XII. A system of infinite linear equations.

Introduction. The main issue in this section is the construction of a unique solution to the system

(*)
$$\sum_{q\neq p} \frac{x_q}{p-q}=0$$

where (*) hold for all positive integers p and $\{x_q\}$ is a sequence of real numbers for which the series

(**)
$$\sum_{q=1}^\infty \frac{x_q}{q}$$

is convergent. The fact that (*) ha a non-trivial solution is far from evident. Before the study of (*) in Section 1 we discuss a general situation described by Carleman in his major lecture at the international congress in Zürich 1932 where one of his topics was devoted to linear systems of equations in an infinite number of variables. A homogeneous system takes the form

(***)
$$\sum_{q=1}^\infty c_{pq}x_q=a_p \quad : \quad p=1,2,\ldots$$

where $\{c_{pq}\}$ is a matrix with an infinite number of elements. A sequence $\{x_q\}$ of complex numbers is a solution to (***) if the sum in the left hand side converges for each p and has value a_p. Notice that one does not require that the series are absolutely convergent.

The generic case. To every $p\geq 1$ we have the linear form L_p defined on finite sequences $\{x_1,x_2,\ldots\}$, i.e. where $x_q=0$ when $q>>0$ by:

$$L_p(x_\bullet)=\sum_{q\geq 1} c_{pq}\cdot x_q$$

The generic case occurs when $c_{1q}\neq 0$ for every q and the linear forms $\{L_p\}_1^\infty$ are linearly independent. The last condition means that for every positive integer M there exists a strictly increasing sequence $q_1<\ldots<q_M$ so that the $M\times M$-matrix with elements $a_{p\nu}=c_{pq_\nu}$ has a non-zero determinant.

The R-functions. Assume that the system (***) is generic and let $\{a_p\}$ be a given sequence for which we seek a solution $\{x_q\}$. The necessary and sufficient condition for this existence goes as follows: Consider n-tuples of positive integers $m_1,\ldots,m_n$ where $n\geq 2$. For every such n-tuple and $1\leq k\leq n-1$ we set

$$\mathcal{D}(k)=\{\nu\colon\ m_1+\ldots+m_k<\nu\leq m_1+\ldots+m_{k+1}\}$$

Next, with $M=m_1+\ldots+m_N$ we denote by $\mathcal{F}(m_1,\ldots,m_n)$ the family of sequences $(x_1,\ldots,x_M)$ such that the following inequalities hold for every $1\leq k\leq n-1$:

$$\Big|\sum_{q=1}^{q=\nu} c_{pq}x_q-a_q\Big|\leq\frac{1}{k} \quad : \nu\in\mathcal{D}(k) \quad \text{and} \quad 1\leq p\leq k$$

The generic assumption implies that the set $\mathcal{F}(m_1,\ldots,m_n)$ is non-empty provided that we start with a sufficiently large m_1 and for every such n-tuple we set

$$R(m_1,\ldots,m_n)=\min\sum_{\nu=1}^{m_1} x_\nu^2$$

where the minimum is taken over sequences $x_1,\ldots,x_{m_1}$ which give the starting terms of some sequence $x_1,\ldots,x_M\in\mathcal{F}(m_1,\ldots,m_n)$.

Theorem. *The necessary and sufficient condition in order that (***) has a least one solution is that there exists a constant K and an infinite sequence of positive integers $\mu_1,\mu_2,\ldots$ such that*

$$R(\mu_1,\mu_2,\ldots,\mu_r)\leq K \quad \text{hold for every } r$$

Remark. The reader may consult [Carleman] for further remarks about this result and also comments upon the more involved criterion for non-generic linear systems. From now on we study linear systems which arise as follows: Consider a rational function of two variables x, y:

$$f(x,y) = \frac{a_0(x) + a_1(x)y + \ldots a_n(x)y^n}{b_0(x) + b_1(x)y + \ldots b_m(x)y^m}$$

Here n and m are positive integers and $a_\nu(x)$ and $b_j(x)$ polynomials in x. No special assumptions are imposed on these polynomials except that $b_m(x)$ and $a_n(x)$ are not identically zero. For example, it is not neccesary that the degree of b_m is $\geq \deg(b_j)$ for all $0 \leq j \leq m-1$.

