0.6 Taylor series and quasi-analytic functions.

The study of Taylor series of differentiable functions on the real line was considered in a general setting by Borel and Denjoy. These authors established several remarkable facts during the years 1910-1922. Among these one should mention Denjoy's result from 1921. He considered a real-valued C^∞-function f on a bounded open interval (a,b) whose derivatives have have finite maximun norms. For each non-negative integer k we put

$$(*) \qquad C_k(f) = \big(|f^{(k)}|_{a,b}\big)^{\frac{1}{k}}$$

where $|f^{(k)}|_{a,b}$ is the maximum of the k:th order derivative of f taken over (a,b). The question posed by Borel and Denjoy was to find coinditions in order that some a prioro estimates on the sequence $\{C_k(f)\}$ implies that f cannot be flat at a point $x_0 \in (a,b)$, unless it is identically zero. To say that f is flat at x_0 means that $f^{(k)}(x_0) =$ for all non-negative integers k. Denjoy proved that if

$$(**) \qquad \sum_{k=0}^{\infty} \frac{1}{C_k(f)} = +\infty$$

then f cannot be flat at a point $x_0 \in (a,b)$ unless it is identically zero. In 1923 Carleman established necessary and sufficient conditions for quasi.analyticity in lectures held at Sorbonne. They are exposed in § xx. A crucial result in Carleman's work goes as follows:

0.1 Carleman's inequality. Let n be a positive integer and denote by $\mathcal{F}_n$ the family of n times continuously differentiable functions f on the closed unit interval such that

$$(0.1.1) \qquad f^{(k)}(0) = f^{(k)}(1) = 0 \quad : 0 \leq k \leq n-1$$

and the L^2-integral is normalised so that

$$(0.1.2) \qquad \int_0^1 f(t)^2\, dt = 1$$

0.2 Theorem. *For each $n \geq 1$ and every $f \in \mathcal{F}_n$ one has the inequality*

$$\sum_{k=1}^{k=n} \frac{1}{C_k(f)^2} \leq \pi \cdot e$$

where e is Neper's constant.

We prove this result in § xx. It is very instructive since it demonstrates the usefulness of harmonic majorisations of subharmonic functions, i.e. the proof relies upon a detour into the complex domain. In § xx we also demonstrate how Theorem 0.2 gives a simple proof of' Denjoy's result.

0.3 Carleman's reconstruction theorem for real-analytic functions.

A real-valued C^∞-function f on the closed unit interval is real analytic if and only if there exist constant C and M such that

$$(0.3.1) \qquad \max_{0 \leq x \leq 1} |f^{(k)}(x)| \leq M \cdot k! \cdot C^k \quad : k = 1, 2, \ldots$$

The analyticity implies that f is determined by its derivatives at the origin. However, the Taylor series

$$\sum_{k \geq 0} f^{(k)}(0) \cdot \frac{x^k}{k!}$$

is in general only convergent for in a small interval $[0 \leq x < \delta$. In 1921 Borel posed the question how on determines $f(x)$ on the whole interval from the sequence $\{f^{(k)}(0)\}$. An affimative answer was given by Carleman in 1923 via solutions to a family of variational problems which goes as follows: Put $\alpha_k = f^{(k)}(0)$ for each $k \geq 0$. If N is a positive integer we denote by $\mathcal{H}_N$ the Hilbert space whose elements are $N-1$-times continuous differentiabke functions g on $[0,1]$, and in addition $g^{(N)}$ is square integrable, i.e. it belongs to $L^2[0,1]$. In "contemporary mathematics"

this means that H_N is a Sobolevs space. But of course the notion of weak L^2-derivatives was perfectly well understood long before and for example used extensively in work by Weyl before 1910. Inside $\mathcal{H}_N$ we have the subspace $\mathcal{H}_N(f)$ which coinsists of functions g such that

$$g^{(k)}(0) = f^{(k)}(0) \quad : k = 0, \ldots, N-1 \tag{0.3.2}$$

With these notatons one regards the variational problem

$$\min_{g \in \mathcal{H}_N(f)} \sum_{k=0}^{k=N} (\log(k+2))^{-2k} \cdot (k!)^{-2k} \cdot \int_0^1 g^{(k)}(x)^2 \, dx \tag{0.3.3}$$

