Analytic function theory in one complex variable

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Introduction.

The chapters are foremost devoted to analytic function theory of one complex variable. In addition results from real analysis appear. The borderline between complex and real situations is not strict since each profits from the other. An example is the solution of the Dirichlet problem which employs real analysis and is extremely useful during the study of analytic functions. Conversely many problems which from the start are given in a real context are solved via complex function theory.

The level varies in these notes where results of a foundational character appear together with more advanced material. The less prepared reader should first of all study the basic parts and may for example delete advanced studies of series at the end of part B in Chapter 1 and other involved results such as Beurlings conformal mapping theorem in Chapter 6. For the experienced reader material in the chapters gives background to sections in *Special Topics*. An overall comment to these notes is that focus is upon *inequalities* while theoretical concepts require a minor effort to digest.

The reader will recognize that most of the results in these notes are of a relatively old vintage. This is no surprise since analytic function theory is a subject which has been developed for more than two centuries. So my ambition has been to put forward discoveries from the past which I personally find both striking and beautiful, and to this one can add that methods of proofs employed by "old masters in the subject" are of equal or even higher interest than the actual results. A typical result of "old vintage" in these notes is due to Jaques Hadamard. Elementary text-books give the formula for the radius of convergence of a power series $\sum c_n z^n$ expressed by the equality

$$(*) \qquad \frac{1}{\rho} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

and refer to (*) as Hadamard's formula. However, this is a minor tribute compared to the theorems from his thesis *Essai sur l'études des fonctions donnés par leur développement de Taylor* from 1893. Let us recall the major result in [ibid] since it has a wide range of applications. Consider a Taylor series as above whose radius of convergence is a positive number ρ. To every pair $p \geq 1$ and $n \geq 0$ we have the Hankel determinants:

$$\mathcal{D}_n^{(p)} = \det \begin{pmatrix} c_n & c_{n+1} & \dots & c_{n+p} \\ c_{n+1} & c_{n+2} & \dots & c_{n+p+1} \\ \dots & \dots & \dots & \dots \\ c_{n+p} & c_{n+p+1} & \dots & c_{n+2p} \end{pmatrix}$$

To each $p \geq 1$ we set

$$\limsup_{n\to\infty} |\mathcal{D}_n^{(p)}|^{\frac{1}{n}} = \delta(p)$$

From (*) it follows easily that

$$(1) \qquad \delta(p) \leq \rho^{-p-1} \quad \text{holds for every} \quad p \geq 1$$

If strict inequality holds for some positive integer in (1) the following result was proved in [loc.cit]:

Theorem. *Assume there exists a positive integer p and some $\rho_* > \rho$ such that $\delta(q) = \rho^{-q-1}$ when $0 \leq q \leq p-1$ while*

$$\delta(p) = \rho^{-p} \cdot \rho_*^{-1}$$

Then there exists a polynomial $Q(z)$ of degree p such that $Q(z) \cdot f(z)$ extends to an analytic function in the disc $\{|z| < \rho^*\}$.

As we shall see in Chapter I the proof is based upon some ingenious calculations with determinants. A merit in Hadamard's proof is that the coefficients of the Q-polynomial are found via robust limit formulas and explicit linear systems of equations obtained from the given sequence $\{c_n\}$. A special case occurs when:

$$(*) \qquad \lim_{p\to\infty}\big[\limsup_{n\to\infty}|\mathcal{D}_n^{(p)}|^{\frac{1}{n}}\big]^{\frac{1}{p}}=0$$

Then $f(z)$ extends to a meromorphic function defined in the whole complex plane. Using this result, Carleman proved in an article from 1917 that spectral functions associated to an extensive class of integral operators are meromorphic functions defined in the whole complex plane. This consolidates fact about eigenvalues and their associated eigenfunctions. Carleman 's proof is presented in a section from *Special Topics* and demonstrates the fruitful interplay between analytic functions, PDE-theory and probabilistic considerations related to the Brownian motion.

The study of *inequalities* is the major issue in these notes. For students aiming to learn "big theories" this may appear to be a modest and almost trivial issue. But the fact is that more or less all deep results in mathematics rely upon inequalities. Consider as an example the Riemann Hypothesis which is is the utmost challenge among open problems in the world of mathematics. A necessary and sufficient condition for its validity goes as follows: For each real number x we denote by $\{x\}$ the largest integer such that $0\le x-\{x\}<1$. To each positive integer M we get M many functions $\{\rho_j(x)\}$ defined on the open unit interval $(0,1)$ by

$$\rho_j(x)=\frac{j}{Mx}-\{\frac{j}{Mx}\}\quad:\quad 1\le j\le M$$

Set

$$\beta(M)=\min_{c_\bullet}\int_0^1\big(\sum_{j=1}^M c_j\cdot\rho_j(x)-1\big)^2\cdot dx$$

with the minimum taken over all M-tuples of real numbers such that

$$\sum_{j=1}^{j=M} j\cdot c_j=0$$

With these notations Arne Beurling has shown that the Riemann Hypothesis is true if and only if

$$(*) \qquad \lim_{M\to\infty}\beta(M)=0$$

Notice that when M is fixed then the minimum is achieved by a unique M-tuple $c_\bullet$ since we minimize a positive definite quadratic form and theoretically $\beta(M)$ is recovered from Lgrange's multiplier for this minimum problem. However. the ρ-functions become "turbulent" as M increases and mathematics is not yet enough developed to establish bounds of the β-numbers in order to decide if (*) holds or not.

Comments to the Notes

Below follows an extensive discussion which includes material from *Special Topics*. It can be read independently of the chapters and is written in the attempt to inspire readers to study analytic function theory in one complex variable, or in a worse case give some readers less interest of these notes since already the introduction includes formulas and computations where calculus stays in the foreground, rather than more abstract theoretical constructions. Function theory in one complex variable is a central subject in mathematics. Apart from its intrinsic beauty which for example leads to residue theory and Riemann's mapping theorem, it has applications in almost every branch of mathematics ; such as the study of analytic solutions to differential equations or the construction of resolvents of linear operators defined outside their spectra which leads to a powerful operational calculus.

Complex analytic functions are special. For example, an entire function $f(z)$ is uniquely determined by its values in a dense subset of an arbitrary small open interval on the real line. On the other hand the class of entire functions is ample enough to approximate continuous functions on the real x-axis. In section XX from Special Topics we prove that if $g(x)$ is an arbitrary complex valued function on the real line and $\epsilon > 0$, then there exists an entire function $f(z)$ such that $|f(x) - g(x)| < \epsilon$ hold for all real x.

Another result which demonstrates the reward of learning basic facts about analytic functions is a theorem due to Harald Cramér. It asserts that if $\chi_1, \ldots, \chi_N$ is a family of independent random variables whose sum variable $\chi_1 + \ldots + \chi_N$ is normally distributed, then each single χ_ν has a normal distribution. This is a useful fact in statistics. Using the existence of complex log-functions of entire functions without zeros the proof boils down to show that an entire function which only increases like $|z|^2$ is a polynomial of degre ≤ 2.

Complex analysis is also used in linear algebra. Let A be an $n \times n$-matrix with real elements $\{a_{k\nu}\}$. To get zeros of the characteristic polynomial $P_A(\lambda) = \det(\lambda \cdot E_n - A)$ one must include those which are complex. Denote by $\rho(A)$e the maximum of the absolute value of these zeros. Now the resolvent operator $R_A(\lambda) = (\lambda \cdot E_n - A)^{-1}$ is an analytic matrix-valued function defined outside the spectrum $\sigma(A)$. From this it follows that $\rho(A)$ is equal to the spectral radius computed via an an *arbitrary norm* on $\mathbf{R}^n$. This invariance property is for example used to study a matrix A whose elements are positive real numbers where the spectral radius formula is used to prove a theorem due to Perron and Frobenius. It asserts that if A is a matrix with real and positive elements then it has a real eigenvector $\mathbf{x}$ where each $x_k > 0$ and $A(\mathbf{x}) = \rho(A) \cdot \mathbf{x}$. This theorem is a cornerstone in linear programming where one seeks solutions which are constrained to convex sets. Another example where complex analysis is used appears in Thorin's convexity theorem. Let $A = \{a_{\nu k}\}$ be an arbitrary $n \times n$-matrix with complex elements, To each pair of real numbers $0 < a, b < 1$ we set

$$M(a,b) = \max_{x,y} |\sum\sum A_{\nu k} \cdot x_\nu \cdot y_k| \quad : \quad \sum |x_\nu|^{1/a} = \sum |y_k|^{1/b} = 1$$

Via Hadamard's maximum principle for analytic functions defined in strip domains, Thorin proved that the function

$$(a,b) \mapsto \log M(a,b)$$

is convex in the square $0 \leq a, b \leq 1$.

The fundamental theorem of algebra is of course a major motivation why complex numbers are so useful. The variation of the argument for a complex-valued function which is $\neq 0$ along some curve is a central issue in complex analysis. Here is a result which can serve as a first challenge for the reader and will be explained in Chapter 4. Consider a polynomial

$$P(z) = z^{2n+1} + a_{2n}z^{2n} + \ldots a_1 z + a_0$$

with real coefficients and assume that the function

$$P_*(iy) = \sum_{k=0}^{n} a_{2k} \cdot (iy)^{2k}$$

is > 0 for all real y. Then, if n is even it follows that $P(z)$ has n complex zeros counted with multiplicity in the right half-plane $\mathfrak{Re}(z) > 0$, while the number of zeros is $n+1$ if n is odd. For example, if $n = 2$ and

$$P(z) = z^5 + Az^3 + Bz + 1$$

where A, B is any pair of real numbers then P has two roots in $\mathfrak{Re} z > 0$.

An important result which illustrates the usefulness of adopting complex numbers is the *spectral theorem for unbounded self-adjoint operators* on complex Hilbert spaces. The theorem was put forward by Carleman in the note [CR 1920] and the complete proof appears in his book *Sur la théorie des équations intégrales à noyeau reél et symétrique* published by Uppsala university in 1923. We expose Carleman's proof in the appendix devoted to functional analysis and remark that one of the key points during the proof is to move from the real axis to the complex plane in order to construct resolvents of linear operators which from the start are defined in a real context.

Jensen's formula. One can say that modern analytic function theory started with a remarkable discovery by the Danish telephone engineer Jensen. In 1899 he proved that if $f(z)$ is analytic in a disc $|z| \leq R$ where $f(0) \neq 0$, then

$$(*) \qquad \frac{1}{2\pi} \cdot \int_0^{2\pi} \log|f(Re^{i\theta})| \cdot d\theta = \log|f(Re^{i\theta})| + \log \frac{R^n}{|a_1| \cdots |a_n|}$$

where $\{a_k\}$ are the zeros of f in the disc $\{|z| > R$ repeated with their multiplicitites. Today's student learns this theorem with relative ease, i.e. one uses a factorisation to get rid of the zeros by a product of Möbius functions and after the mean value formula for a harmonic functions is applied. But it took almost a century until familarity with complex analysis reached a stage when (*) was announced. Jensen's formula is a cornerstone in analytic function theory and our historic comment illustrates that today's student has to learn a lot of material in order to advance in contemporary mathematics. Another essential result is Herglotz' integral formula which exhibits an analytic function $f(z)$ in the unit disc by its real part. More precisely, let $u(\theta)$ be a continuous function on the unit circle. Then we obtain an analytic function in the unit disc defined by

$$(**) \qquad \mathcal{H}_u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot u(\theta) \cdot d\theta$$

We refer to (**) as the Herglotz extension of u. It is closely related to the given u-function because the real part $\mathfrak{Re}\, \mathcal{H}_u$ is expressed by Poisson's formula

$$\mathfrak{Re}(\mathcal{H}_u)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot u(\theta) \cdot d\theta$$

This means that the real part of $\mathcal{H}_u$ extends to a continuous function on the closed disc and equal to u on the unit circle T. Starting from this construction zero-free functions were analyzed by Jensen and Nevanlinna. Here one takes exponentials and define

$$F_u(z) = e^{\mathcal{H}_u(z)}$$

This yields a zero-free function in D which enable us to construct its complex log-function. On T one has the equality

$$\log|F(e^{i\theta})| = u(\theta)$$

and at the origin we find that

$$\log F(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \cdot d\theta$$

The formulas above can be used to construct analytic functions in D via a limit process. Here follows an example which leads to a theoretically interesting result. Let $\{\omega_\nu\}$ be a sequence of pairwise disjoint open intervals on T whose union is a dense set and the closed complement E has no isolated points, i.e. E is a perfect set. Moreover, we assume that E is a null-set, i.e. its linear Lebesgue measure is zero. Finally, let $\{\ell_\nu\}$ be the lengths of the ω-intervals and assume that

$$(*) \qquad \sum_{\nu=1}^{\infty} \ell_\nu \cdot \log \frac{1}{\ell_\nu} < \infty$$

This metric condition turns out to play a decisive role to study boundary values of analytic functions in the unit disc. Section § from Special Topics will expose constructions due to Lennart Carleson which show that when (* holds, then there exists for every positive integer N an analytic function $f(z)$ in the unit disc which extends to an N times differentiable function on the closed disc and at the same time $f = 0$ on E. Thus, E is contained in null-sets for analytic functions of high regularity. The metric condition (*) is not only is sufficient but also necessary for this to be true. More precisely, Carleson proved that if (*) is divergent and $f(z)$ is analytic in D and extends to be Hölder continuous of some order $\alpha > 0$ then f cannot vanish identically on E, unless it is the trivial zero-function.

Passage to several complex variables. Complex analysis in several variables is more involved compared with the 1-dimensional case. However, much progress has been achieved after pioneering work by Weierstrass and Poincaré. See § xx for some comments. An example of a result in several variables is the extension of Jensen's formula by Deval and Sibony from their joint article [Dev-Sib]. Let $n \geq 2$ and consider a strictly pseudo-convex domain Ω in $\mathbf{C}^n$ with smooth boundary. When $z_0 \in \Omega$ one says that a probability measure μ supported by $\partial\Omega$ is a Poisson-Jensen measure if

$$f(z_0) = \int_{\partial\Omega} f(z) \cdot d\mu(z)$$

hold for all analytic functions f in Ω which extend to be continuous on the closure $\bar{\Omega}$. Given the pair (z_0, μ) one is led to analyze the defect of values taken at z_0 by *pluri-subharmonic* functions $u(z)$ of class C^2 expressed by the difference

$$\mathrm{def}(u) = \int_{|\partial\Omega} u(z) \cdot d\mu(z) - u(z_0)$$

In dimension one this is a central issue where the condition is that u is subharmonic and in Chapter V we shall learn that the defect is expressed by an integral of $\Delta(u)$ times a positive density in Ω derived via the Green's function for the domain. The extension to higher dimension goes as follows: The article [Duval-Sibony] constructs a current T_Ω of bi-degree $(n-1, n-1)$ such that

$$(*) \qquad \mathrm{def}(u) = \int_{\Omega} T_\Omega \wedge \bar{\partial}\partial(u)$$

hold for every pluri-suharmonic function $u$of class C^2 where the right hand side in (*) is the integral of the volume form defined by the exterior product $T_\Omega \wedge \bar{\partial}\partial(u)$. It goes without saying that the construction T_Ω requires more involved methods as compared with the 1-dimensional case.

Logarithmic potentials. A central issue in these notes are logarithmic potentials. Here analytic function theory intervenes with Fourier series and measure theory. A closed subset E of the unit circle T has logarithmic capacity zero if there exists a probability measure μ on E such that

$$\lim_{r \to 1} \int_E \log \frac{1}{|e^{i\theta} - r \cdot e^{i\phi}|} \cdot d\mu(\theta) = +\infty \quad \text{for all points} \quad e^{i\phi} \in E$$

Conversely, E has positive capacity if there exists a probability measure μ on E such that the limit of integrals above are bounded by a constant M_μ for all $e^{i\phi} \in E$. Some general results about logarithmic potentials occur in § xx from special sections while § xx and § xx treat more subtle facts based upon work by Beurling and Carleson. Here one studies the space $\mathcal{D}$ of analytic functions $f(z)$ in the unit disc for which the Dirichlet integral

$$(*) \qquad \frac{1}{\pi} \iint_D |f'(z)|^2 \cdot dxdy < \infty$$

The square root of (*) is called the Dirichlet norm and is denoted by $D(f)$. A result due to Beurling from 1940 asserts that if $f \in \mathcal{D}$ then the radial limits

$$\lim_{r \to 1} f(re^{i\theta})$$

exist for points $e^{i\theta}$ on T outside a set of capacity zero. We prove this in § x in *Special topics* and remark that this is a fine result from a measure theoretic point of view, since the condition for a set to have capacity zero is much more restrictive than to say that the linear Lebesgue measure is zero. Another result by Beurling is concerned with functions $f \in \mathcal{D}$ which yield conformal mappings from D onto simply connected domains. For such a conformal mapping Beurling proved that the radial limits of f cannot vanish on a set of positive capacity. But if f is not requested to be conformal examples show that there exists pairs (f, E) where $\mathrm{Cap}(E) > 0$ and f has radial limit zero at all points in E. This led Carleson to investigate the family of $\mathcal{D}_E$ of functions in $\mathcal{D}$ normalised so that $f(0) = 1$ and with radial limit equal to zero at all points in E, with eventual exception on a subset of zero capacity. In § x from *Special Topics* the reader will learn that when E is null-set with positive capacity and the class $\mathcal{D}_E \neq \emptyset$, then there exists a positive number m_E

such that the Dirichlet norm of every $f \in \mathcal{D}_E$ which is normalised so that $f(0) = 1$ stays above a fixed positive constant. Moreover, there exist a unique f in this class with smallest possible Dirichlet norm. In Section XX we expose material from Carleson's article [Carleson 1952] for this result. The proof is instructive and the reader who can pursue the details with the aid of material from earlier chapters has attained good familarity with analytic function theory.

Remark. We inserted some details above at this early stage in order to illustrate what these notes are striving for. The proofs are not "sophisticated" but contain often a sequence of seemingly unrelated threads which constitute the innovative part. To this one may add that the reader should not expect to meet a "perfectly organised theory". The main lessons consist in studying details of separate proofs and eventually become sufficiently familiar with various methods to be able to make new discoveries.

Riemann's ζ-function.

These notes are foremost directed to students who are familiar with undergraduate Calculus and want to encounter more advanced analysis. While entering such studies it is good to have a goal. I cannot imagine anything more appealing than to learn about Riemann's zeta-function defined by the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \quad \mathfrak{Re}\, s > 1\,.$$

The first result about the zeta-function is due to Euler and asserts that it extends to a meromorphic function defined in the whole complex s-plane with a simple pole at $s = 1$. Moreover, the entire function $(1-s)\zeta(s)$ has simple zeros at all even negative integers while the remaining zeros belong to the strip domain

$$0 < \mathfrak{Re}\, s < 1$$

In a work which dates back to 1859, Bernhard Riemann conjectured that the zeros of the zeta-function in this strip domain all belong to the line $\mathfrak{Re}\, s = \frac{1}{2}$. Numerical calculations confirm the hypothesis up to a very high degree. So it is not likely that one may discover an eventual zero outside the critical line by mere guessing. But there remains to *prove* the hypothesis. The study of the zeta-function has contributed to the development in analytic function theory. An example is the asymptotic formula for the zeros in the critical strip $0 < \mathfrak{Re}(s) < 1$ which was announced by Riemann. It asserts that there exists a constant C_0 such that if $\mathcal{N}(T)$ is the number of zeros counted with multiplicities in the rectangle where $-T < \mathfrak{Im}\, s < T$ and $0 < \mathfrak{Re}\, s < 1$, then

$$(*) \qquad \mathcal{N}(T) = \frac{1}{\pi} \cdot T \cdot \operatorname{Log} T - \frac{1 + \operatorname{Log} 2\pi}{\pi} \cdot T + \rho(T) \cdot \operatorname{Log} T \quad \text{where} \quad |\rho(T)| \leq C_0$$

where the constant C_0 is indepenent of T. We prove (*) in the chapter devoted to Riemann-s ζ-function whose content should be more than sufficient to convince that analytic function theory in one complex variable is as active as ever within the present curriculum of mathematics. Anybody who tries to attack the Riemann Hypothesis must be well aquainted both with function theory and harmonic analysis. This explains why these notes in addition to analytic function theory contain quite extensive material about the Fourier transform. Let us illustrate this interaction by the following formula which at first sight appears to "almost settle" the Riemann hypothesis: When $\mathfrak{Re}\, s > 1$ one can prove the equality:

$$(**) \qquad \zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \cdot \int_0^{\infty} \omega(x) \cdot x^{\frac{s}{2}-1} \cdot dx \quad : \quad \omega(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$$

Here $\Gamma(s)$ is the Gamma-function. The constraint $\mathfrak{Re}\, s > 1$ is needed to ensure that the integral in the right hand side is defined since one has the asymptotic formula

$$\omega(x) \simeq \frac{1}{\sqrt{x}} \quad x \to 0$$

A major fact to be proved in these notes is that $\frac{1}{\Gamma(s)}$ is an entire function of the complex variable s. So (**) can be used to investigate zeros of the zeta-function after its meromorphic continuation is constructed via the Dirichlet integral of the ω-function. The equation (**) was used by G.Hardy in [Ha] to demonstrate that a "considerable amount" of zeros of the zeta-function belong to the critical line $\mathfrak{Re}\, s = 1/2$.

Beurling's criterion. A necessary and sufficient condition for the validity of the Riemann hypothesis was presented by Beurling in the article *A closure problem related to the Riemann zeta-function* and goes as follows: Denote by $\rho(x)$ the numerically smallest remainder term modulo 1, i.e. $\rho(x) = |x|$ if $-1/2 < x < 1/2$ and $\rho(x+1) = \rho(x)$ for all real x. Given some $N \geq 2$ and an N-tuple of real numbers $0 < \theta_\nu < 1$ and real constants $c_1, \ldots, c_N$ such that $\sum c_\nu \cdot \theta_\nu = 0$ we define a function $f(x)$ on the unit interval $(0,1)$ by

$$f(x) = \sum_{\nu=1}^{\nu=N} c_\nu \cdot \rho(\frac{\theta_\nu}{x})$$

Theorem. *The Riemann hypothesis is valid if and only if for each $\epsilon > 0$ there exists some N and a pair of N-tuples $\{\theta_\nu\}$ and $\{c_\nu\}$ as above such that*

$$\int_0^1 (1 - f(x))^2 \cdot dx < \epsilon$$

The sufficieny, i.e. that the existence of f-functions which approximate the identity function in the L^2-norm gives the Riemann Hypothesis is easy to prove, while the proof of necessity relies upon an ingenious blend of functional analysis and analytic function theory. Personally I find Beurling's proof exceedingly instructive and hope that this will be shared by the reader. For details we refer to section xx in Chapter VIII. By continuity Beurling's criterion can be phrased in a simpler form as presented in the introduction to these notes.

Alan Turing and the ζ-function. Original ideas which have led to the creation of computers are foremost due to Turing and Johann von Neumann. Less wellknown is perhaps that Turing also was fascinated by the zeta-function. His article *A method for the calculation of the zeta-function* from 1943 already suggested his ambition to build an analog computer specifically intended for calculating values of $\zeta(s)$. A decade later his article *Some calculations of the Riemann zeta-function* gave the first extensive calculations of zeros of $\zeta(s)$ by a computer. This historic reconcilation illustrates how problems in mathematics has stimulated advancement in applied areas of science. The reader may consult the article by Dennis Hejhal in [Hej] for a further account about Turing's work related to the ζ-function and up-to-date references about the zeta-function.

Real versus complex analysis. In chapter 1 we cite an excerpt from the text-book [Bieb] about the historic development of complex numbers where Bieberbach points out that complex analysis is of a rather recent origin and did not start to develop until the beginning of 1800. One reason for this "delay" might be that ordinary calculus and the extensive use of trigonometric functions were sufficient for most problems prior to 1800. But eventually analytic function theory became an essential tool for problems which initially had a "real character". Let us give such an example. Consider a continuous function $k(x,y)$ on the unit square $\{0 \leq x, y \leq 1\}$. In general k is complex-valued and we do not assume that k is symmetric, i.e. $k(y,x) \neq k(x,y)$ can hold. We get the integral operator $\mathcal{K}$ which sends a function $u \in C^1[0,1]$ into

$$\mathcal{K}(u)(x) = \int_0^1 k(x,s) \cdot u(s) \cdot ds$$

With λ as a complex parameter we consider the operators $T_\lambda = E - \lambda \cdot \mathcal{K}$ where E is the identity operator. If M is the maximum norm of the absolute value of k over the square then it is easily seen that $\{T_\lambda\}$ are invertible linear operators when $|\lambda| < \frac{1}{M}$ and these inverse operators are found by the Neumann series

$$(E - \lambda \cdot \mathcal{K})^{-1} = E + \sum_{m=1}^{\infty} \lambda^m \cdot \mathcal{K}^{(m)}$$

where $\mathcal{K}^{(m)}$ is the integral operator defined by the m-fold convolution of k, i.e starting with $k^{(1)} = k$ one defines inductively

$$k^{(m)}(x,y) = \int_0^1 k^{(m-1)}(x,s) \cdot k(s,y) \cdot ds$$

Now we can study the spectrum of $\mathcal{K}$ where an eigenvalue λ arises if there exists some $u \in C^0[0,1]$ which is not identically zero such that

$$u = \lambda \cdot \mathcal{K}(u)$$

One proves that the set of such λ-numbers is a discrete sequence $\{\lambda_\nu\}$, arranged so that the absolute values $\{|\lambda_\nu|\}$ is non-decreasing. An eigenvalue whose associated eigenspace has dimension ≥ 2 is repeated according to this multiplicity. A result due to Schur asserts that the series

(*) $$\sum |\lambda_\nu|^{-2} < \infty$$

This convergence leads to a description of the entire spectral function $D(\lambda)$. Namely, to every positive integer n we define the function $k^*(s_1, \ldots, s_n)$ on the n-dimensional cube $[0,1]^n$ by the Gram-Schur determinant:

$$k^*(s_1, \ldots, s_n) = \det \begin{pmatrix} k^{(0)}(s_1) & k^1(s_2) & \ldots & k^{(n-1)}(s_n) \\ k^{(1)}(s_2) & k^{(2)}(s_2) & \ldots & k^{(n)}(s_n) \\ \ldots & \ldots & \ldots & \ldots \\ k^{(n-1)}(s_1) & k^{(n)}(s_2) & \ldots & k^{(2n-1)}(s_n) \end{pmatrix}$$

Now the spectral function $D(\lambda) = \sum c_n \cdot \lambda^n$ where

$$c_n = \frac{1}{n!} \cdot \int_{[0,1]^n} k^*(s_1, \ldots, s_n) \cdot ds_1 \cdots ds_n$$

In §§ XX from Special Topics we prove that if $\{\lambda_\nu\}$ is the spectrum of $\mathcal{K}$ then there exists a constant a such that

(*) $$D(\lambda) = e^{a\lambda} \cdot \prod (1 - \frac{\lambda}{\lambda_\nu} \cdot e^{\frac{\lambda}{\lambda_\nu}})$$

Notice that (*) holds even if k is not symmetric. The formula (*) was established by Carleman in the article [Carleman] from 1917. The proof employs analytic function theory where an essential ingredient is a result due to A. Wiman from 1914 about minimum modulos properties of entire functions g with a growth $\leq e^{|\lambda|^\rho}$ for some $\rho \leq 1/2$. The result above illustrates that once analytic function theory intervenes with problems in real analysis one can establish quite striking results.

Kepler's equation. During intense calculations based upon astronomic data in the beginning of 1600, Kepler encountered the equation

(*) $$\zeta_a(z) = a + z \cdot \sin(\zeta_a(z))$$

where $a > 0$ is a positive constant which may vary. At that time analytic function theory did not exist and Kepler could only manage to derive a part of the series expansion of the ζ_a-function. In 1760 Lagrange found the whole series expansion of $z \mapsto \zeta_a(z)$ and proved that $\zeta_a(z)$ is analytic in a disc centered at $z = 0$ whose radius of convergence $\rho(a)$ can be taken independent of a. However, no exact formula for $\rho(a)$ is known, i.e. it can only be found numerically. Personally I find historic examples of this kind both instructive and exciting since they illustrate how "heroic efforts" have preceded a "more developed theory". In these notes the reader will learn how basic analytic function theory can be applied to obtain Lagrange's result about the $\zeta_a(z)$-function.

A problem by Lagrange. Let us give another early example where complex methods appear naturally. In a work from 1782 Lagrange studied the motion of the planets in the solar system which by Newton's Law of gravity amounts to solve the N-body problem with $N \geq 3$. As a first approximation for "la longitude du périhélie de l'orbite d'une planete au temps t" one needs to determine the argument of a complex-valued trigonometric polynomial

$$F(t) = a_1 e^{i\lambda_1 t} + \ldots + a_N e^{i\lambda_N t}$$

where $\lambda_1, \ldots, \lambda_N$ are distinct real numbers and $a_1, \ldots, a_N$ some N-tuple of complex numbers. Assume that $F(t) \neq 0$ for all $t \geq 0$. Then there exists a continuous argument function $\psi(t)$ of F such that:

$$F(t) = |F(t)| \cdot e^{i\psi(t)} \quad : t \geq 0 \,.$$

When $|a_N| > |a_1| + \ldots + |a_{N-1}|$ it is easily seen that

$$(*) \qquad \lim_{T \to \infty} \frac{\psi(T)}{T} = \lambda_N \,.$$

Lagrange posed the question if there exists a limit in the general case when no single a-coefficient has a dominating absolute value. This problem with its source in astronomy has led to interesting results in analytic function theory. In 1908 it was proved by Bohl that (*) always has a limit when $N = 3$. For $N \geq 4$ quite general results were found by Weyl and later work has studied the existence of (*) for the more extensive class of almost periodic analytic functions. Let us also mention that the study of (*) is closely related to find zeros of analytic functions in vertical strip domains. More precisely, given $F(t)$ as above it is the restriction of an entire function $F(\zeta)$ to the imaginary axis in the complex ζ-plane where $\zeta = \sigma + it$. For each real σ it turns out that there exists the limit

$$\phi(\sigma) = \mathrm{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \log |F(\sigma + it)| \cdot dt \,.$$

A result due to B. Jessen and M. Tornehave in their joint article [Je-Torn] from 1945 shows that the ϕ-function is continuous and convex with respect to σ and has an ordinary derivative outside a set which contains at most denumerable many points. Moreover, assuming that ϕ has a derivative at $\sigma = 0$ one has the equality

$$\mathrm{Lim}_{T \to \infty} \frac{\psi(T)}{T} = \phi'(0) \,.$$

Finally the derivatives of the ϕ-function determine the asymptotic number of zeros of $F(z)$. More precisely, for a given pair of real numbers $a < b$ we denote by $N_{a,b}(T)$ the number of zeros of $F(\zeta)$ in the domain $\{\zeta = \sigma + it \,:\, a < \sigma < b \text{ and } 0 < t < T\}$. Then

$$\mathrm{Lim}_{T \to \infty} \frac{N(T)}{T} = \frac{\phi'(b) - \phi'(a)}{2\pi}$$

hold when ϕ has a derivative at a and b. Results of this kind illustrate the rich interplay between analytic function theory and other areas of mathematics.

Conformal mappings. In the 3-dimensional space $\mathbf{R}^3$ with coordinates (x, y, z) we consider a surface S defined by an equation

$$z = g(x, y)$$

where g is a real-valued function which is real-analytic as a function of the two real variables (x, y). One may imagine a portion of such a surface which is defined when (x, y) stays in a small square $\square$ centered at the origin in $\mathbf{R}^2$. Given an arbitrary point $p = (x, y, g(x, y))$ on S we consider a pair of curves γ and ρ passing through p. They intersect at p with some angle denoted by $\langle \gamma, \rho \rangle$. Next, let (ξ, η) be another set of coordinates in $\mathbf{R}^2$ and consider a map

$$\phi\colon \ (x, y, g(x, y)) \mapsto (\xi(x, y), \eta(x, y))$$

Working locally we assume that the map is a diffeomorphism and preserves orientation which means that the Jacobian $\xi_x \cdot \eta_y - \xi_y \cdot \eta_x > 0$. Now we can consider the ρ-images of a pair of curves γ, ρ as above and in the (ξ, η)-plane we regard the angle between these two plane curves. If the angle is preserved for every pair of curves passing through arbitrary points on S one says that ϕ is an *conformal* representation of S onto a plane. The first example of such a conformal map goes back to the astronomer Ptolemy who considered the case when S is a portion of a sphere and found the answer by the *stereographic projection* which has become a useful in analytic function theory. Recall also that Mercator constructed another conformal map in 1568 which has been adopted for the construction of sea-maps. In 1779 Lagrange classified all locally defined conformal maps from a portion of the sphere onto the plane and in 1821 Gauss described the whole family of locally defined conformal maps from an arbitrary surface S onto a planar domain. A turning point which led to the contemporary analytic function theory was discovered by Riemann in his thesis from 1851. He introduced ideas which have been used in later investigations of conformal representation and showed that the problem itself is of fundamental importance for analytic function theory. A major result is that every bounded and simply connected domain in $\mathbf{C}$ is conformal with the open unit disc D. This is called *Riemann's Mapping Theorem.* The first rigorous proof was given in 1865 by Hermann Schwartz. More generally there is a mapping theorem for an arbitrary connected domain Ω in $\mathbf{C}$. For topological reasons we cannot find a conformal map from Ω onto D. But there exist locally conformal maps from Ω onto D expressed by *multi-valued* analytic functions on Ω which are used to obtain a holomorphic function $g(z)$ defined in D such that the complex derivative $g'(z) \neq 0$ for all points and the image $g(D) = \Omega$. This identifies the fundamental group of Ω with a group of Möbius transformations on the unit disc D. Starting from this fact, Poincaré began to investigate general groups of Möbius transformations. He considered to begin with a discontinuous group $\mathcal{F}$ of Möbius transforms on D without a fixed point. For such a group there exists a normal domain U inside D characterized by the property that every orbit under $\mathcal{F}$ intersects U in exactly one point. This result, proved in his article *Théorie des groupes fuchsien* was published in the first volume of Acta Mathematica in 1882. All this may at first glance appear as "classic old stuff". But the truth is that the visions by Riemann and Poincaré continue to serve as an inspiration in contemporary mathematics. To this one should also say that analytic function theory in one variable already had reached a quite advanced level more than a century ago. For example, a result which goes beyond the material in these notes is the study of the differential equation $\Delta(u) = e^u$ where one seeks a solution u defined on a closed Riemann surface with prescribed singularities. This second order differential equation was completeley solved by Poincaré in his article [Poinc] *Les fonctions fuchsiennes et l'équation* $\Delta(u) = e^u$ from 1898.

Of course, many results in these notes are of a more recent origin. An example is an extension of the Riemann Mapping Theorem by Beurling from an article published in 1953. A result which illustrates that more recent theorems also are concerned with "concrete problems" was published in 1968 by Karl Dagerholm who was Beurling's first Ph.d student from Uppsala University in 1938, and thirty years later Dagerholm was finally able to prove the following result which had remained as an open question from his thesis:

Theorem. *Up to multiplication with a real number there exists a unique sequence* $\{x_q\}$ *of real numbers which is not identically zero and solves the infinite system of equations*

$$\sum_{q\neq p} \frac{x_q}{p-q} = 0 \quad : \quad p = 1, 2, \ldots \quad \text{where} \sum_{q=1}^{\infty} \frac{x_q}{q} \text{ is convergent.}$$

As we shall see the proof this result employs many ingredients from analytic function theory.

The Hayman-Wu constant. To illustrate that analytic function theory in one complex variable contains many active research areas we mention the following open problem: Let $\phi \colon D \to \Omega$ be a conformal map from the unit disc onto a simply connected domain Ω. Now $\mathfrak{Im}(\phi)$ is a harmonic function in D whose gradient is everywhere $\neq 0$. As we shall explain later on this implies that the zero set $\{\mathfrak{Im}(\phi) = 0\}$ is a union of disjoint open Jordan arcs $\{\gamma_k\}$ where each single arc has two end-points on the unit circle ∂D. Denote by $\ell(\gamma_k)$ the arc-length of each individual Jordan arc. In 1981 it was proved by Hayman and Wu that there exists an absolute constant C such that

$$(*) \qquad \sum \ell(\gamma_k) \leq C$$

The remarkable fact is that C is independent of the conformal map. To find the best constant in (*) is an open problem. So far one knows that C is strictly smaller than 4π [Rohde 2002] and at least π^2 [Öyma 1993].

Remark. The text-book [GM] *Harmonic measure* by John B. Garnett and Donald E. Marshall has been an inspiration for these notes. Actually I started to write these notes to provide background for the reader who wants to study more advanced topics in [GM]. But as in most projects of this sort my original material eventually expanded and cover subjects in analytic function theory which are not within the main streamline in [GM]. In any case, this text-book is highly recommended. For example, Chapter V offers a brilliant account of results due to Lars Ahlfors and contains also an introduction to the theory of extremal distances which was created by Arne Beurling during the years 1943-44 and presented in his article [Beurling] from 1946. An example of a result which goes beyond the scope of these notes appears in [GM; Chapter V : Theorem 2.1] which gives a quite sharp lower bound for harmonic measures. See also Chapter XX in R. Nevanlinna's text-book [Nev] where the Carleman-Milloux problem is studied via certain solutions which were found independently by Nevanlinna and Beurling in 1933. In these notes we are content to establish results which provide upper bounds for harmonic measures in various situations.

Mathematical physics.

Analytic function theory of one complex variable is often used in mathematical physics. An example is the *Biot-Savart Law* which was discovered in 1800 and inspired the development of analytic function theory. In 1879 the electric engineer Robin posed a quite general problem which led to the study of thin sets and various potential theoretic questions. The special case occurs when Γ is a closed Jordan curve in the complex z-plane. To determine a density function which gives electric equilibrium on Γ amounts to find a positive function μ on Γ such that the logarithmic potential

$$(*) \qquad \int_\Gamma \log\frac{1}{|q-p|}\cdot\mu(p)\cdot ds(p) \quad : \quad q\in\Gamma$$

where $ds(p)$ is the arc-length along Γ and one requirs that (*) is constant as q varies on Γ. Bernard Riemann demonstrated the existence of such a density by his conformal mapping theorem. More precisely, the equilibrium density is unique and equal to a constant times $\frac{1}{|f'(z)|}$ where $f'(z)$ is the complex derivative of the conformal map from the *exterior domain* bordered by Γ to the exterior disc $|w|>1$ in the complex w-plane. We prove this result in the chapter devoted to conformal mappings. For the beginner this example is instructive since it shows that one should not only work in the finite complex plane but also be aware of the point at infinity. This was the reason why Riemann introduced the Riemann sphere in his fundamental work [Rie:xx] from 1857. One should also point out that analytic function theory does not "cover everything". Let us give an example. A classical result due to Helmholtz about stationary fluid motion in the plane asserts that if Ω is a Jordan domain and the function

$$z\mapsto \iint_\Omega \log\frac{1}{|z-\zeta|}\cdot d\xi d\eta$$

is constant on $\partial\Omega$, then Ω must be a disc. This result has a natural physical explanation. Less obvious is that this result, i.e. that Ω must be a disc, remains valid when the integrand is replaced by $f(|z-\zeta|)$ for an arbitrary continuous function $f(r)$ defined when $r>0$ under the sole assumption that it is *strictly decreasing*. To prove this one employs the extended symmetrization process due to W. Gross of the ordinary Steiner's symmetrisation which only applies to convex domains. So here real analysis is needed rather than complex function theory.

Many problems can be handled using series expansions and various majorant series which therefore resembles analytic function theory. Let us give an example where this "philosophy" applies. Consider a bounded open domain Ω in $\mathbf{R}^3$ whose boundary $\partial\Omega$ is of class C^1. Let $p=(x,y,z)$ denote points in $\mathbf{R}^3$. The equation which obeys the Stefan-Bolzmann law for heat conduction of

Black Bodies is to find a function u which is harmonic in Ω outside a point p_* (the source of heat) where $u(p)$ is locally of the form $\frac{1}{|p-p_*|}$ plus a harmonic function. On the boundary its interior normal derivative satisfies:

$$(*) \qquad \partial u/\partial \mathbf{n} = k \cdot u^4 \quad : \ k > 0$$

Green's formula shows that the solution to (*) is unique if it exists. The proof of existence can be achieved by regarding intermediate equations where $u_h(p)$ for every $0 < h < 1$ is a solution with the pole at p_* as above and

$$(**) \qquad \partial u_h/\partial \mathbf{n} = k \cdot ((1-h)u_h + h \cdot u_h^4)$$

Using the robust properties of solutions to this linear boundary value problem one can show that if $0 \leq h_0 < 1$ is given and the unique solution u_{h_0} has been found, then (**) is solved for h-values in some interval $h_0 < h < h_0 + \delta$ via a series expansion $u_h = u_{h_0} + \sum_{\nu=1}^{\infty} (h - h_0)^\nu \cdot w_\nu$. Here $\{w_\nu\}$ are functions which are found inductively by solving a system of *linear* boundary value problems. In this way the main burden to obtain a solution to the non-linear Stefan-Boltzmann equation is to prove that various series admit a positive radius of convergence. In XX we expose Carleman's methods from [Car] to solve the non-linear boundary value problem where one seeks u as above in Ω while

$$(***) \qquad \partial u/\partial \mathbf{n}(p) = F(u(p), p) \quad : p \in \partial\Omega$$

Here the sole condition is that $F(u,p)$ is a on-negative continuous function defined on $\mathbf{R}^+ \times \partial\Omega$, and for each fixed $p \in \partial\Omega$ the map $u \mapsto F(u,p)$ is increasing and tends to $+\infty$ with u.

Quantum mechanics. Analytic function theory as well as extensive use of Fourier integral operators occur also in quantum mechanics while one investigates special equations. For example, confluent hypergeometric functions appear as solutions in a Coulomb field and to continue the study of scattering in a Coulomb field where the quantum-mechanical collision problem can be solved exactly, it is from a physical point of view of interest to describe the asymptotic behaviour of confluent hypergeometric functions. Such precise asymptotic expansions are derived via complex line integrals where attention must be given to the choice of local branches of certain multi-valued analytic functions. Examples of this kind illustrate that even quite classical topics in analytic function theory remain "up-to date" since they provide useful tools in quantum mechancs. It would lead us to far to even try to give a glimpse of the physical theories. For the mathematically oriented reader we recommend the outstanding text-books by L.D. Landau and his former student E.M Lifschitz. The third edition of *Non-relativistic quantum mechanics* is especially recommended where an excellent English translation also is available.

The use of computers.

Today's student is confronted with another world than in the past since computers offer numerical solutions and provide figures which can be illustrated in a mobile way. This is a veritable revolution and one may even ask if a traditional text-book in mathematics is out-dated. Hopefully the answer is a compromise, i.e. pure theory should help to create more accurate computer programs and conversely computers can contribute to new theoretical discoveries. Analytic function theory has inspired the creation of computer programs. An example is the text-book [D-T] about the Schwarz-Christoffel Mapping by T. Driscoll and L. Trefthen. In these notes we give formulas for such maps but this is of course not the whole story. The fact that the function defined by

$$f(z) = \int_0^z \frac{d\zeta}{\sqrt{\zeta^4 - 1}}$$

maps the unit disc to a square with corners at $1, i, -1, -i$ does not give insight about the mapping at the interior of the unit disc. Thanks to computers one gets a picture showing the image curves γ_r of circles $|z| = r$ as $0 < r < 1$. It is instructive to see how these simple closed curves for small r are almost circular while they gradually become less circular and get four "corner points" as

$r \to 1$. Numerical calculations and figures for level sets of Green's functions in multiple connected sets is even more exciting. For such complicated situations the theoretical results are rather implicit so computers are helpful to get a good feeling of the theoretical results. Let us give an example whose theoretical result is due to Rolf Nevanlinna. Let Ω be a connected open domain in $\mathbf{C}$ bordered by p many disjoint and closed Jordan curves $\gamma_1, \ldots, \gamma_p$. On each curve γ_i we pick some finite set of open intervals $\{J_{1,i}, \ldots, J_{\nu_i,i}\}$. There exists the unique harmonic function $\omega(z)$ in Ω which on the boundary is equal to one on the union of the open intervals and zero on the rest of $\partial\Omega$. If $0 < \lambda < 1$ we get the level set $S_\lambda = \{\omega = \lambda\}$. It appears as a finite union of arcs inside Ω where some ramification points may occur. Nevannlinna proved that if the number of removed intervals is k and the number of bordering closed Jordan curves is p, then the number of ramification points which appears on the family of all level curves as $0 < \lambda < 1$ is equal to $k + p - 2$. We will prove this in XX.

