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I. The disc algebra $A(D)$

Contents

Introduction.

Denote by $A(D)$ the subalgebra of continuous functions on the closed unit disc $\bar{D}$ which are analytic in the open disc. One refers to $A(D)$ as the *disc-algebra*. If $f \in A(D)$ we have the Poisson representation

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot f(e^{i\theta}) \cdot d\theta \quad : z \in D$$

Since the polynomials in z is a dense subalgebra of $A(D)$ it follows that a Riesz measure μ on T is $\perp$ to $A(D)$ if and only if

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = 0 \quad : \quad n = 0, 1, 2, \ldots \tag{0.1}$$

In Section 1 we will show that (*) implies that μ is absolutely continuous and deduce some facts about boundary values of analytic functions in the open disc. Section 2 is devoted to properties of the disc algebra. Theorem 3.1 in the last section shows that the disc algebra is maximal in a quite strong sense. The proof relies upon results from several complex variables and has been inserted to give the reader a perspective upon the relevance of analytic functions in several variables even for problems which from the start are formulated in $\mathbf{C}$.

1. Theorem of the Brothers Riesz

At the 4:th Scandinavian Congres held in Stockholm 1916, Friedrich and Marcel Riesz proved the following:

1.1 Theorem *Let $E \subset T$ be a closed null set. Then there exists $\phi \in A(D)$ such that $\phi(e^{i\theta}) = 1$ when $e^{i\theta} \in E$ while $|\phi(z)| < 1$ for every $z \in \bar{D} \setminus E$.*

Before the construction of such peak functions we draw a consequence.

1.2. Theorem *Let μ be a Riesz-measure on T such that*

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = \quad : \ n = 1, 2, \ldots$$

Then μ is absolutely continuous.

Proof. Assume the contrary. Then there exists a closed null set E in T such that

$$\int_E d\mu(\theta) \neq 0 \tag{i}$$

Theorem 1.1 gives $\phi \in A(D)$ which is a peak function for E. For each positive integer m we have $\phi^m \in A(D)$. The hypothesis in Theorem 1.2 and (0.1) give:

$$\int_0^{2\pi} \phi^m(e^{i\theta}) \cdot d\mu(\theta) = 0 \quad : m = 1, 2, \ldots \tag{ii}$$

Now we get a contraction since ϕ was a peak function for E. Namely, this implies that

$$\lim_{m \to \infty} \phi^m(e^{i\theta}) \to \chi_E$$

where χ_E is the characteristic function of E and the dominated convergence theorem applied to $L^1(\mu)$ would give $\int_E d\mu = 0$. But this was not the case by (i) above and this contradiction gives Theorem 1.2.

Proof of Theorem 1.1

Let $E \subset T$ be a closed null set and $\{(\alpha_\nu, \beta_\nu)\}$ is the family of open intervals in $T \setminus E$. Since $b_\nu - a_\nu \to 0$ as ν increases, we can choose a sufficiently spare sequence of positive numbers $\{p_\nu\}$ such that

$$\sum p_\nu(\beta_\nu - \alpha_\nu) < \infty \quad \text{and} \quad \lim_{\nu\to\infty} p_\nu = +\infty$$

To each ν we define a function $g_\nu(\theta)$ on the open interval (α_ν, β_ν) by

$$g_\nu(\theta) = \frac{p_\nu(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} : \quad : \ell_\nu = \frac{\beta_\nu - \alpha_\nu}{2} \quad : \gamma_\nu = \frac{\beta_\nu + \alpha_\nu}{2} \tag{1}$$

Next, for each ν a variable substitution gives:

$$\int_{\alpha_\nu}^{\beta_\nu} \frac{d\theta}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} = \int_0^1 \frac{ds}{\sqrt{\frac{1}{4} - (s - \frac{1}{2})^2}} = C \tag{2}$$

where C is a positive constant which the reader may compute. Next, (2) and the convergence of $\sum p_\nu(\beta_\nu - \alpha_\nu)$ imply the function

$$F(\theta) = \sum g_\nu(\theta) \tag{3}$$

has a finite L^1-norm. Here F is defined outside the null set E and since each single g_ν-function restrics to a real analytic function on (α_ν, β_ν) the same holds for F. Next, we notice that

$$\theta \mapsto \frac{(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} \geq 2 \quad \text{for all} \quad \alpha_\nu < \theta < \beta_\nu \tag{4}$$

