Functional analysis and operator theory

8. Bounded self-adjoint operators.

Introduction. Let $\mathcal{H}$ be a complex Hilbert space. A bounded linear operator S on $\mathcal{H}$ is self-adjoint if $S = S^*$, or equivalently

(*) $$\langle x, Sy \rangle = \text{the complex conjugate of } \langle Sx, y \rangle \quad : \quad x, y \in \mathcal{H}$$

If S is self-adjoint we have the equality of operator norms:

(1) $$||S||^2 = ||S^2||$$

To see this we notice that if $x \in \mathcal{H}$ has norm one then

(i) $$\langle Sx, Sx \rangle = \langle x, S^*Sx \rangle = \langle x, S^2x \rangle$$

By the Cauchy-Schwarz inequality the last term is $\leq ||x| \cdot ||S^2||$. Since (i) holds for every x of norm one we conclude that

$$||S||^2 \leq ||S^2||$$

Now (1) follows from the multiplicative inequality for operator norms. Next, by induction over n we get the equalities

$$||S||^{2n} = ||S^n||^2 \quad : \; n \geq 1$$

Taking the n:th root and passing to the limit the spectral radius formula gives

(*) $$||S|| = \max_{z \in \sigma(S)} |z|$$

Next, we consider the spectrum of self-adjoint operators.

8.1 Theorem. *The spectrum of a bounded self-adjoint operator is a compact real interval.*

Proof. Let λ be a complex number and for a given x we set $y = \lambda x - Sx$. It follows that

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 + \lambda \cdot \langle x, Sx \rangle + \bar{\lambda} \cdot \langle Sx, x \rangle$$

Since S is self-adjoint we get

$$\lambda \cdot \langle x, Sx \rangle + \bar{\lambda} \cdot \langle Sx, x \rangle = 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle Sx, x \rangle$$

Now $|\langle Sx, x \rangle| \leq ||Sx|| \cdot ||x||$ so the triangle inequality gives

(i) $$||y||^2 \geq |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 - 2|\mathfrak{Re}(\lambda)|| \cdot ||Sx|| \cdot ||x||$$

With $\lambda = a + ib$ the right hand side becomes

$$b^2||x||^2 + a^2||x||^2 + ||Sx||^2 - 2a \cdot ||Sx|| \cdot ||x|| \geq b^2||x||^2$$

Hence we have proved that

(ii) $$||\lambda x - Sx||^2 \geq (\mathfrak{Im}\,\lambda)^2 \cdot ||x||^2$$

This implies that $\lambda E - S$ is invertible for every non-real λ which proves Theorem 8.1. Notioce that the proof also gives

(iii) $$||(\lambda E - S)^{-1}|| \leq \frac{1}{|\mathfrak{Im}\,\lambda|}$$

Theorem 8.1 together with general results about uniform algebras in § XX give the following:

8.2 Theorem. *Denote by* $\mathbf{S}$ *the closed subalgebra of* $L(\mathcal{H}, \mathcal{H})$ *generated by* S *and the identity operator. Then* $\mathbf{S}$ *is a sup-norm algebra which is isomorphic to the sup-norm algebra* $C^0(\sigma(S))$.

Exercise. Let T be an arbitary bounded opertor on $\mathcal{H}$. Show that the operator $A = T^*T$ is self-adjoint and that $\sigma(A)$ is a compact subset of $[0, +\infty)$, i.e. every point in its spectrum os real and non-negative. A hint is to use the biduality formula $T = T^{**}$ and if s is real the reader should verify that

$$||sx + T^*Tx||^2 = s^2||x||^2 + 2s \cdot ||Tx||^2 + ||T^*Tx||^2$$

8.3 Normal operators.

A bounded linear operator A is normal if it commutes with its adjoint A^*. Let A be normal and put $S = A^*A$ which yields a self-adjoint by Exercise 8.2. Here (8.0.1) gives

(1) $$||S||^2 = ||S^2|| = ||A^2 \cdot A(^*)^2|| \leq ||A^2|| \cdot ||(A^*)^2||$$

where we used the multiplicative inequality for operator norms. Now $(A^*)^2$ is the adjoint of A^2 and we recall from § xx that the norms of an operator and its adjoint are equal. Hence the right hand side in (1) is equal to $||A^2||^2$. At the same time

$$||S|| = ||A^*A|| = ||A||^2$$

and we conclude that (1) gives

(3) $$||A||^2 \leq ||A^2||$$

Exactly as in the self-adjoint case we can take higher powers and obtain the equality

(*) $$||A|| = \max_{z \in \sigma(A)} |z|$$

Since every polynomial in A again is a normal operator for which (*) holds we have proved the following:

8.4 Theorem *Let A be a normal operator. Then the closed subalgebra* $\mathbf{A}$ *generated by A in $L(\mathcal{H}, \mathcal{H})$ is a sup-norm algebra.*

Remark. The spectrum $\sigma(A)$ is some compact subset of $\mathbf{C}$ and in general analytic polynomials restricted to $\sigma(A)$ do not generate a dense subalgebra of $C^0(\sigma(A))$. To get a more extensive algebra we consider the closed subalgebra $\mathcal{B}$ of $L(\mathcal{H}, \mathcal{H})$ which is generated by A and A^*. Since every polynomial in A and A^* again is a normal operator it follows that $\mathcal{B}$ is a sup-norm algebra and here the following holds:

8.5 Theorem. *The sup-norm algebra $\mathcal{B}$ is via the Gelfand transform isomorphic with $C^0(\sigma(A))$.*

Proof. Let $Q \in \mathcal{B}$ be arbitrary. Now $S = Q + Q^*$ is self-adjoint and Theorem 8.1 entails that its Gelfand transform is real-valued, i.e. the function $\widehat{Q}(p) + \hat{Q}^*(p)$ is real. So if with $\widehat{Q}(p) = a + ib$ we must have $\hat{Q}^* = a_1 - ib$ for some real number a_1. Next, QQ^* is also self-adjoint and hence $(a + ib)(a_1 - ib)$ is real. This gives $a = a_1$ and which shows that the Gelfand transform of Q^* is the complex conjugate function of $\widehat{Q}$. Hence the Gelfand transforms of $\mathcal{B}$-elements is a self-adjoint algebra and the Stone-Weierstrass theorem implies that the Glefand transforms of $\mathcal{B}$-elements is equal to the whole algebra $C^0(\mathfrak{M}_{\mathcal{B}}$. Finally, since $\widehat{A}^*$ is the complex conjugate function of $\widehat{A}$ it follows that the Gelfand transform $\widehat{A}$ separates points on $\mathfrak{M}_{\mathcal{B}}$ which means that this maximal ideal space can be identified with $\sigma(A)$.

8.6 Spectral measures.

Let A be a normal operator and $\mathcal{B}$ is the Banach algebra above. Each pair of vectors x, y in $\mathcal{H}$ yields a linear functional on $\mathcal{B}$ defined by

$$T \mapsto \langle Tx, y \rangle$$

Identifying $\mathcal{B}$ with $C^0(\sigma(A))$, the Riesz representation formula gives a unique Riesz measure $\mu_{x,y}$ on $\sigma(A)$ such that

(8.6.1) $$\langle Tx, y \rangle = \int_{\sigma(A)} \widehat{T}(z) \cdot d\mu_{x,y}(z)$$

hold for every $T \in \mathcal{B}$. Since $\widehat{A}(z) = z$ we have

$$\langle Ax, y \rangle = \int z \cdot d\mu_{x,y}(z)$$

Similarly one has

$$\langle A^* x, y\rangle = \int \bar{z} \cdot d\mu_{x,y}(z)$$

8.7 The operators $E(\delta)$. Notice that (8.6.1) implies that the map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures on $\sigma(A)$ is bi-linear. We have for example:

$$\mu_{x_1+x_2,y} = \mu_{x_1,y} + \mu_{x_2,y}$$

Moreover, since $\mathcal{B}$ is the sup-norm algebra $C^0(\sigma(A))$ the total variations of the μ-measures satisfy the equations:

(8.7.1)
$$||\mu_{x,y}|| \leq \max_{T \in \mathcal{B}_*} |\langle Tx, y\rangle|$$

where $\mathcal{B}_*$ is the unit ball in $\mathcal{B}$. From this we obtain

(8.7.2)
$$||\mu_{x,y}|| \leq ||x|| \cdot ||y||$$

Next, let δ be a Borel subset of $\sigma(A)$. Keeping y fixed in $\mathcal{H}$ we obtain a linear functional on $\mathcal{H}$ defined by

$$x \mapsto \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta)$$

By (8.7.2) it has norm $\leq ||y||$ and is represented by a vector $E(\delta)x$ in $\mathcal{H}$. More precisely

(8.7.3)
$$\langle E(\delta)x, y\rangle = \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta)$$

8.8 Exercise. Show that $x \mapsto E(\delta)x$ is linear and that the resulting linear operator $E(\delta)$ commutes with all operators in $\mathcal{B}$. Moreover, show that it is a self-adjoint projection, i.e.

$$E(\delta)^2 = E(\delta) \quad \text{and} \quad E(\delta)^* = E(\delta)$$

Finally, show that

$$E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$$

holds for every pair of Borel subsets and with $\delta = \sigma(A)$ one gets the identity operator.

8.9 Resolution of the identity. If $\delta_1, \ldots, \delta_N$ is any finite family of disjoint Borel sets whose union is $\sigma(A)$ then

$$1 = E(\delta_1) + \ldots + E(\delta_N)$$

At the same tine we get a decomposition of the operator A:

$$A = A_1 + \ldots + A_N \quad \text{where} \quad A_k = E(\delta_k) \cdot A$$

For each k the spectrum $\sigma(A_k)$ is equal to the closure of δ_k. So the normal operator is represented by a sum of normal operators where the individual operators have small spectra when the δ-partition is fine.

9. Unbounded operators on Hilbert spaces

Let T be a densely defined linear operator on a complex Hilbert space $\mathcal{H}$. We suppose that T is unbounded so that:

$$\max_{x\in\mathcal{D}_*(T)} ||Tx|| = +\infty \quad \mathcal{D}_*(T) = \text{ the set of unit vectors in } \quad \mathcal{D}(T)$$

9.1 The adjoint T^*. If $y \in \mathcal{H}$ we get a linear functional on $\mathcal{D}(T)$ defined by

(i) $$x \mapsto \langle Tx, y\rangle$$

If there exists a constant $C(y)$ such that the absolute value of (i) is $\leq C(y)\cdot||x||$ for every $x \in \mathcal{D}(T)$, then (i) extends to a continuous linear functional on $\mathcal{H}$. The extension is unique because $\mathcal{D}(T)$ is dense and since $\mathcal{H}$ is self-dual there exists a unique vector T^*y such that

(9.1.1) $$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad : \quad x \in \mathcal{D}(T)$$

This gives a linear operator T^* where $\mathcal{D}(T^*)$ is characterised as above. Now we shall describe the graph of T^*. For this purpose we consider the Hilbert space $\mathcal{H}\times\mathcal{H}$ equipped with the inner product

$$\langle (x,y),(x_1,y_1)\rangle = \langle x, x_1\rangle + \langle y, y_1\rangle$$

On $\mathcal{H}\times\mathcal{H}$ we define the linear operator

$$J(x,y) = (-y,x)$$

9.2 Proposition. *For every densely defined operator T one has the equality*

$$\Gamma(T^*) = J(\Gamma(T))^\perp$$

Proof. Let (y, T^*y) be a vector in $\Gamma(T^*)$. If $x \in \mathcal{D}(T)$ the equality (9.1.1) and the construction of J give

$$\langle (y, -Tx\rangle + \langle T^*y, x\rangle = 0$$

This proves that $\Gamma(T^)* \perp J(\Gamma(T))$. Conversely, if $(y,z) \perp J(\Gamma(T))$ we have

(i) $$\langle y, -Tx\rangle + \langle z, x\rangle = 0 \quad : x \in \mathcal{D}(T)$$

This shows that $y \in \mathcal{D}(T^*)$ and $z = T^*y$ which proves Proposition 9.2.

9.3 Consequences. The orthogonal complement of a subspace in a Hilbert space is always closed. Hence Proposition 9.2 entails that T^* has a closed graph. Passing to the closure of $\Gamma(T)$ the decompostion of a Hilbert space into a direct sum of a closed subpace and its orthogonal complement gives

(9.3.1) $$\mathcal{H}\times\mathcal{H} = \overline{J(\Gamma(T))} \oplus \Gamma(T^*)$$

Notice also that

(9.3.2) $$\Gamma(T^*)^\perp = \overline{J(\Gamma(T))}$$

9.4 Closed extensions of operators. A closed operator S is called a closed extension of T if

$$\Gamma(T) \subset \Gamma(S)$$

9.4.1 Exercise. Show that if S is a closed extension of T then

$$S^* = T^*$$

9.4.2 Theorem. *A densely defined operator T has a closed extension if and only if $\mathcal{D}(T^*)$ is dense. Moreover, if T is closed one has the biduality formula $T = T^{**}$.*

Proof. Suppose first that T has a closed extension. If $\mathcal{D}(T^*)$ is not dense there exists a non-zero vector $0 \neq h \perp \mathcal{D}(T^*)$ and (9.3.2) gives

(ii) $$(h, 0) \in \Gamma(T^*)^\perp = J(\Gamma(T))$$

By the construction of J this would give $x \in \mathcal{D}(T)$ such that $(h,0) = (-Tx, x)$ which cannot hold since this equation first gives $x = 0$ and then $h = T(0) = 0$. Hence closedness of T implies that $\mathcal{D}(T^*)$ is dense. Conversely, assume that $\mathcal{D}(T^*)$ is dense. Starting from T^* we construct its adjoint T^{**} and Proposition 9.3.2 applied with T^* gives

(i) $$\Gamma(T^{**}) = J(\Gamma(T^*)^{\perp}$$

At the same time $J(\Gamma(T^*)^{\perp}$ is equal to the closure of $\Gamma(T)$ so (i) gives

(ii) $$\overline{\Gamma(T)} = \Gamma(T^{**})$$

which proves that T^{**} is a closed extension of T.

9.4.3 The biduality formula. Let T be closed. and densely defined operator. from the above T^* also is densely defined and closed. Hence its dual exists. It is denoted by T^{**} and called the bi-dual of T. With these notations one has:

(*) $$T = T^{**}$$

9.4.4 Exercise. Prove the equality (*).

9.5 Inverse operators.

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Denote by $\mathfrak{I}(\mathcal{H})$ the set of closed and densely defined operators T such that T is injective on $\mathcal{D}(T)$ and the range $T(\mathcal{D}(T))$ is dense in $\mathcal{H}$. If $T \in \mathfrak{I}(\mathcal{H})$ there exists the densely defined operator S where $\mathcal{D}(S)$ is the range of T and

$$S(Tx) = x \quad : \quad x \in \mathcal{D}(T)$$

By this construction the range of S is equal to $\mathcal{D}(T)$. Next, on $\mathcal{H} \times \mathcal{H}$ we have the isometry defined by $I(x,y) = (y,x)$, i.e we interchange the pair of vectors. The construction of S gives

(i) $$\Gamma(S) = I(\Gamma(T))$$

Since $\Gamma(T)$ by hypothesis is closed it follows that S has a closed graph and we conclude that $S \in \mathfrak{I}(\mathcal{H})$. Moreover, since I^2 is the identity on $\mathcal{H} \times \mathcal{H}$ we have

(ii) $$\Gamma(T) = I(\Gamma(S))$$

We refer to S is as the inverse of T. It is denoted by T^{-1} and (ii) entails that T is the inverse of T^{-1}, i.e. one has

(*) $$T = (T^{-1})^{-1}$$

9.5.1 Exercise. Let T belong to $\mathfrak{I}(\mathcal{H})$. Use the description of $\Gamma(T^*)$ in Proposition 9.3 to show that T^* belongs to $\mathfrak{I}(\mathcal{H})$ and the equality

(**) $$(T^{-1})^* = (T^*)^{-1}$$

9.6 The operator T^*T

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Each $h \in \mathcal{H}$ gives the vector $(h,0)$ in $\mathcal{H} \times \mathcal{H}$ and (9.3.1) gives a pair $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$. such that

$$(h,0) = (x, Tx) + (-T^*y, y) = (x - T^*y, Tx + y)$$

With $u = -y$ we get $Tx = u$ and obtain

(9.6.1) $$h = x + T^*(Tx)$$

9.6.2 Proposition. *The vector x in (9.6.1) is uniquely determined by h.*

Proof. Uniqueness follows if we show that

$$x + T^*(Tx) \implies x = 0$$

But this is clear since the construction of T^* gives

$$0 = \langle x, x)\rangle + \langle x, T^*(Tx)\rangle = \langle x, x)\rangle + \langle Tx, Tx)\rangle \implies x = 0$$

9.7 The density of $\mathcal{D}(T^*T)$. This is the subspace of $\mathcal{D}(T)$ where the extra condition for a vector $x \in \mathcal{D}(T)$ is that $Tx \in \mathcal{D}(T^*)$. To prove that $\mathcal{D}(T^*T)$ is dense we consider some orthogonal vector h. Proposition 9.6 gives some $x \in \mathcal{D}(T)$ such that $h = x + T^*(Tx)$ and for every $g \in \mathcal{D}(T^*T)$ we have

(i) $$0 = \langle x, g\rangle + \langle T^*Tx, g\rangle = \langle x, g\rangle + \langle Tx, Tg\rangle = \langle x, g\rangle + \langle x, T^*Tg\rangle$$

Here (i) hold for every $g \in \mathcal{D}(T^*T)$ and by another application of Proposition 9.6 we find g so that $x = g + T^*Tg$ and then (i) gives $\langle x, x\rangle = 0$ so that $x = 0$. But then we also have $h = 0$ and the requested density follows.

Conclusion. Set $A = T^*T$. From the above it is densely defined and (9.6.1) entails that the densely defined operator $E + A$ is injective. Moreover, its range is equal to $\mathcal{H}$. Notice that

$$\langle x + Ax, x + Ax\rangle = c + \langle x, Ax\rangle + \langle Ax, x\rangle$$

Here

$$\langle x, Ax\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = ||Tx||^2$$

and from this the reader can conclude that

$$||x + Ax||^2 = ||x||^2 + ||Ax||^2 + 2 \cdot ||Tx||^2 \quad : x \in \mathcal{D}(A)$$

The right hand side is $\geq ||x||^2$ which implies that $E + A$ is invertible in Neumann's sense.

9.8 The equality $A^* = A$. Recall the biduality formula $T = T^{**}$ and apply Proposition 9.6 starting with T^*. It follows that $\mathcal{D}(TT^*)$ also is dense and exactly as in (9.6.1) every $h \in \mathcal{H}$ has a unique representation

$$h = y + T(T^*y)$$

9.9. Exercise. Verify from the above that A is self-adjoint, i.e one has the equality $A = A^*$.

9.10 Unbounded self-adjoint operators.

A densely defined operator A on the Hilbert space $\mathcal{H}$ for which $A = A^*$ is called self-adjoint.

9.11 Proposition *The spectrum of a self-adjoint operator A is contained in the real line, and if λ is non-real the resolvent satisfies the norm inequality*

$$||R_A(\lambda)|| \leq \frac{1}{|\mathfrak{Im}\,\lambda|}$$

Proof. Set $\lambda = a + ib$ where $b \neq 0$. If $x \in \mathcal{D}(A)$ and $y = \lambda x - Ax$ we have

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Ax||^2 - 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle x, Ax\rangle$$

The Cauchy-Schwarz inequality gives

(i) $$||y||^2 \geq b^2||x||^2 + a^2||x||^2 + ||Ax||^2 - 2|a| \cdot ||Ax|| \cdot ||x|| \geq b^2||x||^2$$

This proves that $x \to \lambda x - Ax$ is injective and since A is closed the range of $\lambda \cdot E - A$ is closed. Next, if y is $\perp$ to this range we have

$$0 = \lambda\langle x, y\rangle - \langle Ax, y\rangle \quad : x \in \mathcal{D}(A)$$

From this we see that y belongs to $\mathcal{D}(A^*)$ and since A is self-adjoint we get

$$0 = \lambda\langle x, y\rangle - \langle x, Ay\rangle$$

This holds for all x in the dense subspace $\mathcal{D}(A)$ which gives $\lambda \cdot y = Ay$ Since λ is non-real we have already seen that this entails that $y = 0$. Hence the range of $\lambda \cdot E - A$ is equal to $\mathcal{H}$ and the inequality (i) entails $R_A(\lambda)$ has norm $\leq \frac{1}{|\mathfrak{Im}\lambda|}$.

9.12 A conjugation formula. Let A be self-adjoint. For each complex number λ the hermitiain inner product on $\mathcal{H}$ gives the equation

$$\bar{\lambda} - A = (\lambda \cdot E - A)^*$$

So when we take the complex conjugate of λ it follows that § 9.5 that

$$R_A(\lambda)^* = R_A(\bar{\lambda}) \tag{9.12.1}$$

9.13 Properties of resolvents. Let A be self-adjoint. By Neumann's resolvent calculus the family $\{(R_A(\lambda)\}$ consists of pairwise commuting bounded operators outside the spectrum of A. Since $\sigma(A)$ is real there exist operator-valued analytic functions $\lambda \mapsto R_A(\lambda)$ in the upper- respectively the lower half-plane. Moreover, since Neumann's resolvents commute, it follows from (9.12.1) that $R_A(\lambda)$ commutes with its adjoint. Hence every resolvent is a bounded normal operator.

9.14 A special resolvent operator. Take $\lambda = i$ and set $R = R_A(i)$. So here

$$R(iE - A)(x) = x \quad : \quad x \in \mathcal{D}(A)$$

9.15 Theorem. *The spectrum $\sigma(R)$ is contained in the circle*

$$C_* = \{|\lambda + i/2| = 1/2\}$$

Proof. Since $\sigma(A)$ is confined to the real line, it follows from § 0.0. XX that points in $\sigma(R)$ have the form

$$\lambda = \frac{1}{i-a} \quad : a \in \mathbf{R}$$

This gives

$$\lambda + i/2 = \frac{1}{i-a} + i/2 = \frac{1}{2(i-a)}(2 + i^2 - ia) = \frac{1-ia}{2i(1+ia)}$$

and the last term has absolute value $1/2$ for every real a.

9.B. The spectral theorem for unbounded self-adjoint operators.

The operational calculus in § 8.3-8.6 applies to the bounded normal operator R in § 9.14. If N is a positive integer we set

$$C_*(N) = \{\lambda \in C_* : \mathfrak{Im}(\lambda) \leq -\frac{1}{N}\} \quad \text{and} \quad \Gamma_N = C_*(N) \cap \sigma(R)$$

Let χ_{Γ_N} be the characterstic function of Γ_N. Now

$$g_N(\lambda) = \frac{1-i\lambda}{\lambda} \cdot \chi_{\Gamma_N}$$

is Borel functionon $\sigma(R)$ which by operational calculus in § 8.xx gives a bounded and normal linear operator denoted by G_N. On Γ_N we have $\lambda = -i/2 + \zeta$ where $|\zeta| = 1/2$. This gives

$$\frac{1-i\lambda}{\lambda} = \frac{1/2 - i\zeta}{-i/2 + \zeta} = \frac{(1/2 - i\zeta)(i/2 + \bar{\zeta})}{|\zeta - i/2|^2} = \frac{\mathfrak{Re}\,\zeta}{|\zeta - i/2|^2} \tag{1}$$

By § 8.x the spectrum of G_N is the range of the g-function on Γ_N and (1) entails that $\sigma(G_N)$ is real. Since G_N also is normal it follows that it is self-adjoint. Next, notice that

$$\lambda \cdot (\frac{1-i\lambda}{\lambda} + i) = 1 \tag{2}$$

holds on Γ_N. Hence operational calculus gives the equation

$$R(G_N + i) = E(\Gamma_N) \tag{3}$$

where $E(\Gamma_N)$ is a self-adjoint projection. Notice also that

$$R \cdot G_N = (E - iR) \cdot E(\Gamma_N) \tag{4}$$

Hence (3-4) entail that

(5) $$E(\Gamma_N) - iRE(\Gamma_N) = (E - iR) \cdot E(\Gamma_N)$$

Next, the equation $RA = E - iR$ gives

(*) $$RAE(\Gamma_N) = (E - iR)E(\Gamma_N) = R \cdot G_N$$

9.B.1 Exercise. Conclude from the above that

(*) $$AE(\Gamma_N) = G_N$$

Show also that:

(**) $$\lim_{N \to \infty} AE(\Gamma_N)(x) = A(x) \quad \text{for each} \quad x \in \mathcal{D}(A)$$

9.B.2 A general construction. For each bounded Borel set e on the real line we get a Borel set $e_* \subset \sigma(R)$ given by

$$e_* = \sigma(R) \cap \{\frac{1}{i - a} a \in e\}$$

The operational calculus gives the self-adjoint operator G_e constructed via $g \cdot \chi_{e_*}$. We have also the operstor $E(e)$ given by χ_{e_*} and exactly as above we get

$$AE(e) = G_e$$

The bounded self-adjoint operators $E(e)$ and G_e commute with A and $\sigma(G_e)$ is contained in the closure of the bounded Borel set e. Moreover each $E(e)$ is a self-adjoint projection and for each pair of bounded Borel sets we have

$$E(e_1)E(e_2) = E(e_1 \cap E(e_1)$$

In particular the composed operators

$$E(e_1) \circ E(e_2) = 0$$

when the Borel sets are disjoint.

9.C The spectral measure. Exactly as for bounded self-adjoint operators the results above give rise to a map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures:

$$(x, y) \mapsto \mu_{x,y}$$

For each real-valued and bounded Borel function $\phi(t)$ on the real line with compact support there exists a bounded self-adjoint operator ϕ such that

$$\langle \Phi(x), y \rangle = \int g(t) \cdot d\mu_{x,y}(t)$$

All these Φ operators commute with A. If $x \in \mathcal{D}(A)$ and y is a vector in $\mathcal{H}$ one has

$$\langle A(x), y \rangle = \lim_{M \to \infty} \int_{-M}^{M} t \cdot d\mu_{x,y}(t)$$

§ 10. Symmetric operators

A densely defined and closed operator T on a Hilbert space $\mathcal{H}$ is symmetric if

$$(*) \qquad \langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{hold for all pairs} \quad x, y \in \mathcal{D}(T)$$

The symmetry means that the adjoint T^* extends T, i.e.

$$\Gamma(T) \subset \Gamma(T^*)$$

Recall that adjoints always are closed operators. Hence $\Gamma(T^*)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$ and becomes a Hilbert space equipped with the inner product

$$\{x, y\} = \langle x, y\rangle + \langle T^*x, T^*y\rangle$$

Moreover, since T is closed. it follows that $\Gamma(T)$ appears as a closed subspace of this Hilbert space. Consider the eigenspaces:

$$\mathcal{D}_+ = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = ix\} \quad \text{and} \quad \mathcal{D}_- = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = -ix\}$$

10.1 Proposition. *The following orthogonal decomposition exists in the Hilbert space* $\Gamma(T^*)$*:*

$$(*) \qquad \Gamma(T^*) = \Gamma(T) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$$

Proof. The verification that the three subspaces are pairwise orthogonal is left to the reader. To show that the direct sum above is equal to $\Gamma(T^*)$ we use duality and there remains only to prove that

$$(1) \qquad \Gamma(T)^\perp = \mathcal{D}_+ \oplus \mathcal{D}_-$$

To show (1) we pick a vector $y \in \Gamma(T)^\perp$. Here $(y, T^*y) \in \Gamma(T^*)$ and the definition of orthogonal complements gives:

$$\langle x, y\rangle + \langle Tx, T^*y\rangle = 0 \quad : \quad x \in \mathcal{D}(T)$$

From this we see that $T^*y \in \mathcal{D}(T^*)$ and obtain

$$\langle x, y\rangle + \langle x, T^*T^*y\rangle = 0$$

The density of $\mathcal{D}(T)$ entails that

$$0 = y + T^*T^*y = (T^* + iE)(T^* - iE)(y) \implies$$

$$\xi = T^*y - iy \in \mathcal{D}_- \quad \text{and} \quad \eta = T^*y + iy \in \mathcal{D}_+ \implies$$

$$y = \frac{1}{2i}(\eta - \xi) \in \mathcal{D}_- \oplus \mathcal{D}_+$$

which proves (1).

10.2 The case $\dim(\mathcal{D}_+) = \dim(\mathcal{D}_-)$**.** Suppose that $\mathcal{D}_+$ and $\mathcal{D}_-$ are finite dimensional with equal dimension $n \geq 1$. Then self-adjoint extensions of T are found as follows: Let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathcal{D}_+$ and $f_1, \ldots, f_n$ a similar basis in $\mathcal{D}_-$. For each n-tuple $e^{i\theta_1}, \ldots, e^{i\theta_n}$ of complex numbers with absolute value one we have the subspace of $\mathcal{H}$ generated by $\mathcal{D}(T)$ and the vectors

$$\xi_k = e_k + e^{i\theta_k} \cdot f_k \quad : \quad 1 \leq k \leq n$$

On this subspace we define a linear operator A_θ where $A_\theta = T$ on $\mathcal{D}(T)$ while

$$A_\theta(\xi_k) = ie_k - ie^{i\theta_k} \cdot f_k$$

10.3 Exercise. Verify that A_θ is self-adjoint and prove the converse, i.e. if A is an arbitrary self-adjoint operator such that

$$\Gamma(T) \subset \Gamma(A) \subset \Gamma(T^*)$$

then there exists a unique n-tuple $\{e^{i\theta_\nu}\}$ such that

$$A = A_\theta$$

10.4 Example. Let $\mathcal{H}$ be the Hilbert space $L^2[0,1]$ of square-integrable functions on the unit interval $[0,1]$ with the coordinate t. A dense subspace $\mathcal{H}_*$ consists of functions $f(t) \in C^1[0,1]$ such that $f(0) = f(1) = 0$. On $\mathcal{H}_*$ we define the operator T by

$$T(f) = if'(t)$$

A partial integration gives

$$\langle T(f), g\rangle = i\int_0^1 f'(t) \cdot \bar{g}(t) \cdot dt = \int_0^1 \bar{g}'(t) \cdot f(t)dt = \langle f, T(g)\rangle$$

Hence T is symmetric. Next, an L^2-function h belongs to $\mathcal{D}(T^*)$ if and only if there exists a constant $C(h)$ such that

$$\Big|\int_0^1 if'(t) \cdot \bar{h}(t)dt\Big| \leq C(h) \cdot ||f||_2 \quad : f \in \mathcal{H}_*$$

This means that $\mathcal{D}(T^*)$ consists of all L^2-functions h such that the distribution derivative $\frac{dh}{dt}$ again belongs to L^2.

Exercise. Show that

$$\mathfrak{D}_+ = \{h \in L^2 \quad : \frac{dh}{dt} = ih\}$$

is a 1-dimensional vector space generated by the L^2-function e^{ix}. Similarly, $\mathcal{D}_-$ is 1-dimensional and generated by e^{-ix}.

Self-adjoint extensions of T. For each complex number $e^{i\theta}$ we get the linear space $\mathcal{D}_\theta$ of functions $f(t) \in \mathcal{D}(T^*)$ such that

$$f(1) = e^{i\theta} f(0)$$

Exercise. Verify that one gets a self-adjoint operator T_θ which extends T where is $\mathcal{D}(T_\theta) = \mathcal{D}_\theta$. Conversely, show every self-adjoint extension of T is equal to T_θ for some θ. Hence the family $\{T_\theta\}$ give all self-adjoint extensions of T with their graphs contained in $\Gamma(T^*)$.

10.5 Semi-bounded symmetric operators.

Let T be closed, densely defined and symmetric. It is said to be bounded below if there exists some positive constant k such that

(*) $$\langle Tx, x\rangle \geq k \cdot ||x||^2 \quad : \quad x \in \mathcal{D}(T)$$

On $\mathcal{D}(T)$ we have the Hermitian bilinear form:

(1) $$\{x, y\} = \langle Tx, y\rangle \quad \text{where (*) entails that} \quad \{x, x\} \geq k \cdot ||x||^2$$

In particular a Cauchy sequence with respect to this inner product is a Caichy sequence in the given Hilbert space $\mathcal{H}$. So if $\mathcal{D}_*$ is the completion of $\mathcal{D}(T)$ with respect to the inner product above, then it appears as a subspace of $\mathcal{H}$. Put

$$\mathcal{D}_0 = \mathcal{D}(T^*) \cap \mathcal{D}_*$$

10.5.1 Proposition. *One has the equality*

(*) $$T^*(\mathcal{D}_0) = \mathcal{H}$$

Proof. A vector $x \in \mathcal{H}$ gives a linear functional on $\mathcal{D}_*$ defined by

$$y \mapsto \langle y, x\rangle$$

We have

(i) $$|\langle y, x\rangle| \leq ||x| \cdot ||y|| \leq ||x||| \cdot \frac{1}{\sqrt{k}} \cdot \sqrt{\{y, y\}}$$

where we used (1) above. The Hilbert space $\mathcal{D}_*$ is self-dual. This gives a vector $z \in \mathcal{D}_*$ such that

(iii) $$\langle y, x\rangle = \{y, z\} = \langle Ty, z\rangle$$

Since $\mathcal{D}(T) \subset \mathcal{D}_*$ we have (iii) for every vector $y \in \mathcal{D}(T)$, and the construction of T^* entails that $z \in \mathcal{D}(T^*)$ so that (iii) gives

(iv) $$\langle y, x \rangle = \langle y, T^*(z) \rangle$$

The density of $\mathcal{D}_*$ in $\mathcal{H}$ implies that $x = T^*(z)$ and since $x \in \mathcal{H}$ was arbitrary we get (*) in the proposition.

