Lech's theorem on zeros of a polynomial ideal.

Let $n \geq 3$ and $\mathfrak{p}$ is a prime ideal in $\mathbf{C}[z_1, \ldots, z_n]$ whose locus $\mathfrak{p}^{-1}(0) = V$ is an irreducible algebraic set. We assume that $\dim(V) = d$ where $1 \leq d \leq n-2$. Set $p = n - d$ and Let $g_1, \ldots, g_k$ be a finite family in $\mathfrak{p}$ whose set of common zeros is V. To each polynomial $p(z) = \sum c_\alpha z^\alpha$ we can take complex conjugates of the coefficients and get the polynomial $\widehat{p}(z) = \sum \bar{c}_\alpha \cdot z^\alpha$. In particular we get

$$g_*(z) = \sum \widehat{g_j} \cdot g_j \tag{0.1}$$

If $x = (x_1, \ldots, x_n)$ is a real point in $\mathbf{C}^n$ we have $f_*(x) = \sum |g_j(x)|^2$ and hence

$$\mathbf{R}^n \cap V = \mathbf{R}^n \cap g_*^{-1}(0)$$

Less obvious is the following result due to Christer Lech which gave an affirmative answer to a question raised by by Lars Hörmander in 1956. We remark that Lech's theorem below has applications to over-determined systems of hyperbolic PDE-equations.

Main Theorem *There exists a finite set $f_1, \ldots, f_m$ in $\mathfrak{p}$ and a constant C such that the following inequality hold for each $x \in \mathbf{R}^n$:*

$$\operatorname{dist}(x, V) \leq C \cdot \operatorname{dist}(x, f_*^{-1}(0))$$

The intuitive proof. Let $\mathfrak{p}^*$ be the ideal in $\mathbf{C}[z_1, \ldots, z_n]$ generated by the leading forms of polynomials in $\mathfrak{p}$. The Zariski cone V^* is the set of common zeros of these homogeneous polynomials. It is wellknown that V^* is an algebraic set whose dimension is equal to that of V. Let $p = n - d$ be the codimension and consider the Grassmanian $\mathcal{G}$ of $(p-1)$-dimensional subspaces of $\mathbf{C}^n$. A subspace $\Pi \in \mathcal{G}$ is transversal to V if the intersection $V^* \cap \Pi$ is reduced to the origin in $\mathbf{C}^n$. A classical result due to Max Noether - father of the eminent mathematician Emmy Noether - asserts that for every transversal Π, the set-theoretic sum

$$V + \Pi = \{z + w : \ z \in V \quad \& \quad w \in \Pi$$

is an algebraic hypersurface $P_\Pi^{-1}(0)$ where the degree of P_Π is majorised by the multiplicity $e(V)$ of the given algebraic set. It follows that the family $\{P_\Pi\}$ generates a finite dimensional complex vector space in $\mathbf{C}[z]$ denoted by $\mathcal{N}(\mathfrak{p})$. Let $f_1, \ldots, f_m$ be a basis of $\mathcal{N}(\mathfrak{p})$ and define

$$f_*(z) = \sum \widehat{f_\nu}(z) \cdot f_\nu(z)$$

Lech proved the inequality (*) in the Main Theorem for the pair (V, f_*). The proof requires several steps. A crucial ingredient is Lech's inequality in § 0 expressed by Theorem 0.1, together with Lech's Lemma in 0.2. In addition tothis we shall use some facts due to Zariski and Weil which exhibits a certain polynomial $F(z, t)$ in § 1. It is used to get a parametrisation of Max Noether's $(p-1)$-dimensional subspaces Π above and via certain specialisations a continuity principle which is appears during the proof of the Main Theorem in § 3.

Example. The first non-trivial case arises when $n = 3$ and $\mathfrak{p}^{-1}(0) = S$ is an algebraic curve. An example is the non-singular curve S parametrised by a complex variable t so that

$$S = \{(t, t^2, i \cdot t^3\}$$

It is defined by $z_2 = z_1^2$ and $z_3 = iz_1^3$. So the prime ideal $\mathfrak{p}$ is easily described. But distances between real points and S behave in a rather irregular fashion. For example, let R be a large positive real number and take the point $a = (0, 0, R)$. Now we seek a complex number t so that

$$\mathrm{dist}(a : S)^2 = \min_t |t|^2 + |t|^4 + |R + it^3|^2$$

where the minimum is taken over complex t. It is clear that one should choose t so that it^3 is real and negative which does not affect the first two terms, i.e. (1) becomes

$$\min_x x^2 + x^4 + (R - x^3)^2$$

where x are real and positive. The reader may check that the minimum behaves like $R^{\frac{4}{3}}$. One can continue to analyze different directions as a are real and get large euclidian distances from the origin in $\mathbf{R}^3$. With this kept in mind it is not obvious how to prove that there exists a polynomial f_* in the ideal generated by $z_1 - z_2^2$ and $z_3 - iz_1^3$. The reader is invited to carry out further computations and eventually find f_* which works in the main theorem.

