A.4.4 An example. One is often confronted with severe difficulties in specific cases because the Hahn Banach theorem is not constructive. Consider for example a positive integer N and let $\mathcal{P}_N$ denote the $(N+1)$-dimensional real vector space of polynomials $P(t)$ of degree $\leq N$. Define a linear functional λ_0 on $\mathcal{P}_N$ by

$$\text{(i)} \qquad \lambda_0(P) = \sum_{\nu=1}^{\nu=M} c_\nu \cdot P(t_\nu)$$

where $\{t_\nu\}$ and $\{c_\nu\}$ are real numbers. We can identify $\mathcal{P}_N$ with a subspace of $X = C^0[0,1]$ where $C^0[0,1]$ is the vector space of real-valued and continuous functions on the unit interval $\{0 \leq t \leq 1\}$. On X we get a ρ-function via the maximum norm, i.e.

$$\rho(f) = \max_{0 \leq x \leq 1} |f(x)|$$

Next, there exists a unique smallest constant $C > 0$ such that

$$|\lambda_0(P)| \leq C \cdot \rho(P) \quad : P \in \mathcal{P}_N$$

Notice that C exists even in the case when some of the t-points in (i) are outside $[0,1]$. By the result above we can extend λ_0 to a linear functional λ on X and the Riesz representation theorem gives a real-valued Riesz measure μ on $[0,1]$ such that

$$\lambda_0(P) = \int_0^1 P(t) \cdot d\mu(t) \quad : P \in \mathcal{P}_N$$

and at the same time the total variation of μ is equal to the constant C above. However, to determine C and to find μ requires a further analysis which leads to a rather delicate problem in optimization thoery. Already a case when $M = 1$ and we take $t_1 = 2$ and $c_1 = 1$ is highly non-trivial. Here we find a constant C_N for each $N \geq 2$ such that

$$|P(2)| \leq C_N \cdot \rho(P) \qquad : P \in \mathcal{P}_N$$

I do not know how to determine C_N and how to find a Riesz measure μ_N whose total variation is C_N while

$$P(2) = \int_0^1 P(t) \cdot d\mu_N(t) \quad : P \in \mathcal{P}_N$$

For every $p \geq 1$ one has the Banach space ℓ^p whose vectors are sequences $x_1, x_2, \ldots$ of complex numbers for which

$$\sum |x_\nu|^p < \infty$$

A vector x is said to be elementsry if every x_ν has abosluter value one and only finitely many $x : \nu \neq 0$. Dnote by $\beta(x)$ the number of non-zero components of an elementary vector. It is clear that the ℓ^p norm is equal to the p:th root of $\beta(x)$. Next, let $\mathcal{M}$ be a finite family of elementary vectors $m_1, \ldots m_K$. The integer K is denoted by $\mathbf{c}(\mathcal{M})$ and called the rank of $\mathcal{M}$. If T is a linear operator on ℓ^p its trace with respect to $\mathcal{M}$ is defined by

$$\mathrm{Trace}(T; \mathcal{M}) = \frac{1}{\mathbf{c}(\mathcal{M})} \cdot \sum_{k=1}^{k=K} \beta(m_k)^{-1} \cdot \langle T(m_k), m_k \rangle$$

Next, following Enflo we consider an infinite family $\mathcal{F} = \{m_\nu\}$ of elementary vectors which sre pairwise orthogonoal under the inner prduct from (xx) above. Let B be the closed subspace of ℓ^p generated by the vectors in $\mathcal{F}$. With these notations one has:

Proposition. *If $T \colon B \to B$ is a continous linear operator with finite rank, i.e.* $\dim T(B) < \infty$*, then*

$$\lim_{\mathbf{c}(\mathcal{M}) \to +\infty} \mathrm{Trace}(T; \mathcal{M}) = 0$$

hold for finite subfamiles in $\mathcal{F}$ with increaing rank.

The easy proof of Pripostion xx is left to the reader. To get a Banach space nd compact operator which cannot be approximated eith perstors of finite rank, Enflo condtructed a family $\mathcal{F}$ with special properties. More precisely, he found such a family and a sequence $\{\mathcal{M}_\nu\}$ if pairwise disjoint finite subfamiles whhere

$$\lim \mathbf{c}(\mathcal{M}_\nu) = +\infty$$

In additin there is a sequence of positive numers $\{\lambda_\nu\}$ where the series $\sum \lambda_\nu < \infty$, and a compact set K in the Banach space B sich that

$$\left| \mathrm{Trace}(T; \mathcal{M}_{nu+1}) - \mathrm{Trace}(T; \mathcal{M}_\nu) \right| \leq \lambda_\nu \cdot \max_{x \in K} ||Tx||$$

hod for every $\nu \geq$ and every bounded oinear oeprator T on B.

Topics in functional analysis

Contents

Results from topology, such as Baire's category theorem and Tychonoff's theorem about products of compact spaces are used in many situations. Even if basic results in functional analysis are established in a rather straightforward manner their merit is the generality. Let me also add that much inspiration stems from the text-books by Dunford and Schwartz about Linear Operators. In fact, several proofs below are taken from this impressive source where the reader finds an extensive list of exercises and instructive examples of normed linear spaces which are helpful to grasp the general theory.

Advanced results.

The reader who has mastered the basic theory is able to pursue proofs of results such as the Hille-Phillips-Yosida theorem sbout infinitesmal generators of strongly continuous semi-groups in § xx, and Schauder's fixed point theorem in § xx. Of course many essential results are not covered in these notes. Non-linear maps between topological vector spaces is for exampe an extensive subject, as well as results deal about topological groups which in general need not be commutative. Here is an example of a result due to G. Birkhoff and Kakutani:

Let G be a topological group whose topology is metrizable, i.e. there exists some distance function whose metric is equivalent to the given topology. Then there exists a left invariant metric d, i.e.

$$d(gx, gy) = d(x, y)$$

hold for every triple g, x, y in G.

A notable point above is that the existence of d is valid under the sole hypotheis that the topology is metrizable, i.e. G need not be a locally compact. Here are two results about continuous functions on the real line whose proofs require methods beyond standard functional analysis, i.e. the general theory is not "the end of the story".

The Carleman-Weierstrass approximation theorem. It asserts that if $f(x)$ is a continuous complex-valued function on the real x-line then it can be uniformly approxmated by restrictions of entire functions in the complex z-plane with $z = x + iy$. In other words, for every $\epsilon > 0$ there exists an entire function $G(z)$ such that

$$\max_x |G(x) - f(x)| < \epsilon$$

The proof appears in my notes devoted to the matheramtics by Carleman. Notice that the result above goes beyond the ordinary result by Weierstrass since we obtain a uniform approximation on the whole real line.

Beurling's closure theorem. On the real t-line we have the space of bounded and complex-valued bounded and uniformly continuous functions ψ. It means that to each $\epsilon < 0$ there exists $\delta > 0$ such that

$$\max |\psi(x) - \psi(x')| < \epsilon$$

with the maximum taken over pairs of real numbers such that $|x - x'| < \delta$. Denote this space by $C_*(\mathbf{R})$. If μ is a Riesz measure on the real ξ-line we define the function

$$\mathcal{F}_\mu(x) = \int e^{x\xi}\, d\mu(\xi)$$

It is easily seen that $\mathcal{F}_\mu \in C_*(\mathbf{R})$. Denote by $\mathcal{A}$ be subspace of $C_*(\mathbf{R})$ given by $\mathcal{F}_\mu$-functions as μ varies over all Riesz measures as above. Before we announce Beurling's result below we define the notion of weak-star limits in $\mathfrak{M}$. Let $\{\mu_n\}$ be a bounded sequence of Riesz measures, i.e. there exists a constant such that

$$||\mu_n|| \leq M$$

hold for all n. The sequence $\{\mu_n\}$ converges in the weak-star sense to zero if

$$\lim_{n\to\infty} \int e^{ix\xi} \cdot d\mu_n(x) = 0$$

holds pointwise for every ξ.

Theorem. *A function $\psi \in C_*$ belongs to the closure of $\mathcal{A}$ if and only if*

$$(*) \qquad \lim_{n\to\infty} \int \psi(x) \cdot d\mu_n(x) = 0$$

whenever $\{\mu_n\}$ is a sequence in $\mathfrak{M}$ which converges weakly to zero.

Remark. Beurling's proof relies upon a certain non-linear variational problem which requires a quite deliate analysis and is exposed in my notes entitled *Mathematics by Beurling.*

Introduction. The general theory about normed vector spaces, and more generally locally convex spaces, owes much to pioneering work by Banach, Krein and Smulian. The beginner should pay attentions to notion of dual topological vector spaces since duality arguments often are used to prove existence theorems. The determination of the dual space X^* of a given normed vector space X is therefore an important issue. Familarity with measure theory is assumed. The reader may consult my notes on this topic as background for the present chapter.

Remarks about general topology. The notion of compact topological spaces is essential. Here certain phenomena can arise which differ from wellknown facts in euclidian spaces. For example, start with the set $\mathbf{N}$ of non-negative integers. There exists the commutative Banach algebra B whose vectors are bounded complex valued functions on $\mathbf{N}$ to be denoted by ℓ^∞. Now there exists the maximal ideal space $\mathfrak{M}_B$ of B whose construction is given in § xx. One refers to $\mathfrak{M}_B$ as the Gelfand space of B. It is equipped with the Gelfand topology which is constructed in such a way that the space of continuous and complex-valued functions on $\mathfrak{M}_B$ can be identfied with B. Moreover, it is a compact Hausdorff space where $\mathbf{N}$ appears as a dense and discrete subset. In spite of the compactness $\mathfrak{M}(B)$ fails to be *sequentially compact.* In fact, from the countable subset $\mathbf{N}$ one *cannot* find a subsequence of integers $1 \leq n_1 < n_2 < \ldots$ which converges to a limit point $\xi \in \mathfrak{M}(B)$. To see this we construct the bounded function on f on $\mathbf{N}$ where $f(n_k) = (-1)^k$ while $f = 0$ at integers outside the subsequence. If $\{n_k\}$ converges to a point $\xi \in \mathfrak{M}(B)$ we must have

$$\lim_{k\to\infty} f(n_k) = f(\xi)$$

But this cannot hold since f changes signs along to the subsequence. In the opposite direction we mention that there exist topological spaces X which are non-compact while every countable subset contains at least one convergent subsequence. Thus, X is *sequentially compact but not compact.* See § xx for an example. This shows that one often must be careful about the notion of compactness.

Examples of such "ugly compact topological spaces" illustrate that certain "positive results" due to Eberlein and Smulian are highly non-trivial. In § xxx we prove that if X is a Banach space and X is equipped with the weak topology, then every weakly compact subset is sequentially compact, and conversely, if we start from a subset A in X with the property that every countable sequence of vectors in A has a subsequence which converges in the weak topology to a limit vector in X, then the closure of A taken in the weak topology is compact.

Compact metric spaces. To avoid possible confusion we recall the notion of compactness on a metric space which goes back to Heine and Bolzano. Let S be a metric space where d is the distance function. A closed subset K is totally bounded in Heine's sense if there to every $\epsilon > 0$ exists a finite subset $\{x_1, \ldots, x_N\}$ in K such that K is covered by the union of the open balls $B_\epsilon(x_k) = \{x\colon\ d(x, x_k) < \epsilon\}$. A wellknown fact whose verification is left to the reader asserts that a closed subset K is totally bounded if and only if it is compact in the sense of Bolzano, i.e. every sequence $\{x_n\}$ in K contains at least one convergent subsequence.

Summary of the contents.

Hilbert spaces are treated in a special chapter. One reason is that the geometry in Hilbert spaces leads to results which have no counterpart in general normed spaces, and at the same time the material in the present chapter is often trivial when one regards Hilbert spaces. § 1 studies normed vector spaces over the complex field $\mathbf{C}$ or the real field $\mathbf{R}$. We explain how each norm is defined by a convex subset of V with special properties. If X is a normed vector space such that every Cauchy sequence with respect to the norm $||\cdot||$ converges to some vector in X one says that the norm is complete and refer to the pair $(X, ||\cdot||)$ as a Banach space.

Dual spaces. When X is a normed linear space one constructs the linear space X^* whose elements are continuous linear functionals on X. The Hahn-Banach Theorem identifies norms of vectors in X via evaluations by X^*-elements. More precisely, denote by S^* the unit sphere in X^*, i.e. linear functionals x^* of unit norm. Then one has the equality

$$\text{(i)} \qquad ||x|| = \max_{x^* \in S^*} |x^*(x)| \quad \text{for all } x \in X\,.$$

The dual space X^* is used to construct the *weak topology* on X. The detailed construction appears in § xx. In § 5.5.4 we show that when X has infinite dimension, then the norm topology is strictly stronger than the weak topology. However, for convex sets a special result hold. Using a separation theorems for convex sets it follows that a convex subset in X is weakly closed if and only if it is closed in the norm topology. In §§ we prove a result due to Eberlien and Smulian about weakly compact sets in X. An important consequence of Theorem § xxx is that if A is a subset of X which is compact in the weak topology, then the closure of its convex hull, taken in the weak topology is also compact.

Next, on the dual space X^* there exists the *weak-star topology* where Thychonoff's theorem is used to prove that the unit ball in X^* is compact with respect to the weak-star topology. In § xxx we prove a theorem due to Krein and Smulian about the *bounded weak-star topology* where the major result is that when X^* is equipped with this topology, then its dual can be identfied with the given Banach space X.

Reflexive spaces. Starting from a normed space X we get the Banach space X^* whose dual is denoted by X^{**} and called the bi-dual of X. There is a natural injective map

$$\mathfrak{i}_X \colon X \to X^{**}$$

and (i) above shows that it is an isometry, i.e. the norms $||x||$ and $||\mathfrak{i}_X(x)||$ are equal. But in general the bi-dual embedding map is nor surjective. If it is surjective one says that X is reflexive. It is of course intructive to learn about examples, both of reflexive as well as non-reflexive Banach spaces. An example which was put forward by Stefan Banach is to take $X = \mathbf{c}_0$ whose vectors are sequences of complex numbers $\{c_\nu\}$ which tend to zero as $\nu \to +\infty$ and the norm is defined to be the supremum of thier absolute values. It turns out that the dual $X^* = \ell^1$ whose vectors are absolutely oonvergent sequences, while the bi-dual becomes ℓ^∞ where arbitary bounded sequences of complex numbers yield vectors.

Compact operators. Let X and Y be a pair of Banach spaces. A bounded linear operator $T \colon X \to Y$ is compact if the image of the unit ball $B(X)$ in X is relatively compact in Y. In applications one often encounters compact operators and several facts about these operators occur in § 6 where the crucial results deal with Fredholm operators and culminate with Fredholm's index theorem. When the target space $Y = C^0(S)$ a classic result due to Arzela and Vitali asserts that $T(B(X))$ is relatively compact if and only if this family is equi-continuous, i.e. to each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\max_{x \in B(X)} \omega_{Tx}(\delta) \leq \epsilon$$

where we introduced the modulus of continuity, i.e if $f \in C^0(S)$ and $\delta > 0$ then

$$\omega_f(\delta) = \max |f(y_1) - f(y_2)| \quad : \ d(y_1, y_2) < \epsilon$$

In § xx we prove a result due to Schauder which asserts that if $T\colon X\to Y$ is a compact operator between a pair of Banach spaces, then its adjoint $T^*\colon X^*\to Y^*$ is also compact. The proof relies upon the Krein and Smulian theorem in § xx. Smulian proved this result in 1940 but it was not until 1972 that a perspective upon this result was fully understood when Enflo constructed a pair (X,T) where X is a separable Banach space and $T\colon X\to X$ is a compact operator, and yet T cannot be approximated in the operator norm by linear operators with finite dimesnsional range. Enflo's work [Acta 1972] is one of the greatest achievements about geometry on topological vector spaces and has created a veritable industry where one seeks to determine when a given Banach space has the approximation property in the sense that every compact operator can be approximated by finite range operators. Let us remark that Enflo constructed Banach spaces B for which the approximation property fails, where B can be taken as a closed subspace of ℓ^p for every $p>2$, and scrutinizing Enflo's construction one can conclude that the approximation property fails for "generic subspaces" of ℓ^p. But fortunately most Banach spaces which appear in applicstions have the approximation property and as a consequence also posses a Schauder basis.

Calculus on Banach spaces. Let X and Y be two Banach spaces. In § 7 we define the differential of a C^1-map $g\colon X\to Y$ where g in general is non-linear. Here the differential of g at a point $x_0\in X$ is a bounded linear operator from X into Y. This extends the construction of the Jacobian for a C^1-map from $\mathbf{R}^n$ into $\mathbf{R}^m$ expressed by an $m\times n$-matrix. More generally one constructs higher order differentials and refer to C^∞-maps from one Banach space into another. We shall review this in § 7. Let us remark that Baire's category theorem together with the Hahn-Banach theorem show that if S is an arbitrary compact metric space and ϕ is a continuous function on S with values in a normed space X, then ϕ is *uniformly continuous*, i.e. to every $\epsilon>0$ there exists $\delta>0$ such that

$$d_K(p,q)\leq\delta \implies ||\phi(p)-\phi(q)||\leq\epsilon$$

where d_S is the distance function on the metric space S and in the right hand side we have taken the norm in X. Next, there exists a distinguished class of differentiable Banach spaces. By definition a Banach space X is differentiable at a point x if there exists a linear functional $\mathcal{D}_x$ on X such that

$$(*)\qquad \mathcal{D}_x(y)=||x+\zeta\cdot y||-||x||=\mathfrak{Re}\,(\zeta\cdot\mathcal{D}_x(y))+\text{small ordo}(|\zeta|)$$

hold for every $y\in X$ where the limit is taken over complex ζ which tend to zero. One says that X is differentiable if $\mathcal{D}_x$ exist for every $x\in X$. This notion was introduced by Clarkson in 1936. In § XX we expose a theorem due to Beurling and Lorch concerned with certain non-linear duality maps on uniformly convex and differentiable Banach spaces.

Borel-Stieltjes integrals. Let μ be a Riesz measure on the unit interval $[0,1]$ and consider an X-valued function, which to every $0\leq t\leq 1$ assigns a vector $f(t)$ in X. Assume that there exists a constant M where

$$\max_{0\leq t\leq 1}||f(t)||=M$$

i.e the function is bounded with respect to the norm on X. In addition we assume that the the complex-valued functions $t\mapsto x^*(f(t))$ are Borel functions on $[0,1]$ for every x^* in the dual space X^*. Measure theory teaches that there exist the Borel-Stieltjes integral

$$J(x^*)=\int_0^1 x^*(f(t))\,d\mu(t)$$

for every x^*. The boundedness of f implies that

$$x^*\mapsto J(x^*)$$

is a continuous linear functional on X^*. This gives a vector $\xi(f)$ in the bi-dual X^{**} such that

$$(1)\qquad \xi(f)(x^*)=J(x^*)\quad:\quad x^*\in X^*$$

Notice that (1) is found for every nomed space x, i.e. above X need not be a Banach space. In the special case when X is a reflexive Banach space the f-integrals yields a vector in $x = x(\mu, f) \in X$ which computes (1), i.e.

$$x^*(x(\mu, f))) = \int_0^1 x^*(f(t))\, d\mu(t) \quad : x^* \in X$$

The map

$$(\mu, f) \mapsto x(\mu, f)$$

is bi-additive. The family of X-valued functions $f(t)$ for which (*) hold and in addition $t \mapsto x^*(f(t)$ are Borel functions for every $x^* \in X^*$ are called bounded Borel funcction with vlues in X^*. The fact that one can assign integrals above is used in many applications. Of course, avove one could have replaced the unit interval $[0, 1]$ by other topological spaces on which the notion of Riesz meaaures exists. For example, $[0, 1]$ could be replaced by an arbitrary compact metric space.

Analytic functions. Let X be a Banach space and consider a power series with coefficients in X:

(i) $$f(z) = \sum_{\nu=0}^{\infty} b_\nu \cdot z^\nu \quad b_0, b_1, \ldots \text{ is a sequence in } X\,.$$

Let $R > 0$ and suppose there exists a constant C such that

(ii) $$||b_\nu|| \leq C \cdot R^\nu \quad : \nu = 0, 1, \ldots$$

Then the series (i) converges when $|z| < R$ and $f(z)$ is called an X-valued analytic function in the open disc $|z| < R$. More generally, let Ω be an open set in $\mathbf{C}$. An X-valued function $f(z)$ is analytic if there to every $z_0 \in \Omega$ exists an open disc D centered at z_0 such that the restriction of f to D is represented by a convergent power series

$$f(z) = \sum b_\nu (z - z_0)^\nu$$

Using the dual space X^* results about ordinary analytic functions extend to X-valued analytic functions. Namely, for each fixed $x^* \in X^*$ the complex valued function

$$z \mapsto x^*(f(z))$$

is analytic in Ω. From this one recovers the Cauchy formula. For example, let Ω be a domain in the class $\mathcal{D}(C^1)$ and $f(z)$ is an analytic X-valued function in Ω which extends to a continuous X-valued function on $\bar{\Omega}$. If $z_0 \in \Omega$ there exists the complex line integral

$$\int_{\partial\Omega} \frac{f(z)dz}{z - z_0}$$

It is evaluated by sums just as for a Riemann integral of complex-valued functions. One simply replaces absolute values of complex valued functions by the norm on X in approximating sums which converge to the Riemann integral. This gives Cauchy's integral formula, extended to X-valued analytic functions:

$$f(z_0) = \int_{\partial\Omega} \frac{f(z)dz}{z - z_0}\,.$$

Operational calculus. Commutative Banach algebras are defined and studied in § 10. If B is a commutstive Banach algebra with a unit element e and $x \in B$, then we shgall prove that the spectrum $\sigma(x)$ is a non-empty compact subset of $\mathbf{C}$ and there exists the resolvent map:

(i) $$\lambda \mapsto R_x(\lambda) = (\lambda \cdot e - x)^{-1} \quad : \quad \lambda \in \mathbf{C} \setminus \sigma(x)$$

If $\lambda_0 \in \mathbf{C} \setminus \sigma(x)$ one contructs the local Neumann series which represents $R_x(\lambda)$ when λ stays in the open disc of radius $\text{dist}(\lambda_0, \sigma(x))$. It follows that $R_x(\lambda)$ is a B-valued analytic function of the complex variable λ defined in the open complement of $\sigma(x)$. Starting from this, Cauchy's formula gives vectors in B for every analytic function $f(\lambda)$ defined in some open neighborhood of $\sigma(x)$. More precisely, denote by $\mathcal{O}(\sigma(x))$ the algebra of germs of analytic functions on the compact set

$\sigma(x)$. It turns out that there exists an algebra homomorphism from $\mathcal{O}(\sigma(x))$ into X which sends $f \in \mathcal{O}(\sigma(x))$ into an element $f(x) \in X$. Moreover, the *Gelfand transform* of $f(x)$ is related to that of x by the formula

$$(*) \qquad \widehat{f}(x)(\xi) = f(\widehat{x}(\xi)) \quad : \quad \xi \in \mathfrak{M}_B$$

These general results are used in many applications. An example is when B is the Banach algebra generated by a single bounded linear operator on a Banach space.

