14. Sets of harmonic measure zero

Introduction.

The study of harmonic measures and other areas in potential theory goes back to a problem raised by G. Robin in the article [Rob] from 1886 which had physical background in electric engineering. The problem is: *Let E be a compact set in $\mathbf{C}$. Find a probability measure μ on E such that the function*

$$(*) \qquad U_\mu(z) = \int_E \log \frac{1}{[z-\zeta|} \cdot d\mu(\zeta) \quad : \; z \in \mathbf{C} \setminus E$$

takes constant boundary values on E. For every probability measure μ on E, i.e. a non-negative Riesz measure of unit mass supported by E, the integral (*) is defined for points in E where the value can be finite or infinite. To be precise, if $z_* \in E$ is fixed and n is a positive integer we put $E_n = E \setminus \{|z - z_*| \geq 1/n\}$. Since $\log \frac{1}{|z_*-\zeta|} \geq 0$ when $|\zeta - z_*| \leq 1$ it follows

$$n \mapsto \int_{E_n} \log \frac{1}{|z_* - \zeta|} \cdot d\mu(\zeta)$$

is increasing and by definition the integral (*) taken on E with $z = z_*$ is equal to the limit if (1) which therefore is finite or $+\infty$. Using the limits above to compute $U_\mu(z)$ at points in E it follows that the function

$$z \mapsto U_\mu(z)$$

is superharmonic function and always is harmonic in the open complement of E. One refers to $U_\mu(z)$ as the logarithmic potential of μ.

Energy integrals. We can integrate U_μ with respect to μ, i.e. there exists a well defined Borel integral:

$$J(\mu) = \int U_\mu(z) \cdot d\mu(z)$$

We refer to $J(\mu)$ as the energy integral of μ. Notice that the energy also is expressed by a double integral:

$$J(\mu) = \iiint \log, \frac{1}{|z-\zeta|} \cdot d\mu(\zeta) \cdot d\mu(z)$$

We shall foremost study the case when E is compact and totally disconnected. So Perron's criterium for a solution to the Dirichlet problem fails. But it is still meaningful to speak about harmonic functions defined in open complementary sets to E. Notice also that if $z_* \in E$ then we can construct arbitrary small Jordan domains U which where z_* is an interior point while the closed Jordan curve ∂U has empty intersection with E. This leads to the following:

0.1 Definition. *A compact and totally disconnected set E is a removable singularity for bounded harmonic functions if every bounded harmonic function H in $U \setminus E$ for a pair (z_*, U) as above extends to a harmonic function in the whole Jordan domain U.*

Remark. This is a local condition at each point in E and the extensions should hold at all points in E. The following result is due to Myhrberg.

0.2 Theorem. *A compact and totally disconnected set E is a removable singularity for bounded harmonic functions if and only if there exists a probability measure μ on E such that $U_\mu(z) = +\infty$ for every $z \in E$.*

We prove this result in § A. When the two equivalent conditions hold in Theorem 0.2 we say that E is a harmonic null-set. Denote by $\mathcal{N}_{\text{harm}}$ the family of totally disconnected sets harmonic null-sets. One may ask for metric conditions in order that a given compact and totally disconnected set E belongs to $\mathcal{N}_{\text{harm}}$. To analyze this we introduce h-measures where $h(r)$ is a continuous and non-decreasing function defined for $r > 0$ and $h(0) = 0$. If F is a compact set we consider open coverings of F by discs and define its outer h-measure by

$$h^*(F) = \min \sum h(r_\nu)$$

where the minimum is taken over coverings of F by open discs $\{D_\nu\}$ of radius $\{r_\nu\}$. The family of compact sets whose outer h-measure is zero is denoted by $\mathcal{N}(h)$. The case $h(r) = r^2$ means precisely that F has planar Lebesgue measure zero. If $h(r)$ tends more slowly to zero as $r \to 0$ we get a more restrictive class, i.e. then $\mathcal{N}(h)$ consists of sets which are more thin than sets with planar Lebesgue measure zero. In the case when

$$h(r) = \frac{1}{\log \frac{1}{r}}$$

we say that a compact set in $\mathcal{N}(h)$ has logarithmic capacity zero . The first major result about harmonic null-sets was proved by Lindeberg in 1918 and goes as follows:

