7. The p^*-function.

We construct a special harmonic function which will be used to get solutions to the Dirichlet problem in XXX. Let Ω be an open and connected set in $\mathbf{C}$. Its closed complement has connected components. Let E be such a connected component. To each $a \in E$ we get the winding number $\mathfrak{w}_a(\gamma)$. If b is another point in E which is sufficiently close to a it is clear that

$$|\frac{1}{\gamma(t)-a} - \frac{1}{\gamma(t)-b}| < |\frac{1}{\gamma(t)-a}|$$

Rouche's theorem from 1.4 implies that $\mathfrak{w}_a(\gamma) = \mathfrak{w}_b(\gamma)$, i.e. for every closed curve γ in Ω, the winding number stays constant in each connected component of $\mathbf{C} \setminus \Omega$. This enable us to construct single valued Log-functions in Ω. Namely, let $a \in E$ where E is a connected componen in the complement of Ω. Consider $f = \text{Log}\,(z-a)$ and choose a single valued branch f_* at some point $z_0 \in \Omega$. If $\gamma \subset \Omega$ is a closed curve with initial point at z_0 the analytic continuation along γ of the Log-function gives:

$$T_\gamma(f_*) = f_* + 2\pi i \cdot \mathfrak{w}_a(\gamma) \tag{1}$$

Next, if b is another point in E we consider $g_* = \text{Log}(z-b)$ and obtain

$$T_\gamma(g_*) = g_* + 2\pi i \cdot \mathfrak{w}_b(\gamma) \tag{2}$$

Since $\mathfrak{w}_b(\gamma) = \mathfrak{w}_b(\gamma)$ it follows that

$$T_\gamma(f_*) - T_\gamma(g_*) = f_* - g_* \tag{3}$$

Hence the difference $\text{Log}(z-a) - \text{Log}(z-b)$ is a *single valued* analytic function in Ω. Taking is exponential we find $\Psi(z) \in \mathcal{O}(\Omega)$ such that

$$e^{\Psi(z)} = \frac{z-a}{z-b} \tag{4}$$

Since $a \neq b$ we see that $\Psi(z) \neq 0$ for all $z \in \Omega$. Next, we get the harmonic function defined in Ω by

$$p(z) = \mathfrak{Re}\big(\frac{1}{\Psi(z))}\big) = \frac{\mathfrak{Re}(\Psi(z)}{|\Psi(z)|^2} \tag{*}$$

Notice that $\mathfrak{Re}(\Psi(z)) = \text{Log}\,|z-a| - \text{Log}|z-b|$ and since $\text{Log}\,|z-a| \to -\infty$ as $z \to a$ we see from (*) that

$$\lim_{z \to a} p(z) = 0 \tag{**}$$

Notice also that $\Psi(z)$ exends to a continuous function on $\bar{\Omega} \setminus (a,b)$ and we can choose a small $\delta > 0$ such that

$$\text{Log}|z-a| - \text{Log}\,|z-b| < -1 \quad : \quad |z-a| \leq \delta \tag{ii}$$

Then (i) and (ii) give

7.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the single point a. Then there exists a harmonic function $p^*(z)$ in Ω for which*

$$\lim_{z \to a} p^*(z) = 0$$

and there exists $\delta > 0$ such that

$$\max_{\{|z-a|=r\} \cap \Omega} p^*(z) < 0 \quad : \quad z \in D_a(r) \cap \Omega$$

Chapter 4: Multi-valued analytic functions

Introduction.

In section 1 we define winding numbers of curves in $\mathbf{R}^2$ where no complex variables appear. After this we study complex line integrals where Theorem 1.7 is the first main result in this chapter. The second main result is Theorem 2.1, referred to as the *argument principle*. It gives a gateway to find zeros of an analytic function since the counting function of its zeros is expressed by winding numbers arising from the image under the given function of closed boundary curves to a domain where we seek the number of zeros of f.

Section 3 starts with a construction due to Weierstrass and leads to analytic continuation of germs of analytic functions. To grasp his construction one considers the *total sheaf space* $\widehat{\mathcal{O}}$ whose stalks are germs of analytic functions at points in $\mathbf{C}$. This topological manifold is locally homeomorphic to small discs in $\mathbf{C}$ which express germs of multi-valued functions f. If ρ denotes the local homemorphism from $\widehat{\mathcal{O}}$ onto $\mathbf{C}$. Weierstrass's construction gives for every open and connected subset Ω of $\mathbf{C}$ a 1-1 correspondence between connected open subsets of $\widehat{\mathcal{O}}$ whose ρ-image is equal to Ω, and the class of multi-valued analytic functions in Ω. In this correspondence one does not exclude functions which may have analytic continuation to larger sets. In § 4 we prove the *Monodromy Theorem* and describe multi-valued functions in a punctured disc. In § 7 we construct the $\mathbf{p}^*$-function which will be used to solve the Dirichlet problem in Chapter 5. In § 8 we recall the fundamental *Spiegelungsprinzip* by Hermann Schwarz which is applied in §9 to construct the *modular function.* Finally Section 10 gives a brief exposition about some of the contributions by Poincaré which has played a major role for the theory of Fuchsian groups and automorphic functions.

1. Angular variation and winding numbers

Consider a vector-valued function of the real parameter t:

$$t \mapsto (x(t), y(t)) \quad : \quad 0 \leq t \leq T$$

whose image does not contain the origin, i.e. $x^2(t) + y^2(t) > 0$ hold for each t. Moreover we assume that the functions $x(t)$ and $y(t)$ are continuously differentiable. Set

$$\dot{x}(t) = \frac{dx}{dt} \quad : \quad \dot{y}(t) = \frac{dy}{dt} \quad : \quad r(t) = \sqrt{x(t)^2 + y(t)^2}$$

1.1 Proposition. *There exists a unique continuous map $t \mapsto \phi(t)$ such that*

(*) $$x(t) = r(t) \cdot \cos \phi(t) \quad : \quad y(t) = r(t) \cdot \sin \phi(t) \quad : 0 \leq t \leq T$$

where the ϕ-function satisfies the initial condition:

$$x(0) = r(0) \cdot \cos \phi(0) \quad \text{and} \quad y(0) = r(0) \cdot \sin \phi(0$$

Proof. We solve the first order ODE-equation.

$$\dot{\phi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}$$

with initial condition $\phi(0)$ as above. There remains to show that (*) holds for all t. Let us for example verify the $x(t) = r(t) \cdot \cos \phi(t)$. It suffices to prove that

$$\frac{d}{dt}(r(t) \cdot \cos \phi(t)) = \dot{x} \tag{1}$$

To prove this we notice that the left hand side in (1) is equal to

$$\dot{r} \cdot \cos(\phi) - r \cdot \sin(\phi)\dot{\phi} = \frac{\dot{r} \cdot x}{r} - y \cdot \dot{\phi} \tag{2}$$

Next, since $r = \sqrt{x^2 + y^2}$ we have

$$r\dot{r} = x\dot{x} + y\dot{y}$$

Hence (2) is equal to

$$\frac{x^2\dot{x} + xy\dot{y}}{r^2} - \frac{y(x\dot{y} - y\dot{x})}{r^2} = \frac{(x^2 + y^2)\dot{x}}{r^2} = \dot{x}$$

This proves Proposition 1.1

1.2 The angular variation. Since the sine- and the cosine functions have period 2π, the ϕ-function is only determined up to integer multiples of 2π. However, we get an *intrinsic number* by the difference

$$\phi(T) - \phi(0) \tag{1.2.1}$$

This number is called the *angular variation* of the function $t \mapsto (x(t), y(t))$. If we choose another parametrization where $t = t(\tau)$ is non-decreasing and $0 \leq \tau \leq T^*$, then we start with the vector valued function $\tau \mapsto x(t(\tau), y(t(\tau))$ and find $\phi(\tau)$. Calculus shows that (1.2.1) is the same. Thus, the angular variation of an oriented parametrized C^1-curve is intrinsically defined.

A notation. The angular variation along a parametrized curve γ is denoted by $\mathfrak{a}(\gamma)$. If γ is a curve we can construct the curve γ^* with the opposite direction:

$$\gamma^*(t) = \gamma(T-t)$$

It is clear that

$$\mathfrak{a}(\gamma^*) = -\mathfrak{a}(\gamma)$$

In other words, up to a sign the angular variation is determined by the orientation of the curve.

1.3 The case of closed curves.

If $x(0) = x(T)$ and $y(0) = y(T)$ the variation is an integer multiple of 2π. So if γ is a closed parametrized curve then $\mathfrak{a}(\gamma)$ is an integer times 2π and we set

$$\mathfrak{w}(\gamma) = \frac{\mathfrak{a}(\gamma)}{2\pi}$$

The integer $\mathfrak{w}(\gamma)$ is called the *winding number* of γ. The construction gives

$$\mathfrak{w}(\gamma) = \frac{1}{2\pi}\int_0^T \frac{x\dot{y} - y\dot{x}}{x^2+y^2}\cdot dt \tag{1.3.1}$$

Example. Let m be a positive integer and

$$x(t) = \cos mt \quad : \quad y(t) = \sin mt$$

Notice that $x^2 + y^2 = 1$. It follows that

$$\frac{x\dot{y} - y\dot{x}}{x^2+y^2}\cdot dt = \cos mt \cdot m \cdot \cos mt - \sin mt \cdot (-m \cdot \sin mt) = m$$

Hence the winding number is equal to m.

1.4 Homotopy invariance. Consider a family of closed curves

$$\{\gamma_s \colon\ 0 \le s \le 1\} \quad : \gamma_s(0) = \gamma_s(T) : \quad 0 \le s \le 1$$

Let $t \mapsto (x_s(t), y_s(t))$ be the parametrization of γ_s. For each fixed s the curve $t \mapsto \gamma_s(t)$ has a winding number $\mathfrak{w}(\gamma_s)$. If these C^1-functions depend continuously upon s it follows that $s \mapsto \mathfrak{w}(\gamma_s)$ is a continuous function. Since it is integer-valued it must be a constant. Hence we have proved

1.5 Theorem *Let $\{\gamma_s\}$ In a homotopic family of closed curves, the individual curves have a common winding number.*

Remark. In topology one refers to this by saying that the winding number is the same in each homotopy class of closed parametrized curves which surround the origin. A parametrized curve γ is defined by a map $t \mapsto \gamma(t)$ which need not be 1-1, i.e. we only assume that $\gamma(0) = \gamma(T)$. One may think of an insect which takes a walk on the horisontal (x,y)-plane starting at point p at time $t = 0$ and returns to p after a certain time interval T. During this walk the insect may cross an earlier path several times and even walk in the same path but in opposed direction for a while. The sole constraint is that the insect never attains the origin, i.e. $x(t)^2 + y(t)^2 > 0$ must hold in order to construct the winding number.

1.6 The case of non-closed curves. Let p and q be two points outside the origin. Consider two curves γ_1 and γ_2 where p is the common initial point and q the common end-point. Now we get the closed curve ρ defined by

$$\rho(s) = \gamma_1(2s) \quad : 0 \leq s \leq T/2\text{and} \quad \rho(s) = \gamma_2(2T - 2s) \quad : T/2 \leq s \leq T$$

Here we find that

$$\mathfrak{w}(\rho) = \mathfrak{a}(\gamma_1) - \mathfrak{a}(\gamma_2)$$

Next, keeping p and q fixed we consider a continuous family of curves $\{\gamma_s\}$ where $\gamma_s(0) = p$ and $\gamma_s(T) = q$ for all $0 \leq s \leq 1$. To each s we get the two curves γ_0 and γ_s and construct the closed curve ρ as above. Theorem 1.5 implies that

$$s \mapsto \mathfrak{a}(\gamma_0) - \mathfrak{a}(\gamma_s)$$

is a constant function of s. Since the difference is zero when $s = 0$ we conclude that

$$\mathfrak{a}(\gamma_0) = \mathfrak{a}(\gamma_s) \quad : \; 0 \leq s \leq 1$$

Thus, the angular variation is constant in a homotopic family of curves which join a pair of points p and q.