Neumann's resolvents. A special section is devoted to the powerful and extremely important study of densely defined linear operators on a Banach space which need not be bounded. It turns out that for such an operator T there exists a well-defined closed spectral set $\sigma(T)$ and in its open complement one find an analytic function oif a complex parameter λ which gives inverses to $\lambda \cdot E - T$ in the sense of Carl Neumann, where E denotes the identity operator on X. The constructions in § 10 dealing with commutative Banach algebras is a special case of the Neumann calculus. Spectral theory about - in general unbounded - linear oeprators is a major topics where functional analysis finds useful applications. It is here the reader will encounter challenging problems and remarkable results. The material about semi-groups in § xx is an example where we prove a theorem due to Hille, Phillips and Yosida where unbounded operators appear whose spectral and Neumann resolvents have special properties.

1. Normed spaces.

A norm on a complex vector space X is a map from X into $\mathbf{R}^+$ satisfying:

$$(*) \qquad ||x+y|| \leq ||x|| + ||y|| \quad \text{and} \quad ||\lambda \cdot x|| = |\lambda| \cdot ||x|| \quad : \quad x,y \in X \quad : \ \lambda \in \mathbf{C}$$

Moreover $||x|| > 0$ holds for every $x \neq 0$. A norm gives a topology on X defined by the distance function

$$(**) \qquad d(x,y) = ||x-y||$$

1.1 Real versus complex norms. The field of real numbers is a subfield of $\mathbf{C}$. Hence every complex vector space has an underlying structure as a vector space over $\mathbf{R}$. A norm on a real vector space Y is a function $y \mapsto ||y||$ where (*) holds for real numbers λ. Next, let X be a complex vector space with a norm $||\cdot||$. Since we can take $\lambda \in \mathbf{R}$ in (*) the complex norm induces a real norm on the underlying real vector space of X. Complex norms are more special than real norms. For example, consider the 1-dimensional complex vector space $\mathbf{C}$. When the point 1 has norm one the norm of a complex vector $z = a + ib$ is its absolute value. On the other hand we can define many norms on the underlying real (x,y)-space. For example, we may take the norm defined by

$$\text{(i)} \qquad ||(x,y)|| = |x| + |y|$$

It fails to satisfy (*) under complex multiplication. For example, with $\lambda = e^{\pi i/4}$ we send $(1,0)$ to $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ whose norm from (i) becomes $\sqrt{2}$ while it should remain with norm one if (*) holds.

1.2 Norms defined via convex sets. Consider a real vector space Y. A subset K is convex if the line segment formed by a pair of points in K stay in K, i.e.

$$\text{(i)} \qquad y_1, y_2 \in K \implies s \cdot y_2 + (1-s) \cdot y_1 \in K \quad : \quad 0 \leq s \leq 1$$

Let $\mathbf{o}$ denote the origin in Y. Let K be a convex set which contains $\mathbf{o}$ and is symmetric with respect to $\mathbf{o}$:

$$y \in K \implies -y \in K$$

We say that K is called *absorbing* if there to every $y \in Y$ exists some $t > 0$ such that $ty \in K$. Suppose that K is symmetric and absorbing. To every $s > 0$ we set

$$sK = \{sx : \ x \in K\}$$

Since $\mathbf{o} \in K$ and K is convex these sets increase with s and since K is absorbing we have:

$$\text{(ii)} \qquad \bigcup_{s>0} sK = Y$$

Next, impose also the condition that K does not contain any 1-dimensional subspace, i.e. whenever $y \neq 0$ is a non-zero vector there exists some large t^* such that $t^* \cdot y$ does not belong to K. The condition is equivalent with

$$\text{(iii)} \qquad \bigcap_{s>0} s \cdot K = \mathbf{o}$$

1.3 The norm ρ_K. Let K be convex and symmetric and (ii-iii) hold. To each $y \neq 0$ we set

$$(*) \qquad \rho_K(y) = \min_{s>0} y \in s \cdot K$$

If $y \in K$ then $s = 1$ is competing when we seek the minimum and hence $\rho_K(y) \leq 1$. On the other hand, if y is "far away" from K we need large s-values to get $y \in s \cdot K$ and therefore $\rho_K(y)$ is large. It is clear that

$$\text{(i)} \qquad \rho_K(ay) = a \cdot \rho_K(y) \quad : \ a \text{ real and positive}$$

Morover, since K is symmetric we have $\rho_K(y) = \rho_K(-y)$ and hence (i) gives

$$\text{(ii)} \qquad \rho_K(ay) = |a| \cdot \rho_K(y) \quad : \ a \text{ any real number}$$

Next, the convexity of K gives the inclusion

$$s \cdot K + t \cdot K \subset (s+t) \cdot K$$

for every pair of positive numbers. Indeed, with x and y in K one has

$$\frac{sx}{s+t} + \frac{ty}{s+t} \in K$$

and (iii) follows. The reader can check that the construction of ρ_K gives the triangle inequality

(iii) $$\rho_K(y_1 + y_2) \leq \rho_K(y_1) + \rho_K(y_2)$$

We conclude that ρ_K yields a norm on Y.

1.4 A converse result. If $|| \cdot ||$ is a norm on Y we get the convex set

$$K^* = \{y \in Y \quad : \ ||y| \leq 1\}$$

It is clear that $\rho_{K^*}(y) = ||y||$ holds, i.e. the given norm is recaptured by the norm defined by K^*. We can also regard the set

$$K_* = y \in Y \quad : \ ||y| < 1\}$$

Here $K_* \subset K^*$ but the reader should notice the equality

$$\rho_{K_*}(y) = \rho_{K^*}(y)$$

Thus, the two convex sets define the same norm even if the set-theoretic inclusion $K_* \subset K^*$ may be strict. In general, a pair of convex sets K_1, K_2 satisfying (ii-iii) in § 1.2 are said to be equivalent if they define the same norm. Starting from this norm we get K_* and K^* and then the reader may verify that

$$K_* \subset K_\nu \subset K^* \quad : \nu = 1, 2$$

Summing up we have described all norms on Y and they are in a 1-1 correspondence with equivalence classes in the family $\mathcal{K}$ of convex sets which are symmetric and satisfy (ii-iii) from § 1.2.

1.5 Equivalent norms. Two norms $|| \cdot ||_1$ and $|| \cdot ||_2$ are equivalent if there exists a constant $C \geq 1$ such that

(1.5.1) $$\frac{1}{C} \cdot ||y||_1 \leq ||y||_2 \leq C \cdot ||y||_1 \quad : \quad y \in Y$$

Notice that if the norms are defined by convex sets K_1 and K_2 respectively, then (0.6) means that there exists some $0 < t < 1$ such that

$$tK_1 \subset K_2 \subset t^{-1} K_1$$

1.6 The case $Y = \mathbf{R}^n$. If Y is finite dimensional all norms are equivalent. To see this we consider the euclidian basis $e_1, \ldots, e_n$. To begin with we have the *euclidian norm* which measures the euclidian length from a vector y to the origin:

(i) $$||y||_e = \sqrt{\sum_{\nu=1}^{\nu=n} |a_\nu|^2} \quad : \quad y = a_1 e_1 + \ldots + a_n e_n$$

The reader should verify that (i) satisfies the triangle inequality

$$||y_1 + y_2||_e \leq ||y_1||_e + ||y_2||_e$$

which amounts to verify the Cauchy-Schwartz inequality. We have also the norm $|| \cdot ||^*$ defined by

(ii) $$||y||^* = \sum_{\nu=1}^{\nu=n} |a_\nu| \quad : \quad y = a_1 e_1 + \ldots + a_n e_n$$

This norm is equivalent to the euclidian norm. More precisely the reader may verify the inequality

(iii) $$\frac{1}{\sqrt{n}} \cdot ||y||_e \leq ||y||^* \leq \sqrt{n} \cdot ||y||_e$$

Next, let $||\cdot||$ be some arbitrary norm. Put

(iv) $$C = \max_{1\leq\nu\leq n} ||e_\nu||$$

Then (ii) and the triangle inequality for the norm $||\cdot||$ gives

(v) $$||y|| \leq C\cdot ||y||^*$$

By the equivalence (iii) the norm topology defined by $||\cdot||^*$ is the same as the usual euclidian topology in $Y = \mathbf{R}^n$. Next, notice that (v) implies that the sets

$$U_N = \{y \in Y \quad : \quad ||y|| < \frac{1}{N}\} \quad : \quad N = 1, 2, \ldots$$

are *open* sets when Y is equipped with its usual euclidian topology. Now $\{U_N\}$ is an increasing sequence of open sets and their union is obviously equal t oY. in particular this union covers the compact unit sphere S^{n-1}. This gives an integer N such that

$$S^{n-1} \subset U_N$$

This inclusion gives

$$||y||_e \leq N\cdot ||y||$$

Together with (iii) and (v) we conclude that $||\cdot||$is equivalent with $||\cdot||_e$. Hence we have proved

1.7 Theorem. *On a finite dimensional vector space all norms are equivalent.*

1.8 The complex case. If X is a complex vector space we obtain complex norms via convex sets K which not only are symmetric with respect to scalar multiplication with real numbers, but is also invariant under multiplication with complex numbers $e^{i\theta}$ which entails that

$$\rho_K(\lambda\cdot x) = \lambda\cdot\rho_K(x)$$

hold for every complex number λ, i.e. we get a norm on the complex vector space.

2. Banach spaces.

Let X be a normed space over $\mathbf{C}$ or over $\mathbf{R}$. A sequence of vectors $\{x_n\}$ is called a Cauchy sequence if

(*) $$\lim_{n,m\to\infty} ||x_n - x_m|| = 0$$

We obtain a vector space $\widehat{X}$ whose vectors are defined as equivalence classes of Cauchy sequences. The norm of a Cauchy sequence $\{x_n\}$ is defined by

$$\{x_n\} = \lim_{n\to\infty} ||x_n||$$

One says that the norm on X is complete if every Cauchy sequence converges, or equivalently $X = \widehat{X}$. A complete normed space is called a *Banach space* as an attribution to Stefan Banach whose article [Ban] introduced the general concept of normed vector spaces.

2.1 Separable Banach spaces. A Banach space X which contain a denumerable and dense subset $\{x_n\}$ is called separable. If this holds we get for each n the finite dimensional subapce X_n generated by $x_1, \ldots, x_n$ and one constructs a basis in each X_n which gives the existence of a denumarable sequence of linearly independent vectors $e_1, e_2, \ldots$ such that the increasing sequence $\{X_n\}$ are all contained in the vector space

(i) $$X_* = \oplus\,\mathbf{R}\cdot e_n$$

By this construction X_* is a dense subspace of X. There are many ways to construct a denumerable sequence of linearly independent vectors which give a dense subspace of X and one may ask if it is possible to choose a sequence $\{e_n\}$ such that every $x \in X$ can be expanded as follows:

2.2 Definition. *A denumerable sequence* $\{e_n\}$ *of* $\mathbf{C}$*-linearly independent vectors in a complex vector space* X *is called a Schauder basis if there to each* $x \in X$ *exists a unique sequence of complex numbers* $c_1(x), c_2(x), \ldots$ *such that*

$$\lim_{N\to\infty} ||x - \sum_{n=1}^{n=N} c_\nu(x) \cdot e_\nu|| = 0$$

2.3 Enflo's example. The existence of a Schauder basis in every separable Banach space appears to be natural and Schauder constructed such a basis in the Banach space $\mathbf{C}^0[0,1]$ of continuous functions on the closed unit interval equipped with the maximum norm. For several decades the question of existence of a Schauder basis in every separable Banach space was open until Per Enflo at seminars in Stockholm University during the autumn in 1972 presented an example where a Schauder basis does not exist. For the construction we refer to the article [Enflo-Acta Mathematica]. After [Enflo] it became a veritable industry to verify that various concrete Banach spaces Y do have a Schauder basis and perhaps more important, also enjoy the approximation property , i.e. that the class of linear operators on Y with finite dimensional range is dense in the linear space of all compact operators on Y. Fortunately most Banach spaces do have a Schauder basis. But the construction of a specific Schauder basis is often non-trivial. It requires for example considerable work to exhibit a Schauder basis in the disc algebra $A(D)$ of continuous functions on the closed unit disc which are analytic in the interior.

3. Linear operators.

Let X and Y be two normed spaces and $T\colon\ X \to Y$ a linear operator. We say that T is continuous if there exists a constant C such that

$$||T(x)|| \leq C \cdot ||x||$$

where the norms on X respectively Y appear. The least constant C for which this holds is denoted by $||T||$ and called the operator norm of T. Denote by $\mathcal{L}(X,Y)$ the set of all continuous linear operators from X into Y. From the above it is equipped with the norm

(*) $$||T|| = \max_{||x||=1} ||T(x)|| \quad :$$

Above X and Y are not necessarily Banach spaces. But one verifies easily that if $\widehat{X}$ and $\widehat{Y}$ are their completitions, then every $T \in \mathcal{L}(X,Y)$ extends in a unique way to a continuous linear operator $\hat{T}$ from $\hat{X}$ into $\hat{Y}$. Moreover, if Y from the start is a Banach space and $T \in \mathcal{L}(X,Y)$ then it extends in a unique way to a bounded linear operator from $\widehat{X}$ into Y. Finally the reader may verify the following:

3.1 Proposition. *If Y is a Banach space then the norm on $\mathcal{L}(X,Y)$ is complete, i.e. this normed vector space is a Banach space.*

3.2 Null spaces and the range. Let X and Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$. In X we get the subspace

$$\mathcal{N}(T) = \{x\colon\ T(x) = 0\}$$

Since T is continuous the kernel is a closed subspace of X and we get the quotient space

$$\bar{X} = \frac{X}{\mathcal{N}(T)}$$

It is clear that T yields a linear operator $\bar{T}$ from $\bar{X}$ into Y which by the construction of the quotient norm on $\bar{X}$ has the same norm as T. Next, consider the range $T(X)$ and notice the equality

(i) $$T(X) = \bar{T}(\bar{X})$$

One says that T has *closed range* if the linear subspace $T(X)$ of Y is closed. When this holds the complete norm on Y induces a complete norm on $T(X)$. In §4 we prove the *Open Mapping Theorem* which shows that if T has closed range then there exists a constant C such that for every vector $y \in T(X)$ there exists $x \in X$ with $y = Tx$ and

(3.2.1) $$||x|| \leq C \cdot ||y||$$

delete xxxxxxxx

3.3 The closed graph theorem Let X and Y be Banach spaces and T a linear operator from X into Y. For the moment we do not assume that it is bounded. In the product space $X \times Y$ we get the graph

$$\Gamma_T = \{(x, T(x)) \quad : \, x \in X\}$$

3.4 Theorem. *Let T be a linear operator from one Banach space X into another Banach space Y with a closed graph Γ_T. Then T is continuous.*

Proof. We have the surjective map

$$x \mapsto (x, Tx)$$

from X onto the graph. By assumption Γ_T is a closed subspace of the Banach space $X \oplus Y$ equipped with the norm

$$||(x, Tx)|| = ||x|| + ||Tx||$$

The Open Mapping Theoren gives a constant C such that

$$||x|| + ||Tx|| \leq C||x|| \implies ||Tx|| \leq (C-1)||x||$$

and hence T is bounded.

3.5 Densely defined and closed operators.

Let $X_* \subset X$ be a dense subspace and $T\colon\ X_* \to Y$ a linear operator where Y is a Banach space. As above we construct the graph

$$\Gamma_T = \{(x,y)\colon \quad x \in X_*\,:\, y = T(x)\}$$

If Γ_T is a closed subspace of $X \times Y$ we say that the densely defined operator T is closed.

3.6 Example. Let $X = C^0_*[0,1]$ be Banach space whose elements are continuous functions $f(x)$ on the closed interval $[0,1]$ with $f(0) = 0$. The space $X_* = C^1_*[0,1]$ of continuously differentiable functions appears as a dense subspace of X. Let $Y = L^1[0,1]$ which gives a linear map $T\colon\ X_* \to Y$ defined by

(i) $$T(f) = f' \quad : \quad f \in C^1_*[0,1]$$

Now T has a graph

(ii) $$\Gamma(T) = \{(f, f') \quad : f \in C^1_*[0,1]\}$$

Let $\overline{\Gamma(T)}$ denote the closure taken in the Banach space $X \times Y$. By definition a pair (f,g) belongs to $\overline{\Gamma(T)}$ if and only if

$$\exists\ \{f_n\} \in C^1_*[0,1] \quad : \max_{0 \le x \le 1}\ |f(x) - f_n(x)| \to 0 \quad : \int_0^1 |f'_n(t) - g(t)| \cdot dt = 0$$

The last limit means that the derivatives f'_n converge to an L^1-function g. Since $f_n(0) = 0$ hold for each n we have

$$f_n(x) = \int_0^x f'_n(t \cdot dt \to \int_0^x g(t) \cdot dt$$

Hence the limit function f is a primitive integral

(iii) $$f(x) = \int_0^x g(t) \cdot dt$$

Conclusion. The linear space $\overline{\Gamma(T)}$ consists of pairs (f,g) with $g \in L^1[0,T)$ and f is the g-primitive defined by (iii). In this way we obtain a linear operator $\widehat{T}$ with a closed graph. More precisely, $\mathcal{D}(\widehat{T})$ consists of functions $f(x)$ which are primitives of L^1-functions. Lebesgue theory this means that that the domain of definition of $\widehat{T}$ consists of *absolutely continuous functions*. Thus, by enlarging the domain of definition the linear operator T is extended to a densely defined and closed linear operator.

3.7 Remark. The example above is typical for many constructions where one starts with some densely defined linear operator T and finds an extension $\widehat{T}$ whose graph is the closure of $\Gamma(T)$. The reader should notice that the choice of the target space Y affects the construction of closed extensions. For example, replace above $L^1[0,1]$ with the Banach space $L^2[0,1]$ of square integrable functions on $[0,1]$. In this case we find a closed graph extension S whose domain of definition consists of continuous functions $f(x)$ which are primitives of L^2-functions. Since the inclusion $L^1[0,1] \subset L^2[0,1]$ is strict the domain of definition for S is a proper subspace of the linear space of all absolutely continuous functions. In PDE-theory one starts from a differential operator

(*) $$P(x,\partial) = \sum p_\alpha(x) \cdot \partial^\alpha$$

where $x = (x_1, \ldots, x_n)$ are coordinates in $\mathbf{R}^n$ and ∂^α denote the higher order differential operators expressed by products of the first order operators $\{\partial_\nu = \partial/\partial x_\nu\}$. The coefficients $p_\alpha(x)$ are in general only continuous functions defined in some open subset Ω of $\mathbf{R}^n$, though the case when p_α are C^∞-functions is the most frequent. Depending upon the situation one takes various target spaces Y. For example we let Y be the Hilbert space $L^2(\Omega)$. To begin with one restricts $P(x,\partial)$ to the linear space $C_0^\infty(\Omega)$ of test-functions in Ω and constructs the corresponding graph. With $Y = L^2(\Omega)$ we construct a closed extension as bove. This device is often used in PDE-theory.

§ 4 Three basic principles.

A result due to Baire goes as follows: Let X be a metric space whose metric d is complete, i.e. every Cauchy sequence with respect to the distance function d converges.

A.1 Theorem. *Let $\{F_n\}$ is an increasing sequence of closed subsets of X where each F_n has empty interior. Then the union $F^* = \cup F_n$ is meager, i.e. F^* does not contain any open set.*

Proof. To say that F^* is meager means that if $x_0 \in X$ and $\epsilon > 0$ then

(i) $$B_\epsilon(x_0) \cap (X \setminus F^*) \neq \emptyset$$

To show this we first use that F_1 has empty interior which gives some $x_1 \in B_{\epsilon/2}(x_0) \setminus F_1$ and choose $\delta_1 < \epsilon/3$ so that

(ii) $$B_{\delta_1}(x_1) \cap F_1 = \emptyset$$

Now $B_{\delta_1/3}(x_1)$ is not contained in F_2 which gives a pair x_2 and $\delta_2 < \delta_1/3$ such that

(ii) $$B_{\delta_2}(x_2) \cap F_2 = \emptyset$$

We can continue in this way and to every n find a pair (x_n, δ_n) such that

(iii) $$B_{\delta_n}(x_n) \cap F_n = \emptyset \quad : \quad x_n \in B_{\delta_{n-1}}(x_{n-1}) \quad : \quad \delta_n < \delta_{n-1}/3$$

For each pair $M > N \geq 1$ the triangle inequality gives

(iv) $$d(x_M, x_N) \leq \delta_{N+1} + \ldots + \delta_M \leq \delta_N(3^{-1} + \ldots 3^{-M+1}) \leq \frac{2}{3}\delta_N$$

At the same time we notice that

(v) $$\frac{2}{3}\delta_N \leq 3^{-N+1} \cdot \frac{2}{3}\epsilon$$

Hence $\delta_N \to 0$ and (iv) entails that $\{x_n\}$ is a Cauchy sequence. Since the metric is complete there exists a limit point x_* where

$$\lim_{M\to\infty} d(x_M, x_*) = 0$$

To each $N \geq 1$ we apply (iv) for arbitrary large M and obtain

$$d(x_*, x_N) \leq \frac{2}{3}\delta_N \implies x_* \in B_{\delta_N}(x_N) \subset X \setminus F_N$$

At the same time the the last inequality in (iv) applied with arbitrary large N gives

$$d(x_*, x_0) \leq \frac{2}{3}\epsilon \implies x_* \in B_\epsilon(x_0)$$

So x_* gives the requested point which produces a non-empty set in (i).

A.2 The Banach-Steinhaus Theorem.

Let X be a vector space equipped with a complete norm $||\cdot||$. Hence X is a complete metric spacce with the distance function

$$d(x,y) = ||x-y||$$

Let $||\cdot||_*$ be another norm on X, not necessarily complete. To each positive number a we set

$$F_a = \overline{\{x : ||x||_* \leq a\}}$$

where the closure is taken with respect to the given complete norm. The triangle inequality give the inclusion

$$F_a + F_b \subset F_{a+b}$$

for each pair of positive real numbers. Next, since every vector in X has a finite norm $||x||_*$ it follows that the union of these F-sets taken as a varies over the set of positive integers is equal to X. Baire's theorem applied to the complete metric space (X,d) gives an integer N such that F_N has a non-empty interior. Thus, we find $x_0 \in X$ and $\epsilon > 0$ such that

$$B_\epsilon(x_0) \in F_N$$

Here $||x_0||_* < M$ for some integer M and the triangle inequality gives

$$B_\epsilon(0) \subset F_{N+M}$$

If k is a positive integer such that $k^{-1} \leq \epsilon$ and $K = (N+M)k$ it follows after scaling that the open unit ball

$$(*) \qquad B_1 \subset F_K$$

and since F_K is closed in the given norm-topology it follows that (*) also holds when we replace B_1 with the closed unit ball. Hence, för every $x \in X$ with $||x|| \leq 1$ there exists a sequence $\{x_\nu\}$ where $||x_\nu||^* \leq K$ for every ν and

$$\lim\, ||x_\nu - x|| = 0$$

and by scaling we conclude that

$$(**) \qquad ||x||_* \leq K \cdot ||x|| \quad : \quad x \in X$$

If the norm $|| \cdot ||_*$ also is complete we can reverse the role and find a constant K^* such that

$$(***) \qquad d(x,y) \leq K^* \cdot d_*(x,y)$$

From this we conclude the following:

A.2.1 Theorem. *A pair of complete norms $||\cdot||$ and $||\cdot||_*$ on a vector space are always equivalent, i.e there exists a constant $C \geq 1$ such that*

$$C^{-1}||x||_* \leq ||x|| \leq C||x||_*$$

A.3 The Open Mapping theorem.

