Hyperbolic PDE-equations.

Introduction. We shall expose a result from Friedrichs' article *xxx* dealing with boundary value problems for linear symmetric hyperbolic systems. The proof of our main result in Theorem 0.1 teaches the usefulness of regarding densely defined but unbounded linear operators on Hilbert spaces. Theorem 0.1 is announced for a symmetric hyperbolic first order system in any number of variables. The proof involves quite technical steps. are when one restricts to the case of two variables and a single scalar function. For the reader's convenience we therefore include a self-contained proof in the restricted case for systems in two variables and a single scalar function. Here the Hilbert space methods are transparent, and yet the methods which are used in this restricted situation are crucial, i.e. the general case is verbatim the same except for various technical points. Before we announce the result by Firedrichs we recall some background about hyperbolic equations.

A classic result due to Hadamard gives a vanishing principle for well-posed boundary value problems. With coordinates $(x, s) = (x_1, \ldots, x_n, s)$ in $\mathbf{R}^{n+1}$ we consider a differential operator of the form

$$Q(x, s, \partial_x, \partial_s) = \partial_s^p + \sum_{\nu=0}^{p-1} P_\nu(x, s, \partial_x) \cdot \partial_s^\nu$$

where $\{P_\nu(x, s, \partial_x)$ are differential operators which are independent of ∂_s and coefficients in $C^\infty(\mathbf{R}^{n+1})$ which in general are complex-valued. The Cauchy problem is well posed in Hadamard's sense if there to every $f(x) \in C^\infty(\mathbf{R}^n)$ exists a unique C^∞-function $g(x, s)$ in the half-space $\{s \geq 0\}$ such that $Q(g)(x, s) = 0$ when $s > 0$ and on $s = 0$ one has

$$\partial_s^\nu(g)(x, 0) = 0 : 0 \leq \nu \leq p-1 \quad : \quad \partial_s^p(g)(x, 0) = f(x)$$

Under the hypothesis that Cauchy's problem is well-posed one has:

Hadamard's Theorem. *If K is a compact subset in $\mathbf{R}^{n+1}$ there exists a compact set K in the x-space such that thre unique solution f whose Cauchy data on $s = 0$ vanishes on k must vanish on K.*

This result follows easily from Baire's category theorem. The reader may consult [D-S: Volume 2: page 1649-1652] for details.

We shall not enter an extensive discussion about conditions for a PDE-operator to be hyperbolic. The reader may consult the text-book by Petrowsky for an account where many examples of hyperbolic, elliptic and parabolic equations appear.

An ill-posed equation. Let $n = 1$ and consider the 2×2-matrices

$$A_1(x, s) = \begin{pmatrix} -e^{-x} & 0 \\ 0 & 1 \end{pmatrix} \quad : B(x, s) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad :$$

One seeks pairs of functions $(f_1(x, s), f_2(x, s)$ which satisfy the first order system:

$$\frac{\partial f_1}{\partial s} = -e^{-x} \cdot \frac{\partial f_1}{\partial x} + f_2 \quad : \quad \frac{\partial f_2}{\partial s} = \frac{\partial f_2}{\partial x}$$

For any function C^∞-function h of a single variable we see that

$$f_1(x, s) = h(e^x - s) \quad : f_2 = 0$$

solves the system above and here $f_1(x, 0) = h(x)$. Consider the singleton set $\{0, 1\}$ in $\mathbf{R}^2$. Then $f(A) = 1$ for all h-functions such that $h(0) = 0$. Fix a test-function $\phi(t)$ on the real t-line where $\phi(0) = 1$ while $\phi(t) = 0$ if $|t| \geq 1/2$. For every positive integer N we take $h(t) = \phi(Nt)$. If the Cauchy problem is well posed we get the unique solution with $f_1 = h(s - e^x)$ and now

$$f_1(x, 0) = \phi(e^{Nx})$$

Since the support of ϕ is contained in $[-1/2, 1/2]$ we see that $f_1(x,0) \neq 0$ entails that

$$e^{Nx} \leq 1/2 \implies x \leq -N \cdot \log 2$$

Since N can be arbitrary large this violates Hadamard's vanishing principle and hence the Cauchy problem for this system is not well-posed.

The wave operator.

With two real variables we consider the PDE-operator

$$P = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$$

To each pair of functions $g(x_1), h(x_1)$ one seeks $f(x_1, x_2)$ such that $P(f) = 0$ and Cauchy's boundary value conditions:

(i) $$f(x_1, 0) = g(x_1) \quad : \frac{\partial f}{\partial x_2}(x_1, 0) = h(x_1)$$

This boundary value problem corresponds to a first order system where one seeks a pair $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ such that

(ii) $$\frac{\partial f_1}{\partial x_1} = -\frac{\partial f_1}{\partial x_2} + f_2 \quad : \quad \frac{\partial f_2}{\partial x_1} = \frac{\partial f_2}{\partial x_2}$$

with boundary values

(iii) $$f_1(x_1, 0) = g(x_1) \quad : \quad f_2(x_1, 0) = g'(x_1) + h(x_1)$$

Exercise. Show that if f solves (i) then the boundary value system is solved by the pair

$$f_1 = f \quad : f_2 = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}$$

Show also that if f_1, f_2 solves the system then $f = f_1$ solves the original equation. Next, consider the matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad : \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then the system can be written in matrix form as

$$\frac{\partial}{\partial x_1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = A \frac{\partial}{\partial x_2} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + B \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

This clarifies that the orginal equation is a first order symmetric system to be dsecribed in § xx,

Higher order systems. Consider systems with $n+1$ many variables $x_1, \ldots, x_n, s$ where the variable s is distinguished. In a first order scalar system one seeks a function $f(x,s)$ such that

$$\frac{\partial f}{\partial s} = \sum_{j=1}^{j=n} a_j(x,s) \cdot \frac{\partial f}{\partial x_j} + b(x,s)$$

satisfying the boundary condition

$$f(x,0) = g(x)$$

where the g function is given in the n-dimensional x-space. In a vector-valued system of order $m \geq 2$ the a-functions are replaced by $m \times m$-matrices and b by some $m \times m$-matrix. Here one seeks a vector-valued function $f = (f_1, \ldots, f_m)$ such that

(*) $$\frac{\partial}{\partial s} \begin{pmatrix} f_1 \\ \ldots \\ f_m \end{pmatrix} = \sum_{j=1}^{j=n} A_j(x,s) \cdot \frac{\partial}{\partial x_j} \begin{pmatrix} f_1 \\ \ldots \\ f_m \end{pmatrix} + B(x,s) \begin{pmatrix} f_1 \\ \ldots \\ f_m \end{pmatrix}$$

The boundary conditions are expressed by an m-tuple of functions $\{g_\nu(x)\}$ such that $f_\nu(x,0)=g_\nu(x)$ hold for each ν. The A-matrices and the B-matrix are in general complex valued. A famous example is:

Maxwell's equations of electrodynamics. Let $n=m=3$ where $\{A_\nu\}$ are constant 3×3-matrices

$$A_1=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\quad : A_2=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}\quad : A_3=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$$

Moreover $B=0$ and one seeks a vector-valued function f_1,f_2,f_3 which satisfies (*) and boundary value conditions $f_\nu(x,0)=g_\nu(x)$

The symmetric case.

The system (*) is symmetric if the matrices $\{A_\nu(x,s)\}$ are Hermitian for each $1\leq\nu\leq n$. No special condition is imposed on B, i.e. it can be an arbitrary complex $m\times m$-matrix. Notice that the A-matrices in Maxwell's equations are hermitian. The following result is due to Friedrichs:

0.1 Theorem. *Assume that the A-matrices are hermitian and that the matrix elements of $A_1,\ldots,A_n$ and B are bounded functions in $\mathbf{R}^{n+1}$. Then Cauchy's boundary value problem has a unique C^∞solution $f=(f_1,\ldots,f_m)$ for every m tuple $g=(g_1,\ldots,g_m)$ of C^∞-functions in the n-dimensional x-space.*

Remark. The example after Hadamard's result above shows that this boundedness is needed in order that the Cauchy problem is well-posed.

