin^2\theta \cdot \dot{\phi} = C$$

For the θ-function we get the second order equation

$$\ddot{\theta} = \sin\theta \cdot \cos\theta \cdot \dot{\phi}^2 - g \cdot \sin\theta = C^2 \cdot \frac{\cos\theta}{\sin^3\theta} - g \cdot \sin\theta$$

This gives the first order equation

$$\frac{\dot{\theta}^2}{2} = \frac{C^2}{2} \cdot \frac{1}{\sin^2\theta} + g \cdot cos\,\theta + E$$

where E is another constant.

Example. Suppose we start with initial conditions:

$$\dot{\phi}(0) = v \quad : \quad \theta(0) = \pi/4 \quad : \dot{\theta}(0) = 0 \implies$$

$$C = \frac{v}{2} \quad : \quad \frac{v^2}{4} + g \cdot \sqrt{2} + E = 0$$

Remark. At time $t = 0$ we obtain

$$\ddot{\theta}(0) = \frac{v^2}{2} - \frac{g}{\sqrt{2}}$$

So in the case when

$$v^2 = \sqrt{2} \cdot g$$

it follows that $t \mapsto \theta(t)$ stays constant, i.e. $\theta(t) = \frac{\pi}{4}$. This reflects Huyghen's formula for the centrifugal force, i.e. the initial ϕ-velocity is in balance with the force of gravity. The reader should illustrate this by a figure. When $v^2 < \sqrt{2} \cdot g$ the θ-function starts to decrease. After a wgile it oscillates and the result is that $t \mapsto \theta(t)$ is an oscillating function of t. It is a good exercise to calculate the period, i.e. the time t^* until $\theta(t^*) = \frac{\pi}{4}$ for the first time. The motion of p where one takes the ϕ-function into the account is of course more difficult to grasp. If one regards the projection of p onto the horisontal (x, y)-plane we get a parametrized curve

$$t \mapsto (x(t), y(t))$$

Here we have

$$x(t) = \cos \phi(t) \cdot \sin \theta(t) \quad \text{and} \quad \phi(t) = C \cdot \int_0^t \frac{du}{\sin^2 \theta(u)}$$

with a similar expression for $y(t)$. It is instructive to plot this curve with the aid of a computer.

Foucault's pendelum.

Introduction. When Newton formulated the laws of mechanics he was well aware of the fact that the rotation of the earth around its polar axis affects the motion of falling bodies. For example, if a stone is dropped into a mine on the northern hemi-sphere in the vicinity of Stockholm, then it will tend slightly to the east. But the outcome of this experiment is not easy to measure since the deviation caused by the earth's rotation is small. It was not until 1854 that Focault gave a convincing demonstration of the earth's rotation. His pendelum is still placed at Pantheon in Paris. The length of the bar joining the fixed point of suspension and the moving mass-point is 67 meter. Here a student of mathematics can be taught an instructive lesson where two *different phenomena* can occur. The first experiment is to set pendelum in motion by a sudden impact when the suspending bar at time $t = 0$ is at rest in vertical position. The result is that the plane of the pendelum changes during its oscillation after the impact. At Paris it makes a full turn in approximately 32 hours. Moreover, the rotation of the plane is in the *negative* direction. The second experiment is to start with an initial condition where the vertical angle θ_* is $\neq 0$ at time zero when the suspended bar is is released so that the pendelum starts to swing under the force of gravity. In this case the plane of the oscillating motion will turn in the *positive* direction. For the student of mathematics it is a challenge to understand why this difference. As we shall see it follows because different intitial conditions affect the solution to a system of differential equations which determine the motion of Foucault's pendelum.

How to derive the Equations of motion

The angular velocity of the earth's rotation is small, i.e. given by

$$\omega = \frac{2\pi}{60^2 \cdot 24} \tag{i}$$

As a first approximation one may therefore ignore terms where ω^2 appears. For the same reason one may assume that the force of gravity g is constant and directed in the line which is $\perp$ to the horisontal floor inside Pantheon and as a first approximation we put $g = 9,81$. Next, choose an orthonormal basis e_x, e_y, e_z where e_z is negative in the line $\perp$ to the floor, i.e. it is directed in the negative direction towards the center of the earth. Next, e_x is parallel to the meridian directed to the south. So e_y is therefore pointing in the *eastern* direction. The reader should verify that this gives a positively oriented ON-system, i.e. the formula for vector products is expressed by the equality

$$e_x \times e_y = e_z \tag{1}$$

This is best checked by regarding a globe where some suitable city is chosen such as London or Paris and then use the rule of thumbs to confirm (1). In order to apply Newton's general law (Force equals mass times acceleration) for the relative motion of Foucault's pendelum we need a fixed frame attached to the earth itself. Such a fixed frame is given by an orthonormal system e_ξ, e_η, e_ζ where e_ζ is the polar axis while e_ξ and e_η belong to the plane of the equator. Now the coordinate system e_x, e_y, e_z placed in Pantheon rotates with constant angular velocity ω. Notice that the position of e_ζ in the moving frame is constant. More precisely we see that:

$$e_\zeta = -\sin\lambda \cdot e_z - \cos\lambda \cdot e_x \tag{1}$$

Let $t \mapsto p(t)$ be the time dependent function describing the position of the mass point in Foceault's pendelum with respect to the moving frame. Passing to the fixed coordinates of the earth this corresponds to a time dependent function $p^*(t)$ where

$$p^* = S_t(p)$$

Here S_t is the orthogonal linear map from the moving frame to the fixed frame of the earth. In the present case the situation is easy since S_t corresponds to the constant rotation $\omega \cdot e_\zeta$. So by a special case for the formula of Euler's angular velocity in the next section about rigid bodies, we get the following equality for velocities:

$$\dot{p}^* = S_t(\omega \cdot e_\zeta \times p + \dot{p})$$

Passing to the acceleration vectors we obtain

$$\ddot{p}^* = S_t\big(\omega \cdot e_\zeta \times (\omega \cdot e_\zeta \times p) + 2\omega \cdot e_\zeta \times \dot{p} + \ddot{p}\big)$$

Now Newton's law applies for p^*. So if $\mathbf{F}$ is the force vector expressed in the moving frame, one has the equation

$$\mathbf{F} = \omega \cdot \mathbf{e}_\zeta \times (\omega \cdot \mathbf{e}_\zeta \times \mathbf{p}) + \mathbf{2}\omega \cdot \mathbf{e}_\zeta \times \dot{\mathbf{p}} + \ddot{\mathbf{p}} \tag{2}$$

Here ω^2 is small and we may therefore ignore the double vector product above and consider the approximative equation

$$\mathbf{F} = \mathbf{2}\omega \cdot \mathbf{e}_\zeta \times \dot{\mathbf{p}} + \ddot{\mathbf{p}} \tag{3}$$

Next, express the force vector $\mathbf{F}$ in components, i.e write

$$\mathbf{F} = X \cdot e_x + Y \cdot e_y + Z \cdot e_z$$

Using (1) above we compute the vector product in (2) and obtain:

Proposition. *Ignoring the term ω^2 the equations of motion become*

$$\begin{aligned} \ddot{x} &= X + 2\omega \cdot \sin\lambda \cdot \dot{y} \\ \ddot{y} &= Y - 2\omega \cdot \sin\lambda \cdot \dot{x} + 2\omega \cdot \cos\lambda \cdot \dot{z} \\ \ddot{z} &= Z - 2\omega \cdot \cos\lambda \cdot \dot{y} \end{aligned}$$

Next, the force vector $\mathbf{F}$ consists of two parts. One is $g \cdot e_z$ which is caused by gravity. The second is the force of tension from the bar which is a vector is parallell to the bar. So we have

$$X = -\frac{N}{\ell} \cdot x \quad : Y = -\frac{N}{\ell} \cdot y \quad : Z = g - \frac{N}{\ell} \cdot z$$

where $N = N(t)$ is a time dependent function. Notice that we also have the constraint:

$$x^2 + y^2 + z^2 = \ell^2$$

where ℓ is the length of the suspending bar. Together with the three equations from the Proposition one can then solve the ODE-system to find the four functions $x(t), y(t), z(t)$ and $N(t)$.