10.5.2 A self-adjoint extension. Let T_1 be the restriction of T^* to $\mathcal{D}_0$. We leqve it to the reader to check that T_1 is symmetric and has a closed graph. Moroever, since $\mathcal{D}(T) \subset \mathcal{D}_0$ and T^* is an extension of T we have

$$\Gamma(T) \subset \Gamma(T_1)$$

Next, Proposition 11.2.1 gives

$$T_1(\mathcal{D}(T_1) = \mathcal{H}$$

i.e. the T_1 is surjective. But then T_1 is self-adjoint by the general result below.

10.5.3 Theorem . *Let S be a densely defined, closed and symmetric operator such that*

(*) $$S(\mathcal{D}(S)) = \mathcal{H}$$

Then S is self-adjoint.

Proof. Let S^* be the adjoint of S. When $y \in \mathcal{D}(S^*)$ we have by definition

$$\langle Sx, y \rangle = \langle x, S^* y \rangle \quad : \quad x \in \mathcal{D}(S)$$

If $S^* y = 0$ this entails that $\langle Sx, y \rangle = 0$ for all $x \in \mathcal{D}(S)$ so the assumption that $S(\mathcal{D}(S)) = \mathcal{H}$ gives $y = 0$ and hence S^* is injective. Finally, if $x \in \mathcal{D}(S^*)$ the hypothesis (*) gives $\xi \in \mathcal{D}(S)$ such that

(i) $$S(\xi) = S^*(x)$$

Since S is symmetric, S^* extends S so that (i) gives $S^*(x - \xi) = 0$. Since we already proved that S^* is injective we have $x = \xi$. This proves that $\mathcal{D}(S) = \mathcal{D}(S^*)$ which means that S is self-adjoint.

§ 11. Contractions and the Nagy-Szegö theorem

A linear operator A on the Hilbert space $\mathcal{H}$ is a contraction if its operator norm is ≤ 1, i.e.

(1) $$||Ax|| \leq ||x|| \quad : \quad x \in \mathcal{H}$$

Let E be the identity operator on $\mathcal{H}$. Now $E - A^*A$ is a bounded self-adjoint operator and (1) gives:

$$\langle x - A^*Ax, x\rangle = ||x||^2 - ||Ax||^2 \geq 0$$

From the result in § 8.xx it follows that this non-negative self-adjoint operator has a square root:

$$B_1 = \sqrt{E - A^*A}$$

Next, the operator norms of A and A^* are equal so A^* is also a contraction and the equation $A^{**} = A$ gives the self-adjoint operator

$$B_2 = \sqrt{E - AA^*}$$

Since $AA^* = A^*A$ is not assumed the self-adjoint operators B_1, B_2 need not be equal. However, the following hold:

11.3.1 Propostion. *One has the equations*

$$AB_1 = B_2A \quad \text{and} \quad A^*B_2 = B_1A^*$$

Proof. If n is a positive integer we notice that

(i) $$A(A^*A)^n = (AA^*)^nA$$

Now A^*A is a self-adjoint operator whose compact spectrum is confined to the closed unit interval $[0, 1]$. If $f \in C^0[0, 1]$ is a real-valued continuous function it can be approximated uniformly by a sequence of polynomials $\{p_n\}$ and the operational calculus from § XX yields an operator $f(A^*A)$ where

$$\lim_{n\to\infty} ||p_n(A^*A) - f(A^*A)|| = 0$$

Since the spectrum of AA^* also is confined to $[0, 1]$, the same polynomial sequence $\{p_n\}$ gives an operator $f(AA^*)$ where

$$\lim ||p_n(AA^*) - f(AA^*)|| = 0$$

Now (i) and the two limit formulas above give:

(ii) $$A \circ f(A^*A) = f(AA^*) \circ A$$

In particular we can take $f(t) = \sqrt{1-t}$ and Proposition 11.3.1 follows.

11.2 The unitary operator U_A. On the Hilbert space $\mathcal{H} \times \mathcal{H}$ we define a linear operator U_A represented by the block matrix

(*) $$U_A = \begin{pmatrix} A & B_2 \\ B_1 & -A^* \end{pmatrix}$$

11.3 Proposition. *U_A is a unitary operator on $\mathcal{H} \times \mathcal{H}$.*

Proof. For a pair of vectors x, y in $\mathcal{H}$ we must prove the equality

(i) $$||U_A(x \oplus y)||^2 = ||x||^2 + ||y||^2$$

To get (i) we notice that for every vector $h \in \mathcal{H}$ the self-adjointness of B_1 gives

(ii) $$||B_1h||^2 = \langle B_1h, B_1h\rangle = \langle B_1^2h, h\rangle = \langle h - A^*Ah, h\rangle = ||h||^2 - ||Ah||^2$$

where the last equality holds since we have $\langle A^*Ah, h\rangle = \langle Ah, A^{**}h\rangle = ||Ah||^2$ and the biduality formula $A = A^{**}$. In the same way one has:

(iii) $$||B_2h||^2 = ||h||^2 - ||A^*h||^2$$

Next, by the construction of U_A the left hand side in (i) becomes

(iv) $$||Ax + B_2y||^2 + ||B_1x - A^*y||^2$$

Using (iii) we have

$$||Ax + B_2y||^2 = ||Ax||^2 + ||y||^2 - ||A^*y||^2 + \langle Ax, B_2y\rangle + \langle B_2y, Ax\rangle$$

Similarly, (ii) gives

$$||B_1x - A^*y||^2 = ||x||^2 - ||Ax||^2 + ||A^*y||^2 - \langle B_1x, A^*y\rangle - \langle A^*y, B_x\rangle$$

Adding these two equations we conclude that (i) follows from the equality

(v) $$\langle Ax, B_2y\rangle + \langle B_2y, Ax\rangle = \langle B_1x, A^*y\rangle + \langle A^*y, B_x\rangle$$

To get (v) we use Proposition 11.5.1 which gives

$$\langle Ax, B_2y\rangle = \langle x, A^*B_2y\rangle = \langle x, B_1A^*y\rangle = \langle B_1x, A^*y\rangle$$

where the last equality used that B_1 is self-adjoint. In the same way one verifies that

$$\langle B_2y, Ax\rangle = \langle A^*y, B_x\rangle$$

and (v) follows.

11.4 The Nagy-Szegö theorem.

The constructions above were applied by Nagy and Szegö to give:

11.4.1 Theorem *For every bounded linear operator A on a Hilbert space $\mathcal{H}$ there exists a Hilbert space $\mathcal{H}^*$ which contains $\mathcal{H}$ and a unitary operator U_A on $\mathcal{H}^*$ such that*

$$A^n = \mathcal{P} \cdot U_A^n \quad : \quad n = 1, 2, \ldots$$

where $\mathcal{P}\colon \mathcal{H}^* \to \mathcal{H}$ is the orthogonal projection.

Proof. On the product $\mathcal{H}_1 = \mathcal{H} \times \mathcal{H}$ we have the unitary operator U_A from (*) in 11.3.2. Let $\mathcal{P}(x, y) = x$ be the projection onto the first factor. Then (*) in (11.3.2) gives $A = \mathcal{P}U_A$ and the constructions from the proof of Propostion 11.3.4 imply that $A^n = \mathcal{P} \cdot U^n$ hold for every $n \geq 1$ which finishes the proof.

The Nagy-Szegö result has an interesting consequence. Let A be a contraction. If $p(z) = c_0 + c_1 < + \ldots + c_n z^n$ is an arbitrary polynomial with complex coefficients we get the operator $p(A) = \sum c_\nu A^\nu$ and with these notations one has:

11.4.2 Theorem *For every pair $A, p(z)$ as above one has*

$$||p(A)|| \leq \max_{z \in D} |p(z)|$$

where the the maximum in the right hand side is taken on the unit disc.

Proof. Theorem 11.4.1 gives $p(A) = \mathcal{P} \cdot p(U_A)$. Since the orthogonal $\mathcal{P}$-projection is norm decreasing we get

$$||p(A)(\xi)||^2 \leq ||p(U_A)(\xi, 0)||^2$$

Let ξ be a unit vector such that $||p(A)(\xi)|| = ||p(A)||$. The operational calculus in § 7 XX applied to the unitary operator U_A yields a probablity measure μ_ξ on the unit circle such that

$$||p(U_A)(\xi, 0)||^2 = \int_0^{2\pi} |p(e^{i\theta})|^2 \cdot d\mu_\xi(\theta)$$

The right hand side is majorized by $|p|_D^2$ and Theorem 11.4.2 follows.

11.4.3 An application. Let $A(D)$ be the disc algebra. Since each $f \in A(D)$ can be uniformly approximated by analytic polynomials, Theorem 11.4.2 entails that if a linear operator A on the Hilbert space $\mathcal{H}$ is a contraction then each $f \in A(D)$ gives a bounded linear operator $f(A)$, i.e. we have norm-preserving map from the supnorm algebra $A(D)$ into the space of bounded linear operators on $\mathcal{H}$.

§ 12 Miscellanous results

Before Theorem 12.x is announced we recall that the product formula for matrices in § X asserts the following. Let $N \geq 2$ and T is some $N \times N$-matrix whose elements are complex numbers which as usual is regarded as a linear operator on the Hermitian space $\mathbf{C}^N$. Then there exists the self-adjoint matrix $\sqrt{T^*T}$ whose eigenvalues are non-negative. Notice that for every vector x one has

(i) $$||T^*T(x)||^{|}|Tx||^2 \implies ||\sqrt{T^*T}(x)|| = ||Tx||$$

and since $\sqrt{T^*T}$ is self-adjoint we have an orthogonal decomposition

(ii) $$\sqrt{T^*T}(\mathbf{C}^N) \oplus \mathrm{Ker}(\sqrt{T^*T}) = \mathbf{C}^N$$

where the self-adjointness gives the equality

(iii) $$\mathrm{Ker}(\sqrt{T^*T}) = \sqrt{T^*T}(\mathbf{C}^N)^\perp$$

The partial isometry operator. Show that there exists a unique linear operator P such that

(*) $$T = P \cdot \sqrt{T^*T}$$

where the P-kernel is the orthogonal complement of the range of $\sqrt{T^*T}$. Moreover, from (i) it follows that

$$||P(y)|| = ||y||$$

for each vector in the range of $\sqrt{T^*T}$. One refers to P as a partial isometry attached to T.

Extension to operators on Hilbert spaces.. Let T be a bounded opertor on the Hilbert space $\mathcal{H}$. The spectral theorem for bounded and self-adjoint operators gives a similar equation as in (*) above using the non-negative and self-adjoint operator $\sqrt{T^*T}$. More generally, let T be densely defined and closed. From § XX there exists the densely defined self-adjoint operator T^*T and we can also take its square root.

12.1 Theorem. *There exists a bounded partial isometry P such that*

$$T = P \cdot \sqrt{T^*T}$$

Proof. Since T has closed graph we have the Hilbert space $\Gamma(T)$. For each $x \in \mathcal{D}(T)$ we get the vector $x_* = (x, Tx)$ in $\Gamma(T)$. Now

$$(x_*.y_*) \mapsto \langle x, y \rangle$$

is a bounded Hermitiain bi-linear form on the Hilbert space $\Gamma(T)$. The self-duality of Hilbert spaces gives bounded and self-adjoint operator A on $\Gamma(T)$ such that

$$\langle x, y \rangle = \{Ax_*, y_*\}$$

where the right hand side is the inner product between vectors in $\Gamma(T)$. Let

$$j \colon (x, Tx) \mapsto x$$

be the projection from $\Gamma(T)$ onto $\mathcal{D}(T)$ and for each $x \in \mathcal{D}(T)$ we put

$$Bx = j(Ax_*)$$

Then B is a linear operator from $\mathcal{D}(T)$ into itself where

(i) $$\langle Bx, y \rangle = \{Ax_*, y_*\} = \{x_*, Ay_*\} = \langle x, By \rangle \quad : \quad x, y \in \mathcal{D}(T)$$

We have also

$$\langle Bx, x \rangle = \{A^2 x_*, x_*\} = \{Ax_*, Ax_*\} = \langle Bx, Bx \rangle + \langle TBx, TBx \rangle \implies$$
$$||Bx||^2 = \langle Bx, Bx \rangle \leq \langle Bx, x \rangle \leq ||Bx|| \cdot ||x||$$

where the Cauchy-Schwarz inequality was used in the last step. Hence

$$||Bx|| \leq ||x|| \quad : \quad x \in \mathcal{D}(T)$$

This entails that that the densey defined operator B extends uniquely to $\mathcal{H}$ as a bounded operator of norm ≤ 1. Moreover, since (i) hold for pairs x, y in the dense subspace $\mathcal{D}(T)$, it follows that B is self-adjoint. Next, consider a pair x, y in $\mathcal{D}(T)$ which gives

$$\langle x, y\rangle = \{Ax_*, y_*\} = \{x_*, Ay_*\} = \langle x, By\rangle + \langle Tx, TBy\rangle$$

Keeping y fixed the linear functiional

$$x \mapsto \langle Tx, TBy\rangle = \langle x, y\rangle - \langle x, By\rangle$$

is bounded on $\mathcal{D}(T)$. By the construction of T^* it follows that $TBy \in \mathcal{D}(T^*)$ and we also get the equality

(ii) $$\langle x, y\rangle = \langle x, By\rangle + \langle x, T^*TBy\rangle$$

Since (ii) holds for all x in the dense subspace $\mathcal{D}(T)$ we conclude that

(iii) $$y = By + T^*TBy = (E + T^*T)(By) \quad : \quad y \in \mathcal{D}(T)$$

Conclusion. From the above we have the inclusion

$$TB(\mathcal{D}(T)) \subset \mathcal{D}(T^*)$$

Hence $\mathcal{D}(T^*T)$ contains $B(\mathcal{D}(T))$ and (iii) means that B is a right inverse of $E + T^*T$ provided that the y-vectors are restricted to $\mathcal{D}(T)$.

FINISH ..

12.2 Positive operators on $C^0(S)$

Let S be a compact Hausdorff space and X the Banach space of continuous and complex-valued functions on S. A linear operator T on X is positive if it sends every non-negative and real-valued function f to another real-valued and non-negative function. Denote by $\mathcal{F}^+$ the family of positive operators T which satisfy the following: First

(1) $$\lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = 0$$

hold for all pairs $x \in X$ and $x^* \in X^*$. The second condition is that $\sigma(T)$ is the union of a compact set in a disc $\{|\lambda| \leq r$ for some $r < 1$, and a finite set of points on the unit circle. The final condition is that $R_T(\lambda)$ is meromorphic in the exterior disc $\{|\lambda > r\}$, i.e. it has poles at the spectral points on the unit circle.

12.2.1. Theorem. *If $T \in \mathcal{F}^+$ then each spectral value $e^{i\theta} \in \sigma(T)$ is a root of unity.*

Proof. Frist we prove that $R_T(\lambda)$ has a simple pole at each $e^{i\theta} \in \sigma(T)$. Replacing T by $e^{-i\theta} \cdot T$ it suffices to prove this when $e^{i\theta} = 1$. If $R_T(\lambda)$ has a pole of order ≥ 2 at $\lambda = 1$ we know from § XX that there exists $x \in X$ such that

(i) $$Tx \neq x \quad \text{and} \quad (E - T)^2 x = 0$$

This gives $T^2 + x = 2Tx$ and by an induction

(ii) $$\frac{1}{n} \cdot T^n x = \frac{1}{n} \cdot x + (E - T)x \quad : n = 1, 2, \ldots$$

Condition (1) and (ii) give for each $x^* \in X^*$:

$$0 = \lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = \lim_{n\to\infty} x^*(\frac{1}{n} \cdot x + (E - T)x)$$

It follows that $x^*(E - T)(x) = 0$ and since x^* is arbitrary we get $Tx = x$ which contradicts (i). Hence the pole must be simple.

Next, with $e^{i\theta} \in \sigma(T)$ we have seen that R_T has a simple pole. By the general result in § xx there exists some $f \in C^0(S)$ which is not identically zero and

$$T(f) = e^{i\theta} \cdot f$$

Multiplying f with a complex scalar we may assume that its maximum norm on S is one and there exists a point $s_0 \in S$ such that

$$f(s_0) = 1$$

For each $n \geq 1$ we have a linear functional on X defined by $g \mapsto T^n(g)(s_0)$ whuch gives a Riesz measure μ_n such that

$$\int_S g \cdot d\mu_n = T^n g(s_0) \quad : g \in C^0(S)$$

Since T^n is positive the integrals in the left hand side are ≥ 0 when g are real-valued and non-negative which entails that the measures $\{\mu_n\}$ are real-valued and non-negative. For each $n \geq 1$ we put

$$A_n = \{x : \ e^{-in\theta} \cdot f(x) \neq 1\}$$

Since the sup-norm of f is one we notice that

(iii) $$A_n = \{x : \ \mathfrak{Re}(e^{-in\theta} f(x)) < 1\}$$

Now

(iv) $$0 = f(s_0) - e^{-in\theta} \cdot T^n f(s_0) = \int_S [1 - e^{-in\theta} f(s)] \cdot d\mu_n(s)$$

Taking real parts we get

(v) $$0 = \int_S [1 - \mathfrak{Re}(e^{-in\theta} f(s))] \cdot d\mu_n(s)$$

By (iii) the integrand in (v) is non-negative and since the whole integral is zero it follows that

(vi) $$\mu_n(A_n) = \mu_n(\{\mathfrak{Re}(e^{-in\theta} < 1\} = 0$$

Suppose now that there exists a pair $n \neq m$ such that

(vii) $$(S \setminus A_n) \cap (S_m \setminus A_m) \neq \emptyset$$

A point s_* in this non-empty intersection gives

$$1 = e^{in\theta} f(s_*) = e^{im\theta} \cdot f(s_*) \implies e^{in\theta} = e^{im\theta}$$

and hence $e^{i\theta}$ is a root of unity. $m - n \neq 0$. So the proof of Theorem 6.1 is finished if we have established the following

Sublemma. The sets $\{S \setminus A_n\}$ cannot be pairwise disjoint.

Proof. First, f has maximum norm and by the above:

$$\int_S f \cdot d\mu_n = e^{in\theta}$$

Hence the total mass $\mu_n(S)$ is at least one. Next, for each $n \geq 2$ we set

$$\pi_n = \frac{1}{n} \cdot (\mu_1 + \ldots + \mu_n)$$

Since $\mu_n(S) \geq 1$ for each n we get $\pi_n(S) \geq 1$. Put

$$\mathcal{A} = \bigcap A_n$$

Above we proved that $\mu_n(A_n) = 0$ hold for every n which gives

(*) $$\pi_n(\mathcal{A}) = 0 \quad : \ n = 1, 2, \ldots$$

Next, when the sets $\{S \setminus A_k\}$ are pairwise disjoint one has the inclusions

$$S \setminus A_k \subset A_\nu \quad \forall \, \nu \neq k$$

Keeping k fixed it follows that $\pi_\nu(S\setminus A_k)=0$ for every $\nu\geq 0$. So when n is large while k is kept fixed we obtain

$$(**)\qquad \pi_n(S\setminus A_k))=\frac{1}{n}\cdot\mu_k(S\setminus A_k))\implies\lim_{n\to\infty}\pi_n(S\setminus A_k))=0\quad : k=1,2,\ldots$$

At this stage we use Lemma xx which shows that $R_T(\lambda)$ has at most a simple pole at $\lambda=1$. With $\epsilon>0$ the Neumann series expansion gives

$$E+\sum_{k=1}^\infty\frac{T^k}{(1+\epsilon)^k}=R_T(1+\epsilon))=\frac{1}{\epsilon}\cdot Q+W(1+\epsilon))$$

where $W(\lambda)$ is an operator-valued analytic function in an open disc centered at $\lambda=1$ while Q is a bounded linear operator on $C^0(S)$. Keeping $\epsilon>0$ fixed we apply both sides to the identity function 1_S on S and the construction of the measures $\{\mu_n\}$ gives

$$1+\sum_{k=1}^\infty\frac{\mu_k(S)}{(1+\epsilon)^k}=\frac{1}{\epsilon}\cdot Q(1_S)(s_0)+W(1+\epsilon))(1_S)(s_0)$$

If $n\geq 2$ is an integer and $\epsilon=\frac{1}{n}$ one gets the inequality

$$\sum_{k=1}^{k=n}\frac{\mu_k(S)}{(1+\frac{1}{n})^k}\leq n\cdot|Q(1_S)(s_0)|+|W(1+1/n))(1_S)(s_0)|\leq n||Q||+||W(1+1/n)||\implies$$

$$\frac{1}{n}\cdot\sum_{k=1}^{k=n}\mu_k(S)\leq(1+\frac{1}{n})^n\cdot(||Q||+\frac{||W(1+1/n)||}{n}$$

Since Neper's constant $e\geq(1+\frac{1}{n})^n$ for every n we find a constant C which is independent of n such that

$$\frac{1}{n}\cdot\sum_{k=1}^{k=n}\mu_k(S)\leq C$$

Hence the sequence $\{\pi_n(S)\}$ is bounded and we can pass to a subsequence which converges weakly to a limit measure μ_*. For this σ-additive measure the limit formula in (**) above entails that

$$\text{(i)}\qquad \mu_*(S\setminus A_k)=0\quad : k=1,2,\ldots$$

Moreover, by (*) we also have

$$\text{(ii)}\qquad \pi_*(\mathcal{A})=0$$

Now $S=\mathcal{A}\cup A_k$ so (i-ii) give:

$$\mu_*(S)=0$$

But this is impossible for at the same time we have already seen that $\pi_n(S)\geq 1$ for each n and hence also $\mu_*(S)\geq 1$.

Compact pertubations to finish Kakutani-Yosida !!!

In general, consider some complex Banch space X be a Banach space and denote by $\mathcal{F}(X)$ the family of bounded liner operators T on X such that

$$(*)\qquad \lim_{n\to\infty}\frac{1}{n}\cdot x^*(T^nx)=0$$

hold for all pairs $x\in X$ and $x^*\in X^*$.

1. Exercise. Apply the Banach-Steinhaus theorem to show that if $T\in\mathcal{F}(X)$ then there exists a constant M such that the operator norms satisfy

$$||T^n||\leq M\cdot n\quad : n=1,2,\ldots$$

Since the n:th root of $M \cdot n$ tends to one as $n \to +\infty$, the spectral radius formula entails that the spectrum $\sigma(T)$ is contained in the closed unit disc of the complex λ-plane. So in the exterior disc $\{|\lambda| > 1\}$ there exists the the resolvent

$$R_T(\lambda) = (\lambda \cdot E - T)^{-1}$$

2. The class $\mathcal{F}_*$. It consists of those T in $\mathcal{F}(X)$ for which there exists some $\alpha < 1$ such that $R_T(\lambda)$ extends to a meromorphic function in the exterior disc $\{|\lambda| > \alpha\}$. Since $\sigma(T) \subset \{|\lambda| \leq 1\}$ it follows that when $T \in \mathcal{F}_*$ then the set of points in $\sigma(T)$ which belongs to the unit circle in the complex λ-plane is empty or finite and after we can always choose $\alpha < 1$ such that

$$\sigma(T) \cap \{\alpha < |\lambda| < 1\} = \emptyset$$

3. Proposition. *If $T \in \mathcal{F}_*$ and $e^{i\theta} \in \sigma(T)$ for some θ, then Neumann's resolvent $R_T(\lambda)$ has a simple pole at $e^{i\theta}$.*

Proof. Replacing T by $e^{-i\theta} \cdot T$ it suffices to prove the result when $e^{i\theta} = 1$. If $R_T(\lambda)$ has a pole of order ≥ 2 at $\lambda = 1$ we know from § XX that there exists $x \in X$ such that

(i) $$Tx \neq x \quad \text{and} \quad (E - T)^2\, x = 0$$

The last equation means that $T^2 + x = 2Tx$ and an induction over n gives

(ii) $$\frac{1}{n} \cdot T^n x = \frac{1}{n} \cdot x + (E - T)x$$

Since $T \in \mathcal{F}$ we have

(iii) $$\lim_{n \to \infty} \frac{1}{n} \cdot x^*(T^n x) = 0 \quad : \forall, x^* \in X^*$$

Then (ii) entails that $x^*(E - T)(x) = 0$. Since x^* is arbitrary we get $Tx = x$ which contradicts (i) and hence the pole is simple.

4. Theorem. *Let $T \in \mathcal{F}(X)$ be such that there exists a compact operator K where $||T + K|| < 1$. Then $T \in \mathcal{F}_*$ and for every $e^{i\theta} \in \sigma(T)$ the eigenspace $E_T(\theta) = \{x \in X : Tx = e^{i\theta}x\}$ is finite dimensional.*

Proof. Set $S = T + K$ and for a complex number λ we write $\lambda \cdot E - T = \lambda \cdot E - T - K + K$. Outside $\sigma(S)$ we get

(i) $$R_S(\lambda)(\lambda \cdot E - T) = E + R_S(\lambda) \cdot K$$

The Neumann series for large absolute values $|\lambda|$ applied to $R_S(\lambda)$ gives some $\rho > 0$ and

(ii) $$(E + R_S(\lambda) \cdot K)^{-1} = E + R_S(\lambda) \cdot K\dot{(}E + R_S(\lambda) \cdot K)^{-1} \quad : \ |\lambda| > \rho$$

Next, when $|\lambda|$ is large we notice that (i) gives

(iii) $$R_T(\lambda) = (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Together with (ii) we obtain

(iv) $$R_T(\lambda) = R_S(\lambda) + R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Set $\alpha = ||S||$ which by assumption is < 1. Now $R_S(\lambda)$ is analytic in the exterior disc $\{\lambda| > \alpha\}$ so in this exterior disc $R_\lambda(T)$ differs from the analytic function $R_\lambda(S)$ by

(v) $$\lambda \mapsto R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Here K is a compact operator so the result in § XX entails that this function extends to be meromorphic in $\{|\lambda| > \alpha\}$. There remains to prove that eigenspaces at spectral points on the unit circle are finite dimensional. To prove this we use (iv). Let $e^{i\theta} \in \sigma(T)$. By Proposition 3 it is a simple pole so we have a Laurent series expansion

$$R_T(e^{i\theta} + z) = \frac{A_{-1}}{z} + A_0 + A_1 z + \ldots$$

By the general results from §§ there remains to show that A_{-1} has finite dimensional range. To see this we apply (iv) which gives the equation

$$R_S(e^{i\theta}+z)+R_S(e^{i\theta}+z)\cdot(E+R_S(e^{i\theta}+z)\cdot K)^{-1}\cdot R_S(e^{i\theta}+z)$$

To simplify notations we set $B(z)=R_S(e^{i\theta}+z)$ which by assumption is analytic in a neighborhood of $z=0$. Moreover, the operator $B(0)$ is invertible. So now one has

$$\frac{A_{-1}}{z}+A_0+A_1z+\ldots=B(z)+B(z)(E+B(z)\cdot K)^{-1}B(z)$$

Since $B(0)$ is invertible we have a Laurent series expansion

$$(E+B(z)\cdot K)^{-1}=\frac{A^*_{-1}}{z}+A^*_0+A^*_1z+\ldots$$

and identying the coefficient of z^{-1} gives

$$A_{-1}=B(0)A^*_{-1}B(0)$$

Next, from (xx) one has

$$E=(E+B(z)\cdot K)(\frac{A^*_{-1}}{z}+A^*_0+A^*_1z+\ldots)\implies(E+B(0)\cdot K)A^*_{-1}=0$$

Here $B(0)\cdot K$ is a compact operator and hence Fredholm theory implies that A^*_{-1} has a finite dimensional range. Since $B(0)$ is invertible the same is true for A_{-1} which finishes the proof of Theorem 4.

5. Proposition. *If $T\in\mathcal{F}$ is such that $T^N\in\mathcal{F}_*$ for some integer $N\geq 2$. Then $T\in\mathcal{F}_*$.*

Proof. We have the algebraic equation

$$\lambda^N\cdot E-T^N=(\lambda\cdot E-T)(\lambda^{N-1}\cdot E+\lambda^{N-2}\cdot T+\ldots+T^{N_1})$$

It follows that

$$R_T(\lambda)=(\lambda^{N-1}\cdot E+\lambda^{N-2}\cdot T+\ldots+T^{N_1})\cdot R_{T^N}(\lambda^N)$$

Since $T^NB\in\mathcal{F}_*$ there exists $\alpha<1$ such that

$$\lambda\mapsto R_{T^N}(\lambda^N)$$

extends to be meromorphic in $\{|\lambda|>\alpha\}$. At the same time $(\lambda^{N-1}\cdot E+\lambda^{N-2}\cdot T+\ldots+T^{N_1})$ is a polynomial and hence $R_T(\lambda)$ also extends to be meromorphic in this exterior disc so that $T\in\mathcal{F}_*$.

12.3 Factorizations of non-symmetric kernels.

Recall that the Neumann-Poincaré kernel $K(p,q)$ of a plane C^1-curve $\mathcal{C}$ is given by

$$K(p,q) = \frac{\langle p-q, \mathbf{n}_i(p)\rangle}{|p-q|}$$

This kernel function gives the integral operator $\mathcal{K}$ defined on $C^0(\mathcal{C})$ by

$$\mathcal{K}_g(p) = \int_C K(p,q) \cdot g(q)\, ds(q)$$

where ds is the arc-length measure on C. Let M be a positive number which exceeds the diameter of $\mathcal{C}$ so that $|p-q| < M : \ p,q \in \mathcal{C}$. Set

$$N(p,q) = \int_{\mathcal{C}} K(p,\xi) \cdot \log \frac{M}{|q-\xi|} \cdot ds(\xi)$$

Exercise. Verify that N is symmetric, i.e. $N(p,q) = N(q,p)$ hold for all pairs p,q in $\mathcal{C}$. Moreover,

$$S(p,q) = \log \frac{M}{|p-q|}$$

is a symmetric and positive kernel function and since $\mathcal{C}$ is of class C^1 the reader should verify that it gives a Hilbert-Schmidt kernel, i.e.

$$\iint_{\mathcal{C}\times\mathcal{C}} S(p,q)^2\, ds(p)ds(q) < \infty$$

Hence the Neuman-Poincaré operator $\mathcal{K}$ appears in an equation

$$(*) \qquad \mathcal{N} = \mathcal{K} \circ \mathcal{S}$$

where $\mathcal{S}$ is defined via a positive symmetric Hilbert-Schmidt kernel and $\mathcal{N}$ is symmetric. Following [Carleman: § 11] we give a procedure to determine the spectrum of $\mathcal{K}$.

12.3.1 Spectral properties of non-symmetric kernels.

In general,let $K(x,y)$ be a continuous real-valued function on the closed unit square $\square = \{0 \leq x,y \leq 1\}$. We do not assume that K is symmetric but there exists a positive definite Hilbert-Schmidt kernel $S(x,y)$ such that

$$N(x,y) = \int_0^1 S(x,t)K(t,y)\, dy$$

yields a symmetric kernel function. The Hilbert-Schmidt theory gives an orthonormal basis $\{\phi_n\}$ in $L^2[0,1]$ formed by eigenfunctions to $\mathcal{S}$ where

$$(1) \qquad \mathcal{S}\phi_n = \kappa_n \phi_n$$

where the positive κ-numbers tend to zero. Moreover, each $u \in L^2[0,1]$ has a Fourier-Hilbert expansion

$$(2) \qquad u = \sum \alpha_n \cdot \phi_n$$

We seek eigenfunctions of the integral operator $\mathcal{K}$. Let u be a function in $L^2[0,1]$ such that:

$$(3) \qquad u = \lambda \cdot \mathcal{K}u$$

where λ in general is a complex number. It follows that

$$(4) \qquad \lambda \cdot \int N(x,y)u(y)\, dy = \lambda \iint SA(x,t)K(t,y)u(y)\, dtdy = \int S(x,t)u(t)\, dt$$

Multiplying with $\phi_p(x)$ an integration gives

$$(5) \qquad \lambda \cdot \int \phi_p(x)N(x,y)u(y)\, dxdy = \iint \phi_p(x)S(x,t)u(t)\, dxdt = \kappa_p \int \phi_p(t)u(t)\, dt$$

Next, using the expansion of u from (2) we get the equations:

$$\sum_{q=1}^{\infty} \alpha_q \cdot \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy = \kappa_p\alpha_p \quad : \ p = 1,2,\ldots \tag{6}$$

Set

$$c_{qp} = \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy$$

It follows that $\{\alpha_p\}$ satisfies the system

$$\kappa_p\alpha_p = \lambda \cdot \sum_{q=1}^{\infty} c_{qp}\alpha_q \tag{7}$$

Since $N(x,y) = N(y,x)$ the doubly indexed c-sequence is symmetric. Set

$$\beta_p = \sqrt{\kappa_p} \cdot \alpha_p \implies \beta_p = \lambda \cdot \sum_{q=1}^{\infty} \frac{c_{pq}}{\sqrt{\kappa_p} \cdot \sqrt{\kappa_q}} \cdot \beta_q \tag{8}$$

Next, put

$$k_{p,q} = \iint K(x,y)\phi_p(x)\phi_q(y)\,dxdy \tag{9}$$

From the above the following hold for each pair p, q:

$$c_{pq} = \iiint \phi_q(x)\phi_p(y)S(x,t)K(t,y)\,dxdydt = \kappa_q k_{p,q} = \kappa_p k_{q,p} \implies$$

$$\frac{c_{p,q}^2}{\kappa_p \cdot \kappa_q} \leq |k_{p,q} \cdot k_{q,p}| \leq \frac{1}{2}(k_{p,q}^2 + k_{q,p}^2) \tag{10}$$

Here $\{k_{p,q}\}$ are the Fourier-Hilbert coefficients of $K(x,y)$ which entails that

$$\sum\sum k_{p,q}^2 \leq \iint K(x,y)^2\,dxdy$$

Hence the symmetric and doubly indexed sequence

$$\frac{c_{p,q}}{\sqrt{\kappa_p \cdot \kappa_q}} \tag{11}$$

is of Hilbert-Schmidt type.

