1

Lagrange 1788!

The role of Kinetic Energy

"k = degree of Freedom"

Given $\theta = (\theta_1, \ldots \theta_k)$ $k \geq 1$

and a vector valued function

$$p = p(\theta) =: p(\theta_1 \ldots \theta_k) \quad (*)$$

$p = (x, y, z)$ in $\mathbb{R}^3$.

Thus: A map $\theta \longmapsto p(\theta)$ from $\mathbb{R}^k$ into $\mathbb{R}^3$

(*) is of class C^2, i.e. $\exists$ derivatives

$$\frac{\partial p}{\partial \theta_j} \quad \& \quad \frac{\partial^2 p}{\partial \theta_j \partial \theta_k} \qquad 1 \leq j, k \leq k$$

Dynamics

→ Now t = Time variable $(t \geq 0)$ and

given $t \to \theta_\nu(t)$ k Time dependent functions.

$\Rightarrow t \mapsto p(\theta(t))$ moves p in $\mathbb{R}^3$!

Chain Rule give velocity vector:

velocity $\rightsquigarrow$ $$\dot{p} = \sum_{j=1}^{k} \frac{\partial p}{\partial \theta_j} \cdot \dot{\theta}_j \qquad * \; \dot{\theta}_j =: \frac{d}{dt}[\theta_j(t)]$$

acceleration $\rightsquigarrow$ $$\ddot{p} = \sum_\nu \sum_j \frac{\partial^2 p}{\partial \theta_\nu \partial \theta_j} \dot{\theta}_\nu \dot{\theta}_j + \sum_{j=1}^{k} \frac{\partial p}{\partial \theta_j} \ddot{\theta}_j$$

say t fixed

$\left\langle \frac{\partial p}{\partial\theta_v}(t), \frac{\partial p}{\partial\theta_j}(t)\right\rangle \dot\theta_v(t)\dot\theta_j(t)$ ~~Next~~ The Kinetic energy mass $m=1$

$$T = \frac{1}{2}\|\dot p\|^2 = \frac{1}{2}\sum_{v=1}^{k}\sum_{j=1}^{k}\left\langle \frac{\partial p}{\partial\theta_v}, \frac{\partial p}{\partial\theta_j}\right\rangle \dot\theta_v\,\dot\theta_j$$ $(\, , \,) =$ inner product

Regard $T = T(\theta, \dot\theta)_t$ as a function of

$\theta = \theta(t)$ and the k-tuple $\dot\theta = \dot\theta(t)$

~~Get by calculus~~ Calculus Fixed $1 \le i \le k$

$$\frac{\partial T}{\partial\dot\theta_i} = \sum_{j=1}^{k}\left(\frac{\partial p}{\partial\theta_i}, \frac{\partial p}{\partial\theta_j}\right)\dot\theta_j$$

$$\frac{\partial T}{\partial\theta_i} = \sum_{v=1}^{k}\sum_{j=1}^{k}\left\langle \frac{\partial^2 p}{\partial\theta_i\partial\theta_v}, \frac{\partial p}{\partial\theta_j}\right\rangle \dot\theta_v\,\dot\theta_j$$

Notice factor $\frac{1}{2}$ Disappears uses $\langle \, , \, \rangle$ is symmetric.

The Lagrangean functions ~~With These Notations Following Lagrange we define~~

i fixed

$$\mathcal{L}_i := \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta_i}\right) - \frac{\partial T}{\partial\theta_i} \qquad 1 \le i \le k.$$

$\boxed{\text{THEOREM}}$ found by straight Calculus.

$$\mathcal{L}_i = \left\langle \frac{\partial p}{\partial\theta_i}, \ddot p\right\rangle \qquad 1 \le i \le k \; !!$$

fundamental via Newton! $\delta\theta_i$ small displacement at any Time

$\Rightarrow$ ~~$\delta p = \frac{\partial p}{\partial\theta_i}$~~ $\delta p \approx \frac{\partial p}{\partial\theta_i}\cdot\delta\theta_i$

Idea is to follow

infinitesimal work!