1.7 Rouche's principle. Let γ_* be a parametrized closed curve, and γ another closed curve such that

(i) $$|\gamma_*(t) - \gamma(t)| < |\gamma_*(t)| \quad : \quad 0 \leq t \leq T$$

To each $0 \leq s \leq 1$ we obtain the closed curve $\gamma_s(t) = s \cdot \gamma_* + (1 - s)(\gamma_*(t) - \gamma(t))$ which by (i) also the surrounds the origin. This gives a homotopic family and Theorem 1.5 gives:

(*) $$\mathfrak{w}(\gamma_*) = \mathfrak{w}(\gamma)$$

Variation of vector-valued functions

Let γ be a parametrized C^1-curve. Here we do not exclude that $\gamma(t) = (0, 0)$ for some values of t, i.e. γ is an arbitrary C^1-curve. Consider a pair of C^1-functions $u(x, y)$ and $v(x, y)$ defined in some neighborhood of the compact image set $\Gamma = \gamma([0, T])$. Assume that $u^2 + v^2 \neq 0$ on Γ. This gives a curve γ^* which surrounds the origin defined by

(i) $$t \mapsto (u(\gamma(t)), v(\gamma(t))$$

Write $\gamma(t) = (x(t), y(t))$ and set

$$\xi(t) = u(x(t), y(t)) \quad : \quad \eta(t) = v(x(t), y(t))$$

Then we have

(ii) $$\mathfrak{a}(\gamma^*) = \int_0^T \frac{\xi\dot{\eta} - \eta\dot{\xi}}{\xi^2 + \eta^2} \cdot dt$$

Now $\dot{\xi} = u_x\dot{x} + u_y\dot{y}$ and similarly for $\dot{\eta}$. So the last integral becomes

(*) $$\int_0^T \frac{u(v_x\dot{x} + v_y\dot{y}) - v(u_x\dot{x} + u_y\dot{y})}{u^2 + v^2} \cdot dt$$

This yields an integer called the variation of the vector valued function (u, v) along the closed curve γ. We denote this integer by a subscript notation and write $\mathfrak{a}_{(u,v)}(\gamma)$. When γ is a closed curve we define the winding number

$$\mathfrak{w}_{(u,v)}(\gamma) = \frac{1}{2\pi} \cdot \mathfrak{a}_{(u,v)}(\gamma)$$

Notice that this integer depends upon the pair (u, v) while γ is kept fixed.

1.8 The case of CR-pairs

Let γ be a curve and $f(z) = u + iv$ an analytic in a neighborhood of $\gamma(T)$ where $f(\gamma(t)) \neq 0$ for all t. Hence $u^2 + v^2 \neq 0$ on γ so we can define $\mathfrak{a}_{(u,v)}(\gamma)$. Since (u, v) satisfy the Cauchy-Riemann equations we can express $\mathfrak{a}_{(u,v)}(\gamma)$ in an elegant way. Namely let $t \mapsto (x(t), y(t))$ be a parametrization of γ and write $z(t) = x(t) + iy(t))$. Then

$$\dot{z} = \dot{x} + i\dot{y}$$

Regard the function

$$\text{(i)} \qquad t \mapsto \mathfrak{Im}\,[\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t)\,]$$

Since the complex derivative $f'(z) = u_x + iv_x$ we obtain

$$\text{(ii)} \qquad \frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t) = \frac{(u_x + iv_x)(u - iv)(\dot{x} + i \cdot \dot{y})}{u^2 + v^2}$$

The imaginary part becomes

$$\text{(iii)} \qquad \frac{u_x u\dot{y} - u_x v\dot{x} + v_x u\dot{x} + v_x v\dot{y}}{u^2 + v^2} = \frac{u(u_x\dot{y} + v_x\dot{x}) - v(u_x\dot{x} - v_x\dot{y})}{u^2 + v^2}$$

Next, we can apply the Cauchy-Riemann equations and replace u_x with v_y and $-v_x$ by u_y. Then we see that (iii) is equal to the integrand which appears in (*) in 1.7. Hence we have proved the following:.

1.9 Theorem *Let $f(z) = u + iv$ be holomorphic in a neighborhood of γ and set $\mathfrak{a}_f(\gamma) = \mathfrak{a}_{(u,v)}(\gamma)$. Then*

$$(*) \qquad \mathfrak{a}_f(\gamma) = \int_0^T \mathfrak{Im}\,[\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t)\,] \cdot dt$$

1.10 Remark By the construction of complex line integrals, the integral (*) above can be written as

$$\frac{1}{i} \cdot \int_\gamma \mathfrak{Im}\,[\frac{f'(z)}{f(z)}\,] \cdot dz$$

This complex notation is often used. When γ is a closed curve we get the winding number

$$\mathfrak{w}_f(\gamma) = \frac{1}{2\pi i} \cdot \int_\gamma \mathfrak{Im}\,[\frac{f'(z)}{f(z)}\,] \cdot dz$$

So this complex line integral always is an integer whenever $f(z)$ is analytic and $\neq 0$ in some open neighborhood of the compact set $\gamma([0, T])$.

1.11 Jordan's curve theorem

Let γ be a closed C^1-curve and set $\Gamma = \gamma([0,T])$. To each $a \in \mathbf{C} \setminus \Gamma$ the closed curve

$$t \mapsto \frac{1}{\gamma(t) - a}$$

surrounds the origin. Its winding number denoted by $\mathfrak{w}_a(\gamma)$. From (*) in 1.4 we see that this winding number is constant in every connected component if $\mathbf{C} \setminus \Gamma$.

1.12 The case when γ is 1-1 Assume that $\gamma(t)$ is 1-1 except for the common end-values. This means that the image curve $t \mapsto \gamma(t)$ is a *closed Jordan curve.* For each $a \in \mathbf{C} \setminus \Gamma$ we notice that $t \mapsto \gamma(t) - a$ is 1-1. In the equation from XX which determines the ϕ-function for a given a where we may take $\phi(0) = 0$ as initial value shows that $t \mapsto \gamma(t)$ is 1-1 on the open interval $(0,T)$. Hence $\phi(t)$ cannot be an integer multiple of 2π when $0 < t < T$. Starting with $\phi(0) = 0$ it follows that

$$-2\pi < \phi(t) < 2\pi \quad : \quad 0 < t < 2\pi$$

Hence $\phi(T)$ can only attain one of the values $-2\pi, 0, 2\pi$. The *Jordan curve theorem* tells us that the value zero is never attained. Moreover, the set of points a for which the winding number equals 1 is a conneced open set, called the Jordan domain bounded by Γ. The complementary set is also connected and here $\mathfrak{w}_a(\gamma) = 0$. This can be expressed by saying that the closed Jordan curve Γ divides $\mathbf{C}$ into two component. This topological result was proved by Camille Jordan in 1850 and it is is actually valid under the relaxed assumption that the γ-function is only continuous. In that case the proof of Jordan's Curve Theorem is more demanding. For a detailed proof of the continuous version of Jordan's Curve Theorem we refer [Newmann] where methods of algebraic topology are used. We remark that Jordan's theorem in the plane is subtle in view of a quite remarkable discovery in dimension 3 due to X. Alexander who constructed a *homeomorphic copy* of the unit sphere in R^3 where the analogue of Jordan's theorem is not valid. This goes beyond the scope of these notes. A recommended text-book in algebraic topology is Alexander's classic text-book [Al] which gives an excellent introduction to the subtle parts of the theory.

1.13 The case of a simple polygon. Let $p_1, \ldots, p_N$ be distinct points in $\mathbf{C}$ where $N \geq 3$. To each $1 \leq \nu \leq N-1$ we get a line segment $\ell_\nu = [p_\nu, p_{\nu+1}]$ and we also get the line segment $\ell_N = [p_N, p_1]$. Assume that they do not intersect.Then they give sides of a simple closed curve Γ whose corner points are $p_1, \ldots, p_N$. The circle Γ is oriented where one travels in the positive direction from p_ν to $p_{\nu+1}$ when $1 \leq \nu \leq N-1$ and makes the final positive travel from p_N to p_1. We can imagine a narrow channel $\mathcal{C}_+$ which surrounds Γ and from this one can "escape" to the point at infinity. For example at a corner point p_ν where $|p_\nu|$ is maximal the channel contains points of absolute value > 1. From this picture it is clear the the *outer component* Ω_∞ of Γ is connected - and even simply connected if one adds the point at infinity. Rouche's principle shows that the winding number is zero for all points in the exterior component. If we instead construct a narrow channel $\mathcal{C}_*$ which moves "just inside" Γ then the channel itself is obviously connected. But their remains to see why the whole interior is connected and that the common winding number is equal to one. This, if Ω_* is the open complement of $\Gamma \cup \Omega_\infty$ we must first prove that Ω_* is connected. Since the narrow channel $\mathcal{C}_*$ is connected it suffices to show that when $p \in \Omega_*$ then there exists some curve γ from p which reaches $\mathcal{C}_*$. To obtain γ we consider a point $p^* \in \Gamma$ such that $|p - p^*|$ is the distance of p to Γ, i.e. we pick a point nearest to p. Now we draw the straight line L through p and p^* and by a picture the reader discovers that if we travel along L from p towards p^* then we reach $\mathcal{C}_*$ prior to the arrival at p^*. This proves that Ω_* is connected. The proof that the common winding number for points in Ω_* is equal to one is left as an *exercise* to the reader.

2. The argument principle

Let $\Omega \in \mathcal{D}(C^1)$ and $f(z)$ is an analytic function in Ω which extends to a C^1-function on its closure. Denote by $\mathcal{N}_\Omega(f)$ the number of zeros of f in Ω and assume that $f \neq 0$ on $\partial\Omega$.

2.1 Theorem. *Let $\Gamma_1, \ldots, \Gamma_k$ be the oriented boundary curves of Ω. Then*

$$N_\Omega(f) = \sum_{\nu=1}^{\nu=k} \mathfrak{w}_f(\Gamma_\nu)$$

Proof. By the result in § III one has

$$N_\Omega(f) = \sum \frac{1}{2\pi i} \cdot \int_{\Gamma_\nu} \frac{f'(z)dz}{f(z)}$$

Since $N_\Omega(f)$ is an integer and hence a real number it follows that

$$N_\Omega(f) = \sum \frac{1}{2\pi} \cdot \int_{\Gamma_\nu} \mathfrak{Im}\,[\frac{f'(z)dz}{f(z)}\,]$$

By Theorem 1.9 expressed in the complex notation each term of the sum above is equal to $\mathfrak{w}_f(\Gamma_\nu)$ and Theorem 2.1 follows.

2.2 Rouche's Theorem. *Let Ω and f be as above and let g be another holomorphic function in Ω which extends to be C^1 on the closure. If $|g| < |f|$ holds on $\partial\Omega$, it follows that*

$$\mathcal{N}_{f+g}(\Omega) = \mathcal{N}_f(\Omega)$$

Proof. Apply the result in 1.6.

2.3 An application to trigonometric series. Let $1 \leq m < n$ be a pair of positive integers. Consider a trigonometric polynomial

$$P(\theta) = \sum_{\nu=m}^{\nu=n} a_\nu \text{cos}(\nu\theta) + b_\nu \text{sin}(\nu\theta) \quad : \quad a_\nu, b_\nu \in \mathbf{R}$$

We assume that at least one of the coefficients a_m or b_m is $\neq 0$, and similarly at least one of the numbers a_n or b_n is $\neq 0$. Then one has

2.4 Theorem *P has at least $2m$ zeros on $[0, 2\pi]$ counted with multiplicity.*

Proof Consider the polynomial

$$Q(z) = (a_m - ib_m)z^m + \ldots + (a_n - ib_n)z^n$$

Notice that $\mathfrak{Re}(Q(e^{i\theta}) = P(\theta)$. The polynomial Q has a zero of multiplicity m at the origin. Consider some $r < 1$ chosen so that $Q \neq 0$ on the circle $T_r = \{|z| = r\}$. Since $Q(z)$ has at least m zeros counted with multiplicity in the disc D_r, it follows from Theorem 2.2 that

$$\mathfrak{w}_Q(T_r) \geq m$$

Regarding a picture the reader discovers that the curve $\theta \mapsto Q(re^{i\theta})$ must intersect the imaginary axis line at least $2m$ times which means that the function

$$\theta \mapsto \sum_{\nu=m}^{\nu=n} r^\nu \cdot a_\nu \text{cos}(\nu\theta) + r^\nu \cdot b_\nu \text{sin}(\nu\theta)$$

as at least $2m$ distinct zeros on $[0, 2\pi]$. Passing to the limit as $r \to 1$ we get Theorem 2.4.