11.6.2 Conclusion. The eigenfunctions u in $L^2[0,1]$ associated to the $\mathcal{K}$-kernel have Fourier-Hilbert expansions via the $\{\phi_n\}$-basis which are determined by α-sequences satisfying the system (7)

11.6.3 Remark. When a plane curve $\mathcal{C}$ has corner points the Neumann-Poincaré kernel is unbounded. Here the reduction to the symmetric case is more involved and leads to quite intricate results which appear in Part II from [Carleman]. The interplay between singularities on boundaries in the Neumann-Poincaré equation and the corresponding unbounded kernel functions illustrates the general theory densely defined self-adjoint operators. Much analysis remains to be done and open problems about the Neumann-Poincaré equation remains to be settled in dimension three. So far it appears that only the 2-dimensional case is properly understood via results in [Car:1916]. See also § xx for a studiy of Neumann's boundary value problem both in the plane and $\mathbf{R}^3$.

12.4 Uniqueness results for the exterior Laplace equation

Let Ω be a bounded open set in $\mathbf{R}^3$ whose boundary is a finite union of closed surfaces of class C^1 at least. Set $U = \mathbf{R}^3 \setminus \Omega$. Denote by $\mathcal{S}(U)$ the class of real-valued C^2-functions f in U which extend continuously to the boundary of U which of course is equal to $\partial\Omega$. Moreover, we assume that the the exterior normal derivaties $\frac{df}{dn}$ taken along the boundary exist and give a continuous function on ∂U. Consider large positive R-numbers so that the open ball $B(R)$ of radius R centered at the origin contains the closure of Ω. Set

$$D(f)^2(x,y,z) = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + (\frac{\partial f}{\partial z})^2$$

Green's formula applied to the domain $B(R) \setminus \overline{\Omega}$ gives

(1)
$$\int_{B(R)\setminus\overline{\Omega}} D(f)^2\,dp + \int_{B(R)\setminus\overline{\Omega}} f \cdot \Delta(f)\,dp = \int_{\partial U}$$

where $dp = dxdxydz$. We shall use this to prove:

1. Theorem. *If f and $\Delta(f)$ both belong to $L^2(U)$, it follows that*

$$\int_U D(f)^2\,dp = \int_{\partial U} - \int_U f \cdot \Delta(f)\,dp$$

Proof Since f and $\Delta(f)$ are square integrable, the Cauchy-Schwarz inequality entails that $f \cdot \Delta(f)$ is absolutely integrable over U. Hence (1) gives the theorem if we prove the limit formula

(i)
$$\liminf_{R\to\infty} \int_{\partial U} = 0$$

To prove (i) we consider the function

$$\psi(R) = \int_{B(R)\setminus\overline{\Omega}} u^2\,dp$$

The derivative becomes

$$\psi'(R) = \int_{S(R)} u^2 \cdot d\omega$$

where $S(R)$ is the sphere of radius R and $d\omega$ its area measure. Passing to the second derivative the reader should verify the equation

$$\psi''(R) = \frac{2}{R} \cdot \psi'(R) + 2 \cdot \int_{S(R)} f \cdot \frac{\partial f}{\partial n}\,d\omega$$

Hence (i) follows if

(ii)
$$\liminf_{R\to\infty} \psi''(R) - \frac{2}{R} \cdot \psi'(R) = 0$$

To obtain (ii) we first notice that Ψ is non-decreasing and since f is square integrable it tends to a finite limit as $R \to +\infty$. Hence the first order derivative cannot stay above a positive constant for all large R. So for the derivative ψ' two cases can occur. Either it decreases in a monotone way to zero as $R \to +\infty$. In this case it is eveident that there exists a striuctly increasing sequebce $\{r_n\}$ such that $\psi''(r-n) \to 0$ and (ii) follows. In the second case the function $\psi'(R)$ attain a local minium at an infinite sequence $\{r_n\}$ which again tend to zero. Here $\psi''(r_n) = 0$ and at the same time these local minimum values of the first order derivative tend to zero. So again (ii) holds and Theorem 1 is proved.

A vanishing result.

Let f satisfy the differential equation

(*)
$$\Delta(f) + \lambda \cdot f = 0$$

in U for some real number λ. In addition we assume that

(**) $$\frac{df}{dn}(p) = 0 \quad : p \in \partial U$$

2. Theorem. *If f satisfies (*-**) and belongs to $L^2(U)$, then f is identically zero.*

Proof. Notice that Theorem 1 gives the equality

$$\int_U D(f)^2\, dp = \lambda \cdot \int_U f^2\, dp$$

So if $\lambda \leq 0$ the vanishing of f is obvious. From now on $\lambda > 0$. We shall work with polar coordinates, i.e. employ the Euler's angular variables ϕ and θ where

$$0 < \theta \quad : 0 < \phi < 2\pi$$

The wellknown expression of Δ in the variables r, θ, ϕ shows that the equation (*) corresponds to

(i) $$xxxxx + xxxx = 0$$

Let $n \geq 1$ and $Y_n(\theta, \phi)$ some spherical function of degree n with a normalised L^2-integral equal to one. For each r where $B(r)$ contains $\overline{\Omega}$ we set

(ii) $$Z(r) = \int_0^{2\pi} \int_0^{\pi} Y_n(\theta,\phi) \cdot f(r,\theta,\phi)\, \sin(\theta)\, d\theta d\phi$$

The Cauchy-Schwarz inequality gives

$$Z(r)^2 \leq \int_0^{2\pi} \int_0^{\pi} Y_n^2 \cdot \sin(\theta)\, d\theta d\phi \cdot \int_0^{2\pi} \int_0^{\pi} f^2(r,\theta,\phi) \cdot \sin(\theta)\, d\theta d\phi$$

Since the L^2-integral if Y_n is normalised the last product is reduced to

$$J(r) = \int_0^{2\pi} \int_0^{\pi} f^2(r,\theta,\phi) \cdot \sin(\theta)\, d\theta d\phi$$

Now

$$\int_{r_*}^{\infty} r^2 \cdot J(r)\, dr$$

is equal to the finite L^2-integral of f in the exterior domain taken outside a ball $B(r_*)$. From (ii) we conclude that

(iii) $$\int_{r_*}^{\infty} r^2 \cdot Z(r)^2\, dr < \infty$$

A differential equation. Recall that a spherical function of degree n satisfies

(iv) $$\frac{1}{\sin\theta} \frac{d}{d\theta} \big(\sin\theta \cdot \frac{\partial Y_n}{\partial \theta}\big) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_n}{\partial \phi^2} + (n+1)n \cdot Y_n = 0$$

Exercise. Show via suitable partial integrations that (i) and (iv) imply that $Z(r)$ satisfies the differential equation

(v) $$\frac{1}{r^2} \cdot \frac{d}{dr}(r^2 \cdot \frac{dZ}{dr}) + (\lambda - \frac{n(n+1)}{r^2})Z = 0$$

The second order differential equation has two linearly independent solutions of the form

$$Z_1 = \cos(\sqrt{\lambda} r) \cdot [\frac{1}{r} + \frac{a_2}{r^2} + \ldots]$$

$$Z_2 = \sin(\sqrt{\lambda} r) \cdot [\frac{1}{r} + \frac{b_2}{r^2} + \ldots]$$

It follows that there exist a pair of constants c_1, c_2 such that

(vi) $$Z = c_1 Z_1 + c_2 Z_2 = \frac{c_1 \cdot \cos(\sqrt{\lambda} r) + c_2 \cdot \sin(\sqrt{\lambda} r)}{r} + \frac{B(r)}{r^2}$$

where $r \mapsto B(r)$ stays bounded as $r \to +\infty$.

Exercise. Show that the finite integral in (iii) and (vi) give $c_1 = c_2 = 0$ and hence $Z(r)$ identically zero for large r. Since this hold for all spherical functions we conclude that f is identically zero outside the closed ball $B(r_*)$. Finally, by assumption U is connected and the elliptic equation (*) implies that f is a real-analytic function in U, So the vanishing outside a large ball entails that f is identically zero in U which finishes the proof of Theorem 2.

3. A result about absolute continuity.

We consider functions depending upon a real parameter μ which varies in an interval $[a, b]$. To each μ we are given a function $f(x, y, z; \mu)$ which is square integrable C^2-function in U and the normal derivative along ∂U is zero, i.e just as in the class $\mathcal{S}$ above. Moreover, the $L^2(U)$-valued function

(i) $$\mu \mapsto f(x, y, z; \mu)$$

has a finite total variation on $[a, b]$. Next, assume that for every sub-interval ℓ of $[a, b]$ one has the equality

$$\Delta\Big(\int_\ell \frac{d}{d\mu} f(x, y, z; \mu) + \int_\ell \mu \cdot \frac{d}{d\mu} f(x, y, z; \mu) = 0$$

where the integrals as usual are taken in the sense of Borel-Stieltjes.

Theorem. *Every function from (i) which satisfies the conditions above is absolutely continues with respect to μ.*

About the proof. Using similar methods as in the proof of Theorem 1 one reduces the proof to study functions $g(r; \mu)$ where $r \mapsto g(r; \mu)$ is a C^2-function and square integrable on the interval $[r_*, +\infty)$ for a given $r_* > 0$ while μ as above varies in $[a, b]$. Moreover one has

$$\max_\mu \int_{r_*}^\infty g(r; \mu)\, dr < \infty$$

Next, for each sub-interval $\ell = [\alpha, \beta]$ we set

$$\delta_\ell(g(r, \mu) = g(r, \beta) - g(r, \alpha)$$

With these notations we say that $g(r; \mu)$ is absolutely continuous with respect to μ if there to each $\epsilon > 0$ exists $\delta > 0$ such that

$$\sum \int_{r_*} |\delta_{\ell_\nu}((g(r, \mu)|^2 \cdot r^2\, dr < \epsilon$$

for every finite family of sub-intervals $\{\ell_\nu\}$ when the sum of their lengths is $< \delta$.

Theorem. *Assume in addition to the above that the equation below holds for each sub-interval ℓ*

$$\frac{1}{r^2} \frac{d}{dr} \big[r^2 \frac{d}{dr} (\delta_\ell(g(r, \mu)\big] - \frac{n(n+1)}{r^2} \cdot \delta_\ell(g(r, \mu) + \int_\ell \mu \cdot \frac{d}{d\mu}(g(r, \mu) = 0$$

Then $g(r, \mu)$ is absolutely continuous with respect to μ.

A first example: Moment problems.

In the very impressive and highly original article *Recherches sur les fractions continues* [Ann.Fac. Sci. Toulouse. 1894], Stieltjes studied the moment problem on the non-negative real line. This amounts to find a non-negative Riesz measure μ on $\mathbf{R}^+$ with prescribed moments

$$(*) \qquad c_\nu = \int_0^\infty x^\nu d\mu(x) \quad : \nu = 0, 1, 2, \ldots$$

An obvious necessary condition for the existence of μ is that the quadratic form

$$J(x) = \sum_{p=1}^\infty \sum_{q=1}^\infty c_{pq} x_p x_q$$

is positive definite when it is restricted to vectors x-vectors where only finitely many $x_p \neq 0$. The moment problem is determined with respect to the given sequence $\{c_\nu\}$ if there exists a unique non-negative measure μ such that (*) holds. To prove existence and analyze uniqueness, Stieltjes employed the expansion in continued fractions of the c-sequence. Stieltjes' work was later reconsidered by Hamburger in the article *xxxx*[1922] where one allows solutions by Riesz measures μ on the whole real line, i.e. now integration takes place over $(-\infty, +\infty)$. In § xx we explain that this (extended) moment problem is closely related to the class of linear operators defined by inifinite and symmetric matrices of the form

$$xxx = xxx$$

where $\{a_n\}$ is an arbitrary real sequence while $\{b_n\}$ is a sequence of positive real numbers. A complete description of all solutions μ to a non-determined moment problem was established by Carleman in the article *Sur le probléme des moments* [C.R.Acad. Sci. Paris. 1922], and we shall expose this in § xx.

Returning to the case treated by Stieltjes the existence of non-determined moment problems, leads to a peculiar result for certain densely defined linear operators on the Hilbert space ℓ^2. Following Stieltjes we consider a positive sequence of real numbers $\{b_n\}$ and regard the operator S defined by

$$S(x) = (-b_1 x_2, -b_1 x_1, -b_2 x_4, -b_2 x_3, \ldots)$$

where $x = \{x_n\}$ is a vector in ℓ^2. It means that S is represented by the infinite symmetric matrix whose non-zero elements only appear in position $(j, j+1)$ or $(j+1, j)$ for every integer $j \geq 1$. Now there exists the dense subspace $\mathcal{D}(S)$ of vectors $x \in \ell^2$ such that $S(x)$ also belongs to ℓ^2. The question arises if the symmetry of the S-matrix persists in the following sense: Suppose that x and y both belong to $\mathcal{D}(S)$. Does it follow that

$$(**) \qquad \langle S(x), y \rangle = \langle x, S(y) \rangle$$

where we have used the Hermitian inner product on the complex Hilbert space ℓ^2. So above x and y can be complex vectors. In the cited article above, Carleman proved that (**) hold for every pair in $\mathcal{D}(S)$ if and only if the corresponding moment problem is determined. This means that in spite of the symmetry of the matrix which represents S, the resulting densely defined linear operator can fail to be self-adjoint. In § xx we show that whenever this peculair phenomenon occurs, the series

$$\sum_{n=1}^\infty \frac{1}{b_n} < \infty$$

However, the converse is not true, i.e. there exist determine moment problems where the series above is converges.

A final remark. The study of the Hamburger-Stieltjes moment problem offers an instructive lesson about unbounded but densely defined liear operators on Hilbert spaces. At the same time the material in §xx will teach that even if one starts with a problem expressed in terms of functional analysis, it is often necesssty to employ both results and methods from other disciplines such as analytic function theory and Fourier analysis.

Moment problems.

A great inspiration during the development of unbounded symmetric operators on Hibert spaces emerged from Stieltjes' pioneering article *Recherches sur les fractions continues* [Ann. Fac. Sci. Toulouse. 1894]. The moment problem asks for conditions on a sequence $\{c_0, c_1, \ldots\}$ of positive real numbers such that there exists a non-negative Riesz measure μ on the real line and

$$(*) \qquad c_\nu = \int_{-\infty}^{\infty} t^\nu \cdot d\mu(t)$$

hold for each $\nu \geq 0$. One easily verifies that if μ exists then the Hankel determinants

$$\begin{pmatrix} c_0 & c_1 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & c_{2n} \end{pmatrix}$$

are > 0 for every $n = 0, 1, \ldots$. For brevity we say that $\{c_n\}$ is a positive Hankel sequence when the determinants above are > 0. It turns out that this confition also is sufficient.

Theorem. *For each positive Hankel sequence there exists at least one non-negative Riesz measure such that (*) holds.*

Remark. The result above is due to Hamburger who extended Stieltjes' original results which treated the situation where one seeks non-negative measures supported by $\{\geq 0\}$. The moment problem for a positive Hankel sequence is said to be determined if (*) has a unique solution μ. Hamburger proved that the determined case holds if and only if the the associated continued fractions of $\{c_n\}$ is completely convergent. See § x below for an account about this convergence condition. A "drawback" in the Hamburger-Stieltjes theory is that it is often difficult to check when the associated continued fraction of a positive Hankel sequence is completely convergent. So one is led to seek sufficient conditions in order that (*) has a unique solution μ. Such a sufficient condition was established in Carleman's monograph [Carleman 1923. Page 189-220] and goes as follows:

1. Theorem. *The moment problem of a positive Hankel sequence $\{c_n\}$ is determined if*

$$\sum_{n=1}^{\infty} c_n^{-\frac{1}{n}} = +\infty$$

Another criterion for determined moment problems. Studies of continued fractions and their associated quadratic forms go back to work by Heine. See [Handbuch der theorie der Kugelfunktionen: Vol. 1. part 2]. An excellent account appears also in the article *Zur Einordnung der Kettenbruchentheorie in die theorie der quadratischen Formen von unendlichvielen Veränderlichen* [Crelle J. of math. 1914] by Hellinger and Toeplitz. A positive Hankel sequence $\{c_n\}$ corresponds to a quadratic form in an infinite number of variables:

$$J(x) = \sum_{p=1}^{\infty} a_p x_p^2 - 2 \cdot \sum_{p=1}^{\infty} b_p x_p x_{p+1}$$

where $b_p \neq 0$ for every p. More precisely, the sequences $\{a_\nu\}$ and $\{b_\nu\}$ arise when the series

$$-[\frac{c_0}{\mu} + \frac{c_1}{\mu^2} + \frac{c_2}{\mu^3} + \ldots]$$

is formally expanded into a continuous fraction

$$\frac{c_0}{a_1 - \mu - \frac{b_1^2}{a_1 - \mu - \frac{b_2^2}{a_2 - \mu}}}$$

Now there exists the infinite matrix A with diagonal elements $\alpha_{pp} = a_p$ while

$$\alpha_{p,p+1} = \alpha_{p+1,p} = -b_p \quad : p = 1, 2, \ldots$$

and all the other elements are zero. The symmetric A-matrix gives a densely defined linear operator on the complex Hilbert space ℓ^2. The following result is proved [Carleman:1923]:

2. Theorem. *The densely defined operator A on ℓ^2 is self-adjoint if and only if the moment problem for $\{c_n\}$ is determined.*

3. A study of A-operators.

Ignoring the "source" of the pair of real sequences $\{a_p\}$ and $\{b_p\}$ we consider a matrix A as above where the sole condition is that $b_p > 0$ for every p. If μ is a complex number one seeks infinite vectors $x = (x_1, x_2, \ldots)$ such that

$$\text{(i)} \qquad Ax = \mu \cdot x$$

It is clear that (i) holds if and only if the sequence $\{x_p\}$ satisfies the infinite system of linear equations

$$(a_1 - \mu)x_1 = b_1 x_2$$

$$(a_p - \mu)x_p = b_{p-1}x_{p-1} + b_p x_{p+1} \quad : p \geq 2$$

Since $b_p > 0$ for every p we see that x_1 determines the sequence and $x_2, x_3, \ldots$ depend on x_1 and the parameter μ. Keeping x_1 fixed while μ varies the reader can verify that

$$x_p = \psi_p(\mu) \quad : p \geq 2$$

where $\psi_p(\mu)$ is a polynomial of degree $p-1$ for each $p \geq 2$. These ψ-polynomials depend on the given pair of sequences $\{a_p\}$ and $\{b_p\}$ and with the following result is proved in [ibid]:

3.1 Theorem. *The densely defined operator A on ℓ^2 is self-adjoint if and only if*

$$\sum_{p=1}^{\infty} \psi_p(\mu) = +\infty$$

for every non-real complex number μ.

4. A general result. Let $\{c_\nu\}$ be a positive Hankel sequence, determined or not. Let ρ be a non-negative measure which solves the moment problem (*) and set

$$\widehat{\rho}(\mu) = \int \frac{d\rho(t)}{t - \mu}$$

This yields an analytic function in the upper half-plane. A major result in [Carleman] gives a sharp inclusion for the values which can be attained by these $\widehat{\rho}$ functions while ρ varies in the family of non-negative measures which solve the moment problem. It is expressed via constructions of discs in the upper half-plane which arise via a nested limit of discs constructed from a certain family of Möbius transformations. More precisely, to the given Hankel sequence we have the pair of sequences $\{a_\nu\}$ and $\{b_\nu\}$. For a fixed μ in the upper half-plane we consider the maps:

$$S_\nu(z) = \frac{b_\nu^2}{a_{\nu+1} - \mu - z} \quad : \nu = 0, 1, \ldots$$

Notice that

$$\mathfrak{Im}(S_\nu(z)) = \frac{b_\nu^2 \cdot \mathfrak{Im}(\mu + z)}{|a_{\nu+1} - \mu - z|^2} > 0$$

It follows that S_ν maps the upper half-plane U^+ conformally onto a disc placed in U^+.

Exercise. To each $n \geq 1$ we consider the composed map

$$\Gamma_n = S_0 \circ S_1 \circ \ldots \circ S_{n-1}$$

Show that the images $\{C_n(\nu) = \Gamma_n(U^+)\}$ form a decreasing sequence of discs and there exists a limit

$$C(\mu) = \cap\, C_n(\mu)$$

which either is reduced to a single point or is a closed disc in U^+.

4.1 Theorem. *For each $\mu \in U^+$ and every ρ-measure which solves the moment problem one has the inclusion*

$$\widehat{\rho}(\mu) \subset C(\mu)$$

Moreover, for each point z in this disc there exists ρ such that $\widehat{\rho}(\mu) = z$.

§ 12. Symmetric integral operators.

Consider the domain $\square = \{0 \leq x \leq 1\} \times \{0 \leq y \leq 1\}$ in $\mathbf{R}^2$. Let $K(x,y)$ be a real-valued and Lebesgue mesure function on $\square$ such that the integrals

$$\int_0^1 K(x,y)^2\, dy < \infty$$

for all x outside a null-set on $[0,1]$. In addition we assume that K is symmetric, i.e.$K(x,y) = K(y,x)$. The K-kernel is bounded in Hilbert's sense if there exists a constant C such that

$$\iint_\square K(x,y)u(x)u(y)\, dxdy \leq C^2 \cdot \int_0^1 u(x)^2\, dx$$

for each $u \in L^2[0,1]$. A special case occurs when $K(x,y)$ satisfies

$$\iint_\square |K(x,y)|^2\, dxdy < \infty$$

Then $\mathcal{K}$ is called a Hilbert-Schmidt operator and a crucial fact is that it yields a compact operator on the Hilbert space $\mathcal{H} = L^2[0,1]$.

Exercise. Prove that Hilbert-.Schmidt operators are compact. Tne hint is that Lebesgue theory to begin with entails that when (zz) holds then there exists a seqience of symmwtric and continuous kernel functions $\{K_n(,y)\}$ such that

$$\iint_\square |K(x,y) - K(x,y)|^2\, dxdy = 0$$

Next, the Cauchy-Schwarz inequality gives

$$||\mathcal{K} - \mathcal{K}_n|| \leq \iint_\square |K(x,y) - K(x,y)|^2\, dxdy$$

for each n, where the right hand side refers to operator norms. Now one uses the general fact that a linear operator which can be approximated in the operator norm by compact operators is itself compact. Finally the reader should verify that if $K(x,y)$ is contoinous then $\mathcal{K}$ is compact.

Spectral functions. Let K be a symmetric Hilbert-Schmidt operator. A special case of Holbert's theorem from § 8 and the general facts about spectera of compact operators in § xx, entail that there exists a sequence $\{\phi_n\}$ of pairwise orthogonal functions in $\mathcal{H}$ with L^2-norms equal to one, a real eigenvalues $\{\mu_n\}$ so that

$$\mathcal{K}(\phi_n) = \mu_n \cdot \phi_n$$

The eigenvalues form a discrete set outside zero and arranged so that $\mu_1 \geq |\mu_2 \geq \ldots$ and repeated according to multiplicities, i.e. when the corresponding eigenspace has dimension ≥ 2. Set

$$\lambda_n = \mu_n^{-1}$$

For each $\lambda > 0$ we set

$$\rho(x,y;\lambda) = \sum_{0<\lambda_\nu<\lambda} \phi_\nu(x)\phi_\nu(y)$$

$$\rho(x,y;-\lambda) \sum_{-\lambda<\lambda_\nu<0} \phi_\nu(x)\phi_\nu(y)$$

Notice that the λ-numbers in (x) stay away from zero. Hence function ρ_N vanishes in a neighborhood of zero. The spectral theorem applied to symmetric Hilbert-Schmidt operators entails that

$$\mathcal{K}(h)(y) = \int_{-\infty}^{\infty} \frac{1}{\lambda} \cdot \frac{d}{d\lambda}\, \rho(x,y;\lambda) \cdot h(y)\, dy$$

hold for each $h \in \mathcal{H}$.

The Ω-kernel. If $h \in \mathcal{H}$ we have an expansion

$$h = \sum a_k \cdot \phi_k + h^*$$

where h^* is $|perp$ to the subspace of $\mathcal{H}$ generated by the eigenfunctions and the reader should check that

$$\sqrt{\sum |a_k|^2} \leq ||h||_2$$

Apply (x) to a pair h, g in $L^2[0,1]$. Then the reader should check that

$$\text{(xx)} \qquad |\iint_\square \rho(x,y;\lambda) \cdot h(x)g(y)\,dxdy| \leq ||h||_2 \cdot ||g||_2$$

hold for each λ which may be taken to be positive or negative. For each λ we set

$$\Omega(x,y;\lambda) = \int_a^x \int_a^y \rho(s,t;\lambda)\,dsdt$$

Exercise. Conclude from (xx) that the function

$$(x,y) \mapsto \Omega(x,y;\lambda)$$

is Hölder continuous of order 1/2. More precisely, for each λ and every pair $x_1 < x_2$ and $y_1 < y_2$ one has

$$|\Omega(x_2,y_2;\lambda) - \Omega(x_1,y_1;\lambda)| \leq \sqrt{(x_2-x_1)(y_2-y_1)}$$

A crucial point is that (e.1) holds for all Hilbert-Schmidt kernels, i.e. independent of the size of the Hilbert-Schmidt norms.

The ψ-functions. With λ and $0 < x \leq 1$ kept fixed we set

$$\psi(y;\lambda) = \int_0^x \rho(s,y;\lambda)\,ds$$

We have the characteristic function $\chi(x)$ defined as one on $[0,x_*]$ and zero if $x > x_*$. Considered as a vector in $\mathcal{H}$ one has an expansion

$$\chi(x) = \sum a_k\phi_k(x) + \chi_* \quad : \, a_k = \int_0^{x_*} \phi_k(s)\,ds$$

At the same time the construction of the ρ-function entails that

$$\psi(y;\lambda) = \sum_* \int_0^{x_*} \phi_k(s) \cdot \phi_k(y)$$

with the sum restricited over those k for which $0 < \lambda_k < \lambda$. Bessel's inequality gives

$$\int_0^1 \psi(y;\lambda)^2\,dy \leq \bigl(\sum_* \int_0^{x_*} \phi_k(s)\bigr)^2$$

In the last term we have taken a restricted sum which is majorsised by the sum over all k which again by Bessel's inequality is majorised by the L^2-integral of $|chi$, i.e by x^2. Hnece one has the inequality

$$\int_0^1 \psi(y;\lambda)^2\,dy \leq x \quad : 0 < x \leq 1$$

Exercise. More generally, let E be a sum of disjoint intervals on the x-interval and put

$$\psi_E(y;\lambda) = \int_E \rho(x,y;\lambda)\,dx$$

Show that

$$\int_0^1 \psi_E(y;\lambda)^2\,dy \leq |E|_1$$

where the last term is the Lebesgue measure of E.

Weak limits. Let us first notice that the contruction of the ψ-function means that

$$\frac{\partial\Omega(x,y;\lambda)}{\partial y} = \psi_x(y;\lambda)$$

So (xx) entails that the Hölder contionius Ω-function has a partial y-derivative in the sense of distributions which belongs to L^2 which moreover is absolutely continuu as a fuinction of x since (xx) is constucted as a primitive function in the sense of Lebesgue. It follows that one also has

$$\frac{\partial}{\partial x}(\frac{\partial\Omega(x,y;\lambda)}{\partial y}) = \rho(x,y;\lambda)$$

Let us then consider a sequence of Hilbert-Schmidt kernels $\{K_n\}$ and to each of them we get the spectral function $\rho_n(x,y;\lambda)$. Now $\{\Omega_n(x,y;\lambda)$ is an equi-contonuous family of functions on $\square$. So by the Arzela-Ascli theorem we find a subsequence which converges uniformly to a limit function $\Omega_*(x,y;\lambda)$ for each fixed λ. From (*) Ω is again uniformly Hölder continuous. Moreover, the uniform bound for the L^2-norms of $y \mapsto \psi_x(y;\lambda)$ entail that the partial y-erivatives of Ω_* yield L^2-functions which are absoutely continuous with respect to x. So there exists an almost everywhere defined limit function

$$\rho_*(x,y;\lambda) = \frac{\partial}{\partial x}(\frac{\partial\Omega_*(x,y;\lambda)}{\partial y})$$

for which the inequality (xx) holds.

and its first order partial derivatives are L^2-functions. Moreover, after the passage to the limit one still has Bessel's inequality which entails that

$$\int_{-\infty}^{\infty} \frac{d}{d\lambda} \int_0^1 h(x) \cdot \big(\frac{\partial}{\partial x} \int_0^1 \frac{\partial \Omega(x,y;\lambda)}{\partial y} \cdot h(y))\, dy\big)\, dx \leq \int_0^1 h(x)^2\, dx$$

for every $h \in L^2[0,1]$.

In the monograph [Carelan 1923] the condition (*) is imposed while (**) need not be valid. Then we encounter an unbounded operator. But notice that if $u|in\mathcal{H}$ then (*9 and the CuachySchwarz inequality entails that the functions

$$y \mapsto K(x,y)u(y)$$

are absolutely integrable in Lebegue's sense for all x outside the nullset $\mathcal{N}$ above. Hence it makes sense to refer to L^2-functions ϕ on $[0,1]$ which satisfy an eigenvalue equation

$$\phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\,dy \tag{1}$$

where λ is a compkex number. one is also led to consider the integral equation

$$\phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\,dy + f(x) \tag{2}$$

where $f \in \mathcal{H}$ is given and one seeks ϕ. It turns out that if $\mathfrak{Im}(\lambda) \neq 0$, then the inhomogenous equation (2) has at least one solution. Next, consider the equation (1). Let $\rho(\lambda)$ denote the number of linearly independent solution in (1). Carelan proived that the ρ-function is constant when λ varies in $\mathbf{C} \setminus \mathbf{R}$. Follwing Carelan one says that the kernel (x,y) gives an operator $\mathcal{K}$ of Class I if the ρ-function is zero. It means that (2) has unique solutions for every pair $\lambda, f)$ when λ are non-real.

A limit process. Let $\{G_n(x,y)\}$ be a sequece of symmetric kernel functions which are bounded in Hilbert's sense and approximate K in the sense that

$$\lim_{n\to\infty} \int_0^1 |K(x,y) - G_n(x,y)|^2\,dy = 0$$

for all x outside a null set. Fix some non-real λ. Hilbert's theory entials that if $f(x)$ is a continuous function, in general complex-valued, then we find unqiue continuous functions $\{\phi_n\}$ which satisfy the integral equations

$$\phi_n(x) = \lambda \cdot \int_0^1 G_n(x,y)\phi(y)\,dy \tag{1}$$

For a fixed x we consider the complex numbers $\{phi_n(x)\}$. Now there exists the set $Z(x)$ of all cluster points, i.e. a complex number z belongs to $Z(x)$ if there exists some sequence $1 \leq n_1 < n_2 < \ldots$ such that

$$z = \lim_{k\to\infty} \phi_{n_k}(x)$$

Theorem. *For every approximating sequence $\{G_n\}$ as above the sets $Z(x)$ are either reduced to points or circles in the complex plane. Moreover, each $Z(x)$ is reduced to a siungleton set when $\mathcal{K}$ is of class I.*

Remark. The result below was discovered by Weyl for some special unbounded operators which arise during the study of second order differential equations. See § for a comment-. If $\mathcal{K}$ is of Class I then the theorem sbove shows that each pair of a on-real λ and some $f \in \mathcal{H}$ gives a unique function $\phi(x)$ which satsfies (2) and it can be found via a pointwise limit of solutions $\{\phi_n\}$ to the equations (xx). In this sense the limit process is robust because one can emply an arbitrary approximating sequence $\{G_n\}$ under the sole condition that (xx) holds. This already indficartes that the Case I leads to a "consistent theory" even if $\mathcal{K}$ is unbounded. To make this precise Carleman constructed a unique spectral function when Case I holds. More preciely, if $K(x,y)$ is symmetric and Case I holds, then there exists a unique function $\rho(x,y;\lambda)$ defined for $(x,y) \in \square$ and every real λ such that

$$\mathcal{K}(h)(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda} \cdot \frac{d}{d\lambda} \int_0^1 \theta(x,y;\lambda)\,h(y)\,dy$$

for all L^2-functions h. Concerning the spectral θ-function, Carleman proved that it enjoys the same properties as Hilbert's spectral function for bounded operators.

