Derivatives of functions

In the text-book *Théorie de l'intégration* from 1904, Lebesgue proved that a monotone function defined in a real interval has an ordinary derivative outside a null-set. For an arbitrary continuous function a more general result was established by Young and Denjoy which goes as follows: Let $f(x)$ be a real-valued continuous function defined on some open interval (a,b). For each $a<x<b$ we set

$$D^*(x) = \limsup_{h+k\to 0} \frac{f(x+h)-f(x-k)}{h+k}$$

where h and k are positive when we pass to the limes superior. Similarly

$$D_*(x) = \liminf_{h+k\to 0} \frac{f(x+h)-f(x-k)}{h+k}$$

0.1 Theorem. *Outside a (possibly empty) null-set E of (a,b) the following two possibilities occur for each $x \in (a,b)\setminus E$: Either there exists a common finite limit*

(*) $$D^*(x) = D_*(x)$$

Or else one has

(**) $$D^*(x) = +\infty \quad \text{and} \quad D_*(x) = -\infty$$

Remark. Above the pair $h,k)$ tends to zero under the sole condition that $h+k\to 0$. We can take $k=0$ or $h=0$ and consider one-sided limits:

(i) $$D^+(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{and} \quad d^+(x) = \liminf_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(ii) $$D_+(x) = \limsup_{k\to 0} \frac{f(x)-f(x-k)}{k} \quad \text{and} \quad d_+(x) = \liminf_{k\to 0} \frac{f(x)-f(x-k)}{k}$$

With these notations it is clear that:

$$D_*(x) \le d^+(x) \le D^+(x) \le D^*(x)$$

So the equality $D_*(x) = D^*(x)$ entails that f has an ordinary right derivative. Since

$$D_*(x) \le d_+(x) \le D_+(x) \le D^*(x)$$

we conclude that if (*) holds in the theorem, then f has a ordinary derivative at x. If (**) occurs at a point x, then the graph of f close to x is steep but may also change sign in a small interval around such a point. Take for example $x=0$ and let

$$f(x) = \sqrt{x} \quad \text{when} \quad x>0 \quad : \quad f(x) = \sqrt{-x} \quad \text{when} \quad x<0$$

With $k=0$ and $h>0$ we see that $D^*(0) = +\infty$ and with $h=0$ and $k>0$ we see that $D_*(=) = -\infty$. In § xx we present Weierstrass' construction of a continuous function $f(x)$ which fails to have an ordinary derivative at every point in the interval (a,b). The Denjoy-Young theorem shows that such a continuous function has a "turbulent" graph where $D^*(x) = +\infty$ and $D_*(x) = -\infty$ both hold for all x outside a null-set.

0.2 The case of monotone functions. If the continuous function f is non-increasing or non-decreasing, then case (**) cannot occur. So Theorem 0.1 implies that a monotone continuous function has an ordinary derivative almost everywhere.

1. Riesz intervals.

The interested reader may consult Riesz' plenary talk at the IMU-congress in Zürich (1932) for a historic account about derivatives of functions on the real line, and the subsequent proof follows Riesz' presentation in [ibid]. Let $g(x)$ be a real-valued and continuous function defined on a compact interval interval $[a,b]$, and (a,b) denotes the open interval. The forward Riesz set $\mathcal{F}_g$ consists of all points $x \in (a,b)$ for which there exists some $y \in (x,b)$ such that

$$g(x) < g(y) \tag{1.1}$$

It means that x is outside $\mathcal{F}_g$ if and only if

$$x < y < b \implies g(y) \leq g(x)$$

and from this the reader can check that $\mathcal{F}_g$ is the empty set if and only if If g is a non-increasing function. Excluding this case continuity entails that $\mathcal{F}_g$ is an open subset of (a,b) and hence a disjoint union of open intervals

$$\mathcal{F}_g = \cup\,(\alpha_\nu, \beta_\nu) \tag{i}$$

Each interval in (i) is called a *forward Riesz interval* of g. It may occur that some interval is of the form (α, b) i.e. b is a right end-point. Similarly a can be a left end-point. For example, if g from the start is strictly increasing then $\mathcal{F}_g = (a,b)$.