0.3 Theorem. *Let E be a compact set whose logarithmic measure zero. Then E has harmonic measure zero.*

A result which gives a necessary metric condition for a set E to be a harmonic null-set was proved by Henri Cartan in [Cartan]. First we give:

0.4 Definition *Let $\mathfrak{H}_*$ denote the class of non-decreasing and continuous function $h(r)$ satisfying*

$$\int_0^1 \frac{h(r)}{r} \cdot dr < \infty$$

0.5 Theorem. *For every $E \in \mathcal{N}_{harm}$ it follows that*

$$E \in \mathcal{N}(h) = 0 \quad : \forall\ h \in \mathfrak{H}_*$$

0.6 Remark. Cartan's result is close to Lindeberg's sufficiency result. Namely, if $\eta > 0$ we set

$$h(r) = \frac{1}{[\text{Log}\,\frac{1}{r}]^{1+\eta}}$$

It is clear that $h \in \mathfrak{H}_*$ and hence $h^*(E) = 0$ for every $E \in \mathcal{N}_{\text{harm}}$. With η small this comes close to say that the logarithmic capacity of E is zero. However, Cartan's Theorem does not give sufficient conditions in order that a compact and totally disconnected set E has harmonic measure zero. The search for other metric conditions which are either necessary or sufficient in order that a compact set has harmonic measure zero is unclear and one should perhaps not expect too much. The text-book on *Exceptional Sets* by Carleson contains examples which illustrate the difficulty to get definite answers. However, a metric criterion for harmonic null-sets exists if E is a Cantor set on a line. See XXX below.

0.7 Transfinite diameters and the logarithmic capacity. Let E be a compact set which is assumed to be infinite. Here we do not assume that E is totally disconnected. To each n-tuple of distinct points $z_1,\ldots,z_n$ we put:

$$L_n(z_\bullet)=\frac{1}{n(n-1)}\cdot\sum_{k\neq j}\log\frac{1}{|z_j-z_k|}$$

Then we define the number

$$\mathcal{L}_n(E)=\min\, L_n(z_\bullet)$$

where the minimum is taken over all n-tuples in E. Since $\log\frac{1}{r}$ is large when $r\simeq 0$ this means intuitively that we tryo to choose separated n-tuples in order to minimize the L_n-function. Notice for example that when $n=2$ then the minium is achieved for a pair of points in E whose distance is maximal, i.e. $\mathcal{L}_2$ is the diameter of E. As n increases one has

0.8 Proposition. *The sequence $\{\mathcal{L}_n\}$ is non-decreasing.*

Proof. Let $z_1,\ldots,z_{n+1}$ minimize the L_{n+1}-function which gives

$$\mathcal{L}_{n+1}(E)=\frac{1}{n(n+1)}\cdot\sum_{k\neq j}^{(1)}\log\frac{1}{|z_j-z_k|}+\frac{2}{n(n+1)}\cdot\sum_{k=2}^{k=n+1}\log\frac{1}{z_1-z_k|}$$

where (1) above the sum above means that we only consider pairs k,j which both are ≥ 2. Since $z_2,\ldots,z_{n+1}$ is an n-tuple we get the inequality

$$\mathcal{L}_{n+1}(E)\geq\frac{1}{n(n+1)}\cdot n(n-1)\cdot\mathcal{L}_n(E)+\frac{2}{n(n+1)}\cdot\sum_{k=2}^{k=n+1}\log\frac{1}{z_1-z_k|}$$

The same inequaliiy holds when when we instead of z_1 delete some z_j for $2\leq j\leq n+1$ and taking the sum of the resulting inequalities we obtain

$$(n+1)\mathcal{L}_{n+1}(E)\geq\frac{1}{n}\cdot n(n-1)\cdot\mathcal{L}_n(E)+\frac{2}{n(n+1)}\cdot\sum_{k\neq j}\log\frac{1}{|z_j-z_k|}$$

The last term is $2\cdot\mathcal{L}_{n+1}$ which gives:

$$(n-1)\cdot\mathcal{L}_{n+1}(E)\geq\frac{1}{n}\cdot n(n-1)\cdot\mathcal{L}_n(E)=(n-1)\mathcal{L}_n(E)$$

A division by $n-1$ gives the requested inequality.

