Let U be a utulity function. At time $t = 0$ a consummer has a capital K and if $c8t <)$ is the rate of consumption we have a differential equation

$$\dot{K}(t) = rK(t) - c(t)$$

where r is a positive constant ,i.e. a. discount rate. We seek

$$V)K) = \max_c \int_0^\infty U(c(t)) \cdot e^{-at}\, dt$$

where $a > 0$ is another constant. For a general utiliity function we cannot obtsan a "closed soluion". . But let us consider the "popular" case

$$U(c) = \sqrt{c}$$

In this case it turn out that the rate of consumption at every moment is proportional to the capital, i.e. tghere exists a constsnt $\beta > 0$ such thst

$$c(t) = \beta \cdot K(t)$$

Moreover, the whole maximum probelm is only well-posed when $a > 2r$ and when it holds we have the equation

$$(*) \qquad V(K) = \frac{1}{\sqrt{2a - r}} \cdot \sqrt{K}$$

It merans thst if the discount number r is too large compred to the "negstive attiitude" for futire utility expressed by e^{-at}, then the consumer tends to let cspitsl increase and eventuysl obtsin an srbitrary large value V, even if the in itisl capitsl id very small. To prove (*) one preoceeds as follows. let $c(t) = c$ be constsnt duriung a small initial time interval $0 \leq t \leq \Delta$. Then the value becomes

$$\sqrt{c} \cdot \Delta + V(K + rK\delta - c\delta)e^{-a\Delta} =$$

By a Tayklor expansion this becomes

$$V(K) + \Delta\sqrt{c} - V'(K) \cdot c - rK \cdot V'(K) - a \cdot V(K)] + o(\delta)$$

Now c is determined whrn we tske the maximum of

$$\sqrt{c} - -V'(K) \cdot c \implies \sqrt{c} = V'(K)/2$$

For this it follows thst the value funtion satsfies the ODE:

$$\frac{(}{1} 4V'(K) + rKV'(K) = aV(K)$$

with initial coindition $V(0) = 0$ snd from this the reader can check (*) and after show that (xx) hold for a constant β which the reader should determine.

A more general case. For an arbitary utili8iy function $U(c)$ one first weeks

$$\max_c U(c) - V'(K) \cdot c \implies U'(c) = V'(K)$$

Since $c \mapsto U'(c)$ by assumption is strictly increasing this determines c via $V'(K)$, i.e. if γ is the inverse of the function U' one has

$$c = \gamma \circ V'(K)$$

and V satisfies the ODE:

$$U \circ \gamma(V'(K)) + rKV'(k) - \gamma \circ V'(K) \cdot V'(K) = aV(K)$$

Here one needs a computer to analyze the solution and also to find conditions on the pair r, a in order that the maxium problm is well-posed in the sense thst the value function $V(K)$ is bounded.

Exercise. Consider the utility function

$$U(c) = \log{(c + 1)}$$

So here $U'(c) = \frac{1}{c+1}$ and from the above

$$\frac{1}{1+c} = V'(K) \implies \frac{1}{V'(K} - 1 = c$$

Now the reader is invited to solve the ODE and analyze for which pairs r, a the problem is well-posed. Of course, here it is tempting to firt perform some numericl expåeriments when vlues for r snd a are assigned from the stsrt, ssy thst you take $r = 0,1$ snd firdt try a rather large a to see if $K \mapsto V(K)$ stays bounded. To fullfill this exercise gives considerable credit in the course ! Cases when we for example have a iutility funvtion

$$U(c) = \sqrt{c} + c^{1/4}$$

leads to quite involed computations where even the search for numercial solutions is not easy.

The case of a riusky asset

At time $t = 0$ the consumerr puts intial capital n in a risky asset which means that $t \mapsto K(t)$ is a random vsriable, and while consumption takes place one gets the "stocastic equation"

$$dK = \mu \cdot K - \cdot \sigma K \cdot dW$$

Via Taylor's fornula and a similar argument as in the case of a safe asset with $U(c) = \sqrt{c}$ it follows that the V-function satisfies the ODE:

$$\frac{1}{4V'(K)} + \mu K V'(K) + \frac{\sigma^2 K^2}{2} \cdot V''(K) = aV(K)$$

The solution becomes

$$V(K) = \rho \cdot \sqrt{K}$$

where ρ is a constant and

$$\frac{1}{2\rho} + \frac{\mu\rho}{2} - \sigma^2 \cdot \rho \cdot \frac{1}{8} = a\rho \implies \frac{1}{\rho} = (2a - r + \sigma^2/) \| cdot \rho$$

Hence we find the equation

$$\rho = \frac{1}{2a - \mu + \sigma^2/4}$$

Notice that ρ, and hence also $V(K)$ decreases with σ which reflects that the consumer has an aversion to risk which follows from the concavity of the utiliity function. So compared to a rsiky asset the consumer only will prefer the risky asset if the positive rate $\mu > r$. A risk neutral case occurs when

$$\mu = \sigma^2/4 + r \tag{1}$$

A notable fact is that if the consumer at time zero csn put a fraction $0 < \gamma < 1$ in a risky asset and the rest $(1 - \gamma)K$ in a safe asset when (1) holds, then a clever choice of γ leads to a higher expected value $V(K)$. Let us remark that this optimal portfoiio has become popular in text-books dealing with abstract economic models. Here we remakt thst a more detailed study involves the volatility when the risky asset is chosen, i.e. once the consumer puts an amonunt in a risky asset and after maximises the V-function, even the time-dependent conumption is a random variable and the outcome $V(K)$ is a random variable too, where we so far only have discussed its mean-value, Using a computer and eventual Monte-Carlo simulations one can get a numerical insight about the random variable V, and get numerical numbers for its volatility.

A system of ODE:s

Consider the following ODE-system

$$(*) \qquad \dot{z}_1 = z_1^\alpha z_2^\beta - z_1 \quad \& \quad \dot{z}_2 = z_1^\alpha z_2^\beta - z_2$$

where α, β are positive constants and $\alpha + \beta < 1$. If $z_1 = z_2 = 1$ the solution to (*) with initial conditions $z_1(0) = z_2(0) = p$ stays at $(1, 1)$, i.e. $(1, 1)$ is a stationary point for (*). It turns out that every solution converges to this stationary point as the time variable increases.

1. Theorem. *For every pair of positive numbers a, b the unique solution to (*) with $z_1(0) = a$ and $z_2(0) = b$ tends to $(1, 1)$ as $t \to +\infty$.*

Exercise. Prove this. The hint is as follows: First (*) entails that

$$\dot{z}_1 + z_1 = \dot{z}_2 + z_2$$

Check that this gives a function $f(t)$ defined for $t \geq 0$ where $f(0) = 0$ and

$$\text{(i)} \qquad z_1(t) = e^{-t} \cdot (f(t) + a) \quad \& \quad z_2(t) = e^{-t} \cdot (f(t) + b)$$

and f satisfies the ODE:

$$\text{(ii)} \qquad \dot{f}(t) = e^{t(1-\alpha-\beta)} \cdot (f(t) + a)^\alpha \cdot (f(t) + b)^\beta i$$

This non-linear first order ODE cannot be solved in closed form. But in the speciai case $a = b$ we find a solution. Namely, let $a = b = M$ for some $M > 0$ and put $\gamma = \alpha + \beta$. Then

$$\frac{df}{(f + M)^\gamma} = e^{(1-\gamma)t} \implies$$

$$\frac{1}{1-\gamma} \cdot (f(t) + M)^{1-\gamma} = \frac{1}{1-\gamma} \cdot [e^{(1-\gamma)t} + C]$$

where the reader should determine the constant C from the initial condition $f(0) = 0$. It follows that

$$\text{(iii)} \qquad f(t) + M = [e^{(1-\gamma)t} + C]^{\frac{1}{1-\gamma}} = e^t \cdot [1 + Ce^{(\gamma-1)t}]^{\frac{1}{1-\gamma}}$$