The local analytic case. It is an open question if Lech's result is valid in the local analytic case. Here one starts with a prime ideal $\mathfrak{p}$ in the local ring $\mathcal{O}_n$ of germs of holomorphic functions at the origin in $\mathbf{C}^n$. In a small polydisc D^n we get the analytic set $V = \mathfrak{p}^{-1}(0)$ and one asks if there exists a pair $f \in \mathcal{O}(D^n)$ which vanishes on V and a constant C as in Theorem 1. The difficulty in the local analytic case is to find a sufficently generic family in the given prime ideal and yet consolidate that $\mathcal{M}$ is a finite dimensional complex vector space.

§ 0. Lech's inequality.

Let $\phi(z_1,\ldots,z_n)$ be a polynomial of n variables which is not reduced to a constant, i.e. its degree is >0. If $\alpha \in \mathbf{C}^n \setminus \phi^{-1}(0)$ we put

$$\operatorname{dist}(\alpha,\phi^{-1}(0)) = \min_{z\in\phi^{-1}(0)} |z-\alpha|$$

where $|z-\alpha|$ is the euclidian distance. Next, let $\Theta = (\theta_1,\ldots,\theta_n)$ be a complex vector of unit length, i.e. $|\theta_1|^2 + \ldots + \theta_n|^2 = 1$. This gives a polynomial

$$t \mapsto \phi(\alpha + t\cdot\Theta)$$

and a Taylor expansion yields

$$\phi(\alpha + t\cdot\Theta) = \phi(\alpha) + \sum_{k=1}^{k=m} D_k(\alpha;\Theta)\cdot t^k$$

where m is the degree of ϕ and $\{D_k(\alpha;\Theta)\}$ is an m-tuple of complex numbers. For each $1\leq k\leq m$ we set

$$\mathcal{D}_k^*(\phi;\alpha) = \max_{\Theta} |D_k(\alpha;\Theta)|$$

Definition. *The Lech number of ϕ at the point α is defined by*

$$\mathcal{L}(\phi;\alpha) = \max_{1\leq k\leq m} \Big[\frac{|\phi(\alpha)|}{\mathcal{D}_k^*(\phi;\alpha)}\Big]^{\frac{1}{k}}$$

0.1 Theorem. *For every positive integer m and each polynomial ϕ of degree m the following inequality holds when α is outside the zero-set of ϕ:*

$$(*) \qquad \frac{1}{2} \leq \frac{\operatorname{dist}(\alpha,\phi^{-1}(0))}{\mathcal{L}(\phi;\alpha)} \leq m$$

Proof Replacing ϕ by $c\cdot\phi$ for some constant we may assume that $\mathcal{L}(\phi;\alpha) = 1$ which means that

$$(1) \qquad |\phi(\alpha)| = \max_{k,\Theta}\; D_k(\alpha;\Theta)$$

with the maximum taken over pairs (k,Θ) where $1\leq k\leq m$ and Θ are complex n-vectors of unit length. Every unit vector Θ gives the ζ-polynomial

$$(2) \qquad \phi(\alpha + \Theta\cdot\zeta) = \phi(\alpha) + \sum D_k(\alpha;\Theta)\cdot\zeta^k$$

If the absolute calue $|\zeta| < 1/2$ the triangle inequality gives:

$$|\phi(\alpha+\Theta\cdot\zeta)| \geq |\phi(\alpha)| - \sum_{k=1}^{k=m} |\zeta|^k\cdot|D_k(\alpha,\Theta)|$$