A simplification. When the oscillation stays close to the vertical direction, i.e. when

$$z \simeq \ell$$

we can ignore the z-motion and approximate N with g and are left with the following system for x and y:

$$\begin{aligned} \ddot{x} &= -\frac{g}{\ell} \cdot x + 2\omega \cdot \sin\lambda \cdot \dot{y} \\ \ddot{y} &= -\frac{g}{\ell} \cdot y - 2\omega \cdot \sin\lambda \cdot \dot{x} \end{aligned} \tag{*}$$

Solution of the system(*)

Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ an easy computation gives a pair of constants ω_* and C such that

(i) $$r^2\dot{\theta} = -\omega_* \cdot r^2 + C \quad : \quad \omega_* = \sin\lambda \cdot \omega$$

A first example. Suppose the initial conditions are $\dot{\theta}(0) = 0$ and $\dot{r}(0) > 0$. It means that we initially push pendelum when it is at rest in a vertical position. Then $C = 0$ above and we conclude that $\dot{\theta}$ moves in a negative direction and makes a full turn after

$$\frac{24}{\sin\lambda} \text{ hours}$$

A second example. Here

$$\dot{r}(0) = \dot{\theta}(0) = 0 \quad : \quad r(0) = a > 0$$

To find the solution in this case we introduce the ϕ-angle defined by

$$\phi(t) = \theta(t) + \omega_* \cdot t$$

Then we get Kepler's equality:

$$\dot{\phi} = \frac{C}{r^2} \quad : \quad C = \frac{\omega_*}{a^2}$$

Now $t \to r(t)$ starts to decrease, i.e. $a = r(0)$ is a maximum for the r-function. From this we conclude that

$$t \mapsto \theta(t)$$

is increasing, i.e. in contrast to the first example the plane of the pendelum now moves in a *positive direction.*

2. Rigid Bodies

In a rigid body K distances between points remain constant under motion, even when a sudden force acts on K by an impact. When K is placed in $\mathbf{R^3}$ it has six degrees of freedom. To see this we first choose some point $p \in K$. It has some coordinates in $\mathbf{R}^3$. Keeping p fixed all other points in K are determined by a *rotation* expressed by an orthogonal 3×3-matrix with determinant 1, i.e. a matrix $A \in \mathrm{SO}(3)$. To see this we consider an *orthogonal frame* centered at p, i.e. three pairwise orthogonal unit vectors n_1, n_2, n_3. Notice that we always can add the end-points q_1, q_2, q_3 of these $\mathbf{n}$-vectors to K - if necessary each q_ν has zero mass. Now we consider a point $q \in K$. The vector $q - p$ is uniquely expresed by the orthogonal frame:

$$(*) \qquad q - p = a_1 n_1 + a_2 n_2 + a_3 n_3 \quad (a_1, a_2, a_3) \in \mathbf{R^3}$$

Since K is rigid the euclidian distances of q to the points p, q_1, q_2, q_3 are independent of the position of K in R^3. Hence

$$a_1^2 + a_2^2 + a_3^2 = |q - p|^2$$

is determined. At the same time

$$|q - q_1|^2 = |(q - p) - (q_1 - p)|^2 = (a_1 - 1)^2 + a_2^2 + a_3^3 = -2a_1 + 1 + |q - p|^2$$

This determines a_1. In the same way a_2 and a_3 are determined. This shows that the position of K is determined by that of p and the three vectors of the chosen orthogonal frame. We conclude that a rigid body moving in $\mathbf{R}^3$ has six degrees of freedom. Let us now consider a continuous motion of K. So when t is the time variable we first obtain the vector valued function $p(t)$. The attached orthogonal frame gives for each t an orthogonal 3×3-matrix S_t such that the position of a point $q \in K$ expressed by (*) at a time t becomes:

$$q_\nu(t) = p(t) + S_t(n_\nu) \quad 1 \le \nu \le 3$$

Above $t \mapsto S_t$ is a continuous map with values in $O(3)$. Recall that the group of orthogonal 3×3-matrices contains the subgroup SO(3) whose elements have determinant one. Since $t \mapsto S_t$ is continuous the determinant of S_t cannot jump from 1 to -1. Hence, it keeps the same orientation and $t \mapsto S_t$ is therefore and SO(3)-valued function of the chosen frame at time zero is positively oriented.

2.1 Body spaces A body space attached to K consists of an orthonormal vector space where a specific point $p \in K$ is the origin and n_1, n_2, n_3 is an orthonormal basis which is kept fixed in K during its motion in R^3. Thus, the orthonormal frame is *independent* of the position of K in $\mathbf{R}^3$ and is called the body space. Let us denote it by $\mathcal{V}_K$. When $p \in K$ is the origin in $\mathcal{V}_K$ we say that this orthonormal vector space is *centered* at p. With these notations the previous discussion gives:

2.2 Theorem *A continuous motion of K yields an $SO(3)$-valued map $t \mapsto S_t$ such that for each point $q \in K$ one has*

$$\text{(i)} \qquad q^*(t) = p^*(t) + S_t(q - p)$$

where $p^*(t)$ resp. $q^*(t)$ are the positions of p and q in $\mathbf{R^3}$, while $q - p \in \mathcal{V}_K$.

2.3 Remark Notice that S_t for each t is viewed as a linear map from $\mathcal{V}_K$ into $\mathbf{R^3}$. It is called the *rotation matrix* under the motion. The construction started from a chosen point $p \in K$. Suppose we take another point $\rho \in K$. Choosing an orthonormal frame centered at ρ gives another SO(3)-valued map $t \mapsto R_t$ such that

$$\text{(ii)} \qquad q^*(t) = \rho^*(t) + R_t(q - \rho) \quad : \quad q \in K$$

At the same time $\rho^*(t) = p^*(t) + S_t(\rho - p)$ and hence

$$p^*(t) + S_t(q - p) = q^*(t) = p^*(t) + S_t(\rho - p) + R_t(q - \rho) \implies$$

$$R_t(q - \rho) = S_t(q - p) - S_t(\rho - p) = S_t(q - \rho)$$

Since $q \in K$ is arbitrary it follows that $R_t = S_t$. This shows that the rotation matrix is *intrinsically defined*, i.e. independent of a chosen point in K where $\mathcal{V}_K$ is centered.

2.3 Euler's Angular velocity Consider a motion of the rigid body whose time dependent functions are of class C^2 at least. We can take time derivatives of the elements of S_t. This yields for each t the 3×3-matrix $\dot{S}_t$ which again is a linear map from $\mathcal{V}_K$ into $\mathbf{R^3}$. Let us also regard the *inverse* linear map S_t^{-1}. Since S_t is orthogonal this inverse is the adjoint matrix S_t^*. Now $S_t^* \circ \dot{S}_t$ is a linear map from $\mathcal{V}_K$ into itself.

2.4 Proposition. *The matrices $S_t^* \circ \dot{S}_t$ are anti-symmetric for all t.*

Proof. Since S_t is orthogonal, the function $t \mapsto \langle S_t(q), S_t(p)\rangle$ is constant for all pairs p, q in $\mathcal{V}_K$. Hence the time derivative is zero which gives

$$0 = \langle \dot{S}_t(q), S_t(p)\rangle + \langle S_t(q), \dot{S}_t(p)\rangle = \langle S_t^* \circ \dot{S}_t(q), p\rangle + \langle q, S_t^* \circ \dot{S}_t(p)\rangle$$

where the last equality follows since S_t is orthogonal. This proves the required anti-symmetry.

Recall from Linear Algebra that every anti-symmetric matrix is expressed by a *vector product*. Thus, for each t there exists a unique vector $\omega_t \in \mathcal{V}_K$ such that

$$S_t^* \circ \dot{S}_t(q) = \omega_t \times q \qquad q \in \mathcal{V}_K$$

Next, given a fixed point $p \in K$ chosen as the origin in $\mathcal{V}_K$, we conclude that if $q \in K$ then the velocity of $q^*(t)$ is given by

$$\dot{q}^*(t) = \dot{p}^*(t) + \dot{S}_t(q)$$

Since $S_t \circ S_t^*$ is the identity we obtain

$$\dot{q}^*(t) = \dot{p}^*(t) + S_t \circ S_t^* \circ \dot{S}_t(q) = \dot{p}^*(t) + S_t(\omega_t \times q)$$

Hence we have proved that the function $t \mapsto \omega_t$ together with the rotation matrix S_t determine velocities of points in K under the motion. One refers to ω_t as *Euler's angular velocity*. By the construction this time dependent function takes values in the body space $\mathcal{V}_k$.

