Operators on Hilbert spaces

Introduction. We shall begin by exposing some general facts about normal operators. The reader may postpone the material in below after § 12 have been studied. But we start here with an exposition of Hilbert's spectral theorem for bounded normal operators on a Hilbert space. For each bounded linear operator $T\colon \mathcal{H} \to \mathcal{H}$ there exists the adjoint operator T^* for which

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

hold for each pair of vectors x, y. It is clear that the pasage to dual operators give a biduality formula, i.e. T is the adjoint of T^* which we express by the equality

$$T = T^{**}$$

When T is invertible so that the bounded inverse operator T^{-1} exists, then T^* is also invertible and one has the equality

(i)
$$(T^*)-1 = (T^{-1})^*$$

Next, the specrum of a bounded operator T is the set of complex numbers λ for which $\lambda \cdot ET$ is not invertible. Using (xx) iy follows that one has the equality

(ii)
$$\sigma(T^*) = \overline{\sigma(T)}$$

Normal operators. A bounded operator T is normal if it commutes with its adjoint in the algebra $L(\mathcal{H})$ of bounded linear operators on the given Hilbert space. A crucial result is that when T is normal, then its operator norm is determined by its spectral radius, i.e. one has the equality

$$||T|| = \max\{|\lambda| : \lambda \in \sigma(T)\}$$

We prove this in § xx and then Neumann's general calculus in § xx entails that the closed subalgebra of $L(\mathcal{H})$ generated by T is a sup-norm algebra which we denote by $\mathcal{B}(T)$. This is a commuative Banach algebra whose maximal ideal space is equal to $\sigma(T)$. Next, we get a larger commutative and closed subalgebra of $L(\mathcal{H})$ which is generated by the commuting operators T and T^*. Again $\mathcal{B}(T, T^*)$ is a sup-norm algebra and in § xx we show that its maxinma, ideal space again is equal to $\sigma(T)$. But a new feature is that the Gelfand transform of T^* is the complex conjugate of the Gelfand transform of T and as a consequence one arrives at the following:

0.1 Theorem. *Let T be a normal operator. Then the Gelfand transform yields an isomorphism between the commutative Banach algebra $\mathcal{B}(T, T^*)$ and $C^0(\sigma(T))$.*

This result combined with Riesz' representation formula for the dual space of $C^0(\sigma(T))$ leads to some important constructions. For each operator $S \in \mathcal{B}(T, T^*)$ we associate the Gelfand transform $\widehat{S} \in C^0(\sigma(T))$. If x, y is pair of vectors in $\mathcal{H}$ we get a linear functional on $C^0(\sigma(T))$ defined by

(i)
$$\widehat{S} \mapsto \langle Sx, y\rangle$$

By the Cauchy-Schwarz inequality one has

$$|\langle Sx, y\rangle| \leq ||S|| \cdot ||x|| \cdot ||y||$$

Since the operator norm $||S||$ is equal to the maximum norrm of $\widehat{S}$, it follows that (i) is a continuous linear functional which gives a unique Riesz measure μ on $\sigma(T)$ such that

$$\langle Sx, y\rangle = \int_{\sigma(T)} \widehat{S}(\lambda) \cdot d\mu(\lambda)$$

Above μ depends on the pair x, y and is therefore denoted by $\mu_{x,y}$. From the above its total variation is bounded by $||x|| \cdot ||y||$. In particular the folowing hold for each positive integer m:

$$\langle T^m x, y\rangle = \int_{\sigma(T)} \lambda^m \cdot d\mu_{x,y}(\lambda)$$

0.2 The enlarged algebra $\mathcal{B}^\infty(T,T^*)$. On $\sigma(T)$ we have the Banach algebra of bounded Borel functions. See the chapter on measure theory for its detailed construction. If ϕ is a bounded Borel function its integral with respect to a Riesz measure on $\sigma(T)$ exists. With $x \in \mathcal{H}$ kept fixed this yields a map

$$y \mapsto \int_{\sigma(T)} \phi(\lambda) \cdot d\mu_{x,y}(\lambda)$$

Exercise. Recall that a Hilbert space is self-dual. Usie this and (xx) to conclude that here exists a unique vector denoted by $\Phi(x)$ in $\mathcal{H}$ such that

$$\langle \Phi(x), y \rangle = \int_{\sigma(T)} \phi(\lambda) \cdot d\mu_{x,y}(\lambda)$$

hold for each y. Show also that $x \mapsto \Phi(x)$ is linear and its operator norm is estimated above by the sup-norm of ϕ.

From the exercise we arrive at the following conclusive result:

0.2.1 Theorem. *Let T be a normal operator. By $\phi \mapsto \Phi$ one has an isomorphism between the Banach algebra of bounded Borel functions on $\sigma(T)$ and a commutative algebra of bounded linear operators on $\mathcal{H}$ which we denote by $\mathcal{B}^\infty(T,T^*)$.*

In particular we can take characterstic functions $\chi\delta$ of Borel ses in $\sigma(T)$ and obtain bounded linear operators denoted by $E(\delta)$. So here

$$\langle E(\delta)(x), y \rangle = \int_\delta d\mu_{x,y}(\lambda)$$

Since we have an algebra isomorphism in Theorem 0..2.1 it follows that each $E(\delta)$ is an idempotent, i.e. the composed operator $E(\delta)^2 = E(\delta)$. Moreover, if δ_1 and δ_2 is a pair of Borel sets then

$$E(\delta_1) \cdot E(\delta_2) = E(\delta_1 \cap \delta_2)$$

0.2.2 Remark. The results above summarize Hilbert's spectral theorem for bounded normal operators which Hilbert established in 1904. Actually he first restricted the result to self-adjoint and bounded operators, but the extension of his original result is an easy consequence of Riesz' representation formula and was later put forward by Hilbert in collaboration with his former student E. Schmidt.

0.3 Unbounded operators.

Let us start with a bounded normal operator R with the property that it is injective and has a dense range $R\mathcal{H}$). But the range is not equal to the whole Hilbert space. ewe get a densely defined operator T whose domain of definition $\mathcal{D}(T)$ is equal to $R\mathcal{H}$), where

$$T(R(x)) = x \quad : \quad x \in \mathcal{H}$$

In particular the range of T is equal to $\mathcal{H}$, and we have also the equality

$$R(T(x)) = x \quad : \quad x \in \mathcal{D}(T)$$

As explained in the section devoted to the Neumann calculus in § xx it means that R is the resolvent of the densely defined operator T. Since R by hypothesis is no surjective it follows that $\lambda = 0$ belongs to $\sigma(R)$. Let $\{\mu - x, y\}$ be the Riesz measures on $\sigma(R)$ given by Theorem 0.2. To each $\delta > 0$ there exists the Borel set

$$\omega(\delta) = \sigma(R) \cap |\lambda| \geq \delta$$

Theorem 0.2 gives a bounded linear operator T_δ for which

$$\langle T_\delta(x), y \rangle = \int_{\omega_\delta} \lambda^{-1} \cdot d\mu_{x,y}(\lambda)$$

Moreover, we get

$$R \cdot T_\delta = E(\omega_\delta)$$

0.3.1 A passage to the limit. Recall from § xx that

$$\langle x, y \rangle = \int_{\sigma(T)} d\mu_{x,y}(\lambda)$$

This entails that

$$\langle x - E(\omega_\delta)(x), y \rangle = \int_{\sigma(T)\setminus\omega_\delta} d\mu_{x,y}(\lambda)$$

Keeping x, fixed we are led to ask if the Riesz meaaure $\mu_{x,y}$ has a nonzero point mass at $\lambda = 0$. To analyze this we take the Borel function ϕ_* which is 1 at $\lambda = 0$ and otherwise zero. By the operational calculus in Theorem 0.2 we have

$$\langle R(x), y \rangle = \int_{\sigma(T)} \lambda \cdot d\mu_{x,y}(\lambda)$$

This entials that

$$R \cdot \Phi_* = 0$$

Since R by assumption is injective we conclude that the operator Φ_* is zero which means that the mass of $\mu_{x,y}$ at the singleton set $\{0\}$ is zero. Togehter with (xx) this entials that the oprators $\{E(\omega_\delta\}$ converge weakly to the identity in the sense that

$$\lim_{\delta \to 0} \langle E(\omega_\delta)(x), y \rangle = \langle x, y \rangle$$

hold for all pairs x, y. Together with (xx) one therefore expects that there exists a limit

$$\lim_{\delta \to 0} T_\delta(x) = T(x) \quad : \quad x \in \mathcal{D}(T)$$

Thks is indeed true but some care must be taken into th acount.Moe precisely, the limit above is taken in the weak sense, i.e. for each vector $y \in \mathcal{H}$ it holds that

$$\lim_{\delta \to 0} \langle T_\delta(x), y \rangle = \langle T(x), y \rangle \quad : \quad x \in \mathcal{D}(T)$$

0.3.2 Concluding remarks. Above we have presented some crucial results in the spectral theory for operators on Hilbert spaces. Let us also remark that we could have started with an unbounded densely defined operator T with a closed graph. Assume that its spectrum is not the whole complex plane which gives the family of Neumann's resolvents $\{R_T(\lambda)$ defined outside the spectrum of T. Let Ω be a connected component in $\mathbf{C} \setminus \sigma(T)$. Using the local Neumann series from (xx) one easily verifies that if $R_T(\lambda_0)$ is normal for one point $\lambda_0 \in \mathbf{C} \setminus \sigma(T)$, then the resolvents $R_T(\lambda)$ are normal for all points in Ω. Recall also from § xx that the range of every resolvent is equal to $\mathcal{D}(T)$. With a fixed $\lambda_0 \in \Omega$ we can start with the normal operator $R_T(\lambda_0)$ and as above construct the densely defined operator $T(\lambda_0)$ for which

$$T(\lambda_0) \circ R_T(\lambda_0) = E$$

These constructions can be used to investigate T. We shall not continue the discussion any further since our main concern is about self-adjoint operators which are treated in § 1. But in many applications it is useful to dispose the theory about densely defined operators which have normal resolvents.