In addition to this the reader can verify that

$$\frac{(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\alpha + s - \gamma_\nu)^2}} \geq \frac{\beta_\nu - \alpha_\nu}{\sqrt{s \cdot (\beta_\nu - \alpha_\nu - s)}} \quad : \quad 0 < s < \beta_\nu - \alpha_\nu \tag{5}$$

From (4-5) we can show that $F(\theta)$ gets large when we approach E. Namely, let N be an arbitrary positive integer. Then we find ν_* such that

$$\nu > \nu_* \implies p_\nu > N \tag{i}$$

Next, let $\delta > 0$ and consider the open set E_δ of points with distance $< \delta$ to E. If $\theta \in E_\delta$ we have $\alpha_\nu < \theta < \beta_\nu$ for some ν. If $\nu > \nu*$ then (i) and (4) give

$$F(\theta) > 2N \tag{ii}$$

Next, set

$$\gamma = \min_{1 \leq \nu \leq \nu_*} \rho_\nu \cdot \sqrt{\beta_\nu - \alpha_\nu} \tag{iii}$$

Let us now consider some $1 \leq \nu \leq \nu_*$ and a point $\theta \in E_\delta$. which belongs to (α_ν, β_ν). Since $E \cap (\alpha_\nu, \beta_\nu = \emptyset$ we see that

$$\theta - \alpha_\nu < \delta \quad \text{or} \quad \beta_\nu - \theta < \delta \tag{iv}$$

must hold. In both cases (4) gives:

$$g_\nu(\theta) \geq \frac{\rho_\nu \cdot \sqrt{(\beta - \nu - \alpha - \nu}}{\sqrt{\delta}} \geq \frac{\gamma}{\sqrt{\delta}} \tag{v}$$

With γ fixed we find a small δ such that the right hand side is $> N$ and together with (ii) it follows that

(vi) $$\theta \in E_\delta \setminus E \implies F(\theta) > N$$

The construction of ϕ. The Poisson kernel gives the harmonic function:

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 + \cos(\theta - t)} \cdot F(t)dt \quad : \quad re^{i\theta} \in D$$

Since $F \geq 0$ we have U it is ≥ 0 in D and by (vi) $U(z)$ increases to $+\infty$ as z approaces E. More precisely, the following companion to (vi) holds:

Sublemma To every positive integer N there exists $\delta > 0$ such that

$$U(z) > N \quad : \quad z \in D \cap E^*_\delta$$

where $E^*_\delta = \{z \in D : \ dist(z, E) < \delta\}$.

Now we construct the harmonic conjugate:

$$V(re^{i\theta}) = \frac{1}{\pi} \int_0^{2\pi} \frac{r \cdot \sin(\theta - t)}{1 + r^2 + \cos(\theta - t)} \cdot F(t)dt \quad : \quad re^{i\theta} \in D$$

We have no control for the limit behaviour of $V(re^{i\theta})$ as $r \to 1$ and $e^{i\theta} \in E$. But on the open intervals (α_ν, β_ν) where F restricts to a real analytic function there exists a limit function V^*:

$$\lim_{r \to 1} V(re^{i\theta}) = V^*(e^{i\theta)}) \quad : \quad \alpha_\nu < \theta < \beta_\nu$$

Thus, V^* is a function defined on $T \setminus E$. Similarly, $U(re^{i\theta})$ has a limit function $U^*(e^{i\theta})$ defined on $T \setminus E$. Now we set

(*) $$\phi(z) = \frac{U(z) + iV(z)}{U(z) + 1 + iV(z)} \quad : \quad z \in D$$

This is an analytic function in D. Outside E we get the boundary value function

$$\lim_{r \to 1} \phi(re^{i\theta}) = \frac{U^*(e^{i\theta}) + iV^*(e^{i\theta})}{U^*(e^{i\theta}) + 1 + iV^*(e^{i\theta})}$$

The limit on E. Concerning the limit as $z \to E$ we have:

$$[1 - \phi(z)| = \frac{1}{|1 + U(z) + iV(z)|} \leq \frac{1}{1 + U(z)}$$

By the Sublemma the last term tends to zero as $z \to E$. We conclude that $\phi \in A(D)$ and here $\phi = 1$ on E while $|\phi(z)| < 1$ for al $z \in \bar{D} \setminus E$ which gives the requested peak function.