2.5 The center of mass Let K be a rigid body which consists of a finite set of mass points $p_1, \ldots, p_N$. One can imagine that they are joined by rigid bars with zero mass. Now there exists the unique point $\mathfrak{o}$ which is the *the center of mass*, or simply the mass-point of K. It is determined by the equality

$$\mathfrak{o} = \frac{1}{M} \sum_{\nu=1}^{\nu=N} m_\nu \cdot p_\nu \quad M = \sum m_\nu \text{ is the total mass}$$

2.6 Kinetic energy and momentum When the rigid body K moves in $\mathbf{R^3}$ the velocities varies between individual points because of rotation. Choose $\mathfrak{o}$ as the origin in the body space $\mathcal{V}_K$ which gives

$$\dot{p}^*(t) = \dot{o}^*(t) + S_t(\omega_t \times (p - \mathfrak{o})) \quad p \in K$$

So if $p_1, \ldots, p_N$ are the mass points of K, the kinetic energy becomes

$$T = \sum \frac{m_\nu}{2} \langle \dot{p}^*_\nu, \dot{p}^*_\nu \rangle = \sum m_\nu \langle \dot{p}^*_\nu, \dot{o}^* \rangle + \sum m_\nu \langle \dot{p}^*_\nu, S_t(\omega_t \times (p_\nu - \mathfrak{o})) \rangle$$

Since $\mathfrak{o}$ is the center of mass we get $\sum \dot{p}^*_\nu = M \cdot o$ and in $\mathcal{V}_K$ we notice that $\sum m_\nu (p_\nu - \mathfrak{o}) = 0$ since the body space was centerd at $\mathfrak{o}$. Hence the last term above reduces to

$$\frac{1}{2} \cdot M \cdot \langle \dot{o}^*, \dot{o}^* \rangle + \sum m_\nu \cdot \langle S_t(\omega_t \times (p_\nu - \mathfrak{o})), S_t(\omega_t \times (p_\nu - \mathfrak{o})) \rangle$$

We can express the last term using the body space. Namely, the rotation matrix is *orthogonal* and therefore it preserves the inner product. Hence the last sum above can be written as:

$$\sum m_\nu \cdot \langle \omega_t \times (p_\nu - \mathfrak{o}), \omega_t \times (p_\nu - \mathfrak{o}) \rangle$$

This suggests that we introduce a linear operator on $\mathcal{V}_K$.

2.7 Definition *The central operator of inertia is the linear operator defined on $\mathcal{V}_K$ by*

$$q \mapsto \sum m_\nu \cdot (p_\nu - \mathfrak{o}) \times [(q - \mathfrak{o}) \times (p_\nu - \mathfrak{o})]$$

It is denoted by $\mathcal{M}_\mathfrak{o}$.

Next, recall from Linear Algebra that the vector product on an orthonormal space is anti-commutative but fails to satisfy the associative law. Moreover, for each pair of vectors u, v in $\mathcal{V}_K$ one has

$$\langle u, (u \times v) \times u \rangle = ||u \times v||^2$$

Using this we obtain the following:

2.8 Theorem *The kinetic energy is expressed by*

$$T = \frac{1}{2} M \cdot |\dot{o}^*|^2 + \frac{1}{2} \langle \omega_t, \mathcal{M}_\mathfrak{o}(\omega_t) \rangle$$

One refers to $\frac{1}{2}\langle \omega_t, \mathcal{M}_\mathfrak{o}(\omega_t) \rangle$ as the rotational kinetic energy and it is denoted by T_{rot}.

2.9 Change of center It may occur that it is easier to pursue the motion of another point in K than the mass point. For this purpose we give

2.10 Definition *Let $r \in K$. The inertia operator $\mathcal{M}_r$ in a body space centered at a point $r \in K$ is defined by*

$$q \mapsto \sum m_\nu \cdot (p_\nu - r) \times [(q - r) \times (p_\nu - r)]$$

A straightforward calculation which is left to the reader shows that

$$\mathcal{M}_r(q) = \mathcal{M}_{\mathfrak{o}}(q - r) + M \cdot r \times (q \times r) \qquad q \in K$$

This shows how an operator of inertia changes with a chosen center point. Moreover, the reader may check the following for each $p \in K$:

$$T = \frac{1}{2} M \cdot ||\dot{p}^*||^2 + \frac{1}{2} \langle \omega_t, \mathcal{M}_p(\omega_t) \rangle$$

2.11 Angular momentum When K moves we define for each time value t the vector

$$\mathfrak{M}_{\mathfrak{o}} = \sum m_\nu \cdot (p^*(t) - \mathfrak{o}^*(t)) \times \dot{p}^*_\nu$$

Exercise. Verify the equality:

$$\mathfrak{M}_{\mathfrak{o}} = S_t(\mathcal{M}_o(\omega_t))$$

Taking the time derivative of the vector valued function $\mathfrak{M}_{\mathfrak{o}}$ we get

$$\frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = S_t \left[\mathcal{M}_o(\dot{\omega}_t)) + \omega_t \times \mathcal{M}_o(\omega_t) \right]$$

This expression of the time derivative of $\mathfrak{M}_{\mathfrak{o}}$ is cslled Euler's equation for the time derivative of angular momentum.

2.12 Equations of motion

At this stage we study the effect of fores acting on a rigid body during its motion. To begin with, if F_ν is a force vector acting on p_ν at a given moment the Newton's Law gives $F_\nu = m_\nu \ddot{p}^*_\nu$. Next, since the vector product is anti-commutative we get:

$$\frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = \sum m_\nu \cdot (p^*_\nu(t) - \mathfrak{o}^*(t)) \times \ddot{p}^*_\nu$$

Applying Newton's Law one has

$$\frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = \sum (p^*_\nu(t) - \mathfrak{o}^*(t)) \times F_\nu$$

2.13 External versus inner forces During a motion inner forces keep the body rigid. Inner forces consist of pairs f_{ij} and $-f_{ij}$ where f_{ij} acts on the mass-point p_j while the opposed force vector $-f_{ij} = f_{ji}$ acts on p_i. Apart from these there exist *external forces* acting on each single mass point. They are denoted by F_i^{ext}. Since the vector product is anti-commutative, the total effect of inner forces disappears and hence one has:

$$\frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = \sum (p_\nu^*(t) - \mathfrak{o}^*(t)) \times F_\nu^{\text{ext}}$$

This equation together with Euler's identity for the time derivative of the angular momentum are used to find the motion of a rigid body.

2.14 Example In $\mathbf{R}^3$ we denote the vertical direction by z. So here gravity yields an external force whose strength is g. On a mass-point p_ν we have $F_\nu^{\text{ext}} = -m_\nu g \cdot e_z$. Hence we see that

$$\frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = -g \cdot \sum m_\nu \cdot (p_\nu^*(t) - \mathfrak{o}^*(t)) \times e_z = 0$$

Thus, $\mathfrak{M}_{\mathfrak{o}}$ is constant during the motion when gravity is the sole external force.

3. Rotation around a fixed point

We shall study a rigid body K where one point p_* remains fixed during the motion. Let M be its mass. The motion is described by the matrix S_t which maps a body space $\mathcal{V}_K$ entered at p_* into $\mathbf{R^3}$. The inertia operator centered at p_* is denoted by $\mathcal{M}$, i.e. we drop for simplicity the subscript p_*. We assume that $p_* \neq \mathfrak{o}$, i.e. the center of mass is not fixed. In $\mathbf{R}^3$ the z-axis is vertical and gravity acts in the negative z-direction. We assume that this is the sole external force acting on K. Under this assumption we are going to find the equations of motion. For this purpose we consider the angular momentum $\mathfrak{M}$ centered at p_*. Recall that if $\omega(t)$ is Euler's angular velocity, then

$$\mathfrak{M}(t) = S_t(\mathcal{M}(\omega(t))$$

Since gravity is the sole external force we have

$$\frac{d}{dt}(\mathfrak{M}(t)) = \sum m_\nu \cdot p_\nu^* \times (-g \cdot e_z) = -g \cdot M \cdot \mathfrak{o}^* \times e_z$$

where e_z is the euclidian unit vector for the z-coordinate. From XXX we obtain

$$S_t\,[\,\mathcal{M}(\dot{\omega}(t)) + \omega(t) \times \mathcal{M}(\dot{\omega}(t))\,] = -g \cdot M \cdot \mathfrak{o}^* \times e_z$$

Applying S_t^* on both side we get a system of ordinary differential equations in the body space:

$$\mathcal{M}(\dot{\omega}(t)) + \omega(t) \times \mathcal{M}(\omega(t) = -g \cdot M \cdot \mathfrak{o} \times S_t^*(e_z)$$

This system express the *Euler-Lagrange equations* for the body. Notice that the vector $S_t^*(e_z)$ varies in $\mathcal{V}_K$. This vector-valued function is only known via the rotation of K, which in its turn varies via the vector valued function ω_t. So it is not clear how one should proceed to solve this system of ODE:s. For this purpose we will choose a suitable *orthonormal basis* in $\mathcal{V}_K$ adapted to the linear operator $\mathcal{M}$.

3.1 Principal axes Recall that $\mathcal{M}$ is a symmetric linear operator on $\mathcal{V}_K$. By the *spectral theorem* for symmetric matrices there exists an orthogonal basis e_1, e_2, e_3 which diagonalizes $\mathcal{M}$. Hence there are constants A_1, A_2, A_3 such that

$$\mathcal{M}(e_i) = A_i e_i \quad 1 \leq i \leq 3$$

Above each $A_i > 0$ unless K is a linear body which we ignore to discuss separately since then the subsequent material becomes almost trivial. Now ω_t is expressed in this basis:

$$\omega_t = \omega_1(t)e_1 + \omega_2(t)e_2 + \omega_3(t)e_3$$

The $e-$basis is chosen so that $e_1 \times e_2 = e_3$ and so on, i.e. positively oriented with respect to the vector product. Then

$$\mathcal{M}(\dot{\omega}(t)) + \omega(t) \times \mathcal{M}(\omega(t)$$

can be expressed in the e-basis and using the rules for vector products the three components ofthis vector in $\mathcal{V}_K$ become:

$$A_1\dot{\omega}_1 + (A_3 - A_2)\omega_2\omega_3$$
$$A_2\dot{\omega}_2 + (A_1 - A_3)\omega_1\omega_3$$
$$A_3\dot{\omega}_3 + (A_2 - A_1)\omega_1\omega_2$$

The *homogenous system* where each term above is zero was studied by Euler. The solution is qutie involved since the system is non-linear. A detailed study occurs of homogenous solutions to the system above is presented in the first volume of the famous text-books by Landau-Lifschitz.