1.2 Proposition *For each forward Riesz interval (α,β) one has*

$$g(\beta) = \max_{\alpha \leq x \leq \beta} g(x) \tag{1.2.1}$$

Proof. Assume the contrary. This gives some maximum point $\alpha \leq x^* < \beta$ for the g-function on the closed interval $[\alpha,\beta]$. Now $x^* \in \mathcal{F}_g$ which means that

$$\exists\, y \in (x^*, b) \quad \& \quad g(x^*) > g(y)$$

Since x^* is a maximum point over $[\alpha,\beta]$ we must have $y > \beta$. But then $\beta \in \mathcal{F}_g$ which is impossible since β was a boundary point of the open set $\mathcal{F}_g$.

1.3 Backward Riesz intervals Put

$$\mathcal{B}_g = \{x \in (a,b) \colon\ \exists\, y \in (a,x) \quad : \quad g(x) < g(y)\}$$

Again $\mathcal{B}_g$ is open and hence a disjoint union of open intervals (c_ν, d_ν). They are called backward Riesz intervals. By similar reasoning as above one shows that if (c,d) is a backward Riesz interval then

$$g(c) = \max_{c \leq x \leq d} g(x) \tag{1.3.1}$$

1.4 A study of monotone functions. Let $f(x)$ be a continuous and *non-decreasing* function on $[a,b]$. To each $a < x < b$ we set

$$D^+(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.4.1}$$

where limes superior is taken as $h > 0$ decrease to zero. The function $x \mapsto D^+(x)$ takes values in $[0, +\infty]$.

1.4.2 Proposition. *For each positive number C the following set-theoretic inclusion holds:*

$$\{D^+ > C\} \subset \mathcal{F}_g \quad \text{where} \quad g(x) = f(x) - Cx$$

Proof. Suppose that $D^+(x) > C$ for some $a < x < b$. The definition of limes superior gives some $y \in (x, b)$ such that

(i) $$\frac{f(y) - f(x)}{y - x} > C$$

Then $g(y) - g(x) = f(y) - f(x) - C(y - x) > 0$ and hence $x \in \mathcal{F}_g$.

Next, let $\{\alpha_\nu, \beta_\nu)\}$ be the forward Riesz intervals of g. Applying (1.2.1) one has $g(\beta_\nu) \geq g(\alpha_\nu)$ for every such forwards interval. It follows that

$$0 \leq \sum g(\beta_\nu) - g(\alpha_\nu) \implies \sum f(\beta_\nu) - f(\alpha_\nu) \geq C \cdot \sum (\beta_\nu - \alpha_\nu)$$

Since f in non-decreasing we notice that

$$\sum f(\beta_\nu) - f(\alpha_\nu) \leq f(b) - f(a)$$

It follows thst the measure of the open set $\mathcal{F}_g$ satisfies

$$|\mathcal{F}_g| \leq \frac{f(b) - f(a)}{C}$$

Together with the inclusion from Proposition 1.4.2 we obtain:

1.5 Proposition *For every $C > 0$ the outer Lebesgue measure of the set $\{D^+ > C\}$ satisfies the inequality*

$$\big|\{D^+ > C\}\big|^* \leq \frac{f(b) - f(a)}{C}$$

1.6 The D_+-function. To each $a < x < b$ we put

$$D_+(x) = \liminf_{k \to 0} \frac{f(x + k) - f(x)}{k}$$

Let $A > 0$ and put

$$h(x) = f(x) - A$$

A similar reasoning as in Proposition 1.4.2 gives the inclusion

(1.6.1) $$\{D_+ < A\} \subset \mathcal{B}_h$$

where the right hand side is the backward Riesz set of h.