0.1 Proposition *Let $b_m^{-1}(0)$ be the finite set of zeros of b_m. Let $\zeta_0 \in \mathbf{C} \setminus b_m^{-1}(0)$ be such that*

$$\sum_{j=0}^{j=m} b_j(\zeta_0) \cdot q^j = 0$$

holds for some finite set of positive integers, say $1 \leq q_1 < \ldots < q_k$. Then there exists $\delta > 0$ such that

$$\sum_{j=0}^{j=m} b_j(\zeta)q^j \neq 0 \quad : \text{for all positive integers } q \quad : \quad 0 < |\zeta - \zeta_0| < \delta$$

Exercise. Prove this result.

Next, consider the sequence of polynomials of the complex ζ-variable given by:

$$B_q(\zeta) = b_0(\zeta) + b_1(\zeta)q + \ldots + b_m(\zeta)q^m \quad : q = 1, 2, \ldots$$

0.2 Proposition *Put*

$$W = b_m^{-1}(0) \bigcup_{q \geq 1} B_q^{-1}(0)$$

Then W is a discrete subset of $\mathbf{C}$, i.e its intersection with any bounded disc is finite.

Exercise. Prove this result where a hint is to apply Rouche's theorem.

Next, put $W^* = W \cup a_n^{-1}(0)$, i.e. add the zeros of the polynomial a_n to W.

0.3 Proposition. *Let $\zeta_0 \in C \setminus W^*$ and suppose that $\{x_q\}$ is a sequence such that the series*

(i) $$\sum_{q=1}^{\infty} f(\zeta_0, q) \cdot x_q$$

is convergent. Then the series

(ii) $$\sum_{q=1}^{\infty} f(\zeta, q) \cdot x_q \quad \text{converges for every } \zeta \in C \setminus W^*$$

Moreover, the series sum is a meromorphic function of ζ whose poles are contained in W^*.

Remark. Proposition 0.3 gives a procedure to find solutions $\{x_q\}$ which is not the trivial null solution to a homogeneous system:

(*) $$\sum_{q \neq p} f(p, q) \cdot x_q = 0 \quad : \quad p = 1, 2, \ldots$$

More precisely, assume that the rational function $f(x, y)$ is such that $f(p, q) \neq 0$ when p and q are distinct positive integers. To get a solution $\{x_q\}$ to (iii) it suffices to begin with to verify (i) in Proposition 0.3 for some ζ_0 and then also try to find $\{x_q\}$ so that the meromorphic function

(**) $$\zeta \mapsto \sum_{q=1}^{\infty} x_q \cdot f(\zeta, q)$$

has zeros at every positive integer. Using this criterium for a solution we can show the following:

(Theorem. *For every complex number $a \in \mathbf{C} \setminus (-\infty, 0\,]$ the system*

$$\sum_{q=1}^{\infty} \frac{x_q}{p+aq} = 0 \quad : \quad p = 1, 2, \ldots$$

has no non-trivial solution $\{x_q\}$.

There remains to analyze the case when a is real and < 0. In this case complete answers about possible when f is the rational function have been established by K. Dagerholm starting from his Ph.D-thesis at Uppsala University in 1938 with Beurling as supervisor. The hardest case occurs hen $a = 1$ which will be studied in the next section.

1. The Dagerholm series.

Let $\mathcal{F}$ be the family of all sequences of real numbers $x_1, x_2, \ldots$ such that the series

(i)
$$\sum_{q=1}^{\infty} \frac{x_q}{q} < \infty$$

We only require that the series is convergent, i.e. it need not be absolutely convergent.

1.1 Theorem. *Up to a multiple with a real constant there exists a unique sequence $\{x_q\}$ in $\mathcal{F}$ such that*

$$\sum_{q \neq p} \frac{x_q}{p-q} = 0 \quad : \quad p = 1, 2, \ldots$$

The proof of uniquenss relies upon Jensen's formula and the existence upon a solution to a specific Wiener-Hopf equation. We begin to describe the strategy to obtain Theorem 1.1.To begin with there exists a meromorphic function $h(z)$ defined by

(ii)
$$h(z) = \sum_{q=1}^{\infty} \frac{x_q}{z-q}$$

To see that $h(z)$ is defined we notice that if s_* is the series sum in (i) then

(iii)
$$h(z) + s_* = \sum_{q=1}^{\infty} x_q \cdot [\frac{1}{z-q} + \frac{1}{q}] = z \cdot \sum_{q=1}^{\infty} \frac{x_q}{q(z-q)}$$