1. Bounded self-adjoint operators.

Let $\mathcal{H}$ be a complex Hilbert space. A bounded linear operator S on $\mathcal{H}$ is self-adjoint if $S = S^*$, or equivalently

(*) $$\langle x, Sy \rangle = \text{the complex conjugate of } \langle Sx, y \rangle \quad : \quad x, y \in \mathcal{H}$$

If S is self-adjoint we have the equality of operator norms:

(i) $$||S||^2 = ||S^2||$$

To see this we notice that if $x \in \mathcal{H}$ has norm one then

(ii) $$\langle Sx, Sx \rangle = \langle x, S^*Sx \rangle = \langle x, S^2x \rangle$$

By the Cauchy-Schwarz inequality the last term is $\leq ||x| \cdot ||S^2||$. Since (ii) holds for every x of norm one we conclude that

$$||S||^2 \leq ||S^2||$$

Now (i) follows from the multiplicative inequality for operator norms. Next, by induction over n we get the equalities

$$||S||^{2n} = ||S^n||^2 \quad : \; n \geq 1$$

Taking the n:th root and passing to the limit the spectral radius formula gives

(*) $$||S|| = \max_{z \in \sigma(S)} |z|$$

Next, we consider the spectrum of self-adjoint operators.

1.1 Theorem. *The spectrum of a bounded self-adjoint operator is a compact real interval.*

Proof. Let λ be a complex number and for a given x we set $y = \lambda x - Sx$. It follows that

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 + \lambda \cdot \langle x, Sx \rangle + \bar{\lambda} \cdot \langle Sx, x \rangle$$

Since S is self-adjoint we get

$$\lambda \cdot \langle x, Sx \rangle + \bar{\lambda} \cdot \langle Sx, x \rangle = 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle Sx, x \rangle$$

Now $|\langle Sx, x \rangle| \leq ||Sx|| \cdot ||x||$ so the triangle inequality gives

(i) $$||y||^2 \geq |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 - 2|\mathfrak{Re}(\lambda)|| \cdot ||Sx|| \cdot ||x||$$

With $\lambda = a + ib$ the right hand side becomes

$$b^2||x||^2 + a^2||x||^2 + ||Sx||^2 - 2a \cdot ||Sx|| \cdot ||x|| \geq b^2||x||^2$$

Hence we have proved that

(ii) $$||\lambda x - Sx||^2 \geq (\mathfrak{Im}\,\lambda)^2 \cdot ||x||^2$$

This implies that $\lambda E - S$ is invertible for every non-real λ which proves Theorem 1.1. Notioce that the proof also gives

(iii) $$||(\lambda E - S)^{-1}|| \leq \frac{1}{|\mathfrak{Im}\,\lambda|}$$

Theorem 1.1 together with general results about uniform algebras in § XX give the following:

1.2 Theorem. *Denote by* $\mathbf{S}$ *the closed subalgebra of* $L(\mathcal{H}, \mathcal{H})$ *generated by* S *and the identity operator. Then* $\mathbf{S}$ *is a sup-norm algebra which is isomorphic to the sup-norm algebra* $C^0(\sigma(S))$.

Exercise. Let T be an arbitary bounded opertor on $\mathcal{H}$. Show that the operator $A = T^*T$ is self-adjoint and that $\sigma(A)$ is a compact subset of $[0, +\infty)$, i.e. every point in its spectrum os real and non-negative. A hint is to use the biduality formula $T = T^{**}$ and if s is real the reader should verify that

$$||sx + T^*Tx||^2 = s^2||x||^2 + 2s \cdot ||Tx||^2 + ||T^*Tx||^2$$

1.3 Normal operators.

A bounded linear operator A is normal if it commutes with its adjoint A^*. Let A be normal and put $S = A^*A$ which yields a self-adjoint by Exercise 1.2. Hence (*) above Theorem 1.1 gives

(i) $$||S||^2 = ||S^2|| = ||A^2 \cdot A(^*)^2|| \leq ||A^2|| \cdot ||(A^*)^2||$$

where we used the multiplicative inequality for operator norms. Now $(A^*)^2$ is the adjoint of A^2 and we recall from § xx that the norms of an operator and its adjoint are equal. Hence the right hand side in (1) is equal to $||A^2||^2$. At the same time

$$||S|| = ||A^*A|| = ||A||^2$$

and we conclude that (i) gives

(ii) $$||A||^2 \leq ||A^2||$$

Exactly as in the self-adjoint case we can take higher powers and obtain the equality

(1.3.1) $$||A|| = \max_{z \in \sigma(A)} |z|$$

Since every polynomial in A again is a normal operator for which (1.3.1) holds we have proved the following:

1.4 Theorem *Let A be a normal operator. Then the closed subalgebra* $\mathbf{A}$ *generated by A in $L(\mathcal{H}, \mathcal{H})$ is a sup-norm algebra.*

Remark. The spectrum $\sigma(A)$ is some compact subset of $\mathbf{C}$ and in general analytic polynomials restricted to $\sigma(A)$ do not generate a dense subalgebra of $C^0(\sigma(A))$. To get a more extensive algebra we consider the closed subalgebra $\mathcal{B}$ of $L(\mathcal{H}, \mathcal{H})$ which is generated by A and A^*. Since every polynomial in A and A^* again is a normal operator it follows that $\mathcal{B}$ is a sup-norm algebra and here the following holds:

1.5 Theorem. *The sup-norm algebra $\mathcal{B}$ is via the Gelfand transform isomorphic with $C^0(\sigma(A))$.*

Proof. Let $Q \in \mathcal{B}$ be arbitrary. Now $S = Q + Q^*$ is self-adjoint and Theorem 1.1 entails that its Gelfand transform is real-valued, i.e. the function $\widehat{Q}(p) + \hat{Q}^*(p)$ is real. So if with $\widehat{Q}(p) = a + ib$ we must have $\hat{Q}^* = a_1 - ib$ for some real number a_1. Next, QQ^* is also self-adjoint and hence $(a + ib)(a_1 - ib)$ is real. This gives $a = a_1$ and which shows that the Gelfand transform of Q^* is the complex conjugate function of $\widehat{Q}$. Hence the Gelfand transforms of $\mathcal{B}$-elements is a self-adjoint algebra and the Stone-Weierstrass theorem implies that the Glefand transforms of $\mathcal{B}$-elements is equal to the whole algebra $C^0(\mathfrak{M}_{\mathcal{B}}$. Finally, since $\widehat{A}^*$ is the complex conjugate function of $\widehat{A}$ it follows that the Gelfand transform $\widehat{A}$ separates points on $\mathfrak{M}_{\mathcal{B}}$ which means that this maximal ideal space can be identified with $\sigma(A)$.

1.6 Spectral measures.

Let A be a normal operator and $\mathcal{B}$ is the Banach algebra above. Each pair of vectors x, y in $\mathcal{H}$ yields a linear functional on $\mathcal{B}$ defined by

$$T \mapsto \langle Tx, y \rangle$$

Identifying $\mathcal{B}$ with $C^0(\sigma(A))$, the Riesz representation formula gives a unique Riesz measure $\mu_{x,y}$ on $\sigma(A)$ such that

(1.6.1) $$\langle Tx, y \rangle = \int_{\sigma(A)} \widehat{T}(z) \cdot d\mu_{x,y}(z)$$

hold for every $T \in \mathcal{B}$. Since $\widehat{A}(z) = z$ we have

$$\langle Ax, y \rangle = \int z \cdot d\mu_{x,y}(z)$$

Similarly one has

$$\langle A^* x, y\rangle = \int \bar{z} \cdot d\mu_{x,y}(z)$$

1.7 The operators $E(\delta)$. Notice that (1.6.1) implies that the map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures on $\sigma(A)$ is bi-linear. We have for example:

$$\mu_{x_1+x_2,y} = \mu_{x_1,y} + \mu_{x_2,y}$$

Moreover, since $\mathcal{B}$ is the sup-norm algebra $C^0(\sigma(A))$ the total variations of the μ-measures satisfy the equations:

$$||\mu_{x,y}|| \leq \max_{T \in \mathcal{B}_*} |\langle Tx, y\rangle| \tag{1.7.1}$$

where $\mathcal{B}_*$ is the unit ball in $\mathcal{B}$. From this we obtain

$$||\mu_{x,y}|| \leq ||x|| \cdot ||y|| \tag{1.7.2}$$

Next, let δ be a Borel subset of $\sigma(A)$. Keeping y fixed in $\mathcal{H}$ we obtain a linear functional on $\mathcal{H}$ defined by

$$x \mapsto \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta)$$

By (1.7.2) it has norm $\leq ||y||$ and is represented by a vector $E(\delta)x$ in $\mathcal{H}$. More precisely

$$\langle E(\delta)x, y\rangle = \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta) \tag{1.7.3}$$

1.7.4 Exercise. Show that $x \mapsto E(\delta)x$ is linear and that the resulting linear operator $E(\delta)$ commutes with all operators in $\mathcal{B}$. Moreover, show that it is a self-adjoint projection, i.e.

$$E(\delta)^2 = E(\delta) \quad \text{and} \quad E(\delta)^* = E(\delta)$$

Finally, show that

$$E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$$

holds for every pair of Borel subsets and with $\delta = \sigma(A)$ one gets the identity operator.

1.7.5 Resolution of the identity. If $\delta_1, \ldots, \delta_N$ is any finite family of disjoint Borel sets whose union is $\sigma(A)$ then

$$1 = E(\delta_1) + \ldots + E(\delta_N)$$

At the same tine we get a decomposition of the operator A:

$$A = A_1 + \ldots + A_N \quad \text{where} \quad A_k = E(\delta_k) \cdot A$$

For each k the spectrum $\sigma(A_k)$ is equal to the closure of δ_k. So the normal operator is represented by a sum of normal operators where the individual operators have small spectra when the δ-partition is fine.