Sinc $\gamma < 1$ we see that

$$\lim_{t\to\infty} e^{-t} \cdot f(t) = 1$$

and then (i) give the requested limits in the theorem. Next, suppose that $a \neq b$ sand consider for example the case $0 < a < b$. Now we consider the function $\phi(t)$ which satisfies the ODE:

$$\text{(ii)} \qquad \dot{\phi}(t) = e^{t(1-\alpha-\beta)} \cdot (\phi(t) + b)^\alpha \cdot (\phi(t) + b)^\beta i$$

with initial condition $\phi(0) = 0$. The reader can recognize that the ϕ-function increases faster than f and from the special case we have

$$(1) \qquad \lim_{t\to\infty} e^{-t} \cdot \phi(t) = 1 \implies \limsup_{t\to\infty} e^{-t} \cdot f(t) \leq 1$$

In a similar fashion we regard the ψ-function which satisfies $\psi(0) = 0$ and

$$\text{(ii)} \qquad \dot{\psi}(t) = e^{t(1-\alpha-\beta)} \cdot (\psi(t) + a)^\alpha \cdot (\psi(t) + a)^\beta i$$

This time $\psi \leq f$ and the special case applies with $M = a$ so thast

$$(2) \qquad \lim_{t\to\infty} e^{-t} \cdot \psi(t) = 1 \implies \liminf_{t\to\infty} e^{-t} \cdot f(t) \geq 1$$

Together (1-2) proves the theorem.

Remark. One can confirm the theorem via numerical experiments. In practice one is also interested in the rate towards the equilibrium point (1,1). To be precise, take some small $\delta > 0$ and with a given starting point (a, b) one seeks the time T when

$$|z_1(T] - 1| \leq \delta \,\&\, |z_2(T] - 1| \leq \delta$$

Here no analytic solutions are available so one must use the computer to get numeral solutions. For the special models in economy where asymptotic stability appears, it is of course interesting

- and in a sense very relevant - to find approximations of time values as above. The numeral examples show that T can be quite large even when δ is not too small, say that $\delta = 0.1$. Keeping α and β fixed the initial values a and b effect T.

2. A more general system. Let $\phi(z_1, z_2)$ be a typical utlity funation and consider the system

$$(**) \qquad \dot{z}_1 = \phi(z_1, z_2) - z_1 \quad \& \quad \phi(z_1, z_2) - z_2$$

One gets f as above which now satisfies the ODE

$$(2.1) \qquad \dot{f} = e^t \cdot \phi(e^{-t}(f(t) + a), e^{-t}(f(t) + b) \quad \& \quad f(0) = 0$$

When ϕ is a typical utility function of two variables one can expect that

$$(2.2) \qquad \lim_{t \to +\infty} e^{-t} \cdot f(t) = 1$$

exactly as in the very special case above. The reader is invited to use a computer and perform numerical experiments to check if (2.2) hold for some non-trivial ϕ-functions apart from $z_1^\alpha \cdot z_2^\beta$. Let us notice that with

$$f(t) = e^t \cdot y(t)$$

the y-function satisfies the ODE

$$\dot{y} + y = \phi(y(t) + ae^{-t}, y(t) + be^{-t})$$

which gives an indication that $y(t)$ has a limit when $t \to +\infty$ because ae^{-t} and be^{-t} both tend to zero. Of course, a numercial study using a more general ϕ-function as in (*) goes far beyond standard text-books where the popular choice $z_1^\alpha \cdot z_2^\beta$ appears in hundreds of text-books exposing models in economy. Let us only remark that studies of asymptotic limits to non-linear systems of ODE:s is an extensive subject in mathematics where pioneering work was done by Picard and Poincaré around 1880. But thanks to computers one can nowadsys at least perform numerical experiments in order to analyze the eventual existence of limits of solutions to a stationary point of the system (**) where the stationary point would be $z_1 = z_2 = p$ with

$$p = \phi(p, p)$$

Optimal copnsumption in a dynamic model

Let U be a utiliity function. During a time interval $[0,T]$ a consumer seeks maximal utlitiy expressed by a time dependent function $c(t)$ where $c(t)$ is the rate of consumption. We have also a budget constraint

$$(*) \qquad \int_0^T c(t) = A$$

and seek

$$(**) \qquad V = \max_c \int_0^T U(c(t))e^{-rt}\,dt$$

where $r > 0$ is a discount factor. Euler's equation shows that the maximum is attained when there exists a constant B such that

$$(1) \qquad U'(c(t)) = B \cdot e^{rt}$$

for every $0 \leq t \leq T$. Here B is determined via (*). Only in very special cases $c(t)$ can be expressed in a closed form. A popular example in text-books is to take $U(c) = c^\alpha$ for some $0 < \alpha < 1$. Consider as an example the case $\alpha = 1/2$. Now (1) gives

$$\frac{1}{2\sqrt{c(t)}} = Be^{rt} \implies c(t) = \frac{1}{2B^2}e^{-2rt}$$

Via (*) the reader can calculate B when A is given and find the maximmum in (**), where the value function V depends on the 3 parameters A, r, T. Using a computer one gets numeral solutioins when numerical values are assigned to A, T, r.

Exercise. Let us consider the utility function

$$U(c) = \sqrt{c} + ac^{1/3} + bc$$

where a and b are positive contants. Here Euler's equation becomes

$$\frac{1}{2} \cdot c^{-1/2} + \frac{a}{3} \cdot c^{-2/3} + b = Ce^{rt}$$

where the budget constraint determines C. Even for readers who like to perform algebraic manipulations it is quite hard to even vizualise the time dependent solution $c(t)$ to this equation.

Let us consider the numerial example with $a = b = 1$. Taking the time derivative in Euler's equation (1) we get he first order ODE:

$$U''(c(t)) \cdot \dot{c}(t) = Bre^{rt} \implies$$

$$-[\frac{1}{4} \cdot c^{-3/2} + \frac{2a}{9}c^{-5/3}] \cdot \dot{c} = Bre^{rt}$$

where B is determined via the budget constraint (*) where we take $A = 1$. Keeping r and T as "variable parameters" one may now try to exhibit numerical values for the V-function.

2. Salvage values.

Above the consumer must obey the budget constraint (*). We can also imagine that the consumer is allowed to take a loan which leads to determine

$$(2.0) \qquad \max_c \int_0^T U(c(t)) \cdot e^{-rt}\,dt + G(\int_0^T c(t)\,dt - A)$$

where G is an increasing function. A special, case could be that $G(s) = Ms$ for some positive constant M. So here we study variational problem with a salvage value, Let us consider the case $U(c) = \sqrt{c}$ which gives an analytic soluttion. Namely, the value function V from (**) for fixed T and r while A varies becomes

$$V(A) = \rho(r,T) \cdot \sqrt{A}$$

where the reader is invited to determine the ρ-function. Sdding the salvage value it is profitble to to take a lone if

$$V(A+\xi)-M\xi >= V(A)$$

for some $\xi > 0$. To be precise, for $\xi \geq 0$ we consider the functioin

$$\rho(T,r)\sqrt{A+\xi}-M\xi$$

and one finds the maximum when $0 \leq \xi < \infty$. For example, the derivative at $\xi = 0$ becomes

$$\frac{\rho(T,r)}{\sqrt{A}}-M$$

and if it is > 0 the cosnumer will take a loan. The reader is invited to give specific numerical examples where a loan will give a maximum to (2.1).

