Doubly periodic functions.

Introduction. The first extensive study of integrals of square roots of polynomials, and more general integrals which express primitive functions of algebraic functions is due to the genius Niels Henrik Abel (1802-1829). His article *xxxx* started the whole theory and is even today an inspiration for contemporary research. In [Ab-leg], published in connection with the opening ceremony for the *Abel Prize*, the reader will find articles which expose Abel's original contributions and work in his spirit by later generations of mathematicians. Abel's methods were adapted and extended by Jacobi and C. Hermite during the period 1830-1860. A new era started after Riemann's creation of a complex manifold attached to an algebraic function. The theory of elliptic functions and abelian integrals was put into a final form by Weierstrass. In 1880 the theory about elliptic functions and various generalisations to classes of hyper-elliptic functions, theta-functions and so on, was regarded as the best developed discipline of mathematics at that time and served both as inspiration and as a tool during the development of analytic number theory and algebraic geometry. Elliptic functions are also used to get analytic solutions in classical mechanics. To get a perspective on the level of the subject around 1890 the interested reader may consult the work [Kov] by S. Kovalevsky. There appears an extensive section about hyper-elliptic integrals which give the equations of motion of her discovered gyroscope which is the sole non-trivial rigid body rotating around a fixed point under gravity whose equations of motion can be solved by quadrature.

During the last century the theory had been extended to several complex variables where one for example studies meromorphic functions in $\mathbf{C}^n$ which are periodic with respect to lattice points which belong to $\mathbf{Z}^{2n}$. This theory is presented in the series of text-books by C.L Siegel in [xx]. Much progress has also been obtaied using methods in algebraic geometry. See for example the text-book [Mum] by x. Mumford. In the case $n = 1$ a detailed study of elliptic integrals and related topics in function theory can be found in the encyclopedia article by Fabre from 1920. Apart from this source the reader may of course consult text-books of a more recent date which expose the classic theory.

Summary of results. In section 1 we introduce the class $M(\mathbf{G})$ which consists of meromorphic functions $g(z)$ for which $g(z) = g(z + m + in)$ hold for every pair of integers (m, n). Here $\mathbf{G}$ refers to the Gaussian integers in the complex plane. More generally one can start with any lattice, i.e. a pair of complex numbers ω_1, ω_2 which are not parallell and study meromorphic functions for which $g(z) = g(z+m\omega_1+n\omega_2)$ hold for all integers. Our restriction to the special Gaussian lattice is not serious since all the results below are proved in exactly the same way for a general lattice. So we prefer to expose the results in this special case which otherwise would lead to more involved notations. The major results about $M(\mathbf{G})$ rely upon the series:

$$(*) \qquad \mathfrak{p}(z) = \sum \frac{}{(z+\omega)^2} \quad : \omega \in \mathbf{G}$$

This is Weierstrass's $\mathfrak{p}$-function which has double poles at every Gaussian integer. In Theorem xxx we prove that every meromorphic function $g(z)$ in $M(\mathbf{G})$ can be written in a unique way as

$$(**) \qquad g(z)=\frac{Q_1(\mathfrak{p}(z)}{Q_2(\mathfrak{p}(z)}+\frac{Q_3(\mathfrak{p}(z)}{Q_4(\mathfrak{p}(z)}\cdot\mathfrak{p}'(z)$$

where $\{Q_\nu(\mathfrak{p})\}$ are polynomials in $\mathfrak{p}$. Another important result is Theorem xx where we establish the *addition formula* for the $\mathfrak{p}$-function. The proofs, apart from clever constructions due to Jacobi and Weierstrass, are straightforward applications of residue calculus and the use of Hadamard products.

