Uniform approximation by Blaschke products

Introduction. Denote by $H^\infty(D)$ the space of bounded analytic fiunctions in the open unit disc. Recall from § xx that every such function f has radial limits almost everywhere and hence gives a boundary value function on the unit circle which belongs to $L^\infty(T)$. If the boundary function on T has constant absolute value almost everywhere we say that f is an *inner function*. Denote by $\mathcal{I}(D)$ the set of inner functions. From the material in § xx every Blaschke product is an inner function. It turns out that up to multiplication with complex numbers of absolute value one they form a dense subset. More precisely, , the following result was proved by Frostman in [xxx] :

0.1 Theorem. *Let $w(z) \in \mathcal{I}(D)$. For every $\epsilon > 0$ there exists a real number c and a Blaschke product $B(z)$ such that*

$$\max_{z \in D} |w(z) - e^{ic}B(z)| < \epsilon$$

0.1 The Douglas-Rudin Theorem Let w_1 and w_2 is a pair of inner functions. We regard theie boundsry valued functions and then the quotient $\frac{w_1}{w_2}$ yields an $L^\infty(T)$-functions whose absolute value is one almost everywhere. With this kept in mind the following result is due to Douglas and Rudin in [xx]

0.2.1 Theorem *The convex hull formed by functions $\frac{w_1}{w_2}$ from pairs in $\mathcal{I}(D)$ is dense in thre unit bsll of $L^\infty(T)$.*

We prove this result in §§ xxx. The next result is due to D. Marshall in [Mar] and asserts that the convex hull of Blaschke products is dense in the unit ball of $H^\infty(D)$.

0.3 Theorem. *Let $f(z) \in H^\infty(D)$ with $|f|_D \leq 1$. For each $\epsilon > 0$ there exists a finite family of Blaschke functions $B_1 \ldots, B_N$ and an N-tuple of positive real numbers $a_1, \ldots, a_N$ with $\sum a_\nu = 1$ such that*

$$\big|f(z) - \sum a_\nu \cdot B_\nu(z)\big|_D < \epsilon$$

0.4 Remark. The proof of Marshall's theorem requires secveral steps and is not finished until § xx. Notice that Frostman's theorem reduces Theorem 0.3 to the assertion that the convex hull of inner functions is dense in $H^\infty(T)$. Next, consider some $f \in H^\infty(T)$ with norm $|f|_\infty = 1$. Theorem 0.2.1 gives for each $\epsilon > 0$ an N-tuple of unimodular functions $\{\frac{w_1^\nu}{w_2^\nu}\}$ formed by pairs of inner functions such that the following hold for the maximum norm of the boundary value functions.

$$(*) \qquad \big|f - \sum a_\nu \cdot \frac{w_1^\nu}{w_2^\nu}\big|_\infty < \epsilon \quad : \quad a_1 + \ldots + a_N = 1 \ : \ a_\nu \geq 0$$

However, (*) does not give Theorem 0.3 since the individual quotients $\frac{w_1^\nu}{w_2^\nu}$ in general are not analytic in D. To proceed from (*) Marshall considered the inner function

$$\mathcal{J} = w_2^1 \cdots w_2^N$$

Then we can write

$$(0.4.1) \qquad \sum a_\nu \cdot \frac{w_1^\nu}{w_2^\nu} = \sum a_\nu \cdot \frac{W_\nu}{\mathcal{J}} \quad : W_\nu = w_1^\nu \cdot \prod_{j \neq \nu} w_2^j$$

To profit upon this, Marshall employed a result due to Nevanlinna which implies that (*) gives the existence of another inner function W_* such that the function

$$(0.4.2) \qquad g = \sum a_\nu \cdot \frac{W_\nu}{\mathcal{J}} + \epsilon \cdot \frac{W_*}{\mathcal{J}} \quad \text{belongs to } H^\infty(T)$$

Nevanlinna's result which has indpendent interest is treated in § xx. Grantingt this, it follows from the above that the maximum norm $\big|f - g\big| < 2\epsilon$. To finish the proof of Theorem 0.3 there remains only to approximate the special $H^\infty(T)$-functions g from (0.4.2) by a convex combination of inner functions. This final step in the proof of Theorem 0.3 is given under the heading *Marshall's Lemma* in § xx..