It is clear that the right hand side is a meromorphic function with poles confined to the set of positive integers. Hence we obtain the entire function:

$$H(z) = \frac{1}{\pi} \cdot \sin(\pi z) \cdot h(z)$$

1.2 Proposition. *The following hold for each positive integer:*

(i)
$$H(p) = (-1)^p \cdot x_p$$

(ii)
$$H'(p) = (-1)^q \cdot \sum_{q \neq p} \frac{x_q}{p-q} = 0$$

Proof. Let $p \geq 1$ be an integer. With ζ small we have

$$H(p+\zeta) = \frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) \cdot [\frac{x_p}{\zeta} + \sum_{q \neq p} \frac{x_q}{p+\zeta-q}]$$

A series expansion of the complex sine-function at πp gives

$$\frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) = [\zeta \cdot \cos(\pi p) + O(\zeta^3)] \cdot [\frac{x_p}{\zeta} + \sum_{q \neq p} \frac{x_q}{p+\zeta-q}]$$

Proposition 1.2 follow since $\cos \pi p = (-1)^p$.

Remark. Proposition 1.2 shows that $\{x_p\}$ solves the homogeneous system in Theorem 1.1 if the complex derivative of the entire H-function has zeros on all positive integers. This observation is the gateway towards the proof of Dagerholm's Theorem First we establish the uniqueness part.

2. Proof of uniqueness

Let $\{x_q\}$ be a sequence in $\mathcal{F}$. From the constructions in above it is clear that the meromorphic function $h(z)$ satisfies the following in the left half-plane $\mathfrak{Re}(z) \leq 0$:

(i)
$$\lim_{x \to -\infty} h(x) = 0 : \quad |h(x+iy)| \leq C_* \quad : \; x \leq 0$$

where C_* is a constant. Moreover, in the right half-plane there exists a constant C^* such that

(ii)
$$|h(x+iy)| \leq C^* \cdot \frac{|x|}{1+|y|} \quad : \; |x-q| \geq \frac{1}{2} \text{ for all positive integers}$$

To $h(z)$ we get the entire function $H(z)$ and (i-ii) above give the two the estimates below in the right half-plane:

(iii)
$$|f(x+iy)| \leq Ce^{\pi|y|} \; : \; x \leq 0 \quad : \quad |f(x+iy)| \leq C\frac{|x|}{1+|y|} \cdot e^{\pi|y|}$$

Moreover, the first limit formula in (i) gives

(iv)
$$\lim_{x \to -\infty} f(x) = 0$$

It is easily seen that the same upper bounds hold for the entire function $H'(z)$ and a straightforward application of the Phragmén-Lindelöf theorem gives:

2.1 Proposition. *The complex derivative of $H(z)$ satisfies the growth condition:*

$$\lim_{r \to \infty} e^{-\pi r \cdot |\sin \phi|} \cdot |H'(re^{i\theta}| = 0 \quad : \text{ holds uniformly when } \; 0 \leq \theta \leq 2\pi$$

Now we are prepared to prove the uniqueness part in Theorem 0.1. For suppose that we have two sequences $\{x_q\}$ and $\{x_q^*\}$ which both give solutions to (*) are not equal up to a constant multiple of each other. The two sequences give entire functions H_1 and H_2. Since both are constructed via real sequences their Taylor coefficients are real. We can choose a linear combination

$$G = aH_1 + bH_2 = G$$

where a, b are real numbers and the complex derivative $G'(0) = 0$. The hypothesis that there exists two **R**-linearly independent solutions to (*) leads to a contradiction once we have proved the following

2.2 Lemma *The entire function $G'(z)$ is identically zero.*

Proof. To simplify notations we set $g(z) = G'(z)$. To show that $g = 0$ we first consider the series expansion

$$g(z) = a_p z^p + a_{p+1} z^{p+1} + \ldots$$

where a_p ia the first non-vanishing coefficient. Since $g(0) = G'(0) = 0$ we have $p \geq 1$ and since the two x-sequences both are solutions to (*) it follows from (ii) in Proposition 0.2 that

(i)
$$g(p) = 0 \quad : \; p = 1, 2, \ldots$$

Next, the primitive function G is real-valued $G(x)$ on $x \leq 0$ and since the H-functions are zero for every integer ≤ 0 the same holds for G. Then *Rolle's theorem* implies that for every $n \geq 1$ there exists

(i)
$$-n < \lambda_n < -n+1 \quad : \; g(\lambda_n) = 0$$

So if $\mathcal{N}$ is the counting function for the zeros of the entire g-function one has the inequality