Final remark. The construction of double periodic meromorphic functions via the $\mathfrak{p}$-function is *ad hoc* and does not really explain why the special constructions appear. It is only in connection with the niformaisation theorem that the theory becomes comprehensible. However, the equations which determine *automorphic functions* attached to a Fuchsian group can be given in terms of the group itself once a so called fundamental domain is known. This was already pointed out in Poincaré's article [Poinc] which I think is the sole readable introduction to the whole subject, i.e. very few authors - if any - of more recent text-books explain the core of the theory. For the analytic part which uses the geometric description of Poincaré, the general construction of automorphic functions is carried out in the way it was presented in a brief section from Carleman's lecture at the IMU-meeting in Zürich 1932. Let us illuminate this by the case of double periodic functions whch in any case is the main concern in this chapter.

Construction via an integral equation. Consider the class $\mathcal{P}(\omega_1,\omega_2)$ of double periodic meromorphic functions with periods ω_1 and ω_2. To find this class we first consider the family $\mathcal{M}_1$ of meromorphic functions $H(z)$ with the single period ω_1 and with no poles outside the domain

$$z=\omega_1\cdots+\omega_2\cdot t \quad : \quad -\infty<s<\infty \quad 0\leq t<1$$

In addition we require that

$$\lim_r H(z+r\omega_2)=0$$

holds for all $z\in\mathbf{C}$ as r tends to $+\infty$ or $-\infty$. Then every double periodic function is the solution to the integral equation

$$f(z)=J_f(x)+\frac{H(x)+H(x+\omega_1)}{2}+$$

where $J_f(x)$ is given by

$$J_f(x)=\int_0^{\omega_1}\big[\cot\frac{\pi}{\omega_1}\cdot\big(z-x-\omega_2\big)-\cot\frac{\pi}{\omega_1}\cdot\big(z-x+\omega_2\big)\big]\cdot f(z)\cdot dz$$

1. G-periodic functions

A Gaussian integer is a complex number $m+in$ where m and n are integers. Let $\mathbf{G}$ denote this set. It appears as a lattice in $\mathbf{C}$ and is in 1-1 correspondence with $\mathbf{Z}^2$. A meromorphic function $g(z)$ defined in the complex plane is $\mathbf{G}$-periodic if

$$(0.1) \qquad g(z)=g(z+1)=g(z+i) \quad : \ z\in\mathbf{C}$$

Since (0.1) holds for every z it follows that $g(z+w)=g(z)$ when $w\in\mathbf{G}$. The set of meromorphic $\mathbf{G}$-periodic functions is denoted by $M(\mathbf{G})$. It is clear that sums and products of such meromorphic functions again are $\mathbf{G}$-periodic as well as inverses, i.e. if $g\in M(\mathbf{G})$ then the meromorphic function $\frac{1}{g}$ is $\mathbf{G}$-periodic. Hence $M(\mathbf{G}$ is a field.

1.1 Fundamental regions. If z_* is a complex number and g is **G**-periodic the meromorphic function $g(z+z_*)$ is also periodic. Next, w consider the open unit square

$$\square = \{a+ib \quad : 0 < a, b < 1\}$$

If z_* is a complex number we get the translated square $\square(z_*)$ whose points are $z+z_*$ with $z \in \square$. It is clear that the restriction of a double meromorphic function to some $\square(z_*)$ determines g.

1.2 Zeros and poles. Let $g \in M(\mathbf{G})$ and assume it is not reduced to a constant. The zeros and the poles form a discrete subsets of $\mathbf{C}$ and hence we can always choose z_* so that g has neither zeros or poles on the boundary of $\square(z_*)$. By XX the integral

$$(1) \qquad \frac{1}{2\pi i}\int_{\partial\square(z_*)} \frac{g'(z)\cdot dz}{g(z)}$$

is the difference of the number of zeros and the number of poles in $\square(z_*)$. It is trivial that (1) is zero since integration along the two horisontal lines cancel by the periodicity of g and the same holds for the two vertical lines. Hence the number of poles is equal to the number of zeros. Next, let n be this common number and $a_1,\ldots,a_n$, resp. $b_1,\ldots,b_n$ are the zeros, resp. the poles of g in $\square(z_*)$. Then the difference

$$(2) \qquad \sum a_\nu - \sum b_\nu \in \mathbf{G}$$

To prove (2) we may without loss of generality assume that $z_* = 0$ since the difference does not change if g is replaced by a translate. Now (2) is equal to

$$(3) \qquad \frac{1}{2\pi i}\int_{\partial\square(z_*)} \frac{g'(z)\cdot z\cdot dz}{g(z)}$$