2.1 Consumption from a capital stock. A utility function $U(c)$ is given. At time $t = 0$ he consumer has a capital K_0. Due to consumption the capital changes via the ODE

$$\dot{K}(t) = \mu \cdot K(t) - c(t) \tag{i}$$

where $\mu > 0$ is a rate of interest. Now we seek

$$\max_c \int_0^T U(c(t))e^{-rt}\,dt + K(T) \tag{2.1.1}$$

Thus, saved capital at the terminal time is a salvage value. To solve this optimization probelm we should first express $K(T)$ via a chosen rate of consumption. Here (i) gives

$$\frac{d}{dt}(Ke^{-\mu t} = -c(t) \tag{ii}$$

and from this the reader should check that

$$K_0 - K(T) = e^{\mu T}\cdot \int_0^T e^{-\mu t}c(t)\,dt \tag{iii}$$

Hence 2.1.1 amounts to determine

$$V^* K_0 + \max_c \int_0^T U(c(t))e^{-rt}\,dt - e^{\mu T}\cdot \int_0^T e^{-\mu t}c(t)\,dt \tag{2.1.2}$$

Exercise. Find the maximal value V^* in closed form when $U(c) = \sqrt{c}$ while r, μ, A, T are given parameters.

A problem with a risky asset. Above a negative discount factor e^{-rt} appears. One may also consider a positive discount where the utility from consumption is expressed by

$$\int U(c(t))\cdot e^{rt}\,dt$$

An example could be when the consumtion is used for medical care where illness may increase over time and become more costy. With a budget constraint as in (*) the maximum is now attainded by a function $c(t)$ which increases eith t. Let us now assume that the conumer can put a certain initial capital in a risky asset - like an obligation- which may give extra capital at a random time. More precisely, a profit is received via a Poisson process. It means that one has a positive number λ and the probability to gain during a small time interval $[t, t+\Delta]$ is equal to $e^{-\lambda t}\lambda\cdot\Delta$. If this happens - via a lucky number from the lottery - the consumer gets extra capital K which can be used for consumption up to time T, and a part may also be kept as a salavage value. At time $t = 0$ the consumer can put a fraction a of the initial capital A to by an obligation whose reward becomes $a \cdot K$ if a lucky number comes up during the lottery. In practice K is quite large while λ is small. Now one seeks maximum of *expected utility*. Here the outcome from the initial choice is random. So we are confronted with a *stochastic optimization*. The solution to this optimiziation problem is far more involved compared to determinisitc models. The reason is that before an eventual profit in the lottery, the usual Euler equation cannot be used to determine the optimal plan of consumption because future is random. So we shall not try to carty out the details which lead to a solution of the stochastic optimization above. But in the next section we shal, consider a case where the necessary stochastic analysis can be carried out explicitly.

3. A random rescue by a helicopter.

First we first regard a familiar problem where one seeks

$$\min \int_0^T \dot{x}(t)^2 \, dt \tag{3.1}$$

with fixed end-values $x(0) = 0$ and $x(T) = A$. The Euler equation shows that mimimum is attaiend when $\dot{x}$ is a constant, i.e $\dot{x}(t) = A/T$ and the minimum becomes A^2/T. A stochastic optimization arises as follows: Imagine a person who starts at a place in a dry desert and must arrive to a lake at time T to get water in order to survive. The person, call him or her "Silly Bill" or "Silly Mariy" has a compass and knows the straight way to the lake and the distance A. The squared velociity $\dot{x}^2$ corresponds to kinetic energy which Bill or Mary try to minimize during the travel to the lake. Next, society feels pity for persons who walk in the desert and with no cost a helicopter is looking around to search Bill or Mary. The chance to find a person walking in the desert is ruled by a Poisson process with parameter λ, i.e. the probability to discover the person at a small time interval $[t, t + \Delta]$ is equal to

$$e^{-\lambda t}\lambda \cdot \Delta$$

Once the pilot in has discovered a person we imagine that the helicopeter comes to rescue in a very short time and gives the person free lift to the lake, So if the discovery is made at some time $0 < t < T$ then the loss of energy only comes from walking up to this time, i.e. given by

$$\int_0^t \cdot x(t)^2 \, dt$$

The question arises how Bill or Mary should behave before an eventual discovery. It is tempting to just stay at rest in the desert and hope that the helicopeter comes. But the probability that the pilot never discovers the person ïs $e^{-\lambda T}$. So this strategy is risky because the person dies from thirst if the lake is not reached before time T. So Mary who has studed some mathematics understands immediately thstit is necessary to start walking, but with a speed which initiallly is smaller than A/T since the helicopter might arrive to rescue. It turns out that expected energy is minimized when the person walks with a certain slowly increasing pace until the eventual arrival of the helicopter. To solve this stochastic optimization one uses a device which in the literature is referred to as "dynamici programming" , though one should recall that methods to solve various stochastic optimzation has been known for a long time in matheamticla physics, where can mention deep work Helmholtlz and Bolzmann more than a century ago. ¨

The solution. Introduce the function $V(A, T)$ defined as the minimum of *expected cost of energy* when the person has two wak the distance A during a time interval T. Notice that the V-function is definec without knowing the optimal strategy in advance ! But we can find a PDE for V. Namely, during a small time interval $[0, \Delta]$ we let the person walk with constant speed u, and then expected energy becomes

$$u^2 \cdot \Delta + V(A - u \cdot \Delta, T - \Delta) \cdot e^{-\Delta\lambda} \tag{i}$$

where we have used that $e^{-\Delta\lambda}$ is the probability that the helicopter does not arrive during the small initial interval. Via Taylor expansion (i) becomes

$$V(A, T) + (u^2 - V_A \cdot u - V_T - \lambda \cdot V)\Delta + o(\Delta)$$

The right hand side is minimized when $2u + V_A = 0$ and the definition of the V-function gives the PDE:

$$\frac{V_A^2}{4} + V_T + \lambda V = 0 \tag{ii}$$

To solve (ii) we try

$$V(A,T) = A^2 \cdot g(T)$$

and (ii) corresponds to the ODE:

(iii) $$g(T)^2 + g'(T) + \lambda g(T) = 0$$

To solve (iii) we put

(iv) $$g(T) = e^{-\lambda T} \cdot \phi(T)$$

and get the ODE:

$$\phi'(T) = -e^{-\lambda T} \cdot \phi(T)^2 \implies \frac{d\phi}{\phi^2} = -e^{-\lambda T} dT$$

It follows that there is a constant C such that

(v) $$\frac{1}{\phi(T)} = C - \frac{1}{\lambda} \cdot e^{-\lambda T}$$

Above T is a variable and the reader should confirm that

$$\lim_{T \to 0} g(T) = +\infty$$

and deduce that

$$C = \frac{1}{\lambda} \implies \phi(T) = \frac{\lambda}{1 - e^{-\lambda T}}$$

Hence we have proved that

$$g(T) = \frac{\lambda \cdot e^{-\lambda T}}{1 - e^{-\lambda T}} = \frac{\lambda}{e^{\lambda T} - 1}$$

While Bill complains that he never has studied mathematics and from his starting postion is uncertain how to start walking to save energy while he looks at the sky and hopes thst the heliopter arrives, Mary performs a clever computation and determines the optimal time dependent velocity until an eventual arrival of the helicopter. To begin with she finds from the above that

$$V(A,T) = A^2 \cdot \frac{\lambda}{e^{\lambda T} - 1}$$

So at time $t = 0$ the intial velocity should be

(1) $$\dot{x}(0) = \frac{V_A}{2} = A \cdot \frac{\lambda}{e^{\lambda T} - 1}$$

The reader should check that (1) is $< \frac{A^2}{T}$ and $\dot{x}(0) < \frac{A}{T}$. The last inequality reflects the intuitive fact that the eventual arrival of the helicopter means that the clever Mary initially moves with slower speed compared to the case when no helicopter is present.