0.8 Definition. *Put*

$$\mathfrak{D}(E)=\lim_{n\to\infty}\,e^{-\mathcal{L}_n(E)}$$

This non-negative number is called the transfinite diameter of E.

Remark. The definition means that $\mathfrak{D}(E)=0$ if and only if $\mathcal{L}_n(E)$ tends to $+\infty$ as n increases. Intuitively this means that we are not able to choose large tuples in E separated enough to keep the sum of the log-terms bounded. Another number is associated to E is defined by:

$$\mathcal{J}_*(E)=\min_\mu\, J(\mu)$$

where the minimum is taken over all probability measures in E.

0.9 Definition. *The logarithmic capacity of E is defined by:*

$$\text{Cap}(E)=e^{-J_*(E)}$$

0.10 Theorem. *For each compact set E one has the equality*

$$\text{Cap}(E)=\mathfrak{D}(E)$$

Proof. First, let $n \geq 2$ and $z_1^*, \ldots, z_n^*$ is some n-tuple where $L_n(z_\bullet) = \mathcal{L}_n(E)$. Now we have the probability measure

$$\mu = \frac{1}{n} \cdot \sum_{k=1}^{k=n} \delta_{z_k}$$

It is clear that the energy

$$J(\mu) = \frac{n(n-1)}{n^2} \cdot L_n(z_\bullet)$$

Hence we have the inequality

$$\mathcal{J}_*(E) \leq \frac{n(n-1)}{n^2} \cdot \mathcal{L}_n(E)$$

Since $\frac{n(n-1)}{n^2}$ tends to one as $n \to \infty$ a passage to the limit gives:

$$\mathcal{J}_*(E) \leq \lim_{n\to\infty} \mathcal{L}_n(E)$$

Taking exponentials and recalling the negative signs in Definition x and x we conclude that

$$\text{(i)} \qquad \mathfrak{D}(E) \leq \mathrm{Cap}(E)$$

The opposite inequality follows since we can approximate probability measures on E by discrete measures. Leave this as a TRIVIAL EXERCISE,

A. Proof of Myhrberg's theorem

When E is totally disconnected we can surrender E by open sets Ω such that Dirichlet's problem has a solution in the exterior domain $\mathbf{C} \setminus \bar{\Omega}$. After a suitable passage to the limit as these domains shrink to E we obtain a special measure supported by E. To obtain this we employ a construction which was introduced in the present context by De Vallé Poussin.

A.1 Nested coverings Let E as above be a totally disconnected and compact set and consider some $z_* \in E$. Choose a small Jordan domain U which contains z_* while $\partial U \cap E = \emptyset$. In particular can take U to be contained in the disc of radius $1/2$ centered at z_* which gives

$$|z_1 - z_2| < 1$$

for each pair z_1, z_2 in $E \cap U$. To compensate for the failure of solving Dirichlet's problem we construct a sequence of open sets $\{V_N\}$ as follows. For each positive integer N one has the dyadic grid $\mathcal{D}_N$ of open squares whose sides are 2^{-N}. We get the finite family $\mathcal{D}_N(E)$ of dyadic squares in $\mathcal{D}_N$ which have a non-empty intersection with E. The union of this finite family of open squares gives an open neighborhood V_N of E. We consider only large N so that 2^{-N} is strictly larger than the distance of E to ∂U. Let Ω_N^* be the exterior connected component of $D \setminus \bar{V}_N$ whose closure contains ∂U.

A.2 Exercise. Show by a figure that Ω_N^* is a doubly connected domain whose boundary is the disjoint union of ∂U and a closed Jordan curve Γ_N formed by line segments from squares in the dyadic grid. Notice also that $\{\Omega_N^*\}$ form an increasing sequence of open sets where Γ_N appears as a compact subset of Ω_{N+1}^* for each N and conclude that

$$\cup\, \Omega_N^* = D \setminus E$$

Next, fix some point $z_0 \in D \setminus E$ and from now on N are so large that $z_0 \in \Omega_N^*$ hold. The Dirichlet problem has a solution in each domain Ω_N^*. This gives a unique pair of non-negative measures μ_N, γ_N where μ_N is supported by Γ_N and ρ_N by ∂U such that

