The diffusion equation

Random walk governed by a Brownian motion in continuous time leads to the PDE:

$$(*)\qquad \frac{\partial p}{\partial t}=\frac{1}{2}\cdot\frac{\partial^2 p}{\partial x^2}$$

where $p(x,t)$ is a function of two variables Here t is a time variable which is positve. It appears in many scientific areas, both in medicine and biology, as well as physics and chemistry. In general a diffusion constant σ appers while one solves the PDE:

$$(0.1)\qquad \frac{\partial p}{\partial t}=\frac{\sigma^2}{2}\cdot\frac{\partial^2 p}{\partial x^2}$$

To describe situations where solutions to (0.1) are needed under various boundary value conditions would require a very extensive discussion. The interested reader may turno to numerous lectures by Nobel Price winners in medicine, chemisiry and physics during the last century. One can for example mention August Krough who received the prize in 1920 for his expianation of the penetration of blood through very small walls which border flowing blood with tissues in the body. He gave mathematical expansions of these processes which have been applied to improve medical treatment during the last 90 years. A kinetic theory used upon solutions to (*) was established by Einstein in 1905 and led to a whole scientific area in chemisity devoted to so called colloidal solutions. A profound and very interactive exposition of Brownian motion related yo microscopic phenomena appear in the famous text-book *Les Atom* from 1913, written by Jean Perrin who received the Nobel Prize in 1925 for his empirical verifications of Einstein's kinetic theory.

From a pour mathematical point of view, the PDE is an example of a parabolic equation. It is usually solved when one assigns initial data, i.e. the function

$$x\mapsto p(x,0)$$

is supposed to be known. At the same time the state variable x varies and in many problems the state us confined to intervals, either bounded or an unbounded interval $(0,+\infty)$, while one seeks solutions so that

$$\lim_{x\to 0} p(x,t)=\phi(t)$$

hold for a given time dependent ϕ-function. The heat equation was considered long time ago and specific solutions to (*) were established before 1800 by d'Alembert and Lagrange. More recent construction of transforms due to Laplace and Fourier give a method to solve [*). We shall expose a few of these, and mention that in the so called "Mathematics of finance" these classic solutions have become quite popular, especially in connection with option-pricing where various modlels have been introduced whose analytic solutions are found via the clasic d'Alembert-Fourier method.

A. An example. Here we seek solutions to (*) with the two boundary conditions

$$\text{(i)}\qquad \lim_{x\to 0} p(x,t)=0\quad:\quad t>0\quad\&\quad \lim_{t\to 0} p(x,t)=f(x)\quad:\quad x>0$$

where $f(x)$ is some function defined for $0<x<\infty$. To solve this one employs a transform. Namely, let ξ be a new variable and for each function $\phi(\xi)$ we define $p(x,t)$ by

$$\text{(A.1)}\qquad p(x,t)=\frac{1}{\pi}\cdot\int_0^\infty \phi(\xi)\cdot\sin(x\xi)\cdot e^{-2\xi^2 t}\,d\xi$$

The reader can check that $p(x,t)$ satisfies (*) and the first boundary vale condition in (i) holds. To attain the second boundary value one takes ϕ as the so called sine-transform of f. More precisely, put

$$\text{(A.2)}\qquad \phi(\xi)=\int_0^\infty \sin(x\xi)\cdot f(x)\,dx$$

Then one has the inversion formula

$$f(x) = \frac{1}{\pi} \cdot \int_0^\infty \sin(x\xi) \cdot \phi(\xi)\, d\xi \tag{A.3}$$

Remark. Readers who are not familiar with the inversion formula above can check the case when $f(x) = 1$ for $0 < x < A$ while $f(x) = 0$ for $x \geq A$. Here A is an arbitrary constant and we have

$$\phi(\xi) = \int_0^A \sin(x\xi)\, dx = \frac{1 - \cos(A\xi)}{\xi} \tag{i}$$

When $x > A$ one has

$$\int_0^\infty \sin(x\xi) \cdot \phi(\xi)\, d\xi = 0 \tag{ii}$$

To show (ii) one employs one of the most importent results in mathematics, namely the addition formula for the sine-function which in particular gives

$$\sin(x\xi) \cdot \cos(A\xi) = \frac{1}{2} \cdot [\sin(A + x)\xi - \frac{1}{2} \cdot [\sin(A - x)\xi\,]$$

Now the reader can check (ii) since the integrals

$$\int_0^\infty \frac{\sin(a\xi)}{\xi}\, d\xi = \frac{\pi}{2} \quad : \quad \forall\ a > 0 \tag{iii}$$

The reader should also check that

$$\int_0^\infty \sin(x\xi) \cdot \phi(\xi)\, d\xi = \pi \quad 0 < x < a$$

and conclude that the inversion formula (A.3) holds with the special f above.

A second remark. Above we used that

$$\int_0^\infty \frac{\sin\,\xi}{\xi}\, d\xi = \frac{\pi}{2} \tag{*}$$

The magic equality is proved via a detour to the complex domain. Readers familiar with reside calculus knows how to prove (*). Namely, for every ϵ we put

$$J(\epsilon) = \int_\epsilon^\infty \frac{\sin\,\xi}{\xi}\, d\xi = \frac{1}{2i} \cdot \int_{|\xi| \geq \epsilon} \frac{e^{i\xi}}{\xi}\, d\xi$$

Using Stokes Theorem and properties of the complex exponential function the last integral above is equal to

$$\frac{1}{2i} \cdot \int_0^\pi e^{i\epsilon \cdot e^{i\theta}}\, i \cdot d\theta$$

Finally,

$$\lim_{z \to 0} e^z = 1 \implies \lim_{\epsilon \to 0} J(\epsilon) = \frac{\pi}{2}$$

which gives (*).

The d'Alembert-Fourier formula. The solution of the diffusion equation when the boundary value function $f(x) = 1$ for all $x > 0$ becomes

$$(**) \qquad p(x,t) = \frac{2}{\pi} \cdot \int_0^\infty \frac{\sin(x\xi)}{\xi} \cdot e^{-2\xi^2 t}\, d\xi$$

One refers to (*) as the d'Alembert-Fourier formula for the p-function. It is used extensively in "mathematics of finance". For example to evaluate European options via the Black-.Sholes equation. The point is here the probabilistic interpretation. Namey, the construction of the Brownian motion entails that for every fixed $x > 0$, the function $t \mapsto p(x,t)$ express the survival probability that a particle which starting at x at time zero, stays in $\{x > 0\}$ up to time t, i.e. during the time interval $[0,t]$ it has not reached the barrier $\{x = 0\}$. Various numerical studies of this special p-function are found in *Lab 5* to this course. Keping $x > 0$ e consider the negative time derivative

$$(**) \qquad -\frac{\partial p}{\partial t} = \frac{4}{\pi} \cdot \int_0^\infty \xi \cdot \sin(x\xi) \cdot e^{-2\xi^2 t}\, d\xi$$

which is the frequency function for the first hitting time along the barrier $\{x = 0\}$. Plots of (**) for fixed numerical values of x can be obtained via the computer. The frequency function has a "fat tail". In fact, already the mean-value is infinite.

Many interesting probabilistic result can be attained via (**). For example, let $a > 0$ be given and fr a given $T > 0$ we consider the number $p(a, T =$ which from the above is the probability that a Brownian path starting at a stays in $\{x > 0\}$ up to time T. In this restricted family of paths one can ask how the stage x varies at time T. Another problem is for example to investigate the maximum of the stage during the time interval $(0,T)$ of paths from this restricted family. The reader is invited to try to find equations for these probability densities.