Remark. The proof of Theorem 1.1 above was constructive. There exist proofs using functional analysis and the Hilbert space $L^2(d\mu)$ attached to a Riesz measure on T. See the text-book [Koosis: p. 40-47] for such alternative proofs.

1.3 An application of Theorem 1.1

Let $f(z)$ be analytic in the open unit disc and assume there exists a constant M such that

$$\int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta \leq M \quad : \quad 0 < r < 1$$

Consider the family of measures on the unit circle defined by

$$\{\mu_r = f(re^{i\theta}) \cdot d\theta \ : \ r < 1\}$$

The uniform upper bound for their total variation implies by compactness in the weak topology that there exists a sequence $\{r_\nu\}$ with $r_\nu \to 1$ and a Riesz measure μ such that $\mu_{r_\nu} \to \mu$ holds *weakly.* In particular we have

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = \lim_{r_\nu \to 1} \int_0^{2\pi} e^{in\theta} f(r_\nu e^{i\theta}) \cdot d\theta$$

for every integer n. Since f is analytic the right hand side integrals vanish whenever $n \geq 1$ and hence μ is absolutely continuous by Theorem 1.2. So we have $\mu = f^*(\theta)d\theta$ for an L^1-function f^*. Now we construct the analytic function

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(\theta) \cdot e^{i\theta} d\theta}{e^{i\theta} - z}$$

When $z \in D$ is fixed the *weak* convergence applies to the θ-continuous function $\theta \mapsto \frac{e^{i\theta}}{e^{i\theta} - z}$ and hence

$$F(z) = \lim_{\nu \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r_\nu e^{i\theta}) e^{i\theta} d\theta}{e^{i\theta} - z}$$

At the same time, as soon as $|z| < r_\nu$ one has Cauchy's formula:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r_\nu e^{i\theta}) \cdot r_\nu e^{i\theta} \cdot d\theta}{r_\nu \cdot e^{i\theta} - z}$$

Since this hold for every large ν we can pass to the limit and conclude that $F(z) = f(z)$ olds in D. Hence $f(z)$ is represented by the Cauchy kernel of the $L^1(T)$-function $f^*(\theta)$. At this stage we apply *Fatou's theorem* to conclude that

$$\lim_{r \to 1} f(re^{i\theta}) = f^*(\theta) \quad \text{holds almost everywhere}$$

Moreover, one has convergence in the L^1-norms:

$$\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta} - f^*(\theta)| = 0$$

Thus, thanks to Theorem 1.2 the $L^1(T)$- sequence defined by the functions $\theta \mapsto f(re^{i\theta})$ converges almost everywhere to a unique limit function $f^*(\theta) \in L^1(T)$.

1.4 Exercise. Show that for every Lebesgue point θ_0 of $f^*(\theta)$ there exists a radial limit:

$$\lim_{r \to 1} f(re^{i\theta_0}) = f^*(\theta_0)$$

1.5 Exercise. In general, let K be a compact subset of D and μ a Riesz measure supported by K which is $\perp$ to analytic polynomials, i.e.

$$\int z^n \cdot d\mu(z) = 0$$

hold for all $n \geq 0$. Use the existence of peaking functions in $A(D)$ to conclude that if $E \subset T$ is a null-set for linear Lebesgue measure $d\theta$, then E is a null-set for μ. In particular, if K contains a relatively open set given by an arc α on the unit circle, then the restriction of μ to α is absolutely continuous