Let $T\colon X \to Y$ be a linear operator where X and Y are Banach spaces. Assume that T is surjective and bounded, i.e. there exists a constant C such that

$$\text{(i)} \qquad ||Tx|| \leq C \cdot ||x|| \quad \& \quad T(X) = Y$$

Let $\{B_N(X)) : N = 1, 2, \ldots\}$ be the closed balls of norm N in X. Since T is sirjective we have

$$Y = \cup \overline{T(B_N(X))}$$

Baires Category Theorem can be applied exactly as in (A.2) and gives an integer N such that

$$\text{(i)} \qquad B_1(Y) \subset \overline{T(B_N(X))}$$

Let us then take some $y \in Y$ of norm $\leq 1/2$. Then (i) gives x_1 of norm $\leq N/2$ such that

$$||y - T(x_1)| \leq 1/4$$

Then we find x_2 of norm $\leq N/4$ such that

$$||y - T(x_1) + T(x_2)| \leq 1/8$$

We can continue in this way and since both X and Y are Banach spaces the reader can chek that one gets the inclusion

$$B_{1/2}(Y) \subset T(B_N(X))$$

This means that T yields an open mapping from X onto Y, i.r. after a scaling the inclusion (xx) entails that for every $\epsilon > 0$ the image $T(B_\epsilon(X))$ contains a small open ball centered at $y = 0$.

A.4 The Hahn-Banach theorem.

Let E be a real vector space. A function $\rho\colon E \to \mathbf{R}^+$ is subadditive and positively homogeneous if the following hold for each pair x_1, x_2 in X and every non-negative real number s:

$$\text{(A.4.1)} \qquad \rho(x_1 + x_2) \leq \rho(x_1) + \rho(x_2) \quad : \rho(sx_1) = s \cdot \rho(x_1)$$

Let $E_0 \subset E$ be a subspace and $\lambda_0\colon E_0 \to \mathbf{R}$ a linear functional such that

$$\lambda_0(x_0) \leq \rho(x_0) \quad : x_0 \in E_0$$

Then there exists a linear functional λ on X which extends λ_0 and

(A.4.2) $$\lambda(x) \leq \rho(x) \quad : x \in X$$

To prove this we use Zorn's Lemma. Namely, consider the partially ordered family of pairs (V, μ) where V is a subspace of E which contains E_0 and $\mu \colon V \to \mathbf{R}$ a linear map which extends λ_0 and satisfies.

(i) $$\mu(x) \leq \rho(x) \quad : x \in V$$

Zorn's Lemma gives a maximal pair in this family and there remains to prove that $V = E$. If $V \neq E$ we pick a vector $y \in E \setminus V$ and get a contradiction if we can extend μ to a linear map μ^* on $W = V + \mathbf{R}y$ which satisfies (A.4.1) when $x \in W$. To see that this is possible we set

$$\alpha = \min_{x \in V} \rho(x+y) - \mu(x)$$

$$\beta = \max_{\xi \in V} \mu(\xi) - \rho(\xi - y)$$

Now we show that

(i) $$\alpha \geq \beta$$

To get (i) we consider a pair of vectors x and ξ in V and then

$$\rho(x+y) - \mu(x) - (\mu(\xi) - \rho(\xi - y)) = \rho(x+y) + \rho(\xi - y)) - \mu(x + \xi)$$

The triangle inequality fo the ρ-function gives

$$\rho(x+\xi) = \rho((x+y) + (\xi - y)) \leq \rho(x+y) + \rho(\xi - y))-$$

At the same time the vector $x + \xi$ belongs to V and hence

$$\mu(x+\xi) \leq \rho(x+\xi)$$

This proves (i) and now we pick a real number a such that $\beta \leq a \leq \alpha$. At this stage the reader can check that if we define μ^* by

$$\mu^*(x+sy) = \mu(x) + sa$$

for every real nuymber s, then μ^* is a linear map on W for which (A.4.2) holds.

The case of complex vector spaces. Consider a complex vector space X equipped with a norm $||\cdot||$. Let $X_0 \subset X$ be a complex subspace and μ a $\mathbf{C}$-linear map on X_0 such that

$$|\mu(x)| \leq ||x||$$

hold for every $x \in X_0$. Taking real and imaginary parts of the complex numbers under μ we get a pair of real-valued and $\mathbf{R}$-linear maps f_1, f_2 on X_0 such that

$$\mu(x) = f_1(x) + i f_2(x)$$

Since μ is $\mathbf{C}$-linear we have

$$i(f_1(x) + i f_2(x)) = i\mu(x) = \mu(ix) = f_1(ix) + i f_2(ix)$$

Identifying the real parts we see that

(i) $$f_2(x) = -f_1(ix)$$

Next, with $x \in X_0$ we have

$$f_1(x) \leq |f_1(x) - i f_1(ix)| \leq ||x||$$

The real version of the Hahn Banach theorem gives an $\mathbf{R}$-linear map $F \colon X \to \mathbf{R}$ which extends f_1 and satisfies

(ii) $$F_1(x) \leq ||x|| \quad : x \in X$$

Now we get a $\mathbf{C}$-linear map μ^* on X defined by

$$\mu^*(x) = F_1(x) - i F_1(ix)$$

From the above μ^* extends μ and there remains to prove that

(*) $$|\mu^*(x)| = |F_1(x) - iF_1(ix)| \leq ||x||$$

hold for all $x \in X$. To prove this we apply (ii) to vectors $e^{i\theta}x = \cos\theta \cdot x + \sin\theta \cdot ix$ which gives

(iii) $$\cos\theta \cdot F_1(x) + \sin\theta \cdot F(ix) \leq ||e^{i\theta}x|| = ||x||$$

This hold for all $0 \leq \theta \leq 2\pi$ and choose θ so that

$$\cos\,\theta = \frac{F_1(x)}{|F_1(x) - iF_1(ix)|} \quad : \; \sin\,\theta = \frac{F_2(x)}{|F_1(x) - iF_1(ix)|}$$

Then the left hand side in (iii) becomes $|F_1(x) - iF_1(ix)|$ and (*) follows.

A.4.3 Separating hyperplanes.

Consider a convex set K in a real normed vector space X which contains the origin. With $\delta > 0$ we put

$$K_\delta = K + B(\delta) = \{x + \xi : x \in K\; ||x|| < \delta\}$$

As explained in § xx we get a function ρ defined by

$$\rho(x) = \inf_{s>0}\, x \in s \cdot K_\delta$$

where ρ satisfies (A.4.0). Let x_0 be a vector such that

$$\min_{x \in K} ||x - x_0|| = \delta$$

This entails that

$$\rho(x) = 1$$

On the 1-dimensional real subspace of X generated by x_0 we define the linear functional λ_0 where $\lambda_0(x) = 1$. Now the real version of the Hahn Banach theorem gives an extension λ to X such that

$$\lambda(x) \leq \rho(x)$$

hold for all $x \in X$. In particular we take vectors in the open ball $B(\delta)$ for these vectors the inclusion $B(\delta) \subset K_\delta$ gives $\rho(x) \leq 1$. After scaling we conclude that

$$|\lambda(x)| \leq \delta^{-1} \cdot ||x||$$

hold for all vectors in X which means that the linear functional λ is continuous. Next, since $\rho(x) \leq 1$ for all vectors $x \in K_\delta$ we have

$$\lambda(k) + \lambda(\xi) = \lambda(k + \xi) \leq 1$$

for pairs $k \in K$ and $\xi \in B(\delta)$. Here ξ can be chosen so that

$$\lambda(\xi) = -\delta \cdot ||\lambda||$$

We conclude that

$$\max_{k \in K} \lambda(k) \leq 1 - \delta \cdot ||\lambda||$$

At the same time $\lambda(x) = 1$ which means that the half-space

$$\{\lambda \leq 1 - \delta \cdot ||\lambda||\}$$

contains K while x has distance $\geq \delta \cdot ||\lambda||$ to this half-space.

A.5 A special construction. Let X be a vector space of infinite dimension equipped with a norm and consider an infinite sequence $\{x_1, x_2, \ldots\}$ of linearly independent vectors in X. To each $n \geq 1$ we get the n-dimensional subspaces

$$E_n = \{x_1, \ldots, x_n\}$$

generated by the first n vectors. For a fixed n and every vector $y \in X \setminus E_n$ we set

$$d(y; E_n) = \min_{x \in E_n} ||y - x||$$

A.5.1 Exercise. Show that there exists at least one vector $x_* \in E_n$ such that

$$d(y; E_n) = ||y - x_*||$$

Next, construct via an induction over n a sequence of vectors $\{y_n\}$ where each $||y_n|| = 1$ with the following properties. For every n one has

$$E_n = \{y_1, \ldots, y_n\} \quad : \, d(y_{n+1}, E_n) = 1$$

In particular $||y_n - y_m|| = 1$ when $n \neq m$ which entails that the sequence $\{y_n\}$ cannot contain a convergent subsequence and hence the unit ball in X is non-compact. Thus, only finite dimensional normed spaces have a unit ball which is compact with respect to the metric defined by the norm.

A.5.2 A direct sum decomposition. Let X be a normed space and V some finite dimensional subspace of dimension n. Choose a basis $v_1, \ldots, v_n$ in V and via the Hahn-Banach theorem we find an n-tuple $x_1^*, \ldots, x_n^*$ such that $x_j^*(v_k)$ is Kronecker's delta-function. Set

$$W = \{x \in X : x_\nu^*(x) = 0 : \, 1 \leq \nu \leq n\}$$

Then W is a closed subspace of X and we have a direct sum decompotion

$$X = W \oplus V$$

Thus, every finite dimensional subspace of a normed space X has a closed complement. On the other hand, examples show that a closed infinite dimensional subspace X_0 has in general not closed complement.

A.6 Frechet spaces.

A pseudo-norm on a complex vector space X is a map ρ from X into the non-negative real numbers with the properties

(A.6.1) $$\rho(\alpha \cdot x) = |\alpha| \cdot \rho(x) \quad : \rho(x_1 + x_2) \leq \rho(x_1) + \rho(x_2)$$

where x, x_1, x_2 are vectors in X and $\alpha \in \mathbf{C}$. Notice that (A.7.1) entails that the kernel of ρ is a subspace of X denoted by $\mathrm{Ker}(\rho)$. If it is reduced to the zero vector one says that ρ is a norm.

Suppose now that $\{\rho_n\}$ is a denumerable sequence of pseudo-norms and put

(*) $$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\rho_n(x - y)}{1 + \rho_n(x - y)}$$

Assume that the intersection

$$\cap \ker(\rho_n) = \{0\}$$

Then d is a metric. If the resulning metric space (X, d) is complete, i.e. if every Cauchy-sequence with respect to d converges to a vector in X one refers to X as a Frechet space

The class $\mathcal{K}_X$. It consists of closed and convex subsets K which contrain the origin and are absorbing in the sense that for every vector $x \in X$ there exists some $a > 0$ such that $ax \in K$. If N is a positive integer we set

$$NK = \{Nx : x \in K\}$$

A.6.2 Theorem. *Let (X, d) be a Frechet space. Then every $K \in \mathcal{K}_X$ contains an open neighborhood of the origin.*

Proof. To each integer $N \geq 1$ we have the closed set $F_N = N \cdot K$. Since K is absorbing the union of these F-sets is equal to X. Baire's theorem applied to the complete metric space (X, d) yields some N and $\epsilon > 0$ such that F_N contains an open ball of radius ϵ centered at some $x_0 \in F_N$. If $d(x) < \epsilon$ we write

$$x = \frac{x_0 + x}{2} - \frac{x_0 - x}{2}$$

Now F_N contains $x_0 + x$ and $x_0 - x$. By symmetry it also contains $-(x_0 - x)$ and the convexity entails that $x \in F_N$. Hence one has the implication

$$d(x,0) < \epsilon \implies x \in N \cdot K$$

Now the d-metric is defined by a sequence of pseudo-norms $\{\rho_n\}$. Choose an integer M where $2^{-M} < \epsilon/2$. The construction of d shows that if $\rho_n(x) < \frac{\epsilon}{2M}$ hold for $1 \leq n \leq M$, then $d_\rho(x) < \epsilon$. Hence (i) gives

$$\max_{1 \leq n \leq M} \rho_n(x) < \frac{\epsilon}{2M} \implies x \in N \cdot K$$

After a scaling one has

$$\max_{1 \leq n \leq M} \rho_n(x) < \frac{\epsilon}{2MN} \implies x \in K$$

The reader can check that there exists $\epsilon_* > 0$ such that

$$d(x,0) < \epsilon_* \implies \max_{1 \leq n \leq M} \rho_n(x) < \frac{\epsilon}{2MN}$$

and conclude that K contains an open neighborhood of the origin.

A.6.3 The open mapping theorem. *Consider a pair of Frechet spaces X and Y and let*

$$u \colon X \to Y$$

be a continuous linear and surjective map. Then u is an open mapping, i.e. for every $\epsilon > 0$ the u-image of the ϵ-ball with respect to the Frechet metric on X contains an open neighborhood of the origin in Y.

The proof is left as an exercise to the reader where the hint is to employ Baire's theorem and similar arguments as in the proof of the open mapping theoren for Banach spaces.

A.6.4 Closed Graph Theorem. *Let X and Y be Frechet spaces and $u \colon X \to Y$ is a linear map. Put*

$$\Gamma(u) = \{(x, u(x)\}$$

and suppose it is a closed subspace of the Frechet space $X \times Y$. Then the linear operator u is continuous.

Exercise. Prove this theorem. A hint is that one has a bijective linear map from $\Gamma(u)$ onto X defined by

(i) $$(x, u(x)) \mapsto x$$

Since $\Gamma(u)$ is closed in $X \times Y$ it is a Frechet. Moreover, it is clear that (i) is a continuous linear map and hence it is open by (A.6.3) which entails that it is a homeomorphism between metric spaces. From this one easily checks that u is continuous.

§ 5 Dual vector spaces

Let X be a normed space over the complex field. A continuous linear form on X is a $\mathbf{C}$-linear map γ from X into $\mathbf{C}$ such that there exists a constant C with:

$$\max_{||x||=1} |\gamma(x)| \leq C$$

The smallest constant C above is the norm of γ. In this way the continuous linear forms give vectors in a normed vector space denoted by X^*. Since Cauchy-sequences of complex numbers converge it follows that X^* is a Banach space. Notice that this holds even if X from the start is not complete. The reader may verify that of $\widehat{X}$ is the completition of a normed space X, then its dual is equal to that of X, i.e. one has

$$X^* = \widehat{X}^*$$

Next, let Y be a subspace of X. Every $\gamma \in X^*$ can be restricted to Y and gives an element of Y^*, i.e. there exist a restriction map

(i) $$\mathfrak{res}_Y \colon X^* \to Y^*$$

A restricted linear form cannot increase the norm which gives the inequality

$$||\mathfrak{res}_Y(\gamma)|| \leq ||\gamma|| \quad : \quad \gamma \in X^*$$

5.1 The kernel of $\mathfrak{res}_Y$. The kernel is by definition the set of X^*-elements which are zero on Y. This gives a subspace of X^* denoted by $Y^\perp$. It can be identfied with the dual of a new normed space. Namely, consider the quotient space

$$Z = \frac{X}{Y}$$

Vectors in Z are images of vectors $x \in X$ where a pair of x_1 and x_2 give the same vector in Z if and only if $x_2 - x_1 \in Y$. Let $\pi_Y(x)$ denote the image of $x \in X$. Now Z is is equipped with a norm defined by

$$||z|| = \min_x ||x|| \quad : \quad z = \pi_Y(x)$$

From the constructions above the reader can verify that one has a canonical isomorphism

$$Z^* \simeq \text{Ker}(\mathfrak{res}_y) = Y^\perp$$

5.2 An exact sequence. Let $Y \subset X$ be a closed subspace and put $Z = \frac{X}{Y}$. By the Hahn-Banch Theorem every continuous linear form γ on Y has a *norm preserving extension* to a linear form on X. Thus, if γ has some norm C, there exists $\gamma^* \in X$ with norm C such that

$$\mathfrak{res}_Y(\gamma^*) = \gamma$$

One refers to γ^* as a norm-preserving extension of γ. Identifying Z^* with $Y^\perp$ as in (5.1) gives an exact sequence

(5.2.1) $$0 \to Z^* \to X^* \to Y^* \to 0$$

where the restriction map $X^* \to Y^*$ sends the unit ball in X^* onto the unit ball of Y^*.

5.3 The weak star topology

Let X be a normed space. On the dual X^* there is a topology where a fundamental system of open neighborhood of the origin in the vector space X^* consista of sets

(5.4.1) $$U(x_1, \ldots, x_N; \epsilon) = \{\gamma \in X^* \quad : \ |\gamma(x_\nu)| < \epsilon \quad : x_1, \ldots, x_N \ \text{finite set}\}$$

Let Y be the finite dimensional subspace of X generated by $x_1, \ldots, x_n$. It is clear that the kernel of $\mathfrak{res}_Y$ is contained in the U-set above. If k is the dimension of Y, then linear algebra entails that the kernel of $\mathfrak{res}_Y$ has codimension k in X. So the U-set in (5.4.1) contains a subspace of X^* of finite codimension. Thus, weak-star open neighborhoods of the origin in X^* contain subspaces with a finite codimension which for an infinite dimensional space means that the weak-star topology is rather coarse. However, thanks to the Hahn-Banach theorem it yields a Hausdorff topology. Next, let S^* be the unit ball in X^*, i.e. the set of linear functionals on X with norm ≤ 1. Using Tychonoff's theorem in general toplogy the reader should confirm :

5.4.2 Theorem. *The unit ball S^* is compact in the weak star topology.*

The case when X is separable. This means that there exists a denumerable dense set $x_1, x_2, \ldots$ in X. On S^* we define a metric by

$$d(\gamma_1, \gamma_2) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\big|\gamma_1(x_n) - \gamma_2(x_n)\big|}{1 + \big|\gamma_1(x_n) - \gamma_2(x_n)\big|} \tag{5.4.3}$$

Exercise. Verify that the weak-star topology on S^* is equal to the topology defined by the metric above. So when X is separable, then S^* is a compact metric space in the weak topology. Notice also that the topology defined by the metric in (5.4.3) does not depend upon the chosen dense subsequence in X.

A warning. If X is not separable the weak star topology can be "nasty". More precisely, we can find a non-separable Banach space X such that S^* contains a denumerable sequence $\{x_n^*\}$ which does not contain any convergent subsequence. In other words, S^* is not sequentially compact. To get such an example one employs a wellknown construction in general topology which produces a compact Hausdorff space Z containing a denumerable sequence of points $\{z_n^*\}$ which has no convergent subsequence. Let X be the Banach space of bounded complex-valued functions on Z. The dual X^* is the space of Riesz measures on Z. In S^* we find the sequence of unit point masses $\{\delta(z_n)\}$.

5.4.4 Exercise. Use the fact from topology which assers that a compact Hausdorrf space is normal to show that if $\{z_n\}$ is a sequence of points in Z such that

$$\lim_{n \to \infty} |, f(z_n) = \int_Z f \, d\mu$$

hold for a Riesz measure μ and all $f \in C^0(Z)$, then $\{z_n\}$ must converge to a point z_* in Z and $\mu = \delta(z_*)$. So with an "ugly compact space" Z as above we get an example of a Banach space X for which S^* is not sequentially compact.

5.4.5 Weak hulls in X^*. Assume now that X is separable and choose a denumerable and dense subset $\{x_n\}$. Examples show that in general the dual space X^* is no longer separable in its norm topology. However, there always exists a denumerable sequences $\{\gamma_k\}$ in X^* which is dense in the weak-star topology. This is proved in the next exercise.

5.4.6 Exercise. For every $N \geq 1$ we let V_N be the finite dimensional space generated by $x_1, \ldots, x_N$. It has dimension N at most. Applying the Hahn-Banch theorem the reader should construct a sequence $\gamma_1, \gamma_2, \ldots$ in X^* such that for every N the restricted linear forms

$$\gamma_\nu | V_N \quad 1 \leq \nu \leq N$$

generate the dual vector space V_N^*. Next, let Q be the field of rational numbers. Show that if Γ is the subset of X^* formed by all finite Q-linear combinations of the sequence $\{\gamma_\nu\}$ then this denumerable set is dense in X^* with respect to the weak-star topology.

5.4.7 Another exercise. Let X be a separable Banach space and let E be a subspace of X^*. We say that E point separating if there to every $0 \neq x \in X$ exists some $e \in E$ such that $e(x) \neq 0$. Show first that every such point-separating subspace of X^* is dense with respect to the weak topology. This is the easy part of the exercise. The second part is less obvious. Namely, put

$$B(E) = B(X^*) \cap E$$

Prove now that $B(E)$ is a dense $B(X^*)$. Thus, if $\gamma \in B(X^*)$ then there exists a sequence $\{e_k\}$ in $B(E)$ such that

$$\lim_{k\to\infty} e_k(x) = \gamma(x)$$

hold for all $x \in X$.

5.4.8 An example from integration theory. An example of a separable Banach space is $X = L^1(\mathbf{R})$ whose elements are Lebesgue measurable functions $f(x)$ for which the L^1-norm

$$\int_{-\infty}^{\infty} |f(x)| \cdot dx < \infty$$

If $g(x)$ is a bounded continuous functions on $\mathbf{R}$, i.e. there is a constant M such that $|g(x)| \leq M$ for all x, then we get a linear functional on X defined by

$$g^*(f) = \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx < \infty$$

Let E be the linear space of all bounded and continuous functions. By the previous exercise it is a dense subspace of X^* with respect to the weak topology. Moreover, by the second part of the exercise it follows that if $\gamma \in X^*$ has norm one, then there exists a sequence of continuous functions $\{g_n\}$ of norm one at most such that $g_\nu \to \gamma$ holds weakly. Let us now find γ. For this purpose we define the functions

$$\text{(i)} \qquad G_n(x) = \int_0^x g_n(t) \cdot dt \quad : \quad x \geq 0$$

These primitive functions are continuous and enjoy a further property. Namely, since the maximum norm of every g-function is ≤ 1 we see that

$$\text{(ii)} \qquad |G_n(x) - G_n(x')| \leq |x - x'| \quad : \quad x, x' \geq 0$$

This is means that whenever $a > 0$ is fixed, then the sequence $\{G_n\}$ restricts to an *equi-continuous* family of functions on the compact interval $[0, a]$. Moreover, for each $0 < x \leq a$ since we can take $f \in L^1(\mathbf{R})$ to be the characteristic function on the interval $[0, x]$, the weak convergence of the g-sequence implies that there exists the limit

$$\text{(iii)} \qquad \lim_{n\to\infty} G_n(x) = G_*(x)$$

Next, the equi-continuity in (ii) enable us to apply the classic result due to C. Arzéla in his paper *Intorno alla continua della somma di infinite funzioni contionue* from 1883 and conclude that the point-wise limit in (iii) is uniform. Hence the limit function $G_*(x)$ is continuous on $[0, a]$ and it is clear that G_* also satisfies (ii), i.e. it is Lipschitz continuous of norm ≤ 1. Since the passae to the limit can be carried out for every $a > 0$ we conclude that G_* is defined on $[0, +\infty >)$. In the same way we find G_* on $(-\infty, 0]$. Next, by the result in [XX-measure] there exists the Radon-Nikodym derivative $G_*(x)$ which is a bounded measurable function $g_*(x)$ whose maximum norm is ≤ 1. So then

$$G_*(x) = \int_0^x g_*(t) \cdot dt = \lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} \int_0^x g_n(t) \cdot dt$$

holds for all x. Since finite $\mathbf{C}$-linear sums of characteristic functions is dense in $L^1(\mathbf{R})$ we conclude that the limit functional γ is determined by the L^∞-function g_*. So this shows the equality

$$L^1(\mathbf{R})^* = L^\infty(\mathbf{R})$$

Remark. The result above is of course wellknown. But it is interesting to see how the last duality formula is derived from studies of the Lebesgue integral.