0.2 On the uniqueness. Let us illustrate why the condition that the A-matrices are hermitian gives a certain vanishing principle. Let Ω be a bounded open set in the n-dimensional x-space and $-s_*<s<s_*$ is some open s-interval. Let $\{A_j(x,s)\}$ be Hermitian $m\times m$-matrices whose elements as well as their first order partial derivatives are bounded C^∞-functions in $\Omega\times(-s_*,s_*)$. Similarly, assume that $B(x,s)$ is an $m\times m$-matrix whose elements are bounded $C^\infty-$-functions in Ω. Let $f=(f_1,\ldots,f_m)$ be a vector-valued solution to the system (*) where

$$f(x,0)=0\quad : x\in\Omega \tag{0.2.1}$$

and assume there is a compact subset K of Ω and $f=0$ in $(\Omega\setminus K)\times(-s_*,s_*)$. Then (0.2.1) entails that $f=0$ in $\Omega\times(-s_*,s_*)$. To prove this we introduce the function

$$J(s)=\int_\Omega |f(x,s)|^2\,dx$$

where $|f|^2=\sum f_\nu\cdot\bar f_\nu$. Taking the derivative with respect to s we get

$$\frac{dJ}{ds}=2\cdot\mathfrak{Re}\sum_{j=1}^{j_m}\int\partial_s(f_\nu)\cdot\bar f_\nu\,dx \tag{i}$$

Since f satisfies (*) we have

$$\sum_{j=1}^{j_m}\int\partial_s(f_\nu)\cdot\bar f_\nu=\sum_{j=1}^{j=n}\langle A_j(\frac{\partial f}{\partial x_j}),f\rangle+\langle B(f),f\rangle$$

Since $f_\nu(x,s)$ vanish when $x\in\Omega\setminus K$ Stokes theorem gives

$$0=\int\partial_{x_j}(\langle A_j(f),f\rangle)\,dx\quad : 1\leq j\leq n \tag{ii}$$

Rules for differentiation identifies for each j the integrand with

$$\langle\frac{\partial A_j}{\partial x_j}(f),f\rangle+\langle A_j(\partial_{x_j}(f),f\rangle+\langle A_j(f),\partial_{x_j}(f)\rangle$$

Since A_j is hermitian we have

(iii) $$\mathfrak{Re}\,\langle A_j(\partial_{x_j}(f),f\rangle = \mathfrak{Re}\,\langle A_j(f),\partial_{x_j}(f)\rangle$$

Hence (ii) and (iii) give

$$\mathfrak{Re}\int\langle A_j(\frac{\partial f}{\partial x_j}),f\rangle\,dx = -\frac{1}{2}\int\langle\frac{\partial A_j}{\partial x_j}(f),f\rangle\,dx$$

Introduce the matrix-valued function

$$\text{div}(A) = \sum_{j=1}^{j=n}\frac{\partial A_j}{\partial x_j}$$

From the above we get the equation

(iv) $$\frac{dJ}{ds} = \mathfrak{Re}\int\big[-\frac{1}{2}\langle\text{div}(A)(f),f\rangle + \langle B(f),f\rangle\big]\,dx$$

By assumption the elements of the matrices $\{A_j\}$ and of B as well as $\{\frac{\partial A_j}{\partial x_j}\}$. are bounded C^∞-functions. Hence (iv) and the Cauchy-Schwarz inequality gives a constant C such that the absolute value in the right hand side in (iv) is estimated above by

$$C\cdot\int|f(x,s)|^2\,dx \quad : -s_* \leq s < s^*$$

It follows that

$$|\frac{dJ}{ds}| \leq C\cdot J(s) \quad : 0 \leq s \leq s^*$$

At the same time (i) means that $J(0)=0$. Hence Picard's uniquneness theorem to be exposed in § XX implies that if $J(s)=0$ when $-s_* < s < s_*$, i.e.

(0.3.1) $$J(x,s) = 0 \quad : (x,s)\in\{|x|\leq r\}\times[0,s^*]$$

So we have

$$\int_\Omega|f(x,s)|^2\,dx = 0$$

and $\{f_\nu\}$ are continuous functions they vanish identically in $\Omega\times(-s_*,s_*)$ as requested.

0.3 A semi-global uniqueness result. Let $\{A_j(x,s)\}$ be Hermitian matrices defined in the whole of $\mathbf{R}^{n+1}$ whose elements are bounded C^∞-functions. Similarly $B(x,s)$ is defined in the whole of $\mathbf{R}^{n+1}$.

0.3.1 Proposition. *There exists a positive number ρ which only depends on the matrices above such that if $R>0$ and a vector valued C^∞-function $f(x,s)$ is defined in the ball $\{|x|^2+s^2<R^2\}$ where it is a solution to (*) and satisfies*

$$f(x,0) = 0 \quad : |x| < R$$

Then it follows that

$$f(x,s) = 0 \quad : x^2+s^2 < \rho\cdot R^2$$

Proof. Choose a test-function $\psi(x)$ in $\mathbf{R}^n$ such that $\psi=1$ when $|x|\leq 1$ and vanishes when $|x|>3/2$ while the values stay in $[0,1]$. Set

(i) $$\mu = \max_{1\leq j\leq m}\sup_{(x,s)}||A_j(x,s)||$$

where the supremum is taken over all (x,s) in $\mathbf{R}^{n+1}$ and we have taken the Hibert-Schmidt norms of the A-matrices. With $\epsilon>0$ we set $\phi(x)=\psi(\epsilon\cdot x)$ and construct the following $m\times m$-matrices

$$H(x,s)=\sum_{j=1}^{j=n}\frac{\partial\phi}{\partial x_j}(x)\cdot A_j(x,s)$$

$$\widehat{A_j}(x,s)=\phi(x)\cdot A_j(x,\phi(x)s)\quad:\quad\widehat{B}(x,s)=\phi(x)\cdot B(x,\phi(x)s)$$

Put

$$F(x,s)=f(x,\phi(x)s)$$

Since $0\leq\phi(x)\leq 1$ hold for all x it follows that F is defined in $\{|x|^2+s^2<R^2\}$ and the construction of ϕ gives

$$\text{(ii)}\qquad F(x,s)=f(x,s)\quad:|x|<\epsilon^{-1}$$

Rules for differentiation show that F satisfies the system

$$(E-H(x,s))\partial_s(F)=\sum_{j=1}^{j=n}\widehat{A_j}(x,s)\partial_{x_j}(F)+\widehat{B}(x,s)F\quad:\ x^2+s^2<R^2$$

where E is the identity operator.

A choice of ϵ. The partial derivatives of the test-function ψ are bounded by some constant C and we set

$$\epsilon^*=\frac{1}{2n\cdot C\mu}$$

It follows that

$$|\frac{\partial\phi}{\partial x_j}|=\epsilon\cdot|\frac{\partial\psi}{\partial x_j}|\leq\frac{1}{2n\mu}$$

By (i) this entails that

$$\text{(iii)}\qquad \sup_{(x,s)}||H(x,s]||\leq\frac{1}{2}$$

Next, the support of ϕ is contained in the ball $\{|x|\leq\frac{3}{2\epsilon^*}$ so the vanishing in (xx) entails that $F(x,s)=0$ when

$$\frac{3}{2\epsilon^*}\leq|x|<R$$

By the condition (xx) $R\geq R_*$ entails that

$$\frac{3}{2\epsilon^*}=3nC\mu=R_*/2\leq R/2$$

So $R\geq R_*$ implies that

$$\text{(v)}\qquad F_\epsilon(x,s)=0\quad:\ \frac{R}{2}\leq|x|<\sqrt{R^2-s^2}$$

Now (iv) implies that the hermitian matrix $E-H(x,s)$ is invertible and by (iii) F satisfies system as in (0.2). The vanishing in (v) therefore implies that $F_\epsilon(x,s)=0$ hold when $x^2+s^2<R^2$.

FINISH

§ 1: Symmetric hyperbolic systems.