Remark. For each fixed $\lambda > 0$ one has a bounded self-adjoint operator on $\mathcal{H}$ defined by

$$\Theta_\lambda(h)(x) = \int_0^1 \theta(x, y; \lambda)\, h(y)\, dy$$

Moreover, the operator-valued function $\lambda \mapsto \Theta_\lambda$ has bounded variation over each interval $\{a \leq \lambda \leq b\}$ when $0 < a < b$. It means that there exists a constant $C = C(a, b]$ such that

$$\max \sum_{k=0}^{M} ||\Theta_{\lambda_{k+1}} - \Theta_{\lambda_k}|| \leq C$$

for all partitions $a = \lambda_0 < \lambda_1 < \ldots \lambda_{M+1} = b$ and we have taken operator norms of the differences $\Theta_{\lambda_{k+1}} - \Theta_{\lambda_k}$ in the sum above. These bounded variations entail that one can compute the integrals in the right hand side via the usual method by Borel and Stieltjes.

The construction of spectral functions. When case I holds one constructs the ρ-function via a robust limit process. Following [ibid:Chapitre 4] we expose this in § xx. The strategy is to regard an approximating sequence $\{G_n\}$ of Hilbert-Schmidt operator, i.e.

$$\iint_\square G_N(x, y)^2\, dxdy < \infty$$

hold for each N. In this case $\{G_N\}$ are compact opertors. With N fixed we get a discrete sequence of non-zero real numbers $\{\lambda\nu\}$ which are arragned with increasing avbsolute values and an orthonormal family of eigenfunctions $\{\phi_\nu^{(n)}\}$ where

$$G_N(\phi_\nu^{(N)}) = \lambda nu \cdot \phi_\nu^{(N)}$$

hold for each ν. Of course, the eigenvalues also depend on N. If $\lambda > 0$ we set

$$\rho_N(x, y; \lambda) = \sum_{0 < \lambda_\nu < \lambda} \phi_\nu(x)\phi_\nu(y)$$

$$\rho_N(x, y; -\lambda) \sum_{-\lambda < \lambda_\nu < 0} \phi_\nu(x)\phi_\nu(y)$$

let us notice that for each fixed n, the λ-numbers in (x) stay away from zero, i.e. there is a constant $c_N > 0$ such that $|\lambda_\nu \geq c_N$. So the function ρ_N vanishes in a neighborhood of zero. The spectral theorem applied to symmetric Hilbert-Schmidt operators entails that

$$\int_{-\infty}^{\infty} \frac{1}{\lambda} \cdot \frac{d}{d\lambda}\, \rho_N(x, y; \lambda) \cdot h(y)\, dy$$

The Ω-kernels. For each N we introduce the function

$$\Omega_N(x, y; \lambda) = \int_a^x \int_a^y \rho_N(s, t; -\lambda)\, dsdt$$

Let us notice that Bessel's inequality entails that

$$|\iint_\square \rho_N(x, y; \lambda) \cdot h(x)g(y)\, dxdy\big| \leq ||h||_2 |cdot||g|7_2$$

for each pair h, g in $L^2[0, 1]$.

Exercise. Conclude from the above that the variation of Ω over $[0, x] \times [0, y]$ is bounfed above by

$$\sqrt{x \cdot y}$$

for every pair $0 \leq x, y \leq 1$. In partcular the functions

$$(x, y) \mapsto \Omega(x, y; \lambda)$$

are uniformly Hölder continuous of order $1/2$ in x and y respectively.

Using the inequalities above we elave it to the reader to check that there exists at least one subsequence $\{N_k\}$ such that the functions $\{\Omega_{N_k}(x,y;\lambda)$ converges uniformly with respect to x and y while λ stays in a bounded interval. When Case I holds one proves that the limit is independent of the subsequence, i.e. there exists a limit function

$$\Omega(x,y;\lambda) = \lim_{N\to\infty} \Omega_N(x,y;\lambda)$$

From the above the ω-function is again uniformly Hölder continuous and its first order partial derivatives are L^2-functions. Moreover, after the passage to the limit one still has Bessel's inequality which entails that

$$\int_{-\infty}^{\infty} \frac{d}{d\lambda} \int_0^1 h(x)\cdot\big(\frac{\partial}{\partial x}\int_0^1 \frac{\partial\Omega(x,y;\lambda)}{\partial y}\cdot h(y))\,dy\big)\,dx \leq \int_0^1 h(x)^2\,dx$$

for every $h \in L^2[0,1]$.

Definition. *A Case 1 kernel $K(x,y)$ is closed if equality holds in (*).*

Theorem. *A Case I kernel $K(x,y)$ for which the equation*

$$\int_0^1 K(x,y)\cdot\phi(y)\ dy = 0$$

has no non-zero L^2-solution ϕ is closed.

A representation formula. When (*) hols we can apply to to $h+g$ for every pair of L^2-functiuons anbd since the Hilbert space $L^2[0,1]$ is self-dual it follows that for each $f \in L^2[0,1]$ one has the equality

$$f(x) = \frac{d}{dx}\int_{-\infty}^{\infty} \frac{d}{d\lambda}\int_0^1 \frac{\partial\Omega(x,y;\lambda)}{\partial y}\cdot f(y))\,dy$$

almost everywhere with respect to x on $[0,1]$.

Remark. The formula (**) shows that it often is important to decide when a Case I kernel is closed. Theorem xx gives such a sufficent condition. However, it can be extended to a quite geneeral result where one for can relax the passage to the limit via suitable linear operators. The reader may consult [ibid; page 139-143] for details. Here we are content to announce the conlsisive result ehich appear in [ibid;: page 142].

The $\mathcal{L}$-family. Let ξ dentoe a paramter which in general depends on several variables, or represents points in a manifold or vectors in a normed linear space. To each ξ we are given a linear operator

$$L(\xi)\colon f \mapsto L(\xi)(f)$$

from functions $f(x)$ on $[0,1]$ to new functions on $[0,1]$. The linear map is weakly continuous, i.e. if $\{f_\nu\}$ is a sequence in $L^2[0,1]$ which converges weakly to a init function f in $L^2[0,1]$ then

$$\lim L(\xi,f_\nu) \xrightarrow{w} L(\xi)(f)$$

Next, we are given $K(x,y)$ and the second condition for L to be in $\mathcal{L}$ is that for each pair ξ and $0 \leq y \leq 1$ there exists a constant $\gamma(\xi,y)$ which is independent of δ so that

$$|L(\xi)(K_\delta(\cdot,y))(x) \leq \gamma(\xi;y)$$

where $K_\delta(x,y)$ is the truncated kernel function from (xx) and in the left hand side we have applied $L(\xi$ to the function $x \mapsto K_\delta(x,y)$ for each fixed y. Moreover, we have

$$\lim_{\delta\to 0} L(\xi)(K(\cdot,y))(x) \xrightarrow{w} \lim_{\delta\to 0} L(\xi)(K_\delta(\cdot,y))(x)$$

where the convergence again holds weakly for $L2$-functions on $[0,1]$. Funally the equality below holds for every L^2-function ϕ:

$$xxxx$$

Symmetric integral operators.

Consider the domain $\square = \{0 \leq x \leq 1\} \times \{0 \leq y \leq 1\}$ in $\mathbf{R}^2$. Let $K(x, y)$ be a real-vaued and Lebesgue mesure function on $\square$ such that the integrals

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for all x outside a null-set on $[0, 1]$. In addition K is symmetric, i.e.$K(x, y) = K(y, x)$. The K-kernel is bounded in Hilbert's sens if there exists a constant C such that

$$\iint_\square K(x, y)u(x)u(y) \, dxdy \leq C^2 \cdot \int_0^1 u(x)^2 \, dx$$

for each $u \in L^2[0, 1]$. In the text-book —emphIntyergralxxx [xxxx 1904], Hilbert proved that the linear operator on $L^2[0, 1]$ defined by

$$\mathcal{K}(u)(x) = \int_0^1 K(x, y)u(y)dy$$

is bounded and its operator norm is majorsied by C-Moreover the spectrum is real and contained in $[-C, C]$ and just as for symmetric real matrices there exists a spectral resolution. More precisely, set $\mathcal{H} = L^2[0, 1]$. Then there exists a map from $\mathcal{H} \times \mathcal{H}$ to the space of Riesz measures supported by $[-C, C]$ which to every pair u and v in $\mathcal{H}$ assigns a Riesz measure $\mu_{\{u,v\}}$ and $\mathcal{K}$ is recovered by the equation:

$$\langle \mathcal{K}(u), v \rangle = \int_{-C}^{C} t \cdot d\mu_{\{u,v\}}(t)$$

where the left hand side is the inner product on the complex Hilbert space defined by

$$\iint \mathcal{K}(u)(x) \cdot \bar{v}(x) \, dx$$

In the monograph [Carelan 1923] the condition (*) is imposed while (**) need not be valid. Then we encounter an unbounded operator. But notice that if $u|in\mathcal{H}$ then (*9 and the CuachySchwarz inequality entails that the functions

$$y \mapsto K(x, y)u(y)$$

are absolutely integrable in Lebegue's sense for all x outside the nullset $\mathcal{N}$ above. Hence it makes sense to refer to L^2-functions ϕ on $[0, 1]$ which satisfy an eigenvalue equation

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where λ is a compkex number. one is also led to consider the integral equation

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For a fixed x we consider the complex numbers $\{phi_n(x)\}$. Now there exists the set $Z(x)$ of all cluster points, i.e. a complex number z belongs to $Z(x)$ if there exists some sequence $1 \leq n_1 < n_2 < \ldots$ such that

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Remark. The result below was discovered by Weyl for some special unbounded operators which arise during the study of second order differential equations. See § for a comment-. If $\mathcal{K}$ is of Class I then the theorem sbove shows that each pair of a on-real λ and some $f \in \mathcal{H}$ gives a unique function $\phi(x)$ which satsfies (2) and it can be found via a pointwise limit of solutions $\{\phi_n\}$ to the equations (xx). In this sense the limit process is robust because one can emply an arbitrary approximating sequence $\{G_n\}$ under the sole condition that (xx) holds. This already indficartes that the Case I leads to a "consistent theory" even if $\mathcal{K}$ is unbounded. To make this precise Carleman constructed a unique spectral function when Case I holds. More preciely, if $K(x,y)$ is symmetric and Case I holds, then there exists a unique function $\rho(x,y;\lambda)$ defined for $(x,y) \in \Box$ and every real λ such that

$$\mathcal{K}(h)(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda} \cdot \frac{d}{d\lambda} \int_0^1 \theta(x,y;\lambda)\,h(y)\,dy$$

for all L^2-functions h. Concerning the spectral θ-function, Carleman proved that it enjoys the same properties as Hilbert's spectral function for bounded operators.

Remark. For each fixed $\lambda > 0$ one has a bounded self-adjoint operator on $\mathcal{H}$ defined by

$$\Theta_\lambda(h)(x) = \int_0^1 \theta(x,y;\lambda)\,h(y)\,dy$$

Moreover, the operator-valued function $\lambda \mapsto \Theta_\lambda$ has bounded variation over each interval $\{a \leq \lambda \leq b\}$ when $0 < a < b$. It means that there exists a constant $C = C(a,b]$ such that

$$\max \sum_{k=0}^{M} ||\Theta_{\lambda_{k+1}} - \Theta_{\lambda_k}|| \leq C$$

for all partitions $a = \lambda_0 < \lambda_1 < \ldots \lambda_{M+1} = b$ and we have taken operator norms of the differences $\Theta_{\lambda_{k+1}} - \Theta_{\lambda_k}$ in the sum above. These bounded variations entail that one can compute the integrals in the right hand side via the usual method by Borel and Stieltjes.

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$$\iint_\Box G_N(x,y)^2\,dxdy < \infty$$

hold for each N. In this case $\{G_N\}$ are compact opertors. With N fixed we get a discrete sequence of non-zero real numbers $\{\lambda\nu\}$ which are arragned with increasing avbsolute values and an orthonormal family of eigenfunctions $\{\phi_\nu^{(n)}\}$ where

$$G_N(\phi_\nu^{(N)}) = \lambda nu \cdot \phi_\nu^{(N)}$$

hold for each ν. Of course, the eigenvalues also depend on N. If $\lambda > 0$ we set

$$\rho_N(x,y;\lambda) = \sum_{0<\lambda_\nu<\lambda} \phi_\nu(x)\phi_\nu(y)$$

$$\rho_N(x,y;-\lambda) \sum_{-\lambda<\lambda_\nu<0} \phi_\nu(x)\phi_\nu(y)$$

let us notice that for each fixed n, the λ-numbers in (x) stay away from zero, i.e. there is a constant $c_N > 0$ such that $|\lambda_\nu \geq c_N$. So the function ρ_N vanishes in a neighborhood of zero. The spectral theorem applied to symmetric Hilbert-Schmidt operators entails that

$$\int_{-\infty}^{\infty} \frac{1}{\lambda} \cdot \frac{d}{d\lambda}\, \rho_N(x,y;\lambda) \cdot h(y)\, dy$$

The Ω-kernels. For each N we introduce the function

$$\Omega_N(x,y;\lambda) = \int_a^x \int_a^y \rho_N(s,t;-\lambda)\, dsdt$$

Let us notice that Bessel's inequality entails that

$$|\iint_\square \rho_N(x,y;\lambda) \cdot h(x)g(y)\, dxdy\big| \leq ||h||_2|cdot||g|7_2$$

for each pair h,g in $L^2[0,1]$.

Exercise. Conclude from the above that the variation of Ω over $[0,x] \times [0,y]$ is bounfed above by

$$\sqrt{x \cdot y}$$

for every pair $0 \leq x, y \leq 1$. In partcular the functions

$$(x,y) \mapsto \Omega(x,y;\lambda)$$

are uniformly Hölder continuous of order $1/2$ in x and y respectively.

Using the inequalities above we elave it to the reader to check that there exists at least one subsequence $\{N_k\}$ such that the functions $\{\Omega_{N_k}(x,y;\lambda)$ converges uniformly with respect to x and y while λ stays in a bounded interval. When Case I holds one proves that the limit is independent of the subsequence, i.e. there exists a limit function

$$\Omega(x,y;\lambda) = \lim_{N\to\infty} \Omega_N(x,y;\lambda)$$

From the above the ω-function is again uniformly Hölder continuous and its first order partial derivatives are L^2-functions. Moreover, after the passage to the limit one still has Bessel's inequality which entails that

$$\int_{-\infty}^{\infty} \frac{d}{d\lambda} \int_0^1 h(x) \cdot \big(\frac{\partial}{\partial x} \int_0^1 \frac{\partial\Omega(x,y;\lambda)}{\partial y} \cdot h(y))\, dy\big)\, dx \leq \int_0^1 h(x)^2\, dx$$

for every $h \in L^2[0,1]$.

Definition. *A Case 1 kernel $K(x,y)$ is closed if equality holds in (*).*

Theorem. *A Case I kernel $K(x,y)$ for which the equation*

$$\int_0^1 K(x,y) \cdot \phi(y)\ dy = 0$$

has no non-zero L^2-solution ϕ is closed.

A representation formula. When (*) hols we can apply to to $h+g$ for every pair of L^2-functiuons anbd since the Hilbert space $L^2[0,1]$ is self-dual it follows that for each $f \in L^2[0,1]$ one has the equality

$$f(x) = \frac{d}{dx}\ \int_{-\infty}^{\infty} \frac{d}{d\lambda} \int_0^1 \frac{\partial\Omega(x,y;\lambda)}{\partial y} \cdot f(y))\, dy$$

almost everywhere with respect to x on $[0,1]$.

Remark. The formula (**) shows that it often is important to decide when a Case I kernel is closed. Theorem xx gives such a sufficent condition. However, it can be extended to a quite geneeral result where one for can relax the passage to the limit via suitable linear operators. The reader may consult [ibid; page 139-143] for details. Here we are content to announce the conlsisive result ehich appear in [ibid;: page 142].

The $\mathcal{L}$-family. Let ξ dentoe a paramter which in general depends on several variables, or represents points in a manifold or vectors in a normed linear space. To each ξ we are given a linear operator

$$L(\xi)\colon f \mapsto L(\xi)(f)$$

from functions $f(x)$ on $[0,1]$ to new functions on $[0,1]$. The linear map is weakly continuous, i.e. if $\{f_\nu\}$ is a sequence in $L^2[0,1]$ which converges weakly to a init function f in $L^2[0,1]$ then

$$\lim L(\xi, f_\nu) \xrightarrow{w} L(\xi)(f)$$

Next, we are given $K(x,y)$ and the second condition for L to be in $\mathcal{L}$ is that for each pair ξ and $0 \leq y \leq 1$ there exists a constant $\gamma(\xi, y)$ which is independent of δ so that

$$|L(\xi)(K_\delta(\cdot, y))(x) \leq \gamma(\xi; y)$$

where $K_\delta(x,y)$ is the truncated kernel function from (xx) and in the left hand side we have applied $L(\xi$ to the function $x \mapsto K_\delta(x,y)$ for each fixed y. Moreover, we have

$$\lim_{\delta \to 0} L(\xi)(K(\cdot, y))(x) \xrightarrow{w} \lim_{\delta \to 0} L(\xi)(K_\delta(\cdot, y))(x)$$

where the convergence again holds weakly for $L2$-functions on $[0,1]$. Funally the equality below holds for every L^2-function ϕ:

$$xxxx$$

11.5 Stones theorem.

11.3.2 Unitary semi-groups. Specialize the situation above to the case when B is a Hilbert space $\mathcal{H}$ and $\{U_t\}$ are unitary operators. Set $T = \xi_*$ so that

$$B(x) = \lim_{t\to 0} \frac{U_t x - x}{t} \quad : \quad x \in \mathcal{D}(T)$$

If x, y is a pair in $\mathcal{D}(B)$ we get

$$\langle Bx, y\rangle = \lim_{t\to 0} \langle \frac{U_t x - x}{t}, U_t y\rangle = \lim_{t\to 0} \frac{1}{t} \cdot \big(\langle U_t x, U_t y)\rangle - \langle x, U_t y)\rangle \big)$$

Since U_t are unitary we have $\langle U_t x, U_t y)\rangle = \langle x, y)\rangle$ for each t and conclude that the last term above is equal to

$$\lim_{t\to 0} \frac{\langle x, y - U_t y\rangle}{t} = -\langle x, By\rangle$$

Hence B is anti-symmetric, i.e.

$$\langle Bx, y\rangle = -\langle x, By\rangle$$

Set $A = i \cdot T$ which gives

$$\langle Ax, y\rangle = i \cdot \langle Tx, y\rangle = -i \cdot \langle x, Ty\rangle = -\langle x, i \cdot Ty\rangle = \langle x, Ay\rangle$$

where we used that the inner product is hermitian. Hence A is a densely defined and symmetric operator.

Theorem. *A is self-adjoint, i.e. one has the equality $\mathcal{D}(A) = \mathcal{D}(A^*)$.*

Proof. It suffices to prove that $\mathcal{D}(T) = |mathcalD(T^*)$. To btain this we take a vector y be a vector in $\mathcal{D}(T^*)$ which by definition gives a constant $C(y)$ such that

(i) $$|\langle Tx, y\rangle|| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

Now $\langle Tx, y\rangle$ is equal to

(ii) $$\lim_{t\to 0} \langle \frac{U_t x - x}{t}, U_t y\rangle = -\lim_{t\to 0} \langle x, \frac{U_t y - y}{t}\rangle$$

So if

$$\eta_t = fracU_t y - yt$$

it follows that there exists

$$\lim_{t\to 0}, \langle x, \eta_t\rangle = -\langle Tx, y\rangle$$

for each $x \in \mathcal{D}(T)$.

The adjoint operators $\{U_t^*\}$ give another unitary semi-group with infinitesmal generator A_* where

$$A_*(x) = \lim_{t\to 0} \frac{U_t^* x - x}{t} \quad : \quad x \in \mathcal{D}(A_*)$$

Since U_t is the inverse operator of U_t^* for each t we get

(i) $$A_*(x) = \lim_{t\to 0} U_t(A_* x) = \lim_{t\to 0} \frac{x - U_t x}{t} \quad : \quad x \in \mathcal{D}(A_*)$$

From (i) we see that $\mathcal{D}(A) \subset \mathcal{D}(A_*)$ and one has the equation

(ii) $$A_* x) = -A(x) \quad : \quad x \in \mathcal{D}(A_*)$$

Reversing the role the reader can check the equality

(iii) $$\mathcal{D}(A) = \mathcal{D}(A_*)$$

Next, let x, y be a pair in $\mathcal{D}(A)$. Then

$$\langle Ax, y\rangle = \lim_{t\to 0} \langle \frac{U_t x - x}{t}, U_t y\rangle = \lim_{t\to 0} \frac{1}{t} \cdot \big(\langle U_t x, U_t y)\rangle - \langle x, U_t y)\rangle \big)$$

Since U_t are unitary we have $\langle U_t x, U_t y)\rangle = \langle x, y)\rangle$ for each t and conclude that the last term in (xx) is equal to

$$\lim_{t\to 0} \frac{\langle x, y - U_t y\rangle}{t} = -\langle x, Ay\rangle$$

Hence A is anti-symmetric. Set

$$B = iA$$

Exercise. Show that (i) gives the equality $\mathcal{D}(A_*) = \mathcal{D}(A)$ and that:

$$A_*(x) = -A(x) \quad : \quad x \in \mathcal{D}(A)$$

$$\langle Bx, y\rangle = -\langle x, By\rangle \quad : \quad x, y \in \mathcal{D}(B)$$

Exercise. Conclude from the above that the operator iB is self-adjoint.

Remark. The equations above constitute Stones theorem which was established in 1930. It has a wide range of applications. See for example von Neumann's article *Zur Operatorenmethode in der klassischen Mekanik* and Maeda's article *Unitary equivalence of self-adjoint operators and constant motion* from 1936.

11.3.3 A converse construction. Let A be a densely defined self-adjoint operator. From § 9.B A is approximated by a sequence of bounded self-adjoint operators $\{A_N\}$. With N kept fixed we get a semi-group of unitary operators where

$$U_t^{(N)} = e^{-itA_N}$$

The reader may verify that the inifintesmal generator becomes $-iA_N$. Next, for each $x \in \mathcal{D}(A)$ and every fixed t there exists the limit

$$\lim_{N\to\infty} U_t^{(N)}(x)$$

Remark. This gives a densely defined linear operator U_t whose operator norm is bounded by one which therefore extends uniquely to a bounded linear operator $\mathcal{H}$ and it is clear that this extension becomes a unitary operator. In this way we arrive at a semi-group $\{U_t\}$ and one verifies that its infinitesmal generator is equal to $-iA$. However, it is not clear that $\{U_t\}$ is strongly continuous and one may ask for conditions on the given self-adjoint operator A which ensures that $\{U_t\}$ is strongly continuous.

11.8 Transition probability functions.

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space as defined in § XX. Consider a real-valued function P defined on the product set $\Omega \times \mathcal{B}$ with the following two properties:

(*) $\qquad t \mapsto P(t, E) \quad$ is a bounded measurable function for each $E \in \mathcal{B}$

(**) $\qquad E \mapsto P(t, E) \quad$ is a probability measure for each fixed $t \in \Omega$

When (*-**) hold one refers to P as a transition function. Given P we define inductively the sequence $\{P^{(n)}\}$ by:

$$P^{(n+1)}(t, E) = \int_\Omega P^{(n)}(s) \cdot dP(s, E)$$

It is clear that $\{P^{(n)}\}$ yield new transition functions. The probabilistic interpretation is that one has a Markov chain with independent increments. More precisely, if E and S are two sets in $\mathcal{B}$ and $n \geq 1$. then

$$\int_S P^{(n)}(t; E) \cdot d\mu(t)$$

is the probability that the random walk which starts at some point in E has arrived to some point in S after n steps. One says that the given transition function P yields a stationary Markov process if there exists a finite family of disjoint subets $E_1, \ldots, E_m$ in $\mathcal{B}$ and some $\alpha < 1$ and a constant M such that the following hold: First, for each $1 \leq i \leq m$ one has:

(1) $$P(t, E_i) = 1 \quad : t \in E_i$$

Next, if $\Delta = \Omega \setminus E_1 \cup \ldots \cup E_m$ then

(2) $$\sup_{t \in S} P^{(n)}(t, \Delta) \leq M \cdot \alpha^n \quad : n = 1, 2, \ldots$$

Remark. One refers to Δ as the disippative part of Ω and $\{E_i\}$ are the ergodic kernels of the process. Since $\alpha < 1$ in (2) the probabilistic interpretation of (2) is that as n increase then the disippative part is evacuated with high probability while the Markov process stays inside every ergodic kernel.

11.9 The Kakutani-Yosida theorem.

A sufficient condition for a Markov process to be stationary is as follows: Denote by X the Banach space of complex-valued and bounded $\mathcal{B}$-measurable functions on the real s-line. Now P gives a linear operator T which sends $f \in X$ to the function

$$T(f)(x) = \int_\Omega f(s) \cdot dPx, ds)$$

Kakutani and Yosida proved that the Markov process is stationary if there exists a triple α, n, K where K is a compact operator on X, $0 < \alpha < 1$ and n some positive integer such that the operator norm

(*) $$||T + K|| \leq \alpha$$

The proof relies upon some general facts about linear operators on Banach spaces. First one identifies the Banach space X with the space of continuous complex-valued functions on the compact Hausdorff space S given by the maximal ideal space of the commutative Banach algebra X. Then T is a postive linear operator on $C^0(S)$ and in § 11.xx we shall prove that (*) implies that the spectrum of T consists of a finite set of points on the unit circle together with a compact subset in a disc of radius < 1. Moreover, for each isolated point $e^{i\theta} \in \sigma(T)$ the corresponding eigenspace is finite dimensional. Each such eigenvalue corresponds to an ergodic kernel and when the eigenspace has dimension $m \geq 2$, the corresponding ergodic kernel, say E_1, has a further decomposition into pairwise disjoint subsets $e_1, \ldots, e_m$ where the process moves in a cyclic manner between these sets, i.e.

$$\int_{e_{i+1}} P(s, e_i) = 1 \quad : 1 \leq i \leq m \quad \text{where we put} \quad e_{m+1} = e_1$$

11.9.1 Some results about linear operators.

The Kakutani-Yosida theorem follows from some results about linear operators which we begin to expose. Let X be a Banach space and denote by $\mathcal{F}$ the family of bounded liner operators T on X such that

$$(*) \qquad \lim_{n \to \infty} \frac{1}{n} \cdot x^*(T^n x) = 0$$

hold for all pairs $x \in X$ and $x^* \in X^*$.

1. Exercise. Apply the Banach-Steinhaus theorem to show that if $T \in \mathcal{F}$ then there exists a constant M such that the operator norms satisfy

$$||T^n|| \leq M \cdot n \quad : \ n = 1, 2, \ldots$$

Since the n:th root of $M \cdot n$ tends to one as $n \to +\infty$, the spectral radius formula entails that the spectrum $\sigma(T)$ is contained in the closed unit disc of the complex λ-plane. So in the exterior disc $\{|\lambda| > 1\}$ there exists the the resolvent

$$R_T(\lambda) = (\lambda \cdot E - T)^{-1}$$

2. The class $\mathcal{F}_*$. It consists of those T in $\mathcal{F}$ for which there exists some $\alpha < 1$ such that $R_T(\lambda)$ extends to a meromorphic function in the exterior disc $\{|\lambda| > \alpha\}$. Since $\sigma(T) \subset \{|\lambda| \leq 1\}$ it follows that when $T \in \mathcal{F}_*$ then the set of points in $\sigma(T)$ which belongs to the unit circle in the complex λ-plane is empty or finite and after we can always choose $\alpha < 1$ such that

$$\sigma(T) \cap \{\alpha < |\lambda| < 1\} = \emptyset$$

3. Proposition. *If $T \in \mathcal{F}_*$ and $e^{i\theta} \in \sigma(T)$ for some θ, then $R_T(\lambda)$ has a simple pole at $e^{i\theta}$.*

Proof. Replacing T by $e^{-i\theta} \cdot T$ it suffices to prove the result when $e^{i\theta} = 1$. If $R_T(\lambda)$ has a pole of order ≥ 2 at $\lambda = 1$ we know from § XX that there exists $x \in X$ such that

$$\text{(i)} \qquad Tx \neq x \quad \text{and} \quad (E - T)^2 x = 0$$

The last equation means that $T^2 + x = 2Tx$ and an induction over n gives

$$\text{(ii)} \qquad \frac{1}{n} \cdot T^n x = \frac{1}{n} \cdot x + (E - T)x$$

Since $T \in \mathcal{F}$ we have

$$\text{(iii)} \qquad \lim_{n \to \infty} \frac{1}{n} \cdot x^*(T^n x) = 0 \quad : \ \forall, x^* \in X^*$$

Then (ii) entails that $x^*(E - T)(x) = 0$. Since x^* is arbitrary we get $Tx = x$ which contradicts (i) and hence the pole is simple.

4. Theorem. *Let $T \in \mathcal{F}$ be such that there exists a compact operator K where $||T + K|| < 1$. Then $T \in \mathcal{F}_*$ and for every $e^{i\theta} \in \sigma(T)$ the eigenspace $E_T(\theta) = \{x \in X : Tx = e^{i\theta} x\}$ is finite dimensional.*

Proof. Set $S = T + K$ and for a complex number λ we write $\lambda \cdot E - T = \lambda \cdot E - T - K + K$. Outside $\sigma(S)$ we get

$$\text{(i)} \qquad R_S(\lambda)(\lambda \cdot E - T) = E + R_S(\lambda) \cdot K$$

The Neumann series for large absolute values $|\lambda|$ applied to $R_S(\lambda)$ gives some $\rho > 0$ and

$$\text{(ii)} \qquad (E + R_S(\lambda) \cdot K)^{-1} = E + R_S(\lambda) \cdot K\dot{(}E + R_S(\lambda) \cdot K)^{-1} \quad : \ |\lambda| > \rho$$

Next, when $|\lambda|$ is large we notice that (i) gives

$$\text{(iii)} \qquad R_T(\lambda) = (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Together with (ii) we obtain

(iv) $$R_T(\lambda) = R_S(\lambda) + R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Set $\alpha = ||S||$ which by assumption is < 1. Now $R_S(\lambda)$ is analytic in the exterior disc $\{\lambda| > \alpha\}$ so in this exterior disc $R_\lambda(T)$ differs from the analytic function $R_\lambda(S)$ by

(v) $$\lambda \mapsto R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Here K is a compact operator so the result in § XX entails that this function extends to be meromorphic in $\{|\lambda| > \alpha\}$. There remains to prove that eigenspaces at spectral points on the unit circle are finite dimensional. To prove this we use (iv). Let $e^{i\theta} \in \sigma(T)$. By Proposition 3 it is a simple pole so we have a Laurent series expansion

$$R_T(e^{i\theta} + z) = \frac{A_{-1}}{z} + A_0 + A_1 z + \ldots$$

By the general results from §§ there remains to show that A_{-1} has finite dimensional range. To see this we apply (iv) which gives the equation

$$R_S(e^{i\theta} + z) + R_S(e^{i\theta} + z) \cdot (E + R_S(e^{i\theta} + z) \cdot K)^{-1} \cdot R_S(e^{i\theta} + z)$$

To simplify notations we set $B(z) = R_S(e^{i\theta} + z)$ which by assumption is analytic in a neighborhood of $z = 0$. Moreover, the operator $B(0)$ is invertible. So now one has

$$\frac{A_{-1}}{z} + A_0 + A_1 z + \ldots = B(z) + B(z)(E + B(z) \cdot K)^{-1} B(z)$$

Since $B(0)$ is invertible we have a Laurent series expansion

$$(E + B(z) \cdot K)^{-1} = \frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots$$

and identying the coefficient of z^{-1} gives

$$A_{-1} = B(0) A^*_{-1} B(0)$$

Next, from (xx) one has

$$E = (E + B(z) \cdot K)(\frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots) \implies (E + B(0) \cdot K) A^*_{-1} = 0$$

Here $B(0) \cdot K$ is a compact operator and hence Fredholm theory implies that A^*_{-1} has a finite dimensional range. Since $B(0)$ is invertible the same is true for A_{-1} which finishes the proof of Theorem 4.