5.5 The weak topology on X

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Let X be a Banach space. The weak topology on X has by definition a fundamental system of open neighborhoods of the origin of sets

$$U(x_1^*, \ldots, x_N^*; \epsilon) = \{x \in X : |x_\nu^*(x)| < \epsilon\}$$

where $\{x_\nu^*\}$ are finite subsets of X^*.

5.5.1 Weakly convergent sequences. A sequence $\{x_k\}$ in X converges weakly to a limit vector x if

$$\lim_{k\to\infty} x^*(x_k) = x^*(x) \quad \text{hold for all } x^* \in X^*$$

5.5.2 Exercise. Apply Baire's theorem and show that a weakly convergent sequence $\{x_k\}$ is bounded, i.e. there exists a constant C such that

$$||x_k|| \leq C \quad : k = 1, 2, \ldots$$

Weak versus strong convergence. A weakly convergent sequence need not be strongly convergent. An example is when $X = C^0[0,1]$ is the Banach space of continuous functions on the closed unit interval. By the Riesz representation theorem the dual space X^* consists of Riesz measures. A sequence $\{x_n(t)\}$ of continuous functions converge weakly to zero if

$$\lim_{n\to\infty} \int_0^1 x_n(t) \cdot d\mu(t) = 0 \quad \text{hold for every Riesz measure} \quad \mu \tag{5.5.3}$$

By a result from [Measure] (5.5.3) holds if and only if the maximum norms of the x-functions are uniformly bounded and the sequence converges pointwise to zero. One can construct many such pointwise convergent sequences of continous functions which fail to converge in the maximum norm.

5.5.4 Exercie. Let X be an infinite dimensional Banach space. Then the norm-topology is always strictly stronger than the weak topology. A hint to prove this goes as follows. By the construction of the weak topology on X its equality with the norm topology gives a finite subset $x_1^*, \ldots, x_N^*$ of X^* and a constant C such that one has the implication

$$\max_\nu |x_\nu^*(x)| < C \implies ||x|| < 1 \quad : x \in X$$

But then the Hahn-Banach theorem implies that the complex vector space X^* is generated by the n-tuple $x_1^*, \ldots, x_N^*$ which entails that X has dimension N at most and contradicts the assumption that X has infinite dimension.

5.6 The bidual X^{**} and reflexive spaces.

Let X be a normed space. The dual of X^* is denoted by X^{**} and called the bidual of X. Each $x \in X$ yields a bounded linear functional on X^* by

$$\widehat{x}(x^*) = x^*(x)$$

The Hahn Banach theorem implies that when x is given, then there exists $x^* \in X*$ such that $x^*(x) = ||x||$. This implies that the norm of $\widehat{x}$ taken in X^{**} is equal to x. We can express this by saying that the bidual embedding $x \mapsto \widehat{x}$ is norm preserving and the image of X in X^{**} is denoted by $\mathfrak{i}(X)$. If $\mathfrak{i}$ is surjective one says that X is reflexive. Recall that dual spaces always are complete. So every reflexive normed space must be a Banach space. But the converse is not true, i.e there exists non-reflexive Banach spaces. An example is to take $X = \mathbf{c}_0$ in which case $X^* = \ell^1$ and $X^{**} = \ell^\infty$. An example of a reflexive Banach space is ℓ^p where $1 < p < \infty$. The vectors are sequences of complex numbers $x_1, x_2, \ldots$ for which

$$||x||_p = \big(\sum_{\nu=1}^\infty |x_\nu|^p\big)^{\frac{1}{p}} < \infty$$

Hölder's inequality entails that the dual space of ℓ^p is ℓ^q where $q = \frac{p-1}{p}$, and from which it is clear that ℓ^p is reflexive.

5.6.1 Condition for X to be reflexive. Let X be a normed space whose unit ball is denoted by S. The bidual embedding identifies S with a subset of S^{**}. We shall analyze how S can deviate from S^{**}. For this purpose we consider a finite set $x_1^*, \ldots, x_n^*$ in X^* and if $\gamma \in S^{**}$ we get complex numbers $\{c_i = \gamma(x_i^*)\}$. Then, if $\alpha_1, \ldots, \alpha_n$ is some n-tuple of complex numbers we have

$$|\sum \alpha_i c_i| = |\gamma(\sum \alpha_i x_i^*)| \leq ||\gamma|| \cdot ||\sum \alpha_i x_i^*|| \leq ||\sum \alpha_i x_i^*||$$

where the last inequality follows since γ has norm one at most. The general result in § xx gives for each $\epsilon > 0$ some $x \in X$ with $||x|| < 1 + \epsilon$ and

$$\widehat{x}(x_i^*) = x_i^*(x) = c_i = \gamma(x_i^*)$$

Let us now assume that S equipped with the weak topology is compact. The same holds for $2S$ and taking from (x) we find a sequence $\{x_n\}$ wih norms $||x_n|| < 1 + 1/n$. The assumed compactness gives a limit vector $x_* \in X$ of norm ≥ 1 where $x_n \overset{w}{\to} x$. Moreover, this weak convergence entials that

$$\widehat{x_*}(x_i^*) = \gamma(x_i^*)$$

hold for each $1 \leq i \leq n$. Here we started with a finite subset of X^*. At this stage the reader should verify that when S is compact in the weak topology then we can find $x \in S$ such that $\widehat{x} = \gamma$. This proves that the bidual embdedding is surjective, i.e. X is reflexive. Summing up we have proved:

5.6.2 Theorem. *If S is compact with respec to the weak toplogy, then X is reflexive.*

5.7 The Eberlein-Smulian theorem.

Let X be a Banach space. A subset A is called weakly sequentially compact if every countable sequence $\{x_n\}$ in A contains at least one subsequence which converges weakly to a limit vector x. Here it is not required that x belongs to A. Now we prove a result due to Eberlein and Smulian.

5.7.1 Theorem. *Let A be a subset of a Banach space X. Then A is weakly sequentially compact if and only if its closure taken in the weak topology is weakly compact.*

Before we enter the proof we notice that the example in § xx. shows that a similar result does not hold when we regard the weak star topology on X^*. Let us also point out that we have not assumed that X is separable, so the weak topology on X need not be metrizable. The proof of Theorem 5.7.1 requires several steps. Assume first that A is weakly sequentally compact. We leave it to the reader that this implies that A is bounded. So without loss of generality we can assume that A is contained in the unit ball S in X. Next, denote by $w(A)$ its weak closure. To prove that $w(A)$ is compact in the weak topology on X we shall use the bidual embedding $X \mapsto X^{**}$. By definition the weak star topology on X^{**} restricts to the weak topology on the subspace $\mathfrak{i}(X)$. Hence the bi-dual embedding is continuous when X is equipped with the weak topology and X^{**} with the weak star topology. It follows that

(i) $$\mathfrak{i}(w(A)) \subset \overline{\mathfrak{i}(A)}$$

where the right hand side is the weak star closure of $\mathfrak{i}(A)$ taken in X^{**} Since $A \subset S$ and $\mathfrak{i}$ is norm preserving we have

(ii) $$\overline{\mathfrak{i}(A)} \subset S^{**}$$

By Theorem xx the unit ball S^{**} is compact in the weak star topology. So if we have proved the inclusion

(iii) $$\overline{\mathfrak{i}(A)} \subset \mathfrak{i}(X)$$

it follows that $\overline{\mathfrak{i}(A)}$ is weakly compact in X and together wtih the inclusion (i) the set $w(A)$ is weakly compact.

Proof of (iii)

.

Let λ be a vector in $\overline{\mathfrak{i}(A)}$ and set

$$N(\lambda) = \{x^* \in X^* : \ \lambda(x^*) = 0\}$$

This is a hyperplane in X^*. If $N(\lambda)$ is closed with respect to the weak star topology on X^* then the observation in § xx gives a vector $x_0 \in X$ such that $\lambda = \mathbf{i}(x_0)$ and (iii) follows. So there remains to show that $N(\lambda)$ is weak star closed in X^*. To prove this we shall use a result which goes as follows:

Lemma. *Let Z be a bounded subset of X and λ a vector in $\overline{\mathfrak{i}(Z)}$ with norm ≤ 1. Then, if y_0^* a vector in the weak star closure of $N(\lambda)$ and $\epsilon > 0$ there exist sequences $\{y_\nu^*\}$ in $N(\lambda)$ and $\{x_\nu\}$ in Z such that the following hold for every $n \geq 1$:*

(1) $$|y_n^*(x_k) - \lambda(y_0^*)| < \epsilon \quad : 1 \leq k \leq n$$

(2) $$|y_n^*(x_k)| < \epsilon \quad : 1 \leq n < k$$

Exercise. Prove this via an inductive construction.

Next, apply the Lemma with $Z = A$. The hypothesis on A entails that $\{x_n\}$ has a convergent subsequence with a limit vector x_*. Passing to a subsequence we may assume that (1-2) hold above and that $x_n \xrightarrow{w} x_*$. By the resut in § xx the weak convergence entails that x_* is a strong limit of convex combinations of $\{x_\nu\}$. So with $\epsilon > 0$ kept fixed we find a large positive integer N and real non-negative numbers $a_1, \ldots, a_N$ whose sum is one such that

(iv) $$||\xi - x_*|| < \epsilon \quad : \ \xi = a_1 x_1 + \ldots + a_N x_N$$

Now (1) gives

(v) $$|\lambda(y_0^*) - y_N^*(\xi)| \leq \sum_{k=1}^{k=N} a_k \cdot ||\lambda(y_0^*) - y_N^*(x_k)| < \epsilon$$

Next, since y_N^* has norm ≤ 1 we have

$$|y_N^*(\xi) - y_N^*(x_*)| \leq ||\xi - x_*|| < \epsilon$$

From (iv-v) the triangle inequality gives

(vi) $$|\lambda(y_0^*)| < 2\epsilon + |y_N^*(x_*)|$$

Funally, since $x_k \xrightarrow{w} x_*$ it follows that we can take $k > N$ so large that

$$|y_N^*(x_*) - |y_N^*(x_k)| < \epsilon$$

Above $k > N$ and (2) entails that $|y_N^*(x_k)| < \epsilon$. So another application of the triangle inequality gives

$$|\lambda(y_0^*)| < 2\epsilon + 2\epsilon = 4\epsilon$$

In the lemma we can take ϵ arbitrarily small and conclude that $\lambda(y_0^* = 0$. Hence $y_0^* \in N(\lambda)$ and since y_0^* was an arbitrary vector in the weak star closure of $N(\lambda)$ we have proved that this hyperplane is weak star closed.

Proof of the converse.

There remains to show that if $w(A)$ is weakly compact then it is sequentially compact. DO IT ...

5.7.2 Applications of Theorem 5.7.1 Let A be a subset of the Banach space X. We construct its convex hull $\text{co}(A)$ and pass to its closure in the norm topology, i.e. we get the norm-closed set $\overline{\text{co}(A)}$. With these notations one has the result below which also is due to Eberlein and Smulian.

5.7.3 Theorem. *If A is weakly compact it follows that $\overline{\text{co}(A)}$ also is weakly compact.*

Proof. By xx the norm closed convex set $\overline{\text{co}(A)}$ is weakly closed and even equal to the weak closure of $\text{co}(A)$. By Theorem 5.7.1 there remains to show that $\text{co}(A)$. is weakly sequentially compact. So let u sconsider a suence of points $\{p_n\}$ in $\text{co}(A)$. Each p_n is a finite convex combination of points in A denoted by B_n and we set

$$B^* = \cup B_n$$

The countable set B^* generates a separable closed subspace X_0 of X. CONTINUE PROOF ....

5.8 The Krein-Smulian theorem.

Articles by these authors from the years around 1940 contain a wealth of results. A major theorem from their work goes as follows. Let X be a Banach space and X^* its dual. In (B.2) we constructed the weak star topology. Next, the *bounded weak-star topology* is defined as follows. Let S^* be the open ball of vectors in X^* with norm < 1. If n is a positive integer we get the ball nS^* of vectors with norm $< n$. A subset V of X^* is open in the bounded weak-star topology if and only if the interesections $V \cap nS^*$ are weak-star open for every positive integer n. In this way we get a new topology on X^* whose corresponding topological vector space is denoted by X^*_{bw}, while X^*_w denotes the topological vector space when X^* is equipped with the weak topology. Notice that the family of open sets in X^*_{bw} contains the open sets in X^*_w, i.e. the bounded weak-star topology is stronger. Examples show that the topologies in general are not equal.

Next, let λ be a linear functional on X^* which is continuous with respect to the weak-star topology. This gives by definition a finite set $x_1, \ldots, x_M$ in X such that if $|x^*(x_\nu)| < 1$ for each ν, then $\lambda(x^*)| < 1$. This implies that the subspace of X^* given by the common kernels of $\widehat{x}_1, \ldots \widehat{x}_M$ contains the λ-kernel and linear algebra gives an M-tuple of complex numbers such that

$$\lambda = \sum c_\nu \cdot \widehat{x}_\nu$$

We can express this by saying that the dual space of X^*_w is equal to $\widehat{X}$, i.e. every linear functional on X^* which is continuous with respect to the weak-star topology is of the form $\widehat{x}$ for a unique $x \in X$. Less obvious is the following:

5.8.1. Theorem. *The dual of X^*_{bw} is equal to $\widehat{X}$.*

Proof. For each finite subset A of X we put

$$\widehat{A} = \{x^* : \max_{x \in A} |x^*(x)| \leq 1\}$$

Let U be an open set in X^*_{bw} which contains the origin and S^* is the closed unit ball in X^*. The construction of the bounded weak-star topology gives a finite set A_1 in X such that

(i) $$S^* \cap \widehat{A}^0 \subset U$$

Next, let $n \geq 1$ and suppose we have constructed a finite set A_n where

(ii) $$nS^* \cap \widehat{A_n} \subset U$$

To each finite set B of vectors in X with norm $\leq n^{-1}$ we notice that

(iii) $$\widehat{A_n \cup B} \subset \widehat{A_n}$$

Put

$$F(B) = (n+1)S^* \cap \widehat{A_n \cup B} \cap (X^* \setminus U)$$

It is clear that $F(B)$ is weak-star closed for every finite set B as above. If these sets are non-empty for all B, it follows from the weak-star compactness of $(n+1)S^*$ that the whole intersection is non-empty. So we find a vector

$$x^* \in \bigcap_B F(B)$$

Notice that $F(B) \subset \widehat{B}$ for every finite set B as above which means that $|x^*(x)||leq1$ for every vector x in in X of norm $\leq n^{-1}$. Hence the norm

$$||x^*|| \leq n$$

But then (iii) gives the inclusion

(iv) $$x^* \in nS^* \widehat{A_n}) \cap (X \setminus U)$$

This contradicts (ii) and hence we have proved that there exists a finite set B of vectors with norm $\leq n^{-1}$ such that $F(B) = \emptyset$.

From the above it is clear that an induction over n gives a sequence of sets $\{A_n\}$ such that (ii) hold for each n and

(v) $$A_{n+1} = A_n \cup B_n$$

where B_n is a finite set of vectors of norm $\leq n^{-1}$.

Final part in the roof of the Krein-Smulian theorem. Let θ be a linear functional on X^* which is continuous with respect to the bounded weak-star topology. This gives an open neighborhood U in X^*_{bw} such that

(i) $$|\theta(x^*) \leq 1 \quad : \, x^* \in U$$

To the set U we find a sequence $\{A_n\}$ as above. Let us enumerate the vectors in this sequence of finite sets by $x_1, x_2, \ldots$, i.e. start with the finite string of vectors in A_1, and so on. By the inductive construction of the A-sets we have $||x_n|| \to 0$ as $n \to \infty$. If x^* is a vector in X^* we associate the complex sequence

$$\ell(x^*) = \{x^*(x_n)\}$$

which tends to zero since $||x_n|| \to 0$ as $n \to \infty$. Then

$$x^* \mapsto \ell(x^*)$$

is a linear map from X^* into the Banach space $\mathbf{c}_0$. If

$$\max_n |x^*(x_n)| \leq 1$$

we have by definition $x^* \in A_n^0$ for each n. Choose a positive integer N so that $||x^*|| \leq n$. Thus entails that

$$x^* \in NS^* \cap A_N^0$$

From (ii) during the inductive construction of the A-.sets, the last set is contained in U. Hence $x^* \in U$ which by (i) gives $\theta(x^*)| \leq 1$. We conclude that θ yields a linear functional on on the image space of the ρ-map with norm one at most. The Hahn-Banach theoren gives $\lambda \in \mathbf{c}_0^*$ of norm one at most such that

$$\theta(x^*) = \lambda(\ell(x^*))$$

Next, by a wellknown result due to Banach the dual of $\mathbf{c}_0$ is ℓ^1. Hence there exists a sequence $\{\alpha_n\}$ in ℓ^1 such that

$$\theta(x^*) = \sum \alpha_n \cdot x^*(x_n)$$

In X we find the vector $x = \sum \alpha_n \cdot x_n$ and conclude that $\theta = \widehat{x}$ which proves the Krein-Smulian theorem.

5.9 A result by Pietsch .

The result below illustrates the usefulness of regarding various weak topologies. Let T be a bounded linear operator on a Banach space X and $\{p_n(z)\}$ is a sequence of polynomials with complex coefficients where $p_n(1) = 1$ for each n. We get the bounded operators

$$A_n = p_n(T) \quad : \, n = 1, 2, \ldots$$

Suppose that

(i) $$\lim_{n\to\infty} A_n(x) - A_n(T(x)) \xrightarrow{w} 0$$

hold for every $x \in X$ where the superscript w mens that we regard weak convergence. In addition to (i) we assume that for every $x \in X$, the sequence $\{A_n(x)\}$ is relatively compact with respect to the weak topology. Under these two assumptions one has:

5.9.1 Theorem. *For every $x \in X$ the sequence $\{A_n(x)\}$ converges weakly to a limit vector $B(x)$ where B is a bounded linear operator on X. Moreover B is an idempotent, i.e. $B = B^2$ and one has a direct sum decomposition*

$$X = \overline{(E - T)(X)} \oplus \ker(E - T)$$

where E is the identity operator on X and $\overline{(E-T)(X)}$ is the closure taken in the norm topology of the range of $E-T$. Finally, $\ker(E-T)$ *is equal to the range $B(X)$ while* $\ker(B) = \overline{(E-T)(X)}$.

The proof in § xx and gives an instructive lesson of "duality methods" while infinite dimensional normed spaces are considered.

6. Fredholm theory.

Throughout this section X and Y are Banach spaces with dual spaces X^* and Y^*.

6.1 Adjoint operators. Let $u\colon X \to Y$ be a bounded linear operator . The adjoint u^* is the linear operator from Y^* to X^* defined by

(1) $$u^*(y^*)\colon\, x \mapsto y^*(u(x)) \quad : \quad y^* \in Y \quad : x \in X$$

Exercise. Show that the Hahn-Banach theorem gives the equality of operator norms:

$$||u|| = ||u^*||$$

6.2 The operator $\bar{u}$. The bounded linear operator u has a kernel denoted by $\mathcal{N}(u)$ in X, often called the null space of u. Since u is bounded it is clear that the null space is and gives the Banach space $\frac{X}{\mathcal{N}(u)}$. Now one gets the induced linear operator

(6.2.1) $$\bar{u}\colon\, \frac{X}{\mathcal{N}(u)} \to Y$$

By construction $\bar{u}$ is an *injective* linear operator with the same range as u:

(6.2.2) $$u(X) = \bar{u}(\frac{X}{N_u})$$

6.3 The image of u^*. In the dual space X^* we get the subspace

(i) $$\mathcal{N}(u)^\perp = \{x^* \in X^* \quad : \; x^*(\mathcal{N}(u)) = 0\}$$

Consider a pair $y^* \in Y^*$ and $x \in \mathcal{N}(u)$. Then

$$u^*(y^*)(x) = y^*(u(x)) = 0$$

The proves the inclusiuon

(ii) $$u^*(Y^*) \subset \mathcal{N}(u)^\perp$$

Next, the Hahn-Banach theorem gives the canonical isomorphism

(iii) $$[\frac{X}{\mathcal{N}(u)}]^* \simeq \mathcal{N}(u)^\perp$$

Next, consider the linear operator $\bar{u}$ from (6.2.1). The canonical isomorphism (iii) gives a linear map

(iv) $$\bar{u}^*\colon\; Y^* \mapsto \mathcal{N}(u)^\perp$$

From this and (ii) we get the equality

(6.3.1) $$\text{Im}(\bar{u}^*) = \text{Im}(u^*)$$

where both sides appear as subspaces of $\mathcal{N}(u)^\perp$.

6.4 The closed range property

A bounded linear operator $u\colon X \to Y$ is said to have closed range if $u(X)$ is a closed subspace of Y. When this holds

$$\bar{u}\colon \frac{X}{\mathcal{N}(u)} \to u(X)$$

is a bijective map between Banach spaces. The Open Mapping Theorem implies that this is an isomorphism of Banach spaces. We use this to prove:

6.4.1 Proposition. *If u has closed range then u^* has closed range and one has the equality*

$$\text{Im}(u^*) = \mathcal{N}(u)^\perp$$

Proof. Using (6.3.1) we can replace u by $\bar{u}$ and assume that $u\colon X \to Y$ is injective. Then $\mathcal{N}U(u)^\perp = Y^*$ and there remains to show that u^* is surjective, i.e. that

(i) $$u^*(Y^*) = X^*$$

To prove (i) this we use the assumption that u has closed range which by the Open Mapping theorem gives a constant $c > 0$ such that

(ii) $$||u(x)|| \geq c \cdot ||x|| \quad : x \in X$$

If $x^* \in X^*$ the injectivity of u gives a linear functional ξ on $u(X)$ defined by

(iii) $$\xi(u(x)) = x^*(x)$$

Now (ii) entails that ξ belong to $u(X)^*$ with norm $\leq c \cdot ||x^*||$. The Hahn-Banach theorem applied to the subspace $u(X)$ of Y gives a norm preserving extension $y^* \in Y^*$ where (iii) entails that

$$u^*(y^*)(x) = y^*(u(x)) = x^*(x)$$

This means that $u^*(y^*) = x^*$ and the requested surjectivity follows.