The main result in this section appears in Theorem 1.xx. Before it can be announced we need several preliminaries. We will study periodic functions. Let n be a positive integer and consider the $(n+1)$-dimensional torus T^{n+1} with variables $(x,s) = (x_1,\ldots,x_n,s)$. Denote by $C^\infty(T^{n+1})$ the space of complex-valued C^∞-functions which are 2π-periodic in all the variables. Passing to the L^2-norm the closure of these functions give the complex Hilbert space $L^2(T^{n+1})$ whose vectors are complex-valued functions $f(x,s)$ which are square integrable on the $(n+1)$-dimensional 2π-periodic torus. Next, to each multi-index $\alpha = (\alpha_1,\ldots,\alpha_{n+1})$ one associates the differential operator

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\cdot\partial_s^{\alpha_{n+1}}$$

If k is a positive integer an inner product is defined on $C^\infty(T^{n+1})$ by

$$\langle f,g\rangle_{(k)} = \sum_{|\alpha|\leq k}\int \partial^\alpha(f)\cdot\overline{\partial^\alpha(g)}\,dxds \tag{1.1}$$

Passing to the closure we obtain a Hibert space denoted by $H^{(k)}$ whose elements are L^2-functions $g(x,s)$ such that the distribution derivatives $\partial^\alpha(g)$ are square integrable when $|\alpha|\leq k$. From § XX we recall:

The Fourier-Sobolev Lemma. *If $k\geq xx$ every $g\in H^{(k)}$ is a periodic function of class C^1 at least on $\mathbf{T}^{n+1}$.*

More generally, if $m\geq 2$ we consider vector-valued functions $f=(f_1,\ldots,f_m)$ and get the Hilbert space $H^{(k)}[m]$ whose vectors are m-tuples of functions in $H^{(k)}$. With $f=(f_1,\ldots,f_m)$ and $g=(g_1,\ldots,g_m)$ the inner product is defined as in (1,1):

$$\langle f,g\rangle_{(k)} = \sum_{\nu=1}^{\nu=m}\sum_{|\alpha|\leq k}\int \partial^\alpha(f_\nu)\cdot\overline{\partial^\alpha(g_\nu)}\,dxds \tag{1.2}$$

With $m\geq 1$ we consider a matrix-valued functions $\{A_j(x,s)\ldots A_n(x,s)\}$ where each $A_j(x,s)$ is an $m\times m$-matrix whose elements are periodic complex-valued C^∞-functions on T^{n+1}. Let $B(x,s)$ be another matrix-valued functions whose elements also are periodic and of class C^∞. Set

$$P(x,s,\partial_x,\partial_s) = E_m\cdot\partial_s - \sum_{j=1}^{j=n} A_j(x,s)\cdot\partial_{x_j} + B(x,s) \tag{1.3}$$

This differential operator acts on vector-valued functions f. Identifying the space of vector-valued and periodic C^∞-functions with a subspace of $H^{(k)}[m]$ one has the linear map

$$P\colon f\to P(f)$$

from $C^\infty[m]$ into $H^{(k)}[m]$. Keeping k and m fixed we denote this linear map by T_0. It means that T_0 is densely defined linear operator on $H^{[k)}[m]$ whose domain of definition $\mathcal{D}(T_0) = C^\infty_[m]$. We have also the densely defined linear operator T_1 where

$$\mathcal{D}(T_1) = \{f\in H^{(k)}[m] : P(f)\in H^{[k)}[m]\}$$

By the general result in § XX the graph of T_1 taken in the product $H^{(k)}[m]\times H^{(k)}[m]$ is closed. Next, since T_0 is densely defined there exists the adjoint operator T_0^*. By definition $\mathcal{D}(T_0^*)$ consists of vectors $g\in H^{(k)}[m]$ for which there exists a constant $C(g)$ such that

$$|\langle T_0(f),g\rangle| \leq C(g)||f||_k \quad : f\in\mathcal{D}(T_0)$$

and for such g-vectors we get a unique vector $T_0^*(g)$ such that

$$\langle T_0(f),g\rangle = \langle f,T_0^*(g)\rangle$$

1.4 The case when $\{A_j\}$ are hermitian. An $m \times m$-matrix $A(x,s) = \{a_{\nu,\mu}(x,s)\}$ whose elements are periodic C^∞-functions is hermitian if

$$a_{\mu,\nu}(x,s) = \overline{a_{\nu,\mu}(x,s)}$$

hold for all pairs $1 \leq \nu, \mu \leq m$.

1.5 Proposition. *If $A_1, \ldots, A_n$ are hermitian it follows that $\mathcal{D}(T_0^*) = \mathcal{D}(T_1)$ and there exists a bounded and self-adjoint linear operator $\mathcal{B}$ on $H^{(k)}[m]$ such that*

$$(*) \qquad T_0^* + T_1 = \mathcal{B}$$

Before we enter the proof we need some constructions. Repeated use of Stokes Theorem gives the equality below for every pair of functions f, g in $C^\infty(T^{n+1})$ and every multi-index α:

$$(-1)^{|\alpha|} \cdot \int \partial^{2\alpha}(f) \cdot \bar{g} \; dxds = \int \partial^\alpha(f) \cdot \overline{\partial^\alpha(g)} \; dxds$$

More generally, let $Q = Q(x, s, \partial_x, \partial_s)$ be a differential operator. With Q given as

$$Q = \sum q_\alpha(x,s) \cdot \partial^\alpha$$

where $q_\alpha \in C^\infty(T^{n+1})$ one gets the differential operator

$$Q^* = \sum (-1)^\alpha \cdot \partial^\alpha \circ q_\alpha(x,s)$$

where $\partial^\alpha \circ q_\alpha(x,s)$ is the product taken in the ring of differential operators with $\{q_\alpha(x,s)\}$ regarded as zero-order differential operators. Stokes theorem gives

$$(1.6) \qquad \int Q(f) \cdot \bar{g} \, dxds = \int f \cdot Q^*(\bar{g}) \, dxds$$

Let us write out

$$Q^* = \sum r_\alpha(x,s) \cdot \partial^\alpha$$

We take the complex conjugates of the r-functions and put

$$\overline{Q^*} = \sum \bar{r}_\alpha(x,s) \cdot \partial^\alpha$$

Using the hermitian inner product on $L^2(T^{n+1})$ we can express (1.6) by the equation

$$(1.7) \qquad \langle Q(f), g \rangle = \langle f, \overline{Q^*}(g) \rangle$$

1.8 The Γ-operator. Let us introduce the differential operator

$$\Gamma = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \cdot \partial^{2\alpha}$$

If $m \geq 2$ we denote by Γ_m the operator given by the diagonal $m\times$-matrix where whose diagonal elements are Γ. Stokes theorem entails that if f and g is a pair of vector-valued functions in $C^\infty(T^{n+1})$ then

$$(1.9) \qquad \langle f, g \rangle_{(k)} = \int \Gamma_m(f) \cdot \bar{g} \, dxds$$

1.10 Exercise. Both sides in (1.9) are defined under the relaxed condition that $g \in H^{(k)}[m]$ while $f \in C^\infty[m]$. Show by continuity that (1.9) remains valid for such pairs.

Next, in the algebra of $m \times m$-matrices whose elements are differential operators we consider the product $\Gamma_m \cdot P$.