$$(*) \qquad h(z_0) = \int_{\Gamma_N} h(\zeta) \cdot d\mu_N(\zeta) + \int_{\partial U} h(\zeta) \cdot d\rho_N(\zeta)$$

hold for every h-function which is harmonic in Ω_N^* with continuous boundary values. In particular we let h be the harmonic measure function $\mathfrak{m}_N$ with respect to Γ_N, i.e. it is 1 on Γ_N and zero on ∂U. Then

$$(**) \qquad \mathfrak{m}_N(z_0) = \int_{\Gamma_N} d\mu_N(\zeta) = ||\mu_N||$$

Since $\Gamma_N \subset \Omega_{N+1}^*$ we have $\mathfrak{m}_{N+1} \leq \mathfrak{m}_N$ in Ω_N^* which implies that

$$||\mu_{N+1}|| \leq ||\mu_N||$$

for each N. Hence there exists the limit

$$(***) \qquad \alpha = \lim_{N \to \infty} ||\mu_N||$$

A.3 The case $\alpha = 0$. When this holds the mass of ρ_N tends to one and since (*) in particular hold for h-functions which are harmonic in the whole set U with continuous boundary values on ∂U the reader may verify:

A.4 Proposition. *If $\alpha = 0$ the sequence $\{\rho_N\}$ converges weakly to the representing measure $m(z_0)$ for which*

$$H(z_0) = \int_{\partial U} H(\zeta) \cdot dm(z_0, \zeta)$$

when H is harmonic in U and continuous on $\bar{U}$.

Keeping the case A.3 we can apply the result in Exercise A.4 and obtain the following crucial result.

A.5 Proposition. *When A.3 holds there exists a pair of positive numbers $0 < a < A$ such that*

$$a \leq \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) \leq A$$

hold for all $w \in E$ and every N.

A.6 Exercise. Prove this result. Here is a hint in the case when U is the unit disc so that $\partial U = T$ is the unit circle. If $w \in E$ we have

$$|e^{i\theta} - w| = |1 - \bar{w} \cdot e^{i\theta}|$$

This gives the equality

$$\text{(i)} \qquad \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta) = \int_T \log \frac{1}{|1 - \bar{w}\zeta|} \cdot d\rho_N(\zeta)$$

Next, in D we have the harmonic function $H(z) = \log \frac{1}{|1 - \bar{w}z|}$ in D and Exercise A.4 together with (i) give

$$\text{(ii)} \qquad \log \frac{1}{|1 - \bar{w}z_0|} = \lim_{N \to \infty} \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta)$$

At the same time (**) applied with $h(z) = \log \frac{1}{|z - w|}$ gives

$$\text{(iii)} \qquad \log \frac{1}{|z_0 - w|} = \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) + \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta)$$

By (ii) the last integral converges to $\log \frac{1}{1-\bar{w}z_0|}$ which entails that

$$\lim_{N\to\infty}\int_{\Gamma_N}\log\frac{1}{|\zeta-w|}\cdot d\mu_N(\zeta)=\log\frac{1-\bar{w}z_0|}{|z_0-w|}$$

Since $|z_0|<1$ and there is some $r<1$ such that $|w|\leq r$ for every $w\in E$ the last term is between a and A for a pair of positive numbers which proves Proposition A.5 in the case when $U=D$. For a general Jordan domain the reader can deduce the requested result using a conformal map or suitable Green's functions in U adapted to the point z_0.

A.7 The limit measure μ_*. Keeping A.3 we have the probability measures

$$\nu_N=\frac{1}{||\mu_N||}\cdot\mu_N$$

We can extract a subsequence which converges weakly to a probability measure μ which by (xx) is supported by E. For this limit measure the following holds:

A.8 Proposition. *The potential function U_μ is $+\infty$ on E except possible at a finite set of points.*

Proof. GIVE IT is easy ...