To this one can add that numerical solutions to more involved problems nowadays can be handled by computers. Let us describe a problem whose theoretical solution appears to be more or less hopeless to attain, while computers give numerical solution. For simplicity we pose the problem in dimension 2. Let Ω be a multiply connected domain in $\mathbf{C}$ with a C^1-boundary. Pick some arc γ on $\partial\Omega$. If $p_* \in \Omega$ we start the standard Browninan motion at time zero at p_*. The probability to hit a point in γ on the first arrival of the path to $\partial\Omega$ is the value of the harmonic measure function ω_γ at p_*. Here the time of arrival is not considered. So a more involved problem is to find the joint probability distribution expressed by a frequency function $f(p,t)$ expressing the probability to hit a point $p \in \gamma$ at time t within the restricted family of Brownian paths which have survived up to time t, i.e. those paths which stay in Ω for time values $< t$. Next, one may imagine a "living object" where the arc γ may changes. A situation could be as follows: Let $\lambda > 0$ and suppose that a random variable which is independent of the Brownian motion can close γ at a time t via Poisson's distribution, i.e. the probability that γ stays open until time t is $e^{-\lambda t}$. Let us also suppose that if γ becomes close at a certain time t_* then it stays closed during a time interval T and after time $t_* + T$ it stays open for good. Now Brownian paths which starts at p_* when $t = 0$ may hit γ during a time when γ is closed. So these paths are lost and there remains to determine the probability for a Brownian path to hit γ before it is has been absorbed on $\partial\Omega \setminus \gamma$, or on γ while this interval happens to be closed. In problems of this kind the theoretical base is to determine upper bounds for statistical confidence when a Monte Carlo box is used to simulate Browninan paths in a discrete grid. So many new areas where numerical analysis is combined with pure theory are open for future research.

Remark about the contents in these notes.

I have tried to make the notes reasonably self-contained. For example, a detailed proof of *Stokes Theorem* is given together with the formulas of Green and Gauss. Here attention is given to the notion of differential forms. With $z = x + iy$ we identify the complex z-plane with $\mathbf{R}^2$ and consider differential 1-forms $W = f \cdot dx + g \cdot dy$ where f and g is a pair real-valued C^1-functions. The 1-form W is closed when

$$(*) \qquad dW = (-f'_y + g'_x) \cdot dx \wedge dy = 0$$

In Chapter II we establish a quite general result expressing the difference of two line integrals of a 1-form W which is not necessarily closed, taken over a pair of curves Γ_1 and Γ_0 with common end-points which are linked by a continuous family of similar curves. The formula which appears in Theorem 5.7 from Chapter II is a result in real 2-variable analysis which superseeds the subsequent study of complex line integrals where the integrands are complex analytic functions. So the reader should be aware of the fact that ordinary Calculus provide us with sufficient tools to attain integral formulas in analytic function theory. However, we shall prove results in the complex analytic set-up for its own sake. One reason for doing this extra job is that complex line integrals are nicely expressed by taking complex Riemann sums. Another is the special flavour which arises from complex arguments. A typical case occurs in Section 2 from Chapter 4 where the argument principle is used to count zeros of an analytic function via its complex logarithmic derivative.

Other background material includes linear algebra with special attention given to complex matrices. In the Appendix devoted to complex vector spaces we prove the existence of Jordan's normal form for matrices since this result is needed to describe multi-valued analytic functions in punctured discs.

Fredholm determinants. Determinants are very useful and become particularly interesting when one moves from finite-dimensional situation to a more analytic context. An example are the Fredholm determinants. Let us describe the simplest case where $f(x, y)$ is a given continuous function on the closed unit square $[0, 1]^2$. To each $p \geq 1$ we obtain a function defined in $[0, 1]^p$ by

$$\mathcal{F}_p(s_1, \ldots, s_p)\text{det}XXX$$

If λ is a complex variable the associated Fedholm determinant function is defined by

$$\mathcal{D}_f(\lambda) = 1 + \frac{\lambda^p}{p\,!} \cdot \sum_{p=1}^{\infty} \int_{[0,1]^p} \mathcal{F}_p(s_1, \ldots, s_p) \cdot ds_1 \cdots ds_p$$

It turns out that this series converges for all complex λ and a more precise description of this entire function will be given in section §§§ from special topic. Both the construction and the proof that $\mathcal{D}_f(\lambda)$ is an entire function illustrate a typical interaction between linear algebra and analytic function theory.

One appendix treats Lebesgue's measure theory and the more extensive class of Riesz measures. This appendix contains a proof of Friedrich Riesz' famous result from 1907 which asserts that a non-decreasing and continuous function on the real line has a classical derivative outside a null-set. This is a good example of high standard from classical analysis where no "general theory" is able to improve the original proof. Let us also mention that Riesz' result was used by Fatou in his article [Fat] from 19xx when he proved that if $f(z)$ is a bounded analytic function in the unit disc, then the radial limit $\lim_{r \to 1} f(re^{i\theta})$ exists for almost all θ. Of course this is nowadays a consequence of far more general results. But it is interesting to turn back to the original proof which uses the primitive function F of f. Here F has an extension to the closed disc and Fatou's proved that if the function $F(e^{i\theta})$ of the angular variable has an ordinary derivative at a point θ_0, then the radial limit for f exists at this θ-value.

One should also be aware of the fact that results which are useful in analytic function theory sometimes are derived via "ordinary calculus". An example is a convexity theorem from 1906 by Jensen which goes as follows: Let $\{\alpha_\nu\}$ be a decreasing sequence of positive real numbers with $\alpha_1 \leq 1$ and let $\{p_\nu\}$ another sequence of positive numbers. Assume that there exists some $\delta > 0$ such that the series $\Phi(r) = \sum p_\nu \cdot r^{\alpha_\nu}$ converges when $0 < r < \delta$. Then Jensen proved that $\log \Phi(r)$ is a convex function of $\log r$. This can be used to prove convexity theorems expressed by integrals connected to analytic functions and more generally to subharmonic functions. For example, Jensen's result was used when Hardy in his work *The Mean Value of the Modulus of an Analytic function* from 1915 proved that the function $r \mapsto \int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta$ is a convex function of $\log r$ when $f(z)$ is analytic in some disc centered at the origin.

Series. Since analytic functions are locally represented by convergent power series the study of series is an imporant issue. Starting from basic facts treated in an elementary section about series we proceed to prove results of a more involved nature in XXX. An example is the Tauberian Theorem by Hardy and Littlewood which asserts that if $\{a_n\}$ is a sequence of positive real numbers such that there exists the limit

$$\lim_{x \to 1} (1 - x) \cdot \sum_{n=1}^{\infty} a_n \cdot x^n = A$$

then there exists the mean-value limit

$$\lim_{n\to\infty}\frac{a_1+\ldots+a_n}{n}=A$$

It is instructive to follow the single steps in the proof which are put together in an ingenious fashion. One might say that the theorem above illustrates what analysis is all about, i.e. *to put small seemingly unrelated threads together.* The proof by Hardy-Littlewood is "elementary" in the sense that no functional analysis is used. Actually the proof cannot be derived from general principles in functional analysis since no a priori bounds occur for the sequence $\{a_n\}$. On the other hand, results in functional analysis are used to prove *Ikehara's Theorem* which asserts that if μ is a non-negative Riesz measure on $[1,+\infty)$ such that the integrals

$$\int_1^\infty x^{-\delta}\cdot d\mu(x)<\infty\quad\text{for all}\quad \delta>1$$

then there exists the limit

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x d\mu(t)=A$$

under the assumption that there exists a continuous function $G(u)$ such that

$$\lim_{\epsilon\to 0}\Big[\int_0^\infty x^{iu-(1+\epsilon)}\cdot d\mu(x)-\frac{A}{\epsilon+iu}\Big]=G(u)$$

holds uniformly over all bounded intervals $-b\leq u\leq b$. The proof uses results about Wiener-Beurling algebras in XXX. Let us finish this discussion with a result proved by Volterra in 1913. He considered an L^1-function $K(x)$ supported by $[0,+\infty)$ and defined the function

$$\Phi(x)=\sum_{n=1}^\infty(-1)^n\cdot K^n(x)\quad K^n(x)\text{ is the n-fold convolution of }\ K(x)\,.$$

The question arises when $\Phi(x)$ also belongs to $L^1(\mathbf{R}^+)$. To settle this we consider the Laplace transform

$$\hat{K}(w)=\int_0^\infty e^{-wx}\cdot K(x)\cdot dx\,,\quad \mathfrak{Re}\,w\geq 0\,.$$

Volterra proved that Φ is integrable if and only if

$$\frac{\hat{K}(w)}{1+\hat{K}(w)}\neq 0\quad\text{for all}\quad \mathfrak{Re}\,w\geq 0\,.$$

In 1913 the proof was quite involved. Today, with the aid of general results about Banach algebras Volterra's result is an easy exercise which we explain in the Chapter XX. So this example illustrates that it is profitable to learn abstract theories.

The use of geometry. Many results in complex analysis are proved by geometric considerations. An example is *Poisson's formula* expressing the values taken by a harmonic function in the unit disc from its boundary values. Actually it was one of the great pioneers in function theory, namely Hermann Schwarz, who established this formula in the context of analytic function theory. Even more important is that his geometric description of the Poisson kernel leads to various invariance principles for the spherical, respectively the hyperbolic metric. An example where geometric considerations are used in analytic function theory is a result due to Julia and Caratheodory which goes as follows:

Let $f(z)=u(x,y)+iv(x,y)$ be an analytic function in the right half-plane $U=\mathfrak{Re}(z)>0$ where the real part u is a positive function and satisfies:

$$\inf_{x+iy\in U}\frac{u(x+iy)}{x}=0$$

Then it follows that

$$\lim_{x\to+\infty}\frac{f(x+iy)}{x}=0\quad:\text{holds uniformly every sector }\{y\leq N\cdot x\}\quad: N=1,2,\ldots$$

This is a remarkable result since no assumption is imposed on the imaginary part of f. As we shall see in XX the proof of the Julia-Caratheodory theorem relies upon a general principle for the spherical measure on domains in $\mathbf{C}$. The result above was used when Julia studied iterated maps which has led to Julia sets and fractals. This subject has nowadays become "fashion" since many phenomena can be illustrated by computers. Deeper results arise when geometric ideas and hard analysis are put together. An example is the notion *Carleson measures* which is essential in the proofs of two famous theorems due to Lennart Carleson: *The Corona problem and the interpolation by bounded analytic functions.* These results are exposed in a section from *Special Topics* which may be regarded as the "high-light" of these notes.

Measure theory. It plays a central role in analytic function theory. Conversely problems from complex analysis has initiated results in "pure measure theory". An example comes from the study of Hardy spaces which appear in many places, foremost in Section 2 and 3 from Special Topics where we for example prove a remarkable result due to C. Fefferman and E.Stein which assets that if $F(\theta)$ a measurable function on the unit circle has bounded mean oscillation if and only if the squared length of the gradient vector $\nabla(H_F) = (\partial H_F/\partial x, \partial H_F/\partial y$ of its harmonic extension $H_F(z))$ to the unit disc yields a Carleson measure in the disc $|z| < 1$, in the sense that there exists a constant C such that

$$\frac{1}{h} \cdot \iint_{S_h(\theta)} |z| \cdot \log \frac{1}{|z|} \cdot \big|\nabla(H_f)\big|^2 \cdot dxdy \leq C$$

hold for every $0 < h < 1/2$ where $S_h(\theta)$ is the subset of D defined by

$$S_h(\theta) = \{z = re^{i\phi} \quad : 1 - h < r < 1 \quad : |\phi - \theta| < h\}$$

This result illustrates how geometric ideas together with analytic estimates which often rely upon Green's formula are used to attain deep results where measure theoretic concepts are used to express the results.

A result which tells about the distinction between absolutely continuous, respectively singular measures on the unit circle T arises via a closure theorem due to Beurling. Consider a non-negative measure μ on T which gives a bounded analytic function $g(z)$ in the unit disc D defined by

$$g(z) = \exp\big[-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \cdot d\mu(t)\big]$$

The g -function is quite special. It has no zeros in D and $g(0) = e^{-a}$ where $a = \frac{1}{2\pi}\int_0^{2\pi} d\mu(t)$ is a positive constant. At the same time $\log|g|$ is the Poisson integral of $-\mu$ and by a result which goes back to F. Riesz and Fatou this implies that the radial limit

$$\lim_{r \to 1} |g(re^{i\theta})| = 1$$

exists for almost every θ in the sense of Lebesgue. Next, let $H^2(T)$ be the Hardy space whose elements are square integrable functions on T which are boundary values of analytic functions in D. This is a Hilbert spaces where the exponential functions $\{e^{in\theta} \colon n = 0, 1, 2, \ldots\}$ is an orthonormal basis. One might expect that multiplication with g_μ yields a linear operator on $H^2(T)$ whose range is dense. But this turns out to be false. In fact, $g_\mu \cdot H^2(T)$ is a closed proper subspace of $H^2(T)$ as soon as the singular part of the measure μ is $\neq 0$. On the other hand, if $\mu = \phi(t) \cdot dt$ is absolutely continuos, i.e when $\phi(t)$ is an L^1-function on T, then $g_\mu \cdot H^2(T)$ is a dense subspace of $H^2(T)$. The proof appears in Section XX and gives an instructive lesson since one employs a mixture of measure theory, functional analysis and analytic function theory.

A general remark. These notes do not strive to present the theory in its most perfect form and I often try to give a historic perspective, perhaps a bit too affected by my own experience as undergraduate student at Stockholm University in the early 60:s when I was taught function theory by Otto Frostman in a way I appreciated very much.

Several results in these notes are proved in a "pedestrian manner" rather than using the full force of concepts from functional analysis. This is not just a matter of a "historic reconcilation" since many results are better understood if one finds a constructive proof rather than to rely upon the axiom of choice via the Hahn Banach theorem or the existence of maximal ideals which appear in commutative Banach algebras. A typical case is when we start from a function $f(x) \in L^1(\mathbf{R})$ and regard its Fourier transform $\hat{f}(\xi)$. Suppose that $a \leq \xi \leq b$ is an interval on the real ξ-line and let $\Phi(z)$ be an analytic function defined in an open neighborhood of the compact image set $\hat{f}([a,b])$. Then there exists $g(x) \in L^1(\mathbf{R})$ whose Fourier transform $\hat{g}(\xi)$ is equal to $\Phi(\hat{f}(\xi))$ when $a \leq \xi \leq b$. The mere existence of g follows easily by an abstract reasoning. However, one can construct g in a rather explicit manner using the local power series expansions of the analytic Φ-function. We shall give this "pedestrain proof" of existence which has the merit that it is constructive. Of course, this proof is more technical and requires more effort as compared to the "abstract proof". But I think it teaches the student more. So we have included details of such a constructive proof based upon a lecture by Carleman at Institute of Mittag Leffler in 1935.

The reader of these notes will recognize that repetitions of similar arguments appear in slightly different contexts. A typical case occurs in studies of various maximal functions. It would be possible to concentrate all this to some few general results which can be established in a general context where one regards Riesz measures on metric spaces whose distribution of mass satisfies certain conditions with respect to the given metric. This would lead to shorter proofs but often with less control of a prior constants which appear. In these notes we prefer to repeat similar arguments in various specific situations and refer to the Appendix on measure theory for an account about general maximal functions taken from the elegant article [Smith] which becomes useful when analytic function theory in one complex variable is replaced by studies of harmonic and subharmonic functions in $\mathbf{R}^d$ for $d \geq 3$.

A (nasty ?) remark. Personally I do not understand the contemporary trend in the teaching of mathematics where subjects tend to be isolated from each other into small separate courses devoted to measures, function theory, Fourier series and so on. Not to mention the *crazy borderline* between algebra and analysis. To give an example. In the past Eisenstein's famous theorem from 1852 was demonstrated to most students entering graduate studies. The elegant proof of Heine from 1854 (presented in Chapter XX) combines algebraic facts about irreducible polynomials in two variables and algebraic functions with analytic function theory. Eisenstein's theorem says that if the power series expansion $\sum c_\nu(z-z_0)^\nu$ of a regular local branch of an algebraic function is such that every c_ν is a rational number, then there exists a positive integer k so that $k^\nu \cdot c_\nu \in \mathbf{Z}$ for all ν. My question to contemporary teaching is whether any course in mathematics for beginning graduate students is suited to present Eisenstein's theorem. A result of a similar nature due to Fatou in [Fat] asserts the following:

Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function with no poles in the unit disc $|z| < 1$ and assume that its Taylor series $\sum c_\nu \cdot z^\nu$ at the origin has integer coefficients. Then every pole is a root of unity, i.e. equal to $e^{\frac{2\pi i}{m}}$ for some integer m.

Again it is a matter of taste if this result belongs to algebra or analytic function theory.

Acknowledgement.

The reader will recognize that numerous results in these notes are due to Torsten Carleman based upon material from articles in his collected work [xx] and the books [Car.1,2,3]. This reflects my personal taste since I regard Carleman as one of the most prominent mathematicians ever in analysis. The chapter devoted to functional analysis contains Carleman's theorem from 1923 of the spectral resolution for unbounded self-adjoint operators on a Hilbert space which is a very important result with a wide range of applications. Concerning *harmonic measures* it was Carleman who first recognized its power to study limits and growth properties of harmonic or analytic functions. See page xx in Nevanlinna's text-book [Nevanlinna] for comments about *Carleman's principle for extensions of domains.* A first lesson how to use harmonic majorisation is Carleman's proof of a result originally due to Ernst Lindelöf. It asserts that if $f(z)$ is a bounded analytic function in a half-disc $D_+ = \{|z| < 1 \cap \mathfrak{Re}(z) > 0\}$ and if there exists a Jordan curve γ

which except for one end-point at the origin belongs to D_+ such that $\lim_{z\to 0}\, f(z) = 0$ holds along γ, then $f(z)$ tends uniformly to zero in every Fatou sector $\{|z| < 1 \cap \{-a < \arg(z) < a\}$ where $0 < a < \pi/2$. The beginner should look at the proof in Chapter XX to get a first glimpse of the usefulness to introduce the subharmonic function $\log |f|$ and employ the harmonic measure.

Several sections also expose work due to Arne Beurling which not only offer elegant results but have the merit that the proofs are instructive. Many of Beurling's original ideas have led to new areas in analytic function theory, harmonic analysis and potential theory. We have not tried to include the deepest results by Beurling but refer to the two volumes of his Collected Works whose foreword includes a very informative account about his major contributions written by Lars Ahlfors and Lennart Carleson. However, several sections from his collected work, including results from his seminars held at Uppsala University during the years 1937-1954 are presented in these notes.

Much inspiration in these notes, including sources for historic accounts, come from Ludwig Bieberbach's text-books [Bi:1,2] which cover analytic function theory up to 1925. In several sections I borrow proofs from these text-books where my presentation is only slightly different. An example is the Uniformisation Theorem for multiply connected domains in **C** where I have followed Bieberbach's presentation of the elegant proof due to Caratheodory. The text book [Cartan] by Henri Cartan has inspired the way I have organised the chapters devoted to analytic functions. Here the reader may find exercises which give very good illustrations to theoretical results. The book [Ko] by Paul Koosis has also been a valuable source. At several places material from Koosis' original text are presented with minor changes, such as in the proof of Donald Marshall's convexity theorem for inner functions in the unit disc. I have also profited upon material from the two volumes about the logarithmic potential by Koosis which is a good reference for further studies in analytic function theory. Material devoted to Runge's Theorem is inspired by Raghavan Narasimhan's text-book [Na] where the reader also may find a nice introduction to Riemann surfaces. Basic facts about the harmonic measure and the Lindelöf-Pick principle is inspired by R. Nevanlinna's book [xx]. E. Landau's book [xx] has contributed to sections devoted to series. Topics from the book [PE] by Raymond Paley and Norbert Wiener also appear. Apart from the text-books above there also occur some problems taken from [Po-Sz] by Polya and Szegö. An example is a result due to Siegel which gives an upper bound for products of roots to polynomial in any degree expressed by the size of its coefficients. Siegel's proof gives a good lesson since it teaches how complex analysis can be used in an unexpected fashion.

In addition to the references above I have been inspired by the over-all presentation in the text-book [Kr] by Steven G. Krantz. Here the reader will find a number of results about conformal maps of multiply connected domains and their associated automorphism groups together with a study of the Bergman kernel and constructions of various metrics. The reader may also consult the text-boks by John Conway, especially part II which contains material of considerable interest. For example a chapter is devoted to De Brange's proof of the Bieberbach conjecture which for more than half a century was an open problem in a analytic function theory. Chapters devoted to harmonic and subharmonic functions also merit attention such the exposition of the Fine Topology in **C** which by definition is the weakest topology making every subharmonic function with values in the extended interval $-\infty, \infty]$ into continuous functions. Another recommended text-book is [Andersson] which presents elegant proofs of advanced results such as Carleson's Corona theorem.

Finally I must mention the text-books [1-2] by Lars Ahlfors. The masterful presentation of basic complex analysis in [1] is recommended to everybody who enters the study in complex analysis. Since [1] was a veritable "bible" for me as a graduate student, it is clear that the text by Ahlfors has inspired much material in these notes, foremost in sections devoted to the foundations in analytic function theory. Concerning the more advanced book [Ahl-2] there occur references in these notes to the general Uniformisation Theorem. But I have not tried to pursue this in detail. One reason is that this book does not treat potential theoretic methods. But hopefully these notes may inspire readers to study [Ahl:2]. One must also mention Beurling's theory of extremal length and the deep studies of moduli problems in articles such as [Ahl-Beu] where one seeks invariants to decide when two multiple connected domains are conformally equivalent. Beurling's article [Beu:xx] from 1946 opened a new era in analytic function theory whose content goes beyond these notes, and from the work by Ahlfors one should mention his far reaching studies of quasi-conformal maps. His article [Ah. Acta 1935] is even today an inspiration for contemporary research. The lyric comments when Caratheodory presented this work at the IMU-congress in 1936 where Ahlfors was the first mathematician to receive the first Fields medal is a recommended reading. Twenty years later Ahlfors discovered an explicit analytic solution to the Beltrami equation and soon after, in collaboration with Lipman Bers, quasiconformal variations were applied to study Kleinian groups. Among more recent topics of advanced nature we mention complex dynamics where Sullivan's theorem about Julia sets is an example. A proof of this result appears in the text-book [Ca-Ga] by L. Carleson and J. Garnett. Finally, we already mentioned the text-book [Ga-Ma] by J. Garnett and Marshall devoted to recent discoveries about harmonic measure and extremal length.

A final comment.

My personal opinion while entering graduate studies in mathematics is that it is rewarding to pursue specific proofs of a more involved nature, rather than to "swallow a mass of general concepts". My favourite example is due to Carleman. In the article [Ca] it is proved that there exists an absolute constant $\mathcal{C}$ with the following property: *For every pair (f,n) where n is a positive integer and f a non-negative real-valued C^∞- functions defined on the closed unit interval* $[0,1]$ *whose derivatives up to order n vanish at the two end points, one has the inequality*

$$(*)\qquad \sum_{\nu=1}^{\nu=n} \frac{1}{[\beta_\nu]^{\frac{1}{\nu}}} \leq \mathcal{C}\cdot\int_0^1 f(x)dx \quad : \quad \beta_\nu = \sqrt{\int_0^1 [f^{(\nu)}(x)]^2 \cdot dx}$$

The remarkable fact is of course that $\mathcal{C}$ is independent of n. The result is sharp in the sense that there exists a constant C_* such that one for every $n \geq 2$ can find functions $f_n(x)$ as above so that the opposed inequality (*) holds with C_*. Hence Carleman's inequality demonstrates that the standard cut-off functions which in many applications are used to keep maximum norms of derivatives small up to order n small are optimal up to a constant. So this theoretical result has an obvious role in more applied areas of mathematics. The proof employs a Laplace transform of f and after certain majorants together with estimates for harmonic measures.

An open problem. Let me finish this introduction by describing an open problem in the spirit of these notes. Let $n \geq 2$ and consider a polynomial

$$f(z) = a_0 + a_1\cdot x + \ldots + a_n \cdot z^n$$

where the sole assumption is that a_0 and a_n both are $\neq 0$. The coefficients are in general complex numbers. To each $0 < \theta < \pi$ we count the number of zeros in the sector of the complex z-plane whose points have argument in the interval $(-\theta,\theta)$. Denote this integer by $n_f(\theta)$ where multiple zeros are repeated according to their multiplicities. By the fundamental theorem of algebra f has n complex zeros and if n is large one expects that in average

$$(*)\qquad \frac{n_f(\theta)}{n} \simeq \frac{\theta}{\pi}$$

This "statistical result" is indeed true and goes back to work by Schur. The search for a more precise control of the deviation in (*) stimulated further research. A quite conclusive result has been achieved in work by Erdös, Turan and Ganelius.

Theorem. *There exists a constant C^* which is independent of n such that the following hold for every $0 < \theta < \pi$ and every polynomial f as above:*

$$(*)\qquad \left|\frac{n_f(\theta)}{n} - \frac{\theta}{\pi}\right| \leq C^* \cdot n^{-1/2}\cdot \log\frac{|a_0|+|a_1|+\ldots+|a_n|}{\sqrt{|a_0\cdot a_n|}}$$

Remark. In [Ganelius] it is proved that one can take

$$C^* = \sqrt{\frac{2\pi}{\mathcal{G}}}$$

where $\mathcal{G}$ is Catalani's constant defined by

$$\mathcal{G} = 1 + \sum_{\nu=1}^{\infty} (-1)^\nu \cdot \frac{1}{(2\nu+1)^2}$$

However, this upper bound is not sharp so the question remains if one can determine the best constant C^* above, or at least "as sharp as possible" which eventually only has to work for all sufficiently large n.

A related problem about Fourier series. The major step to find C^* comes from a result about trigonometric polynomials which goes as follows: Let H and K be positive real numbers. Consider all pairs of finite sequences of real numbers $\{a_k\}$ and $\{b_k\}$ indexed by positive integers k such that the following two inequalities hold for every $0 \leq \phi \leq 2\pi$:

(i)
$$\sum a_k \cdot \cos k\phi + b_k \cdot \sin k\phi \leq K$$

(ii)
$$\sum k \cdot (a_k \cdot \sin k\phi - b_k \cdot \cos k\phi) \leq H$$

To this pair of sequences we associate the conjugate Fourier series

$$v(\phi) = \sum a_k \cdot \sin k\phi - b_k \cdot \cos k\phi$$

In [xx: Lemma 11.6.3] the following is proved:
B. Theorem. *Put $C_* = 2\pi^{\frac{3}{2}} \cdot \mathcal{G}^{-1/2}$. Then (i-ii) entail that*

$$\left|v(\beta) - v(\alpha)\right| \leq C_* \cdot \sqrt{HK}$$

hold for all pairs $0 \leq \beta < \alpha < 2\pi$.

However, this constant is not sharp and an "opimal" choice of C_* in Theorem B would improve the constant C^* in Theorem A.

Analytic function theory in one complex variable

Contents

Introduction.

The chapters treat basic results in analytic function theory of one complex variable. More advanced material appears in *Special Topics*. An Appendix covers background with sections devoted to measure theory, distribution theory and functional analysis. So apart from the complex case these notes treat results from real analysis where proofs often rely upon analytic function theory. However, the borderline between complex and real situations is not strict, each profits from the other. For example, solutions to the Dirichlet problem employ real analysis and is extremely helpful for the study of analytic functions.

Above we have just given titles of the chapters which start with a summary of their contents. The level varies since we do not only describe foundational results but also treat more advanced material. The less prepared reader may first pay attention to basic material and delete advanced studies which for example occur at the end of the the section about series in Chapter I part B and certain results in Chapter 6 such as Beurlings mapping theorem at the end of that chapter. For the more experienced reader these chapters serve as background *Special Topics*.

At several occasions details of proofs are left as exercises to the reader. Chapter 7 contains both examples and exercises where residue calculus gives many examples which are instructive for the beginner.

Chapter I: Basic complex analysis

Introduction. The chapter consists of sections A-E where separate section starts with a list of their contents. Section A is devoted to properties of complex numbers. The fundamental theorem of algebra is a central result and we also introduce Möbius functions and expose various geometric constructions which rely upon the calculus with complex numbers where their arguments play an important role. Section B is devoted to series. Even though this study from start appear to be elementary, the study of series is quite involved and many technically hard results appear in this section. Linear algebra is exposed in Section C where focus is upon finite dimensional complex vector space and matrices representing linear operators on these. Section D contains material concerned with zeros of polynomials where we remark that the study of complex zeros is far more involved compared to the read case. Section E contains basic material about Fourier series and we remark that further material about Fourier series and their relation to analytic function theory appears in some sections from *Special Topics*.

Some results in this chapter use analytic functions where we rely upon later material, foremost from chapter 3. But the remaining study is self-contained and it is both essential and instructive for the beginner to become well familiar with the various phenomena which appear when complex numbers are used.

I:A Complex numbers

Content

Introduction.

Bieberbach's text-book [Bi:1] starts with a historic account about the origin of complex numbers and reflections upon how they are introduced to the beginner. Here follow an excerpt where I am responsible for the rather free translation:

Every school pupil who learns about complex numbers and how to compute with them follows the same path as mathematical science did in the past. One first becomes familiar with the new and unpleasant concepts which surround complex numbers and by the natural inertia of the human mind it is not obvious why one should learn about all formal rules from the start. It is only later that one learns about the usefulness of complex numbers which makes it possible to answer previously unsolved problems. The "miracle" is that many problems which are phrased in the real number system can be solved by a detour over complex numbers. Once such examples have been understood the strength and beauty of complex numbers becomes clear.

The active role of complex numbers in algebra and analysis appeared quite late in the history of mathematics. An explanation might be that *conceptual thinking* (Begriffliches Denken) was rather remote to most mathematicians until the end of 1700. Even in his thesis from 1799, C.F. Gauss still did not fully break to the traditions in using complex numbers. Not until 1831 "würde Volle

Klarheit nachbeweisbar" in his mathematical work. It is in [Ga:xx] that one finds the Gaussian plane which is used to give the geometric description of complex numbers. Here one must also give credit to the Norwegian mathematician Caspar Wessel who already in 1799 presented a work at the Danish Academy where "Eine ausfürliche Theorie der Komplexen Zahlen auf Geometrischen Grundlage ist enwickelt".

Remark. Wessel's article became most likely familiar to Niels Henrik Abel (1801-1829) when he visited in Copenhagen as a student in 1822. Two years later he demonstrated that the general algebraic equation of degree ≥ 5 cannot be solved by roots and radicals. Apart from the result, Abel's proof laid the foundations for the modern theory of algebraic number fields. Complex analysis appears in Abel's famous article [Ab.2] from 1827 where pioneering methods were introduced which for example demonstrated that if $f(z)$ is a doubly-periodic meromorphic function in $\mathbf{C}$, i.e. $f(z) = f(z+1) = f(z+i)$ hold for all $z = x+iy$, then the sum of its zeros in the open unit square $\{0 < x, y < 1\}$ minus the sum of its poles is equal to $p+qi$ where p, q are integers, and conversely there exist a doubly periodic meromorphic function with these zeros and poles when the integral condition holds. Let me finish by recalling the following result from the introduction in [Ab]: *Let $n \geq 2$ be an integer and $f(x_1, \ldots, x_n)$ a function on n variables such that every partial derivative $\partial f/\partial x_j$ is a rational function, i.e. a quotient of two polynomials. Then $f = R(x) + Q(x) \cdot Log\, P(x)$ where P is a polynomial and R and S are rational functions.* Abel's proof uses an induction over n via the fundamental theorem of algebra and an induction over n. The reader may consult the book [Abel Legacy] where many articles give an account of Abel's impressive work.

About the contents. Subsection A treats basic material about complex numbers and B is devoted to the fundamental theorem of algebra where Theorem B.2 was proved by Augustine Cauchy in 1815. Interpolation formulas for polynomials appear subsection C and in D contains results about extremal Tchebyscheff polynomials. Let us remark that already at the end of section B we expose more advanced facts due to Ostrowski and Polya about absolute values of roots to complex polynomials.

Section 2 studies Möbius functions which give first examples of conformal mappings. The Laplace operator and the complex logarithm are introduced where the results are expressed in the real (x, y)-coordinates which enable us to apply ordinary calculus . Later we shall learn more about the complex Log-function after analytic functions have been introduced in Chapter III. The stereographic projection is constructed in section 5 together with the spherical and the hyperbolic metrics which both are used to study analytic functions.

Remark. In section A-E the reader is often asked to supplement the text with examples and computers are helpful since plots give insight about the geometry which for example arises when Möbius transforms are studied.

1.A Basic facts.

A complex number z is expressed by $x+iy$ where (x, y) is a point in $\mathbf{R}^2$. In this way the complex plane $\mathbf{C}$ is identified with $\mathbf{R}^2$. When $z = x+iy$ we set

$$\mathfrak{Re}(z) = x \quad : \mathfrak{Im}(z) = y$$

and refer to x as the real part and y as the imaginary part of z. The sum of two complex numbers is defined by

(i) $$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Complex multiplication is defined by:

(ii) $$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_1 - y_1 y_2) + i(x_1 y_2 + ix_2 y_1)$$

One verifies that the product satisfies the associative law. Moreover, if $z = x+iy$ its multiplicative inverse becomes

(*) $$z^{-1} = \frac{x - iy}{x^2 + y^2}\,.$$

1.1 Conjugation and absolute value. If $z = x+iy$ its complex conjugate is $x-iy$ and denoted by $\bar{z}$. The absolute value of z is defined as $\sqrt{x^2+y^2}$ and is denoted by $|z|$. Notice that the map $z \mapsto \bar{z}$ corresponds to reflection of plane vectors with respect to the x-axis and (*) gives

$$(**) \qquad z^{-1} = \frac{\bar{z}}{|z|^2}$$

1.2. The complex argument. In $\mathbf{R}^2$ we have polar coordinates (r, ϕ). If z is non-zero we can write:

$$(1.2) \qquad z = r \cdot \cos\phi + i \cdot r \cdot \sin\phi \quad : \quad r = |z|\,.$$

The angle is denoted by $\arg(z)$ and called the argument of z. Since trigonometric functions are periodic, the number $\arg(z)$ is only determined up to an integer multiple of 2π. Specific choices of $\arg(z)$ appear in different situations. As an example, consider the upper half-plane $\mathfrak{Im}(z) > 0$. Here one usually takes $0 < \phi < \pi$ for $\arg(z)$. In the right half plane $\mathfrak{Re}(z) > 0$ one takes $-\pi/2 < \phi < \pi/2$. Another case occurs when we consider the polar representation of complex numbers z outside the *negative* real axis $(-\infty, 0]$. Then trigonometric formulas show that every z has a unique polar representation in (1.2) with $-\pi < \phi < \pi$.

1.3. The complex number $e^{i\phi}$. This is the complex number with absolute value one and argument ϕ. Thus

$$(1.3) \qquad e^{i\phi} = \cos\phi + i \cdot r\sin\phi\,,$$

where e as *Neper's constant* defined by the limit

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

The notation (1.3) comes from the Taylor series expansions of the sine- and the cosine functions. Recall from *Calculus* that

$$(i) \qquad \sin\phi = \sum_{\nu=0}^{\infty} (-1)^\nu \cdot \frac{\phi^{2\nu+1}}{(2\nu+1)!} \quad : \quad \cos\phi = \sum_{\nu=0}^{\infty} (-1)^\nu \cdot \frac{\phi^{2\nu}}{(2\nu!)}$$

Adding these series and using that $i^2 = -1$ which gives $i^4 = 1$ and so on, we get:

$$(ii) \qquad \cos\phi + i \cdot \sin\phi = \sum_{\nu=0}^{\infty} \frac{i\phi)^\nu}{\nu\,!}$$

The last series resembles the series of the real exponential function from *Calculus*:

$$(iii) \qquad e^x = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu\,!} \quad : \quad x \in \mathbf{R}$$

The similarity of (ii) and (iii) explains the notation $e^{i\phi}$.

1.4 Addition formula for $\arg(z)$. From *Euclidian geometry* one has the two addition formulas for the sine-and the cosine functions:

$$(1) \qquad \sin(\phi_1 + \phi_2) = \sin(\phi_1)\cos(\phi_2) + \sin(\phi_2)\cos(\phi_1)$$

$$(2) \qquad \cos(\phi_1 + \phi_2) = \cos(\phi_1)\cos(\phi_2) - \sin(\phi_1)\sin(\phi_2)$$

for any pair ϕ_1, ϕ_2.

There exist several proofs of this wellknown result from euclidian geometry. The reader should be well aware of at least one such a proof and if necessary consult a text-book on euclidian geometry for details. An analytic proof using differential equations goes as follows. The sine-and the cosine-functions satisfy the differential equation $y'' + y = 0$. With ϕ as independent variable the general solution to

$$y''(\phi) + y(\phi) = 0$$

is given by

$$y(\phi) = A \cdot \sin\phi + B \cdot \cos\phi$$

for a pair of constants. Since the differential equation (*) has constant coefficients it follows that when ϕ_2 is fixed then

$$\sin(\phi + \phi_2) = A \cdot \sin\phi + B \cdot \cos\phi$$

for some pair of constants. Regarding the value when $\phi = 0$ and also the first order derivative the reader recovers the addition formula (i) and (ii) is derived in a similar fashion.

Next, the construction of complex multiplication and (1.3) yields the equality

$$r_1 \cdot e^{\phi_1} \cdot r_2 \cdot e^{\phi_2} = r_1 r_2 \cdot e^{\phi_1 + \phi_2}$$

for all pairs of positive numbers r_1, r_2 and a pair of ϕ-angles. So when complex arguments are identified up to integer multiples of 2π we get:

(3) $$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$$

for each pair of non-zero complex numbers. By an induction over k the following hold for every k-tuple of complex numbers:

(*) $$\sum_{\nu=1}^{\nu=k} \arg(z_\nu) = \arg(\prod_{\nu=1}^{\nu=k} z_\nu)\,.$$

We refer to (*) as the addition formula for the argument function. It plays a fundamental role in complex analysis.

1.5 Associated matrices. Let $z = a + ib$ be a complex number. Identifying $\mathbf{C}$ with $\mathbf{R}^2$ the complex multiplication with z yields a linear operator represented by a matrix. More precisely, the euclidian basis vectors e_1, e_2 correspond to the complex numbers 1 and i. Since $z \cdot i = ia - b$ the 2×2-matrix M_z associated to multiplication with z becomes

$$M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Notice that the determinant of M_z is $a^2 + b^2$ and the inversion formula (*) from 1.1 corresponds to the matrix identity

$$M_z^{-1} = \frac{1}{a^2 + b^2} \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

1.6 Exercise By Neper's limit formula in (1.3.2) the left hand side in (1.3.1) is the limit of ϕ-polynomials

(i) $$P_n(\phi) = \big(1 + \frac{i\phi}{n}\big)^n$$

Notice that

$$\arg(1 + \frac{i\phi}{n}) = \frac{\phi}{n}$$

The addition formula (*) in 1.4 therefore gives

$$\arg(P_n(\phi)) = \phi$$

for every n. Next, regarding absolute values we have

$$|1 + \frac{i\phi}{n}|^2 = 1 + \frac{\phi^2}{n^2} \implies |(P_n(\phi)|^2 = \big(1 + \frac{\phi^2}{n^2}\big)^n$$

From calculus we know that

$$\lim_{n\to\infty} \big(1 + \frac{\phi^2}{n^2}\big)^n = 1$$

This confirms that the complex number $e^{i\phi}$ has absolute value one.

B. The fundamental theorem of algebra.

Let $k \geq 2$ and consider a polynomial

$$P(z) = z^k + c_{k-1}z^{k-1} + \ldots + c_1 z + c_0 \quad : \quad c_0, \ldots, c_{k-1} \text{ are complex numbers}$$

A root of P is a complex number α such that:

$$\alpha^k + c_{k-1}\alpha^{k-1} + \ldots + c_1\alpha + c_0 = 0$$

If P has a root α one gets a factorisation

$$P(z) = (z - \alpha)(z^{k-1} + d_{k-2}z^{k-2} + \ldots + d_1 z + d_0)$$

where the d-coefficients are found by algebraic identities. One has for example

$$d_{k-2} = c_{k-1} - \alpha \quad : \quad d_{k-3} = c_{k-2} - \alpha d_{k-2}$$

and so on. If the factor polynomial of degree $k-1$ also has some complex root we can continue and conclude

B.1 Proposition. *Assume that every polynomial $P(z)$ has at least one complex root. Then it has a factorisation*

$$P(z) = \prod_{\nu=1}^{\nu=k} (z - \alpha_\nu)$$

Here k is the degree of P and $\alpha_1, \ldots, \alpha_k$ is a k-tuple of complex numbers where repetitions occur when P has multiple roots.

Now we prove that the hypothesis in the Proposition B.1 holds.

B.2 Theorem *Every polynomial $P(z)$ has at least one root.*

Proof. Put $M = |c_0| + \ldots + |c_{k-1}|$. If $|z| \geq 1$ the triangle inequality gives

(i) $$|P(z)| \geq |z|^k - M \cdot |z|^{k-1} \geq |z| - M$$

With $R = M + 2 \cdot |c_0|$ it follows that

(ii) $$|z| \geq R \implies |P(z)| \geq R - M \geq 2 \cdot |c_0| = 2 \cdot |P(0)|$$

Next, the restriction of $P(z)$ to the closed disc $|z| \leq R$ is a continuous function and therefore the absolute value takes a minimum at some point z_0 which in particular gives $|P(z_0)| \leq |P(0)|$. Hence (ii) implies that we have a global minimum, i.e.

(iii) $$|P(z_0)| \leq |P(z)|$$

hold for all z. There remains to show that (iii) entails $P(z_0) = 0$. To show this we argue by contradiction, i.e suppose that $P(z_0) \neq 0$ and with a new variable ζ we get the polynomial

(iv) $$P(z_0 + \zeta) = P(z_0) + d_m\zeta^m + d_{m+1}\zeta^{m+1} + \ldots + d_k\zeta^k$$

where $1 \leq m \leq k$ and $d_m \neq 0$. We find real numbers α, β such that

(v) $$P(z_0) = |P(z_0)|e^{i\alpha} \quad \text{and} \quad d_m = |d_m|e^{i\beta}$$

Next, with $\epsilon > 0$ we set

(vi) $$\zeta = \epsilon \cdot e^{i \cdot \frac{\pi + \alpha - \beta}{m}}$$

Since $e^{i\pi} = -1$ this choice of ζ together with (v) gives

(vi) $$P(z_0) + d_m\zeta^m = (1 - |d_m| \cdot \epsilon^m)P(z_0)$$

Put $M^* = |d_{m+1}| + |d_{m+2}| + \ldots |d_k|$. When $\epsilon < 1$ the triangle inequality gives

(vii) $$|d_{m+1}\zeta^{m+1} + d_{m+2}\zeta^{m+2} + \ldots d_k z^k| \leq M \cdot \epsilon^{m+1}$$

Together with (vii) another application of the triangle inequality gives:

$$|P(z_0 + \epsilon \cdot e^{i \cdot \frac{\pi+\alpha-\beta}{m}})| \leq |P(z_0)|(1 - |d_m|\epsilon^m| + M \cdot \epsilon^{m+1} =$$

(viii) $$|P(z_0)| - \epsilon^m\big(|d_m| \cdot |P(z_0)| - M \cdot \epsilon\big)$$

Now we can take

$$0 < \epsilon < \frac{|d_m| \cdot |P(z_0)|}{M}$$

and then (viii) gives a strict inequality

$$|P(z_0 + \zeta)| < |P(z_0)|$$

This contradicts that z_0 gave a minimum for the absolute value of P and the proof is finished.

Remark. The proof above is due to Cauchy and was given in 1815. If the degree of $P(z)$ is ≤ 4 one can find the roots by *Cardano's formula*. See xx below. But as soon as the degree is ≥ 5 it is in general not possible to find the zeros of a polynomial by roots and radicals. This was proved by Niels Henrik Abel in 1823. The article [Ab:1] contains the first general account about algebraic field extensions and Abel used this pioneering theory to prove that the general algebraic equation of degree ≥ 5 cannot be solved by roots and radicals from a system of 120 linear equations expressed by the coefficients of a polynomial in degree ≥ 5. For an account about Abel's great contributions the reader should consult articles from *The Abel Legacy* published in 2004 on the occasion of the first Abel Prize. After Abel's decease in 1829, Everiste Galois constructed a group to every polynomial which gives a general criterion when zeros of a polynomial can be found by roots and radicals. This leads to *Galois theory* which brings the theory about field extensions together with group theory and has become a central topic in algebra. Of special interest are complex numbers which are *algebraic* over the field Q of rational numbers, i.e. complex numbers α which are roots to some polynomial whose coefficients are rational numbers. One shows easily that the set of all such complex numbers is a subfield of $\mathbf{C}$ denoted by A. Inside A there occur smaller subfields generated by roots to a finite family of polynomials. Such subfields K are finite dimensional vector spaces over Q and called finite algebraic fields. Given such a field K one then gets a subring $\mathcal{D}(K)$ which consists of all $\alpha \in K$ such that α is a root of a monic polynomial with integer coefficients, i.e. α satisfies an equation

$$\alpha^m + c_{m-1}\alpha^{m-1} + \ldots + c_1\alpha + c_0 \quad : \quad c_0, \ldots, c_{m-1} \text{ are integers}$$

The ring $\mathcal{D}(K)$ is a Dedekind ring and enjoys nice properties which are exposed in text-books devoted to algebraic number fields. Let us finish this more algebraic discussion by a typical problem which arises when one is asking for more precise positions of roots to polynomials with rational coefficients. In general, consider a pair of polynomials

$$p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n$$
$$q(z) = z^n + b_1 z^{n-1} + \ldots + b_{n-1}z + b_n$$

where $\{a_\nu\}$ and $\{b_\nu\}$ are rational numbers. We also assume that the roots of p and q are simple so we can write

$$p(z) = \prod (z - \alpha_j) \quad \text{and} \quad q(z) = \prod (z - \beta_j)$$

Each root α_j of p is an algebraic number and it generates a field $K = Q[\alpha_j]$ which as a vector space of Q has dimension n and a basis is given by $1, \alpha_j, \ldots, \alpha_j^{n-1}$. One may ask when there exists a pair of roots α_j, β_ν for p and q respectively such that the two fields $K[\alpha_j]$ and $K[\beta_\nu]$ are equal. It means for example that we can write

$$\beta_\nu = q_0 + q_1 \cdot \alpha_j + \ldots + q_{n-1}\alpha_j^{n-1}$$

for some n-tuple of rational numbers. A necessary and sufficient condition for the existence of such a pair of roots goes as follows:

Theorem. *A pair of roots exist such that $K(\alpha_j] = K[\beta_\nu]$ if and only if there is some integer $1 \leq k \leq n$ such that*

$$\alpha_1^k \cdot \beta_1 + \ldots + \alpha_n^k \cdot \beta_n$$

is a rational number.

