Chapter 5.B. Subharmonic functions

Introduction.

The theory of subharmonic functions is foremost due to F. Riesz whose article [Ri] from 1926 contains the essential facts about subharmonic functions. See also the text-book *Subharmonic functions* by Rado from 1937 for an account about subharmonic functions. We shall not discuss subharmonic functions in $\mathbf{R}^n$ when $n \geq 3$ which for example are treated in the text-book [Hayman]. The basic results about subharmonic functions appear in § 1-4 while the remaining sections are devoted to special topics. For example, a delicate result due to Beurling appears in Theorem 8.2 which gives a sufficent condition for the existence of a largest subharmonic minorant.

0.0 Subharmonic functions in complex domains. We identify $\mathbf{R}^2$ with $\mathbf{C}$ where real-valued functions $u(x,y)$ become functions of the single complex variable $z = x + iy$. If $u(x,y)$ is a C^2-function defined in an open set Ω then the Laplacian $\Delta(u)$ is a non-negative function if and only if u satisfies the *local mean-value inequality*. In other words, for every point $p \in \Omega$ there exists some $\delta > 0$ such that the disc $\{|z-p| < \delta\} \subset \Omega$ and

$$(*) \qquad u(p) \leq \frac{1}{\pi r^2} \cdot \iint_{D_r(p)} u(x,y) \cdot dxdy \quad \text{for all} \quad 0 < r < \delta$$

where $D_r(p)$ is the disc of radius r centered at p. These local mean-value inequalities make sense for non-differentiable functions. Thus, let $u(x,y)$ be a real-valued and continuous function defined in some open set Ω. We say that u satisfies the local mean-value inequality if there for every $p \in \Omega$ exists some $\delta > 0$ with $\delta < \text{dist}(p, \partial\Omega)$ such that (*) above holds. This family of continuous functions is denoted by $\text{SH}_c(\Omega)$ where the prefix c indicates that we restrict the attention to continuous functions.

0.1 The majorant principle. The local mean value inequality for a function u in $\text{SH}_c(\Omega)$ gives the following majorisation principle. Consider a pair (U,h) where $U \subset \Omega$ is an open subset and h a continuous function on the compact closure $\bar{U}$ and harmonic in U. Then

$$(**) \qquad u|\partial U \leq h|\partial U \implies u \leq h \quad \text{in the whole set } \bar{U}$$

The proof of (**) relies upon the mean-value equality for harmonic functions and solving the Dirichlet problm for a disc one gets the converse result, i.e. a continuous function u in Ω for which the majorisation (**) holds belongs to $\text{SH}_c(\Omega)$. Hence there is an equivalent condition for a continuous function to be subharmonic !

0.2 Logarithmic potentials. Let μ be a non-negative Riesz measures with compact support in $\mathbf{C}$. Since the function log $|z|$ is locally integrable there exists the convolution integral:

$$(***) \qquad L_\mu(z) = \frac{1}{2\pi} \int \log|z-\zeta| \cdot d\mu(\zeta)$$

We refer to L_μ as the logarithmic potential of μ. It can be attained as the pointwise limit of a decreasing sequence of C^∞-functions. Namely, when $\epsilon > 0$ we set

$$L_\mu^\epsilon(z) = \frac{1}{4\pi}\int \log\big(|z-\zeta|^2+\epsilon\big)\cdot d\mu(\zeta)$$

The reader should verify that these functions are of class C^∞ and with z kept fixed we have

$$\frac{dL_\mu^\epsilon(z)}{d\epsilon} = \frac{1}{4\pi}\int \frac{1}{|z-\zeta|^2+\epsilon}\cdot d\mu(\zeta)$$

It follows that $\{L_\mu^\epsilon\}$ is a decreasing sequence of C^∞-functions whose pointwise limit function is L_μ. Hence the L^1_{loc}-function L_μ takes values in $[-\infty,+\infty)$. and is upper semi-continuous. Put

$$\text{Polar}(\mu) = \{z\colon\ L_\mu(z) = -\infty\}$$

This polar set is the intersection of the open sets $\{L_\mu < -N\}$ taken over all positive integer N and is therefore a so called G_δ-set. Moreover, its 2-dimensional Lebesgue measure is zero since L_μ is locally integrable. It turns out that the polar set of a given non-negative measureμ belongs to a more restricted family of sets than the family of null sets in the sense of Lebesgue. See § XX from Special Topics which is devoted to complex potential theory.

0.3 A mean-value function. With μ fixed we define for a given $r>0$

$$M_{\mu,r}(z) = \frac{\pi}{r^2}\int_0^r\int_0^{2\pi} L_\mu(z+se^{i\theta})\cdot sds\cdot d\theta \tag{1}$$

It means that we take the mean value of U_μ over the disc of radius r centered at z. The function of z in (1) is continuous for each fixed $r>0$. Set

$$\Phi_r(z,\zeta) = \frac{\pi}{r^2}\int_0^r\int_0^{2\pi} \log|z+se^{i\theta}-\zeta|\cdot sds\cdot d\theta \tag{2}$$

Fubini's theorem gives

$$M_{\mu,r}(z) = \int \Phi_r(z,\zeta)\cdot d\mu(\zeta) \tag{3}$$

It is clear that the ϕ-function only depends upon $|z-\zeta|$. If a is real and positive we set

$$\phi_r(a) = \frac{\pi}{r^2}\int_0^r\int_0^{2\pi} \log|a+se^{i\theta}|\cdot sds\cdot d\theta =$$

$$\log|a| + \frac{a^2\pi}{r^2}\int_0^{\frac{r}{a}}\int_0^{2\pi}\log|1+te^{i\theta}|\cdot tdt\cdot d\theta \tag{4}$$

where the last equality follows after the variable substitution $s\to at$. Denote the last function in (4) by $\psi_r(a)$. Then we have

$$M_{\mu,r}(z) - U_\mu(z) = \int \psi_r(|z-\zeta|)\cdot d\mu(\zeta) \tag{5}$$

Exercise. Consider the function defined for $s>0$ by

$$g(s) = \frac{1}{\pi}\cdot\int_0^s\int_0^{2\pi}\log|1+te^{i\theta}|\cdot tdt\cdot d\theta \tag{i}$$

Show that $g(s) =$ when $s\le 1$ and if $s>1$ the reader may verify that

$$\frac{dg}{ds} = 2\cdot s\log s \implies g(s) = s^2\cdot\log s - \frac{s^2}{2}+\frac{1}{2}\quad : s>1 \tag{ii}$$

Now

$$\psi_r(|z-\zeta| = \frac{|z-\zeta|^2}{r^2}\cdot g(\frac{r}{|z-\zeta|}) \tag{iii}$$

Together (5) and (ii-iii) express the non-negative difference $M_{\mu,r}(z) - U_\mu(z)$. The reader may analyze the the non-decreasing function

$$r \mapsto M_{\mu,r}(z)$$

while z is fixed. For example use (ii) to find the derivative with respect to r.

Now we enlarge the class of sub-harmonic functions to include U_μ-functions which arise from non-negative measures μ.

0.4 Definition. *Let Ω be an open set in* $\mathbf{C}$*. The the family of functions $u \in L^1_{loc}(\Omega)$ for which the distribution $\Delta(u)$ is a non-negative Riesz measure is denoted by $SH(\Omega)$.*

Remark. The condition for a locally integrable function to be subharmonic can be phrased in another way where we do not use distribution derivatives. Namely, let $u(x,y)$ be a function in $L^1_{\text{loc}}(\Omega)$. Denote by $\mathfrak{Leb}(u)$ the set of its Lebesgue points. If $p \in \mathfrak{Leb}(u)$ and $0 < r < \text{dist}(p, \partial\Omega)$ we consider the mean value

$$M_r(p) = \frac{1}{\pi r^2} \cdot \iint_{D_r(p)} u(x,y) \cdot dxdy \tag{1}$$

We say that u satisfies the local mean value inequality if there to every Lebesugue point p exists some $\delta > 0$ such that

$$u(p) \leq M_r(p) \quad : \quad 0 < r < \delta \tag{2}$$

In § xx we prove that this is equivalent to the condition that u is subharmonic in the sense of Definition 0.4.

0.5 The Riesz representation formula. It turns out that every subharmonic function in the sense of Definition 0.4 is locally expressed by the logarithmic potential of a Riesz measure plus some harmonic function. More precisely, let u be subharmonic in an open set Ω and put $\mu = \Delta(u)$. Let Ω_0 be a relatively compact subset of Ω and denote by K its compact closure. Extending μ to be zero outside K we get the compactly supported measure μ_K and its logarithmic potential L_{μ_K}. In XXX we prove that there exists a harmonic function H in Ω_0 such that the equality below holds in Ω_0:

$$u = L_{\mu_K} + H \tag{*}$$

0.6 Rado's inequality. Riesz' representation formula gives an a priori inequality which goes as follows: Let D be the open unit disc and denote by $\mathcal{F}$ the class of subharmonic functions u in D such that $u(0) = 0$ and $u(z) < 1$ for all $z \in D$. In § xx we prove that there exists a constant C such that

$$\iint_{|z|\leq 1/2} e^{-u(z)}\, dxddy \leq C \tag{0.6.1}$$

for every $u \in \mathcal{F}$. This gives a conrol on negative values taken by u. For example, set

$$A_n = \{n|z| \leq 1/2\} \cap \{-(n+1) \leq u(z) \leq -n\}$$

Then

$$\sum_{n=1}^{\infty} |A_n|_2 \cdot e^n < \infty \tag{0.6.2}$$

0.7 Examples of subharmonic functions. Let $f(z)$ be analytic in the open unit disc where $f(0) = 0$ and $|f(z)| < 1$ for all $z \in D$. To each $0 < r < 1$ there exists the function $\mathcal{N}_r(w)$ in the disc $|w| < 1$, defined by

$$\mathcal{N}_r(w) = \sum \log \frac{r}{|\zeta|} \quad : \text{sum taken over all } \zeta \in D_r \quad : \; f(\zeta) = w \tag{0.7.1}$$

In the sum one repeats zeros of $f(z) - w$ with their multiplicity. In XXX we show that $\mathcal{N}_r(w)$ is a subharmonic function in the unit disc of the complex w-plane. This class of subharmonic functions was introduced by Nevanlinna when he developed the value distribution theory for meromorphic functions. See [Nev. page xx-xx] which describes the usefulness of this class of subharmonic functions. Extending the construction of these $\mathcal{N}$-functions to universal covering spaces of the image domains $f(D_r)$ these subharmonic functions can be used in value distribution theory on Riemann surfaces where the interested reader can consult the article [Lehto] by O. Lehto for details.