Elementary Hilbert space methods yield a unique minimziing function denoted by f_N. These succesive solutions give a sequence $\{f_N\}$ where each $f_N)$ has at least $N-1$ cointinous derivatives. Less obvious is the following:

0.3.4 Theorem. *For each real-analytic function f the sequence $\{f_N\}$ converges uniformly together with all derivatives to f, i.e. for every $m \geq 0$ it holds that*

$$\lim_{N \to \infty} |f_N^{(m)} - f^{(m)}|_{0,1} = 0$$

Remark. Since every indivdual funtion f_n is determined by derivative of f up to order $N-1$ at $x = 0$ it means that one has attained a reconstruction of the real-analytic function f via these derivatives.

§ 0.4 General quasi-analytic classes.

Let $\mathcal{A} = \{\alpha_\nu\}$ be a non-decreasing sequence of positive real numbers. Denote by $\mathcal{C}_\mathcal{A}$ the family of all $f \in C^\infty[0,1)$ for which there exists a constants M and k which may depend on f such that

$$\max_{0 \leq x \leq 1} |f^{(\nu)}(x)| \leq M \cdot k^\nu \cdot \alpha_\nu^\nu \quad : \quad \nu = 0, 1, \ldots \tag{0.4.1}$$

One says that $\mathcal{C}_\mathcal{A}$ is a quasi-analytic class if every $f \in C_\mathcal{A}$ whose Taylor series is identically zero at $x = 0$ vanishes identically on $[0,1)$. A conclusive result about quasi-analyticy was proved by Carleman in 1922 and asserts the following:

0.4.1Theorem. *The class $C_\mathcal{A}$ is quasi-analytic if and only if*

$$\int_1^\infty \log\Big[\sum_{\nu=1}^\infty \frac{r^{2\nu}}{a_\nu^{2\nu}}\Big] \cdot \frac{dr}{r^2} = +\infty$$

'The proof of Theorem 0.4.1 is rather involved It is presented in the separate section § xx.

§ 0.5 . Quasi-analytic boundary values.

Another problem is concerned with boundary values of analytic functions whose set of non-zero Taylor-coefficients is sparse. In general, consider a power series $\sum a_n z^n$ whose radius of convergence equal to one. Assume that there exists an interval ℓ on the unit circle such that the analytic function $f(z)$ defined by the series extends to a continuous function in the closed sector where $\arg(z) \in \ell$. So on ℓ we get a continuous boundary value function $f^*(\theta)$. Let f be given by the series

$$f = \sum a_n \cdot z^n$$

Suppose that gaps occur and write the sequence of non-zero coefficients as $\{a_{n_1}, a_{n_2} \ldots\}$ where $k \mapsto n_k$ is a strictly increasing sequence. With these notations the following result is due to Hadamard:

0.5.1 Theorem. *Let $f(z)$ be as above and assume it has a continuous extension to some open interval on the unit circle where the boundary function $f^*(\theta)$ is real-analytic. Then there exists an integer M such that*

$$n_{k+1} - n_k \leq M \quad : k = 1, 2, \ldots$$

Hadamard's result was extended to the quasi-analytic case in [Carleman] . In particukar we may consider the case when f^* belongs to a Denjoy class $\mathcal{D}_{\mathcal{A}}$ for a sequence $\mathcal{A} = \{\alpha_\nu\}$ where the series in § xx diverges.