Exercise. Show that when P is as in (1.3) where $\{A_j\}$ are hermitian then there exists an $m \times m$-matrix Q whose elements are differential operators of order $\leq 2k$ such that the following hold in the algebra above:

$$\Gamma_m \circ P + \overline{P^*} \circ \Gamma_m = Q \tag{1.11}$$

Now (xx) and (xx) give the equation

$$\langle T_0(f), g \rangle_{(k)} = \int Q(f) \cdot \bar{g}\, dxds - \int \overline{P^*} \circ \Gamma_m(f) \cdot \bar{g}\, dxds$$

Apply (xx) to the pair of vector-valued functions $\Gamma_m(f)$ and g and the differential operator $\overline{P^*}$. Notice that the complex conjugate of the adjoint $(\overline{P^*})^*$ is equal to P and from this the reader can check form the above that the last term in (xx) is equal to

$$-\int \Gamma_m(f) \cdot \overline{P(g)}\, dxds$$

Applying (xx) this entails that the following hold for each pair f, g in $C^\infty(T^{n+1})$.

$$\langle T_0(f), g \rangle_{(k)} = -\langle f, T_0(g) \rangle_{(k)} + \int Q(f) \cdot \bar{g}\, dxds$$

Exercise. Since the differential operator Q has degree $\leq 2k$ the reader should verify the existence a bounded linear operator $\mathcal{B}_k$ on the Hilbert space $H^{(k)}[m]$ such that

$$\int Q(f) \cdot \bar{g}\, dxds = \langle \mathcal{B}_k(f), g \rangle_{(k)}$$

hold when f is a vector-valued C^∞-function and $g \in H^{(k)}[m]$. In particular we can take a pair f, g in C^∞ and notice that

$$(f, g) \mapsto \langle T_0(f), g \rangle_{(k)} + \langle T_0(g), f \rangle_{(k)}$$

is symmtric in f and g. Form this the reader can conclude that the bounded operator $\mathcal{B}_k$ is self-adjoint.

Proof of Proposition 1.5

Assume first that $g \in \mathcal{D}(T_0)$ which gives

$$\langle T_0(f), g\rangle_{(k)} = \langle f, T_0^*(g)\rangle_{(k)}$$

Here $g \in H^{(k)}$ and regarding g as a distribution we get the vector-valued distribution $P(g)$. Now xx hold for all vector-valued periodic C^∞-functions f. From the above the trinsgle inequality and Cauchy-Schwarz gives

$$\big|\langle f), P(f)g\rangle_{(k)}\big| \leq (||g||_k + ||\mathcal{B}_k(g)||_k) \cdot ||f||_k$$

Since this inequslity hold for all f in the dense subspace $C^\infty[m]$ it follows that the distribution $P(f)g$ belongs to $H^{(k)}[m]$ so by the construction of T_1 one has $g \in \mathcal{D}(T_1)$. Hence one has the inclusion

$$\text{(i)} \qquad \mathcal{D}(T_1) \subset \mathcal{D}(T_0^*)$$

Conveersley, if $g \in \mathcal{D}(T_1)$ the absolute value in the right hand side of (xx) is majorized by

$$||T_1(g)||_k + ||\mathcal{B}_k(g)||_k) \cdot ||f||_k$$

The construction of T_0^* entails that $g \in \mathcal{D}(T_0^*$ and hence equality holds in (i) Finally it is clear that this equality and (xxx) gives the operator equation

$$T_0^* = -T_1 + \mathcal{B}_k^*$$

Since we already proved that $\mathcal{B}_k$ is self-adjoint the proof of Proposition 1.5 is finished.

§ 2. A study of T_1 .

In § 1 we constructed the densely defined and closed operator T_1 on $H^{(k)}[m]$. Consider some $f \in C^\infty[m]$ and a real number λ. Now

$$||T_1(f) + \lambda \cdot f - \frac{1}{2}\mathcal{B}_k^*(f)||_{(k)}^2 =$$

$$||T_1(f) - \frac{1}{2}\mathcal{B}_k^*(f)||_{(k)}^2 + \lambda^2 ||f||_{(k)}^2 + \lambda \cdot \langle T_1(f) - \frac{1}{2}\mathcal{B}_k^*(f), f\rangle_{(k)} + \lambda \cdot \langle f, T_1(f) - \frac{1}{2}\mathcal{B}_k^*(f)\rangle_{(k)}$$

Since f is C^∞ we have $T_1(f) = T_0(f)$ and since $\mathcal{B}_k^*$ is self-adjoint it follows that the sum of the last two terms above becomes

$$\text{(i)} \qquad \lambda \cdot (\langle\, f, T_0^*(f) - \frac{1}{2}\mathcal{B}_k^*(f)\rangle_{(k)} + \langle f, T_0(f) - \frac{1}{2}\mathcal{B}_k^*(f)\rangle_{(k)})$$

The operator equation in Proposition 1.5 gives

$$T_0(f) + T_0^*(f) = \mathcal{B}_k^*$$

which proves that (i) is zero. Hence we have proved the equality

$$(2.1) \qquad ||T_1(f) + \lambda \cdot f - \frac{1}{2}\mathcal{B}_k^*(f)||_{(k)}^2 = ||T_1(f) - \frac{1}{2}\mathcal{B}_k^*(f)||_{(k)}^2 + \lambda^2 ||f||_{(k)}^2$$

Since $||T_1(f) - \frac{1}{2}\mathcal{B}_k^*(f)||_{(k)} \geq 0$ the triangle inequality gives the inequality below for every real number λ

$$(2.2) \qquad ||T_1(f) + \lambda \cdot f||_{(k)} \geq |\lambda| \cdot ||f||_{(k)} - \frac{1}{2} \cdot ||\mathcal{B}_k^*(f)||_{(k)} \quad : f \in C^\infty_{\text{per}}[m]$$

Above the real number λ can be both positive or negative and the inequality is of interest when $|\lambda|$ exceeds the operator norm of $\frac{\mathcal{B}_k^*}{2}$. We have for example

$$(2.3) \qquad ||T_1(f) + \lambda \cdot f||_{(k)} \geq \frac{|\lambda|}{2} \cdot ||||f||_{(k)} \quad : |\lambda| \geq ||\mathcal{B}_k^*||$$

2.4 The operator $\widehat{T_0}$. Recall that T_1 is closed and extends T_0 in the sense that its graph contains that of T_0. Taking the closure of $\Gamma(T_0)$ we get the densely defined and closed operator $\widehat{T_0}$ whose graph is contained in $\Gamma(T_1)$. When f are C^∞-functions we have $T_0(f) = \widehat{T_0}(f) = T_1(f)$ so (2.3) holds with T_1 replaced by $\widehat{T_0}$. Since $C^\infty[m]$ is dense in $H^{(k)}[m]$ the inequality below holds for every $g \in \mathcal{D}(\widehat{T_0})$:

$$||\widehat{T_0}(g) + \lambda \cdot g||_{(k)} \geq \frac{|\lambda|}{2} \cdot ||||g||_{(k)} \quad : |\lambda| \geq ||\mathcal{B}_k^*|| \tag{2.4}$$

Since $\widehat{T_0}$ is closed it follows that the range of $\widehat{T_0}(g) + \lambda \cdot E$ is closed when $|\lambda| \geq ||\mathcal{B}_k^*||$.

2.5 Density of the range. From now on $|\lambda| \geq ||\mathcal{B}_k^*||$. Recall from the general material in § XX that the adjoint of $\widehat{T_0}$ is equal to T_0^*. If the range of $\widehat{T_0}(g) + \lambda \cdot E$ is not dense there exists $0 \neq g \in H^{(k)}[m]$ such that

$$\langle \widehat{T_0}(f) + \lambda \cdot f, g \rangle_{(k)} = 0 \quad : f \in C^\infty[m]$$

Here $\widehat{T_0}(f) = P(f)$ for C^∞-functions and (x) gives

$$|\langle P(f), g \rangle_{(k)}| \leq |\lambda| \cdot |\langle f, g \rangle_{(k)}|$$

It follows that $g \in \mathcal{D}(T_0^*)$ so (i) in gives

$$\langle f, T_0^*(g) \rangle_{(k)} + \lambda \cdot \langle f, g \rangle_{(k)} = 0$$

This hold for all $f \in C^\infty[m]$ and since λ is real we get hence

$$T_0^*(g) + \lambda \cdot g = 0$$

Now the operator equation in Proposition XX gives

$$T_1(g) = \lambda \cdot g + \mathcal{B}_k^*(g)$$

At this stage we assume that k is so large that the Sobolev inequslity entails that $H^{(k)}[m]$ consists of vector-valued functions of class C^1 at least which in addition are peridic on the whole torus T^{n+1}. Now (xx) means that

$$P(g) = \lambda \cdot g + \mathcal{B}_k^*(g)$$

From this we shall prove that $g = 0$ if the real number λ is sufficiently large. To attin this we consider the function

$$G(s) = \int_{T^n} |g(x,s)|^2 \, dx$$

It follows that

$$\frac{dG}{ds} = 2 \cdot \mathfrak{Re} \int_{T^n} \partial_s(g)(x,s) \cdot \overline{g(x,s)} \, dx$$

Then conclude...