2. Unbounded operators on Hilbert spaces

Let T be a densely defined linear operator on a complex Hilbert space $\mathcal{H}$. We suppose that T is unbounded so that:

$$\max_{x\in\mathcal{D}_*(T)} ||Tx|| = +\infty \quad \mathcal{D}_*(T) = \text{ the set of unit vectors in } \quad \mathcal{D}(T)$$

2.1 The adjoint T^*. If $y \in \mathcal{H}$ we get a linear functional on $\mathcal{D}(T)$ defined by

(i) $$x \mapsto \langle Tx, y\rangle$$

If there exists a constant $C(y)$ such that the absolute value of (i) is $\leq C(y)\cdot||x||$ for every $x \in \mathcal{D}(T)$, then (i) extends to a continuous linear functional on $\mathcal{H}$. The extension is unique because $\mathcal{D}(T)$ is dense and since $\mathcal{H}$ is self-dual there exists a unique vector T^*y such that

(2.1.1) $$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad : \quad x \in \mathcal{D}(T)$$

This gives a linear operator T^* where $\mathcal{D}(T^*)$ is characterised as above. Now we shall describe the graph of T^*. For this purpose we consider the Hilbert space $\mathcal{H}\times\mathcal{H}$ equipped with the inner product

$$\langle (x,y),(x_1,y_1)\rangle = \langle x,x_1\rangle + \langle y,y_1\rangle$$

On $\mathcal{H}\times\mathcal{H}$ we define the linear operator

$$J(x,y) = (-y,x)$$

2.2 Proposition. *For every densely defined operator T one has the equality*

$$\Gamma(T^*) = J(\Gamma(T))^\perp$$

Proof. Let (y, T^*y) be a vector in $\Gamma(T^*)$. If $x \in \mathcal{D}(T)$ the equality (2.1.1) and the construction of J give

$$\langle (y, -Tx\rangle + \langle T^*y, x\rangle = 0$$

This proves that $\Gamma(T^)* \perp J(\Gamma(T))$. Conversely, if $(y,z) \perp J(\Gamma(T))$ we have

(i) $$\langle y, -Tx\rangle + \langle z, x\rangle = 0 \quad : x \in \mathcal{D}(T)$$

This shows that $y \in \mathcal{D}(T^*)$ and $z = T^*y$ which proves Proposition 2.2.

2.3 Consequences. The orthogonal complement of a subspace in a Hilbert space is always closed. Hence Proposition 2.2 entails that T^* has a closed graph. Passing to the closure of $\Gamma(T)$ the decompostion of a Hilbert space into a direct sum of a closed subpace and its orthogonal complement gives

(2.3.1) $$\mathcal{H}\times\mathcal{H} = \overline{J(\Gamma(T))} \oplus \Gamma(T^*)$$

Notice also that

(2.3.2) $$\Gamma(T^*)^\perp = \overline{J(\Gamma(T))}$$

2.4 Closed extensions of operators. A closed operator S is called a closed extension of T if

$$\Gamma(T) \subset \Gamma(S)$$

2.4.1 Exercise. Show that if S is a closed extension of T then

$$S^* = T^*$$

2.4.2 Theorem. *A densely defined operator T has a closed extension if and only if $\mathcal{D}(T^*)$ is dense. Moreover, if T is closed one has the biduality formula $T = T^{**}$.*

Proof. Suppose first that T has a closed extension. If $\mathcal{D}(T^*)$ is not dense there exists a non-zero vector $0 \neq h \perp \mathcal{D}(T^*)$ and (2.3.2) gives

(ii) $$(h,0) \in \Gamma(T^*)^\perp = J(\Gamma(T))$$

By the construction of J this would give $x \in \mathcal{D}(T)$ such that $(h,0) = (-Tx, x)$ which cannot hold since this equation first gives $x = 0$ and then $h = T(0) = 0$. Hence closedness of T implies that $\mathcal{D}(T^*)$ is dense. Conversely, assume that $\mathcal{D}(T^*)$ is dense. Starting from T^* we construct its adjoint T^{**} and Proposition 2.3.2 applied with T^* gives

(i) $$\Gamma(T^{**}) = J(\Gamma(T^*)^{\perp}$$

At the same time $J(\Gamma(T^*)^{\perp}$ is equal to the closure of $\Gamma(T)$ so (i) gives

(ii) $$\overline{\Gamma(T)} = \Gamma(T^{**})$$

which proves that T^{**} is a closed extension of T.

2.4.3 The biduality formula. Let T be closed. and densely defined operator. from the above T^* also is densely defined and closed. Hence its dual exists. It is denoted by T^{**} and called the bi-dual of T. With these notations one has:

(*) $$T = T^{**}$$

2.4.4 Exercise. Prove the equality (*).

2.5 Inverse operators.

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Denote by $\mathfrak{I}(\mathcal{H})$ the set of closed and densely defined operators T such that T is injective on $\mathcal{D}(T)$ and the range $T(\mathcal{D}(T))$ is dense in $\mathcal{H}$. If $T \in \mathfrak{I}(\mathcal{H})$ there exists the densely defined operator S where $\mathcal{D}(S)$ is the range of T and

$$S(Tx) = x \quad : \quad x \in \mathcal{D}(T)$$

By this construction the range of S is equal to $\mathcal{D}(T)$. Next, on $\mathcal{H} \times \mathcal{H}$ we have the isometry defined by $I(x,y) = (y,x)$, i.e we interchange the pair of vectors. The construction of S gives

(i) $$\Gamma(S) = I(\Gamma(T))$$

Since $\Gamma(T)$ by hypothesis is closed it follows that S has a closed graph and we conclude that $S \in \mathfrak{I}(\mathcal{H})$. Moreover, since I^2 is the identity on $\mathcal{H} \times \mathcal{H}$ we have

(ii) $$\Gamma(T) = I(\Gamma(S))$$

We refer to S is as the inverse of T. It is denoted by T^{-1} and (ii) entails that T is the inverse of T^{-1}, i.e. one has

(*) $$T = (T^{-1})^{-1}$$

2.5.1 Exercise. Let T belong to $\mathfrak{I}(\mathcal{H})$. Use the description of $\Gamma(T^*)$ in Proposition 2.3 to show that T^* belongs to $\mathfrak{I}(\mathcal{H})$ and the equality

(**) $$(T^{-1})^* = (T^*)^{-1}$$

2.6 The operator T^*T

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Each $h \in \mathcal{H}$ gives the vector $(h,0)$ in $\mathcal{H} \times \mathcal{H}$ and (2.3.1) gives a pair $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$. such that

$$(h,0) = (x, Tx) + (-T^*y, y) = (x - T^*y, Tx + y)$$

With $u = -y$ we get $Tx = u$ and obtain

(2.6.1) $$h = x + T^*(Tx)$$

2.6.2 Proposition. *The vector x in (2.6.1) is uniquely determined by h.*

Proof. Uniqueness follows if we show that

$$x + T^*(Tx) \implies x = 0$$

But this is clear since the construction of T^* gives

$$0 = \langle x, x)\rangle + \langle x, T^*(Tx)\rangle = \langle x, x)\rangle + \langle Tx, Tx)\rangle \implies x = 0$$

2.7 The density of $\mathcal{D}(T^*T)$. This is the subspace of $\mathcal{D}(T)$ where the extra condition for a vector $x \in \mathcal{D}(T)$ is that $Tx \in \mathcal{D}(T^*)$. To prove that $\mathcal{D}(T^*T)$ is dense we consider some orthogonal vector h. Proposition 2.6 gives some $x \in \mathcal{D}(T)$ such that $h = x + T^*(Tx)$ and for every $g \in \mathcal{D}(T^*T)$ we have

(i) $$0 = \langle x, g\rangle + \langle T^*Tx, g\rangle = \langle x, g\rangle + \langle Tx, Tg\rangle = \langle x, g\rangle + \langle x, T^*Tg\rangle$$

Here (i) hold for every $g \in \mathcal{D}(T^*T)$ and by another application of Proposition 2.6 we find g so that $x = g + T^*Tg$ and then (i) gives $\langle x, x\rangle = 0$ so that $x = 0$. But then we also have $h = 0$ and the requested density follows.

2.8 Conclusion. Set $A = T^*T$. From the above it is densely defined and (2.6.1) entails that the densely defined operator $E + A$ is injective. Moreover, its range is equal to $\mathcal{H}$. Notice that

$$\langle x + Ax, x + Ax\rangle = c + \langle x, Ax\rangle + \langle Ax, x\rangle$$

Here

$$\langle x, Ax\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = ||Tx||^2$$

and from this the reader can conclude that

$$||x + Ax||^2 = ||x||^2 + ||Ax||^2 + 2 \cdot ||Tx||^2 \quad : x \in \mathcal{D}(A)$$

The right hand side is $\geq ||x||^2$ which implies that $E + A$ is invertible in Neumann's sense.

2.9 The equality $A^* = A$. Recall the biduality formula $T = T^{**}$ and apply Proposition 2.6.2 starting with T^*. It follows that $\mathcal{D}(TT^*)$ also is dense and exactly as in (2.6.1) every $h \in \mathcal{H}$ has a unique representation

$$h = y + T(T^*y)$$

2.10. Exercise. Verify from the above that A is self-adjoint, i.e one has the equality $A = A^*$.

§ 2.B Unbounded self-adjoint operators.

A densely defined operator A on the Hilbert space $\mathcal{H}$ for which $A = A^*$ is called self-adjoint.

2.B.1 Proposition *The spectrum of a self-adjoint operator A is contained in the real line, and if λ is non-real the resolvent satisfies the norm inequality*

$$||R_A(\lambda)|| \leq \frac{1}{|\mathfrak{Im}\,\lambda|}$$

Proof. Set $\lambda = a + ib$ where $b \neq 0$. If $x \in \mathcal{D}(A)$ and $y = \lambda x - Ax$ we have

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Ax||^2 - 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle x, Ax\rangle$$

The Cauchy-Schwarz inequality gives

(i) $$||y||^2 \geq b^2||x||^2 + a^2||x||^2 + ||Ax||^2 - 2|a| \cdot ||Ax|| \cdot ||x|| \geq b^2||x||^2$$

This proves that $x \to \lambda x - Ax$ is injective and since A is closed the range of $\lambda \cdot E - A$ is closed. Next, if y is $\perp$ to this range we have

$$0 = \lambda\langle x, y\rangle - \langle Ax, y\rangle \quad : x \in \mathcal{D}(A)$$

From this we see that y belongs to $\mathcal{D}(A^*)$ and since A is self-adjoint we get

$$0 = \lambda\langle x, y\rangle - \langle x, Ay\rangle$$

This holds for all x in the dense subspace $\mathcal{D}(A)$ which gives $\lambda \cdot y = Ay$ Since λ is non-real we have already seen that this entails that $y = 0$. Hence the range of $\lambda \cdot E - A$ is equal to $\mathcal{H}$ and the inequality (i) entails $R_A(\lambda)$ has norm $\leq \frac{1}{|\mathfrak{Im}\lambda|}$.