Next, when $0 < t < T$ and the helicopter has not arrived up to time t, the time dependent function $t \mapsto x(t)$ satisfies the following ODE: At time t Mary has walked a distance $x(t)$ so there remains the distance $A - x(t)$ to the lake which must be reached before time $T - t$. Now one has the definite result:

1. Theorem. *When Mary walks in order to minimize expected energy the optimal velocity prior to the eventual arrival of the helicopter satisfies the differential equation*

(1.1) $$\dot{x}(t) = \frac{\lambda \cdot (A - x(t))}{e^{\lambda(T-t)} - 1} = (A - x(t)) \cdot \frac{\lambda \cdot e^{\lambda t}}{e^{\lambda T} - e^{\lambda t}}$$

and the intial condition $x(0) = 0$ (1.1) has the solution

(1.2) $$A - x(t) = \frac{A}{e^{\lambda T} - 1} \cdot (e^{\lambda T} - e^{\lambda t}) \implies x(t) = A \cdot \frac{e^{\lambda t} - 1}{e^{\lambda T} - 1}$$

Remark. So if the helicopter never arrives then the velocity at time T becomes

$$\dot{x}(T) = \frac{\lambda \cdot A \cdot e^{\lambda T}}{e^{\lambda T} - 1} = \frac{\lambda \cdot A}{1 - e^{-\lambda T}} \tag{1.3}$$

The reader should check that (1.3) is strictly larger than $\frac{A}{T}$.

Notice also that with t and A fixed, the function in (1.3) taken with respect to λ is striclty increasing where the limit value as $\lambda \to 0$ is equal to $\frac{A}{T}$ On the other hand (1.3) becomes quite large when λ increases while A and T are kept fixed. This is reflected by the intuitive fact that with a large λ the person is optimistic and starts with a slow speed in the hope to get a free lift before time T, i.e. it is only when $t \to T$ that the person must start running with higher speed in order to arrive at $x = A$ at time T.

Exercise. Give details of the proof of (1.1-1.2) in the Theorem above !

2. Optimization with a salvage funtion. Above the peron is obliged to arrive at A when $t = T$. In addition to the probability of an arriving helicopter - where no charge occurs it it happens that the hrlicopter gives a free lift - we suppose that the person can rent a helicopter from a private company cost and get an immediate rescue. For example, if the person moves with a rather slow speed compared with (1.1) in Theorem 1, it may occur that $x(T) < A$. But the person has now the option to rent a helicopter at a time $T - \delta$ with small δ and quickly arrive at $x = A$ at time T, i.e. we imagine that a helicopter which picks up the person moves very fast to the terminal point $x = A$ If the person rents the helicopter we suppose that the cost is $C \cdot (A - x(T)$ where C is a fairly large constaant. To solve this new optimization problem we regard the value function $V(A, T)$ which was computed above. Now the person can choose a strategy with optimal speed $\dot{x}(t)$ before the free helicoppter eventually arrives where the new end-value $x(T)$ is equal to a for some $a < A$ and chosen to minimize the sum:

$$V(a, T) + C \cdot (A - a)$$

So it means that

$$2a\lambda = (e^{\lambda T} - 1) \cdot C \tag{2.1}$$

Only when $a < A$ holds in (2.1) the person is willing to rent a helicopter. Thus, if

$$2A\lambda \leq (e^{\lambda T} - 1) \cdot C$$

then the company who has a "saving helicopter" never gets a demand from persons moving as above, while the eventual resuce by the helicopter which is free of charge may appear. This salvage prblem leads to a further optimization problem. Namely, the company which has a helicopter which can provide a free lift at any moment should decide a price C in order to maximize *expected profit* while clever persons are moving as above and choose optimal strategies to minimize their expected cost of energy, plus eventual charge from the company's helicopter when $a < A$ holds in (2.1). The analysis of this new stochastic optimization problem, which has to be solved by the company which rents a helicopter is quite interesting. Let us only remark that the solution leads to a quite involved calculation where computers help to provide numerical solutions in order to get a feeling for this optimization problem. It goes wothout sayung that a solution to a problem of this nature wuold give a top-rate in the present course.

Another optimization problem.

Imagine a company where research is essential to discover a new item. A typical case would be to produce a new medicine which requires extensive research before it can be found and lead to profitable production. Let $[0, T]$ be a time interval and suppose that the company only gets profit if the discovery is found during this time interval. Let $t \mapsto \rho(t)$ be the intensity of research. So

$$R(t) = \int_0^t \rho(s)\, ds$$

is ackumulated research up to time t. We suppose that the non-decreasing R-function affect a time dependent Poisson parameter whose random process governs the probability that a new item is discovered. Thus, there exists a constant $a > 0$ such that

$$\lambda(t) = a \cdot R(t)$$

hold before an eventual discovery. By Poisson's equation this means that if the company invests in research with a given rate, then the probability for a discovery during a small time interval $[\tau, \tau + \Delta]$ becomes

$$a \cdot e^{-a \cdot \int_0^\tau R(s)\, ds} \cdot R(\tau) \cdot \Delta$$

If the discovery of the new item is found at a time t the company recieves a profit

$$\Pi(t) = A \cdot e^{-rt}$$

where r is some positive constant, i.e. the sooner a discovery is found the higher is the reward. At the same time the company has a cost for performed research up to this time. So net profit is given by an equation

$$\Pi_*(t) = \Pi(t) - B \cdot R(t)$$

where B is another positive constant. The company now tries to maximize expected profit, i.e. the problem is to find the optimal rate of research in order to maximize expected net profit. Above we assumed that the cost of research is constant, i.e. during the time period we imagine that the company already has employed people whose salaries remain fixed. It is intuitively clear that the company should invest in research as quick as possible in order to increase Poisson's λ-parameter. For a company with large resources the time derivative $\dot{R} = \rho$ can be made large. However, in practice limitations occur. One can imagine chemical processes or examinations of biological bodies which imply that the function $\rho(t)$ is bounded above by some constant. So one is led to consider a constrained optimization problem where one seeks to maximize expected net profit when

$$0 \leq \rho(t) \leq M$$

hold for every t. Adding this constraint the optimization problem has a Bang-bang solution. More precisely, during a time interval $[0, \tau]$ the company decides to choose $\rho(t) = M$ and after $\rho(t) = 0$ when $t > \tau$. If one regards such a strategy one can compute expected profit as a function of the switch time τ and solve the optimization problem. One also finds Π_* epxressed as a function of the parameters A, B, a, r and M.

Remark. The formal proof that Π_* is maximized by a Bang-bang strategy relies upon Pontryagin's general results in OCT.