The geometric case. So far we have discussed the additive Brownian motion. We can also analyze the geometric case and in *Lab 5* there appear some numerical studies related to the geometric case. Suppose for example that we start at $x = 1$ at time zero. Under a geometric Brownian motion we seek the the probability that the geometric Brownian path does not reach an upper barrier $\{x = A\}$ during a time interval $(0,t)$, where $A > 1$. It turns out that this probability, denoted by $p^*(A,T)$ can be found for all pairs $T > 0$ and $A > 1$. Moe precisely, one regards the logarithm of the geometric Brownian motion denoted by X_t^* which imlies that

$$t \mapsto \log X_t^*$$

is the usual additive motion. The requested probability $p^*(A,T)$ is then equal to $q(\log A, T)$ where $\log A >=$ is a barrier while the additive Brownian motion starts at $x =$ at time zoo and moves in the interval $(-\infty, \log A)$. Hence one gets

$$p^*(A,T) = \frac{2}{\pi} \cdot \int_0^\infty \frac{\sin(\log A \cdot \xi)}{\xi} \cdot e^{-2\xi^2 T}\, d\xi$$

Equations of this kind show that the use of computers is essential since it is not easy to grasp the p^*-function above !

The lognormal distribution

For each $t > 0$ the normal distribution with variance t ha the frequency function

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} \cdot e^{-x^2/2t}$$

To be precise one has the equality

$$t = \int_{-\infty}^{\infty} x^2 \cdot g_t(x)\, dx \tag{0.1}$$

The reader should check (0.1) where the hint is to employ the variable substation $x \mapsto \sqrt{y} \cdot u$. The distribution function becomes

$$G_t(x) = \int_{-\infty}^{x} g_t(u)\, du$$

It is instructive too ploy g_t and G_t for some different values of t, In particular to see how they change when t is small respectively large,

Exercise. Take $t = 1$ and seek the higher even monuments

$$\mathbf{m}_{2k} = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} x^{2k} \cdot e^{-x^2/2}\, dx \quad : \ k \geq 2$$

A hint is to use partial integration and verify the recursion formula

$$\mathbf{m}_{2k} = (2k-1)\mathbf{m}_{2k-2}$$

The central limit theorem. It asserts that suitably normalized sum variables of a sequence of independent random variables converge to the normal distribution. The classic version was established by de Moivre in 1733 who considered a sequence oif independent Bernoulli variables $B_1, B_2, \ldots$ and proved that the sum variables

$$S_N = \frac{B_1 + \ldots + B_N}{\sqrt{N}}$$

converge to the nomal distribution. More generally, if $X_1, X_2, \ldots$ are independent random variables, each with mean value zero and a finite variances $\{\sigma_k\}$, which by defintionis the square root of the second moment, then the sum variables

$$S_N = \frac{X_1 + \ldots + X_N}{\sqrt{N}}$$

converge to the normal distribution if one first has

$$\lim_{N \to \infty} \frac{\sigma_1^2 + \ldots + \sigma_N^2}{N} = 1$$

and in addition the tails of the these variables satisfy the Lindeberg condition.

B. The log-normal distribution.

For each $t > 0$ we have the distribution function defined by

$$L_t(x) = \int_{-\infty}^{\log x} g_t(u)\, du \quad : \quad x > 0 \tag{B.1}$$

Its frequency function $f_t(x)$ is zero when $x < 0$ and if $x > 0$ we have

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} \cdot \frac{1}{x} \cdot e^{-(\log x)^2/2t} \tag{B.2}$$

The reader should check that even if $1/x$ appears as a term above, the expontential log-function

$$x \mapsto e^{-(\log x)^2/2t}$$

tends to zero more quite rapidly as $x \to 0$ a which implies that

$$\lim_{x \to 0} f_t(x) = 0$$

It is instructive to plot the frequency function $f_t(x)$ for some different values of t.

Moment formulas. For each real number $a > 0$ we consider the moment

$$\mathbf{m}_a(t) = \frac{1}{\sqrt{2\pi t}} \cdot \int_0^\infty x^a \cdot f_t(x)\, dx$$

The variable substation $x \mapsto e^u$ identifies the integral by

$$\frac{1}{\sqrt{2\pi t}} \cdot \int_{-\infty}^\infty e^{ua} \cdot e^{-u^2/2t}\, du$$

To compute this integral one rewrites the integrand and get

$$\mathbf{m}_a(t) = e^{a^2 t/2} \cdot \frac{1}{\sqrt{2\pi t}} \cdot \int_{-\infty}^\infty e^{-(u-at)^2/2t}\, du = e^{a^2 t/2}$$

We find for example the mean value

$$\mathbf{m}_1(t) = e^{t/2}$$

The second moment becomes e^{2t} and hence the central variance σ is given by

$$\sigma^2 = \mathbf{m}_2(t) - \mathbf{m}_1(t)^2 = e^{2t} - e^t = e^t(e^t - 1)$$

One has also the multiplicative version of the CLT. Namely, let $t > 0$ and for each N we get the random variable

$$Q_N(t) = \prod_{k=1}^{k=[tN]} (1 + \frac{B_k}{\sqrt{N}})$$

Passing to logarithms we get

$$\text{(i)} \qquad \log L_N(t) = \sum_{k=1}^{k=[tN]} \log(1 + \frac{B_k}{\sqrt{N}})$$

Taylor expansion of the log-function with a remainder shows that (i) is equal to

$$\sum_{k=1}^{k=[tN]} \frac{B_k}{\sqrt{N}} - \frac{1}{2N} \sum_{k=1}^{k=[tN]} +O(N^{-1/2}$$

De Moivres theorem entiuasls that this converges to to the normal distribution with variance t minus $t/2$. It follows that

$$\lim_{N \to \infty} Q_N(t) = e^{-t/2} \cdot L(t)$$

The right hand side is a log-normal distribution whose mean-value is zero.

Given a positive real number μ we can also consider the sequence of random variables

$$Q_N^\mu(t) == \prod_{k=1}^{k=[tN]} (1 + \frac{\mu}{N} + \frac{B_k}{\sqrt{N}})$$

The reader should confirm that

$$\lim_{N \to \infty} Q_N^\mu(t) = e^{\mu t - t/2} \cdot L(t)$$

So this time the mean-values tend to $e^{\mu t}$.

Remark. The PDE above implies that the stochastic process in continuous time with frequency functions $\{f_t\}$ is a diffusion process. More precisely, recall that if $\{X_t\}$ is a stochastic process with white noise as increment, i.e. satisfying the stochastic PDE: '

$$dX = a(t,X) \cdot dt + b(t,X) \cdot dW$$

then the frequency functions $\{f_t(x)\}$ satisfy the PDE:

$$\partial_t(f) = \partial_x(af) + \frac{1}{2} \cdot \partial_x^2(b^2 \cdot f(x,t))$$

So via (v) one has $a = x/2$ and $b = x$, i.e.the stochastic PDE becomes

$$dX = \frac{X}{2} \cdot dt + X dW$$

Introducing the mean values $E(X_t)$ gives the ODE;

$$\frac{d}{dt}(E(X_t) = \frac{X_t}{2} \implies E(X_t) = e^{t/2}$$

which is in accordance with the mean-value formula found in (0.4)

The solution to the parabolic PDE in (v) expressed by the function $f(x,t)$ has been determined via the initial condition that $X_0 = 1$, i.e.

$$\lim_{t \to 0} f_t(x) = \delta(0)$$

where $\delta(0)$ is the nit point mass at $x = 0$. In general one can consider an initial condition expressed by a frequency function $\phi_0(x)$ and then the PDE in (v) has the solution

$$\phi(x,t) = \int f(x-y,t) \cdot \phi(y)\, dy \quad : \ t > 0$$

It is instructive to plot such ϕ-functions when $\phi_0(x)$ is specified. Take as an example the case when $\phi_0(x) = a/2$ on an interval $[-a,a]$ while $\phi_0(x) =$ for $x| \geq a$. Since we have found a closed formula for the frequency functions $f_t(x)$ it is possible to get accurate plots of the frequency functions $\phi_t(x)$ when one assigns a numerical value $t > 0$ Let us also remark that one can pass to vector-valued PDE:s and also consider cases of several x-variables. With $n = 2$ one takes $x = (x_1, x_2)$ and regards the PDE:

$$\partial_t(f) = \frac{1}{2} \cdot \Delta(|x|^2 \dot{f})$$

where $|x|^2 = x_1^2 + x_2^2$ and Δ is the Laplace operator. With initial condition $f(0,x) = \delta(0)$ the solution is a radial function of x which therefore admits a plot when $t > 9$. The reader is invited to obtain this, where the hint is to express Δ in polar coordinates which reduces to a second order operator in r with $r = |x| \geq 0$. Of course one can try to solve more involved PDE:s. Conider as an example with $n = 1$ the equation

$$\partial_t(f) = \frac{1}{2} \cdot \partial_x^2(x^4 f)$$

So here the stochastic PDE becomes

$$\text{(i)} \qquad dX = X^2 \cdot dW$$

In § xx we use the Central Limit Theorem to approximate this by a stochastic process in discrete time where Monte Carlo simulations provide approximate solutions. Notice that the mean-values $E(X_t) = 1$ for all $t > 0$, i.e. no drift appears.