The major step to prove this result uses an interpolation formula where one treats the coefficients of p and q as parameters. Namely, for a given $1 \leq k \leq n$ there exists a monic polynomial $Q(z)$ whose degree is $n\,!$ and expressed via Newton's logarithmic formula:

$$\frac{Q'(z)}{Q(z)} = \sum \frac{1}{z - \alpha_{j_i}^k \cdot \beta_\nu}$$

where the sum extended over all permutations $j_1, \ldots, j_n$. The coefficients of this Q-polynomial are rational functions of the $2n$-many parameters $a_1, \ldots, a_n, b_1, \ldots, b_n$. The explicit formula for these rational functions of a and b which yield the coefficients of Q are rather cumbersome to obtain. For lower values of n a complete investigation has been carried out in work by Delannay and Tschebotaröw. The interested reader should consult the plenary talk by Tschebotaröw from the IMU-congress at Zürich in 1932.

Remark. We have inserted the comments above to illustrate that the focus upon analytic problems in complex analysis in these notes in many situations is insufficient for more subtle number theoretic aspects where many important formulas are proved by involved combinatorical methods.

B.3 Solutions by roots and radicals. We give a specific construction of roots to a polynomial of degree 3 which goes back to Tarignalo and Cardano. Consider the polynomial

$$P(x) = x^3 + x + 1$$

The derivative $P'(x) = 3x^2 + 1 > 0$ for all x. Hence $P(x)$ is strictly increasing on the real x-axis and has one real root x_* which is < 0. To find x_* Taraglino and Cardano proceeded as follows. Let u and v be a pair of *independent* variables and regard the function

(*) $$f(u, v) = (u + v)^3 + u + v + 1 = u^3 + v^3 + (u + v)(3uv + 1) + 1$$

When $v = -\frac{1}{3u}$ we see that (*) is zero if

(**) $$u^3 - \frac{1}{27u^3} + 1 = 0$$

With $\xi = u^3$ this yields the second order algebraic equation

$$27\xi^2 - 1 + 27\xi = 0$$

It is rewritten as

$$(\xi + \frac{1}{2})^2 = \frac{1}{27} + \frac{1}{4}$$

Here we find the positive root

$$\xi_* = \sqrt{\frac{1}{27} + \frac{1}{4}} - \frac{1}{2}$$

Hence $P(x)$ has the real root

(***) $$x_* = \xi_*^{\frac{1}{3}} - \frac{1}{3} \cdot \xi_*^{-\frac{1}{3}}$$

There remains to find the two complex roots. Since P has real coefficients they occur in a conjugate pair, i.e. the complex roots are of the form $a + ib$ and $a - ib$ where we may take $b > 0$. Since the real root ξ_* has been found one can determine a and b as follows: By a wellknown algebraic identity the sum of the three roots of P is zero, i.e. it holds in our specific example since the x^2-coefficient of P is zero. We conclude that

(1) $$a = -\frac{x_*}{2}$$

Next, the product of the three roots is equal to -1. This gives

$$(a^2 + b^2) \cdot x_* = -1 \implies b^2 = -\frac{1}{x_*} - \frac{x_*^2}{4}$$

The reader should verify that the last term is > 0 and hence b is the real root

$$b = \sqrt{-\frac{1}{x_*} - \frac{x_*^2}{4}}$$

B.4 Real roots

In many situations one is foremost interested in real roots. Consider a polynomial $p(x)$ with real coefficients and let us study their eventual real rots on the x-line. Classical rules give conditions for the presence of real roots where the derivative $p'(x)$ intervenes via Rolle's mean-value theorem. More precisely, the derivative $p'(x)$ must have at least one zero between two consequtive zeros of p. For example, if p has k many simple real zeros $\{\alpha_\nu\}$ where $k \geq 2$ which are arranged with increasing order then $p'(x)$ has a least one zero in each interval $(\alpha_\nu, \alpha_{\nu+1})$ when $1 \leq \nu \leq k-1$.

More generally one can study the collection of real roots in a finite family of polynomials. For this purpose certain sign-symbols have been introduced by Lars Hörmander which leads to Theorem B.5 below. The combined sign of an m-tuple of polynomials is denoted by $\text{SIGN}(p_1, \ldots, p_m)$ and registers, in increasing order, all the zeros of these polynomials and the signs of all polynomials at each zero and every interval, including the intervals which stretch to $+\infty$ or $-\infty$. For a single polynomial $p(x)$ the sign is given by a finite ordered sequence of + or -and 0 expressing eventual zeros of p and signs of $p(x)$ just before or after one zero. For example, if $p(x) = x^2 - 1$ one writes:

$$+; 0; -; 0; +$$

which reflects that $p > 0$ for large negative x, has a zero at $x = -1$ and is < 0 in the interval $(-1, 1)$ and is > 0 after the zero at $x = 1$. The sign sequence of a polynomial p without any real zeros is reduced to a single + if it is > 0 or a single minus sign if $p(x) < 0$ holds on the x-line. Next, if p and q is a pair of polynomials the sign chain is expressed by pairs at each stage. For example, if $p(x) = x^2 - 1$ and $q(x) = x$ then $\text{SIGN}(p, q)$ is given by

$$+/-; 0/-; -/0; -/+; 0/+ : +/+$$

Above the first symbol $+/-$ indicates that $p(x) > 0$ when $x < -1$ while $q(x) < 0$. The second term $0/-$ is the zero of p at $x = -1$ and the extra minus sign indicates that $q(-1) < 0$. The third term $-/0$ is the zero of q at $x = 0$ where the minus sign above 0 appears since $p(0) < 0$.

Sign-chains become more and more involved when the number m of polynomials increases. But they can always be found in an algorithmic way.

B.5 Theorem. *Let $p(x)$ be a real polynomial and $r(x)$ the polynomial after an euclidian division*

$$p = A \cdot p' + r$$

where $p'(x)$ is the derivative and $deg(r) \leq deg(p) - 2$. Then $SIGN(p', r)$ determines $SIGN(p)$.

Proof. Let $p(x) = a_n x^n + \ldots + a_0$ where $a_n \neq 0$. Replacing p by $-p$ reverse all signs for p but also for the pair (p', r). So without loss of generality we can assume that $a_n > 0$. Next, when $x << 0$ then $\text{SIGN}(p)$ starts with + if n is even and at the same time $p'(x) < 0$ for $x << 0$ so the sign-chain of (p', r) decides if n is even or not.

Consider for example the case when n is even and let x_0 be the first zero of p' which must exist since $p'(x) < 0$ when $x << 0$ while $p'(x) > 0$ when $x >> 0$. Now $p'(x) < 0$ for all $x < x_0$ which means that that $x \to p(x)$ is strictly decreasing for $x < x_0$ and therefore the sign-sequence of p is determined on this interval, i.e there only occurs + if p has no zero or otherwise it attains a zero and its sign-sequence starts eith $+; 0; -$ prior to x_0. Next, at x_0 the sign of $r(x_0)$ determines that of $p(x_0)$ where one does not exclude the case when $r(x_0) = 0$ which would give a zero for p at x_0. Now we pass to the (eventual) next zero $x_1 > x_0$ and whatever is the sign of $p'(x)$ on $(x_0, x_1$ we know at least that $x \mapsto p(x)$ is strictly increasing or strictly decreasing on this interval and since the sign or an eventual zero of p is known at x_0, we see that the sign-sequence of p is determined on $(x_, x_1)$. Arriving at x_1 we use that the sign of $r(x_1)$ is known and hence the sign-sequence for p is determined on $(-\infty, x_1]$. One can continue in this way and conclude that $\text{SIGN}(p)$ is determined on the whole line.

Exercise. Let p_1 and p_2 be a pair of real polynomials were $\deg(p_2) \geq \deg(p_1)$. and perform the two euclidian divisions:

$$p_2 = A \cdot p'_\nu + r \quad \text{and} \quad p_2 = B \cdot p + s$$

Show that SIGN(p_1, p_2) is determined by SIGN(p_1, p_2', r_1, r_2)

Hint. To begin with SIGN$(p_2', r$ determines SIGN(p_2) by Theorem B.5. After this the relative position of SIGN(p) is determined via the values of the remainder polynomial s at the zeros of q and the monotonicity intervals of q and since SIGN(p_2', s) is assumed to be known, one can determine SIGN(p, q) by a similar procedure as in the proof of Theorem B.5.

Remark. Theorem B.5 extends to arbitrary finite families of polynomials and leads to a proof of the fundamental result which asserts that the class of semi-algebraic sets is preserved under any polynomial map from one euclidian space to another. This theorem is due to Tarski and Seidenberg and has a wide range of applications in PDE-theory in several variables and is also used to establish the existence of various asymptotic expansions. In addition to Seidenberg's article [Seidenberg 1954] we refer to the Appendix in [Hörmander: XX] for a further details and applications of the preservation of semi-algebraic sets under polynomial maps from one euclidian space to another.

B.5 Absolute values of complex roots.

We shall expose some results due to Ostrowski and Polya. Let

(*) $$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

be a polynomial of some degree $n \geq 2$ where the the sole assumption is that a_0 and a_n both are $\neq 0$. The roots are arranged with non-decreasing absolute values, i.e. $\{|\zeta_1| \leq |\zeta_2| \leq \cdots\}$. The article *Recherches sur la méthode de Graeffe et les zeros des polynomes et des series de Laurent* by Ostrowski covers 150 pages in vol. 72 in Acta Mathematica (1940) and contains many interesting results. Considerable tribute should be given to Graeffe. In 1826 he presented some fundamental methods to obtain numerical solutions of algebraic equations. His ingenious idea was to consider certain sign numbers which cover the classic the search for real roots by Descartes and Newton as a special case. The original work by Graeffe was consolidated and extended into modern context by Polya in 1913.

Newton's diagram. A fundamental tool in Graeffe's original work employs the Newton diagram and the Newton polynomial $\mathfrak{M}_p(z)$ associated to an arbitrary polynomial $p(z)$. Let us recall their constructions. Let $p(z)$ be given by (*) and in the (x, y)-plane one associates points as follows: For each $0 \leq \nu \leq n$ such that $a_\nu \neq 0$ we put:

$$\xi_\nu = (\nu, \log \frac{1}{|a_\nu|})$$

Starting from ξ_0 we find the unique piecewise linear convex curve ℓ_* which stays below all the ξ-points. The reader discovers ℓ_* by drawing a picture. Corner points of ℓ_* appear as a subset of the ξ-points where ξ_0 and ξ_n always are present. To each integer $0 \leq \nu \leq n$ we get real numbers $\{\chi_\nu\}$ which give y-coordinates of the points on ℓ_* whose x-coordinate is an integer in $[0, n]$.

$$(\nu, \chi_\nu) \in \ell_* \quad : \quad 0 \leq \nu \leq n$$

Newtons majorizing polynomial is defined by

$$\mathfrak{M}_p(z) = \sum e^{-\chi_\nu} \cdot z^\nu$$

The polynomial $\mathfrak{M}_p$ is special since the coefficients satisfy Newton's convexity law:

(1) $$e^{-2\chi_\nu} \geq e^{-\chi_{\nu+1}} \cdot e^{-\chi_{\nu-1}}$$

The numerical inclination and the deviation number. The differences $\{\chi_\nu - \chi_{\nu-1}\}$ are called logarithmic inclinations and the numerical inclination number at place ν is defined by

$$R_\nu = \frac{T_{\nu-1}}{T_\nu} \quad : \quad T_\nu = e^{-\chi_\nu}$$

Finally the deviation number at place ν is given by:

$$D_\nu = \frac{R_{\nu+1}}{R_\nu} = \frac{T_\nu^2}{T_{\nu+1} \cdot T_{\nu-1}}$$

Notice that $D_\nu \geq 1$ follows from the the convexity in (1).

Remark. The reader should illustrate the construction by figures. Consider as an example the case when $a_0 = 1$ which means that $\xi_0 = (0,0)$ and that $|a_n| < 1$ $\xi_n = (n, \eta_n)$ where

$$\eta_n = \log \frac{1}{|a_n|} > 0$$

To find the first corner point of ℓ_* we seek the minimum

$$\min_k \frac{\eta_k}{k} \quad : \quad 1 \leq k \leq n$$

The minimum is attained at a smallest k which gives the first corner point of ℓ_*. Now one starts at this point and when $k \leq n-1$ the reader should express the analytic formula for the next corner-point with the aid of a figure.

Exercise. Show with the aid of a figure that if there exists some $1 \leq k \leq n$ such that

$$|a_{k-1} = |a_k| = 1$$

then it follows that $|a_\nu| \leq 1$ for all other coefficients. Moreover, if $0 \leq k_* \leq k-1$ is the smallest integer such that $|a_{k_*}| = 1$ then $(k_*, 0)$ is a corner point of ℓ_*. Finally if we start from an arbitrary polynomial $p(z)$ where we assume that a_0 and a_n both are $\neq 0$, then there exists positive numbers B and b such that (*) above occurs for the scaled polynomial

$$q(z) = B \cdot p(bz)$$

Exercise. Show that the construction of the Newton's diagram gives the inequalities

$$(1) \qquad |a_\nu| \leq |a_0| \cdot \frac{T_\nu}{T_0} = \frac{|a_0|}{R_1 \cdots R_\nu} \quad : \quad \nu \geq 1$$

Now we establish a result from Ostrowski's article [ibid].

B.5.1 Theorem. *Let p be a polynomial of degree $n \geq 2$ where $p(0) \neq 0$. Then*

$$\frac{|\zeta_k|}{R_k} \geq 1 - 2^{-1/k} \quad \text{hold for each} \quad 1 \leq k \leq n$$

Proof. By scaling we may assume that $R_k = 1$ which by the Exercise gives $|a_\nu| \leq 1$ for each ν and an integer $k_* < k$ such that $|a_{k_*}| = 1$. There remains to prove the inequality

$$(1) \qquad |\zeta_k| \geq 1 - 2^{-1/k}$$

To show (1) we may assume from the start that $|\zeta_k| < 1$. Put

$$(1) \qquad F(z) = \frac{z^k}{(z-\zeta_1) \cdot (z - \zeta_k)} = \frac{1}{(1 - \zeta_1/z) \cdots 1 - \zeta_k/z)}$$

The last expression gives a Lauren series expansion

$$(2) \qquad F(z) = 1 + \sum_{\nu=1}^{\infty} \sigma_\nu \cdot z^{-\nu}$$

Next, put

$$F^*(z) = \frac{1}{(1 - |\zeta_1|/z) \cdots 1 - |\zeta_k|/z)} = 1 + \sum_{\nu=1}^{\infty} \sigma_\nu^* \cdot z^{-\nu}$$

It is clear that one has the majorisations

$$|\sigma_\nu| \leq \sigma_\nu^*$$

Taking the sum over all $\nu \geq 1$ we get

$$\text{(3)} \qquad \sum |\sigma_\nu| \leq \sigma^*_\nu = F^*(1) - 1 = \frac{1}{(1-|\zeta_1|)\cdots 1-|\zeta_k|)} - 1$$

The first expression of F in (1) shows that $f \cdot F$ is analytic and is of the form

$$f \cdot F = z^k \cdot G(z)$$

where G is analytic. Hence the coefficient of z^{k_*} is zero which entails that

$$a_{k_*} + \sum_{\nu \geq 1} a_{k_*+\nu} \cdot \sigma_\nu = 0$$

Since $|\alpha_\nu| \leq 1$ hold for all ν and $|a_{k_*}| = 1$ the triangle inequality gives

$$\sum_{\nu \geq 1} |\sigma_\nu| \geq 1$$

Together with (3) we obtain

$$2 \leq \frac{1}{(1-|\zeta_1|)\cdots 1-|\zeta_k|)} \implies (1-|\zeta_1|)\cdots(1-|\zeta_k|) \leq \frac{1}{2}$$

Finally we have $|\zeta_\nu| \leq |\zeta_k|$ when $\nu < k$ which gives:

$$(1-|\zeta_k|)^p \leq \frac{1}{2}$$

Taking the p:th root the requested inequality in Theorem B.5.1 follows.

The next result is attributed to Polya in Ostrowski's article.

B.5.3 Theorem. *Let p be a polynomial as above. Then the following hold for each $1 \leq k \leq n$*

$$\frac{R_1 \cdot R_k}{|\zeta_1 \cdots \zeta_k|} \leq \sqrt{(k+1)\cdot(1+\frac{1}{k})^k} \leq \sqrt{(k+1)e}$$

Proof. Write $p(z) = \sum a_\nu z^\nu$. Landau's inequality from Exercise §§ in chapter III gives the following inequality for every $r > 0$ and $k \geq 1$:

$$(*) \qquad \frac{r^k}{|\zeta_1 \cdots \zeta_k|} \leq \frac{1}{|a_0|} \cdot \sqrt{\sum |a_\nu|^2 \cdot r^{2\nu}}$$

Next, the inequality in Exercise XX gives:

$$\text{(1)} \qquad |a_\nu| \leq |a_0| \cdot \frac{T_\nu}{T_0} = \frac{|a_0|}{R_1 \cdots R_\nu} \quad : \quad \nu \geq 1$$

Taking the square in Landau's inequality and using (1) we get the inequality

$$\text{(2)} \qquad \frac{r^{2k}}{|\zeta_1 \cdots \zeta_k|^2} \leq 1 + \sum_{\nu \geq 1} \frac{r^2}{R_1^2} \cdots \frac{r^2}{R_\nu^2}$$

Keeping k fixed we set

$$r = \theta \cdot R_k \quad \text{where} \quad \theta = \sqrt{\frac{k}{k+1}}$$

Then we see that (2) gives

$$\text{(3)} \qquad \frac{R_k^{2k}}{|\zeta_1 \cdots \zeta_k|^2} \cdot \theta^{2k} \leq 1 + \sum_{\nu \geq 1} \theta^{2\nu} \cdot \frac{R_k^2}{R_1^2} \cdots \frac{R_k^2}{R_\nu^2}$$

It follows that

$$\left(\frac{R_1 \cdots R_k}{|\zeta_1 \cdots \zeta_k|}\right)^2 \leq \theta^{-2k} \cdot \Big[\sum_{\nu=0}^{\nu=k-1} \theta^{2\nu} \cdot \left(\frac{R_{\nu+1}}{R_k} \cdots \frac{R_{k-1}}{R_k}\right)^2 + \theta^{2k} + \sum_{\nu > k} \theta^{2\nu} \cdot \left(\frac{R_k}{R_{k+1}} \cdots \frac{R_k}{R_\nu}\right)^2\Big]$$

Since the sequence $\{R_\nu\}$ is increasing the right hand side is majorized by

$$\theta^{-2k} \cdot \sum_{\nu \geq 0} \theta^{2\nu} = \theta^{-2k} \cdot \frac{1}{1-\theta^2} = (k+1) \cdot (1 + \frac{1}{k})^k$$

Taking square roots we get Polya's inequality.

B.5.4 Further results. Ostrowski's cited article contains many other interesting results. Among these occur a study of zeros for Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \cdot z^n$$

which converges in some annulus $R_* < |z| < R^*$.

C. Interpolation formulas

Given an arbitrary k-tuple of complex numbers $c_0, \ldots, c_{k-1}$ the general monic polynomial of degree k takes the form:

$$P(z) = z^k + c_{k-1} z^{k-1} + \ldots + c_1 z + c_0$$

Let $\alpha_1, \ldots, \alpha_k$ be the roots. Here multiple zeros may occur. In contrast to sets of real numbers there is no obvious procedure to order a set of complex numbers. Thus, the the roots should be regarded as an *unordered* k-tuple of complex numbers. But there exist *symmetric polynomials* of this unordered k-tuple. In particular we construct the symmetric sums

(i) $$\sigma_m = \alpha_1^m + \ldots + \alpha_k^m \quad : \ 1 \leq m \leq k$$

C.1 Theorem. *For each $m \geq 1$ there exists a polynomial $Q_m(c_0, \ldots, c_{k-1})$ of the independent c-variables such that*

$$\sigma_m = Q_m(c_0, \ldots, c_{k-1})$$

Exercise. Prove this result and show also that the Q-polynomials have integer coefficients. After we have learnt residue calculus we find the σ_m-numbers as follows: Euclidian division gives a unique pair of polynomials $A_m(z)$ and $\Gamma_m(z)$ such that

(i) $$z^m \cdot P'(z) = A_m(z) \cdot P(z) + \Gamma_m(z)$$

where Γ has degree $\leq k-1$. Residue calculus gives

$$\sigma_m = \frac{1}{2\pi i} \cdot \int_{|z|=R} \frac{z^m \cdot P'(z) \cdot dz}{P(z)} = \frac{1}{2\pi i} \cdot \int_{|z|=R} \frac{\Gamma_m(z)) \cdot dz}{P(z)} = \gamma_*(m)$$

where $\gamma_*(m)$ is the coefficient of z^{k-1} in Γ_m. Assume that $k \geq 2$ and let $\gamma_{k-2}(m)$ be the coefficient of z^{k-2} in Γ_m. Now the reader can verify the recursion formula

$$\gamma_*(m+1) = c_{k-1}\gamma_*(m) + \gamma_{k-2}(m)$$

From this one easily concludes that $\gamma_*(m)$ is expressed as a polynomial in $c_0, \ldots, c_{k-1}$ with integer coefficients. The reader may consult some text-book in algebra for the explicit expression of these polynomials. We have for example

$$\sigma_2 = c_{k-1}^2 - 2 \cdot c_{k-2}$$

C.2 The discriminant. It is defined by:

(*) $$\mathfrak{D}_P = \prod_{i \neq \nu} (\alpha_i - \alpha_\nu)$$

In the product appears $k(k-1)/2$ many terms. Since the k-tuple of roots appear in a symmetric fashion it follows from Theorem 1.10 that there exists a polynomial $Q^*(c_0, \ldots, c_{k-1})$ such that

(iii) $$\mathfrak{D}_P = Q^*(c_0, \ldots, c_{k-1})$$

The reader may consult a text-book in algebra for the expression of the Q^*-polynomial.

Example If $k = 2$ we have $P(z) = z^2 + c_1 z + c_0$ and if α_1, α_2 are the roots, it follows that

$$\mathfrak{D}_P = -(\alpha_1 - \alpha_2)^2 = 2\alpha_1 \cdot \alpha_2 + (c_1\alpha_1 + c_0 + c_1\alpha_2 + c_0) = 4c_0 - c_1^2$$

In general, let $k \geq$. Every k-tuple $(c_0, \ldots, c_{k-1})$ for which $Q^*(c_0, \ldots, c_{k-1}) \neq 0$ gives a polynomial with *distinct* roots. Since the Q^*-polynomial is not identically zero, it follows that the *generic* polynomial $P(z)$ of degree k has simple roots. The exception occurs when the point $(c_0, \ldots, c_{k-1})$ in $\mathbf{C}^k$ belongs to the algebraic hypersurface $\{Q^* = 0\}$ where Q^* is regarded as a complex-valued function of the k many complex variables $c_0, \ldots, c_{k-1}$. The detailed study of this algebraic hypersurface is a topic within algebraic geometry. Using euclidian divisions in the polynomial ring $\mathbf{C}[z]$ one can find an expression for Q_*. This is explained below.

C.3 The polynomial ring $\mathbf{C}[z]$. Given a monic polynomial $P(z)$ of some degree $k \geq 2$ its derivative $P'(z)$ is a polynomial of degree $k-1$. The condition that the roots of $P(z)$ are simple means that P and P' have no root in common. When this holds euclidian division gives a unique pair of polynomials $A(z)$ and $B(z)$ such that

$$(*) \qquad A(z)P(z) + B(z)P'(z) = 1 \quad : \quad \deg(B) \leq k-1$$

Above $B(z)$ is the unique polynomial of degree $k-1$ such that

$$B(\alpha_\nu) = \frac{1}{P'(\alpha_\nu)} \quad : \quad \alpha_1, \ldots, \alpha_k \text{ are the distinct roots of } P(z)$$

Exercise. Verify the formula below for $B(z)$ which already appeared in Newton's text-books in algebra and analysis from 1666:

$$B(z) = \sum_{\nu=1}^{\nu=k} \frac{1}{\prod_{i\neq\nu}(\alpha_\nu - \alpha_i) \cdot P'(\alpha_\nu)} \cdot \frac{P(z)}{z - \alpha_\nu}$$

C.4 Conditions for simple roots. Let $P(z)$ be a monic polynomial of degree $k \geq 2$ and assume that it has simple zeros so that the equation (*) above can be solved.

Exercise. Show that to every integer $0 \leq \nu \leq 2k-2$ there exists a unique pair of polynomials $A_\nu(z)$ of degree $\leq k-2$ and $B_\nu(z)$ of degree $\leq k-1$ such that

$$\text{(i)} \qquad A_\nu(z)P(z) + B_\nu(z)P'(z) = z^\nu$$

Next, the vector space of all polynomials of degree $\leq 2k-2$ has dimension $2k-1$. This criterion and the calculus with determinants implies that the polynomial $P(z)$ has simple roots if and only if a certain determinant of an $2k-1 \times 2k-1$-matrix is non-zero. If $k = 3$ the condition for simple roots is:

$$\det \begin{pmatrix} c_0 & 0 & c_1 & 0 & 0 \\ c_1 & c_0 & 2c_2 & c_1 & 0 \\ c_2 & c_1 & 3 & 2c_2 & c_1 \\ 1 & c_2 & 0 & 3 & 2c_2 \\ 0 & 1 & 0 & 0 & 3 \end{pmatrix} \neq 0$$

The reader is invited to find matrices for higher k-values.

C.5 Newton's interpolation. Let $k \geq 2$ and consider a pair of k-tuples $w_1, \ldots, w_k$ and $z_1, \ldots, z_k$. Assume that the z-numbers are distinct, i.e. $z_j \neq z_\nu$ hold when $j \neq \nu$. The w-numbers are arbitrary and it may even occur that all w-numbers are equal. Then there exists a *unique* polynomial $P(z)$ of degree $k-1$ at most such that

$$\text{(i)} \qquad P(z_\nu) = w_\nu \quad : 1 \leq \nu \leq k$$

One refers to P as Newton's interpolating polynomial. One has the formula:

(ii)
$$P(z) = \sum_{j=1}^{j=k} w_j \cdot \frac{\prod_{\nu \neq j}(z - z_\nu)}{\prod_{\nu \neq j}(z_j - z_\nu)}$$

It is clear that P satisfies (i). Another procedure is to seek a polynomial

$$Q(z) = c_{k-1}z^{k-1} + \ldots + c_0$$

Here (i) gives a system of equations:

(iii)
$$c_0 + c_1 z_j + \ldots + c_{k-1}z_j^{k-1} = w_j \quad : 1 \leq j \leq k$$

Since $z_1, \ldots, z_k$ are distinct the *van der Monde determinant* of the $k \times k$-matrix whose rows are $(1, z_j, \ldots, z_j^{k-1})$ is non-zero. Hence (iii) has a unique solution $(c_0, \ldots, c_{k-1})$. Notice that when the k-tuple $z_1, \ldots, z_k$ is kept fixed, it follows that the c-numbers are linear functions of $w_1, \ldots, w_k$ whose coefficients depend on the k-tuple $\{z_j\}$. Cramer's rule is used to describe these coefficients. More precisely, for each $0 \leq \nu \leq k-1$ we can write

(iv)
$$c_\nu = \sum_{i=1}^{i=k} G_{\nu,i}(z_1, \ldots, z_k) \cdot \omega_i$$

Remark. Above we treat $z_1, \ldots, z_k$ as independent complex variables. The G-functions have been determined under the assumption that the k-tuple is distinct, i.e. $z_i \neq z_\nu$ hold when $i \neq \nu$. At the same time we recall from the above that $c_0, \ldots, c_{k-1}$ can be solved via Cramer's rule. From this it follows that every doubly indexed G-function is a rational function of the k-many variables $z_1, \ldots, z_k$. The reader can consult a text-book in algebra for explicit descriptions of these G-polynomials.

C.6 Exercise. Let $k \geq 2$ and consider the family of k-tuples $z_1, \ldots, z_k$ for which $\sum |z_\nu^2 = 1$. When the k-tuple is distinct we get the positive number

$$\delta(z_\bullet) = \min_{j \neq \nu} |z_j - z_\nu|$$

Let us then consider a polynomial $Q(z)$ of degree $\leq k-1$ with coefficients $c_0, \ldots, c_{k-1}$. Now the c-coefficients can be estimated by the maximum norm

$$|Q|_{z_\bullet} = \max_\nu |Q(z_\nu)|$$

Newton's interpolation formula gives for each $0 \leq \nu \leq k-1$ a constant C_ν which is independent of Q such that

(*)
$$|c_\nu| \leq C_\nu \cdot \delta(z_\bullet)^{-k+1} \cdot |Q|_{z_\bullet}|$$

The reader is invited to analyze how sharp these inequalities are. For example, is it necessary to the rather high negative powers above of $\delta(z_\bullet)$ for a general polynomial Q. We remark that *a priori* inequalities of this kind are of interest from an analytic point of view and precise results rely upon a mixture of algebraic computations and solutions to suitable variational problems, i.e. there is no fixed borderline between algebra and analysis for questions dealing with a priori estimates.

D. Tchebyscheff polynomials and transfinite diameters

Let $N \geq 2$ and $E = (z_1, \ldots, z_N)$ is some N-tuple of distinct points. For each integer $n \geq 0$ we denote by $\mathcal{P}(n)$ the set of polynomials of degree $\leq n$. For every $p(z) \in \mathcal{P}(n)$ we define the maximum norm

$$|p|_E = \max_k |p(z_k)|$$

If $n \leq N-1$ then the maximum norm must be positive since the polynomial p has at most n distinct zeros and therefore cannot vanish on the N-tuple of points in E. For each integer $n \leq N-1$ we put:

$$\mathfrak{Tch}_E(n) = \min_{q \in \mathcal{P}(n-1)} |z^n + q(z)|_E$$

D.1 Proposition. *For each $n \leq N-1$ there exists a unique $q_* \in \mathcal{P}(n-1)$ such that*

(*) $$\mathfrak{Tch}_E(n) = |z^n + q_*(z)|_E$$

Proof. A polynomial $q \in \mathcal{P}(n-1)$ is said to be extremal if equality holds in (*). By (*) in Exercise C.6 there exists a uniform upper bound for the coefficients of competing extremal polynomials and since bounded sets of complex numbers are relatively compact there exists at least one extremal polynomial. It remains to show that it is unique. To see this we consider some extremal polynomial q and denote by $\mathcal{E}_q^*$ the set of points $z_k \in E$ such that

(i) $$\mathfrak{Tch}_E(n) = |z^n + q(z_k)|$$

Suppose that $\mathcal{E}_q^*$ consists of $\leq n-1$ many points, say $z_1, \ldots, z_m$ for some $m \leq n-1$. Then we can find $\phi \in \mathcal{P}(n-1)$ such that

$$\phi(z_k) = z_k^n + q(z_k) \quad : \quad 1 \leq k \leq m$$

Now the reader can verify that if $\epsilon > 0$ is sufficiently small, then

$$|z^n + q(z) - \epsilon \cdot \phi(z)| = (1-\epsilon) \cdot \mathfrak{Tch}_E(n)$$

which cannot occur since q was extremal. So when q is extremal then $\mathcal{E}_q^*$ contains at least n many points. Suppose now that q_1 and q_2 are two extremal polynomials and set $q = \frac{1}{2}(q_1 + q_2)$ which gives

$$z^n + q = \frac{1}{2}(z^n + q_1) + \frac{1}{2}(z^n + q_2)$$

The triangle inequality for the maximum norm over E entails that q is extremal. Hence we find at least n many points, say $z_1, \ldots, z_n$ in $\mathcal{E}_q^*$ which means that

(ii) $$\mathfrak{Tch}_E(n) = |z^n + \frac{1}{2}(q_1(z_k) + q_2(z_k))|$$

Since q_1 and q_2 are extremal we also have

(iii) $$|z_k^n + q_\nu(z_k)| \leq \mathfrak{Tch}_E(n) \quad : \quad \nu = 1, 2$$

It follows from (ii-iiii) that we must have the equality $q_1(z_k) = q_2(z_k)$ for each k. Hence the polynomial $q_1 - q_2$ has at least n zeros which only can occur if they are identical which finishes the proof of uniqueness.

D.2 The case when E is infinite. Let E be a compact set in $\mathbf{C}$ which is not reduced to a finite set. Let $\{z_\nu\}$ be a denumerable dense subset which for each N gives the finite set $E_N = (z_1, \ldots, z_N)$. Next, fix some positive integer n. Proposition D.1 gives for each $N \geq n+1$ a unique extremal $q_N \in \mathcal{P}(n-1)$ such that

(i) $$\mathfrak{Tch}_{E_N}(n) = |z^n + q_N(z)|_{E_N}$$

It is obvious that

(ii) $$N \mapsto \mathfrak{Tch}_{E_N}(n)$$

increases with N. Notice that we can take q_N as the zero polynomial for every N and get the inequality

$$\mathfrak{Tch}_{E_N}(n) \leq \max_{z \in E} |z|^n$$

which is finite because E is compact. Hence (ii) is bounded above and there exists a limit

(iii) $$\lim_{N \to \infty} \mathfrak{Tch}_{E_N}(n)$$

At the same time we have the sequence $\{q_N\}$ in $\mathcal{P}(n-1)$. For each N we write

$$q_N(z) = c_0(N) + c_N(1) \cdot z + \ldots + c_N(n-1) \cdot z^{n-1}$$

Using §§ XX the reader may verify that there is a constant M such that

$$\sum_{\nu=0}^{\nu=n=1} |c_N(\nu)| \leq M$$

hold for every $N \geq n+1$.

Exercise. Use the above to show that there always exist a subsequence $N_1 < N_2 < \ldots$ such that

$$\lim_{j \to \infty} c_{N_j}(\nu) = c_*(\nu) \quad : \quad 1 \leq \nu \leq n-1$$

From this extracted subsequence we obtain the polynomial

$$q_*(z) = \sum_{\nu=0}^{\nu=n-1} c_*(\nu) \cdot z^\nu$$

Use Proposition D.1 to show that this limit polynomial is the same for any chosen subsequence $\{N_k\}$ so one has unrestricted limits

$$\lim_{j \to \infty} c_N(\nu) = c_*(\nu) \quad : \quad 1 \leq \nu \leq n-1$$

Finally, show that q_* is the unique extremal polynomial for which one has the equality

$$|z^n + q_*(z)|_E = \min_{q \in \mathcal{P}(n-1)} |z^n + q(z)|_E$$

D.3 Tchebyscheff norms. With q_* as the unique extremal in $\mathcal{P}(n-1)$ above we set

$$T_n^E(z) = z^n + q_*(z)$$

and refer to this monic polynomial as the Tchebyscheff polynomial of degree n attached to the compact set E. The Tchebyscheff norm of order n over E is defined by:

$$\mathfrak{Tch}_E(n) = |T_n^E|_E$$

D.4 Exercise. Let E be an infinite and compact set. For each $n \geq 1$ we put

$$\rho(n) = \log \mathfrak{Tch}_E(n)$$

Show that the function

$$n \mapsto \frac{\rho(n)}{n}$$

is convex, i.e. that

$$\rho(n+m) \leq \frac{m}{n+m} \cdot \rho(m) + \frac{n}{n+m} \cdot \rho(n)$$

holds for each pair $m, n \geq 1$. *Hint.* Use that for each $q \in \mathcal{P}(n+m-1)$ we find $q_1 \in \mathcal{P}(m-1)$ and $q_2 \in \mathcal{P}(n-1)$ such that

$$z^{n+m} + q(z) = (z^m + q_1(z))(z^n + q_2(z))$$

The convexity above entails by a general and easily proved result about non-decreasing sequences of real numbers which is bounded above, that there exists the limit

$$\lim_{n\to\infty} \log \frac{\mathfrak{Tch}_E(n)}{n}$$

Passing to exponential functions we get the limit number

$$\mathfrak{Tch}(E) = \lim_{n\to\infty} \left[\mathfrak{Tch}_E(n)\right]^{\frac{1}{n}}$$

We refer to this number as the Tchebyscheff diameter of the compact set E. This terminology stems for the fact that $\mathfrak{Tch}(E)$ is equal to a number called the transfinite diameter of E which is defined in a geometric manner in the next section.

D. 5 The transfinite diameter.

To each n-tuple of distinct points $z_1, \ldots, z_n$ in $\mathbf{C}$ we set

$$L_n(z_\bullet) = \frac{1}{n(n-1)} \cdot \sum_{k\neq j} \log \frac{1}{|z_j - z_k|}$$

Let us now consider an infinite and compact set E. Then we define the number

$$\mathcal{L}_n(E) = \min\, L_n(z_\bullet)$$

where the minimum is taken over all n-tuples in E. Since $\log \frac{1}{r}$ is large when $r \simeq 0$ this means intuitively that we tryo to choose separated n-tuples in order to minimize the L_n-function. Notice for example that when $n = 2$ then the minium is achieved for a pair of points in E whose distance is maximal, i.e. $\mathcal{L}_2$ is the diameter of E. As n increases one has

D.6 Proposition. *The sequence $\mathcal{L}_n\}$ is non-decreasing.*

Proof. Let $z_1^*, \ldots, z_{n+1}^*$ minimize the L_{n+1}-function. We get

$$\mathcal{L}_{n+1}(E) = \frac{1}{n(n+1)} \cdot \sum_{k\neq\nu}^{(1)} \log \frac{1}{|z_\nu^* - z_k^*|} + \frac{2}{n(n+1)} \cdot \sum_{k=2}^{k=n+1} \log \frac{1}{z_1^* - z_k^*|}$$

where (1) above the first summation means that $2 \leq \nu \neq k\}$ holds. Here $z_2^*, \ldots, z_{n+1}^*$ is an n-tuple competing to maximize $\mathcal{L}_E(n)$ which gives the inequality:

$$\mathcal{L}_{n+1}(E) \geq \frac{1}{n(n+1)} \cdot n(n-1) \cdot \mathcal{L}_n(E) + \frac{2}{n(n+1)} \cdot \sum_{k=2}^{k=n+1} \log \frac{1}{z_1^* - z_k^*|}$$

The same inequaliiy holds when when we instead of z_1 delete some z_j for $2 \leq j \leq n+1$. Taking the sum of the resulting inequalities we obtain

$$(n+1)\mathcal{L}_{n+1}(E) \geq \frac{1}{n} \cdot n(n-1) \cdot \mathcal{L}_n(E) + \frac{2}{n(n+1)} \cdot \sum_{k\neq j} \log \frac{1}{|z_j - z_k|}$$

The last term is $2 \cdot \mathcal{L}_{n+1}$ which gives:

$$(n-1) \cdot \mathcal{L}_{n+1}(E) \geq \frac{1}{n} \cdot n(n-1) \cdot \mathcal{L}_n(E) = (n-1)\mathcal{L}_n(E)$$

A division by $n-1$ gives the requested inequality.

D.7 Definition. *The limit number defined by*

$$\mathfrak{D}(E) = \lim_{n\to\infty} e^{-\mathcal{L}_n(E)}$$

is called the transfinite diameter of E.

Remark. The definition means that $\mathfrak{D}(E) = 0$ if and only if $\mathcal{L}_n(E)$ tends to $+\infty$ as n increases. Intuitively this means that we are not able to choose large tuples in E separated enough to keep the sum of the log-terms bounded.

D.8 Example. Consider the interval $E = [-1, 1]$. Using the concavity of the Log-function in the positive real one one shows that $L_n(z_\bullet)$ is minimized when $\{z_\nu\}$ are equi-distributed and from this a passage to the limit gives:

$$(*) \qquad \lim_{n\to\infty} \mathcal{L}_n(E) = \iint \log \frac{1}{|s-t|} \cdot dsdt$$

with the double integral taken over the square $-1 \leq s, t \leq 1$. Indeed, the reader may verify this by approximating the double integral by Riemann sums.

Exercise. Show that the double integral has the value $\log 2$ and conclude that

$$\mathfrak{D}(E) = \frac{1}{2}$$

At the same time Tchebysheff found the unique extremal polynomials which determine $L_n(E)$ for each $n \geq 1$, i.e the Tchebyscheff polynomials described in XX which gives the equalities:

$$\mathfrak{Tch}_E(n) = 2^{-n+1}$$

Passing to the limit and taking the n:th root we get

$$\mathfrak{Tch}_E = \frac{1}{2}$$

and hence this number is equal to $\mathfrak{D}(E)$. It turns out that this equality holds in general.

D.9 Theorem. *For every compact set E one has the equality*

$$\mathfrak{D}(E) = \mathfrak{Tch}(E)$$

Remark. Theorem D.9 is due to Szegö in [Szegö: 1924 Bemerkungen]. The construction of the Tchebysheff number was earlier found by Faber in the article [Faber: 1920] who also studied many other extremal problems in the complex domain related to polynomials.

E. Exercises.

E.1 Lagrange's identity. Let $n \geq 2$ and $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ are two n-tuples of complex numbers. Show that

$$\Big|\sum_{j=1}^{j=n} z_j w_j\Big|^2 = \sum_{j=1}^{j=n} |z_j|^2 \cdot \sum_{j=1}^{j=n} |w_j|^2 - \sum_{1\leq j<k\leq n} |z_j\bar{w}_k - \bar{w}_j z_k|^2$$

Conclude that one has the inequality

$$\Big|\sum_{j=1}^{j=n} z_j w_j\Big| \leq \sqrt{\sum_{j=1}^{j=n} |z_j|^2 \cdot \sum_{j=1}^{j=n} |w_j|^2}$$

where equality holds if and only if here exists a complex number λ such that

$$w_\nu = \lambda \cdot z_\nu \quad : \quad 1 \leq \nu \leq n$$

E.2 Zeros of derivatives. Let $P(z)$ be a polynomial of some degree n whose zeros are $\alpha_1, \ldots, \alpha_n$ where eventual multiple zeros are repeated so in general the number of distinct zeros may be $< n$.

E.3 Theorem. *Let K be a convex set in $\mathbf{C}$ which contains all zeros of P. Then the zeros of P' belong to K.*

Exercise. Prove this result using the following hint: Suppose first that all zeros of P have real part ≤ 0. To find zeros of P' we use the fractional decomposition

$$\text{(i)} \qquad \frac{P'(z)}{P(z)} = \sum \frac{1}{z - \alpha_\nu}$$

If $\mathfrak{Re}(z) = b > 0$ we get

$$\mathfrak{Re}(\frac{P'(z)}{P(z)}) = \sum \frac{b - \mathfrak{Re}(\alpha_\nu)}{|z - \alpha_\nu|^2}$$

By assumption $\mathfrak{Re}(\alpha_\nu) \leq 0$ for every ν and hence the right hand side is a sum of positive terms which implies that the derivative P' has no zeros in the open right half-plane $\mathfrak{Re}(z) > 0$. At this stager the reader can finish the proof using the wellknown fact that a convex set is the intersection of half-planes.

E.4 Kakeya's theorem. *Let $c_n > c_{n-1} > \ldots > c_0 > 0$ be a strictly decreasing sequence of positive real numbers. Then the zeros of the polynomial*

$$P(z) = c_n z^n + \ldots + c_0$$

all have absolute value > 1.

Exercise. Prove this result. The *hint* is to write

$$(1-z)P(z) = c_n - \left[(c_n - c_{n-1}) \cdot z + (c_{n-1} - c_{n-2}) \cdot z^2 + \ldots + (c_1 - c_0) \cdot z^n + c_0 \cdot z^{n+1}\right]$$

After this one applies the triangle inequality when $|z| \leq 1$ and the hypothesis on the c-sequence.

E.5 Exercise. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with real coefficients whose zeros belong to the left half-plane $\mathfrak{Re}\, z < 0$. Show that all a-coefficients are > 0. The hint is that the fundamental theorem of algebra gives a factorisation

$$P(z) = \prod (z + q_j) \cdot \prod (x + \alpha_\nu)^2 + \beta_\nu)$$

where $\{-q_j\}$ are the strictly negative real roots while the second product takes into the account that complex roots appear in conjugate pairs and by the hypothesis each α_ν above is real and > 0.