By Newton work to displace

$p \mapsto p + \delta p$ is

$$\underset{=1}{\pm}\, m \cdot \langle \delta p\,,\, \ddot{p} \rangle = \langle \frac{\partial p}{\partial \theta_i}, \ddot{p} \rangle \cdot \delta\theta_i$$

Scale factor for work

when $\theta_i \mapsto \theta_i + \delta\theta_i$

Special case External force is

a potential $U = U(\theta_1, \ldots \theta_k)$

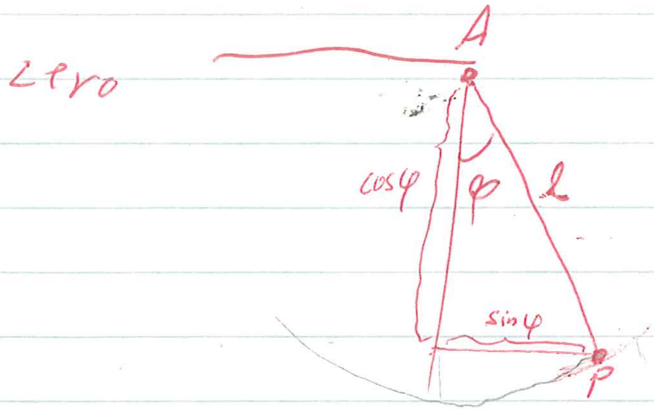
Then this work $\pm \dfrac{\partial U}{\partial \theta_i}$!!

$$\mathcal{L}_i = \pm \frac{\partial U}{\partial \theta_i}$$

$\boxed{\text{sign understood by examples}}$

Simple Pendelum

(4)



$\varphi = \theta_1 \quad k = 1$

$\downarrow g$

$\underline{m_P = 1}$

$$\underline{\underline{T = \frac{1}{2} l^2 \dot{\varphi}^2}}$$

We have only one masspoint P at the tip of the s.p. of length l.
So, angular velocity $= \dot{\varphi}$
$\therefore$ The K.E. by dfn is the euclidean length of the velocity vector

$T := \frac{1}{2} \| \dot{P} \|^2$

$= \frac{1}{2} (l \cdot \dot{\varphi})^2 = \frac{1}{2} l^2 \dot{\varphi}^2$

$$\mathcal{L} := \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right)$$

$$= \frac{d}{dt}\left(l^2 \dot{\varphi}\right)$$

$$= l^2 \ddot{\varphi}$$

Newtons gravite

$$U(\varphi) = -g l \cos \varphi$$

"sign"
Potential Energy (Dfn)

$$\Rightarrow \frac{\partial U}{\partial \varphi} = \underline{\underline{g \cdot l \cdot \sin \varphi}} > 0$$

$U(\varphi) :=$

$\frac{\partial}{\partial \varphi} \cos \varphi = -\sin \varphi$

Notice

step up

$l \cdot \sin \varphi$

So $\underline{gl \cdot \sin \varphi}$

Raise to P $\simeq \delta\varphi \cdot l \cdot \sin \varphi$

"factor for work"

(5)

Need work to raise φ!

By Lagrange + Newton

So $\mathcal{L}_\varphi = l^2 \ddot{\varphi} = - g l \sin\varphi$!!!

$$\ddot{\varphi} = -\frac{g}{l} \sin\varphi$$

Remain To solve $\varphi(0) = 0$

$\dot{\varphi}(0) = v > 0$

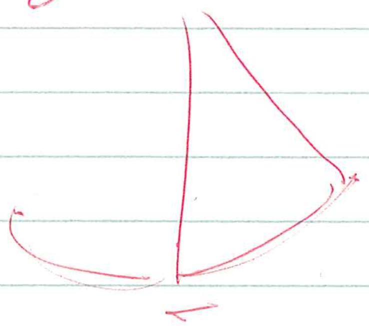
(v Not Too large)

$t \longrightarrow \varphi(t)$ oscillates.