1. Blaschke products and inner functions

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Before we enter the proofs in this section we we recall some facts about inner functions and Blaschke products which asppear in Chapter XX devoted to the Jensen-Nevanlinna class of analytic functions in D. A sequence $\{z_n\}$ of non-complex numbers in D arranged so that $0 < |z_1| \leq |z_2| \leq ...$ satisfies Blaschke's condition if

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$$

Blaschke's theorem asserts that the infinite product

$$B(z) = \prod_{n=1}^{\infty} \frac{z - z_n}{1 - \bar{z}_n z} \cdot \frac{\bar{z}_n}{z_n} \tag{1}$$

converges in D and yields an analytic function such that

$$\lim_{r \to 1} |B(re^{i\theta})| = 1 \quad \text{holds almost everywhere } 0 \leq \theta \leq 2\pi$$

In particular every Blaschke product belongs to $\mathcal{I}(D)$. Next, let μ be a singular non-negative Riesz measure on the unit circle, i.e. μ carries all mass on a null set in the sense of Lebesgue. Then there exists the analytic function in D defined by

$$G_\mu(z) = \exp\big[-\frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{e^{i\theta} - z}{e^{i\theta} + z} \cdot d\mu(\theta)\big] \tag{2}$$

The Factorisation Theorem from XX gives:

1.1. Theorem *Every $w \in \mathcal{I}(D)$ is a unique product*

$$w(z) = e^{ic} \cdot B(z) \cdot G_\mu(z) \quad : \quad c = \text{a real number}$$

where $B(z)$ is the Blascke product formed by the zeros of w in D and μ a singular and non-negative measure.

We will also need Theorem XXX from XXX which characterizes when an inner function is a Blaschke product.

1.2 Theorem. *An inner function $w(z)$ is of the form $e^c \cdot B(z)$ if and only if*

$$\limsup_{r \to 1} \int_0^{2\pi} \log \big|w(re^{i\theta})\big| \cdot d\theta = 0$$

2. Proof of Frostman's theorem

Let $0 < \rho < 1$ and γ is a complex number with $|\gamma| \leq 1$. Then one has the equality

$$\frac{1}{2\pi} \int_0^{2\pi} \log \big| \frac{\gamma - \rho e^{i\theta}}{1 - \rho e^{-i\theta}\gamma} \big| \cdot d\theta = \max(\rho, \log|\gamma|) \tag{*}$$

The verification is left to the reader. Now we consider some $w \in \mathcal{I}(D)$. Apply (*) with $\gamma = w(re^{it})$ for pairs $0 < r < 1$ and $0 \leq t \leq 2\pi$. Integration with respect to t gives the equality:

$$\frac{1}{2\pi} \int_0^{2\pi} \big[\int_0^{2\pi} \log \big| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \big| \cdot d\theta \big] dt = \int_0^{2\pi} \max\big[\rho, \log|w(re^{it})|\big] \cdot dt \tag{*}$$

Since $w \in \mathcal{I}(D)$ we have

$$\lim_{r \to 1} \log|w(re^{it})| = 0 \quad : \quad \text{for almost all } t \tag{i}$$

Keeping $0 < \rho < 1$ fixed the function $t \mapsto \max\left[\rho, \log|w(re^{it}|\right]$ is bounded so by *dominated convergence* under the intergral sign we have:

$$\text{(ii)} \qquad \lim_{r\to 1} \int_0^{2\pi} \max\left[\rho, \log|w(re^{it}|)\right] \cdot dt = 0$$

Replace this limit by the double integral which comes from the equality (*) and apply *Fubini's theorem* to interchange the order of integration. Hence (ii) gives:

$$\text{(iii)} \qquad \lim_{r\to 1} \frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} \left[\int_{t=0}^{t=2\pi} \log\left| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \right| \cdot dt\right] d\theta = 0$$

Now we use that the integrand

$$\text{(iv)} \qquad \log\left| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \right| \leq 0 \quad : \quad 0 \leq \theta, t \leq 2\pi$$

Then (iii) and *Fatou's theorem* gives:

$$\text{(v)} \qquad \limsup_{r\to 1} \int_0^{2\pi} \log\left| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \right| \cdot dt] = 0 \quad : \text{almost everywhere for } \theta$$