1.7 Some inequalitites. Consider a pair $0 < c < C$ and the intersection

$$E = \{D_+ < c\} \cap \{D^+ > C\}$$

Now (1.6.1) gives the inclusion

(i) $$E \subset \{D^+ > C\} \cap \mathcal{B}_h$$

Let $\{(\alpha_\nu, \beta_\nu)\}$ be the interval decomposition of $\mathcal{B}_h$. For each ν we consider the restriction of f to the interval (α_ν, β_ν) and Proposition 1.4.2 gives the inequality

(ii) $$|\{D^+ > C\} \cap (\alpha_\nu, \beta_\nu)\,|^* \leq \frac{f(\beta_\nu) - f(\alpha_\nu)}{C}$$

Since (α_ν, β_ν) is a backward Riesz interval of $f(x) - c$ we have $f(\beta_\nu) - f(\alpha_\nu) \leq c(\beta_\nu - \alpha_\nu)$. Hence (i) gives:

(iii) $$|\{D^+ > C\} \cap (\alpha_\nu, \beta_\nu)\,|^* \leq \frac{c}{C} \cdot (\beta_\nu - \alpha_\nu)$$

Since the backward Riesz intervals are disjoint a summation over ν and the inclusion ((1.6.1) give:

(1.7.1) $$|\{D^+ > C\} \cap \{D_+ < c\}|^* \leq \frac{c}{C}(b - a)$$

2. Proof of Lebesgue's theorem.

The function f restricts to a non-decreasing function on an arbitrary open subinterval (a_*, b_*) of (a, b) and since both D^+ and d_+ are constructed by limits close to a point we get the same inequality as in (1.7.1), i.e. one has the inequality

$$|\{d_+ < c\} \cap \{D^+ > C\} \cap (a_*, b_*)|^* \leq \frac{c}{C} \cdot (b_* - a_*)$$

Now the criterion from §XX implies that $\{d_+ < c\} \cap \{D^+ > C\}$ is a null-set. Apply this for pairs $c = q < r = C$ where q, r are rational numbers. Since a denumerable union of null-sets is a null-set we conclude that the equality

$$\text{(i)} \qquad d_+(x) = D^+(x)$$

holds almost everywhere. In the same way one proves that the equality

$$\text{(ii)} \qquad d^+(x) = D_+(x) \quad \text{holds almost everywhere}$$

Finally, it is obvious that when (i-ii) hold then f has an ordinary derivative which proves that every monotone function has a derivative almost everywhere.

2.1 An extension of Lebesgue's theorem. Let f be a continuous function on the closed unit interval $[0, 1]$. Suppose that E is a measurable subset of $(0, 1)$ such that the restriction of f to E is non-decreasing. Removing an eventual zero set we also assume that $E = \mathcal{L}(E)$, i.e. every $x \in E$ is a point of density for E as explained in § XX. Using exactly the same methods as above it follows that there is a (possibly empty) null-set $S \subset E$, there exists a derivative at every $x \in E$ in the sense that

$$\text{(1)} \qquad \lim_{h+k \to 0} \frac{f(x+h) - f(x-k)}{h+k} = f'_E(x)$$

exists for each $x \in E \setminus S$ where the limit is restricted in the sense that $x + h$ and $x + k$ belong to E during the passage to $h + k \to 0$. But since x is a point of density (1) holds without this restriction, i.e. $f'_E(x)$ gives an ordinary derivative of f. Let us supply the details for this assertion. We may take $x = 0$ and replacing f by $f - f'_E(0)x - f(0)$ we can assume that $f'_E(0) = f(0) = 0$. Next, let $0 < \epsilon < 1/4$ which gives some $\delta > 0$ such that if $0 < x < \delta$ and $x \in E$ then

$$f(x) \leq \epsilon \cdot x$$

At the same time the density condition entails that if δ is small enough then

$$|E \cap (-x, x)| \geq 2x(1 - \epsilon) \quad : \quad 0 < x < \delta$$

If we consider some $0 < x < \delta/2$ we see that (xx) implies the interval $(x + 4\epsilon \cdot x, x)$ must intersect E and if $x^* \in E$ is in this interval we get

$$f(x) \leq f(x^*) \leq \epsilon \cdot x^* \leq \epsilon \cdot 2x$$

Since $\epsilon > 0$ this proves that $D^+(0) = 0$ and in the same way the reader can verify that the right derivative at $x = 0$ vanishes.