A more involved case. Tne model above is not always realistic. It may occur that the input of research at an early time eventually does not affect the Poisson parameter. One can imagine fragile items which are studied during the research and if no disvovery has been made after a certain time, the information from early research becomes diminished, or even lost. Above we assumed that

$$\lambda(t) = a \cdot \int_0^t \rho(s)\, ds \tag{*}$$

Let us replace this by an equation of the form

$$\lambda(t) = a \cdot \int_0^t e^{\mu(s-t)} \cdot \rho(s)\, ds \tag{**}$$

where μ is a positive constant. It means that research performed during a small early time interval $[s, s + \delta]$ will no enlarge $\lambda(t)$ so much when $t > s$. When (*) is replaced by (**) the optimization problem becomes more involved and it is no longer clear if the company will choose a Bang bang solution. For example, if μ is large one may expect that the company will decide to perform some reseach at a late time value if no discovery has appeared since invested early research has diminished the Poisson parameter. For example, one may get a Bang-bang solution where several switch times appear. So the discussion shows that more *realistic optimization* *problems* become quite complicated. A further scenarium appears in a competitive world where our company is not alone in the search for a new item. But thanks to ”clever spies” we suppose that our company knows input of resarch at every moment performed by another company C^* and the resulting time dependent Poisson parsmrter which C^* has at every time. If a reward for a discovery only is given once, this means that our company is faced with a new optimization problem since expected net profit from reseach decays while the opponent is active. Models of this kind appear in ”real life” and require a considerable effort to solve a stochastic optimization of this kind.

Optimizing consumption of two commodities.

A consumer can share a given capital to buy two commodities. Let x snd y be the amount of them which gives a utility expressed by $U(x,y)$. Here U is a function defined for pairs of positive numbers. We assume that the first order partial derivatives U_x and U_y both are > 0 for all points $(x,y) \in \mathbf{R}_+^2$. In addition U is such that for every $A > 0$ the level curve $\{U = A\}$ can be expresed by a decreasing function

$$x \mapsto y_A(x)$$

whose second derivative $\frac{d^2 y_A}{dx^2} > 0$ for all $x > 0$. Examples of such utilitiy functions arise as follows: For every $m \geq 1$ and a pair of m-tuples $\{0 < a_k < 1\}$ and $0 < b_k < 1\}$ and an arbitrary m-tuple $\{c_k\}$ of positive numbers, we set

$$U(x,y) = \sum c_k \cdot x^{a_k} \cdot y^{b_k}$$

Next, for each $A > 0$ wee consider the open set $\{U > A\}$ and the reader should verify that this set is convex.

Given a utility function U ss above we consider the optimmzation problem

$$(*) \qquad \max_{xy} U(x,y)$$

where the maximum is taken over pairs (x,y) which satisfy a budget constraint

$$|px + qy = A$$

Here p and q are prices, i.e. both are > 0. The maximum problem has a unique solution, i.e. for every triple p,q,A there exists a unique pair $(x^*, y*)$ which maximises (*). Keeping p and q fixed we denote the maximum in (*) by $V(A)$ and then $A \mapsto V(A)$ is an increasing functioin. It turns out that the derivstive $V'(A)$ coincides with Lagrange's multiplier. More precisely, when (*) is maximized there exists $\lambda > 0$ and

$$U_x(x^*, y^*) = \lambda \cdot q \quad \& \quad U_y(x^*, y^*) = \lambda \cdot p$$

and then one has the equality

$$(**) \qquad \lambda = V'(A)$$

The reader should verify (**) and we remark that (**) appears as a popular equation in text-books devoted to economics since it gives an interpretation of Lagrange's multiplier.

Nujmerical experiments. Explicit solutions are easily found when

$$U(x,y) = x^a y^b$$

for a pair $0 < a, b < 1$. The reader is invited to treat the case $a = b = 1/2$ and analyze how $(x^*, y*)$ changes with a price vector (p,q) while A is kept fixed. Passing to a more general utilitiy function one must rely upon the computer which gives numerical solutions. A specific case is when

$$U(x,y)\sqrt{xy} + ax + by$$

where a, b are teo positive constants. With $A = 1$ and fixed equal prices $p = q = B$ the reader is invited to see how (x^*, y^*) changes with the pair a, b and evaluate numerically the maximal value V which with A and B kept fixed becomes a function of a and b.

Remark. It goes without saying that the results above can be extended to the case when $n \geq 3$ and the consumer distributes the incomle to buy n commodities where a orice vector $p_1, \ldots, p_n$ is given. So here a function $U(x_1, \ldots, x_n)$ is defined when the n-vector has $x_k >=$ for each k. The optmisation problem for a given price vector and income has a unique solution x^* when U is such that the first order partial derivatives are > 0. and the sets $\{U > C\}$ are strictly convex for every $C > 0$.

Composed ODE:s

Consider the variational problem

$$\text{(i)} \qquad \min \int_0^T [e^{rt}x(t)^2 + \dot{x}(t)^2]\, dt$$

with fixed end-values $x(0) = 0$ and $x(1) = 1$, while r is a constant which can be < 0 or > 0. The Euler equation becomes

$$\text{(i)} \qquad \ddot{x}(t) = e^{rt}x(t)$$

Without solving (i) explicitly one cank keep trace of this function and introduce another ODE where a time dependent function $z(t)$ satsfies

$$\dot{z}(t) = e^{rt}x(t)^2 + \dot{x}(t)^2 \quad \& \quad z(0) = 0$$

Now one gets a numerial solution of the z-function and obtsan plot over given time intervals $[0, T]$. In this way the minmal values in our variational problem can be found numerically while the time intevral T varies.

The reader is invited to perform numerical experiments and also confirm them after having computed (by hand) the analytic solutions to our given problem. To be precise, the Euler equstion has the solution

$$x(t)C(e^{\sqrt{r}t} - 1)$$

where the constant C is determined by the end-value $x(T) = 1$, and after this the reader can try to exhibit a "closed formula" for the z-function. '

A secomd example. Here one regsrds an optimization with a salvage value:

$$(*) \qquad \max \int_0^T \sqrt{\dot{x}(t)} \cdot e^{-rt}\, dt + A(K - x(T))$$

where r and A are > 0 and the maximum is found in the family of non-negative functions $c(t)$ defined on $[0, T]$ with the constraint

$$x(T) = \int_0^T \cdot x(t)\, dt \leq A$$

The Euler equstion gives the following ODE for an extremal x-function:

$$\dot{x}(t) = C \cdot e^{-2rt}$$

where C is a constant which must satisfy

$$C \cdot \int_=^T e^{-2rt}\, dt \leq A$$

and then C is found via the salvage term while one maximizes (*). To carry out the computations and get an equation for trhe maximal value V when r, T, A are given is rather involved. Using the computer the reader is invited to perform numerical experiments. To be orecise, using the *transversality equation* when a salvage value is present, the reader can get hold of the constabnt C and after find a numerical vaklue fior V by regarding the z-function which satisfies the ODE:

$$\dot{z} = e^{rt} \cdot \dot{x}(t)^2$$

This example illustrates a general situation where one can perform numerical investigations in more general cases. Let us also remakr that apart from the standard device which for example is implemented in Mathemacitca, it often requires extra "organisation" to express the compound ODE:s which evaluates $z(T)$ while r, T, A are nuumerically assigned from the start.

ODE:s and the use of computers.