At this stage the proof is almost finished. Namely, we notice that the functions

$$F_\theta(z) = \frac{w(z) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(z)}$$

belong to $\mathcal{I}(D)$ for every θ. Moreover, (v) gives a null set $\mathcal{N}$ in T such that

$$\text{(vi)} \qquad \limsup_{r\to 1} \int_0^{2\pi} \log|F_\theta(re^{it})|\, dt = 0 \quad : \quad \theta \in T \setminus \mathcal{N}$$

Finally, for every $\theta \in T \setminus \mathcal{N}$, Theorem 1.2 implies that $F_\theta(z) = e^{ic_\theta} B_\theta(z)$ for some Blaschke product B_θ and a constant c_θ. With such a choice of θ we have

$$\text{(vii)} \qquad |w(z) - e^{ic} \cdot B_\theta(z)| = \left|\frac{\rho e^{i\theta} - \rho e^{-i\theta}(w(z))^2}{1 - \rho e^{-i\theta} w(z)}\right| \leq \frac{2\rho}{1-\rho}$$

Here we can choose any $\rho < 1$. So with $\epsilon > 0$ we choose ρ so small that $\frac{2\rho}{1-\rho} < \epsilon$ and Frostman's Theorem follows.

3. A theorem by Nevanlinna

Theorem 3.1 below was proved by R. Nevanlinna in his article [Nev:xx.] from 1919. The Banach space $L^\infty(T)$ which contains the closed subspace $H_0^\infty(T)$ of boundary values from bounded analytic functions $h(z)$ in D which vanish at $z = 0$. Recall from XX that a bounded Lebesgue measurable function f on T belongs to $H_0^\infty(T)$ if and only if

$$\text{(0.1)} \qquad \int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) d\theta = 0 \quad : \quad n = 0, 1, 2, \ldots$$

Since $H_0^\infty(T)$ is a closed subset of $L^\infty(T)$ there exists the Banach space

$$\text{(0.2)} \qquad \mathcal{B} = \frac{L^\infty(T)}{H_0^\infty(T)}$$

For each $F \in L^\infty(T)$ we set:

$$\text{(0.3)} \qquad \mathfrak{nev}(F) = \min ||F - h||_\infty \quad : \; h \in H_0^\infty(D)$$

and refer to $\mathfrak{nev}(F)$ as the Nevanlinna norm of F. Since $\mathfrak{nev}(F)$ is the norm in the quotient space $\mathcal{B}$ one has

$$(*) \qquad \mathfrak{nev}(F) \leq |F|_\infty$$

When strict inequality holds in (*) one has the following remarkable conclusion:

3.1 Theorem *Let $F \in L^\infty(T)$ be such that $\mathfrak{nev}(F) < |F|_\infty$. Then there exists $h^* \in H^\infty(T)$ such that*

$$|F(e^{i\theta}) - h^*(e^{i\theta})| = |F|_\infty \quad : \quad \text{holds almost everywhere on } T$$

Proof. By a change of scale, i.e. replacing F by F times the inverse of $|F|_\infty$ we may assume that $|F|_\infty = 1$. Set

(i) $$\mathcal{H}_F = \{h \in H^\infty(T) \quad : \quad |F - h| \leq 1\}$$

The triangle inequality gives $|h|_\infty \leq 2$ for every $h \in \mathcal{H}_F$. So we have a uniform bound and hence $\mathcal{H}_F$ is a normal family of analytic functions in the unit disc D. By the general result due to Montel there exists some $h^* \in \mathcal{H}_F$ such that

(ii) $$h^*(0) = \max_{h \in \mathcal{H}_F} |h(0|$$

Notice that $h^*(0) > 0$. In fact, since $\mathfrak{nev}(F) < 1$ is assumed there exists to begin with some $g \in H_0^\infty(T)$ with $|F - g|_\infty \leq 1 - \delta$. for some $\delta > 0$. Hence $\mathcal{H}_F$ contains the functions $g(z) + \delta$ which is $\neq 0$ at the origin. We shall use this to show that (*) holds in Theorem 3.1 when we take h^* as above. A first step to get this is the following:

Sublemma. With h^* chosen so that (ii) holds it follows that

$$|F - h^* - \phi|_\infty \geq 1 \quad : \quad \phi \in H_0^\infty(T)$$

Proof. We argue by a contradiction. Suppose there exists $\phi \in H_0^\infty(T)$ and some $\delta > 0$ such that

$$|F - h^* - \phi|_\infty = 1 - \delta$$

If $a > 0$ it follows that

$$|F - (1+a)h^* - \phi| \leq 1 - \delta + a|h^*|_\infty \leq 1 - \delta + 2a$$

Hence $h_1 = (1 + \delta/2)h^* + \phi$ belongs to $\mathcal{H}_F$. Here $h_1(0) > h^*(0)$ which contradicts the maximality of $h^*(0)$ and the Sublemma follows.

