The heat equation and Brownian motion

Introduction. The heat equation has a long history whose physical relevance has inspired mathematicians to develop tools to get solutions. A major step was introduced by Fourier which gives a recipe to solve the heat equation with prescribed boundary values. We restrict the discussion to the 2-dimensional case where the basic PDE-equation is

$$(*) \qquad \frac{\partial u}{\partial t}(t,z) = \Delta(u)(z,t)$$

Here t is a time variable and $z = x + iy$ moves in a domain Ω of the complex plane and is of class $\mathcal{D}(C^1)$. The PDE-equation corresponds to a Brownian motion which takes place in Ω. Suppose that the density of particles observed at time $r = 0$ is observed and expressed by a density function $g_0(z)$, i.e. g is real-valued and non-negative and its area integral over Ω is one. When $t > 0$ classical laws in physics mean that the observed density at some time $t > 0$ is expressed by a density function $x \mapsto u(z,t)$ where u solves (*) and $u(0,z) = g_0(z)$ and the following boundary condition hold for every $t > 0$:

$$(*) \qquad \frac{\partial u}{\partial n}(z,t) = 0 \quad : \quad z \in \partial\Omega$$

Above (*) is an example of a parabolic equation for which uniqueness and existence of solutions with prescribed initial conditions were proved by Gevrey, Hadamard and Holmgren in work around 1900. The heat equation can be rewritten as an integral equation and this was used in pioneering work by Ivar Fredholm which leads to existence theorems and estimates for associated eigenvalues and eigenfunctions under various boundary value conditions. There also exists a close interplay betwee the heat equation and the Brownian motion. This was pointed out by Poincaré and made explicit by Bachelier in the article *La Bourse* which contains the essential foundations for the subject which nowadays bears the popular name "mathematics of finance". For an account and further historic comments about the Brownian motion we refer to § XX in *Appendix Measure.*

Let us now discuss the heat equation in $\mathbf{C}$. Consider a domain Ω of class $\mathcal{D}(C^1)$ whose boundary consists of p many closed and disjoint differentiable Jordan curves. Here p is some positive integer and the case $p = 1$ is not excluded. In §§ from Chapter V we constructed the Green's function $G(z,\zeta)$ defined in the closed product $\bar{\Omega} \times \bar{\Omega}$. Denote by $C_*(\Omega)$ the Banach space of continuous functions in $\bar{\Omega}$ which are zero on the boundary. There exists the linear operator from this Banach space into itself defined by

$$(1) \qquad Tu(z) = \iint_\Omega G(z,\zeta)u(\zeta) \cdot d\xi d\eta$$

Recall from §§ in Chapter V that if ϕ is a C^2-function with compact support in Ω then

$$\iint_\Omega \Delta(\phi)(z) \cdot u(\zeta) dxdy = \iint_\Omega \phi(z) \cdot Tu(\zeta) dxdy$$

This means that Tu regarded as a distribution has a Laplacian expressed by the continuous density u, i.e.

$$(2) \qquad \Delta(Tu)(z) = u(z) \quad : \quad z \in \Omega$$

Let $u \in C^0_*(\Omega)$ be an eigenfunction where

$$u(z) = \mu \cdot Tu(z)$$

holds for some non-zero constant μ. From (2) it follows that

$$\Delta(u) = \frac{1}{\mu} \cdot u \tag{3}$$

which entails that u is of class C^2 in Ω. Moreover, Green's formula gives:

$$\mu \cdot \iint_\Omega \Delta(u)(z) \cdot u(z) dxdy = \iint_\Omega \nabla(u)^2(z) \cdot dxdy = 0 \tag{4}$$

where $\nabla(u)^2 = u_x^2 + u_y^2$. Hence the eigenvalue μ is real and strictly negative.

1. The spectrum of T and the function $\mathcal{D}(\lambda)$. Set $G^{(0)} = G$ and define inductively

$$G^{(m)}(z, \zeta) = \mu \cdot \iint_\Omega G(z, \zeta) \cdot G^{(m-1)}(z, \zeta)\, d\xi d\eta$$

Let λ be a new complex parameter and put

$$\mathcal{D}(\lambda) = \sum_{m=0}^{\infty} \lambda^m \cdot G^{(m)}(z, \zeta)$$

We regard $\mathcal{D}(\lambda)$ as a function with values in the Hilbert space of square integrable functions on the product $\Omega \times \Omega$, i.e. we use that

$$\iint_{\Omega \times \Omega} |G(z, \zeta)|^2\, d\xi d\eta dxdy < \infty$$

and similar finite double integrals occur for the functions $\{G^{(m)}\}$. The general result in §§ gives

2. Theorem. *The function $\mathcal{D}(\lambda)$ extends to a meromorphic function in the whole complex λ-plane whose poles are confined to a sequence of strictly negative real numbers.*

