The Stieltjes measure in $\mathbf{R}^2$.

Modern integration theory started in 1890 when Stieltjes introduced measures on the closed real interval $[0,1]$. More precisely, let $s(x)$ be a non-decreasing continuous function on this interval where we assume that $s(0)=0$ and $s(1)=1$. Recall that every open subset U of $[0,1]$ is a disjoint union of at most denumarably many open intervals $\{\alpha_\nu,\beta_\nu)\}$. Among these may occur half-open intervals $[0,\beta)$ or $[\alpha,1]$. Following Stieltjes we assign the measure

$$\mu_s(U)=\sum(s(\beta_\nu)-s(\alpha_\nu))$$

For a closed, i.e. a compact subset K of $[0,1]$ we define

$$\mu_s(K)=1-\mu_s([0,1]\setminus K)$$

The continuity of s entails that

$$\mu_s(K)=\lim_{\delta\to 0}\mu_s(K_\delta)\quad : K_\delta=\{x\colon\ dist(x,K)<\delta\}$$

From this it follows that μ_s extends to a σ-additive measure defined on the σ-algebra $\mathcal{B}$ of Borel sets, i.e. the smallest σ-algebra of subsets of $[0,1]$ generated by intervals. Moreover Stieltjes constructed integrals

$$\int_0^1 f(x)d\mu_x\quad : f\in C^0[0,1]$$

and more generally there exist well-defined integrals for every *bounded Borel-function* ϕ, i.e. those real-valued functions for which the sets

$$\{\phi<a\}\in\mathcal{B}\quad : a\in\mathbf{R}$$

A *null set* with respect to μ_s is a set such that

$$0=\inf\,\mu_s(U)\colon\ A\subset U$$

where the infimum is taken over open sets U which contain A. This family of sets which are negligable with respect to μ_s is denoted by $\mathcal{N}(\mu_s)$. A set A is said to be μ_s-measurable if its inner and outer measures are equal, i.e. if

$$\sup\,\mu_s(K)=\inf\,\mu_s(U)\quad : K\subset A\subset U$$

One proves easily that if A is measurable then

$$A=F^*\cup N\quad N\in\mathcal{N}(\mu_s)\quad \&\quad F=\cup K_\nu$$

where the last term is given via by an increasing sequence $\{K_\nu\}$ of compact sets.

The family of all μ_s-measurable sets is denoted by $\mathfrak{M}(\mu_s)$.

Vitali's differential theorem. Let A be μ_s-measurable. For each point $p\in A$ and every $\epsilon>0$ we put

$$\delta_*(\epsilon,p)=\frac{1}{s(b)-s(a)}\cdot\inf\,\mu_s(A\cap(a,b))\quad : a<p<b\quad\&\quad a-b<\epsilon$$

Notice that above we compete with *all* open intervals which contain p, i.e. p need not be the mid-point of (a,b).

Definition. *A point $p\in A$ is a point of density in the sense of Vitali if*

$$\lim_{\epsilon\to 0}\delta_*(\epsilon,p)=1$$

The set of points in A for which (*) hold is denoted by $\mathcal{V}(A)$ and called Vitali-points of A.

Using his famous *Covering Lemma*, Vitali proved the following:

Theorem. *For every $A\in\mathfrak{M}(\mu_s)$ it follows that*

$$A\setminus\mathcal{V}(A)\in\mathcal{N}(\mu_s)$$

The passage to $\mathbf{R}^2$.

Consider the closed square

$$\square = \{(x,y)\colon\ 0 \le x, y \le 1\}$$

Let $s(x,y) \in C^0(\square)$ be doubly non-decreasing, i.e. for every freezed y the function $x \mapsto s(x,y)$is non-decreasing, and vice versa $y \mapsto s(x,y)$ is non-decreasing while x is freezed. In addition

$$s(0,0) = 0 \quad \& \quad s(1,1) = 1$$

In order to construct the measaure μ_s we employ *dyadic grids*, Thus, to a positive integer N we divide $\square$ into 2^{2N} many squares

$$\delta_N(p,q) = \{2^{-N} \cdot p \le x \le 2^{-N}(p+1) \,\&\, 2^{-N} \cdot q \le y \le 2^{-N}(q+1)\}$$

where p, q run over integers from zero to $2^N - 1$.

Next, for every square with sides parallell to the coordinate axes we define

$$\mu_s(\square) = s(a^*, b^*) + s(a_*, b_*) - s(a^*, b_*) - s(a_*, b^*)$$

where the square has its upper corner point at (a^*, b^*) and its lower corner point at $(a_*.b_*)$.

Let us now consider an - arbitrary - closed subset K of $\square$. When $N \ge 1$ we denote by $\mathcal{D}_N(K)$ the family of squares $\delta_N(p,q)$ for which

$$\delta_N(p,q) \cap K \ne \emptyset$$

Next, put

$$\rho_N(K) = \sum^{*} \mu_s(\delta_N(p,q)) \qquad \delta_N(p,q) \in \mathcal{D}_N(K)$$

One checks that these ρ-numbers *decrease* as N increases. Passing to the limit we put

$$\mu_s(K) = \lim_{N \to \infty} \rho_N(K) \tag{1}$$

In a similar fashion we construct $\mu_s(U)$ for open sets U, i.e. here we put

$$\rho_N(U) = \sum \mu_s(\delta_N(p,q)) \quad : \quad \delta_N(p,q) \subset U$$

Thus, we add measures from those cubes which are contained in U. It is clear that $N \mapsto \rho_N(U)$ increase with N and we put

$$\mu_s(U) = \lim \rho_N^*(U) \tag{2}$$

Via (1-2) one easily verifies that the measure μ_s extends to be σ-additive on $\mathcal{B}(\square)$. and we also remark that the Vitali theorem from § 1 remains valid where one now regard a point p in a μ_s-measurable set A and take limits of quotients

$$\frac{\mu_s(\delta \cap A)}{\mu_s(\delta)}$$

where δ are small squares tending to the singleton set $\{p\}$.

Remark. The construction of μ_s entails that when $f(x,y)$ is twice continously differentiable then

$$\iint f \cdot d\mu_s = \iint \frac{\partial^2 f}{\partial x \partial y} \cdot s(x,y)\, dx dy$$

In other words, $d\mu_s$ is the second order mixed distribution derivative $\partial_x \partial_y(s)$.

It goes withput saying that the construction for $n = 2$ extends verbatim to every $n \ge 3$. Concerning the measures which are found above we notice that since $\{s(x,y)\}$ is continuous, it follows that every vertical line $\{x = a\}$ or horiszontal line $\{y = b\}$ are nullsets. *Conversely* every σ-additive and non-negative measure γ on $\square$ for which this family of lines are null-sets is equal to μ_s for a unique $s(x,y)$. So one has - at least essentially - recovered all probability measures on $\square$ via the constructions perfomed above.