1. The subharmonic Log-function

1.0 The function L_ϵ. For each $\epsilon > 0$ we set

$$F_\epsilon(x,y) = \log(x^2+y^2+\epsilon)$$

The partial derivative with respect to x becomes:

$$\partial_x(F_\epsilon) = \frac{2x}{x^2+y^2+\epsilon} \quad : \quad \partial_x^2(F_\epsilon) = \frac{2}{x^2+y^2+\epsilon} - \frac{4x^2}{(x^2+y^2+\epsilon)^2}$$

and similarly for the partial y-derivative. A summation of the second order partial derivatives gives:

(i)
$$\Delta(F_\epsilon) = \frac{4\epsilon}{(x^2+y^2+\epsilon)^2}$$

The double integral over $\mathbf{R^2}$ becomes:

$$\iint \frac{4\epsilon}{(x^2+y^2+\epsilon)^2}\,dxdy = \int_0^\infty \int_0^{2\pi} \frac{4\epsilon}{(r^2+\epsilon)^2} \cdot rd\theta =$$

(ii)
$$4\pi\epsilon \cdot \int_0^\infty \frac{2rdr}{(r^2+\epsilon)^2} = -4\pi\epsilon \cdot \frac{1}{r^2+\epsilon}\Big|_0^\infty = 4\pi$$

Put

$$L_\epsilon(x,y) = \frac{1}{2\pi} \cdot \mathrm{Log}\,\sqrt{x^2+y^2+\epsilon} = \frac{1}{4\pi} \cdot F_\epsilon(x,y)$$

Then (ii) entails that its double integral is equal to one and (i) gives

(iii)
$$\Delta(L_\epsilon) = \frac{1}{\pi}\frac{\epsilon}{(x^2+y^2+\epsilon)^2}$$

With $z = x+iy$ we can write

$$L_\epsilon(z) = \frac{1}{2\pi} \cdot \log\,\sqrt{|z|^2+\epsilon}$$

Outside the origin we get the limit formula:

(*)
$$\lim_{\epsilon\to 0} L_\epsilon(z) = \frac{1}{2\pi} \cdot \log|z|$$

1.1 L_ϵ as distributions. Let $\phi \in C_0^\infty(\mathbf{R}^2)$ be a test-function. Green's formula gives

(1)
$$\iint \Delta(L_\epsilon) \cdot \phi\,dxdy = \iint L_\epsilon \cdot \Delta(\phi)\,dxdy \quad : \quad \epsilon > 0$$

The left hand side has a limit as $\epsilon \to 0$. Namely, (iii) above gives:

(ii)
$$\lim_{\epsilon\to 0} \frac{1}{2\pi} \iint \frac{\epsilon}{(x^2+y^2+\epsilon)^2} \cdot \phi(x,y)\,dxdy = \phi(0)$$

The reader should confirm the limit formula (ii) by computing the integral in polar coordinates. In distribution theory this is expressed as follows:

1.2 Theorem. *The distribution densities $\Delta(L_\epsilon)$ converge to the unit point mass δ_0.*

Remark. Above we constructed a *regularisation* of the Dirac measure. Using (*) it means that the locally integrable function $\log|z|$ considered as a distribution is such that its Laplacian - taken in the distribution sense - is equal to $2\pi i \cdot \delta_0$. One therefore says that $\frac{1}{2\pi} \cdot \log|z|$ is a fundamental solution to the Δ-perator.

Next, let μ be a Riesz measure in $\mathbf{R^2}$ with a compact support which defines a distribution by

$$\phi \mapsto \int \phi \cdot d\mu$$

To each $\epsilon > 0$ we construct the convolution

$$L_\epsilon * \mu(x,y) = \int L_\epsilon(x-t, y-s) \cdot d\mu(t,s)$$

Here $L_\epsilon * \mu$ are C^∞-functions and taking the Laplacian we get

$$\Delta(L_\epsilon * \mu)(x,y) = \frac{1}{2\pi}\int \frac{\epsilon}{(x-t)^2+(y-s)^2+\epsilon}\cdot d\mu(t,s)$$

Next, when $\phi \in C_0^\infty$ we perform integration with respect to (x,y) and obtain

$$\iint \Delta(L_\epsilon * \mu)(x,y)\cdot\phi(x,y)dxdy = \frac{1}{2\pi}\int \Big[\iint \frac{\epsilon\cdot\phi(x,y)dxdy}{(x-t)^2+(y-s)^2+\epsilon}\Big]\cdot d\mu(t,s)$$

Here the inner double intergal is a function of (s,t) which converges uniformly to ϕ as $\epsilon \to 0$. Hence a passage to the limit gives

$$\lim_{\epsilon\to 0}\iint \Delta(L_\epsilon * \mu)(x,y)\cdot\phi(x,y)\,dxdy = \int \phi\cdot d\mu$$

This is expressed by saying the the distribution densities $\Delta(L_\epsilon * \mu)$ converge to the distribution defined by μ. Next, recall from distribution theory that convolution commutes with differentiation. Hence we get

$$(*)\qquad \lim_{\epsilon\to 0}\Delta(L_\epsilon * \mu) = \lim_{\epsilon\to 0}\Delta(L_\epsilon) * \mu = \mu$$

where the last equality follows from Theorem 1.2.

1.3 The logarithmic potential. Recall from measure theory that one can define the convolution of a compactly supported Riesz measure μ with an L^1-functions. We apply this with the locally integrable function $\log\sqrt{x^2+y^2}$ which in complex notation is written as $\log|z|$. The convolution integral

$$U_\mu(z) = \frac{1}{2\pi}\int \log|z-\zeta|\cdot d\mu(\zeta)$$

is called the logarithmic potential of the Riesz measure μ. Notice that $U(z)$ belongs to $L^1_{\text{loc}}\mathbf{C})$. The limit formulas from Theorem 1.2 and (*) above yield

1.4 Theorem. *The Laplacian of U_μ taken in the distribution sense is equal to μ.*

Remark. To confirm Theorem 1.4 we consider a test-function g. Fubini's theorem gives the equality

$$\frac{1}{2\pi}\cdot\int \Delta\, g(z)\cdot\Big]\int \log|z-\zeta|\cdot d\mu(\zeta)\Big]\cdot dxdy = \frac{1}{2\pi}\cdot\Big[\int \Delta\, g(z)\cdot\log|z-\zeta|\cdot dxdy\Big]\cdot d\mu(\zeta)$$

By Theorem 1.2 the last term becomes

$$\int g(\zeta)\cdot d\mu(\zeta)$$

Hence the definition of distribution derivatives gives Theorem 1.4.
Next, recall that $\partial = \frac{1}{2}(\partial_x - i\partial_y)$. A differentiation gives:

$$\partial(L_\epsilon) = \frac{1}{4\pi}\cdot\frac{x-iy}{x^2+y^2+\epsilon} = \frac{1}{4\pi}\cdot\frac{\bar{z}}{|z|^2+\epsilon}$$

So outside the origin we get the limit formula

$$(**)\qquad \lim_{\epsilon\to 0}\partial(L_\epsilon)(z) = \frac{1}{4\pi z}$$

1.5 The Cauchy transform.

The function $\frac{1}{z}$ is locally integrable and can therefore be convolved with a compactly supported Riesz measure. Put

$$\mathcal{C}_\mu(z) = \int \frac{d\mu(\zeta)}{z-\zeta}$$

Since this is a convolution of a measure with compact support and the locally integrable function $\frac{1}{z}$ it belongs to L^1_{loc}. We refer to $\mathcal{C}_\mu$ as the Cauchy transform of μ. Notice that $\mathcal{C}_\mu$ is an analytic function outside the support of μ. For example, its complex derivative becomes

$$\frac{d\mathcal{C}_\mu(z)}{dz} = -\int \frac{d\mu(\zeta)}{(z-\zeta)^2}$$

1.6 The equality $\bar{\partial}(\mathcal{C}_\mu) = \pi \cdot \mu$.

Recall from XX that the Laplacian Δ can be expressed as the product of the first order differential operators

$$\partial = \frac{1}{2}(\partial_x - i\partial_y) \quad : \quad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

More precisely we have

$$\Delta = 4 \cdot \bar{\partial}_z \cdot \partial_z$$

Exercise. Apply ∂ to the locally integrable function $U_\mu(z)$ and show the equality

$$\partial(U_\mu) = \frac{1}{4\pi} \cdot \mathcal{C}_\mu$$

By Theorem 1 .2 we also have

$$\mu = \Delta(U_\mu) = 4 \cdot \bar{\partial}(\partial U)) = 4 \cdot \frac{1}{4\pi} \cdot \bar{\partial}(\mathcal{C}_\mu)$$

From the above the equality (1.6) follows. Since it is so important we state

1.7 Theorem. *One has the equality*

$$\bar{\partial}(\mathcal{C}_\mu) = \pi \cdot \mu$$

Example. Let $\mu = \delta_0$ be the unit mass at the origin. In this case $\mathcal{C}_\mu(z) = \frac{1}{z}$ and the definition of distribution derivatives means that

(i) $$g(0) = -\frac{1}{\pi} \int \frac{\bar{\partial}(g)\, dxdy}{z} \quad : \quad g \in C_0^\infty(\mathbf{C})$$

Recall from XX that $dz \wedge d\bar{z} = -2i \cdot dxdy$. So the minus sign is changed and the right and side becomes

(ii) $$\frac{1}{2\pi i} \int \frac{\bar{\partial}(g) \cdot dz \wedge d\bar{z}}{z} \quad : \quad g \in C_0^\infty(\mathbf{C})$$

Since $\frac{1}{z}$ is locally integrable this integral is equal to

(iii) $$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|>\epsilon} \frac{\bar{\partial}(g) \cdot dz \wedge d\bar{z}}{z} \quad : \quad g \in C_0^\infty(\mathbf{C})$$

Now we regard the differential 1-form $\alpha = \frac{g(z) \cdot dz}{z}$ and as explained in XX we have

(iv) $$d\alpha = \frac{\bar{\partial}(g) \cdot d\bar{z} \wedge dz}{z} = -\frac{\bar{\partial}(g) \cdot dz \wedge d\bar{z}}{z}$$

where we used the the exterior product of two 1-forms is anti-commutative. When Stokes formula is applied the outer normal with respect to the exterior domain is minus the usual outer normal with respect to the open disc $|z| < \epsilon$. So when Stokes Theorem is applied to differential 1-form α we see that (iii) above is equal to

(v) $$-\frac{1}{2\pi i} \cdot \lim_{\epsilon \to 0} \int_{|z|>\epsilon} d\alpha = \frac{1}{2\pi i} \cdot \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{g(z)dz}{z} = g(0)$$

This confirms the equality in Theorem 1.7 when $\mu = \delta_0$.

1.8 Approximation theorems

We shall apply Theorem 1.7 to deduce certain approximation theorems. Let K be a compact null set in $\mathbf{C}$, i.e. its 2-dimensional Lebesgue measure is zero. We have the Banach space $C^0(K)$ of continuous and complex valued functions on K. If $z \in \mathbf{C} \setminus K$ the rational function $\frac{1}{z-\zeta}$ restricts to a continuous function on K. Taking finite linear combinations we get functions

$$\zeta \mapsto \sum c_\nu \frac{1}{z_\nu - \zeta} \quad : z_1, \ldots, z_N \in \setminus K \quad : c_1, \ldots, c_N \in \mathbf{C}$$

Denote by $R(K)$ the closure of this linear subspace of $C^0(K)$. Thus, functions in $R(K)$ consist of continuous functions on K which can be uniformly approximated by rational functions with poles outside K.