(iii)
$$\mathcal{N}(r) \geq [2r]$$

where $[2r]$ is the largest integer $\leq 2r$. Next, recall that a_p is the first non-zero term in the series expansion of g. Hence Jensen's formula gives:

$$(*) \qquad \log|a_p| + p\cdot\log\, r + \int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Log}|g(re^{i\theta}|\cdot d\theta$$

Proposition 2.1 applied to $g(z)$ gives:

$$(\text{iv}) \qquad \int_0^{2\pi} \operatorname{Log}|g(re^{i\theta}|\cdot d\theta = 2r - m(r) \quad \text{where } \lim_{r\to\infty} m(r) = +\infty$$

At this stage we get the contradiction as follows. First (iii) gives

$$\int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} \geq 2r - \operatorname{Log}(r) - 1$$

Now (*) and (iii) give the inequality

$$(\text{vi}) \qquad \log|a_p| + p\cdot\log\, r + 2r - 1 - \log r \leq 2r - m(r) \quad : \ r \geq 1$$

Here $p \geq 1$ which therefore would give:

$$\log|a_p| - 1 + m(r) \leq 0$$

But this is impossible since we have seen that $m(r) \to +\infty$.

3. Proof of existence

We start with a general construction. Let $\phi(z)$ be analytic in the unit disc D which extends to a continuous function on T except at the point $z = 1$. We also assume that there exists some $0 < \beta < 2$ and a constant C such that

$$(1) \qquad |\phi(\zeta)| \leq C|1-\zeta|^{-\beta}$$

This implies that the function

$$\theta \mapsto \theta\cdot\phi(e^{i\theta})$$

is integrable on the unit circle. Hence there exists the entire function

$$(2) \qquad f(z) = \frac{-i}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta z}\cdot\theta\cdot\phi(e^{i\theta})\cdot d\theta$$

Next, with $\epsilon > 0$ small we let γ_ϵ be the interval of the circle $|z-1| = \epsilon$ with end-points at the intersection with $|z| = 1$. So on γ_ϵ we have

$$z = 1 + \epsilon\cdot e^{i\theta} \quad : \ -\frac{\pi}{2} + \epsilon_* < \theta < \frac{\pi}{2} - \epsilon_*$$

where ϵ_* is small with ϵ. We obtain the entire function

$$F(z) = \frac{1}{2\pi}\int_\epsilon^\pi e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi}\int_{-\pi}^{-\epsilon} e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi i}\int_{\gamma_\epsilon} \frac{e^{-z\cdot\operatorname{Log}\zeta}\cdot\phi(\zeta)d\zeta}{\zeta}$$

If $z = n$ is an integer we have

$$e^{-in\theta} = \zeta^{-n} \quad : \quad e^{-n\cdot\operatorname{Log}\zeta} = \zeta^{-n}$$

Hence we get

$$(*) \qquad F(n) = \frac{1}{2\pi i}\cdot\int_{\Gamma_\epsilon} \frac{\phi(\zeta)\cdot d\zeta}{\zeta^{n+1}}$$

where Γ_ϵ is the closed curve given as the union of γ_ϵ and the interval of T where $|\theta| \geq \epsilon$. Cauchy's formula applied to ϕ gives:

2.1 Proposition. *Let* $\phi(\zeta) = \sum c_n \zeta^n$. *Then*

$$F(n) = c_n \quad : n \geq 0 \quad \text{and} \quad F(n) = 0 \quad n \leq -1$$

Next, using (i) above we also have:

2.2 Proposition. *The complex derivative of* F *is equal to* f*.*

Proof. With $\epsilon > 0$ the derivative of the sum of first two terms from the construction of $F(z)$ above become

$$\text{(i)} \qquad \frac{1}{2\pi}\int_{|\theta|\geq\epsilon} -i\theta \cdot e^{-iz\theta}\phi(e^{i\theta})d\theta$$

In the last integral derivation with respect to z gives

$$\text{(ii)} \qquad -\frac{1}{2\pi i}\int_{\gamma_\epsilon} \frac{e^{-z\cdot \mathrm{Log}\zeta}\cdot\phi(\zeta)d\zeta}{\zeta}$$

Now $\zeta = 1 + \epsilon \cdot e^{i\theta}$ during the integration along γ_ϵ which gives:

$$|\mathrm{Log}\,(1 + \epsilon \cdot e^{i\theta})| \leq \epsilon$$

At the same time the circle interval γ_ϵ has length $\leq \epsilon$ and hence the growth condition (i) shows that the integral (iii) tends to zero when $\epsilon \to 0$. Finally, since we assumed that the function $\theta \mapsto \theta \cdot \phi(e^{i\theta})$ is absoutely integrable on T a passage to the limit as $\epsilon \to 0$ gives $F' = f$ as requested.