6.4.2 A converse result. Let $u\colon X \to Y$ be a bounded linear operator and assume that u^* has closed range. Then we shall prove that u has closed range. To begin with we reduce the proof to the case when u is injective. For if $X_0 = \frac{X}{\ker(u)}$ we have the induced linear operator

$$u_0\colon X_0 \to Y$$

where $u_0(X_0) = u(X)$ and at the same time

$$u_0^*\colon Y \to X_0^*$$

where we recall that

$$X_0^* = \ker(u)^\perp = \{x^* \in X^*\colon x^*(\ker(u)) = 0\}$$

Here $u_0^*(Y^*)$ can be identified with the closed subspace $u^*(Y)$ in X_0^* which entails that we reduce the proof to the case when u is injective. From now on u is injective and consider the image space $u(X)$ whose closure yields a Banach space $\overline{u(X)}$. Here

$$u\colon X \to \overline{u(X)}$$

is a linear operator whose range is dense. Let us denote this operator with T. The adjoint

$$T^*\colon\ \overline{u(X)} \to X^*$$

and we have seen that

$$\overline{u(X)} = \frac{Y^*}{\ker(u^*)}$$

In particular the T^*-image is equal to $u^*(Y^*)$ and hence T^* has a closed range. The requuested closedness of $u(X)$ follows if we show that

$$T\colon X \to \overline{u(X)}$$

is surjective. Hence, we have reduced the proof of to the following:

6.4.3 Proposition. *Let $T\colon X \to Y$ be injective where $T(X)$ is dense in Y and T^* has closed range. Then $T(X) = Y$.*

Proof. Let y be a non-zero vector in Y and put

$$\{y\}^\perp = \{y^* \in Y^*\colon\ y^*(y) = 0\}$$

Consider also the image space

$$V = T^*(\{y\}^\perp)$$

Let us first show that

(i) $$V \neq X^*$$

To prove (i) we choose y^* in Y^* such that $y^*(y) = 1$ and get the vector $T^*(y^*)$. If $V = X^*$ this gives some $\eta \in \{y\}^\perp$ such that $T^*(y) = T^*(\eta)$. This means that

$$y^*(Tx) = \eta(Tx) \quad : x \in X$$

The density of $T(X)$ implies that $y^* = \eta$ which is a contradiction since $\eta(y) = 0$.

Next we show that V is closed in the weak-star topology on X^*. By the Krein-Smulian theorem the weak-star closedness follows if V is closed in X^*_{bw}. So let S be the unit ball in X and $\{\xi_n\}$ is a sequence in $V \cap S^*$ where $\xi_n \xrightarrow{w} x^*$ for some limit vector x^*. The Open Mapping Theorem applies to the operator $T^* \colon \{y\}^\perp \to X^*$ and gives a constant C and a sequence $\{y_n^* \in \{y\}^\perp\}$ such that $||y_n^*|| \leq C$ and $T^*(y_n) = \xi_n$. By weak-star compactness for bounded sets in $\{y\}^\perp$ we can pass to a subsequence and assume that y_n^* converge in $\{y\}^\perp$ to a limit vector y^* in the weak star topology. In particular we can apply this to every vector Tx with $x \in X$ and get

$$y^*(Tx) = \lim y_n^*(Tx)$$

This entails that

$$T^*(y^*)(x) = y^*(Tx) = \lim y_n^*(Tx) = \lim T^*(y_n)(x) = \lim \xi_n(x) = x^*(x)$$

Hence $x^* = T^*(y^*)$ which proves that V is weak-star closed in X^*.

We have proved that V is closed in the weak-star topology on X^* and not equal to the whole of X^*. This gives the existence a non-zero vector $x \in X$ such that $\widehat{x}(V) = 0$. So if $y^* \in Y^*$ is such that $y^*(y) = 0$ we have by definition $T^*(y^*) \in V$ and obtain

(ii) $$y^*(Tx) = \widehat{x}(T^*(y^*)) = y^*(Tx) = 0$$

Hence we have the implication:

(iii) $$y^*(y) = 0 \implies y^*(Tx) = 0$$

Finally, since T is injective we have $Tx \neq 0$ and then (iii) gives a complex number α such that $y = \alpha \cdot T(x)$, i.e. the vector y belongs to $T(X)$ as requested.

6.5 Compact operators.

A linear operator $T\colon X \to Y$ is compact if the the image under T of the unit ball in X is relatively compact in Y. An equivalent condition for T to be compact is that if $\{x_k\}$ is an arbitrary sequence in the unit ball $B(X)$ then there exists a subsequence of $\{T(x_k)\}$ which converges to some $y \in Y$.

6.5.1 Exercise. Let $\{T_n\}$ be a sequence of compact operators which converge to another operator T, i.e.

$$\lim_{n\to\infty} ||T_n - T|| = 0$$

where we employ the operator norm on the Banach space $L(X,Y)$. Verify that T also is a compact operator.

6.5.2 Theorem. *A bounded linear operator T is compact if and only if its adjoint T^* is compact.*

Proof. Assume first that T is compact and let B be the unit ball in X. From the material in § xx this entails that for each positive integer N there exists a finite set F_N in B which is N^{-1}-dense in $T(B)$, i.e. for each $y \in T(B)$ there exists $x \in F_N$ and

(i) $$||T(x) - y|| < N^{-1}$$

Let us then consider a sequence $\{y_n^*\}$ in the unit ball of Y^*. By the standard diagonal procedure we find a subsequence $\{\xi_j = y_{n_j}^*\}$ such that

(ii) $$\lim_{j\to\infty} \xi_j(Tx) \quad : x \in \bigcup_{N\geq 1} F_N$$

Next, if $x \in B$ and $\epsilon > 0$ we choose N so large that $N^{-1} < \epsilon/3$. Since (i) hold for the finite set of points in F_N there exists an integer w such that

(iii) $$|\xi_j(Tx) - \xi_i(Tx)| < \epsilon/3 \quad : j, i \geq w$$

hold for each $x \in F_N$. Since the ξ-vectors have unit norm it follows from (i) and the triangle inequality that (iii) hold for each $x \in B$. Above $\epsilon > 0$ is arbitrary small which entials that $\{\xi_j(Tx)\}$ is a Cauchy seqence of complex numbers for every $x \in B$ and then the same hold for each $x \in X$. Since Cauchy sequences of complex numbers converge there exist limits:

(iv) $$\lim_{j\to\infty} \xi_j(Tx) \quad : x \in X$$

It is clear that these limits values are linear with respect to x. So by the construction of T^* there exist the pointwise limits

(v) $$\lim_{j\to\infty} T^*\xi_j(x) \quad : x \in X$$

which means that there exists $x^* \in X^*$ such that

(vi) $$x^*(x) = \lim_{j\to\infty} T^*\xi_j(x) \quad : \in x \in X$$

The requested compactness of T^* follows if we have proved that the pointwise convergence in (vi) is uniformwhen x stays in B, i.e. that

(vii) $$\lim_{j\to\infty} ||T^*\xi_j - x^*|| = 0$$

To prove that (vi) gives (viii) we apply the Arzela-Ascoli theorem which shows that pointwise convergence on the relatively compact set $T(B)$ of the equicontinuous family of functions $\{\xi_j\}$ gives the uniform convergence in (vii).

Above we proved that if T is compact, so is T^*. To prove the the opposed implication we employ the bi-dual space X^{**}. From the above the compactness of T^* implies that T^{**} is compact. At this stage the reader can check that the restriction of T^{**} to the the closed subspace $j(X)$ of X^{**} under the bi-dual embedding is compact which entails that T is compact.

6.5.3 Operators with finite dimensional range. Suppose that the image space $u(X)$ has a finite dimension N and choose an N-tuple $x_1, \ldots, x_N$ in X such that $\{u(x_k)\}$ is a basis for $u(X)$. In Y^* we can find an N-tuple $y_1^*, \ldots, y_N^*$ such that

$$j \neq k \implies y_j^*(u(x_k)) = 0 \quad \text{and} \quad y_j^*(u(x_j)) = 1$$

So if u^* is the adjoint operator then

$$u^*(y_j^*)(x_k) = \text{Kronecker's delta function}$$

If $y^* \in Y^*$ we can therefore find an N-tuple of complex numbers such that

$$u^*(y^* - \sum c_j \cdot y_j^*)(x_k) = 0 \quad : k = 1, \ldots, N$$

This entails that the vector $y^* - \sum c_j \cdot y_j^*)$ belongs to the kernel of u^* and hence the range of u^* is the N-dimensional subapace of X^* generated by $\{u^*(y_j^*)\}$. in particular the adjoint u^* has finite dimensional range.

6.6 Compact pertubations.

We shall prove the following:

6.6.1 Theorem. *Let $u \colon X \to Y$ be an injective operator with closed range and $T \colon X \to Y$ a compact operator. Then the kernel of $u + T$ is finite dimensional and $u + T$ has closed range.*

Proof. To begin with the Open Mapping Theoren gives gives a poaitive number c such that

(i) $$||u(x)|| \geq c \cdot ||x||$$

Now we show that $\mathcal{N}(u + T)$ is finite dimensional. By the result in § xx it suffices to show that the set

$$V = \mathcal{N}(u + T) \cap B(X)$$

is relatively compact in the norm topology on X, where we recall that $B(X)$ is the closed unit ball in X. Let $\{x_n\}$ be a sequence in V. Since T is compact there is a subsequence $\{\xi_j = x_{n_j}\}$ and some vector y such that $\lim T\xi_j = y$. Since $u(\xi_j) = -T(\xi_j)$ it follows that $\{u(\xi_j)\}$ is a Cauchy sequence and (i) entails that $\{\xi_j\}$ is a Cauchy sequence and hence has a limit vector. This proves that V is relatively compact.

The closedness of Im$(u+T)$. Since $\mathcal{N}(u+T)$ is finite dimensional the result in § xx gives a direct sum decomposition

$$X = \mathcal{N}(u + T) \oplus X_*$$

Here $(u + T)(X) = (u + T)(X_*)$ so it suffices that the last image is closed and we can restrict both u and T to X_* where T_* again is compact. Hence we may assume that the operator $u + T$ is *injective*, i.e. that

(i) $$\mathcal{N}(u + T) = \{0\}$$

If a vector y belongs to the closure of Im$(u + T)$ there exists a sequence $\{x_n\}$ in X such that

(ii) $$\lim\,(u + T)(x_n) = y$$

Suppose first that the norms of $\{x_n\}$ are unbounded. Passing to a subsequence if necessary we may assume that $||x_n|| \to \infty$. With $\xi_n = \frac{x_n}{||x_n||}$ it follows that

(iii) $$\lim\, u(\xi_n) + T(\xi_n) = 0$$

Now $\{\xi_n\}$ is bounded and since T is compact we can pass to another subsequence and assume that $T(\xi_n) \to y$ holds for some $y \in Y$. Then (ii) entails that $u(\xi_n)$ also has a limit and (i) implies that $\{\xi_n\}$ converges in X to a limit vector ξ_*. Here $\xi_* \neq 0$ since $||\xi_n|| = 1$ for all n. Moreover, (iii) entails that $u(\xi_*) + T(\xi_*) = 0$. This gives a contradiction since (i) was assumed.

Hence $\{x_n\}$ is a bounded sequence in (ii) and since T is compact we can pass to a subsequence and assume that $T(x_n) \to \eta$ holds for some $\eta \in Y$, and (ii) entails that

(iv) $$\lim\, u(x_n) = y - \eta$$

Finally, by assumption u is injective and has a closed range, and then the Open Mapping Theorenm together with (iv) entail that $\{x_n\}$ converges to a vector ξ. Passing to the limit in (ii) we get

$$u(\xi) + T(\xi) = y$$

Hence y belongs to $\mathrm{Im}(u + T)$ and Theorem 6.6.1 is proved.

6.7 Spectra and resolvents of compact operators.

Let $T \colon X \to X$ be a compact operator. For each complex number $\lambda \neq 0$ we set

$$\mathcal{N}(\lambda) = \{x \colon Tx = \lambda \cdot x\}$$

6.7.1 Theorem. *The set of non-zero λ for which $N(\lambda)$ contains a non-zero vector is discrete.*

Proof. Suppose that there exists a non-zero cluster point $\lambda_0 \neq 0$, i.e. a sequence $\{\lambda_n\}$ where $\lambda_n \to \lambda_0$ and for each n a non-zero vector $x_n \in \mathcal{N}(\lambda_n)$. Since the numbers $\{\lambda_n\}$ are distinct and $T(x_n) = \lambda_n \cdot x_n$ hold one easily verifies that the vectors $\{x_n\}$ are linearly independent. By the result in § xx we find a sequence of unit vectors $\{y_n\}$ satisfying the separation in Theorem § xx and

$$\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$$

hold for each n. Now

(i) $$T(\lambda_n^{-1} y_n) - \lambda_m^{-1} T(y_m) = y_n - y_m \in \{y_1, \ldots, y_{n-1}\} \quad : n > m$$

At the same time the separation gives

(ii) $$||y_n - y_m|| \geq 1 \quad : n > m$$

Since λ_n converge to the non-zero number λ_0 this entails that $\{\lambda_n^{-1} y_n\}$ is a bounded sequence and from (i-ii) we see that $\{T(\lambda_n^{-1} y_n)\}$ cannot contain a convergent subsequence. This contradiction proves Theorem 6.7.1

6.7.2 Theorem. *The spectrum of a compact operator T is discrete outside the origin.*

Proof. Consider a non-zero $\lambda_0 \in \sigma(T)$. By Schauder's result in (xx) the adjoint T^* is also compact. Hence Theorem 6.7.1 applies to T and T^* which gives a small punctured disc $\{0 < |\lambda - \lambda_0| < \delta\}$ such that $\lambda \cdot E - T$ and $\lambda \cdot E^* - T^*$ both are injective when λ belongs to the punctured disc. Next, $\lambda \cdot E - T$ has a closed range by Theorem 5.X. If it is a proper subspace of X find a non-zero $x^* \in X^*$ which vanishes on this range. The construction of T^* entails that $T^*(x^*) = \lambda \cdot x^*$. But this was not the case and hence $\lambda \cdot E - T$ is surjective which shows that λ is outside $\sigma(T)$ and finishes the proof of Theorem 6.7.2.

6.7.3 Spectral projections. Let λ_0 be non-zero in $\sigma(T)$. By Theorem 6.7.2 it is an isolated poiunt in $\sigma(T)$ which gives the spectral projection $E_T(\lambda_0)$ where we recal, from § xx that this operator commutes with T. So if $V = E_T(\lambda_0)(X)$ then T restricts to a bounded linear operator on V denoted by T_V where we recall from § xx that the spectrum of T_V is reduced to the singleton set $\{\lambda_0\}$. Since $\lambda_0 \neq 0$ it means that T_V is an invertible and compact operator on V. So by the result in § xx V is finite dimensional. This finiteness and linear algebra applied to T_V gives an integer $m \geq 1$ such that

(6.7.3.1) $$(Tx - \lambda_0 x)^m = 0 \quad : x \in V$$

6.8 Fredholm operators.

A bounded linear operator $u\colon X \to Y$ is called a Fredholm operator if it has closed range and the kernel and the cokernel of u are both finite dimensional. When u is Fredholm its index is defined by:

$$\mathfrak{ind}(u) = \dim N_u - \dim\Big[\frac{Y}{u(X)}\Big]$$

6.8.1 Theorem. *Let u be of Fredholm type and $T\colon X \to Y$ a compact operator. Then $u + T$ is Fredholm and one has the equality*

$$\mathfrak{ind}(u) = \mathfrak{ind}(u + T)$$

The proof requires several steps where the crucial point is to regard the case $X = Y$ and a compact pertubation of the identity operator. Thus we begin with:

6.8.2 Theorem. *Let $T\colon X \to X$ be compact. Then $E - T$ is Fredholm and has index zero.*

Proof. Apply (6.7.3) with $\lambda_0 = 1$ which gives the decompostion

(i) $$X = E_T(1)(X) \oplus (E - E_T(1))(X)$$

Theorem 6.6.1 implies that $E - T$ has closed range. Next, from the spectral decomposition in (6.7.3) it follows that $E - T$ restricts to a bijective operator on $(E - E_T(1))(X)$ which by (i) entails that the codimension of $(E - T)(X)$ is at most the dimension of the finite dimensional vector space $V = E_T(1)(X)$. Moreover, the kernel of $E - T$ is finite dimensional by Theorem 6.6.1.

Hence we have proved that $E - T$ is a Fredholm operator and there reamins to show that its index is zero. To obtain this we notice again that the decomposition (i) impies that this index is equal to that of the restricted operator $E - T$ to the finite dimensional vector space V. Finally, recall from linear algebra that the index of a linear operator on a finite dimensional vector space always is zero which finishes the proof.

6.8.3 The general case. Consider first the case when the kernel of the Fredholm operstor u is zero. Now there exists a finite dimensional subspace W of Y such that

(*) $$Y = u(X) \oplus W$$

where $u\colon X \to u(X)$ is an isomorphisn between the Banach spaces X and $u(X)$ which gives the existence of a bounded inverse operator

$$\phi\colon u(X) \to X$$

So here $\phi \circ u$ is the identity on X. Next, given a compact operator T we consider the projection operator $\pi\colon Y \to u(X)$ whose kernel is W and notice that $\pi \circ T$ is a compact operator. Now we can regard the operator

$$u + \pi \circ T\colon X \to u(X)$$

From the above and Theorem 6.8.2 the reader can verify that this Fredholm operator has index zero. Next, we notice that

(1) $$\mathfrak{ind}(u) = -\dim W$$

We have also the operator

$$T_* = (E_Y - \pi) \circ T\colon X \to W$$

The direct sum decomposition (*) entails that

(2) $$\ker(u + T) = \ker(u + \pi \circ T) \cap \ker T_* \quad \& \quad \frac{Y}{(u + T)(X)} = \frac{Y}{(u + \pi \circ T)(X)} \oplus \frac{W}{T_*(X)}$$

From (1-2) we leave it as an exercise to show that the index of $u + T$ is equal to that if u given by (1).

Above we treated the case when u is injective. In general, since $\mathcal{N}(u)$ is finite dimensional one has a decompostion

$$X = \mathcal{N}(u) \oplus X_*$$

whewre the restricted operator $u\colon X_* \to Y$ is injective. For a given compact operator T we can also consider the restricted compact operator

$$T_*\colon X_* \to Y$$

and the previous special case gives the equality

$$\mathfrak{ind}(u_* + T_*) = \mathfrak{ind}(u_*) \tag{3}$$

Next, let $\pi\colon X \to \mathcal{N}(u)$ be the projection with kernel X_* which gives

$$T = T_* + T \circ \pi$$

At this stage we leave it as an exercise to verify that (3) gives the requested index formula

$$\mathfrak{ind}(u + T) = \mathfrak{ind}(u)$$

7. Calculus on Banach spaces.

Let X and Y be two real Banach spaces and $g\colon X \to Y$ some map. Here g is not assumed to be linear. Since the Banach spaces are real the dual space Y^* consists of continuous $\mathbf{R}$-linear maps from Y into $\mathbf{R}$ and every $y^* \in Y^*$ yields the real-valued function $y^* \circ g$ on X. With x_0 kept fixed we can impose the condition that there exist limits

$$(1) \qquad \lim_{\epsilon \to 0} \frac{y^* \circ g(x_0 + \epsilon \cdot x) - y^*(g(x_0))}{\epsilon}$$

for each vector $x \in X$. These limits resemble directional derivatives in calculus and we can impose the extra condition that the limits above depend linearly upon x. Thus, assume that each $y^* \in Y$ yields a linear form $\chi(y^*)$ on X such that (1) is equal to $\chi(y^*)(x)$ for every $x \in X$. When this holds it is clear that

$$(2) \qquad y^* \mapsto \chi(y^*)$$

is a linear mapping from Y^* into X^*. When both Y and X are finite dimensional real vector spaces this linear operator corresponds to the usual Jacobian in calculus. In the infinite-dimensional case it is not always true that (2) is continuous with respect to the norms on the dual spaces. As an extra condition for differentaiblity at x_0 we impose the condition that there exists a constant C such that

$$(3) \qquad ||\chi(y^*)|| \leq C \cdot ||y^*||$$

When (3) holds we have a bounded linear operator $\chi\colon Y^* \to X^*$ associated to g and the given point $x_0 \in X$. It may occur that χ is the adjoint of a bounded linear operator from X into Y which means that there exists a bounded linear operator $L\colon X \to Y$ such that

$$(*) \qquad \lim_{\epsilon \to 0} \frac{||g(x_0 + \epsilon \cdot x) - g(x_0) - \epsilon \cdot L(x)||}{\epsilon} = 0$$

Concerning the passage to the limit the weakest condition is that it holds pointwise, i.e. (*) holds for every vector x. A stronger condition is to impose that the limits above hold uniformly which means that

$$(**) \qquad \lim_{\epsilon \to 0} \max_{x \in B(X)} \frac{||g(x_0 + \epsilon \cdot x) - g(x_0) - \epsilon \cdot L(x)||}{\epsilon} = 0$$

where the maximum is taken over X-vectors with norm ≤ 1. In applications the the condition (**) is often taken as a definition for g to be differentiable at x_0 and the uniquely determined linear map L above is denoted by $D_g(x_0)$ and called the differential of g at x_0. If g is a map from some open subset Ω of X with values in Y we can impose the condition that g is differentiable at each $x_0 \in \Omega$ and add the condition that $x \mapsto D_g(x)$ is continuous in Ω where the values are taken in the Banach space of continuous linear maps from X into Y. When this holds we get another map $x \to D_g(x)$ from Ω into $\mathcal{L}(X, Y)$ and can impose the condition that it also is differentiable in the strong sense above. This leads to the notion of k-times continuosly differentiable maps from a Banach space into another for every positive integer k.

Remark. We shall not dwell upon a general study of differentiable maps between Banach spaces which is best illustrated by various examples. For a concise treatment we refer to Chapter 1 in Hörmander's text-book [PDE:1] which contains a proof of the implicit function theorem for differentiable maps between Banach spaces in its most general set-up.

7.1 Line integrals

Let Y be a Banach space. Consider a continuous map g from some open set Ω in $\mathbf{C}$ with values in Y. Let $t \mapsto \gamma(t)$ be a parametrized C^1-curve whose image is a compact subset of Ω. Then there exists the Y-valued line integral

$$(*) \qquad \int_\gamma g \cdot dz = \int_0^T g(\gamma(t)) \cdot \dot{\gamma}(t) \cdot dt$$

The evaluation is performed exactly as for ordinary Riemann integrals, Namely, one uses the fact that the Y-valued function

$$t \mapsto g(\gamma(t))$$

is uniformly continuous with respect to the norm on Y, i.e. the Bolzano-Weierstrass theorem gives:

$$\lim_{\epsilon \to 0} \max_{|t-t'| \leq \epsilon} ||g(t) - g(t')|| = 0$$

Then (*) is approximated by Riemann sums and since Y is complete this gives a unique limit vector in Y.

7.2 Uniformly convex Banach spaces.

From now on X is a complex Banach space. One says that X is uniformly convex if there corresponds to each $0 < \epsilon < 1$ some $\delta(\epsilon)$ tending to zero with ϵ such that

$$(7.2.1) \qquad \frac{||x+y||}{2} \geq 1 - \epsilon \implies ||x-y|| \leq \delta(\epsilon)$$

for all pair of vectors of norm one at most. This condition was introduced by Clarkson in the article [Clarkson] from 1936.

Exercise. Show that in a uniformly convex Banach space each closed convex set contains a unique vector of minimal norm.

7.3 Directional derivatives.

Let $p \in X$ be a non-zero vector. If $x \neq 0$ is another vector we get the function of a real variable a defined by:

$$a \mapsto ||p + ax|| - ||p||$$

We say that a directional x-derivative exists at p if there exists the limit

$$\lim_{a \to 0} \frac{||p + ax|| - ||p||}{a} = D_p(x)$$

Notice that in this limit a can tend to zero both from the negative and the positive side. The following result is due to Clarkson:

7.3.1 Theorem *Let p be a non-zero vector in X such that the directional derivatives above exist for every $x \in X$. Then $x \mapsto D_p(x)$ is $\mathbf{R}$-linear. Moreover*

$$|D_p(x)| \leq ||x||$$

hold for every x and $D_p(p) = 1$.

Exercise. Prove Clarkson's Theorem.