$$\sum \frac{1}{\alpha_\nu} = +\infty$$

In [ibid] it is proved that when this hold and $f(z)$ is not identically zero, then the gaps of its Taylor coefficients cannot be too sparse. However, in contrast to Hadamard's theorem the result is more involved and and the rate of increase depends upon $\{\alpha_\nu\}$. Up to the present date it appears that no precise descriptions of the growth of $k \mapsto n_k$ which would ensure unicity is known while one regards arbitrary Denjoy classes as above. So there remains many basic questions concerned with quasi-analyticity, and readers who would like to pursue this should first consult the subtle analysis which appears in Carleman's original work

§ 1. Proof of Theorem 0.2

We shall first establish a general inequality of independent interest. Let $0 < b_1 < \ldots < b_n$ be a strictly increasing sequence of positive real numbers where $n \geq 1$ is some integer. Let $\phi(z)$ be an analytic function in the right half-plane $\mathfrak{Re}\, z > 0$ which in addition extends to a continuous function on the imaginary axis. Assume that its maximun norm over the right half-plane is ≤ 1 and in addition

$$|z|^k \cdot \phi(z) \leq b_k^k \quad : k = 1, \ldots, n \tag{1.1}$$

1.2 Theorem. *For each ϕ as above and every real $a > 0$ one has the inequality*

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq \log \frac{1}{\phi(a)} \tag{1.2.1}$$

Proof. On the imaginary axis we consider the intervals

$$\ell_k = [e \cdot b_k, e \cdot eb_{k+1}] \quad : \ k = 1, \ldots, n-1 \quad : \ \ell_n = [eb_n, +\infty) \tag{i}$$

Since $\log e^{-1} = -1$ it is clear that (1.1) gives

$$\log |\phi(iy)| \leq -k \quad : y \in \ell_k \tag{ii}$$

Taking the negative intervals $-\ell_k = [-e \cdot b_{k+1}, -e \cdot b_k]$ and $-\ell_n = (-\infty, -eb_n$ we also have

$$\log |\phi(iy)| \leq -k \quad : y \in -\ell_k \tag{iii}$$

Moreover, since the maximum norm of ϕ is ≤ 1 one has

$$\log |\phi(iy)| \leq 0 \quad : \ -b_1 \leq y \leq b_1 \tag{iv}$$

Next, solving the Dirichlet problem we find the harmonic function u in the open right half-plane whose boundary values on $(-b_1, b_1)$ is zero ,while $u = -k$ in the the open intervals ℓ_k and $-\ell_k$ for every k. The principle of harmonic majorisation applied to the subharmonic function $\log\, |\phi(z)|$ entails that

$$\log\, |\phi(a) \leq u(a) \tag{v}$$

Now we evaluate $u(a)$ using Poisson's formula to represent harmonic functions in the right half-plane. For each $1 \leq k \leq n-1$ we denote by $\theta_a(k)$ the angle between the two vectors which join a to the end-points ieb_k and ieb_{k+1}. Computing the area of the triangle with corner points at a, ieb_k, ieb_{k+1} the reader may check that

$$\sqrt{a^2 + e^2 b_k^2} \cdot \sqrt{a^2 + e^2 b_{k+1}^2} \cdot \sin \theta_a(k) = a \cdot e \cdot (b_{k+1} - b_k) \tag{vi}$$

Finally, let $\theta_a(n)$ be the angle between the vector which joins a with ieb_n and the vertical line $\{x = a$. The reader may check with the aid of a figure that

$$\sin \theta_a(n) = \frac{a}{\sqrt{a^2 + e^2 b_n^2}} \tag{vii}$$

Exercise. Confirm via Poisson's formula that

$$u(a) = -\frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \theta_a(k)$$

Together with (v) it follows that

$$\frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \theta_a(k) \leq \log \frac{1}{\phi(a)|} \tag{viii}$$

The inequality $\sin t \leq t$ for every $t > 0$ implies that

(ix) $$\frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \sin(\theta_a(k)) \leq \log \frac{1}{\phi(a)|}$$