Since $g(z) = g(z+1) = g(z+i)$ we see that (3) is equal to

$$(4) \qquad -i\cdot\frac{1}{2\pi i}\cdot\int_0^1 \frac{g'(x)\cdot dx}{g(x)} + \frac{1}{2\pi i}\int_0^1 \frac{g'(iy)\cdot idy}{g(iy)}$$

Notice that i can be divided out in both terms. Since $g(1) = g(0)$ the logarithmic integral

$$\int_0^1 \frac{g'(x)\cdot dx}{g(x)} = 2\pi\cdot m \quad : m \in \mathbf{Z}$$

Similarly, we see that the second term is an integer times the imaginary unit.

1.3 The σ-function There exists the entire function given by a Hadamard product:

$$(1) \qquad \sigma_*(z) = z\cdot\prod_*(1-\frac{z}{\omega})\cdot e^{\frac{z}{\omega}} \quad : \omega \in \mathbf{G}\setminus 0$$

Indeed, this follows from XX which also shows that σ_* is an entire function of exponential type with simple zeros at the Gaussian integers. The translated function $\sigma_*(z+1)$ has the same zeros so the quotient $\sigma_*(z)/\sigma_*(z+1)$ is entire without zeros. By the division theorem for the class $\mathcal{E}$ of entire functions of exponential type in XX we get a pair of constants A, B such that

$$\sigma_*(z+1) = Ae^{Bz}\cdot\sigma_*(z)$$

To get rid of the exponential factor e^{Bz} we consider the entire function

(2) $$\sigma(z) = \sigma_*(z) \cdot e^{\tau \cdot z^2} \quad : \; \tau = \sum_* \frac{1}{8 \cdot \omega^2}$$

A computation which is left to the reader gives:

(3) $$\sigma(z+1) = -\sigma(z) \quad : xxx\sigma(z+i) = -\sigma(z) \quad : xxxx$$

We can use the σ-function to produce period meromorphic functions whose poles and zeros are preassigned. Namely, let $n \geq 2$ and consider two n-tuples $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of complex numbers - where repetitions may occur. Assume that

(4) $$\sum a_\nu = \sum b_\nu$$

Set

(5) $$g_*(z) = \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}$$

The two equalites in (3) and the condition (4) show that $g_*(z) \in M(\mathbf{G})$. *Conversely*, let $g(z)$ be a meromorphic function in $M(\mathbf{G})$. We pick zeros and poles from a square as in 1.2 and by (2) from 1.2 we have the condition (4) above. Now g_* and g have the same poles and zeros so g/g_* is entire and also periodic. Hence it is bounded and therefore reduced to a constant. So we get

(6) $$g(z) = c_0 \cdot g_*(z)$$

for some constant c_0. Hence we have a complete description of **G**-periodic functions. Notice that we always can pick zeros and poles form a representative domain given by the semi-closed square

$$\square_0 = \{(x,y) \colon \; 0 \leq x, y < 1$$

2. The $\mathfrak{p}$-function.

Consider the series

(1) $$\mathfrak{p} = \sum \frac{1}{(z+\omega)^2} \quad : \; \omega \in \mathbf{G}$$

By the general result in xx the series is convergent and gives a meromorphic function which obviously is **G**-periodic. We see that $\mathfrak{p}$ has a double pole at every every Gaussian integer. Notice also that $\mathfrak{p}$ is an even function, i.e.