2.5 A special estimate. Theorem 2.1 can be used to give upper bounds for the counting function $\mathcal{N}_\Omega(f)$. Suppose that Ω is a rectangle

$$\{z = x + iy\colon\ a < x < b\colon\ 0 < y < T\}$$

Here $\partial\Omega$ contains the vertical line $\ell = \{x = b : \ 0 < y < T\}$. The line integral along ℓ contributes to the evaluation of $\mathcal{N}_\Omega(f)$ by

$$\frac{1}{2\pi}\cdot\int_\ell \mathfrak{Im}\,|\frac{f'(z)dz}{f(z)}] \tag{2.5.1}$$

Now $dz = idy$ along ℓ and therefore the integral above is equal to

$$\frac{1}{2\pi}\cdot\int_0^T \mathfrak{Re}[\frac{f'(b+iy)}{f(b+iy)}]\cdot dy$$

Let us now assume that $\mathfrak{Re}\, f(b+iy) \geq c_0 > 0$ for all $0 \leq y \leq T$. Then there exists a single valued branch of the complex Log-function, i.e.

$$\log f(b+iy) = \log|f(b+iy)| + i\cdot\arg(f(b+iy))\ :\ -\pi/2 < \arg(f(b+iy)) < \pi/2$$

Since $f'(z) = \frac{1}{i}\cdot\partial_y(f)$ it follows that

$$\frac{f'(b+iy)}{fb+iy)} = \frac{1}{i}\cdot[\partial_y(\log|f(b+iy)|) + i\cdot\partial_y(\arg(f(b+iy)))]$$

Hence we obtain

2.6 Proposition. *One has the equality*

$$\mathfrak{Re}\,\frac{f'(b+iy)}{f(b+iy)} = \partial_y(\arg(f(b+iy)))$$

2.7 Remark. Proposition 2.6 gives therefore

$$\frac{1}{2\pi}\cdot\int_\ell \mathfrak{Im}\,|\frac{f'(z)dz}{f(z)}] = \frac{1}{2\pi}\cdot\arg(f(b+iT))) - \arg(f(b))) \tag{*}$$

The right hand side is a real number in $(-1/4, 1/4)$ and hence we get a small contribution from the line integral in the left hand side when we regard whole line integral over $\partial\Omega$ which evaluates $\mathcal{N}_f(\Omega)$. This will be used to study the zeros of Riemann's ζ-function.

2.8 A local implicit function theorem. Let $m \geq 2$ and $g_2(z), \ldots, g_m(z)$ are analytic functions defined in an open disc D centered at $z = 0$ where $g_\nu(0) = 0$ for every ν. Let $\phi(z)$ be another analytic function in D with $\phi(0) = 0$ and consider the equation

$$y + g_2(z)y^2 + \ldots + g_m(z)y^m = \phi(z) \tag{2.8.1}$$

Thus, we seek $y(z)$ so that (2.8.1) holds. It turns out that there exists a unique analytic function $y(z)$ defined in some open disc D_* centered at $z = 0$ where $y(0) = 0$ and (2.8.1) hold for every $z \in D_*$. To prove this we set

$$P(y, z) = y + g_2(z)y^2 + \ldots + g_m(z)y^m$$

Since ϕ and the g-functions are zero at $z = 0$ there exists some $\delta > 0$ such that if $z \in D(\delta)$ then

$$|\phi(z)| < |P(e^{i\theta}, z)| \quad \text{for all} \quad 0 \leq \theta \leq 2\pi \tag{i}$$

Next, let us put

$$P'_y(y,z) = 1 + 2g_2(z)y + \ldots + mg_m(z)y^{m-1}$$

From (i) there exists the integral

$$\frac{1}{2\pi i} \cdot \int_{|y|=1} \frac{P'_y(y,z)}{P(y,z) - \phi(z)} \cdot dy \quad : z \in D(\delta) \tag{1}$$

By Rouche's Theorem this integer-valued function is constant when z varies in $D(\delta)$. When $z = 0$ the integrand is $\frac{1}{y}$ and hence the constant integer is 1. This means that when $z \in D(\delta)$ is fixed, then the analytic function

$$y \mapsto P(y,z) - \phi(z)$$

has exactly one simple zero in $|y| < 1$. Denote this zero by $y(z)$. The residue formula gives:

$$y(z) = \frac{1}{2\pi i} \cdot \int_{|y|=1} \frac{y \cdot P'_y(y,z)}{P(y,z) - \phi(z)} \cdot dy \tag{2}$$

It is clear that $y(z)$ is analytic in $D(\delta)$ and by the construction $P(y(z), z) = 0$. Thus, $y(z)$ is the required solution.

2.9 The case of higher multiplicity. This time we consider the equation

$$P(z,y) = g_1(z)y + g_2(z)y^2 + \ldots + g_m(z)y^m = 0$$

where $g_k(0) \neq 0$ for some $k \geq 2$ while $g_m(0) = \ldots = g_{k+1}(0) = 0$. No special, assumption is imposed on $g_1(0), \ldots, g_{k-1}(0)$, i.e. some of these numbers may be $\neq 0$. Since $g_k(z) \neq 0$ in some disc around $z = 0$ and we study a homogeneous equation we can divide out g_k and assume that it is 1 from the start. Let us then consider the polynomial

$$Q(y) = y^k + g_{k-1}(0)y^{k-1} + g_1(0)y$$

It has k zeros counted with multiplicity and we choose R so large that these zeros all belong to $|y| < R$. With R kept fixed it is clear that there exists $\delta_* > 0$ such that

$$|z| < \delta_* \implies |g_m(z)y^m + \ldots + g_{k+1}(z)y^{m+1}| < |y^k + g_{k-1}(z)y^{k1} + \ldots + g_0(z)|$$

when $|y| = R$. Rouche's theorem implies that $y \mapsto P(z,y)$ has m zeros in $|y| < R$ for each $|z| < \delta_*$. This m-tuple of zeros need be single-valued analytic functions in $|z| < \delta_*$. In XX we describe the multi-valued behaviour of these root functions in more detail.

2.10 Images of closed curves. Let γ be a closed Jordan curve parametrized by arc-length. So we have a map $s \to z(s)$ where $\gamma(0) = \gamma(L)$ and we assume that this function is C^1 with a non-zero derivative, i.e. with $z = x+iy$ the two functions $x(s)$ and $y(s)$ are both of class C^1 and $x'(s)^2 + y'(s)^2 > 0$ hold when $0 \leq s \leq L$. Let $f(z)$ be analytic in some open neighborhood of γ and assume that the absolute value $|f(z)| = c$ for some constant $c > 0$ holds on γ. This gives a continuous function $\theta(s)$ such that

$$f(z(s)) = c \cdot e^{i\theta(s)} \tag{i}$$

where $\theta(0)$ is determined by the equality $f(z(0)) = c \cdot e^{i\theta(0)}$. Taking the derivative with respect to s we get

$$\text{(i)} \qquad f'(z(s)) \cdot z'(s) = c \cdot i\theta'(s) \cdot e^{i\theta(s)} \implies \frac{f'}{f} \cdot z'(s) = i \cdot \theta'(s)$$

It follows that

$$(*) \qquad \frac{1}{2\pi i} \cdot \int_\gamma \frac{f' \cdot dz}{f} = \frac{1}{2\pi} \int_0^{2\pi} \theta'(s) \cdot ds = \frac{\theta(L) - \theta(0)}{2\pi}$$

Let us now assume now that the left hand side is 1 which gives

$$\text{(ii)} \qquad \theta(L) = \theta(0) + 2\pi$$

In addition we assume that the complex derivative $f' \neq 0$ on γ. Then (i) shows that the s-derivative of the real-valued θ-function is always $\neq 0$ and since the value increases by (ii) we have $\theta'(s) > 0$ for all s. From this we conclude

2.10 Theorem. *Assume that $f' \neq 0$ on γ and that the left hand side in (*) is one. Then f yields a bijective map from γ onto the circle of radius c centered at the origin.*

Next, with the assumptions in Theorem 2.10 we consider a complex number w of absolute value < 1. Since γ is a closed curve we know that

$$(1) \qquad \frac{1}{2\pi i} \cdot \int_\gamma \frac{f' \cdot dz}{f - w}$$

is an integer and just as in Rouche's theorem we conclude that (1) is equal to one for every $|w| < 1$. The reader may also verify that

$$(2) \qquad |w| > 1 \implies \frac{1}{2\pi i} \cdot \int_\gamma \frac{f' \cdot dz}{f - w} = 0$$

Remark. Notice that (1-2) hold without the hypothesis that f extends to be analytic in the Jordan domain Ω bordered by γ. If we in addition assume that f from the start is analytic in a neighborhood of $\bar{\Omega}$ then (1) shows that $f(z) - w$ has exactly one zero in Ω for each $|w| < 1$ which means that f yields a conformal map from Ω onto the disc $|w| < c$. Thus, to check when a given $f \in \mathcal{O}(\Omega)$ which extends to γ where $|f| = c$ and $f' \neq 0$, it suffices to check that the left hand side in (*) is equal to one in order that f yields a conformal map.

2.11 A conformal condition. Let γ and f be as above where f extends to be analytic in a neighborhood of γ and assume that the curve γ is real-analytic, i.e. $z(s)$ is a real-analytic function of s. We can find a closed Jordan curve γ_* contained in the Jordan domain Ω which together with γ borders a doubly-connected domain U. Here γ_* can be close to γ as illustrated by figure XXX,

2.12 Theorem. *Assume that the restriction of f to U is 1-1. Then f yields a bijective map from γ onto the circle of radius c.*

Proof. We may assume that $\theta'(0) > 0$ and Theorem 2.12 follows if we prove that $\theta'(s) > 0$ for all s. If $\theta'(s_0) = 0$ for some $0 < s_0 < L$ and $z_0 = z(s_0)$ we have $f'(z_0) = 0$. Hence we have a Taylor series with

$$f(z_0 + \zeta) = f(z_0) + c_2\zeta^2 + c_3\zeta^3 + \dots$$

Let m be the smallest integer such that $c_m \neq 0$. We claim that $m = 2$ must hold. To see this we use tha γ is of class C^1 which means that if $\epsilon > 0$ is sn´mall then

$$\{|z - z_0| < \epsilon\} \cap U = V$$

is almost a small half-disc.

Exercise. Show that if $m \geq 3$ with $c_m \neq 0$ then the restriction of f to V cannot be 1-1.

We conclude that if $\theta'(s_0) = 0$ then the second order derivative $f''(z_0) \neq 0$. At the same time we notice that another derivation in (ii) above at $s = s_0$ where $f'(z_0) = 0$ holds

$$\frac{f''(z_0)}{f(z_0)} \cdot z'(s_0(^2) = i \cdot \theta''(s_0)$$

So whenever $\theta'(s_0) = 0$ it follows that $\theta''(s_0) \neq 0$.

Exercise. Show that when $f|U$ is 1-1 then $\theta'(s) \neq 0$ for all s and hence $f|\gamma$ is bijective.

2.13 The case when the winding number is zero. We keep the assumption that γ is a real-analytic closed Jordan curve and f extends to an analytic function which is 1-1 in a domain U as above while $|f(z)| = c$ has constant absolute value along γ. But this time we suppose that

$$(*) \qquad \int_\gamma \frac{f'}{f}\, dz = 0$$

So here $\theta(L) = \theta(0)$ and after a rotation we may assume that this common value is zero. while $\theta'(0) > 0$. It follows that $\theta(s)$ takes a maximum > 0 for some s^* where we therefore get $\theta'(s^*)$. So we always find the smallest $0 < s_0 < L$ where $\theta'(s_0) = 0$. So $s \to \theta(s)$ is strictly increasing on $[0, s_0)$ and it is clear that the hypothesis that $f|U$ is 1-1 entails that the range $\theta[0, s_0]$ is an interval $[0, \theta_0|$ where $\theta_0 < 2\pi$, i.e. the θ-function has not made a full turn so the image set $f(\gamma[0, s_0])$ is an interval on the circe $|w| = c$ where we write $w = f(z)$.