1.9 Theorem. *For every compact null set K one has the equality $C^0(K) = R(K)$.*

Proof. Suppose that $R(K) \neq C^0(K$. Riesz' representation formula gives the existence of a non-zero measure μ supported by K such that $\mu \perp R(K)$. Consider the Cauchy transform

$$\mathcal{C}_\mu(z) = \int_K \frac{d\mu(\zeta)}{z - \zeta}$$

Since $\mu \perp R(K)$ it is identically zero outside K. Now K is a null set so the L^1_{loc}-function $\mathcal{C}_\mu$ is identically zero and so is its distribution derivative $\bar{\partial}(\mathcal{C}_\mu)$. But then we get a contradiction from Theorem 1.7 since this distribution derivative must recapture $\pi \cdot \mu$. which by assumption is non-zero.

Remark. The proof of Theorem 1.8 relies upon the Hahn-Banach theorem which gives the existence of a non-zero Riesz measure carried by K when $R(K) \neq C^0(K)$. The drawback of this proof is that it does not give any hint about how one actually approximates a given continuous function on K by rational functions having poles in the complement. So instead of the easy proof above which is based on an "argument by contradiction" one would like to have a *constructive proof* of Theorem 1.7, i.e. given a null-set K one may ask for some sort of algorithm to approximate every given continuous function on K. This point of view was put forward by E. Bishop in his book [Bish]. His objection to the proof above is not just a question of taste and philosophy. In fact, Erret Bishop is one of the most prominent analysists in complex function theory. It is therefore good to keep in mind that various theoretical results do not give the whole story if one really wants to apply them in more concrete situations.

1.10 Megelyan's swiss-cheese. Following [Merg] we construct a compact set K in C with empty interior where $R(K) \neq C^0(K)$. Take the unit disc D and remove a finite number of discs $D_1, \ldots, D_N$ inside D. They are chosen so that the closed discs $\{\bar{D}_\nu\}$ are pairwise disjoint and stay inside D. For each $1 \leq \nu \leq N$ we let μ_ν be the measure supported by ∂D_ν and given by $d\zeta$, i.e. if D_ν has a radius r_ν it is simply the measure whose Cauchy transform becomes

$$\mathcal{C}_\nu(z) = \int_0^{2\pi} \frac{i r_\nu e^{i\theta}}{z - (a_\nu + r_\nu)e^{i\theta}} \cdot d\theta$$

Notice that $\mathcal{C}_\nu(z) = 0$ outside $\bar{D}_\nu$. Indeed, this follows from the trivial observation from XXX. Next, on ∂D we get the measure μ^* defined by $d\zeta$ restricted to $|\zeta| = 1$. Let $\mathcal{C}^*(z)$ denote its Cauchy transform which now is identically zero outside D. In addition to this it has a compensating influence. For if $z_0 \in D_\nu$ for some ν, we have by Cauchys formula

$$\int_{\partial D_\nu} \frac{d\zeta}{z_0 - \zeta} = \int_{\partial D} \frac{d\zeta}{z_0 - \zeta}$$

Consider the measure

$$\rho_N = \mu^* - (\mu_1 + \ldots + \mu_N)$$

The previous observations show that $\mathcal{C}_{\rho_N}$ is zero in the set $\Omega_N = \cup D_\nu \cup |\zeta| > 1$. In other words, if

$$K_N = \bar{D} \setminus D_1 \cup \ldots \cup D_N$$

then this compact set is the support of the L^1-function $\mathcal{C}_{\rho_N}$.

At this stage we see how one should proceed to construct a swiss-cheese. Namely, inside D we construct a denumerable sequence of discs $D_1, D_2, \ldots$ so that the compact set

$$K = \bar{D} \setminus \cup D_\nu$$

has no interior points, i.e. just make sure that the center points of the discs $\{D_\nu\}$ appears as a dense subset of D. Moreover, let r_ν be the radius of D_ν and perform the construction so that

$$\sum r_\nu < \infty$$

The total variation of μ_ν becomes $2\pi \cdot r_\nu$ and hence we get a measure

$$\rho_* = \mu^* - \sum_{\nu=1}^{\infty} \mu_\nu$$

By the construction it is clear that $\rho_* \perp R(K)$ and hence $R(K) \neq C^0(K)$.

1.11 Wermer's example. In [We] appears a Jordan arc Γ whose 2-dimensional Lebesgue measure is positive whose existence goes back to work by Peano. The restriction of polynomial $P(z)$ to Γ gives a **C**-subalgebra of $C^0(\Gamma)$. Let $P(\Gamma)$ be its uniform closure. Then

$$P(\Gamma) \neq C^0(\Gamma)$$

This inequality relies upon results about analytic capacity. The reader may consult the book [We] about uniform algebras for a detailed account about this example. See also the text-book [Ga] by T. Gamelin which is devoted to uniform algebras.

2. Subharmonic functions

If Ω is an open subset of $\mathbf{C}$ we denote by $\mathrm{SH}^2(\Omega)$ the class of C^2-functions u in Ω such that the continuous function $\Delta(u)$ is non-negative. If $u \in \mathrm{SH}^2(\Omega)$ and Ω_0 is a domain in the class $\mathcal{D}(C^1)$ which appears as a relatively compact in Ω, then Green's formula gives:

$$\iint_{\Omega_0} \Delta(u)\, dxdx = \int_{\partial\Omega_0} u_{\mathbf{n}} \cdot ds$$

Hence the subharmonicity entails that the integral of the outer normal derivative is ≥ 0 for any such domain Ω_0. Consider the case when $\Omega = D_R$ is a disc centered at the origin and $\Omega_0 = D_r$ for some $r < R$. In polar coordinates we get

$$\int_{\partial D_r} u_{\mathbf{n}} \cdot ds = \int_0^{2\pi} [\cos\theta \cdot u_x + \sin\theta \cdot u_y] \cdot r d\theta$$

Next, define the mean-value function

$$M_u(r) = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(r,\theta) \cdot d\theta$$

Since $\frac{d}{dr}(u(r,\theta) = \cos\theta \cdot u_x + \sin\theta \cdot u_y$ we obtain

$$\frac{d}{dr}(M_u(r)) = \frac{1}{\pi r} \int_{\partial D_r} u_n ds = \frac{1}{2\pi r} \cdot \iint_{D_r} \Delta(u) dxdx$$

Hence the function $r \mapsto M_u(r)$ is non-decreasing and when $\Delta(u) > 0$ it is even strictly increasing. Since u is continuos we have

$$\lim_{r \to 0} M_u(r) = u(0,0)$$

It follows that u satisfies the mean-value inequality

$$(*) \qquad u(0) \leq M_u(r) \quad : \quad 0 < r < R$$

2.1 Harmonic majorization. Let $u \in \mathrm{SH}^2(\Omega)$. If $\delta > 0$ and $u_\delta(x,y) = u(x,y) + \delta(x^2+y^2)$ then $\Delta(u_\delta) = \Delta(u) + 4\delta > 0$. Since $u_\delta \to u$ when $\delta \to 0$ we can always approximate a subharmonic function by a decreasing sequence of strictly subharmonic functions. Next, recall the following result from from *Calculus*. Let $f(x,y)$ be a C^2-function defined in some open subset of $\mathbf{R^2}$ which has a (not necessarily strict) maximum at some point (x_0, y_0) , i.e.

$$f(x,y) \leq f(x_0,y_0) \quad : \quad (x-x_0)^2 + (y-y_0)^2 < \epsilon$$

hold for some small ϵ. Then the *Hessian* of f at (x_0, y_0) must be negative semi-definite. In particular the trace $f_{xx} + f_{yy} = \Delta(f) \leq 0$. This elementary facts gives

2.2 Proposition. *Let u be a strictly subharmonic function of class C^2 defined in an open set Ω. Then u cannot have any local maximum in Ω. Thus, if U is a relatively compact subset of Ω then u takes its maximum on the boundary of U, i.e.*

$$\max_{\bar{U}} u = \max_{\partial U} u$$

Next, Proposition 2.2. together with the mean-value property of harmonic functions give the following:

2.3 Theorem. *Let $u \in SH^2(\Omega)$ and let h be a harmonic function in Ω. Then the following implication hold for every relatively compact open subset U of Ω;*

$$u \leq h \text{ on } \partial U \implies u \leq h \text{ on } U$$

Proof. Follows from Proposition 2.2 since $u - h$ is subharmonic. We refer to this as the principle of *harmonic majorization.*

2.4 Subharmonic functions in L^1_{loc}.

Now we relax the C^2-hypothesis. Let $u \in L^1_{\text{loc}}(\Omega)$ where u is real-valued. We get the distribution $\Delta(u)$ and impose the condition that it is equal to a non-negative Riesz measure μ. By the definition of distribution derivatives this means that

$$(*) \qquad \iint_\Omega \Delta(\phi) \cdot u dx dy = \iint_\Omega \phi \cdot d\mu \quad : \quad \phi \in C_0^\infty(\Omega)$$

2.5 Definition. *A function u in $L^1_{loc}(\Omega)$ is called subharmonic if the distribution $\Delta(u) \geq 0$. The class of subharmonic functions in Ω is denoted by $SH(\Omega)$.*

2.6 Regularisations.

Definition 2.5 is a bit abstract since it is not easy to discover the distribution $\Delta(u)$ when u is just assumed to be locally integrable. So we shall find other conditions in order that a function in L^1_{loc} is subharmonic. For this we use regularisations. In general, let $u \in \text{SH}(\Omega)$ where Ω is a bounded open set. If $\delta > 0$ we set

$$\Omega[-\delta] = \{z \in \Omega : \text{dist}(z, \partial\Omega) \geq \delta\}$$

Notice that $\Omega[-\delta]$ is a compact subset of Ω. For every test-function ϕ with compact support in the disc D_δ the convolution $\phi * u$ exists in $\Omega[-\delta]$. By the general formula from XXX we have

$$\Delta(\phi * u) = \phi * \Delta(u) = \phi * \mu$$

where μ by assumption is a non-negative Riesz measure.

2.7 The case when ϕ is radial. We shall use test-functions which depend on $x^2 + y^2$ only. Let us recall the construction. Start from a test-function $\phi_*(z)$ which is > 0 in $|z| < 1$ and has compact support in $|z| \leq 1$ and depends on $|z|$ only while

$$\iint_D \phi_*(z) dx dy = 1$$

Then, to every $0 < \delta < 1$ we get the test-function

$$\phi_\delta(z) = \frac{\phi_*(\frac{z}{\delta})}{\delta^2}$$

which has support in $\bar{D}_\delta$. Next, recall from XXX that for any L^1_{loc}-function f it follows that the convolution $\phi_\delta * f$ is a C^∞.function. We apply this with u above and conclude that $\phi_\delta * u$ is a C^∞-function defined in some open neighborhood of $\Omega[-\delta]$. By (*) in 2.6 we have

$$\text{(i)} \qquad \Delta(\phi_\delta * u) = \phi_\delta * \mu$$

Since both μ and ϕ_δ are ≥ 0, it follows that the convolution is ≥ 0. Hence the Laplacian of the C^∞-function $\phi_\delta * u$ is ≥ 0 so we can apply the results from the C^2-case. In particular $\phi_\delta * u$ satisfies the mean-value inequality

$$\phi_\delta * u(p) \leq \frac{1}{\pi r^2} \cdot \int_{D_r(p)} \phi_\delta(z - p) \cdot u(z) \cdot dx dy \quad : p \in \Omega[-2\delta] \quad : \quad 0 < r < \delta$$

If $p \in \Omega[-2\delta]$ is a Lebesgue point for u we can pass to the limit as $\delta \to 0$ and conclude that

$$(*) \qquad u(p) \leq \frac{1}{\pi r^2} \cdot \int_{D_r(p)} u(z) \cdot dx dy \quad : p \in \Omega[-2\delta] \quad : \quad 0 < r < \delta$$

This shows that u satisfies the local mean-value inequality in $\Omega[-\delta]$. Since δ an be arbitrary small, it follows that u satisfies the local mean-value inequality in the whole of Ω, i.e. we have proved:

2.8 Theorem. *Let $u \in SH(\Omega)$. Then the following holds for each Lebesgue point of u in Ω:*

$$(*) \qquad u(p) \leq \frac{1}{\pi r^2} \cdot \int_{D_r(p)} u(z) \cdot dx dy \quad : 0 < r < \text{dist}(p, \partial\Omega)$$

Remark. Above we have recovered the definition of subharmonic functions from the introduction via Definition 2.5. The *converse* also holds, i.e. if we from start assume that the L^1_{Loc}-function u satisfies the local mean value inequality then it is subharmonic in the sense of Theorem 2.8.