3. The heat equation. Let $\{\lambda_k\}$ be the poles of $\mathcal{D}$. If the pole has multiplicity $e_k \geq 2$ the corresponding eigenspace is e_k-dimensional. Repeating eigenvalues with eventual multiplicities we obtain a sequence of eigenfunctions $\{u_k\}$ with eigenvalues $\{\lambda_k\}$ and for each k the eigenfunction u_k is normalised so that

$$\iint u_k^2(x, y) \cdot dxdy = 1$$

and chosen so that they form an orthonormal set in the Hilbert space $L^2(\Omega$. Notice that every u-function is real-valued. Next, let t be a new real parameter which serves as a time variable. If $\{c_k\}$ is a sequence of complex numbers we set

$$p(t, z) = \sum_{k=1}^{\infty} c_k \cdot e^{-\lambda_k t} \cdot u_k(z)$$

The series converges nicely when $t > 0$ if $\{c_k\}$ do not increase too rapidly and the p-function satisfies the PDE-equation

$$\frac{\partial p}{\partial t} = p_{xx} + p_{yy} = \Delta(p)$$

when $t > 0$ and $z \in \Omega$. Next, the sequence $\{c_k\}$ determines an initial condition which usually is interpretated via a limit

$$\lim_{t \to 0} p(t, z) = p_*(z)$$

where $p_*(z)$ is a distribution. If p_* belongs to L^2 we have for example

$$c_k = \iint p_*(z) u_k(z)\, dxdy$$

4. The Brownian motion. Solutions to the heat equation correspond to probability densities for a particle whose time-dependent change of position is governed by a Brownian motion. If $z \in \Omega$ is given and the particle starts at z at time zero then we consider the probability distribution:

$$t \mapsto \text{prob}(z,t)$$

which gives the probability that the particle stays in Ω up to time t. Since Ω is bounded the particle eventually hits the boundary where it is absorbed. It means that

$$\text{(i)} \qquad \lim_{t\to\infty} \text{prob}(z,t) = 0$$

On the other hand the particle stays in Ω with high probability under short time intervals, i.e.

$$\text{(ii)} \qquad \lim_{t\to 0} \text{prob}(z,t) = 1$$

Above (i-ii) hold for every $z \in \Omega$. The function

$$p(t,z) = \text{prob}(z,t)$$

satisfies the heat equation and by the results in § 3 given by the series

$$p(t,z) = \sum c_k \cdot e^{-\lambda_k t} \cdot u_k(z) \quad \text{where} \quad c_k = \iint_\Omega u_k(z) \cdot dxdy \tag{4.0}$$

4.1 The E-function. When the particle starts at a point z we consider the expected time before it hits the boundary which is expressed by the integral

$$E(z) = -\int_0^\infty t \cdot \frac{\partial p}{\partial t}(t,z) \cdot dt$$

Since p satsifies the heat equation and the differential operators ∂_t and the Laplacian of the z-variable commute it follows that

$$\Delta(E)(z) = -\int_0^\infty t \cdot \frac{\partial^2 p}{\partial t^2}(t,z) \cdot dt \tag{4.1.1}$$

A partial integration gives

$$\Delta(E)(z) = -1$$

Hence the function

$$E(z) + \frac{|z|^2}{2}$$

is harmonic in Ω and since $E = 0$ on the boundary we conclude that

$$E(z) = \int_{\partial\Omega} P_z(\zeta) \cdot \frac{|\zeta|^2}{2}\, d\xi d\eta - \frac{|z|^2}{2}$$

where $P_z(\zeta)$ is the Poisson kernel which exhibits solutions to the Dirichlet problem.

4.2 Example. Let $\Omega = \{|z| < R\}$ be a disc. Then

$$E(z) = \frac{1}{2}(R^2 - |z|^2)$$

Next, let $\Omega = \{1 < |z| < R\}$ be an annulus. Then the reader may verify that

$$E(z) = \frac{R^2-1}{2} \cdot \frac{\log|z|}{\log R} + \frac{1-|z|^2}{2} \tag{4.2.1}$$

Notice that E takes its maxium over the circle of radius r^* where

$$r^* = \sqrt{\frac{R^2 - 1}{2\ddot{\log}R}} \tag{4.2.2}$$

The reader is invited to interpretate (4.2.1-4.2.2) by probabilistic considerations.

4.3 Points of arrival. Let ω be a finite union of subintervals of $\partial\Omega$. Starting the Brownian motion at a point $z \in \Omega$ we consider the paths which at the first arrival to the boundary hits points in ω. Again we get a p-function satisfying the heat equation and the initial condition depends upon ω. More precisely, the probability that a Brownian path escapes for the first time at a point in ω is equal to the value of the harmonic measure function $\mathfrak{m}(\omega, z)$. Set

$$p_\omega(t,z) = \sum c_k(\omega) \cdot e^{-\lambda_k \cdot t} \cdot u_k(z)$$

where $\{c_k(\omega)\}$ are determined by

$$c_k(\omega) = \iint_\Omega \mathfrak{m}(\omega, z) \cdot u_k(z) \cdot dxdy$$

4.4 A joint probability distribution. Let $\omega \subset \partial\Omega$ be as above and $t > 0$ some fixed time-value. With δt small we seek the probability that the particle which starts at some z, escapes at some point in ω for the first time during the interval $[t, t + \delta t]$. From the above this probability up to small ordo of δt is given by:

$$\big[\sum c_k(\omega) \cdot \lambda_k \cdot e^{-\lambda_k \cdot t} \cdot u_k(z)\big] \cdot \delta t$$