Now (1) entails that the right hand side above majorises

$$|\phi(\alpha)|\cdot(1 - \sum_{k=1}^{k=m} |\zeta|^k)$$

Since $|\zeta| < 1/2$ the last factor above is >0 and hence $\zeta \mapsto \phi(\alpha+\Theta\cdot\eta)$ is zero-free in the disc $\{|\zeta| < 1/2$ for every unit vector Θ. Hence $\operatorname{dist}(\alpha,\phi^{-1}(0)) \geq 1/2$ which proves the lower bound in (*) from Theorem 0.1. To get the upper bound we choose a pair k_*, Θ_* such that

$$|\phi(\alpha)| = D_{k_*}(\alpha; \Theta_*)$$

and consider the polynomial

$$g(\zeta) = \zeta^m \cdot \phi(\alpha + \Theta_* \cdot \zeta^{-1}) \tag{3}$$

Write

$$g(\zeta) = c_m \zeta^m + \ldots + c_0$$

Now (1) implies that the absolute values of c_m and c_{m-k} are equal. Let $\beta_1, \ldots, \beta_m$ be the zeros of g where eventual multiple zeros are repeated. The symmetric polynomial of order $m - k$ of this m-tuple is a sum of monomials in the roots of degree k and equal to

$$(-1)^{m-k} \cdot \frac{c_{m-k}}{c_m}$$

whose absolute value is 1. If all the zeros have absolute value $\leq 1/m$ the absolute value of the symmetric sum above is majorised by

$$m^{-k} \binom{m}{k}$$

Since this term is < 1 we conclude that g must have a zero of absolute value $> 1/m$ which by (3) implies that the right hand side in (*) is $\leq m$.

An application. For each finite family of polynomials $\{\phi_1, \ldots, \phi_k\}$ and every real point a we set

$$\mathrm{dist}(a, \phi_\bullet^{-1}(0)) = \min_\nu \mathrm{dist}(a, \phi_\nu^{-1}(0))$$

0.2 Lech's Lemma *Let $\mathcal{M}$ be a finite dimensional subspace of $\mathbf{C}[z_1, \ldots, z_n]$ and $f_1, \ldots, f_m$ some basis of $\mathcal{M}$. Then, if $\{g_1, \ldots, g_m\}$ is another basis in $\mathcal{M}$ there exists a constant $c > 0$ such that the inequality below holds for every real point a:*

$$\mathrm{dist}(a, f_*^{-1}(0)) \geq c \cdot \mathrm{dist}(a, g_\bullet^{-1}(0))$$

where $f_* = \sum \bar{f}_\nu \cdot f_\nu$.

Proof. Applying Lech's inequality to the the g-functions and f_* we can reformulate Lech's Lemma as follows: There exists a constant C which is independent of the real point a such that the following hold: If

$$|g_\nu(\alpha) \geq A^k \cdot D_k(g_\nu)(\alpha) \tag{1}$$

hold for all $k \geq 1$ and all ν and some constant A, then

$$|f_*(\alpha) \geq (CA)^k \cdot D_k(f_*)(\alpha) \tag{2}$$

To get (2) we proceed as follows. First, since $f_1, \ldots, f_m$ is a k-basis in $\mathcal{M}$ there exists a constant C_0 which is independent of α and

$$|g_\nu(\alpha) \leq C_0 \cdot \sum_{k=1}^{k=m} |f_k(\alpha)| \quad : \ 1 \leq \nu \leq m$$

Conversly, since the g-polynomials also is a basis of $\mathcal{M}$ it is clear that (1) gives a constant $C_1 > 0$ which again is independent of α such that

$$C_1 \cdot \max_\nu |f_\nu(\alpha)| \geq A^k \cdot \sum_{\nu=1}^{\nu=m} D_k(f_\nu; \alpha) \tag{3}$$

Next, since α is real we have

$$f_*(\alpha) = \sum |f_\nu(\alpha)|^2 \implies C_1 \cdot \sqrt{f_*(\alpha)} \geq A^k \cdot \sum_{\nu=1}^{\nu=m} D_k(f_\nu;\alpha) \tag{4}$$

Put

$$D_k^*(\alpha) = \sum_{\nu=1}^{\nu=N} D_k(f_\nu;\alpha) \tag{5}$$

Notice that when f_ν is replaced by $\bar{f}_\nu$ one has

$$D_k(f_\nu;\alpha) = D_k(\bar{f}_\nu;\alpha) \quad : \ k = 1, 2, \ldots$$

These Taylor expansions give the inequality

$$D_k(f_*;\alpha) \leq N(k+1) \max_{i+j=k} D_i^*(\alpha) \cdot D_j^*(\alpha) \tag{6}$$

Finally, with $k = i + j$ it is clear that (4) gives

$$A^k \cdot D_i^*(\alpha) \cdot D_j^*(\alpha) \leq C_1^2 |f_*(\alpha)| \tag{7}$$

Then (6-7) give

$$A^k \cdot D_k(f_*;\alpha) \leq N(k+1) \cdot C_1^2 \cdot f_*(\alpha)| \tag{8}$$

Since this hold for each k we get the requested positive constant C in (2) above.