5. Proposition. *If $T \in \mathcal{F}$ is such that $T^N \in \mathcal{F}_*$ for some integer $N \geq 2$. Then $T \in \mathcal{F}_*$.*

Proof. We have the algebraic equation

$$\lambda^N \cdot E - T^N = (\lambda \cdot E - T)(\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1})$$

It follows that

$$R_T(\lambda) = (\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1}) \cdot R_{T^N}(\lambda^N)$$

Since $T^N B \in \mathcal{F}_*$ there exists $\alpha < 1$ such that

$$\lambda \mapsto R_{T^N}(\lambda^N)$$

extends to be meromorphic in $\{|\lambda| > \alpha\}$. At the same time $(\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1})$ is a polynomial and hence $R_T(\lambda)$ also extends to be meromorphic in this exterior disc so that $T \in \mathcal{F}_*$.

A result about positive operators.

Let S be a compact Hausdorff space and X is the Banach space of continuous and complex-valued functions on S. A linear operator T on X is positive if it sends every non-negative and real-valued function f to another real-valued and non-negative function.

6. Theorem. *If T is positive and belongs to $\mathcal{F}_*$ then each $e^{i\theta} \in \sigma(T)$ is a root of unity.*

Proof. The hypothesis gives $f \in C^0(S)$ which is not identically zero and

$$T(f) = e^{i\theta} \cdot f$$

Multiplying f with a complex scalar we may assume that the maximum norm on S is one and there exists $s_0 \in S$ such that

$$f(s_0) = 1$$

The dual space X^* consists of complex-valued Riesz measures on X. If $n \geq 1$ we get the measure μ_n such that the following hold for every $g \in C^0(S)$:

$$\int_S g \cdot d\mu_n = T^n g(s_0) \geq 0$$

Since T^n is positive the integrals in the left hand side are ≥ 0 when g are real-valued and non-negative. This entails that each μ_n is a real-valued and non-negative measure. Next, for each n we put

$$A_n = \{e^{-in\theta} \cdot f \neq 1\} = \{\mathfrak{Re}(e^{-in\theta} f) < 1\} \tag{6.0}$$

where the last equality follows since the sup-norm of f is one. Now

$$0 = f(s_0) - e^{-in\theta} \cdot T^n f(s_0) = \int_S [1 - e^{-in\theta} f(s)] \cdot d\mu_n(s) \tag{6.1}$$

Taking real parts we get

$$0 = \int_S [1 - \mathfrak{Re}(e^{-in\theta} f(s))] \cdot d\mu_n(s)$$

By (6.0) the integrand above is non-negative and since the whole integral is zero it follows that

$$\mu_n(A_n) = \mu_n(\{\mathfrak{Re}(e^{-in\theta} < 1\} = 0 \tag{6.2}$$

Hence (6.2) A_n is a null set with respect to μ_n. Suppose now that there exists a pair $n \neq m$ such that

$$(S \setminus A_n) \cap (S_m \setminus A_m) \neq \emptyset$$

A point s_* in this non-empty intersection gives

$$1 = e^{in\theta} f(s_*) = e^{im\theta} \cdot f(s_*) \implies e^{in\theta} = e^{im\theta}$$

and hence $e^{i\theta}$ is a root of unity since $m - n \neq 0$. So the proof of Theorem 6.1 is finished if we have established the following

Sublemma. The sets $\{S \setminus A_n\}$ cannot be pairwise disjoint.

Proof. First, f has maximum norm and by the above:

$$\int_S f \cdot d\mu_n = e^{in\theta}$$

Hence the total mass $\mu_n(S)$ is at least one. Next, for each $n \geq 2$ we set

$$\pi_n = \frac{1}{n} \cdot (\mu_1 + \ldots + \mu_n)$$

Since $\mu_n(S) \geq 1$ for each n we get $\pi_n(S) \geq 1$. Put

$$\mathcal{A} = \bigcap A_n$$

Above we proved that $\mu_n(A_n) = 0$ hold for every n which gives

$$\pi_n(\mathcal{A}) = 0 \quad : n = 1, 2, \ldots \tag{*}$$

Next, when the sets $\{S \setminus A_k\}$ are pairwise disjoint one has the inclusions

$$S \setminus A_k \subset A_\nu \quad \forall \, \nu \neq k$$

Keeping k fixed it follows that $\pi_\nu(S \setminus A_k) = 0$ for every $\nu \geq 0$. So when n is large while k is kept fixed we obtain

$$\pi_n(S \setminus A_k)) = \frac{1}{n} \cdot \mu_k(S \setminus A_k)) \implies \lim_{n \to \infty} \pi_n(S \setminus A_k)) = 0 \quad : k = 1, 2, \ldots \tag{**}$$

At this stage we use the hypothesis that T belongs to $\mathcal{F}_*$ which by Proposition XX entails that the resolvent $R_T(\lambda)$ has at most a simple pole at $\lambda = 1$. With $\epsilon > 0$ the Neumann series expansion gives

$$E + \sum_{k=1}^{\infty} \frac{T^k}{(1+\epsilon)^k} = R_T(1+\epsilon)) = \frac{1}{\epsilon} \cdot Q + W(1+\epsilon))$$

where $W(\lambda)$ is an operator-valued analytic function in an open disc centered at $\lambda = 1$ while Q is a bounded linear operator on $C^0(S)$. Keeping $\epsilon > 0$ fixed we apply both sides to the identity function 1_S on S and the construction of the measures $\{\mu_n\}$ gives

$$1 + \sum_{k=1}^{\infty} \frac{\mu_k(S)}{(1+\epsilon)^k} = \frac{1}{\epsilon} \cdot Q(1_S)(s_0) + W(1+\epsilon))(1_S)(s_0)$$

If $n \geq 2$ is an integer and $\epsilon = \frac{1}{n}$ one gets the inequality

$$\sum_{k=1}^{k=n} \frac{\mu_k(S)}{(1+\frac{1}{n})^k} \leq n \cdot |Q(1_S)(s_0)| + |W(1+1/n))(1_S)(s_0)| \leq n||Q|| + ||W(1+1/n)|| \implies$$

$$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq (1+\frac{1}{n})^n \cdot (||Q|| + \frac{||W(1+1/n)||}{n}$$

Since Neper's constant $e \geq (1+\frac{1}{n})^n$ for every n we find a constant C which is independent of n such that

$$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq C$$

Hence the sequence $\{\pi_n(S)\}$ is bounded and we can pass to a subsequence which converges weakly to a limit measure μ_*. For this σ-additive measure the limit formula in (**) above entails that

(i) $$\mu_*(S \setminus A_k) = 0 \quad : \ k = 1, 2, \ldots$$

Moreover, by (*) we also have

(ii) $$\pi_*(\mathcal{A}) = 0$$

Now $S = \mathcal{A} \cup A_k$ so (i-ii) give:

$$\mu_*(S) = 0$$

But this is impossible for at the same time we have already seen that $\pi_n(S) \geq 1$ for each n and hence also $\mu_*(S) \geq 1$.

Stone's theorem.

Here follows an example where unbounded operators appear. The reader has begun to master unbounded operators and their spectral properties by learning the details which appear during the proof of Stone's theorem below. Let $\mathcal{H}$ be a complex Hilbert space. A strongly continuous semi-group of unitary operators is a family $\{U_t : t \geq 0\}$ where each U_t is unitary, i.e. they preserve the Hermitian inner products:

$$\langle U_t x, U_t y \rangle = \langle x, y \rangle$$

Moreover, U_0 is the identity operator E and $U_t \circ U_s = U_{t+s}$ hold for every pair of non-negative real numbers. Finally, the strong continuity means that

$$\lim_{t \to 0} ||U_t x - x|| = 0$$

for every $x \in \mathcal{H}$. Stone's theorem asserts that there exists a densely defined and self-adjoint operator A on $\mathcal{H}$ which is called the infinitesmal generator. Its domain of defintion $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$ which consists of vectors x such that there exists a limit

$$\lim_{\delta \to 0} \frac{U_\delta(x) - x}{\delta} = y$$

and when it holds we set

$$A(x) = iy$$

where i is the imaginary unit. The proof that such an inifintesmal generator exists relies upon Neumann's calculus in § 0.0. Here follow the crucial steps which lead to the construction of A.

The operators $R(\lambda)$. When λ are complex numbers with positive real part we obtain bounded linear operators

$$R(\lambda)(x) = \int_0^\infty e^{-\lambda t} \cdot U_t(x)\, dt$$

The integrals exists because of the strong continuity and the convergence with respect to the norm in $\mathcal{H}$ is evident since $||U_t(x)||| \leq ||x||$ for each fixed vector x. Approximating the integrals by Riemann sums the bounded linear operator $R(\lambda)$ is found by the limit equation

$$\text{(i)} \qquad R(\lambda) = \lim_{\delta \to 0} \delta^{-1} \cdot (E - e^{-\delta\lambda} U_\delta)^{-1}$$

Notice that $\mathfrak{Re}(\lambda) > 0$ implies that the operator norm

$$||e^{-\delta\lambda} U_\delta)|| = e^{-\mathfrak{Re}(\lambda)} < 1$$

and therefore linear operators $E - e^{-\delta\lambda} U_\delta)$ are invertible which explains how one proceeds to the limit in the right hand side above. A notable point is that (i) enable us to describe the compact spectra of the operators $R(\lambda)$. Consider for example the case $\lambda = 1$. Since the spectrum of a unitary opertor is contained in the unit circle it follows that

$$\sigma(E - e^{-\delta} \cdot U_\delta) \subset \{z \in \mathbf{C} : \ |z - 1| = e^{-\delta}\}$$

Next, general formulas for spectra of bounded operators to be explained in §xx give

$$\sigma(\delta^{-1} \cdot (E - e^{-\delta\lambda} U_\delta)^{-1}) \subset \{z\colon \ |z - \frac{\delta}{1 - e^{-2\delta}}| = \frac{\delta}{1 - e^{-\delta}}\}$$

Passing to the limit as $\delta \to 0$ it follows that

$$\text{(ii)} \qquad \sigma(R(1)) \subset \{z\colon \ |z - 1/2| = 1/2\}$$

The infinitesmal generator. If $h > 0$ and $x \in \mathcal{H}$ then the semi-group equations $U_h \circ U_t = U_{h+t}$ give

$$\frac{U_h - E}{h} \circ R(\lambda)(x) = h^{-1} \cdot \int_0^\infty e^{-\lambda t} \cdot (U_{t+h}(x) - U_t(x))\, dt$$

A variable substitution shows that the last term is equal to

$$\text{(iii)} \qquad \frac{e^{\lambda h} - 1}{h} \cdot \int_h^\infty e^{-\lambda t} \cdot U_t(x)\, dt - \frac{1}{h} \int_0^h e^{-\lambda t} \cdot U_t(x)\, dt$$

Passing to the limit as $h \to 0$ it is readily seen that the vectors in (iii) converge in the norm topology on $\mathcal{H}$ to the vector

$$\lambda \cdot R(\lambda)(x) - x$$

The conclusion is that if $y = R(\lambda)(x)$ then there exists the limit

$$\text{(iv)} \qquad \lim_{h \to 0} \frac{U_h(y) - y}{h} = \lambda \cdot R(\lambda)(x) - x$$

This suggests that we regard the subspace $\mathcal{H}_0$ of $\mathcal{H}$ which consists of vectors y such that there exists a limit vector $S(y)$ where

$$\lim_{h \to 0} ||\frac{U_h(y) - y}{h} - S(y)|| = 0$$

It is clear that $S\colon \mathcal{H}_0 \to \mathcal{H}$ is a linear operator and (iv) entails that the range of $R(\lambda)$ is contained in $\mathcal{H}_0$ when $\mathfrak{Re}(\lambda) > 0$. Moreover

$$\text{(v)} \qquad S \circ R(\lambda)(x) = \lambda \cdot R(\lambda)(x) - x \quad : x \in \mathcal{H}$$

We can rewrite (v) and get

$$(\lambda \cdot E - .S) \circ R(\lambda)(x) = x \quad : x \in \mathcal{H}$$

In § 0.0 we shall learn that this means that $\{R(\lambda)\}$ appear as resolvent operators to the densely defined linear operator S. Applying the inclusion (ii) above and the general result in (0.0.5.1) from Neumann's calculus, it follows that $\sigma(S)$ consists of purely imaginary numbers. Set

$$A = iS$$

From the above $\sigma(A) \subset \mathfrak{R}$ and in § xx we shall learn that A is a densely defined and self-adjoint operator which gives the requested the infinitesmal generator of the unitary semi-group $\{U_t\}$.

Remark. Stone's theorem has a wide range of applications, especially in ergodic theory. For applications in quantum mechanics we refer to the article *xxxx* by J. von Neumann. Let us also remark that an analogue of Stone's theorem hold for a strongly continuous semi-group $\{V_t\}$ satisfying $V_t^*V_t = E$ for every t, but not necessarily $V_tV_t^* = E$, i.e. the V-operators need not be unitary. In § xx we also expose a result due to J. von Neumann about uniqueness of solutions to a certain Schrödinger equation which arises via the uncertainty principle in quantum mechanics. See also § A:x for a "concrete version" of Stones theorem.

More precisely, in the unbounded open complement $U = \mathbf{R}^3 \setminus \overline{\Omega}$ there exists the symmetric Green's function $G(p, q)$ associated to Neumann's boundary value problem, i.e.

$$G(p,q) = \frac{1}{4\pi \cdot |p-q|} + H(p,q)$$

where $|p - q|$ is the euclidian distance between a pair of pointsp, q in U. Moreover, $q \mapsto H(p, q)$ is harmonic as a function of q for each fixed $p \in U$, and for each $p \in U$ we have

$$\frac{\partial G}{\partial n_q}(p,q) = 0 \quad : q \in \partial\Omega$$

where n_q is the outwards normal derivative. Now G gives a densely defined operator on the Hilbert space $L^2(U)$ defined by

$$\mathcal{G}_f(p) = \int_U G(p,q) \cdot f(q)\, dq$$

To be precise, the volume integrals in the right hand side give L^2-functions of p for a dense subset of L^2-functions f in U. We shall learn that this unbounded and densely defined operator has a spectrum in the sense of Neumann which is confined to the non-negative real line. Moreover, there exists an associated spectral function $\theta(p, q; \lambda)$ which for each $\lambda > 0$ yields a bounded integral operator on $L^2(U$:

$$\Theta(\lambda)_f(p) = \int_U \theta(p,q;\lambda) \cdot f(q)\, dq$$

and here

$$\mathcal{G}_f(p) = \lim_{\delta \to 0} \int_\delta^{1/\delta} \lambda^{-1} \cdot \big[\int_U \frac{d}{d\lambda}\theta(p,q;\lambda) \cdot f(q)\, dq\big]\, d\lambda$$

for those L^2-functions f which belong to the domain of definition of $\mathcal{G}$. To learn about this construction and how one employs the spectral function to fiond solutions to an assocaited wave equation can be regarded as one of the major goals for this chapter. As expected measure theoretic considerations intervene, i.e. "abstract functional analysis" is to a large extent related to measure theory. In the situation above one also considers the total variation of operator norms for the Θ-operators, i.e. on a closed interval $\{a \leq \lambda \leq b\}$ where $0 < a < b$ we put

$$\Theta^*[a,b] = \max \sum ||\Theta(\lambda_{\nu+1}) - \Theta(\lambda_\nu)||$$

with the maximum taken over all partitions $a = \lambda_0 < \lambda_1 < \ldots < \lambda_N = b$ and we have taken a sum of operator norms for the corresponding difference of the Θ-operators which are parametrized with respect to λ. It turns out that the operator valued function $\lambda \mapsto \Theta(\lambda)$ is absolutely continuous. It means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum \Theta^*[a_j.b_j] < \epsilon$$

for every finite familly of closed and disjoint intervals $\{[a_j, b_j]\}$ arranged so that $b_j < a_{j+1}$ for each j. To learn the proof which is presented in § xx gives the reader a good insight in spectral theory since one arrives at a specific result in a concrete situation which therefore consolidates the general theory.

Let Ω be a bounded open set in $\mathbf{R}^3$ whose boundary is a finite union of closed surfaces of class C^1. In the unbounded open complement $U = \mathbf{R}^3 \setminus \overline{\Omega}$ there exists the symmetric Green's function $G(p, q)$ associated to Neumann's boundary value problem, i.e.

$$G(p, q) = \frac{1}{4\pi \cdot |p - q|} + H(p, q)$$

where $|p - q|$ is the euclidian distance between a pair of pointsp, q in U. Moreover, $q \mapsto H(p, q)$ is harmonic as a function of q for each fixed $p \in U$, and for each $p \in U$ we have

$$\frac{\partial G}{\partial n_q}(p, q) = 0 \quad : q \in \partial\Omega$$

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and here

$$\mathcal{G}_f(p) = \lim_{\delta \to 0} \int_\delta^{1/\delta} \lambda^{-1} \cdot \big[\int_U \frac{d}{d\lambda}\theta(p, q; \lambda) \cdot f(q)\, dq\big]\, d\lambda$$

for those L^2-functions f which belong to the domain of definition of $\mathcal{G}$. To learn about this construction and how one employs the spectral function to fiond solutions to an assocaited wave equation can be regarded as one of the major goals for this chapter. As expected measure theoretic considerations intervene, i.e. "abstract functional analysis" is to a large extent related to measure theory. In the situation above one also considers the total variation of operator norms for the Θ-operators, i.e. on a closed interval $\{a \leq \lambda \leq b\}$ where $0 < a < b$ we put

$$\Theta^*[a, b] = \max \sum ||\Theta(\lambda_{\nu+1}) - \Theta(\lambda_\nu)||$$

with the maximum taken over all partitions $a = \lambda_0 < \lambda_1 < \ldots < \lambda_N = b$ and we have taken a sum of operator norms for the corresponding difference of the Θ-operators which are parametrized with respect to λ. It turns out that the operator valued function $\lambda \mapsto \Theta(\lambda)$ is absolutely continuous. It means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum \Theta^*[a_j.b_j] < \epsilon$$

for every finite familly of closed and disjoint intervals $\{[a_j, b_j]\}$ arranged so that $b_j < a_{j+1}$ for each j. To learn the proof which is presented in § xx gives the reader a good insight in spectral theory since one arrives at a specific result in a concrete situation which therefore consolidates the general theory.

Unitary groups. There exists also a 1-1 correspondence exists between the family of self-adjoint operators and unitary groups of transformations which are strongly continuous on a dense subspace of the given Hilbert space $\mathcal{H}$. Moreo rpecisley, if A is self-adjoint there exists for each real number t a unitary transformation U_t such that

$$U_t(x) = e^{itA(x)}$$

hold for a dense subspace of the domain of definition of A and since A is self-adnoint one eaily verifies the gruop law:

$$U_{t+s} = U_t \circ U_s$$

Conversly, let $\{U_t\}$ be a group of untary transformations and assume that it is strongly continuos on a dense subapace $\mathcal{H}_*$ of $\mathcal{H}$, i.e. for vectors in this subspace one has

$$\lim_{t \to 0} ||U_t(x) - x|| = 0$$

When this holds one constructs opeprtator-valued Stieltjes integrals for complex numbers in the open right half-plane. More precisely, each vector $x \in \mathcal{H}_*$ there exists the Stieltjes' integral

$$R(\lambda)(x) = \int_0^\infty e^{-\lambda t} \cdot U_t(x)\, dt$$

One verifies that

$$||R(\lambda(x)|| \leq e^{-\mathfrak{Re}(\lambda)} \cdot ||x|| \quad : x \in \mathcal{H}_*$$

It follows that $R(\lambda)$ extends in a unique fashion to a bounded linear operator. We can specialize λ and get the bounded operator

$$R = R(1)$$

Since $\{U_t\}$ is a unitary group an easy calculation to be given in § xx entails that R is normal and one has the inclusion

$$\sigma(R) \subset$$

Spectral functions.

In § xx we show that under the sole condition (1), there exist a distinguished family of spectral functions $\theta(x, y; \lambda)$ associated to the symmetric function $K(x, y)$. For each spectral function, $(x, y) \mapsto \theta(x, y; \lambda)$ is symmetric and yields a bounded linear operarator on $L^2[0, 1]$ denoted by $\Theta(\lambda)$. Moreover, the operator valued function $\lambda \mapsto \Theta(\lambda)$ has a finite total variation with respect to λ over every bounded interval the real λ-line and the unbounded operator $\mathcal{K}$ is recaptured from the spectral function by the integral formula

$$\mathcal{K}(h) = \int_{-\infty}^\infty \frac{1}{\lambda} \cdot \frac{d\Theta}{\lambda}(h)$$

for every L^2-function h in the domain of definition for K. The precise calculus and constructions of operator valued integrals above are exposed in § xx. But let us briefly recall their construction. Let K satisfy (1). Now we can approximate $\mathcal{K}$ in a weak sense by sequences of Hilbert-Schmidt operators, i.e. linear operators on $L^2[0, 1]$ of the form

$$\mathcal{G}(f)(x) = \int_0^1 G(x, y) \cdot f(y)\, dy$$

where G are real-valued symmetric kernel functions such that

$$\iint G(x, y)^2\, dxdy < \infty$$

A sequece $\{G_n\}$ approximates K weakly if

$$\lim_{n \to \infty} \int_0^1 |K(x, y) - G_n(x, y)|^2\, dy = 0$$

hold outside a null set on the x-interval. Next, if $\{G_n\}$ is an approximates K as above one constructs spectral θ-functions as follows: First, each fixed G_n is compact and has therefore a discrete spectrum. This gives an orthonormal family $\{\phi_\nu^{(n)}\}$ in $L^2[0, 1$ and to each n a non-zero real number $\lambda_\nu^{(n)}$ such that

$$\mathcal{G}_n(\phi) = \lambda_\nu^{(n)} \cdot \phi_\nu^{(n)}$$

The λ-numbers are enumerated with increasing absolute values. For each n and every $\lambda > 0$ we set

$$\rho_n(x, y; \lambda) = \sum_{0 < \lambda_\nu^{(n)} < \lambda} \phi_\nu^{(n)}(x) \phi_\nu^{(n)}(y)$$

$$\rho_n(x,y;-\lambda)=\sum_{-\lambda<\lambda_\nu^{(n)}<0}\phi_\nu^{(n)}(x)\phi_\nu^{(n)}(y)$$

Applying Bessel inequalities several times, it is proved in [ibid:Chapter 1-2] that one can always find subsequences $\{\rho_{n_1},\rho_{n_2}\ldots\}$ which converge to a spectral ρ-function associated to K via the limit equation (xx). So every K-kernel which satisfies (1) has a non-empty family of associated spectral ρ-functions. The size of this ρ-family is governed by Weyl's limit circles which arise as follows: Let f be a non-zero real-valued continuous on $[0,1]$. If $\{G_n\}$ approximates K we find for each n a unique function ϕ_n which solves the integral equation

$$\phi_n(x)=i\cdot\int_0^1 G_n(x,y)\cdot\phi_n(y)\,dy+f(x)$$

Here ϕ_n is automatically continuous. Keeping f and x fixed we get a sequence of complex numbers $\{\phi_n(x)\}$. Let $Z(x)$ be the closed set of cluster points of its closure. It turns out that the union of all such cluster sets as we take sequences $\{G_n\}$ as above, is a closed subset of $\mathbf{C}$ which either is reduced to a single point or equal to a circle. In the latter case the circle is denoted by by $W_K(f:x)$ and called a Weyl circle associated to the triple K,f,x. When Weyl circles appear it truns out that there exist many disctinct spectral functions of K.

The self-adjoint case. Following [Carleman 1923] one says that the $K(x,y)$ is of Class I if (*) has non non-trivial solution. In this case no Weyl circles appear and $\mathcal{K}$ is self-adjoint with a unique spectral function. Another fundamental result from [Carleman 1923] asserts that if K is such that there does not exist a non-zero $\phi\in L^2[0,1]$ such that $\mathcal{K}(\phi)=0$, then every L^2-function f on $[0,1]$ can be represented as a limit

$$f(x)=\lim_{\ell\to+\infty}\int_{-\ell}^{\ell}\frac{d}{d\lambda}\Big(\int_0^1\theta(x,y;\lambda)\cdot h(y)\,dy\Big)$$

where equality hold for all x outside a nullset in $[0,1]$.

Remark. The restriction to the unit interval is harmless, i.e by a change of variables the theory holds verbatim for L^2-spaces over domains in $\mathbf{R}^n$ for every positive integer n, and of course also on manifolds equipped with some volume density. Except for some minor technical points one can also extend the whole study to cover case where infinite number of variables appear. But to save notations we often prefer to work with kernel functions on intervals in $\mathbf{R}$.

Let us finally recall that An alternative proof of Carleman's spectral theorem for unbounded operators on Hilbert spaces was later given by J. von Neumann in the article *Eigenwerttheorie Hermitescher Funktionaloperatoren*[xxx: 1929]. from 1929. A merit in von Neumann's account is that some measure theoretic technicalities in Carleman's proof can be omitted and his proof covers the case for operators on non-separable Hilbert spaces and von Neumann's more abstract version of the spectral theorem is presented in § 9. On the other hand it is often necessary to establish formulas of a measure theoretic character in order to apply the spectral theorem. An example is Carleman's solution to the Bohr-Schrödinger equation above which will be exposed in § xx.

H. Neumann's resolvent operators

.

The less experienced reader might prefer to start with § 1-6 whose contents "start from scratch". However, it is essential to become familiar with unbounded linear operators " as soon as possible" since they arise naturally in many applications. So already in § 0.0 we introduce Neumann's resolvent operators and the construction of spectral sets of densely defined but unbounded linear operators. Here analytic function theory plays a crucial role such as in the proof of Theorem 0.0.6.3 which enable us to construct bounded linear operators from a given densely defined operator. Let us also remark that Neumann's calculus for densely defined linear operators extends several results about matrices. For example, let $T\colon X\to X$ be a densely defined linear operator on a Banach space with a closed graph in $X\times X$ and suppose also that the adjoint T^* has a dense domain of definition. In § xx we prove the equality

$$\sigma(T) = \sigma(T^*)$$

under the extra assumption that the Banach space X is reflexive. Here $\sigma(T)$ is the spectrum of T which will be constructed in § 0.0. for each densely defined and closed linear operator on a Banach space. In the case when λ_0 is an isolated point in the spectrum of T we prove in § 0.0.6 that there exists a bounded linear map $E(\lambda_0)$ which commutes with T and the spectrum of the operator $E(\lambda_0)T$ is reduced to the singleton set $\{\lambda_0\}$. We remark that this extends the classical constructions of Cayley matrices in linear algebra.

The operator $i \cdot \frac{d}{dx}$. A "first lesson" about densely defined linear operators occurs when we consider the densely defined derivation operator $A = i \cdot \frac{d}{dx}$ acting on the Hilbert space of square integrable functions on the real x-line. Using Fourier transforms and Parseval's equality we prove in § xx that A is self-adjoint. The resder should keep this example in mind while the general theory is exposed. Another densely defined operator B on $L^2(\mathbf{R})$ is given by multiplication with $-ix$. Here the domain of definition for B consists of L^2-functions $f(x)$ such that $x \cdot f(x)$ also is square integrable. The chain rule for derivatives and the equality $i^2 = -1$ show that if $f(x)$ is a test-function on $\mathbf{R}$ then

$$(*) \qquad A \circ B(f) - B \circ A(f) = f$$

One refers to (*) as Weyl's identity. The equation (*) appears in relation to the uncertainty principle in quantum mechanics. in 1926 Schrödinger discovered that when the pair above are identified with infinitesmal generators of unitary groups on $L^2)\mathbf{R})$, then one arrives at an irreducible representation. In §xx we expose von Neumann's uniqueness theorem which gives an affirmative answer to a question which was originally posed by Schrödinger. Namely, that every pair of densely defined self-adjoint operators which satisfy (*) and as infinitesmal generators of unitary groups give riose to an irreducible representation of $L^2)\mathbf{R})$, is equal to the special pair above up to a unitary equivalence.

NEW TITLE FOR SCATTERED MATERIAL

A product formula. Let us announce a theorem due to Nagy and F. Riesz whose proof relies upon results in § 8-9 and will be proved in § 11.xx. Let $\mathcal{H}$ be a Hilbert space and $T\colon \mathcal{H} \to \mathcal{H}$ a densely defined and closed linear operator. Then we shall prove that the adjoint T^* also is densely defined, as well as the composed operator TT^*. Moreover, TT^* is self-adjoint with a spectrum confined to the non-negative real axis and we shall learn that there exists the densely defined self-adjoint operator

$$A = \sqrt{TT^*}$$

where $A^2 = TT^*$. Next, a bounded linear operator P is called a partial isometry attached to A if the norms

$$||P(Ax)|| = ||A(x)|| \quad : x \in \mathcal{D}(A)$$

while the P-kernel is the orthogonal complement to the range of A, i.e. if y is a vector such that $\langle Ax, y\rangle = 0\colon\ x \in \mathcal{D}(A)$, then $P(y) = 0$.

Theorem. *For every densely defined and closed operator T there exists a partial isometry P attached to $\sqrt{TT^*}$ such that*

$$T(x) = P \circ \sqrt{TT^*}(x) \quad : x \in \mathcal{D}(TT^*)$$

Summing up, results as above illustrate the aim of this chapter which apart from presenting basic facts about normed vector spaces pays considerable attention to densely defined operators on Hilbert spaces.

Hellinger's integrals. One often needs specific results from calculus, especially about integrals where one encounters both those of Lebesgue and integrals in the sense of Borel and Stieltjes. Let us recall some constructions and results due to Hellinger's in *Die Orthogonalinvarianten quadratischer Formen von unendlichen vielen Variabeln* [Dissertation. Göttingen 1907]. Here Stieltjes' constructions are extended to second order differences. Fix a bounded and closed interval $J = [a, b]$. For each subintervall $\ell = [\alpha, \beta]$ and a continuous real-valued function ϕ on J we set

$$D_\ell(\phi) = \phi(\beta) - \phi(\alpha)$$

Next,let g and h be a pair of continuous and non-decreasing functions which are kept fixed from now on. Denote by $\mathcal{S}$ the class of continuous functions f such that

$$(D_\ell(f))^2 \leq D_\ell(g) \cdot D_\ell(h) \tag{*}$$

hold for every subinterval ℓ as above. Next, take some arbitrary $u \in C^0(J)$ while $f \in \mathcal{S}$. If $\{[\alpha_\nu, \beta_\nu]\}$ is a partition of $[a, b]$ into an increasing sequence of intervals, i.e $\alpha_{\nu+1} = \beta_\nu$ hold, we define the upper respectively the lower sums

$$S^* = \sum \max_{x \in \ell_\nu} u(x) \cdot \frac{(D_{\ell_\nu}(f))^2}{D_{\ell_\nu}(g)} \cdot \ell_\nu$$
$$S_* = \sum \min_{x \in \ell_\nu} u(x) \cdot \frac{(D_{\ell_\nu}(f))^2}{D_{\ell_\nu}(g)} \cdot \ell_\nu$$

where $\ell_\nu = \beta_\nu - \alpha_\nu$. If $\omega_u(\delta) = \max\{|u(x) - u(x')| : |x - x'| \leq \delta\}$ is the modulos of continuity for u we see that (*) gives

$$S^* - S_* \leq \omega_u(\delta) \cdot (h(b) - h(a))|quad\colon \delta = \max_\nu \beta_\nu - \alpha_\nu$$

From this the reader can deduce that there exists a limit when the partitions get fine, i.e. the δ-numbers tend to zero in above. The result is Hellinger's integral denoted by

$$\int_a^b \frac{df^2}{dg} \tag{1}$$

In a similar fashion the reader can verify that if f_1, f_2 is a oaiin $\mathcal{S}$, then there exists the integral

$$\int_a^b \frac{df_1 df_2}{dg} \tag{2}$$

Next, with $f \in \mathcal{S}$ and $u \in C^0(J)$ we put

$$S^* = \sum \max_{x \in \ell_\nu} u(x) \cdot D_{\ell_\nu}(f) \cdot \ell_\nu$$

$$S_* = \sum \min_{x \in \ell_\nu} u(x) \cdot (D_{\ell_\nu}(f)) \cdot \ell_\nu$$

Now

$$|S^* - S_*| \leq \omega_u(\delta) \cdot \sum |D_{\ell_\nu}(f)| \cdot \ell_\nu$$

Using (*) and the Cauchy Schwartz inequality we see that the last sum is majorised by

$$\omega_u(\delta) \cdot \sqrt{(g(b) - g(a))(h(b) - h(a)}$$

From this the reader can conclude that there exists a welldefined integral

$$\int_a^b u \cdot df \tag{3}$$

Exercise. Let ϕ and ψ be some pair of continuous functions while f_1 and $f - 2$ belong to $\mathcal{S}$. Put

$$\Phi(x) = \int_a^x \phi \cdot df \quad : \Psi(x) = \int_a^x \psi \cdot df$$

Show the equality below for every $u \in C^0(J)$:

$$\int_a^b u \cdot \frac{dF_1 dF_2}{dg} = \int_a^b u\phi\psi \cdot \frac{df_1 df_2}{dg}$$

Exercise. Let u and v be a pair in $C^0(J)$ while f_1, f_2 belong to $\mathcal{S}$. Show the equality

$$\int_a^b v(x) \cdot (\int_a^x u \cdot \frac{df_1 df_2}{dg})\, dx$$

Picard's equation.