3.2 The Kovalevsky case

Assume that $A_1 = A_2 = 2A_3$ and $\mathfrak{o} = e_1$, i.e. the center of mass is placed in the plane of symmetry at a unit distance from the fixed point. To find the Euler-Lagrange equations we regard the time dependent function $S_t^*(e_z)$ in the body space. Set

$$S_t^*(e_z) = \gamma_1(t)e_1 + \gamma_2(t)e_2 + \gamma_3(t)e_3$$

To simplify the notations a bit further we assume that the total mass M is such that $gM = 1$ which does not affect the general structure of solutions to the equations if motion. With $\mathfrak{o} = e_1$ we notice that

$$\mathfrak{o} \times \gamma = -\gamma_3 e_2 + \gamma_2 e_3$$

So with $A_1 = A_2 = 2$ and $A_3 = 1$ the Euler equations for the angular velocity are:

$$\begin{aligned} &2\dot{\omega}_1 - \omega_2\omega_3 = 0 \\ &2\dot{\omega}_2 + \omega_1\omega_3 = -\gamma_3 \\ &\dot{\omega}_3 = \gamma_2 \end{aligned}$$

In addition to these three equations we have a first order system for the γ-vector. Namely, since e_z is fixed in $\mathbf{R}^3$ we get

$$0 = \frac{d}{dt}(e_z) = S_t(\omega \times \gamma + \dot{\gamma}) \implies \dot{\gamma} = \gamma \times \omega$$

The last equaliy yields the system:

$$\begin{aligned} \dot{\gamma}_1 &= \gamma_2\omega_3 - \gamma_3\omega_2 \\ \dot{\gamma}_2 &= -\gamma_1\omega_3 + \gamma_3\omega_1 \\ \dot{\gamma}_3 &= -\gamma_1\omega_2 - \gamma_2\omega_1 \end{aligned}$$

Invariant integrals Above appear six differential equations for the two 3-vectors ω and γ. In addition there exist three algebraic identities. First, since γ is a unit vector one has

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

Next, the sum of kinetic and potential energy is constant which gives

$$\omega_1^2 + \omega_2^2 + \frac{\omega_3^3}{2} + \gamma_1 = E$$

Finally, the time derivative of the angular momentum $\mathfrak{M}$ is $\perp$ to e_z. Hence $\langle \mathfrak{M}, e_z \rangle$ is a constant and since $\mathfrak{M} = S_t(\mathcal{M}(\omega))$ there is a constant F such that $\langle \mathcal{M}(\omega), \gamma \rangle = F$. Hence we have a third algebraic equation:

$$2\omega_1\gamma_1 + 2\omega_2\gamma_2 + \omega_3\gamma_3 = F$$

Kovalevsky's fourth integral. It turns out that there exists one more integral to the ODE-system. To obtain it we introduce the imaginary unit. The first two equations for the ω-system give:

$$2(\dot{\omega}_1 + i\dot{\omega}_2) = i\omega_3(\omega_1 + i\omega_2) - i\gamma_3$$

Next, the first two equations for the γ-system give:

$$\dot{\gamma}_1 + i\dot{\gamma}_2 = i\gamma_3(\omega_1 + i\omega_2) - i\omega_3(\gamma_1 + i\gamma_2)$$

Together these two equations yield

$$\frac{d}{dt}(\dot{\omega}_1 + i\dot{\omega}_2)^2 = i\omega_3(\omega_1 + i\omega_2)^2 - i\gamma_3(\omega_1 + i\omega_2) \implies$$

$$\frac{d}{dt}[\dot{\omega}_1 + i\dot{\omega}_2)^2 + (\gamma_1 + i\gamma_2)] = i\omega_3[(\omega_1 + i\omega_2)^2 + (\gamma_1 + i\gamma_2)]$$

Put

$$\phi = (\omega_1 + i\omega_2)^2 + (\gamma_1 + i\gamma_2)$$

Then ϕ satisfies the differential equation

$$\dot{\phi} = i\omega_3 \cdot \phi$$

Since the function $\omega_3(t)$ is real -valued this differential equation implies that the t-derivative of $\log(\phi(t))$ is *purely imaginary*. Recall now that for the complex log-function one has

$$\mathfrak{Re}[\log(\phi(t))) = \log|\phi(t)]$$

Hence the *absolute value* of ϕ is constant. This gives a real constant k such that

$$|(\omega_1 + i\omega_2)^2 + \gamma_1 + i\gamma_2|^2 = k^2 \implies$$
$$(\omega_1^2 - \omega_2^2 + \gamma_1)^2 + \omega_1^2\omega_2^2\gamma_2^2 = k^2$$

Remark The algebraic equation above gives the *fourth integral* which is used to solve the equations of motion can be solved by quadrature. But let us remark that in contrast to the "classical case " by Euler where the quadrature solution is achieved by an elliptic integral of the first kind, the solution for the time dependent Euler angles in the Kovalesvky gyroscope require a more extensive class of so called *hyper-elliptic integrals.* For details we refer to Kovalevsky's original article which in addition to the example of a special gyroscope contains about 20 pages devoted to quite involved calculations. Personally I think that every beginner should study original work, i.e. my opinion is that the presentation in Kovalevsky's article offers an optimal introduction to some specific families of hyper-elliptic functions.

3.3 Eulerian angles

Given two angles $0 < \theta < \pi$ and $0 \le \phi \le 2\pi$ we put

$$e_3 = \cos\theta \cdot e_z + \sin\theta(\cos\phi \cdot e_x + \sin\phi \cdot e_y)$$

To this vector we associate the unit vectors

$$\xi_2 = -\sin\theta \cdot e_z + \cos\theta\,[\cos\phi \cdot e_x + \sin\phi \cdot e_y] \quad : \quad \xi_1 = \sin\phi \cdot e_x - \cos\phi \cdot e_y$$

We notice that $\xi_1 \times \xi_2 = e_3$ and hence the ordered triple ξ_1, ξ_2, e_3 is a positively oriented orthonormal frame. A general ON-frame (e_1, e_2, e_3) arises when we consider another angular variable ψ and set:

$$e_1 = \cos\psi\,\xi_1 + \sin\psi\,\xi_2 \quad : \quad e_2 = \sin\psi\,\xi_1 + \cos\psi\,\xi_2$$

We refer to θ, ϕ, ψ as the Euler angles defining this ON-frame. Let us now consider a rigid body which rotates around the origin and let e_1, e_2, e_3 be some postively oriented ON-frame in the body space. We get the time dependent vectors in $\mathbf{R}^3$:

$$e^*_\nu(t) = S_t(e_\nu)$$

For each t they give an ON-frame in $\mathbf{R^3}$. We assume that $e^*_3(t)$ is not parallell to the z-axis. Then there exist time dependent functions $\theta(t)$ and $\phi(t)$ such that

$$e^*_3(t) = \cos\theta(t) \cdot e_z + \sin\theta(t)(\cos\phi(t) \cdot e_x + \sin\phi(t) \cdot e_x)$$

The remaining vectors $e^*_1(t)$ and $e^*_2(t)$ are now determined by the two angular functions $\theta(t), \phi(t)$ and a ψ-function which corresponds to a rotation of K around e^*_3. The result is that the time dependent rotation matrix S_t is a function of the three angular variables. Now we can take time derivatives and in this way express Euler's angular velocity by the three angle functions and their time derivatives. A straightforward calculation which is left to the reader gives:

$$\omega_1 = \dot\phi \cdot \sin\theta\sin\psi + \dot\theta\cos\psi$$

$$\omega_2 = \dot\phi \cdot \sin\theta\cos\psi - \dot\theta\sin\psi$$

$$\omega_3 = \dot\phi \cdot \cos\theta + \dot\psi$$

The kinetic energy. Suppose that e_1, e_2, e_3 yield principal axes for the operator of inertia. Hence

$$T = \frac{1}{2}[A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2]$$

Inserting the equations above we express T as a function of the angle functions and their time derivatives.