2.9 Exercise. Prove the converse. The hint is that if ϕ_δ as above are radial test-functions and (*) is assumed, then the local mean value inequality hold for $\phi_\delta * u$. Here we have C^2-functions and by Green's formula one shows that $\delta(\phi_\delta * u) \geq 0$ follows. Finally one takes the limit as $\delta \to 0$ and the reader should now confirm that $\delta(u) \geq 0$ holds in the distribution sense. Show also that this family of functions is monotone, i.e. verify the following:

2.10 Proposition. *Let $u \in SH(\Omega)$. Then the sequence of functions $\{\phi_\delta * u\}$ decrease, i.e.*

$$\phi_{\delta_1} * u(p) \leq \phi_{\delta_2} * u(p) \quad : \quad \delta_1 < \delta_2$$

Moreover, this decreasing sequence converges almost everywhere to the measurable function u.

Proposition 2.10 implies u is almost everywhere equal to the pointwise limit if a monotone sequence of continuous functions and therefore we can always take u to be an upper semi-continuous function. The set where it becomes $-\infty$ is a null set. Thus, every subharmonic function enjoys similar properties as logarithmic potentials from § 1.

3. Riesz representation formula

Let $u \in \mathrm{SH}(\Omega)$ where Ω is a bounded open set. Now $\Delta(u)$ exists as a distribution and we have by assumption

(*) $$\iint \Delta(\phi) \cdot u\, dxdy \geq 0 \quad : \phi \in C_0^\infty(\Omega)$$

We can take regularisations of u as in § 2, i.e. construct convolutions $\phi_k * u$ where $\{\phi_k\}$ is a sequence of non-negative test functions with smaller and smaller compact support in some disc $|x| \leq \delta$ while their integrals are one for every k. Let $g \in C_0^\infty(\Omega)$ with compact support in $\Omega[-\delta]$. Green's formula gives:

(**) $$\iint \Delta(g) \cdot \phi_k * u\, dxdy = \iint g \cdot \Delta(\phi_k * u)\, dxdy$$

Since u is subharmonic the functions $\Delta(\phi_k * u) \geq 0$ in $\Omega[-\delta]$. Hence they become non-negative measures. Let us fix a compact subset K in $\Omega[-\delta]$. For example, we can take

$$K = \text{closure of } \Omega[-2\delta]$$

Now we can regard the *total mass*

$$\rho_k = \iint_K \Delta(\phi_k * u)\, dxdy$$

3.1. An inequality. Since u is subharmonic the mean-value inequality from § XX gives the inequality:

$$\phi_k * u(z) \leq u(z) \quad : \quad z \in \Omega[-2\delta]$$

From this we conclude that if $g \in C_0^\infty(\Omega)$ is non-negative and identically one on $\Omega[-2\delta]$ then (**) above gives

(i) $$\rho_k \leq \int \Delta(g) \cdot u\, dxdy \quad : \quad k = 1, 2, \ldots$$

Together with a general result about positive distributions in to be proved in § XX it follows that:

3.2. Proposition. *There exists a constant C_K such that*

$$\rho_k \leq C_K \quad : , k = 1, 2, \ldots$$

The uniform bound in Proposition 3.2 gives the existence of a subsequence of the non-negative measures $\{\Delta(\phi_k * u)\}$ which converges weakly to non-negative Riesz measure μ in K. At the same time we notice that

$$\lim_{k \to \infty} \phi_k * u \to u$$

where the limit holds in L^1. Hence we have proved:

3.3. Proposition *For every test-function $g \in C_0^\infty(\Omega)$ with support contained in K one has:*

$$\iint \Delta(g) \cdot u\, dxdy = \int g\, d\mu$$

3.4. Constructing Log-potentials. With $\delta > 0$ we construct a test-function χ satisfying

$$\chi = 1 \text{ in } \Omega[-3\delta] \quad : \chi \in C_0^\infty(\Omega[-\delta]$$

Next, we use the C^∞-functions L_ϵ from § 2 in V:A and keeping $\delta > 0$ fixed we define the functions:

$$g_\epsilon = \chi \cdot L_\epsilon \quad : \quad 0 < \epsilon < \delta$$

Passing to the limit as $\epsilon \to 0$ while χ is kept fixed, Proposition 3.3 and Theorem 1.4 imply that the function

$$w(z) = u(z) - \int \log|\zeta - z| \cdot d\mu(\zeta) \quad : z \in \Omega[-3\delta]$$

has a Laplacian in the *distribution sense* which is equal to zero in $\Omega[-3\delta]$.

3.5. Conclusions. So are we have not proved anything definitive. But we have demonstrated that one should regard two problems. The first is to explain why the ρ-numbers stay bounded, i.e. to verify Proposition 3.2. The second is to show that if ϕ is some L^1_{loc}-function such that $\Delta(\phi) = 0$ holds in the distribution sense, then ϕ is *automatically* a nice function, i.e. at least C^2 and hence harmonic. If this has been achieved the results above show that the subharmonic function u is represented as a logarithmic potential of a measure plus a harmonic functions inside $\Omega[-3\delta]$. Since $\delta > 0$ can be made arbitrary small this gives a representation in any relatively compact subset of Ω. So there remains to establish two general results from distribution theory.

3.6. Positive distributions.

Consider an open square $\square = \{(x, y)\colon\ 0 < x, y < 1\}$. Let $\mathcal{L}$ be a linear form on $C_0^\infty(\square)$ and assume that there exists some integer $k \geq 0$ and a constant C such that

$$|L(g)| \leq C \cdot ||g||_k \quad : g \in C_0^\infty(\square) \text{ where } ||g||_k = \text{norm in } C^k(\square)$$

We say that L is positive if

$$g \geq 0 \implies L(g) \geq 0$$

3.7 Theorem. *Let L be defined and positive as above. Then, for every $0 < r < 1$ there is a constant C_r such that*

$$|L(g)| \leq C_r \cdot ||g||_0 \quad : \text{Supp}(g) \subset \square_r$$

Proof Given $r < 1$ we construct $\phi \in C_0^\infty(\square)$ where $\phi = 1$ on $\square_r$ and is non-negative. Now, if g ha support in $\square_r$ it follows that the function

$$||g||_0 \cdot \phi - g \geq 0$$

Since L is positive we get

$$L(g) \leq ||g||_0 \cdot L(\phi)$$

So we can take $C_r = L(\phi)$ and Theorem 3.7 follows.

3.8. The elliptic property of Δ.

Let $w \in L^1_{\text{loc}}(\Omega)$ for some bounded open set. Assume that

$$\iint \Delta(g) \cdot w \, dxdy = 0 \quad : g \in C_0^\infty(\Omega)$$

3.9 Theorem. *Under the assumption above w is a harmonic function in Ω.*

Proof. We use similar regularisations as above. With $\delta > 0$ we choose the sequence $\{\phi_k\}$ and now $\phi_k * w \in C^\infty(\Omega[-\delta]$. Since convolution commutes with Δ, it follows that these functions are harmonic in $\Omega[-\delta]$. Moreover, since w b assumption has a finite L^1-norm over the relatively compact subset $\Omega[-\delta]$, the L^1-norms of $\{\phi_k * w\}$ are uniformly bounded in $\Omega[-2\delta]$, i.e. we have a constant C so that

$$\iint_{\Omega(-2\delta)} |\phi_k * w| \, dxdy \leq C$$

Poisson's formula implies that we get a uniform bound for the *maximum norms* in $\Omega[-3\delta]$, i.e. with another constant C_1 one has

$$\max |\phi_k * w(z)| \leq C_1 \quad : z \in \Omega[-3\delta]$$

At this stage we apply Montel's results for normal families of harmonic functions in XX. Passing to a subsequence if necessary, it follows that

$$\lim_{k \to \infty} \phi_k * w = G \text{ holds uniformly in compact subsets of } \Omega[-3\delta]$$

where the limit function G is harmonic. At the same time $w \in L^1_{\text{loc}}$. From Lebesgue theory we know that $\phi_k * w \to w$ and hence w must be equal to the "true" harmonic function G in $\Omega(-\delta)$. Since δ can be arbitrary small we conclude that w is a true harmonic function in the whole of Ω.

3.10 Remark. Above we gave a "pedestrian proof" which could have been given in a quicker way if one admits further results in distribution theory. Moreover, the elliptic property of Δ holds for distributions, i.e. if w is replaced by *any* distribution μ defined in some open set Ω where $\Delta(\mu) = 0$ holds in the sense of distributions, then μ is a "true" harmonic function. Thus can be shown by using regularisations as above. Namely, exactly as above $\phi_k * \mu \in C^\infty(\Omega[-\delta]$. Next, the distribution μ restricted to the relatively compact set $\Omega[-\delta]$ has a finite order k say. Using this one can proceed exactly as in the proof of Theorem 3.9 except that one has to be a bit more careful and take into the account growth of the derivatives of the ϕ-functions up to order k. We leave the details to the reader who also may consult text-books devoted to distribution theory which show that Theorem 3.9 holds with w replaced by a distribution.

3.11 Analytic expansions of harmonic functions

The elliptic character of Δ is made more precise by a result due to L. Ehrenpreis which shows how to express distributions via absolutely convergent integrals taken in $\mathbf{C}^2$ via the Fourier transform of μ. This result go beyond these notes since the proofs rely upon several complex variables. Ehrenpreis' integral formulas appear in [Björk: Chapter 8] and also in [Hö-complex] as well as in [Hö:2 Chapter PDE]. Let as explain the result for harmonic functions $w(x, y)$ in the unit disc.