A.9 Final part of the proof of Theorem 0.2. Suppose first that E is a removable singularity for bounded harmonic functions. Working locally as above around some $z_*\in E$ and a Jordan domain U, we obtain for each N the harmonic measure function $\mathfrak{m}_N$ in Ω_N^*. They take values in $(0,1)$ and passing to a subsequence we obtain a bounded harmonic limit function $\mathfrak{m}_*$ defined in $U\setminus E$. By construction $\mathfrak{m}_*=0$ on ∂U so if it extends to a harmonic function in U it must be identically zero. This entails that

$$\lim_{N\to\infty}\mathfrak{m}_N(z_0)=0$$

Now we recall that $\mathfrak{m}_N(z_0)=||\mu_N||$ and conclude that A.3 holds which by Theorem XX gives the existence of a probability measure μ on E such that $U_\mu(w)=+\infty$ for all $w\in E\cap U$.

The converse. Assume that there exists μ on E for which $U_\mu(w)=+\infty$ for all $w\in E$. Given a large positive constant we consider the open subset of U defined by

$$\{z\in U\quad:\quad U_\mu(z)>C\}$$

By the hypothesis this is an open neighborhood of $E\cap U$ so when N is large we have $U_\mu\geq C$ on Γ_N. Next, U_μ restricts to a continuous function on ∂U and we find its harmonic extension $H(z)$ to the whole Jordan domain U. Now $U_\mu-H$ is harmonic in Ω_N^* and if C_* is the maximum of H on Γ_N it follows that $U_\mu-H\geq C-C_*$ on Γ_n. This gives the inequality

$$\mathfrak{m}_N(z_0)\leq\frac{1}{C-C_*}\cdot[U_\mu(z_0)-H(z_0)]$$

Here we can start with arbitrary large C and get (xx) for sufficiently large N which entails that

$$\lim_{N\to\infty}\mathfrak{m}_N(z_0)=0$$

The conclusion is valid for any starting point $z_0\in U\setminus E$ and let us consider a bounded harmonic function h in $U\setminus E$ with continuous boundary values on ∂U. It can be restricted to Ω_N^* for every N. In A.2 we could have started with an arbitrary point $z_0\in U\setminus E$ and by (*) from A.2 we have

$$h(z_0)=$$

Passing to the limit as $N \to \infty$ it follows from (x) and the discussion in (xx) that

$$h(z_0) = \int_{\partial U} h(\zeta) \cdot dm(z_0, \zeta)$$

Since this hold for every z_0 in $U \setminus E$ we get the requested harmonic extension H given by

$$H(z) = \int_{\partial U} h(\zeta) \cdot dm(z, \zeta) \quad : \quad z \in U$$

B. Equilibrium distributions and Robin's constant.

Let E be a compact set in $\mathbf{C}$. To each probability measure μ supported by E we get the potential function

$$U_\mu(z) = \int \log \frac{1}{|z - \zeta|} \cdot d\mu(\zeta)$$

We are going to construct a special μ for which U_μ either is identically $+\infty$ or else takes a constant value almost everywhere on E with respect to μ. First we carry out the construction in the special case when E is a finite union of pairwise disjoint and closed Jordan domains $U_1, \ldots, U_m$ for some $m \geq 1$. We also assume that each Jordan curve ∂U_k is of class C^1. When this holds we get the connected exterior domain

$$\Omega^* = \mathbf{C} \cup \{\infty\} \setminus \cup \bar{U}_k$$

Here we can solve Dirichlet's problem. in particular we obtain the unique probability measure μ on $\partial \omega^*$ such that

$$H(\infty) = \int H \cdot d\mu$$

for every harmonic function H in Ω^* with continuous boundary values. If z_1 and z_2 are two points in $/cup\, U_k$ which may or may not belong to the same Jordan domain then we notice that the function

$$H(z) = \log |z - z_1| - \log |z - z_2|$$

is harmonic i Ω^*. Moreover, as $|z| \to \infty$ we notice that

$$H(z) = \log |1 - \frac{z_1}{z}| - \log |1 - \frac{z_2}{z}|$$

and in the limit we have $H(\infty) = 0$. Since $\log r = -\log \frac{1}{r}$ for each $r > 0$ it follows that

$$U_\mu(z_1) = U_\mu(z_2)$$

Hence the function $z \mapsto U_\mu(z)$ is constant in the interior of E Since the boundary curves $\{\partial U_k\}$ are C^1 it follows that U_μ extends to a continuous function with constant value on the whole set E. Of course, U_μ is also continuous outside E where it is harmonic. In fact, we conclude that U_μ is a globally defined and continuous super-harmonic function in $\mathbf{C}$. The measure μ is called the equilibrium distribution of E.