2.6 Conclusive results. If $k \geq xxx$ we have proved that there exists a positive constant μ_k such that if $|\lambda| \geq \mu_l$ then the densely defined operator $\lambda \cdot E - \widehat{T_0}$ is surjective and at the same time one has the inequality XX. This means that there exists the resolvent operator $R(\lambda; \overline{T_0})$ for such real λ. Keeping k fixed we get the closed spectrum of $\widehat{T_0}$ which by the general result in § xx is a closed subset of $\mathbf{C}$. Since $\widehat{T_0}$ is an unbounded operator one cannot expect that the spectrum is compact. Moreover in contrast to the more favourable cases for elliptic equations the resolvent operators are in general not compact.

§ 1.Differential inequalities.

Let $M(s)$ be a non-negative real-valued continuous function on a closed interval $[0, s^*]$. To each $0 \leq s < s^*$ we set

$$d_M^+(s) = \limsup_{\Delta s \to 0} \frac{M(s+\Delta s) - M(s)}{\Delta s}$$

where Δs are positive during the limit.

1.1 Proposition. *If there exists a real number B such that $d_M^+(s) \leq B \cdot M(s)$ holds in $[0, s^*)$ then*

$$M(s) \leq M(0) \cdot e^{Bs} \quad : 0 < s \leq s^*$$

The proof of this result is left as an exercise to the reader. The hint is to consider the function $N(s) = M(s)e^{-Bs}$ and show that $d_N^+(s) \leq 0$ for all s. Notice that B is an arbitrary real number, i.e. it may also be < 0.

More generally, let $k(s)$ be some non-decreasing continuous function with $k(0) = 0$. suppose that

$$d_M^+(s) \leq B \cdot M(s) + k(s) \quad : 0 \leq s < s^*$$

Now the reader may verify that

$$M(s) \leq M(0) \cdot e^{Bs} + \int_0^s k(t)\, dt \tag{1.1.1}$$

Next, consider a product set $\square = [0, \pi] \times [0, s^*]$ where $0 \leq x \leq \pi$ and consider functions $g(x, s)$ which are periodic in x, i.e.

$$g(0, s) = g(\pi, s) \quad : 0 \leq s \leq s^*$$

A C^1-function g is periodic C^1-function when g and the partial derivatives $\partial_s(g)$ and $\partial_x(g)$ are periodic in x.

1.2 Theorem. *Let g be a C^1-function which satisfies the PDE-equation*

$$\partial_s(g)(x, s) = a \cdot \partial_x(g) + b \cdot g \tag{*}$$

in $\square$ where a and b are x-periodic real-valued continuous functions. Set

$$M_g(s) = \max_x |g(x, s)| \quad : B = \max_{x,s} |b(x, s)|$$

Then one has the inequality

$$M_g(s) \leq M_g(0) \cdot e^{Bs}$$

Proof. Consider some $0 < s < s^*$ and let $\epsilon > 0$. Put

$$m^*(s) = \{x : g(x, s) = M_g(s)\}$$

The continuity of g entials that the function $M(s)$ is continuous and the sets $m^*(s)$ are compact. If $x^* \in m^*(s)$ the periodicity of the C^1-function $x \mapsto g(x, s)$ entails that $\partial_x(x^*, s) = 0$ and (*) gives

$$\partial_s(g)(x, s) = b(x, s)g(x, s) \quad : x \in m^*(s)$$

Next, let $\epsilon > 0$. We find an open neighborhood U of $m^*(s)$ such that

$$|\partial_x(g)(x, s)| \leq \epsilon \quad : x \in U$$

Now there exists $\delta > 0$ such that

$$|g(x, s)| \leq M(s) - 2\delta \quad : x \in [0, \pi] \setminus U$$

Continuity gives some $\rho > 0$ such that if $0 < \Delta s < \rho$ then the inequalities below hold:

(i) $\quad |g(x, s + \Delta s)| \leq M(s) - \delta \quad : x \in [0, \pi] \setminus U \quad : M(s + \Delta s) > M(s) - \delta$

(ii) $\qquad M(s+\Delta s) \leq M(s)+\epsilon \quad : |\partial_x(g)(x,s+\Delta s)| \leq 2\epsilon \quad : x \in m^*(s)$

If $0 < \Delta s < \rho$ we see that (i) gives $x \in m^*(s+\Delta s) \subset U$ and for such x-values Rolle's mean-value theorem and the PDe-equation give

$$M_g(x,s+\Delta s) - g(x,s) = \Delta s \cdot \partial_s(g(x,s+\theta\cdot\Delta s) =$$

(iii) $\qquad \Delta s \cdot \big[a(x,s+\Delta s)\cdot\partial_x(g)(x+\theta\cdot\Delta s) + b(x,s+\Delta s)\cdot g(x,s+\theta\cdot\Delta s)\big]$

Let A be the maximum norm of $|a(x,s)|$ taken over $\square$. Since $|g(x,s)| \leq M(s)$ the triangle inequality and (iii) give

$$M(s+\Delta s) \leq M(s) + \Delta s[\cdot A\cdot 2\epsilon + B\cdot M(s+\theta\cdot\Delta s)]$$

Since the function $s \mapsto M(s)$ is continuous it follows that

$$\limsup_{\Delta s\to 0} \frac{M(s+\Delta s)-M(s)}{\Delta s} \leq A\cdot 2\epsilon + BM(s)$$

Above ϵ can be arbitrary small and hence

$$d^+(s) \leq B\cdot M(s)$$

Then Proposition 1.1 gives (*) in the theorem.

1.3 Higher order derivatives. Supose that g is a C^2-function satisfying the PDE-equation (*) where a and b have a continuous partial x-derivative. Set $h = \partial_x(g)$. Since the differential operators ∂_x and ∂_s commute we obtain

(1.3.1) $\qquad \partial_s(h) = \partial_x(a\cdot h) + \partial_x(b\cdot g) = a\cdot\partial_x h + (\partial_x(a)+b)h + \partial_x(b)g$

L^2-inequalities. Let $g(x,s)$ be a C^1-function satisfying (*) in Theorem 1.2. Set

$$J_g(s) = \int_0^\pi g^2(x,s)\,dx$$

Diffeernation with respect to s and (*) give

$$\frac{dJ_g}{ds} = 2\cdot\int_0^\pi (a\partial_s(g)\cdot\partial g + b\cdot g)\,dx$$

By periodicity $\int_0^\pi \partial_x(ag^2)\,dx = 0$ which entials that the right hand side becomes

$$\int_0^\pi (-\partial_x(a)+b)\cdot g^2\,dx$$

So if K is the maximum norm of $-\partial_x(a)+b$ over $\square$ it follows that

$$\frac{dJ_g}{ds}(s) \leq K\cdot J(s)$$

Hence Theorem xx gives

$$\int_0^\pi g^2(x,s)\,dx \leq e^{Ks}\cdot\int_0^\pi g^2(x,0)\,dx \quad : 0 < s \leq s^*$$

Integration with respect to s entails that

$$\iint_\square g^2(x,s)\,dxds \leq \int_0^{s^*} e^{Ks}\,ds\cdot\int_0^\pi g^2(x,0)\,dx$$

This, the L^2-integral of $x \to g(x,0)$ majorizes both the area integral and each slice integral when $0 < s \leq s^*$.

Higher order derivatives. Same procedure gives majorisations of these integrals when higher order x-derivatives are inserted. Simialrly we regard PDE-equations when g is replaced by $h = \partial_s(g)$ and so on.

§ 2. A boundary value equation

Let $a(x,s)$ and $b(x,s)$ be real-valued C^∞-functions on $\square$ which are periodic in x. Consider the PDE-operator

$$P = \partial_s - a \cdot \partial_x - b$$

Given a periodic C^1-function $f(x)$ on $[0,\pi]$ we seek a function $g(x,s)$ in $\square$ which satisfies $P(g) = 0$ and the initial condition

$$g(x,0) = f(x)$$

Above we require that g is at least of class C^1. From § xx we see that g is unique if it exists. There remains to prove existence.