The Central Limit Theorerm

Let $\{f_n(x)\}$ be sequence of frequency functions, i.e. each function is non-negative and

$$\int f_n(x)\,dx = 1$$

hold for every n, We get the distribution functions

$$F_n(x) = \int_{-\infty}^{x} f_n(s)\,ds$$

Next, consider the Gaussian density function

$$g(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

and let $G(x)$ be its distribution function. We shall find a sufficient condition in order that

$$(*) \qquad \lim_{n\to\infty} \max_x |F_n(x) - G(x)| = 0$$

To achieve this we introduce the Fourier transforms

$$\widehat{f_n}(\xi) = \int e^{-ix\xi} \cdot f(x)\,dx$$

and recall that the Fourier transform

$$\widehat{g}(xi) = e^{-\xi^2/2}$$

Theorem. *Assume that for every $A > 0$ one has*

$$\lim_{n\to\infty} \int_{-A}^{A} |\widehat{f}_n(\xi) - e^{-\xi^2(2}|\,d\xi = 0$$

Then (*) follows.

First we prove Theirem A under the extra assumption that there exists a constant C such that

$$(1) \qquad |\widehat{f}_n(\xi)| \leq \frac{C}{\xi^2+1}$$

hold for every n. Nameöly when (1) holds and $A >=$, then Fouier's inversion formula and the triangle inequality give the inequality below for every x:

$$|f_n(x) - g(x)| \leq \frac{1}{2\pi} \cdot \int_{-A}^{A} |\widehat{f}_n(\xi) - e^{-\xi^2(2}|\,d\xi + \frac{1}{2\pi} \cdot \int_{|\xi|>A} (|\widehat{f}_n(\xi)| + e^{-\xi^2(2})\,d\xi$$

By (1) the last term is majored by

$$\frac{1}{2\pi} \cdot \frac{2C}{A} + \frac{1}{2\pi} \cdot \int_{|\xi|>A} e^{-\xi^2(2})\,d\xi \leq \frac{1}{2\pi A} \cdot (2C + 2e^{-A^2/2})$$

In particular the last term tends to zero as $A \to +\infty$ and then it is clear that (*) entails that

$$\lim_{n\to\infty} \max_x |f_n(x) - g(x)| = 0$$

Thus, the sequence of frequency functions converge uniformly to the Guassain density. Passing to the distribution functions ne easily checks that (ii) gives true requested limit in Theorem A.,

The use of convolutions. There remains to see that chart the specie case when (x) holds suffices to obtain Theorem A., The idea its to use suitable bubble functions. More preciously, for every $\delta > 0$ we find a non-negative C^∞-function $\phi_\delta(x)$ with support in $[-\delta, \delta]$ and the integral $\int \phi_\delta(x)\,dx = 1$. Put

$$f_n^\delta(x) = \int \phi_\delta(x-y) \cdot f_n(y)\,dy$$

which gives

$$\widehat{f^\delta}_n(\xi) = \hat{f}_n(\xi) \cdot \hat{\phi}_\delta(\xi)$$

Since the function $\xi \mapsto \hat{\phi}_\delta(\xi)$ is rapifly decreasing we find in particular a constant $C(\delta)$ such that

$$|\hat{f}_n^\delta(\xi)| \leq \frac{C(\delta)}{\xi^2+1}$$

Next, we also get the convoilutions

$$g^\delta(x) = \phi_\delta) * g(x)$$

where

$$\widehat{g^\delta}(\xi) = \hat{\phi}_\delta(\xi) \cdot e^{-\xi^2/2}$$

From this we see that the special case applies and hence

$$\lim_{n\to\infty} \max_x |F_n^\delta(x) - G^\delta(x)| = 0$$

Here (zz) hold for every $\P delta > 0$. Now we use that ϕ_δ is supported by $[-\delta, \delta]$ which gives

$$F_n^\delta(x-\delta) \leq F_n(x) \leq F_n^\delta(x+\delta)$$

for all x sand every $n \geq 1$, and a similar inequality hold when we take G. So for every x one has

$$F_n(x) - G(x) \leq F_n^\delta(x+\delta) - G(x) \leq |F_n^\delta(x+\delta) - G(x+\delta)| + G(x+\delta) - G(x)$$

and similarly

$$F_n(x) - G(x) \geq F_n^\delta(x-\delta) - G(x) \geq |F_n^\delta(x-\delta) - G(x-\delta)| + G(x-\delta) - G(x)$$

Finally, it is clear that

$$\lim_{\delta|\to 0} \max_x \; G(x+\delta) - G(x-\delta) = 0$$

and taking arbitrary small,l δ we get the requested limit in Theorem A.

Applications. Let $X_1, X_2, \ldots$ be a sequence oif ubsepoebdebt rabdoim variuabkes where each $X_:k$ has mean value zero band a finite variance σ_k, i.e.

$$\int x^2 \cdot f_k(x)\, dx = \sigma_k^2$$

hold for every k. To each $N \geq 1$ we get the sum variable

$$S_N = \frac{X_1 + \ldots + X_N}{\sqrt{N}}$$

Let $\phi_N(x)$ be its frequency function. Then we have

$$\hat{\phi}_N(\xi) = \prod_{k=1}^{k=N} \widehat{f}_n(\xi/\sqrt{N})$$

Theorem A entails that $\{S_N\}$ converge to the normal distributions in the sense of (*), provided that the integrals from Theorem A tend to zero for each fixed $A > 0$. A necessarty condiot is firt thst one has a limit for the variance, i.e.

$$\lim_{N\to\infty}, \frac{\sigma_1^2 + \ldots \sigma_N^2}{N} = 1$$

must hold. The question arises if this condition also is sufficient. Examples show that this is not so. More precisely one must impose a further condition which wads introduced Lindeberg in a famous work from 1919 which gives an almost optimal version of the Central Limit Theorem. We shall discuss Lindeberg's theorem in § x. But let us first porove a classic versdioln of the CLt where we impose the ext ran condition that the 3rd-moments are bounded by a fixed constant ,i.e, suppode that yjrtr rcidyd as consdtsnt C such that

$$\int |x|^3 \cdot f_n(x)\, dx \leq C$$

hold for every n. Now we prove that (xx) implies that the convergence criterion from Theorerm A is valid. Namely, with a fixed $A > 0$ we notice that $\xi/\sqrt{N}$ stays close to zero when $-A \leq \xi \leq A$. Hence we can use a ixck Taykoir exobasion fro the funcvtyions $\widehat{f}_k(\xi)$. More precisely, (xx) imply that they are of class C^3 and the maximum norms of the derivsties up tom order 3 are uniofrmula bounded. So when N is large one has via Tayklorās formula with a rest-term

$$\widehat{f}_k(\xi/\sqrt{N}) = -\frac{\sigma_k^2}{2N} + \rho_k(\xi) \cdot N^{-3/2}$$

where there exists a constant $C(A))$ such that

$$|\rho_k(\xi)| \leq \frac{C(A)}{N^{3/2}} \quad : \quad -A \leq \xi \leq A$$

for every k. From (x-xx) and (x) it follows that

$$\sum_{k=1}^{k=N} \log\, f_k(\xi/\sqrt{N}) \to -\xi/2$$

holds uniformly on the interval $[-A, A]$. Passoing to the exponent ails we get

$$\lim_{N\to\infty} \widehat{\phi}_N(\xi) - e^{-\xi^2/2} = 0$$

holds uniformly in the interval $[-A, A]$. This gives in particular (*) and hence Theorem A entails that the random variables $\{S_N\}$ converge to the normal distribution in the strong sense expressed by (A.1).