Next we use (vi-vii) to estimate $\{\sin(\theta_a(k))\}$. When $1 \leq k \leq n-1$ we have from (vi)

$$e^2 \cdot b_k \cdot b_{k+1} \cdot \sqrt{1 + \frac{a^2}{e^2 b_k^2}} \cdot \sqrt{1 + \frac{a^2}{e^2 b_{k+1}^2}} \cdot \sin \theta_a(k) = a \cdot e \cdot (b_{k+1} - b_k) \implies$$
$$e \cdot (1 + \frac{a^2}{e^2 b_1^2}) \cdot \sin\, \theta_a(k) \leq a \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}})$$

where the last inequality follows since $b_k \geq b_1$ for every k. We conclude that the left hand side in (ix) majorizes

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n-1} k \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}}) + \frac{2}{\pi} \cdot n \cdot \sin \theta_a(n)$$

Finally, (vii) gives

$$\sin \theta_a(n) = \frac{a}{eb_n} \cdot \frac{1}{\sqrt{1 + \frac{a^2}{e^2 b_n^2}}} \geq \frac{a}{eb_n} \cdot \frac{1}{1 + \frac{a^2}{e^2 b_1^2}}$$

From this we see that the left hand side in (ix) majorizes

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot (\sum_{k=1}^{k=n-1} k \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}}) + n \cdot \frac{1}{b_n})$$

Abel's summation formula identifies the last term with $\sum_{k=1}^{k=n} \frac{1}{b_k}$. Hence we have proved the requested inequality

(x) $$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq \log \frac{1}{\phi(a)}$$

1.3 A special case. Assume in addition to (1.1) that

(1.3.1) $$\phi(a) \geq e^{-a} \quad : a > 0$$

This gives

$$\log \frac{1}{\phi(a)} \leq a$$

So here Theorem 1.2 after division with a gives

(1.3.2) $$\frac{2}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq 1$$

Passing to the limit as $a \to 0$ it follows that

(1.3.3) $$\sum_{k=1}^{k=n} \frac{1}{b_k} \leq \frac{e\pi}{2}$$

Proof of Theorem 0.2. Final part.

We are given $f \in \mathcal{F}_n$ and put

$$\phi(z) = \int_0^1 e^{-zt} \cdot f(t)^2 \, dt$$

When $\mathfrak{Re} z \geq 0$ the absolute value $|e^{-zt}| \leq 1$ for all t on the unit interval. The normalisation in (0.1.2) implies that the maximum norm of ϕ is ≤ 1. Next, if $1 \leq k \leq n$ the vanishing in (0.1.1) and partial integration give

(i) $$z^k \cdot \phi(z) = \sum_{\nu=0}^{\nu=k} \binom{k}{\nu} \int_0^1 f^{(\nu)}(t) \cdot f^{(k-\nu)}(t)(t) \, dt$$

The Cauchy-Schwarz inequality estimates the absolute value of the right hand side by

(ii) $$\sum_{\nu=0}^{\nu=k} \binom{k}{\nu} \cdot ||f^{(\nu)}||_2 \cdot ||f^{(k-\nu)}||_2$$

At this stage we use a wellknown resut from calculus which entails that

$$||f^{(\nu)}||_2 \leq ||f^{(k)}||_k \quad : 0 \leq \nu \leq k$$

and from this the reader can check that (ii) is majorised by $2^k \cdot ||f^{(\nu)}||_k^2$. Hence

(iii) $$|z|^k \cdot |\phi(z)| \leq 2^k \cdot (||f^{(k)}||_2)^2 \quad : \ k = 1, 2, \ldots$$

Put

(iv) $$b_k = 2 \cdot (||f^{(k)}||_2)^{\frac{2}{k}} \implies |z|^k \cdot |\phi(z)| \leq b_k^k$$

Next, if $a > 0$ we have

$$\phi(t) = \phi(z) = \int_0^1 e^{-at} \cdot f(t)^2 \, dt \geq e^{-a} \cdot \int_0^1 f(t)^2 \, dt = e^{-a}$$

where the last equality holds by the normalsation in (0,xx). Now the special case in (1.3.3) gives