2.B.2 A conjugation formula. Let A be self-adjoint. For each complex number λ the hermitiain inner product on $\mathcal{H}$ gives the equation

$$\bar{\lambda} - A = (\lambda \cdot E - A)^*$$

So when we take the complex conjugate of λ it follows that § 2.5 that

$$R_A(\lambda)^* = R_A(\bar{\lambda}) \tag{2.5.1}$$

2.B.3 Properties of resolvents. Let A be self-adjoint. By Neumann's resolvent calculus the family $\{(R_A(\lambda)\}$ consists of pairwise commuting bounded operators outside the spectrum of A. Since $\sigma(A)$ is real there exist operator-valued analytic functions $\lambda \mapsto R_A(\lambda)$ in the upper- respectively the lower half-plane. Moreover, since Neumann's resolvents commute, it follows from (2.5.1) that $R_A(\lambda)$ commutes with its adjoint. Hence every resolvent is a bounded normal operator.

2.B.4 A special resolvent operator. Take $\lambda = i$ and set $R = R_A(i)$. So here

$$R(iE - A)(x) = x \quad : \quad x \in \mathcal{D}(A)$$

2.B.5 Theorem. *The spectrum $\sigma(R)$ is contained in the circle*

$$C_* = \{|\lambda + i/2| = 1/2\}$$

Proof. Since $\sigma(A)$ is confined to the real line, it follows from § 0.0. XX that points in $\sigma(R)$ have the form

$$\lambda = \frac{1}{i - a} \quad : a \in \mathbf{R}$$

This gives

$$\lambda + i/2 = \frac{1}{i-a} + i/2 = \frac{1}{2(i-a)}(2 + i^2 - ia) = \frac{1 - ia}{2i(1 + ia)}$$

and the last term has absolute value $1/2$ for every real a.

2.C. The spectral theorem for unbounded self-adjoint operators.

The operational calculus in § 1.3-1.6 applies to the bounded normal operator R in § 2.14. If N is a positive integer we set

$$C_*(N) = \{\lambda \in C_* :\ \mathfrak{Im}(\lambda) \leq -\frac{1}{N}\} \quad \text{and} \quad \Gamma_N = C_*(N) \cap \sigma(R)$$

Let χ_{Γ_N} be the characterstic function of Γ_N. Now

$$g_N(\lambda) = \frac{1 - i\lambda}{\lambda} \cdot \chi_{\Gamma_N}$$

is Borel functionon $\sigma(R)$ which by operational calculus in § 1.xx gives a bounded and normal linear operator denoted by G_N. On Γ_N we have $\lambda = -i/2 + \zeta$ where $|\zeta| = 1/2$. This gives

$$\text{(1)} \qquad \frac{1 - i\lambda}{\lambda} = \frac{1/2 - i\zeta}{-i/2 + \zeta} = \frac{(1/2 - i\zeta)(i/2 + \bar{\zeta})}{|\zeta - i/2|^2} = \frac{\mathfrak{Re}\,\zeta}{|\zeta - i/2|^2}$$

By § 1.x the spectrum of G_N is the range of the g-function on Γ_N and (1) entails that $\sigma(G_N)$ is real. Since G_N also is normal it follows that it is self-adjoint. Next, notice that

$$\text{(2)} \qquad \lambda \cdot (\frac{1 - i\lambda}{\lambda} + i) = 1$$

holds on Γ_N. Hence operational calculus gives the equation

$$\text{(3)} \qquad R(G_N + i) = E(\Gamma_N)$$

where $E(\Gamma_N)$ is a self-adjoint projection. Notice also that

$$\text{(4)} \qquad R \cdot G_N = (E - iR) \cdot E(\Gamma_N)$$

Hence (3-4) entail that

$$\text{(5)} \qquad E(\Gamma_N) - iRE(\Gamma_N) = (E - iR) \cdot E(\Gamma_N)$$

Next, the equation $RA = E - iR$ gives

$$\text{(*)} \qquad RAE(\Gamma_N) = (E - iR)E(\Gamma_N) = R \cdot G_N$$

2.C.1 Exercise. Conclude from the above that

$$\text{(*)} \qquad AE(\Gamma_N) = G_N$$

Show also that:

$$\text{(**)} \qquad \lim_{N \to \infty} AE(\Gamma_N)(x) = A(x) \quad \text{for each} \quad x \in \mathcal{D}(A)$$

2.C.2 A general construction. For each bounded Borel set e on the real line we get a Borel set $e_* \subset \sigma(R)$ given by

$$e_* = \sigma(R) \cap \{\frac{1}{i - a}\, a \in e\}$$

The operational calculus gives the self-adjoint operator G_e constructed via $g \cdot \chi_{e_*}$. We have also the operstor $E(e)$ given by χ_{e_*} and exactly as above we get

$$AE(e) = G_e$$

The bounded self-adjoint operators $E(e)$ and G_e commute with A and $\sigma(G_e)$ is contained in the closure of the bounded Borel set e. Moreover each $E(e)$ is a self-adjoint projection and for each pair of bounded Borel sets we have

$$E(e_1)E(e_2) = E(e_1 \cap E(e_1)$$

In particular the composed operators

$$E(e_1) \circ E(e_2) = 0$$

when the Borel sets are disjoint.

2.C.3 The spectral measure. Exactly as for bounded self-adjoint operators the results above give rise to a map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures:

$$(x, y) \mapsto \mu_{x,y}$$

For each real-valued and bounded Borel function $\phi(t)$ on the real line with compact support there exists a bounded self-adjoint operator ϕ such that

$$\langle \Phi(x), y \rangle = \int g(t) \cdot d\mu_{x,y}(t)$$

All these Φ operators commute with A. If $x \in \mathcal{D}(A)$ and y is a vector in $\mathcal{H}$ one has

$$\langle A(x), y \rangle = \lim_{M \to \infty} \int_{-M}^{M} t \cdot d\mu_{x,y}(t)$$

§ 3. Symmetric operators

A densely defined and closed operator T on a Hilbert space $\mathcal{H}$ is symmetric if

$$(*) \qquad \langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{hold for all pairs} \quad x, y \in \mathcal{D}(T)$$

The symmetry means that the adjoint T^* extends T, i.e.

$$\Gamma(T) \subset \Gamma(T^*)$$

Recall that adjoints always are closed operators. Hence $\Gamma(T^*)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$ and becomes a Hilbert space equipped with the inner product

$$\{x, y\} = \langle x, y\rangle + \langle T^*x, T^*y\rangle$$

Moreover, since T is closed. it follows that $\Gamma(T)$ appears as a closed subspace of this Hilbert space. Consider the eigenspaces:

$$\mathcal{D}_+ = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = ix\} \quad \text{and} \quad \mathcal{D}_- = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = -ix\}$$

3.1 Proposition. *The following orthogonal decomposition exists in the Hilbert space $\Gamma(T^*)$:*

$$(*) \qquad \Gamma(T^*) = \Gamma(T) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$$

Proof. The verification that the three subspaces are pairwise orthogonal is left to the reader. To show that the direct sum above is equal to $\Gamma(T^*)$ we use duality and there remains only to prove that

$$(1) \qquad \Gamma(T)^\perp = \mathcal{D}_+ \oplus \mathcal{D}_-$$

To show (1) we pick a vector $y \in \Gamma(T)^\perp$. Here $(y, T^*y) \in \Gamma(T^*)$ and the definition of orthogonal complements gives:

$$\langle x, y\rangle + \langle Tx, T^*y\rangle = 0 \quad : \quad x \in \mathcal{D}(T)$$

From this we see that $T^*y \in \mathcal{D}(T^*)$ and obtain

$$\langle x, y\rangle + \langle x, T^*T^*y\rangle = 0$$

The density of $\mathcal{D}(T)$ entails that

$$0 = y + T^*T^*y = (T^* + iE)(T^* - iE)(y) \implies$$

$$\xi = T^*y - iy \in \mathcal{D}_- \quad \text{and} \quad \eta = T^*y + iy \in \mathcal{D}_+ \implies$$

$$y = \frac{1}{2i}(\eta - \xi) \in \mathcal{D}_- \oplus \mathcal{D}_+$$

which proves (1).

3.2 The case $\dim(\mathcal{D}_+) = \dim(\mathcal{D}_-)$**.** Suppose that $\mathcal{D}_+$ and $\mathcal{D}_-$ are finite dimensional with equal dimension $n \geq 1$. Then self-adjoint extensions of T are found as follows: Let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathcal{D}_+$ and $f_1, \ldots, f_n$ a similar basis in $\mathcal{D}_-$. For each n-tuple $e^{i\theta_1}, \ldots, e^{i\theta_n}$ of complex numbers with absolute value one we have the subspace of $\mathcal{H}$ generated by $\mathcal{D}(T)$ and the vectors

$$\xi_k = e_k + e^{i\theta_k} \cdot f_k \quad : \quad 1 \leq k \leq n$$

On this subspace we define a linear operator A_θ where $A_\theta = T$ on $\mathcal{D}(T)$ while

$$A_\theta(\xi_k) = ie_k - ie^{i\theta_k} \cdot f_k$$

3.3 Exercise. Verify that A_θ is self-adjoint and prove the converse, i.e. if A is an arbitrary self-adjoint operator such that

$$\Gamma(T) \subset \Gamma(A) \subset \Gamma(T^*)$$

then there exists a unique n-tuple $\{e^{i\theta_\nu}\}$ such that

$$A = A_\theta$$

3.4 Example. Let $\mathcal{H}$ be the Hilbert space $L^2[0,1]$ of square-integrable functions on the unit interval $[0,1]$ with the coordinate t. A dense subspace $\mathcal{H}_*$ consists of functions $f(t) \in C^1[0,1]$ such that $f(0) = f(1) = 0$. On $\mathcal{H}_*$ we define the operator T by

$$T(f) = if'(t)$$

A partial integration gives

$$\langle T(f), g\rangle = i\int_0^1 f'(t) \cdot \bar{g}(t) \cdot dt = \int_0^1 \bar{g}'(t) \cdot f(t)dt = \langle f, T(g)\rangle$$

Hence T is symmetric. Next, an L^2-function h belongs to $\mathcal{D}(T^*)$ if and only if there exists a constant $C(h)$ such that

$$|\int_0^1 if'(t) \cdot \bar{h}(t)dt| \leq C(h) \cdot ||f||_2 \quad : f \in \mathcal{H}_*$$

This means that $\mathcal{D}(T^*)$ consists of all L^2-functions h such that the distribution derivative $\frac{dh}{dt}$ again belongs to L^2.