The CLT holds in particular when $\{X_k\}$ are independent Bernoulli variables In this special case one cash also get a quite sharp upper bound for rthe rate of convergence. More precisely, there exists a constant Csuch that

$$\max_x\, |\Phi_N(x) - G(x)| \leq \frac{C}{\sqrt{N}}$$

where $\{\Phi_N\}$ are rthe distribution funtyions of the sum variables in (x) when each X_l is a Bernoulii variable. let us remark that estimates for C were established by Gustav Essen in his doctor's thesis from Uppsala in 1942, with Arne Beurllng as adviser. With the power of today's computer one can perform very extensive Monte Carlo simulations and in this way it is tempting to try to predict some value of the constant C_* for which (*) hold for very large N.

An optimization problem

Let $[0, T]$ be a time interval and U is a utility function, i.e. it is defined for $x \geq 0$ and striclty increasing and stricly concave. If an invidual has a capital K_0 st time $t = 0$ and $t \mapsto c(t)$ is the rate of cunsuyption, then t we get the function $K(t)$ which satisfies the ODE:

$$\dot{K}(t) = r \cdot K(t) - c(t)$$

where $r > 0$ is an interest rate. We seek a c-function in order to maximise

$$(*) \qquad \int_0^T U(c(t))\, dt + e^{-rT} K(T)$$

Above the last term is the discounted capital at time T, i.e. it appears as a salvage value.

Solution The maximum is attained when the time dependent c-function satisfies the equation

$$U'(c(t)) = e^{-rt} \quad : \quad 0 \leq t \leq T(i)$$

Example. Tajke

$$U(c) = 2A \cdot \sqrt{c}$$

where A is a positive constant. Then (i) holds with

$$c(t) = A^2 \cdot e^{2rt}$$

Exerixe. Find the maximum in (*) with $8t9$ given as above. Next, in this solution it may occur thst $K(T) < 0$. This leads to a restriited optiimization problem where one in addition resuires thst $K(T) \geq 0$. For example, the solution above gives $k(T) < 0$ if K_0 is rsther small. The reader should find the solution to the OCT in this restricted case. Next, we cn also discount utiliity and consider ome positive constant μ and seek the maximum of

$$(**) \qquad \int_0^T e^{-\mu t} \cdot U(c(t))\, dt + e^{-rT} K(T)$$

Tge reader is invited to determine the optimal consmuption $c(t)$. Let us finally remark that we later on will consider a stochastic situstion where the chnge of $K(t)$ is not only governed by a fixed interest rate, but may also be disposed in a risky asset. In this stochastic optimization one seeks a strsgey of time dependent consumption over the time interval $[0, T]$ in order to maximise utility plus the salvage value at $t = T$. Of course, here the analysis becomes more involved.

About linear ODE:s

We shall discuss linear ODE:s of order one or two. The independent variable is denoted by t and in appplications regarded as a time variable. A first example is the exponential function $y(t) = e^t$ which solves the ODE:

$$\dot{y} = y \tag{*}$$

where $\dot{y}$ denotes the time derivative $\frac{dy}{dt}$. It is important to grasp solution to (*). Without knowing about the expoential funtion one solves (*) by a series expansion. Namely, suppose thst $y(t)$ solves (*) with the initial condition $y(0) = 1$ and where the function has a series expansion

$$y(t) = 1 + c_1 t + c_2 t^2 + \ldots$$

Since the time derivative of t^k is kt^{k-1} for every positive integer we see that (*) gives the recursive fomulas:

$$k \cdot c_k = c_{k-1} \quad : \ k \geq 1$$

By an induction over k this gives

$$c_k = \frac{1}{k!}$$

which mesns that the solution has the series expansion

$$y(t) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \tag{0.1}$$

This series represents by definition the exponential function e^tm where e is Neper's constant found by

$$e = \lim_{n \to \infty} (1 + 1/n)^n \tag{0.2}$$

Remark. Let us briefly recall how Isaac Newton and his collegue John Wallis at Oxford University perceived the result above around 1680. First one defines a function $L(x)$ when $x \geq 1$ by

$$L(x) = \int_1^x \frac{ds}{s}$$

Thus, following Newton we have taken the primitive function of $1/x$ when $x \geq 1$. Notice that $L(1) = 0$ and the L-function is strictly increasing with the derivative

$$L'(x) = \frac{1}{x}$$

Next, let x snd y be > 1. Then

$$L(xy) = \int_1^x \frac{ds}{s} + \int_x^{xy} \frac{ds}{s}$$

By a variable substituion the reader checks that the last integral is equal to $L(y)$, Hence

$$L(xy) = L(x) + L(y) \tag{i}$$

hold for every pair $x, y > 1$. Next, one verifies easily that

$$\lim_{x \to \infty} \int_1^x \frac{ds}{s} = +\infty$$

This means that the range of the L-function is $[0, +\infty)$ and since it is strictly increasing we find an inverse function denoted by E. So here $[0, +\infty)$ is the domain of definition for E and

$$E(L(x)) = x \quad : \ x \geq 1 \tag{ii}$$

The e-number will now be defined by the equation

$$1 = \int_1^e \frac{ds}{s} = L(e) \tag{iii}$$

Next, using (i) and (ii) the reader can check that the E-function satisfies

(iv) $$E(t+s) = E(t)E(s)$$

for every pair of non-negative resl numbers. Keeping t fixed while $s \to 0$ it follows that

$$\dot{E}(t) = \lim_{s \to 0} \frac{E(t+s) - E(t)}{(s} = E(t) \cdot \lim_{s \to 0} \frac{E(s) - 1}{(s}$$

From the construction of E via (ii) the reader can finally check that

$$\lim_{s \to 0} \frac{E(s) - 1}{s} = \lim_{x \to 1} \frac{1}{1-x} \cdot \int_1^x \frac{ds}{s} = 1$$

Hence the E-function satsfies the ODE from (*)-

Finally, from the construction of e above and the functional equation for the E-function expressed via (iv) the reader should confirm that

$$E(t) = e^t$$

where the right hand side as a function of t is found via rules for exponential of real numbers with e kept as a base while the exponentials are constructed for positive values of t.

Exercise. Show via formulas for binomial expansions that when we define the function $y(t)$ via (0.1), then it satisfies the functional equation

$$y(t+s) = y(t)y(s)$$

Exercise. When $x \geq 1$ one puts

$$\log(x) = L(x)$$

i.e. the log-function is the primitive of $1/x$. Show that $\log x$ has a series expansion

$$\log(1+s) = s - s^2/2 + s^3/3 - s^4/4 + \ldots$$

which converges when $0 < s < 1$. One can also define the log-function when $0 < x <$ by the equation

$$\log x = -\log(x^{-1})$$

Show that when $0 < s < 1$ this gives the series expansion

$$\log(1-s) = -[s + s^2/2 + s^3/3 + s^4/4 + \ldots$$

Exercise. Use the above to show thst with e defined via (iii)n above it is equal to Neper's limit in (0.2).

1. First order ODE:s

Let $a(t)$ be a real-valued function defined for $t \geq 0$. Now one seeks a function $y(t)$ where $y(0) = 1$ and

(1.1) $$\dot{y} = a \cdot y$$

Suppose thst $a(t)$ has a series expansion

$$a(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$

An example is when the series is finite, i.e. when a is a polynomial. Now y is found via a series

$$y(t) = 1 + c_1 t + c_2 t^2 + \ldots$$

Namely, the reader can check that the c-coefficients satisfy the reursive equations

(1.2) $$k \cdot c_k = a_{k-1} + a_{k-2} c_1 + \ldots + a_0 \cdot c_{k-1}$$

Here the c-numbers become uniquely determined which means that the solution to the ODE in (1.1) is unique.