$$\mathfrak{p}(z) = \mathfrak{p}(-z)$$

2.1 The complex derivative $\mathfrak{p}'(z)$. It is clear that the derivative is a periodic function with a triple pole at $z = 0$. Thus, at the origin its expansion starts with

$$\mathfrak{p}'(z) = \frac{-2}{z^3} + c_1 z + c_3 z^3 + c_5 z^5 + \ldots$$

Here no constant term appears since $\mathfrak{p}$ is even and therefore has a series expansion at $z = 0$ of the form

$$\mathfrak{p}'(z) = \frac{1}{z^2} + d_0 + d_2 z^2 + \ldots$$

Next, we claim that $\mathfrak{p}'$ has simple zeros at $1/2, i/2, (1+i)/2$. To see this we use that the function is odd which gives

$$\mathfrak{p}'(1/2) = -\mathfrak{p}'(-1/2) = -\mathfrak{p}'(1/2)$$

where the last equality follows since $\mathfrak{p}'(z) = \mathfrak{p}'(z+1)$. Hence we get $\mathfrak{p}'(1/2) = 0$. In the same way one sees that it has a simple zero at $i/2$ and $(1+i)/2$. Since it has a sole triple pole in a fundamental region it follows from .2 that this gives all zeros of $\mathfrak{p}'$ in each fundamental region.

2.2 An algebraic equation. Using the two series expansions from (2.1) one easily verifies:

2.3 Proposition. There exists a complex numbers g_2 such that

$$\mathfrak{p}'^2 - 4 \cdot \mathfrak{p}^3 - g_2 \cdot \mathfrak{p}$$

has no pole at $z = 0$.

It follows that $g = \mathfrak{p}'^2 - 4 \cdot \mathfrak{p}^3 - g_2 \cdot \mathfrak{p}$ is periodic and has no poles. Hence it is an entire function and since it is periodic it is in bounded and therefore reduced to a constant, say g_3. Hence we have proved:

2.4 Theorem. There exist constants g_2, g_3 such that

$$(\mathfrak{p}')^2 = 4 \cdot \mathfrak{p}^3 + g_2 \cdot \mathfrak{p} + g_3$$

Remark. The constants g_2, g_3 can be found by computing terms of the series expansions at $z = 0$ from 2.1. Another procedure is to use that we already know the zeros of $\mathfrak{p}'$. Put

$$e_1 = 1/2 \quad : e_2 = i/2 \quad : e_3 = 1/2 + i/2$$

Then we must have

$$\mathfrak{p}^3(e_\nu) + g_2 \cdot \mathfrak{p}(e_\nu) + g_3 = 0 \quad : \ 1 \leq \nu \leq 3$$

From these equations we can determine the two g-constants by computing $\mathfrak{p}_\nu$ at the three e-points.

2.5. The ϕ-functions We shall construct periodic functions with preassigned zeros and poles using $\mathfrak{p}$ and its derivative. Fix a point $z_* \in \mathbf{C} \setminus \mathbf{G}$. Put

$$\phi(z; z_*) = \frac{1}{2} \cdot \frac{\mathfrak{p}'(z) - \mathfrak{p}'(z_*)}{\mathfrak{p}(z) - \mathfrak{p}(z_*)}$$

It is clear that this is a periodic meromorphic function and (2.1) shows that it has a simple pole at $z = 0$ whose residue is -1. At $z = z_*$ both the numerator and the denominator are zero and one shows easily that no pole appears at this point. But at $z = -z_*$ we get a pole since $\mathfrak{p}$ is even. An easy computation which is left to the reader shows that $\phi(z; z_*)$ has a simple pole at $-z_*$ whose residue is 1. No further poles can occur, hence $\phi(z : z_*)$ is a periodic function with a minimal set of poles, i.e. just two simple poles at $z = 0$ and at $z = -z_*$.