Proof continued. Put $G = F - h^*$. The Sublemma means that the norm of the G-image in the Banach space $\mathcal{B}$ is at least 1. At the same time the $L^\infty(T)$-norm of G is one since $h^* \in \mathcal{H}_F$. It follows that the $\mathcal{B}$-norm of G is 1. Now we use the duality between $H_0^\infty(T)$ and $H^1(T)$ from § XX. This gives a sequence $\{\phi_n\}$ in $H^1(T)$ whose L^1-norms are equal to one such that:

(1) $$\lim_{n \to \infty} \int_0^{2\pi} G \cdot \phi_n \cdot d\theta = 1$$

Put $c_n = \phi_n(0)$. Since $h^* \in H^\infty(T)$ we notice that

(2) $$\frac{1}{2\pi} \cdot \int_0^{2\pi} h^* \cdot \phi_n \cdot d\theta = h^*(0) \cdot c_n$$

Sublemma 2. There exists some positive constant a such that

$$|c_n| \geq a \quad : \quad n = 1, 2, \ldots$$

Proof. Assume the contrary. If $c_n \to 0$ then (1) and (2) give

(3) $$\lim_{n \to \infty} \frac{1}{2\pi} \cdot \int_0^{2\pi} F \cdot \phi_n \cdot d\theta = 1$$

But this is impossible since $\mathfrak{nev}(F) < 1$. Indeed, this gives the existence of some $\psi \in H_0^\infty(T)$ with $[F - \psi|_\infty = 1 - \delta$ for some $\delta > 0$. Here

$$\int_0^{2\pi} \psi \cdot \phi_n \cdot d\theta = 0$$

hold for every n and then (3) cannot hold since

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} |F - \psi| \cdot \phi_n \cdot d\theta \leq |F - \psi|_\infty \cdot |\phi_n|_1 = 1 - \delta$$

Final part of the proof. Sublemma 2 gives some $a > 0$ such that

$$(4) \qquad |\phi_n(0)| \geq a \quad : \quad n = 1, 2, \ldots$$

At the same time (1) above holds and we also know that $|G|_\infty = 1$. Using this we will show that

$$(5) \qquad |G(e^{i\theta})| = 1 \quad : \text{holds almost everywhere}$$

To prove (5) we use Jensen's inequality from § XX which by (4) gives:

$$(6) \qquad 2\pi \cdot \text{Log}\,|a| \leq \int_0^{2\pi} \text{Log}\,\big|\phi_n(e^{i\theta})\big| \cdot d\theta \quad : \ n = 1, 2, \ldots$$

Suppose now that (5) does not hold. This gives the existence of some $\rho < 1$ and a set E of positive Lesbesgue measure such that the maximum norm $|G|_E \leq \rho$. Since $|G|_\infty = 1$ it follows that

$$(7) \qquad \int_0^{2\pi} G \cdot \phi_n \cdot d\theta \leq \rho \cdot \int_E |\phi_n| \cdot d\theta + \int_{T \setminus E} |\phi_n| \cdot d\theta$$

By (1) the left hand side tends to one and since the L^1-norms of ϕ_n are all equal to one, we conclude that

$$(8) \qquad \lim_{n \to \infty} \int_E |\phi_n| \cdot d\theta = 0$$

Since E has positive Lebegue measure it follows from (8) that

$$(9) \qquad \lim_{n \to \infty} \int_E \text{Log}\,|\phi_n| \cdot d\theta = -\infty$$

But this cannot hold by the Nevanlinna-Jensen theory in XX. Namely, the ϕ-functions have L^1-norms equal to one and at the same time (4) holds which by the material from § XXX gives a constant C which is independent of n so that

$$(10) \qquad \int_0^{2\pi} \text{Log}^+\,|\phi_n| \cdot d\theta \leq C$$

and hence (9) cannot hold. This contradiction finishes the proof of Nevanlinna's theorem.