Exercise. Show by a similar reasoning as in XxX that $\theta''(s_0) \neq 0$ which means that $\theta(s)$ starts to decrease on some interval $s_0 \leq s \leq s_1$ until $\theta'(s_1) = 0$ which must occur for some $s_1 < L$ since $\theta'(L) \neq 0$ was assumed. Keeping this in mind the reader should supply a figure and verify the details of the following result:

2.14 Theorem. *f maps γ onto a proper circular interval of $|w| = c$ where it yields a double cover except at the two end-points of the interval given by $f(\theta(s_0)$ and $f(\theta(s_1))$ where the θ-function achieves its maxium respectively its minium.*

2.15 Koebe's function. Consider the analytic function

$$f(z) = z + \frac{1}{z}$$

When γ is the unit circle we have $f(e^{i\theta}) = 2 \cdot \cos\, \theta$. This yields a double-cover onto the interval $[-2, 2]$. At the same time one easily verifies f restricts to a 1-1 map in the punctured disc $0 < |z| < 1$ and it is also 1-1 in the exterior disc $|z| > 1$. In fact, in the extended w-plane we have the simple connected domain $\Omega^* = \mathbf{C} \cup \infty \setminus [-2, 2]$ and f is a conformal map from the union of the exterior disc $|z| > 1$ and $z = \infty$ onto Ω^*. The double cover of $[-2, 2]$ arises from this conformal map and Ω^* is an example of a so called slit-domain.

2.16 A counting of fixed points. In the open unit disc D we remove a finite number of disjoint Jordan domains $\{U_\nu\}$ bordered by closed Jordan curves $\gamma_1,\ldots\gamma_k$ where each γ-curve is of class C^1. Let $\Omega = D \setminus \cup \bar{U}_\nu$, i.e. we remove the family of Jordan domains. Now $\partial\Omega$ is th union of the γ-curves and the unit circle T. Let $f(z)$ be an analytic function in Ω which extends continuously to the γ-curves and in addition across the unit circle, i.e. f is analytic in a domain $\{|z| < 1+\delta\} \setminus \cup \bar{U}_\nu$ for some $\delta > 0$. We also assume that $|f(z)| < 1$ in the annulus $\{1 < |z| < 1+\delta\}$. Set

$$\phi(z) = z - f(z)$$

and assume that it has a finite number of zeros on the unit circle and a finite number of zeros in the domain Ω, while $\phi \neq 0$ on each γ-curve. Without essential loss of generality we suppose that $f(1) \neq 1$ and now the zeros of ϕ appear at points $e^{i\theta_\nu}$ for a finite set $0 < \theta_1 < \ldots < \theta_k < 2\pi$. Let e_ν be the multiplicity of the zero θ_ν. With a small $\epsilon > 0$ we get the simple closed curve T_ϵ by replacing intervals along T centered at the points θ_ν by the circular arcs which belong to the intersection of D and the circle $\{|z - e^{i\theta_\nu}| = \epsilon\}$. Under the conditions above we have:

Theorem. *One has the equality*

$$\frac{1}{2\pi i} \cdot \int_{T_\epsilon} \frac{\phi'(z)}{\phi(z)}\, dz =$$

Proof. Let N be the number of zeros of ϕ in the domain *omega*. with ϵ small it follows that the integral in star is equal to N plus the line integral over gamma-curves. now we take the larger T upper ϵ curve. line integral over that is enlarged with sum of multiplicities. at the same time a direct computation shows that the integral is 2 pi. residue calculus shows also that outer integral minus that over inner T_ϵ is equal to the sum of multiplicities. so in all the integral over inner part is a half-sum of multiplicities minus 2 pi. and at the same time the global formula ....

3. Multi-valued functions

Let Ω be an open connected subset of $\mathbf{C}$ and $D \subset \Omega$ is an open disc of some radius r centered at a point z_0. The material about power series in Chapter XX shows that $\mathcal{O}(D)$ is identified with convergent power series

$$\sum c_\nu (z-z_0)^\nu \quad : \quad \text{radius of convergenece } \geq r$$

So if $f \in \mathcal{O}(\Omega)$ its restriction to any disc $D \subset \Omega$ determines a convergent power series. These power series must be matching when two discs have a non-empty intersection which is the starting point for a general construction due to Weierstrass.

3.1 Analytic continuation along paths Let $s \mapsto \gamma(s)$ be a continuous and complex valued function with values in Ω. We do not require that $\gamma(0) = \gamma(1)$ or that γ is 1-1. The points $p = \gamma(0)$ and $q = \gamma(1)$ are called the terminal points of γ. Let $f_0 \in \mathcal{O}(D_r(p))$ for some $r > 0$, i.e. f is analytic in a small disc centered at p. Consider a strictly increasing sequence $0 = s_0 < s_1 < \ldots < s_N = 1$ and to each $p_\nu = \gamma(s_\nu)$ we choose a small disc $D_{p_\nu}(r_\nu)$ such that:

$$D_{p_\nu}(r_\nu) \cap D_{p_{\nu+1}}(r_{\nu+1}) \neq \emptyset \qquad 0 \leq \nu \leq N-1$$

Assume that for each $1 \leq \nu \leq N$ exists $f_\nu \in \mathcal{O}(D_{r_\nu}(p_\nu))$ such that

$$f_\nu = f_{\nu+1} \text{ holds in } D_{p_\nu}(r_\nu) \cap D_{p_{\nu+1}}(r_{\nu+1})$$

After N many *direct analytic continuations over pairs of intersecting discs* we arrive at f_N which is analytic in an open disc centered at $\gamma(1)$. The uniqeness of each direct analytic continuation entails that the locally defined analytic function f_N at $\gamma(1)$ is the same if we have chosen a *refined* partition of $[0,1]$. Since two coverings of γ via finite families of discs have a common refinement, we conclude that locally defined analytic function at the end-point is unique. Thus, the construction yields a map T_γ map which sends an analytic function f defined in a disc around $\gamma(0)$ to an analytic function $T_\gamma(f)$ defined is some disc centered at $\gamma(1)$. Of course, here T_γ is only defined on those f at $\gamma(0)$ which have an analytic continuation along γ in the sense of Weierstrass.

3.2 The class $M_\Omega(\mathcal{O})$. Let Ω be an open subset of $\mathbf{C}$. At each point $z \in \Omega$ we denote by $\mathcal{O}(z_0)$ the germs of analytic functions at z_0 and recall that this set is identified with power series $\sum c_\nu (z-z_0)^\nu$ with some positive radius of convergence. In $\mathcal{O}(z_0)$ we can consider those germs which have analytic continuation along *every* curve in Ω whose initial point is z_0 while the end-point is arbitrary. This leads to:

3.3 Definition *A germ $f \in \mathcal{O}(z_0)$ generates a multi-valued analytic function in Ω if it can be extended in the sense of Weierstrass along every curve $\gamma \subset \Omega$ which has z_0 as initial point. The set of all these germs is denoted by $M_\Omega(\mathcal{O})(z_0)$.*

3.4 Remark. Notice that $M_\Omega(\mathcal{O})(z_0)$ contains those germs at z_0 which are induced by *single-valued* analytic functions in Ω. If $f \in M_\Omega(\mathcal{O})(z_0)$ and γ is a curve in Ω with z_0 as initial point and z_1 as end-point, then the germ $T_\gamma(f)$ at z_1 belongs to $M_\Omega(\mathcal{O})(z_1)$. This is obviuos since if γ_1 is a curve starting at z_1 with end-point at z_2, then f extends along the composed curve $\gamma_1 \circ \gamma_1$ and one has the composition formula:

$$(*) \qquad T_{\gamma_1}(T_\gamma(f)) = T_{\gamma_1 \circ \gamma}(f)$$

3.5 The total sheaf space $\widehat{\mathcal{O}}$. We construct a big topological space $\widehat{\mathcal{O}}$ as follows: One has a map ρ from $\widehat{\mathcal{O}}$ onto $\mathbf{C}$. The inverse fiber $\rho^{-1}(z) = \mathcal{O}(z)$ for each $z \in \mathbf{C}$. An open neighborhood of a "point" $f \in \rho^{-1}(z_0)$ consists of a pair (f, D) where D is a small disc centered at z_0 such that the germ f extends to an analytic function in D. Then its induced germ at a point $z \in D$ belongs to $\rho^{-1}(z)$. The set of points in $\widehat{\mathcal{O}}$ obtained in this way yields the subset (f, D) and as D shrinks to z_0 they give by definition a fundamental system of open neighborhoods of the point f in $\widehat{\mathcal{O}}$. With this topology on $\widehat{\mathcal{O}}$ the map ρ is a *local homeomorphism* and each inverse fiber $\rho^{-1}(z_0)$ appears as a *discrete* subset of $\widehat{\mathcal{O}}$.

Remark $\widehat{\mathcal{O}}$ is the first example of a sheaf which later led to the general construction of sheaves. The construction of the sheaf topology on $\widehat{\mathcal{O}}$ yields the following elegant description of multi-valued functions.

3.6 Proposition. *Let Ω be an open and connected subset of $\mathbf{C}$. Let $z_0 \in \Omega$ and $f \in M_\Omega(\mathcal{O})(z_0)$. Then f appears in the inverse fiber $\rho^{-1}(z_0)$ of an open and connected set $\mathcal{W}(f)$ of $\rho^{-1}(\Omega)$ called Weierstrass Analytische Gebilde of the germ of this multi-valued funtion. For each $z \in \Omega$ the set $\mathcal{W}(f) \cap \rho^{-1}(z)$ consists of all germs at z obtained by analytic continuation of f along some curve with end-point at z.*

Some notations. Let f be as above. If $z \in \Omega$ we denote by $W(f : z)$ the set of germs at z which arise via all analytic continuations of f. Thus, $W(f : z)$ is equal to $\mathcal{W}(f) \cap \rho^{-1}(z)$. In addition to this we can consider the set of values at z which are attained by these germs. We have also the set

$$R_f(z) = \{T_\gamma(f)(z) \quad : \; T_\gamma(f) \in W(f : z)\}$$

Exercise. Let Ω be an open and connected subset of $\mathbf{C}$. Conclude from ther above that there exists a 1-1 correspondence between open and connected subsets of $\rho^{-1}(\Omega)$ whose ρ-image is Ω, and the family of multi-valued analytic functions in Ω.

Example Let $\Omega = \mathbf{C}$ minus the origin, i.e. the punctured complex plane. Then we have the multi-valued Log-function. At each point $z \in \Omega$ it has an infinite set of local branches which differ by integer multiples of $2\pi i$. The resulting connected set $\mathcal{W}(\log(z))$ can be regarded as a 2-dimensional connected manifold. In topology one learns that this is the *universal covering space* of Ω. In particular $\mathcal{W}(\log(z))$ is a *simply connected* manifold.

3.7 Normal families.

Let Ω be some connected open set in $\mathbf{C}$. Let $x_0 \in \Omega$ and consider some germ $f \in M\mathcal{O}(\Omega)(x_0)$. We say that f yields a bounded multi-valued function if there exists a constant K such that

$$(*) \qquad \left|T_\gamma(f)(x)\right| \leq K$$

holds for all pairs x, γ where $x \in \Omega$ and γ is any curve from x_0 to x. Suppose that $\{f_\nu\}$ is a sequence of germs in $M\mathcal{O}(\Omega)(x_0)$ which are uniformly bounded, i.e. (*) above holds for some constant K and every ν. If we to begin with consider a small open disc D centered at x_0 we get the unique single-valued branches of each f_ν in $\mathcal{O}(D)$. This family in $\mathcal{O}(D)$ is normal by the results in XXX. Passing to a subsequence we may assume that there exists a limit function $g \in \mathcal{O}(D)$, i.e. shrinking D if necessary we may assume that

(i) $$\lim_{\nu\to\infty} ||f_\nu - g||_D \to 0$$

Next, if γ is a curve in which starts at x_0 and has some end-point x we cover γ with a finite number of open discs and each f_ν has its analytic continuation along γ by the Weierstrass procedure. From the material in XXX it is clear that during these analytic continuations the local series expansions of the sequence $\{f_\nu\}$ converge uniformly and as a result we find that g has an analytic extension along γ. Hence the germ of g at x_0 belongs to $M\mathcal{O}(\Omega)(x_0)$. Moreover, the uniform convergence "propagates". For example, if γ is a closed curve at x_0 we get the sequence $\{T_\gamma(f_\nu)\}$ after the analytic continuation along γ, and similarly $T_\gamma(g)$. Then

(ii) $$\lim_{\nu\to\infty} ||T_\gamma(f_\nu) - T_\gamma(g)||_D \to 0$$

holds for a small disc D centered at the end point of γ.