2. Principal ideals in the disc algebra.

Let $A(D)$ be the disc algebra. The point $z = 1$ gives a maximal ideal in $A(D)$:

$$\mathfrak{m} = \{f \in A(D) \quad : f(1) = 0\}$$

Let $f \in A(D)$ be such that $f(z) \neq 0$ for all z in the closed disc except at the point $z = 1$. The question arises if the principal ideal generated by f is dense in $\mathfrak{m}$. This is not always true. A counterexample is given by the function

$$f(z) = e^{\frac{z+1}{z-1}}$$

Following the appendix in [Carleman: Note 3] we give a sufficient condition on f in order that its principal ideal is dense in $\mathfrak{m}$. Namely, since $f(z(\neq 0$ except when $z = 1$ there exists the analytic function

$$f^*(z) = \exp\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \log|\frac{1}{f(e^{i\theta})}| \cdot d\theta\}$$

We say that f has no logarithmic reside a $z = 1$ if $f = f*$ and now the following holds:

2.2 Theorem. *If f has no logarithmic residue then $A(D)f$ is dense in $\mathfrak{m}$.*

Proof. With $\delta > 0$ we choose a continuous function $\rho_\delta(\theta)$ on T which is equal to $\log|\frac{1}{f(e^{i\theta})|}$ outside the interval $(-\delta, \delta)$ while

(i) $$0 < \rho_\delta(\theta) < \log|\frac{1}{f(e^{i\theta})}| \quad : -\delta < \theta < \delta$$

Next, let $\phi \in \mathfrak{m}$ and set

(ii) $$\omega_\delta(z) = \phi(z) \cdot \exp\{-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \rho_\delta(\theta) \cdot d\theta\}$$

It follows that

(iii) $$\big|\omega_\delta(z) \cdot f(z) - \phi(z)\big| = |f(z)| \cdot |\phi(z)| \cdot \big| 1 - \exp\{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \big[\log\frac{1}{|f(e^{i\theta})|} - \rho_\delta(\theta)\big] \cdot d\theta\}\, |$$

Exercise. Show that the limit of the right hand side is zero when $\delta \to 0$ and conclude that ϕ belongs to the closure of the principal ideal generated by f.

2.6 Some facts about $A(D)$

The disc algebra $A(D)$ is a uniform algebra, where the spectral radius norm is equal to the maximum over the closed disc. By the maximum principle for analytic functions in D one has $|f|_D = |f|_T$. One therefore calls T the *Shilov boundary* of $A(D)$. A notable point is that $A(D)$ is a Dirichlet algebra which means that the linear space of real parts of functions restricted to T is a dense subspace of all real-valued and continuous functions on T. From XX we recall that if $\rho(\theta)$ is real-valued and continuous on T then $\rho = \mathfrak{Re}(f)$ on T for some $f \in A(D)$ if and only if the function

$$z \mapsto \int_0^{2\pi i} \frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta} - z|^2} \cdot \rho(\theta) d\theta$$

extends to a continuous function on the closed disc. For example, every C^1-function on T belongs to $\mathfrak{Re}(A(D))$.

2.7 Wermer's maximality theorem. A result due to J. Wermer asserts that $A(D)$ is a maximal uniform algebra. It means that if $f \in C^0(T)$ is such that the closed subalgebra of $C^0(T)$ generated by f and z is not equal to $C^0(T)$, then f must belong to $A(D)$. Another way to phrase the result is that whenever $f \in C^0(T)$ is such that

$$\int_0^{2\pi} e^{ik\theta} \cdot f(e^{i\theta}) \cdot d\theta \neq 0$$

holds for at least one positive integer k, then $[z, f]_T = C^0(T)$.

Outline of the proof. Let $f \in C^0(T)$ and consider the uniform algebra $B = [z, f]_T$ on the unit circle. Now there exists the maximal ideal space $\mathfrak{M}_B$ whose points correspond to multiplicative functionals on B. If $p \in \mathfrak{M}_B$ and p^* is the corresponding multiplicative functional it is clear that there exists a unique point $z(p) \in D$ such that $p^*(f) = f(z(p))$ for every f in the subalgebra $A(D)$ of B. If $z(p) \in T$ holds for every p then the B-element z is invertible. But this means that B contains both $e^{i\theta}$ and $e^{-i\theta}$ and by Weierstrass theorem they already generate a dense subalgebra of $C^0(T)$. So if $B \neq C^0(T)$ there must exist at least one point $p \in \mathfrak{M}_B$ such that $z(p)$ stays in the open unit disc. In fact, *every point* $z_0 \in D$ is of the form $z(p)$ for some p for otherwise $\frac{1}{z-z_0}$ belongs to B and one verifies easily that the two functions on T given by $e^{i\theta}$ and $\frac{1}{e^{i\theta}-z_0}$ also generate a dense subalgebra of $C^0(T)$. There remains to consider the case when $p \mapsto z(p)$ sends $\mathfrak{M}_B$ onto the closed disc.