The symmetric case. Assume that $A_1 = A_2$. Then a simple reduction yields:

$$T = \frac{1}{2}A_3(\dot\psi + \dot\phi\cos\theta) + \frac{1}{2}A_1(\dot\theta^2 + \dot\phi^2\sin^2\theta)$$

3.4 The Kovalevsky's gyroscope in Eulerian angles Here $A_1 = A_2 = 2$ and $A_3 = 1$ and the center of mass $\mathfrak{o}$ is placed at ae_1 for some $a > 0$. The kinetic energy becomes

$$T = \frac{1}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + (\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

To study the effect of the external gravity force we express the inner product

$$\langle e_1^*(t), e_z\rangle = -\cos\psi\langle\xi_1, e_z\rangle + \sin\psi\langle\xi_2, e_z\rangle = -\sin\psi\cdot\sin\theta$$

Preservation of energy gives a constant E such that

$$\frac{1}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + (\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - ag\cdot\sin\psi\cdot\sin\theta = E$$

In addition to this we have the Lagrangean equations. Here ϕ is a cyclic variable. Hence there is a constant C such that

$$\cos\theta\cdot(\dot{\psi} + \dot{\phi}\cos\theta) + 2\dot{\phi}\cdot\sin^2\theta = C$$

From this equation we can eliminate $\dot{\phi}$ so that the constant energy integral is a function of ψ, θ and its time derivatives. We have also the Lagrangean equation for the ψ-variable which yields

$$\frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta) = -ag\cdot\cos\psi\cdot\sin\theta$$

Hence we have found a system of ODE-equations which has a unique solution for any given initial condition. The fact that it can be solved by quadrature is not evident when the equations are expressed as above. In a work by Königsberg (Acta Mathematica 189x] the solution was expressed by quadrature formulas where Königsberg implemented Kovalevsky's fourth integral. But the final result is is expressed by complicated functions , i.e. not as easy as in Euler's case.

Poincare's theory of Fuchsian groups

The theory of Fuchsian groups was created by Poincaré. His two articles *Théorie des groupes fuchsiens* and *Memoire sur les fonctions fuchsiennes* were published 1882 in the first first volume of Acta Mathematica. The article *Memoire sur les groupes kleinéens* appeared in volume III of Acta Mathematica. This article is more advanced and we shall not discuss Kleinan groups here. Nor do we discuss the article *Memoire sur les fonctions zétafuchsiennes.* The connection to arithmetic was presented in a later article *Les fonctions fuchsiennes et l'Arithmétique* from 1887. One should also mention the more recent article *Les fonctions fuchsiennes et l'équation* $\Delta(u) = e^u$ where Poincaré proved that this second order differential equation has a subharmonic solution with prescribed singularities on every closed Riemann surface attached to an algebraic equation. The last work started potential theoretic analysis on complex manifolds. Here we only discuss more elementary material from the first two articles above.

Remark. Of course Poincaré was inspired by earlier work, foremost by Bernhard Riemann, Hermann Schwarz and Karl Weierstrass. For example, he used the construction of multi-valued analytic extensions by Weierstrass which leads to the *Analytische Gebilde* of a multi-valued function f defined in some connected open subset Ω of $\mathbf{C}$. This *Analytische Gebilde* is a connected complex manifold X on which f becomes a single valued analytic function f^*. More precisely, there exists a locally biholomorphic map

$$\pi\colon\ X \mapsto \Omega$$

When $U \subset \Omega$ is simply connected the inverse image $\pi^{-1}(U)$ is a union of pairwise disjoint open sets U^*_γ where the single-calued analytic function f^* is determined by a branch T_γ of f, i.e. one has

$$T_\gamma(f)(\pi(x) = f^*(x) \quad : x \in U^*_\gamma$$

Major contributions are also due to Schwarz. In 1869 he used the reflection principle and calculus of variation to settle the Dirichlet problem and used this to prove the uniformisation theorem for connected domains bordered by p many real analytic and closed Jordan curves where p in general is ≥ 2. Of special interest is the multi-valued $\mathfrak{m}$-function defined in $\mathbf{C} \setminus \{(0,1)\}$ which is related to the elliptic integral of the first kind and hence to Jacobi's $\mathfrak{sn}$-function which appears in the equation of motion when a rigid body rotates around a fixed point. We shall recall the uniformisation theorem below and describe how Poincaré attached a certain group of Möbius transformations to realise the fundamental group $\pi_1(\Omega)$ for domains Ω as above.

An extra comment. Poincaré's theory of Fuchsian functions was not restricted to analytic function theory. His main concern was the theory of differential systems, both linear and non-linear and also directed towards to the general theory about abelian functions and their integrals inspired by Abel's pioneering work from the years between 1823 and 1828. Hundreds of text-books have appeared after Poincaré's original work. Personally I find that his own and quite personal presentation superseeds standard text-books which in most cases just treat some of the fraction from the great visions by Poincaré. So Poincaré's original work offers a very good introduction for the student who wants to enter studies about linear and non-linear differential systems in an algebraic context, together with function theory which leads to Fuchsian as well as Kleinian groups and there associated functions, See in particular the book *Analyse de ses travaux scientifiques* which contains a survey the scientific work by Poincaré. In several chapters Poincaré describes his own research from the period between 1880 until 1907 which offers not only a summary of results but also explanations of the the main ideas and methods which led to the theories.

Of course there exists more recent advancement in function theory. Here one should foremost mention work by Lars Ahlfors. So in addition to the cited reference above I recommend text-books by Ahlfors, especially his book *Conformal Invariants* which contains material about the theory of *extremal length* created by Arne Beurling around 1945. So from an analytic point of view the more recent discoveries by Ahlfors and Beurling have a wider scope and leads to many still unsolved problems in complex analysis including the study of so called quasi-conformal mappings.

Fuchsian groups.

They are constructed via Möbius transforms which give conformal mappings of the unit disc D onto itself. The set up is as follows: To each $a \in D$ we get the Möbius transform

$$\text{(i)} \qquad M_a(z) = \frac{z+a}{1+\bar{a}\cdot z}$$

If a, b is a pair of points in D a trivial computation shows that the composed map

$$\text{(ii)} \qquad M_b \circ M_a = M_c \quad : \quad c = \frac{a+b}{1+\bar{a}\cdot b}$$

In particular M_{-a} is the inverse to M_a and by (ii) the points in the unit disc are with elements in the group $\mathcal{M}$ of all Möbius transforms of the form (i). Let $\mathcal{F}$ be a subgroup of $\mathcal{M}$. To each $z \in D$ we get the orbit:

$$\mathcal{F}_z = \{M_a(z) \quad : \; M_a \in \mathcal{F}\}$$

We say that $\mathcal{F}$ is a discrete Fuchsian group if every orbit is a discrete subset of D. It means that if $r <$ then $\mathcal{F}_z$ only contains a finite set of points in the disc D_r of radius r. Notice that if $z = 0$ is the origin then the orbit $\mathcal{F}_0$ corresponds to the set of of points $a \in D$ for which the Möbius transform M_a belongs to $\mathcal{F}$.

Fundamental domains. Let $\mathcal{F}$ be a discrete Fuchsian group. We seek open subsets $\mathfrak{U}$ of the unit disc where every pair of distinct are *non-equivalent*. To find such domains we use a distance function on the unit disc introduced by Hermann Schwarz.

Definition. The δ-distance in the unit disc D is defined by

$$\delta(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - z_1\bar{z}_2|} \quad : \quad z_1, z_2 \in D$$

Let us see how a Möbius transformation affects the δ-function. If $a \in D$ a computation gives:

$$\delta(M_a(z_1), M_a(z_1)) = \frac{1}{1-|a|^2} \cdot \delta(z_1, z_2)$$

So when $|a| \to 1$ then the Möbius transform M_a tends to increase the δ-distance. Given a Fuchsian group $\mathcal{F}$ as above we set

$$(*) \qquad \mathfrak{D} = \{z \in D \quad : \quad \delta(z,0) < \delta(z,a) \quad \forall\, a \in \mathcal{F}_0 \setminus \{0\}\}$$

where $\mathcal{F}_0$ is the orbit which contains the origin.