3 Proof of Theorem 0.1

For each non-negative integer $n = 0, 1, 2, \ldots$ and every rational number $r \in (a, b)$ we denote by $E_{n,r}$ the set of all $r < x < b$ such that

$$\frac{f(x) - f(\xi)}{x - \xi} > -n \quad : \quad r < \xi < x$$

Exercise. Show the set-theoretic inclusion

$$\{D_*(x) > -\infty\} \subset \bigcup E_{n,r}$$

where the union is taken over all $n \geq 0$ and every rational number $a < r < b$.

3.1 Proposition. *For each pair* (n, r) *the equality*

$$D^*(x) = D_*(x)$$

holds almost everywhere in the measurable set $E_{n,r}$.

Proof. Replacing the interval (a, b) by (r, b) and f by $f(x - r) + nx$ we can assume that $r = n = 0$ and now $E_{0,0} \subset (0, b - r)$ where the restriction of f to this measurable set is monotone, i.e.

$$0 < \xi < x \implies f(x) > f(\xi)$$

holds for every pair $\xi < x$ in $E_{0,0}$. To simplify notations we set $E = E_{0,0}$. Let E_* be the set of density for E as defined in XX and recall from XX that $E \setminus E_*$ is a null-set. Ignoring this null-set we consider the restriction of f to E_* which again is a non-decreasing function. The extended version from 2.1 of Lebegue's theorem applies and shows that after removing another null-set from E_* if necessary, then the limit below exists for each $x \in E_*$:

$$(*) \qquad D(x) = \lim_{h+k \to 0} \frac{f(x + h) - f(x - k)}{h + k}$$

In the same way one proves that if a null-set is removed from the set

$$D^*(x) = +\infty\}$$

then f has an ordinary derivative so that $D^*(x) = D_*(x)$. This finishes the proof of the Denjoy-Young theorem.

4 Derivatives of jump functions.

Above we studied monotone and continuous functions. There also exist non-decreasing jump functions which arise as follows: Let $\{\xi_n\}$ be a sequence of real numbers in $(0,1)$. They are not ordered and may give a dense set. For example, we can take some enumeration of all rational numbers in $(0,1)$. Next, let $\{\delta_n\}$ be a sequence of positive numbers such that $\sum \delta_n < \infty$. To each n we get the jump function $H_n(x)$ where

$$H_n(x) = 0 \quad : \quad x < \xi_n \quad \text{and} \quad H_n(x) = \delta_n \quad : \quad x \geq \xi_n$$

Now

$$s(x) = \sum H_n(x)$$

is a non-decreasing function which has jump-discontinuites at each ξ_n.

Exercise. Show that s is pointwise continuous at every x outside the set $\{\xi_n\}$, i.e. show that if $\epsilon > 0$ then there exists $\delta > 0$ such that

$$s(x+\delta) < s(x) + \epsilon \quad \& \quad s(x-\delta) > s(x) - \epsilon$$

Less evident is the following:

4.1 Theorem. *$s(x)$ has an ordinary derivative which is equal to zero almost everywhere.*

Proof. Let $\alpha > 0$ and denote by E be the subset of $(0,1)$ which consists of numbers $0 < x < 1$ such that

$$\limsup_{h+k\to 0} \frac{s(x+h) - s(x-k)}{h+k} > \alpha$$

It suffices to show that E is a null-set. To prove this we consider some $\epsilon > 0$ and choose N so large that