E.6 A second order differential equation Let $n \geq 2$. Show that there exists a unique monic polynomial $p(z)$ of degree n which satisfies the second order differential equation

$$(z^2 - 1)p''(z) + 2zp'(z) = n(n+1)p(z) \tag{*}$$

It turns out that the zeros of p are real. To prove this we argue by contradiction. If not all the zeros are real we choose a zero α whose imaginary part is maximal, Consider the factorisation

$$p(z) = (z - \alpha)q(z) \tag{1}$$

where the polynomial $q(z)$ has degree $n - 1$. Notice that

$$p'(z) = q(z) + (z - \alpha)q'(z) \quad \text{and} \quad p''(z) = 2q'(z) + (z - \alpha)q'(z) \implies$$
$$p'(\alpha) = q(\alpha) \quad \text{and} \quad p''(\alpha) = 2q'(\alpha)$$

So with $z = \alpha$ in the differential equation we obtain

$$(\alpha^2 - 1)2q'(\alpha) + 2\alpha \cdot q(\alpha) = 0 \implies \frac{q'(\alpha)}{q(\alpha)} = \frac{\alpha}{1 - \alpha^2}$$

Let $\beta_1, \ldots, \beta_{n-1}$ be the zeros of q. Newton's fractional decomposition gives

$$\frac{q'(\alpha)}{q(\alpha)} = \sum \frac{1}{\alpha - \beta_\nu} = \sum \frac{\bar{\alpha} - \bar{\beta}_\nu}{|\alpha - \beta_\nu|^2} = \frac{\alpha}{1 - \alpha^2} \tag{2}$$

The choice of α entails that $\mathfrak{Im}(\beta_\nu) \leq \mathfrak{Im}(\alpha)$ hold for each ν which gives:

$$\mathfrak{Im}(\bar{\alpha}_\nu - \bar{\beta}_\nu) \leq 0$$

Hence the imaginary part in the right hand side of (2) is ≤ 0. But this gives a contradiction for if $\alpha = x + iy$ with $y > 0$ we see that

$$\mathfrak{Im}\,(\frac{\alpha}{1 - \alpha^2}) = \frac{y(x^2 + y^2)}{|1 - \alpha^2|^2} > 0$$

E.6.1 Exercise. Use the reality of the roots to show that they are all bounded and stay in the interval $[-1, 1]$. The hint is to use Rolle's mean-value theorem, or rather the classic rule of Descartes for real zeros of polynomials with real coefficients.

Asymptotic distribution of zeros. When n increases we study how the zeros of the polynomial of degree n which solves the differential equation above are distributed on $[-1, 1]$. Let $\alpha_1, \ldots, \alpha_n$ be the zeros of the polynomial p_n and set

$$\phi_n(z) = \frac{1}{n} \cdot \sum \frac{1}{z - \alpha_\nu}$$

As we shall learn later on this is the Cauchy transform of the probability measure on $[-1, 1]$ which assign the mass $\frac{1}{n}$ and each α_ν. By Newtons formula we have

$$n \cdot \phi_n = \frac{p_n'}{p_n}$$

Taking the complex derivative on both sides it follows that

$$n \cdot \phi_n' = \frac{p_n''}{p_n} - n^2 \cdot \phi_n^2 \tag{1}$$

At the same time the differential equation satisfied by p_n entails that

$$(z^2 - 1)\frac{p_n''}{n(n+1)p_n} + 2z\frac{p'}{n(n+1) \cdot p} = 1 \implies$$

$$(z^2 - 1) \cdot \frac{n^2}{n(n+1)} \cdot \phi_n^2 + (z^2 - 1) \cdot \frac{n}{n(n+1)} \cdot \phi_n' + z\frac{n}{n(n+1)}\phi_n = 1$$

Passage to a limit. Above we have found a sequence of probability measures $\{\mu_n\}$. The weak-start compactness entails that some subsequence converges weakly to a limit measure μ_* which again is a probability measure on $[-1, 1]$. next, passing to a oimit as $n \to \infty$ in (xx) we see that

$$\lim \phi_n^2(z) = \frac{1}{z^2 - 1}$$

When the absolute value of z is > 1 we find a square root of $z^2 - 1$ and it follows that

$$\int_{-1}^{1} \frac{d\mu_*(s)}{z - s} = \frac{1}{z} \cdot \frac{1}{\sqrt{(1 - z^{-2}}} \quad : \quad |z| > 1$$

Th last equation determines the limit measure μ_* which implies that the whole sequence $\{\mu_n\}$ converges weakly to μ_*, i.e there exists an asymptotic limit distribution for the zeros of $\{p_n\}$. There remains to find the density function of s which defines the probability measure μ_*. It means that one must exhibit an inversion formula from (xx). We shall, discuss this in more detail in (xx).

E. Zeros of $\mathfrak{Im}\, Q$. Let $k \geq 2$ and consider a monic polynomial of degree k:

$$Q(z) = z^k + c_{k-1}z^{k-1} + \ldots + c_1 z + c_0$$

Write $Q = u + iv$ where $u = \mathfrak{Re}\, Q$ and $v = \mathfrak{Im}\, Q$. Assume that the zeros f Q are simple and given by the k-tuple $\alpha_1, \ldots \alpha_k$. Next, the derivative

$$Q'(z) = kz^{k-1} +)k - 1)c_{k-1}z^{k-2} + \ldots + c_1$$

has some complex zeros. Assume that the value $v(\beta) \neq 0$ at every zero of Q'. Now $v(x, y)$ is a polynomial with real coefficients and we get the zero set

$$S = \{v = 0\}$$

A point $p \in S$ is critical if the gradient vector $\nabla(v)$ is zero a p. Later we shall learn that the Cauchy-Riemann equations applied to the polynomial $Q(z)$ entails that its complex derivative

$$Q'(z) = v_y(x,y) + iv'_x(x,y)$$

at every $z = x + iy$. The hypothesis that $v \neq 0$ at the zeros of $Q*$ therefore implies that $\nabla(v) \neq 0$ on S. It follows by calculus that S is a a union of pairwise disjoint smooth curves $\{\gamma_\nu\}$ in the (x,y)-plane.

Theorem. *The number of curves in S is equal to k and they can be ordered so that the curve γ_ν passes the zero α_ν of Q.*

Remark. The result above means that we can order the zeros of Q so that $j(\nu) = \nu - 1$ hold for each $1 \leq \nu \leq k$. Moreover, we can restrict the real polynomial u to each curve $\gamma\nu$. the Cauchy-Rieman equations show that the gradient vectors $\nabla(u($ and $\nabla(v)$ are perpendicular. From this it follows that u restricts to a strictly monotone function along each curve γ_ν where its value is zero at the corresponding zero α_ν of Q.

Exercise. Let us illustrate the theorem above with some specific examples. Consider a polynomial

$$Q(z) = z^2 - 2az - i$$

where a is real and positive. It follows that

$$v(x,y) = 2xy - 2ay - 1$$

Since $Q'(z) = 2z - 2a$ its zero is $z = a$ and here $v \neq 0$. Now $\{v = 0\}$ consists of two disjoint curves. One has the equation

$$x = a + \frac{1}{2y} \quad : \quad y > 0$$

When $y \to +\infty$ then $x \to a$. Next, consider the real part

$$u(x,y) = x^2 - y^2 - 2ax$$

Its restriction to the curve can be expressed as a function of y and becomes

$$y \mapsto (a + \frac{1}{2y})^2 - y^2 - 2a^2 - \frac{a}{y} = \frac{1}{4y^2} - y^2 - a^2$$

The last expression shows that u restricts to a strictly monotone function on the curve which is a branch of a 2-sheeted hyperbel. The second curve of S has the same equation as (x) above but here $y < 0$.

A more involved example. This time we take

$$Q(z) = z^3 + 3z - i$$

It follows that

$$v = 3x^2y - y^3 + 3y - 1$$

Next, $Q'(z) = 3z^2 + 3$ with zeros at i and $-i$ and we see that $v \neq 0$ at these zeros. So we get 3 γ-curves, each of which passes through one of the simple roots of Q. The reader is invited to draw a picture of these γ-curves with the aid of a computer.

2. Möbius functions

Let a, z be a pair of complex numbers where $|a| < 1$ and $|z| \leq 1$. Set

(1) $$M_a(z) = \frac{z-a}{1-\bar{a}z}$$

Using polar coordinates we write:

$$a = se^{i\phi} \quad : \quad z = re^{i\theta}$$

Then $\bar{a}z = sre^{i(\theta-\phi)}$ and from this it is clear that

(2) $$M_a(z) = M_s(e^{-i\phi}z)$$

Thus, up to a rotation of z the study of M-functions is reduced to the case when a is real and positive. Let $0 \leq a < 1$ and set

(3) $$w = \frac{z-a}{1-az}$$

Notice that $z = re^{i\theta}$ gives

(i) $$|z-a|^2 = (r\cos\theta - a)^2 + r^2\sin^2\theta = r^2 + a^2 - 2ar\cos\theta$$

Similarly we find that

(ii) $$|1-az|^2 = 1 + a^2r^2 - 2ar\cos\theta$$

It follows that

(iii) $$|1-az|^2 - |z-a|^2 = 1 - r^2 + a^2(r^2-1) = (1-r^2)(1-a^2) > 0$$

From this we conclude that

(*) $$|z| < 1 \implies |w| < 1$$

Next, we can solve out z in (3) and obtain:

(4) $$z = \frac{w+a}{1+aw}$$

It follows from (*) above that $z \mapsto M_a(z)$ is a *bijective map* of the open unit disc D onto itself. Next, if $0 < a, b < 1$ we construct the composed map $M_b \circ M_a$. A calculation gives

(5) $$M_b(M_a(z)) = M_c(z) \colon \quad c = \frac{a+b}{1+ab}$$

Remark. The formula (5) shows that the Möbius transforms M_b and M_a commute. Hence, the map

(*) $$(a,b) \mapsto \frac{a+b}{1+ab} \quad : \quad 0 \leq a, b < 1$$

gives a product rule which via (5) satisfies the associate law where $a = 0$ is the neutral element. To obtain a *commutative group* we need inverses. If we allow a, b to vary in the open interval $(-1, 1)$ we get a commutative group of bijective maps on D defined by

(**) $$a \mapsto M_a \quad : \quad -1 < a < 1$$

Next, let us study the map M_a-map keeping $0 < a < 1$ fixed.

2.1 Proposition. *The absolute value* $|M_a(z)| \leq 1$ *with equality if and only if* $|z| = 1$.

Proof. By the remark after (2) it suffices to prove this when $a = s > 0$ is real and positive. With $z = re^{i\theta}$ we get

$$|z-s|^2 = r^2 + s^2 - 2sr\cos\theta \quad : \; |1-sz|^2 = 1 + s^2r^2 - 2sr\cos\theta$$

Hence

(i) $$|1-sz|^2 - |z-s|^2 = (1-r^2)(1-s^2)$$

Since $s^2 < 1$ we see from (i) that $|1-sz| \geq |z-s|$ with equality if and only if $|z| = r = 1$.

Warning. The whole group of Möbius transformations is denoted by $\mathcal{M}$. It is not commutative. To see this we study rotations given by $M_\phi(z) = e^{i\phi} \cdot z$ where $0 \leq \phi \leq 2\pi$. Next, let T be the unit circle. We get a map from the product set $T \times (-1, 1)$ which sends a pair (ϕ, a) to the Möbius transform

$$M_a \circ M_\phi(z) = \frac{e^{i\phi}z - a}{1 - e^{i\phi}az} = e^{i\phi} \cdot M_\alpha(z) \quad : \alpha = e^{-i\phi}a$$

At the same time we notice that

$$M_\phi \circ M_a(z) = e^{i\phi} \cdot M_a(z)$$

This shows that the pair M_ϕ and M_a do not commute unless $\phi = 0$. It turns out that the group $\mathcal{M}$ of all Möbius transforms is quite extensive. For example, the Uniformisation Theorem will show that $\mathcal{M}$ contains the fundamental group $\pi_1(\Omega)$ where $\Omega = \mathbf{C} \setminus (p_1, \ldots p_N)$, i.e. we have removed N points form the complex plane. This fundamental group is free with N generators and since N can be arbitrary large this means that $\mathcal{M}$ is a large group which for every positive integer N contains a free group of rank N.

2.2 Some image curves.

Let $0 < a < 1$ and $0 < r < 1 - a$. Consider the image under $M_a(z)$ when $|z - a| = r$, i.e. when z moves on the circle of radius r centered at a. The image is a simple closed curve γ. With $z = a + re^{i\theta}$ the image curve is defined by

(i)
$$\theta \mapsto \frac{re^{i\theta}}{1 - a^2 - are^{i\theta}}$$

Exercise. Prove that (i) is a circle and determine its center and radius.

Next, consider images of circles centered at the origin. Here we get image curves

(ii)
$$w_r(\theta) = \frac{re^{i\theta} - a}{1 - are^{i\theta}} \quad : \ 0 \leq \theta \leq 2\pi \quad : \ 0 < r < 1$$

To obtain a more transparent formula we multiply with the complex conjugate of $1 - are^{i\theta}$ and get

(iii)
$$w_r(e^\theta) = \frac{-(ar^2 + a) + re^{i\theta} - a^2re^{-i\theta}}{1 + a^2r^2 - 2ar\cos\theta}$$

Exercise. Prove that the image curve is a circle and express its center and radius by a and r.

2.3 Images of general circles. Let $0 < a < 1$ and consider the Möbius transform $M_a(z)$. Consider an arbitrary circle $\mathcal{C}$ defined by $|z - z_0| = r$ where $z_0 \in D$ and $r < 1 - |z_0|$. As above the image is a circle. To prove this we express the map as follows:

(*)
$$M_a(z) = \frac{z - \frac{1}{a}}{1 - a \cdot z} + \frac{\frac{1}{a} - a}{1 - a \cdot z} = -\frac{1}{a} + \frac{1 - a^2}{a} \cdot \frac{1}{1 - a \cdot z}$$

Since translates of circles are circles and a dilation of the scale preserve circles it suffices to consider the image of

$$z \mapsto \frac{1}{1 - a \cdot z}$$

With $\zeta = 1 - az$ the reader may first verify that $\mathcal{C}$ is mapped into a circle $\mathcal{C}^*$ in the complex ζ-plane. Then there only remains to show that the inversion map $\zeta \mapsto \frac{1}{\zeta}$ sends circles which do not contain $\zeta = 0$ into circles.

Example. Use a computer to plot images of the circles defined by $|z - i/2| = r$ where $0 < r < 1/2$ while $0 < a < 1$ holds as above.

The case $r = 1$. Now we study how $M_a(z)$ maps the unit circle onto itself. So here we consider the map

(iv)
$$\theta \mapsto \frac{e^{i\theta}-a}{1-ae^{i\theta}} \quad : \, 0 \leq \theta \leq 2\pi$$

We already know that complex numbers of absolute value 1 appear. Hence there exists a function $\rho(\theta)$ such that

(*)
$$e^{i\rho(\theta)} = \frac{e^{i\theta}-a}{1-ae^{i\theta}} \quad : \, 0 \leq \theta \leq 2\pi$$

We see that $\rho(0)=0$ and $\rho(\pi)=\pi$ and identifying the imaginary parts we get:

$$\sin\rho(\theta) = \frac{(1-a^2)\cdot\sin\theta}{1+a^2-2a\cos(\theta)} \quad : \, 0 \leq \theta \leq \pi$$

From this the reader should verify that the function $\rho(\theta)$ is *strictly increasing* over the interval $0 \leq \theta \leq \pi$. Check also that the derivative at $\theta = 0$ becomes $\frac{1+a}{1-a}$. Starting at π one verifies in the same way that the ρ-function increases from π to 2π when $\pi \leq \theta \leq 2\pi$.

Remark. Later we encounter Möbius transforms as conformal maps on the unit disc. Here the ρ-function tells how the unit circle is mapped onto itself via the Möbius transform. It is therefore instructive to have a picture of the ρ-function. Today's student should use computers and plot the ρ-functions for different real $0<a<1$. In particular, investigate the case when $a \to 1$. Here one finds that the ρ-function initially increases rapidly and becomes almost equal to $\pi/2$ until θ gets close to π where the ρ-function again has a rapid increase until $\rho(\pi)=\pi$.

2.4 Derivatives of $|M_a(z)|$. Let $0<a<1$. If $z = re^{i\theta}$ we get

(i)
$$|M_a(re^{i\theta})|^2 = \frac{|re^{i\theta}-a|^2}{|1-\bar{a}re^{i\theta}|^2} = \frac{r^2+a^2-2ar\cos\theta}{1+a^2r^2-2ar\cos\theta}$$

Keeping θ fixed (i) is a function of r. Applying the real Log-function we get

(ii)
$$\log|M_a(re^{i\theta})| = \frac{1}{2}\log\Big[\frac{r^2+a^2-2ar\cos\theta}{1+a^2r^2-2ar\cos\theta}\Big]$$

Using one-variable calculus we take the r-derivative of the right hand side. To find the result we apply the usual formula for the real Log-function so that (ii) becomes

(iii)
$$\frac{1}{2}\cdot[\mathrm{Log}\,(r^2+a^2-2ar\cos\theta) - \mathrm{Log}\,(1+a^2r^2-2ar\cos\theta)]$$

Using (iii) an easy calculation gives:

2.5 Proposition. *The r-derivative evaluated at $r=1$ becomes:*

$$\frac{1-a^2}{1+a^2-2a\cdot\cos\theta}$$

Above we assumed that a is positive and real. But recall from (2) in the beginning of this section that if $a = se^{i\phi}$ then we only have to rotate z to get a similar M-function. Deduce from this and the exercise above that one has:

2.6 Proposition. *With $a = se^{i\phi}$, the r-derivative evaluated when $r=1$ of the function*

$$r \mapsto |M_a(re^{i\theta})| = \frac{|re^{i\theta}-ae^{i\phi}|}{|1-ae^{-i\phi}re^{i\theta}|}$$

is equal to

(*)
$$\frac{1-s^2}{1+s^2-2s\cdot\cos(\theta-\phi)}$$

3. The Laplace operator.

The second order differential operator $\partial_x^2 + \partial_y^2$ is denoted by Δ and called the Laplace operator. A C^2-function $u(x,y)$ satisfying $\Delta(u) = 0$ is called a harmonic function. The vanishing of $\Delta(u)$ is closely related to a mean-value property. To se this we first consider an arbitrary real-valued function $u(x,y)$ of class C^2 and at the origin it has a Taylor expansion

(i) $$u(x,y) = u(0,0) + ax + by + Ax^2 + By^2 + Cxy + O(x^2 + y^2)$$

where the remained is small ordo of $x^2 + y^2$.

Exercise. Show that when $r > 0$ is small then

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} u(re^{i\theta}) \cdot d\theta = u(0,0) + \frac{A+B}{2} \cdot r^2 + o(r^2)$$

At the same time we notice that

$$\Delta u(0,0) = 2A + 2B$$

So if the Laplacian is > 0 at the origin then we locally obtain a strict mean-value inequality in the sense that

$$\lim_{r \to 0} \frac{M_u(r) - u(0,0)}{r^2} = \frac{\Delta u(0,0)}{4}$$

where $M_u(r)$ is the mean-value taken over the circle $\{|z| = r\}$. On the other hand we get an infinitesmal mean-value equality when $\Delta u(0,0) = 0$. A major result about harmoinic functions is that they are charaterised by the mean-value formuka. More precisely we shal later prove the following:

Theorem. *Let u be a C^2-function defined in an open set Ω. Then u is harmonic if and only if*

$$u(p) = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(p + re^{i\theta}) \cdot d\theta$$

hold for each point $p \in \Omega$ and every $r < dist(p, \partial\Omega)$.

Poisson's harmonic function Keeping θ fixed while $a = x + iy$ varies in the open unit disc D we get the function

$$(x,y) \mapsto \frac{1 - x^2 - y^2}{1 + x^2 + y^2 - 2s \cdot \cos(\theta - \phi)} \quad : \; s = \sqrt{x^2 + y^2} \quad : \; x = s \cdot \cos(\phi) \; : \; y = s \cdot \sin(\phi)$$

The addition formula for the cosine function gives

(1) $$u(x,y) = \frac{1 - x^2 - y^2}{1 + x^2 + y^2 - 2\cos(\theta) \cdot x + 2\sin(\theta) \cdot y}$$

It turns out that u is harmonic. A direct verification by taking derivatives in x and y is a bit messy so we are going to reduce the calculation using special properties of the Laplace operator Δ. Namely, it commutes with rotations, i.e. if $g(x,y)$ is a C^2-function we set

$$g_\theta(x,y) = g(x\cos\theta + y\sin\theta, x\sin\theta + y\cos\theta) \quad : \quad 0 \leq \theta \leq 2\pi$$

With this notation one has

(*) $$\Delta(g_\theta) = (\Delta(g))_\theta$$

The verification is left to the reader. So to prove that u is harmonic we can take a rotation and assume that $\theta = 0$ in (1). There remains to consider the function

(2) $$h(x,y) = \frac{1 - x^2 - y^2}{1 + x^2 + y^2 - 2x} = \frac{1 - x^2 - y^2}{(1-x)^2 + y^2}$$

Here a further simplification is possible since Δ also commutes with *translations.* So by the linear map $x \to \xi + 1$ and $y \to \eta$ there remains to regard the function

$$k(\xi,\eta) = \frac{-2\xi - \xi^2 - \eta^2}{\xi^2 + \eta^2} = -1 - 2 \cdot \frac{\xi}{\xi^2 + \eta^2} \tag{3}$$

Notice that we only regard h when $x^2 + y^2 < 1$. This means that we only consider the k-function when $\xi^2 + \eta^2 \neq 0$. Now it is obvious that:

$$(\partial_\xi^2 + \partial_\eta^2)(\frac{\xi}{\xi^2 + \eta^2}) = 0 \quad : \xi^2 + \eta^2 > 0 \tag{4}$$

Hence Poisson's u-function is harmonic.

Remark. Later on we will give " a complex proof" that (1) yields a harmonic function. But it is instructive to see that all the necessary calculations can be achieved by calculus.

3.1 The function $\log((x-a)^2 + (y-b)^2)$**.** Let a, b be two real numbers. In $\mathbf{R}^2 \setminus (a, b)$ we have $(x-a)^2 + (y-b)^2) > 0$ where the real-valued Log-function above is defined. We shall study its partial derivatives. First we get:

$$\partial_x(\log((x-a)^2 + (y-b)^2)) = \frac{2(x-a)}{(x-a)^2 + (y-b)^2} \tag{i}$$

Taking the second order partial derivative we obtain

$$\partial_x^2(\log((x-a)^2 + (y-b)^2)) = \frac{2}{(x-a)^2 + (y-b)^2} - \frac{4(x-a)^2}{[(x-a)^2 + (y-b)^2]^2} \tag{ii}$$

A similar result holds when we apply ∂_y^2. With $\Delta = \partial_x^2 + \partial_y^2$ we add up the result and obtain:

$$\Delta(\log (x-a)^2 + (y-b)^2) = 0 \tag{iii}$$

Thus, the Log-function satisfies the Laplace equation in $\mathbf{R}^2 \setminus (a, b)$.

3.2 The $\mathcal{L}_\epsilon$-functions. There remains to understand what occurs at (a, b). Since the situation is invariant under translation we can take (a, b) as the origin and with $\epsilon > 0$ we consider the function

$$\mathcal{L}_\epsilon(x, y) = \log[\ (x-a)^2 + (y-b)^2 + \epsilon\,]$$

Here the derivatives are defined in the whole of $\mathbf{R}^2$ and a calculation gives

$$\Delta(\mathcal{L}_\epsilon)(x, y) = \frac{4\epsilon}{(x^2 + y^2 + \epsilon)^2} \tag{1}$$

Let us calculate the area integral over $\mathbf{R}^2$. Using polar coordinates it becomes

$$4\epsilon \cdot 2\pi \int_0^\infty \frac{r dr}{(r^2 + \epsilon)^2} = 8\epsilon \cdot \pi \cdot \frac{1}{2} \frac{1}{(r^2 + \epsilon)} |_\epsilon^\infty = 4 \cdot \pi \tag{2}$$

Taking one half of the $\mathcal{L}$-function which means that we take a square root of the Log-function we have therefore proved:

3.3 Proposition. *For every $\epsilon > 0$ one has*

$$\frac{1}{2\pi} \cdot \iint_{\mathbf{R}^2} \Delta(\log \sqrt{x^2 + y^2 + \epsilon}) \cdot dxdy = 1$$

Remark. Thus, for every $\epsilon > 0$ we have the nice function $\log \sqrt{x^2 + y^2 + \epsilon})$. We have proved that its Laplacian is a positive function and its integral taken over $\mathbf{R}^2$ is equal to 2π. The passage to the limit as $\epsilon \to 0$ leads to an important conclusion. Namely, let $\phi(x,y)$ be an arbitrary C^2-function with compact support. In chapter II we shall learn that Green's formula entails that

$$\text{(i)} \qquad \iint \Delta(\phi)\cdot\mathcal{L}_\epsilon\cdot dxdy = \iint \phi\cdot\Delta(\mathcal{L}_\epsilon)\cdot dxdy$$

hold for each $\epsilon > 0$. Using (1-2) the reader may verify the limit formula

$$\lim_{\epsilon\to 0}\iint \phi\cdot\Delta(\mathcal{L}_\epsilon)\cdot dxdy = 4\pi\cdot\phi(0,0)$$

Hence the left hand side in (i) also has a limit. In the section about distributions we shall learn that the limit formula above means that the Laplacian taken in the distribution sense of the locally integrable function

$$\log|z| = \log\sqrt{x^2+y^2}$$

is equal to 2π times the Dirac measure at the origin, i.e. one has the equality

$$(*) \qquad \Delta(\log|z|) = 2\pi\cdot\delta_{(0,0)}$$

The result in (*) is fundamental and will appear frequently later on where and we remark that (*) is a real version of the complex residues which we shall encounter in Chapter III.

3.5 Subharmonic functions. A C^2-function u is called subharmonic if $\Delta(u)\neq 0$. By Green's formula and using the log-function one gets a formula for the deviation between values of u at a point p and mean-values taken over discs centered at p. More precisely, suppose that p is the origin and u is defined in some disc $\{|z|<R\}$. of $0<r<R$ we consider the function $\log\frac{|z|}{r}$ which is zero on $|z|=r$. Using Green's formula we shall learn in Chapter V:B that one has the equation:

$$u(0,0) = M_u(r) - \frac{1}{2\pi}\iint_{[z|<r}\log\frac{|z|}{r}\cdot\Delta(u)(x,y)\cdot dxdy$$

3.4 Radial functions. Let $\phi(r)$ be a function defined for the positive real numbers where it has at least two derivatives. In the punctured complex plane with the origin is removed we get the function

$$z\mapsto\phi(|z|)$$

Since $|z|=\sqrt{x^2+y^2}$ rules of differentiation give:

$$\text{(i)} \qquad \partial_x(\phi(\sqrt{x^2+y^2}) = \frac{x}{\sqrt{x^2+y^2}}\cdot\phi'(\sqrt{x^2+y^2})$$

The second order partial x-derivative becomes:

$$\text{(ii)} \qquad \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}\cdot\phi'(\sqrt{x^2+y^2}) + \frac{x^2}{x^2+y^2}\phi''(\sqrt{x^2+y^2})$$

Similar formulas hold for the partial y-derivatives and adding the result we get

3.5 Proposition. *One has the equality*

$$\Delta(\phi(|z|) = \frac{1}{|z|}\cdot\phi'(|z|) + \phi''(|z|)$$

Remark. This equation is often used to construct subharmonic functions via radial functions.

4. Some complex mappings

4.1 The inversion $z \to \frac{1}{z}$. Let w be a new complex variable. Consider the map

$$z \mapsto \frac{1}{z} = w \tag{1}$$

Denote by $\mathcal{C}$ the family of all circles in the complex z-plane which do not contain the origin, i.e. a circle in this family is of the form

$$|z - z_*| = r \quad : \quad 0 < r < |z_*| \tag{*}$$

It turns out that every image of a circle in (*) gives a circle in the w-plane. To show this we prefer to express the equation for circles in $\mathcal{C}$ in another way. Namely, the equation for a circle in $\mathcal{C}$ is of the form:

$$az\bar{z} + bz + \bar{b}\bar{z} + c = 0 \quad : a, c \text{ real and both } \neq 0 \tag{2}$$

The proof of (2) is left as an exercise. From (2) it is easily seen that the equation for the image with $w = \frac{1}{z}$ becomes

$$cw\bar{w} + \bar{b}w + b\bar{w} + a = 0 \tag{3}$$

Here (3) is the equation for a circle in the w-plane which again does not contain the origin. Hence we have proved that the inversion map gives a *1-1 correspondence* between the class of circles which do not contain the origin.

Exercise. Consider a circle in $\mathcal{C}$ defined by an equation

$$|z - A| = s \quad : \quad 0 < s < A$$

Thus, the center is the real point A and the radius is s. Show that the equation for the image circle becomes

$$|w - \frac{A}{A^2 - s^2}| = \frac{s}{A^2 - s^2} \tag{*}$$

Next, let $\mathcal{C}_*$ be the family of circles in the z-plane which contain $z = 0$. The equation for a circle in $\mathcal{C}_*$ is given as in (2) above where

$$a, b \neq 0 \quad : \quad c = 0$$

The image circle gets the equation

$$\bar{b}w + b\bar{w} + a = 0$$

With $b = re^{i\theta}$ and a real we get

$$\mathfrak{Re}\, e^{i\theta} \cdot w = -\frac{a}{2r} \tag{4}$$

This is the equation for a line in the w-plane and the reader should contemplate upon some specific examples to see how these lines are affected when θ varies and also verify that if C_1 and C_2 are two different circles in $\mathcal{C}_*$ then the image lines are *different*. Indeed, θ determines the direction of a line and once $\arg(b)$ is fixed, the quotients $\frac{a}{r}$ are never equal for a pair of $\mathcal{C}$-circles defined by (2) with $c = 0$. Hence we have proved that there is a 1-1 correspondence between circles in the z-plane which contain the origin and lines in the w-plane which do not contain the origin. Moreover, under this correspondence a pair of circles C_1 and C_2 from $\mathcal{C}_*$ gives two *parallell lines* in the w-plane if and only if

$$\arg(b_1) = \arg(b_2) \tag{5}$$

The reader should also verify that (5) holds if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent to each other at the origin and illustrate this by figures.

4.2 A specific example. Consider the family of circles

$$\mathcal{C}_r = \{(x-r)^2 + y^2 = r^2\} \quad : r > 0$$

In the complex notation the equations are:

$$z\bar{z} - rz - r\bar{z} = 0$$

So here $a = 1$ and $b = -r$. The image line has therefore the equation

$$\mathfrak{Re}\, w = \frac{1}{2r} \tag{6}$$

Take as an example $r = 1$. On $\mathcal{C}_1$ we find the point $(2, 0)$ and its image is the real w-pont $\frac{1}{2}$ is okay by (6) since $r = 1$. Moving along $\mathcal{C}$ in the positive direction we set

$$z = 1 + e^{i\theta} \quad : 0 < \theta < \pi$$

and get the image points $w(\theta) = \frac{1}{1+e^{i\theta}}$ where

$$\mathfrak{Im}\, w(\theta) = -\frac{\sin\theta}{2 + 2\cos\theta}$$

Notice the minus sign, i.e. $w(\theta)$ travels in the negative direction on the vertical line $\mathfrak{Re}\, w = \frac{1}{2}$ while z moves along $\mathcal{C}$ in the positive sense. This is no surprise because the inversion changes the orientation.

4.3 The map $z \mapsto z + \frac{1}{z}$

Here the geometric description is more involved. But let us give some examples of images under this map from the z-plane to the w-plane where we recall that $w = u + iv$.

4.4 Proposition 1 *Let $\mathcal{C}$ be a circle of radius $r > 1$. Then its image $\mathcal{C}_*$ in the w-plane is an ellipse defined by the equation*

$$\frac{u^2}{4r^2} + \frac{v^2}{(r - \frac{1}{r})^2} = 1$$

The easy verification is left to the reader. in addition to the equation above the ellipse has focal points and again we leave as an exercise to show that they are placed at $(1, 0)$ and $(-1, 0)$ which means that

$$|w - 1| + |w + 1| = 4$$

for all points on the ellipse. Notice that the two focal points do not depend on r. When $r \to 1$ the ellipse approaches the real interval $\{-2 \le u \le 2\}$.

4.5 Proposition *Let $\ell = \{se^{i\theta} : -\infty < s < \infty\}$ be a line but not the real or the imaginary axis. Then ℓ_* is the hyperbel defined by*

$$\frac{u^2}{\cos^2(\theta)} - \frac{v^2}{\sin^2(\theta)} = 1$$

4.6 Remark. We leave the proof as an exercise and the reader should also verify that the focal points are the same as for the ellipses above, i.e. $(1, 0)$ and $-1, 0)$. Let is also recall a result in euclidian geometry which asserts that when an ellipse and a hyperbel have common focal points, then they intersect at right angles. Above the lines and the circles intersect with the angle $\pi/2$ in the z-plane and we shall learn that the map $z \to z + \frac{1}{z}$ is conformal which means that angles are preserved. Hence the classical result from euclidian geometry is rediscovered via Proposition 4.4-4.5 which both are derived using complex computations.

4.7 The complex exponential e^z

With $x = z + iy$ we consider the strip domain defined by

$$\square = \{(x, y) \quad : -\infty < x < \infty \quad \text{and} \quad 0 < y < 2\pi\}$$

Let $\zeta = \xi + i\eta$ be a new complex variable. We construct a map from the (x, y)-plane into the (ξ, η)-plane by

$$(*) \qquad \xi = e^x \cdot \cos(y) \quad \text{and} \quad \eta = e^x \cdot \sin(y)$$

It is clear that this gives a 1-1 map from $\square$ onto the (ξ, η)-plane where the non-negative axis $\{\xi \geq 0\} \cap \{\eta = 0\}$ has been removed.

4.8 Some geometric images. The vertical line $\ell^*(a) = \{x = a\} \cap \{0 < y < 2\pi\}$ is mapped to a circle $\xi^2 + \eta^2 = e^{2a}$ where the point $(e^{2a}, 0)$ has been excluded. The image of a horizontal line $\ell_*(b) = \{y = b\}$ becomes a half-ray defined by

$$\xi = r \cdot \cos(b) \quad \text{and} \quad \eta = r \cdot \sin(b)$$

Notice that the image circles and the half-rays intersect at a right angle. The same is true for the corresponding families of vertical, respectively horizontal line in the strip. As we shall seen this is no accident, i.e. it reflects the conformality of the complex analytic function e^z which precisely corresponds to the map above and later we learn that $z \mapsto \zeta = e^z$ is a conformal map from $\square$ onto $\mathbf{C} \setminus \mathbf{R}^*+$. But it is instructive to see some geometric pictures at this early stage. Let us now describe images of a more involved nature.

4.9 Images of circles. Let $0 < r < \pi/2$ and consider the circle defined by

$$\mathcal{C}(r) \quad : \quad x^2 + (y - \pi/2)^2 = r^2$$

In polar coordinates we write $x = r \cdot \cos(\phi)$ and $y = \pi/2 + r \cdot \sin(\phi)$ ann now the image is a closed curve parametrized by ϕ where

$$\xi(\phi) = e^{r \cdot \cos(\phi)} \cdot \cos(\pi/2 + r \cdot \sin(\phi)) \quad \text{and} \quad \eta(\phi) = e^{r \cdot \cos(\phi)} \cdot \sin(\pi/2 + r \cdot \sin(\phi))$$

Using *Mathematica* the reader can plot these closed curves as $0 < r < \pi/2$ varies. When $r \to 0$ the curves become more and more circular. Let us notice that

4.10 The inverse map. From (*) we get

$$\xi^2 + \eta^2 = e^{2x} \implies x = \text{Log}(\sqrt{\xi^2 + \eta^2})$$

Using the complex argument we see that

$$y = \arg(\xi + i\eta)$$

where the argument is determined when it takes values in $(0, 2\pi)$. In this way we recognize that the inverse map is described by the complex log-function $\text{Log}(\zeta)$ and we shall learn that in the present situation we have chosen a single-valued branch of this function since the non-negative real ξ-axis has been removed.

Remark. The discussion above shows that on can start from constructions in real analysis dealing with vector-valued functions in $\mathbf{R}^2$ and after make complexifications of these. So it is foremost a matter of notations to use z and ζ instead. But in the long run the complex notations are more convenient when we begin to construct complex derivatives and power series expansions of analytic functions. However, one should not forget the underlying real picture. As an example, from (*) we can pay attention to the function $\xi = \xi(x, y)$ and notice that its partial derivatives become

$$(1) \qquad \xi'_x = e^x \cdot \cos(y) \quad \text{and} \quad \xi'_y = -e^x \cdot \sin(y)$$

A similar computation of the partial derivatives of the η-function gives the two identities

(2) $$\xi'_x = \eta'_y \quad \text{and} \quad \xi'_y = -\eta'_x$$

We shall learn in Chapter 3 that this expresses the Cauchy-Riemann equations which hold because e^z is an analytic function. Notice also that $\xi(x,y)$ satisfies the Laplace equation $\Delta(\xi) = 0$, i.e. it yields a harmonic function. Similarly $\Delta(\eta) = 0$.

4.11 The harmonic angle function.

With $z = x + iy$ we refer to $\{y < 0\}$ as the upper half-plane in $\mathbf{C}$. It is denoted by U_+ and the boundary is the real x-axis.If $z \in U_+$ we use polar coordinates and write:

$$z = x + iy = r \cdot \cos\theta + i \cdot r \cdot \sin\theta$$

where $r = |z| = \sqrt{x^2 + y^2}$ and $0 < \theta < \pi$. Here the θ-angle is determined in a unique way and this means that we have a well defined angle-function

(1) $$(x, y) \mapsto \theta$$

of course, introducing the argument of complex numbers this is the same as

(2) $$\arg(z) = \theta$$

Now $\theta = \theta(x, y)$ is regarded as a function of x an y.

4.12 Exercise. Prove that the partial derivatives of the θ-function are given by

$$\theta'_x = -\frac{y}{x^2 + y^2} \quad \text{and} \quad \theta'_y = \frac{x}{x^2 + y^2}$$

Deduce from this that $\Delta(\theta) = 0$, i.e. the θ-function is harmonic in U_+.

4.13 Exercise. Let $a < b$ be two real numbers. In U_+ we define the function

$$H_{a,b}(z) = \arg(z - b) - \arg(z - a)$$

Use Exercise 4.12 to show that $\Delta(H) = 0$ and show the limit formulas:

(1) $$\lim_{y \to 0} H(x + iy) = \pi \quad \text{for all} \quad a < x < b$$

(2) $$\lim_{y \to 0} H(x + iy) = 0 \quad x < a \quad \text{or} \quad x > b$$

4.14 Remark. The harmonic H-functions above are very important. For example, they can be used to solve the Dirichlet problem in a quite explicit fashion where one starts with a bounded continuous function $f(x)$ on the real x-line and seeks a harmonic function $H(x, y)$ in U_+ whose boundary values give f, i.e.

(*) $$\lim_{y \to 0} H(x, y) = f(x)$$

We return to this in XXX.

5. The sterographic projection.

Introduction. A sphere Σ of diameter one is placed above the origin in the complex z-plane. Let (x, y, h) be the coordinates in $\mathbf{R}^3$. The center of the sphere is $(0, 0, 1/2)$ and Σ is defined by the equation

(0.1) $$x^2 + y^2 + (h - 1/2)^2 = \frac{1}{4}$$

The point $N = (0, 0, 1)$ is called the north pole and the origin the south pole. If $(x, y, 1/2) \in \Sigma$ then (0.1) entails that

(0.2) $$x^2 + y^2 = h - h^2$$

The stereographic projection sends a point $p \in \Sigma \setminus N$ to the point p_* in $\mathbf{C}$ which arises when we draw the line through N and p which hits p_*. If $p = (x, y, h) \in \Sigma$ the reader may verify that one has the equality:

$$p_* = \frac{1}{1-h} \cdot (x + iy) \tag{0.3}$$

In particular the the southern hemi-sphere where $h < 1/2$ is mapped onto the unit disc D and the northern hemi-sphere where $1/2 < h < 1$ is mapped onto the exterior disc $|z| > 1$. The equator circle $h = 1/2$ is mapped onto the circle of radius one centered at the origin in the z-plane.

Images of great circles.

A great circle $\mathcal{C}_\Pi$ on Σ arises when we take the intersection of Σ and a 2-dimensional plane Π in $\mathbf{R}^3$ which contains the center $(0, 0, 1/2)$. In the special case when Π contains the north-pole the image of $\mathcal{C}_\Pi$ under the map (0.2) becomes a straight line The easy verification is left to the reader. Next, consider the case when $\mathcal{C}_\Pi$ does not contain the northpole. Then the image $\Pi_*$is a bounded closed curve. It turns out that it is a circle, i.e. one has

5.1. Theorem. *For every great circle $\mathcal{C}$ which does not contain N, the image $\mathcal{C}_*$ under (0.3) is a circle.*

Exercise. Prove this. The hint is that up to a rotation it suffices to consider a plane defined by the equation

$$x = A(h - 1/2) \quad : \; A > 0$$

The points on $\Pi \cap \Sigma$ whose stereographic projection has maximal resp. minimal distance to the origin has y-coordinate zero, i.e. we seek points

$$p = (x, 0, h) \quad : \quad x^2 = h - h^2 \quad \text{and} \quad x = A(h - 1/2)$$

It means that h satisfies the second order algebraic equation

$$A^2(h - 1/2)^2 + (h - 1/2)^2 = 1/4$$

The two solutions become

$$h^* = \frac{1}{2}[1 + \frac{1}{\sqrt{1 + A^2}}] \quad : \quad h_* = \frac{1}{2}[1 - \frac{1}{\sqrt{1 + A^2}}]$$

The two projected points become

$$x^* = A \cdot \sqrt{\frac{h^*}{1 - h^*}} x_* = A \cdot \sqrt{\frac{h_*}{1 - h_*}}$$

At this stage the reader can finish the proof of Theorem 1.1.

The spherical metric.

In the complex z-plane we define a σ-metric as follows:

5.2 Definition. *The spherical metric is defined by*

$$d\sigma = \frac{ds}{1 + |z|^2}$$

where ds denotes the usual euclidian metric.

This means that if γ is some parametrised C^1-curve then its total σ-length becomes

$$\sigma(\gamma) = \int_0^T \frac{|\dot{z}(t)| \cdot dt}{1 + |z(t)|^2}$$

where we have put $z(t) = x(t) + iy(t)$ so that $|\dot{z}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$.

5.3 Geodesic curves. Given a pair of points p and q in the z-plane one seeks the minimum of $\sigma(\gamma)$ taken over all closed curves γ whose end-points are p and q. This leads to a variational problem. It turns out that the curve γ which minimises the σ-distance between two points p and q

which both are outside the origin is a circular arc. In the case when p is the origin the minimizing curve is the straight line from p to q. To prove this one uses the stereograhic projection. First, on the sphere Σ one employs the usual metric induced by the euclidian metric in $\mathbf{R}^3$ and it is wellknown that geodesic curves on Σ are consist of arcs on great circles. For example, if p is the south pole then the shortest air-borne flight from p to another point q on the earth is to follow the great circle which passes through p and q. The notable fact is that the stereographic projection is an *isometry* when we use the σ-metric in the complex z-plane. To prove this we first analyse the case when p is the south pole and $q = (x, 0, h)$ is a point on the southern hemisphere with $y = 0$. Now $q_* = (x_*, 0)$ and the geodesic curve with respect to the σ-metric is obviously given by the line from $z = 0$ to q_*. Its σ-distance becomes

$$\int_0^{x_*} \frac{dt}{1+t^2}$$

Next, the distance from the south pole to q is given by

$$\frac{\theta}{2} \quad \text{where} \quad \sin(\theta) = \frac{x}{2}$$

where θ is the angle between the vertical line from the north pole to the south pole and the line from the center $(0, 0, 1/2)$ to q. The reader may draw a picture to illustrate this and verify that

$$1 - 2h = \cos\theta \quad \text{and} \quad x = \frac{\sin\theta}{2}$$

The trigonometric formula $\cos\theta = 1 - 2 \cdot \sin^2(\theta/2)$ gives

$$h = \sin^2(\theta/2) \implies 1 - h = \cos^2(\theta/2)$$

Now $q_* = (x_*, 0)$where

$$x_* = \frac{x}{1-h} = \frac{1}{2} \cdot \frac{\sin(\theta)}{\cos^2(\theta/2)}$$

Since $\sin(\theta) = 2\sin(\theta/2) \cdot \sin(\theta/2)$ we finally obtain

$$x_* = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \mathrm{tg}(\theta/2)$$

Hence the σ-distance becomes

$$\int_0^{x_*} \frac{dt}{1+t^2} = \mathrm{arctg}(x_*) = \frac{\theta}{2}$$

where the last term is the air-borne distance on the earth from the south pole to q. This proves that the stereograhic projection is an isometry in the special case when p is the south pole so that p_* is the origin, The general case is proved using a family of isometric maps with respect to the σ-metric given below.