Proposition. *Every $\mathcal{F}$-orbit intersects $\mathfrak{D}$ in at most one point.*

Proof. Assume the contrary, i.e. there exists some $b \in D$ and $\neq a \in \mathcal{F}_0$ such that both b and $M_a(b)$ belong to $\mathfrak{D}$. Since $\mathcal{F}$ is a group we also have $-a \in \mathcal{F}_0$. Now

$$\delta(b,0) < \delta(b,-a)$$

From (xx) we get

$$\delta(M_a(b), M_a(0)) < \delta(M_a(b), M_a(-a)) = \delta(M_a(b), 0)$$

This gives a contradiction since $M_a(0) = a$ and the inclusion $M - a(b) \in \mathfrak{D}$ means that we have the opposite inequality

$$\delta(M_a(b), 0) < \delta(M_a(b), M_a(0))$$

The boundary of $\mathfrak{D}$. If z is a boundary point of $\mathfrak{D}$ it follows by continuity that there exists at least some $0 \neq a \in \mathcal{F}_0$ such that

$$(*) \qquad \delta(z,0) = \delta(z,a)$$

The converse also holds, i.e. the reader should verify

Proposition. *The set $\partial\mathfrak{D} \cap D$ is equal to the set of points $z \in D$ for which there exists some $0 \neq a \in \mathcal{F}_=$ for which (*) above holds.*

The sets $K(a,b)$. The Proposition above suggests that we consider sets of the form:

$$K(a,b) = \{z \in D \quad : \quad \delta(z,a) = \delta(z,b)\} \quad : \quad a, b \in D$$

Proposition. *The set $K(a,b)$ is an arc of a circle which intersects the unit circle at a right angle. Moreover, a and 0 "liegen spiegelbildlich zueinander".*

Proof. The assertion is invariant under a Möbius transform. So it suffices to consider the case when $b = -a$ with $0 < a < 1$, i.e. the pair are real and symmetrically placed with respect to the origin. Then

$$K(a,b) = \{z : \frac{|z-a|}{1-az|} = \frac{z+a|}{1+az|}$$

With $z = x + iy$ an easy computation shows that (i) holds if and only if

$$4ax(1-x^2-y^2) = 0$$

It follows that $K(A,-A) \cap D$ is the lie segment $(-i,i)$ on the imaginary axis. It is regarded as a circle which has a $\perp$-intersection with the unit circle and the tow real points a and $-a$ are mutually reflected with each other along the imaginary axis.

Remark. Of course one should make a picture and in particular describe the sets $K(a,0)$ as a varies. So here one seeks all $z \in D$ such that

$$|z| = \frac{|z-a|}{1-\bar{a}z|}$$

Up to rotation enough to treat the case when $0 < a < 1$ is real and positive.

The favourable case. In most applications $\partial\mathfrak{D}$ consists of a finite union of circular arcs which belong to $K(0,a)$ for a finite set of points $0 \neq a \in \mathcal{F}_0$. Moreover, every such circular arc has end-points on the unit circle. So on T there exists a finite set of corner points which appear as common end-points of two circular arcs in the boundary of $\mathfrak{D}$. The simplest case is the Fuchsian group which corresponds to the fundamental group of $\mathbf{C} \setminus \{0,1\}$ which was already described by Schwarz after his explicit construction of the modular function with the aid of the reflection principle. For further examples we refer to the cited articles by Poincaré. Of course, the reader may also consult text-books of more recent origin for further illustrations.

Automorphic functions and the uniformisation theorem.

Let $\mathcal{F}$ be a discrete Fuchsian group. An analytic function $F(z)$ in D is called $\mathcal{F}$-automorphic if

$$F(z) = F(M_a(z)) \quad : \quad a \in \mathcal{F}$$

The existence of such automorphic functions is automatic if $\mathcal{F}$ is defined via a uniformisation of a connected open domain in $\mathbf{C}$. We describe this below. But let us first recall the Uniformisation Theorem restricted to planar domains. Here is the set up: Let Ω be a connected open set in $\mathbf{C}$ whose complement has at least two points. Fix some $x_0 \in \Omega$ and denote by $M\mathcal{O}(x_0)$ the set of germs of analytic functions at x which can be extended in the sense of Weierstrass to multi-valued analytic functions defined in Ω.

Remark. Thus, if $f \in M\mathcal{O}(x_0)$ and γ is a curve in Ω whose initial point is x_0, then f can be extended along γ via the procedure initiated by Heine and fully clarified by Weierstrass. If x is the end-point of γ we obtain a germ $T_\gamma(f) \in \mathcal{O}(x)$ which by the *Mondromy Theorem* only depends upon the homotopy class in the curve family joining x_0 and x. The fundamental group $\pi_1(\Omega)$ is as usual identified with the homotopy classes of closed curves at x_0. If γ is a closed curve we let $\{\gamma\}$ denote its homotopy class. So when $f \in M\mathcal{O}(x_0)$ we get a map from $\pi_1(\Omega)$ into $M\mathcal{O}(x_0)$ defined by:

$$(*) \qquad \{\gamma\} \mapsto T_{\{\gamma\}}(f)$$

We can evaluate each such germ which gives a map from $\pi_1(\Omega)$ into $\mathbf{C}$ defined by:

(**) $$\{\gamma\} \mapsto T_{\{\gamma\}}(f)(x_0)$$

By analytic continuation of f we can study its germs at any other point $x \in \Omega$ and identify $\pi_1(\Omega$ with homotopy classes of closed curves at x. This gives two similar maps as in (*) and (**) where we now regard local branches of f at x.

The case when (**) are injective. Denote by $M^*\mathcal{O}(x_0)$ the set of f in $M\mathcal{O}(x_0)$ such that (**) are injective for every $x \in \Omega$ and the complex derivatives of all its local branches are $\neq 0$ at all points in Ω.

The image of f. Let $f \in M^*\mathcal{O}(x_0)$. The complete image of f is the union of the set of values taken by all the local branches of f at points in Ω. We denote this set by $f(\Omega)$. Notice that $f(\Omega)$ is j the range of the map from the connected manifold given by *Die vollständige analytische Gebilde* on which f exists as a single-valued analytic function. With these notations one has:

Theorem Let $x_0 \in \Omega$. Then there exists a unique $f \in M^*\mathcal{O}(x_0)$ where $f(\Omega)$ is the open unit disc and $f(x_0) = 0$. Moreover, f has a distinguished germ f_* at x_0 whose complex derivative is real and positive and has the following extremal property:

$$\frac{df_*}{dx}(x_0) \geq |g'(x_0)| \quad : \forall g \in M\mathcal{O}(x_0) \quad : g(x_0) = 0 : \; g(\Omega) \subset D$$

Remark. This result is due to Hermann Schwarz. The proof is left as an exercise to the reader who is supposed to be familiar with the ordinary mapping theorem for simply connected domains. The *hint* is to employ the modular function $\mathfrak{m}$-function constructed explicitly by Hermann Schwarz in 1869. The $\mathfrak{m}$-function belongs to $M\mathcal{O}(i)$ where $\Omega = \mathbf{C} \setminus \{(0,1)\}$. For a general domain Ω we may assume that $\Omega \subset \mathbf{C} \setminus \{(0,1)\}$ and that $x_0 = i$. Given this one regards the variational problem where we for a fixed $x_0 \in \Omega$ put:

$$A = \max |f'(x_0)| \quad : \; f \in M\mathcal{O}(x_0) \quad : f(x_0) = 0 \quad : f(\Omega) \subset D$$

The variational problem has a solution since the existence of the $\mathfrak{m}$-function implies that the competing class is non-empty One verifies easily that this competing class is normal family of multi-valued functions and hence there exists an extremal function f^* which will be unique when we choose f so that the derivative at x_0 is real and positive. Using Möbius transforms one also verifies that $f^*(\Omega) = D$.

The inverse function F Let $z \in D$. Since f^* yields a covering map from Ω onto D the Covering Lemma yields a unique curve γ_z in Ω whose initial point is x_0 and which lifts above the line in the sense that

(i) $$T_{\gamma_z(t)}(f^*)(\gamma_z(t) = tz \quad : \quad 0 \leq t \leq 1$$

Let $x^*(z)$ denote the end-point $\gamma_z(1)$. In this way we have constructed an analytic function x^* in D. We refer to x^* as the inverse of the multi-valued f^*-function and prefer to write

$$F^*(z) = x^*(z)$$

The complex derivative $F*(z) \neq 0$ for all points and the image set $F(D) = \Omega$. However, F is not 1-1. So we introduce the following subset of D:

$$\mathfrak{U}_0 = \{z \in D \quad : \quad F^*(z) = x_0\}$$

Here $\mathfrak{U}_0$ is a discrete subset of D. More generally, if a and b is a pair of points in D we say that they are equivalent if

$$F^*(a) = F^*(b)$$

In this way D is the union of pairwise disjoint equivalence classes. Next, the monodromy theorem applied to the multi-valued f^*-function gives:

Theorem. *Let $a \in \mathfrak{U}_0$. Then the Möbius transform M_a maps each equivalence class in D into itself. In particular M_a restricts to a bijective map on $\mathfrak{U}_0$.*

The proof is left as an exercise to the reader. If necessary,consult the cited article by Poincaré for details or try to find a readable text-book in function theory which explains the proof.

The group $\mathcal{F}$ This is is the group of Möbius transforms whose elements are

$$M_a \quad : \quad a \in \mathfrak{U}_0$$

The monodromy theorem applied to f shows that $\mathcal{F}$ is isomorphic to $\pi_1(\Omega)$.
Fundamental domains. Starting from the group $\mathcal{F}$ one constructs a fundamental domain by the procedure from section 1. In this way we obtain a certain open subset of D on which the inverse F^*-function is 1-1 and has an image which is "almost equal" to Ω. This is best illustrated for the special case when $\Omega = \mathbf{C} \setminus \{(0,1)\}$. Let us remark that here one often prefers to use the upper half-plane instead of the unit disc.