7.4 Conjugate vectors.

For brevity we say that X is differentiable if directional derivatives exist for all pairs p, x. Let S be the unit sphere in X and S^* the unit sphere in X^*. A pair $x \in S$ and $x^* \in S^*$ are said to be conjugate if

$$x^*(x) = 1$$

7.4.1 Theorem. *Let X be uniformly convex and differentiable. Then every $x \in S$ has a unique conjugate given by $x^* = D_x(x)$ and the map $x \to x^*$ from S to S^* is bijective.*

Exercise. Prove this result.If necessary, consult Clarkson's article or some text-book.

7.5. Duality maps.

Assume that X be uniformly convex and differentiable. As usual S denotes the unit sphere in X. Let $\phi(r)$ be a strictly increasing and continuous function on $r \geq 0$ where $\phi(0) = 0$ and

$$\lim_{r \to +\infty} \phi(r) = +\infty$$

Each vector in X is of the form $r \cdot x$ with $x \in S$ while $r \geq 0$. Clarkson'a conjugate in Theorem 7.4.1 yields a function $\mathcal{D}_\phi$ from X into X^* defined by

$$\mathcal{D}_\phi(rx) = \phi(r) \cdot x^* \quad \text{when} \quad x \in S \quad \text{and} \quad r \geq 0$$

If C is a closed subspace in X we put:

$$C^\perp = \{\xi \in X^* : \xi(C) = 0\}$$

7.5.1 Theorem. *For each closed and proper subspace $C \neq X$ the following hold: For every pair of vectors $x_* \in X$ and $y^* \in X^*$ the intersection*

$$\mathcal{D}_\phi(C + x_*) \cap \{C^\perp + y*\}$$

is non-empty and consists of a single point in X^*.

Proof. Introduce the function

$$\Phi(r) = \int_0^r \phi(s)\, ds$$

Since ϕ is strictly increasing, Φ is a strictly convex function and since $\phi(r) \to +\infty$ we have $\frac{\Phi(r)}{r} \to +\infty$. Consider the functional defined on $C + x_*$ by

$$F(x) = \Phi(||x||) - y^*(x)$$

If $||x|| = r$ we have

$$F(x) \geq \Phi(r) - r||y^*||$$

The right hand side is a strictly convex function of r which tends to $+\infty$ and is therefore bounded below. Hence there exists a number

$$\delta = \inf_{x \in C + x_*} F(x)$$

Let $\{x_n\}$ be a minimizing sequence for F. The strict convexity of Φ entails that the norms $\{||x_n||\}$ converge to some finite limit α. Since the set $C + x_*$ is convex we have

$$F(\frac{x_n + x_m}{2}) \geq \delta$$

Next, the convexity of ϕ entails that

$$0 \leq \frac{1}{2}\big[\Phi(||x_n||) + \Phi(||x_m||) - \Phi(||\frac{x_n + x_m}{2}||) =$$

$$\frac{1}{2}\big(F(x_n) + F(x_m)\big) - F(\frac{x_n + x_m}{2})$$

Since $\{x_n\}$ is F-minimizing the last terms tend to zero when n and m increase which gives

$$\lim_{n,m \to \infty} \Phi(||\frac{x_n + x_m}{2}||) = \Phi(\alpha) \tag{1}$$

where we recall that

$$\alpha = \lim_{n \to \infty} ||x_n||$$

Now (1) and the strict convexity of Φ gives

$$\lim_{n,m \to \infty} ||\frac{x_n + x_m}{2}||) = \alpha$$

The unifom convexity entails that $\{x_n\}$ is a Cauchy sequence which gives a limit point p where $F(p) = \delta$ and consequently

$$F(p + tx) - F(p) \geq 0 \quad : \quad x \in C \tag{2}$$

Since p belongs to $C + x_*$ the existence part in Theorem 7.14 follows if we have proved the inclusion

$$\mathcal{D}_\phi(p) \in C^\perp + y^* \tag{3}$$

To get (3) we use that the Banach space is differentiable and since the Φ-function as a primitive of a continuous function is of class C^1 one has

$$\Phi(||p + tx||) - \Phi(||p||) = \Phi'(||p||) \cdot \mathfrak{Re}\, tD_p(x) + o(|t|) = \phi(\alpha) \cdot \mathfrak{Re}\, tD_p(x) + o(|t|)$$

where t is a small real or complex number. Together with (2) this gives

$$\phi(\alpha) \cdot \mathfrak{Re}\, tD_p(x) + o(|t|) \geq \mathfrak{Re}\, y^*(tx) \quad : \quad x \in C$$

By linearity it is clear that this implies that

$$\phi(\alpha) \cdot D_p(x) = y^*(x)$$

This means precisely that the linear functional $y^* - \phi(\alpha) \cdot D_p$ belongs to $C^\perp$, or equivalently that

$$\phi(||p||) \cdot D_p \in C^\perp + y^*$$

Finally we recall from (7.13) that

$$\mathcal{D}_\phi(p) = \phi(||p||) \cdot D_p(p)$$

and the requested inclusion (3) is proved.

The uniqueness part. Above we proved the existence of at least one point in the intersection from Theorem 7.12. The verification that this set is reduced to a single point is left as an exercise to the reader.

§ 8. Locally convex spaces

Introduction. We expose facts about real vector spaces equipped with a locally convex topology. The definition is given in § 2. A crucial result is the Hahn-Banach theorem for lcoally convex spaces in § 1. It has several important consequences such as Hörmander's result in Theorem 3.6.

§ 1. Convex sets and their ρ-functions.

Let E be a real vector space. A convex set U which contains the origin is said to be absorbing if there for each vector $x \in E$ exists some real $s > 0$ such that $s \cdot x \in U$. It may occur that the whole line $\mathbf{R}x$ is contained in U, and then we say that x is fully absorbed by U. The convexity of U entails that the set of fully absorbed vectors is a linear subspace of E which we denote by $\mathcal{L}_U$.

1.1 The function ρ_U. Let x be a non-absorbed vector x. Then there exists a positive real number

$$\mu(x) = \max\{s : sx \in U\}$$

If x is absorbed we put $\mu(x) = +\infty$ and for every non-zero vector x we set

$$\rho_U(x) = \frac{1}{\mu(x)}$$

it is clear that if $x \in U$ then $\mu(x) \geq 1$ and hence $\rho_U(x) \leq 1$. Notice that we also have

$$\rho_U(x) = \min\{s : \ x \in s^{-1}U\}$$

1.2 Exercise. Show that the convexity of U entails that ρ_U satisfies the triangle inequlity

$$\rho_U(x_1 + x_2) \leq \rho_U(x_1) + \rho_U(x_2) \tag{1.2.1}$$

for all pairs of vectors in E. Moreover, check also that $\rho_U(x) = 0$ if and only if x belongs to $\mathcal{L}_U$ and that ρ_U is positively homogeneous, i.e. the equality below holds when a is real and positive:

$$\rho_U(ax) = a\rho_U(x) \quad : a > 0 \tag{1.2.2}$$

1.3 The Hahn-Banach theorem. Keeping U fixed we set $\rho(x) = \rho_U(x)$. An $\mathbf{R}$-linear map λ from E to the 1-dimensional real line is majorised by ρ if

$$\lambda(x) \leq \rho(x) \tag{1.3.1}$$

hold for every vector x. More generally, let E_0 be a subspace of E and $\lambda_0 \colon E_0 \to \mathbf{R}$ a linear map such that (1.3.1) hold for vectors in E_0. Then there exists a linear map $\lambda \colon E \to \mathbf{R}$ which extends λ_0 and is again majorised by ρ. This result was proved in § 4.A.4.

8.2 Locally convex topologies.

Denote by $\mathcal{C}_E$ the family of convex sets U as in § 1. Let $\mathfrak{U} = \{U_\alpha\}$ be a family in $\mathcal{C}_E$ such that

$$\bigcap \mathcal{L}_{U_\alpha} = \{0\}$$

i.e. the intersction is reduced to the origin. Now there exists a topology on E where a basic for open neighborhoods of the origin consists of sets:

$$\cap \{\rho_{U_{\alpha_i}}(x) < \epsilon\} \tag{1}$$

where $\epsilon > 0$ and $\{\alpha_1, \ldots, \alpha_k\}$ is a finite set of indices defining the U-family. If x_0 is a vector in XS, then a basis for its open neihghborhoods are given by sets of the for $x_0 + U$ where U is a set from (1). In general, a subset Ω in E is open if there to eah $x_0 \in \Omega$ exists some U from (1) such that $x_0 + U \subset \Omega$. it is clear that this gives a topology and (1) entaills that it is separated, i.e. a Huasdorff topology on E. Notice also that eqch set in 82) is convex. One therefore refers to a locally con vex topology on E.

2.1 Remark. The locally convex topology above depends upon the family $\mathfrak{U}$-toplogy. Its topology is not changed if we enlarge the family to consist of all finite intersection of its convex subsets. When this has been done we notice that if $U_1, \ldots, U_n$ is a finite family in $\mathfrak{U}$ then the norm defined by $U = U_1 | \cap \ldots \cap U_n$ is stronger than the individual ρ_{U_i}-norms. Hence a fundamental system of neighborhoods consists of single ρ-balls:

$$\Omega = \{\rho_U < \epsilon\} \quad : U \in \mathfrak{U}$$

2.2 The dual space E^*. Let E be equipped with a locally convex $\mathfrak{U}$-topology. As above $\mathfrak{U}$ has been enlarged so that the balls above give a basis for neighborhoods of the origin. A linear functional ϕ on E is $\mathfrak{U}$-continuous if there exists some $U \in \mathfrak{U}$ and a constant C such that

$$|\phi(x)|| \leq C \cdot \rho_U(x)$$

2.3 Closed half-spaces. To each pair $\phi \in E^*$ and a real number a one assigns the closed half-space

$$H = \{x \in X : \ \phi(x) \leq a\}$$

Notice that $a < 0$ can occur in which case H does not contain the origin.

2.4 The separation theorem. *Each closed convex set K in E is the intersection of closed half-spaces.*

2.5 Exercise. Show (2.4) using the Hahn-Banach theorem.

Next, let K_1 and K_2 be a pair of closed and disjoint convex sets. Then they can be spearated by a hyperplane. More precisley, there exists some $\phi \in E^*$ and a positive number δ such that

$$\max_{x \in K_1} \ \phi(x) + \delta \leq \min_{x \in K_2} \ \phi(x) \tag{2.6}$$

Again we leave the proof as an exercise to the reader.

8.3. Support functions of convex sets.

Let E be a locally convex space as above. Vectors in E are denoted by x, while y denote vectors in E^*. To each closed and convex subset K of E we define a function $\mathcal{H}_K$ on the dual E^* by:

$$\mathcal{H}_K(y) = \sup_{x \in K} \ y(x)$$

Notice that $\mathcal{H}_K$ take values in $(-\infty, +\infty]$, i.e. it may be $+\infty$ for some vectors $y \in E^*$. For example, let $K = \{\mathbf{R}^+ x_0$ be a half-line. Then $\mathcal{H}_K(y) = +\infty$ when $y(x_0) > 0$ and otherwise zero. So here the range consisists of 0 and $+\infty$. It is clear that

$$\mathcal{H}_K(sy) = s\mathcal{H}_K(y)$$

hold when s is a positive real number, i.e $\mathcal{H}_K$ is positively homogeneous.

3.1 Exercise. Show that the convexity of K entails that

$$\mathcal{H}_K(y_1 + y_2) \leq \mathcal{H}_K(y_1) + \mathcal{H}_K(y_2)$$

for each pair of vectors in E^*.

3.2 Upper semi-continuity. For each fixed vector $x \in E$ the function

$$y \mapsto y(x)$$

is weak-star continuous on E^*. Since the supremum function attached to an arbitrary family of weak-star continuoes functions is upper semi-continuous, it follows that $\mathcal{H}_K$ is upper semi-contiuous.

3.4 Exercise. Let K and K_1 be a pair of closed convex sets such that $\mathcal{H}_K = \mathcal{H}_{K_1}$. Show that this entails that $K = K_1$. The hint is to use the separation theorem.

3.5 The class $\mathcal{S}(E)$. It consids of all all upper semi-continuous functions G on E^* with values in $(-\infty, +\infty]$ which satisfy (x) and (xx). The next result was proved by Hörmander in the article *Sur la fonction d'appui des ensembles convexes dans un espaces localementt convexe* [Arkiv för mat. Vol 3: 1954].

3.6 Theorem. *Each $G \in \mathcal{S}(E)$ is of the form $\mathcal{H}_K$ for a unique closed convex subset K in E.*

Remark. As pointed out by Hörmander in [ibid] this result is closely related to earlier studies by Fenchel in the article *On conjugate convex functions* Canadian Journ. of math. Vol 1 p. 73-77) where Legendre transforms are studied in infinite dimensional topological vector spaces. The novely in Theorem 3.3 is the generality and we remark that various separation theorems in text-books dealing with notions of convexity are easy consequences of Theorem 5.C.2.

Proof of Theorem 3.6 Put $F = E \oplus \mathbf{R}$ which is a new vector space where the 1-dimensional real line is added. It dual space $F^* = E^* \oplus \mathbf{R}$. We are given $G \in \mathcal{S}(E)$ and put

(i) $$G_* = \{(y, \eta) \in E^* \oplus \mathbf{R} \quad : \; G(y) \leq \eta\}$$

Condition in (*) entails that G_* is a convex cone in F^* and the semi-continuous hypothesis on G implies that G_* is closed with respect to the weak-star toplogy on F^*. Next, in F we define the set

(ii) $$G_{**} = \{(x, t) \in E \oplus \mathbf{R}^+ : \; y(x) \leq \eta t : \; (y, \eta) \in G_*\}$$

This gives a set $\widehat{C}$ in F^* which consists of vectors (y, η) such that

$$\max_{(x,t) \in G_{**}} \; y(x) - \eta t \leq 0$$

It is clear that $G_* \subset \widehat{C}$. Now we prove the equality

(*) $$G_* = \widehat{C}$$

To get (*) we use Theorem 2.4. Namely, since the two sets in (*) are weak-star closed a strict inequality gives a separating vector $(x_*, t_*) \in E$, i.e. there exists $(y_*, \eta_*) \in \widehat{C}$ and a real number α such that

(iv) $$y_*(x_*) - \eta_* t_* > \alpha \quad \text{and} \quad (y, \eta) \in D_K \implies y(x_*) - \eta t_* \leq \alpha$$

Since G_* contains $(0, 0$ we have $\alpha \leq 0$. and since it also is a cone the last implication gives $(x_*, t_*) \in G_{**}$. Now the construction of $\widehat{C}$ in (iii) contradicts the strict inequality in the left hand side of (iv). Hence there cannot exist a separating vector and (*) follows.

Next, in E we consider the convex set

$$K = \{x : (x, 1) \in G_{**}\}$$

Using (*) the reader can check that

$$\mathcal{H}_K(y) = G(y)$$

for all $y \in E^*$ which proves that G has the requested form. The uniqueness of K follows from Exercise 3.4.

3.7 The case of normed spaces. If X is a normed vector space Theorem 3.6 leads to a certain isomorphism of two families. Denote by $\mathcal{K}$ the family of all convex subsets of E which are closed with respect to the norm topology. A topology on $\mathcal{K}$ is defined when we for each $K_0 \in \mathcal{K}$ and $\epsilon > 0$ declare an open neighborhod

$$U_\epsilon(K_0) = \{K \in \mathcal{K} : \; \text{dist}(K, K_0) < \epsilon\}$$

where the norm defines the distance between K and K_0 in the usual way. Denote by $\mathfrak{H}$ the family of all functions G on E^* which satisfy (*) in 5.B.1 and are continuous with respect to the norm topology on E^*. A subset M of $\mathfrak{H}$ is equi-continuous if there to each $\epsilon > 0$ exists $\delta > 0$ such that

$$||y_2 - y_1|| < \delta \implies ||G(y_2) - G(y_1)|| < \epsilon$$

for every $G \in M$ and all pairs y_1, y_2 in E^*. The topology on $\mathfrak{H}$ is defined by uniform convergence on equi-continuous subsets.

3.8 Theorem. *If E is a normed vector space the set-theoretic bijective map $K \to \mathcal{H}_K$ is a homeomorphism when $\mathcal{K}$ and $\mathfrak{H}$ are equipped with the described topologies.*

3.9 Exercise. Deduce this result from Theorem 3.6

§ 9. Fixed point theorems.

A compact topological space S has the fixed point property if every continuous map $T\colon S \to S$ has at least one fixed point. An example is the closed unit ball in $\mathbf{R}^n$ whose fixed-point property is proved in § 5.xx. More generally, consider a locally convex vector space X. In § 8 we constructed of its dual space X^* whose vectors are continuous linear functionals on X. Now one equips X with the weak topology whose open sets are generated by pairs $x^* \in X^*$ and positive numbers δ) of the form:

$$B_\delta(x^*) = \{x \in X : |x^*(x)| \leq \delta\}$$

Denote by $\mathcal{K}(X)$ the family of convex subsets of X which are compact with respect to the X^*-topology. We are going to prove the following two results.

9.1 The Schauder-Tychonoff fixed point theorem. *Each K in $\mathcal{K}(X)$ has the fixed point property.*

The merit of this result is of course that one allows non-linear maps. The next result is due to Kakutani. Here X is a normed space over the field of real numbers. By a semi-group of linear transformations $\mathbf{G}$ on a real vector space X we mean a family of linear maps $g\colon X \to X$ such that composed maps $g_2 \circ g_1$ again belong to $\mathbf{G}$.

9.2 Kakutani's theorem. *Let $K \in \mathcal{K}(X)$ be $\mathbf{G}$-invariant, i.e. $g(K) \subset K$ for every $g \in \mathbf{G}$. Assume in addition that the family of the restricted $\mathbf{G}$-maps to K is equicontinuous. Then there exists at least some vector $k \in K$ such that $g(k) = k$ for every $g \in \mathbf{G}$.*

Remark. The equi-continuous assumption means that to each pair every (x^*, ϵ) with $x^* \in X^*$ and $\epsilon > 0$, there exists a finite family $x_1^*, \ldots, x_M^*$ and some $\delta > 0$ such that the following hold: If p and q is a pair of points in K such that $p - q$ belongs to $\cap\, B_\delta(x_\nu^*)$, then

$$g(p) - g(q) \in B_\epsilon(x^*)$$

hold for all $g \in \mathbf{G}$.

9.3 Haar measures. Let G be a compact topological group which means that the group is equipped with a Hausdorff topology where the group operations are continuous, i.e the map from $G \times G$ into G which sends a pair of group elements g, h to the product gh is continuous, and the inverse map $g \mapsto g^{-1}$ is bi-continuous. Let $C^0(G)$ be the Banach space of continuous real-valued functions on G. Recall from basic measure theory that the dual space consists of Riesz measures. Denote by $P(G)$ the family of non-negative measures with total mass one, i.e. probability measures on G. If $\phi \in C^0(G)$ and $g \in G$ we get the new continuous function $S_g(\phi)$ defined by

$$S_g(\phi)(h) = \phi(gh) \quad : h \in G$$

Next, every $\mu \in P(G)$ gives the probability measure $T_g(\mu)$ for which

$$\int_G S_g(\phi)\, d\mu = \int_G \phi\, dT_g(\mu)$$

In this way G is identified with a group of linear maps on $P(G)$. Next, $P(G)$ is equipped with the weak-star topology where open neighborhoods of a given $\mu \in P(G)$ consists of finite intersections of sets $\{\gamma \in P(G) : |\gamma(\phi) - \mu(\phi)| < \delta\}$ for pairs $\delta > 0$ and $\phi \in C^0(G)$. The uniform continuity of every $\phi \in C^0(G)$ entails that that the group action on $P(G)$ is equi-continuous with respect to the weak-star topology. Kakutani's theorem applies and yields a fixed point. Hence there is a probability measure μ_* such that

$$(*) \qquad \int_G \phi(gh)\, d\mu_*(h) = \int_G \phi(h)\, d\mu_*(h)$$

hold for every pair $g \in G$ and $\phi \in C^0(G)$. This is expressed by sayung that μ_* is left invariant. It turns out that μ_* is unique, i.e. only one probability measure enjoys the invariance above. Moreover, starting with the operators

$$S_g^*(\phi)(h) = \phi(hg) \quad : h \in G$$

one finds a probability measure μ^* such that

$$(**) \qquad \int_G \phi(hg)\, d\mu^*(h) = \int_G \phi(h)\, d\mu^*(h)$$

hold for every pair $g \in G$ and $\phi \in C^0(G)$. A wellknown fact about compact topological groups asserts $\mu^* = \mu_*$ and this unique probability measure os called the Haar measure on G.

Preliminary results.

Using Stokes theorem one has the classical result:

9.4 Theorem. *The closed unit ball in $\mathbf{R}^n$ has the fixed point property.*

Proof. By Weierstrass approximation theorem every continuous map from B into itself can be approxiated unifomly by a C^∞-map. Together with the compactness of B the reader can check that it suffices to prove every C^∞-map $\phi\colon B \to B$ has at least one fixed point. We are going to derive this by contradiction, i.e suppose that $\phi(x) \neq x$ for all $x \in B$. Each $x \in B$ gives the quadratic equation in the variable a

$$\text{(i)} \qquad 1 = |x + a(x - \phi(x)|^2 = |x|^2 + 2a(1 - \langle x, \phi(x)\rangle) + a^2|x - \phi(x)|^2$$

Exercise 1. Use that $\phi(x) \neq x$, to check that (i) has two simpe roots for each $x \in S$, and if $a(x)$ is the larger, then the function $x \mapsto a(x)$ belongs to $C^\infty(B)$. Moreover

$$\text{(E.1)} \qquad a(x) = 0 \quad : x \in S$$

Next, for each real number t we set

$$f(x,t) = x + ta(x)(x - \phi(x))$$

This is a vector-valued function of the $n+1$ variables $t, x_1, \ldots, x_n$ where x varies in B. With $f = (f_1, \ldots, f_n)$ we set

$$g_i(x) = a(x)(x_i - \phi_i(x)$$

Taking partial derivatives with respect to x we get

$$\text{(ii)} \qquad \frac{\partial f_i}{\partial x_k} = e_{ik} + t\frac{\partial g_i}{\partial x_k}$$

where $e_{ii} = 1$ and $e_{ik} = 0$ if $i \neq k$. Let $D(x;t)$ be the determinant of the $n \times n$-matrix whose elements are the partial derivatives in (ii) and put

$$\text{(iii)} \qquad J(t) = \int_B D(x;t)\, dx$$

When $t = 0$ we notice that the $n \times n$-matrix above is the identy matrix and hence $D(x;0)$ has constant value one so that $J(0)$ is the volume of B. Next, (i) entails that $x \mapsto f(x;1)$ satisfies the functional equation

$$|f(x;1)|^2 = 1$$

which implies that $x \mapsto D(x;1)$ is identically zero and hence $J(1) = 0$. The requested contradiction follows if $t \mapsto J(t)$ is a constant function of t. To attain this we shall need:

2. Exercise. Use Leibniz's rule and that determinants of matrices with two equal columns are zero to conclude that

$$\text{(E.2)} \qquad \frac{d}{dt}(D(x;t) = \sum\sum (-1)^{j+k} \cdot \frac{\partial g_i}{\partial x_k}$$

where the double sum extends over all pairs $\leq j, k \leq n$.