Exercise. Show that

$$\mathfrak{D}_+ = \{h \in L^2 \quad : \frac{dh}{dt} = ih\}$$

is a 1-dimensional vector space generated by the L^2-function e^{ix}. Similarly, $\mathcal{D}_-$ is 1-dimensional and generated by e^{-ix}.

Self-adjoint extensions of T. For each complex number $e^{i\theta}$ we get the linear space $\mathcal{D}_\theta$ of functions $f(t) \in \mathcal{D}(T^*)$ such that

$$f(1) = e^{i\theta} f(0)$$

Exercise. Verify that one gets a self-adjoint operator T_θ which extends T where is $\mathcal{D}(T_\theta) = \mathcal{D}_\theta$. Conversely, show every self-adjoint extension of T is equal to T_θ for some θ. Hence the family $\{T_\theta\}$ give all self-adjoint extensions of T with their graphs contained in $\Gamma(T^*)$.

3.5 Semi-bounded symmetric operators.

Let T be closed, densely defined and symmetric. It is said to be bounded below if there exists some positive constant k such that

(*) $$\langle Tx, x\rangle \geq k \cdot ||x||^2 \quad : \quad x \in \mathcal{D}(T)$$

On $\mathcal{D}(T)$ we have the Hermitian bilinear form:

(1) $$\{x, y\} = \langle Tx, y\rangle \quad \text{where (*) entails that} \quad \{x, x\} \geq k \cdot ||x||^2$$

In particular a Cauchy sequence with respect to this inner product is a Caichy sequence in the given Hilbert space $\mathcal{H}$. So if $\mathcal{D}_*$ is the completion of $\mathcal{D}(T)$ with respect to the inner product above, then it appears as a subspace of $\mathcal{H}$. Put

$$\mathcal{D}_0 = \mathcal{D}(T^*) \cap \mathcal{D}_*$$

3.5.1 Proposition. *One has the equality*

(*) $$T^*(\mathcal{D}_0) = \mathcal{H}$$

Proof. A vector $x \in \mathcal{H}$ gives a linear functional on $\mathcal{D}_*$ defined by

$$y \mapsto \langle y, x\rangle$$

We have

(i) $$|\langle y, x\rangle| \leq ||x| \cdot ||y|| \leq ||x||| \cdot \frac{1}{\sqrt{k}} \cdot \sqrt{\{y, y\}}$$

where we used (1) above. The Hilbert space $\mathcal{D}_*$ is self-dual. This gives a vector $z \in \mathcal{D}_*$ such that

(iii) $$\langle y, x\rangle = \{y, z\} = \langle Ty, z\rangle$$

Since $\mathcal{D}(T) \subset \mathcal{D}_*$ we have (iii) for every vector $y \in \mathcal{D}(T)$, and the construction of T^* entails that $z \in \mathcal{D}(T^*)$ so that (iii) gives

(iv) $$\langle y, x \rangle = \langle y, T^*(z) \rangle$$

The density of $\mathcal{D}_*$ in $\mathcal{H}$ implies that $x = T^*(z)$ and since $x \in \mathcal{H}$ was arbitrary we get (*) in the proposition.

3.5.2 A self-adjoint extension. Let T_1 be the restriction of T^* to $\mathcal{D}_0$. We leqve it to the reader to check that T_1 is symmetric and has a closed graph. Moroever, since $\mathcal{D}(T) \subset \mathcal{D}_0$ and T^* is an extension of T we have

$$\Gamma(T) \subset \Gamma(T_1)$$

Next, Proposition 4.2.1 gives

$$T_1(\mathcal{D}(T_1) = \mathcal{H}$$

i.e. the T_1 is surjective. But then T_1 is self-adjoint by the general result below.

3.5.3 Theorem . *Let S be a densely defined, closed and symmetric operator such that*

(*) $$S(\mathcal{D}(S)) = \mathcal{H}$$

Then S is self-adjoint.

Proof. Let S^* be the adjoint of S. When $y \in \mathcal{D}(S^*)$ we have by definition

$$\langle Sx, y \rangle = \langle x, S^*y \rangle \quad : \quad x \in \mathcal{D}(S)$$

If $S^*y = 0$ this entails that $\langle Sx, y \rangle = 0$ for all $x \in \mathcal{D}(S)$ so the assumption that $S(\mathcal{D}(S)) = \mathcal{H}$ gives $y = 0$ and hence S^* is injective. Finally, if $x \in \mathcal{D}(S^*)$ the hypothesis (*) gives $\xi \in \mathcal{D}(S)$ such that

(i) $$S(\xi) = S^*(x)$$

Since S is symmetric, S^* extends S so that (i) gives $S^*(x - \xi) = 0$. Since we already proved that S^* is injective we have $x = \xi$. This proves that $\mathcal{D}(S) = \mathcal{D}(S^*)$ which means that S is self-adjoint.

§ 4. Contractions and the Nagy-Szegö theorem

A linear operator A on the Hilbert space $\mathcal{H}$ is a contraction if its operator norm is ≤ 1, i.e.

$$(1) \qquad ||Ax|| \leq ||x|| \quad : \quad x \in \mathcal{H}$$

Let E be the identity operator on $\mathcal{H}$. Now $E - A^*A$ is a bounded self-adjoint operator and (1) gives:

$$\langle x - A^*Ax, x\rangle = ||x||^2 - ||Ax||^2 \geq 0$$

From the result in § 1.xx it follows that this non-negative self-adjoint operator has a square root:

$$B_1 = \sqrt{E - A^*A}$$

Next, the operator norms of A and A^* are equal so A^* is also a contraction and the equation $A^{**} = A$ gives the self-adjoint operator

$$B_2 = \sqrt{E - AA^*}$$

Since $AA^* = A^*A$ is not assumed the self-adjoint operators B_1, B_2 need not be equal. However, the following hold:

4.3.1 Propostion. *One has the equations*

$$AB_1 = B_2A \quad \text{and} \quad A^*B_2 = B_1A^*$$

Proof. If n is a positive integer we notice that

$$\text{(i)} \qquad A(A^*A)^n = (AA^*)^nA$$

Now A^*A is a self-adjoint operator whose compact spectrum is confined to the closed unit interval $[0, 1]$. If $f \in C^0[0, 1]$ is a real-valued continuous function it can be approximated uniformly by a sequence of polynomials $\{p_n\}$ and the operational calculus from § XX yields an operator $f(A^*A)$ where

$$\lim_{n\to\infty} ||p_n(A^*A) - f(A^*A)|| = 0$$

Since the spectrum of AA^* also is confined to $[0, 1]$, the same polynomial sequence $\{p_n\}$ gives an operator $f(AA^*)$ where

$$\lim ||p_n(AA^*) - f(AA^*)|| = 0$$

Now (i) and the two limit formulas above give:

$$\text{(ii)} \qquad A \circ f(A^*A) = f(AA^*) \circ A$$

In particular we can take $f(t) = \sqrt{1-t}$ and Proposition 4.3.1 follows.

4.2 The unitary operator U_A. On the Hilbert space $\mathcal{H} \times \mathcal{H}$ we define a linear operator U_A represented by the block matrix

$$(*) \qquad U_A = \begin{pmatrix} A & B_2 \\ B_1 & -A^* \end{pmatrix}$$

4.3 Proposition. *U_A is a unitary operator on $\mathcal{H} \times \mathcal{H}$.*

Proof. For a pair of vectors x, y in $\mathcal{H}$ we must prove the equality

$$\text{(i)} \qquad ||U_A(x \oplus y)||^2 = ||x||^2 + ||y||^2$$

To get (i) we notice that for every vector $h \in \mathcal{H}$ the self-adjointness of B_1 gives

$$\text{(ii)} \qquad ||B_1h||^2 = \langle B_1h, B_1h\rangle = \langle B_1^2h, h\rangle = \langle h - A^*Ah, h\rangle = ||h||^2 - ||Ah||^2$$

where the last equality holds since we have $\langle A^*Ah, h\rangle = \langle Ah, A^{**}h\rangle = ||Ah||^2$ and the biduality formula $A = A^{**}$. In the same way one has:

$$\text{(iii)} \qquad ||B_2h||^2 = ||h||^2 - ||A^*h||^2$$

Next, by the construction of U_A the left hand side in (i) becomes

$$\text{(iv)} \qquad ||Ax + B_2y||^2 + ||B_1x - A^*y||^2$$

Using (iii) we have

$$||Ax + B_2y||^2 = ||Ax||^2 + ||y||^2 - ||A^*y||^2 + \langle Ax, B_2y\rangle + \langle B_2y, Ax\rangle$$

Similarly, (ii) gives

$$||B_1x - A^*y||^2 = ||x||^2 - ||Ax||^2 + ||A^*y||^2 - \langle B_1x, A^*y\rangle - \langle A^*y, B_x\rangle$$

Adding these two equations we conclude that (i) follows from the equality

$$\text{(v)} \qquad \langle Ax, B_2y\rangle + \langle B_2y, Ax\rangle = \langle B_1x, A^*y\rangle + \langle A^*y, B_x\rangle$$

To get (v) we use Proposition 4.5.1 which gives

$$\langle Ax, B_2y\rangle = \langle x, A^*B_2y\rangle = \langle x, B_1A^*y\rangle = \langle B_1x, A^*y\rangle$$

where the last equality used that B_1 is self-adjoint. In the same way one verifies that

$$\langle B_2y, Ax\rangle = \langle A^*y, B_x\rangle$$

and (v) follows.

4.4 The Nagy-Szegö theorem.

The constructions above were applied by Nagy and Szegö to give:

4.4.1 Theorem *For every bounded linear operator A on a Hilbert space $\mathcal{H}$ there exists a Hilbert space $\mathcal{H}^*$ which contains $\mathcal{H}$ and a unitary operator U_A on $\mathcal{H}^*$ such that*

$$A^n = \mathcal{P} \cdot U_A^n \quad : \quad n = 1, 2, \ldots$$

where $\mathcal{P} \colon \mathcal{H}^* \to \mathcal{H}$ is the orthogonal projection.