Sonja Kovalevsky, Emmy Noether och Marie Curie

Även om Sonja Kovalevsky gav betydelsefulla bidrag till matematiken, främst i doktorsavhandlingen från 1874 och lösningen av ett klassiskt problem i mekanik som renderade henne Bordinpriset 1888, bör det tillfogas att hon inte tillhör den exklusiva skaran av de allra främsta samtida matematikerna med namn som Karl Weierstrass och Henri Poincaré. Detta bör dock inte uppfattas som något förklenande. Under åren i Stockholm från 1884 fram till hennes bortgång i januari 1891 var hon den mest framstående matematikern i Sverige. Hennes föreläsningar vid Stockholms Högskola under elva terminer inspirerade yngre matematiker, bland annat Bendixson och Phragmén som båda kom att bli professorer i Stockholm. Som medredaktör i Acta Mathematica gjorde hon betydelsefulla insatser och på ett internationellt plan var redan doktorsavhandlingen från 1874 mycket uppmärksammad. Den intresserade läsaren hänvisas till en intressant artikel om Cauchy-Kovalevsky teoremet författad av Harold Shapiro som ingår i boken [XX] från Kovalevsky-sympositet som ägde rum i juni 2000 vid Stockholms Universitet.

Den främsta kvinnliga matematikern någonsin är *Emmy Noether* (1879-1935). Hon gav under åren 1915 fram till 1935 många fundamentala bidrag, främst inom algebra där hon på 1920-talet som docent i Göttingen under sina föreläsningar bokstavligen trollband sina åhörare med namn som Emil Artin, Helmuth Hasse och van der Waerden i skolbänkarna. Hon gav även viktiga bidrag inom variationskalkyl, bland annat genom en mer generell matematisk tolkning av resultat som ingår i Albert Einsteins allmänna relativitetsteori. Emmy Noether var också en av huvudtalarna vid den internationella matematikerkongressen i Zürich 1932.

Den i särklass främste kvinnliga naturvetaren genom tiderna är naturligtvis *Marie Curie* (1868-1934). Hon växte upp i Polen och flyttade i början av 1890-talet till Frankrike där hon kom att vistas under återstoden av sitt liv. Det finns en viss koppling mellan den yngre Marie Curie och Sonja Kovalevsky, även om de på grund av Sonjas alltför tidiga bortgång aldrig kom att mötas. Marie inledde nämligen sin karriär som studentska vid Sorbonne i början av 1890-talet när hon i dåtidens "Concours" blev tvåa i matematik och etta fysik, en bedrift som naturligtvis väckte stor uppmärksamhet. Sonja Kovalevskys matematiska kollega Paul Appel, som bland annat erhöll Bordin priset 1884 samt delade Oscar II:s matematikpris i Stockholm med Henri Poincaré i början av 1890-talet, var Marie Curies lärare i matematik och mekanik vid Sorbonne. Måhända att Sonja Kovalvskys framgångar som visat att kvinnor kan vara framstående inom naturvetenskliga discipliner, bidrog, eller i varje fall underlättade, beslut av dåvarande professorer i Paris att ge Marie Curie möjlighet att börja bedriva forskning efter avslutad unversitetsexamen.. Detta ledde till att hon 1894 blev assistent vid dåtidens främsta laboratorium i kemi där två av hennes handledare senare blev Frankrikes första Nobelpristagare i början av 1900-talet. Marie själv fick ju Nobelpriset i kemi 1911. Tidigare hade hon delat priset i fysik för upptäckten av radium 1904 med Bequerel och hennes make Jaques Curie som omkom vid en tragisk trafikolycka i Paris 1906. Marie var också förtrogen inom fysik, bl.a. var hon sakkunig när Albert Einstein fick sin professur i Zürich i janauri 1912. Deras möten under åren fram till första världskriget inspirerade Einstein när han utformade sin allmänna relativitetsteori. Särskilt då Einstein besökte Paris 1913 där han förutom sine egna flera föreläsningar kunde ta del av de allra senaste framstegen inom dåtida kärnfysik kopplat till instabila strukturer som radium där Curie och Rutherford var de världsledande forskarna. Mer fantisfullt har annars möten mellan Einstein och Curie skildrats av Maries dotter Eve från deras besök hemma hos Einstein i Zürich där vandringar i alperna har fått ge bakgrund till tankar om den allmänna relativitetsteorin som ännu låg i sin linda 1913.

Einsteins hyllning till Marie Curie som person och vetenskapsidkare i en minnesartikel för New York Times skriven i juli 1934 kort efter hennes bortgång torde ha få motsvarigheter. Förutom banbrytande upptäckter inom kemi var Marie Curie också pionjär inom radiologi. Det allra första radiuminstitutet som kunde ge medicinsk hjälp mot svårartade cancertumörer byggdes åren 1911-1914 under Marie Curies och medicine professor Renaults ledning. På grund av första världskrigets utbrott samma sommar kunde dock inte verksamheten inledas förrän 1919. Under sina sista femton levnadsår var Curie tillsammans med Renault föreståndare vid institutet. I en omfattande minnesartikel från 1934 har Renault beskrivit betydelsen av Curies insatser inom cancervård. Sedan 1980-talet är Marie Curie priset som delas ut till prominenta kvinnliga läkare inom radiologi ett av de mest prestigefyllda medicinpriserna i världen.

Marie Curies liv och vetenskapliga karriär är skildrade av hennes yngsta dotter Eve i en bok som utgavs 1937 och torde vara den mest lästa biografin någonsin som handlar om en naturvetenskaplig forskare. Den finns översatt till i stort sett alla världens språk. Förutom ständigt nya upplagor på franska, engelska och polska kanske den intresserade läsaren lyckas finna exemplar på antikvariat av den svenska översättningen som utkom 1937. Eve föddes 1904 och är bosatt i New York. Förhoppningsvis får hon uppleva sin 104-årsdag nu i november. Eve mottog bland annat ett stort pris av Unesco så sent som i juli 2006 där hon i bilder från ceremonin i FN-huset i New York förefaller både yngre och mer vital än de som då gav henne utmärkelsen för alla hennes insatser för barn i tredje världen sedan början av 1950-talet. Här bör också nämnas att i motsats till sin storasyster Helène var Eve inte alls intresserad av naturvetenskap. Hon uppmuntrades i stället att ägna sig åt musik och debuterade som konsertpianist i 20-årsåldern. Senare kom Eve främst att senare bli verksam som journalist och författare. Under andra världskriget var hon krigskorrespondent, bland annat i Burma.

Så sent som för fem år sedan upptäcktes en samling anteckningar skrivna för nära hundra år sedan av en av Marie Curies elever då Marie vid sidan av sin professur vid Sorbonne undervisade en grupp barn i 10-årsåldern, bl.a. äldsta dottern Helene som ju också är Nobelpristagare i kemi. Hon delade priset med sin make 1934 för banbrytande upptäckter som bidragit till att på ett mer ekonomiskt och effektivt sätt kunna utföra Curieterapi vid behandling av svåra cancertumörer. Anteckningarna från Maries lektioner handlar om olika fysikaliska och kemiska fenomen och en rad laborativa experiment som ibland "fick smälla till ordentligt" vid olika kemiska reaktioner. Tack vare Maries genialitet är all detta kopplat med välgrundade teoretiska förklaringar och finns numera utgivet i boken *Lecons de Marie Curie* som kan beställas genom Marie Curies museum i Paris. Kanske vore detta något att ta del av vid undervisning i kemi och fysik för mellanstadiets elever i svenska skolor ?

Non-classical mechanics.

Quantum mechanics was created around 1925. From a historic point of view it is interesting to note that some essential mathematical foundation which are needed to solve equations in quantum mechanics were established *before* the physical laws in quantum mechanics had been formulated. One may compare this situation with Clerk Maxwell's early discoveries around 1860 phrased within pure mathematics which three decades later started the technology based upon electro-magnetic fields. Today relatively few persons in the world reflect upon Maxwell's discoveries while the majority take it for granted that one can communicate all over the world by mobile telephones. Returning to quantum mechanics we describe an achievement by Carleman in pure mathematics which appeared before quantum mechanics was born but a few years later became an essential tool to solve some f the equations which appear in quantum mechanics.

Torsten Carleman (1892-1949). He entered university studies in mathematics at Uppsala in 1910. After his PHD-exam in 1916 he became docent in Uppsala and in 1924 he was appointed as professor at Stockholm University in 1924. From 1927 he also served as the director of Institute of Mittag-Leffler and led the institute for more than two decades. From 1920 until his decease in 1949 he was considered as one of the world-leading mathematicians. For an account about Carleman's work the reader may consult the chapter in Lars Gårding's book about mathematics in Sweden before 1950. Some of his major results were established in 1923 and became a few years later closely related to quantum mechanics. The pure mathematics appeared in his article *Sur la théorie des equations intégrales à noyeau reél et symmetrique* published in [Årskrift 17, University of Uppsala 1923]. It contains the essential ingredients to solve the equations in quantum mechanics which a few years later were stated by Dirac, Heisenberg and Schrödinger. Carleman was also the first to give a rigorous solution to Schrödinger's equation at lectures in Paris in 1931.