5.4 Distance preserving maps. Let a be a complex number with $|a| < 1$. and consider the map

(*) $$M_a^*(z) = \frac{z-a}{1+\bar{a}z} \quad : \; |z| < 1$$

Set $w(z) = M_a^*(z)$. Differentiation gives

$$\frac{dw}{dz} = \frac{1}{1+\bar{a}z} - \frac{\bar{a}(z-a)}{(1+\bar{a}z)^2} = \frac{1+|a|^2}{(1+\bar{a}z)^2}$$

We have also

$$\frac{1}{1+|w|^2} = \frac{|1+\bar{a}z|^2}{|z-a|^2 + |1+\bar{a}z|^2}$$

It follows that

(i) $$\frac{|dw|}{1+|w|^2} = \frac{(1+|a|^2) \cdot |dz|}{|z-a|^2 + |1+\bar{a}z|^2}$$

Next, we notice the identity

(ii)
$$(1+|a|^2)(1+|z|^2) = |z-a|^2 + |1+\bar{a}z|^2 \quad : |a|<1$$

We conclude that

(*)
$$\frac{|dw|}{1+|w|^2} = \frac{|dz|}{1+|z|^2}$$

Hence the map $z \to M_a^*(z)$ preserves the σ-metric.

Exercise Suppose that $0 < a < 1$ is real and positive. Set

$$w^* = \frac{1+a}{1-a} \quad \text{and} \quad w_* = \frac{1-a}{1+a}$$

We get the middle point

$$w_0 = \frac{w^* + w_*}{2} = \frac{1+a^2}{1-a^2}$$

Now we get the circle centered at w_0 whose radius us

$$r = w^* - w_0 = \frac{2a}{1-a^2}$$

With these notations the reader should verify that the map $z \to M_a^*(z)$ is a bijective from the unit disc $|z| < 1$ onto the disc $|w - w_0| < r$.

Remark. Notice that the transformations defined by (*) in 2.3 are not Möbius transforms since we have changed the sign for the term $\bar{a}z$.

5.5 The hyperbolic metric.

The spherical metric must not be confused with the hyperbolic metric in D whose metric is defined in the open unit disc by

(*)
$$d\sigma_{hyp} = \frac{ds}{1-r^2} \quad : \quad |z| = r$$

Here Möbius transforms preserve the hyperbolic metric. More precisely, let $|a| < 1$ and consider the transform

$$M_a(z) = \frac{z-a}{1-\bar{a}\cdot z} \quad : \quad z \in D$$

Then one has the quality

(**)
$$d\sigma_{hyp}(z_1, z_2) = d\sigma_{hyp}(M_a(z_1), M_a(z_2)) \quad : \quad z_1, z_2 \in D$$

To prove (**) we take some point $z_0 \in D$ and with a small $\Delta\, z$ we set $z_1 = z_0 + \Delta\, z$. Then we have

$$M_a(z_0 + \Delta\, z) - M_a(z_0) = \frac{1-|a|^2}{(1-\bar{a}z)^2} \cdot \Delta\, z + O(|\Delta\, z|^2)$$

It follows that

$$d\sigma_{hyp}(M_a(z_0+\Delta z), M_a(z_0)) = \frac{1-|a|^2}{|1-\bar{a}z|^2} \cdot |\Delta\, z| \cdot \frac{1}{1-|M_a(z_0)|^2} + O(|\Delta z|^2) =$$

(i)
$$\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2 - |z-a|^2} \cdot |\Delta\, z| + O(|\Delta z|^2)$$

Now the reader may verify that the first term in (i) becomes

$$\frac{|\Delta z|}{1-|z|^2} \simeq d\sigma_{hyp}(z+\Delta z, z)$$

When $\Delta z \to 0$ we get (**).

5.6 Geodesic curves. It is easily seen that if $z_0 \neq 0$ is a point in D, then the geodesic curve from the origin to z_0 is the straight line. The distance becomes

$$(*) \qquad \int_0^r \frac{ds}{1-s^2} = \frac{1}{2} \cdot \text{Log}\, \frac{1+r}{1-r} \quad : \quad |z| = r$$

Thus, in the hyperbolic metric points close to the unit circle are remote from the origin.

5.7 The general linear group. It is defined by maps

$$(*) \qquad z \mapsto \frac{az+b}{cz+d} \quad : \quad ad-bc \neq 0$$

This is a group of transformations which can be identified with the multiplicative group of invertible complex 2×2-matrices divided by the central subgroup of diagonal matrises $\lambda \cdot E_2 \quad : \lambda \neq 0$. For a detailed study of this group we refer to Chapter 1 in [Ahl]. Let us only remark that via the stereographic projection every transformation in (*) yields a bijective map on Σ.

5.8 Schwarz derivatives

Let $0 < r < 1$ and $z = re^{\phi}$ for some ϕ. If $w = e^{i\theta}$ is a point on the unit circle T we get the point $w^* \in T$ by drawing the line from w through z which hits T at w^*. Euclidian geometry shows that

$$(1) \qquad 1 - |z|^2 = |w-z| \cdot |w^* - z|$$

By this construction we get a map from T onto itself defined by

$$w \mapsto w^*$$

With $w = e^{i\theta}$ we set $w^* = e^{i\beta(\theta)}$. So now $\beta(\theta)$ is a 2π-period function.

Exercise. Show that the β-derivative is given by the formula:

$$(*) \qquad \frac{d\beta}{d\theta} = \frac{1-|z|^2}{|w-z|^2} = \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}$$

Remark. In the right hand side we recognize the Poisson kernel which therefore is expressed by the θ-derivative of $\beta = \beta(\theta)$ when $z = re^{i\phi}$ is kept fixed. Regarding the geometric picture for the construction of the β-function it is evident that when θ makes a full turn around the unit circle, then the β-angle also makes a full turn. Hence (**) gives:

$$(**) \qquad \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} = 2\pi$$

Remark. H. Schwarz used this geometric construction of the Poisson kernel to show that various integrals are invariant under Möbius transformations.

5.9 Die kreisgeometrischen Massbestimmung We shall introduce a distance function in D which for example is used to construct of *normal domains* attached to locally conformal maps from D onto domains which are not simply connected. To each pair of points z_1, z_2 in D we set

$$\delta(z_1, z_2) = |\frac{z_1 - z_2}{1 - \bar{z}_1 \cdot z_2}|$$

In 5.10 we shall prove that δ satisfies the triangle inequality. Notice that the distance is < 1 for each pair of points. When $z_2 = 0$ we see that $\delta(z_1, 0) = |z_1|$, i.e. we recover the ordinary euclidian distance from the origin to z_1. On the other hand, if $0 < a < b < 1$ are real and positive we see that

$$\delta(b,a) = \frac{b-a}{1-ab} > b-a$$

So for such pair the δ distance is larger than the euclidian. Next, let $\zeta \in D$ and consider the Möbius transform

$$z \mapsto \frac{z-\zeta}{1-\bar{\zeta}\cdot z}$$

Consider a pair z_1, z_2 and put

$$w_1 = \frac{z_1-\zeta}{1-\bar{\zeta}\cdot z_1} \quad : \quad w_2 = \frac{z_2-\zeta}{1-\bar{\zeta}\cdot z_2}$$

A computation which is left to the reader shows that

$$(*) \qquad \delta(z_1,z_2) = \delta(w_1,w_2)$$

Hence Möbius transforms preserve the δ-distance.

5.10 The triangle inequality. To prove that δ satisfies the triangle inequality we use (*) which reduces the proof to the case when $z_1 = 0$ and $z_2 = a$ is real and positive. Now we must show

$$\delta(0,z_3) \leq \delta(0,z_2) + \delta(a,z_3) \quad : \quad z_3 \in D$$

This amounts to show that

$$\text{(i)} \qquad |z_3| \leq a + \Big|\frac{z_3-a}{1-az_3}\Big|$$

Here (i) is obvious if $|z_3| \leq a$. Next, with b kept fixed we consider for some $a < r < 1$ all z_3 of absolute value r. Notice that

$$\min_\theta \Big|\frac{re^{i\theta}-a}{1-re^{i\theta}}\Big|$$

is attained when $\theta = 0$, i.e. when $z_3 = r$ is real and positive. In this case we have

$$a + \frac{r-a}{1-ar} = \frac{a-a^2r+r-a}{1-ar} - r = \frac{a-a^2r+r-a-r+ar^2}{1-ar} = \frac{ar(r-a)}{1-ar} > 0$$

This proves the triangle inequality for the δ-function.

Remark. The δ-distance is closely attached to Möbius transforms. For example, if $0 < \rho < 1$ and $w_* \in D$ is fixed then the set

$$\{w \quad : \quad \Big|\frac{w-w_*}{1-\bar{w}_*\cdot w}\Big| = \rho$$

is a circle strictly contained in D.

Exercise. Let $0 < a < 1$ and consider the set

$$E = \{w \in \mathbf{C} \quad : \quad \Big|\frac{w+a}{1+aw}\Big| = \Big|\frac{w-a}{1-aw}\Big|\}$$

Show that with $w = u + iv$ the equation for E becomes

$$4au(1-a^2)(1-u^2-v^2) = 0$$

Hence E is the union of the unit circle and the imaginary axis. Use this result together with the invariance property (*) to investigate sets of the form

$$E_{w_1,w_2} = \{w \in D \quad : \quad \Big|\frac{w-w_1}{1-\bar{w}_1 w}\Big| = \Big|\frac{w-w_2}{1-\bar{w}_2 w}\Big|$$

for a pair of points w_1, w_2 in D.

Chapter I:B. Series

Content of chapter

Introduction.

In Chapter 3 we establish Cauchy's integral formula and conclude that analytic functions are locally represented by a convergent power series. For this reason the study of series is important in analytic function theory. We start with additive series and after we treat product series and power series. Here various *Counting functions* are introduced which are used in many situations. An example is the proof by Kolmogorov Section XX about Hardy spaces in *Special Topics.* For a power series $\sum c_n z^n$ the radius of convergence ρ is determined by the formula

$$(*) \qquad \frac{1}{\rho} = \limsup_{n\to\infty} |\rho_n|^{\frac{1}{n}}$$

The study of convergence at boundary points where $|z| = \rho$ is quite involved. To begin with one studies existence of radial limits. For a fixed $0 \leq \theta < 2\pi$ one says that a power series (*) has a radial limit in the θ-direction if there exists

$$(*) \qquad \lim_{r\to 1} \sum c_n \cdot e^{in\theta} \cdot r^n$$

One can also consider less restricted limit conditions and we prove a result due to Landau of this kind at the end of section 2. *Blaschke products* play a central role in analytic function theory and they are studied in section 4.

The final sections contains more advanced material which is not needed for the overall study of analytic functions. Section 6 starts with Abel's theorem which gives a sufficient condition in order that an additive series $\sum a_n$ converges and has a sum s_* under the hypothesis that there exists

$$(*) \qquad \lim_{x\to 1} \sum a_n \cdot x^n = s_*$$

Abel's sufficiency condition is that

$$(**) \qquad \lim_{n\to\infty} \frac{a_n}{n} = 0$$

Under relaxed assumptions the convergence of $\sum a_n$ when (*) holds we shall prove a famous theorem due to Hardy and Littlewood. The result is that if $\{a_n\}$ is a sequence for which there exists a constant C such that

$$a_n \leq \frac{C}{n} \quad \text{hold for every} \quad n \geq 1$$

then (*) implies that the series $\sum a_n$ is convergent. The proof contains several ingenious steps which demonstrates that the study of series is a rich and often quite hard subject. Thorin's convexity theorem in section 10 illustrates the usefulness of complex methods to prove inequalities which from the start are given in a real setting. In Section 11 we study summations in the sense of Cesaro and Hölder and prove that they lead to equal limits whenever one of them exists. The material in section 8,9 and 12 contains results due to Carleman. Section 13 announce results about quasi-analytic functions and these more advanced results are not needed for the basic study of analytic functions.

Remark. The material in this section is not entirely self-contained since we often appeal to analytic functions whose theory is not presented until chapter III. But the reader can enter these more involved parts at a later occasion after analytic function theory has been studied in chapter III.

I. Additive series

1. Partial sums and convergent series. To a sequence $\{a_\nu\}$ of complex numbers indexed by non-negative integers one constructs the partial sums:

$$S_N = \sum_{\nu=0}^{\nu=N} a_\nu \quad : \quad N = 1, 2, \ldots$$

If the sequence of partial sums converges converge to a limit S_* one writes

$$S_* = \sum_{\nu=0}^{\infty} a_\nu \tag{1}$$

and say that $\{a_\nu\}$ yields a convergent series.

2. Absolute convergence. The series is absolutely convergent if

$$\sum_{\nu=0}^{\infty} |a_\nu| < \infty$$

Absolute convergence implies that $\sum a_\nu$ converges. For if $\{S_N\}$ are the partial sums of the series the triangle inequality gives

$$|S_M - S_N| = |\sum_{\nu=N+1}^{M} a_\nu| \leq \sum_{\nu=N+1}^{M} |a_\nu| \quad : \quad M > N \geq 0$$

The absolute convergence therefore implies that the sequence of partial sums is a *Cauchy sequence* of complex numbers and hence has a limit. The converse is false, i.e. there exist convergent series which are not absolutely convergent. For example, the series given by $\{a_\nu = \frac{(-1)^\nu}{\nu} : \ \nu \geq 1\}$ is not absolutely convergent since

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu} = \infty$$

On the other hand the series is convergent by the general result in 1.4 below.

3. A majorant principle. Let $\{a_\nu\}$ be a bounded sequence and $\{b_\nu\}$ a sequence such that $\sum |b_\nu| < \infty$. Then it is obvious that

$$\sum |a_\nu \cdot b_\nu| < \infty$$

4. Alternating series. Let $a_0, a_1, \ldots$ be a sequence of positive real numbers which is strictly decreasing, i.e. $a_0 > a_1 > \ldots$. Assume also that $\lim a_n = 0$. Then one gets a convergent series by taking alternating signs, i.e. the series below converges:

$$\sum_{\nu=0}^{\infty} (-1)^\nu \cdot a_\nu \tag{*}$$

Proof. Even partial sums are expressed by a positive series:

$$S_{2N} = (a_0 - a_1) + \ldots + (a_{2N} - a_{2N-1}) > 0 \quad : N \geq 1$$

At the same time one has

$$S_{2N} = a_0 - [(a_1 - a_2) + \ldots + a_{2N-1} - a_{2N}]$$

where the last term is the negative of a positive series. Hence $\{S_{2N}\}$ is bounded and non-decreasing sequence of positive real numbers. This implies that there exists a limit:

$$\lim_{N\to\infty} S_{2N} = S_*$$

Finally, since $a_n \to 0$ we get the same limit using odd indices and conclude that (*) is convergent.

5. The partial sum formula. Consider two sequences $\{a_\nu\}$ and $\{b_\nu\}$. Set

$$S_N = \sum_{\nu=0}^{\nu=N} a_\nu \quad : \quad T_N = \sum_{\nu=0}^{\nu=N} a_\nu \cdot b_\nu$$

Since $a_\nu = S_\nu - S_{\nu-1}$ it follows that

$$(*) \qquad T_N = \sum_{\nu=0}^{\nu=N} (S_\nu - S_{\nu-1}) \cdot b_\nu = a_0 b_0 + \sum_{\nu=1}^{\nu=N} S_\nu \cdot (b_\nu - b_{\nu-1}) + S_N \cdot b_{N-1}$$

This formula which resembles partial integration of functions is quite useful.

6. Exercise Let $b_1 \geq b_2 \geq \ldots$ be a non-increasing sequence of positive real numbers. Show that for every finite n-tuple $a_1, \ldots, a_n$ of complex numbers one has the inequality

$$\big|b_1 a_1 + \ldots + b_n a_n\big| \leq b_1 \cdot M \quad \text{where} \quad M = \max_{1 \leq k \leq n} |a_1 + \ldots + a_k|$$

7. Theorem by Abel. *Assume that the partial sums $\{S_N\}$ of $\{a_\nu\}$ is a bounded sequence and that the positive series $\sum |b_\nu - b_{\nu-1}| < \infty$ where $b_\nu \to 0$ as $\nu \to \infty$. Then the series below is convergent*

$$\sum_{\nu=0}^{\infty} a_\nu \cdot b_\nu$$

Proof. By (3) the series $\sum_{\nu=0}^{\infty} S_\nu \cdot (b_\nu - b_{\nu-1})$ is absolutely convergent and by the hypothesis we also have $S_N \cdot b_{N-1} \to 0$ as $N \to \infty$. Hence (*) in (5) shows that $\{T_N\}$ has a limit T_* expressed by the absolutely convergent series

$$T_* = a_0 b_0 + \sum_{\nu=1}^{\infty} S_\nu \cdot (b_\nu - b_{\nu-1})$$

Next, let $\{b_\nu(p)\}$ be a doubly indexed sequence of non-negative numbers which satisfies

$$\nu \mapsto b_\nu(p) \quad \text{is non-increacing for each} \quad p = 0, 1, \ldots$$

$$\lim_{p \to \infty} b_\nu(p) = 1 \quad \text{for each} \quad \nu = 0, 1, \ldots$$

8. Theorem. *For each convergent series $\sum a_\nu$ it follows that*

$$\sum_{\nu=0}^{\infty} a_\nu \cdot b_\nu(p)$$

converges and one has the limit formula

$$\lim_{p \to \infty} \sum_{\nu=0}^{\infty} a_\nu \cdot b_\nu(p) = \sum_{\nu=0}^{\infty} a_\nu$$

9. Exercise. Prove this result. The hint is to employ the partial sum formula. We finish with another useful result.

10. Theorem *Let $\{a_\nu\}$ be a non-decreasing sequence of real numbers such that*

$$(*) \qquad \sum_{k=1}^{\infty} 2^{-k}(a_{k+1} - a_k) < \infty$$

Then

$$(**) \qquad \lim_{k \to \infty} 2^{-k} a_k \to 0$$

Proof. Let S_N denote partial sums of (*). If $M > N$ the partial summation formula gives:

$$\text{(i)} \qquad S_M - S_N = 2^{-M} a_{M+1} - 2^{-N} a_N + \sum_{\nu=N+1}^{M} 2^{-\nu} a_\nu$$

By the assumption $a_N \leq a_\nu$ when $\nu > N$ which gives:

$$S_M - S_N \geq 2^{-M} a_{M+1} - 2^{-N} a_N + a_N \cdot \sum_{\nu=N+1}^{\nu=M} 2^{-\nu} =$$

$$2^{-M} a_{M+1} - 2^{-N} a_N + a_N \cdot 2^{-N}(1 - 2^{-M+N}) = 2^{-M} a_{M+1} - a_N \cdot^{-M}$$

Now we can argue as follows. Since (*) holds the partial sums is a Cauchy sequence. So if $\epsilon > 0$ there exists N_* such that $S_M < S_{N_*} + \epsilon$ for every $M > N_*$. With $N = N_*$ above we therefore get

(iii) $$2^{-M} a_{M+1} \leq \epsilon + a_{N_*} \cdot 2^{-M} \quad : \quad M > N_*$$

With N_* fixed we find a large M such that (iii) entails

$$2^{-M-1} a_{M+1} \leq 2 \cdot \epsilon$$

Since ϵ is arbitrary we get the limit (**).

B. Counting functions.

A counting function $N(s)$ is an integer valued function with jumps at some strictly increasing sequence $0 < s_1 < s_2 < \ldots$. Suppose that $N(s)$ is defined for $0 < s < 1$ and assume that:

(*) $$\int_0^1 (1 - s) \cdot dN(s) < \infty$$

1.B Theorem. When (*) holds it follows that

(**) $$\lim_{s \to 1} (1 - s) N(s) = 0$$

Proof. Put $a_k = N(1-2^{-k})$. The interval $[0, 1]$ can be divided into the intervals $[1-2^{-k}, 1-2^{-k-1})$. It is easily seen that (*) implies that

$$\sum 2^{-k}(a_{k+1} - a_k) < \infty$$

Hence Theorem 7 gives

(i) $$\lim_{k \to \infty} 2^{-k} N(1 - 2^{-k}) = 0$$

Next, if $s < 1$ we choose k such that $1 - 2^{-k} \leq s < 1 - 2^{-k-1}$ and then

(ii) $$(1 - s) N(s) \leq 2^{-k} \cdot N(1 - 2^{-k-1}) = 2 \cdot 2^{-k-1} \cdot N(1 - 2^{-k-1})$$

Hence (i) implies that (**) tends to zero as required.

B.2 A study of $\sum (1 - a_k)$

.

Let $0 < a_k < 1$ and suppose that the series

(1) $$\sum (1 - a_k) < \infty$$

Let $N(s)$ be the counting function with jumps at $\{a_k\}$. So (1) means that

(2) $$\int_0^1 (1 - s) dN(s) < \infty$$

With $s = 1 - \xi$ close to 1, i.e. when ξ is small the Taylor expansion of the Log-function at $s = 1$ gives

$$\log \frac{1}{s} = \log \frac{1}{1 - \xi} = \xi + \text{higher order terms in } \xi$$

Using this it is easily seen that (2) holds if and only if

(3) $$\int_0^1 \log(\frac{1}{s}) \cdot dN(s) < \infty$$

Next, for each $0 < r < 1$ we set

$$S(r) = \int_0^r \log\frac{1}{s}\cdot dN(s) \quad\text{and}\quad T(r) = \int_0^r \log\frac{r}{s}\cdot dN(s) \tag{4}$$

Since $\log(\frac{1}{s}) - \log(\frac{r}{s}) = \log\frac{1}{r}$ it follows that

$$S(r) - T(r) = \log\frac{1}{r}\int_0^r dN(s) = \log\frac{1}{r}\cdot N(r) \tag{5}$$

Now $\log\frac{1}{r} \simeq 1 - r$ as $r \to 1$ and since (2) above is assumed, it follows from Theorem 1.B that we have:

$$\lim_{r\to 1} \log\frac{1}{r}\cdot N(r) = 0$$

Hence (5) gives

$$\lim_{r\to 1} S(r) - T(r) = 0 \tag{6}$$

With the notations above we have therefore proved

3.B Theorem. *Assume that (2) holds. Then the following limit formula holds*

$$\lim_{r\to 1}\int_0^r \log\left(\frac{r}{s}\right)\cdot dN(s) = \int_0^1 \log\left(\frac{1}{s}\right)\cdot dN(s)$$

4.B Remark. By the multiplicative property of the Log-function the last term becomes

$$\sum \log\frac{1}{a_k} = \log\prod\frac{1}{a_k} \tag{i}$$

In the right hand side there appears the infinite product series defined by the a-sequence. In section III we study product series in more detail but already here we have seen an example of the interplay between additive series and product series. Notice also that via the equivalence of (1) and (2) above, Theorem 3.B gives the following:

5.B Theorem. *Let $\{a_k\}$ be a sequence with each $0 < a_k < 1$. Then the additive series $\sum(1 - a_k)$ is convergent if and only if the product series*

$$\prod_{k=1}^{\infty}\frac{1}{a_k} < \infty \tag{i}$$

Moreover, when (i) holds one has the limit formula

$$\lim_{r\to 1}\prod_r \frac{r}{a_k} = \prod_{\nu=1}^{\infty}\frac{1}{a_k} < \infty \tag{ii}$$

where $\prod_r$ is extened over those k for which $a_k \leq r$.

II. Power series.

Starting with a sequence $\{a_\nu\}$ and a complex number $z \neq 0$ we get the sequence $\{a_\nu \cdot z^\nu\}$. If this sequence is convergent the sum is denoted by $S(z)$.

1. Definition *The set of all $z \in \mathbf{C}$ for which the series*

$$\sum_{\nu=0}^{\infty} a_\nu \cdot z^\nu$$

converges is denoted by $\mathfrak{conv}(\{a_\nu\})$ and called the set of convergence for the a-sequence.

Remark. It may occur that $\mathfrak{conv}(\{a_\nu\})$ just contains $z = 0$. An example is when $a_\nu = \nu\,!$. But if the absolute values $|a_\nu|$ do not increase too fast the domain of convergence is non-empty. Since the terms of a convergent sequence must be uniformly bounded there exists a constant M such that

$$(1) \qquad |a_\nu| \cdot |z_0|^\nu \leq M \quad : \quad \nu = 0, 1, \ldots \quad : z_0 \in \mathfrak{conv}(\{a_\nu\})$$

If $|z| < |z_0|$ it follows that the series defined by $\{a_\nu \cdot z^\nu\}$ is absolutely convergent. Indeed, we have

$$|a_\nu \cdot z^\nu| \leq M \cdot \frac{|z|^\nu}{|z_0|^\nu}$$

Here $r = \frac{|z|}{|z_0|} < 1$ and the geometric series $\sum r^\nu$ is convergent. Hence the *Majorant principle* from I.3 yields the the absolute convergence of $\{a_\nu \cdot z^\nu\}$.

2. The radius of convergence. Above we saw that if $z_0 \in \mathfrak{conv}(\{a_\nu\})$ then the domain of convergence contains the open disc of radius $|z_0|$. Put

$$(*) \qquad \mathfrak{r} = \max |z| \quad : \quad z \in \mathfrak{conv}(\{a_\nu\})$$

Assume that $\mathfrak{conv}(\{a_\nu\})$ is not reduced to $z = 0$. Then $\mathfrak{r}$ is a positive number or $+\infty$. It is called the radius of convergence for $\{a_\nu\}$). The case $\mathfrak{r} = +\infty$ means that the series

$$\sum a_\nu \cdot z^\nu$$

converges for all $z \in \mathbf{C}$.

3. Hadamard's formula for $\mathfrak{r}$. Given a sequence $\{a_\nu\}$ its radius of convergence $\mathfrak{r}$ is found by taking a limes superior. More precisely, one has

$$(*) \qquad \frac{1}{\mathfrak{r}} = \text{Lim.sup}_{\nu \to \infty} \, |a_\nu|^{\frac{1}{n}}$$

The proof of this wellknown result is left to the reader.

Example. A sufficient condition in order that $\mathfrak{r} \geq 1$ for a given sequence $\{a_\nu\}$ can be checked as follows. Suppose that

$$|a_\nu| \leq e^{\rho(\nu)}$$

for some sequence $\{\rho(\nu)\}$. With $r < 1$ we can write $r = e^{-\delta}$ for some $\delta > 0$ and obtain

$$|a_n| \cdot r^n \leq \exp(\rho(n) - \delta \cdot n) \quad : n = 1, 2, \ldots$$

From this we conclude that $\mathfrak{r} \geq 1$ holds if

$$(**) \qquad \lim_{n \to \infty} \rho(n) - \delta \cdot n = -\infty \quad \text{holds for all} \quad \delta > 0$$

4. Application. Let $\sum a_n \cdot z^n$ be a power series whose radius of convergence is one. Let $\{b_n\}$ be some other sequence of complex numbers. We seek for conditions in order that the series $\sum b_n a_n \cdot z^n$ also converges when $|z| < 1$. The result below gives a sufficient condition for this to hold.

5. Theorem. *Let $\{\gamma_n\}$ be a sequence of positive numbers such that*

(i) $$\lim_{n\to\infty}\frac{\gamma_n}{n}\cdot\log(n)=0$$

Then the $\mathfrak{r}$-number of $\{b_\nu\cdot a_\nu\}$ is ≥ 1 for every b-sequence such that

$$|b_n|\leq n^{\gamma_n}\quad:\quad n=1,2,\ldots$$

6. Exercise. Prove the result above. Notice that it in particular applies if $\gamma_n=k$ for some positive integer k, i.e. this means that $\{\nu^k\cdot a_\nu\}$ at least has $\mathfrak{r}$-number ≥ 1 when the a-sequence has it.

7. Another exercise. Let $a_0,a_1,\ldots$ be a sequence of positive real numbers. Suppose there exists an integer m and a constant C such that

$$a_k\leq\frac{a_{k-1}+\ldots+a_{k-m}}{k}\quad\text{for all}\quad k\geq m$$

Show that no matter how $a_0,\ldots,a_{m-1}$ are determined initially it follows that the power seres

$$\sum a_\nu\cdot z^\nu$$

has an infinite radius of convergence, i.e. for every $R>0$ the positive series $\sum a_\nu\cdot R^\nu<\infty$.

II.B Convergence at the boundary

Let $\{a_\nu\}$ be a sequence with $\mathfrak{r}=1$. If $e^{i\theta}$ is a complex number whose absolute value is one it is not always true that the series

(1) $$\sum_{\nu=}^{\infty}a_\nu\cdot e^{i\nu\theta}$$

converges. So we have a possibly empty subset of $[0,2\pi]$ defined by

(2) $$\mathcal{F}=\{0\leq\theta\leq 2\pi\}\quad:\quad\text{The series (1) converges for }\theta$$

We can also impose the condition that (1) is absolutely convergent and set

(3) $$\text{abs}(\mathcal{F})=\{0\leq\theta\leq 2\pi\}\quad:\quad\text{The series X converges absolutely for }\theta$$

Here the inclusion $\mathcal{F}\subset\text{abs}(\mathcal{F})$ is in general strict.

1. Example Let $\{a_\nu=\frac{1}{\nu}\}$. Here $\mathfrak{r}=1$ and abs$(\mathcal{F})$ is empty since the series $\sum\frac{1}{\nu}$ is divergent. On the other hand the series

$$\sum\frac{e^{i\nu\theta}}{\nu}$$

converges for many θ. In fact we have

$$\mathcal{F}=(0,2\pi)$$

To see this we notice that if $a_\nu=e^{i\nu\theta}$ with $0<\theta<2\pi$ then the partial sums are:

$$S_N=\frac{1-e^{i(N+1)\theta}}{e^{i\theta}-1}$$

This sequence is bounded and since the positive series $\sum(\frac{1}{\nu}-\frac{1}{\nu+1})$ converges, the inclusion above follows from Abel's theorem.

2. Radial limits

Let $\{a_\nu\}$ be a sequence whose radius of convergence is 1. If $0<r<1$ and $0\leq\theta\leq 2\pi$ we get the convergent series

$$S(r,\theta)=\sum_{\nu=0}^{\infty}a_\nu\cdot r^\nu e^{i\nu\theta}$$

Keeping θ fixed we say that one has a radial limit if there exists

(*) $$\lim_{r\to 1}S(r,\theta)=S_*(\theta)$$

Denote by Let $\mathfrak{rad}(\{a_\nu\})$ the set of θ for which the limit above exists. The question arises if $\theta \in \mathfrak{rad}(\{a_\nu\})$ implies that the series $\sum a_\nu e^{i\nu\theta}$ converges. This is not true in general. The simplest example is to take $a_\nu = (-1)^\nu$ and $\theta = 1$. Here $S(r, 0) = \frac{1}{1+r}$ whose limit is $\frac{1}{2}$ while $\sum a_\nu$ must diverge since the a-sequence does not tend to zero. Sufficient conditions for convergence at a boundary point where the radial limit exists will be studied in secion VI.

3. A converse result.

Let $\{a_\nu\}$ give a convergent additive series, i.e. the partial sums

$$S_N = \sum_{\nu=0}^{\nu=N} a_\nu$$

converge to a limit S_*. In this case there exists a radial limit when $x \to 1$ for the power series $\sum a_\nu x^\nu$. To prove this we can always modify a_0 and assume that $S_* = 0$. Set

(i) $$\rho_N = \max_{\nu \geq N} |S_\nu|$$

So the hypothesis is now that $\rho_N \to 0$ as $N \to \infty$. For each $0 < x < 1$ we set:

$$S_N(x) = \sum_{nu=0}^{\nu=N} a_\nu \cdot x^\nu$$

When $0 < x < 1$ is fixed the infinite power series

(ii) $$S_*(x) = \sum_{nu=0}^{\infty} a_\nu \cdot x^\nu$$

converges. Next, when $0 < x < 1$ then the sequence $\{b_\nu = x^\nu - x^{\nu+1}\}$ is non-increasing.

3.1 Exercise. Use the remarks above and Exercise 6 from Additive Series to show that that for each pair $M > N$ and every $0 < x < 1$ one has

$$[S_M(x) - S_N(x)| \leq \rho_N$$

Since this holds for every $M > N$ and the series (i) converges we obtain

$$[S_*(x) - S_N(x)| \leq \rho_N$$

Next, the triangle inequality gives:

$$|S_*(x)| \leq |S_*(x) - S_N(x)| + |S_N(x) - S_N| + |S_N| \leq$$

$$2 \cdot \rho_N + |S_N(x) - S_N|$$

Finally, if $\epsilon > 0$ we first choose N so that $2 \cdot \rho_N < \epsilon/2$ and with N fixed we have

$$\lim_{x \to 1} S_N(x) = S_N$$

This proves the requested limit formula

(*) $$\lim_{x \to 1} S_*(x) = 0$$

4. A theorem of Landau.

One can also study limits on sparse sets which converge to a boundary point. Results of this nature appear in the article *Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande des Konvergenzgebietes* by Landau from 1907. Here we announce and prove one of these results. Consider a sequence of complex numbers $\{z_k\}$ in the open unit disc D which converge to 1. We say that the sequence of the Landau type if there exists a constant $\mathbf{L}$ such that

(i) $$\frac{|1 - z_k|}{1 - |z_k|} \leq \mathbf{L} \quad : \quad \frac{1}{\mathbf{L}} \leq k \cdot |1 - z_k| \leq \mathbf{L} \quad : \; k = 0, 1, 2, \ldots$$

Remark. The first inequality means that z_k come close to the real axis as $|z_k| \to 1$. The second condition means that the sequence of absolute values $1-|z_k|$ decreases in a relatively regular manner.

4. Theorem. *Let $\{a_\nu\}$ be a sequence such that $\nu \cdot a_\nu \to 0$ as $\nu \to +\infty$ and suppose there exists a sequence $\{z_k\}$ of the Landau type such that there exists a limit*

$$\lim_{k\to\infty} \sum a_\nu \cdot z_k^\nu = A$$

Then the series $\sum a_\nu$ is convergent and the series sum is equal to A.

Proof. Since $\nu \cdot a_\nu \to 0$ it follows that

$$\text{(i)} \qquad \lim_{k\to\infty} \frac{1}{k} \cdot \sum_{\nu=1}^{k} a_\nu = 0$$

Next, set

$$\text{(ii)} \qquad f(k) = \sum_{\nu=1}^{\nu=k} a_\nu z_k^\nu \quad \text{and} \quad S_k = \sum_{\nu=1}^{\nu=k} a_\nu$$

The triangle inequality gives

$$|S_k - f(k)| \leq \Big|\sum_{\nu=1}^{\nu=k} a_\nu(1-z_k^\nu) - \sum_{\nu>k} a_\nu z_k^\nu\Big| \leq$$

$$\text{(iii)} \qquad \sum_{\nu=1}^{\nu=k} |a_\nu|(1-z_k|\cdot\nu + \sum_{\nu>k} |a_\nu|\cdot|z_k|^\nu = W(k)_* + W(k)^*$$

Put

$$\text{(iv)} \qquad \epsilon(k) = \max\{\nu\cdot|a_\nu| \colon\ \nu \geq k+1\} \implies |a_\nu| \leq \frac{\epsilon(k)}{k} \quad : \nu \geq k+1$$

Since we also have $|z_k|^{k+1} \leq 1$ it follows from (iv) that

$$\text{(v)} \qquad W^*(k) \leq \frac{\epsilon(k)}{k} \cdot \frac{1}{1-|z_k|} \leq \frac{\mathbf{L}\cdot\epsilon(k)}{k\cdot|1-z_k|} \leq \mathbf{L}^2 \cdot \epsilon(k)$$

At the same time we have

$$\text{(vi)} \qquad W_*(k) \leq k\cdot|1-z_k| \cdot \frac{\sum_{\nu=1}^{\nu=k} \nu\cdot|a_\nu|}{k} \leq \mathbf{L} \cdot \frac{\sum_{\nu=1}^{\nu=k} \nu\cdot|a_\nu|}{k}$$

Now we are done, i.e. $W_*(k) \to 0$ by the observation in (*) and $W^*(k) \to 0$ since the hypothesis on $\{a_\nu\}$ gives $\epsilon(k) \to 0$.

III. Product series

Consider a sequence of positive real numbers $\{q_\nu\}$. To each $N \geq 1$ we define the partial product

$$\Pi_N = \prod_{\nu=1}^{\nu=N} q_\nu$$

If $\lim_{N\to\infty} \Pi_N$ exists we say that the infinite product converges and put

$$\Pi_* = \prod_{\nu=1}^{\infty} q_\nu$$

It is obvious that if the product converges then $\lim_{\nu\to\infty} q_\nu = 1$. The function $\log r$ has the Taylor expansion close to $r = 1$ given by $\log r = (r-1) + (r-1)^2/2 + \ldots$. Using this one gets following:

1. Theorem. *Let $\{q_\nu\}$ be a sequence where $0 < q_\nu < 1$ hold for all ν. Then the following three conditions are equivalent:*

$$\sum (1 - q_\nu) < \infty \quad : \quad \sum \operatorname{Log} \frac{1}{q_\nu} < \infty \quad : \quad \prod_{\nu=1}^{\infty} q_\nu > 0$$

Exercise Prove Theorem 1.

Next, when $|z| < 1$ the complex log-function has the series expansion

$$\log(1+z) = z - z^2/2 + z^3/3 + \ldots$$

From this one easily gets

2. Proposition *One has the inequality*

$$|\operatorname{Log}(1+z) - z| \leq |z|^2 \quad : \; |z| \leq 1/2$$

Next, consider a complex sequence $a(\cdot)$ where $|a_\nu| \leq \frac{1}{2}$ hold for all ν and put:

$$\Pi_N = \prod_{\nu=1}^{\nu=N} (1 - a_\nu) \implies \log(\Pi_N) = \sum_{\nu=0}^{\nu=N} \log(1 - a_\nu)$$

Proposition 2 gives the inequality

$$(*) \qquad |\log(1 - a_\nu) + a_\nu| \leq |a_\nu|^2$$

This enable us to investigate the convergence of the product series with the aid of the additive series for $\{a_\nu\}$. We get for example

$$(**) \qquad |\log(\Pi_N) + \sum_{\nu=1}^{\nu=N} a_\nu| \leq \sum_{\nu=1}^{\nu=N} |a_\nu|^2$$

From (**) we can conclude:

3. Theorem. *Let $\{a_\nu\})$ be a sequence where each $|a_\nu| \leq \frac{1}{2}$ and $\sum |a_\nu|^2 < \infty$. Then $\sum a_\nu$ converges if and only if the product series $\Pi(1 - a_\nu)$ converges. Moreover, when convergence holds one has the equality*

$$\log(\Pi_*) = \sum_{\nu=1}^{\infty} \log(1 - a_\nu)$$

IV. Blaschke products.

Let $\{a_\nu\}$ be a sequene in the open unit disc D which are enumerated so that their absolute values are non-decreasing. But repetitions may occur, i.e. several a-numbers can be equal. We always assume that $|a_\nu| \to 1$ as $\nu \to +\infty$. Hence $\{a_\nu\}$ is a discrete subset of D. To each ν we set

$$\beta_\nu(\theta) = \frac{e^{i\theta} - a_\nu}{1 - e^{i\theta} \cdot \bar{a}_\nu} \cdot \frac{\bar{a}_\nu}{|a_\nu|} \quad : \quad 0 \leq \theta \leq 2\pi \tag{1}$$

The *Blaschke product of order* N is the partial product

$$B_N(\theta) = \prod_{\nu=1}^{\nu=N} \beta_\nu(\theta) \tag{2}$$

The question arises when the product series converges and gives a limit

$$B_*(\theta) = \prod_{\nu=1}^{\infty} \beta_\nu(\theta) \tag{3}$$

To analyze this we use polar coordinates and put

$$a_\nu = r_\nu e^{i\theta\nu}$$

Each β-umber has absolute value one and if $\theta \neq \theta_\nu$ for every ν we have

$$\beta_\nu(\theta) = e^{i\cdot\gamma(r_\nu, \theta - \theta_\nu)} \quad : \quad 0 < \gamma(r_\nu, \theta - \theta_\nu) < 2\pi \tag{4}$$

Exercise. Show that when $-\pi/2 < \theta - \theta_\nu < \pi/2$ then the construction of the arctan-function gives

$$\gamma(r, \theta - \theta_\nu) = \text{arctg}\,[\frac{(1 - r^2) \cdot \sin(\theta_\nu - \theta)}{1 + r^2 - 2r\cos(\theta_\nu - \theta)}\,] \tag{4}$$

4.2. Blashke's condition We impose the condition that the positive series

$$\sum (1 - r_\nu) < \infty \tag{*}$$

Later we shall consider the analytic function defined in $|z| < 1$ by

$$B(z) = \prod_{\nu=0}^{\infty} \frac{z - a_\nu}{1 - \bar{a}_\nu \cdot z} \cdot e^{-i\arg(a_\nu)} \tag{2}$$

A major result to be proved later on asserts that the Blaschke condition implies that the radial limit

$$\lim_{r \to 1} B(re^{i\theta}) = B_*(\theta)$$

exists almost everywhere. Moreover, the absolute value of the limit $B_*(\theta)$ is equal to one almost everywhere. But the determination of the set of all $0 \leq \theta \leq 2\pi$ for which the radial limit exists is not clear since no special assumption is imposed on the $\{\theta_\nu\}$-sequence. For example, divergence may appear when many θ_ν:s are close to θ even if $\{r_\nu\}$ tend rapidly to 1.

4.3 Exercise. Let $\{r_\nu\}$ be given where the positive series in 4.2 converges. Next, if x is a real number we set

$$\{x\} = \min_{k \in \mathbf{Z}} [x - 2\pi k|$$

Show that the Blaschke product has a radial limit at $\theta = 0$ if and only if there exists the limit

$$(5) \qquad \lim_{N\to\infty} \Big\{ \sum_{\nu=1}^{\nu=N} \frac{(1-r_\nu)\cdot\theta_\nu}{(1-r_\nu)^2+\theta_\nu^2} \Big\}$$

Notice that θ_ν may be < 0 or > 0 and it is not necessary that all of them become close to 0. However, the convergence in 4.2 shows that only those θ_ν which are rather close to 0 matter. But to determine all sequence of pairs (r_ν, θ_ν) where 4.1 holds and $\theta_\nu \to 0$ appears to be a very difficult problem and is perhaps not even solvable in generality, i.e. one can find various sufficient conditions but they are not necessary.

V. Estimates using the counting function.

We establish some inequalities which will be used to study entire functions of exponential type in XXX. Let $\{\alpha_\nu\}$ be a complex sequence where $0 < |\alpha_1| \leq |\alpha_2| \leq \ldots\}$. This time the absolute values tend to $+\infty$. We get the counting function $N(R)$ which for every $R > 0$ is the number of α_ν with absolute value $\leq R$. We consider the situation when there exists a constant C such that

$$(*) \qquad N(R) \leq C\cdot R \quad \text{for all} \quad R > 0$$

5.1. The first estimate. To each $R > 0$ we set

$$(2) \qquad S(R) = \prod (1 + \frac{R}{|\alpha_\nu|}) \quad : \text{product taken over all } |\alpha_\nu| \leq 2R$$

Then we have

$$(*) \qquad S(R) \leq e^{KR} \quad \text{where} \quad K = 2C(1 + \text{Log}\,\frac{3}{2})$$

To prove this we consider log $S(R)$. A partial integration gives:

$$\log S(R) = \int_0^{2R} \log(1 + \frac{R}{t})\cdot dN(t) = \log(1+\frac{1}{2})\cdot N(2R) + \int_0^{2R} \frac{R\cdot N(t)}{t(t+R)}\cdot dt$$

Since $\frac{R}{t+R} \leq 1$ for all t, the last integral is estimated by $2R\cdot C$ and (*) follows.

5.2. The second estimate. For each $R > 0$ we consider infinite tail products:

$$(i) \qquad S^*(R) = \prod (1 + \frac{R}{\alpha_\nu})\cdot e^{-\frac{R}{\alpha_\nu}} \quad : \text{product taken over all } |\alpha_\nu| \geq 2R$$

To estimate (i) we notice that the analytic function $(1+\zeta)e^{-\zeta} - 1$ has a double zero at the origin. This gives a constant A such that

$$(ii) \qquad |(1+\zeta)e^{-\zeta} - 1| \leq A\cdot|\zeta|^2 \quad : \ |\zeta| \leq \frac{1}{2}$$

Since $|\alpha_\nu| \geq 2R$ for every ν we obtain:

$$(iii) \qquad \log^+ |(1+\frac{R}{\alpha_\nu})\cdot e^{-\frac{R}{\alpha_\nu}}| \leq \log[1 + A\frac{R^2}{|\alpha_\nu|^2}) \leq A\cdot\frac{R^2}{|\alpha_\nu|^2}$$

From (6) we get

$$(iv) \qquad \log^+(S^*(R)) \leq AR^2 \int_{2R}^\infty \frac{dN(t)}{t^2} = A\cdot N(2R) + 2AR^2\cdot\int_{2R}^\infty \frac{N(t)}{t^3}$$

The last term is estimated by

$$(8) \qquad 2AR^2\cdot C\cdot\int_{2R}^\infty \frac{dt}{t^2} = AC\cdot R$$

Adding up the result we get

5.3 Theorem. *One has the inequality*

$$S^*(R) \leq \frac{5A}{4}\cdot C\cdot R$$

VI. Convergence on the boundary.

Introduction. Consider a power series $f(z) = \sum a_n z^n$ whose radius of convergence is one. If $r < 1$ and $0 \leq \theta \leq 2\pi$ we are sure that the series

$$f(re^{i\theta}) = \sum a_n r^n e^{in\theta}$$

is convergent. In fact, it is even absolutely convergent since the assumption implies that

$$\sum |a_n| \cdot r^n < \infty \quad \text{for all} \quad r < 1$$

Passing to $r = 1$ it is in general not true that the series $\sum a_n e^{in\theta}$ is convergent. An example arises if we consider the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

So here $a_n = 1$ for all n and hence the series $\sum a_n$ is divergent. At the same time we notice that when $0 < \theta < 2\pi$ there exists the limit

$$\lim_{r \to 1} \sum r^n e^{in\theta} = \frac{1}{1 - re^{i\theta}}$$

while the series $\sum a_n e^{in\theta}$ is divergent. This leads to the following problem where we without loss of generality can take $\theta = 0$. Consider as above a convergent power series and assume that there exists the limit

$$(*) \qquad \lim_{r \to 1} \sum a_n r^n$$

When can we conclude that the series $\sum a_n$ also is convergent and that one has the equality

$$(**) \qquad \sum a_n = \lim_{r \to 1} \sum a_n r^n$$

The first result in this direction was established by Abel in a work from 1823:

A. Theorem *Let $\{a_n\}$ be a sequence such that $\frac{a_n}{n} \to 0$ as $n \to \infty$ and there exists*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Then $\sum a_n$ is convergent and the sum is A.