One can solve (1.1) in another way using exponential functions. Namely, if we seek y in the form

$$y(t) = e^{f(t)}$$

then (1.1) means that

$$\dot{f} = a$$

i.e. f is a primitive function of a. With the inital condition $y(0) = 1$ this means that we take the primitive function

$$A(t) = a_0 \cdot t + a_2 \cdot t^2/2 + \ldots$$

and the requested y-solution becomes

$$y(t) = e^{A(t)}$$

1.2 The equation $\dot{y} = af + b$. Let $b(t)$ be another function of t defined for $t \geq 0$. One seeks y which solves the equation above with initial condition $y(0) = 0$. With $A(t)$ as above we try a solution of the form

$$y(t) = e^{A(t)} \cdot f(t)$$

Then

$$\dot{y} = a \cdot y + e^{A(T)} \cdot \dot{f}$$

f Hence y gives the requested solution if

$$e^{A(T)} \cdot \dot{f} = b$$

We conclude that a solution to the inhomogenous equation is given by

$$y(t) = e^{A(t)} \cdot \int_0^t e^{-A(s)} \cdot b(s)\, ds$$

2. Second order ODE:s

Given a pair of time dependent functions a, b we seek $y(t)$ which satisifes

$$\ddot{y} = a\dot{y} + by \tag{2.1}$$

To solve (2.1) one proceeds as follows. First we regard a first order system where one seeks a pair f, g so that

$$\dot{f} = g \quad \& \quad \dot{g} = ag + bf \tag{2.2}$$

The reader can check that if f, g satisfy the system (2,2) then f alone satisfies (2.1). Now the idea is to employ 2×2-matrices and get solutions to (2.2) via series expansions exactly as for first order ODE:s. Namely, let us introduce the t-dependent 2×2-matrices

$$A(t) = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$$

Now we seek a t-dependent 2×2-matrix $\phi(t)$ where $\Phi(0) = E_2$ is the identity matrix and

$$\dot{\Phi} = A(t)\dot{\Phi}(t)$$

We can find ϕ via a reursion. Namely, expanding the functions a, b, c as a series of t we can write

$$A(t) = A_0 + tA_1 + t^2 A_2 + \ldots$$

where $\{A_k\}$ are constant 2×2-matrices. Then ϕ is found by a series

$$\Phi(t) = E_2 + t \cdot \Phi_1 + t2 \cdot \Phi_2 + \ldots$$

where the constant matrices $\{\Phi_k\}$ satisfy

$$k\dot{\Phi}_k = A_{k-1} + A_{k-2}\Phi_1 + \ldots + A_0 \Phi_{k-1}$$

Remark. One refers to $\Phi(t)$ as the fundamental solution to the systrm defined via the matrix-valued function $A(t)$.

Exercise. Introducing the elements of the Φ-matrix we get four time dependent functions where

$$\Phi(t) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

Show that the function

$$y_1(t) = \phi_{11}(t)$$

satisfies (2.1) where y_1 satisfies the initial conditions

$$y_1(0) = 1 \quad \& \quad \dot{y}(0) = 0$$

Show also that the function

$$y_2(t) = \phi_{12}(t)$$

satisfies (2.1) where y satisfies the initial conditions

$$y_2(0) = 0 \quad \& \quad \dot{y}_2(0) = 1$$

These two functions constitute a base for the 2-dimensional solution space to the ODE in (2,1) where a general solution y with intial conditions $y(0) = \alpha$ and $\dot{y}(0) = \beta$ is found by

$$y = \alpha \cdot y_1 + \beta \cdot y_2$$

3. Inhomogeneous equations. Given a third function c we seek y so that

$$\ddot{y} = a\dot{y} + by + c \tag{3.1}$$

It turns out that (3.1) can be solved by a similar procedure as for 1st order ODE:s. To achieve this one first proves thst the time dependent fundamental solution give invertible matrices for every t. To see this we use the product formula for determinants. This implies that

$$\frac{d}{dt}(\det \Phi) = \det A \cdot \det \Phi$$

So the real-vslued function defined by $g(t) = \det \Phi(t)$ satisfies the ODE:

$$\dot{g} = \det A \cdot g$$

Recall that $\phi_2(0) = E_2$ which gives $\det \phi(O) = 1$ and hence $g(0) = 1$. It follws that

$$\det \phi(t) = e^{\int_0^t \det A(s)\, ds}$$

hold for every t. So every determinant is $\neq 0$ and hence the matrices $\Phi(t)$ are invertible.

Now we can apply the inverse 2×2-matrices $\Phi^{-1}(t)$ to time dependent 2-vectors. in this way we get for example a vector valued funvtion defined by

$$t \mapsto V(t) = \int_0^t \Phi^{-1}(s) \begin{pmatrix} c(s) \\ 0 \end{pmatrix} ds$$

Now we apply $\Phi(t)$ to this vector-valued function and regsrd

$$t \mapsto W(t)\Phi(t)(V(t))$$

Rules for differentation give

$$\frac{dW}{dt} = \dot{\Phi}(t)(V(t)) + \Phi(t)(\dot{V}(t)) = A(t)\Phi(t) + \begin{pmatrix} c(t) \\ 0 \end{pmatrix}$$

Exericse. Use the above to express a solution to (3.1) which satisfies $y(0) = \dot{y}(0) = 0$ where the two solutions $y_1(t)$ and $y_2(t)$ can be used.

A special case. Cnosder the ODE

$$\ddot{y} + y = c(t)$$

where $c(t)$ is a givn function. With $c = 0$ the teo homogeneous solutions y_1 and y_2 are known, i.e. the reader can check that

$$y_1(t) = \cos t \quad \& \quad y_2(t) = \sin t$$

To solve (x) we should firdt find the fundamental solution $\phi(t)$. To begin with we notice that the matrix $A(t)$ from (xx) is constant and becomes

$$A(t) = A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Now $\Phi(t)$ is given by rhe exponetial matrix-valued function

$$t \mapsto e^{tA} = E_2 + tA + t^2 A^2/2! + \dots$$

In order to evaluate the sum above we first notice that

$$A^2 = -E_2$$

i.e. minus the identy matrix. Recalling the poer series formulas for the cosine- and the sine funvtion the reader can check thst

$$\Phi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Exercise. Compute the inverse matrix $\phi^1(t)$ and use this to find the solution to the inhomogeneous equation (xx) where $y(0) = \dot{y}(0) = 0$.

Some further examples.

Of course one can try to solve a second order ODE directly. Let us give some examples. Consider the ODE:

(i) $$\ddot{y} = ty$$

and seek a solution $y(t)$ with initial conditions $y(0) = 0$ and $\ddot{y}(0) = 1$. To find $y(t)$ we consider a power series expansion

$$y(t) = t + c_2 t^2 + c_3 t^3 + \dots$$

Then (i) holds if the c-numbers satisfy the recursive equations

$$k(k-1)c_k = c_{k-3} \quad : \quad k \geq 3$$

while the reader csn check that $c_2 = c_3 = 0$. We find for example

$$c_4 = \frac{1}{2} \quad \& \quad c_7 = \frac{1}{42} c_4$$

and so on. The solution $y(t)4 isnotexpressedbyan"elementaryfunction", i.e.notrealisedbythodwhichoccurinordinarycalcul$ from the ODE in (i). For example, since $\dot{y}(0) = 1$ it follows that $\ddot{y}(t) > 0$ and hence $y(t)$ is a strictly increasing and convex funvtion of t. Using a computer one csn plot $y(t)$ over an interval $[0, T]$ snd confirm this observation.