Next, for all pairs $i.k$, Stokes theorem gives

$$\text{(iv)} \qquad \int_B \frac{\partial g_i}{\partial x_k}\, dx = \int_S g_i \cdot \mathbf{n}_k\, d\omega$$

where ω is the area measure on the unit sphere. From (E.1) we have $g_i = 0$ on S for each i. Hence (E.2) and (iv) imply that

$$\frac{dJ}{dt} = \int_B \frac{d}{dt}(D(x;t)\,dx = 0$$

So $t \mapsto J(t)$ is constant which is impossible because $J(0) = 1$ and $J(1) = 0$ which finishes the proof.

9.5 The Hilbert cube $\mathcal{H}_\square$. This is the closed subset of the Hilbert space ℓ^2 which consists of vectors $x = (x_1, x_2, \ldots)$ such that $|x_k| \leq 1/k$ for each k.

9.5.1 Proposition. *Every closed and convex subset of $\mathcal{C}$ has the fixed point property.*

9.5.2 Exercise. Deduce this result from Theorem 9.4.

Next, let X be a locally convex vector space and X^* its dual. Denote by $\mathcal{K}(X)$ the family of convex subsets which are compact with respect to the weak topology on X. Let $K \in \mathcal{K}(X)$ and $T\colon K \to K$ a continuous map with respect to the weak topology. For each fixed $f \in X^*$, it follows from our assumptions that the complex-valued function on K defined by

$$p \mapsto f(T(p))$$

is uniformly contnuus with respect to the weak topology. So for each positive integer n there exists a finite set $G_n = (x_1^*, \ldots, x_N^*)$ and some $\delta > 0$ such that the following implication holds for each pair of points p, q in K:

(i) $$p - q \in \cap B_\delta(x_\nu^*) \implies |f(T(p)) - f(T(q))| \leq n^{-1}$$

We can attain this for each positive integer n and get a denumerable set

$$G = \cup G_n$$

From (i) it is clear that if p, q is a pair in K and $g(p) = g(q)$ hold for every $g \in G$, then $x^*(T(p)) = x^*(T(q)$. We refer to G as a determining set for the map T. In a similar way we find a denumerable determining set $G^{1)}$ for g_1, By a standard diagonal argument the reader may verify the following:

9.6 Proposition. *There exists a denumerable subset G in X^* which contains f and is self-determining in the sense that it determines each of its vectors as above.*

9.7 An embedding into the Hilbert cube. During the construction of the finite G_m-sets which give (i), we can choose small δ-numbers and take $\{x_\nu^*\}$ such that the maximum values

$$\max_{p \in K} |x_\nu^*(p)|$$

are small. From this observation the reader should confirm that in Proposition 9.6 we can construct the sequence $G = (g_1, g_2, \ldots)$ in such a way that

$$\max_{p \in K} |g_n(p)| \leq n^{-1}$$

hold for every n. Hence each $p \in K$ gives the vector $\xi(p) = (g_1(p), g_2(p), \ldots)$ in the Hilbert cube and now

$$K_* = \{\xi(p) : \, p \in K\}$$

yields a convex subset of $\mathcal{C}$. Since G is self-determininig we have $T(p) = T(q)$ whenever $\xi(p) = \xi(q)$. Hence there exists a map from K_* into itself defined by

(*) $$T_*(\xi(p)) = \xi(T(p)$$

Exercise. Use the compact property of K to show that K_* is closed in the Hibert cube and that T_* is a continuous map with respect to the induced strong norm topology on K_* derived from the complete norm on ℓ^2.

9.8 A consequence. Suppose from the start that we are given a pair of points p_1, p_2 in K and some $f \in X^*$ where $f(p_1) \neq f(p_2)$. From the above f appears in the G-sequence and put

$$K_0 = \{p \in K:\ \xi(p) = \xi(p_1)\}$$

Then K_0 is a convex subset of K, and since f appears in the G-sequence it follows that p_2 does not belong to K_0. Moreover, since G is self-determining with respect to T it is clear that

$$T(K_0) \subset K_0$$

Hence we have proved:

9.9 Proposition. *For each pair K and T as above where K is not reduced to a single point, there exists a proper and X^*-closed convex subset K_0 of K such that $T(K_0) \subset K_0$.*

9.10 Proof of the Schauder-Tychonoff theorem.

Let $T\colon K \to K$ be a continuous map where K belongs to $\mathcal{K}(X)$. Consider the family $\mathcal{F}$ of all closed and convex subsets which are T-invariant. It is clear that intersections of such sets enjoy the same property. So we find the minimal set

$$K_* = \bigcap K_0$$

given by the intersection of all sets K_0 in $\mathcal{F}$. If K_* is not reduced to a single point then Proposition 6.1 gives a proper closed subset which again belongs to $\mathcal{F}$. This is contradicts the minimal property. Hence $K_* = \{p\}$ is a singleton set and p gives the requested fixed point for T.

9.11 Proof of Kakutani's theorem.

With the notations from the introduction we are given a semi-group $\mathbf{G}$ whose linear operators preserve the convex set K in $\mathcal{K}(X)$. Zorn's lemma gives a minimal closed and convex subset K_* of K which again is $\mathbf{G}$-invariant. Kakutani's theorem follows if K_* is a singleton set. To prove this we argue by contradiction. For K_* is not a singleton set then

$$K_* - K_* = \{p - q\colon p, q \in K_*\}$$

contains a non-zero vector and we find a convex open neighborhood V of the origin such that

(i) $$(K_* - K_*) \setminus \overline{V} \neq \emptyset$$

Since $\mathbf{G}$ is equicontinuous on K and hence also on K_* there exists an open convex neighborhood U of the origin such that whenever k_1, k_2 is a pair in K_* with $k_1 - k_2 \in U$, then the orbit

(ii) $$\mathbf{G}(k_1 - k_2) \subset V$$

Let U^* be the convex hull of $\mathbf{G}(U)$. Since the $\mathbf{G}$-maps are linear, the set U^* is $\mathbf{G}$-invariant and continuity gives the equality

(iii) $$\mathbf{G}(\overline{U^*}) = \overline{U^*}$$

We find the unique positive number δ such that the following hold for every $\epsilon > 0$:

(iv) $$(K_* - K_*) \subset (1+\epsilon) \cdot \delta \cdot \overline{U^*} \neq \emptyset \quad \& \quad (K_* - K_*) \setminus (1-\epsilon) \cdot \delta \cdot \overline{U^*} \neq \emptyset$$

Next, $\{k + \frac{\delta}{2} \cdot U\colon\ k \in K_*\}$ is an open covering of the compact set K_* and Heine-Borel's Lemma gives a finite set $k_1, \ldots, k_n$ in K_* such that

(v) $$K_* \subset \bigcup (k_\nu + \frac{\delta}{2} \cdot U^*)$$

Put

(vi) $$K_{**} = K_* \cap \bigcap_{k \in K_*} (k + (1 - 1/4n)\delta \cdot \overline{U}^*)$$

Since $\overline{U}^*$ is $\mathbf{G}$-invariant and the intersection above is taken over all k in the invariant set K_*, we see that K_{**} is a closed convex and $\mathbf{G}$-invariant set. The requested contradiction follows if we prove that K_{**} is non-empty and strictly contained in K_*. To get the strict inclusion we take some $0 < \epsilon < 1/4n$. Then (iv) gives a pair k_1, k_2 in K_* such that $k_1 - k_2$ does not belong to $(1-\epsilon)\delta \cdot \overline{U^*}$. At the same time the eventual inclusion $k_1 \in K_{**}$ would entail that

(v) $$k_1 \in (k_2 + (1-1/4n)\delta \cdot \overline{U}^*) \implies k_1 - k_2 \in (1-1/4n)\delta \cdot \overline{U}^*$$

But this cannot hold since $1 - 1/4n < 1 - \epsilon$ and hence $k_1 \in K_* \setminus K_{**}$ which proves the requested strict inclusion. The proof of Kakutani's theorem is therefore finished if we have shown that

(vi) $$K_{**} \neq \emptyset$$

To see this we take an arbitrary $k \in K_*$. From (v) we find some $1 \leq \nu \leq n$ such that

(vii) $$k_\nu - k \in \frac{\delta}{2} \cdot U^*$$

Without loss of generality we can assume that $\nu = 1$ and get a vector $u \in U^*$ such that

(viii) $$k_1 = k + \frac{\delta}{2} \cdot u$$

It follows that

(ix) $$\frac{k_1 + \ldots + k_n}{n} = k + \frac{\delta}{2n} \cdot u + \sum_{i=2}^{i=n} \frac{1}{n}(k_i - k)$$

Next, for each $\epsilon > 0$ the left hand inclusion in (iv) and the convexity of U^* give

(x) $$\sum_{i=2}^{i=n} \frac{1}{n}(k_i - k) \subset \frac{n-1}{n}(1+\epsilon) \cdot \delta \cdot U^*$$

It follows that

(xi) $$\frac{\delta}{2n} \cdot u + \sum_{i=2}^{i=n} \frac{1}{n}(k_i - k) \in (\frac{\delta}{2n} + \frac{n-1}{n}(1+\epsilon)\delta) \cdot U^*$$

Above we can choose ϵ so small that

$$\frac{n-1}{n}(1+\epsilon) + \frac{1}{2n}) < 1 - 1/4n$$

and then we see that the vector

$$p = \frac{k_1 + \ldots + k_n}{n} \in k + (1-1/4n)\delta \cdot \overline{U}^*$$

Above $k \in K_*$ was arbitrary so by (tvi) we get the inclusion $p \in K_{**}$ and hence (vi) holds which finishes the proof.

§ 10. Semi-groups and infinitesmal generators.

We shall expose a result due to Hille, Phillips and Yosida. Let X be a Banach space. A family of bounded operators $\{T_t\}$ indexed by non-negative real numbers is a semi-group if $T_0 = E$ is the identity and

$$T_{t+s} = T_s \circ T_t$$

for all pairs of non-negative real numbers. In particular the T-operators commute. The semi-group is said to be *strongly continuous* if the vector-valued functions

$$x \mapsto T_t(x)$$

are continuous with respect to the norm in X for each $x \in X$.

1. Proposition. *Let $\{T_t\}$ be a strongly continuous semi-group and set*

$$\omega = \log |||T_1||$$

Then the operator norms satisfy

$$||T_t|| \leq e^{\omega t} \quad : t \geq 0$$

Exercise. Prove this using calculus applied to sub-multiplicative functions.

With ω as above we consider the open half-plane

$$U = \{\mathfrak{Re}(\lambda) > \omega\}$$

Let $\lambda \in U$ and x is a vector in X. The Borel-Stieltjes construction of integrals with values in a Banach space gives the X-valued integral

$$\int_0^\infty e^{-\lambda t} \cdot T_t(x)\, dt \tag{1.1}$$

whose value is denoted by $\mathcal{T}(\lambda)(x)$. It is clear that

$$x \mapsto \mathcal{T}(\lambda)(x)$$

is linear and the triangle inequality gives

$$||\mathcal{T}(\lambda)(x)|| \leq \int_0^\infty e^{-\mathfrak{Re}\lambda t} \cdot e^{\omega t}\, dt \cdot ||x|| = \frac{1}{\mathfrak{Re}\,\lambda - \omega} \cdot ||x|| \tag{1.2}$$

2. Infinitesmal generators. Let $\{T_t\}$ be a a strongly continuous semi-group. We are going to prove that there exists a densely defined and closed operator which is called its infinitesmal generator which is related to the semi-group as follows:

2.1 Theorem. *There exists a dense subspace $\mathcal{D}$ in X such that*

$$\lim_{h \to 0} \frac{T_h(x) - x}{h} \tag{*}$$

exists for each $x \in \mathcal{D}$. Moroever, if $A(x)$ is the limit value in (*) then A is a densely defined operator and

$$\sigma(A \subset \{\mathfrak{Re}\,\lambda \leq \omega\} \tag{**}$$

and in the open half-space U one has the equality

$$R_A(\lambda) = \mathcal{T}(\lambda) \tag{***}$$

where the left hand side are Neumann's resolvent operators of A.

Theorem 2.1 produces infinitesmal generators of strongly continuous semi-groups. It turns out that this class of densely defined and closed operators can be described via properties of their spectra and their resolvent operators. Denote by $\mathcal{HPY}$ the family of densely defined and closed linear operators A with the property that $\sigma(A)$ is contained in a half-space $\{\mathfrak{Re}\,\lambda \leq a\}$ for some real number a, and if $a^* > a$ there exists a constant M such that

$$||R_A(\lambda)|| \leq M \cdot \frac{1}{\mathfrak{Re}\,\lambda - \omega} \quad : \mathfrak{Re}\,\lambda \geq a^* \tag{2.2}$$

Such A-operators are called Hille-Phillips-Yosida operators.

2.2 Theorem. *Each $A \in \mathcal{HPY}$ is the infinitesmal generator of a uniquely determined strongly continuous semi-group.*

Remark. Notice that (***) in Theorem 2.1 together with (1.2) imply that the infinitesmal generator of a strongly continuous semi-group belongs to $\mathcal{HPY}$. Hence Theorem 2.2 gives a 1-1 correspondence between $\mathcal{HPY}$ and infinitesmal generators of strongly contionuous semi-groups.

3. The case of bounded operators.

Before we give the proofs of the two theorems above we shall consider special cases arising via bounded operators. Every bounded linear operator B on X gives the strongly continuous semi-group where

$$T_t = e^{tB} = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \cdot B^n$$

If $h > 0$ and $x \in X$ we have

$$|| \lim_{h \to 0} \frac{T_h(x) - x}{h} - B(x)|| = || \sum_{n=2}^{\infty} | \frac{h^n}{n!} \cdot B^n(x)||$$

The triangle inequality entails that the last term is majorised by

$$\frac{h^2}{2} \cdot e^{||B(x)||}$$

We conclude that B is the infinitesmal generator of the semi-group $\mathcal{T} = \{T_t\}$. Next, if $\omega = ||B||$ then $\sigma(B)$ is contained in the disc of radius ω and hence in the half-plane $U = \{\mathfrak{Re}\,\lambda \leq \omega\}$. If $\lambda \in U$ the operator-valued integral

$$\int_0^{\infty} e^{-\lambda t} \cdot e^{Bt}\, dt = \frac{1}{\lambda} \cdot E + \sum_{n=1}^{\infty} \big(\int_0^{\infty} e^{-\lambda t} \cdot t^n\, dt \big) \cdot \frac{B^n}{n!}$$

Evaluating the integrals the right hand side becomes

$$\frac{1}{\lambda} \cdot E + \sum_{n=1}^{\infty} \frac{B^n}{\lambda^{n+1}}$$

The last sum is equal to the Neumann series for $R_B(\lambda)$ from § xx. which gives the equation

$$R_B(\lambda) = \mathcal{T}(\lambda)$$

Hence Theorem 2.1 is confirmed for the special semi-group defined via B.

4. Uniformly continuous semi-groups.

Let B as above be a bounded operator. The semi-group $\{T_t = e^{tB}\}$ has the additional property that

$$\lim_{t \to 0} |||T_t - E|| = 0 \tag{4.1}$$

In fact, the triangle inequality gives

$$||T_t - E|| \leq \sum_{n=1}^{\infty} \frac{t^n}{n!} \cdot ||B||^n$$

and it is clear that the right hand side tends to zero as $t \to 0$. This leads us to give:

4.2 Definition. *A semi-group* $\{T_t\}$ *is called uniformly continuous if*

(4.2.1) $$\lim_{t\to 0}\,|||T_t - E|| = 0$$

When (4.2.1) holds we find $h_* > 0$ such that

$$||T_t - E|| \leq 1/2 \quad : \, 0 \leq t \leq h_*$$

As explained in § xx this gives bounded operators $\{S_t : \, 0 \leq t \leq h_*\}$ such that

$$T_t = e^{S_t}$$

This entails that if $0 < h \leq h_*$ then

$$\frac{T_h - E}{h} = \frac{S_h}{h} + \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!} \cdot S_h^n$$

4.3 Exercise. Verify that the semi-group equations entail that if N is a positive integer and h so small that $Nh \leq h_*$ one has the equation

(4.3.1) $$\frac{S_{Nh}}{N} = S_h$$

Next, for each small and positive $h < h_*$ we choose the largest positive integer N_h such that

$$\frac{h_*}{N_h + 1} < h \leq \frac{h_*}{N_h}$$

With this choice of N_h we apply (4.3.1) and get

$$\frac{S_h}{h} = \frac{1}{N_h \cdot h} \cdot S_{N_h \cdot h}$$

Passing to the limit as $h \to 0$ the reader can check the limit equation

$$\lim_{h\to 0} \frac{S_h}{h} = h_*^{-1} \cdot S(h_*)$$

4.4 Exercise. Show from the above that

$$\lim_{h\to 0} \frac{T_h - E}{h} = h_*^{-1} \cdot S(h_*)$$

which means that the infinitesmal generator of $\{T_t\}$ is given by the bounded operator $h_*^{-1} \cdot S(h_*)$.

4.5 Conclusion. There exists a 1-1 correspondence between uniformly continuous semi-groups and bounded linear operators on X.

5. Proof of the main theorems.

First we prove Theorem 2.1. If $x \in X$ and $\delta > 0$ we put

$$x_\delta = \int_0^\delta T_t(x)\, dt$$

For every $h > 0$ the semi-group equation gives

$$\frac{T_h(x_\delta) - x_\delta}{h} = \frac{1}{h} \cdot \int_\delta^{\delta+h} T_t(x)\, dt$$

The strong continuity entails that the limit in the right hand side exists as $h \to 0$ and gives a vector $T_\delta(x)$. Hence the space $\mathcal{D}$ contains x_δ. The continuity of $t \mapsto T_t(x)$ at $t = 0$ implies that $||x_\delta - x|| \to 0$ which proves that $\mathcal{D}$ is dense and the construction of the infinitesmal generator A gives for every $\delta > 0$:

(5.1) $$A(x_\delta) = T_\delta(x)$$

Next, consider some vector $x \in \mathcal{D}(A)$. Now there exists the integral which defines $\mathcal{T}(\lambda)(Ax)$ when λ belongs to the half-plane U. If λ is real and $h > 0$ a variable substitution gives

$$\mathcal{T}(\lambda)(T_h(x)) = \int_0^\infty e^{-\lambda t} \cdot T_{t+h}(x)\,dt = e^{\lambda h} \cdot \int_h^\infty e^{-\lambda s} T_s(x)\,ds$$

It follows that

$$\mathcal{T}(\lambda)(\frac{T_h(x)) - x}{h}) = \frac{e^{\lambda h} - 1}{h} \cdot \int_h^\infty e^{-\lambda s} T_s(x)\,ds - \frac{1}{h} \cdot \int_0^h e^{-\lambda s} T_s(x)\,ds$$

Passing to the limit as $h \to 0$ the reader can check that the right hand side becomes

$$\lambda \cdot \mathcal{T}(\lambda)(x) - x$$

So with $x \in \mathcal{D}(A)$ we have the equation

$$\mathcal{T}(\lambda)(A(x)) = \lambda \cdot \mathcal{T}(\lambda)(x) - x$$

which can be written as

$$\mathcal{T}(\lambda)(\lambda \cdot E - A)(x) = x$$

Exercise. Conclude from the above that

$$\mathcal{T}(\lambda) = R_A(\lambda)$$

and deduce Theorem 2.1.

5.2 proof of Theorem 2.2

Let A belong to $\mathcal{HPY}$. So here $\sigma(A)$ is contained in $\{\mathfrak{Re}(\lambda) \leq a\}$ for some real number a, and when λ varies in the open half-plane $U = \{\mathfrak{Re}(\lambda) \leq a\}$ there exist the resolvents $R(\lambda)$ where the subscript A is deleted while we consider the operator A. By assumption there exists a constant K and some $a^* \geq a$ such that

(5.2.0) $$||R(\lambda)|| \leq \frac{K}{\lambda}$$

when $\mathfrak{Re}(\lambda \geq a^*$.

The operators B_λ. For each $\lambda \in U$ we set

(ii) $$B_\lambda = \lambda^2 \cdot R(\lambda) - \lambda \cdot E$$

Notice that (5.2.0) gives

(iv) $$||B_\lambda|| \leq (K+1)|\lambda|$$

Consider a vector $x \in \mathcal{D}(A)$. Now

$$B_\lambda(x) = \lambda \cdot (\lambda R(\lambda)(x) - x) = \lambda \cdot R(\lambda)(Ax)$$

We have also

$$\lambda \cdot R(\lambda)(Ax) - R(\lambda(A(x)) = A(x)$$

Hence

$$B_\lambda(x) - A(x) = R(\lambda)(A(x))$$

Hence (5.2.0) gives

$$||B_\lambda(x) - A(x)|| \leq \frac{K}{\lambda} \cdot ||A(x)||$$

Hence we have the limit formula

$$\lim_{\lambda \to \infty} ||B_\lambda(x) - A(x)|| = 0 \quad : x \in \mathcal{D}(A)$$

The semi-groups $\mathcal{S}_\lambda = \{e^{tB_\lambda}\,:\, t \geq 0\}$. To each $t \geq 0$ and $\lambda \in U$ we set

(5.2.1) $$S_\lambda(t) = e^{tB_\lambda} = E + \sum_{n=1}^\infty \frac{t^n}{n!} \cdot B_\lambda^n$$

By (iv) the series converges and with t fixed termwise differentiation with respect to λ gives

$$\frac{d}{d\lambda}(S_\lambda(t)) = t \cdot \frac{d}{d\lambda}(B_\lambda) \cdot S_t(\lambda) = -t \cdot S_\lambda(t) + t \cdot (\lambda \cdot R(\lambda)^2 - R(\lambda)) \cdot S_\lambda(t)$$

Keeping t fixed we get after an integration

(5.2.2) $$S_\mu(t) - S_\lambda(t) = t \cdot \int_\lambda^\mu (\xi \cdot R(\xi)^2 - R(\xi)) S_\xi(t)\, ds \quad : \mu > \lambda \geq a^*$$

Next, we notice that (5.2.0) gives

(5.2.3) $$\lim_{\xi \to \infty} ||(\xi \cdot R(\xi)^2 - R(\xi)|| = 0$$

Together with the general differential inequality from (xx) we conclude that if t stays in a bounded interval $[0, T)$ and $\epsilon > 0$, then there exists some large ξ^* such that for every $0 \leq t \leq T$ the operator norms satisfy

(5.2.4) $$||S_\mu(t) - S_\lambda(t)|| < \epsilon \quad : \mu \geq \lambda \geq \xi^*$$

5.2.5 The semi-group $\{S(t)\}$. Since the bounded operators on X is a Banach space, it follows from the above that each t gives a bounded operator $S(t)$ given by

$$\lim_{\lambda \to +\infty} S_\lambda(t) = S(t)$$

where the limit is taken in the operator norm and the convergence holds uniformly when t stays in a bounded interval. Since $t \mapsto S_\lambda(t)$ is a semi-group for each large λ, the same holds for the family $\{S(t)\}$. Moreover, the convergence properties entail that the semi-group $\{S(t)\}$ is strongly continuous. Neumann's differential equation from (§ 0.x) gives

(iii) $$\frac{d}{d\lambda}(B_\lambda) = R(\lambda) - E - \lambda \cdot R(\lambda)^2$$

The infinitesmal generator of $\{S(t)\}$. Returning to the series (5.2.1) it is clear that

(5.2.6) $$\frac{d}{dt}(S_\lambda(t) = S_\lambda(t) \cdot B_\lambda$$

An integration and the equality $S_\lambda(0) = E$ give for every $t > 0$ and each vector $x \in X$:

(5.2.7) $$S_\lambda(t)(x) - x = \int_0^t S_\lambda(\xi) \circ B_\lambda(\xi)(x)\, d\xi$$

When $x \in \mathcal{D}(A)$ we have the limit formula (xx) and together with the limit which produces the semi-group $\{S(t)\}$ we conclude that

$$S(t)(x) - x = \int_0^t S(\xi) \circ A(x)\, d\xi \quad : x \in \mathcal{D}(A)$$

We can take a limit as $t \to 0$ where the strong continuity of the semi-group $\{S(t)\}$ applies to vectors $A(x) : x \in \mathcal{D}(A)\}$. Hence

$$\lim_{t \to 0} \frac{S(t)(x) - x}{t} = A(x) \quad : x \in \mathcal{D}(A)$$

So if $\widehat{A}$ is the infinitesmal generator of the semi-group $\{S(t)\}$ then its graph contains that of A, i.e. $\widehat{A}$ is an extension of the densely defined and closed operator A. However, we have equality because $\widehat{A}$ being an infinitesmal generator of a strongly continuous semi-group has its spectrum confined to a half-space $\{\mathfrak{Re}(\lambda) \leq b\}$ for some real number b, i.e. here we used Theorem 2.1. In particular there exist points outside the union of $\sigma(A)$ and $\sigma(\widehat{A})$ and then the equality $A = \widehat{A}$ follows from the general result in § xx. Hence A is an infinitesmal generator of a semi-group which finishes the proof of Theorem 2.2.