3.12 Integrals over harmonic exponentials Let ζ and w be two complex numbers and set:

$$\mathfrak{e}(x, y) = e^{i(x\zeta + y\eta)}$$

We see that $\Delta(\mathfrak{e}) = -(\zeta^2 + w^2) \cdot \mathfrak{e}$. Put

$$S = \{(\zeta, w) \in C^2 \quad : \zeta^2 + w^2 = 0\}$$

Points on this algebraic hypersurface in $\mathbf{C}^2$ produce harmonic $\mathbf{e}$-functions in the (x, y)-plane. It is therefore tempting to consider a complex-valued Riesz measure μ in the 4-dimensional real (ζ, w)-space with support in S and define the function

$$(*) \qquad U(x, y) = \int_S e^{i(x\zeta + yw)} \cdot d\mu(\zeta, w)$$

With $\zeta = \xi + i\eta$ and $w = u + iv$ we have

$$\text{(i)} \qquad |e^{i(x\zeta + yw)}| = e^{-(x\eta + yv)}$$

If $z = x + iy \in D$ so that $x^2 + y^2 < 1$, the Cauchy-Schwartz inequality gives

$$\text{(ii)} \qquad |x\eta + yv| \leq \sqrt{\eta^2 + v^2}$$

Assume that the mass distribution of μ satisfies

$$\text{(iii)} \qquad \int_S e^{\sqrt{\eta^2 + v^2}} \cdot |d\mu(\zeta, w)| < \infty$$

Under this hypothesis we see from (i-iii) that the integral defining $U(x, y)$ in (*) converges for every point $(x, y) \in D$ and gives a harmonic function.

3.13 Extension to the complex Levi ball. From the real pair (x, y) we can take pass to complex numbers z_1, z_2 with $\mathfrak{Re}(z_1) = x$ and $\mathfrak{Re}(z_2) = y$. Let us then try to evaluate the integral

$$\mathcal{U}(z_1, z_2) = \int_S e^{i(z_1\zeta + z_2 w)} \cdot d\mu(\zeta, w)$$

With $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ we get

$$|e^{i(z_1\zeta + z_2 w)}| = e^{a_1\mathfrak{Re}(\zeta) - b_1\mathfrak{Im}(\zeta) + a_2\mathfrak{Re}(w) - b_2\mathfrak{Im}(w)}$$

Let us put

$$\mathcal{L} = \{(z_1, z_2) \quad : |z_1| + z_2| < 1\}$$

This open subset of $\mathbf{C}^2$ is called the Levi ball. Notice that its intersection with the real subspace where $\mathfrak{Im}(z_1) = \mathfrak{Im}(z_2)$ is equal to the unit disc D in the (x, y)-plane. The Cauchy-Schwartz inequality gives

$$|a_1\mathfrak{Re}(\zeta) - b_1\mathfrak{Im}(\zeta) + a_2\mathfrak{Re}(w) - b_2\mathfrak{Im}(w)| \leq |z_1| \cdot \sqrt{\xi^2 + \eta^2} + |z_2| \cdot \sqrt{u^2 + v^2}$$

At the same time we stay on S so that $\zeta^2 + w^2 = 0$. Regarding the real part this gives

$$\xi^2 - \eta^2 + u^2 - v^2 = 0$$

At this stage the reader discovers the picture. More precisely we obtain

3.14 Lemma. *When $(z_1, z_2) \in \mathcal{L}$ one has the inequality*

$$|z_1| \cdot \sqrt{\xi^2 + \eta^2} + |z_2| \cdot \sqrt{u^2 + v^2} \leq \sqrt{\eta^2 + v^2} \quad : \ (\xi + i\eta, u + iv) \in S$$

The easy proof is left to the reader. Using this inequality and assuming that the integral (iii) above is finite, it follows that one has *absolutely convergent integrals*:

$$(**) \qquad \mathcal{U}(z_1, z_2) = \int_S e^{i(z_1\zeta + z_2 w)} \cdot d\mu(\zeta, w) \quad : \ (z_1, z_2) \in \mathcal{L}$$

Moreover, when $|z_1| + |z_2| < 1$, i.e. when we stay inside the open Levi ball we can take complex derivatives with respect to z_1 and z_2 to conclude that $\mathcal{U}$ is an analytic function in the open Levi ball. Its restriction to the real subspace is the harmonic function $U(x, y)$ which by (**) extends to an analytic function of two complex variables in the Levi ball.

3.15 Application. For each integer $N \geq 0$ we put

$$\mathcal{H}_N = \{u \in C^N(\bar{D}) \quad : \ u \text{ harmonic in } D\}$$

This is a Banach space where the norm $||u||_N$ can be taken as the sum of the maximum norm of its derivatives up to order N plus the maximum norm of u itself. With this notation the following result holds - i.e. a special case from the L^2-estimates from [Hör]. See also Chapter 4 in the text-book [Hör:xx.]

3.16 Theorem. *There exists a fixed integer m^* and for every $N \geq 0$ a constant C_N such that every $u \in \mathcal{H}_N$ can be represented inside D by an absolutely convergent integral*

$$u(x, y) = \int_S e^{i(x\zeta + yw)} d\mu(\zeta, w)$$

and the measure μ satisfies

$$\int_S [1 + \sqrt{\eta^2 + v^2}\,]^{m^* - N} \cdot e^{i\sqrt{\eta^2 + v^2}} \cdot |d\mu(\zeta, w)| \leq C_N \cdot ||u||_N$$

Remark. Hörmander's proof shows that the constants C_N have polynomial growth, i.e. there are constants A, B such that

$$C_N \leq A(1 + N)^B \quad : \ N = 1, 2, \ldots$$

3.17 Question. It would be interesting to determine the best possible m^* for which constants C_N as above exist for all N. For this question one need not insist that m^* is an integer.

3.18 Expansions by Hayman The analytic extension of a harmonic function $u(x, y)$ in D to the Levi ball can also be proved via expansions of $u(x, y)$ into harmonic polynomials. This is done by Hayman in [xx] without any use complex analysis in several variables. Using Hayman's expansions it seems reasonable to get a good upper bound for m^* and perhaps even find the best possible choice.

4. Perron families.

Le Ω be a bounded open set in $\mathbf{C}$. No further assumptions are imposed, i.e. Ω need not be connected and its boundary can be "ugly". For example, it may have positive two-dimensional Lebesgue measure. On $\partial\Omega$ we have a function $\phi(x)$ which takes values between 0 and some $M > 0$. No other conditions are imposed, i.e. ϕ need not even be measurable. Given ϕ we denote by $\mathcal{P}(\phi)$ the family of all $u \in \mathrm{SH}^0(\Omega)$ for which

$$\limsup_{z\to w} u(z) \leq \phi(w) \quad : \quad w \in \partial\Omega$$

In Ω we get the function

$$H^*_\phi(z) = \max_{u\in\mathcal{P}(\phi)} u(z)$$

It is called Perron's maximal function of ϕ. With these notations one has

4.1 Theorem. *Perron's maximal function is harmonic in Ω.*

Proof. The constant functions 0 belons to $\mathcal{P}(\phi)$ and the maximum principle gives $u(z) \leq M$ for every $u \in \mathcal{P}(\phi)$. It suffices to show that $H^*_\phi(z)$ is harmonic in a disc D inside Ω. When $u \in \mathcal{P}(\phi)$ its restriction to ∂D is a continuous function where the solution to Dirichlet problem gives the harmonic function u^* in D. The function defined as u^* in D and u in $\Omega \setminus D$ belongs to $\mathcal{P}(\phi)$. At the same time $u \leq u^*$ holds in D. We conclude from this that the values of H^*_ϕ inside D are obtained when we take u:s from the restricted class of $\mathcal{P}(\phi)$ which are harmonic and ≥ 0 in D. Denote this restricted class with $\mathcal{P}_*(\phi)$. So inside D we have

$$\text{(i)} \qquad H^*_\phi(z) = \max_{u\in\mathcal{P}_*(\phi)} u(z)$$

The harmonic functions in D which come from $\mathcal{P}_*(\phi)$ take values between 0 and M and hence their restrictions to D give a normal family of harmonic functions in D. Let a be the center of D. The normal family property yields a sequence $\{u_n\}$ in $\mathcal{P}_*(\phi)$ such that

$$\text{(ii)} \qquad H^*(a) = \lim_n u_n(a)$$

and $\{u_n\}$ converge uniformly to a harmonic function $U(z)$ in D. We claim that

$$\text{(iii)} \qquad H^*(z) = U(z) \quad : z \in D$$

For assume the contrary. To begin with, since $H^*(z)$ is the maximal function it is clear that $U(z) \leq H^*(z)$ holds in D. So if (iii) fails there exists

$$\text{(iv)} \qquad z_0 = a + re^{\theta_0} \in D \quad : \quad U(z_0) < H^*_\phi(z_0)$$

To see that (iv) cannot occur we regard the point z_0 and again use again the normal family to obtain a sequence $\{v_n\}$ in $\mathcal{P}_*(\phi)$ which converges uniformly to a harmonic function $V(z)$ in D where

$$\text{(v)} \qquad V(z_0) = H^*_\phi(z_0)$$

Now we can derive a contradiction if (iii) fails. Namely, in Ω we have the subharmonic function $w_n = \max(u_n, v_n)$ and taking its harmonic majorant inside D we get a new sequence $\{w^*_n\}$ in $\mathcal{P}^*_\phi$. Passing to a subsequence if necessary w^*_n converges to a limit function $W(z)$ which is harmonic in D and by construction it is $\geq$ both to U and to V. But from (ii) we have

$$\text{(vi)} \qquad U(a) = H^*_\phi(a) \implies U(a) = W(a)$$

At the same time $V \leq W$ and $V(z_0) = W(z_0) > U(z_0)$. Hence we get a strict inequality

$$\text{(vii)} \qquad \frac{1}{2\pi}\int_0^{2\pi} U(a + re^{i\theta}) \cdot d\theta < \frac{1}{2\pi}\int_0^{2\pi} W(a + re^{i\theta}) d\theta$$

But this contradicts the equality $U(a) = W(a)$ from (vi) since both terms in (vii) by the mean value equality express $U(a)$ and $W(a)$. Hence equality holds in (iii) and Theorem 4.1 is proved.

5. Maximum of several harmonic functions

Let $H_1, \ldots, H_k$ be a finite family of harmonic functions defined in some open set Ω. Put

$$u(x,y) = \max \{H_1(x,y), \ldots, H_k(x,y)\}$$

Since harmonic functions satisfy the mean-value condition it follows that the mean-value inequality holds for u and hence u is subharmonic. We are going to describe the non-negative measure $\Delta(u)$. To attain this we introduce the set

$$\Gamma = \bigcup_{i \neq \nu} \{H_i = H_\nu\}$$

Recall from XXX that the zero set of an arbitrary harmonic function consists of a union of smooth real analytic curves $\{\gamma_\alpha\}$ where each pair of these curves may interest in a discrete set and when it occurs this intersection is transveral. Since Γ is a finite union of zero sets of harmonic functions it enjoys the same description. There appears also the discrete set $\sigma(\Gamma)$ where at least two curves intersect. Now $\Gamma \setminus \sigma(\Gamma)$ is a disjoint union of smooth and connected real-analytic curves $\{\gamma_k\}$ called the *regular branches* of Γ. If Ω_0 is a relatively compact subset of Ω only finitely many regular branches intersect Ω_0.