Very few ODE:s can be solved "nalayitcslly", i.e. via expressions of established functions. So in most applications one must rely upon numerical solutions. At the same time experiments by computers can suggest new theoretical results. Let us illuminate this by discussing a second order ODE of the form

$$\ddot{y} = y + b(t)$$

where we seek solutions defined for $t \geq 0$. If we for example takee $b(t) = a$ for some constant a then we get the soluti

$$y(t) = -a + a \cdot e^{-t}$$

where $y(0) = 0$ and

$$\lim_{t \to +\infty} y(t) = -a$$

So this function is decreasing but converges to a finite negative number. Suppose now thst $b(t)$ is defined on $[0, +\infty)$ where

$$0 \leq b(t) \leq 1$$

hold for every t. Now one can ask if the ODE:

$$\ddot{y} = y + b(t)$$

has a solution where $y(0) = 0$ and a finite limit as in (xx) exist, i.e.

$$\lim_{t \to +\infty} y(t) = A$$

hold for some real number A. This problem is a challenge since we cannot solve the ODE explicitly. It is therefore tempting to perform experiments by a computer. One such experiment can be done as follows: With a large positive integer, say $N = 10^3$ we define $b(t)$ as follows. on the interval $[1, N]$ in a random way , i.e. on each interval $[k, k+1]$ we let $b(t)$ be $+1$ or 0, where the choice of the sign is random and picked form a Monte Carlo box. So here $b(t)$ is picewise constant with jumps at the integers $1, 2, \ldots, N$. When $t > N$ we declare that $b(t) = 0$. Fir each such b-function we find a solution $y(t)$ where $y(t)$ is contonoud on $[0, +\infty)$ with $y/(0) = 0$ and

$$y(t) = y(N)e^{N-t} \quad : t \geq N$$

Calculus of variation.

We are given a function $f(x,z,t)$ of three variables. For every C^1-function $x(t)$ of the variable t we assign its derivative $\dot{x}(t)$ and obtain the function

$$t \mapsto f(x(t), \dot{x}(t), t)$$

Let $[0,T]$ be a given interval and A is a constant. Denote by $\mathcal{C}^1(A)$ the family of all C^1-functions $x(t)$ on $[0,T]$ such that $x(0)=0$ and $x(T)=A$. To each such function we set

$$(*) \qquad \mathcal{F}(x) = \int_0^T f(x(t), \dot{x}(t), t) \cdot dt$$

Here $x \mapsto \mathcal{F}(x)$ is a function whose domain of definition is $\mathcal{C}^1[A]$ and one refers to $\mathcal{F}$ as the evaluating functional. Next, a *bubble function* is a C^1-function $\phi(t)$ with $\phi(0) = \phi(T) = 0$.

A. Exercise. Use Taylor expansion to show that if $x \in \mathcal{C}^1[A]$ and ϕ is a bubble function, then

$$\text{(i)} \qquad \frac{\mathcal{F}(x+\epsilon\cdot\phi) - \mathcal{F}(x)}{\epsilon} = \int_0^T \big[\phi(t)\cdot f'_x(x(t),\dot{x}(t),t) + \dot{\phi}(t)\cdot f'_{\dot{x}}(x(t),\dot{x}(t),t)\,\big]\cdot dt + o(\epsilon)$$

where $o(\epsilon)$ is small ordo, and show that partial integration gives

$$\text{(ii)} \qquad \int_0^T \dot{\phi}(t)\cdot f'_{\dot{x}}(x(t),\dot{x}(t),t)\,\cdot dt = \phi\cdot f'_{\dot{x}}|_0^T - \int_0^T \phi(t)\cdot \frac{d}{dt}\big(f'_{\dot{x}}(x(t),\dot{x}(t),t)\big)\,\cdot dt$$

Since ϕ is a bubble function the first term above vanishes and (i-ii) give

$$\frac{\mathcal{F}(x+\epsilon\cdot\phi) - \mathcal{F}(x)}{\epsilon} =$$

$$\text{(iii)} \qquad \int_0^T \big[\phi(t)\cdot[f'_x(x(t),\dot{x}(t),t) - \frac{d}{dt}\big(f'_{\dot{x}}(x(t),\dot{x}(t),t)\,\big)\,\big]\cdot dt + o(\epsilon)$$

Passing to the limit $\epsilon \to 0$ we can ignore the small ordo-term and obtain

$$\text{(iv)} \qquad \lim_{\epsilon\to 0} \frac{\mathcal{F}(x+\epsilon\cdot\phi) - \mathcal{F}(x)}{\epsilon} = \int_0^T \big[\phi(t)\cdot[f'_x(x,\dot{x},t) - \frac{d}{dt}\big(f'_{\dot{x}}(x,\dot{x},t)\,\big)\,\big]\cdot dt$$

A.1 Extremal x-functions. Suppose that $x^*(t)$ is extremal in the sense that

$$\mathcal{F}(x) \le \mathcal{F}(x^*)$$

hold for all competing $\mathcal{C}^1(A)$-functions. Then (iv) must be zero for all bubble-functions ϕ, i.e. the right hand side vanishes for all such ϕ. This implies that the function

$$\text{(A.1.1)} \qquad t \mapsto f'_x(x^*(t),\dot{x}^*(t),t) - \frac{d}{dt}\big(f'_{\dot{x}}(x^*(t),\dot{x}^*(t),t) = 0 \quad \text{for all} \quad 0 \le t \le T$$

One refers to (A.1.1) as the Euler equation which from the above must to be satisfied by every extremal $\mathcal{F}$-function $x^*(t)$.

Notice that the same conclusion holds if we instead suppose that $x_*(t)$ is an extremal function in the sense that

$$\mathcal{F}(x) \ge \mathcal{F}(x^*)$$

hold for all competing functions. Thus, the Euler equation is necessary for an extremal which yields a maximum or a minimum to the variational problem.

A.2 Exercise. Consider the case when $f = f(x,\dot{x})$ is independent of t. Use rules for differentiation to show the equaluity below for every function $x)t)$:

$$\dot{x}(t)\cdot(f'_x(x(t),\dot{x}(t)) - \frac{d}{dt}\big(f'_{\dot{x}}(x(t),\dot{x}(t))) =$$

$$\frac{d}{dt}[(f(x(t),\dot{x}(t)) - \dot{x}(t)\cdot f'_{\dot{x}}(x(t),\dot{x}(t))]$$

So ignoring very special solutions when $\dot{x}(t) = 0$ on some intervsl, the EWuler equation for an extremal solution is equivalent to the existence of a constant C such that

$$f(x(t), \dot{x}(t) - \dot{x}(t) \cdot f'_{\dot{x}}(x(t), \dot{x}(t)) = C \tag{A.2.1}$$

B. Examples. Consider the variatioinal problem with

$$\mathcal{F}(x) = \int_0^T (\dot{x}^2 + ax^2)\, dt \tag{B.1}$$

where a is a real number which can be > 0 or < 0, and the end-values are $x(0) = 0$ and $x(T) = 1$. It is tempting to seek a minimum. To begin with the reader csn check thst the Euler equation becomes

$$\ddot{x} = ax$$

If $a > 0$ this second order ODE has the solutlion

$$x(t) = \frac{1}{e^T - e^{-T}} \cdot (e^t - e^{-t})$$

which satisfies the end-value conditions. The case when $a < 0$ leads to a more involved analysis. To begin eith Euler's equation has the general solution

$$x(t) = c_1 \cdot \sin \sqrt{a}t + c_2 \cdot \cos \sqrt{a}t$$

The conditiion that $x(0) = 0$ gives $c_2 = 0$ and then

$$1 = x(T) = c_1 \cdot \sin \sqrt{a}T \tag{i}$$

If we spoecify $a = -1$ and $T = \pi$ then (i) *cannot be satsified* because $\sin \pi = 0$. So this example shows that a variational problem can be ill-posed and fail to have extremal solutions.