Back-shooting. Above one gets the eries soluion from the two initial conditions $y(0) =$ snd $\dot{y}(0) = 1$. Suppose now that we only impose $y(0) = 0$ snd instead of prescribing $\dot{y}(0)$ we fix some $T > 0$ and require thst $y(T) = A$. So on the interval $[0, T]$ we prescribe end-values of the y-function. It turns out that there exists a unique positive number a with $\dot{y}(0) = a$ for which $y(T) = A$. To find - or st least approximate a numerically i not an easy affair. Using a computer one can employ experiments to approach the solution. Namely, start with some initial value α for $\dot{y}(0) = \alpha$ and plot this solution and inspect if the value at $t = T$ is $>$ or $<$ than A. If it is $< A$ one trie a new initial value for $\dot{y}(0)$ which is larger than α, and so on. In this way it is always possible to attsin a solution where $y(T)$ is very close to A.

Non-linear ODE:s

An example is the first order ODE:

$$\dot{y} = y^2$$

Even though the right hand side is a polynomial of y, the solution has a finite life-span. More precisely, let us find the solution defined for $t \geq 0$ with initial value $y(0) = 1$. The ODE csn be written as

$$\frac{dy}{y^2} = dt \implies -\frac{1}{y} = t + C$$

for some constant C. With $y(0) = 1$ it follows that

$$y(t) = \frac{1}{1-t}$$

This solution explodes when $t \to 1$, for then $y(t) \to +\infty$. More generally, consider an ODE of the form

$$\dot{y} = P(y)$$

where $P(y)$ is a polynomial of y which is > 0 when $y \geq 0$. An example could be

$$\dot{y} = (y+1)(y+2) \tag{0.1}$$

with initial value $y(0) = 0$. Via numerical experiments on a computer one csn approximate the life-span, i.e find the postive number t^* for which $y(t)$ exists on $[0, t^*)$ while $y(t) \to +\infty$ as $t \to t^*$. At the same time one can solve the ODE. Namely, use the algebraic identity

$$\frac{1}{(y+1)(y+2)} = \frac{1}{y+1} - \frac{1}{y+2}$$

Hence (0.2) has a solution

$$\log(1+y) - \log(2+y) = t + C$$

for a constant C. Since $y(0) = 0$ we get $C = -\log 2$ and get

$$\log \frac{1+y}{2+y} = t - \log 2 \implies t^* = \log 2$$

It is instructive to check this theoretical discovery via the computer where plots approximate the life-span t^*.

Another example. Consider the ODE:

$$\dot{y} = (y+1)(y^2+1)$$

and seek the life span when $y(0) = 0$. To get the solution one uses Newton's classic decomposition which gives constants $A; B; C$ so that

$$\frac{1}{(y+1)(y^2+1)} = \frac{A}{y+1} + \frac{By + C}{y^2+1}$$

A computation shows that

$$A = 1/2 \ \& \ B = -A \& \ C = A$$

Friom this the reader can deduce thst the solution becomes

$$\frac{1}{2}\dot{\log}(y+1) - \frac{1}{4}\log(1+y^2) + \frac{1}{2} \cdot \arctan y = t$$

for a constant c. Notice thát the left hand side can be written as

$$\frac{1}{4} \cdot \log \frac{y^2 + 2y + 1}{y^2 + 1} + + \frac{1}{2} \cdot \arctan y$$

Now the arctan-function tends to $\pi/2$ as $y \to +\infty$. From this the reader should dduce that the life span becomes

$$t^* = \frac{\pi}{4}$$

Again it is instructivle to check this result by numerical experiments on a computer, i.e. simply plot solutions to the given ODE where the length of the time interval is chosen in a careful way until one gets an "explosive character" of solutions when $t \to \pi/4$.

Remark. Above one gets formulas for t^* since we have started with relatively simple ODE:s. in general one cannot get explciit solutions and must therefore rely upon numerical experiments. Consider as an example the ODE

$$\dot{y} = y^2 + 1 + 3ty + t^2$$

with initial condition $y(0) = 0$. The reader is invited to approximate the life span by numerical experiments.

A problem for Homework. In general one can consider a polynomial $P(y)$ where we assume that $P(y) > 0$ for every $y \geq 0$. So the zerios of P are either strictly negative real numbers or complex, where complex roots $a + ib$ appear in conjugate pairs. Under the assumption that the zeros of P are all simple one has a decomposition

$$\frac{1}{P(y)} = \sum \frac{C_\nu}{y + c_\nu} + \sum \frac{A_k + B_k y}{(y - a_k)^2 + b_k^2)}$$

where $-c_1, \ldots, -c_m$ are the real negative roots and the conjugate complex roots are $a_k + ib_k$ and $a_k - ib_k$. The solution to $\dot{y} = P(y)^{-1}$ with $y(0) = 0$ becomes

(i) $y(t) = \sum C_\nu \cdot \log(y - c_\nu) + \sum B_k \cdot \log \sqrt{y - a_k)^2 + b_k^2)} + \sum A_k \cdot \arctan((y - a_k)^2 + b_k^2) = t + D$

for a constant D determined by $y(0) = 0$. Next, by a general fact when P^1 is decomposed as above one has

$$\sum C : \nu + \sum B_k = 0$$

From this it folows that the sum in (i) converges to

$$\frac{\pi}{2} \cdot \sum A_k$$

when $y = y(t) \to +\infty$.. Hence the life span t^* is given by

(ii) $$t^* = \frac{\pi}{2} \cdot \sum A_k - C$$

Using a comuter one csn perform epxeriments via various polynomials $P(y)$. The reader csn for example analyze the case when

$$P(y) = y^5 + y + 1$$

and use the computer to find apprximate eros of P and the constsnts which sppesr in (i) . After one can check (ii) numerically, i.e. a first stsge is simply to plot solutiond to the ODE and recognize at which time value the solution explodes. Nticie also thst in a converse way the numerial solution to the ODE gives insight sbout the algebraically determined numbers in (i). let us remark thst the polynomial $P(y)$ above was chosen to illustrate a deep discovery due to Niels Henrik Abel. In 1823 he demonstrated that the roots of this equation of degree 5 cannot be found by extracting roots and radicals as in the case for algebraic equation of degree 2,3 or 4. So whether the algebraically minded student likes it or not, it is necessary to employ numerical analysis in almost all situations.

A general existence theorem for ODE:s

Let $f(x,y)$ be a real-valued and continuous function defined on the closed rectangle $\square$:

$$0 \leq x \leq A \quad \& \quad -B \leq y \leq B$$

Put

$$M = \max_{x \in \square} |f(x,y)|$$

We shall assume that

$$AM \leq B \tag{0.0}$$

Under these conditions there exists a solution $y(x)$ to the ODE

$$y'(x) = f(x, y(x)) \tag{0.1}$$

which is defined for $0 \leq x \leq A$ and where $y(0) = 0$ is the initial condition. Before we enter the proof we give an exampe which shows thst such a solution in general is not unique. Namely, consider the ODE;

$$y' = y^{1/3}$$

So here $f(x,y) = y^{1/3}$ onlyl dpends upon y. Now we get a solution defined for $x \geq 0$ by

$$y(x) = \frac{2}{3} x^{3/2}$$

More genrally for every positive number a we find a solution $y_a(x)$ which is identically zero when $0 \leq x \leq a$ while

$$y(x) = \frac{2}{3}(x-a)^{3/2} \quad : \quad x \geq a$$

To ensure uniqueness of a solution to (0.1) we impose an extra condition. Namely, that there exists a constant C so that

$$|f(x,y_1) - f(x,y_2)| \leq C \cdot |y_1 - y_2| \quad : \quad 0 \leq x \leq A \quad \& \; -B \leq y_1, y_2 \leq B \tag{0.2}$$

Theorem. *When (0.2) holds there exists a unique solution $y(x)$ to the ODE wirh $y(0) = 0$ and for every $0 \leq x \leq A$ one has*

$$|y(x)| \leq$$

Remark. This general result is attributed to Picard whose classic text-books contain a wealth of results about ODE:s. Prior to Picard one treated the case when $f(x,)$ is a polynomial, or more generlly an analytic function of t x snd y. Many specific situations, where the classic analyris from Newton's famous text-books from 1665 confirmed the existence and uniqueness in Picard's theorem. Another, and for numerical applications important study was performed by Euler who found approximate solutions by decomposition the x-interval and associated difference equations. Euler's approximate solutions were later been refined by several authors. Conclusive. - and for numerical approximations - fundamental results were established by Germund Dahlquist whose Phd-thesis from Stockholm university in 1958 is a veritable classic. Here we shall not try to expose the analysis by Dahlquiet. Today's student can profit upon implemented programs which solve ODE:s numerically via fast algorithms.