Next, the open complement $\Omega \setminus \Gamma$ has connected components denoted by $\{\Omega_\alpha\}$. To every such component it is clear from the definition of u that there exists an integer $1 \leq i(\alpha) \leq k$ such that

$$u = H_{i(\alpha)} \quad \text{holds in} \quad \Omega_\alpha \tag{1}$$

Let us now consider a regular branch γ_k of Γ. It borders two connected components, say Ω_α and Ω_β. From (1) we get the pair $H_{i(\alpha)}$ and $H_{i(\beta)}$ where

$$H_{i(\alpha)} = H_{i(\beta)} \quad \text{holds on } \gamma_k$$

Since γ_k is a regular branch it follows that the two gradient vectors $\nabla(H_{i(\alpha)}$ and $\nabla)H_{i(\beta)}$ are not equal at any point on γ_k. Let $\mathfrak{n}_\alpha$ be the normal to γ_k which is directed into Ω_α. This means that

$$H_{i(\alpha)} > H_{i(\beta)} \quad \text{holds in} \quad \Omega_\alpha$$

and with this choice of $\mathfrak{n}_\alpha$ we have

$$\partial_{\mathfrak{n}_\alpha}(H_{i(\alpha)}) > \partial_{\mathfrak{n}_\alpha}(H_{i(\beta)}) \quad \text{on } \gamma_k$$

Thus, if ds is the arc-length measure on γ_k we get the positive measure

$$\mu_{\gamma_k} = [\partial_{\mathfrak{n}_\alpha}(H_{i(\alpha)}) - \partial_{\mathfrak{n}_\alpha}(H_{i(\beta)})] \cdot ds \tag{*}$$

5.1 Proposition. *Along γ_k the measure $\Delta(u)$ is given by the positive density above.*

Exercise. Prove this result where the hint us to apply Stokes formula.

Proposition 5.1 gives the following conclusive result:

5.2 Theorem. *The non-negative Riesz measure $\Delta(u)$ is equal to*

$$\sum \mu_{\gamma_k}$$

where the sum is taken over all regular branches of Γ.

Proof. By Propostion 5.1. there only remains to show that $\Delta(u)$ cannot contain a discrete part from discrete point masses in $\sigma(\Gamma)$. But this is clear for if $\Delta(u)$ contains a discrete measure $c \cdot \delta_p$ with $c \neq 0$ and $p \in \sigma(\Gamma)$ then the logarithimc potential of this point mass yields a discontinuous function while u from the start obviously is a continuos function.

5.3 Subharmonic configurations. Above we have clarified that every finite set of harmonic functions $H_1, \ldots, H_k$ yields a subharmonic maximum function. One may ask if this k-tuple can be used to construct other subharmonic functions u than the maximum function. Let us give

5.4 Definition. *A subharmonic configuration of $H_1, \ldots, H_k$ is a subharmonic function u in Ω whose Laplacian is supported by Γ and for every connected component Ω_α of of $\Omega \setminus \Gamma$ one has:*

$$(*) \qquad u = H_{i(\alpha)} \quad \text{for some} \quad 1 \leq i(\alpha) \leq k$$

Thus, when u is a subharmonic configuration then $\Omega \setminus \Gamma$ is covered by by a k-tuple of pairwise disjoint open sets $W_1, \ldots, W_k$ such that $u = H_i$ holds in W_i. Moreover, every W-set is a union of connected components of $\Omega \setminus \Gamma$.

An example. It turns out that there exist subharmonic configurations which which are not given by the maximum function. The following example is due to Borsea and Bögvad in [B-B]:

GIVE Example.

Local uniqueness. The example above leads us to find conditions on the harmonic functions in order that they only admit the obvious subharmonic configuration. In the article [B-B-B] a local uniqueness result is proved which goes as follows: Let $p \in \Omega$ be a point such that k-tuple of gradient vectors $\{\nabla(H_i)(p)\}$ all are extreme points in the convex hull they generate in $\mathbf{R}^2$. Under this condition one has

5.5 Theorem. *Let u be a subharmonic configuration of $H_1, \ldots, H_k$ defined in some open neighborhood of p where all the H-functions are active, i.e. the closure of the open set where $u = H_i$ contains $\{p\}$ for each $1 \leq i \leq k$. Then*

$$u = \max\left(H_1, \ldots, H_k\right)$$

holds in a neighborhood of p.

5.6 A study of Cauchy transforms Let μ be a non-negative Riesz measure whose support is a compact null set K. Now we get the Cauchy transform

$$\mathcal{C}_\mu(z) = \int_K \frac{d\mu(\zeta)}{z - \zeta}$$

Let Ω be an open set which contains K and $g_1, \ldots, g_k$ is some k-tuple of holomorphic functions in Ω. We can impose the condition that for every connected component Ω_α of there exists $1 \leq i(\alpha) \leq k$ such that

$$(*) \qquad \mathcal{C}_\mu | \Omega_\alpha = g_{i(\alpha)}$$

Let us also consider the logarithmic potential

$$U_\mu(z) = \int_K \operatorname{Log}\left(|z - \zeta|\right) \cdot d\mu(\zeta)$$

It turns out that the subharmonic function U_μ is locally piecewise harmonic in Ω. To prove this it suffices to work locally inside Ω so without loss of generality we may assume that Ω from the start is simply connected Then there exist primitive analytic functions $G_1, \ldots, G_k$ of the g-functions. The formulas from XXX show that (*) is equivalent to the condition that for every Ω_α there exists a constant c_α such that

$$(2) \qquad U_\mu(z) | \Omega_\alpha = \mathfrak{Re}(G_{i(\alpha)}) + c_\alpha$$

Here $\{H_i = \mathfrak{Re}(G_i)\}$ are harmonic functions in Ω. If the number of the constants $\{c_\alpha\}$ which appear above is finite it follows that U_μ is piecewise harmonic with respect to a finite set of harmonic functions, i.e. given by the family $\{H_i\}$ plus eventual constants via (2) above. In this *favourable* case we get the similar result as in Theorem 5.2. In particular we conclude that the support of $\Delta(\mu)$ is a union of real-analytic γ-arcs. In [BV-B-B] the following affirmative result is proved without any initial assumption of the local finiteness of the c-constants.

5.7 Theorem. *When (*) holds above it follows that the logarthmic potential U_μ is locally piecewise harmonic and hence the support of $\Delta(\mu)$ consists of a locally finite union of real-analytic curves.*

Remark. The proof Theorem 5.7 is quite involved and we refer to [B-B-B] for details. The difficulty is to show that (*) implies that the number of c-constants from (**) is locally finite.

5.8 Cauchy transforms and algebraic functions. As above μ is a non-negative measure supported by a compact null set K in $\mathbf{C}$. The Cauchy transform $\mathcal{C}_\mu(z)$ is analytic in $\mathbf{C}\setminus K$. Suppose it satisfies an algebraic equation, i.e. there exists some $m \geq 1$ and polynomials $p_0(z), \ldots, p_m(z)$ such that

(*) $$p_m(z) \cdot \mathcal{C}_\mu^m(z) + \ldots p_1(z) \cdot \mathcal{C}_\mu(z) + p_0(z) = 0 \quad : \quad z \in \mathbf{C} \setminus K$$

Using Theorem 5.7 it is proved in [B-B-B] that (*) implies that K is a finite union of real-analytic curves which are related to roots of the algebraic equation

(**) $$p_m(z) \cdot y^m + \ldots p_1(z) \cdot y + p_0(z) = 0 \quad : \quad z \in \mathbf{C} \setminus K$$

5.9 Remark. The result in 5.8 is illustrated by examples from the article [Bergquist-Rullgård). Here asymptotic expansions for distributions of roots of eigenpolynomials which appear for a class of ODE-equations which extend the usual hypergeometric equation. The asymptotic distributions of roots are given by probability measures μ whose Cauchy transforms satisfy an algebraic equation of the form

$$\mathcal{C}_\mu^m(z) = \frac{1}{Q(z)} (*)$$

where $Q(z)$ is a monic polynomial of degree m with simple zeros $\alpha_1, \ldots, \alpha_k$. It is proved in [B-R] that there exists a *unique* probability measure μ with compact support whose Cauchy transform satisfies (*). Moreover, the support of μ is an analytic tree Γ, i.e. a connected set given by a finite union of real-analytic Jordan arcs which meet at some corner points. Moreover. $\mathbf{C} \setminus \Gamma$ is connected.

6. On zero sets of subharmonic functions.

Let Ω in $\mathbf{C}$ be a bounded open set and denote by $\mathrm{SH}_0(\Omega)$ the set of subharmonic functions in Ω whose Laplacian is a Riesz measure supported by a compact null set. Every such function v is locally a logarithmic potential of $\Delta(V)$ plus a harmonic function and can therefore be taken to be upper semi-continuous. Moreover, the distribution derivatives $\partial V/\partial x$ and $\partial V/\partial y$ belong to $L^1_{\mathrm{loc}}(\Omega)$. Before we announce Theorem 6.1 we introduce a geometric construction. If U is an open subset of Ω we construct its forward star-domain as follows: To each $\zeta \in U$ we find the largest $s(\zeta) > 0$ such that the line segment

$$\ell_\zeta(s(\zeta)) = \{\zeta + x :\ 0 \leq x < s(\zeta)\} \subset \Omega$$

Now we put

$$(*) \qquad \mathfrak{s}(U) = \bigcup_{\zeta \in U} \ell_\zeta(s(\zeta))$$

and refer to this open set as the forward star domain of U.

6.1 Theorem. *Let $V \in SH_0(\Omega)$ and put $K = Supp(\Delta(V))$. Suppose that $V = 0$ in an open subset U of $\Omega \setminus K$ and furthermore*

$$(*) \qquad \partial V/\partial x(z) < 0, \quad \text{holds in} \quad \Omega \setminus (K \cup U)\,.$$

Then $V = 0$ in $\mathfrak{s}(U)$.

Proof. It is clear that it suffices to show the following: Let $z_0 \in U$ and consider a horisontal line segment

$$\ell = \{z = z_0 + s :\ 0 \leq s \leq s_0\}$$

which is contained in Ω. Then, if $0 < \delta < \mathrm{dist}(\ell, \partial\Omega)$ and the open disc $D_\delta(z_0)$ of radius δ centered at z_0 is contained in U, it follows that V vanishes in the open set

$$(1) \qquad \{z :\ \mathrm{dist}(z, \ell) < \delta\}$$

Notice that (1) is a relatively compact subset of Ω. Consider the complex derivative

$$\partial V/\partial z = \frac{1}{2}(\partial V/\partial x - i\partial V/\partial y)$$

This yields a complex-valued and locally integrable function and since $V = 0$ in $D_\delta(z_0)$ it is clear that $V = 0$ in the open set from (1) if we prove that $\partial V/\partial z = 0$ holds almost everywhere in (1) To prove this we take some $\epsilon > 0$ and put

$$(2) \qquad \Psi_\epsilon(z) = \log\left(\partial V/\partial z - \epsilon\right)$$

where the single-valued branch of the complex Log-function is chosen so that

$$(3) \qquad \pi/2 < \mathfrak{Im}\,\Psi_\epsilon < 3\pi/2$$

Hence we can write

$$(4) \qquad \Psi_\epsilon(z) = \mathrm{Log}|\epsilon - \partial V/\partial z| + i\tau(z) \quad : \quad \pi/2 < \tau(z) < 3\pi/2$$

A regularisation. Choose a non-negative test-function ϕ with compact support in $|z| \leq \delta$ while $\phi(z) > 0$ if $|z| < \delta$ and $\iint \phi(z)dxdy = 1$. We construct the convolution $\sigma * \Psi_\epsilon$ which is defined in the subset of Ω whose points have distance $> \delta$ to $\partial\Omega$. Rules for first order derivations of a convolution give

$$\bar\partial/\bar\partial\bar z\big(\phi * \Psi_\epsilon\big) = \frac{\phi * \bar\partial\partial(V)}{\partial V/\partial z) - \epsilon} = \frac{1}{4} \cdot \frac{1}{\partial V/\partial z - \epsilon} \cdot \phi * \Delta(V)$$