Proof. On the product $\mathcal{H}_1 = \mathcal{H} \times \mathcal{H}$ we have the unitary operator U_A from (*) in 4.3.2. Let $\mathcal{P}(x, y) = x$ be the projection onto the first factor. Then (*) in (4.3.2) gives $A = \mathcal{P}U_A$ and the constructions from the proof of Propostion 4.3.4 imply that $A^n = \mathcal{P} \cdot U^n$ hold for every $n \geq 1$ which finishes the proof.

The Nagy-Szegö result has an interesting consequence. Let A be a contraction. If $p(z) = c_0 + c_1 < + \ldots + c_n z^n$ is an arbitrary polynomial with complex coefficients we get the operator $p(A) = \sum c_\nu A^\nu$ and with these notations one has:

4.4.2 Theorem *For every pair $A, p(z)$ as above one has*

$$||p(A)|| \leq \max_{z \in D} |p(z)|$$

where the the maximum in the right hand side is taken on the unit disc.

Proof. Theorem 4.4.1 gives $p(A) = \mathcal{P} \cdot p(U_A)$. Since the orthogonal $\mathcal{P}$-projection is norm decreasing we get

$$||p(A)(\xi)||^2 \leq ||p(U_A)(\xi, 0)||^2$$

Let ξ be a unit vector such that $||p(A)(\xi)|| = ||p(A)||$. The operational calculus in § 7 XX applied to the unitary operator U_A yields a probablity measure μ_ξ on the unit circle such that

$$||p(U_A)(\xi, 0)||^2 = \int_0^{2\pi} |p(e^{i\theta})|^2 \cdot d\mu_\xi(\theta)$$

The right hand side is majorized by $|p|_D^2$ and Theorem 4.4.2 follows.

4.4.3 An application. Let $A(D)$ be the disc algebra. Since each $f \in A(D)$ can be uniformly approximated by analytic polynomials, Theorem 4.4.2 entails that if a linear operator A on the Hilbert space $\mathcal{H}$ is a contraction then each $f \in A(D)$ gives a bounded linear operator $f(A)$, i.e. we have norm-preserving map from the supnorm algebra $A(D)$ into the space of bounded linear operators on $\mathcal{H}$.

§ 5 Miscellanous results

Recall the product formula for matrices in § X which asserts the following. Let $N \geq 2$ and T is some $N \times N$-matrix whose elements are complex numbers which as usual is regarded as a linear operator on the Hermitian space $\mathbf{C}^N$. Then there exists the self-adjoint matrix $\sqrt{T^*T}$ whose eigenvalues are non-negative and for every vector x one has

(i) $$||T^*T(x)|| = ||Tx||^2 \implies ||\sqrt{T^*T}(x)|| = ||Tx||$$

Moreover, since $\sqrt{T^*T}$ is self-adjoint we have an orthogonal decomposition

(ii) $$\sqrt{T^*T}(\mathbf{C}^N) \oplus \mathrm{Ker}(\sqrt{T^*T}) = \mathbf{C}^N$$

where the self-adjointness gives the equality

(iii) $$\mathrm{Ker}(\sqrt{T^*T}) = \sqrt{T^*T}(\mathbf{C}^N)^\perp$$

The partial isometry operator. Show that there exists a unique linear operator P such that

(*) $$T = P \cdot \sqrt{T^*T}$$

where the P-kernel is the orthogonal complement of the range of $\sqrt{T^*T}$. Moreover, from (i) it follows that

$$||P(y)|| = ||y||$$

for each vector in the range of $\sqrt{T^*T}$. One refers to P as a partial isometry attached to T.

Extension to operators on Hilbert spaces.. Let T be a bounded operator on the Hilbert space $\mathcal{H}$. The spectral theorem for bounded and self-adjoint operators gives a similar equation as in (*) above using the non-negative and self-adjoint operator $\sqrt{T^*T}$. More generally, let T be densely defined and closed. From § XX there exists the densely defined self-adjoint operator T^*T and we can also take its square root.

5.1 Theorem. *There exists a bounded partial isometry P such that*

$$T = P \cdot \sqrt{T^*T}$$

Proof. Since T has closed graph we have the Hilbert space $\Gamma(T)$. For each $x \in \mathcal{D}(T)$ we get the vector $x_* = (x, Tx)$ in $\Gamma(T)$. Now

$$(x_*, y_*) \mapsto \langle x, y\rangle$$

yields a bounded Hermitian bi-linear form on the Hilbert space $\Gamma(T)$. The self-duality of Hilbert spaces gives bounded and self-adjoint operator A on $\Gamma(T)$ such that

$$\langle x, y\rangle = \{Ax_*, y_*\}$$

where the right hand side is the inner product between vectors in $\Gamma(T)$. Let

$$j\colon (x, Tx) \mapsto x$$

be the projection from $\Gamma(T)$ onto $\mathcal{D}(T)$ and for each $x \in \mathcal{D}(T)$ we put

$$Bx = j(Ax_*)$$

Then B is a linear operator from $\mathcal{D}(T)$ into itself where

(i) $$\langle Bx, y\rangle = \{Ax_*, y_*\} = \{x_*, Ay_*\} = \langle x, By\rangle \quad : \quad x, y \in \mathcal{D}(T)$$

We have also

$$\langle Bx, x\rangle = \{A^2x_*, x_*\} = \{Ax_*, Ax_*\} = \langle Bx, Bx\rangle + \langle TBx, TBx\rangle \implies$$

$$||Bx||^2 = \langle Bx, Bx\rangle \leq \langle Bx, x\rangle \leq ||Bx|| \cdot ||x||$$

where the Cauchy-Schwarz inequality was used in the last step. Hence

$$||Bx|| \leq ||x|| \quad : \quad x \in \mathcal{D}(T)$$

This entails that that the densey defined operator B extends uniquely to $\mathcal{H}$ as a bounded operator of norm ≤ 1. Moreover, since (i) hold for pairs x, y in the dense subspace $\mathcal{D}(T)$, it follows that B is self-adjoint. Next, consider a pair x, y in $\mathcal{D}(T)$ which gives

$$\langle x, y\rangle = \{Ax_*, y_*\} = \{x_*, Ay_*\} = \langle x, By\rangle + \langle Tx, TBy\rangle$$

Keeping y fixed the linear functiional

$$x \mapsto \langle Tx, TBy\rangle = \langle x, y\rangle - \langle x, By\rangle$$

is bounded on $\mathcal{D}(T)$. By the construction of T^* it follows that $TBy \in \mathcal{D}(T^*)$ and we also get the equality

(ii) $$\langle x, y\rangle = \langle x, By\rangle + \langle x, T^*TBy\rangle$$

Since (ii) holds for all x in the dense subspace $\mathcal{D}(T)$ we conclude that

(iii) $$y = By + T^*TBy = (E + T^*T)(By) \quad : \quad y \in \mathcal{D}(T)$$

Conclusion. From the above we have the inclusion

$$TB(\mathcal{D}(T)) \subset \mathcal{D}(T^*)$$

Hence $\mathcal{D}(T^*T)$ contains $B(\mathcal{D}(T))$ and (iii) means that B is a right inverse of $E + T^*T$ provided that the y-vectors are restricted to $\mathcal{D}(T)$.

FINISH ..

5.2 Transition probability functions.

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space as defined in § XX. Consider a real-valued function P defined on the product set $\Omega \times \mathcal{B}$ with the following two properties:

(*) $\qquad t \mapsto P(t, E) \quad$ is a bounded measurable function for each $E \in \mathcal{B}$

(**) $\qquad E \mapsto P(t, E) \quad$ is a probability measure for each fixed $\quad t \in \Omega$

When (*-**) hold one refers to P as a transition function. Given P we define inductively the sequence $\{P^{(n)}\}$ by:

$$P^{(n+1)}(t, E) = \int_\Omega P^{(n)}(s) \cdot dP(s, E)$$

It is clear that $\{P^{(n)}\}$ yield new transition functions. The probabilistic interpretation is that one has a Markov chain with independent increments. More precisely, if E and S are two sets in $\mathcal{B}$ and $n \geq 1$. then

$$\int_S P^{(n)}(t; E) \cdot d\mu(t)$$

is the probability that the random walk which starts at some point in E has arrived to some point in S after n steps. One says that the given transition function P yields a stationary Markov process if there exists a finite family of disjoint subets $E_1, \ldots, E_m$ in $\mathcal{B}$ and some $\alpha < 1$ and a constant M such that the following hold: First, for each $1 \leq i \leq m$ one has:

(1) $$P(t, E_i) = 1 \quad : t \in E_i$$

Next, if $\Delta = \Omega \setminus E_1 \cup \ldots \cup E_m$ then

(2) $$\sup_{t \in S} P^{(n)}(t, \Delta) \leq M \cdot \alpha^n \quad : n = 1, 2, \ldots$$

Remark. One refers to Δ as the disippative part of Ω and $\{E_i\}$ are the ergodic kernels of the process. Since $\alpha < 1$ in (2) the probabilistic interpretation of (2) is that as n increase then the disippative part is evacuated with high probability while the Markov process stays inside every ergodic kernel.

The Kakutani-Yosida theorem.