§ 11. Commutative Banch algebras

Let B be a complex Banach space equipped with a commutative product whose norm satisfies the multiplicative inequality

$$(*) \qquad ||xy|| \leq ||x|| \cdot ||y|| \quad : \, x, y \in B$$

We also assume that B has a multiplicative unit element e where $ex = xe$ hold for all $x \in B$ and $||e|| = 1$. When this hold we refer to B as a commutative Banach algebra with a multiplicative unit.

Neumann series. A non-zero element x in B gives a bounded linear operator $T(x)$ on the underlying Banach space defined by

$$y \mapsto x \cdot y$$

i.e. we simply employ the multiplication on B. Now we apply Carl Neumann's general theory about resolvent operators from § xx. To begin with we have seen that $T(x)$ has a non-empty compact spectrum denoted by $\sigma(x)$ and in the open complement we put

$$R(\lambda) = (\lambda \cdot e - x)^{-1}$$

Notice that the norm of the linear operstor $T(x)$ is equal to the norm of the B-vector x. So by the general result in § xx applied with $T(x)$ the exterior Neumann series expansion

$$R(\lambda) = \frac{e}{\lambda} + \sum_{n=1}^{\infty} \frac{x^n}{\lambda^{n+1}} \quad : \, |\lambda| > ||x||$$

converges and gives an analytic function of the complex parameter λ when

$$|\lambda| > ||x||$$

Now there exists a unique largest $r_* \geq 0$ such thst $R(\lambda)$ extends to an analytic B-valued function in the exterior disc $\{|\lambda| > r_*\}$. If $r > r_*$ and n a positive integer, Cauchy's residie calculus gives

$$x^n = \frac{1}{2\pi i} \int_{|\zeta|=r} \zeta^n \cdot R(\zeta) \, d\zeta$$

Here

$$\max_{\theta} ||R(re^{i\theta}|| = M < \infty$$

for some constant M. The triangle inequality entails that

$$\text{(i)} \qquad ||x^n|| \leq M \cdot r^n$$

At this stage we shall need the following:

1. Theorem. *For every $x \in B$ there exists the limit*

$$(1.1) \qquad \lim_{n \to \infty} ||x^n||^{\frac{1}{n}}$$

This follows from a general fact about sub-multiplicative sewuences. See § xxx. Let $\rho(x)$ denote the oimit in (1.1). If $r > r_*$ we take roots in (i) and obtain

$$\text{(ii)} \qquad ||x^n||^{\frac{1}{n}} \leq M^{\frac{1}{n}} \cdot r$$

Since the n:th roots of M converge to one a passage to thr limit gives

$$\rho(x) \leq r$$

Above $r > r_*$ was arbitrsry and hence we have proved the inequality

$$(*) \qquad \rho(x) \leq r_*$$

The opposite inequality in (**) is clear. For if $r > \rho(x)$ we find a large positive integer n such that

$$n \geq N \implies ||x^n|| \leq r^n$$

and then it is obvious that the exterior Neumann series above converges when $|\lambda| > r$. Since tihs hold for every $r > \rho(x)$ we conclude that (*) is an equality and then the Neumann calculus from § xx gives:

2. The spectral radius formula. *For every $x \in X$ one has the equality*

$$\rho(x) = \max\{|\lambda| : \lambda \in \sigma(x)\} \tag{2.1}$$

3. Multiplicative functionals. A **C**-linear form λ on B is multiplicative if:

$$\lambda(xy) = \lambda(x) \cdot \lambda(y) \quad \text{for all pairs} \quad x, y \in B \tag{3.1}$$

When λ satisfies (3.1) and is not identically zero it is clear that $\lambda(e) = 1$.

3.2 Theorem. *Every multiplicative functional λ on B is automatically continuous, i.e. an element in the normed dual space B^* and its norm is equal to one.*

Proof. Ignoring the topology we use that **C** is a field and conclude that the λ-kernel is a *maximal ideal* in the commutative **C**-algebra B which we denote by $\mathfrak{m}$. When $x \in B$ has norm < 1 the B-valued power series

$$e + \sum_{n=1}^{\infty} x^n$$

converges and gives the inverse to $(e - x)$. Since λ is multiplicative with $\lambda(e) = 1$ it follows that $\lambda \cdot e - x) \neq 0$, So if B_0 is the open unit ball in X then

$$\mathfrak{m} = \ker(\lambda) \cap \{e + B_0\} = \emptyset$$

By the remark in § xx this implies that the hyperplane $\mathfrak{m}$ is closed in the Banach space B and therefore a continuous linear functional. Hence there exists a constant C such that

$$|\lambda(x)| \leq C \cdot ||x|| \quad : x \in X$$

Finally, since λ is multiplicative this entails that

$$|\lambda(x)|^n \leq C \cdot ||x^n|| \quad : x \in X$$

Taking n:th rooots and passing to infinity the reader can check that this gives

$$|\lambda(x)| \leq \rho(x) \leq ||x||$$

4. The Gelfand transform. Denote by $\mathcal{M}(B)$ the family of sll multiplicative functionals. Keeping an element $x \in B$ fixed we get the complex-valued function on $\mathcal{M}(B)$ defined by:

$$\lambda \mapsto \lambda(x)$$

The resulting function is denoted by $\widehat{x}$ and called the Gelfand transform of the b-element x.

Exercise. Apply the specteral radius formuka to show that

$$\max_{\lambda \in \mathcal{M}(B)} |\widehat{x}(\lambda)| = \rho(x) \tag{4.1}$$

Following a device by Gelfand we use that $\mathcal{M}(B)$ is a subset of the dual space B^* which is equipped with the weak-star topology and we know from § xx that the unit ball S^* in B^* is compact in this topology. The reader can check that $\mathcal{M}(B)$ appears as a weak-star closed subset of S^* and hence becomes a compact space where every Gelfand transform $\widehat{x}$ is a continuous function. This gives an algebra homomorphism from B into the commutative algebra $C^0(\mathcal{M}(B))$:

$$x \mapsto \widehat{x} \tag{*}$$

called the Gelfand transform. The inequality in (xx) shows that the map (*) is continuous with norm ≤ 1 where we employ the sup-norm on the Banach space $C^0(\mathcal{M}(B))$.

5. Semi-simple algebras. The spectral radius formula shows that $\widehat{x}$ is the zero function if and only if $\rho(x) = 0$. One says that the Banach algebra B is *semi-simple* if the Gelfand transform in (*) is injective. An equivalent condition is that

$$0 \neq x \implies \rho(x) > 0 \tag{5.1}$$

6. Uniform algebras. If B is semi-simple the Gelfand transform identifies B with a subalgebra of $C^0(\mathcal{M}(B))$. In general this subalgebra is not closed. The reason is that there can exist B-elements of norm one while the ρ-numbers can be arbitrarily small. If the equality below holds for every $x \in B$:

$$||x|| = \rho(x) = |\widehat{x}|_{\mathcal{M}(B)} \tag{*}$$

one says that B is a uniform algebra.

Remark. Multiplicative functionals on specific Banach algebras were used by Norbert Wiener and Arne Beurling where the focus was on Banach algebras which arise via the Fourier transforms. Later Gelfand, Shilov and Raikow established the abstract theory which has the merit that it applies to general situations such as Banach algebras generated by linear operators on a normed space. Moreover, Shilov applied results from the theory of analytic functions in several complex to construct *joint spectra* of several elements in a commutative Banach algebra. See [Ge-Raikov-Shilov] for a study of commutative Banach algebras which include results about joint spectra. One should also mention the work by J. Taylor who used integral formulas in several complex variables to analyze the topology of Gelfand spaces which arise from the Banach algebra of Riesz measures with total bounded variation on the real line, and more generally on arbitrary locally compact abelian groups.

SPECIAL TOPICS

12.2 Positive operators on $C^0(S)$

Let S be a compact Hausdorff space and X the Banach space of continuous and complex-valued functions on S. A linear operator T on X is positive if it sends every non-negative and real-valued function f to another real-valued and non-negative function. Denote by $\mathcal{F}^+$ the family of positive operators T which satisfy the following: First

$$\text{(1)} \qquad \lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = 0$$

hold for all pairs $x \in X$ and $x^* \in X^*$. The second condition is that $\sigma(T)$ is the union of a compact set in a disc $\{|\lambda| \leq r$ for some $r < 1$, and a finite set of points on the unit circle. The final condition is that $R_T(\lambda)$ is meromorphic in the exterior disc $\{|\lambda > r\}$, i.e. it has poles at the spectral points on the unit circle.

12.2.1. Theorem. *If $T \in \mathcal{F}^+$ then each spectral value $e^{i\theta} \in \sigma(T)$ is a root of unity.*

Proof. Frist we prove that $R_T(\lambda)$ has a simple pole at each $e^{i\theta} \in \sigma(T)$. Replacing T by $e^{-i\theta} \cdot T$ it suffices to prove this when $e^{i\theta} = 1$. If $R_T(\lambda)$ has a pole of order ≥ 2 at $\lambda = 1$ we know from § XX that there exists $x \in X$ such that

$$\text{(i)} \qquad Tx \neq x \quad \text{and} \quad (E-T)^2 x = 0$$

This gives $T^2 + x = 2Tx$ and by an induction

$$\text{(ii)} \qquad \frac{1}{n} \cdot T^n x = \frac{1}{n} \cdot x + (E-T)x \quad : n = 1, 2, \ldots$$

Condition (1) and (ii) give for each $x^* \in X^*$:

$$0 = \lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = \lim_{n\to\infty} x^*(\frac{1}{n} \cdot x + (E-T)x)$$

It follows that $x^*(E-T)(x) = 0$ and since x^* is arbitrary we get $Tx = x$ which contradicts (i). Hence the pole must be simple.

Next, let $e^{i\theta} \in \sigma(T)$ which form the above gives a simple pole of R_T. By the result in § xx there exists some $f \in C^0(S)$ which is not identically zero and

$$T(f) = e^{i\theta} \cdot f$$

Multiplying f with a complex scalar we may assume that its maximum norm on S is one and there exists a point $s_0 \in S$ such that

$$f(s_0) = 1$$

For each $n \geq 1$ we have a linear functional on X defined by $g \mapsto T^n(g)(s_0)$ which gives a Riesz measure μ_n such that

$$\int_S g \cdot d\mu_n = T^n g(s_0) \quad : g \in C^0(S)$$

Since T^n is positive the integrals in the left hand side are ≥ 0 when g are real-valued and non-negative. Hence $\{\mu_n\}$ are real-valued and non-negative measures. For each $n \geq 1$ we put

$$\text{(iii)} \qquad A_n = \{x : \; e^{-in\theta} \cdot f(x) \neq 1\} = \{x : \; \mathfrak{Re}(e^{-in\theta} f(x)) < 1\}$$

where the last equality follows since the sup-norm of f is one. Now

$$\text{(iv)} \qquad 0 = f(s_0) - e^{-in\theta} \cdot T^n f(s_0) = \int_S [1 - e^{-in\theta} f(s)] \cdot d\mu_n(s)$$

Taking real parts we get

$$\text{(v)} \qquad 0 = \int_S [1 - \mathfrak{Re}(e^{-in\theta} f(s))] \cdot d\mu_n(s)$$

By (iii) the integrand in (v) is non-negative and since the whole integral is zero it follows that

$$\text{(vi)} \qquad \mu_n(A_n) = \mu_n(\{\mathfrak{Re}(e^{-in\theta} < 1\} = 0$$

Suppose now that there exists a pair $n \neq m$ such that

(vii) $$(S \setminus A_n) \cap (S_m \setminus A_m) \neq \emptyset$$

A point s_* in this non-empty intersection gives

$$1 = e^{in\theta} f(s_*) = e^{im\theta} \cdot f(s_*) \implies e^{in\theta} = e^{im\theta}$$

and hence $e^{i\theta}$ is a root of unity. $m - n \neq 0$. So the proof of Theorem 6.1 is finished if we have established the following

Sublemma. The sets $\{S \setminus A_n\}$ cannot be pairwise disjoint.

Proof. First, f has maximum norm and by the condtrtuction of the μ-measires we have

$$\int_S f \cdot d\mu_n = e^{in\theta} \quad : n = 1, 2, \ldots$$

In particular the total mass $\mu_n(S) \geq 1$ for every n. Next, for each $n \geq 2$ we set

$$\pi_n = \frac{1}{n} \cdot (\mu_1 + \ldots + \mu_n)$$

Since $\mu_n(S) \geq 1$ for each n we get $\pi_n(S) \geq 1$. Put

$$\mathcal{A} = \bigcap A_n$$

By(vi) we have $\mu_\nu(A_\nu) = 0$ for every $\nu \geq 1$ which gives

(*) $$\pi_n(\mathcal{A}) = 0 \quad : n = 1, 2, \ldots$$

Next, when the sets $\{S \setminus A_k\}$ are pairwise disjoint one has the inclusions

$$S \setminus A_k \subset A_\nu \quad \forall \, \nu \neq k$$

Keeping k fixed it follows that $\pi_\nu(S \setminus A_k) = 0$ for every $\nu \geq 0$. So when n is large while k is kept fixed we obtain

(**) $$\pi_n(S \setminus A_k)) = \frac{1}{n} \cdot \mu_k(S \setminus A_k)) \implies \lim_{n \to \infty} \pi_n(S \setminus A_k)) = 0 \quad : k = 1, 2, \ldots$$

At this stage we use that $R_T(\lambda)$ in particular has at most a simple pole at $\lambda = 1$. So with $\epsilon > 0$ the Neumann series expansion gives

$$E + \sum_{k=1}^{\infty} \frac{T^k}{(1+\epsilon)^k} = R_T(1+\epsilon)) = \frac{1}{\epsilon} \cdot Q + W(1+\epsilon))$$

where $W(\lambda)$ is an operator-valued analytic function in an open disc centered at $\lambda = 1$ while Q is a bounded linear operator on $C^0(S)$. Keeping $\epsilon > 0$ fixed we apply both sides to the identity function 1_S on S and the construction of the measures $\{\mu_n\}$ gives

$$1 + \sum_{k=1}^{\infty} \frac{\mu_k(S)}{(1+\epsilon)^k} = \frac{1}{\epsilon} \cdot Q(1_S)(s_0) + W(1+\epsilon))(1_S)(s_0)$$

If $n \geq 2$ is an integer and $\epsilon = \frac{1}{n}$ one gets the inequality

$$\sum_{k=1}^{k=n} \frac{\mu_k(S)}{(1+\frac{1}{n})^k} \leq n \cdot |Q(1_S)(s_0)| + |W(1+1/n))(1_S)(s_0)| \leq n||Q|| + ||W(1+1/n)|| \implies$$

$$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq (1+\frac{1}{n})^n \cdot (||Q|| + \frac{||W(1+1/n)||}{n}$$

Since Neper's constant $e \geq (1+\frac{1}{n})^n$ for every n we find a constant C which is independent of n such that

(***) $$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq C$$

It is clear that (***) entails that

$$\liminf_{n\to\infty} \mu_n(S) < \infty$$

So by weak-star compacntness there exists a subsequence $\{\mu_{n_k}\}$ which converges weakly to a limit measure μ_*. For this σ-additive measure the limit formula in (**) above entails that

(i) $$\mu_*(S \setminus A_k) = 0 \quad : \ k = 1, 2, \ldots$$

Moreover, by (*) we also have

(ii) $$\mu_*(\mathcal{A}) = 0$$

Now $S = \mathcal{A} \cup A_k$ so (i-ii) give:

$$\mu_*(S) = 0$$

But this is impossible for at the same time we have already seen that $\pi_n(S) \geq 1$ for each n and hence $\mu_*(S) \geq 1$.

A result about compact pertubations.

Let X be a complex Banch space and denote by $\mathcal{F}(X)$ the family of bounded liner operators T on X such that

$$(*)\qquad \lim_{n\to\infty}\frac{1}{n}\cdot x^*(T^nx)=0$$

hold for all pairs $x\in X$ and $x^*\in X^*$.

1. Exercise. Apply the Banach-Steinhaus theorem to show that if $T\in\mathcal{F}(X)$ then there exists a constant M such that the operator norms satisfy

$$||T^n||\leq M\cdot n\quad : \ n=1,2,\ldots$$

Since the n:th root of $M\cdot n$ tends to one as $n\to+\infty$, the spectral radius formula entails that the spectrum $\sigma(T)$ is contained in the closed unit disc of the complex λ-plane.

2. The class $\mathcal{F}_*$. It consists of those T in $\mathcal{F}(X)$ for which there exists some $\alpha<1$ such that $R_T(\lambda)$ extends to a meromorphic function in the exterior disc $\{|\lambda|>\alpha\}$. Since $\sigma(T)\subset\{|\lambda|\leq 1\}$ it follows that when $T\in\mathcal{F}_*$ then the set of points in $\sigma(T)$ which belongs to the unit circle in the complex λ-plane is empty or finite and if necessary we can enlarge $\alpha<1$ so that

$$(2.1)\qquad \sigma(T)\cap\{\alpha<|\lambda|<1\}=\emptyset$$

2.2. Proposition. *If $T\in\mathcal{F}_*$ and $e^{i\theta}\in\sigma(T)$ for some θ, then Neumann's resolvent $R_T(\lambda)$ has a simple pole at $e^{i\theta}$.*

Proof. Replacing T by $e^{-i\theta}\cdot T$ it suffices to prove the result when $e^{i\theta}=1$. If $R_T(\lambda)$ has a pole of order ≥ 2 at $\lambda=1$ we know from § XX that there exists $x\in X$ such that

$$\text{(i)}\qquad Tx\neq x\quad\text{and}\quad (E-T)^2x=0$$

The last equation means that $T^2+x=2Tx$ and an induction over n gives

$$\text{(ii)}\qquad \frac{1}{n}\cdot T^nx=\frac{1}{n}\cdot x+(E-T)x$$

Since $T\in\mathcal{F}$ we have

$$\text{(iii)}\qquad \lim_{n\to\infty}\frac{1}{n}\cdot x^*(T^nx)=0\quad : \ \forall, x^*\in X^*$$

Then (ii) entails that $x^*(E-T)(x)=0$. Since x^* is arbitrary we get $Tx=x$ which contradicts (i) and hence the pole is simple.

2.3 Theorem. *Let $T\in\mathcal{F}(X)$ be such that there exists a compact operator K where $||T+K||<1$. Then $T\in\mathcal{F}_*$ and for every $e^{i\theta}\in\sigma(T)$ the eigenspace $E_T(\theta)=\{x\in X : Tx=e^{i\theta}x\}$ is finite dimensional.*

Proof. Put $S=T+K$. Outside $\sigma(S)$ we get

$$\text{(i)}\qquad R_S(\lambda)(\lambda\cdot E-T)=E+R_S(\lambda)\cdot K$$

The Neumann series for large absolute values $|\lambda|$ applied to $R_S(\lambda)$ gives some $\rho>0$ and

$$\text{(ii)}\qquad (E+R_S(\lambda)\cdot K)^{-1}=E+R_S(\lambda)\cdot K\dot{(}E+R_S(\lambda)\cdot K)^{-1}\quad : \ |\lambda|>\rho$$

When $|\lambda|$ is large we notice that (i) gives

$$\text{(ii)}\qquad R_T(\lambda)=R_S(\lambda)\cdot(E+R_S(\lambda)\cdot K)^{-1}$$

At the same time we have the algebraic equality

$$(E+R_S(\lambda)\cdot K)^{-1}=E+R_S(\lambda)\cdot K\dot{(}E+R_S(\lambda)\cdot K)^{-1}iii$$

when $|\lambda|$ is large. Together with (ii) we obtain

$$\text{(iv)}\qquad R_T(\lambda)=R_S(\lambda)+R_S(\lambda)\cdot(E+R_S(\lambda)\cdot K)^{-1}\cdot R_S(\lambda)$$

Set $\alpha = ||S||$ which by assumption is < 1. Now $R_S(\lambda)$ is analytic in the exterior disc $\{\lambda| > \alpha\}$ so in this exterior disc $R_\lambda(T)$ differs from the analytic function $R_\lambda(S)$ by

$$\text{(v)} \qquad \lambda \mapsto R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Here K is a compact operator so the result in § XX entails that (v) extends to a meromorphic function in $\{|\lambda| > \alpha\}$. There remains to prove that eigenspaces at spectral points on the unit circle are finite dimensional. Let $e^{i\theta} \in \sigma(T)$ which by Proposition 2.2 is a simple pole and hence gives a Laurent series expansion

$$R_T(e^{i\theta} + z) = \frac{A_{-1}}{z} + A_0 + A_1 z + \ldots$$

By the general results from §§ there remains to show that A_{-1} has finite dimensional range. To see this we apply (iv) which gives the equation

$$R_S(e^{i\theta} + z) + R_S(e^{i\theta} + z) \cdot (E + R_S(e^{i\theta} + z) \cdot K)^{-1} \cdot R_S(e^{i\theta} + z)$$

To simplify notations we set $B(z) = R_S(e^{i\theta} + z)$ which is analytic in a neighborhood of $z = 0$ since $||S|| < 1$. Moreover, the operator $B(0)$ is invertible which give

$$\frac{A_{-1}}{z} + A_0 + A_1 z + \ldots = B(z) + B(z)(E + B(z) \cdot K)^{-1} B(z)$$

Since $B(0)$ is invertible we also have a Laurent series

$$\text{(vi)} \qquad (E + B(z) \cdot K)^{-1} = \frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots$$

and identifying the coefficient of z^{-1} gives

$$\text{(vii)} \qquad A_{-1} = B(0) A^*_{-1} B(0)$$

Next, from (vi) one has

$$\text{(viii)} \qquad E = (E + B(z) \cdot K)\left(\frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots\right) \implies A^*_{-1} + B(0) \cdot K \cdot A^*_{-1} = 0$$

Here $B(0) \cdot K$ is a compact operator and from (viii) and Fredholm theory it follows that A^*_{-1} has a finite dimensional range. Since $B(0)$ is invertible the same is true for A_{-1} by (vii) above. This finishes the proof of Theorem 2.3.