An extension of Abel's result was established by Tauber in 1897.

B.1 Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists the limit*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Set

$$\omega_n = \frac{a_1 + \ldots + a_n}{n} \quad : \, n \geq 1$$

If $\lim_{n \to \infty} \omega_n = 0$ it follows that the series $\sum a_n$ is convergent and the sum is A.

C. Results by Hardy and Littlewood.

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In their joint article *xxx* from 1913 the following extension of Abel's result was proved by Hardy and Littlewood:

C.1 Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a constant C so that $\frac{a_n}{n} \leq C$ for all $n \geq 1$. Assume also that the power series $\sum a_n z^n$ converges when $|z| < 1$. Then the same conclusion as in Abel's theorem holds.*

Remark. The proof of Theorem C.1 requires several steps where an essential ingredient is a result about positive series from the cited article which has independent interest.

C.2 Theorem. *Assume that each $a_n \geq 0$ and that there exists the limit:*

(*) $$A = \lim_{r \to 1} (1-r) \cdot \sum a_n r^n$$

Then there exists the limit

(**) $$A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Notice that we do not impose any growth condition on $\{a_n\}$ above,i.e. the sole assumption is the existing limit (*).

Remark. The proofs of Abel's and Tauber's results are quite easy while the two cited results by Hardy and Littlewood are more demanding. Here we need some results from calculus in one variable. So before we enter the proofs of the cited theorems insert some preliminaries.

1. Results from calculus

Below $g(x)$ is a real-valued function defined on $(0,1)$ and of class C^2 at least.

1.1 Lemma *Assume that there exists a constant $C > 0$ such that*

$$g''(x) \leq C(1-x)^{-2} \quad : 0 < x < 1 \quad \text{and} \quad \lim_{x \to 1} g(x) = 0$$

Then one has the limit formula:

$$\lim_{x \to 1} (1-x) \cdot g'(x) = 0$$

1.2 Lemma *Assume that the second order derivative $g''(x) > 0$. Then the following implication holds for each $\alpha > 0$:*

$$\lim_{x \to 1} (1-x)^{\alpha} \cdot g(x) = 1 \implies \lim_{x \to 1} (1-x)^{\alpha+1} \cdot g'(x) = \alpha$$

Remark. If $g(x)$ has higher order derivatives which all are > 0 on $(0,1)$ we can iterate the conclusion in Lemma 1.2 where we take α to be positive integers. More precisely, by an induction over ν the reader may verify that if

$$\lim_{x \to 1} (1-x) \cdot g(x) = 1$$

exists and if $\{g^{(\nu)}(x) > 0\}$ for all every $\nu \geq 2$ then

(*) $$\lim_{x \to 1} (1-x)^{\nu+1} \cdot g^{(\nu)}(x) = \nu\,! \quad : \nu \geq 2$$

Next, to each integer $\nu \geq 1$ we denote by $[\nu - \nu^{2/3}]$ the largest integer $\leq (\nu - \nu^{2/3})$. Set

$$J_*(\nu) = \sum_{n \leq [\nu - \nu^{2/3}]} n^{\nu} e^{-\nu} \quad : \quad J^*(\nu) = \sum_{n \geq [\nu + \nu^{2/3}]} n^{\nu} e^{-\nu}$$

1.3 Lemma *There exists a constant C such that*

$$\frac{J^*(\nu) + J_*(\nu)}{\nu\,!} \leq \delta(\nu) \quad : \quad \delta(\nu) = C \cdot \exp\left(-\frac{1}{2} \cdot \nu^{\frac{1}{3}}\right) \quad : \nu = 1, 2, \ldots$$

Proofs

We prove only Lemma 1.1 which is a bit tricky while the proofs of Lemma 1.2 and 1.3 are left as exercises to the reader. Fix $0 < \theta < 1$. Let $0 < x < 1$ and set

$$x_1 = x + (1-x)\theta$$

The mean-value theorem in calculus gives

(i) $$g(x_1) - g(x) = \theta(1-x)g'(x) + \frac{\theta^2}{2}(1-x)^2 \cdot g''(\xi) \quad \text{for some} \quad x < \xi < x_1$$

By the hypothesis

$$g''(\xi) \leq C(1-\xi)^{-2} \leq C)1 - x_1)^{-2}$$

Hence (i) gives

$$(1-x)g'(x) \geq \frac{1}{\theta}(g(x_1)-g(x)) - C\cdot\frac{\theta}{2}\frac{(1-x)^2}{1-x_1)^2} =$$

$$\frac{1}{\theta}(g(x_1)-g(x)) - \frac{C\cdot\theta}{2(1-\theta)^2}$$

Keeping θ fixed we have by assumption

$$\lim_{x\to 1} g(x) = 0$$

Notice also that $x\to 1 \implies x_1 \to 1$. It follows that

$$\liminf_{x\to 1}\ (1-x)g'(x) \geq -\frac{C\cdot\theta}{2(1-\theta)^2}$$

Above $0<\theta<1$ is arbitrary, i.e. we can choose small $\theta>0$ and hence we have proved that

(*) $$\liminf_{x\to 1}\,(1-x)g'(x)\geq 0$$

Next we prove the opposed inequality

(**) $$\limsup_{x\to 1}\,(1-x)g'(x)\leq 0$$

To get (**) we apply the mean value theorem in the form

(ii) $$g(x_1)-g(x) = \theta(1-x)g'(x_1) - \frac{\theta^2}{2}(1-x)^2\cdot g''(\eta) \quad : x<\eta<x_1$$

Since $(1-x_1) = \theta(1-x)(1-\theta)$ we get

(iii) $$(1-x_1)g'(x_1) = \frac{1-\theta}{\theta}\cdot(g(x_1)-g(x)) + \frac{(1-\theta)\theta}{2}\cdot(1-x)^2 g''(\eta)$$

Now $g''(\eta)\leq C(1-\eta)^{-2}\leq C(1-x_1)^{-2}$ so the right hand side in (iii) is majorized by

$$\frac{1-\theta}{\theta}\cdot(g(x_1)-g(x)) + C\cdot\frac{(1-\theta)\theta}{2}\cdot(1-x)^2(1-x_1)^2 =$$

(iv) $$\frac{1-\theta}{\theta}\cdot(g(x_1)-g(x)) + C\cdot\frac{\theta}{2(1-\theta}$$

Keeping θ fixed while $x\to 1$ we obtain:

$$\liminf_{x\to 1}\,(1-x)g'(x)\leq C\cdot\frac{\theta}{2(1-\theta}$$

Again we can choose arbitrary small θ and hence (**) holds which finishes the proof of Lemma 1.1.

2. Proof of Abel's theorem.

Without loss of generality we can assume that $a_0=0$ and set $S_N = a_1+\ldots+a_N$. Given $0<r<1$ we let $f(r)=\sum a_n r^n$. For every positive integer N the triangle inequality gives:

$$\big|S_N - f(r)\big| \leq \sum_{n=1}^{n=N} |a_n|(1-r^n) + \sum_{n\geq N+1} |a_n| r^n$$

Set $\delta(N) = \max_{n\geq N}\frac{|a_n|}{n}$. Since $1-r^n) = (1-r)(1+\ldots+r^{n-1} \leq (1-r)n$ the last sum is majorised by

$$(1-r)\cdot\sum_{n=1}^{n=N} n\cdot|a_n| + \delta(N+1)\cdot\sum_{n\geq N+1}\frac{r^n}{n}$$

Next, the obvious inequality $\sum_{n\geq N+1}\frac{r^n}{n}\leq\frac{1}{N+1}\cdot\frac{1}{1-r}$ gives the new majorisation

$$(1)\qquad (1-r)\cdot\sum_{n=1}^{n=N}\frac{|a_n|}{n}+\frac{\delta(N+1)}{N+1}\cdot\frac{1}{1-r}$$

This hold for all pairs N and r. To each $N\geq 2$ we take $r=1-\frac{1}{N}$ and hence the right hand side in (1) is majorised by

$$\frac{1}{N}\cdot\sum_{n=1}^{n=N}\frac{|a_n|}{n}+\delta(N+1)\cdot\frac{N}{N+1}$$

Here both terms tend to zero as $N\to\infty$. Indeed, Abel's condition $\frac{a_n}{n}\to 0$ implies that $\frac{1}{N}\cdot\sum_{n=1}^{n=N}\frac{|a_n|}{n}$ tends to zero as $N\to\infty$. Hence we have proved the limit formula:

$$(*)\qquad \lim_{N\to\infty}\big|s_N-f(1-\frac{1}{N})\big|=0$$

Finally it is clear that (*) gives Abel's result.

3. Proof of Tauber's theorem.

We may assume that $a_0=0$. Notice that

$$a_n=\frac{\omega_n-\omega_{n-1}}{n}\quad : n\geq 1$$

It follows that

$$f(r)=\sum\frac{\omega_n-\omega_{n-1}}{n}\cdot r^n=\sum\omega_n(\frac{r^n}{n}-\frac{r^{n+1}}{n+1})$$

Next, we use the equality $\frac{1}{n}=\frac{1}{n+1}=\frac{1}{n(n+1)}$ and rewrite the right hand side as follows:

$$\sum\omega_n(\frac{r^n-r^{n+1}}{n+1}+\frac{r^n}{n(n+1)})$$

Set

$$g_1(r)=\sum\omega_n(\frac{r^n-r^{n+1}}{n+1}=(1-r)\cdot\sum\frac{\omega_n}{n+1}\cdot r^n$$

By the hypothesis $\lim_{n\to\infty}\frac{\omega_n}{n+1}=0$ and then it is clear that we get

$$\lim_{r\to 1}g_1(r)=0$$

Since we also have $f(r)\to 0$ as $r\to 1$ we conclude that

$$(1)\qquad \lim_{r\to 1}\sum\frac{\omega_n}{(n+1)}\cdot r^n=0$$

Next, with $b_n=\frac{\omega_n}{n(n+1)}$ we notice that $nb_n=\frac{\omega_n}{n+1}\to 0$. Hence Abel's theorem applies so (1) gives convergent series

$$(2)\qquad \sum\frac{\omega_n}{n(n+1)}=0$$

If $N\geq 1$ we have the partial sum

$$S_N=\sum_{n=1}^{n=N}\frac{\omega_n}{n(n+1)}=\sum_{n=1}^{n=N},\omega_n\cdot(\frac{1}{n}-\frac{1}{n+1})$$

The last term becomes

$$\sum_{n=1}^{n=N}\frac{1}{n}(\omega_n-\omega_{n-1})-\frac{\omega_N}{N+1}=\sum_{n=1}^{n=N}a_n-\frac{\omega_N}{N+1}$$

Again, since $\frac{\omega_N}{N+1}\to 0$ as $N\to\infty$ we conclude that the convergent series from (2) implies that the series $\sum a_n$ also is converges and has sum equal to zero. This finishes the proof of Tauber's result.

4. Proof of Theorem C.2

Set $f(x) = \sum a_n x^n$ which is defined when $0 < x < 1$. Notice that

$$(1-x)f(x) = \sum s_n x^n \quad \text{where} \quad s_n = a_1 + \ldots + a_n$$

Set $g(x) = \sum s_n x^n$ which is defined when $0 < x < 1$. Since $s_n \geq 0$ for all n all the higher order derivatives

$$g^{(p)}(x) = \sum_{n=p}^{\infty} n(n-1)\cdots(n-p+1)a_n x^{n-p} > 0$$

when $0 < x < 1$. The hypothesis that $\lim_{x \to 1} g(x) = A$ and Lemma 1.1 and the inductive result in the remark after Lemma 1.2 give:

(1) $$\lim_{x \to 1} (1-x)^{\nu+2} \cdot \sum s_n \cdot n^\nu x^n = (\nu+1)! \quad : \nu \geq 1$$

We shall use the substitution $e^{-t} = x$ where $t > 0$. Since $t \simeq 1 - x$ when $x \to 1$ we see that (1) gives

(2) $$\lim_{t \to 0} t^{\nu+2} \cdot \sum s_n \cdot n^\nu e^{-nt} = (\nu+1)! \quad : \nu \geq 1$$

Let us put

$$J_*(\nu, t) = \frac{t^{\nu+2}}{(\nu+1)!} \cdot \sum_{n=1}^{\infty} s_n \cdot n^\nu e^{-nt}$$

So for each fixed ν one has

(3) $$\lim_{t \to 0} J_*(\nu, t) = 1$$

Next, for each pair $\nu \geq 2$ and $0 < t < 1$ we define the integer

(*) $$N = [\frac{\nu - \nu^{2/3}}{t}]$$

Since the sequence $\{s_n\}$ is non-decreasing we get

(i) $$s_N \cdot \sum_{n \geq N} n^\nu e^{-nt} \leq \sum_{n \geq N} s_n \cdot n^\nu e^{-nt} \leq \frac{(\nu+1)! \cdot J_*(\nu, t)}{t^{\nu+2}}$$

Next, the construction of N and Lemma 1.3 give:

(ii) $$\sum_{n \geq N} n^\nu e^{-nt} \geq \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta(\nu))$$

where the δ function is independent of ν and tends to zero as $\nu \to \infty$. Hence (i-ii) give

(iii) $$s_N \leq \frac{(\nu+1)}{t} \cdot \frac{1}{1-\delta(\nu)} \cdot J_*(\nu, t)$$

Next, by the construction of N one has

$$N + 1 \geq \frac{\nu - \nu^{2/3}}{t} = \frac{\nu}{t} \cdot (1 - \nu^{-1/3})$$

It follows that (iii) gives

(iv) $$\frac{s_N}{N+1} \leq \frac{\nu+1}{\nu} \cdot \frac{1}{1-\nu^{-1/3}} \cdot \frac{1}{1-\delta(\nu)} \cdot J_*(\nu, t)$$

Since $\delta(\nu) \to 0$ it follows that for any $\epsilon > 0$ there exists some ν_* such that

(v) $$\frac{\nu_*+1}{\nu_*} \cdot \frac{1}{1-\nu_*^{-1/3}} \cdot \frac{1}{1-\delta(\nu_*)} < 1 + \epsilon$$

Keeping ν_* fixed we now consider pairs t_n, N such that (*) above hold with $\nu = \nu_*$. Notice that

(vi) $$N \to +\infty \implies t_N \to 0$$

It follows from (iv) and (v) that we have:

(vii) $$\frac{s_N}{N+1} < (1+\epsilon)\cdot J_*(\nu_*, t_N) \quad : N \geq 2$$

Now (vi) and the limit in (3) which applies with ν_* while $t_N \to 0$ entail that

$$\lim_{N\to\infty} J(\nu_*, t_N) = 1$$

We have also that $\frac{N}{N+1} \to 1$ and since $\epsilon > 0$ was arbitrary we see that (vii) proves the inequality

(I) $$\text{Lim.sup}_{N\to\infty} \frac{s_N}{N} \leq 1$$

So Theorem 2 follows if we also prove that

(II) $$\text{Lim.inf}_{N\to\infty} \frac{s_N}{N} \geq 1$$

The proof of (II) is similar where we now define the integers N by:

$$N = [\frac{\nu + \nu^{2/3}}{t}]$$

Then we have

$$S_N \cdot \sum_{n\leq N} n^\nu e^{-nt} \geq \frac{(\nu+1)!\cdot J_*(\nu,t)}{t^{\nu+2}} - \sum_{n>N} s_n \cdot n^\nu e^{-nt}$$

Here the last term can be estimated above since the Lim.sup inequality (I) gives a constant C such that $s_n \leq Cn$ for all n and then

$$\sum_{n>N} s_n \cdot n^\nu e^{-nt} \leq C \cdot \sum_{n>N} n^{\nu+1} e^{-nt} \leq C \cdot \delta^*(\nu) \cdot \frac{(\nu+1)!}{t^{\nu+2}}$$

where Lemma 1.3 entails that $\delta^*(\nu) \to 0$ as ν increases. At the same time Lemma 1.3 also gives

$$\sum_{n\leq N} n^\nu \cdot e^{-nt} = \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta_*(\nu)$$

where $\delta(\nu_*) \to 0$. At this stage the reader can verify that (II) by similar methods as in the proof of (I).

5. Proof of Theorem C.1

Set $f(x) = \sum a_n x^n$. Notice that it suffices to prove Theorem C.1 when the limit value

$$\lim_{x\to 1} \sum a_n x^n = 0$$

Next, the assumption that $a_n \leq \frac{c}{n}$ for a constant c gives

$$f''(x) = \sum n(n-1)a_n x^{n-2} \leq c \sum (n-1)x^{n-2} = \frac{c}{1-x)^2}$$

The hypothesis $\lim_{x\to 1} f(x) = 0$ and Lemma xx therefore gives

(i) $$\lim_{x\to 1} (1-x)f'(x) = 0$$

Next, notice the equality

(ii) $$\sum_{n=1}^{\infty} \frac{na_n}{c} x^n = \frac{x}{c} \cdot f'(x)$$

At the same time $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ and hence (i-ii) together give:

$$\lim_{x\to 1} (1-x) \cdot \sum (1 - \frac{na_n}{c}) \cdot x^n = 1$$

Here $1 - \frac{na-n}{c} \geq 0$ so Theorem 2 gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n=N} (1 - \frac{na_n}{c}) = 1$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{n=N} na_n = 0$$

This means precisely that the condition in Tauber's Theorem holds and hence this Tauber's result proves that $\sum a_n$ converges and has series sum equal to 0.

VII. An example by Hardy

Consider the series expansion of

$$(1-z)^\alpha = \sum b_n z^n$$

where α in general is a complex number. One has Newton's binomial formula:

$$(*) \qquad b_n = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n\,!}$$

Let us apply this with $\alpha = i$. If $n \geq 1$ we find

$$|n \cdot b_n| = \frac{|(i+1)|\dots|i+n-1|}{n\,!} = \sqrt{(1+\frac{1}{1^2})\cdot(1+\frac{1}{2^2})\cdots(1+\frac{1}{(n-1)^2}}$$

It follows that

$$\lim_{n\to\infty} |n\cdot b_n| = \sqrt{\prod_{\nu=1}^{\infty}(1+\frac{1}{\nu^2})}$$

In particular $|b_n| \simeq \frac{1}{n}$ when n is large which implies that the series

$$\text{(i)} \qquad \sum_{n=2}^{\infty} \frac{|b_n|}{\operatorname{Log} n} = +\infty$$

In spite of the divergence above one has:

Theorem 7.1. *The series*

$$\sum_{n=2}^{\infty} \frac{b_n}{\operatorname{Log} n} \cdot e^{in\phi}$$

converges uniformly when $0 \leq \phi \leq 2\pi$.

Remark. Theorem 7.1 amounts to prove the following: To each $m \geq 2$ we consider the partial sum series

$$S_m(\phi) = \sum_{n=2}^{n=m} \frac{b_n}{\operatorname{Log} n} \cdot e^{in\phi}$$

Now Theorem 7.1 asserts that to every $\epsilon > 0$ there exists an integer M such that

$$(1) \qquad \max_{0\leq\phi\leq 2\pi} |S_m(\phi) - S_M(\phi)| < \epsilon \quad : \quad \forall\ m > M$$

To prove (1) we employ the partial sums

$$(2) \qquad B_n(\phi) = \sum_{\nu=1}^{\nu=n} b_\nu \cdot e^{i\nu\phi}$$

For each pair $m > M \geq 2$, the partial summation formula in gives

$$S_m(\phi) - S_M(\phi) =$$

$$(3) \qquad \sum_{n=M}^{n=m} B_n(\phi)\cdot\big[\,\frac{1}{\operatorname{Log} n} - \frac{1}{\operatorname{Log}(n+1)}\big] - \frac{B_{M-1}(\phi)}{\operatorname{Log} M} + \frac{B_m(\phi)}{\operatorname{Log}(m+1)}$$

To get (1) we need shall need:

Sunlemma *There exists a constant* K *such that*

$$(*) \qquad |B_n(\phi)| \leq K \quad >: \quad n \geq 2 : 0 \leq \phi \leq 2\pi$$

Proof of (1). If the Sublemma is admitted we see that (3) gives

$$\big|S_m(\phi) - S_M(\phi)\big| \leq K \cdot \sum_{n=M}^{n=m} \big[\frac{1}{\operatorname{Log} n} - \frac{1}{\operatorname{Log}(n+1)}\big] + \frac{1}{\operatorname{Log} M} + \frac{1}{\operatorname{Log}(m+1)} = \frac{2K}{\operatorname{Log} M}$$

Hence the proof of Theorem 7.1 is finished provided we have established the inequality (*) in the Sublemma. To prove this we study the analytic function defined in the open unit disc D by the convergent power series:

$$(*) \qquad (1-z)^i = \sum c_n \cdot z^n$$

Since $\mathfrak{Re}(1-z) > 0$ when $|z| < 1$ there exists a single valued branch of $\log(1-z)$ and the function above can be written as

$$g(z) = e^{i \cdot \log(1-z)}$$

Now the argument of $\log(1-z)$ stays in $(-\pi/2, \pi/2$ and we conclude that

$$|g(z)| \leq e^{\pi/2} \quad : z \in D$$

Hence the g-function is bounded in D. Now

$$g(z) = \sum b_n z^n$$

and we know by the construction that

$$|b_n| \leq \frac{C}{n} \quad : n \geq 1$$

Finally (*) above follows from the following general result:

7.2. Theorem. *Let $\{a_n\}$ be a sequence of complex numbers such that $|a_n| \leq \frac{C}{n}$ hold for all n and some constant C and the analytic function $f(z) = \sum a_n z^n$ is bounded in D, i.e.*

$$|f(z)| \leq M \quad : z \in D$$

Then, if $B_n(z) = a_0 + a_1 + \ldots + a_n z^n$ are the partial sums one has the inequality

$$\max_\theta |B_m(e^{i\theta})| \leq M + 2C \quad \text{for all} \quad m = 1, 2, \ldots$$

Proof. When $0 < r < 1$ and θ are given we have

$$\big|B_m(\theta) - f(re^{i\theta})\big| = \sum_{n=0}^{n=m} a_n e^{in\theta}(1-r^n) - \sum_{n=m+1}^{\infty} a_n e^{in\theta} \cdot r^n \leq$$

$$(1-r) \sum_{n=0}^{n=m} n \cdot |a_n| + \sum_{n=m+1}^{\infty} |a_n| \cdot r^n$$

With m given we apply this when $r = 1 - 1/m$. Then the last sum above is estimated above by

$$(*) \qquad \frac{1}{m} \cdot \sum_{n=1}^{n=m} n \cdot |a_n| + \frac{c}{m} \cdot \sum_{n=m+1}^{\infty} r^n \leq C + \frac{c}{m} \cdot \sum_{n=0}^{\infty} r^n = 2C$$

Finally, since the maximum norm of f is $\leq M$ the triangle inequality gives

$$|B_m(e^{i\theta})| \leq 2C + M$$

Here m and θ are arbitrary so Theorem 7.2 follows.

8. Convergence under substitution.

Introduction. Let $\{a_k\}$ be a sequence of complex numbers where $\sum a_k$ is convergent. This gives an analytic function $f(z)$ defined in the open disc by

(1) $$f(z) = \sum a_n \cdot z^n$$

If $0 \neq z_0 \in D$ we can expand f around z_0 and obtain another series

(2) $$f(z_0 + z) = \sum c_n \cdot z^n$$

In general this series is only convergent when $|z| < 1| - |z_0|$. Suppose that $z_0 = b$ is real and positive. From the convergence of $\sum a_k$ one expects that the passage to $z = 1 - b$ entails the convergence in (2) entails that the series

(3) $$\sum c_n \cdot (1-b)^n$$

also is convergent. This is indeed true and was proved by Hardy and Littlewood in (H-L].

Example. With $b = 1/2$ the result by Hardy and Littlewood means that the double series

$$\sum_{\nu \leq n} 2^{-n+\nu} \cdot \binom{n}{\nu} \cdot a_\nu$$

is convergent. Theorem 8.1 below gives this result as a special case. In general, consider some other power series

(1) $$\phi(z) = \sum b_\nu \cdot z^\nu$$

which converges for all $|z| = 1$ and represents an analytic function $\phi(z)$ in D which extends to be continuous on the closed disc, i.e. ϕ belongs to the disc algebra $A(D)$. Moreover, assume that $|\phi(z)| < 1$ when $|z| < 1$. Then there exists the composed power series

(*) $$f(\phi(z)) = \sum_{k=0}^{\infty} c_k \cdot z^k$$

We seek conditions on ϕ in order that the series

(**) $$\sum c_k$$

converges and consider first the case when the series for ϕ has real and non-negative coefficients.

8.1. Theorem. *Assume that $\{b_\nu \geq 0\}$ and that $\sum b_\nu = 1$. Then (**) converges and the sum is equal to $\sum a_k$.*

Proof. Since $\{b_\nu\}$ are real and non-negative the Taylor series for ϕ^n also has non-negative real coefficients for arbitrary $k \geq 2$. Put

$$\phi^k(z) = \sum B_{k\nu} \cdot z^\nu$$

where the B-numbers are real and non-negative. To each pair of integers k, p we set

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=p} B_{k\nu}$$

Then our assumptions give:

(i) $$\lim_{N \to \infty} \Omega_{N,p} = 0 \quad \text{for every} \quad p$$

(ii) $$k \mapsto \Omega_{k,p} \quad \text{decreases for every} \quad p$$

(iii) $$\sum_{\nu=0}^{\infty} B_{k\nu} = 1 \quad \text{hold for every} \quad k$$

The verification of (i-iii) is left to the reader. Next, the Taylor series of the composed analytic function $f(\phi(z))$ is given by

$$\sum a_k \cdot \phi^k(z) = \sum_{\nu=0}^{\infty} [\sum_{k=0}^{\infty} a_k \cdot B_{k\nu}] \cdot z^\nu$$

It follows that the series

$$\sum_{k=0}^{\infty} a_k \cdot B_{k\nu}$$

converges for every ν. For each positive integer n^* we set

(1) $$\sigma_p[n^*] = \sum_{\nu=0}^{\nu=p} [\sum_{k=0}^{k=n^*} a_k \cdot B_{k,\nu}]$$

(2) $$\sigma_p(n^*) = \sum_{\nu=0}^{\nu=p} [\sum_{k=n^*+1}^{\infty} a_k \cdot B_{k,\nu}]$$

Notice that

$$\sigma_p[n^*] + \sigma_p(n^*) = \sum_{k=0}^{k=p} c_k$$

hold for each p. Our aim is to show that the last partial sums have a limit. To get this we study the σ-terms separately. Introduce the partial sums

$$s_n = \sum_{k=0}^{k=n} a_k$$

By assumption $s_n \to s$ so for a given $\epsilon > 0$ we find n^* such that

(3) $$n \geq n^* \implies |s_n - s| \leq \epsilon$$

Next, using the limit (i) above a partial summation gives

(3) $$\sigma_p(n^*) = \sum_{k=n^*+1}^{k=N} (s_k - s_{n^*}) \cdot (\Omega_{k,p} - \Omega_{k+1,p})$$

Then (i-ii) obviously give

(4) $$|\sigma_p(n^*)| \leq \epsilon \cdot \Omega_{n^*,p} \leq \epsilon$$

where the last inequality holds since all the Ω -numbers are ≤ 1.

A study of $\sigma_p[n^*]$. Keeping n^* and ϵ fixed we apply (iii) for each $0 \leq k \leq n^*$ and find an integer p^* such that

$$1 - \sum_{\nu=0}^{\nu=p} B_{k,\nu} \leq \frac{\epsilon}{n^*+1} \quad \text{for all pairs} \quad p \geq p^* : 0 \leq k \leq n^*$$

The the triangle inequality gives

(5) $$|\sigma_p(n^*) - s_{n^*}| \leq \frac{\epsilon}{n^*+1} \cdot \sum_{k=0}^{k=n^*} |a_k| \quad \text{for all} \quad p \geq p^*$$

Since the series $\sum a_k$ converges the sequence $\{a_k\}$ is bounded, i.e. we have a constant M such that $|a_k| \leq M$ for all k. Hence (3) and (5) give

(6) $$|\sigma_p(n^*) - s| \leq \epsilon + \epsilon \cdot M \quad : \quad p \geq p^*$$

Together with (4) this entails that

$$n \geq n^* \implies |\sum_{k=0}^{n^*} |c_k - s| \leq 2\epsilon + M \cdot \epsilon$$

Since we can chose ϵ arbitrary small we conclude that $\sum c_k$ converges and the limit is equal to s. This finishes the proof of Theorem 1.

Now we try to relax the condition that $\{b_\nu\}$ are real and nonnegative. To begin with we impose the condition that $\phi(1) = 1$ while $|\phi(z)| < 1$ for all $z \in \bar{D} \setminus \{1\}$ which means that $z = 1$ is a peak point for the continuous function ϕ on the closed unit disc. Consider also the function $\theta \mapsto \phi(e^{i\theta})$ where θ is close to zero and impose the condition that there exists some positive real number β and a constant C such that

$$|\phi(e^{i\theta}) - 1 - i\beta| \leq C \cdot \theta^2 \tag{1}$$

holds in some interval $\ell \leq \theta \geq \ell$. This implies that for every integer $n \geq 2$ we get another constant C_n so that

$$|\phi^n(e^{i\theta}) - 1 - in\beta| \leq C_n \cdot \theta^2 \tag{2}$$

Hence the following integrals exist for all pairs of integers $p \geq 0$ and $n \geq 1$:

$$J(n.p) = \int_{-\ell}^{\ell} \frac{\phi(e^{i\theta})^n \cdot (1 - \phi(e^{i\theta})}{e^{ip\theta} \cdot (1 - e^{i\theta})} \cdot d\theta \tag{3}$$

With these notations one has

8.2. Theorem. *Let 1 be a peak point for ϕ and assume also that (1) above holds. Then, if there exists a constant C such that*

$$\sum_{n=0}^{\infty} |J(n,p)| \leq C \quad \text{for all} \quad p \geq 0 \tag{*}$$

it follows that the series (**) from the introduction converges and the sum is equal to $\sum a_k$.

Proof With similar notations as in the previous proof we introduce the Ω-numbers by:

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=k} B_{k\nu}$$

Repeating the proof of Theorem 8.1 the reader may verify that the series $\sum c_k$ converges and has the limit s if the following two conditions hold:

$$\lim_{N \to \infty} \Omega_{N,p} = 0 \quad \text{holds for every} \quad p \tag{i}$$

$$\sum_{k=0}^{\infty} \big|\Omega_{k+1,p} - \Omega_{k,p}\big| \leq C \quad \text{for a constant} \quad C \tag{ii}$$

where C is is independent of p. Here (i) is clear since $\{g_N(z) = \phi^N(z)\}$ converge uniformly to zero in compact subsets of the unit disc and therefore their Taylor coefficients tend to zero with N. To check (ii) we use residue calculus which gives:

$$\Omega_{k+1,p} - \Omega_{k,p} = \frac{1}{2\pi i} \int_{|z|=1} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz \tag{iii}$$

Let ℓ be a small positive number and T_ℓ denotes the portion of the unit circle where $\ell \leq \theta \leq 2\pi - \ell$. Since 1 is a peak -point for ϕ there exists some $\mu < 1$ such that

$$\max_{z \in T_\ell} |\phi(z)| \leq \mu$$

This gives

$$\frac{1}{2\pi} \cdot \Big|\int_{z \in T_\ell} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz\Big| \leq \mu^k \cdot \frac{2}{|e^{i\ell} - 1}\Big| \tag{iv}$$

Since the geometric series $\sum \mu^k$ converges it follows from (iii) and the construction of the J_ℓ-functions in Theorem 8.2 that (ii) above holds precisely when

$$\sum_{k=0}^{\infty} |J_\ell(k,p]| \leq C$$

hold for a constant which is independent of p which finishes the the proof of Theorem 8.2.

8.3. The use of oscillatory integrals. Since condition (*) Theorem 8.2 is a bit implicit we shall give a sufficient condition in order that the J-integrals satisfy (*) expressed by extra local conditions on the ϕ-function close to $z = 1$. The assumption (1) above entails that we can write

$$\text{(i)} \qquad \phi(e^{i\theta}) = e^{i\beta\theta + \rho(\theta)}$$

holds in a neighborhood of $\theta = 0$ where the ρ-function behaves like big ordo of θ^2 when $\theta \to 0$. The next result gives the requested convergence of the composed series expressed by an additional condition on the ρ-function in (i) above.

8.4. Theorem. *Assume that $\rho(\theta)$ is a C^2function on some interval $-\ell < \theta < \ell$ and that the second derivative $\rho''(0)$ is real and negative. Then (*) in Theorem 8.2 holds.*

Remark. We leave the proof as a (hard) exercise to the reader. If necessary, consult Carleman's article [Car] which contains a detailed proof.

IX. The series $\sum \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}}$

Introduction. We shall prove a result from [Carleman:xx. Note V page 112-115]. Let $\{a_\nu\}$ be a sequence of positive real numbers such that $\sum a_\nu < \infty$ and e denotes Neper's constant.

9.1 Theorem. *Assume that the series $\sum a_\nu$ is convergent and let S be the sum. Then one has the strict inequality*

$$(*) \qquad \sum_{\nu=1}^{\infty} \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} < e \cdot S$$

Remark. The result is sharp in the sense that e cannot be replaced by a smaller constant. To see this we consider a large positive integer N and take the finite series $\{a_\nu = \frac{1}{\nu} \quad : 1 \leq \nu \leq N\}$. Stirling's limit formula gives:

$$\left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} \simeq e\nu \quad : \nu >> 1$$

Since the harmonic series $\sum \frac{1}{\nu}$ is divergent we conclude that for every $\epsilon > 0$ there exists some large integer N such that $\{a_\nu = \frac{1}{\nu}\}$ gives

$$\sum_{\nu=1}^{\nu=N} \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} > (e - \epsilon) \cdot \sum_{\nu=1}^{\nu=N} \frac{1}{\nu}$$

There remains to prove the strict upper bound (*) when $\sum a_\nu$ is a convergent positive series. To attain this we first establish inequalities for finite series. Given a positive integer m we consider the variational problem

$$(1) \qquad \max_{a_1,\ldots,a_m} \sum_{\nu=1}^{\nu=m} \left[a_1 \cdots a_\nu\right]^{\frac{1}{\nu}} \quad \text{when} \quad a_1 + \ldots + a_m = 1$$

Let $a_1^*, \ldots, a_m^*$ give a maximum and set $a_\nu^* = e^{-x_\nu}$. The Lagrange multiplier theorem gives a number $\lambda^*(m)$ such that if we set

$$y_\nu = \frac{x_\nu + \ldots + x_m}{\nu}$$

then

$$(2) \qquad \lambda^*(m) \cdot e^{-x_\nu} = \frac{1}{\nu} \cdot e^{-y_\nu} + \ldots + \frac{1}{m} \cdot e^{-y_m} \quad : \quad 1 \leq \nu \leq m$$

A summation over all ν gives

$$\lambda^*(m) = e^{-y_1} + \ldots + e^{-y_m} = \sum_{\nu=1}^{\nu=m} \big[a_1^* \cdots a_\nu^*\big]^{\frac{1}{\nu}}$$

In other words, $\lambda^*(m)$ is the maximum for the variational problem which is no surprise since $\lambda^*(m)$ is Lagrange's multiplier. Now we shall prove a strict inequality

(3) $$\lambda^*(m) < e$$

We prove (3) by contradiction. To begin with, subtracting the successive equalities in (2) we get the following equations:

(4) $$\lambda^*(m) \cdot [e^{-x_\nu} - e^{-x_{\nu+1}}] = \frac{1}{\nu} \cdot e^{-y_\nu} \quad : \quad 1 \leq \nu \leq m-1$$

(5) $$m \cdot \lambda^*(m) = e^{x_m - y_m}$$

Next, set

(6) $$\omega_\nu = \nu(1 - \frac{a_{\nu+1}}{a_\nu}) : \quad 1 \leq \nu \leq m-1$$

With these notations it is clear that (4) gives

(7) $$\lambda^*(m) \cdot \omega_\nu = e^{x_\nu - y_\nu} \quad : \quad 1 \leq \nu \leq m-1$$

Let us notice that (7) implies that

(8) $$\big(\lambda^*(m) \cdot \omega_\nu\big)^\nu = e^{\nu(x_\nu - y_\nu)} = \frac{a_1 \cdots a_{\nu-1}}{a_\nu^{\nu-1}}$$

By an induction over ν which is left to the reader it follows the ω-sequence satisfies the recurrence equations:

(9) $$\omega_\nu^\nu = \frac{1}{\lambda^*(m)} \cdot \big(\frac{\omega_{\nu-1}}{1 - \frac{\omega_{\nu-1}}{\nu-1}}\big)^{\nu-1} \quad : \quad 1 \leq \nu \leq m-1$$

Notice that we also have

(10) $$\omega_1 = \frac{1}{\lambda^*(m)}$$

A special construction. With λ as a parameter we define a sequence $\{\mu_\nu(\lambda)\}$ by the recursion formula:

(**) $$\mu_1(\lambda) = \frac{1}{\lambda} \quad \text{and} \quad [\mu_\nu(\lambda)]^\nu = \frac{1}{\lambda} \cdot \big[\frac{\mu_{\nu-1}(\lambda)}{1 - \frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1} \quad : \quad \nu \geq 2$$

From (5) and (9) it is clear that $\lambda = \lambda^*(m)$ gives the equality

(***) $$\mu_m(\lambda^*(m)) = m$$

Now we come to the key point during the whole proof.

Lemma *If $\lambda \geq e$ then the $\mu(\lambda)$ -sequence satisfies*

$$\mu_\nu(\lambda) < \frac{\nu}{\nu+1} \quad : \quad \nu = 1, 2, \ldots$$

Proof. We use an induction over ν. With $\lambda \geq e$ we have $\frac{1}{\lambda} < \frac{1}{2}$ so the case $\nu = 1$ is okay. If $\nu \geq 1$ and the lemma holds for $\nu - 1$, then the recursion formula (**) and the hypothesis $\lambda \geq e$ give:

$$[\mu_\nu(\lambda)]^\nu = \frac{1}{\lambda} \cdot \big[\frac{\mu_{\nu-1}(\lambda)}{1 - \frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1} < \frac{1}{e} \cdot \big[\frac{\frac{\nu-1}{\nu}}{1 - \frac{\nu-1}{\nu(\nu-1)}}\big]^{\nu-1}$$

Notice that the last factor is 1 and hence:

$$[\mu_\nu(\lambda)]^\nu < e < \big(1 + \frac{1}{\nu}\big)^{-\nu}$$

where the last inequality follows from the wellknown limit of Neper's constant. Taking the ν:th root we get $\mu_\nu(\lambda) < \frac{\nu}{\nu+1}$ which finishes the induction.

Conclusion. If $\lambda^*(m) \geq e$ then the lemma above and the equality (***) would entail that

$$m = \mu(\lambda^*(m)) < \frac{m}{m+1}$$

This is impossible when m is a positive integer and hence we must have proved the strict inequality $\lambda^*(m) < e$.

The strict inequality for an infinite series. It remains to prove that the strict inequality holds for a convergent series with an infinite number of terms. So assume that we have an equality

$$\text{(i)} \qquad \sum_{\nu=1}^{\infty} \big[a_1 \cdots a_\nu\big]^{\frac{1}{\nu}} = e \cdot \sum_{\nu=1}^{\infty} a_\nu$$

Put as as above

$$\text{(ii)} \qquad \omega_n = n(1 - \frac{a_{n+1}}{a_n})$$

Since we already know that the left hand side is at least equal to the right hand side one can apply Lagrange multipliers and we leave it to the reader to verify that the equality (i) gives the recursion formulas

$$\text{(iii)} \qquad \omega_n^n = \frac{1}{e} \cdot \big[\frac{\omega_{n-1}}{1 - \frac{\omega_{n-1}}{n-1}}\big]^{n-1}$$

Repeating the proof of the Lemma above it follows that

$$\text{(iv)} \qquad \omega_n < \frac{n}{n+1} \implies \frac{a_{n+1}}{a_n} > \frac{n}{n+1}$$

where (ii) gives the implication. So with $N \geq 2$ one has:

$$\frac{a_{N+1}}{a_1} > \frac{1 \cdots N}{1 \cdots N(N+1)} = \frac{1}{N+1}$$

Now $a_1 > 0$ and since the harmonic series $\sum \frac{1}{N}$ is divergent it would follows that $\sum a_n$ is divergent. This contradiction shows that a strict inequality must hold in Theorem 9.1.

10. Thorin's convexity theorem.

Introduction. In the article [Thorin] a convexity theorem was established which goes as follows: Let $N \geq 2$ be a positive integer and $\mathcal{A} = \{A_{\nu k}\}$ a complex $N \times N$-matrix. To each pair of real numbers a, b in the square $\square = \{0 < a, b < 1\}$ we set

$$M(a,b) = \max_{x,y} |\sum\sum A_{\nu k} \cdot x_k \cdot y_\nu| \quad : \quad \sum |x_\nu|^{1/a} = \sum |y_k|^{1/b} = 1$$

10.1 Theorem *The function $(a,b) \mapsto \log M(a,b)$ is convex in $\square$.*

The proof relies upon Hadamard's inequality for maximum norms of bounded analytic functions in strip domains. More precisely, let $f(w)$ be an entire function which is bounded in the infinite strip domain

$$\Omega = \{\sigma + is \quad : 0 \leq \sigma \leq 1 : \, -\infty < s < \infty\}$$

So we assume that there is a constant B such that $|f(z)| \leq B$ for all $z \in \Omega$. Set

$$M_f(\sigma) = \max_s \, f(\sigma + is)| \quad : \quad 0 \leq \sigma \leq 1$$

Then the following is proved in XX:

$$(*) \qquad M_f(\sigma) \leq M_f(0)^{1-\sigma} \cdot M_f(1)^\sigma$$

Proof of Theorem 1. With $0 < a, b < 1$ fixed we consider N-tuples $x_\bullet$ and $y_\bullet$ in $\mathbf{C}^N$ and write

$$x_\nu = c_\nu^a \cdot e^{i\theta_\nu} \quad \text{and} \quad y_k = d_k e^{i\phi_k}$$

where the c-and the d-numbers are real and positive whenever they are $\neq 0$. It is is clear that

(1) $$M(a,b) = \max_{c,d,\theta,\phi} \Big|\sum\sum A_{\nu k} \cdot c_\nu^a \cdot d_k^b \cdot e^{i\theta_\nu} e^{i\phi_k}\Big|$$

where the maximum is taken over N-tuples $\{c_\bullet\}$ and $\{d_\bullet\}$ of non-negative real numbers such that

(2) $$\sum c_\nu = \sum d_k = 1$$

and $\{\theta_\nu\}$ and $\{\phi_k\}$ are arbitrary N-tuples from the periodic interval $[0, 2\pi]$. Consider a pair (a_1, b_1) and (a_2, b_2) in $\square$ and let $(\bar{a}, \bar{b})$ be the middle point. Then we find $c^*, d^*, \theta^*, \phi^*$ so that

(3) $$M(\bar{a}, \bar{b}) = \Big|\sum\sum A_{\nu k} \cdot (c_\nu^*)^a \cdot (d_k^*)^b \cdot e^{i\theta_\nu^* + i\phi_k^*}\Big|$$

Let $w = \sigma + is$ be a complex variable and define the analytic function f by

(4) $$f(w) = \sum\sum A_{\nu k} \cdot (c_\nu^*)^{a_1 + w(a_2 - a_1)} \cdot (d_k^*)^{b_1 + w(b_2 - b_1)} \cdot e^{i\theta_\nu^* + i\phi_k^*}$$

It is clear that $f(w)$ is an entire analytic function and $|f(1/2)| = M(\bar{a}, \bar{b})$. Next, with $w = is$ purely imaginary we have

(5) $$f(is) = \sum\sum A_{\nu k} \cdot (c_\nu^*)^{a_1} \cdot (d_k^*)^{b_1} \cdot e^{is(a_2-a_1)\log c_\nu^* + is(b_2-b_1)\log d_k^*} \cdot e^{i\theta_\nu^* + i\phi_k^*}$$

For each pair ν, k the exponential product

$$e^{is(a_2-a_1)\log c_\nu^* + is(b_2-b_1)\log d_k^*} \cdot e^{i\theta_\nu^* + i\phi_k^*} = e^{i(\theta_\nu(s) + \phi_k(s))}$$

for some pair $\theta_\nu(s), \phi_k(s)$. From (1) we see that

(6) $$\max_s |f(is)| \leq M(a_1, b_1)$$

In the same way the reader can verify that

(7) $$\max_s |f(1+is)| \leq M(a_2, b_2)$$

Now Hadamard's inequality (*) entails that

$$\log M(\bar{a}, \bar{b}) \leq \frac{1}{2} \cdot [\log M(a_1, b_1) + \log M(a_2, b_2)$$

This proves the required convexity.