4. The Monodromy Theorem

Let $f \in M\mathcal{O}(\Omega)$. If $z_0 \in \Omega$ and γ is a curve starting at z_0 we obtain the germ $T_\gamma(f)$ at the end-point z_1 of γ. The analytic continuation is obtained by the Weierstrass procedure and since γ is a compact subset of Ω it can be covered by a finite set of discs $D_0, D_1, \ldots, D_N$ where $D_\nu \cap D_{\nu+1}$ are non-empty and the analytic continuation of f is achieved by succesive direct continuations of analytic functions $\{g_\nu \in \mathcal{O}(D_\nu)\}$ where $g_\nu = g_{\nu+1}$ holds in $D_\nu \cap D_{\nu+1}$. The discs are chosen so small that they are relatively compact in Ω. If γ_1 is another curve from z_0 to z_1 which stays so close to γ that the discs $D_0, \ldots, D_N$ again can be used to perform the analytic continuation of f along γ_1, then it is clear that $T_\gamma(f) = T_{\gamma_1}$. This observation gives:

4.1 Theorem *Let (z_0, z_1) be a pair in Ω and $\Gamma(s,t)$ a continuous map from the unit square in the (s,t)-space into Ω where*

$$\Gamma(s,0) = z_0 \quad : , \gamma(s,1) = z_1 \quad : \quad 0 \le s \le 1$$

Then, if $\{\gamma_s\}$ is the family of curves defined by $t \mapsto \Gamma(s,t)$, it follows that

$$T_{\gamma_s}(f) = T_{\gamma_0}(f) \quad : \quad 0 \le s \le 1$$

Remark This is called the monodromy theorem and can be expressed by saying that analytic continuation along a curve which joins a given pair of points only depends on the *homotopy* class of the curve, taken in the family of all curves which joint the two given points. Of course, when we deal with some multi-valued function in an open set Ω we are obliged to use curves inside Ω only.

4.2 The case of finite determination. Let $f \in M(\Omega)$. If $z_0 \in \Omega$ we get the set of germs $W(f : z_0)$ at z_0. This is a subset of $\mathcal{O}(z_0)$ and we can regard the complex vector space it generates. It is denoted by $\mathcal{H}_f(z_0)$. Suppose that this complex vector space has a finite dimension k. Then we can choose a k-tuple of germs $g_1, \ldots, g_k$ in $W(f : z_0)$ which give a basis of $\mathcal{H}_f(z_0)$. Thus, one has to begin with

$$\mathcal{H}_f(z_0) = Cg_1 + \ldots + Cg_k$$

Let z_1 be another point in Ω and fix some curve γ which joins z_0 and z_1. At z_1 we get the germs $T_\gamma(g_1), \ldots, T_\gamma(g_k)$. By the remark in XXX T_γ is a *bijective map* from $W(f : z_0)$ to $W(f : z_1)$. Moreover, if $\phi = c_1 g_1 + \ldots + c_k g_k$ belongs to $\mathcal{H}_f(z_0)$ we have

$$T_\gamma(\phi) = c_1 T_\gamma(g_1) + \ldots + c_k T_\gamma(g_k)$$

Hence the k-tuple $\{T_\gamma(g_\nu)\}$ generates the vector space $\mathcal{H}_f(z_1)$. Since we also can use the inverse map $T_{\gamma^{-1}}$ it follows that the k-tuple $\{T_\gamma(g_\nu)\}$ yields a basis of $\mathcal{H}_f(z_1)$. In particular the vector space $\mathcal{H}_f(z)$ have common dimension k as z varies in the connected open set Ω. *Summing up*, we can conclude the following:

4.3 Proposition *If $f \in M(\Omega)$ has finite determination the complex vector spaces $\mathcal{H}_f(z)$ have common dimension. Moreover, one gets a basis of these by starting at any point z_0 and choose some k-tuple of C-linearly germs $g_1, \ldots, g_k$ in $W(f : z_0)$. Then we obtain a basis in $\mathcal{H}_f(z)$ for any point $z \in \Omega$ by a k-tuple $\{T_\gamma(g_\nu)\}$ where γ is any curve which joins z_0 and z.*

4.4 The case of a punctured disc Let $\dot{D} = \{0 < |z| < R\}$ be a punctured disc centered at the origin. Consider some $f \in \mathcal{MO}(\dot{D})$ of finite determination and let k be its rank. In a punctured open disc every closed curve is homotopic to a closed circle parametrized by $\theta \mapsto re^{i\theta}$. Another way to express this is that the fundamental group $\pi_1(\dot{D})$ is isomorphic to the abelian group of integers. Thus, the multi-valuedness is determined by a sole T-operator which arises when we let γ be a circle surrounding the origin in the positive sense. Given $z_0 \in \dot{D}$ we consider the **C**-linear operator

$$T_\gamma \colon \mathcal{H}_f(z_0) \mapsto \mathcal{H}_f(z_0)$$

By Jordan's decomposition theorem we can choose a basis in $\mathcal{H}_f(z_0)$ such that the matrix representing T_γ is of Jordan's normal form. This means that we have a direct sum

$$\mathcal{H}_f(z_0) = \oplus \mathcal{K}_\nu(z_0)$$

where $\{\mathcal{K}_\nu(z_0)\}$ are T_γ-invariant subspaces and the restriction of T_γ $\mathcal{K}_\nu(z_0)$ is represented by an elementary Jordan matrix $J(m, \lambda)$ for some complex number λ and $m \geq 1$. Given the pair m, λ we consider a local branch of the function

$$f(z) = z^\alpha \cdot [\operatorname{Log} z]^{m-1} \quad : \ e^{2\pi i\alpha} = \lambda$$

which for example is defined close to $z = 1$ where $f(1) = 0$. After one turn around the origin we get a new local branch of the form

$$f_1(z) = \lambda \cdot z^\alpha \cdot [\operatorname{Log} z + 2\pi i]^{m-1}$$

Continuing in this way m times we see that the local branches of f generate an m-dimensional complex vector space whose monodromy is determined by the matrix $J(m, \lambda)$. Using this fact it follows that if $f(z)$ is any local branch of a multi-valued function of finite determination, then it can be expressed as:

$$(*) \qquad f(z) = \sum_{\nu=1}^{k} \sum_j \ g_{\alpha_\nu, j}(z) \cdot z^{\alpha_\nu} \cdot [\operatorname{Log} z]^j$$

Here $0 \leq \mathfrak{Re}(\alpha_1) < \ldots < \mathfrak{Re}(\alpha_k) < 1$ and $\{j\}$ is a finite set of non-negative integers and the g-functions are *single-valued* in the punctured disc $\dot{D}$. Moreover these g-functions are uniquely determined provided a specific local branch of the Log-function is chosen. For example, when f is a local branch at some real point $0 < a < R$ where $\operatorname{Log} a$ is chosen to be real.

5. Homotopy and Covering spaces

Let X be a metric space, i.e. the topology is defined by some distance function. By a curve in X we mean a continuous map γ from the closed unit interval $[0, 1]$ into X. In general γ need not be 1-1. The initial point is $\gamma(0)$ and the end point is $\gamma(1)$. If $\gamma(0) = \gamma(1)$ we say that γ is a *closed* curve. We say that X is *arcwise connected* if there to each pair of points x_0, x_1 exists some curve γ with $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$.

A notation. Given a point $x_0 \in X$ we denote by $\mathcal{C}(x_0)$ the family of all closed curves γ where $\gamma(0) = \gamma(1) = x_0$.

5.1 Definition. *A pair of closed curves γ_0 and γ_1 in $\mathcal{C}(x_0)$ are homotopic if there exists a continuos map Γ from the unit square $\square = \{(t, s) \;\; : 0 \leq t, s \leq 1\}$ into X such that*

$$\Gamma(t, 0) = \gamma_0(t) \text{ and } \Gamma(t, 1) = \gamma_1(t) \quad \Gamma(0, s) = \Gamma(1, s) = x_0 \;:\; 0 \leq s \leq 1$$

It is clear that homotopy yields an equivalence relation on $\mathcal{C}(x_0)$. If $\gamma \in \mathcal{C}(x_0)$ then $\{\gamma\}$ denotes its homotopy class. Next, if γ_0 and γ_1 are two closed curves at x_0 we get a new closed curve γ_2 defined by

$$\gamma_2(t) = \gamma_1(2t) : \; 0 \leq t \leq \frac{1}{2} \text{ and } \gamma_2(t) = \gamma_2(2t - 1) : \; \frac{1}{2} \leq t \leq 1$$

We refer to γ_2 as the composed curve and it is denoted by $\gamma_1 \circ \gamma_0$. One verifies easily that the homotopy class of γ_2 depends upon $\{\gamma_1\}$ and $\{\gamma_1\}$ only. In this way we obtain a composition law on the set of homotopy classes of closed curves at x_0 defined by

$$\{\gamma_1\} \cdot \{\gamma_0\} = \{\gamma_1 \circ \gamma_0\}$$

One verifies easily that this composition satisfies the associative law. A neutral element is the closed curve γ_* for which $\gamma_*(t) = x_0$ for every t. Finally, if $\gamma(t)$ is any closed curve at x_0 we get a new closed curve by reversing the direction, i.e. set

$$\gamma^{-1}(t) = \gamma(1 - t)$$

Exercise. Show that the composed curve $\gamma^{-1} \circ \gamma$ is homotopic to γ_*.

5.2 The fundamental group. The construction of composed closed curves and the exercise above show that homotopy classes of closed curves at x_0 give elements of a group to be denoted by $\pi_1(X : x_0)$.

Remark. The group $\pi_1(X : x_0)$ is intrinsic in the sense that it does not depend upon the chosen point x_0. Namely, let x_1 be another point in X and fix a curve λ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$. Then we obtain a map from $\mathcal{C}(x_1)$ to $\mathcal{C}(x_0)$ defined by

(i) $$\gamma \mapsto \lambda^{-1} \circ \gamma \circ \lambda$$

One verifies that (i) sends homotopic curves to homotopic curves and by considering homotopy classes we obtain an isomorphism between $\pi_1(X : x_0)$ and $\pi_1(X : x_1)$. Hence there exists an intrinsically defined group denoted by $\pi_1(X)$. It is called the fundamental group of the metric space X. If $\pi_1(X)$ is reduced to a single element, i.e. when all closed curves in $\mathcal{C}(x_0)$ are homotopic we say that X is *simply connected*.

Exercise. Let x_0 and x_1 be two distinct points in X. Denote by $\mathcal{C}(x_0, x_1)$ the family of curves γ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Two such curves γ_0 and γ_1 are homotopic if there exists a continuos map Γ from the square $\square$ such that

$$\Gamma(t,0) = \gamma_0(t) \text{ and } \Gamma(t,1) = \gamma_1(t) \quad \Gamma(0,s) = x_0 \text{ and } \Gamma(1,s) = x_1 \ : \ 0 \leq s \leq 1$$

Show that a pair γ_0 and γ_1 are homotopic in $\mathcal{C}(x_0, x_1)$ if and only if the closed curve $\gamma_1^{-1} \circ \gamma_0$ is homotopic to γ^* in $\mathcal{C}(x_0)$ where

$$\gamma_1^{-1} = \gamma_1(1-t)$$

In particular each pair of curves in $\mathcal{C}(x_0, x_1)$ are homotopic if X is simply connected.

5.3 Covering maps.

Let X and Y be two arcwise connected metric spaces. A continuous map ϕ from X onto Y is a local homeomorphism if the following hold: For each $y_0 \in Y$ there exists an open neighborhood U such that the inverse image $\phi^{-1}(U)$ is a union of pairwise disjoint open sets $\{U^*_\alpha\}$ and the restriction of ϕ to each U^*_α is a homeomorphism from this set onto U.

5.4 Lifting of curves. Let $\phi\colon X \to Y$ be a local homeomorphism where we assume that $\phi(X) = Y$. Let γ be a curve in Y defined by a continuous map $t \to \gamma(t)$ from the closed unit interval $[0,1]$ into Y with some initial point $y_0 = \gamma(0)$ and some end-point $y_1 = \gamma(1)$. The case $y_0 = y_1$ is not excluded, i.e. γ may be a closed curve. Next, in X we chose a point x_0 such that $\phi(x_0) = y_0$. By assumption there exists an open neighborhood U of y_0 in X a unique open neighborhood U^* of x_0 such that $\phi\colon U^* \to U$ is a homeomorphism. Since $t \to \gamma(t)$is continuous there exists some $\delta > 0$ such that

(i) $$\gamma(t) \in U\,, \quad 0 \leq t \leq \delta$$

Then we get a *unique* curve γ^* in X defined for $0 \leq t \leq \delta$ such that

(ii) $$\phi(\gamma^*(t)) = \gamma(t)\,, \quad 0 \leq t \leq \delta \quad \text{and } \gamma^*(0) = x_0\,.$$

If this lifting process can continued for all $0 \leq t \leq 1$ we say that γ has a lifted curve γ^*. This means that there exists a curve $t \mapsto \gamma^*(t)$ from $[0,1]$ into X such that

(*) $$\phi(\gamma^*(t)) = \gamma(t)\,, \quad 0 \leq t \leq 1 \quad \text{and } \gamma^*(0) = x_0\,.$$

Exercise. Show that the curve γ^* is unique if it exists. The hint is to use that ϕ is a local homeomorphism.