2.1 Theorem. *For every positive integer p and each periodic $f \in C^p[0,\pi]$ there exists a unique periodic $g \in C^p(\square)$ where $P(g) = 0$ and $g(x,0) = f(x)$.*

The proof requires several steps and employs Hilbert space methods. To each non-negtive integer k we get the complex Hilbert space $\mathcal{H}^{(k)}$ from § xx, i.e. we complete the space of complex-valued C^k-functions on $\square$ which are periodic with respect to x. By the Soboloev inequality from § xx we know that if $k \geq 2$ every function in $\mathcal{H}^{(k)}$ is continuous and more generally one has the inclusion

$$\mathcal{H}^{(k)} \subset C^{k-2}(\square) \quad : k \geq 3$$

Moreover, the first order PDE-operator P maps $\mathcal{H}^{(k+1)}$ into $\mathcal{H}^{(k)}$. From now on we only consider k-integers which are ≥ 2. On the periodic x-interval $[0,\pi]$ we get the Hilbert spaces $\mathcal{H}^k[0,\pi]$. The result in § xx shows that if $f(x) \in \mathcal{H}^k[0,\pi]$ is such that there exists some $F(x,s) \in \mathcal{H}^{(k)}$ such that $P(F) = 0$ and $F(x,0) = f(x)$ then F is unique and there exists a constant C which only depends upon the C^∞-functions a and b such that

$$||F||_k \leq C \cdot ||f||_k$$

where we have taken norms in $\mathcal{H}^{(k)}$ $\mathcal{H}^k[0,\pi]$. Moreover (*) shows that C can be chosen such that the function $f^*(x) = F(x,s^*)$ satisfies

$$||f^*||_k \leq C \cdot ||f||_k$$

Let $\mathcal{D}_k(P)$ be the set of all $f(x) \in \mathcal{H}^k[0,\pi]$ for which $F(x,s)$ above exists.

2.2 Density Lemma. *If $\mathcal{D}_k(P)$ is dense in $\mathcal{H}^k[0,\pi]$ then it is equal to this Hilbert space.*

Proof. Let f be in $\mathcal{H}^k[0,\pi]$ and by density we find a sequence $\{f_n\}$ in $\mathcal{D}_k(P)$ where $||f_n - g||_k \to 0$. By xx above we have

$$||F_n - F_m||_k \leq C||f_n - f_m||_k$$

hence $\{F_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}^{(k)}$ and converges to a limit F. Since each $P(F_n) = 0$ it follows that $P(F) = 0$ and it is clear that the continuous boundary value function $F(x,0) = f(x)$ which entails that f belongs to $\mathcal{D}_k(P)$.

2.3 The operators S_k. To each $f \in \mathcal{D}_k(P)$ we get $F(x,\pi)$ in $\mathcal{H}^k[0,\pi]$ and set

$$S_k(f) = F(x,\pi)$$

So here the domain of definition of S_k is equal to $\mathcal{D}_k(P)$ and by (xx) above there exists a conatant M_k such that

$$||S_k(f)|| \leq M_k \cdot ||f||_k \quad : f \in \mathcal{D}_k(P)$$

2.4 Proposition. *For each k there exists some $\alpha(k) < 0$ such that the range of the operator $E - \alpha \cdot S_k$ contains all periodic C^∞-functions on $[0,\pi]$.*

We prove Propostion 2.4 in § xx below. Let us now show that it gives the density of $\mathcal{D}_k(P)$. Namely, if $\mathcal{D}_k(P)$ fails to be dense there exists a non-zero $f_0 \in \mathcal{D}_k(P)$ which is $\perp$ to $\mathcal{D}_k(P)$. In Proposition 2.4 we choose $0 < \alpha \leq \alpha(k)$ so small that we also have

(i) $$\alpha < M_k/2$$

Since periodic C^∞-functions are dense in $\mathcal{H}^k[0,\pi]$ it follows from Proposition 2.4 that there exists a sequence $\{h_n\}$ in $\mathcal{D}_k(P)$ such that

(ii) $$\lim_{n\to\infty} ||h_n - \alpha \cdot S_k(h_n) - f_0||_k \to 0$$

It follows that

(iii) $$\langle f_0, f_0\rangle = 1 = \lim \langle f_0, h_n - \alpha \cdot S_k(h_n)\rangle = -\alpha \cdot \lim \langle f_0, S_k(h_n)\rangle$$

Next, the triangle inequality and (ii) give

(iv) $$||h_n||_k \leq 1 + \alpha \cdot ||(S_k(h_n)|| \leq 1 + 1/2 \cdot ||h_n|| \implies ||h_n||_k \leq 2$$

Funally, by the Cauchy-Schwarz inequality the absolute vaölue in the right hand side of (iii) is majorized by

$$\alpha \cdot M_K \cdot 2 < 1$$

which contradicts (iii). Hence the orthogonal complement of $\mathcal{D}_k(P)$ is zero which proves the requiested density and by the above we get the following conclusive result:

2.5 Theorem. *If $k \geq 2$ and $f(x) \in \mathcal{H}^k[0,\pi]$ there exists a unique $F(x,s) \in \mathcal{H}^{(k)}$ such that $F(x,0) = f(x)$.*

§ 3. A class of inhomogeneous PDE-equations.

Before Theorem 3.1 is announced we introduce some notations. Put

$$\square = \{0 \leq x \leq \pi\} \times \{0 \leq s \leq 2\pi\}$$

We shall consider doubly periodic functions $g(x, s)$ on $\square$, i.e.

$$g(\pi, s) = g(0, s) \quad : g(x, 0) = g(x, 2\pi)$$

If $k \geq 0$ we denote by $C^k(\square)$ the space of k-times doubly-periodic continuously differentiable functions. If $g \in C^k(\square)$ we set

$$||g||^2_{(k)} = \sum_{j,\nu} \int_\square \Big| \frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}(x, s)\Big|^2 \, dxds$$

with the double sum extended pairs $j+\nu \leq k$. This gives the complex Hilbert space $\mathcal{H}^{(k)}$ after a completion of $C^k(\square)$ with respect to the norm above. Recall from § xx that every function $g \in \mathcal{H}^{(2)}$ is automatically continuous and doubly periodic on the closed square. More generally, if $k \geq 3$ each $g \in \mathcal{H}^{(k)}$ has continuous and doubly periodic derivatives up to order $k - 2$. Next, consider a first order PDE-operator

$$P = \partial_s - a(x, s)\partial_x - b(x, s)$$

where a and b are real-valued doubly periodic C^∞-functions. It is clear that P maps $\mathcal{H}^{(k)}$ into $\mathcal{H}^{(k+1)}$ for every $k \geq 2$. Keeping $k \geq 2$ fixed we set

$$\mathcal{D}_k(P) = \{g \in \mathcal{H}^{(k)} : \ P(g) \in \mathcal{H}^{(k)}\}$$

Since $C^\infty(\square)$ is dense in $\mathcal{H}^{(k)}$ this yields for each $k \geq 2$ a densely defined operator

$$\text{(i)} \qquad P \colon \mathcal{D}_k(P) \to \mathcal{H}^{(k)}$$

In $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$ we get the graph

$$\Gamma_k = \{(g, P(g) \colon \ g \in \mathcal{D}_k(P)\}$$

Since P is a differential operator the general result in § xx entails that Γ_k is a closed subspace so the densely defined operator in (i) has a closed graph. Thus. for each $k \geq 2$ we have a densely defined linear operator and closed operator on $\mathcal{H}^{(k)}$ denoted by $\mathcal{T}_k$. So its domain of definition $\mathcal{D}(\mathcal{T}_k) = \mathcal{D}_k$. Next, we consider the graph

$$\gamma_k = \{(g, P(g) \colon \ g \in C^\infty(\square)\}$$

This is a subspace of Γ_k and the closure $\bar{\gamma}_k$ yields the graph of another densely defined linear operator denoted by T_k. We remark that the inclusion

$$\mathcal{D}(T_k) \subset \mathcal{D}(\mathcal{T}_k)$$

in general is strict. Let E be the identity operator. With these notations one has

3.1 Theorem. *For each integer $k \geq 2$ there exists a positive real number $\rho(k)$ such that $T_k - \lambda \cdot E$ is surjective on $\mathcal{H}^{(k)}$ for every $\lambda > \rho(k)$ and its kernel is zero.*

3.2 Remark. The result is remarkable since T_k is only densely defined while Theorem 3.1 asserts that

$$T_k - \lambda \cdot E \colon \mathcal{D}_k \to \mathcal{H}^{(k)}$$

is bijective. Hence the closed and densely defined operator T_k has a non-empty resolvent set so by the general results in §§ x there exists resolvent operators $R_k(\lambda)$ defined outside the closed specrum $\sigma(T_k)$ where the composed operators

$$(\lambda \cdot E - T_k) \circ R_k(\lambda) = E$$

for all λ outside $\sigma(T_k)$. The determination of these spectra is unclear and most likely one needs extensive numerical studies to grasp these closed sets which in addition depend upon k.