The reader may consult Carleman's article *Sur la mathématique de l'équation de Schrödinger* [Arkiv för matematik, astronomoy and physics, vol. 24 (1934)]. which gives proof of existence for an extensive class of Schrödinger equation, including the spectral decomposition of associated eigenfunctions. Let us also mention that Carleman was very concerned with the interplay between pure mathematics and experimental sciences. See his lecture held at a meeting of the Academy of Sciences in Stockholm in 1944 which in its printed French version is entitled: *Sur l'action réciproque entre les mathematiques et les sciences éxperimentales exactes.*

Abel's inversion formula

Linear motion in a potential field. Before we expose the general equations by d'Alembert and Lagrange we shall consider a single particle systems and derive Abel's inversion formula. Here is the situation. A particle of unit mass can move on the non-negative x-axis. Let t be the time variable. Under a motion we get the time-depdendent function $x(t)$ and the velocity $\dot{x} = \frac{dx}{dt}$. The second order derivative $\ddot{x}$ is acceleration. A strictly increasing function $U(x)$ with $U(0)$ is also given. It is a potential function where $-U'(x)$ is the force acting at any point $x > 0$. Notice that it is negative. By Newton's law we have

$$\text{(i)} \qquad \ddot{x} = -U'(x)$$

It follows that:

$$\text{(ii)} \qquad \frac{d}{dt}[\frac{\dot{x}^2}{2}] + U(x(t)] = \dot{x}(\ddot{x} + U'(x(t)) = 0$$

Hence $\frac{\dot{x}^2}{2} + U(x(t))$ stays constant under the motion which reflects the energy principle, i.e. the sum of kinetic and potential energy is constant. Suppose the particle at time zero is at $x = 0$ and gets an initial velocity $\dot{x}(0) = v$ caused by a sudden impact. By (ii) the function $\dot{x}(t)$ decreases and we assume that $U(x)$ increases so fast that the particle eventually comes to rest at some time T_* where the position $x(T_*) > 0$. After the force caused by the potential function will push the particle back to $x = 0$. By symmetry we get the equality:

$$\text{(iii)} \qquad x(T_* + t) = x(T_* - t) \quad T_* \leq t \leq 2T_* \,.$$

At time $T_{**} = 2T_*$ the particle arrives at $x = 0$, but this time with velocity $-v$ in the negative x-direction. In many physical applications one does not know $U(x)$. But one may perform experiments, i.e. give the particle an initial velocity v and with the aid of clock measure the return time T_{**}. Notice that we do not assume that one can follows the particle during its travel on the x-axis. For example, one may imagine it is invisible except at $x = 0$. So by an experiment one cannot measure $x^*(T_*)$, nor make any intermediate observations during the motion.

The question arises if it is still possible to determine the U-functions by measuring T_{**} for many initial velocities. It turns out that there exists an *inversion formula* which determines U. Before we announce the result we introduce some notations. Set

$$w = \frac{v^2}{2} \quad \text{where} \quad v = \dot{x}(0)\,.$$

Next, by succesive experiments the observer placed at $x = 0$ with a clock is able to determine the function

$$w \mapsto T_{**}(w)$$

and in this way also calculate the function defined by:

$$\text{(iv)} \qquad \mathcal{J}(A) = \int_0^A \frac{T_{**}(w) \cdot dw}{\sqrt{A - w}} \quad : A > 0$$

Next, since $U(x)$ is strictly increasing it suffices to determine the inverse function U^{-1} which we denote by $\xi(x)$, So here

(v) $$U(\xi(x)) = x \quad : x > 0$$

With these notations one has

Abel's inversion formula. For every $A > 0$ one has

$$\xi(A) = \frac{\mathcal{J}(A)}{\sqrt{2} \cdot \pi}$$

Proof. During the time interval $0 \leq t \leq T_*$ while the particle moves with positive velocity we have by (ii)

(1) $$\dot{x} = \sqrt{2} \cdot \sqrt{w - U(x)}$$

Now $dx = \dot{x} \cdot dt$ so $dt = \frac{dx}{\dot{x}}$ and an integration gives

(2) $$T_* = \frac{1}{\sqrt{2}} \cdot \int_0^{x(T_*)} \frac{dx}{\sqrt{w - U(x)}}$$

Now $T_* = T_*(w)$ is a function of w and so $x(T_*(w)) = x(w)$. Next, we have the ξ-function and a change of variables in (2) gives

(3) $$T_*(w) = \frac{1}{\sqrt{2}} \cdot \int_0^{\xi(w)} \frac{d\xi}{\sqrt{(w - \xi)}}$$

Next, since $T_{**}(w) = 2T_*(w)$ the definition of $\mathcal{J}(w)$ in (iv) gives:

$$\mathcal{J}(A) = \sqrt{2} \cdot \int_0^A [\int_0^{\xi(w)} \frac{d\xi}{\sqrt{(w - \xi)}}] \cdot \frac{dw}{\sqrt{A - w}}$$

Interchanging the order of integration we get:

(4) $$\sqrt{2} \cdot \int_0^A [\int_\xi^A \frac{dw}{\sqrt{A - w} \cdot \sqrt{w - \xi}}] \cdot d\xi$$

Now we can finish the proof using the following

Sublemma. One has

$$\int_\xi^A \frac{dw}{\sqrt{A - w} \cdot \sqrt{w - \xi}} = \pi \quad : \quad 0 < \xi < A$$

Proof With the substitution $w \to u + \xi$ the integral becomes

(i) $$\int_0^{A - \xi} \frac{du}{\sqrt{A - \xi - u} \cdot \sqrt{u}}$$

With $u = (A - \xi)s$ it follows that (i) becomes

(ii) $$\int_0^1 \frac{ds}{\sqrt{1 - s} \cdot \sqrt{s}}$$

and the reader verifies that the value is π.

Next, the Sublemma and (4) give $\mathcal{J}(A) = \sqrt{2} \cdot \pi \cdot \xi(A)$. Then division with $\sqrt{2}\pi$ gives Abel's inversion formula.

The harmonic oscillator

.

It arises when $U(x) = kx^2$ for some $k > 0$. In this case $\xi(A)$ is a constant times $\sqrt{A}$. Abel's inversion formula shows that this holds if and only if the T_{**}-function is independent of w. The fact that T_{**} is independent of w when $U(x) = k \cdot x^2$ was of course known long before. It was for example wellknown to R. Descartes around 1640 and a few years later Christian Huyghens used the isochronic property to construct reliable clocks which apart from daily life use, gave scientists a new powerful tool. For example, thanks to accurate time measure the Danish mathematician and astronomer Ole Brömer gave a reasonable estimate for the speed of light in 1676. His result was a bit slower than today, but his assertion that light travels with a speed exceeding 240 000 kilometers per second was a veritable achievement at that time. The interested reader may consult text-books in physics which explain how Brömer performed the measure by studying the moon Ion which moves around Jupiter. His method to approximate the speed of light used device which later was used to construct so called winding numbers of plane curves and led to residue calculus of complex analytic functions.

Inversion formulas with fixed intervals. Here follows a more recent example of an inversion forumla in the spirit of Abel. In his work *Abelsche Integralgleichung mit konstanten Integrationsgrenzen* from 1922 Carleman studied inversion formulas for integrals of the form

$$(*) \qquad \int_0^1 \frac{1}{|x-y|^\alpha} \cdot \phi(y)dy = f(x) \quad : \quad 0 < \alpha < 1$$

Here $f(x)$ is supposed to be a known function defined on the unit interval $0 \leq x \leq 1$ and one seeks a formula for the ϕ-function. In contrast to Abel's inversion formula the inverse formula which determines ϕ requires a more involved proof where complex analysis is used. Another inversion formula arises when we regard the equation

$$(**) \qquad \int_0^1 \operatorname{Log}|x-y| \cdot \psi(y)dy = f(x)$$

To see an example from the world of mathematics we recall Carleman's inversion formula for the ψ-function in (**):

Theorem. One has the formula:

$$\psi(x) =$$

$$\frac{1}{\pi^2} \cdot \frac{1}{\sqrt{x(1-x)}} \cdot \int_0^1 \frac{f'(s) \cdot \sqrt{s(1-s)}}{s-x} \cdot ds - \frac{1}{2\pi^2 \cdot \operatorname{Log} 2 \cdot \sqrt{x(1-x)}} \cdot \int_0^1 \frac{f(s)}{\sqrt{s(1-s)}} \cdot ds$$

The proof uses residue calculus and is of course beyond the scope of these lectures.