Reach φ^* max during motion

when $\dot{\varphi}(T) = 0$

$T = \frac{1}{4}$ full Revolution



find T find $\varphi(T)$ = amplitude

Determined by $\dot{\varphi}(T) = 0$

Solution Multiply by $\dot{\varphi} \Rightarrow$

$$\frac{d}{dt}\left(\frac{\dot{\varphi}^2}{2}\right) = \dot{\varphi}\,\ddot{\varphi} = -\frac{g}{l}\sin\varphi\cdot\dot{\varphi} =$$

$$= \frac{g}{l}\cdot\frac{d}{dt}(\cos\varphi)$$

use $\frac{d}{dt}(\cos\varphi) = -\sin\varphi\cdot\dot{\varphi}$

equality of time derivatives implies just additive constant

Conclude from the fact that the $\frac{d}{dt}\left(\frac{\dot{\varphi}^2}{2}\right) = \frac{g}{l}\frac{d}{dt}(\cos\varphi)$

$$\frac{\dot{\varphi}^2}{2} - \frac{g}{l}\cos\varphi = E \text{ . constant}$$

$$\frac{\dot{\varphi}^2}{2} = gl\cos\varphi + E \quad \leftarrow (*)$$

$\varphi(0) = 0$ $\quad \dot{\varphi}(0) = V$ $\quad$ at Time $t=0$

$$\frac{V^2}{2} = gl + E \Rightarrow E = \frac{V^2}{2} - gl \quad \leftarrow$$

E at initial condition is the same for all t.

$$(**) \quad \frac{\dot{\varphi}^2}{2} = \frac{V^2}{2} + gl(\cos\varphi - 1) \quad \leftarrow$$

Plug E back in (*)

~~1st~~ Case 1 $\dot{\varphi} = 0 \Rightarrow 1 - \cos\varphi = \frac{V^2}{2gl}$

WANT $0 < \varphi < \frac{\pi}{2}$ $\qquad \varphi = \frac{\pi}{2}$

Need $\frac{V^2}{2gl} < 1 \Rightarrow V < \sqrt{2gl}$ $\qquad \cos\left(\frac{\pi}{2}\right) = 0$

Now problem start.

1st Amplitude φ^* given by

$$\boxed{\cos\varphi^* = 1 - \frac{v^2}{2gl}}$$

$l \mapsto \varphi^*(l) \downarrow$

$l \mapsto a(l) \uparrow$

Seek T !!

from $(**)$

$$\dot\varphi^2 = v^2 + 2gl\cos\varphi - 2gl$$

$= \cancel{2gl\,[\ \cos\varphi]}$ Put $A = 2gl - v^2 > 0$

$$\dot\varphi^2 = 2gl\cos\varphi - A$$

$$\dot\varphi = \sqrt{2gl\cos\varphi - A}$$

Legendre 1800

$$= \sqrt{2gl}\cdot\sqrt{\cos\varphi - \frac{A}{2gl}} \qquad \Big| \quad a = \frac{A}{2gl}$$

$$\dot\varphi = \sqrt{2gl}\,\sqrt{\cos\varphi - a}$$

$$a = \frac{A}{2gl} = \frac{2gl - v^2}{2gl} = 1 - \frac{v^2}{2gl}$$

$$\frac{d\varphi}{\sqrt{\cos\varphi - a}} = \sqrt{2gl}\cdot dt \Longrightarrow$$

integrator is φ

integrator is time

$$\sqrt{2gl}\cdot T = \int_0^{\varphi^*} \frac{d\varphi}{\sqrt{\cos\varphi - a}} \quad ; \quad \underline{\underline{\cos\varphi^* = a}}$$

implicit ~~analytical formula~~ numerical method

but has classical name "Legendre integral"

Known to Euler (probably earlier too).

Case of special Interest

Analyze $l \mapsto T(l)$

when v fixed "modest" $\simeq 0$

To derive it via our

Implicit formulas is not easy!

Objection concerning V. ~~..~~ initial ~~angular~~ ~~velocity~~ speed

$\forall l$ give kick $V := \underbrace{\alpha \cdot l}$

$\alpha :=$ Initial angular velocity.

Play with α and l ~~fas~~ to

Determine φ^* and T

Complete Picture. To

express $\left.\begin{array}{l} \varphi^* = \varphi^*(\alpha, l) \\ T = T(\alpha, l) \end{array}\right\}$ "final answer!)