About the proof of Picard's theorem. It will be presented during the lectures in the course and detsils will be included as Homework where answers which those provide a "nice details " gets credit for the exam.

Pertubations. Picard's theorem yieds both existence and uniqueness and has several consequences. Let us expose one of them. Consider the fsmily $\mathcal{F}$ of all functions $f(x,y)$ which satisfy (0.0) in the given rectsngel $\square$, and for each f in this family there also exists a Lipschitz constant

$C(f)$ in (0.2) which can depend on f. To each $f \in \mathcal{F}$ we get the unique solution denoted by $y_f(x)$. Now

$$f \mapsto y_f$$

is a functional whose domain of definition is $\mathcal{F}$ where each f produces the unique solution y_f. One expects continuity propertites of this map. The actual proof of Picard's Theorem reveals why "nice properties" hold. In paricular there exists a certain differential which we begin to explain. Consider a fixed $f \in \mathcal{F}$ where we assume that its maximum norm f is such that one has a strict inequslity

$$AM < B$$

Then, for every Lipshitz function ϕ on $\square$, it follows that $f + \epsilon \cdot \phi$ belongs to $\mathcal{F}$ when ϵ is sufficiently small. So for small ϵ we find the solution

$$y(f + \epsilon \cdot \phi)$$

It turns out that there exists a function $\rho(x)$ defined on $[0, A]$ so that

$$\lim_{\epsilon \to 0} \max_{0 \leq x \leq A} \Big| \frac{y(f + \epsilon \cdot \phi)(x) - y(f)(x)\big|}{\epsilon} - \rho(x)\Big| = 0$$

Moreover, $\rho(x)$ is the unique solution to the linear ODE

$$\rho'(x) = \phi(x) \cdot f'_y(, y(x))$$

The ODE $\dot{y} = f(t, x, y)$.

In many applications one starts from a function $f(t, x, y)$ if 3 variables. Here t is regarded as a time parameter. Given a time dependent function $x(t)$ one seeks $y(t)$ which solves the ODE

(i) $$\dot{y}(t) = f(t, x(t), y(t))$$

Keeping f fixed one can solve this ODE for different time dependent x-functions. Under suitable Lipschitz cionditions one gets unique solutions to (i), and again one can analyze pertubed solutions when $x(t)$ is replaced by functions of the form

$$t \mapsto x(t) + \epsilon \cdot \phi(t)$$

Keeping ϕ fied we get a solution $y_\epsilon(t)$ which satisfies the ODE

(ii) $$\dot{y}_\epsilon(t) = f(t, x(t) + \epsilon \cdot \phi(t),), y_\epsilon(t))$$

It turns out that there exists a limit function

$$\rho(t) = \lim_{\epsilon \to 0} \frac{y_\epsilon(t) - y(t)}{\epsilon}$$

where ρ satisfies the linear ODE:

(iii) $$\dot{\rho}(t) = f'_x(t, x(t), y(t) \cdot \phi(t) + f'_y(t, x(t), y(t)) \cdot \rho(t)$$

This equation is often used in optimal control theory.

Let $n \geq 2$ and $A = (a_{pq})$ is an $n \times n$-matrix whose elements in general are complex numbers. The determinatn is defined by

$$\det A = \sum \operatorname{sign}(i_1 \ldots i_n) \cdot a_{1i-1} \cdots$$

To each pair $1 \leq p, q \leq n$ we obtain an $(n-1) \times (n-1)$-matrix by removing the p.th row and the q:th column. it ios deboted by $A[p,q]$. Now $\det(A)$ can be cpomuted via an exanbiusion along rows. For every $1 \leq p \leq n$ this entials that

$$\det(A) = \sum_{q=1}^{q=n} a_{p,q} \cdot (-1)^{p+q} \cdot \det(A[p,q])$$

We degbiun e the $n \times$n-matrix $\mathfrak{C}_A$ with elements

$$c_{pq} = (-1)^{p+q} \cdot \det A[q,p]$$

and refer to $\mathfrak{C}_A$ as Cremer's matrix associated to A. From (i-ii) one has

$$A \cdot \mathfrak{C}_A = \mathfrak{C}_A \cdot A = \det A \cdot E'$$

where E is the dientity matrix. Next, let s be a variable and wiuth A kepts fixed we put

$$C(s) = (C)_{sE-A}$$

QWe have also the characterstic piolynomial

$$\phi(s) = \det(sE - A)$$

which has degree n. With these nottiuons

$$(sE - A) \cdot C(s) = \phi(s) \cdot E$$

Next, let

$$r^* = \max \lambda$$

wiuth the maximum taken over the roots of ϕ. Then the inverse matrix $/(sE_a)^{-1}$ exists when $|s| > r^*$ and (xx) gives

$$(sE - A)^{-1} = \frac{1}{\phi(s)} \cdot C(s)$$

Letr us now assume that ϕ has a simple root at $s = r^*$ while the absolute values of the remaning roots are all $\geq r_*$ for some $r_* < r^*$. We can write

$$\phi(s) = (s - r^*)\psi(s)$$

and newton'äs fractional series gives

$$\frac{1}{\phi(s)} = \frac{1}{(s-r) \cdot \phi'(r)} + \frac{g(s)}{\psi(s)}$$

where $g(s)$ is a polynomial of degree $\leq n-2$. It follows that

$$(sE - A)^{-1} = \frac{1}{(s-r)\phi'(r)} \cdot C(s) + \frac{g(s)}{\psi(s)} \cdot C(s)$$

next, we have the neumann series expansion

$$(sE - A)^{-1} = \frac{E}{s} + \frac{A}{s^2} +$$

At the same time one has the geometric series expansion

$$\frac{1}{s-r} = \frac{1}{s} + \frac{r}{s^2} + \ldots$$

Next, since the rooots of $\psi(s)$ stay in the disc $|s| \leq r_*$, it follows that one has a expansion

$$\frac{g(s)}{\psi(s)} \cdot C(s) = \sum \frac{B_k}{s^{k+1}}$$

which converges in the exterior disc $|s| > r_*$. Inserting $s = r$ in (xx) the expnaions abopve give

$$A^k = r^k \cdot C(r)/\phi'(r) + B_k$$

§ 0. The normal distribution.

The normal distribution is defined by the function

(0.1)
$$\mathcal{N}(x)=\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{x}e^{-t^2/2}\,dt$$

The derivative becomes

$$\mathfrak{g}(x)=\frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}$$

This is an even function of x and one has

(0.2)
$$\int_{-\infty}^{\infty}\mathfrak{g}(x)\,dx=1$$

Recall the proof of (0.2). First, since $\mathfrak{g}$ is an even function we have

$$\int_{-\infty}^{\infty}\mathfrak{g}(x)\,dx=2\cdot\int_{0}^{\infty}\mathfrak{g}(x)\,dx$$

The idea is now to consider the double integral

$$J=\iint e^{-x^2/2-y^2/2}\,dxdy$$

where integration takes place when $0\le x,y<\infty$, i.e over the non-negative quarter-plane $\mathbf{R}^2_+$. Usng polar coordinates we see that

$$J=\int_0^\infty\int_0^{\pi/2}e^{-r^2/2}\cdot r\cdot drd\theta=\frac{\pi}{2}\cdot\int_0^\infty e^{-r^2/2}\cdot r\cdot dr=\frac{\pi}{2}\implies$$

$$\sqrt{\frac{\pi}{2}}=\int_0^\infty e^{-x^2/2}\,dx$$

and from this the reader can check (0.2).