The whole discussion above leads to

5.5 Definition. *A local homeomorphism $\phi\colon X \to Y$ is called a covering map of $\mathcal{L}$-type if the following hold: For each pair of points $y_0 \in Y$ and $x_0 \in \phi^{-1}(0)$, every* *curve γ in Y with initial point y_0 and arbitrary end-point y can be lifted to a curve in X with initial point x_0.*

5.6 The case when X is simply connected. Assume this and let $\phi\colon X \to Y$ be a covering map of $\mathcal{L}$-type. Let $y_0 \in Y$ and choose some point $x_0 \in \phi^{-1}(y_0)$. Next, let γ be a closed curve in Y with $\gamma(0) = \gamma(1) = y_0$. By assumption there exists a unique lifted curve γ^* in X with $\gamma^*(0) = x_0$. Suppose that $\gamma^*(1) = x_0$, i.e. the lifted curve is closed. Since X is simply connected it is homotopic to the trivial curve which stays at x_0, i.e. there exists a continuos map Γ^* from $\square$ into X such that

(i) $\quad \Gamma^*(t,0) = \gamma^*(t) \,\text{and}\, \Gamma^*(t,1) = x_0 \quad \Gamma^*(0,s) = \Gamma(1,s) = x_0 \,:\, 0 \leq s \leq 1$

Now $\Gamma(t,s) = \phi(\Gamma^*(t,s)$ is a continuous map from $\square$ into Y and from (i) we see that Γ yields a homotopy between γ_0 and γ_1. Using this observation we arrive at:

5.7 Proposition. *Let γ_0 and γ_1 be two closed curves at y_0. Then they are homotopic if and only if $\gamma^*(1) = \gamma^*(1)$.*

Proof. We have already seen that if $\gamma^*(1) = \gamma^*(1)$ then the two curves are homotopic. Conversely, if they are homotopic we get a continus map $\Gamma(s,t)$ from $\square$ into Y and for each $0 \leq s \leq 1$ we have the closed curve $\gamma_s(t) = \Gamma(t,s)$ at y_0. Since the inverse fiber $\phi^1{}_1(y_0)$ by assumption is a discrete set in X, it follows by continuity and the unique path lifting that $s \mapsto \gamma_s^*(1)$ is constant and hence $\gamma_0^*(1) = \gamma_1^*(1)$.

5.8 Conclusion. Proposition 5.7 shows that homotopy classes of closed curves γ at y_0 are in a 1-1 correspondence with their end-points in X. Notice also that if x belongs to $\phi^{-1}(y_0)$ then the arc-wise connectivity of X gives a curve ρ where $\rho(0) = x_0$ and $\rho(1) = x$. Now $\gamma(t) = \phi(\rho(t))$ is a closed curve at y_0 and here $\gamma^*(t) = \rho(t)$ and hence x appears as an end-point for at least one closed curve at y_0. Identifying $\pi_1(Y)$ with homotopy classes of closed curves at y_0 we have therefore proved the following:

5.9 Theorem. *The map $\gamma \to \gamma^*(1)$ yields a bijective correspondence between the fundamental group $\pi_1(Y)$ and the inverse fiber $\phi^{-1}(y_0)$.*

Exercise. Let X and Z be two simply connected metric spaces. Suppose that $\phi\colon X \to Y$ and $\psi\colon Z \to Y$ are two covering maps which both belong to the class $\mathcal{L}$. Fix some $y_0 \in Y$. Choose $x_0 \in \phi^{-1}(y_0)$ and $z_0 \in \psi^{-1}(y_0)$. Next, let $y \in Y$ and consider some curve γ in Y with $\gamma(0) = y_0$ and $\gamma(1) = y$. Its unique lifted curve to X is denoted by γ^* and we get the end-point

$$\gamma^*(1) \in \phi^{-1}(y)$$

Similarly. we get a unique lifted curve γ^{**} in Z and the end-point

$$\gamma^{**}(1) \in \psi^{-1}(y)$$

From the above these two end-points only depend on the homotopy class of γ. Use this to conclude that we obtain a *unique and bijective* map from the discrete fiber $\phi^{-1}(y)$ to $\psi^{-1}(y)$. Moreover, as y varies in Y this gives a unique homeomorphism G from X onto Z with $G(x_0) = z_0$.

6. The uniformisation theorem.

Introduction. Let Ω be a connected open subset of $\mathbf{C}$. If the closed complement contains at least two points there exists a a covering map $f\colon D \to \Omega$ of $\mathcal{L}$-type given by an analytic function. This will be proved in Chapter 6. Here we take this existence for granted and analyze some consequences. More precisely, in Chapter VI we prove Riemann's mapping theorem for connected domains which asserts the following:

6.1 Theorem. *For every $z_0 \in \Omega$ there exists a unique analytic covering map f of $\mathcal{L}$-type where $f(0) = z_0$ and $f'(0)$ is real and positive.*

6.2 The multi-valued inverse to f. We take the theorem above for granted and discuss some consequences. Let f be a an analytic covering map as above. To distinguish the z-coordinate in Ω from D we let w be the complex coordinate in D. To begin with f yields a biholomorphic map from a small open disc D_* centered at the origin in D to a small open neighborhood U_0 of z_0. It gives the inverse analytic function $F(z)$ defined in U_0 such that

$$F(f(w)) = w \quad w \in D\,.$$

Next, let γ be a curve in Ω with $\gamma(0) = z_0$. Since f is of $\mathcal{L}$-type there exists a unique lifted curve γ^* in D with $\gamma^*(0) = 0$. Now the germ of F at z_0 can be continued analytically along γ where

(i) $$T_{\gamma(t)}(F(\gamma(t)) = \gamma^*(t) \quad : \ 0 \leq t \leq 1$$

Hence we get a multi-valued analytic function F in Ω. It gives an inverse to f in the following sense: Let $w \in D$ and consider the curve $t \mapsto t \cdot w$ in D. Now $t \mapsto f(t \cdot w)$ is a curve γ in Ω and by the construction (i) we have

(ii) $$T_{\gamma(t)}(F(f(t \cdot w)) = tw \quad : \ 0 \leq t \leq 1$$

We may express this by saying that the composed function $F \circ f$ is the identity on D.

6.3 Constructing single-valued functions. Consider the situation in Theorem 6.2, i.e. f is a covering map of $\mathcal{L}$-type from D onto Ω. Let $g(w)$ be some analytic function in D whose range $g(D) \subset \Omega$ and $g(0) = z_0$. We use F to construct a single-valued analytic function $F \circ g$ in D. Namely, let $w \in D$ which gives the curve γ_w parametrized by $t \mapsto g(t \cdot w)$ in Ω where $\gamma_w(0) = z_0$. We can continue F along this curve and when $t = 1$ we get the value

$$T_{\gamma_w(1)}(F(g(w))$$

It is clear that this gives an analytic function in D defined by

$$F \circ g(w) = T_{\gamma_w(1)}(F(g(w))$$

This construction can be performed for every $g \in \mathcal{O}(D)$ such that $g(0) = z_0$ and $g(D) \subset \Omega$. Hence we have proved the following:

6.4 Proposition. *Let $\mathcal{O}_*(D : \Omega)$ denote the family of analytic functions g in D where $g(0) = z_0$ and $g(D) \subset \Omega$. Then there exists a map from $\mathcal{O}_*(D : \Omega)$ into $\mathcal{O}(D)$ given by:*

$$g \mapsto F \circ g\,.$$

Here $F \circ g(0) = 0$ and the range $(F \circ g)(D) \subset D$.

6.5 Möbius transforms. Let f be a as in Theorem 6.1 and identify the fundamental group $\pi_1(\Omega)$ with homotopy classes of closed curves at z_0. Theorem xx gives a bijective map between elements in the group $\pi_1(\Omega)$ and the discrete subset $f^{-1}(z_0)$ of D. Let a be a point in this fiber, i.e. here $f(a) = z_0$. For each $0 \leq \theta < 2\pi$ we get an analytic function in D defined by

$$g(w) = f(e^{i\theta} \cdot \frac{w+a}{1+\bar{a}\cdot w})$$

Here $g(0) = f(a) = z_0$ and the complex derivative at $w = 0$ becomes

$$g'(0) = f'(a) \cdot e^{i\theta} \cdot (1 - |a|^2)$$

We can choose θ so that $f'(a) \cdot e^{i\theta}$ is real and positive. With this choice of θ it follows from the uniqueness in Theorem 6.1 that $g = f$. Hence the function f satisfies

$$(*) \qquad f(w) = f(e^{i\theta} \cdot \frac{w+a}{1+\bar{a}\cdot w})$$

This means that f enjoys certain invariance properties. We return to a discussion at the end of section xx.

6.6 Inverse multi-valued functions.

Let ϕ be an analytic function defined in some open and connected subset U of $\mathbf{C}$. We assume that the derivative is $\neq 0$ at every point and get the open image domain $\Omega = \phi(U)$. Since ϕ is locally conformal it is in particular a local homeomorphism. We add the hypothesis that ϕ yields a covering map of $\mathcal{L}$-type. Consider some $\zeta_0 \in \Omega$ and put $x_0 = \phi(\zeta_0)$. We get a germ $f(x)$ of an analytic function at x_0 using the local inverse of ϕ, i.e. since $\phi'(\zeta_0) \neq 0$ there exists a small open disc $D_\delta(\zeta_0)$ such that

$$f(\phi(\zeta) = \zeta \quad : \quad |\zeta - \zeta_0| < \delta$$

In fact, we simply find the convergent power series

$$f(x) = \sum c_\nu (x - x_0)^\nu$$

where $c_0, c_1, \ldots$ are determined so that

$$\sum c_\nu \big(\phi(\zeta) - \phi(\zeta_0)\big)^\nu = \zeta$$

6.7 Proposition. *The germ f at x_0 extends to a multi-valued analytic function in U.*

Proof. Let γ be a curve in U having x_0 as initial point. The lifting lemma gives a unique curve γ^* in Ω. The required analytic continuation of f along γ now follows when we apply the Heine-Borel Lemma cover the compact set γ with a finite set of discs which are homemorphic images of discs in Ω whose consequtive union covers γ^*. Then we use that ϕ is everywhere analytic. The result is that the germ $T_\gamma(f)$ at the end-point $\zeta_1 = \gamma(1)$ satisfies

$$T_\gamma(f)(\phi(x)) = x$$

where x is close to the point $\phi(\gamma^*(1))$ in Ω. So in particular

$$T_\gamma(f)(\gamma(1)) = \gamma^*(1)$$

which clarifies how to determine values of the multi-valued analytic function.

6.8 Remark. It is instructive to consider some specific cases. Consider the entire function $\phi(\zeta) = e^\zeta$. With $\Omega = \mathbf{C}$ the image domain U is the punctured complex plane. If we take $x_0 = 1$ and $\zeta_0 = 0$ we find that f is the multi-valued Log-function where we start with the local branch at $x_0 = 1$ for which $\log 1 = 0$. Next, let us regard the polynomial $\phi(\zeta) = \zeta^2$. in order to get a covering we must exclude the origin to ensure that $\phi'(\zeta) \neq 0$. So if $\Omega = \mathbf{C} \setminus \{0\}$ we get a covering whose image set U also becomes the punctured complex plane. In this case the inverse fiber consists of two points and the function $f(z)$ is the multi-valued square-root of z.