Taking the real part we get

$$(5) \qquad \mathfrak{Re}(\bar\partial/\bar\partial\bar z(\phi * \Psi_\epsilon)) = \frac{\partial V/\partial x - \epsilon}{4|\epsilon - \partial V/\partial z|^2} \cdot \phi * \Delta(V)$$

To simplify notations we set

$$\sigma(z) = \text{Log}|\epsilon - \partial V/\partial z| \tag{6}$$

The definition of the $\bar{\partial}$-derivative and the decomposition $\Psi_\epsilon = \sigma + i\cdot\tau$ together with the inequality (5) give

$$\partial_x(\phi * \sigma) \leq \partial_y(\phi * \tau) \tag{7}$$

In the right hand side we use the partial y-derivative on ϕ, i.e. we use the general formula:

$$\partial_y(\phi * \tau) = \partial_y(\phi) * \tau$$

Since $\pi/2 \leq \tau \leq 3\pi/2$ the absolute value of this function is majorized by

$$M = \frac{3\pi}{2} \cdot ||\partial_y(\phi)||_1 \tag{8}$$

where $||\partial_y(\phi)||_1$ denotes the L^1-norm. Next, consider the function $s \mapsto \phi * \sigma(z_0 + s)$ where $0 \leq s \leq s_0$ whose s-derivative becomes $\partial_x(\phi * \sigma)(z + s)$. Hence (7-8) give:

$$\frac{d}{ds}(\phi * \sigma(z + s)) \leq M$$

$$\implies \phi * \sigma(z_0 + s_0) \leq \phi * \sigma(z_0) + M \cdot s_0 \tag{9}$$

From now on $\epsilon < 1$ so that $\log \epsilon < 0$. Since $V = 0$ in $D_\delta(z_0)$ we also have $\partial V/\partial z = 0$ in this disc and conclude that

$$\phi * \sigma(z_0) = \log\, \epsilon \tag{10}$$

Next, we have

$$\sigma = \log\, \epsilon + \log\, |1 - \frac{\partial V/\partial z}{\epsilon}|$$

Put

$$f_\epsilon = \log\, |1 - \frac{\partial V/\partial z}{\epsilon}| \tag{11}$$

Then (9-10) give the inequality

$$\phi * f_\epsilon(z_0 + s_0) \leq M \cdot s_0 \tag{12}$$

So from the construction of a convolution this means that

$$\iint_{|\zeta|\leq\delta} f_\epsilon(z_0 + s_0 + \zeta) \cdot \phi(\zeta) \cdot d\xi d\eta \leq M \cdot s_0 \tag{13}$$

Next, we can write

$$f_\epsilon(z) = \frac{1}{2} \cdot \log\big((1 - \frac{\partial V/\partial x}{\epsilon})^2 + (\frac{\partial V/\partial y}{\epsilon})^2\big) \tag{14}$$

Since $\partial V/\partial x \leq 0$ holds almost everywhere we have $f \geq 0$ almost everywhere and at each point z where $\partial V/\partial z \neq 0$ we have

$$\lim_{\epsilon\to 0} f_\epsilon(z) = +\infty \tag{15}$$

Since (13 holds for each $\epsilon > 0$ and $\phi > 0$ in the open disc $|\zeta| < \delta$ we conclude that $\partial V/\partial z = 0$ must hold almost everywhere in the disc $D_\delta(z_0 + s_0)$. Here we considered the largest s-value along the horisontal line ℓ. Of course, we get a similar conclusion for each $0 < s < s_0$ and hence $\partial V/\partial z = 0$ holds almost everywhere in the open set (1). At the same time $V = 0$ in $D_\delta(z_0)$ and we conclude that $V = 0$ in the open set from (1) as requested.

Remark. The method used in the proof above is due to Bergqvist-Rullgård in [Be-Ru] where the Theorem was proved under the assumption that the range of V is a finite set. But the essential idea to regard the complex log-function from (2) in the proof and employ regularisations already occur in [Be-Ru].

7. A subharmonic majorization.

Let $\{0 < \beta_1 < \beta_2 < \ldots\}$ be a sequence of positive real numbers. Given $a > 0$ we construct harmonic functions in the right half-plane $\Omega = \{\mathfrak{Re}\, z > a\}$ as follows: Set $q_\nu = a + i\beta_\nu$ and to each $z \in \Omega$ we get the triangle with corner points at the three points $z, eq_\nu, eq_{\nu+1}$ where e is Neper's constant. Denote the angle at z by $\theta_\nu(z)$. Notice that $0 < \theta_\nu(z) < \pi$. As explained in § XX $\theta_\nu(z)$ is a harmonic function with the boundary value π on $\mathfrak{Re}\, z = a$ when $\beta_\nu < y < \beta_{\nu+1}$ while the boundary value is zero outside the closed y-interval $[\beta\nu, \beta_{\nu+1}]$. If we instead consider the points $\{q_\nu^* = a - ie\beta_\nu\}$ we get similar harmonic angle functions $\{\theta_\nu^*\}$ when we regard the angle at z formed by the triangle with corner points at $z - eq_\nu, -eq_{\nu+1}$.

Exercise. Show by euclidian geometry that if $b < 0$ is real and positive then

$$\sin\, \theta_\nu(a+b) = \frac{eb(\beta_{\nu+1} - \beta_\nu)}{\sqrt{(\beta_\nu^2 + b^2)(\beta_{\nu+1}^2 + b^2)}} \tag{1}$$

Use also that $\beta_\nu < \beta_{\nu+1}$ and show that (1) gives

$$\theta_\nu(a+b) > \frac{eb\beta_1^2 \cdot (\beta_{\nu+1} - \beta_\nu)}{\beta_{\nu+1} \cdot \beta_\nu \cdot (e^2\beta_1^2 + b^2)} = C(b, \beta_1) \cdot (\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) \quad : \quad C(b, \beta_1) = \frac{eb\beta_1^2}{e^2\beta_1^2 + b^2}$$

In addition to these θ_ν-functions we get the angle functions $\{\theta_n^*\}$ where we for each $n \geq 2$ consider the harmonic extension to the half-plane whose boundary values are π on $y > \beta_n$ and zero when $y < \beta_n$. This harmonic function is denoted by $\theta_n^*(z)$ and by a figure the reader can verify that

$$\sin\, \theta_n^*(a+b) = \frac{b}{\beta_n^2 + b^2)} \implies \theta_n^*(a+b) > \frac{eb\beta_1^2}{e^2\beta_1^2 + b^2} \cdot \frac{1}{\beta_n} \tag{2}$$

7.1 A class of harmonic functions. Let us also consider a sequence of positive real numbers $\{\lambda_\nu\}$. To each $n \geq 2$ we get the harmonic function $u_n(x, y)$ in Ω defined by

$$u_n(x, y) = \frac{1}{\pi} \cdot \sum_{\nu=1}^{\nu=n-1} \lambda_\nu \cdot (\theta_\nu + \theta_\nu^*) + \lambda_n \cdot (\theta_n + \theta_n^*) \tag{3}$$

If $z = a + b$ is real with $b > 0$ the inequalities in (1-2) give

$$u_n(a+b) \geq \frac{C(b, \beta_1)}{\pi} \cdot [\sum_{\nu=1}^{\nu=n-1} \lambda_\nu(\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) + \frac{\lambda_n}{\beta_n}] \tag{4}$$

From the above we get:

7.2 Proposition. *Let $\{\lambda_\nu\}$ and $\{\beta_\nu\}$ be such that*

$$\lim_{n\to\infty} [\sum_{\nu=1}^{\nu=n-1} \lambda_\nu(\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}}) = +\infty \tag{*}$$

Then the sequence $\{u_n(a+b)\}$ increases to $+\infty$ for every $b > 0$.

Remark. If the λ-sequence increases a partial summation gives

$$\sum_{\nu=1}^{\nu=n} \frac{\lambda_\nu - \lambda_{\nu-1}}{\beta_\nu} = \sum_{\nu=1}^{\nu=n-1} \lambda_\nu(\frac{1}{\beta_\nu} - \frac{1}{\beta_{\nu+1}})$$

where we have put $\lambda_0 = 0$. Hence (*) is equivalent to the divergence of the positive series

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu - \lambda_{\nu-1}}{\beta_\nu} \tag{**}$$

7.3 An application. Let $\{\beta_\nu\}$ and $\{\lambda_\nu\}$ be two strictly increasing sequences of positive real numbers. Consider an analytic function $\Phi(z)$ defined in the half-plane $\mathfrak{Re}\, z > a$ with continuous boundary values on $\mathfrak{Re}\, z = a$ which satisfies the inequalities

$$(*) \qquad |\Phi(z)| \leq \Big(\frac{\beta_\nu}{|z|}\Big)^{\lambda_\nu} \quad : \quad \nu = 1, 2, \ldots$$

7.4 Exercise. Denote by $u_*(z)$ the harmonic function in the half-plane whose boundary values are one on $-\beta_1 < y < \beta_1$ and otherwise zero. Show that (*) gives the following inequality on $\mathfrak{Re}\, z = a$ for every $n \geq 0$ and $-\infty < y < +\infty$

$$(7.4.1) \qquad \log|\Phi(a + iy)| + u_n(a + iy) \leq \log K \cdot u_*(a + iy) \quad : \quad K = \max_{-\beta_1 \leq y \leq \beta_1} |\Phi(a + iy)|$$

Since u_n and u_* are harmonic functions while $\log|\Phi|$ is subharmonic, the principle of harmonic majorization implies that (7.4.1) holds in Ω. In particular, for every real $b > 0$ we have

$$(7.4.2) \qquad \log|\Phi(a + ib)| + u_n(a + ib) \leq \log K \cdot u_*(a + ib)$$

When Φ is not identically zero we can fix some $b > 0$ where $\Phi(a + ib) \neq 0$ and (7.4.2) entails that the sequence $\{u_n(a + ib)\}$ is bounded. Hence Proposition 7.2 gives

7.5 Theorem. *Assume there exists a non-zero analytic function $\Phi(z)$ in the half-plane $\mathfrak{Re}\, z > a$ such that (*) holds in (7.3). Then*

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu - \lambda_{\nu-1}}{\beta_\nu} < \infty$$

Remark. We can rephrase the result above and get a vanishing principle. Namely, if the positive series in Theorem 7.5 is divergent an analytic function $\Phi(z)$ in the half-plane satisfying (7.3) must be identically zero.