0.3 Exercise. Show that

(0.3.1)
$$\int_{-\infty}^{\infty}x^2\cdot\mathfrak{g}(x)\,dx=1$$

More generally, let $k\ge 2$ and put

(0.3.2)
$$\mathfrak{m}_{2k}^2=\int_{-\infty}^{\infty}x^{2k}\cdot\mathfrak{g}(x)\,dx=\frac{2}{\sqrt{2\pi}}\cdot\int_0^\infty x^{2k}\cdot e^{-x^2/2}\,dx$$

The variable substituion $x^2=2u$ identifies the last integral by

$$\frac{2}{\sqrt{2\pi}}\cdot\int_0^\infty u^k\cdot e^{-u}\,\frac{du}{\sqrt{u}}$$

The integral above is found via the Γ-function which is defined by

(0.3.3)
$$\Gamma(\lambda)=\int_0^\infty u^\lambda\cdot e^{-u}\,du$$

for every non-negative real number λ. In the special case when λ is a positive integer m the reader can check that succesive partial integration gives

(0.3.4)
$$\Gamma(m)=m\,!$$

From the above one has

(0.3.5)
$$\mathfrak{m}_{2k}^2=\frac{2}{\sqrt{2\pi}}\cdot\Gamma(k-1/2)$$

Hence (0.3.4) entails that the sequence $\{\mathfrak{m}_{2k}\}$ increases like $\{m\,!\}$ as $m\to+\infty$.

Later on we shall need the following:

0.4 Proposition. *For every real number a one has the equaity*

$$e^{a^2/2} \cdot \int_0^\infty \cos(ax) \cdot e^{-x^2/2}\, dx = \sqrt{2\pi} \tag{0.4.1}$$

Proof. Put

$$\psi(a) = \int_0^\infty \cos(ax) \cdot e^{-x^2/2}\, dx$$

The derivative with respect to a becomes

$$\psi'(a) = -\int_0^\infty \sin(ax) \cdot x \cdot e^{-x^2/2}\, dx$$

Partial integration shows that the right hand side is equal to

$$\sin(ax) \cdot e^{-x^2/2}|_0^\infty - a \cdot \int_0^\infty \cos(ax) \cdot e^{-x^2/2}\, dx = -a \cdot \psi(a)$$

So the ψ-function satisfies the differential equation

$$\psi'(a) = -a \cdot \psi(a)$$

At the same time the $a-$derivative of $\phi(a) = e^{a^2/2}$ is equal to $a \cdot \phi(a)$. Hence Leibniz's rule gives

$$\frac{d}{da}(\phi(a) \cdot \psi(a) = 0$$

Hence (0.4.1) is a constant function of a as $a > +$. Finally, since $\cos 0 = 1$ we see that this constant function is equal to $\sqrt{2\pi}$ by (0.2).

0.4.2 Remark. When $a \to +\infty$ we notice that $e^{a^2/2}$ gets large. At the same time the function

$$x \mapsto \cos(ax)$$

oscillates with values taken between -1 and $+1$ and as a increases the osciallations becomes rapid which entails that the integrals $\int_0^\infty \cos(ax) \cdot e^{-x^2/2}\, dx$ tend to zero as $a \to +\infty$.

0.5. The general normal distribution. For each $\sigma > 0$ and every real number m we define the density function

$$x \mapsto \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-(x-m)^2/2\sigma^2} \tag{0.5.1}$$

By the variable subsituion $x \mapsto m + \sigma \cdot u$ one verifies that the integral taken over the real x-line is one. The distribution function

$$x \mapsto \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \int_{-\infty}^x e^{-(t-m)^2/2\sigma^2}\, dt$$

is given by

$$x \mapsto \mathcal{N}(\frac{x-m}{\sigma})$$

with $\mathcal{N}$ as in (0.1). Here (0.5.1) is the frequency) function of a normally distributed stochastic variable whose mean value is m. The variance is defined by

$$\frac{1}{\sigma \cdot \sqrt{2\pi}} \int (x-m)^2 \cdot \cdot e^{-(x-m)^2/2\sigma^2}\, dx$$

After a variable substitution the reader can check that this integral is equal to σ^2.

0.6 Remark. It is instructive to plot graphs of the functions

$$x \mapsto \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-x^2/2\sigma^2}$$

for various choice of σ. When σ is small the area below the graph is concentrated to small intervals around $x = 0$. For example, let $\epsilon > 0$ be a small positive number. Now

$$\int_{|x|\geq\epsilon} \frac{1}{\sigma} \cdot e^{-x^2/2\sigma^2}\, dx = \int_{|x|\geq\epsilon/\sigma} e^{-u^2/2}\, du$$

Keeping $\epsilon > 0$ fixed while $\sigma \to 0$ we have $\epsilon/\sigma \to +\infty$, and when R is a large positive number

$$\int_{x\geq R} e^{-u^2/2}\, du \leq \frac{1}{R} \cdot \int_R^{+\infty} u \cdot e^{-u^2/2}\, du = \frac{1}{R} \cdot e^{-R^2/2} \tag{0.6.1}$$

where the last term gets very small as $R \to +\infty$.

0.7 Log-normal distributions .

We define a distribution funtion $\mathcal{L}$ which is zero if $x \leq 0$ and

$$\mathcal{L}(x) = \mathcal{N}(\log x) \quad : \quad x > 0$$

The frequency function becomes

$$\mathcal{L}'(x) = \frac{1}{\sqrt{2\pi} \cdot x} \cdot e^{-(\log x)^2/2} \tag{0.7.1}$$

It is instructive to plot the graph of this function, and the reader should check that (0.6.1) tends to zero as $x \to 0$ and that the maximum is taken when $x = 1$ where

$$\mathcal{L}'(1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-(\log 1)^2/2} = \frac{1}{\sqrt{2\pi}} \tag{0.7.2}$$

Next, the mean value

$$E(\mathcal{L}) = \int_0^\infty x \cdot \mathcal{L}'(x)\, dx = \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-(\log x)^2/2}\, dx$$

The substitution $x = e^u$ gives $dx = e^u \cdot du$ and the right hand side above is equal to

$$\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^\infty e^{-u^2/2} \cdot e^u\, du \tag{0.7.3}$$

The reader can check that (0.7.3) becomes

$$\frac{\sqrt{e}}{\sqrt{2\pi}} \cdot \int_{-\infty}^\infty e^{-(u-1)^2/2}\, du = \sqrt{e} \tag{0.7.4}$$

For the second moment one has

$$\mathfrak{m}_2(\mathcal{L}) = \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty s \cdot e^{-(\log s)^2/2}\, ds = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^\infty e^{-u^2/2} \cdot e^{2u}\, du$$

The reader can check that the last integral is equal to e^2. It follows that the central variance

$$\sigma^2(\mathcal{L}) = \mathfrak{m}^2(\mathcal{L}) - E(\mathcal{L})^2 = e^2 - e = (e-1)e \tag{0.7.5}$$

If $k \geq 3$ the moment

$$\mathfrak{m}_k(\mathcal{L}) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^\infty e^{-u^2/2} \cdot e^{ku}\, du =$$

$$e^{k^2/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^\infty e^{-(u-k)^2/2}\, du = e^{k^2/2} \tag{0.7.6}$$

0.8 Other distribution functions. In § xx we study stochastic processes and encounter situations where the distribution functions no longer are normal. An example is the stochastic variable χ defined by a sum:

$$\chi = \chi_* + \mathcal{N}$$

where χ_* and $\mathcal{N}$ are independent stochastic variables and $\mathcal{N}$ the standard normal distribution. Let $f_*(x)$ be the frequency function of χ_*. In § xx we shall learn that the frequency function of χ becomes:

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot \int f_*(x+s) \cdot e^{-s^2/2}\, ds$$

Consider as an example the case when χ_* takes the values +1 or -1 with probability 1/2. Then

$$f(x) = \frac{1}{\sqrt{8 \cdot \pi}} \cdot [e^{-(s+1)^2/2} + e^{-(s-1)^2}] \tag{0.8.1}$$