Next,R recall from § xx that the closed and densely defined operator T_k has an adjoint T_k^*. A crucial step in the proof of Theorem 3.1 is the following:

3.3 Proposition. *One has the equality $\mathcal{D}(T_k^*) = \mathcal{D}_k$ and there exists a bounded self-adjoint operator B_k on $\mathcal{H}^{(k)}$ such that*

$$T_k^* = -\mathcal{T}_k + B_k$$

Proof of Proposition 3.3

Keeping $k \geq 2$ fixed we set $\mathcal{H} = \mathcal{H}^{(k)}$. For each pair g, f in $\mathcal{H}$ their inner product is defined by

$$\langle f, g \rangle = \sum \int_{\square} \frac{\partial^{j+\nu} f}{\partial x^j \partial s^\nu}(x,s) \cdot \overline{\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}}(x,s)\, dxds$$

where the sum is taken when $j + \nu \leq k$. Introduce the differential operator

$$\Gamma = \sum_{j+\nu \leq k} (-1)^{j+\nu} \cdot \partial_x^{2j} \cdot \partial_s^{2\nu}$$

Partial integration gives

$$\text{(i)} \qquad \langle f, g \rangle = \int_{\square} f \cdot \Gamma(\bar{g})\, dxds = \int_{\square} \Gamma(f) \cdot \bar{g}\, dxds \quad : f, g \in C^\infty$$

Now we consider the operator $P = \partial_s - a \cdot \partial_x - b$ and get

$$\text{(ii)} \qquad \langle P(f), g \rangle = \int_{\square} P(f) \cdot \Gamma(\bar{g})\, dxds$$

Partial integration identifies (ii) with

$$\text{(iii)} \qquad -\int_{\square} f \cdot \big(\partial_s - \partial_x(a) - a \cdot \partial_x - b\big) \circ \Gamma(\bar{g})\, dxds$$

1.1 Exercise. In (iii) appears the composed differential operator

$$\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma$$

Show that in the ring of differential operators with C^∞-coefficients this differential operator can be written in the form

$$\Gamma \circ (\partial_s - a \cdot \partial_x - b) + Q(x, s, \partial_x, \partial_s)$$

where Q is a differential of order $\leq 2k$ with coefficients in $C^\infty(\square)$. Conclude from the above that

$$\text{(1.1.1)} \qquad \langle Pf, g \rangle = -\langle f, Pg \rangle + \int_{\square} f \cdot Q(\bar{g})\, dxds$$

1.2 Exercise. With Q as above we have a bilinear form which sends a pair f, g in $C^\infty(\square)$ to

$$\text{(1.2.1)} \qquad \int_{\square} f \cdot Q(\bar{g})\, dxds$$

Use partial integration and the Cauchy-Schwarz inequelity to show that there exists a conatant C which depends on Q only such that the absolute value of (1.2.1) is majorized by $C_Q \cdot ||f||_k \cdot ||g||_k$. Conclude that there exists a bounded linear operator B_k on $\mathcal{H}$ such that

$$\text{(1.2.2)} \qquad \langle f, B_k(g) \rangle = \int_{\square} f \cdot Q(\bar{g})\, dxds$$

1.3 Proof that B_k is self-adjoint From the above we have

$$\text{(1.3.1)} \qquad \langle Pf, g \rangle = -\langle f, Pg \rangle + \langle f, B_k(g) \rangle$$

Keeping f in $C^\infty(\Box)$ we notice that $\langle f, B_k(g)\rangle$ is defined for every $g \in \mathcal{H}$. From this the reader can check that (1.3.1) remains valid when g belongs to $\mathcal{D}(\mathcal{T}_k)$ which means that

$$\langle Pf, g\rangle = -\langle f, \mathcal{T}_k g\rangle + \langle f, B_k(g)\rangle \quad : f \in C^\infty(\Box) \tag{1.3.2}$$

Moreover, when both f and g belong to $C^\infty(\Box)$ we can reverse their positions in (*) which gives

$$\langle Pg, f\rangle = -\langle g, Pf\rangle + \langle g, B_k(f)\rangle \tag{1.3.3}$$

Since a and b are real-valued it is clear that

$$\langle Pg, f\rangle = -\langle f, Pg\rangle \tag{1.3.4}$$

It follows that

$$\langle f, B_k(g) = \langle g, B_k(f) \quad : f, g \in C^\infty(\Box) \tag{1.3.5}$$

Since this hold for all pairs of C^∞-functions and B_k is a bounded linear operator on $\mathcal{H}$ the density of $C^\infty(\Box)$ entails that B_k is a bounded self-adjoint operator on $\mathcal{H}$.

1.4 The equality $\mathcal{D}(T_k^*) = \mathcal{D}_k$. The density of $C^\infty(\Box)$ in $\mathcal{H}$ entails that a function $g \in \mathcal{H}$ belongs to $\mathcal{D}(T_k^*)$ if and only if there exists a constant C such that

$$|\langle Pf, g\rangle| \le C \cdot ||f|| \quad : f \in C^\infty(\Box) \tag{1.4.1}$$

Since B_k is a bounded operator, (1.3.2) gives the inclusion

$$\mathcal{D}_k \subset \mathcal{D}(T_k^*) \tag{1.3.3}$$

To prove the opposite inclusion we use that the Γ-operator is elliptic. If $g \in \mathcal{D}(T_k^*)$ we have from (i) in § 1.1:

$$\langle Pf, g\rangle = \langle f, T_k^* g\rangle = \int \Gamma(f) \cdot \overline{T_k^*(g)}\, dxds \quad : f \in C^\infty(\Box)$$

Similarly

$$\langle f, B_k(g)\rangle = \int \Gamma(f) \cdot \overline{B_k(g)}\, dxds$$

Treating $\mathcal{T}_k(g)$ as a distribution the equation (1.3.2) entails that the elliptic operator Γ annihilates $T_k^*(g) - \mathcal{T}_k(g) + B_k(g)$. Since both $T_k^*(g)$ and $B_k(g)$ belong to $\mathcal{H}$ this implies by the general result in § xx that $\mathcal{T}_k(g)$ belongs to $\mathcal{H}$ which proves the requested equality (1.4) and at the same time the operator equation

$$T_k^* = -\mathcal{T}_k(g) + B_k \tag{1.4.2}$$

1.5 An inequality. Let $f \in C^\infty(\Box)$ and λ is a positive real number. Then

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 =$$

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f)||^2 + \lambda^2 \cdot ||f||^2 - \lambda\big(\langle \mathcal{T}_k(f) - \frac{1}{2}B_k(f), f\rangle + \langle f, \mathcal{T}_k(f) - \frac{1}{2}B_k(f)\rangle\big)$$

The last term is λ times

$$\langle \mathcal{T}_k(f), f\rangle + \langle f, \mathcal{T}_k(f)\rangle - \langle f, B_k f\rangle \tag{i}$$

where we used that B_k is symmetric. Now $T_k = \mathcal{T}_k$ holds on $C^\infty(\Box)$ and the definition of adjoint operators give

$$\langle \mathcal{T}_k(f), f\rangle = \langle f, T_k^*\rangle \tag{ii}$$

Then (1.4.2) implies that (i) is zero and hence we have proved

$$||T_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 = \lambda^2 \cdot ||f||^2 + ||T_k(f) - \frac{1}{2}B_k(f)||^2 \ge \lambda^2 \cdot ||f||^2 \tag{iii}$$