Exercise. Take $a = -1$ and let $T = \alpha \cdot \pi$ for some $0 < \alpha < \pi$. Kepinig the end-values $x(0) = 0$ and $x(T) = 1$ we find Euler's solution

$$x_*(t) = \frac{\sin t}{\sin T}$$

and the reader can check that

$$\mathcal{F}(x_*) = \int_0^T (\cos^2 t - \sin^2 t)\, dt$$

Now

$$\int_0^T (\cos^2 t - \sin^2 t)\, dt = \int_0^T \cos 2t\, dt = \frac{\sin 2T}{2}$$

As an example we take

$$T = \pi - \delta$$

where $0 < \delta < \pi/2$. When $\delta \to 0$ the reader should verify that

$$\lim_{\delta \to 0} \mathcal{F}(x_*) = -\infty$$

From this one can conclude that the varistional problem is ill-posed when $T = \pi$ in the sense that

$$\min_x \mathcal{F}(x) = -\infty$$

when we take competing functions $x(t)$ for which $x(0) = 1$ and $x(\pi) = 1$.

Legendre's test.

With $0 < T < \pi$ we have found the Euler solution $x_*(t)$ above. Recall that Euler*s equation only asets a necessary condition for an extremal. So there remains to investigate if $x_*(t)$ is a global minimum, which amounts to show that

$$\mathcal{F}(x_*) < \mathcal{F}(x_* + \phi)$$

hold for every non-zero bubble function ϕ on $[0,T]$. To prove that this indeed holds the reader should fiurt verify the equation

$$\mathcal{F}(x_* + \phi) = \mathcal{F}(x_*) + \int_0^T (\dot{\phi}^2 - \phi^2)\, dt$$

Hence (x) holds if

$$(*) \qquad \int_0^T \phi^2\, dt < \int_0^T \dot{\phi}^2\, dt$$

hold for every non-zero bubble function. To prove that (*) holds one can empoy series examnsiuons of bubble functions. More precisely, every function $\phi(t)$ defined on the interval $[0,\pi]$ for ehich $\phi(0) = \phi(\pi)$ has a sine-series expansion

$$\phi(t) = \sum_{k=1}^{\infty} c_k \cdot \sin kt$$

where the c-numbers are determined by

$$c_k = \frac{2}{\pi} \cdot \int_0^\pi \sin kt \cdot \phi(t)\, dt$$

The reader is invited to deduce (*) from this wellknown fact from the theory about Fourier series. One can also confirm (*) by experiments on a computer. Namely, consider polynopmials $P(t)$ eith real coefficiets of the form

$$P(t) = t(t-1) \cdot p(t)$$

where $p(t)$ is an arbitrary polynomial and via a Monte-Carlo box one picks a fsmily of polynomials $\{p(t)\}$ and check that (*) holds numerically.

Legendre's conditions.

The examples above show thst in general one ust be careful while a variational problem is studied, i.e. the Euler equstiongives only a necessary condition for an extremal, and it may even occur that no extremal solution exists which satisfied prescribed end-values. But in many applications there exists a unique extremal solution which satisfies Euler's equation. Of course, one should distinguish between maxima or minima. Follwing a classic stufy by Legendre we impose the following condition on the function $f(t,x,y)$ which where (x,y) vary in $\mathbf{R}^2$ while $0 \leq t \leq T$.

$$\text{(L.1)} \qquad a^2 \cdot f''_{xx}(t,x,y) + 2b \cdot f''_{xy}(x,y,t) + c^2 \cdot f''_{yy}(t,x,y) > 0$$

hold for every triple of real numbers a, b, c where at least one is $\neq 0$ and for all (x,y,t).

Exercise. Suppose that $x_*(t)$ solve Euler's equation and ket ϕ be a bubble function. For every real number s we put

$$J(s) = \int_0^T f(t, x(t) + s\phi(t), \dot{x}(t) + s\dot{\phi}(t))|, dt$$

Show that (L.1) entals that the second derivative $J''8s) > 0$ for every $0 \leq s \leq 1$. At the same time Euers equation for $x_*(t)$ entails that the derivative $J'(0) = 0$. Conclude via the general result below that $x_*(t)$ yields a globsl minium.

Exercise. Let $g(s)$ be a C^2-.function on thr unit interval $[0,1]$ with $g'(0) = 0$ and $g'')s) > 0$ for every s. Show that

$$g(0) < g(1)$$

Remark. If > 0 is replaced by < 0 we find exactly as above a unique maximizing function $x^*(t)$. Let us remark that (L.1) means that the symmetric 2×2-matrix

$$xxx$$

is positive definite for all $t, x, y)$. In the literature one often refers to this as the positive Hessian condition for the function $(x, y) \to f(t, x, y)$ which is regarded for every fixed time-vslue. Geometrically it means that the function

$$(x, y) \mapsto f(t, x, y)$$

is stricly concave for every fixed t. Notice also that unique minimimal funcions $x_*(T)$ csn exist even when Legendreäs condition fails. We have seen such examples with

$$f(t, x, y) = x^2 - ay^2$$

where $a > 0$. But in most applicstions one of the teo legendre conditions are satisfied which gives a unique minimizing .or maximizing - solution to the variatinsl problem. Excepy for a few cases one cannot find x^* or X_* explicitly, i.e. one must rely upon numerical experminetns.

An example. Let

$$f(t, x, y) = y^2 + x^4$$

and with $T > 0$ and $A > 0$ we regard the variational probelm

$$\min_x \int_=^T (\dot{x}^2 + x^4)\, dt$$

with $x(0) = 0$ and $x(T) = A$. Euler's equation becomes

$$\ddot{x} = 4x^3$$

This is a non-linear second order ODE. If one assigns a numerical value for the time derivative,

$$\ddot{x}(0) = a$$

for some $a > 0$ then the computer plots a solution and one evaluates numerically $x(T)$. usin the so called Back-Shooting one determines a via the end-value condition $x(T) = A$, and the reader is invited to find numerical solutions for various pairs of A and T.

The value function $V(T, A)$. Keeping f as above one gets $x_*(t)$ and evaluate $\mathcal{F}(x_*)$ which depends on A and T. In many spplications one os foremost interested in this V-function. So as an extra exercise the reader is invited to evalue $V(T, A)$ numericslly with the aid of solutions $x_*(t)$ while varoius numercaik vakues for $\ddot{}A$ snd T are regarded. A speciic case os to let $T = 1$ be fixed while A varies. it is thrn vlerar thst the function

$$A \mapsto V(1, A)$$

increases with A but its shape can only be revealed numerically.

A more special case. it occurs if $f(x, y)x^+y^2$, i.e with $T = 1$ we consider

$$\min_x \int_0^1 (\dot{x}^2 + x^2)\ dt$$

Here we find the oinear Euler equation

$$\ddot{x} = x$$

and it can be solvled expocityl with end-values $x(0) = 0$ and $X(1) = A$ which after yields an explicit equation for $V(1, A)$ as a function of A.

Exercise, In the previous examples no discount factors were introduced. A typcal example in economis is to regard

$$\min_x \int_0^T e^{rt} \cdot \dot{x}^2\, dt$$

with $x(t) = 0$ and $x(T) = A$. The Euler equation entails that

$$t \mapsto e^{rt} \cdot x(t)$$

is a constant function of t. From this the reader can check that

$$x(t) = C \cdot (1 - e^{-rt})$$

where the constant C satisfies

$$C(1 - e^{-rT}) = A$$

Let us instad regard the variational problem

$$\min_x \int_0^T (e^{rt} \cdot \dot{x}^2 + ax)\, dt$$

where $a > 0$ with end-valued $x(0) = 0$ snd $x(T) = A$. The reader is invited to find the minimizing solution $x_*(t)$ and investigate numerically the value function V where we take $T = 1$ while r, A and a vary. For example, hos is V affected by a ?