§ 1. Specialisations and linear systems.

We shall use results from the Zasriski-Weil theory. The approach by these two masters in algebraic geometry has the merit that various geometric are carried over to algebraic calculations which are not easily found in an "intuitive fashion". We shall only employ the Zariski-Weil theory in characteristic zero and apply results from Weil's book *Foundations of algebraic geometry* published in 1943. Personally I think the material in this outstanding text should be introduced at an early stage to students interested in systems of algebraic equations, and I have never understood the point in putting so much emphasis upon "soft sheaf theory" and trivial yoga about schemes which tend to hide relevant calculations. See Weil's critical comments about these matters in the reprinted version of his text-book from 1962.

Material from the first two chapters in Weil's text-book is taken for granted during the subsequent proof of Lech's theorem, But we will state results which are needed below. Let k be a subfield of $\mathbf{C}$ which is finitely generated over Q and $\mathfrak{p}$ a prime ideal in the polynomial ring $k[t_1,\ldots,t_n]$ where $n \geq 2$. Generic specialisations of $\mathfrak{p}$ consist of points

$$\xi = (\xi_1,\ldots,\xi_n) \in \mathbf{C}^n$$

with the property that a polynomial $f(t)$ in $k[t]$ belongs to $\mathfrak{p}$ if and only if $f(\xi) = 0$. This means that one simply evaluates f at the point $\xi \in \mathbf{C}^n$. The existence of generic specialisations is not difficult to prove and is explained in Chapter 1 in [ibid]. An invariant of $\mathfrak{p}$ is the degree of trancendency of ξ over k, i.e. this degree does not depend upon the chosen generic specialisation.

The Zariski-Weil linear system.

Let $\mathfrak{p}$ be a prime ideal in $k[t]$ and choose a specialisation ξ. Suppose that the degree of trancendency of ξ over k is some integer $p \geq 2$. Let

$$\bar{\tau}_k = (\tau_k^1 \ \ldots, \tau_k^n) \quad : 1 \leq k \leq p-1$$

be $(p-1)$ many n-vectors where the double indexed family $\{\tau_\nu^i\}$ gives $n(p-1)$ algebraically independent elements over k. Consider also a $(p-1)$-tuple $\lambda_1,\ldots,\lambda_{p-1}$ where $\{\tau_\nu^i\}$ and $\{\lambda_\nu\}$ together give $(n+1)(p-1)$ algebraically independent elements over k. Next, we have the n-vector ξ and put

$$(1) \qquad \zeta_i = \xi_i + \sum_{k=1}^{k=p-1} \lambda_k \cdot \tau_k^i \quad : \quad 1 \leq i \leq n$$

Denote by K the field extension of k generated by the n-tuple $\{\xi_\nu\}$, the $(p-1)$-tuple $\{\lambda_j\}$ and the $n(p-1)$-tuple $\{\tau_\nu^i\}$. Its degree of transcendency becomes

$$(\mathrm{i}) \qquad n(p-1) + (p-1) + (n-p) = np - 1$$

Notice that K contains the subfield $k(\zeta,\bar{\tau})$ generated by $\zeta_1,\ldots,\zeta_n$ and $\bar{\tau} = \{\tau_\nu^i\}$. From (i) it follows that the n-tuple ζ and the $n(p-1)$ type $\bar{\tau}$ are not algebraically independent. Less obvious is the following result whose detailed proof appears in [ibid:Chapter 1]:

1.2 Theorem. *The field K is algebraic over the subfield $k(\zeta,\bar{\tau})$.*

From the above it follows that the degree of trancendency of $k(\zeta,\bar{\tau})$. over k is equal to $n(p-1$. With n new variables $\{z_\nu\}$ and arranging the $n(p-1)$ many trancendental τ-variables as $t=(t_1,\ldots,t_{np-n})$, it follows that that there exists an irreducible polynomial $F(z,t)$ in $k[z,t]$ such that

$$F(\zeta,\bar{\tau})=0$$

We can express F in the form:

$$F(z,t)=\sum_{\nu=1}^{\nu=m} f_\nu(z)\cdot\phi_\nu(t) \tag{1.2.1}$$

where m is some positive integer, and $\{f_\nu(z)\}$ are k-linearly independent in $k[z]$ while $\{\phi_\nu(t)\}$ are k-linearly inependent in the polynomial ring of the t-variables. The polar decomposition is not unique. However, the vector space over k generated by the f-polynomials in $k[z]$ does not depend upon the polar representation of F. It is denoted by $\mathcal{W}(\mathfrak{p})$ and called the Weil-space attached to the prime ideal $\mathfrak{p}$.