2.3 Conclusion. If n is a positive integer in Proposition 2.3 we have:

$$(**) \qquad F'(n) = \frac{-i}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\cdot\theta\cdot\phi(e^{i\theta})\cdot d\theta$$

These integrals are zero for every $n \geq 1$ if and only if $\theta \cdot \phi(e^{i\theta})$ is the boundary value function of some $\psi(z)$ which is analytic in the exterior disc $|z| > 1$. In 2.X we will show that this is true for a specific ϕ-function satisfying the growth condition (1) above and in addition the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{c_n}{n}$$

converges.

2.4 How to deduce a solution $\{x_p\}$. Suppose we have found ϕ satisfying the conditions above which gives the entire function $F(z)$ whose derivatives are zero for all $n \geq 1$. Now we set

$$x_p = (-1)^p \cdot c_p$$

By (***) this sequence belongs to $\mathcal{F}$ and we construct the associated entire function $H(z)$. From (i) in Proposition 0.1 and Proposition 2.1 we get

$$H(p) = (-1)^p \cdot x_p = c_p = F(p)$$

In addition both H and F have zeros at all integers ≤ 0. Next, by the construction of F it is clear that this is an entire function of exponential type and by the above the entire function $G = H - F$ has zeros at all integers. We leave as an exercise to the reader to show that G must be identically zero. The hint is to use similar methods as in the proof of the uniqueness. It follows that

$$H'(q) = F'(q) = 0$$

for all $q \geq 1$. By (ii) in Proposition 0.2 this means precisely that $\{x_p\}$ is a solution to the requested equations in (*) which gives the existence in Dagerholm's Theorem.

2.5 The construction of ϕ.

There remains to find ϕ such that the conditions above hold. To obtain ϕ we start with the integrable function on T defined by:

$$u(\theta) = \frac{1}{2} \cdot \log \frac{1}{|\theta|} \quad : \; -\pi < \theta < \pi$$

We get the analytic function

$$g(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} - \zeta}{e^{i\theta} + \zeta} \cdot u(\theta) \cdot d\theta$$

In the exterior disc we find the analytic function

$$\psi(\zeta) = \exp \; -\bar{g}(\frac{1}{\bar{\zeta}})$$

Let us also put $\phi_*(\zeta) = e^{g(\zeta)}$. Now we have

$$\log |\phi(e^{i\theta})| = \mathfrak{Re}\, g(e^{i\theta}) = u(\theta)$$

In the same way we see that

$$\log |\psi(e^{i\theta})| = -\mathfrak{Re}\, g(e^{i\theta}) = -u(\theta)$$

Since $2u(\theta) = -\log\, |\theta|$ it follows that

$$\log\, |\theta| + \log |\phi_*(e^{i\theta})| = \log |\psi(e^{i\theta})|$$

Taking exponentials we obtain

$$|\theta| \cdot |\phi_*(e^{i\theta})| = |\psi(e^{i\theta})|$$

Exercise. Check also arguments and verify that we can remove absolute values in the last equality to attain

$$(*) \qquad |\theta| \cdot \phi_*(e^{i\theta}) = \psi(e^{i\theta})$$

Here (*) is not precisely what we want since our aim was to construct ϕ so that $\theta| \cdot \phi(e^{i\theta})$ is equal to the boundary value of an analytic function in $|z| > 1$. So in order to get rid of the absolute value for θ in (*) we modify ϕ_* as follows: Set

$$\rho(\theta) = \frac{\pi i}{2} \cdot \operatorname{sign} \theta \cdot e^{-i\theta} \quad : \; -\pi < \theta < \pi$$

Next, consider the two analytic functions in D, respectively in $|z| > 1$ defined by:

$$\phi_1(z) = \frac{1}{\sqrt{1 - z^2}} \quad \text{and} \quad \psi_1(z) = \frac{1}{\sqrt{1 - z^{-2}}}$$

Exercise. Show that one has the equality

$$\rho(\theta) = \frac{\phi_1(e^{i\theta})}{\psi_1(e^{i\theta})}$$

when $-\pi < \theta < \pi$ and $\theta \neq 0$.