C. Variation of end-values.

Assume Legendre's concavity condition (l.1) which gives a unique function $x_*(t)$ minimizing $\mathcal{F}$. Now we shall study variations with respect to A and the time horizon T while the minimum value is taken. So for each pair A, T Legendre's result gives a unique extremal function $x^*_{A,T}$ defined on $[0, T]$ and we define the value function

$$V(A, T) = \mathcal{F}(x^*_{A,T})$$

When δA is small, pertubations of solutions to an ODE of the Picard type give the existence of a function $\xi(t)$ such that

$$\text{(i)} \qquad x^*_{A+\delta A,T} = x^*_{A,T} + \delta A \cdot \xi(t) + o(\delta A)$$

Using (i) a first order Taylor expansion gives:

$$\frac{V(A + \delta A, T) - V(A, T)}{\delta A} =$$

$$\text{(ii)} \qquad \int_0^T [\xi(t) \cdot f'_x(x^*, \dot{x}^*, t) + \dot{\xi}(t) \cdot f'_{\dot{x}}(x^*, \dot{x}^*, t)] \cdot dt + o(\delta A)$$

A partial integration and using that x^* satisfies the Euler's equation implies that the integral above becomes

$$\text{(iii)} \qquad \xi(T) \cdot f'_{\dot{x}}(x^*(T), \dot{x}^*(T), T)$$

At the same time (i) and the equality $x^*_{A+\delta A,T} = A + \delta A$ entail that $\xi(T) = 1$. Passing to the limit as $\delta A \to 0$ we have therefore proved:

C.1 Theorem. *One has the equality*

$$\frac{\partial V}{\partial A}(A, T) = f'_{\dot{x}}(x^*_{A,T}, \dot{x}^*_{A,T}, T)$$

C.2 Exercise Use Theorem C.1 and similar arguments as above to show that

$$\frac{\partial V}{\partial T}(A, T) = f(x^*_{A,T}, \dot{x}^*_{A,T}, T) - \dot{x}^*_T(A, T) \cdot f'_{\dot{x}}(x^*_{A,T}, \dot{x}^*_{A,T}, T)$$

C.3 Exericise. Consider the variational problem

$$\min_x \int_0^1 (x^2 + \dot{x}^2)\, dt$$

Let $x(0)=1$ while the end-value $x(1)$ is free. The Euler equation becomes

$$\ddot{x} = x$$

With $x(0)=1$ the general solution is

(i) $$x(t) = Ae^t + (1-A)e^{-t}$$

Since $x(1)$ is free the partial derivative $\frac{\partial V}{\partial A}$ in Theorem C.1 is zero a $T=1$. Since $f_{\dot{x}} = 2\dot{x}$ it follows that A should be chosen so that

$$\dot{x}(1) = Ae + Ae^{-1} = 0 \implies A = 0$$

Hence the extremal function becomes

(ii) $$x_*(t) = e^{-t}$$

If we try to prove (ii) without Theorem C.1 we consider some function from (i) and evaluate

$$\int_0^1 [(Ae^t + (1-A)e^{-t})^2 + (Ae^t + Ae^{-t})^2]\,dt$$

This integral is a quadratic polynomial of A and the reader can check that it takes a minimum when $A=0$ which confirms the solution (ii) and at the same time illustrates that Theorem C.1 simplifies computations in situations of a free terminal value.

D. Constrained Euler equations.

A new clas of variational problems arise when we in addition to boundary values $x(0)=0$ and $x(T)=A$ impose a further constraint that

(D.1) $$\int_0^1 g(x,\dot{x},t)\,dt = B$$

for a constant B and a given $g$function of three variables. An example is the isoperimetric problem where t is replaced by x and we seek a curve $y(x)$ with $y(-1)=y(1)=0$ and the constraint

$$\int_0^1 \sqrt{1+y'(x)^2}\,dx = L$$

while $f=y$. Before we treat this special case we consider the general problem where the pair f,g are arbitrary. Exactl as before one works with bubble-functions ϕ and given some $x(t)$ for which (D.1) holds we take $x_\epsilon(t) + \epsilon\cdot\phi(t)$. The new feature is that (D.1) again should be satisfied. With a small ϵ a Taylor expansion gives

$$\int_0^1 g(x_\epsilon,\dot{x}_\epsilon,t)\,dt = B + \int_0^1 [\,g'\_x(x,\dot{x},t)\cdot\phi + g'\_\dot{x}(x,\dot{x},t)\cdot\dot{\phi}]\,dt$$

After a partial integration of the last term we get the equation

(D.2) $$\int_0^1 [\,g'_x(x,\dot{x},t) - \frac{d}{dt}(g'_{\dot{x}}(x,\dot{x},t)]\cdot\phi\,dt = 0$$

When this holds we say tha ϕ is an admissable bubble-function with respec to $x(t)$. If $x_*(t)$ is an extremal for the variational problem one employs admissable bubble-functions and repeating the previous arguments from when Euler's equation is derived it follows that

(D.3) $$\int_0^1 [\,f'_x(x,\dot{x},t) - \frac{d}{dt}(f'_{\dot{x}}(x,\dot{x},t)]\cdot\phi\,dt = 0$$

So the vanishing in (D.2) implies that (D. 3) also vanishes and then an elementary argument which is left as an exercise to the reader gives the conclusive result:

D.4 Theorem. *If $x^*(t)$ is an extremal there exists a constant c such that*

$$f'_x(x^*,\dot{x}^*,t) - \frac{d}{dt}(f'_{\dot{x}}(x^*,\dot{x}^*,t) = c\cdot[g'_x(x^*,\dot{x}^*,t) - \frac{d}{dt}(g'_{\dot{x}}(x^*,\dot{x}^*,t)]$$

Specific problems.

Problem A. Consider the variational problem

$$\min_x \int_0^1 [\dot{x}^2(t) + e^{\mu t} x(t)]\, dt$$

with end-values $x(0) = 0$ and $x(1) = 1$. Above μ is a real number. Use Euler's equation to find the minimizing x-function which amounts to solve the ODE:

$$2 \cdot \ddot{x} = 2e^{\mu t} x$$

The minimium value is denoted by $V(\mu)$ which the reader should express as a funtion of μ, where μ can be either > 0 or < 0.

Problem A.1 Do the same problem where we now seek

$$\min_x \int_0^1 [\dot{x}^2(t) + A\dot{x} \cdot x \cdot e^{\mu t} x(t)]\, dt$$

where A is another constant. A hinit is that Euler's ODE for the differential equation is the same as when $A = 0$. while the value function can depend on A.

Problem B. Seek

$$\max_c \int_0^\infty \sqrt{c(t)} \cdot e^{-rt}\, dt$$

where the competing c-functions are ≥ 0 and

$$\int_0^\infty c(t)\, dt = A$$

hold for a positive constant A.

Problem C. Here we regard an optimization with a salvage value and seek

$$\max_x \int_0^1 \sqrt{\dot{x}}\, dt + A(K - x(T))$$

where $x(0) = 0$ and the end value satisfies $0 \leq x(T) \leq K$, while A och K are positive constants.

Problem D (harder) Consider the variational problem

$$\min_x \int_0^\pi [\dot{x}(t)^2 - (\pi - t)x^2(t)]\, dt$$

with end-value conditions $x(0) = 0$ and $x(\pi) = 1$.