At this stage one employs a general result from uniform algebras. Namely, since every multiplicative functional has norm one it follows that that for every $p \in \mathfrak{M}_B$ there exists a probability measure μ_p on the unit circle such that

$$(*) \qquad p^*(g) = \int_T g(e^{i\theta}) \cdot d\mu_p(\theta) \quad \text{hold for all} \quad g \in B$$

Now we use that $A(D)$ is a Dirichlet algebra. Namely, (*) holds in particular for $A(D)$-functions and since μ_p is a real measure we conclude that it must be equal to the Poisson kernel of the point $z(p)$. This proves to begin with that the map $p \to z(p)$ is *bijective.* So for every $g \in B$ we get a continuous function on the closed unit disc defined by

$$g^*((z(p) = p^*(g)$$

But (*) above means that g^* is the harmonic extension to D of the boundary function g on T. Finally, since B is algebra one easily verifies that when every B-function is harmonic in D, then B consists of complex analytic functions only. This means precisely that $B = A(D)$. At this stage we conclude that when $B = [z, f]_T$ and $B \neq C^0(T)$ is assumed, then $f \in A(D)$ holds which is the assertion in Wermer's maximality theorem.

3. Relatively maximal algebras

Introduction. An extension of Wermer's maximality theorem was proved in [Björk] and goes as follows. Let K be a closed subset of $\bar{D}$ whose planar Lebesgue measure is zero. We also assume that K contains T and that $\bar{D} \setminus K$ is connected. Finally we assume that there exists some some open interval on T which does no belong to the closure of $K \setminus T$. In this situation the following holds:

3.1. Theorem. *Let $f \in C^0(K)$ be such that the uniform algebra $[z, f]_K \neq C^0(K)$. Then f extends from K to an analytic function in $D \setminus K$.*

Remark. The case when K is the union of T and a finite set of Jordan arcs where each arc has one end-point on T and the other in the open disc D is of special interest. If these Jordan arcs are not too fat, then f extends analytically across each arc which means that the restriction of f to T must belong to the disc-algebra. This case was a motivation for Theorem 3.1 since it is connected to the problem of finding conditions on a Jordan arc J in order that it is locally a removable singularity for continuous functions g which are analytic in open neighborhoods of J. The interested reader may consult [Björk:x] for a further discussion about this problem where comments are given by Harold Shapiro about the connection to between Theorem 3.1 and results by Privalov concerning analytic extensions across a Jordan arc.

Proof of Theorem 3.1. The proof will employ the *Local maximum Principle* by Rossi which is a powerful tool to study uniform algebras whose Shilov boundary is a proper subset of the maximal ideal space. Let us then start the proof. Set

$$B = [z, f]_K$$

Since $B \neq C^0(K)$ is assumed there exists a non-zero Riesz measure μ on K which annihilates B. Notice that μ can be complex-valued. Let π be the projection from $\mathfrak{M}_B$ into D which means that when z is regarded as an element in B then its Gelfand transform $\widehat{z}$ satisfies

$$\widehat{z}(p) = \pi(p) \quad : \, p \in \mathfrak{M}_B$$

As usual K is identified with a compact subset of $\mathfrak{M}_B$. If $e^{i\theta} \in T$ we use that it is a peak point for $A(D)$ and hence also for B. This entails that the fiber $\pi^{-1}(e^{i\theta})$ is reduced to the natural point $e^{i\theta} \in K$. Next, since we assume that K has planar measure zero we know from XX that the uniform algebra on K generated by rational functions with poles outside K is equal to $C^0(K)$. Since $z \in B$ and $B \neq C^0(K)$ it follows that $\pi^{-1}(D \setminus K) \neq \emptyset$. We are going to prove that the fiber above every point in $D \setminus K$ is reduced to a single point and for this purpose we define the following two analytic functions in the open set $D \setminus K$:

$$(*) \qquad W(z) = \int_K \frac{f(\zeta) \cdot d\mu(\zeta}{\zeta - z} \quad \text{and} \quad R(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z}$$