7.6 Example. Let $\{c_n\}$ be a sequence of positive real numbers. Suppose that $\Phi(z)$ is analytic in the half-space and satisfies

$$|\Phi(z)| \leq \frac{c_n}{|z|^n} \quad : \quad n = 1, 2, \ldots$$

Then Φ must vanish identically if the series

$$\sum_{n=1}^{\infty} \frac{1}{c_n^{\frac{1}{n}}} = +\infty$$

7.7 A determined moment problem.

In a moment problem one asks for the existence of a non-negative measure μ on the real x-axis such that all higher order moments

$$(*) \qquad \int_0^\infty x^n \cdot d\mu(x) = c_n$$

where the sequence $\{c_n\}$ is given in advance. In 1894 Stieltjes settled this problem. Namely, a necessary and sufficient condition for the existence of μ is that if $\{\beta_\nu\}$ are the coefficients in the continued fractions associated to $\{c_n\}$, then the series

$$\sum (-1)^\nu \cdot \frac{\beta_\nu}{x^{\nu+1}}$$

is positive. We shall now prove a uniqueness in relation to the moment problem. Namely, let $\{c_n\}$ be given and suppose that (*) has at least one solution μ. Following Stieltjes one says that the moment problem is determined if two solutions μ and γ only differ by a discrete measure. A necessary and sufficient condition for the moment problem to be determined was established by

Hamburger in [XX]. The condition is that the continued fractions of $\{c_n\}$ is completely convergent. For this notion we refer to [ibid] and also to [Japanese] as well as the article *sur le probleme des moments* by M. Riesz in [Arkiv 1923].

Since the calculation of a continued fraction is quite involved one may ask for growth conditions on $\{c_n\}$ which at least are are sufficient for the moment problem to be determined. Using the previous results the following sufficiency result was proved in [Carleman-Chapter VIII]:

7.8 Theorem. *Hamburger's moment problem is determined if*

$$\sum \frac{1}{c_n^{\frac{1}{2n}}} = +\infty$$

Proof. Recall from § xx that a signed measure ν without atoms. is determined by the transform

$$V(z) = \int_{-\infty}^{\infty} \frac{d\nu(x)}{z-x}$$

More precisely, if $V(z)$ vanishes identically in the half-plane $\mathfrak{Im}\, z > 0$ then $\nu = 0$. from this we can deduce Theorem 7.8 as follows. Suppose that μ_1 and μ_2 are two non-negative measures which solve the moment problem and that the difference $\nu = \mu_1 - \mu_2$ has no atoms. Set

$$F_k(z) = \int_{-\infty}^{\infty} \frac{d\mu_k(x)}{z-x} \quad : \quad k = 1,2$$

So now $V = F_1 - F_2$. Next, or each $n \geq 1$ we can write

$$\frac{1}{z-x} = -XXX$$

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It follows that

$$F_k(z) = -\sum \frac{c_\nu}{z^{\nu+1}} - XXX$$

Taking the difference it follows that

$$V(z) = F_1(z) - F_2(z) = \frac{1}{z^{2n+1}} \int \frac{x^{2n} \cdot d\nu(x)}{1 - \frac{x}{z}}$$

Next, notice that if z is in the upper half-plane then

$$|1 - \frac{x}{z}| \leq \frac{|z|}{\mathfrak{Im}\, z)}$$

So if $\mathfrak{Im}\, z \geq 1$, the triangle inequality gives

$$|V(z)| \leq \frac{1}{|z|^{2n}} \cdot \int x^{2n} \cdot |d\nu(x)| \leq \frac{2c_{2n}}{|z|^{2n}}$$

Now the assumed divergence in the theorem and (xx) imply that V is identically zero and Theorem XX is proved.

after a TRICK to get Stieltjes case. see page 81 in carleman.

8. Subharmonic minorant of a given function.

Introduction. Let Ω be a bounded open and connected subset of $\mathbf{C}$ and F a non-negative and upper semi-continuous function in Ω which may take values $+\infty$ at some points. Denote by $\mathcal{F}_*$ the class of subharmonic functions u in Ω such that $u(x) \leq F(x)$ hold for every $x \in \Omega$. Set

$$\mathcal{S}_F(x) = \max_{u \in \mathcal{F}_*} u(x)$$

We refer to $\mathcal{S}_F$ as Sjöberg's maximal function associated to F. Recall that every subharmonic function u is upper semicontinuous which implies that for every compact subset K of Ω there is a constant C such that $u(x) \leq C$ on K. It turns out that this is the sole obstruction in order that $\mathcal{S}_F$ itself subharmonic. More precisely the following result was proved by Sjöberg in [1938; Scand. Congress Helsinki]

8.1. Theorem. *If $\mathcal{S}_F$ is bounded on every compact subset of Ω then it is subharmonic and gives therefore the largest subharmonic function in the class $\mathcal{F}_*$.*

Remark. In addition to [Sjöberg[we refer to Domar's article *Subharmonic minorants of a given function* [Arkiv 1954] where Sjöberg's resuts is extended in a wider context and a similar result is proved for subharmonic functions in $\mathbf{R}^k$ when $k \geq 3$. One expects that if $F(x)$ enjoys some finiteness condition then Sjöberg's condition holds so that $\mathcal{S}_F$ gives the largest subharmonic minorant to F. The following sufficency result was proved by Beurling in [Beurling xx]:

8.2. Theorem. *The function $\mathcal{S}_F$ is subharmonic if there exists $\epsilon > 0$ such that*

$$(*) \qquad \iint_\Omega \left[\log^+ F(x,y)\,\right]^{1+\epsilon} < \infty$$

Proof. Assume (*) and with ϵ kept fixed the integral is denoted by $J(F)$. We shall prove that if K is a compact subset of Ω then there exists an integer n such that

$$(1) \qquad \max_{x \in K} u(x) \leq e^n$$

hold for every $u \in \mathcal{F}_*$ and Sjöberg's result entails that $\mathcal{S}_F$ is subharmonic. To prove (1) we fix a positive integer λ and a positive constant C such that

$$(2) \qquad \frac{e}{\pi C^2} + e^{-\lambda} \leq 1$$

Let us take some $u \in \mathcal{F}_*$ and to each integer ν we set

$$(3) \qquad U_\nu = \{e^\nu \leq u < e^{\nu+1}\}$$

Here $\{U_\nu\}$ is a family of disjoint sets whose union is Ω and to each ν we denote by ℓ_ν the area of U_ν. Next, suppose that for some integer $n > \lambda$ there exists a point $z_n \in \Omega$ such that

$$(4) \qquad u(x_n) \geq e^n \quad \text{and} \quad \operatorname{dist}(x_,\partial\Omega) > C \cdot \sqrt{\ell_{n-\lambda} + \ldots + \ell_n}$$

Consider the disc $D_n^* = \{|z - z_n| \leq C \cdot \sqrt{\ell_{n-\lambda} + \ldots + \ell_n}\}$.

Sublemma.The inequality (4) entails that

$$(5) \qquad \max_{z \in D_n^*} u(z) \geq e^{n+1}$$

Proof. We argue by a contradiction. Set

$$\rho = C \cdot \sqrt{\ell_{n-\lambda} + \ldots + \ell_n}$$

If (5) fails the upper semi-continuity of u gives some $\rho_* > \rho$ such that the disc $\Delta = \{|z - z_n| \leq \rho^*$ is contained in Ω and $u \leq e^{n+1}$ holds in Δ. The mean-value inequality gives

$$(6) \qquad e^n \leq u(z_n) \leq \frac{1}{\pi\rho_*^2} \iint_\Delta u(x,y)\,dxdy$$

Now $U_\nu \cap \Delta = \emptyset$ if $\nu > n$ and since the U-sets are disjoint the right hand side in (6) is majorised by

$$\text{(7)} \qquad \frac{1}{\pi\rho_*^2} \cdot e^{n+1}(\ell_n + \cdots + \ell_{n-\lambda}) + e^{n-\lambda} = e^n \cdot \frac{\rho^2}{\rho_*^2} \cdot [e \cdot \frac{\ell_n + \cdots + \ell_{n-\lambda}}{\pi} + e^{-\lambda}]$$

The last factor is $\frac{e}{\pi C^2} + e^{-\lambda}$ which is ≤ 1 by (2) above. Hence (6-7) would give

$$e^n \leq e^n \cdot \frac{\rho^2}{\rho_*^2}$$

Thus is a contradiction since $\rho_* > \rho$ and hence (4) $\implies$ (5) holds.

Proof continued. Next, given $z_n \in \Omega$ where $u(z_n) \geq e^n$ we set

$$\xi(m) = C \cdot \sum_{\nu=n}^{\nu=m} \sqrt{\ell_{\nu-\lambda} + \ldots + \ell_\nu} \quad : \forall\, m > n$$

Repeated use of (4) $\implies$ (5) shows that when $m > n$ and the disc $\{|z - z_n| \leq \xi(m)\}$ stays in Ω then it contains a point z_m where $u(z_m) \geq e^m$. Since u is bounded over compact subsets of Ω we must have

$$\text{(*)} \qquad \mathrm{dist}(z_n, \partial\Omega) \leq C \cdot \sum_{\nu=n}^{\infty} \sqrt{\ell_{\nu-\lambda} + \ldots + \ell_\nu}$$

Let us majorize the right hand side in (*). Since $\sqrt{a_1 + \ldots + a_k} \leq \sqrt{a_1} + \ldots + \sqrt{a_k}$ hold for all tuples of positive numbers the right hand side in (*) is majorized by

$$C \cdot \sum_{\nu=n}^{\infty} [\sqrt{\lambda_{\nu-\lambda} + \ldots + \lambda_\nu}] \leq C \cdot (\lambda + 1) \sum_{\nu=n-\lambda}^{\infty} \sqrt{\lambda_\nu}$$

To estimate the sum of the square roots we pick the positive ϵ in the Theorem and write

$$\sum_{\nu=n-\lambda}^{\infty} \sqrt{\lambda_\nu} = \sum_{\nu=n-\lambda}^{\infty} \nu^{-1/2-\epsilon/2} \cdot \sqrt{\lambda_\nu} \cdot \nu^{1/2+\epsilon/2} \leq \sqrt{\sum_{\nu=n-\lambda}^{\infty} \nu^{-1-\epsilon}} \cdot \sqrt{\sum_{\nu=n-\lambda} \lambda_\nu \cdot \nu^{1+\epsilon}}$$

where the Cauchy-Schwarz inequality was used. Since $\epsilon > 0$ the first factor is a function $n \mapsto \tau(n)$ given as a square root of tail sums of a convergent series and hence $\tau(n) \to 0$ when $n \to +\infty$. Now (*) is majorised by

$$\text{(**)} \qquad C \cdot \tau(n) \cdot \sqrt{\sum_{\nu=n-\lambda} \lambda_\nu \cdot \nu^{1+\epsilon}}$$

Next, we have

$$\log^+ u(z) \geq \nu : |, z \in U_\nu\}$$

Since the sets $\{U_\nu\}$ are disjoint it follows that

$$\sum_{\nu=n-\lambda} \lambda_\nu \cdot \nu^{1+\epsilon} \leq \iint_\Omega [\log^+ u(x,y)|^{1+\epsilon}\, dxdy \leq J(F)$$

where the last inequality follows since $u \leq F$. Hence (*) gives

$$\mathrm{dist}(z_n, \partial\Omega) \leq C \cdot \sqrt{J(F)} \cdot \tau(n)$$

Finally, if K is a compact subset of Ω its distance to $\partial\Omega$ is a positive number a_K and since $\tau(n) \to 0$ we find a large integer N such that $C \cdot \sqrt{J(F)} \cdot \tau(n) \leq a_K$ and by the above $u \leq e^n$ holds on K. Since $u \in \mathcal{F}_*$ was arbitrary we have proved that $\mathcal{S}_F$ is bounded on K and Theorem 2 is proved.