(i) $$\sum_{n>N} \delta_n < \alpha \cdot \epsilon$$

Set $s_*(x) = s(x) - (H_1(x) + \ldots + H\_N(x))$. If E_* is the corresponding set in (x) with s replaced by s_* then E and E_* only differ by the finite set $\xi_1, \ldots, \xi_N$ so the measures of e and E_* are the same. Now we apply Vitali's covering theorem using s_* and obtain a sequence of disjoint intervals $\{a_n, b_n)\}$ which yields a Vitali covering of E_* and at the same time

$$\frac{s_*(b_\nu - s_*(a_\nu)}{b_\nu - a_\nu} \geq \alpha$$

It follows that

(ii) $$s_*(1) - s_*(0) \geq \alpha \cdot \sum (b_\nu - a_\nu)$$

At the same time (i) entails that $s_*(1) - s_*(0) \leq \alpha \cdot \epsilon$ and hence we have

$$|E|^* = |E_*|^* \leq \sum (b_\nu - a_\nu) \leq \epsilon$$

Since ϵ was arbitrary we get $|E|^* = 0$ as requested.

5. Stieltjes' Moment problem.

Let f be a real-valued and continuous function defined on $x \geq 0$ such that the integrals

$$(*) \qquad \int_0^\infty x^n \cdot |f(x)|\, dx < \infty$$

for all positive integers n. At first sight one may expect that if

$$\int_0^\infty x^n \cdot f(x)\, dx = 0$$

hold for all n, then f must be identically zero. However, this is not true. We shall give examples below. But first we insert the following

Exercise. Let $g(x)$ be an arbitrary continuous function on $[0,1]$. Then the following two equalities hold:

$$\int_0^1 g^2(x)\, dx = \frac{2}{\pi} \lim_{R\to\infty} \int_0^R \big[\int_0^1 \cos(st) g(t)\, dt\big]^2\, ds$$

$$\int_0^1 g^2(x)\, dx = \frac{2}{\pi} \lim_{R\to\infty} \int_0^R \big[\int_0^1 \sin(st) g(t)\, dt\big]^2\, ds$$

This follows from Parseval's formula for L^2-functions on the real line.

Next, consider a test-function $\phi(s)$ supported by $[0,1]$. Now there exists the cosine integral

$$\Phi(x) = \int_0^1 \cos(sx) \cdot \phi(s)\, ds$$

This function satisfies the integrability condition (*). For example, if $m \geq 1$ we perform $2m$-many partial integrations and find that

$$x^{2m} \Phi(x) = (-1)^m \cdot \int_0^1 \cos(sx) \cdot \phi^{(2m)}(s)\, ds$$

Similarly we have the sine-transform:

$$\Psi(x) = \int_0^1 \sin(sx) \cdot \phi(s)\, ds$$

For each positive integer p a partial integration gives

$$\Phi(x) + i\Psi(x) = \frac{(-1)^p}{(is)^p} \int_0^1 e^{ixs} f^{(p)}(s)\, ds$$

Using this and the exercise above the reader may verify the equalities below for every non-negative integer p:

$$\frac{2}{\pi} \int_0^\infty x^{2p} \cdot \Phi(x)^2\, dx = \frac{2}{\pi} \int_0^\infty x^{2p} \cdot \Psi(x)^2\, dx = m_p^2$$

where we have put

$$m_p^2 = \int_0^1 [f^{(p)}(s)]^2\, ds$$

Now we consider the non-decreasing functions

$$G(x) = \frac{2}{\pi} \int_0^{\sqrt{x}} \Phi(s)^2\, ds \quad \text{and} \quad H(x) = \frac{2}{\pi} \int_0^{\sqrt{x}} \Psi(s)^2\, ds$$

From the above we obtain

$$m_p^2 = \int_0^\infty x^p \cdot G(x)\, dx = \int_0^\infty x^p \cdot H(x)\, dx$$

Since $G \neq H$ this gives an example where the moment problem has no unique solution with $c_p = m_p^2$.