Remark. If E is contained in the unit disc it is clear that the constant value of U_μ is positive. On the other hand, let $R > 1$ and E is the disc $|z \leq R$. here μ is the measure $\frac{1}{\pi} \cdot d\theta$ on the circle of radius R and we find that the constant value is $-\log R$.

Notation. If a is the constant value of U_μ we set

$$\text{cap}(E) = e^{-a}$$

and refer to this as the capacity of E. For example, if E is the disc $|z| \leq r$ where r is small we see that the capacity becomes r.

The general case. Now E is an arbitrary compact set. To construct a special probability measure μ_E we use a similar construction as in section A. Thus, for $N \geq 1$ we get the family of cubes in $\mathcal{D}_N$ which have a non-empty intersection with E and then we construct the outer boundary curves of thus set which borders a connected exterior domain Ω_N^* whose boundary now will be a union of closed and piecewise linear Jordan curves where two of these may interest at corner points. We solve the Dirichlet problem and exactly as above we find the equilibrium measure μ_N supported by $\partial\Omega_N^*$.

C. Cartan's theorem

We shall actually establish an inequality in Theorem C.1 below which has independent interest since it applies to compact sets E which are not necessarily harmonic null sets. Consider a pair (h, μ) where μ is a probability measure with compact support in a compact set E of $\mathbf{C}$ with planar Lebesgue measure zero while $h \in \mathfrak{H}_*$. To each point $a \in E$ and every $r > 0$ we have the open disc $D_r(a)$ centered at a and can regard its μ-mass. This gives anon-decreasing function

$$r \mapsto \mu(D_r(a)) \quad : r > 0$$

Put

$$\mathcal{U}^* = \{a \in E \quad : \exists\, r > 0 \quad : \mu(D_r(a)) > h(r)\} \tag{1}$$

We assume that the pair (h, μ) is such that this set is non-empty. Since μ is a Riesz measure one has the limit formula

$$\lim_{\rho \to r} \mu(D_\rho(a) = \mu(D_r(a)$$

for each $r > 0$ where the limit is taken as ρ increases to r. From this it is obvious that $\mathcal{U}^*$ is a relatively open subset of E and in the closed complement we have

$$a \in E \setminus \mathcal{U}^* \implies \mu(D_r(a)) \leq h(r) \quad : \forall\, r > 0 \tag{2}$$

Now the size of $\mathcal{U}^*$ is controlled as follows:

C.1 Cartan's Covering Lemma. *There exists a sequence $\{a_\nu\}$ in E and a sequence of positive numbers $\{r_\nu\}$ such that the following hold:*

$$\mathcal{U}^* \subset \cup \bar{D}_{r_\nu}(a_\nu) \quad \text{and} \quad \sum h(r_\nu) \leq 6$$

Moreover, for each $z \in \mathbf{C}$ at most five discs from the family $\{D_{r_\nu}(a_\nu)\}$ contains z.

Proof We may assume that $\mathcal{U}^* \neq \emptyset$. Set

$$\lambda_1^*(r) = \max_{a \in E} \mu(D_r(a)) \tag{1}$$

Since the functions $r \mapsto \mu(D_r(a))$ are lower semi-continuous for each a, it follows that the maximum function $\lambda_1^*(r)$ also is lower semi-continuous. Hence the set $\{r\colon \lambda_1^*(r) > h(r)\}$ is open and we find its least upper bound r_1^*. Thus,

$$\lambda_1^*(r_1^*) = h(r_1^*) \quad : \lambda_1^*(r) < h(r) \text{ for all } r > r_1^* \tag{2}$$