3.2 A consequence of Nevanlinna's Theorem. Let $g \in H^\infty(T)$ and w is an inner function which gives $\frac{g}{w}$ in $L^\infty(T)$ But in general this quotient does not belong to $H^\infty(T)$. To compensate for this we can study its deviation from $H^\infty(T)$ and here the following hold:

3.3 Theorem. *Let $0 < \epsilon < |g|_\infty$ be such that there exists $f \in H^\infty(T)$ of norm one for which*

$$(*) \qquad \big|f - \frac{g}{\mathcal{J}}\big|_\infty = \epsilon$$

Then there exists an inner function w^* such that

$$(**) \qquad \frac{\epsilon \cdot w^* + g}{w} \in H^\infty(T)$$

Proof. Nevanlinna's Theorem gives some $h^* \in H^\infty(T)$ such that

$$|\frac{g}{w} - h^*| = \epsilon \quad : \ \text{almost everywhere}$$

Since the inner function w has absolute value one almost everywhere, it follows that the analytic function $g - h^* w$ has absolute value ϵ almost everywhere on T. This this gives an inner function w^* such that

$$\text{(iii)} \qquad w \cdot h^* - g = \epsilon \cdot w^*$$

After division with w we get (**) in Theorem 3.3

4. Proof of Theorem 0.2.1

Let $\mathcal{U}(T)$ be the unimodular functions on T, i.e. L^∞-functions with absolute value one almost everywhere. Elementary geometry shows that the convex hull of $\mathcal{U}(T)$ is dense in the unit ball of $L^\infty(T)$, i.e. to every G with $|G|_\infty \leq 1$ there exist $g_1, \ldots, g_M$ in $\mathcal{U}(T)$ such that

$$(*) \qquad |G - \sum a_\nu \cdot g_\nu|_\infty < \epsilon \quad : \; a_1 + \ldots + a_M = 1$$

Since products of inner functions are inner, Theorem 0.3 follows if we show that every $g \in \mathcal{U}(T)$ can be uniformly approximated by convex combinations of quotients of inner functions. Again, since products of inner functions again are inner, it suffices to achieve this approximation for a generating set in the multiplicative group of unimodular functions on T. Hence it suffices to consider g-functions which only take two values. After a rotation we may assume that they are +1 and -1. So now we have a measurable set E where $g = 1$ on E and $g = -1$ on $T \setminus E$. Let K be some positive number. Solving the Dirichlet problem with a real valued function u which is K on E and and 0 on $T \setminus E$ we take the bounded analytic function

$$f(z) = e^{-(u+iv)}$$

where v is the harmonic conjugate of u. Here the absolute value $|f| = e^{-K}$ on E and $|f| = 1$ on $T \setminus E$. To be precise, equality holds almost everywhere in the sense of Lebesgue. Next, let $\epsilon > 0$. Let $\mathbf{C}^e$ denote the extended complex plane. By the general result from XX there exists for a suitable K a conformal map Φ from the annulus $1 < |z| < e$ to the doubly connected domain

$$\Omega = \mathbf{C}^e \setminus [-\epsilon, 0] \cup [\ell, \ell*]$$

where ℓ can be made large and $\ell^* \leq \ell + \epsilon$. Here $\Phi(z_0) = \infty$ for some $e^{-K} < |z_0| < 1$. Now we put

$$\Psi = \frac{i + \Phi \circ f}{i - \Phi \circ f}$$

This yields a meromorphic in D with poles at the points $a \in D$ for which $f(a) = z_0$. On T the composed function $\Phi \circ f$ takes values in the union of the real intervals $[-\epsilon, 0]$ and $[\ell, \ell^*]$ which implies that $|\Psi| = 1$ holds almost everywhere on T. With ϵ small we see that $\Psi \simeq 1$ holds on E and when ℓ is large we have $\Psi \simeq -1$ on $T \setminus E$. So the Ψ-function approximates the given unimodular function uniformly up to a small number of order ϵ. Now Ψ may have some poles and w take the Blaschke product B for this so that $B \cdot \Psi$ is analytic in D and since $|B| = 1$ holds almost everywhere this yields an inner function denoted by ψ. Now

$$\Psi = \frac{\psi}{B}$$

is a quotient of inner functions which gives the requested approximation.

5. Marshall's Approximation Lemma

Consider a function $g \in H^\infty(T)$ expressed as

$$(*) \qquad g = \sum_{k=1}^{k=N} a_k \cdot \frac{w_k}{J}$$

where $w_1, \ldots, w_N$ and J are inner functions while $a_1, \ldots, a_N$ are some real numbers. Here we do not assume that they are ≥ 0 or that the sum is one. But we assume that the maximum norm $|g|_\infty < 1$.