The ϕ-function. it is defined by

$$\phi(z) = \frac{z}{\sqrt{1 - z^2}} \cdot \phi_*(z)$$

From (*) above and the construction of ρ it follows that

$$\theta \cdot \phi(e^{i\theta}) = \frac{\pi}{2} \cdot \psi_1(e^{i\theta}) \cdot \psi(e^{i\theta})$$

The right hand side is the boundary function of an analytic function in $|z| > 1$ and hence ϕ satisfies (**) from XX. Consider its Taylor expansion

$$\phi(z) = \sum c_n \cdot z^n$$

There remains to verify that the series (**) converges and that ϕ satisfies the growth condition in XX. To prove this we begin to analyze the function

$$\text{(i)} \qquad \phi_*(z) = e^{g(z)}$$

Rewrite the u function as a sum

$$\text{(ii)} \qquad u(\theta) = \frac{1}{2}\log\left|\frac{1}{1-e^{i\theta}}\right| + k(\theta) \quad \text{where} \quad k(\theta) = \frac{1}{2}\log\left|\frac{1-e^{i\theta}}{\theta}\right|$$

When θ is small we have an expansion

$$\text{(iii)} \qquad \frac{1-e^{i\theta}}{\theta} = -i + \theta/2 + \ldots$$

From this we conclude that the k-function is at least twice differentiable as a function of θ. So the Fourier coefficients in the expansion

$$\text{(iv)} \qquad k(e^{i\theta}) = \sum b_\nu e^{i\nu\theta}$$

have a good decay. For example, there is a constant C such that

$$\text{(v)} \qquad |b_\nu| \leq \frac{C}{\nu^2} \quad : \nu \neq 0$$

This implies that the analytic function

$$\text{(vi)} \qquad \mathcal{K}(z) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot k(e^{i\theta}) \cdot d\theta$$

yields a bounded analytic function in the unit disc. Next, the construction of the g-function gives:

$$\text{(vii)} \qquad g(z) = \frac{1}{2} \cdot \log\frac{1}{1-z} + \sum b_\nu z^\nu \implies$$

$$\phi_*(z) = \frac{1}{\sqrt{1-z)}} \cdot e^{\mathcal{K}(z)}$$

We conclude that

$$\phi(z) = \frac{z}{1-z} \cdot \frac{1}{\sqrt{1+z)}} \cdot e^{\mathcal{K}(z)}$$

Since $\mathcal{K}(z)$ extends to a continuous function on the closed disc it follows that ϕ satisfies the growth condition (1) with $\beta = 1$. Moreover, the function $\theta \cdot \phi(e^{i\theta})$ belongs to $L^1(T)$ since $\frac{1}{\sqrt{1+e^{i\theta}}}$ is integrable. There remains only to prove:

Lemma. The series

$$\sum_{p=1}^{\infty} (-1)^n \cdot \frac{c_n}{n}$$

is convergent.

Proof. let us put

$$A(z) = \frac{z}{\sqrt{1+z)}} \cdot e^{\mathcal{K}(z)}$$

This gives

$$\phi(z) = \frac{A(1)}{1-z} + \frac{A(z)-A(1)}{1-z}$$

From (v) it follows that $K(z)$, and hence also $e^{K(z)}$ is differentiable at $z01$ which gives the existence of a constant C such that

$$|\frac{A(z)-1}{1-z}| \leq C \cdot \frac{1}{|\sqrt{1+z}|}$$

Here the function $\theta \mapsto \frac{1}{|\sqrt{1+e^{i\theta}}|}$ belongs to $L^p(T)$ for each $p < 2$ which by the inequality for L^p-norms between functions and their Fourier coefficients in XX for example implies that if $\{c_\nu^*\}$ give the Taylor series for $\frac{A(z)-1}{1-z}$ then

$$\sum |c_\nu^*|^3 < \infty$$

Now Hölder's inequality gives

$$\sum \frac{|c_\nu^*|}{\nu} \leq \big(\sum |c_\nu^*|^3\big)^{\frac{1}{3}} \cdot \big(\sum \nu^{-3/2}\big)^{\frac{2}{3}} < \infty \tag{8}$$

We conclude that the Taylor series for ϕ becomes

$$A(1) \cdot (1 + z + z^2 + \ldots) + \sum c_\nu^* z^\nu$$

Hence $c_n = A(1) + c_\nu^*$ and now Lemma xx follows since the alternating series $\sum (-1)^n \frac{1}{n}$ is convergent and we have the absolute convergence in XX above.