Pick $a_1 \in E$ so that

$$\lambda_1^*(r_1^*) < \mu(D_{r_1^*}(a_1)) + 1/2 \tag{3}$$

Next, set $E_1 = E \setminus D_{r_1^*}(a_1)$ and define

$$\lambda_2^*(r) = \max_{a \in E_1} \mu(D_r(a))$$

If $\lambda_2^*(r) \leq h(r)$ for every r we stop the process. Otherwise we find the unique largest r_2^* such that

$$\lambda_2^*(r_2^*) = h(r_2^*)$$

Notice that $r_2^* \leq r_1^*$ holds since h is non-decreasing while it is obvious that $\lambda_2^* \leq \lambda_1^*$. This time we pick $a_2 \in E$ so that

$$\lambda_2^*(r_2^*) < \mu(D_{r_2^*}(a_2)) + 2^{-2}$$

Put $E_2 = E_1 \setminus D_{r_2^*}$ and continue as above, i.e. inductively we get E_n and set

$$\lambda_{n+1}(r) = \max_{a \in E_n} \mu(D_r(a))$$

The process continues if we have found r_{n+1}^* so that $\lambda_{n+1}(r_{n+1}^*) = h(r_{n+1}^*)$, then we pick $a_{n+1} \in E_n$ where

$$\lambda_{n+1}(r_{n+1}^*) \leq \mu(D_{r_{n+1}^*}(a_{n+1})) + 2^{-n-1} \tag{4}$$

In this way we get the sequence $r_1^* \geq r_2^* \geq \ldots$ and a family of discs $\{D_{r_\nu^*}(a_\nu)\}$. To simplify notations we set

$$D_\nu^* = D_{r_\nu^*}(a_\nu)$$

Sublemma Every point $a \in E$ belongs to at most five many D^*-discs.

Proof. If some a belongs to six discs then elementary geometry gives a pair a_k, a_ν such that the angle between the lines $[a, a_k]$ and $[a.a_\nu]$ is $< \pi/3$. Suppose that for example that $|a - a_k| \geq [a - a_\nu|$. Euclidian geometry gives

$$|a_k - a_\nu| < |a - a_k|$$

But this is impossible. For say that $k < \nu$. Now the disc D_k^* was removed and a_ν is picked from the subset E_ν of E_k while $E_k \cap \Delta_k = \emptyset$.

Proof continued. The Sublemma implies that

$$\sum \mu(D_\nu^*) \leq 5 \cdot \mu(E) = 5 \tag{5}$$

The convergence of (5) and (4) imply that $\lim_{\nu \to \infty} r_\nu^* = 0$. From this it follows that

$$\mathcal{U}^* \subset \cup \bar{D}_{r_\nu}(a_\nu) \tag{6}$$

Finally we have

$$\sum h(r_\nu^*) = \sum \lambda_\nu^*(r_\nu^*) \leq \sum [\mu(D_\nu^*) + 2^{-\nu}] \leq 5 \cdot \mu(E) + \sum 2^{-\nu} = 6 \tag{7}$$

This completes the proof of Cartan' s Covering Lemma.

The family $\mathcal{G}_h$. Let $g(r)$ be a positive function defined on $(0, +\infty)$ which satisfies:

$$\lim_{r \to 0} g(r) = +\infty$$

In this family we get those g-functions for which

$$\int_0^1 g(r) \cdot dh(r) < \infty \tag{*}$$

This family is denoted by $\mathcal{G}_h$. With this notation we have:

C.2 Lemma *For each $g \in \mathcal{G}_h$ and every point $a \in E \setminus \mathcal{U}^*$ one has*

$$\int_E g(|z - a|) d\mu(z) \leq \int_0^\rho g(r) dh(r) \quad \text{where } h(\rho) = 1$$

Proof. Since a is outside $\mathcal{U}^*$ we have

$$\mu(D_r(a)) \leq h(r)$$

for every $r > 0$. Moreover, we recall that μ has total mass one and now the reader can verify the inequality in Lemma C.2 b using a partial integration.

C.3 A special choice of g. Let us take

$$g(r) = \text{Log}\,\frac{1}{r} \quad : \; 0 < r < 1 \quad : \quad g(r) = 0 \quad : \; r \geq 1$$

This g-function belongs to $\mathcal{G}_h$ by the condition on h-functions in Cartan's theorem. Next, for every $\lambda > 1$ we get the function $h_\lambda = \lambda \cdot h$ in $\mathfrak{H}_*$ and set:

$$E \setminus \mathcal{U}^*(\lambda) = \{a \in E \quad : \; \mu(D_r(a)) \leq \lambda \cdot h(r) \; : \; \forall \; r > 0\}$$

Proposition XX(measure general) applied with h_λ gives:

$$\text{(1)} \qquad \int_E g(|z-a|)d\mu(z) \leq \lambda \cdot \int_0^{\rho/\lambda} g(r)dh(r) \quad : \; a \in E \setminus \mathcal{U}^*(\lambda)$$