5.1 Proposition. *For every $\epsilon > 0$ there exists a convex sum of inner functions V such that*

$$\big|g - V\big|_\infty < \epsilon$$

Proof. For the inner functions $\{w_k\}$ and J we have the equalities below on T:

(i)
$$\frac{1}{w_k} = \bar{w}_k \quad : \frac{1}{J} = \bar{J}$$

It follows that almost everywhere on T:

(ii)
$$\bar{g}(e^{i\theta}) = \sum_{k=1}^{k=N} a_k \cdot \frac{J(e^{i\theta})}{w_k(e^{i\theta})}$$

Next, $W = w_1 \cdots w_N$ is an inner function and (ii) implies that the product

(iii)
$$W \cdot \bar{g} \in H^\infty(T)$$

Now we shall use (iii) to get another expression for g. By assumption $|g|_\infty \leq 1 - \delta$ for some $\delta > 0$. Since $\int_0^{2\pi} e^{i\nu t} dt = 0$ for every $\nu \geq 1$ we have the equality below for every complex number λ of absolute value < 1:

(iv)
$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\lambda e^{it} + g(z)}{1 + \lambda e^{it} \bar{g}(z)} \cdot dt$$

Next, for each $z \in D$ we can take $\lambda = W(z)$ and obtain

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{W(z)e^{it} + g(z)}{1 + e^{it} \cdot W(z) \cdot \bar{g}(z)} \cdot dt \quad : \quad z \in D$$

The family of functions

$$t \mapsto \Phi_t(z) = \frac{W(z)e^{it} + g(z)}{1 + e^{it} \cdot W(z) \cdot \bar{g}(z)}$$

is equi-continuous with respect to t since $|g|_\infty < 1 - \epsilon$ is assumed. So we can evaluate the integral which defines $g(z)$ by Riemann sums in a uniform manner, i.e. to every $\epsilon > 0$ there exists a finite set $0 \leq t_1 < \ldots t_M < 2\pi$ and positive numbers $\{b_\nu\}$ with $\sum b_\nu = 1$ such that

(v)
$$\big|g(z) - \sum b_\nu \cdot \frac{W(z)e^{it_\nu} + g(z)}{1 + e^{it_\nu} W(z) \cdot \bar{g}(z)}\,\big| < \epsilon$$

Next, since $W(z)$ is inner we notice that the functions

$$\phi_\nu(z) = \frac{W(z)e^{it_\nu} + g(z)}{1 + e^{it_\nu} W(z) \cdot \bar{g}(z)} \quad : 1 \leq \nu \leq M$$

are all inner. Since $\epsilon > 0$ can be made arbitrary small it follows that $g(z)$ can be uniformly approximated by a convex combination of inner functions.

6. Proof of Theorem 0.3

Consider some $f \in H^\infty(T)$ of norm 1. Theorem 0.2.1 gives a convex combination of quotients of inner functions such that:

$$\big|f - \sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{w_1^\nu}{w_2^\nu}\big|_\infty < \epsilon$$

Set

$$\mathcal{J} = \prod w_\nu^2$$

This is an inner function and we also get the inner functions

(1)
$$W_\nu = \frac{w_1^\nu}{w_2^\nu} \cdot \mathcal{J} \implies \big|f - \sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{W_\nu}{\mathcal{J}}\big|_\infty < \epsilon$$

Theorem 3.3 applies to the pair f and $g = \sum a_\nu W_\nu$ and gives an inner function W_* such that

$$\sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{W_\nu}{\mathcal{J}} + \epsilon \cdot \frac{W_*}{\mathcal{J}} \in H^\infty(T) \tag{2}$$

Let g be this analytic function. From (1) and the triangle inequality we have

$$|f - g|_\infty < 2 \cdot \epsilon \tag{3}$$

Finally, with $N = k + 1$ we can apply Marshall's lemma to the function $\frac{1}{1+\epsilon} \cdot g$ which is expressed by a convex combination (*) from (4). This gives a convex sum V formed by inner functions such that

$$|V - g|_\infty < 2 \cdot \epsilon \tag{4}$$

Since $\epsilon > 0$ is arbitrary we get Theorem 0.3 via (3) and (4) above.