Picard's article *Sur une équation intègrales singuliére* [Ann. Ecole, Norm. Sup.1911) is devoted to the integral equation

$$\phi(x) = \lambda \cdot \int_{-\infty}^{\infty} e^{-|x-y|}) \cdot \phi(y)\, dy + f(x) \tag{*}$$

where λ is a complex parameter. To begin with one seeks solutions ϕ when $f(x)$ is a C^2-function on the real line with compact support and requests that ϕ belongs to $L^2(\mathbf{R})$. By analyzing solutions to certain second order differential equations, Picard proved that (*) has unique L^2-solutions ϕ for each f as above when the complex parameter λ stays outside the real interval $[1/2, +\infty)$. Moreover, the solution is given by

$$\phi(x) = f(x) + \frac{\lambda}{\sqrt{1 - 2\lambda}} \cdot \int_{-\infty}^{\infty} e^{-|x-y|}) \cdot f(y)\, dy$$

where a single valued branch of $\sqrt{1 - 2\lambda}$ in $\mathbf{C} \setminus [1/2, \infty)$ appears. Let us express this via operators. Picard's integral operator is defined by

$$\mathcal{K}(f)(x) = \int_{-\infty}^{\infty} e^{-|x-y|} \cdot f(y)\, dy$$

It is densely defined on the Hilbert space $L^2(\mathbf{R}]$ and Picard's uniqueness above entails that when λ is not real, then there does not exist a non-zero L^2-function ϕ such that

$$\phi + \lambda \cdot \mathcal{K}(\phi) = 0$$

General results about densely defined and symmetric operators on Hilbert spaces which are exposed later on, imply that $\mathcal{K}$ is self-adjoint.

and from the above it follows that if E is the identity operator,. then $E - \lambda \cdot \mathcal{K}$ is invertible in Neumann's sense if λ stays outside $[1/2, +\infty)$.

Starting from this, Carleman applied the general theory for symmetric and densely defined operators and proved that the densely defined operator $\mathcal{K}$ defined by

$$\mathcal{K}_g(x) = \int_{-\infty}^{\infty} e^{-|x-y|} \cdot g(y)\, dy$$

belongs to the favourable class I which means that it is self-adjoint and the spectrum is $[1/2, +\infty)$.
we shall also give explicit formulas for the spectral function of Picard's operator. More generally one can consider densely defined operators of the form

$$\mathcal{K}_g(x) = \int_{-\infty}^{\infty} \big(H_1(|x-y|) + H_2(x+y)\big) \cdot g(y)\, dy$$

where H_1 and H_2 is a pair of real-valued functions such that the L^2-integrals

$$\int_0^{\infty} H_1(s)^2\, ds \quad : \int_{-\infty}^{\infty} H_2(s)^2\, ds$$

both exist. Again one proves that $\mathcal{K}$ is self-adjoint with a spectrum confined to the real line. See § xx for further details., Let us remark that $\mathcal{K}$-operators as above gives a gateway to analyse integral operators with kewrnel functions

$$K(x,y) = \frac{1}{x}\Omega(\frac{x}{y})$$

which are defined when x and y both a > 0 and

$$\Omega(t) = t^{-1} \cdot \Omega(\frac{1}{t})$$

Here K is a kernel for a densely defined operator on the Hilbert space $L^2(\mathbf{R}^+)$ and we shall prove that it is self-adjoint under the condition that

$$\int_0^{\infty} \Omega(t)^2\, dt < \infty$$

Symmetric integral operators.

Consider the domain $\square = \{0 \leq x \leq 1\} \times \{0 \leq y \leq 1\}$ in $\mathbf{R}^2$. Let $K(x,y)$ be a real-vaued and Lebesgue mesure function on $\square$ such that the integrals

$$\int_0^1 K(x,y)^2\, dy < \infty$$

for all x outside a null-set on $[0,1]$. In addition K is symmetric, i.e.$K(x,y) = K(y,x)$. The K-kernel is bounded in Hilbert's sens if there exists a constant C such that

$$\iint_\square K(x,y)u(x)u(y)\, dxdy \leq C^2 \cdot \int_0^1 u(x)^2\, dx$$

for each $u \in L^2[0,1]$. In the text-book —emphIntyergralxxx [xxxx 1904], Hilbert proved that the linear operator on $L^2[0,1]$ defined by

$$\mathcal{K}(u)(x) = \int_0^1 K(x,y)u(y)dy$$

is bounded and its operator norm is majorsied by C-Moreover the spectrum is real and contained in $[-C, C]$ and just as for symmetric real matrices there exists a spectral resolution. More precisely, set $\mathcal{H} = L^2[0,1]$. Then there exists a map from $\mathcal{H} \times \mathcal{H}$ to the space of Riesz measures supported by $[-C, C]$ which to every pair u and v in $\mathcal{H}$ assigns a Riesz measure $\mu_{\{u,v\}}$ and $\mathcal{K}$ is recovered by the equation:

$$\langle \mathcal{K}(u), v \rangle = \int_{-C}^{C} t \cdot d\mu_{\{u,v\}}(t)$$

where the left hand side is the inner product on the complex Hilbert space defined by

$$\iint \mathcal{K}(u)(x) \cdot \overline{v}(x)\, dx$$

In the monograph [Carelan 1923] the condition (*) is imposed while (**) need not be valid. Then we encounter an unbounded operator. But notice that if $u|in\mathcal{H}$ then (*9 and the CuachySchwarz inequality entails that the functions

$$y \mapsto K(x,y)u(y)$$

are absolutely integrable in Lebegue's sense for all x outside the nullset $\mathcal{N}$ above. Hence it makes sense to refer to L^2-functions ϕ on $[0,1]$ which satisfy an eigenvalue equation

$$\phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\, dy \tag{1}$$

where λ is a compkex number. one is also led to consider the integral equation

$$\phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\, dy + f(x) \tag{2}$$

where $f \in \mathcal{H}$ is given and one seeks ϕ. It turns out that if $\mathfrak{Im}(\lambda) \neq 0$, then the inhomogenous equation (2) has at least one solution. Next, consider the equation (1). Let $\rho(\lambda)$ denote the number of linearly independent solution in (1). Carelan proived that the ρ-function is constant when λ varies in $\mathbf{C} \setminus \mathbf{R}$. Follwing Carelan one says that the kernel (x,y) gives an operator $\mathcal{K}$ of Class I if the ρ-function is zero. It means that (2) has unique solutions for every pair $\lambda, f)$ when λ are non-real.

A limit process. Let $\{G_n(x,y)\}$ be a sequece of symmetric kernel functions which are bounded in Hilbert's sense and approximate K in the sense that

$$\lim_{n\to\infty} \int_0^1 |K(x,y) - G_n(x,y)|^2\, dy = 0$$

for all x outside a null set. Fix some non-real λ. Hilbert's theory entials that if $f(x)$ is a continuous function, in general complex-valued, then we find unqiue continuous functions $\{\phi_n\}$ which satisfy the integral equations

$$\phi_n(x) = \lambda \cdot \int_0^1 G_n(x,y)\phi(y)\, dy \tag{1}$$

For a fixed x we consider the complex numbers $\{phi_n(x)\}$. Now there exists the set $Z(x)$ of all cluster points, i.e. a complex number z belongs to $Z(x)$ if there exists some sequence $1 \leq n_1 < n_2 < \ldots$ such that

$$z = \lim_{k\to\infty} \phi_{n_k}(x)$$

Theorem. *For every approximating sequence $\{G_n\}$ as above the sets $Z(x)$ are either reduced to points or circles in the complex plane. Moreover, each $Z(x)$ is reduced to a siungleton set when $\mathcal{K}$ is of class I.*

Remark. The result below was discovered by Weyl for some special unbounded operators which arise during the study of second order differential equations. See § for a comment-. If $\mathcal{K}$ is of Class I then the theorem sbove shows that each pair of a on-real λ and some $f \in \mathcal{H}$ gives a unique function $\phi(x)$ which satsfies (2) and it can be found via a pointwise limit of solutions $\{\phi_n\}$ to the equations (xx). In this sense the limit process is robust because one can emply an arbitrary approximating sequence $\{G_n\}$ under the sole condition that (xx) holds. This already indficartes that the Case I leads to a "consistent theory" even if $\mathcal{K}$ is unbounded. To make this precise Carleman constructed a unique spectral function when Case I holds. More preciely, if $K(x,y)$ is symmetric and Case I holds, then there exists a unique function $\rho(x, y; \lambda)$ defined for $(x, y) \in \Box$ and every real λ such that

$$\mathcal{K}(h)(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda} \cdot \frac{d}{d\lambda} \int_0^1 \theta(x, y; \lambda)\, h(y)\, dy$$

for all L^2-functions h. Concerning the spectral θ-function, Carleman proved that it enjoys the same properties as Hilbert's spectral function for bounded operators.

Remark. For each fixed $\lambda > 0$ one has a bounded self-adjoint operator on $\mathcal{H}$ defined by

$$\Theta_\lambda(h)(x) = \int_0^1 \theta(x, y; \lambda)\, h(y)\, dy$$

Moreover, the operator-valued function $\lambda \mapsto \Theta_\lambda$ has bounded variation over each interval $\{a \leq \lambda \leq b\}$ when $0 < a < b$. It means that there exists a constant $C = C(a, b]$ such that

$$\max \sum_{k=0}^{M} ||\Theta_{\lambda_{k+1}} - \Theta_{\lambda_k}|| \leq C$$

for all partitions $a = \lambda_0 < \lambda_1 < \ldots \lambda_{M+1} = b$ and we have taken operator norms of the differences $\Theta_{\lambda_{k+1}} - \Theta_{\lambda_k}$ in the sum above. These bounded variations entail that one can compute the integrals in the right hand side via the usual method by Borel and Stieltjes.

The construction of spectral functions. When case I holds one constructs the ρ-function via a robust limit process. Following [ibid:Chapitre 4] we expose this in § xx. The strategy is to regard an approximating sequence $\{G_n\}$ of Hilbert-Schmidt operator, i.e.

$$\iint_{\Box} G_N(x, y)^2\, dxdy < \infty$$

hold for each N. In this case $\{G_N\}$ are compact opertors. With N fixed we get a discrete sequence of non-zero real numbers $\{\lambda\nu\}$ which are arragned with increasing avbsolute values and an orthonormal family of eigenfunctions $\{\phi_\nu^{(n)}\}$ where

$$G_N(\phi_\nu^{(N)}) = \lambda nu \cdot \phi_\nu^{(N)}$$

hold for each ν. Of course, the eigenvalues also depend on N. If $\lambda > 0$ we set

$$\rho_N(x, y; \lambda) = \sum_{0 < \lambda_\nu < \lambda} \phi_\nu(x)\phi_\nu(y)$$

$$\rho_N(x, y; -\lambda) \sum_{-\lambda < \lambda_\nu < 0} \phi_\nu(x)\phi_\nu(y)$$

let us notice that for each fixed n, the λ-numbers in (x) stay away from zero, i.e. there is a constant $c_N > 0$ such that $|\lambda_\nu \geq c_N$. So the function ρ_N vanishes in a neighborhood of zero. The spectral theorem applied to symmetric Hilbert-Schmidt operators entails that

$$\int_{-\infty}^{\infty} \frac{1}{\lambda} \cdot \frac{d}{d\lambda} \rho_N(x, y; \lambda) \cdot h(y)\, dy$$

The Ω-kernels. For each N we introduce the function

$$\Omega_N(x, y; \lambda) = \int_a^x \int_a^y \rho_N(s, t; -\lambda)\, dsdt$$

Let us notice that Bessel's inequality entails that

$$|\iint_{\Box} \rho_N(x, y; \lambda) \cdot h(x)g(y)\, dxdy\big| \leq ||h||_2 |cdot||g|7_2$$

for each pair h, g in $L^2[0, 1]$.

Exercise. Conclude from the above that the variation of Ω over $[0,x]\times[0,y]$ is bounfed above by

$$\sqrt{x\cdot y}$$

for every pair $0\leq x,y\leq 1$. In partcular the functions

$$(x,y)\mapsto \Omega(x,y;\lambda)$$

are uniformly Hölder continuous of order $1/2$ in x and y respectively.

Using the inequalities above we elave it to the reader to check that there exists at least one subsequence $\{N_k\}$ such that the functions $\{\Omega_{N_k}(x,y;\lambda)$ converges uniformly with respect to x and y while λ stays in a bounded interval. When Case I holds one proves that the limit is independent of the subsequence, i.e. there exists a limit function

$$\Omega(x,y;\lambda)=\lim_{N\to\infty}\Omega_N(x,y;\lambda)$$

From the above the ω-function is again uniformly Hölder continuous and its first order partial derivatives are L^2-functions. Moreover, after the passage to the limit one still has Bessel's inequality which entails that

$$\int_{-\infty}^{\infty}\frac{d}{d\lambda}\int_0^1 h(x)\cdot\big(\frac{\partial}{\partial x}\int_0^1\frac{\partial\Omega(x,y;\lambda)}{\partial y}\cdot h(y))\,dy\big)\,dx\leq\int_0^1 h(x)^2\,dx$$

for every $h\in L^2[0,1]$.

Definition. *A Case 1 kernel $K(x,y)$ is closed if equality holds in (*).*

Theorem. *A Case I kernel $K(x,y)$ for which the equation*

$$\int_0^1 K(x,y)\cdot\phi(y)\;dy=0$$

has no non-zero L^2-solution ϕ is closed.

A representation formula. When (*) hols we can apply to to $h+g$ for every pair of L^2-functiuons anbd since the Hilbert space $L^2[0,1]$ is self-dual it follows that for each $f\in L^2[0,1]$ one has the equality

$$f(x)=\frac{d}{dx}\int_{-\infty}^{\infty}\frac{d}{d\lambda}\int_0^1\frac{\partial\Omega(x,y;\lambda)}{\partial y}\cdot f(y))\,dy$$

almost everywhere with respect to x on $[0,1]$.

Remark. The formula (**) shows that it often is important to decide when a Case I kernel is closed. Theorem xx gives such a sufficent condition. However, it can be extended to a quite geneeral result where one for can relax the passage to the limit via suitable linear operators. The reader may consult [ibid; page 139-143] for details. Here we are content to announce the conlsisive result ehich appear in [ibid;: page 142].

The $\mathcal{L}$-family. Let ξ dentoe a paramter which in general depends on several variables, or represents points in a manifold or vectors in a normed linear space. To each ξ we are given a linear operator

$$L(\xi)\colon f\mapsto L(\xi)(f)$$

from functions $f(x)$ on $[0,1]$ to new functions on $[0,1]$. The linear map is weakly continuous, i.e. if $\{f_\nu\}$ is a sequence in $L^2[0,1]$ which converges weakly to a init function f in $L^2[0,1]$ then

$$\lim L(\xi,f_\nu)\xrightarrow{w}L(\xi)(f)$$

Next, we are given $K(x,y)$ and the second condition for L to be in $\mathcal{L}$ is that for each pair ξ and $0\leq y\leq 1$ there exists a constant $\gamma(\xi,y)$ which is independent of δ so that

$$|L(\xi)(K_\delta(\cdot,y))(x)\leq\gamma(\xi;y)$$

where $K_\delta(x,y)$ is the truncated kernel function from (xx) and in the left hand side we have applied $L(\xi$ to the function $x \mapsto K_\delta(x,y)$ for each fixed y. Moreover, we have

$$\lim_{\delta\to 0} L(\xi)(K(\cdot,y))(x) \xrightarrow{w} \lim_{\delta\to 0} L(\xi)(K_\delta(\cdot,y))(x)$$

where the convergence again holds weakly for $L2$-functions on $[0,1]$. Funally the equality below holds for every L^2-function ϕ:

$$xxxx$$

Moment problems.

A great inspiration during the development of unbounded symmetric operators on Hibert spaces emerged from work by Stieltjes' pioneering article *Recherches sur les fractions continues* [Ann. Fac. Sci. Toulouse. 1894]. The moment problem asks for conditions on a sequence $\{c_0, c_1, \ldots\}$ of positive real numbers such that there exists a non-negative Riesz measure μ on the real line and

$$(*) \qquad c_\nu = \int_{-\infty}^{\infty} t^\nu \cdot d\mu(t)$$

hold for each $\nu \geq 0$. One easily verifies that if μ exists then the Hankel determinants

$$\begin{pmatrix} c_0 & c_1 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & c_{2n} \end{pmatrix}$$

are > 0 for every $n = 0, 1, \ldots$. For brevity we say that $\{c_n\}$ is a positive Hankel sequence when the determinants above are > 0. It turns out that this confition also is sufficient.

Theorem. *For each positive Hankel sequence there exists at least one non-negative Riesz measure such that (*) holds.*

Remark. The result above is due to Hamburger which extended Stieltjes' original results where one seeks non-negative measures supported by $\{\geq 0\}$. The moment problem for a positive Hankel sequence is said to be determined if (*) has a unique solution μ. Hamburger proved that the determined case holds if and only if the the associated continued fractions of $\{c_n\}$ is completely convergent. See § x below for an account about this convergence condition. A "drawback" in the Hamburger-Stieltjes theory is that it is often difficult to check when the associated continued fraction of a positive Hankel sequence is completely convergent. So one is led to seek weaker, but sufficient conditions in order that (*) has a unique solution μ. Such a sufficient condition was established in Carleman's monograph [Carleman 1923. Page 189-220] and goes as follows:

Theorem. *The moment problem of a positive Hankel sequence $\{c_n\}$ is determined if*

$$\sum_{n=1}^{\infty} c_n^{-\frac{1}{n}} = +\infty$$

Another criterion for determined moment problems. Constructions of continued fractions and their associated quadratic forms go back to work by Heine. See [Handbuch der theorie der Kugelfunktionen: Vol. 1. part 2]. An excellent account appears also in the article *Zur Einordnung der Kettenbruchentheorie in die theorie der quadratischen Formen von unendlichvielen Veränderlichen* [Crelle J. of math. 1914] by Hellinger and Toeplitz. Let us recall that a positive Hankel sequence $\{c_n\}$ corresponds to a quadratic form in an infinite number of variables:

$$J(x) = \sum_{p=1}^{\infty} a_p x_p^2 - 2 \cdot \sum_{p=1}^{\infty} b_p x_p x_{p+1}$$

where $b_p \neq 0$ for every p. More precisely, the sequences $\{a_\nu\}$ and $\{b_\nu\}$ arise when the series

$$-[\frac{c_0}{\mu} + \frac{c_1}{\mu^2} + \frac{c_2}{\mu^3} + \ldots]$$

is formally expanded into a continuous fraction

$$\frac{c_0}{a_1 - \mu - \frac{b_1^2}{a_1 - \mu - \frac{b_2^2}{a_2 - \mu}}}$$

Now there exists the infinite matrix A with diagonal elements $\alpha_{pp} = a_p$ while

$$\alpha_{p,p+1} = \alpha_{p+1,p} = -b_p \quad : p = 1, 2, \ldots$$

and all the other elements are zero. The symmetric A-matrix gives a densely defined linear operator on the complex Hilbert space ℓ^2. The following result is proved [Carleman:1923]:

Theorem. *The densely defined operator A on ℓ^2 is self-adjoint if and only if the moment problem for $\{c_n\}$ is determined.*

A general study of A-operators.

Ignoring the "source" of the pair of real sequences $\{a_p\}$ and $\{b_p\}$ we consider a matrix A as above where the sole condition is that $b_p \neq 0$ for every p. If μ is a complex number one seeks infinite vectors $x = (x_1, x_2, \ldots)$ such that

(i) $$Ax = \mu \cdot x$$

It is clear that (i) holds if and only if the sequence $\{x_p\}$ satisfies the infinite system of linear equations

$$(a_1 - \mu)x_1 = b_1 x_2$$
$$(a_p - \mu)x_p = b_{p-1}x_{p-1} + b_p x_{p+1} \quad : p \geq 2$$

Since $b_p \neq 0$ is assumed for every p we see that x_1 determines the sequence. Here $x_2, x_3, \ldots$ depend on x_1 and the parameter μ. Keeping x_1 fixed while μ varies the reader can verify that

$$x_p = \psi_p(\mu) \quad : p \geq 2$$

where $\psi_p(\mu)$ is a polynomial of degree $p-1$ for each $p \geq 2$. These ψ-polynomials depend on the given pair of sequences $\{a_p\}$ and $\{b_p\}$ and with these notations the following results is proved in [ibid]:

2. Theorem. *The densely defined operator A on ℓ^2 is self-adjoint if and only if*

$$\sum_{p=1}^{\infty} \psi_p(\mu) = +\infty$$

for every non-real complex number μ.

A more general result. Let $\{c_\nu\}$ be a positive Hankel sequence. which is determined or not. If $|rho$ is a non-negative measure which solves the moment problem (*) we set

$$\widehat{\rho}(\mu) = \int \frac{d\rho(t)}{t - \mu}$$

It is defined outside the rell axis and yields in particular an analytic funcvtion in the upper half-plane. A major result in [Carleman] gives a sharp inclusion for the values which can be attained by these $\widehat{\rho}$ functions while ρ are non-negative measures which solve the moment problem. It is expressed via constructions of discs in the upper half-plane which arise via a nested limit of discs constructed from a certain family of Möbius transformations. More precisely, to the given Hankel sequence we have the pair of sequences $\{a_\nu\}$ and $\{b_\nu\}$. For a fixed μ in the upper half-plane we consider the maps:

$$S_\nu(z) = \frac{b_\nu^2}{a_{\nu+1} - \mu - z} \quad : \nu = 0, 1, \ldots$$

Notice that

$$\mathfrak{Im}(S_\nu(z)) = \frac{b_\nu^2 \cdot \mathfrak{Im}(\mu + z)}{|a_{\nu+1} - \mu - z|^2} > 0$$

It follows that S_ν maps the upper half-plane U^+ conformally onto a disc placed in U^+.

Exercise. To each $n \geq 1$ we consider the composed map

$$\Gamma_n = S_0 \circ S_1 \circ \ldots \circ S_{n-1}$$

Show that the images $\{C_n(\nu) = \Gamma_n(U^+)\}$ form a decreasing sequence of discs and there exists a limit

$$C(\mu) = \cap C_n(\mu)$$

which either is reduced to a single point or is a closed disc in U^+.

Theorem. *For each $\mu \in U^+$ and every ρ-measure which solves the moment problem one has the inclusion*

$$\widehat{\rho}(\mu) \subset C(\mu)$$

The Neumann-Poincaré kernel

Operator theory was used at an early stage by Neumann and Poincaré. So for historic reasons we expose some facts about a specific boundary value problem. Let Ω be a bounded and connected open set in $\mathbf{C}$ of class $\mathcal{D}(C^1)$, i.e. the boundary $\mathcal{C}$ is the disjoint union of closed and continuously differentiable Jordan curves. To each $p \in \mathcal{C}$ there exists the inner normal $\mathbf{n}_i(p)$ of unit length. Set

$$K(p,q) = \frac{\langle p-q, \mathbf{n}_i(p)\rangle}{|p-q|}$$

It is clear that the C^1-hypothesis entails that K is a continuous function on $\mathcal{C} \times \mathcal{C}$. Next, let M be a positive real number. If u is a real-valued continuous function on $\mathcal{C}$ there exists a harmonic function U in Ω defined by

$$U(p) = \int_{\mathcal{C}} \log \frac{M}{|p-q|} \cdot u(q)\, ds(q)$$

where ds is the arc-length measure on $\mathcal{C}$. We choose M so large that it exceeds the diamater of $\mathcal{C}$ defined by max $|q_1 - q_2| : q_1, q-2 \in \mathcal{C}$. As explained in §xx the inner normal derivative of U is given by the equation

$$\frac{\partial U}{\partial \mathbf{n}_i}(p) = -\pi \cdot u(p) - \int_{\mathcal{C}} K(p,q) \cdot u(q)\, ds(q)$$

To solve the boundary value problem (*) from the introduction therefore amonunts to find u which satisfies the integral equation

$$(1) \qquad -\pi \cdot u(p) - \int_{\mathcal{C}} K(p,q) \cdot u(q)\, ds(q) = a(p) \cdot \int_{\mathcal{C}} \log \frac{M}{|p-q|} \cdot u(q)\, ds(q) + f(p)$$

Following the device by Neumann and Poincare we choose M so large that that the function below defined on $\mathcal{C} \times \mathcal{C}$ is everywhere positive:

$$S(p,q) = a(p) \cdot \log \frac{M}{|p-q|} + K(p,q)$$

To S one associaes the integral operator $\mathcal{S}$ which sends $u \in C^0(\mathcal{C})$ into

$$\mathcal{S}_u(p) = \int_{\mathcal{C}} S(p,q) \cdot u(q)\, ds(q)$$

Now (1) amounts to find u such that

$$(2) \qquad \mathcal{S}_u(p) + \pi \cdot u(p) = -f(p)$$

Since the kernel function S is everywhere positive one verifies easily that the linear operator on the Banach space $C^0(\mathcal{C})$ defined by $u \mapsto \mathcal{S}_u + \pi \cdot u$ is invertible, i.e. $-\pi$ does not belong to the spectrum of the bounded operator $\mathcal{S}$. Introducing the inverse operator we solve (2) by

$$(3) \qquad u = -(\pi \cdot E + \mathcal{S})^{-1}(f)$$

Remark. From the above we have solved the boundary value problem (*) in the introduction by operator methods. Notice that the solution is found without using solutionds to the Dirichlet problem.

A study of spectra. The Neumann-Poincaré solution above shows that the study of the integral operator $\mathcal{K}$ defined by

$$\mathcal{K}_u(p) = \int_{\mathcal{C}} K(p,q) \cdot u(q)\, ds(q)$$

is crucial. In § xx we shall learn that $\mathcal{K}$ is a compact operator on the Hilbert space $\mathcal{H}$ of square integrable functions on $\mathcal{C}$ with respect to the arc-length measure. Here one seeks real-valued eigenfunctions u which satisfy

$$(**) \qquad u(p) = \lambda \cdot \mathcal{K}_u(p)$$

for some real number λ. An obstacle is that K is not symmetric, i.e. $K(p,q) \neq K(q,p)$. Following a construction invented by Poincaré one can find eigenvalues and eigenfunctions via a symmetric kernel function.

The kernel $N(p,q)$. Let M be a positive number which as above exceeds the diameter of $\mathcal{C}$. Set

$$N(p,q) = \int_{\mathcal{C}} K(p,\xi) \cdot \log \frac{M}{|q-\xi|} \cdot ds(\xi)$$

Green's formula applied to a pair of harmonic functions H_1, H_2 in Ω with continuously differentiable boundary values gives the equlity

$$\int_{\mathcal{C}} \frac{\partial H_1}{\partial \mathbf{n}_i}(p) \cdot H_2(p)\, ds(p) = \int_{\mathcal{C}} \frac{\partial H_2}{\partial \mathbf{n}_i}(p) \cdot H_1(p)\, ds(p)$$

From this it follows that the the kernel function N is symmetric, i.e. $N(p,q) = N(q,p)$. At the same time we have the operator $\mathcal{S}$ defined by

$$\mathcal{S}(u) = \int_{\mathcal{C}} \log \frac{M}{|q-\xi|} \cdot u(\xi)\, ds(\xi)$$

The construction of N entails that its associated integral operator $\mathcal{N}$ satisfies the equation

$$\mathcal{N} = \mathcal{S} \circ \mathcal{K}$$

This gives a gateway to find eigenfuntions to the non-symmetric operator $\mathcal{K}$. To begin with $\mathcal{S}$ is special since the kernel function $S(p,q)$ is both symmetric and positive. In the Hilbert space $\mathcal{H}$ above we shall learn in § x that there exists an orthonormal basis $\{\phi_n\}$ formed by eigenfunctions which satisfy

(1) $$\mathcal{S}\phi_n = \kappa_n \phi_n$$

where the positive κ-numbers tend to zero. Moreover, each $u \in \mathcal{H}$ has a Fourier-Hilbert expansion

(2) $$u = \sum \alpha_n \cdot \phi_n$$

Next, the symmetric N-function yields a doubly indexed and symmtric sequence $\{c_{jk}\}$ given by

$$c_{jk} = \iint \phi_j(p)\phi_k(q) N(p,q)\, ds(p) ds(q)$$

Theorem. *A pair (u,λ) solves (**) if and only if u has an expansion*

$$u = \sum_{n=1}^{\infty} \alpha_n \cdot \phi_n(p)$$

where the sequence $\{\alpha_n\}$ satisfies the system of linear equations

$$\kappa_p \cdot \alpha_p = \lambda \cdot \sum_{q=1}^{\infty} c_{qp} \alpha_q \quad : p = 1, 2, \ldots$$

Remark. The proof is given in § xx where a crucial step is to show that

$$\sum \sum \frac{c_{j,k}^2}{\kappa_j \cdot \kappa_k} < \infty$$

The case when $\mathcal{C}$ has corner points. When $\mathcal{C}$ has corner points, for example if $\partial\Omega$ is bordered by piecewise linear and closed Jordan curves, then the Neumann-Poincaré kernel is unbounded. Here the reduction to the symmetric case is more involved and relies upon quite intricate results which appear in [Carleman : Part II: 1916]. Further analysis remains to be done and open problems about the Neumann-Poincaré equation remains to be settled in dimension ≥ 3. So far it appears that only the 2-dimensional case is well understood by results from [Carleman :1916].

The self-adjoint operator $i \cdot \frac{d}{dx}$.

On the real x-line we have the Hilbert space $\mathcal{H} = L^2(\mathbf{R})$ of complex-valued and square integrable functions. It contains $C_0^\infty(\mathbf{R})$ as a dense subspace on which the operator

$$\frac{d}{dx} \colon f \mapsto f'(x)$$

is defined where $f'(x)$ is the derivtive of f. Set $A = i \cdot \frac{d}{dx}$. This gives a densely defined operator on $\mathcal{H}$ whose domain of defintion $\mathcal{D}(A)$ consists of all square-integerable functions f for which the distribution derivative f' again belongs to $\mathcal{H}$. Let us remark that every $f \in \mathcal{D}(A)$ therefore is locally the primitive of an L^1-function and therefore locally absolutely continuous. So in particular $\mathcal{D}(A) \subset C^0(\mathbf{R})$. The graph of A is the set of pairs $(f, if') : f \in \mathcal{D}(A)\}$. Basic facts in distribution theory show that $\Gamma(A)$ is a closed subset of $\mathcal{H} \times \mathcal{H}$. See § xx for details. Hence A is densely defined and closed. Now we seek Neumann's spectrum $\sigma(A)$. To begin with, if λ is a complex number and $A(f) = \lambda \cdot f$ for some $f \in \mathcal{H}$ we have

$$i \cdot f' = \lambda \cdot f$$

Basic facts about ordinary differential equations entail that f is a constant times $e^{-i\lambda x}$ and this exponential function has absolute value one and cannot belong to $\mathcal{H}$. So if E is the identity operator then $\lambda \cdot E - A$ is injective for all complex numbers λ, i.e. the "point spectrum" of A is empty. However, the invertibility in Neumann's sense requires more conditions on the densely defined operator $\lambda \cdot E - A$, i.e its range must be dense and there exists a constant $c > 0$ such that

$$(1) \qquad ||\lambda \cdot f - if'||_2 \geq c \cdot ||f||_2 \quad : f \in \mathcal{D}(A)$$

To analyze when these two conditions hold we use the Fourier transform. When $f \in C_0^\infty(\mathbf{R})$ we have the inversion formula

$$i \cdot f'(x) = -\frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot \xi \cdot \widehat{f}(\xi)\, d\xi$$

Fourier's inversion formula gives

$$(\lambda \cdot E - A)(f)(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot (\lambda + \xi) \cdot \widehat{f}(\xi)\, d\xi$$

If $\lambda = a + ib$ with $b \neq 0$ we have $|\lambda + \xi)|^2 \geq b^2$ for all real ξ. Applying Plancherel's equality we concludue that

$$||\lambda \cdot f - if'||_2 \geq b \cdot ||f||_2 \colon f \in \mathcal{D}(A)$$

Hence (1) hols for every non-real λ. There remains to show that the range of $(\lambda \cdot E - A)$ is dense in the Hilbert space $\mathcal{H}$ when $\mathfrak{m}(\lambda) \neq 0$. If the density fails the orthogonal complement of the range is non-zero and we find a non-zero function $g \in \mathcal{H}$ such that

$$(3) \qquad \langle \lambda \cdot f - if', g \rangle = \int (\lambda \cdot f - if') \cdot \bar{g}\, dx = 0 \quad : f \in \mathcal{D}(A)$$

Thjs holds in particular when f are test-functions and using thr notion of distribution derivatives we see that (2) entails that the g-function satisfies the equation

$$\lambda \cdot g = i \cdot \frac{dg}{dx}$$

From (xx) above it follows that $g = 0$ and hence the range of $(\lambda \cdot E - A)$ must be dense which finishes the proof of the inclusion

$$\sigma(A) \subset \mathbf{R}$$

The equality $A = A^*$. By (2) and Parseval's equality for L^2-norms under the Fourier transform it follows that $|mathcalD(A)$ consists of all $f \in L^2$ such that $\xi \cdot \widehat{f}(\xi)$ is square integrable on the ξ-line. Next, for a pair f, g in C_0^∞ a partial integration gives the equality

$$\langle Af, g \rangle = \langle f, Ag \rangle$$

which means that the densely defined operator A is symmetric. Next, by definition a function g belongs to $\mathcal{D}(A^*)$ if there exists a constant $C(g)$ such that

$$|\langle Af, g\rangle \leq C(g) \cdot ||f||_2 \quad : f \in \mathcal{D}(A)$$

Passing to the Fourier transform and using Parseval's equality this is equivalent to the condition that there exists a constant C such that

$$||\xi \cdot \widehat{f} \cdot \widehat{g}||_2 \leq C \cdot ||\widehat{f}||_2 \quad : f \in \mathcal{D}(A)$$

Since the family $\{\widehat{f} : f \in \mathcal{D}(A)$ is dense in the L^2-space on the ξ-line this entails that $\xi \cdot \widehat{g}$ is locally square integrable and hence g belongs to $\mathcal{D}(A)$. Hence we have proved the equality

$$\mathcal{D}(A) = \mathcal{D}(A^*)$$

which means that A is a densely defined and self-adjoint operator on $\mathcal{H}$.