6.9 Constructing single-valued functions.

Let Ω be a connected open set and consider some multi-valued analytic function F in Ω. Let U be some open and *simply connected* set. Consider some $h \in \mathcal{O}(U)$ whose image set $h(U)$ is contained in Ω. No further conditions on h are imposed, i.e. the inclusion $h(U) \subset \Omega$ may be strict and the derivative of h may have zeros. Using F we produce single valued analytic functions in U by the following procedure. Let us fix a point $\zeta_0 \in U$ and put $x_0 = h(\zeta_0)$. At x_0 we haver the family of local branches of F. Let f_* be one such local branch. Next, let γ be a curve in U where x_0 is the initial point and $x = \gamma(1)$ denotes the end-point. In Ω we get the image curve

(i) $$t \mapsto h(\gamma(t))$$

Now f_* has an analytic continuation along the curve in (i). When $t = 1$ we arrive at the endpoint $\gamma(1)$ which we denote by x. At x we can evaluate the local branch $T_\gamma(f_*)$. Next, let $\gamma_1(t)$ be another curve in U with the same end-point x as γ. By assumption U is simply connected which means that the curves γ and γ_1 are homotopic. It is clear that the homotopy in U implies that the two image curves obtained via (i) are homotopic in the curve family in Ω which joint x_0 and x. It follows that the image curves constructed via (i) are homotopic The monodromy theorem applied to F implies that

(ii) $$T_\gamma(f_*)(x) = T_{\gamma_1}(f_*)(x)$$

We conclude that (ii) gives an analytic function in U. Denote by $\mathcal{O}(U)_\Omega$ the family of analytic functions in U whose image is contained in Ω. With these notations the discussion above gives:

6.10 Proposition. For each point $\zeta_0 \in U$ there exists a map

$$\rho\colon\ \mathcal{O}(U)_\Omega \times M\mathcal{O}(\Omega)(x_0) \to \mathcal{O}(U)$$

where $x_0 = h(\zeta_0)$ and for a pair $h \in \mathcal{O}(U)_\Omega$ and $f_* \in M\mathcal{O}(\Omega)(x_0)$ the analytic function $\rho(h, f_*)$ satisfies

$$\rho(h, f_*)(\zeta) = T_\gamma(f_*)(h(\zeta)) \quad : \quad \zeta \in U$$

where γ is the h-image of any curve in U which joins ζ_0 with ζ.

Remark. Keeping h fixed we notice that the map $f_* \to \rho(h, f_*)$ is a $\mathbf{C}$-algebra homomorphism from the complex $\mathbf{C}$-algebra $M\mathcal{O}(\Omega)(x_0)$ into $\mathcal{O}(U)$.

7. The p^*-function.

We construct a special harmonic function which will be used to get solutions to the Dirichlet problem in XXX. Let Ω be an open and connected set in $\mathbf{C}$. Its closed complement has connected components. Let E be such a connected component. To each $a \in E$ we get the winding number $\mathfrak{w}_a(\gamma)$. If b is another point in E which is sufficiently close to a it is clear that

$$|\frac{1}{\gamma(t)-a} - \frac{1}{\gamma(t)-b}| < |\frac{1}{\gamma(t)-a}|$$

Rouche's theorem from 1.4 implies that $\mathfrak{w}_a(\gamma) = \mathfrak{w}_b(\gamma)$, i.e. for every closed curve γ in Ω, the winding number stays constant in each connected component of $\mathbf{C} \setminus \Omega$. This enable us to construct single valued Log-functions in Ω. Namely, let $a \in E$ where E is a connected componen in the complement of Ω. Consider $f = \text{Log}\,(z-a)$ and choose a single valued branch f_* at some point $z_0 \in \Omega$. If $\gamma \subset \Omega$ is a closed curve with initial point at z_0 the analytic continuation along γ of the Log-function gives:

(1) $$T_\gamma(f_*) = f_* + 2\pi i \cdot \mathfrak{w}_a(\gamma)$$

Next, if b is another point in E we consider $g_* = \text{Log}(z-b)$ and obtain

(2) $$T_\gamma(g_*) = g_* + 2\pi i \cdot \mathfrak{w}_b(\gamma)$$

Since $\mathfrak{w}_b(\gamma) = \mathfrak{w}_b(\gamma)$ it follows that

(3) $$T_\gamma(f_*) - T_\gamma(g_*) = f_* - g_*$$

Hence the difference $\text{Log}(z-a) - \text{Log}(z-b)$ is a *single valued* analytic function in Ω. Taking is exponential we find $\Psi(z) \in \mathcal{O}(\Omega)$ such that

(4) $$e^{\Psi(z)} = \frac{z-a}{z-b}$$

Since $a \neq b$ we see that $\Psi(z) \neq 0$ for all $z \in \Omega$. Next, we get the harmonic function defined in Ω by

(*) $$p(z) = \mathfrak{Re}\big(\frac{1}{\Psi(z))}\big) = \frac{\mathfrak{Re}(\Psi(z)}{|\Psi(z)|^2}$$

Notice that $\mathfrak{Re}(\Psi(z)) = \text{Log}\,|z-a| - \text{Log}|z-b|$ and since $\text{Log}\,|z-a| \to -\infty$ as $z \to a$ we see from (*) that

(**) $$\lim_{z \to a} p(z) = 0$$

Notice also that $\Psi(z)$ exends to a continuous function on $\bar{\Omega} \setminus (a,b)$ and we can choose a small $\delta > 0$ such that

(ii) $$\text{Log}|z-a| - \text{Log}\,|z-b| < -1 \quad : \quad |z-a| \leq \delta$$

Then (i) and (ii) give

7.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the single point a. Then there exists a harmonic function $p^*(z)$ in Ω for which*

$$\lim_{z \to a} p^*(z) = 0$$

and there exists $\delta > 0$ such that

$$\max_{\{|z-a|=r\} \cap \Omega} p^*(z) < 0 \quad : \quad z \in D_a(r) \cap \Omega$$

8. Extensions by reflection

Introduction. *Das Spiegelungsprinzip* is due to H. Schwartz. First we describe the standard case. Let $f(z)$ be analytic in the upper half plane $U_+ = \{\mathfrak{Im}\, z > 0\}$. Let $J(a,b) = \{a < x < b\}$ be an interval, on the real axis. Suppose that f extends to a continuous function to this open interval and takes real values. In the lower half-plane U_- we get the analytic function

$$\text{(i)} \qquad f_*(z) = \bar{f}(\bar{z})$$

By the result in XX the two functions are analytic continuations of each other over (a,b). This means that f has an analytic extension to the open set $\Omega = \mathbf{C} \setminus J$, where $J_* = (-\infty, a] \cup [b, +\infty)$ is the closed complement of (a,b) on the x-axis. Next, suppose that $e^{i\theta} f(z)$ extends to a real-valued function on (a,b) for some θ. After multiplication with $e^{-i\theta}$ we get an extension of f. That is, one has only to require that the argument of f is constant to obtain an analytic continuation. Suppose now that the argument of f is constant over a family of pairwise disjoint intervals $\{J(a_\nu, b_\nu)\}$. Then we get analytic continuations across each interval. In particular one has:

8.1 Theorem. *Let $a_1 < \ldots < a_N$ be a finite set of real numbers and assume that f extends to a continuous function on each of the intervals*

$$J_0 = (-\infty, a_1) \quad : \; J_\nu = (a_\nu, a_{\nu+1}) : 2 \leq \nu \leq N_1 \quad : \; J_N = (a_N, +\infty)$$

and on every such interval the argument of f is some constant. By successive reflections over there intervals we obtain a in general multi-valued analytic of f_*^ν defined in $\mathbf{C} \setminus (a_1, \ldots, a_N)$.

Example. In the upper half-plane U_+ we consider the analytic function

$$f(z) = \sqrt{z} \cdot \sqrt{1-z}$$

The single-valued branches of the root functions are chosen so that

$$\sqrt{z} = \sqrt{r} \cdot e^{i\theta/2} \quad : \quad \sqrt{z-1}) = \sqrt{1 + r^2 - 2r \cdot \cos\theta} \cdot e^{i\phi} \quad : z = re^{i\theta}$$

where $0 < \theta < \pi$ and ϕ is the outer angle of the triangle in figure xx. So here $0 < \phi < \pi$ holds. As we approach a point $0 < x < 1$ we get the boundary value

$$f(x) = \sqrt{x} \cdot i \cdot \sqrt{1-x}$$

Now get the analytic continuation f_*^1 across the interval $J_1 = (0,1)$ which becomes an analytic function defined in the lower half-plane U_- by

$$f_*^1(z) = -\bar{f}(\bar{z})$$

Notice that the minus-sign appears in order that $f(x) = f_*^1(x)$ holds for $0 < x < 1$. Suppose now that $x > 1$. Then we get

$$\lim_{y \to 0} f_*^1(x - iy) = \lim_{y \to 0} -\bar{f}(1(x+iy) = -\sqrt{x} \cdot \sqrt{x-1}$$

So f and f_*^1 do not agree on the real interval $(1, +\infty)$. At the same time f_*^1 can be continued analytically across $(1, +\infty$ and gives an analytic function f_+** defined in U_+ where we obtain

$$f_+^{**}(z) = -f(z) \quad : \quad z \in U_+$$

Another analytic continuation of f_*^1 takes place across $(-\infty, 0)$. When $x < 0$ we have

$$\lim_{y \to 0} f_*^1(x - iy) = \lim_{y \to 0} -f(1(x + iy) =$$

After f has been extended to the lower half-plane where we get an analytic function denoted by f_* we notice that f_* by the construction also has boundary values with a constant argument as we approach points on the real axis form below. So f_* also extends to the upper half-plane where we encounter a new analytic function $f^*(z)$. Next, we can continue f^* to the lower half-plane and so on. The result is that f extends to a multi-valued function in $\mathbf{C} \setminus (0, 1)$.

8.2 The use of conformal maps. For local extensions there exists a general result. Let D be an open disc and γ a Jordan arc which joins two points on ∂D and separates $D \setminus \gamma$ into a pair of disjoint Jordan domains. Let f is analytic in one of the Jordan domains, say D^*. Assume also that f extends to a continuous and real-valued function on γ. Now there exists a conformal map from D^* to the upper half-plane and using this it follows that f extends analytically across γ. of course, the extension f_* will in general only exist in a small domain close to γ, i.e. this is governed via the conformal mapping. But here exists at least a locally defined analytic continuation across γ.

8.3 Boundary values on circles Let $f(z)$ be as above and suppose it extends continuously to γ where the absolute value is constant, say 1. Using a conformal map from D_* to the unit disc we may assume that γ is an interval of the unit disc D and f is analytic in a small region $U \subset D$ where γ appears as a relatively open subset of ∂U. By hypothesis $f(\gamma)$ is a subset of another unit circle and using a conformal map from the disc bordered by this unit circle we get the situation in 7.2. and conclude that f continues analytically across γ. More generally, using a locally defined conformal map there exists an analytic extension of f across γ if we only assume that the continuous boundary values of f on γ are contained in some locally defined *real-analytic curve*. Finally, by a two-fold application of conformal mappings we get the following quite general result:

8.4 Theorem. *Let $f(z)$ be analytic in a Jordan domain Ω and suppose that $\gamma$is an open arc of $\partial\Omega$ such that f extends continuosly from Ω to $\Omega \cup \gamma$ and the restriction $f|\gamma$has a range $f(\gamma)$ contained in a simple real-analytic curve γ^*. Then f extends analytically across γ, i.e. there exists an open and connected neighborhood U of γ such that the original f-function extends to the connected domain $\Omega \cup U$.*

Remark. Theorem 8.4 follows from the fact that if Ω_2 and Ω_2 are two Jordan domains whose boundaries both are *real analytic* closed Jordan curves, then a conformal map from Ω_1 to Ω_2 extends to a conformal map from an open neighborhood of $\bar{\Omega}_1$ to an open neighborhood of $\bar{\Omega}_2$.