Next, by Theorem 1.2. each ξ_ν is algebraic over the field $k(\zeta,\tau)$ which implies that

$$\sum f_\nu(\xi)\cdot\phi_\nu(t)=0 \quad : \forall t\in\mathbf{C}^{n(p-1)} \tag{*}$$

Since the polynomials $\{\phi_\nu(t)\}$ are k-linearly independent we conclude that $f_\nu(\xi)=0$ for each ν, i.e. these polynomials belong to $\mathfrak{p}$. Hence $\mathcal{W}(\mathfrak{p})$ appears as a finite dimensional k-subspace of $\mathfrak{p}$. We shall later on use the complex vector space

$$\widehat{\mathcal{W}}(\mathfrak{p})=\mathbf{C}\otimes_k\,\mathcal{W}(\mathfrak{p})$$

1.3 A special family of polynomials. We can fix $(p-1)$ many complex τ-vectors which gives a family $\{\bar{\tau}^j:\ j=1,\ldots,p-1\}$ of vectors in $\mathbf{C}^n$ and the polynomial:

$$g(z)=F(z,\bar{\tau})\in\mathbf{C}[z_1,\ldots,z_n]$$

Another result from the Zariski-Weil theory asserts that for every $(p-1)$-tuple $\bar{\lambda}$ there exists a specialisaton $(\zeta,\tau)\mapsto(\bar{\zeta},\bar{\tau})$ which give the equations

$$\bar{\zeta}_i=\xi_i+\sum_{j=1}^{j=p-1}\bar{\lambda}^j\cdot\bar{\tau}_i^j \quad : 1\leq i\leq n$$

It follows that if $\mathcal{L}$ is the subspace of $\mathbf{C}^n$ generated by the $(p-1)$-many n vectors $\{\bar{\tau}^j=(\bar{\tau}_1^j,\ldots,\bar{\tau}_n^j)\}$ then g-polynomial vanishes at each point in the set

$$\mathfrak{p}^{-1}(0)+\mathcal{L}$$

In other words, g vanishes on the subset of $\mathbf{C}^n$ whose points are of the form $\alpha+\beta$ where $\alpha\in\mathfrak{p}^{-1}(0)$ and $\beta\in\mathcal{L}$.

1.4 A basis for $\widehat{\mathcal{W}}(\mathfrak{p})$. With m from (1,2,1) we consider an m-tuple of specialized τ-vectors, i.e. a family $\{\bar{\tau}^j(k)\colon 1\leq k\leq m\}$. Evaluating the ϕ-polynomials from (1.2.1) give an $m\times m$-matrix with elements $\{\phi_\nu(\bar{\tau}(k)\}$. If the determinant of this matrix is $\neq 0$ the Zariski-Weil theory asserts that the m-tuple

$$g_k(z)=F(z,\bar{\tau}(k)) \quad : 1\leq k\leq m$$

is a basis of the complex vector space $\widehat{\mathcal{W}}(\mathfrak{p})$.

§ 2. Proof of the Main Theorem

From § 1 we have the m-tuple $f_1, \ldots, f_m$ in $k[z]$ and construct the polynomial f_* as in (0.1). Let $\{a_\mu\}$ be a sequence in $\mathbf{R}^n$ which stays outside V. For each μ we set

(i) $$c_\mu = \frac{1}{\sum |f_\nu(\alpha_\mu)|}$$

Take a subsequence $1 \leq \mu_1 < \mu_2 < \ldots$ such that there exist limits

(ii) $$d_\nu = \lim_{j \to \infty} c_{\mu_j} \cdot f_\nu(\alpha_{\mu_j}) \quad : 1 \leq \nu \leq m$$

Notice that (i) entails that $|d_1| + \ldots + |d_m| = 1$. To simplify notations we set

$$\beta_j = \alpha_{\mu_j}$$

From § 1.1 we have the polynomial $F(z, t)$ with its polar decomposition. Put

(iii) $$\phi_*(t) = \sum d_\nu \cdot \phi_\nu(t)$$

Here $\phi_*(t)$ is not identically zero since $\{\phi_\nu(t)\}$ are linearly independent and the d-vector is non-zero. Next, we can choose an m-tuple of specialised τ-vectors $\{\bar{\tau}(k)\}$ such that the determinant of the matrix $\{\ \phi_\nu(\bar{\tau}(k)\}$ from § 1.4 is non-zero and in addition