A partial integration shows that the right hand side in (1) becomes

$$g(\rho) + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r}$$

Hence we have the inequality

$$\text{(2)} \qquad \int g(|z-a|) \cdot d\mu(z) \leq g(\rho) + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r} \quad : \quad a \in E \setminus \mathcal{U}^*(\lambda)$$

In addition to this, the Covering Lemma gives an inclusion

$$\text{(3)} \qquad \mathcal{U}^*(\lambda) \subset \cup \, D_{r_\nu}(a_\nu) \quad \text{where} \sum \; h_\lambda(r_\nu) < 6$$

Since $h_\lambda = \lambda \cdot h$ this means that the outer h-measure

$$\text{(4)} \qquad h^*(\mathcal{U}^*(\lambda)) \leq \frac{6}{\lambda}$$

Hence we have proved the following where we recall that $g(r) = \text{Log}\,\frac{1}{r}$:

C.4 Theorem. *For every triple (E, μ, h), where μ is a probability measure supported by E and $h \in \mathfrak{H}_*$, and any $\lambda > 1$ there exists a relatively open subset $\mathcal{U}^*(\lambda) \subset E$ such that the following two inequalities hold:*

$$\text{(i)} \qquad \int_E \log \frac{1}{|z-a|} \cdot d\mu(z) \leq \log \frac{1}{\rho} + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r} \quad : \quad a \in E \setminus \mathcal{U}_*(\lambda)$$

$$\text{(ii)} \qquad h^*(U_*(\lambda)) < \frac{6}{\lambda}$$

C.11 Proof of Theorem 0.5. Let $E \in \mathcal{N}_{\text{harm}}$ which by Theorem 0.2 gives a probability measure μ supported by E such that that the left hand side in (i) is $+\infty$ for every $a \in E$. It follows that the set $E \setminus \mathcal{U}^*(\lambda)$ is empty for every $\lambda > 1$. With a fixed λ we apply Cartan's covering Lemma and since $E = \mathcal{U}^*(\lambda)$ it follows that

$$h^*(E) \leq \frac{6}{\lambda}$$

Here $\lambda > 1$ is arbitrary which gives $h^*(E) = 0$ as required and of Cartan's theorem follows.

D. Cantor sets.

We construct a family of closed subsets of $[0, 1]$ as follows. Let $1 < p_1 < p_2 < \ldots$ be some strictly increasing sequence of real numbers such that the products $\{p_1 \cdots p_n\}$ tend to $+\infty$ as n increases. Then we can construct a decreasing sequence of closed sets $E_1, E_2, \ldots$ where each E_n is the union of 2^n-many closed intervals with equal length

$$\ell_n = 2^{-n} \cdot \frac{1}{p_1 \cdots p_n}$$

D.1 The construction. First E_1 is any closed interval $[a_1, b_1]$ with

$$b_1 - a_1 = \frac{1}{2p_1}$$

Inside this closed interval we pick two pairwise disjoint closed interval both of length ℓ_2 and let E_2 be their union. In the next step we pick a pair of closed intervals both of length ℓ_3 from each of the two intervals in E_2. Their union gives the set E_3 and we continue in the same way for every n and arrive at a closed set

$$\mathcal{E} = \cap E_n$$

We refer to $\mathcal{E}$ as a Cantor set. The construction is flexible since we do not impose any condition on specific positions while we at stage n pick pairs of intervals of length ℓ_{n+1} from each of the 2^n many intervals of E_n. Thus, for a given p-sequence we obtain a whole family of Cantor sets denoted by $\text{Cantor}(p_\bullet)$. The next result gives a condition for such Cantor sets to have harmonic measure zero.

D.2 Theorem. *The following are equivalent for an arbitrary sequence $p_\bullet$ as above*:

$$\text{Cantor}(p_\bullet) \subset \mathcal{N}_{\text{harm}} \text{ holds if and only if } \sum_{\nu=1}^{\infty} \frac{\operatorname{Log} p_\nu}{2^\nu} = +\infty$$

The proof uses the explicit formulas for Robin constants of intervals on the real line. For the detailed proof we refer to page xx-xx in [Nevanlinna].