A sufficient condition for a Markov process to be stationary is as follows: Denote by X the Banach space of complex-valued and bounded $\mathcal{B}$-measurable functions on the real s-line. Now P gives a linear operator T which sends $f \in X$ to the function

$$T(f)(x) = \int_\Omega f(s) \cdot dPx, ds)$$

Kakutani and Yosida proved that the Markov process is stationary if there exists a triple α, n, K where K is a compact operator on X, $0 < \alpha < 1$ and n some positive integer such that the operator norm

(*) $$||T + K|| \leq \alpha$$

The proof relies upon some general facts about linear operators on Banach spaces. First one identifies the Banach space X with the space of continuous complex-valued functions on the compact Hausdorff space S given by the maximal ideal space of the commutative Banach algebra X. Then T is a postive linear operator on $C^0(S)$ and in § 11.xx we shall prove that (*) implies that the spectrum of T consists of a finite set of points on the unit circle together with a compact subset in a disc of radius < 1. Moreover, for each isolated point $e^{i\theta} \in \sigma(T)$ the corresponding eigenspace is finite dimensional. Each such eigenvalue corresponds to an ergodic kernel and when the eigenspace has dimension $m \geq 2$, the corresponding ergodic kernel, say E_1, has a further decomposition into pairwise disjoint subsets $e_1, \ldots, e_m$ where the process moves in a cyclic manner between these sets, i.e.

$$\int_{e_{i+1}} P(s, e_i) = 1 \quad : 1 \leq i \leq m \quad \text{where we put} \quad e_{m+1} = e_1$$

The proof of the assertions above rely upon two results announced in the two theorem in 5.2.1 and 5.2.6 below. We begin with

A result about positive operators on $C^0(S)$. Let S be a compact Hausdorff space and X the Banach space of continuous and complex-valued functions on S. A linear operator T on X is positive if it sends every non-negative and real-valued function f to another real-valued and non-negative function. Denote by $\mathcal{F}^+$ the family of positive operators T which satisfy the following: First

$$\lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = 0 \tag{1}$$

hold for all pairs $x \in X$ and $x^* \in X^*$. The second condition is that $\sigma(T)$ is the union of a compact set in a disc $\{|\lambda| \leq r$ for some $r < 1$, and a finite set of points on the unit circle. The final condition is that $R_T(\lambda)$ is meromorphic in the exterior disc $\{|\lambda > r\}$, i.e. it has poles at the spectral points on the unit circle.

5.2.1. Theorem. *If $T \in \mathcal{F}^+$ then each spectral value $e^{i\theta} \in \sigma(T)$ is a root of unity.*

Proof. First we prove that $R_T(\lambda)$ has a simple pole at each $e^{i\theta} \in \sigma(T)$. Replacing T by $e^{-i\theta} \cdot T$ it suffices to prove this when $e^{i\theta} = 1$. If $R_T(\lambda)$ has a pole of order ≥ 2 at $\lambda = 1$ we know from § XX that there exists $x \in X$ such that

$$Tx \neq x \quad \text{and} \quad (E - T)^2 x = 0 \tag{i}$$

This gives $T^2 + x = 2Tx$ and by an induction

$$\frac{1}{n} \cdot T^n x = \frac{1}{n} \cdot x + (E - T)x \quad : n = 1, 2, \ldots \tag{ii}$$

Condition (1) and (ii) give for each $x^* \in X^*$:

$$0 = \lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = \lim_{n\to\infty} x^*(\frac{1}{n} \cdot x + (E - T)x)$$

It follows that $x^*(E - T)(x) = 0$ and since x^* is arbitrary we get $Tx = x$ which contradicts (i). Hence the pole must be simple.

Next, let $e^{i\theta} \in \sigma(T)$. Since R_T has a simple pole at this spectral point, the general result in gives some $f \in C^0(S)$ which is not identically zero and

$$T(f) = e^{i\theta} \cdot f$$

Multiplying f with a complex scalar we may assume that its maximum norm on S is one and there exists a point $s_0 \in S$ such that

$$f(s_0) = 1$$

For each $n \geq 1$ we have a linear functional on X defined by $g \mapsto T^n(g)(s_0)$ which gives a Riesz measure μ_n such that

$$\int_S g \cdot d\mu_n = T^n g(s_0) \quad : g \in C^0(S)$$

Since T^n by the hypothesis is positive, the integrals in the left hand side are ≥ 0 when g are real-valued and non-negative. This entails that the measures $\{\mu_n\}$ are real-valued and non-negative. For each $n \geq 1$ we put

$$A_n = \{x : \, e^{-in\theta} \cdot f(x) \neq 1\}$$

Since the sup-norm of f is one we notice that

$$A_n = \{x : \, \mathfrak{Re}(e^{-in\theta} f(x)) < 1\} \tag{iii}$$

Now

(iv) $$0 = f(s_0) - e^{-in\theta} \cdot T^n f(s_0) = \int_S [1 - e^{-in\theta} f(s)] \cdot d\mu_n(s)$$

Taking real parts we get

(v) $$0 = \int_S [1 - \mathfrak{Re}(e^{-in\theta} f(s))] \cdot d\mu_n(s)$$

By (iii) the integrand in (v) is non-negative and since the whole integral is zero it follows that

(vi) $$\mu_n(A_n) = \mu_n(\{\mathfrak{Re}(e^{-in\theta} < 1\} = 0$$

Suppose now that there exists a pair $n \neq m$ such that

(vii) $$(S \setminus A_n) \cap (S_m \setminus A_m) \neq \emptyset$$

A point s_* in this non-empty intersection gives

$$1 = e^{in\theta} f(s_*) = e^{im\theta} \cdot f(s_*) \implies e^{in\theta} = e^{im\theta}$$

and hence $e^{i\theta}$ is a root of unity if $m - n \neq 0$. So the proof of Theorem 5.2.1 is finished if we have established the following

Sublemma. The sets $\{S \setminus A_n\}$ cannot be pairwise disjoint.

Proof. First, f has maximum norm and by the above:

$$\int_S f \cdot d\mu_n = e^{in\theta}$$

Hence the total mass $\mu_n(S)$ is at least one. Next, for each $n \geq 2$ we set

$$\pi_n = \frac{1}{n} \cdot (\mu_1 + \ldots + \mu_n)$$

Since $\mu_n(S) \geq 1$ for each n we get $\pi_n(S) \geq 1$. Put

$$\mathcal{A} = \bigcap A_n$$

Above we proved that $\mu_n(A_n) = 0$ hold for every n which gives

(*) $$\pi_n(\mathcal{A}) = 0 \quad : n = 1, 2, \ldots$$

Next, when the sets $\{S \setminus A_k\}$ are pairwise disjoint one has the inclusions

$$S \setminus A_k \subset A_\nu \quad \forall \, \nu \neq k$$

Keeping k fixed it follows that $\pi_\nu(S \setminus A_k) = 0$ for every $\nu \geq 0$. So when n is large while k is kept fixed we obtain

(**) $$\pi_n(S \setminus A_k)) = \frac{1}{n} \cdot \mu_k(S \setminus A_k)) \implies \lim_{n \to \infty} \pi_n(S \setminus A_k)) = 0 \quad : k = 1, 2, \ldots$$

next, recall that we already proved that $R_T(\lambda)$ has at most a simple pole at $\lambda = 1$. With $\epsilon > 0$ the Neumann series expansion gives

$$E + \sum_{k=1}^{\infty} \frac{T^k}{(1+\epsilon)^k} = R_T(1+\epsilon)) = \frac{1}{\epsilon} \cdot Q + W(1+\epsilon))$$

where $W(\lambda)$ is an operator-valued analytic function in an open disc centered at $\lambda = 1$ while Q is a bounded linear operator on $C^0(S)$. Keeping $\epsilon > 0$ fixed we apply both sides to the identity function 1_S on S and the construction of the measures $\{\mu_n\}$ gives

$$1 + \sum_{k=1}^{\infty} \frac{\mu_k(S)}{(1+\epsilon)^k} = \frac{1}{\epsilon} \cdot Q(1_S)(s_0) + W(1+\epsilon))(1_S)(s_0)$$

If $n \geq 2$ is an integer and $\epsilon = \frac{1}{n}$ one gets the inequality

$$\sum_{k=1}^{k=n} \frac{\mu_k(S)}{(1+\frac{1}{n})^k} \leq n \cdot |Q(1_S)(s_0)| + |W(1+1/n))(1_S)(s_0)| \leq n||Q|| + ||W(1+1/n)|| \implies$$

$$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq (1+\frac{1}{n})^n \cdot (||Q|| + \frac{||W(1+1/n)||}{n}$$

Since Neper's constant $e \geq (1+\frac{1}{n})^n$ for every n we find a constant C which is independent of n such that

$$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq C$$

Hence the sequence $\{\pi_n(S)\}$ is bounded and we can pass to a subsequence which converges weakly to a limit measure μ_*. For this σ-additive measure the limit formula in (**) above entails that

(i) $$\mu_*(S \setminus A_k) = 0 \quad : \ k = 1, 2, \ldots$$

Moreover, by (*) we also have

(ii) $$\pi_*(\mathcal{A}) = 0$$

Now $S = \mathcal{A} \cup A_k$ so (i-ii) give:

$$\mu_*(S) = 0$$

But this is impossible for at the same time we have already seen that $\pi_n(S) \geq 1$ for each n and hence also $\mu_*(S) \geq 1$. This finishes the proof of Theorem 5.2.1.

5.2.2 The family $\mathcal{F}(X)$.

Let X be a Banach space. Then $\mathcal{F}(X)$. consists of bounded liner operators T on X such that

(*) $$\lim_{n \to \infty} \frac{1}{n} \cdot x^*(T^n x) = 0$$

hold for all pairs $x \in X$ and $x^* \in X^*$. The Banach-Steinhaus theorem implies that if $T \in \mathcal{F}(X)$, then there exists a constant M such that the operator norms satisfy

$$||T^n|| \leq M \cdot n \quad : \ n = 1, 2, \ldots$$

Since the n:th root of $M \cdot n$ tends to one as $n \to +\infty$, the spectral radius formula entails that the spectrum $\sigma(T)$ is contained in the closed unit disc.

5.2.3 The family $\mathcal{F}_*(X)$. It consists of those T in $\mathcal{F}(X)$ for which there exists some $\alpha < 1$ such that $R_T(\lambda)$ extends to a meromorphic function in the exterior disc $\{|\lambda| > \alpha\}$. Recall from the beginning of § 5.2 that

$$\sigma(T) \subset \{|\lambda| \leq 1\}$$

So if $T \in \mathcal{F}_*(X)$ then the set of points in $\sigma(T)$ which belongs to the unit circle in the complex λ-plane is empty or finite and there exists $\alpha < 1$ such that

$$\sigma(T) \cap \{\alpha < |\lambda| < 1\} = \emptyset$$

Exactly as in the beginning of the proof of Theorem 5.2.1 one has

5.2.4 Exercise. Show by the same reasoning as in the proof of Theorem 5.2.1 that if $T \in \mathcal{F}_*$ and $e^{i\theta} \in \sigma(T)$ for some θ, then Neumann's resolvent $R_T(\lambda)$ has a simple pole at $e^{i\theta}$.