4.5 Example. Suppose that the "open window" which the particle wants to hit on the boundary changes with time. The probability that it will escape through the changing window becomes

$$\sum \big[\int_0^\infty c_k(\omega_t) e^{-\lambda_k t} \cdot dt\,\big] \cdot \lambda_k \cdot u_k(z) \tag{*}$$

4.6 A special case. Let Ω be the unit disc and $z = 0$ the starting point. Let $0 < a < \pi$ and suppose that the interval ω_t is $(-a \cdot |\sin\gamma t|, a \cdot |\sin\gamma t|)$ where $\gamma > 0$ is a constant. So the window is closed when $t = 0$ and has maximal width at time values when $|\sin\gamma t| = 1$. Here we have:

$$c_k(\omega_t) = \frac{a \cdot |\sin(\gamma t|}{\pi}$$

4.7 Remark. The reader may consult text-books for the classic formulas which determine the sequence of eigenvalues $\{\lambda_k\}$ and the sequence $\{u_k(0)\}$ in a disc. So above we obtain a closed formula for the probability to escape the changing window but a computer should be used to obtain a numerical value. One can employ Monte Carlo simulations to determine (*) above. More precisely, instruct the computer to change the size of the open window and now the computer provides accurate approximations and there is not difficulty to extend the situation when one starts from an arbitrary point in D. Of course one can replace the chosen "opening function" $|a| \cdot |\sin\gamma t|$ by other time dependent functions and in general cases one gets numerical solutions via Monte Carlo simulations, i.e. one finds numerical values for the probability to escape a moving window on the boundary of an arbitrary domain in $\mathcal{D}(C^1)$.

Passing to dimension $n = 3$ is of special interest one can still employ Mone Carlo simulations and establish numerical values for many different expected values as well as higher moments and other joint distributions.

Schrödinger's equation.

Quantum mechanics raised new questions which give rise to non-linear equations. A precise question was raised by Schrödinger in the article *Théorie relativiste de l'électron et l'interpretation de la mécanique quantique* from 1932. In the non-classical case the previous solution $u(x,t)$ is no longer valid. Instead another density $z \mapsto f_1(z) \neq u(t_1, z)$ is observed at some time $t_1 > 0$. Thus, something highly improbable but nevertheless possible has occurred during the time interval $[0, t_1]$. Schrödinger's problem was to find the most likely density a time t_1 and he concluded that the requested time-dependent density $w(t,z)$ which substitutes $u(t,z)$ above, is found in a non-linear class of functions $\mathcal{W}$ formed by products

$$u_0(t,z) \cdot u_1(z,t)$$

where $u_0(z,t)$ is a solution to the same heat equation as in (*) while u_1 solves the adjoint equation

$$\frac{\partial u_1}{\partial n}(z,t) = -\Delta(u_1)(t,z) \quad \text{for time values} \quad t < t_1$$

and satisfies the same boundary condition (1). Now one seeks a function $w(t,z)$ in the family $\mathcal{W}$ which satisfies the two boundary value conditions

$$w(0,z) = f_0(z) \quad : \quad w(t_1, z) = f_1(z)$$

where f_0, f_1 is a pair of *prescribed* density functions. The solution to this problem can be transformed into a system of non-linear integral equations. Namely, for the given domain Ω there exists the Poisson-Greens function $K(t,z,\zeta)$ and we have

$$u_0(t,z) = \iint_\Omega K(t,z,\zeta) \cdot g_0(\zeta) \cdot d\xi d\eta$$

$$u_1(t,z) = \iint_\Omega K(t_1 - t, z, \zeta) \cdot g_1(\zeta) \cdot d\xi d\eta$$

Next, the boundary conditions for w yield

$$f_0(z) = g_0(z) \cdot \iint_\Omega K(-t_1, z, \zeta) \cdot g_1(\zeta) \cdot d\xi d\eta$$

$$f_1(z) = g_1(z) \cdot \iint_\Omega K(-t_1, z, \zeta) \cdot g_1(\zeta) \cdot d\xi d\eta$$

Remark. The solvability of the system above was left open by Schrödinger and put forward to the mathematical community at the IMU-congress in Zürich in 1932 by Serge Bernstein. The article *Résolution d'un systeme d'équations de Schrödinger* by Fortet from 1940 gave a method for succesive approximations which led to solutions under some specific conditions imposed on the boundary data. A general method was introduced by Beurling in the article [Beurling] which is exposed in § XX from Special sections and gives existence of solutions to the system above for smooth domains above ad boundary data expressed by the functions f_0, f_1, g_0, g_1. Beurling's solution is established via a non-linear variational problem for product measures.