(v) $$\sum_{k=1}^{k=n} \frac{1}{b_k} \leq \frac{e\pi}{2}$$

Finally, we have the trivial inequality

$$||f^{(k)}||_2 \leq \max_{0 \leq t \leq 1} |f^{(k)}(t)|$$

It follows from (v) that

$$\frac{1}{b_k} \geq \frac{1}{2 \cdot C_k(f)}$$

and hence (v) gives the requested inequality in Theorem 0.2

Proof of Theorem 0.3.4

For each N we denote by $J_*(N)$ the minimum in the variational problem from (xxx). Among the competing functions we can choose f and hence

$$J_*(N) \leq J_N(f)$$

Now there exist constants C and M from (0.3.1) which entails that

$$J_N(f) \leq M \cdot \sum_{k=0}^{N} (\log(k+2))^{-k} C^{2k}$$

Since $\log(k+2)$ tends to $+\infty$, it is clear that the series

$$\sum_{k=0}^{\infty} (\log(k+2))^{-k} C^{2k} < \infty$$

We conclude that there exists a constant J_* such that

(i) $$J_*(N) \leq J_* \quad : N = 1, 2, \ldots$$

So if m is some positive integer and $N \geq m$ we have

(ii) $$\sum_{k=0}^{k=m} (\log(k+2))^{-2} \int_0^1 f_N^{(k)}(x)^2 \, dx \leq J_N \leq J_*$$

Now we recall the classic resut due to Arzela-Ascoli which implies that bounded sets in H_m give relatively compact subsets of $C^{m-1}[0,1]$. Since (ii) hold for each m, it follows by a standard diagonal procedure which is left to the reader that we can find a subsequence $\{g_\nu = f_{N_\nu}\}$ such that the sequence of derivatives $\{g_\nu^{(m)}\}$ converge uniformly for every m, i.e $g_\nu \to g_*$ holds in the space $C^\infty[0,1]$. Next, by (0.3.2) we have for each fixed integer $k \geq 0$:

$$f^{(k)}(0) = f_N^{(k)}(0) \quad : N \geq k+1$$

From this it follows that

(iii) $$f^{(k)}(0) = g_*^{(k})(0) \quad : k = 0, 1, 2 \ldots$$

Hence the C^∞-function

$$\phi = f - g_*$$

is flat at $x = 0$. Next, for a fixed integer k the uniform bound in (ii) gives

(iv) $$\int_0^1 \phi^{(k)}(x)^2 \, dx \leq J_* \cdot (\log(k+2))^{2k} \cdot (k!)^2$$

Moreover, for each $0 < x \leq 1$ the Cauchy-Schwartz inequality gives

$$\phi^{(k)}(x) = \int_0^x \phi^{(k+1)}(t) \, dt \leq \sqrt{\int_0^1 \phi^{(k)}(x)^2 \, dx}$$

and since (iv) hold for every k it follows that

$$\max_x |\phi^{(k)}(x)| \leq J_* \cdot (\log(k+2))^k \cdot k!$$

Since $k! \leq k^k$ this entails that

$$\mathcal{C}_k(\phi) =\leq J_*^{\frac{1}{k}} \cdot k \cdot (\log(k+2))$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k \log k}$ is divergent we conclude that

$$\sum_{k=1}^{\infty} \frac{1}{\mathcal{C}_k(\phi)} = +\infty$$

Hence Denjoy's result in xxx proves that ϕ is identically zero.

Final part of the proof. We have proved that $\phi = 0$ which means that

$$\lim_{k\to\infty} f_{N_k} = f$$

where the convergence holds in the space $C^\infty[0,1]$. Finally, by the compactness above this limit for an arbitrary convergent subsequence entails that the whole sequence $\{f_N\}$ converges to f which finishes the proof of Theorem 0,xx.