11. Cesaro and Hölder limits

Introduction. Around 1880 Cesaro introduced a certain summation procedure which which is a substitute for divergent series and leads to the notion of Cesaro summability which is defined below. Another summability was introduced by Hölder and later Knopp and Schnee proved that the conditions for Cesaro-respectively Hölder are equivalent. Here we do not expose their original proof but follow the elegant proof due to Schur which also appears in Chapter 2 in [Landau]. Now we define the two summation methods and the main result appears in Theorem 11.8 below. To each sequence of complex numbers $a_0, a_1, a_2, \ldots$ we get the new sequence

$$S_n = a_0 + \ldots + a_n$$

If $k \geq 0$ we define inductively

$$S_n^{(k+1)} = S_0^{(k)} + \ldots + S_n^{(k)}$$

where we set $S_n^{0)} = S_n$.

11.1 Definition. *For a given integer $k \geq 0$ we say that the sequence $\{a_n\}$ is Cesaro summable of order k if there exists a limit*

(*) $$s_* = \lim_{n\to\infty} \frac{k!}{n^k} \cdot S_n^{(k)}$$

The next result shows that Cesaro summability of some order implies the summability for every higher order.

11.2 Proposition. *If (*) holds for some k_* then a limit exists for every $k \geq k_*$ with the same limit value s_*.*

Proof. Cesaro summability of some order k with a limit s_k means that

$$S_n^{(k)} = \frac{n^k}{k!} \cdot s_k + o(n^k)$$

where the last term is small ordo. If it holds we get

$$S_n^{(k+1)} = \frac{s_k}{k!} \sum_{\nu=0}^{n} n^\nu + o(\sum_{\nu=0}^{n} n^\nu) = \frac{s_k}{k!} \cdot [\frac{n^{k+1}}{k+1} + o(n^{k+1}] + o(n^{k+1})$$

From this the reader discovers the requested induction step and Proposition 11.2 follows.

11.3 Exercise. Assume that $\{a_n\}$ is Cesaro summable of some order k. Show that this entails that

$$a_n = O(n^k)$$

In other words, the sequence $\{n^{-k} \cdot a_n\}$ is bounded.

11.4 The power series $f(x) = \sum a_n x^n$. Assume that $\{a_n\}$ is cecaro summable of some order k. 11.3 it follows that the power series $f(x)$has a radius of convergence which is at least one. For every integer $k \geq 0$ the reader should verify that equality

$$f(x) = \sum a_n x^n = (1-x)^{k+1} \cdot \sum S_n^{(k)} \cdot x^n$$

11.5 Exercise. Deduce from 11.4 that if $\{a_n\}$ is Cesaro summable of some order k_* with limit value s_* then one has the limit formula:

$$s_* = \lim_{x \to 1} f(x)$$

11.6 Hölder's summation. To each sequence of complex numbers $a_0, a_1, a_2, \ldots$ we put

$$H_n^{(0)} = a_0 + \ldots + a_n$$

and if $k \geq 0$ we define inductively

$$H_n^{(k+1)} = \frac{H_0^{(k)} + \ldots + H_n^{(k)}}{n+1}$$

11.7 Definition. *The sequence $\{a_n\}$ is Hölder summable of order k if there exists a limit*

(**) $$\lim_{n \to \infty} H_n^{(k)}$$

11.8 Theorem *A sequence $\{a_n\}$ is Cesaro summable of of some order k if and only if it is Hölder summable of the same order and there respectively limits are the same.*

The proof of Theorem 11.8 requires several steps. First we introduce arithmetic mean value sequences attached to every sequence $\{x_0, x_1, \ldots\}$:

$$M(\{x_\nu\})[n] = \frac{x_0 + \ldots + x_n}{n+1}$$

Next, to each $k \geq 1$ we construct the sequence $T_k(\{x_\nu\})$ by

$$T_k(\{x_\nu\})[n] = \frac{k-1}{k} \cdot M(\{x_\nu\})[n] + \frac{x_n}{k}$$

So above M and $\{T_k\}$ are linear operators which send a complex sequence to another complex sequence. The reader may verify that these operators commute, i.e.

$$T_k \circ M = M \circ T_k$$

hold for every k and similarly the T-operators commute.

For a given k we can also regard the passage to the Cesaro sequence $\{S_n^{[k)}\}$ as a linear operator which we denote by $\mathcal{C}^{(k)}$. Similarly we get the Hölder operators $\mathcal{H}^{(k)}$ for every $k \geq 1$.

11.9 Proposition. *The following identities hold*

(i) $$T_k \circ \mathcal{C}^{(k-1)} = M \circ \mathcal{C}^{(k)} \quad : \quad k \geq 1$$

(ii) $$\mathcal{H}^{(k)} = T_2 \circ \ldots \circ T_k \circ \mathcal{C}^{(k)} \quad : \quad k \geq 2$$

11.10 Exercise. Prove (i) and (ii) above.

As a last preparation towards the proof of Theorem 11.8 we some limit formulas below which show that the T-operators have robust properties. First we have:

11.11 Lemma *Let $\{x_1, x_2, \ldots\}$ be a sequence of complex numbers and q a positive integer such that*

$$\lim_{n \to \infty} q \cdot \frac{x_1 + \ldots + x_n}{n} + x_n = 0$$

Then it follows that

$$\lim_{n \to \infty} x_n = 0$$

Proof. Set $y_n = q(x_1 + \ldots + x_n) + nx_n$. By an induction over n one verifies that

(1) $$\sum_{\nu=1}^{\nu=n} (\nu+1) \cdots (\nu+q-1) \cdot y_\nu = (n+1) \cdots (n+q) \cdot \sum_{\nu=1}^{\nu=n} x_\nu$$

hold for every $n \geq 1$. By the hypothesis $y_n = o(n)$ where $o(n)$ is small ordo of n. It follows that the left hand side in (1) is $o(n^{q+1})$ and since the product $(n+1) \cdots (n+q) \simeq n^q$ we conclude that

(2) $$\sum_{\nu=1}^{\nu=n} x_\nu = o(n)$$

Finally, we have

$$nx_n = y_n - q \cdot \sum_{\nu=1}^{\nu=n} x_\nu$$

and by (2) and the hypothesis the right hand side is $o(n)$ which after division with n gives $x_n = o(1)$ as required.

11.12 Proposition. *Let $\{x_\nu\}$ be a sequence and $k \geq 1$ an integer such that there exists*

$$\lim_{n \to \infty} T_k(\{x_\nu\}][n] = s$$

Then it follows that $\{x_n\}$ converges to s.

11.13 Exercise. Deduce Proposition 11.12 from Lemma 11.11.

11.14 Proof of Theorem 11.8. The easy case $k = 1$ is left to the reader and we proceed to prove the theorem when $k \geq 2$. Assume first that $\{a_n\}$ is Cesaro summable of some order $k \geq 2$ with a limit s. Exercise 11.12 implies that $T_k \circ \mathcal{C}^{(k)}$ sends $\{a_n\}$ to a convergent sequence with limit s. If $k \geq 3$ we apply the exercise to T_{k-1} and continue until the composed operator

$$T_2 \circ \cdots \circ T_k \circ \mathcal{C}^{(k)}$$

sends the a-sequence to a convergent sequence with limit s. By (ii) in Proposition 11.8 this entails that $\{a_k\}$ is Hölder summable of order k with limit k.

Conversely, assume that $\{a_n\}$ is Hölder summable of some order $k \geq 2$. The equality (ii) from Proposition 11.xx gives

$$\mathcal{H}^{(2)} = T_2 \circ \mathcal{C}^{(2)}$$

Hence Proposition 11.13 applied to T_2 shows that Hölder summability of order 2 entails Cesaro summability of the same order. Next, if $k \geq 3$ we again use (ii) in 11.xx and conclude that the sequence

$$T_3 \circ \ldots T_k \circ \mathcal{C}^{(k)}(\{a_n\})$$

is convergent. By repeated application of (ii) in 11.18 applied to $T_3, \ldots, T_k$ we conclude that the a-sequence is Cesaro summable of order k and has the same limit as the Hölder sum.

12. Power series and arithmetic means.

Consider a power series

$$f(x) = \sum a_n \cdot x^n$$

which converges when $|x| < 1$ and assume also that

$$(*) \qquad \lim_{x \to 1} \sum a_n \cdot x^n = 0$$

For each $k \geq 1$ we get the sequence $\{S_n^{(k)}\}$ from the previous section and we prove the following:

12.1 Theorem. *Assume (*) and that there exists some integer $k \geq 1$ such that*

$$\lim_{n \to \infty} S_n^{(k)} = 0$$

Then the series $\sum a_n$ converges.

Example. Consider the case $r = 1$ where

$$S_n^{(1)} = \frac{na_0 + (n-1)a_1 + \ldots + a_n}{n}$$

The sole assumption that $S_n^{(1)} \to 0$ does not imply $\sum a_n$ converges. But in addition (*) is assumed in Theorem 12.1 which will give the convergence. The proof of Theorem 12.1 is based upon the following convergence criterion where (*) above is tacitly assumed.

12.2 Proposition. *The series $\sum a_n$ converges if there to every $\epsilon > 0$ exists a pair (p_0, n_0) such that*

$$p \geq p_0 \implies J(n_0, p) = \Big| \int_0^1 \frac{\sin 2p\pi(x-1)}{x-1} \cdot \sum_{n=n_0}^{\infty} a_n x^n \cdot dx \Big| < \epsilon$$

Exercise. Prove this classic result which already was wellknown to Abel.

Proof of theorem 12.1. To profit upon Proposition 12.2 we need the two inequalities below which are valid for all pairs of positive integers p and n:

$$\text{(i)} \qquad \Big| \int_0^1 \sin(2p\pi x) \cdot x^k(1-x)^n \cdot dx \Big| \leq 2\pi(k+2)! \cdot \frac{p}{n^{k+2}}$$

$$\text{(ii)} \qquad \Big| \int_0^1 \sin 2p\pi x \cdot x^k(1-x)^n \cdot dx \Big| \leq \frac{C(k)}{p \cdot n^k}$$

where the constant $C(k)$ in (ii) as indicated only depends upon k. The verification of (i-ii) is left to the reader. Next, recall from the previous section that:

$$\text{(iii)} \qquad f(x) = \frac{(1-x)^{k+1}}{(k+1)!} \cdot \sum S_n^{(k)} n^k \cdot x^n$$

Let $\epsilon > 0$ and choose n_0 such that

$$\text{(iv)} \qquad n \geq n_0 \implies |S_n^{(k)}| < \epsilon$$

which is possible from the assumption in the Theorem. Notice that (iii) gives the equality

$$\sum_{n=n_0}^{\infty} a_n x^n = \frac{(1-x)^{k+1}}{(k+1)!} \cdot \sum_{n=n_0}^{\infty} S_n^{(k)} n^k \cdot x^n \tag{iii}$$

Hence (iv) and the triangle inequality shows that with n_0 kept fixed, the absolute value of the integral in Proposition 12.2 is majorized as follows for every p:

$$J(n_0, p) \leq \epsilon \cdot \sum_{n=n_0}^{\infty} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big|$$

In (iv) we have chosen n_0 and for an arbitrary $p \geq p_0 = n_0 + 1$ we decompose the sum from n_0 up to p and after we take a sum with $n \geq p+1$ which means that $J(n_0, p)$ is majorized by ϵ times the sum of the following two expressions:

$$\sum_{n=n_0}^{n=p} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big| \tag{1}$$

$$\sum_{n=p+1}^{\infty} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big| \tag{2}$$

Using (i) above it follows that (1) is estimated by

$$2\pi \cdot (k+2)! \cdot \frac{C(k)}{p} \cdot (p - n_0) \leq 2\pi \cdot (k+2)! \cdot C(k) = K_1$$

Next, using (ii) it follows that (2) is estimated by

$$\pi \cdot \frac{k+2}{k+1} \cdot p \cdot \sum_{n=p+1}^{\infty} n^{-2} \leq \pi \cdot \frac{k+2}{k+1} = K_2$$

So with $K = K_1 + K_2$ we have

$$J(n_0, p) \leq 2K \cdot \epsilon$$

for every $p \geq n_0 + 1$ and since $\epsilon > 0$ was arbitrary the proof of Theorem 12.2 is finished.

13. Taylor series and quasi-analytic functions.

Introduction. Let $f(x)$ an infinitely differentiable function defined on the interval $(-1,1)$. At $x = 0$ we can take the derivatives and set

$$C_\nu = f^{(\nu)}(0)$$

In general the sequence $\{C_\nu\}$ does not determine $f(x)$. The standard example is the C^∞-function defined for $x > 0$ by $e^{-1/x}$ and zero on $x \leq 0$. Here $\{C_\nu\}$ is the null sequence and yet the function is no identically zero. But if we impose sufficiently strong growth conditions on the derivatives of f over the whole interval $(-1,1)$ then $\{C_\nu\}$ determines f. In general, let $\{\alpha_\nu\}$ be an increasing sequence of positive real numbers and denote by $\mathcal{C}_\alpha$ the class of C^∞-functions on $(-1,1)$ where the maximum norms of the derivatives satisfy

$$\max_x |f^{(\nu)}(x)| \leq k^\nu \cdot \alpha_\nu \quad : \quad \nu = 0, 1, \ldots$$

for some $k > 0$ which may depend upon f. Denjoy proved that if the series

$$\sum \frac{1}{\alpha_\nu} \tag{*}$$

diverges then $\mathcal{C}_\alpha$ is quasi-analytic which means that every function in this class is determined by its Taylor series at $x = 0$. A complete solution was found by Carleman who gave necessary and sufficient conditions on a sequence $\{\alpha_\nu\}$ in order that $\mathcal{C}_\alpha$ is quasi-analytic. This theory is presented in the book [Carleman]. See also chapter 1 in [Hörmander] for an account of Carleman's results.

Here we restrict the attention to the situation considered by Denjoy, i.e, when (*) is divergent. Since quasi-analytic functions resemble analytic functions in the sense that they are determined by their Tayor series at a single point there remains the question how to determine $f(x)$ in a quasi-analytic class when the sequence $\{C_\nu\}$ is known. Such a reconstruction theorem was announced by Carleman in [CR-1923] and a detailed proof appears in his famous book [Carleman]. The reconstruction arises via variational problems which goes as follows. Let $\{\alpha_\nu\}$ give a divergent series in (*) and put

$$\beta_\nu = \alpha_\nu \cdot \sum_{k=0}^{k=\nu} \alpha_k$$

Next, let $\{C_\nu\}$ be a sequence of positive numbers. For each positive integer n we consider the variational problem

$$\min_f J_n(f) = \sum_{\nu=0}^{\nu=n} \beta_\nu^{-2\nu} \cdot \int_0^1 [f^{(\nu)}(x)]^2 \cdot dx$$

where the competing family consist of n-times differentiable functions on $[0,1]$ such that

$$f^{(\nu)}(0) = C_\nu \quad : \quad 0 \leq \nu \leq n$$

The strict convexity of L^2-norms imply that the variational problem has a unique minimizing function f_n which depends linearly upon $C_0, \ldots, C_n$. In other words, there exists a unique doubly indexed sequence of functions $\{\phi_{p,n}\}$ defined for pairs $0 \leq p \leq n$ such that

$$f_n(x) = \sum_{\nu=0}^{\nu=n} C_p \cdot \phi_{p,n}(x)$$

where the functions $\{\phi_{0,n}, \ldots \phi_{n,n}\}$ only depends upon $\beta_0, \ldots, \beta_n$. With these notations a major result in Carleman's theory goes as follows:

13.1 Theorem. *Let $\{C_\nu\}$ be a sequence such that there exists a function $F(x)$ in the class C_α with $C_\nu = F^{(\nu)}(0)$ for every ν. Then*

$$\lim_{n\to\infty} f_n(x) = F(x) \quad \text{hold for all} \quad 0 < x < 1$$

13.2 A series formula. Using the result above Carleman proved that if the series (*) diverges then there exists a doubly indexed sequence $\{a_{\nu,n}\}$ defined for pairs $0 \leq \nu \leq n$ which only depends on $\{\alpha_\nu\}$ such that whenever $F(x)$ belongs to $\mathcal{C}_\alpha$ in some interval $(-b, b)$ centered at $x = 0$, then

$$F(x) = \lim_{n\to\infty} \sum_{\nu=0}^{\nu=n} a_{\nu,n} \cdot \frac{F^{(\nu)}(0)}{\nu\,!} \cdot x^\nu$$

13.3 Quasi-analytic boundary values. Another delicate problem studied in [Carleman] is about boundary values in power series where the number of non-zero coefficients is sparse. First, consider a power series $\sum a_n z^n$ which has radius of convergence equal to one. Assume that there exists an interval ℓ on the unit circle such that the analytic function $f(z)$ defined by the series extends to a continuous function in the closed sector where $\arg(z) \in \ell$. So on ℓ we get a continuous boundary value function $f^*(\theta)$. Suppose that f^* belongs to some quasi-analytic class on ℓ. in [Carleman] it is proved that this hypothesis entails that the sequence of non-zero coefficients cannot have to many gaps, i.e. if we have gaps and arrange the non-zero coefficients in order $\{a_{\nu_1}, a_{\nu_2}, \ldots\}$ then the integer-valued function

$$k \mapsto n_k$$

cannot increase too rapidly.

13.4 Hadamard's criterion. Let $f(z)$ be as above and assume this time that the boundary function $f^*(\theta)$ is real-analytic on the interval ℓ which in general can be arbitrary small. Then a famous result due to Hadamard entails that one must have an upper bound

$$\limsup_{k \to \infty} n_{k+1} - n_k < \infty$$

In other words, when the sequence of non-zero coefficients is too sparse the analytic function f cannot posses a boundary function which is real-analytic on some small interval on the unit circle.

I:C Complex vector spaces

Contents

Introduction.

Linear Algebra is foremost exposed in real vector spaces. Here the focus will be the complex case where special results appear since $\mathbf{C}$ is an algebraically closed field. Basic facts such as the notion of dimensions for finite dimensional vector spaces, the construction of determinants and Cramer's rule to solve inhomogeneous linear systems are assumed.

To each integer $n \geq 2$ we denote by $M_n(\mathbf{C})$ the set of $n \times n$-matrices with complex elements. As a complex vector space $M_n(\mathbf{C})$ has dimension n^2 and it is an associative $\mathbf{C}$-algebra defined by the usual matrix product where the identity E_n is the matrix whose elements outside the diagonal are zero while $e_{\nu\nu} = 1$ for every $1 \leq \nu \leq n$. When $n \geq 2$ a pair of $n \times n$-matrices A and B do not commute in general which means that $M_n(\mathbf{C})$ is a non-commutative algebra over the complex field.

Outline of the content. In section 1 we prove Wedderburns theorem which asserts that $\{M_n(\mathbf{C})\}$ are the sole complex algebras with the properties above. Section 2 studies *resolvents* which consist of the inverse matrices $R_\lambda(A) = \lambda \cdot E_n - A)^{-1}$ when λ is outside the spectrum $\sigma(A)$ which by definition is the set of zeros in the characteristic polynomial

$$P_A(\lambda) = \det(\lambda \cdot E_n - A) \tag{0.1}$$

A fundamental fact is that $P_A(\lambda)$ only depends upon the associated linear operator defined by the A-matrix. More precisely, if S is an invertible matrix the product formula for determinants give the equality

$$P_A(\lambda) = P_{SAS^{-1}}(\lambda) \tag{0.2}$$

In section 3 we prove *Jordans theorem* which is used to describe multi-valued analytic functions in punctured discs. Special classes of matrices are studied in section 4 and section 5 shows how to obtain fundamental solutions to systems of linear differential equations. In section 6 we prove an inequality due to Carleman which gives a bound for the operator norms of resolvents. The proof uses some results of independent interest such as an inequality by Hadamard in 6.2 and a result in Theorem 6.4 about matrices A for which the trace is zero.

Remark. In section 2 we use analytic function theory to attain results which after can be extended to the operational calculus on linear operators on infinite dimensional vector spaces. This analytic approach is also useful when constructions depend upon parameters. Here is an example. Let A be an $n \times n$-matrix whose characteristic polynomial $P_A(\lambda)$ has n simple roots $\alpha_1, \ldots, \alpha_n$. When λ is outside the spectrum $\sigma(A)$. we get an elegant expression for the resolvents $(\lambda \cdot E_n - A)^{-1}$:

$$(*) \qquad (\lambda \cdot E_n - A)^{-1} = \sum_{k=1}^{k=n} \frac{1}{\lambda - \alpha_k} \cdot \mathcal{C}_k(A)$$

where each matrix $\mathcal{C}_k(A)$ is a polynomial in A:

$$\mathcal{C}_k(A) = \frac{1}{\prod_{\nu \neq k} (\alpha_k - \alpha_\nu)} \cdot \prod_{\nu \neq k} (A - \alpha_\nu E_n)$$

This formula goes back to work by Sylvester, Hamilton and Cayley. The resolvent $R_A(\lambda)$ is also used to construct the Cayley-Hamilton polynomial of A which by definition this is the unique monic polynomial $p_A(\lambda]$ in the polynomial ring $\mathbf{C}[\lambda]$ of smallest possible degree such that the associated matrix $p_A(A) = 0$. It is found as follows: Let $\alpha_1, \ldots, \alpha_k$ be the distinct roots of $P_A(\lambda)$ so that

$$P_A(\lambda) = \prod_{\nu=1}^{\nu=k} (\lambda - \alpha_\nu)^{e_\nu}$$

where $e_1 + \ldots + e_k = n$. Now the meromorphic and matrix-valued resolvent $R_A(\lambda)$ has poles at $\alpha_1, \ldots, \alpha_k$. If the order of a pole at root α_j is denoted by ρ_j one has the inequality

$$\rho_j \leq e(\alpha_j)$$

which in general can be strict. In Section 2 we prove that

$$(**) \qquad p_A(\lambda) = \prod_{\nu=1}^{\nu=k} (\lambda - \alpha_\nu)^{\rho_\nu}$$

0.A Determinants.

In analytic function theory as well as many other subjects calculations of determinants is essential. A wealth of results exist and it would take us too far to expose the theory of determinants in detail. But we recall some facts which for example are needed for the proof of Hadamard's theorem in section 7. The first result is due to Sylvester and gives a procedure to calculate determinants. Let $n \geq 2$ and consider some $n \times n$-matrix A with elements $\{a_{ik}\}$. Put

$$b_{rs} = a_{11}a_{rs} - a_{r1}a_{1s} \quad : \quad 2 \leq r, s \leq n$$

These b-numbers give an $(n-1) \times (n-1)$-matrix where b_{22} is put in position $(1,1)$ and so on. This matrix is denoted by $\mathcal{S}^1(A)$ and called the first order Sylvester matrix. Under the condition that $a_{11} \neq 0$ one has the equality

$$(1) \qquad a_{11}^{n-2} \cdot \det(A) = \det(\mathcal{S}^1(A))$$

Exercise. Prove (1) or consult a text-book in linear algebra which treats determinants in a more serious fashion such as Gerhard Kovalevski's excellent text-book *Determinantenheorie* from 1909 where results about determinants and other topics in linear algebra are proved in a very elegant and detailed fashion.

Next, repeat the construction starting from the $(n-1 \times (n-1)$-matrix $\mathcal{S}^1(A)$ which gives an $(n-2) \times (n-2)$ -matrix $\mathcal{S}^2(A) = \mathcal{S}^1(\mathcal{S}^1(A)$ whose determinant becomes be

$$(2) \qquad b_{22}^{n-3} \cdot a_{11}^{n-2} \cdot \det A)$$

Sylvester proved that (2) is equal to the determinant of the $(n-2)\times(n-2)$-matrix $\mathcal{S}^{(2)}(A)$ whose elements are given by

$$b_{rs}^{(2)} = \det XXX \tag{3}$$

More generally, for every $h \geq 3$ one construct an $n-h)\times(n-h)$-matrix $\mathcal{S}^{(h)}(A)$ with elements

$$b_{rs}^{(h)} = XXX$$

and the following equality holds for every $1 \leq h < n$:

$$\det(\mathcal{S}^{(h)}(A)) = \det(A)\cdot \det XXX$$

The Sylvester-Franke theorem. Let $n \geq 2$ and A is some $n\times n$-matrix with elements $\{a_{ik}\}$. Let $m < n$ and consider the family of minors of size m, i.e. one picks m columns and m rows which give an $m\times m$-matrix whose determinant is called a minor of size m of the given matrix A. The total number of such minors is equal to

$$N^2 \quad \text{where} \quad N = \binom{n}{m}$$

We have N many strictly increasing sequences $1 \leq \gamma_1 < \ldots \gamma_m \leq n$ where a γ-sequence corresponds to preserved columns when a minor is constructed. Similarly we have N strictly increasing sequences which correspond to preserved rows. With this in mind we get for each pair $1 \leq r, s \leq N$ a minor $\mathfrak{M}_{rs}$ where the enumerated r:th γ-sequence preserved columns and similarly s corresponds to the enumerated sequence of rows. Now we obtain the $N\times n$-matrix

$$xxxx$$

whose determinant is denoted by $\mathcal{A}_m$.

Theorem. *For every $1 \leq m < n$ one has the equality*

$$\mathcal{A}_m = \det(A)^{\binom{n-1}{m-1}}$$

Example. Consider the diagonal 3×3-matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

With $m = 2$ we have 9 minors of size 2 and the reader can recognize that when they are arranged so that we begin to remove the first column, respectively the first row, then the resulting $\mathfrak{M}$-matrix becomes

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Its determinant is $4 = 2^2$ which is in accordance with the general formula since $n = 3$ and $m = 2$ give $\binom{n-1}{m-1} = 2$. For the proof of Theorem xx the reader can consult [Kovalevski: page102-105].

0.B Hankel determinants. Let $\{c_0, c_1, \ldots\}$ be a sequence of complex numbers. For each integer $p \geq 0$ and every $n \geq 0$ we obtain the $(p+1)\times(p+1)$-matrix:

$$\mathcal{C}_n^{(p)} = \begin{pmatrix} c_n & c_{n+1} & \ldots & c_{n+p} \\ c_{n+1} & c_{n+2} & \ldots & c_{n+p} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+p} & c_{n+p+1} & \ldots & c_{n+2p} \end{pmatrix}$$

Let $\mathcal{D}_n^{(p)}$ denote the determinant. One refers to $\{\mathcal{D}_n^{(p)}\}$ as the recursive Hankel determinants associated to $\{c_n\}$. They are used to describe various properties of the given c-sequence. To begin with we define a rank number r^* as follows: To every non-negative integer n one has the infinite vector

$$\xi_n = (c_n, c_{n+1}, \ldots)$$

We say that $\{c_n\}$ has finite rank if there exists a number r^* such that r^* many ξ-vectors are linearly independent and the rest are linear combinations of these.

Remark. The sequence $\{c_n\}$ gives the formal power series

$$(*) \qquad f(x) = \sum_{\nu=0}^{\infty} c_\nu x^\nu$$

If $n \geq 1$ we set

$$\phi_n(x) = x^{-n} \cdot (f(x) - \sum_{\nu=0}^{n-1} c_\nu x^\nu)$$

where $\phi_0(x) = f(x)$. The condition that $\{c_\nu\}$ has finite rank means that the ϕ-functions generate a finite dimensional complex subspace of $\mathbf{C}[[x]]$. If this dimension is p we therefore find a non-zero $(p+1)$-tuple $a_0, \ldots, a_p)$ such that the power series

$$a_0 \cdot \phi_0(x) + \ldots + a_p \cdot \phi_p(x) = 0$$

Multiplying this equation with x^p it follows that

$$(a_p + a_{p-1}x + \ldots + a_o x^p) \cdot f(x) = q(x)$$

where $q(x)$ is a polynomial. Hence the finite rank entails that the formal power series (*) above represents a rational function.

B.1 Exercise. Conversely, assume that

$$(1) \qquad \sum c_\nu x^\nu = \frac{q(x)}{g(x)}$$

for some pair of polynomials. Show that $\{c_n\}$ has finite rank.

B.2 Proposition. *A sequence $\{c_n\}$ has a finite rank if and only if there exists an integer p such that*

$$(4) \qquad \mathcal{D}_0^{(p)} \neq 0 \quad \text{and} \quad D_0^{(q)} = 0 \quad : \quad q > p$$

Moreover, one has the equality $p = r^*$.

Again we leave the proof as an exercise.

B.3 A specific example. Suppose that the degree of q is strictly less than that of g and that the rational function is expressed by a sum of simple fractions which means that

$$(2) \qquad \sum c_\nu x^\nu = \sum_{k=1}^{k=p} \frac{d_k}{1 - \alpha_k x}$$

where $\alpha_1, \ldots, \alpha_p$ are distinct and every $d_k \neq 0$. Then we see that

$$(3) \qquad c_n = \sum_{k=1}^{k=p} d_k \cdot \alpha_k^n$$

where we have put $\alpha_k^0 = 1$ so that $c_0 = \sum d_k$.

Exercise. Compute the determinant $\mathcal{D}_0^{(p)}$ when B.2 above holds and show in particular that $\mathcal{D}_n^{(p)} \neq 0$ for all integers $n \geq 0$. .

B.4 The reduced rank. Assume that $\{c_n\}$ has finite rank. To each $k \geq 0$ we denote by r_k the dimension of the vector space generated by $\xi_k, \xi_{k+1}, \ldots$. it is clear that $\{r_k\}$ decrease and we find a non-negative integer r_* such that $r_k = r_*$ for large k. We refer to r_* as the reduced rank. By the construction $r_* \leq r^*$. The relation between r^* and r_* s closely related to the representation

$$f(x) = \frac{q(x)}{g(x)}$$

where q and g are polynomials without common factor. We shall not pursue this discussion any further but refer to the literature. Further results and details of proofs appear in the exercises from [Polya-Szegö : Chapter VII]. See in particular problems 17-34.

B.5 Hankel's formula for Laurent series. Consider a rational function of the form

$$R(z) = \frac{q(z)}{z^p - [c_1 z^{p-1} + \ldots + c_{p-1}z + c_p]}$$

where the polynomial q has degree $\leq p-1$. At ∞ we have a Laurent series

$$R(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \ldots$$

Consider the $p \times p$-matrix

$$A = \begin{pmatrix} 0 & 0 & & \ldots & 0 & c_p \\ 1 & 0 & 0 & \ldots & 0 & c_{p-1} \\ 0 & 1 & 0 & \ldots & \ldots & c_{p-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & c_1 \end{pmatrix}$$

Prove the following for every $n \geq 1$:

$$\mathcal{D}_n^{(p)} = \mathcal{D}_0^{(p)} \cdot \big(\det(\,A)\big)^n$$

B.6 Hadamard's identity. For all pairs of positive integers p and n one has the equality:

$$(*) \qquad \mathcal{D}_n^{(p)} \cdot \mathcal{D}_{n+2}^{(p)} = \mathcal{D}_n^{(p+1)} \mathcal{D}_{n+2}^{(p-1)} + (\mathcal{D}_{n+1}^{(p)})^2$$

The proof of (*) is left as an exercise to the reader where the hint is to use the product formula for determinants and suitable expansions along rows or column vectors to compute the determinant of a matrix.

0.C Hadamard's theorem.

Several fundamental results were established in [Hadamard] with a wide range of applications. Theorem C.1 below is proved in section § XX. Consider a power series

$$f(z) = \sum c_n z^n$$

which initially is analytic in some open disc $\{|z| < \rho\}$ where $0 < \rho < \infty$ is the radius of convergence given by the wellknown limit formula

$$\frac{1}{\rho} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$$

Next, we have the Hankel determinants $\{\mathcal{D}_n^{(p)}\}$ and for each $p \geq 1$ we set

$$(1) \qquad \delta(p) = \limsup_{n \to \infty} \big[\, |\mathcal{D}_n^{(p)}|\, \big]^{\frac{1}{n}}$$

C.1 Theorem. *The series $\sum c_n z^n$ represents a meromorphic function of the complex variable z defined in the whole complex plane if and only if*

$$\lim_{p \to \infty} [\delta(p)]^{\frac{1}{p}} = 0$$

0.D The Gram-Fredholm formula.

A result whose discrete version is due to Gram and later extended to integrals by Fredholm goes as follows: Let $\phi_1, \ldots, \phi_p$ and $\psi_1, \ldots, \psi_p$ be two p-tuples of continuous functions on the unit interval. We get the $p \times p$-matrix with elements

$$a_{\nu k} = \int_0^1 \phi_\nu(x)\psi_k(x) \cdot dx$$

At the same time we define the following functions on $[0,1]^p$:

$$\Phi(x_1, \ldots, x_p) = \det \begin{pmatrix} \phi_1(x_1) & \cdots & \phi_1(x_p) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \phi_p(x_1) & \cdots & \phi_1(x_p) \end{pmatrix} \quad : \quad \Psi(x_1, \ldots, x_p) = \det \begin{pmatrix} \psi_1(x_1) & \cdots & \psi_1(x_p) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \psi_p(x_1) & \cdots & \psi_1(x_p) \end{pmatrix} \quad :$$

Product rules for determinants give the Gram-Fredholm equation

$$(*) \qquad \det(a_{\nu k}) = \frac{1}{p\,!} \int_{[0,1]^p} \Phi(x_1, \ldots, x_p) \cdot \Psi(x_1, \ldots, x_p) \cdot dx_1 \ldots dx_p$$

Exercise. Prove (*) or consult a text-book. See in particular the classic text-book [Bocher] which contains a detailed account about Fredholm determinants and their role for solutions to integral equations.

D.1 Application to integral equations. Let $f(x)$ be a continuous complex-valued function on the closed unit interval $[0,1]$ and $k(x,y)$ is continuous on the product $\square = [0,1] \times [0,1]$. Inductively we define functions $\{f_n\}$ on $[0,1]$ starting with $f_0 = f$ and if $n \geq 1$ one has

$$f_n(x) = \int_0^1 k(x,y) \cdot f_{n-1}(y) \cdot dy$$

If M is $\geq$ the maximum norms of f and k it is clear that

$$\max_{0 \leq x \leq 1} |f_n(x)| \leq M^n \quad \text{hold for every} \quad n \geq 1$$

This enable us to construct the series:

$$u_\lambda(x) = \sum_{n==}^{\infty} \lambda^n \cdot f_n(x)$$

when λ are complex numbers with absolute value $< M$. Using Hadamard's Theorem C.1 and the Gram-Fredholm identities the following was proved by Carleman in [Carleman CR. Acad 1919]:

D.2 Theorem. *The $C^0[0,1]$-valued function $\lambda \mapsto u_\lambda$ of the complex parameter λ extends to a meromorphic function defined in the whole complex plane.*

Theorem D.2 will be proved in section §§ XX from Special Topics.

0.E Hadamard's inequality.

Let us finish our exposition about determinants by another result due to Hadamard.

E.1 Theorem. *Let $A = \{a_{\nu k}\}$ be some $p \times p$-matrix with complex numbers as elements. To each $1 \leq k \leq p$ we set*

$$\ell_p = \sqrt{|a_{1k}|^2 + \ldots + |a_{pk}|^2}$$

Then

$$\big|\det(A)\big| \leq \ell_1 \cdots \ell_p$$

The proof is left as an exercise to the reader.

1. Wedderburn's theorem.

Let us begin with some exercises before Theorem 1.1 is announced.

A. Exercise. Prove that the center of $M_n(\mathbf{C})$ is isomorphic to $\mathbf{C}$. In other words, the sole matrices which commute with all other matrices are of the form $\lambda \cdot E_n$ for some complex number λ.

B. Exercise. Set $\mathcal{A} = M_n(\mathbf{C})$ and identify every matrix with a $\mathbf{C}$-linear operator on $\mathbf{C}^n$. Show that every left ideal L is of the form $\mathcal{A} \cdot E$ where E is an idempotent matrix, i.e. $E = E^2$ and if $\mathcal{N}(E)$ is the null space of E then the principal left ideal generated by E is described as follows:

$$\mathcal{A} \cdot E = \{B \in \mathcal{A} \quad : B(\mathcal{N}(E)) = 0\}$$

Conclude that the family of left ideals in $\mathcal{A}$ is in a 1-1 correspondence with the family of subspaces of $\mathbf{C}^n$.

C. Exercise. Prove that $\mathcal{A}$ is a simple ring, i.e. the only two-sided ideals are the trivial zero-ideal and the whole ring.

The theorem below gives a converse to Exercise C.

1.1 Theorem. *Let A be an associative and finite dimensional $\mathbf{C}$-algebra with a unit element. Then, if $\mathcal{A}$ is simple as a ring there exists an integer n such that*

$$A \simeq M_n(\mathbf{C})$$

Proof. A left ideal L in A is minimal if there does not exist any non-zero left ideal which is strictly smaller than L. Denote by $\mathcal{L}_*$ the family of all minimal left ideals. Notice that if $0 \neq x \in L$ for some minimal ideal then we must have $A \cdot x = L$, i.e. the single element x generates L. Moreover it is clear that a left principal ideal $A \cdot x$ belongs to $\mathcal{L}_*$ if and only if the left annihilator:

$$\ell(x) = \{a \in A \quad : ax = 0\}$$

is a maximal left ideal. So when $A \cdot x \in \mathcal{L}_*$ and $a \in A$ is such that $xa \neq 0$, then xa also generates a minimal left ideal and the maximality of $\ell(x)$ gives the equality

$$\ell(x) = \ell(xa)$$

It follows that the left A-modules Ax and Axa are isomorphic. Next, every left ideal is in particular a complex subspace. If N is the dimension of the complex vector space A then every increasing sequence of complex subspaces has at most N strict inclusions. This shows that there exist minimal left ideals. Choose some $a_0 \neq 0$ where Aa_0 is a minimal left ideal. By left multiplication every $a \in A$ gives a $\mathbf{C}$-linear operator on Aa_0 defined by

$$(1) \qquad a^*(xa_0) = a \cdot x \cdot a_0 \quad : \quad x \in A$$

If a and b is a pair of elements in A the composed $\mathbf{C}$-linear operator $b^* \circ a^*$ is given by

$$(2) \qquad b^* \circ a^*(xa) = b^*(axa_0) = baxa_0 = (ba) \cdot x \cdot a_0 = (ba)^*(xa_0)$$

Hence $a \mapsto a^*$ is a homomorphism from A into the algebra $\mathcal{M} = \mathrm{Hom}_{\mathbf{C}}(Aa_0, Aa_0)$

Sublemma. The map $a \mapsto a^*$ is injective.

Proof. To say that $a^* = 0$ means that

$$axa_0 = 0 \quad \text{for all} \quad x \in A$$

Hence the *two-sided* ideal generated by a is contained in $\ell(a_0)$. So if $a \neq 0$ this two-sided ideal would be the whole ring and then $1 \cdot a_0 = a_0 = 0$ which is a contradiction.

Proof continued. Let k be the dimension of the complex vector space Aa_0 which after a chosen basis identifies $\mathcal{M}$ with the algebra of $k \times k$-matrices. Wedderburn's Theorem follows from the Sublemma if prove that the map (1) is surjective. Counting dimensions this amounts to show the equality

$$(3) \qquad \dim_{\mathbf{C}} A = k^2$$

To prove (3) we consider the the two-sided ideal generated by the family $\{Aa_0x : x \in A\}$. Since A is simple it gives the whole ring and we find a finite set $\{x_1, \ldots, x_m\}$ such that

$$A = Aa_0x_1 + \ldots + \ldots + Aa_0x_m \tag{4}$$

Here we can choose m to be minimal which gives a direct sum in (4), i.e. now

$$A = Aa_0x_1 + \oplus \ldots \oplus Aa_0x_m \tag{5}$$

By previous observations the left ideal $Aa_0x_i \simeq Aa_0$ for each i. So the direct sum decomposition (5) entails that

$$\dim_{\mathbf{C}} A = m \cdot k \tag{6}$$

Hence (3) follows if we can show the equality $m = k$. To attain this we consider the unit element 1_A in the ring A which by (5) has an expression:

$$1_A = \xi_1 + \ldots + \xi_m \quad : \xi_i = b_i a_0 x_i \tag{7}$$

for some m-tuple $b_1, \ldots, b_m$. Since (5) is a direct sum it is easily seen that the ξ-elements satisfy:

$$\xi_i^2 = \xi_i \quad \text{and} \quad \xi_i \cdot \xi_k \;:\; i \neq k \tag{8}$$

Thus, $\{\xi_i\}$ are mutually orthogonal idempotents. Moreover, from the previous observations we have the isomorphism of left A-modules

$$A\xi_i \simeq Aa_0 \quad : \quad 1 \leq i \leq m \tag{9}$$

Next, since ξ is an idempotent we notice that $\xi_i A \xi_i$ is a $\mathbf{C}$-algebra which is naturally identified with the Hom-space

$$\mathrm{Hom}_A(A\xi_i, A\xi_i) \tag{10}$$

Since $A\xi_i$ is a simple left A-module this Hom-algebra is a division ring and since $\mathbf{C}$ is algebraically closed it follows that

$$\xi_i A \xi_i \simeq \mathbf{C} \quad : 1 \leq i \leq m \tag{11}$$

Next, for each pair i, j consider the Hom-space:

$$E_{ij} = \mathrm{Hom}_A(A\xi_i, A\xi_j) \tag{12}$$

By the isomorphisms in (9) and the equality (11) it follows that E_{ij} are one-dimensional complex vector spaces for all pairs i, j. Moreover, the reader may verify the following equality:

$$E_{ij} = \{\xi_i x \xi_j \quad : x \in A\} \tag{13}$$

Next, consider ξ_1. From the expression of 1_A in (7) we obtain

$$A\xi_1 = \sum_{i=1}^{i=m} \xi_i \cdot A \cdot \xi_1 = \sum E_{i1}$$

Since $\{E_{i1}\}$ are 1-dimensional we get the inequality

$$k = \dim_{\mathbf{C}}(A\xi_1) \leq m \tag{*}$$

At this stage we are done since the injectivity in the Sublemma and (6) give

$$m \cdot k \leq k^2 \implies m \leq k$$

Together with (*) it follows that $m = k$ and the requested equality (3) follows.

2. Resolvents

Let A be some matrix in $M_n(\mathbf{C})$. Its characteristic polynomial is defined by

$$(*) \qquad P_A(\lambda) = \det(\lambda \cdot E_n - A)$$

By the fundamental theorem of algebra P_A has n roots $\alpha_1, \ldots, \alpha_n$ where eventual multiple roots are repeated. The union of distinct roots is denoted by $\sigma(A)$ and called the spectrum of A. Since matrices with non-zero determinants are invertible we obtain a matrix valued function defined in $\mathbf{C} \setminus \sigma(A)$ by:

$$(**) \qquad R_A(\lambda) = (\lambda \cdot E_n - A)^{-1} \quad : \quad \lambda \in \mathbf{C} \setminus \sigma(A)$$

One refers to $R_A(\lambda)$ as the resolvent of A. The map

$$\lambda \mapsto R_A(\lambda)$$

yields a matrix-valued analytic function defined in $\mathbf{C} \setminus \sigma(A)$. To see this we take some $\lambda_* \in \mathbf{C} \setminus \sigma(A)$ and set

$$R_* = (\lambda_* \cdot E_n - A)^{-1}$$

By Exercise D from the introduction R_* is a 2-sided inverse which gives the equality

$$E_n = R_*(\lambda_* \cdot E_n - A) = (\lambda_* \cdot E_n - A) \cdot R_* \implies R_* A = A R_*$$

Hence the resolvent R_* commutes with A. Next, construct the matrix-valued power series

$$(1) \qquad \sum_{\nu=1}^{\infty} (-1)^\nu \cdot \zeta^\nu \cdot (R_* A)^\nu$$

which is convergent when $|\zeta|$ are small enough.

2.1 Exercise. Prove the equality

$$R_A(\lambda_* + \zeta) = R_* + \sum_{\nu=1}^{\infty} (-1)^\nu \cdot \zeta^\nu \cdot R_* \cdot (R_* A)^\nu$$

These local series expansions show that the resolvent yields a matrix-valued analytic function in $\mathbf{C} \setminus \sigma(A)$.

2.2 A differential equation. When $|\lambda|$ is strictly larger than the absolute values of the roots of $P_A(\lambda)$, then the resolvent is given by the geometric series

$$(*) \qquad R_A(\lambda) = \frac{E_n}{\lambda} + \sum_{\nu=1}^{\infty} \lambda^{-\nu-1} \cdot A^\nu$$

Taking the complex derivative of $\lambda \cdot R_A(\lambda)$ we get

$$(1) \qquad \frac{d}{d\lambda}(\lambda R_A(\lambda)) = -\sum_{\nu=1}^{\infty} \nu \cdot \lambda^{-\nu-1} \cdot A^\nu$$

Exercise. Use (1) to prove that if $|\lambda|$ is large then $R_A(\lambda)$ satisfies the differential equation:

$$(2) \qquad \frac{d}{d\lambda}(\lambda R_A(\lambda)) + A[\lambda^2 R_A(\lambda) - E_n - \lambda A] = 0$$

Now (2) and the analyticity of the resolvent give:

2.3 Theorem *Outside the spectrum $\sigma(A)$ the matrix-valued function $R(\lambda)$ satisfies the differential equation*

$$\lambda \cdot R_A'(\lambda) + R_A(\lambda) + \lambda^2 \cdot A \cdot R_A(\lambda) = A + \lambda \cdot A^2$$

2.4 Some residue calculus. Since the resolvent is analytic we can construct complex line integrals and apply results in complex residue calculus. Start from the Neumann series (*) from (2.2) and let us perform integrals over circles $|\lambda| = w$ where w is large.