The main step in the proof is to show that if $z \in D \setminus K$ and $\xi \in \pi^{-1}(z)$ then the Gelfand transform $\widehat{f}$ satisfies:

$$(**) \qquad \widehat{f}(\xi) \cdot R(z) = W(z) \quad : \forall \, \xi \in \pi^{-1}(z)$$

Here $R(z)$ it cannot be identically zero in $D \setminus K$ for then the Riesz measure μ would be identically zero. If $R(z) \neq 0$ for some $z \in D \setminus K$ then (**) entails that the fiber $\pi^{-1}(z)$ is reduced to a single point. This hold for all points outside the eventual discrete zero-set of R and when a fiber $\pi^{-1}(z)$ is reduced to a single point the meromorphic function $\frac{W}{R}$ has a value taken by the continuous Gelfand transform of f at this unique fiber-point. This implies that $\frac{W}{R}$ is bounded outside the zeros of R and therefore analytic in the whole set $D \setminus K$. From this it follows easily that (**) implies that al fibers are reduced to single points and the analytic function $\frac{W}{R}$ in $D \setminus K$ is identified with the restriction of $\widehat{f}$ to this open set in the maximal ideal space of B. So there remains to give:

Proof of (**). Since μ annihilates the functions z^N and $z^N \cdot f(z)$ for every $N \geq 0$ we have

$$\int_K \frac{\bar{z} \cdot d\mu(\zeta)}{1 - \bar{z} \cdot \zeta} = \int_K \frac{\bar{z} \cdot f(\zeta) \cdot d\mu(\zeta)}{1 - \bar{z} \cdot \zeta} = 0 \quad \text{for every} \quad z \in D$$

Adding these zero-functions in (*) it follows that

$$(1) \qquad W(z) = \int_K \frac{(1 - |z|^2| \cdot f(\zeta) \cdot d\mu(\zeta)}{(\zeta - z)(1 - \bar{z}\zeta)} \quad \text{and} \quad R(z) = \int_K \frac{(1 - |z|^2 \cdot d\mu(\zeta)}{(\zeta - z)(1 - \bar{z}\zeta)}$$

The assumption that the closure of $K \setminus T$ does not contain T gives some open arc $\alpha = (\theta_0, \theta_1)$ on T which is disjoint from the closure of $K \setminus T$. The local version of the Brother's Riesz theorem from Exercise 1.5 implies that the restriction of μ to α is absolutely continuous. Hence, by Fatou's theorem there exist the two limits

$$(2) \qquad \lim_{r \to 1} W(re^{i\phi}) = W(e^{i\phi}) \quad : \lim_{r \to 1} R(re^{i\phi}) = R(e^{i\phi})$$

almost every on $\theta_0 < \phi < \theta_1$. Let us fix $\theta_0 < \phi_0 < \phi_1 < \theta_1$ where the radial limits in (2) exist for ϕ_0 and ϕ_1. Next, consider a point $z_0 \in D \setminus K$ and choose a closed Jordan curve Γ which is the union of the T-interval $[\phi_0, \phi_1]$ and a Jordan arc γ which is disjoint to the closure of $K \setminus T$ while z_0 belongs to the Jordan domain Ω bordered by Γ. We can always choose a nice arc Γ which is of class C^1 and hits T at $e^{i\phi_0}$ and $e^{i\phi_1}$ at right angles. Since Γ has a positive distance from $K \setminus T$ there exists $r_* < 1$ such that if $r_* < r < 1$ then the functions

$$(3) \qquad W_r(z) = W(rz) \quad : \, R_r(z) = R(rz)$$

are analytic in a neighborhood of the closure of Ω. Now we consider the set $\pi^{-1}(\Omega) = \Omega^*$ in $\mathcal{M}_B$ whose boundary in $\mathcal{M}_B$ is contained in $\pi^{-1}(\Gamma) = \Gamma^*$. If $Q(z)$ is an arbitrary polynomial the *Local Maximum Principle* gives

$$|Q(z_0)| \cdot [\hat{g}(\xi) \cdot R_r(z_0) - W_r(z_0)| \leq \big|Q \cdot (\widehat{f} \cdot R - W_r)\big|_{\Gamma^*} \tag{4}$$