Further examples.

2. An ugly example. Let $\{a_{pq}\}$ be a doubly-indexed and symmetric sequence of real numbers, i.e. $a_{pq} = a_{qp}$ hold for each pair of positive integers. Moreover we assume that

$$(*) \qquad \sum_{q=1}^{\infty} a_{pq}^2 < \infty \quad : p = 1, 2, \ldots$$

Consider the Hilbert space ℓ^2 whose vectors are complexes sequences $\{\xi_p\}$ such that

$$\text{(i)} \qquad \sum_{q=1}^{\infty} |\xi_q|^2 < \infty$$

The Cauchy-Schwarz inequality entails that if $\{\xi_p\} \in \ell^2$ then the series

$$\sum_{q=1}^{\infty} a_{pq} \cdot \xi_q$$

converges for each p. However, the imposed conditions on the a-sequence does in general not imply that

$$\sum_{p=1}^{\infty} |\sum_{q=1}^{\infty} a_{pq} \cdot \xi_q|^2 < \infty$$

In an article from 1920, Carleman constructed a sequence $\{a_{pq}\}$ whch satisfies (*) and at the same time violates the wellknown fact in linear algebra which asserts that eigenvalues of a symmetric matrix with real elements are real. Namely, in Carleman's sequence $\{a_{pq}\}$ there exists a complex vector $\xi \in \ell^2$ with norm one such that

$$\text{(ii)} \qquad \xi_p = i \cdot \sum_{q=1}^{\infty} a_{pq} \cdot \xi_q \quad : p = 1, 2, \ldots$$

while the series (i) has value one. We give the construction in § xx and remark that this example shows that care must be taken when one considers linear equations in an inifinite number of variables. Namely, with the ξ-vector as in (ii) we have the limit formula:

$$\lim_{N\to\infty} \sum_{p=1}^{p=N} |\xi_p|^2 = 1$$

Next, (ii) entails that

$$1 = \lim_{N\to\infty} i \cdot \sum_{p=1}^{p=N} \sum_{q=1}^{\infty} a_{pq}\xi_q \cdot \bar{\xi}_p = i \cdot \sum_{q=1}^{\infty} \xi_q \cdot \sum_{p=1}^{p=N} a_{pq}\bar{\xi}_p$$

For each q we set

$$\xi_q(N) = i \cdot \sum_{p=1}^{p=N} a_{qp}\xi_p$$

Since $\{a_{pq}\}$ is symmetric and real we have

$$\sum_{p=1}^{p=N} a_{pq}\bar{\xi}_p = i \cdot \sum_{p=1}^{p=N} a_{qp}\bar{\xi}_p = -\bar{\xi}_q(N)$$

and the equality $i^2 = -1$ entails that

$$\text{(iii)} \qquad -1 = \lim_{N\to\infty} \sum_{q=1}^{\infty} \xi_q \cdot \bar{\xi}_q(N)$$

At the same time we have limit formulas:

$$\text{(iv)} \qquad \lim_{N\to\infty} \bar{\xi}_q(N) = \bar{\xi}_q \quad : q = 1, 2, \ldots$$

But in spite of (iii) the results above give

$$\lim_{N\to\infty} \xi_q \cdot (\bar{\xi}_q + \bar{\xi}_q(N) = 0$$

which at first sight seems to contradict the limit formulas above. However, the convergence in (iv) holds for each fixed q, but not uniformly with respect to q. This is the reason why the "obscure limits" above appear when (ii) holds.

3. The dual of ℓ^∞. Consider the Banach space ℓ^1 whose vectors are sequences of $\{x_n\}$ of complex numbersi indexed by positive integers with the norm $||x|| = \sum |x_n|$. The dual space is ℓ^∞ which consists of bounded vectors. It contains the subspace ℓ_0^∞ of sequences $\{\xi_n\}$ where only finitely many $\xi_n \neq 0$, This is not a dense subspace for if ξ^* is the sequence with $\xi_n^* = 1$ for every n its distance to ℓ_0^∞ is one. On the other hand ℓ_0^∞ is dense in the *weak-star topology* which by definition is the weakest - or coarsest - topology on ℓ^∞ for which the functions

$$\xi \mapsto \xi(x) = \sum \xi_n x_n$$

are continuous for every ℓ^1-vector x. This leads us to consider the dual space of ℓ^∞. It turns out that it consists of Riesz measures on the compact space $\beta\mathbf{Z}$ obtained as the maximal ideal space of the Banach algebra ℓ^∞. One refers to $\beta\mathbf{Z}$ as the Stone-Cech compactification of the integers where $\mathbf{Z}$ appears as a dense subset and at the same time there exist non-zero Riesz measurs μ on $\beta\mathbf{Z}$ without any point mass at the integers. Every such Riesz measure, regarded as an element in the dual of ℓ^∞ vanishes identically on the subspace ℓ_0^∞. The result is that the bi-dual

$$(\ell^1)^{**} = \mathfrak{i}(\ell^1) \oplus \mathfrak{M}(\beta\mathbf{Z} \setminus \mathbf{Z})$$

where the last term is the space is Riesz measures on the compact topoligical space $\beta\mathbf{Z} \setminus \mathbf{Z}$).

4. Domains of definition for the Laplace operator. Let D be the unit disc in $\mathbf{R}^2$ and consider the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$. In the Hilbert space $L^2(D)$ of square-integrable complex valued functions there exists the subspace V of $L^2(D)$ with the property that for each $\phi \in V$ its Laplacian taken in the distribution sense belongs to $L^2(D)$. The necessary and sufficient condition for this to hold is that there is a constant $C(\phi)$ such that

$$|\iint_D \Delta(f) \cdot \phi \, dxdy| \leq C(\phi) \cdot ||f||_2 \quad : f \in C_0^\infty(D) \tag{1}$$

where $||f||_2$ is the L^2-norm and as indicated f are test-functions with compact support in D. In fact, since Hilbert spaces are self-dual and $C_0^\infty(D)$ is dense in $L^2(D)$ the inequality (1) gives a unique $\psi \in L^2(D)$ such that

$$\iint_D \Delta(f) \cdot \phi \, dxdy = \iint_D f \cdot \psi \, dxdy \quad : f \in C_0^\infty(D) \tag{2}$$

which means that $\psi = \Delta(\phi)$ in the distribution sense. Hence one has a linear operator

$$T \colon \phi \mapsto \Delta(\phi)$$

The domain of definition $V = \mathcal{D}(T)$ contains test-functions and is therefore dense in $L^2(D)$. One easily verifies that the graph of T taken in the product $L^2(D) \times L^2(D)$ is closed. Hence T is a densely defined and closed linear operator. The kernel consists of harmonic functions in D which are square integrable. So if one wants an injective and densely defined operator it is necessary to shrink the domain of definition. To achieve this we introduce the logarithmic kernel

$$L(z,\zeta) = \frac{1}{2\pi} \cdot \log \frac{|z-\zeta|}{|1-\bar{\zeta}z|}$$

It gives the linear operator $\mathcal{L}$ defined on $L^2(D)$ by:

$$\mathcal{L}(\phi)(z) = \iint_D L(z,\zeta) \cdot \phi(\zeta) \, d\xi d\eta \tag{3}$$

Here $L(z,\zeta)$ is square integrable on the product $D \times D$ which means that $\mathcal{L}$ is a Hilbert-Schmidt operator on $L^2(D)$ and in particular compact. Moreover, since $L(z,\zeta) = 0$ when $|z| = 1$ and

$|\zeta| < 1$ one easily verifies that $\mathcal{L}(\phi)$ is a continuous function on the closed disc with boundary values zero on the unit circle. Moreover one has the equality

$$\Delta(\mathcal{L}(\phi)) = \phi \quad : \phi \in L^2(D) \tag{4}$$

Hence the composed operator $\Delta \circ \mathcal{L}$ is the identity on $L^2(D)$ and Stokes theorem gives

$$f = \mathcal{L}(\Delta(f)) \quad : f \in C_0^\infty(D)$$

In particular the range of $\mathcal{L}$ is a dense subspace of $\mathcal{L}(L^2(D))$ and from the above we see that if $g = \mathcal{L}(\phi)$ for some $\phi \in L^2(D)$ then

$$\iint_D \Delta(f) \cdot g \, dxdy = \iint_D f \cdot \phi \, dxdy \quad : f \in C_0^\infty(D)$$

We conclude that $\mathcal{L}(L^2(D))$ is a subspace of $\mathcal{D}(T)$ and we have the linear operator

$$\Delta \colon g \mapsto \Delta(g) \quad : g \in \mathcal{L}(L^2(D))$$

With this restricted domain of definition one has

$$\mathcal{L} \circ \Delta(g) = g \quad : g \in \mathcal{L}(L^2(D))$$

So here Δ becomes injective and at the same time (4) holds.

The closure property. There remains to show that the chosen domain of definition yields a closed operator which amounts to show that the graph

$$\Gamma = \{(g, \Delta(g)) : g \in \mathcal{L}(L^2(D))\}$$

The closed graph property relies upon the following crucial result:

4.1 Theorem. *An L^2-function g belongs to $\mathcal{L}(L^2(D))$ if and only if there exists a constant $C(g)$ such that*

$$|\iint_D \Delta(f) \cdot \phi \, dxdy| \leq C(g) \cdot ||f||_2 \quad : f \in C^\infty(\bar{D}) \tag{*}$$

Remark. Notice the distinction from (1) where (*) only is imposed when f have compact support in D. So (*) is more restrictive and the inclusion $\mathcal{L}(L^2(D)) \subset V$ is strict. The theorem above will be proved in § xx and entails that Δ with the chosen domain of definition is an unbounded self-adjoint operator.

5. The complex Hilbert transform. Identify the complex z-plane with the 2-dimensional real (x, y)-space. Now we have the Hilbert space $L^2(\mathbf{C})$ whose vectors are complex-valued and square integrable functions in the (x, y)-plane. If g is a continuously differentiable function with compact support there exists a limit

$$G(z) = \lim_{\epsilon \to 0} \iint_{|z-\zeta|>\epsilon} \frac{g(\zeta)}{(z-\zeta)^2} \, d\xi d\eta \tag{5.1}$$

exists for every $z \in \mathbf{C}$. In fact, using Stokes theorem we show in § xx tht

$$G(z) = \iint \frac{\frac{\partial g}{\partial \zeta}(\zeta)}{z - \zeta} d\xi d\eta$$

where the last integral exists since $\frac{\partial g}{\partial \zeta}$ is a continuous function with compact support and $(z - \zeta)^{-1}$ is locally integrable in the (ξ, η)-space for each fixed z. The complex Hilbert transform is defined by the linear operator:

$$T_g(z) = \frac{1}{\pi} G(z)$$

It turns out that T is an isometry with respect to the L^2-norm, i.e.

$$\iint |T_g(z)|^2 \, dxdy = \iint |g(z)|^2 \, dxdy \quad : g \in C_0^1(\mathbf{C}) \tag{5.2}$$

There are several proofs of (5.2). One relies upon the Fourier transform and Parseval's theorem which is exposed in § xx. Another proof employs an abstract reasoning and goes as follows: Let $\mathcal{H}$ be a Hilbert space and Φ us a subset of vectors with the property that the subspace of vectors in $\mathcal{H}$ of the form

$$c_1\phi_1 + \ldots + c_k\phi_k \quad : \langle \phi_j, \phi_i\rangle = 0 \colon j \neq i$$

is dense in $\mathcal{H}$. Thus, we assume that finite linear combinations of pairwise orthogonal Φ-vectors is dense. Let us now consider a linear operator T on $\mathcal{H}$ such that the following hold:

(*) $$||T(\phi)|| = ||\phi|| \quad : \phi \in \Phi$$

(**) $$\langle \phi_1, \phi_2\rangle = 0 \implies \langle T(\phi_1), T(\phi_2)\rangle = 0 \quad : \phi_1, \phi_2 \in \Phi$$

Exercise Show that (*)-(**) imply that T is an isometry, i.e. $||T(g)|| = ||g||$ for every $g \in \mathcal{H}$.

Apply the result above to the complex Hilbert transform where Φ is the family of vectors in $L^2(\mathbf{C})$ given by characteristic functions of dics in $\mathbf{C}$. It is clear that this family satisfies the density condition above. Now we study the action by the Hilbert transform on such functions. Let $z_0 \in \mathbf{C}$ and $r_0 > 0$ be given and ϕ is the characteristic function of the disc of radius r_0 centered at z_0.

5.1 Proposition. *One has* $T_\phi(z) = 0$ *when* $|z - z_0| < r_0$ *while*

$$T_\phi(z) = \frac{r_0^2}{(z-z_0)^2} \quad : |z - z_0| > r_0$$

Proof. After a translation we can take $z_0 = 0$ and via a dilation of the scale reduce the proof to the case $r_0 = 1$. If $|z| > 1$ we have

$$T_\phi(z) = \frac{1}{\pi}\iint_D \frac{d\xi d\eta}{(z-\zeta)^2} = \frac{1}{\pi z^2}\iint_D \frac{d\xi d\eta}{(1-\zeta/z)^2}$$

Expand $(1-\zeta/z)^{-2}$ in a power series of ζ and notice that

$$\iint_D \zeta^m \, d\xi d\eta = 0 \quad : m = 1, 2, \ldots$$

Then we see that $T_\phi(z) = z^{-2}$. There remains to show that $|z| < 1$ gives $T_\phi(z) = 0$. To prove this we use the differential 2-form $d\zeta \wedge d\bar{\zeta}$ and there remains to show that

(i) $$\lim_{\epsilon \to 0} \iint_{D_\epsilon} \frac{d\zeta \wedge d\bar{\zeta}}{(z-\zeta)^2} = 0$$

where $D_\epsilon = D \setminus \{|\zeta - z| \leq \epsilon\}$. Now $\partial_\zeta((z-\zeta)^{-1}) = (z-\zeta)^{-2}$ and Stokes theorem gives (i) if

$$\int_{|\zeta|=1} \frac{d\bar{\zeta}}{z-\zeta} - \lim_{\epsilon \to 0}\int_{|\zeta - z| = \epsilon} \frac{d\bar{\zeta}}{z-\zeta} = 0$$

In the first integral we use the series expansion $(z-\zeta)^{-1} = -\sum_{n=0}^\infty \zeta^{-n-1} \cdot z^n$ and get a vanishing since

$$\int_{|\zeta|=1} \zeta^{-n-1} \, d\bar{\zeta} = 0 \quad : n = 0, 1, \ldots$$

The verification that the limit as $\epsilon \to 0$ in the second integral vanishes is left to the reader.

The Proposition entails that

$$||T(\phi)||^2 = r_0^4 \cdot \iint_{|z|>r_0} \frac{dxdy}{|z|^4} = r_0^4 \cdot 2\pi \int_{r_0}^\infty \frac{dr}{r^3} = \pi \cdot r_0^2$$

Since $||\phi||^2 = \pi \cdot r_0^2$ we conclude that $||T(\phi)|| = ||\phi||$. The verification that (**) holds when ϕ_1, ϕ_2 are characteristic functions of disjoint discs is left as an exercise.

A. Some results in functional analysis.

The subsequent material exposes some results in functional analysis with special attention given to linear operators on Hilbert spaces. Proofs of some of the assertions below appear in § 11.

A.1 The Riezs-Nagy theorem.

Let $\mathcal{H}$ be a Hilbert space. A linear operator $U\colon \mathcal{H} \to \mathcal{H}$ is unitary if

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

hold for all pairs x, y in $\mathcal{H}$. Next, a linear operator A on $\mathcal{H}$ of norm one is called a contraction. A family of contractions arises as follows: Let $\mathcal{H}_1$ be another Hibert space which contains $\mathcal{H}$ as a closed subspace. There exists the orthogonal projection P from $\mathcal{H}_1$ onto $\mathcal{H}$ and if U is a unitary operator on $\mathcal{H}_1$ we get a linear operator on $\mathcal{H}$ defined by

$$A = P \circ U$$

In other words, we take $x \in \mathcal{H}$ and apply U to get $U(x) \in \mathcal{H}_1$ and P projects this vector back to $\mathcal{H}$. Since the unitary operator U preserves norms it follows that A is a contraction. A result due to F. Riesz and Nagy asserts that every contraction on $\mathcal{H}$ arises from a pair U, P as above. Moreover, they proved that the pair can be chosen so that

$$p(A) = P \circ p(U)$$

hold for every polynomial in A. We prove this in § 11.xx.

A.2 The infinitesmal generator of a unitary semi-group

.

A result due to M.H. Stone about strongly continuous semi-groups of unitary operators goes as follows: There is given a doubly-index sequence of functions $\{c_{pq}(t)\}$, where p, q are positive integers and the c-functions are defined on $\{t \geq 0\}$ which satisfies the following: First the Hermitian condition holds, i.e. $\bar{c}_{qp}(t) = c_{p,q}(t)$ for all pairs of integers. In addition the following hold for all $s, t \geq 0$ and every pair p, q.

(i) $$\sum_{q=1}^{\infty} |c_{qp}(t)|^2 = 1 \quad : \quad p = 1, 2, \ldots \quad \text{and} \quad \sum_{q=1}^{\infty} c_{qp}(t) \cdot \overline{c_{qr}}(t) = 0 \quad : \quad p \neq r$$

(ii) $$c_{qp}(t+s) = \sum_{j=1}^{\infty} c_{qj}(s) \cdot c_{jp}(t)$$

(iii) $$\lim_{t \to 0} \sum_{q=1}^{\infty} \bigl|\, \xi_q - \sum_{p=1}^{\infty} c_{qp}(t)\xi_p \,\bigr|^2 = 0 \quad \text{for every} \quad \xi \in \ell^2$$

where ℓ^2 is the Hilbert space ℓ^2 whose vectors are complex sequences $\{\xi_p\}$ such that $\sum |\xi_p|^2 < \infty$. From now on Stone's condition hold. For every pair of positive numbers $s.t$ and each $\xi \in \ell^2$ we set:

$$D_\xi(s,t) = \sum_{q=1}^{\infty} \Bigl|\, \frac{1}{t} \cdot \sum_{p=1}^{\infty} c_{qp}(t)\xi_p - \frac{1}{s} \cdot \sum_{p=1}^{\infty} c_{qp}(s)\xi_p + (\frac{1}{s} - \frac{1}{t})\xi_q \,\Bigr|^2$$

Stones Theorem. *There exists a dense subspace $\mathcal{H}_*$ of the Hibert space ℓ^2 such that*

$$\lim_{(s,t)\to(0,0)} D_\xi(s,t) = 0 \quad : \quad \xi \in \mathcal{H}_*$$

Moreover, to each $\xi \in \mathcal{H}_*$ we get the limit vector $\eta = \{\eta_q\}$ where

$$\eta_q = \lim_{t\to 0} \frac{1}{t} \cdot [\sum_{p=1}^{\infty} c_{qp}(t)\xi_p - \xi_q\,]$$

Finally, the linear operator

$$A(\xi) = i \cdot \eta$$

is a densely defined self-adjoint operator.

A.3 Unbounded self-adjoint operators.

Let $\mathcal{H}$ be a Hilbert space and consider a closed and densely defined linear operator A which in general is unbounded. One says that A is symmetric if

(0.1) $$\langle Ax, y\rangle = \langle x, Ay\rangle \quad : \ x, y \in \mathcal{D}(A)$$

Let E be the identity operator on $\mathcal{H}$. If $x \in \mathcal{D}(A)$ and λ is a complex number the symmetry gives:

$$||(\lambda\dot{E} - A)(x)||^2 = |\lambda|^2 \cdot ||x||^2 + ||Ax||^2 - 2 \cdot \mathfrak{Re}\,\lambda \cdot \langle x, Ax\rangle$$

The Cauchy-Schwarz inequality gives:

(0.2) $$|\langle x, Ax\rangle| \leq ||x|| \cdot ||Ax|| \implies ||(\lambda\dot{E} - A)(x)||^2 \geq |\mathfrak{Im}\,\lambda|^2 \cdot ||x||^2$$

Since A has a closed graph the inequality (0.2) implies that if $\mathfrak{Im}\,\lambda \neq 0$ then the operator $B = \lambda \cdot E - A$ sends $\mathcal{D}(A)$ into a closed subspace of $\mathcal{H}$. Since Hilbert spaces are self-dual the range $B(\mathcal{D}(A))$ is therefore $\neq \mathcal{H}$ if and only if its orthogonal complement is $\neq \emptyset$. A vector y belongs to $B(\mathcal{D}(A))^\perp$ if

(*) $$\lambda \cdot \langle x, y\rangle = \langle Ax, y\rangle \quad : \quad x \in \mathcal{D}(A)$$

So if (*) has no solutions $y \neq 0$ when $\mathfrak{Im}\,\lambda \neq 0$ then $\lambda \cdot E - A$ is surjective for every non-real complex number and together with the inequality (0.2) we see that $\lambda \cdot E - A$ is invertible in the sense of (0.0.1). Hence it has a resolvent and we find the operator valued analytic function $R_A(\lambda)$ defined outside the real axis in the complex λ-plane.

Remark. In § 9 we study symmetric and densely defined operators satisfying the above. A notable fact is that if (*) has no solution $x \neq 0$ when λ is not real, then the symmetric operator A is equal to its adjoint A^*. The detailed proofs appear in § 9.

A.4 Differential operators

.

Pioneering work by G.D. Birkhoff carried out around 1905 motivated and inspired subsequent studies of unbounded densely defined operators on Hibert spaces. Let

$$P(t, \frac{d}{dt}) = \sum_{j=0}^{j=m} a_j(t) \cdot \frac{d^j}{dt^j}$$

be a differential operator where we $\{a_j(t)\}$ are complex-valued polynomials. The adjoint operator P^* is defined by the equation

$$\int P(f) \cdot \bar{g}\, dt = \int f \cdot \overline{P^*(g)}\, dt$$

for each pair of test-functions f and g. If $P = P^*$ we say that P is symmetric. Assume this and consider some bounded interval (a, b) which gives the Hilbert space $L^2(a, b)$. Now P yields a densely defined but unbounded operator on this Hilbert space denoted by T_{ab}. We shall learn that there exists the adjoint operator T^*_{ab} which also is densely defined on $L^2(a, b)$. However, the symmetry may be lost under this construction and the search for self-adjoint extensions of P on

$L^2(a,b)$ leads to an involved theory. Results concerned with spectral theory of ordinary differential operators are foremost due to Birkhoff, Weyl, Titschmarsh and Kodaira. These authors also found various boundary conditions in order to achieve self-adjoint extensions of P. A notable fact is that when P is a given symmetric ODe-operator, then the class of self-adjoint extensions to the Hilbert space $L^2(a,b)$ can depend upon the interval (a,b). We refrain from a detailed discussion. The reader may consult chapter XIII from volume 2 in [Dunford-Schwartz] which offers an extensive account about spectral theory for ordinary differential equations.

A.7 Moment problems.

In 1894 Stieltjes established necessary and sufficient conditions for the existence of solutions to the following equation: There is given a sequence $\{c_n\}$ of non-negative real numbers. One asks if there exists a non-negative measure μ on the real x-line such that

$$\int_{-\infty}^{\infty} x^n \cdot d\mu(x) = c_n$$

hold for each $n = 1, 2, \ldots$. No specific bound on $\{c_n\}$ is imposed and Stieltjes solution relies upon properties of continued fractions associated to $\{c_n\}$. The moment problem inspired later studies in operator theory. Elegant results appear in Stone's book *Linear operators on Hilbert spaces and their applications to analysis* (AMS: 1932). For example, Stone proved a uniqueness result which goes as follows: Denote by $\mathcal{P}_*$ the class of non-negative measures μ on the real line such

$$\int x^{2n} \cdot d\mu(x) < \infty \quad : \quad n = 1, 2, \ldots$$

To each such measure μ one gets a unique sequence of polynomials $\{P_n(x)\}$, where P_n has degree n and leading coefficient > 0 while

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) \cdot d\mu(x) = 0 \quad : \quad n \neq m$$

A measure $\mu \in \mathcal{P}_*$ is of the Stone type if

$$\sum_{n=0}^{\infty} |P_n(x+iy]^2 = +\infty$$

hold for every non-real complex number $z = x + iy$ with $y \neq 0$. Now Stone's uniqueness theorem asserts the following:

A pair μ, ν of Stone type with equal moments for all n must be identical, i.e. $\mu = \nu$.

A.8 Optimal control theory.

Problems in the calculus of variation can lead to severe doubts about the existence of extremal solutions. This was put forward by Weierstrass and a historic example is Lamé's equation where tentative proofs of existence of solutions occured in work by Kovavelsky and Volterra but It was not until 1920 that Zeilon settled the existence of solutions to Lamé's equations using advanced Fourier analysis. In retrospect we mention that Zeilon's proof already used distribution-theoretic methods. There also occur optimization problems where the control is restricted to vector-valued functions with range in a compact polyhedron Π in $\mathbf{R}^n$. In such problems one often arrives at Bang-bang solutions where the extremal control switches a finite number of times between extreme points of the polyhedron Π. In this pioneering work which led to a general maximum principle in control problems, Pontryagin employed functional analysis where one allows competing families of control functions in spaces of measurable functions. The outcome is a Bang-bang solution which is piecewise constant while its existence is demonstrated via a "detour into larger spaces". So functional analysis has a wide range of applications in optimization theory.

A.10 L^p-inequalities.

Many results which originally were constrained to specific situations can be extended when notions in functional analysis are adopted. An example is a theorem from the article [Hörmander-1960] where singular integrals are treated in a general context. Here is the set up: Let B_1 and B_2 be two Banach spaces and n some positive integer. Let $K(x)$ be a function which for every $x \in \mathbf{R}^n$ assigns a vector in the Banach space $L(B_1, B_2)$ of continuous linear operators from B_1 to B_2. One only assumes that K is a continuous function, i.e. it need not be linear. If $f(x)$ is a continuous B_1-valued function defined in $\mathbf{R}^n$ with compact support there exists the convolution integral

$$\mathcal{K}f(x) = \int_{\mathbf{R}^n} K(x-y)(f(y))\, dy$$

To be precise, with x fixed in $\mathbf{R}^n$ the right hand side is a welldefined B_2-valued integral which exists because $K(x-y)$ as a linear operator sends the B_1-vector $f(y)$ into B_2, i.e. the integrand in the right hand side is a B_2-valued function

$$y \mapsto K(x-y)(f(y))$$

which can be integrated with respect to y and the resulting integral yields the B_2-vector in the left hand side. When x varies, $x \mapsto \mathcal{K}f(x)$ becomes a B_2-valued function. Following [ibid] we impose two conditions on K where norms on the three Banach spaces $L(B_1, B_2), B_1, B_2$ appear. We say that K satisfies the Hörmander condition if there exist positive constants A and C such that the following hold for every real number $t > 0$:

$$(1) \qquad \int_{|x|\geq 1} ||K(t(x-y)) - K(tx)||\, dx \leq C \cdot t^{-n} \quad \text{for all} \quad |y| \leq 1$$

where the left hand side employs the operator norms of K defined for pairs of points tx and $t(x-y)$ in $\mathbf{R}^n$. In addition to (1) we impose the following L^2-inequality for every function f with the same constant C as above:

$$(2) \qquad \int_{\mathbf{R}^n} ||\mathcal{K}f(x)||_2^2\, dx \leq C^2 \cdot \int_{\mathbf{R}^n} ||f(x)||_1^2\, dx$$

Here one has used norms on B_1 respectively B_2 during the integration. Hörmander extended Vitali's Covering Lemma in $\mathbf{R}^n$ to a similar covering principle on arbitrary normed spaces to prove the following weak-type inequality:

Theorem. *There exists an absolute constant C_n which depends on n only such that for every pair of Banach spaces B_1, B_2 and every linear operator K which satisfies (1-2), the following hold for every $\alpha > 0$:*

$$\text{vol}_n(\{x \in \mathbf{R}^n : ||\mathcal{K}f(x)||_2 > \alpha\}) \leq \frac{C_n \cdot C}{\alpha} \cdot \int_{\mathbf{R}^n} ||f(x)||_1\, dx$$

Remark. This result has a wide range of applications when it is combined with interpolation theorems due to Markinkiewicz and Thorin. For example, classical results which involve L^p-inequalties during the passage to Fourier transforms or other kernel functions can be put into a general frame where one employs vector-valued functions rather than scalar-valued functions. Typical examples occur when K sends real-valued functions to vectors in a Hilbert space, i.e. here B_1 is the 1-dimensional real line and $B_2 = \ell^2$. So Hörmander's result constitutes a veritable propagande for learning general notions in functional analysis. The idea to regard vector-valued functions was of course considered at an early stage. An example is a theorem due to Littlewood and Paley which goes as follows: Let $f(x)$ be a function on the real line such that both f and f^2 are integrable. To each integer $n \geq 0$ we set

$$f_n(x) = \Big| \int_{2^{n-1}}^{2^n} e^{ix\xi} \cdot \widehat{f}(\xi)\, d\xi \,\Big| + \Big| \int_{-2^{-n}}^{-2^{n-1}} e^{ix\xi} \cdot \widehat{f}(\xi)\, d\xi \,\Big|$$

Then it is proved in [L-P] that for each $1 < p \leq 2$ there exists a constant $C_p \geq 1$ such that

$$(*) \qquad C_p^{-1}||f||_p \leq \cdot \int_{-\infty}^{\infty} \big(\sum_{n=0}^{\infty} f_n(x)^2\big)^{\frac{2}{p}}\, dx \leq C_p \cdot ||f||_p$$

where $||f||_p = \big(\int |f(x)|^p\, dx\big)^{\frac{1}{p}}$ is the L^p-norm of f. Hörmander's theorem can be used to give an elegant proof of (*) which has the merit that the use a suitable vector-valued operator during the proof becomes transparent.

B. Hilbert's spectral theorem for bounded self-adjoint operators.

The theory about integral equations created by Fredholm led to Hilbert's result from 1904 which we begin to describe. Let $\mathcal{H}$ be a Hilbert space and denote by $L(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. Every $T \in L(\mathcal{H})$ has its operator norm

$$||T|| = \max_x ||T(x)|| \quad \text{maximum over vectors of norm} \quad \leq 1$$

Next, let A be a bounded self-adjoint operator on $\mathcal{H}$ whose compact spectrum is denoted by $\sigma(A)$. A crucial fact is that the self-adjointness implies that

$$||A||^2 = ||A^2||$$

and passing to higher powers it follows that the operator norm $||A||$ is equal to the spectral radius defined by

$$\lim_{n\to\infty} ||A^n||^{\frac{1}{n}}$$

Using this equality the Operational Calculus in § 7 will show that there exists an algebra isomorphism from the sup-norm algebra $C^0(\sigma(A))$ into a closed subalgebra $\mathcal{A}$ of $L(\mathcal{H})$, i.e. each continuous function g on the compact spectrum $\sigma(A)$ gives a bounded linear operator G. Moreover, it is an isometry which means that

$$(*) \qquad |g|_{\sigma(A)} = ||G||$$

where the left hand side is the maximum of $|g|$ over $\sigma(A)$. Next, since A is self-adjoint $\sigma(A)$ is a compact subset of the real line where we use t as the variable. If $g(t) = c_0 + c_1 t + \ldots + c_m t^m$ is a polynomial, the operational calculus shows that $G = E + c_1 A + \ldots + c_m A^m$ where E is the identity operator on $\mathcal{H}$. By Weierstrass' theorem the set of polynomials a dense subalgebra of $C^0(\sigma(A))$ and hence $\mathcal{A}$ is the closure in $L(\mathcal{H})$ of the algebra formed by all polynomials of A.