2.6 The addition theorem. The following hold for every pair z, w in $\mathbf{C}$.

$$\mathfrak{p}(z+w)+\mathfrak{p}(z)+\mathfrak{p}(w)=\frac{1}{4}\cdot\frac{(\mathfrak{p}'(z)-\mathfrak{p}'(w))^2}{(\mathfrak{p}(z)-\mathfrak{p}(w))^2}$$

Proof. From (2.5) we have

(i) $$\phi(z;w)=\frac{1}{2}\cdot\frac{\mathfrak{p}'(z)-\mathfrak{p}'(w)}{\mathfrak{p}(z)-\mathfrak{p}(w)}$$

Moreover, $\phi(z;w)$ has simple poles at $z=0$ and $z=-w$, both with residue 1. So ϕ^2 has double poles at these points. Next, the function $\mathfrak{p}(z)+\mathfrak{p}(z+w)$ has also double poles at these two points and the polar expansion starts with $\frac{1}{z^2}$ at the origin and $\frac{1}{(z+w)^2}$. We conclude that the function

(ii) $$g(z)=\phi^2(z;w)-\mathfrak{p}(z)-\mathfrak{p}(z+w)$$

has no poles at all and hence this periodic entire function is a constant C_w which depends on w, i.e. we have

(iii) $$\phi^2(z;w)-\mathfrak{p}(z)-\mathfrak{p}(z+w)=C_w \quad : \ (z,w)\in\mathbf{C}^2$$

Above the role of z and w can be interchanged and we find a constant C_z such that

(iv) $$\phi^2(z;w)-\mathfrak{p}(w)-\mathfrak{p}(z+w)=C_z \quad : \ (z,w)\in\mathbf{C}^2$$

Subtracting the two equations gives:

(v) $$C_z-\mathfrak{p}(z)=C_w-\mathfrak{p}(w) \quad : \ (z,w)\in\mathbf{C}^2$$

Here the left hand side depends on z only and the right hand side on w only. So both sides must be reduced to a constant C^* which is independent of both z and w. In particular $C_z=\mathfrak{p}(z)+C^*$. Inserting this in (iv) gives:

(vi) $$\mathfrak{p}(w)+\mathfrak{p}(z+w)+\mathfrak{p}(z)+C^*=\phi^2(z;w)$$

Here $\phi^2(z;w)$ is the right hand side in (2.6) so there remains only to show that $C^*=0$. To get this we take $z=1/2$ and $w=1/2$. Since $\mathfrak{p}'$ is zero at these points we have $\phi(e_1;e_2)=0$. At the same time xxx shows that

$$\mathfrak{p}(e_1)+(\mathfrak{p}(e_2)+\mathfrak{p}(e_1+e_2)=0$$

Hence $C^*=0$ and the proof of (2.6) is finished.

3. The field $M(\mathbf{G})$

Using the ϕ-functions from 2.5 and the algebraic equation for the derivative in Theorem 2.4 one has:

3.1Theorem. Let $g\in M(\mathbf{G})$. Then there exist two unique rational functions $A(t)$ and $B(t)$ the substitution $t=\mathfrak{p}$ gives

$$g=A(\mathfrak{p})+B(\mathfrak{p})\cdot\mathfrak{p}'$$

The easy proof is left as an exercise.

Remark. Theorem 3.1 means that the field $M(\mathbf{G})$ is generated by $\mathfrak{p}$ and its derivative. At the same time we have Theorem 2.4. So starting from the field

$\mathbf{K} = \mathbf{C}(\mathfrak{p})$ of rational functions in $\mathfrak{p}$ we adjoin $\mathfrak{p}'$ which by Theorem 2.4 is algebraic over $\mathbf{K}$. Moreover, since $\mathfrak{p}'$ satisfies an equation of degree 2, it follows that $M(\mathbf{G})$ is a *quadratic field extension* of $\mathbf{K}$.