Recall that $\pi^{-1}(T)$ is a copy of T Identifying the subinterval $[\phi_0, \phi_1]$ with a closed subset of $\mathcal{M}_B$ we can write

$$\Gamma^* = \gamma^* \cup [\phi_0, \phi_1] \quad : \gamma^* = \pi^{-1}(\Gamma \setminus (\phi_0, \phi_1)) \tag{5}$$

Now (4) and the continuity of the Gelfand transform $\widehat{f}$ give a constant M which is independent of r such that the maximum norms

$$\big|\widehat{f} \cdot R - W_r)\big|_{\Gamma^*} \leq M \quad : r_* < r < 1 \tag{6}$$

Since $\widehat{f}(e^{i\theta}) = f(e^{i\theta})$ holds on T it follows from (2) that the maximum norms:

$$\delta(r) = \big|\hat{g} \cdot R_r - W_r|_{[\phi_0, \phi_1]} = 0 \tag{7}$$

tend to zero as $r \to 1$. Next, let $\epsilon > 0$. Runge's theorem gives a polynomial $Q(z)$ such that

$$Q(z_0) = 1 \quad : |Q|_\gamma < \frac{\epsilon}{M} \tag{8}$$

When $\xi \in \pi^{-1}(z_0)$ it follows from (6) that

$$|\widehat{f}(\xi)R(z_0) - W(z_0)| \leq \text{Max}\,(\epsilon, \big||Q|\big|_{[\phi_0, \phi_1]} \cdot \delta(r)) \tag{9}$$

Passing to the limit as $r \to 1$ we use that $\delta(r) \to 0$ together with the obvious limit formulas $R_r(z_0) \to R(z_0)$ and $W_r(z_0) \to W(z_0)$, and conclude that that

$$\big|\widehat{f}(\xi) \cdot R(z_0) - W(z_0)\big| \leq \epsilon \tag{10}$$

Since we can choose ϵ arbitrary small we get

$$\widehat{f}(\xi) \cdot R(z_0) = W(z_0) \quad : \xi \in \pi^{-1}(z_0) \tag{11}$$

Since $z_0 \in D \setminus k$ was arbitrary we have proved (**) and as explained after (**) it follows that

$$\pi^{-1}(D \setminus K) \simeq D \setminus K \tag{12}$$

3.2 The extension to K. At this stage we can easily finish the proof of Theorem 3.1. We have already found the analytic function $\widehat{f}(z)$ in $D \setminus K$ and it is clear that it extends to f on the free circular arc (θ_0, θ_1) of T. To see that $\widehat{f}$ extends to K and gives a continuous function on the whole closed unit disc we solve the Dirichlet problem for the continuous functions $\mathfrak{Re}\, f$ and $\mathfrak{Im}\, f$ on K and conclude that $\widehat{f}$ extends and moreover its boundary value function on K is equal to the restriction of f to K. The proof of Theorem 3.1 is therefore finished if we have shown the equality:

$$\mathcal{M}_B \simeq D$$

To see that this holds we put $U = \pi^{-1}(\mathcal{M}_B \setminus D)$ and notice that its boundary in $\mathcal{M}_B$ is contained in the closure of $K \setminus T$. Call, this compact set K_*. Since we have the free arc (ϕ_0, ϕ_1) and $D \setminus K$ is connected it follows that $\mathbf{C} \setminus K_*$ is connected, i.e. only the unbounded component exists. So by Mergelyan's Theorem polynomials in z generate a dense subalgebra of $C^0(K_*)$. But then the Local Maximum Principle implies that U must be empty and the proof of Theorem 3.1 is finished.