B.1 The spectral measure. The algebra isomorphism gives a map from the product $\mathcal{H} \times \mathcal{H}$ to the space of Riesz measures on $\sigma(A)$ which to every pair (x, y) in $\mathcal{H}$ assigns a Riesz measure $\mu_{x,y}$ such that

$$(**) \qquad \langle g(A)x, y\rangle = \int_{\sigma(A)} g(t) \cdot d\mu_{x,y}(t)$$

holds for every $g \in C^0(\sigma(A))$. The isometry (*) implies that the total variation of $\mu_{x,y}$ is bounded by $||x|| \cdot ||y||$ for every pair x, y. Now Borel's integrals give a larger subalgebra of $L(\mathcal{H})$. Namely, for every bounded Borel function $g(t)$ on $\sigma(A)$ the integrals in the sense of Borel exist in the right hand side of (**) for each pair x, y in $\mathcal{H}$. In this way the g-function gives a bounded linear operator G such that

$$(***) \qquad \langle G(x), y\rangle = \int_{\sigma(A)} g(t) \cdot d\mu_{x,y}(t) \quad \text{hold for all pairs} \quad x, y$$

This yields an algebra isomorphism from the algebra $\mathcal{B}^\infty(\sigma(A))$ of bounded Borel functions to a subalgebra of $L(\mathcal{H})$ denoted by $B(\mathcal{A})$. Again the map $g \mapsto G$ is an isometry and in this extended algebra we can construct an ample family of self-adjoint operators. Namely, for every Borel subset γ of $\sigma(A)$ we can take its characteristic function and get the bounded linear operator Γ. By the Operational Calculus the spectrum of Γ is equal to the closure of γ. Moreover, Γ is a self-adjoint operator and commutes with A. In particular we can consider partitions of $\sigma(A)$. Namely, choose $M > 0$ so that $\sigma(A) \subset [-M, M]$ and M is outside $\sigma(A)$. With a large integer N we consider the half-open intervals

$$\gamma_\nu = \big[-M + \frac{\nu}{N} \cdot M, -M + \frac{\nu+1}{N} \cdot M\big] \quad : 0 \leq \nu \leq 2N-1$$

Then $\Gamma_0 + \ldots + \Gamma_{2N-1} = E$ and we also get the decomposition

$$(1) \qquad A = A_0 + \ldots + A_{2N-1} \quad : A_\nu = A\Gamma_\nu$$

Above $\{\Gamma_\nu\}$ gives a *resolution of the identity* where (1) means that A is a sum of self-adjoint operators where every individual operator has a spectrum confined to an interval of length $\leq \frac{1}{N}$. This constitutes Hilbert's Theorem for bounded self-adjoint operators.

C. Carleman's spectral theorem for unbounded self-adjoint operators

In a note from May 1920 [Comptes rendus], Carleman described a procedure to handle unbounded symmetric operators expressed via integral kernels which fail to satisfy Fredholm's conditions. The theory was presented in the monograph *Sur les équations singulières à noyau reèl et symmétrique* from 1923 published by Uppsala University. In addition to [ibid] the reader may also consult Carleman's plenary talk [Carleman] at the Scandinavian Congress held at Copenhagen in 1925 where the relation to the new spectral theory with Fredholm's pioneering discoveries is discussed and illuminated by various examples. For historic reasons we give a citation from the introduction in [ibid] where Carleman expressed his admiration for Fredholm's work: *Peu du découvertes mathématiques ont été données par leur auteur sous une forme si achevée que la découverte de la solution de l´équation intégrale*

$$\phi(x) = \lambda \cdot \int_a^b K(x,y)\phi(y)\,dy + f(x)$$

par M. Fredholm. Les nombreuses application de cette théorie admirable ont pourtant améné un grand nombere de rechereches concerntant le cas óu le noyaue $K(x,y$ est non borneé.

C.1 An ugly example. Following in [Carleman 1923] we give an example which illustrates that new phenomena can occur for unbounded symmetric operators. On the unit interval $[0,1]$ we have an orthonormal family of functions $\psi_0, \psi_1, \ldots$ where $\psi_0 = 1$ is the identity and $\psi_1(x) = -1$ on $(0,1/2)$ and $+1$ on $(1/2,1)$. For each $n \geq 2$ we set:

$$\psi_n(x) = -2^{\frac{n-1}{[}2} : 1 - 2^{-n+1} \leq x < 1 - 2^{-n} \quad \text{and} \quad \psi_n(x) = 2^{\frac{n-1}{2}} : 1 - 2^{-n} < x < 1$$

while $\psi_n(x) =$ when $0 < x < 1 - 2^{-n+1}$. If $\{a_p\}$ is a sequence of real numbers we define the kernel function on $[0,1) \times [0,1)$ by

$$\text{(i)} \qquad K(x,y) = \sum a_p \cdot \psi_p(x)\psi_q(x)$$

It is clear that K is symmetric and the associated operator

$$\mathcal{K}(u)(x) = \int_0^1 K(x,y)u(y)\,dy$$

is defined on L^2-functions u supported by $0 \leq x \leq x_*$ for every $x_* < 1$ and therefore densely defined. Choose a sequence $\{a_p\}$ such that the positive series.

$$\text{(ii)} \qquad \sum_{p=0}^{\infty} \frac{2^p}{1 + a_p^2} < \infty$$

This convergence means that the sequence $\{a_p\}$ must have a high rate of growth so the kernel function (i) becomes turbulent when x and y approach 1.

Exercise. Show that (ii) entails that there exists an L^2-function $\phi(x)$ such that $\mathcal{K}(\phi)$ also is an L^2-function and

$$\phi = i \cdot \mathcal{K}(\phi)$$

If necessary, consult [ibid: page 62-66] for a demonstration.

A favourable case.

We begin to expose Carleman's result for "wellbehaved" integral operators. Let $K(x,y)$ be a complex-valued function defined in the open unit square$\{0 < x, y < 1\}$ satisfying the hermitian condition

$$K(y,x) = \bar{K}(x,y)$$

Assume that for all x outside a nullset $\mathcal{N}$ one has a finite integral

$$\text{(1)} \qquad \int_0^1 |K(x,y)|^2\,dy$$

If $\phi(y)$ is an L^2-function, the Cauchy-Schwartz inequality gives:

$$\int_0^1 K(x,y)\phi(y)\,dy \quad \text{exists for all} \quad x \in (0,1) \setminus \mathcal{N}$$

C.2 Definition. *The hermitian kernel K is of type I if the homogeneous equation*

$$\phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\,dy$$

has no non-trivial L^2-solution $\phi(y)$ when $\mathfrak{Im}\,\lambda \neq 0$.

C.3 Theorem. *If K is of type I the equation*

$$\text{(*)} \qquad \phi(x) = \lambda \cdot \int_0^1 K(x,y)\phi(y)\,dy + f(x)$$

has a unique L^2-solution ϕ for every $f \in L^2$ and every non-real λ. Moreover the ϕ-solution satisfies:

$$\text{(**)} \qquad \int_0^1 |\phi(x)|^2\,dx \leq \frac{|\lambda|^2}{\mathfrak{Im}(\lambda)^2} \cdot \int_0^1 |f(x)|^2\,dx$$

Proof. To get the existence we use the truncated kernels $\{K_n(x,y)\}$ where $K_n(x,y) = K(x,y)$ when $K(x,y)| \leq n$ and otherwise zero. Given $f(x) \in L^2[0,1]$ and a non-real λ, the Hilbert-Schmidt theory from § xx gives a unique $\phi_n \in L^2$ such that

$$\text{(i)} \qquad \phi_n(x) = \lambda \cdot \int_0^1 K_n(x,y)\phi_n(y)\,dy + f(x)$$

Mutltiplying with $\bar{\phi}_n$ it follows that

$$\frac{1}{\lambda} \int_0^1 \bar{\phi}_n(x)(\phi_n(x) - f(x))\,dx = \iint_\square K_n(x,y)\bar{\phi}_n(x)\phi_n(y)\,dxdy$$

Since K_n is hermitian the right hand side is real and taking imaginary parts we obtain

$$\text{(ii)} \qquad (\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}) \cdot \int_0^1 |\phi_n(x|^2\,dx) = \frac{1}{\lambda} \cdot \int_0^1 \bar{\phi}_n(x)f(x)dx) - \frac{1}{\bar{\lambda}} \cdot \int_0^1 \phi_n(x)\bar{f}(x)dx)$$

and the Cauchy-Schwarz inequality gives

$$\text{(iii)} \qquad \int_0^1 |\phi_n(x)|^2\,dx \leq \frac{|\lambda|^2}{\mathfrak{Im}(\lambda)^2} \cdot \int_0^1 |f(x)|^2\,dx$$

Hence the L^2-norms of $\{\phi_n\}$ are uniformly bounded and we can find a subsequence which converges weakly to an L^2-function ϕ whose L^2-norm again is bounded by the right hand side in (iii). Now the assumption that K is of type I entials that the weak limit is unique. Hence the whole sequence $\{\phi_n\}$ is weakly convergent and one verifies that ϕ solves the inhomogeneous equation (*).

C.4 Spectral resolution. Let $K(x,y)$ be a hermitian kernel of type I. Hilbert's theorem applies to the bounded kernels $\{K_n\}$ and give for each pair $f, g \in L^2[0,1]$ a Riesz measure $\mu^n_{f,g}$ supported by the compact real spectrum of the kernel operator defined by K_n. In [Carleman] it is proved that the sequence $\{\mu^n_{f,g}\}$ converges weakly to a Riesz-measure $\mu_{f,g}$ supported by the real line and Hilbert's operational calculus applies when one intergates continuous functions $\phi(t)$ with compact support, i.e. every such ϕ yields a bounded linear operator Φ on $L^2[0,1]$ such that

$$\int_0^1 \Phi(f) \cdot \bar{g}\,dx = \int \phi(t)d\mu_{f,g}(t)$$

hold for every pair f, g in $L^2[0,1]$.

D. The Bohr-Schrödinger equation.

In 1923 quantum mechanics had not yet appeared so the studies in [Carleman] were concerned with singular integral equations, foremost inspired from work by Neumann, Poincaré, Fredholm and Volterra. The creation of quantum mechanics gave new challenges for mathematicians. The interested reader should consult the lecture held by Niels Bohr at the Scandianavian congress in mathematics held in Copenhagen 1925 where he speaks about the interplay between the new physics and "pure" mathematics. Bohr's lecture presumably inspired Carleman when he some years later resumed work from [Car 1923]. Recall that the fundamental point in Schrödinger's theory is the hypothesis on energy levels which correspond to orbits in Bohr's theory of atoms. For a very good account about the physical background the reader may consult Bohr's plenary talk when he received the Nobel Prize in physics 1923. Mathematically the Bohr-Schrödinger theory leads to the equation

$$(*)\qquad \Delta\phi + 2m\cdot\big(E-U\big)\big(\frac{2\pi}{h}\big)^2\cdot\phi = 0$$

Here Δ is the Laplace operator in the 3-dimensional (x,y,z)-space, m the mass of a particle and h Planck's constant while $U(x,y,z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Let us cite an excerpt from Carlemans lectures in Paris at Institut Henri Poincaré held in 1930:

Dans ces dernières années l'ínteret de la question qui nous occupe a considérablement augmenté. C'est en effet un instrument mathématique indispensable pour development de la mechanique moderne crée par M.M. de Brogile, Heisenberg et Schrödinger. Etude de l'équation integrale:

$$\phi(x) = \lambda\cdot\int_a^b K(x,y)\phi(y)dy + f(x) \quad : \lambda\in\mathbf{C}\setminus\mathbf{R}$$

The theory from [Carleman :1923] applies to the following PDE-equations attached to a second order differential operator

$$(**)\qquad L = \Delta + c(x,y,z) \quad : \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

where $c(x,y,z)$ is a real-valued function. The L-operator is densely defined and symmetric on the dense subspace formed by test-functions u in $\mathbf{R}^3$. The problem is to find conditions on the c-function in order that the Case I from § B occurs. The following sufficiency result is proved in [ibid]:

D.1 Theorem *Let $c(x,y,z)$ be a continuous and real-valued function such that there is a constant M for which*

$$\limsup_{x^2+y^2+z^2\to\infty} c(x,y,z) \leq M$$

Then Case I holds for the operator $\Delta + c(x,y,z)$.

Example. Considers a potential function:

$$W(p) = \sum \frac{\alpha_k}{|p-q_k|} + \beta$$

where $\{q_k\}$ is a finite subset of $\mathbf{R}^3$ and the α-numbers and β are real and positive. With $c = W$ we get Case I and hence solutions to the Schrödinger equation can be established via a robust limit process. See §§ xx for details.

Further comments. The literature about the Schrödinger equation and other equations which emerge from quantum mechanics is extensive. For the source of quantum mechanics the reader should first of all consult the plenary talks by Heisenberg, Dirac and Shrödinger when they received the Nobel prize in physics. Apart from physical considerations the reader will find expositions where explanations are given in a mathematical framework. Actually Heisenberg was sole winner 1931 while Dirac and Schrödinger shared the prize in 1932. But they visited Stockholm together in December 1932.

For mathematician who wants to become acquainted with quantum physics the eminent text-books by Lev Landau are recommended. Especially *Quantum mechanics: Non-relativistic theory* in Vol. 3. The relativistic case is treated in later volums of [L-L]. Here one employs Heisenberg's matrix representation and Dirac's equations are used to study radiation phenomena. In the introduction to [ibid: Volume 3] Landau inserts the following remark: *It is of interest to note thst the complete mathematical formalisim of quantum mechnaics was constructed by W. Heisenberg and E. Schrödinger in 1925-26, before the discovery of the uncertainty principle which revealed the physical contents of this formalism.*

In the non-relativistic situation one studies foremost wave equations of Schrödinger's type. The mathematical foundations were laid in Schrödinger's article *Quantizierung als Eigenwertproblem* from 1926 In § XX from Special Topics we describe another of Schrödinger's equations which led to a veritable challenge in the "world of mathematics" which remains quite open.

Final remark. After [Carleman 1923] an alternative proof of the spectral theorem for unbounded operators on Hilbert spaces was given by J. von Neumann's in the article *Eigenwerttheorie Hermitescher Funktionaloperatoren* from 1929. Neumann"s account seems to have become more fashion by readers who prefer a "formal style". A merit in Neumann's work is that some measure theoretic technicalities in Carleman's proof can be omitted. On the other hand it is often necessary to establish formulas of a measure theoretic character in order to apply the spectral theorem. An example is Carleman's solution to the Bohr-Schrödinger equation which he presented during lectures at Sorbonne in 1930. Here the Laplace operator in $\mathbf{R}^3$ is transformed to a suitable integral operator. After this estimates which lead to solutions in $L^2)\mathbf{R}^3)$ are obtain via solutions to Neumann's boundary value problem in increasing balls in $\mathbf{R}^3$. Let us also remark that separable Hilbert spaces can be embedded into L^2-spaces. So the restriction to integral operators covers most applications. In Chapter 1 from [Carleman] the reader may find a very general construction of a family of spectral functions associated with Hermitian kernels which need not satisfy the condition from Definition B.2 above.

Examples. From a historic point of view we mention that various unbounded symmetric operators were studied in connecttion with differential operators in work by Birkhoff, Picard and Weyl around 1910. Picard's equation was already described in the introduction. More generally one can consider kernel functions of the form

$$K(x,y) = H_1(|x-y|) + H_2(|x+y|)$$

which give linear operators on $L^2(\mathbf{R}$. In [Carleman: Chapitre 4] it is proved that K is of class I when

$$\text{(i)} \qquad \int_{-\infty}^{\infty} H_\nu(s)^2\, ds < \infty$$

hold for $\nu = 1, 2$. To prove this one seeks solutions to the integral equation

$$\phi(x) = \lambda \cdot \int_{-\infty}^{\infty} K(x,y)\phi(y)\, dy + f(x)$$

where f is a given L^2-function on the real line and λ in genereal is a complex number. In § xx we shall expose material from [Carleman 1923] and construct Neumann's sresolvent to the integral operator $K(x,y)$ which after settles why (i) entails that it is self-adjoint. We remark that in this construction one employs specific solutions to Picard's equation. The result above can then be applied to analyze integral operators in $\mathbf{R}^3$ defined by kernel functions

$$S(p,q) = F(|p-q|)$$

where

$$\int_0^{\infty} F(r)^2 \cdot r\, dr < \infty$$

Another interesting class of integral operators are given by

$$K(x,y) = \frac{P(x,y)}{Q(x,y)}$$

Here P is a polynomial of some degree $m \geq 1$, and $Q(x,y)$ is homogeneous of degree $m+1$ which takes real and positive values when x and y both are > 0. Already the special case when

$$K(x,y) = \frac{1}{x+y}$$

leads to some non-trivial conclusions. In § xx we prove that the spectrum on the Hilbert space of square integrable functions in the positive quadrant of $\mathbf{R}^2$ is the unbounded interval $[\pi^{-1}, +\infty)$.

E. Application to a dynamical system.

Using the spectral theorem a rigorous proof of the Ergodic Hypothesis in Statistical Mechanics was given by Carleman at seminars held at Institute Mittag-Leffler in May 1931. Here is the situation: There is given an n-tuple of C^1-functions $A_1(x), \ldots, A_n(x)$ where $x = (x_1, \ldots, x_n)$ are points in $\mathbf{R}^n$. Let t be a time variable and consider the differential system

$$\frac{dx_k}{dt} = A_k(x_1(t), \ldots, x_n(t)) \quad : \quad 1 \leq k \leq n \tag{1}$$

Assume that there exists a compact hypersurface S in $\mathbf{R}^n$ such that if $p \in S$ and $\mathbf{x}_p(t)$ is the vector-valued solution to (1) with initial condition $\mathbf{x}_p(0) = p$, then $\mathbf{x}_p(t)$ stay in S for every t. The uniqueness for solutions to the differential systems above gives for every t a bijective map $p \mapsto \mathbf{x}_p(t)$ from S onto itself. It is denoted by $\mathcal{T}_t$ and we notice that

$$\mathcal{T}_s \circ \mathcal{T}_t = \mathcal{T}_{s+t}$$

In addition to this we assume that there exists an invariant measure σ on S for the $\mathcal{T}$-maps. In other words, a non-negative measure σ such that

$$\sigma(\mathcal{T}_t(A)) = \sigma(A)$$

hold for every σ-measurable set. For the applications it suffices to consider the case when σ is absolutely continuos, i.e. a positive continuous function times the area measure on S. So now we have the Hilbert space $L^2(\sigma)$ of complex-valued measurable functions U on S for which

$$\int_S |U(p)|^2 \cdot d\sigma(p) < \infty$$

next, on the Hilbert space $L^2(\sigma)$ there exists the following densely defined symmetric operator:

$$U \mapsto i \cdot \sum_{\nu=1}^{\nu=n} A_\nu \cdot \frac{\partial U}{\partial x_\nu} \tag{*}$$

It is easy to verify that *Case 1* holds for this operator and hence we can apply the spectral theorem. In particular, to each a pair of L^2-functions U and V we consider the following mean-value integrals over time intervals $[0, T]$:

$$J_T(U, V) = \frac{1}{T} \cdot \int_0^t \left[U(\mathcal{T}_t(p)) \cdot V(p) \cdot d\sigma(p)\right] \cdot dt \tag{*}$$

Using the spectral theorem from [Carleman 1923], the result below was proved in [Carleman 1931]:

Theorem. *Let $\{\omega_\nu\}$ be an orthonormal basis in $\mathcal{H}$. For each pair U, V in $L^2(\sigma)$ one has the equality*

$$\lim_{T \to \infty} J_T(U, V) = \sum_{\nu=1}^{\infty} \langle \omega_\nu, U \rangle \cdot \langle \omega_\nu, V \rangle$$

where

$$\langle \omega_\nu, U \rangle = \int_S \omega_\nu(p) \cdot U(p) \cdot d\sigma(p)$$

and similarly with U replaced with V.

Remark. Let $\mathcal{H}_*$ be the space of $L^2(\sigma)$-functions which are $\mathcal{T}$-invariant, i.e. L^2-functions ω satisfying $\mathcal{T}_t(\omega) = \omega$ for all t. When $\mathcal{H}_*$ is reduced to the one-dimensional space of constant functions the theorem above implies that almost every trajectory comes close to every point in S which confirms the ergodic condition.

Birkhoff's almost everywhere convergence result. As pointed out in [1932: page 80] where Carleman gave an affirmative answer to a question raised by Poincaré in his lecture *L'avénir dés mathematiques* held at Rome in 1908, the article [xxx] by Birkhoff published in December 1931 contains deeper results which establish properties which hold almost everywhere. For example, given a point $p_0 \in S$ and a measurable subset $\Omega \subset S$ one can study the avergage time interval when the particle which starts at p_0 when $t = 0$ visits Ω. Thus, for $T > 0$ we denote by $\tau_{p_0;\Omega}(T)$ the total time of visits in Ω as $0 \leq t \leq T$. With these notations Birkhoff proved that

$$(*) \qquad \lim_{T \to \infty} \frac{\tau_{p_0;\Omega}(T)}{T}$$

exists for almost all $p_0 \in S$. This impressive result was the starting point for extensive studies in Ergodic Theory which has stayed as an active subject. Many subtle problems dealing with almost everywhere convergence properties have been established,. Studies of semi-groups of transformations which are adapated to stochastic differential equations is an extensive subject where the scenario is almost "unlimited" since global considerations occur while a process can be governed by many different differential systems. A typical example from recent literature is the article *Ergodicity of the 2D Navier Stokes equations with degenerate stochastic forcing* by Hairer and Mattingley.

11.6 Numerical range.

Let A be a closed and densely defined linear operator on the Hilbert space $\mathcal{H}$. Its numerical range was introduced by Toeplitz and Hausdorff:

11.6.1 Definition. *The numerical range is the subset of* $\mathbf{C}$ *defined by*

$$\text{Num}(A) = \{\langle Ax, x\rangle \quad : x \quad \text{unit vector in} \quad \mathcal{H}\}$$

11.6.2 Proposition. *The numerical range is a convex set.*

Proof. If α, β are complex numbers and $B = \alpha A + \beta E$ it is clear that $\text{Num}(B)$ is the image of $\text{Num}(A)$ under the linear map $z \mapsto \alpha z + \beta$. So it siuffices to show that when 0 and 1 belong to $\text{Num}(A)$, then the real interval $[0, 1]$ stays in the numerical range. So now we have two vectors f, g for which

(i) $$\langle Af, f\rangle = 0 \quad : \quad \langle Ag, g\rangle = 1$$

To prove this we consider the family of vectors

$$\xi = f + se^{i\theta}g$$

where $s \geq 0$ is real and $0 \leq \theta \leq 2\pi$. It follows that

$$\langle A\xi, \xi\rangle = s^2 + s\big(e^{i\theta}\langle Ag, f\rangle + e^{-i\theta}\langle Af, g\rangle\big)$$

Above we have the complex-valued function

$$\theta \mapsto e^{i\theta}\langle Ag, f\rangle + e^{-i\theta}\langle Af, g\rangle$$

Since $e^{i\pi} = -1$ its value has opposed signs at 0 and π which implies that the imaginary part by the usual mean-value theorem has some zero $0 \leq \theta_* \leq \pi$. So with $\xi_* = \xi = f + se^{i\theta_*}g$ it follows that

$$\langle A\xi_*, \xi_*\rangle = s^2 + s \cdot \rho \quad \text{where} \quad \rho \in \mathbf{R}$$

The right hand side is zero when $s = 0$ and tends to $+\infty$ when s increases and hence this its range contains $[0, 1]$ which give s-values for which $f + se^{i\theta_*}g$ and hence numerical values at every point in $[0, 1]$.

Next, let A be a bounded linear operator. In this case it is clear that $\text{Num}(A)$ is a bounded set. However, it is in general not closed. See § below for an example. The following inclusions hold:

11.6.3 Theorem. *For every bounded operator A one has*

$$\sigma(A) \subset \overline{\text{Num}(A)} \subset \widehat{\sigma(A)}$$

where the right hand side is the convex hull of $\sigma(A)$.

Proof. TO BE GIVEN....

11.5 Stones theorem.

11.3.2 Unitary semi-groups. Specialize the situation above to the case when B is a Hilbert space $\mathcal{H}$ and $\{U_t\}$ are unitary operators. Set $T = \xi_*$ so that

$$B(x) = \lim_{t \to 0} \frac{U_t x - x}{t} \quad : \quad x \in \mathcal{D}(T)$$

If x, y is a pair in $\mathcal{D}(B)$ we get

$$\langle Bx, y\rangle = \lim_{t \to 0} \langle \frac{U_t x - x}{t}, U_t y\rangle = \lim_{t \to 0} \frac{1}{t} \cdot \big(\langle U_t x, U_t y)\rangle - \langle x, U_t y)\rangle\,\big)$$

Since U_t are unitary we have $\langle U_t x, U_t y)\rangle = \langle x, y)\rangle$ for each t and conclude that the last term above is equal to

$$\lim_{t \to 0} \frac{\langle x, y - U_t y\rangle}{t} = -\langle x, By\rangle$$

Hence B is anti-symmetric, i.e.

$$\langle Bx, y\rangle = -\langle x, By\rangle$$

Set $A = i \cdot T$ which gives

$$\langle Ax, y\rangle = i \cdot \langle Tx, y\rangle = -i \cdot \langle x, Ty\rangle = -\langle x, i \cdot Ty\rangle = \langle x, Ay\rangle$$

where we used that the inner product is hermitian. Hence A is a densely defined and symmetric operator.

Theorem. *A is self-adjoint, i.e. one has the equality $\mathcal{D}(A) = \mathcal{D}(A^*)$.*

Proof. It suffices to prove that $\mathcal{D}(T) = |mathcalD(T^*)$. To btain this we take a vector y be a vector in $\mathcal{D}(T^*)$ which by definition gives a constant $C(y)$ such that

(i) $$|\langle Tx, y\rangle|| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T)$$

Now $\langle Tx, y\rangle$ is equal to

(ii) $$\lim_{t \to 0} \langle \frac{U_t x - x}{t}, U_t y\rangle = -\lim_{t \to 0} \langle x, \frac{U_t y - y}{t}\rangle$$

So if

$$\eta_t = fracU_t y - yt$$

it follows that there exists

$$\lim_{t \to 0}, \langle x, \eta_t\rangle = -\langle Tx, y\rangle$$

for each $x \in \mathcal{D}(T)$.

The adjoint operators $\{U_t^*\}$ give another unitary semi-group with infinitesmal generator A_* where

$$A_*(x) = \lim_{t \to 0} \frac{U_t^* x - x}{t} \quad : \quad x \in \mathcal{D}(A_*)$$

Since U_t is the inverse operator of U_t^* for each t we get

(i) $$A_*(x) = \lim_{t \to 0} U_t(A_* x) = \lim_{t \to 0} \frac{x - U_t x}{t} \quad : \quad x \in \mathcal{D}(A_*)$$

From (i) we see that $\mathcal{D}(A) \subset \mathcal{D}(A_*)$ and one has the equation

(ii) $$A_* x) = -A(x) \quad : \quad x \in \mathcal{D}(A_*)$$

Reversing the role the reader can check the equality

(iii) $$\mathcal{D}(A) = \mathcal{D}(A_*)$$

Next, let x, y be a pair in $\mathcal{D}(A)$. Then

$$\langle Ax, y\rangle = \lim_{t \to 0} \langle \frac{U_t x - x}{t}, U_t y\rangle = \lim_{t \to 0} \frac{1}{t} \cdot \big(\langle (U_t x, U_t y)\rangle - \langle x, U_t y)\rangle\big)$$

Since U_t are unitary we have $\langle U_t x, U_t y)\rangle = \langle x, y)\rangle$ for each t and conclude that the last term in (xx) is equal to

$$\lim_{t \to 0} \frac{\langle x, y - U_t y\rangle}{t} = -\langle x, Ay\rangle$$

Hence A is anti-symmetric. Set

$$B = iA$$

Exercise. Show that (i) gives the equality $\mathcal{D}(A_*) = \mathcal{D}(A)$ and that:

$$A_*(x) = -A(x) \quad : \quad x \in \mathcal{D}(A)$$

$$\langle Bx, y\rangle = -\langle x, By\rangle \quad : \quad x, y \in \mathcal{D}(B)$$

Exercise. Conclude from the above that the operator iB is self-adjoint.

Remark. The equations above constitute Stones theorem which was established in 1930. It has a wide range of applications. See for example von Neumann's article *Zur Operatorenmethode in der klassischen Mekanik* and Maeda's article *Unitary equivalence of self-adjoint operators and constant motion* from 1936.

11.3.3 A converse construction. Let A be a densely defined self-adjoint operator. From § 9.B A is approximated by a sequence of bounded self-adjoint operators $\{A_N\}$. With N kept fixed we get a semi-group of unitary operators where

$$U_t^{(N)} = e^{-itA_N}$$

The reader may verify that the inifintesmal generator becomes $-iA_N$. Next, for each $x \in \mathcal{D}(A)$ and every fixed t there exists the limit

$$\lim_{N\to\infty} U_t^{(N)}(x)$$

Remark. This gives a densely defined linear operator U_t whose operator norm is bounded by one which therefore extends uniquely to a bounded linear operator $\mathcal{H}$ and it is clear that this extension becomes a unitary operator. In this way we arrive at a semi-group $\{U_t\}$ and one verifies that its infinitesmal generator is equal to $-iA$. However, it is not clear that $\{U_t\}$ is strongly continuous and one may ask for conditions on the given self-adjoint operator A which ensures that $\{U_t\}$ is strongly continuous.

POSTPONE

An exercise. One often applies Newton's decomposition of rational functions and we shall employ his formulas without hesitation. Here is an example. Let $n \geq 2$ and $t_1 < \ldots < t_n$ is a strictly invreasing sequence of real numbers. With a variable z we consider the rational function

$$P(z) = \sum_{\nu=1}^{\nu=n} \frac{A_\nu}{t_\nu - z}$$

where $\{A_\nu\}$ are positive real numbers. Put $\mathcal{A} = \sum A_\nu$ and

$$a = \mathcal{A}^{-1} \cdot \sum A_\nu t_\nu$$

Show that $P(z)$ has $n-1$ simpe zeros $\tau_1, \ldots, \tau_{n-1}$ where $t_1 < \tau_1 < t_2 < \ldots < \tau_{n-1} < t_n$, and there exist positive numbers $B_1, \ldots, B_{n-1}$ such that

$$P(z) = \frac{\mathcal{A}}{a - z - Q(z)} \quad : \mathcal{A} = \sum A_\nu \quad \text{where} \quad Q(z) = \sum_{\nu=1}^{\nu=n-1} \frac{B_\nu}{\tau_\nu - z}$$

Finally, use residue calculus to prove that the B-numbers are given by the equations

$$B_p^{-1} = \mathcal{A}^{-1} \cdot \sum_{\nu=1}^{\nu=n} \frac{A_\nu}{(t_\nu - \tau_p)^2}$$

We discuss two of the most problems whose solutions inspired later development of operator theory.

The Dirichlet problem Let Ω be a bounded domain in $\mathbf{R}^3$ whose boundary is a finite union of closed C^1-surfaces. The Dirichlet problem amounts to find a harmonic extension F of each $f \in C^0(\partial\Omega)$, i.e. here $\Delta(F) = 0$ in Ω while $F = f$ on the boundary. The maximum principle for harmonic functions implies that the Dirichlet problem has a unique solution if it exists. Moreover

$$(*) \qquad \max_{p \in \overline{\Omega}} |F(p)| = \max_{p \in \partial\Omega} |f(p)|$$

It suffices to get solutions for a dense subapace of $C^0(\partial\Omega)$. For if $\{f_n\}$ is a sequence which convege uniformyl to a limit function f and Dirichet's solutions $\{F_n\}$ exists, then (*) entails that this sequence converges uniformly on $\overline{\Omega}$ to a limit function F which solves the Dirichlet probelm for f. To solve the Dirchlet problem for a dense set of boundary value functions one can proceed as follows: Each real-vaued continuous boundary function ϕ gives via Newton's potential a harmonic function in Ω:

$$\Phi(p) = \frac{1}{4\pi} \cdot \int_{\partial\Omega} \frac{\phi(q)}{|p-q|} \, dA(q)$$

where dA is the area measure on the boundary. If ϕ is of class C^1 on the C^1-manifold $\partial\Omega$ it is easily verified that Φ extends continuously to the closure. So if $\phi_* = \Phi|\partial\Omega$ we obtain a continuous function which has Φ as a harmonic extension, i.e. the Dirichlet problem is solved for ϕ_*. Hence one is led to consider the linear operator defined by

$$T(\phi)(p) = \frac{1}{4\pi} \cdot \int_{\partial\Omega} \frac{\phi(q)}{|p-q|} \, dA(q)$$

where p varies on $\partial\Omega$. Since

$$\max_{p \in \partial\Omega} \int_{\partial\Omega} \frac{1}{|p-q|} \, dA(q) < \infty$$

elementary measure theory entails that T is a bounded linear operator on the Banach space $C^0(\partial\Omega)$ and from the above the Dirichlet problem is well-posed if the range of the linear operator T is dense. To prove this density one uses that the dual space of $C^0(\partial\Omega)$ consists of Riesz measures.

Baisc meaasure theory shows that for evety such measure μ the function defined on the boundary by

$$p \mapsto \int_{\partial\Omega} \frac{1}{|p-q|}\, d\mu(q)$$

is integrable in Lebegue's sense with respect to the area measure dA. The requested density of T amounts to show that this L^1-function cannot be zero unless $\mu = 0$. This can be proved in several ways. See § xx for details. Hence the Dirichlet problem can be settled in the spirit of Neumann and Poincaré via operator methods.