(iv) $$\phi_*(\bar{\tau}(k)) \neq 0 \quad : 1 \leq k \leq m$$

To the specialised m-tuple of $\bar{\tau}$-vectors we assign polynomials

(v) $$g_k(z) = F(z, \bar{\tau}(k))$$

By the result in § 1.4 the non-vanishing determinant above entails that $\{g_k\}$ is a basis in $\widehat{\mathcal{W}}(\mathfrak{p})$. Notice that

(vi) $$c_{\mu_j} \cdot g_k(\beta_j) = \sum c_{\mu_j} \cdot f_\nu(\beta_j) \cdot \phi_\nu(\bar{\tau}(k))$$

The limit in (ii) entails that (vi) close to

$$\sum d_\nu \cdot \phi_\nu(\bar{\tau}(k)) = \phi_*(\bar{\tau}(k))$$

This approximative equality will be used to prove the following:

3.1 Lemma. *For every k it holds that*

$$\limsup_{j \to \infty} \frac{\operatorname{dist}(\beta_j, g_k^{-1}(0)}{\operatorname{dist}(\beta_j, V)} > 0$$

Proof. Consider some k, say $k = 1$. and denote by $\mathcal{L}$ the linear subspace of $\mathbf{C}^n$ spanned by the specialized vector $\bar{\tau}(1)$. As explained in § 1.4

$$g_1^{-1}(0) = V + \mathcal{L}$$

Next one has

$$\text{(i)} \qquad \mathrm{dist}(\beta_j, g_1^{-1}(0)) = \min_{z,\zeta} ||\beta_j - z - \zeta||$$

where the minimum is taken over pairs $z \in V$ and $\zeta \in \mathcal{L}$. Notice that the quotient in Lemma 3.1 for each j is majorizes

$$\text{(ii)} \qquad \min_{z,\zeta} \frac{||\beta_j - z - \zeta||}{||\beta_j - z||}$$

Since $\mathcal{L}$ is a subspace the minimum is the same as

$$\text{(iii)} \qquad \min_{z,\zeta} ||\frac{\beta_j - z}{||\beta_j - z||} - \zeta||$$

Above we measure distances of unit vectors $\frac{\beta_j - z}{||\beta_j - z||}$ to the subspace $\mathcal{L}$. Lemma 3.1 amounts to prove that this distance is $\geq c_*$ for a positive constant c_* and all sufficently large j. If no such c_* exists we can find specialisations $\bar{\tau}$ where the corresponding vector space $\mathcal{L}(\bar{\tau}))$ is close to $\mathcal{L}$ in the Grassmannian of $(p-1)$-dimensional complex subspaces pf $\mathbf{C}^n$.Moreover, there exist large j and points $z_j \in V$ such that

$$\text{(iv)} \qquad \frac{\beta_j - z_j}{||\beta_j - z_j||} \in \mathcal{L}(\bar{\tau})$$

This entails that

$$\beta_j \in V + \mathcal{L}(\bar{\tau})$$

As explained in § 1.4 it follows that

$$0 = F(\beta_j, \bar{\tau}) = \sum f_\nu(\beta_j) \cdot \phi_\nu(\bar{\tau}) \implies$$

$$\text{(v)} \qquad 0 = \sum c_{\mu_j} \cdot f_\nu(\beta_j) \cdot \phi_\nu(\bar{\tau})$$

Next, recall that

$$\lim_{j \to \infty} c_{\mu_j} \cdot f_\nu(\beta_j) = d_\nu$$

hold for every ν and hence (v) would entail that

$$\phi_*(\bar{\tau}) = \sum d_\nu \cdot \phi_\nu(\bar{\tau}) \simeq 0$$

This gives a contradiction since continuity implies that $\phi_*(\bar{\tau})$ stays away from zero when $\bar{\tau}$ is close to $\bar{\tau}(k)$ and Lemma 3.1 is proved.

3.2 Final part of the proof.

If the Main Theorem fails there exists a sequence of real points $\{a_\mu\}$ for which

$$\lim_\mu \frac{\mathrm{dist}(a_\mu, f_*^{-1}(0))}{\mathrm{dist}(a_\mu, V)} = 0$$

But this is impossible in view of Lech's Lemma in § 0 and Lemma 3.1 which finishes the proof of the Main Theorem.