Next, we shall prove:

5.2.5 Proposition. *If $T \in \mathcal{F}$ is such that $T^N \in \mathcal{F}_*$ for some integer $N \geq 2$. Then $T \in \mathcal{F}_*$.*

Proof. We have the algebraic equation

$$\lambda^N \cdot E - T^N = (\lambda \cdot E - T)(\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1})$$

It follows that

$$R_T(\lambda) = (\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1}) \cdot R_{T^N}(\lambda^N)$$

Since $T^N B \in \mathcal{F}_*$ there exists $\alpha < 1$ such that

$$\lambda \mapsto R_{T^N}(\lambda^N)$$

extends to be meromorphic in $\{|\lambda| > \alpha\}$. At the same time $(\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1})$ is a polynomial and hence $R_T(\lambda)$ also extends to be meromorphic in this exterior disc so that $T \in \mathcal{F}_*$.

The next result is due to Kakutani.

5.2.6 Theorem. *Let $T \in \mathcal{F}(X)$ be such that there exists a compact operator K where $||T+K|| < 1$. Then $T \in \mathcal{F}_*$ and for every $e^{i\theta} \in \sigma(T)$ the eigenspace $E_T(\theta) = \{x \in X : Tx = e^{i\theta}x\}$ is finite dimensional.*

Proof. Set $S = T + K$ and for a complex number λ we write $\lambda \cdot E - T = \lambda \cdot E - T - K + K$. Outside $\sigma(S)$ we get

(i) $$R_S(\lambda)(\lambda \cdot E - T) = E + R_S(\lambda) \cdot K$$

The Neumann series for large absolute values $|\lambda|$ applied to $R_S(\lambda)$ gives some $\rho > 0$ and

(ii) $$(E + R_S(\lambda) \cdot K)^{-1} = E + R_S(\lambda) \cdot K\dot{(}E + R_S(\lambda) \cdot K)^{-1} \quad : \; |\lambda| > \rho$$

Next, when $|\lambda|$ is large we notice that (i) gives

(iii) $$R_T(\lambda) = (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Together with (ii) we obtain

(iv) $$R_T(\lambda) = R_S(\lambda) + R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Set $\alpha = ||S||$ which by assumption is < 1. Now $R_S(\lambda)$ is analytic in the exterior disc $\{\lambda| > \alpha\}$ so in this exterior disc $R_\lambda(T)$ differs from the analytic function $R_\lambda(S)$ by

(v) $$\lambda \mapsto R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Here K is a compact operator so the result in § XX entails that this function extends to be meromorphic in $\{|\lambda| > \alpha\}$. There remains to prove that eigenspaces at spectral points on the unit circle are finite dimensional. To prove this we use (iv). Let $e^{i\theta} \in \sigma(T)$. By Proposition 5.2.3 it is a simple pole so we have a Laurent series expansion

$$R_T(e^{i\theta} + z) = \frac{A_{-1}}{z} + A_0 + A_1 z + \ldots$$

By the general results from §§ there remains to show that A_{-1} has finite dimensional range. To see this we apply (iv) which gives the equation

$$R_S(e^{i\theta} + z) + R_S(e^{i\theta} + z) \cdot (E + R_S(e^{i\theta} + z) \cdot K)^{-1} \cdot R_S(e^{i\theta} + z)$$

To simplify notations we set $B(z) = R_S(e^{i\theta} + z)$ which by assumption is analytic in a neighborhood of $z = 0$. Moreover, the operator $B(0)$ is invertible. So now one has

$$\frac{A_{-1}}{z} + A_0 + A_1 z + \ldots = B(z) + B(z)(E + B(z) \cdot K)^{-1} B(z)$$

Since $B(0)$ is invertible we have a Laurent series expansion

$$(E + B(z) \cdot K)^{-1} = \frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots$$

and identying the coefficient of z^{-1} gives

$$A_{-1} = B(0) A^*_{-1} B(0)$$

Next, from (xx) one has

$$E = (E + B(z) \cdot K)(\frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots) \implies (E + B(0) \cdot K) A^*_{-1} = 0$$

Here $B(0) \cdot K$ is a compact operator and hence Fredholm theory implies that A_{-1}^* has a finite dimensional range. Since $B(0)$ is invertible the same is true for A_{-1} which finishes the proof of Theorem 5.2.6.

5.3 Factorizations of non-symmetric kernels.

Let $K(x,y)$ be a continuous real-valued function on the closed unit square $\square = \{0 \leq x,y \leq 1\}$. We do not assume that K is symmetric but assume that there exists a positive definite Hilbert-Schmidt kernel $S(x,y)$ where

$$N(x,y) = \int_0^1 S(x,t)K(t,y)\,dy$$

yields a symmetric kernel function. The Hilbert-Schmidt theory gives an orthonormal basis $\{\phi_n\}$ in $L^2[0,1]$ formed by eigenfunctions to $\mathcal{S}$ where

$$\mathcal{S}\phi_n = \kappa_n\phi_n \tag{1}$$

and the positive κ-numbers tend to zero. Moreover, each $u \in L^2[0,1]$ has a Fourier-Hilbert expansion

$$u = \sum \alpha_n \cdot \phi_n \tag{2}$$

We seek eigenfunctions of the integral operator $\mathcal{K}$ defined by the kernel function K. Let u be a function in $L^2[0,1]$ such that:

$$u = \lambda \cdot \mathcal{K}u \tag{3}$$

where λ in general is a complex number. It follows that

$$\lambda \cdot \int N(x,y)u(y)\,dy = \lambda \iint SA(x,t)K(t,y)u(y)\,dtdy = \int S(x,t)u(t)\,dt \tag{4}$$

Multiplying with $\phi_p(x)$ an integration gives

$$\lambda \cdot \int \phi_p(x)N(x,y)u(y)\,dxdy = \iint \phi_p(x)S(x,t)u(t)\,dxdt = \kappa_p \int \phi_p(t)u(t)\,dt \tag{5}$$

Next, using the expansion of u from (2) we get the equations:

$$\sum_{q=1}^{\infty} \alpha_q \cdot \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy = \kappa_p\alpha_p \quad : \; p = 1,2,\ldots \tag{6}$$

Set

$$c_{qp} = \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy$$

It follows that $\{\alpha_p\}$ satisfies the system

$$\kappa_p\alpha_p = \lambda \cdot \sum_{q=1}^{\infty} c_{qp}\alpha_q \tag{7}$$

Since $N(x,y) = N(y,x)$ the doubly indexed c-sequence is symmetric. Set

$$\beta_p = \sqrt{\kappa_p} \cdot \alpha_p \implies \beta_p = \lambda \cdot \sum_{q=1}^{\infty} \frac{c_{pq}}{\sqrt{\kappa_p} \cdot \sqrt{\kappa_q}} \cdot \beta_q \tag{8}$$

Next, put

$$k_{p,q} = \iint K(x,y)\phi_p(x)\phi_q(y)\,dxdy \tag{9}$$

From the above the following hold for each pair p,q:

$$c_{pq} = \iiint \phi_q(x)\phi_p(y)S(x,t)K(t,y)\,dxdydt = \kappa_q k_{p,q} = \kappa_p k_{q,p} \implies$$

$$\frac{c_{p,q}^2}{\kappa_p \cdot \kappa_q} \leq |k_{p,q} \cdot k_{q,p}| \leq \frac{1}{2}(k_{p,q}^2 + k_{q,p}^2) \tag{10}$$

Here $\{k_{p,q}\}$ are the Fourier-Hilbert coefficients of $K(x,y)$ which entails that

$$\sum\sum k_{p,q}^2 \leq \iint K(x,y)^2\,dxdy$$

Hence the symmetric and doubly indexed sequence

$$\frac{c_{p,q}}{\sqrt{\kappa_p\cdot\kappa_q}} \tag{11}$$

is of Hilbert-Schmidt type.

5.3.2 Conclusion. The eigenfunctions u in $L^2[0,1]$ associated to the $\mathcal{K}$-kernel have Fourier-Hilbert expansions via the $\{\phi_n\}$-basis which are determined by α-sequences satisfying (7).

An applicaion.

Recall that the Neumann-Poincaré kernel $K(p,q)$ of a plane C^1-curve $\mathcal{C}$ is given by

$$K(p,q) = \frac{\langle p-q, \mathbf{n}_i(p)\rangle}{|p-q|}$$

This kernel function gives the integral operator $\mathcal{K}$ defined on $C^0(\mathcal{C})$ by

$$\mathcal{K}_g(p) = \int_C K(p,q)\cdot g(q)\,ds(q)$$

where ds is the arc-length measure on C. Let M be a positive number which exceeds the diameter of $\mathcal{C}$ so that $|p-q|<M:\ p,q\in\mathcal{C}$. Set

$$N(p,q) = \int_{\mathcal{C}} K(p,\xi)\cdot\log\frac{M}{|q-\xi|}\cdot ds(\xi)$$

Exercise. Verify that N is symmetric, i.e. $N(p,q)=N(q,p)$ hold for all pairs p,q in $\mathcal{C}$. Moreover,

$$S(p,q) = \log\frac{M}{|p-q|}$$

is a symmetric and positive kernel function and since $\mathcal{C}$ is of class C^1 the reader should verify that it gives a Hilbert-Schmidt kernel, i.e.

$$\iint_{\mathcal{C}\times\mathcal{C}} S(p,q)^2\,ds(p)ds(q)<\infty$$

Hence the Neuman-Poincaré operator $\mathcal{K}$ appears in an equation

$$\mathcal{N} = \mathcal{K}\circ\mathcal{S} \tag{*}$$

where $\mathcal{S}$ is defined via a positive symmetric Hilbert-Schmidt kernel and $\mathcal{N}$ is symmetric. Now one can apply the previous results to study the spectrum of $\mathcal{K}$. Let us remark that the results which led to (5.3.2) and the present example were given by Carleman in his thesis from 1916. Here Carleman also treated the more involved case when the plane curve $\mathcal{C}$ has corner points. Then the Neumann-Poincaré kernel is unbounded and the reduction to the symmetric case is more involved which leads to quite intricate results in Part II from [Carleman]. So the interplay between singularities on boundaries in the Neumann-Poincaré equation and the corresponding unbounded kernel functions illustrates the general theory densely defined self-adjoint operators. It appears that further analysis remains to be done where various open problems about the Neumann-Poincaré equation remain open to in dimension three. So far only the 2-dimensional case is properly understood via the results in [Carleman].