From (iii) and the triangle inequality for norms we obtain

(iv) $$||T_k(f) - \lambda \cdot f|| \geq \lambda \cdot ||f|| - \frac{1}{2}||B_k(f)||$$

Now B_k has a finite operator norm and if $\lambda \geq ||B_k||$ we see that

(v) $$||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f||$$

Finally, since $C^\infty(\square)$ is dense in $\mathcal{D}(T_k)$ it is clear that (v) gives

(1.5.1) $$||T_k(f) - \lambda \cdot f|| \geq \frac{\lambda}{2} \cdot ||f|| \quad : f \in \mathcal{D}(T_k)$$

§ *2. Proof of Theorem 3.1*

Suppose we have found some $\lambda^* \geq \frac{1}{2} \cdot ||B||$ such that $T_k - \lambda$ has a dense range in $\mathcal{H}$ for every $\lambda \geq \lambda^*$. If this is so we fix $\lambda \geq \lambda^*$ and take some $g \in \mathcal{H}$. The hypothesis gives a sequence $\{f_n \in \mathcal{D}(T_k)$ such that

$$\lim_{n\to\infty} ||T(f_n) - \lambda \cdot f_n - g|| = 0$$

In particular $\{||T_k(f_n) - \lambda \cdot f_n\}$ is a Cauchy sequence in $\mathcal{H}$ and (1.5.x) implies that $\{f_n\}$ is a Cacuhy sequence in the Hilbert space $\mathcal{H}$ and hence converges to a limit f_*. Since the operator T_k is closed we conclude that $f_* \in \mathcal{D}(T)$ and we get the equality

$$T(_k f_*) - \lambda \cdot f_* = g$$

Finally, since the graph of T is contained in T_1 we have the requested equation

$$P(f_*) - \lambda \dot{f}_* = g$$

Thus finishes the proof of Theorem 1.6 provided we have established the existence of λ_* above

2.1 Density of the range. By the construction of adjoint operators the range of $T_k - \lambda \cdot E$ fails to be dense if and ony if $T_k^* - \lambda$ has a non-zero kernel. So assume that

(i) $$T_k^*(f) - \lambda \cdot f = 0$$

for some $f \in \mathcal{D}(T_k^*)$ which is not identically zero. Notice that T_k sends real-vaölued functions into real-valued functions. So above we can assume that f is real-valued and also assume that f is normalised so that

$$\int_\square f^2(x,s)\,dxds = 1$$

By (**) the equation (xx) gives

(ii) $$\mathcal{T}_k(f) + \lambda \cdot f - B(f) = 0$$

Let us then consider the function

$$V(s) = \int_0^\pi f^2(x,s)\,dx$$

Recall from § xx that the $\mathcal{H}$-function f is of class C^1. Now

(iii) $$\frac{1}{2} \cdot V'(s) = \int_0^\pi f \cdot \frac{\partial f}{\partial s}\,dx$$

By (ii) we have

$$\frac{\partial f}{\partial s} - a(x)\frac{\partial f}{\partial x} - b \cdot f = B(f) - \lambda \cdot f$$

Hence the right hand side in (iii) becomes

$$-\lambda \cdot V(s) + \int_0^\pi f(x,s) \cdot B(f)(x,s)\,dx + + \int_0^\pi a(x,s) \cdot f(x,s) \cdot \frac{\partial f}{\partial x}(x,s)\,dx$$

By partial integration the last term is equal to

$$-\frac{1}{2}\int_0^\pi \partial_x(a)(x,s) \cdot f^2(x,s)\,dx$$

Set

$$M = \frac{1}{2} \cdot \max_{(x,s)\in\square} |\partial_x(a)(x,s)|$$

Then we get the inequality

$$\frac{1}{2} \cdot V'(s) \leq (M - \lambda) \cdot V(s) + \int_0^\pi f(x,s) \cdot B(f)(x,s)\,dx$$

Set

$$\Phi(s) = \int_0^\pi |f(x,s)| \cdot |B(f)(x,s)|\,dx$$

Since the L^2-norm of f is one the Cauchy-Schwarz inequality gives

$$\int_{-\pi}^\pi \Phi(s)\,ds \leq \sqrt{\int_\square |B(f)(x,s)|^2\,dxds} \leq ||B(f)||$$

where the last equality follows since the squared integral of $B(f)$ is majorized by its squared norm in $\mathcal{H}$. When $\lambda > M$ it follows from (xx) that

$$(\lambda - M) \cdot V(s) + \frac{1}{2} \cdot V'(s) \leq \Phi(s)$$

Next, since f is double periodic we have $V(-\pi) = V(\pi)$ so after an integration (xx) gives

$$(\lambda - M) \cdot \int_\pi^\pi V(s)\,ds = \int_{-\pi}^\pi \Phi(s)\,ds \leq ||B(f)||$$

By (xx) we have $\int_\pi^\pi V(s)\,ds = 1$ which gives a contradiction if $\lambda > M + ||B(f)||$.

Remark. Set

$$\tau = \min_f ||B(f)||$$

with the minimum taken over funtions $f \in \mathcal{D}(T_0^*)$ whose L^2-integral is normalised by (xx). The proof has shown that the kernel of $T_0^* - \lambda$ is zero for all $\lambda > M + \tau$.

A special solution.

Let $f(x)$ be a periodic C^∞-function on $[0,\pi]$. Put

$$Q = a(x,s)\cdot\frac{\partial}{\partial x} + b(x,s)$$

Let $\eta(s)$ be a C^∞-function of s and m a postive integer If $\lambda > 0$ is a real number. we set

(i) $$g_\lambda(x,s) = \eta(s)\cdot f + \eta(s)\cdot\sum_{j=1}^{j=m}\frac{(s-\pi)^j}{j!}\cdot(Q-\lambda)^j(f) \quad : 0 \le s \le \pi$$

We choose η to be a real-valued C^∞-function such that $\eta(s) = 0$ when $s \le 1/4$ and -1 if $s \ge 1/2$. Hence $g_\lambda(x,s) = 0$ in (i) when $0 \le s \le 1/4$ and we extend the function to $[-\pi \le s \le \pi$ where $g_\lambda(x,-s) = g_\lambda(x,s)$ if $0 \le s \le \pi$. So now g_λ is π-periodic with respect to s and vanishes when $|s| \le 1/4$.

Exercise. If $1/2 \le s \le \pi$ we have $\eta(s) = 1$. Use (i) to show that

$$(P+\lambda)(g_\lambda) = \frac{\partial g_\lambda}{\partial s} - (Q-\lambda)(g_\lambda) = \frac{(s-\pi)^m}{m!}\cdot(Q-\lambda)^{m+1}(f)$$

hold when $1/2 \le s \le \pi$. At the same time $g_\lambda(s) = 0$ when $0 \le s \le 1/4$. So $(P+\lambda)(g)$ is a function whose derivatives with respect to s vasnish up to order m at $s = 0$ and $s = \pi$ and is therefore doubly periodic of class C^m in $\square$. Now Theorem 2.2 applies. For a given $k \ge 2$ we choose a sufficently large m and find $h(x,s)$ so that

$$P(h) + \lambda\cdot h = (P+\lambda)(g_\lambda)(x,s)$$

where h is s-periodic, i.e.

$$h(x,0) = h(x,\pi)$$

Notice also that $g_\lambda(x,0) = 0$ while $g_\lambda(x,\pi) = f(x)$. Set

$$g_*(x) = h - g_\lambda$$

Then $P(g_*) + \lambda\cdot g_* = 0$ and

$$g_*(x,0) - g_*(x,\pi) = f(x)$$

Above we started with the C^∞-function. Given $k \ge 2$ we can take m sufficiently large during the constructions above so that g_* belongs to $\mathcal{H}^{(k)}(\square)$.