9. The elliptic modular function

Introduction. We shall construct an analytic function $\phi(z)$ in the upper half-plane $U_+ = \mathfrak{Im}\, z > 0$ whose complex derivative of ϕ is everywhere $\neq 0$ and the image $\phi(U_+)$ is equal to the connected open set $\Omega = \mathbf{C} \setminus \{0,1\}$, i.e. the two points 0 and 1 are removed from the complex plane. So ϕ is locally conformal but not 1-1. Moreover the function is invariant under a group of Möbius maps which preserve U_+. Consider first the map

$$z \mapsto \frac{z}{2z+1} = w \tag{1}$$

Then

$$\mathfrak{Im}(w) = \frac{)\mathfrak{m}(z)}{|2z+1|^2} > 0$$

so (i) is a map from U_+ into itself asnd is conformal because we have the inverse map

$$z = \frac{w}{1-2w}$$

Another conformal mapping on U_+ is $z \mapsto z+1$. Together with (i) it generates a group $\mathcal{F}$ of conformal mappings on U_+. We are going to construct an analytic function ϕ in U_+ which is $\mathcal{F}$-invariant, i.e.

$$\phi(z) = \phi(z+1) = \phi(\frac{z}{2z+1}) \quad : \, z \in U_+ \tag{2}$$

and the range

$$\phi(U_+) = \mathbf{C} \setminus \{0,1\} \tag{3}$$

When (ii) holds one says that ϕ is an $\mathcal{F}$-automorphic function and the subsequent construction of ϕ will show that for each point $w \in U_+$ one has the equality

$$\mathcal{F}(w) = \{z \in U_+ : \, \phi(z) = w\} \tag{4}$$

where the left hand side is the $\mathcal{F}$-orbit of ϕ.

The construction of ϕ. Consider the simply connected domain:

$$V_0 = U_+ \cap \, |z - 1/2| > 1/2 \cap \{0 < \mathfrak{Re}\, z < 1\}$$

Here ∂V_0 consists of three pieces: The vertical half-lines $\ell_0 = \{x = 0 : \, y > 0\}$, and $\ell_0 = \{x = 1 : \, y > 0\}$, together with half-circle

$$T_0^+ = \{1/2 + 1/2 \cdot e^{i\theta}\} \quad : \quad 0 < \theta < \pi$$

Riemann's mapping theorem gives a conformal mapping ϕ_0 from V_0 onto U_+ such that ϕ_0 yields bijective maps from ℓ_0 onto $(-\infty, 0)$ and T_0 onto $(0,1)$, and finally ℓ_1 onto $(1, +\infty)$. In particular $\phi_0(0) = 0$ and $\phi_0(1) = 1$ hold and as $z \to \infty$ in V_0 then $\phi_0(z) \to \infty$ in U_+. See figure 1 for an illustration of this conformal mapping. Following Schwarz we shall perform reflections to obtain an analytic function ϕ_* defined in the domain

$$V_* = \{0 < \mathfrak{Re}\, z < 1\} \cap \mathfrak{Im}\, z > 0\}$$

whose image is equal to $\mathbf{C} \setminus \{0,1\}$.

The first reflection. Consider the open set W_0 defined by

$$W_0 = \{z \quad : \frac{\bar{z}}{2\bar{z}-1} \in V_0\} \tag{1}$$

Exercise. Show that W_0 is a simply connected domain bordered by three circular arcs, i.e.

$$W_0 = \{|z - 1/4| > 1/4\} \cap \{|z - 3/4| > 1/4\} \cap \{|z - 1/2| < 1/2\}$$

A hint is that if $z = iy \in \ell_0$ thrn

$$\text{(i)} \qquad w = \frac{-iy}{-2iy - 1} = \frac{iy}{2iy - 1} \implies |w - 1/4| = 1/4$$

Similarly, if $z = 1 + iy \in \ell_1$ we find

$$\text{(ii)} \qquad w = \frac{1 - iy}{2 - 2iy - 1} = \frac{1 - iy}{1 - 2iy} \implies |w - 3/4| = 1/4$$

Finally we have the half-circle $T_0^+ = \{|z - 1/2| = 1/2 \cap U_+\}$ and with $z = 1/2 + 1/2e^{i\theta}$ we get

$$\text{(iii)} \qquad w = \frac{1/2 + 1/2e^{-i\theta}}{1 + e^{-i\theta} - 1} = \frac{e^{i\theta} + 1}{2} \implies |w - 1/2| = 1/2$$

Next, in W_0 a two-fold complex conjugation gives the analytic function $g_0(z)$ defined by

$$g_0(z) = \bar{\phi}_0(\frac{\bar{z}}{2\bar{z} - 1})$$

At the same time we notice that ϕ_0 is real-valued on T_0^+ and then (iii) above gives

$$\text{(iv)} \qquad g_0(z) = \phi_0(z) : \; z \in T_0^+$$

By Schwarz' reflection principle the pair $\phi_0, g_0)$ yields an analytic function ϕ_1 defined in the simply connected set

$$V_1 = V_0 \cup T_0^+ \cup W_0$$

The second step. Above we constructed an analytic function ϕ_1 in V_1. Since $\phi_0(iy)$ and $\phi_0(1 + iy$ takes real values for all $y > 0$, we see that ϕ_1 takes real values on the half-circles

$$T_{10}^+ = \{|z - 1/4| = 1/4 \cap U_+\} \quad : T_{11}^+ = \{|z - 3/4| = 1/4 \cap U_+\}$$

Again we can apply Schwarz reflection principle to get an analytic extensions of ϕ_1 across each of these half-circles. More precisely, put

$$W_{10} = \{\bar{z} \quad : \quad \frac{\bar{z}}{4\bar{z} - 1} \in V_1\}$$

If $z = 1/4 + e^{i\theta}/4$ belongs to T_{10}^+ we obtain

$$\frac{1/4 + e^{-i\theta}/4}{e^{-i\theta}} = 1/4 + e^{i\theta}/4$$

Reflection gives an analytic function g_{10} in W_{10} defined by

$$g_{10}(z) = \bar{\phi}_1(\frac{\bar{z}}{4\bar{z} - 1})$$

The reader may also verify that

$$W_{10} = \{|z - 1/8| > 1/8\} \cap \{|z - 3/8| > 1/8\} \cap \{|z - 1/4| < 1/4\}$$

To get an extension across T_{11}^+ we put

$$W_{11} = \{\bar{z} \quad : \quad \frac{\bar{z}}{4\bar{z}-3} \in V_1\}$$

If $z = 3/4 + e^{i\theta}/4$ we obtain

$$\frac{3/4 + e^{-i\theta}}{3 + e^{-i\theta} - 3} = 3/4 + e^{i\theta}/4$$

This gives an analytic extension where $g_{11}(z)$ is defined in W_{11} by

$$g_{11}(z) = \phi_1(\frac{\bar{z}}{4\bar{z}-3})$$

As a result we get an analytic function ϕ defined in the simply connected domain bordered by ℓ_0 and ℓ_1 and the union of four half-circles of radius $1/8$ centered at the points $1/8, 3/8, 5/8, 7/8$. See figure § xx for an illustration.

At this stage it is clear how one proceeds to construct larger and larger domains $\{W_k\}$ and analytic functions ϕ_k via reflections over suitable half-circles. See figure XX. The result is an analytic function $\phi_*(z)$ defined in the half-strip

$$\square_+ = 0 < \mathfrak{Re}\, z < 1\} \cap \{\mathfrak{Im}\, z < 0\}$$

The construction shows that the derivative of the analytic function ϕ_* in $\square_+$ is everywhere $\neq 0$ and the range is $\mathbf{C} \setminus \{0,1\}$ Next, since ϕ_* is real-valued on ℓ_0 and ℓ_1 it extends by reflection over these two vertical lies and the reader may verify that we obtain an analytic function ϕ defined in the whole upper half-plane which is 1-periodic, i.e.

$$\phi(z+1) = \phi(z)$$

Exericse. Verify that the constructions imply thaty

$$\phi(z) = \phi(\frac{z}{2z+1}) = \phi(z+1) \quad : z \in U_+$$

and that the orbit equation (4) holds. Finally, show that the construction entails that the range of ϕ is $\mathbf{C} \setminus \{0,1\}$.

9.1 The multi-valued inverse. Since ϕ is locally conformal there exists a multi-valued inverse function denoted by $\mathfrak{m}$. Namely, set $\Omega = \mathbf{C} \setminus \{0,1\}$ and consider the point $\zeta_0 = i$. We first find the unique point $z_0 \in V_0$ such that $\phi(z_0) = i$. At ζ_0 we get a unique germ $\mathfrak{m}_0(\zeta) \in \mathcal{O}(\zeta_0)$ such that

$$\mathfrak{m}_0(\phi(z)) = z$$

hold for z close to i. Next, let γ be a curve in Ω which starts at i and has some end-point ζ_1. Since λ is locally conformal there exists a unique curve γ^* in U_+ such that

$$\phi(\gamma^*(t)) = \lambda(t) \quad : \ 0 \leq t \leq 1$$

As explained in xx there exists an analytic extension of $\mathfrak{m}_0$ along γ which locally produces inverses of the ϕ-function. The resulting multi-valued $\mathfrak{m}$-function gives the set of values $W(\mathfrak{m}, \zeta)$ for every $\zeta \in \Omega$ which is in a 1-1 correspondence with the inverse fiber $\lambda^{-1}(\zeta)$ and hence equal to an orbit under the group $\mathcal{F}$.

10. Poincare's theory of Fuchsian groups

The theory of Fuchsian groups was created by Poincaré. His articles *Théorie des groupes fuchsiens* and *Memoire sur les fonctions fuchsiennes* were published 1882 in the first first volume of Acta Mathematica and the article *Memoire sur les groupes kleinéens* appeared in volume III. The last article is more advanced and we shall not discuss Kleinan groups here. Nor do we discuss the article *Memoire sur les fonctions zétafuchsiennes.* The connection to arithmetic was presented in a later article *Les fonctions fuchsiennes et l'Arithmétique* from 1887. One should also mention the article *Les fonctions fuchsiennes et l'équation* $\Delta(u) = e^u$ where Poincaré proved that this second order differential equation has a subharmonic solution with prescribed singularities on every closed Riemann surface attached to an algebraic equation. The last work started potential theoretic analysis on complex manifolds. Here we only discuss material from the first two cited articles.

Poincaré was inspired by earlier work, foremost by Bernhard Riemann, Hermann Schwarz and Karl Weierstrass. For example, he used the construction of multi-valued analytic extensions by Weierstrass which leads to the *Analytische Gebilde* of a multi-valued function f defined in some connected open subset Ω of $\mathbf{C}$. This *Analytische Gebilde* is a connected complex manifold X on which f becomes a single valued analytic function f^*. More precisely, there exists a locally biholomorphic map

$$\pi\colon\ X \mapsto \Omega$$

When $U \subset \Omega$ is simply connected the inverse image $\pi^{-1}(U)$ is a union of pairwise disjoint open sets U^*_γ where the single-calued analytic function f^* is determined by a branch T_γ of f, i.e. one has

$$T_\gamma(f)(\pi(x) = f^*(x) \quad : x \in U^*_\gamma$$

Major contributions are also due to Schwarz. In 1869 he used the reflection principle and calculus of variation to settle the Dirichlet problem and used this to prove the uniformisation theorem for connected domains bordered by p many real analytic and closed Jordan curves where p in general is ≥ 2. Of special interest is the multi-valued $\mathfrak{m}$-function from § 8 defined in $\mathbf{C} \setminus \{0,1\}$. It is related to the elliptic integral of the first kind and hence to Jacobi's $\mathfrak{sn}$-function which appears in the equation of motion when a rigid body rotates around a fixed point.

A comment. The study of Fuchsian groups was not restricted to analytic function theory. Poincaré's main concern was to develop the theory of differential systems, both linear and non-linear. His research was also directed towards to the general theory about abelian functions and their integrals, inspired by Abel's pioneering work. Hundreds of text-books have appeared after Poincaré. Personally I find that his own and often quite personal presentation superseeds most text-books which individually only treat some fraction from the great visions by Poincaré. His original work offers therefore a good introduction for the student who enters studies about linear and non-linear differential systems in an algebraic context, together with function theory which leads to Fuchsian as well as Kleinian groups and there associated functions. See in particular the book *Analyse de ses travaux scientifiques* which contains a survey of his the scientific work. In several chapters Poincaré describes in his own words various research areas from the period between 1880 until 1907, which has the merit that it not only contains a summary of results but also explanations of the the main ideas and methods which led to the theories.

Of course there exists more recent advancement in function theory. Here one should foremost mention work by Lars Ahlfors. So in addition to the cited reference above I recommend text-books by Ahlfors, especially his book *Conformal Invariants* which contains material about the theory of extremal length which was created by Arne Beurling in the years 1942-1946. From a complex analytic point of view the discoveries by Ahlfors and Beurling have a wider scope and has led to many still unsolved problems in complex analysis. including the study of quasi-conformal mappings. In addition to this we refer to the excellent material in the text-book [A-S] by Ahlfors and Sario about Riemann surfaces.