2.5 Exercise. Show that when w is strictly larger than the absolute value of every root of $P_A(\lambda)$ then

$$A^k = \frac{1}{2\pi i}\int_{|\lambda|=w} \lambda^k \cdot R_A(\lambda)\cdot d\lambda \quad : \quad k=1,2,\ldots$$

It follows that when $Q(\lambda)$ is an arbitrary polynomial then

(*) $$Q(A) = \frac{1}{2\pi i}\int_{|\lambda|=w} Q(\lambda)\cdot R_A(\lambda)\cdot d\lambda$$

In particular we take the identity $Q(\lambda)=1$ and obtain

(**) $$E_n = \frac{1}{2\pi i}\cdot\int_{|\lambda|=w} R_A(\lambda)\cdot d\lambda$$

Show also that if $Q(\lambda)$ is a polynomial which has a zero of order $\geq e(\alpha_\nu)$ at every root. then residue calculus entails that

(***) $$Q(A) = 0$$

2.6 Residue matrices. Let $\alpha_1,\ldots,\alpha_k$ be the distinct zeros of $P_A(\lambda)$. For a given root, say α_1 of multiplicity $p\geq 1$ we have a local Laurent series expansion

(i) $$R_A(\alpha_1+\zeta) = \frac{G_p}{\zeta^p}+\ldots+\frac{G_1}{\zeta}+B_0+\zeta\cdot B_1+\ldots$$

We refer to $G_1,\ldots,G_p$ as the residue matrices at α_1. Choose a polynomial $Q(\lambda)$ in $\mathbf{C}[\lambda]$ which vanishes up to the multiplicity at all the the remaining roots $\alpha_2,\ldots,\alpha_k$ while it has a zero of order $p-1$ at α_1, i.e. locally

(i) $$Q(\alpha_1+\zeta) = \zeta^{p-1}(1+q_1\zeta+\ldots)$$

2.7 Exercise. Use residue calculus and the formula from Exercise 2.5 to show that:

(*) $$Q(A) = \frac{1}{2\pi}\int_{|\lambda-\alpha_1|=\epsilon} Q(\lambda)\cdot R_A(\lambda)\cdot d\lambda = G_p$$

Hence the matrix G_p is a polynomial of A. In a similar way one proves that every G-matrix in the Laurent series (i) is a polynomial in A.

2.7 Some idempotent matrices. Consider a zero α_j and choose a polynomial Q_j such that $Q_j(\lambda)-1$ has a zero of order $e(\alpha_j)$ at α_j while Q_j has a zero of order $e(\alpha_\nu)$ at the remaining roots. Set

(1) $$E_A(\alpha_j) = \frac{1}{2\pi i}\int_{|\lambda|=w} Q_j(\lambda)\cdot R_A(\lambda)\cdot d\lambda$$

where w is large as in 2.5. Since the polynomial $S=Q_j-Q_j^2$ vanishes up to the multiplicities at all the roots of $P_A(\lambda$ we have $S(A)=0$ from (***) in 2.5 which entails that

(*) $$E_A(\alpha_j) = E_A(\alpha_j)\cdot E_A(\alpha_j)$$

In other words, we have an idempotent matrix.

2.8 Cayley-Hamilton decomposition. Recall the equality

$$E_n = \frac{1}{2\pi i}\cdot\int_{|\lambda|=w} R_A(\lambda)\cdot d\lambda$$

where the radius w is so large that the disc D_w contains the zeros of $P_A(\lambda)$. The previous construction of the E-matrices at the roots of $P_A(\lambda)$ entail that

$$E_n = E_A(\alpha_1)+\ldots+E_A(\alpha_k)$$

Identifying A with a $\mathbf{C}$-linear operator on $\mathbf{C}^n$ we obtain a direct sum decomposition

(*) $$\mathbf{C}^n = V_1\oplus\ldots\oplus V_k$$

where each V_ν is an A-invariant subspace given by the image of $E_A(\alpha_\nu)$. Here $A-\alpha_\nu$ restricts to a *nilpotent* linear operator on V_ν and the dimension of this vector space is equal to the multiplicity of the root α_ν of the characteristic polynomial. One refers to (*) as the *Cayley-Hamilton decomposition* of $\mathbf{C}^n$.

2.9 The vanishing of $P_A(A)$. Consider the characteristic polynomial $P_A(\lambda)$. By definition it vanishes up to the order of multiplicity at every point in $\sigma(A)$ and hence (***) in 2.5 gives $P_A(A)=0$. Let us write:

$$P_A(\lambda)=\lambda^n+c_{n-1}\lambda^{n-1}+\ldots+c_1\lambda+c_0$$

Notice that $c_0=(-1)^n\cdot\det(A)$. So if the determinant of A is $\neq 0$ we get

$$A\cdot[A^{n-1}+c_{n-1}A^{n-2}+\ldots+c_1]=(-1)^{n-1}\det(A)\cdot E_n$$

Hence the inverse A^{-1} is expressed as a polynomial in A. Concerning the equation

$$P_A(A)=0$$

it is in general not the minimal equation for A, i.e. it can occur that A satisfies an equation of degree $<n$. More precisely , if α_ν is a root of some multiplicity $k\geq 2$ there exists a Jordan decomposition which gives an integer $k_*(\alpha_\nu)$ for the largest Jordan block attached to the nilpotent operator $A-\alpha_\nu$ on V_{α_ν}. The *reduced* polynomial $P_*(\lambda)$ is the product where the factor $(\lambda-\alpha_\nu)^{k_\nu}$ is replaced by $(\lambda-\alpha_\nu)^{k_*(\alpha_\nu)}$ for every α_ν where $k_\nu<k_*(\alpha_\nu)$ occurs. Then P_* is the polynomial of smallest possible degree such that $P_*(A)=0$. One refers to P_* as the *Hamilton polynomial* attached to A. This result relies upon Jordan's result in Section 3.

2.10 Similarity of matrices. Recall that the determinant of a matrix A does not change when it is replaced by SAS^{-1} where S is an arbitrary invertible matrix. This implies that the coefficients of the characterstic polynomial $P_A(\lambda)$ are intrinsically defined via the associated linear operator, i.e. if another basis is chosen in $\mathbf{C}^n$ the given A-linear operator is expressed by a matrix SAS^{-1} where S effects the change of the basis. Let us now draw an interesting consequence of the previous operational calculus. Let us give the following:

2.11 Definition. *A pair of $n\times n$-matrices A,B are* similar *if there exists some invertible matrix S such that*

$$B=SAS^{-1}$$

Since the product of two invertible matrices is invertible this yield an equivalence relation on $M_n(\mathbf{C})$ and from 2.2 above we conclude that $P_A(\lambda)$ only depends on its equivalence class. The question arises if to matrices A and B whose characteristic polynomials are equal also are similar in the sense of Definition 2.6. This is not true in general. More precisely, *Jordan normal form* determines the eventual similarity between a pair of matrices with the same characteristic polynomial.

3. Jordan's normal form

Introduction. Theorem 3.1 below is due to Camille Jordan. It plays an important role when we discuss multi-valued analytic functions in punctured discs and is also used in ODE-theory. Jordan's theorem says that every equivalence class in $M_n(\mathbf{C})$ contains a matrix which is built up by Jordan blocks which are defined below.

Before we enter Jordan's Theorem we discuss some consequences of the material in the previous section. The Cayley-Hamilton decomposition from 2.7. shows that an arbitrary $n\times n$-matrix A has a similar matrix $B=S^{-1}AS$ which is represented in a block form. More precisely, to every root α_ν of some multiplicity $e(\alpha_\nu)$ there occurs a square matrix B_ν of size $e(\alpha_\nu)$ and α_ν is the only root of $P_{B_\nu}(\lambda)$. It follows that for every fixed ν one has

$$B_\nu=\alpha\cdot E_{k_\nu}+S_\nu$$

where E_{k_ν} is an identity matrix of size k_ν and S_ν is nilpotent, i.e. there exists an integer m such that $S_\nu^m = 0$. Jordan's theorem gives a further description of these nilpotent S-matrices which therefore yields a refinement of the Cayley-Hamilton decomposition.

3.0 Jordan blocks. An *elementary* Jordan matrix of size 4 is matrix of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}$$

where λ is the eigenvalue. For $k \geq 5$ one has similar expressions. In general several elementary Jordan block matrices build up a matrix which is said to be in Jordan's normal form.

3.1 Theorem. *For every matrix A there exists an invertible matrix u such that UAU^{-1} is in Jordan's normal form.*

Proof. By the remark after Proposition 2.12 it suffices to prove Jordan's result when A has a single eigenvalue α and replacing A by $A - \alpha$ there remains only to consider the nilpotent case, i.e when $P_A(\lambda) = \lambda^n$ so that $A^n = 0$. In this nilpotent case we must find a basis where A is represented in Jordan's normal form. This is done below.

3.2 The case of nilpotent operators. Let S be a nilpotent $\mathbf{C}$-linear operator on some n-dimensional complex vector space V. So for each non-zero vector in $v \in V$ there exists a unique integer m such that

$$S^m(v) = 0 \quad \text{and} \quad S^{m-1}(v) \neq 0$$

The unique integer m is denoted by $\text{ord}(S, v)$. The case $m = 1$ occurs if $S(v) = 0$. If v has order $m \geq 2$ one verifies that the vectors $v, S(v), \ldots, S^{m-1}(v)$ are linearly independent. The vector space generated by this m-tuple is denoted by $\mathcal{C}(v)$ and called a *cyclic* subspace of V. When $S(v) =$ the corresponding cyclic space $\mathcal{C}(v)$ is reduced to $\mathbf{C} \cdot v$. Now Jordan's theorem amounts to prove the following:

3.3 Proposition *Let S be a nilpotent linear operator. Then V is a direct sum of cyclic subspaces.*

Proof. Set

$$m^* = \max_{v \in V} \text{ord}(S, v)$$

Choose $v^* \in V$ such that $\text{ord}(S, v^*) = m^*$ and construct the quotient space $W = \frac{V}{\mathcal{C}(v^*)}$ on which S induces a linear operator denoted by $\bar{S}$. By an induction over $\dim(V)$ we may assume that W is a direct sum of cyclic subspaces. This implies that we can pick a finite set of vectors $\{v_\alpha\}$ in V such that if $\{\bar{v}_\alpha\}$ are the images in W, then

(1) $$W = \oplus \mathcal{C}(\bar{v}_\alpha)$$

To each α we have the integer $k_\alpha = \text{ord}(\bar{S}, \bar{v}_\alpha)$. By the construction of a quotient space this means that

(2) $$S^{k_\alpha}(v_\alpha) \in \mathcal{C}(v^*)$$

Hence there exists some m^*-tuple $c_0, \ldots, c_{m-1}$ in $\mathbf{C}$ such that

(3) $$S^{k_\alpha}(v_\alpha) = c_0 \cdot v^* + c_1 \cdot S(v^*) + \ldots + c_{m-1} \cdot S^{m^*-1}(v^*)$$

Next, put

(4) $$k_\alpha^* = \text{ord}(S, v_\alpha)$$

It is obvious that $k_\alpha^* \geq k_\alpha$. If this inequality is strict we use (3)and obtain we can write

$$0 = S^{k_\alpha^*}(v_\alpha) = \sum c_\nu \cdot S^{k_\alpha^* - k_\alpha + \nu}(v^*)$$

The maximal choice of m^* entails that $k^*_\alpha \leq m^*$ which gives

(5) $$c_0 = \ldots = c_{k_\alpha - 1} = 0$$

Hence (3) enable us to find $w_\alpha \in \mathcal{C}(v^*)$ such that

(6) $$S^{k_\alpha}(v_\alpha) = S^{k_\alpha}(w_\alpha)$$

Now the images of v_α and $v_\alpha - w_\alpha$ are equal in $\mathcal{C}(v^*)$. So if we replace $\{v\alpha\}$ by $\{\xi_\alpha = v_\alpha - w_\alpha\}$ we still have

(7) $$W = \oplus\, \mathcal{C}(\bar{\xi}_\alpha)$$

Moreover, the construction of the ξ-vectors entail that

(8) $$\operatorname{ord}(\bar{S}, \bar{\xi}_\alpha) = \operatorname{ord}(S, v_\alpha)$$

hold for each α. At this stage an obvious counting of dimensions give the requested direct sum decomposition

$$V = \mathcal{C}(v^*) \,\oplus \mathcal{C}(\xi_\alpha)$$

Remark. The proof was bit cumbersome. The reason is that the direct sum decomposition in Jordan's Theorem is not unique. Only the individual *dimensions* of the cyclic subspaces which appear in a direct sum decomposition are unique. It is instructive to perform Jordan decompositions using an implemented program which for example can be found in *Mathematica*.

4. Hermitian and Normal operators.

The n-dimensional vector space $\mathbf{C}^n$ is equipped with the hermitian inner product:

$$\langle x, y \rangle = x_1 \bar{y}_1 + \ldots + x_n \bar{y}_n$$

A basis $e_1, \ldots, e_n$ is orthonormal if $\langle e_i, e_k \rangle =$ Kronecker's delta function. A linear operator U is called unitary if it preserves the inner product:

$$\langle U(x), U(y) \rangle = \langle x, y \rangle$$

for all x and y. A unitary operator U has the property that it sends an orthonormal basis to another orthonormal basis.

4.0.1 Adjoint operators. Let A be a linear operator. Its adjoint A^* is the linear operator for which

$$\langle A(x), y \rangle = \langle x, A^*(y) \rangle$$

4.0.2 Exercise. Show that if $e_1, \ldots, e_n$ is an arbitrary orthonormal basis in the inner product space $\mathbf{C}^n$ where A is represented by a matrix with elements $\{a_{p,q}\}$, then A^* is represented by the matrix whose elements are

$$a^*_{pq} = \bar{a}_{qp}$$

Show also that a linear operator U is unitary if and only if

$$U^{-1} = U^*$$

4.0.3 Hermitian operators. A linear operator A is called Hermitian if

$$\langle A(x), y \rangle = \langle x, A(y) \rangle$$

holds for all x and y. An equivalent condition is that A is equal to its adjoint A^*. Therefore one also refers to a self-adjoint operator, i.e the notion of a hermitian respectively self-adjoint matrix or operator is the same.

4.0.4 Self-adjoint projections. Let V be a subspace of $\mathbf{C}^n$ of some dimension $1 \leq k \leq n-1$. Its orthogonal complement is denoted by $V^\perp$ and we have the direct sum decomposition

$$\mathbf{C}^n = V \oplus V^\perp$$

To V we associate the linear operator E whose kernel is $V^\perp$ while it restricts to the identity on V. Here

$$E = E^2 \quad \text{and} \quad E = E^*$$

One refers to E as a self-adjoint projection.

4.0.5 Exercise. Show that if E is some $n \times n$-matrix which is idempotent in $M_n(\mathbf{C})$ and Hermitian in the sense of 4.0.1 then E is the self-adjoint projection attached to the subspace $V = E(\mathbf{C}^n)$.

4.0.6 On orthonormal bases. In general, let $V_1 \subset V_2 \subset \ldots V_n = \mathbf{C}^n$ be a strictly increasing sequence of subspaces. So here each V_k has dimension k. By the *Gram-Schmidt orthogonalisation* there exists an orthonormal basis $\xi_1, \ldots, \xi_n$ such that

$$V_k = \mathbf{C} \cdot \xi_1 + \ldots + \mathbf{C} \cdot \xi_k$$

hold for every k. The verification of this wellknown construction is left to the reader. Next, if A is an arbitrary $n \times n$-matrix we know that there exists a sequence $\{V_k\}$ as above such that every V_k is A-invariant, i.e.

$$A(V_k) \subset V_k$$

hold for each k. Now we find the orthonormal basis $\{\xi_k\}$ and get the unitary operator U which sends the standard basis in $\mathbf{C}^n$ onto this ξ-basis. In this ξ-basis we see that the linear operator A is represented by an upper triangular matrix. hence we have

4.0.7 Theorem. *For every $n \times n$-matrix A there exists a unitary matrix U such that U^*AU is upper diagonal.*

4.1 The spectral theorem.

If A is Hermitian there exists an orthonormal basis $e_1, \ldots, e_n$ in $\mathbf{C}^n$ where each e_k is an eigenvector to A whose eigenvalue is a real number. Thus, A can be diagonalised in an orthonormal basis and expressed by matrices this means that there exists a unitary matrix U such that

$$(*) \qquad U^*AU = S$$

where S is a diagonal matrix and every s_{ii} is a real number. In particular the roots of the characteristic polynomial $\det(P_A(\lambda))$ are all real.

Proof. Since A is self-adjoint we have a real-valued function on $\mathbf{C}^n$ defined by

$$(1) \qquad x \mapsto \langle Ax, x \rangle$$

Let m^* be the maximum of (1) as x varies over the compact unit sphere of unit vectors in $\mathbf{C}^n$. The maximum is attained by some complex vector x_* of unit length. Suppose y is a unit vector where that $y \perp x_*$ and let λ be a complex number. Since A is self-adjoint we have:

$$(2) \qquad \langle A(x_* + \lambda y), x_* + \lambda y \rangle = m^* + 2 \cdot \mathfrak{Re}\big(\lambda \cdot \langle Ax_*, y \rangle\big) + |\lambda|^2 \cdot \langle Ay, y \rangle$$

Now $x + \lambda y$ has norm $\sqrt{1 + \lambda|^2}$ and the maximality gives:

$$(3) \qquad m^* + 2 \cdot \mathfrak{Re}\big(\lambda \cdot \langle Ax_*, y \rangle\big) + |\lambda|^2 \cdot \langle Ay, y \rangle \leq \sqrt{1 + |\lambda|^2} \cdot m^*$$

Suppose now that $\langle Ax_*, y \rangle \neq 0$ and set

$$\langle Ax_*, y \rangle = s \cdot e^{i\theta} \quad : \quad s > 0$$

With $\delta > 0$ we take $\lambda = \delta \cdot e^{-i\theta}$ and (3) entails that

$$(4) \qquad 2s \cdot \delta \leq (\sqrt{1 + \delta^2} - 1) \cdot m^* - \langle Ay, y \rangle \cdot \delta^2$$

Next, by calculus one has $2 \cdot \sqrt{1 + \delta^2} - 1) \leq \delta^2$ so after division with δ we get

$$(5) \qquad 2s \leq \delta \cdot \big(\frac{m_*}{2} - \langle Ay, y \rangle\big)$$

But this is impossible for arbitrary small δ and hence we have proved that

$$(6) \qquad y \perp x_* \implies \langle Ax_*, y \rangle = 0$$

This means that $x_*^\perp$ is an invariant subspace for A and the restricted operator remains self-adjoint. At this stage the reader can finish the proof to get a unitary matrix U such that (*) holds.

Remark. In text-books the spectral theorem is usually announced and proved for real and symmetric matrices. So above we have established the general complex version for Hermitian matrices.

4.2 Normal operators.

An $n \times n$-matrix R is normal if it commutes with its adjoint, i.e.

$$(*) \qquad A^*A = AA^* \quad \text{holds in} \quad M_n(\mathbf{C})$$

4.2.0 Exercise. Let A be a normal matrix. Show that every equivalent matrix is normal, i.e. if S is invertible then SAS^{-1} is also normal. The hint is to use that

$$(S^{-1})^* = (S^*)^{-1}$$

holds for every invertible matrix. Conclude from this that we can refer to normal linear operators on $\mathbf{C}^n$.

4.2.1 Exercise. Let A and B be two Hermitian matrices which commute, i.e. $AB = BA$. Show that the matrix $A + iB$ is normal.

Next, let R be normal and assume that the its characteristic polynomial has simple roots. This means that there exists a basis $\xi_1, \ldots, \xi_n$ formed by eigenvectors to R with eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus:

$$(*) \qquad R(\xi_\nu) = \lambda_\nu \cdot \xi_\nu \quad : \quad 1 \leq \nu \leq n$$

Notice that R is invertible if and only if al the eigenvalues are $\neq 0$. It turns out that the normality gives a stronger conclusion.

4.3 Proposition. *Assume that the eigenvalues are $\neq 0$. Then the ξ-vectors in (*) are orthogonal.*

Proof. Consider some eigenvector, say ξ_1. Now we get

$$\text{(i)} \qquad R(R^*(\xi_1)) = R^*(R(\xi_1)) = \lambda_1 \cdot R^*(\xi_1)$$

Hence $R^*(\xi_1)$ is an eigenvector to R with eigenvalue λ_1. By hypothesis this eigenspace is 1-dimensional which gives

$$R^*(\xi_1) = \mu \cdot \xi_1 \implies$$
$$\lambda_1 \cdot \langle \xi_1, \xi_1 \rangle = \langle R(\xi_1), \xi_1 \rangle = \langle \xi_1), R^*(\xi_1) \rangle = \bar{\mu} \cdot \langle \xi_1, \xi_1 \rangle$$

Hence $\mu = \bar{\lambda}_1$ which shows that the eigenvalues of R^* are the complex conjugates of the eigenvalues of R. There remains to show that the ξ-vectors are orthogonal. Consider two eigenvectors, say ξ_1, ξ_2. Then we obtain:

$$\bar{\lambda}_2 \lambda_1 \cdot \langle \xi_1, \xi_2 \rangle = \langle R\xi_1, R\xi_2 \rangle = \langle \xi_1, R^*R\xi_2 \rangle \langle \xi_1, RR^*\xi_2 \rangle =$$

$$\text{(ii)} \qquad \langle R^*\xi_1, R^*\xi_2 \rangle = \bar{\lambda}_1 \cdot \lambda_2 \cdot \langle \xi_1, \xi_2 \rangle \implies (\bar{\lambda}_2 \lambda_1 - \lambda_2 \bar{\lambda}_1) \cdot \langle \xi_1, \xi_2 \rangle = 0$$

By assumption $\lambda_1 \neq \lambda_2$ and both are $\neq 0$. It follows that $\bar{\lambda}_2 \lambda_1 - \lambda_2 \bar{\lambda}_1) \neq 0$ and then (ii) gives $\langle \xi_1, \xi_2 \rangle = 0$ as required.

4.4 Remark. Proposition 4.3 shows that if R is an invertible normal operator with n distinct eigenvalues then there exists a unitary matrix U such that U^*RU is a diagonal matrix. But in contrast to the Hermitian case the eigenvalues are in general complex.

4.5 Exercise. Let R as above be an invertible normal operator with distinct eigenvalues. Show that R is self-adjoint, i.e. a Hermitian matrix if and only if the eigenvalues are real numbers.

4.6 Theorem. *Let R be an invertible normal operator with distinct eigenvalues. Then there exists a unique pair of Hermitian operators A, B such that $AB = BA$ and*

$$R = A + iB$$

4.7 Exercise. Prove Theorem 4.6.

4.8 The operator R^*R. Let R as above be an invertible normal operator with eigenvalues $\lambda_1,\ldots,\lambda_n$. From Remark 4.4 it is clear that R^*R is a Hermitian operator whose eigenvalues all are given by the positive numbers $\{|\lambda_\nu|^2\}$ and if $A.B$ are the Hermitian operators in Theorem 4.6 then we have

$$R^*R = A^2 + B^2$$

Thus, R^*R is represented as a sum of squares of two pairwise commuting Hermitian operators.

4.9 The normal operator $(A + iE_n)^{-1}$. Let A be a arbitrary Hermitian $n \times n$-matrix. We have already seen that its eigenvalues are real . Let us denote them by $r_1,\ldots,r_n$. The spectral theorem gives a unitary matrix U such that U^*AU is diagonal with elements $\{r_\nu\}$. It follows that the matrix $A + iE_n$ is invertible and its inverse

$$R = (A + iE_N)^{-1}$$

is a normal operator with eigenvalues $\{\frac{1}{r_\nu+i}\}$.

4.10 The case of multiple roots

The assumption that the eigenvalues of a normal operator are all distinct can be relaxed. Thus, for every normal and invertible operator R there exists a unitary operator U such that U^*RU is diagonal.

4.11 Exercise. Prove the assertion above. The hint is to establish the following which has independent interest:

4.12 Proposition. *Let R be normal and nilpotent. Then* $R = 0$

Proof. By Jordan's Theorem it suffices to prove this when R is a single Jordan block represented by a special S-matrix whose elements below the diagonal, are 1 while all the other elements are zero. If $n = 2$ we have for example

$$S = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \implies S^* = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

The reader verifies that $S^*S \neq SS^*$ and a similar calculation gives Proposition 4.12 for every $n \geq 3$.

4.13 Remark. The result above means that if R is normal then there never appear Jordan blocks of size > 1 and hence there exists an invertible matrix S such that SRS^{-1} is diagonal.

5. Fundamental solutions to ODE:s.

Recall from Calculus that every ordinary differential equation can be expressed as a system of first order equations. The fundamental issue is therefore to consider a matrix valued function $A(t)$, i.e. an $n \times n$-matrix whose elements $\{a_{ik}(t)\}$ are functions of t. Given $A(t)$ there exists at least locally close to $t = 0$, a unique $n \times n$-matrix $\Phi(t)$ such that

$$\frac{d\Phi}{dt} = A(t) \cdot \Phi(t)$$

with the initial condition $\Phi(0) = E_n$. One refers to Φ as a fundamental solution. The columns of the Φ-matrix give solutions to the homogenous system defined by $A(t)$. Moreover, the determinant of $\Phi(t)$ is $\neq 0$ for every t. In fact his follows from the equality (*) below:

Exercise. The trace function of A is defined by:

$$\mathrm{Tr}(A)(t) = a_{11}(t) + \ldots + a_{nn}(t)$$

Show that the function $t \mapsto \det(\,\Phi(t))$ satisfies the ODE.equation

$$\frac{d}{dt}(\det \Phi(t)) = \det \Phi(t) \cdot \mathrm{Tr}(A)(t)$$

Hence we have the formula

$$(*) \qquad \det \Phi(t) = e^{\int_0^t \mathrm{Tr}(A)(s) \cdot ds} \quad : t \geq 0$$

For example, if the trace function is identically zero then $\det \Phi(t) = 1$ for all t.

5.1 Inhomogeneous equations. From (*) it follows that the matrix $\Phi(t)$ is invertible for all t. This gives a formula to solve a inhomogeneous equation:

$$\frac{d\mathbf{x}}{dt} = A(t)(\mathbf{x}(t)) + \mathbf{u}(t) \tag{1}$$

Here $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t)$ is a given vector-valued function and one seeks a vector-valued function $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)$ such that (1) holds and in addition satisfies the initial condition:

$$\mathbf{x}(0) = \mathbf{b} \quad \text{where } \mathbf{b} \text{ is some vector} \tag{2}$$

Exercise. Show that the unique solution to (1) is given by

$$\mathbf{x}(t) = \Phi(t)(\mathbf{b}) + \Phi(t)\big(\int_0^t \Phi^{-1}(s)\big(\mathbf{u}(s)\big) \cdot ds\big) \tag{**}$$

In other words, for every t we first evaluate the matrix $\Phi(t)$ on the n-vector $\mathbf{b}$ which gives the first time dependent vector in the right hand side. In the second term the inverse matrix $\Phi^{-1}(s)$ is applied to $\mathbf{u}(s)$ for every $0 \leq s \leq t$. After integration over $[0, t]$ we get a time-dependent n-vector on which $\Phi(t)$ is applied.

6. Carleman's inequality

Introduction Theorem 6.1 below was proved by Carleman in the article *Sur le genre du denominateur $D(\lambda)$ de Fredholm* from 1917. At that time the result was used to study non-singular integral equations of the Fredholm type. For more recent applications of Theorem 6.1 we refer to Chapter XI in [Dunford-Schwartz].

The Hilbert-Schmidt norm. It is defined for an $n \times n$-matrix $A = \{a_{ik}\}$ by:

$$||A|| = \sqrt{\sum\sum |a_{ik}|^2}$$

where the double sum extends over all pairs $1 \leq i, k \leq n$. Notice that this norm is the same as

$$||A||^2 = \sum_{i=1}^{i=n} ||A(e_i)||^2$$

where $e_1, \ldots, e_n$ can be taken as an arbitrary orthogonal basis in $\mathbf{C}^n$. Next, for a linear operator S on $\mathbf{C}^n$ its *operator norm* is defined by

$$\text{Norm}[S] = \max_x ||S(x)||$$

with the maximum taken over unit vectors.

6.1 Theorem. *Let $\lambda_1, \ldots, \lambda_n$ be the roots of $P_A(\lambda)$ and $\lambda \neq 0$ is outside $\sigma(A)$. Then one has the inequality:*

$$\big|\prod_{i=1}^{i=n} \big[1 - \frac{\lambda_i}{\lambda}\big] e^{\lambda_i/\lambda}\big| \cdot \text{Norm}\big[R_A(\lambda)\big] \leq |\lambda| \cdot \exp\big(\frac{1}{2} + \frac{||A||^2}{2 \cdot |\lambda|^2}\big)$$

The proof requires some preliminary results. First we need inequality due to Hadamard which goes as follows:

6.2 Hadamard's inequality. *For every matrix A with a non-zero determinant one has the inequality*

$$\big|\det(A)\big| \cdot \text{Norm}(A^{-1}) \leq \frac{||A||^{n-1}}{(n-1)^{n-1)/2}}$$

Exercise. Prove this result. The hint is to use expansions of certain determinants while one considers $\det(A) \cdot \langle A^{-1}(x), y\rangle$ for all pairs of unit vectors x and y.

6.3 Traceless matrices. Let A be an $n \times n$-matrix. The trace is by definition the complex number

(i) $$\text{Tr}(A) = b_{11} + \ldots + b_{nn}$$

Recall that $-\text{Tr}(A)$ is equal to the sum of the roots of $P_A(\lambda)$. In particular the trace of two equivalent matrices are equal. This will be used to prove the following:

6.4 Theorem. *Let A be an $n \times n$-matrix whose trace is zero. Then there exists a unitary matrix U such that the diagonal elements of U^*AU all are zero.*

Proof. Consider first consider the case $n = 2$. By Theorem 4.0.7 it suffices to consider the case when the 2×2-matrix A is upper diagonal and since the trace is zero it has the form

$$A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$$

where a, b is a pair of complex numbers. If $a = 0$ then the two diagonal elements are zero and wee can take $U = E_2$ to be the identity in Lemma 6.5. If $a \neq 0$ we consider a vector $\phi = (1, z)$ in $\mathbf{C}^2$. Then $A(\phi)$ is the vector $(a + bz, -az)$ and hence the inner product becomes:

(i) $$\langle A(\phi), \phi \rangle = a + bz - a|z|^2$$

We can write

$$\frac{b}{a} = re^{i\theta}$$

where $r > 0$ and then (i) is zero if

(ii) $$|z|^2 = 1 + se^{i\theta} \cdot z$$

With $z = se^{-i\theta}$ it amounts to find a positive real number s such that $s^2 = 1 + s$ which clearly exists. Now we get the vector

$$\phi_* = \frac{1}{1 + s^2}(1, se^{-i\theta})$$

which has unit length and

(ii) $$\langle A(\phi_*), \phi_* \rangle = 0$$

By 4.0.6 we find another unit vector ψ_* so that ϕ_*, ψ_* is an orthonormal base in $\mathbf{C}^2$ and hence there exists a unitary matrix U such that $U(e_1) = \phi_*$ and $U(e_2) = \psi_*$. If $B = U^*AB$ the vanishing in (ii) gives $b_{11} = 0$. At the same time the trace is unchanged, i.e. $\text{tr}(B) = 0$ holds and hence we also get $b_{22} = 0$. This means that the diagonal elements of U^*AU are both zero as required.

The case $n \geq 3$. For the induction the following is needed:

Sublemma. Let $n \geq 3$ and assume as above that Tr$(A) = 0$. Then there exists some non-zero vector $\phi \in \mathbf{C}^n$ such that

(*) $$\langle A(\phi), \phi \rangle = 0$$

Proof. If (*) does not hold we get the positive number

$$m_* = \min_{\phi} \big|\langle A(\phi), \phi \rangle\big|$$

where the minimum is taken over unit vectors in $\mathbf{C}^n$. The minimum is achieved by some unit vector ϕ_*. Let $\phi_*^\perp$ be its orthonormal complement and E the self-adjoint projection from $\mathbf{C}^n$ onto $\phi_*^\perp$. On the $(n-1)$-dimensional inner product space $\phi_*^\perp$ we get the linear operator $B = EA$, i.e.

(i) $$B(\xi) = E(A(\xi)) \quad : \quad \xi \in \phi_*^\perp$$

If $\psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\phi_*^\perp$ then the n-tuple $\phi_*, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and since the trace of A is zero we get

(ii) $$0 = \langle A(\phi_*), \phi_* \rangle + \sum_{\nu=1}^{\nu=n-1} \langle A(\psi_\nu), \psi_\nu \rangle = m + \sum_{\nu=1}^{\nu=n-1} \langle B(\psi_\nu), \psi_\nu \rangle$$

where we used that $E(\psi_\nu) = \psi_\nu$ for each ν and that E is self-adjoint so that

$$\langle A(\psi_\nu), \psi_\nu\rangle = \langle A(\psi_\nu), E(\psi_\nu)\rangle = \langle E(A(\psi_\nu)), \psi_\nu\rangle = \langle B(\psi_\nu), \psi_\nu\rangle$$

Now (ii) gives

$$\mathrm{Tr}(B) = -m$$

Hence the $(n-1)\times(n-1)$-matrix which represents $B + \frac{m}{n-1}\cdot E$ has trace zero. By an induction over n we find a unit vector $\psi \in \phi_*^\perp$ such that

$$\langle B(\psi_*), \psi_*\rangle = -\frac{m}{n-1}$$

Finally, since E is self-adjoint we have already seen that

$$\langle A(\psi_*), \psi_*\rangle = \langle B(\psi_*), \psi_*\rangle \implies \big|\langle A(\psi_*), \psi_*\rangle\big| = \big|\frac{m}{n-1}\big| = \frac{m_*}{n-1}$$

Since $n \geq 3$ the last number is $< m_*$ which contradicts the minimal choice of m_*. Hence we must have $m_* = 0$ which proves lemma 6.5

Final part of the proof. Let $n \geq 3$. The Sublemma gives unit vector ϕ such that $\langle A(\phi), \phi\rangle = 0$. Consider the hyperplane $\phi^\perp$ and the operator B from the Sublemma which now has trace zero on this $(n-1)$-dimensional space. So by an induction over n there exists an orthonormal basis $\psi_1, \ldots, \psi_{n-1}$ in $\phi^\perp$ such that $\langle B(\psi_\nu), \psi_n u\rangle = 0$ for every ν. Now $\phi, \psi_1, \ldots, \psi_{n-1}$ is an orthonormal basis in $\mathbf{C}^n$ and if U is the unitary matrix which has this n-tuple as column vectors it follows that the diagonal elements of U^*AU all vanish. This finishes the proof of Theorem 6.4.

Proof Theorem 6.1

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Set $B = \lambda^{-1}A$ so that $\sigma(B) = \{\lambda_i/\lambda\}$ and $\mathrm{Tr}(B) = \sum \frac{\lambda_i}{\lambda}$. We also have

$$||B||^2 = \frac{||A||^2||}{|\lambda|^2} \quad \text{and} \quad \big|\lambda\big| \cdot \mathrm{Norm}[R_A(\lambda)] = \mathrm{Norm}[(E-B)^{-1}]$$

Hence Theorem 6.1 follows if we prove the inequality

$$(*) \qquad \big|e^{\mathrm{Tr}(B)}\big| \cdot \big|\prod_{i=1}^{i=n}\big[1 - \frac{\lambda}{\lambda_i}\big] \cdot \mathrm{Norm}\big[E - B)^{-1}\big] \leq \exp\big[\frac{1+||B||^2}{2}\big]$$

To prove (*) we choose an arbitrary integer N such that $N > \big|\mathrm{Tr}(B)\big|$ and for each such N we define the linear operator B_N on the $n+N$-dimensional complex space with points denoted by (x,y) with $y \in \mathbf{C}^N$ as follows:

$$(**) \qquad B_N(x,y) = (Bx\,,\, -\frac{\mathrm{Tr}(B)}{N}\cdot y)$$

The eigenvalues of the linear operator $E - B_N$ is the union of the n-tuple $\{1 - \frac{\lambda_i}{\lambda}\}$ and the N-tuple of equal eigenvalues given by $1 + \frac{\mathrm{Tr}(B)}{N}$. This gives the determinant formula

$$(1) \qquad \det(E - B_N) = \big(1 + \frac{\mathrm{Tr}(B)}{N}\big)^N \cdot \prod_{i=1}^{i=n}(1 - \frac{\lambda_i}{\lambda})$$

The choice of N implies that (1) is $\neq 0$ so the inverse $(E - B_N)^{-1}$ exists. Moreover, the construction of B_N gives for any pair (x,y) in $\mathbf{C}^{N+n}$:

$$(E - B_N)^{-1}(x,y) = \big(E - B)^{-1}(x), \frac{y}{1 + \frac{1}{N}\cdot\mathrm{Tr}(B)}\big)$$

It follows that

$$\mathrm{Norm}\big[(E-B)^{-1})\big] \leq \mathrm{Norm}\big[(E - B_N)^{-1}\big] \implies$$

$$(2) \qquad \big|\det(E - B_N)\big| \cdot \mathrm{Norm}\big[(E-B)^{-1})\big] \leq \big|\det(E - B_N)\big| \cdot \mathrm{Norm}\big[(E - B_N)^{-1}\big]$$

Hadarmard's inequality estimates the hand side in (2) by:

(3) $$\frac{||E-B_N||^{N+n-1}}{(N+n-1)^{N+n-1)/2}}$$

Next, the construction of B_N implies that its trace is zero. So by the result in 6.3 we can find an orthonormal basis $\xi_1,\ldots,\xi_{n+N}$ in $\mathbf{C}^{n+N}$ such that

$$\langle B_N(\xi_k),\xi_k\rangle = 0 \quad : 1\leq k\leq n+N$$

Relative to this basis the matrix of $E-B_N$ has 1 along the diagonal and the negative of the elements of B_N elsewhere. It follows that the Hilbert-Schmidt norm satisfies the equality:

(4) $$||E-B_N||^2 = N+n+||B_N||^2 = N+n+||B||^2+N^{-1}\cdot|\mathrm{Tr}(B)|^2$$

Hence, (1) and the inequalities from (2-3) give:

$$\big(1+\frac{\mathrm{Tr}(B)}{N}\big)^N\cdot\prod_{i=1}^{i=n}(1-\frac{\lambda_i}{\lambda})\cdot\mathrm{Norm}\big[(E-B)^{-1}\big]\leq$$

$$\frac{\big(N+n+||B||^2+N^{-1}\cdot|\mathrm{Tr}(B)|^2\big)^{(N+n-1)(2}}{\big(N+n-1\big)^{N+n-1/2}} = \frac{\big(1+\frac{||B||^2}{N+n}+\frac{|\mathrm{Tr}(B)|^2}{N(N+n)}\big)^{(N+n-1)/2}}{\big(1-\frac{1}{N+n}\big)^{N+n-1/2}}$$

This inequality holds for arbitrary large N. Passing to the limit as $N\to\infty$ the definition of Neper's constant e give

$$\lim_{N\to\infty}\big(1+\frac{\mathrm{Tr}(B)}{N}\big)^N = e^{\mathrm{Tr}(B)}$$

and the reader may also verify that the limit of the last term above is equal to $\exp\big[\frac{1+||B||^2}{2}\big]$ which finishes the proof of (*) above and hence also of Theorem 6.1.

7. Hadamard's radius theorem.

Hadamard's thesis *Essais sur l'études des fonctions donnés par leur dévelopment d Taylor* contains many interesting results. Here we expose material from Section 2 in [Hadamard]. We are given a power series

$$f(z)=\sum c_nz^n$$

whose radius is a positive number ρ. So f is analytic in the open disc $\{|z|<\rho\}$ but has at least one singular point on the circle $\{|z|=\rho\}$. Hadamard found a condition in order that these singularities consist of a finite set of poles only so that f extends to be meromorphic in some disc $\{|z|<\rho_*\}$ with $\rho_*>\rho$. The condition is expressed via properties of the Hankel determinants defined as before by:

$$\mathcal{D}_n^{(p)} = XXXX$$

For each $p\geq 1$ we set

$$\delta(p) = \big[\limsup_{n\to\infty}[\mathcal{D}_n^{(p)}]^{\frac{1}{n}}$$

In the special case $p=0$ we have

$$\delta(0)=\frac{1}{\rho}=\limsup_{n\to\infty}|c_n|^{\frac{1}{n}}$$

This entails that for every $\epsilon>0$ there exists a constant C_ϵ such that

$$|c_n|\leq C\cdot(\rho-\epsilon)^{-n} \quad \text{hold for every} \quad n$$

It follows trivially that

$$|\mathcal{D}_n^{(p)}| \leq p! \cdot C^p(\rho-\epsilon)^{-pn}$$

Passing to limes superior where while high n:th roots are taken we get

$$\delta(p) = \limsup_{n\to\infty} \left[\mathcal{D}_n^{(p)}\right]^{\frac{1}{n}} \leq \rho^{-(p+1)} \tag{1}$$

Suppose now that there exists some $p \geq 1$ such that one has a strict inequality:

$$\delta(p) < \rho^{-(p+1)} \tag{2}$$

Let p be the smallest integer ≥ 1 for which such a strict inequality holds. Hence there exists $\rho_* > \rho$ such that

$$\delta(p) = \rho_*^{-1} \cdot \rho^{-p} \tag{3}$$

7.1 Theorem. *With p chosen to be minimal as above one has the limit formula*

$$\lim_{n\to\infty} \left[\mathcal{D}_n^{(p-1)}\right]^{\frac{1}{n}} = \rho^{-p} \tag{*}$$

Moreover, $f(z)$ extends to a meromorphic function in the disc of radius ρ_* where the number of poles counted with multiplicity is at most p.

The proof requires several steps. First we show the limit formula (*) in the theorem.

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7.2 The meromorphic extension to $\{|z| < \rho_*\}$. The sequence $\{c_n\}$ gives the Hankel matrices $\{\mathcal{C}_n^{(p)}\}$ from O.XX. For all sufficently large n the result XX above entails that the determinants $\{\mathcal{D}_n^{(p-1)}\}$ are $\neq 0$. If n is large we find unique p-vectors $(A_n^{(1)}, \ldots, A_n^{(p)})$ which solve the inhomogeneous system

$$\sum_{k=0}^{k=p-1} c_{n+k+j} \cdot A_n^{(p-k)} = -c_{n+p+j} \quad : \quad 0 \leq j \leq p-1 \tag{1}$$

Next, put

$$\delta_n^k = A_{n+1}^{(k)} - A_n^{(k)} \quad : \quad 1 \leq k \leq p \tag{2}$$

Solving (1) for n and $n+1$ an easy computation shows that the δ-numbers satisfy the system

$$\sum_{k=0}^{k=p-1} c_{n+j+k+1} \cdot \delta_n^{(p-k)} = 0 \quad : \quad 0 \leq j \leq p-2 \tag{3}$$

and the last equation is

$$\sum_{k=0}^{k=p-1} c_{n+p+k} \cdot \delta_n^{(p-k)} = -(c_{n+2p} + A_n^{(1)} \cdot c_{n+2p-1} + \ldots + A_n^{[(p)} \cdot c_{n+p}) \tag{4}$$

Let us write

$$H_n = c_{n+2p} + A_n^{(1)} \cdot c_{n+2p-1} + \ldots + A_n^{[(p)} \cdot c_{n+p} \tag{5}$$

Expressed in matrix form (3-5) mean that

$$\mathcal{C}_{n+1}^{p-1)}(xxx) == xxx$$

Exercise. Show that the evaluation of $\mathcal{D}_n^{(p)}$ via an expansion of the last column gives the equality:

$$H_n = \frac{\mathcal{D}_n^{(p)}}{\mathcal{D}_n^{(p-1)}} \tag{6}$$

The limit inequality (3) above Theorem 7.1 together with the lower bounds for the absolute values of $\mathcal{D}_n^{(p-1)}$ in §§ XX give for every $\epsilon > 0$ a constant C_ϵ such that the following hold for all sufficiently large n:

$$H_n \leq C_\epsilon \cdot \big(\frac{\rho + \epsilon}{\rho_* - \epsilon}\big)^n \tag{7}$$

Next, the δ-numbers are solved via Cramer's rule. The minors of degree $p-1$ in $\mathcal{C}_{n+1}^{p-1}$ have elements from the given c-sequence and (§§ xx) above implies that every such minor has an absolute value majorized by

$$C \cdot (\rho - \epsilon)^{-(p-1)n}$$

where C is a constant which is independent of n. We conclude that the δ-numbers satisfy

$$|\delta_n^{(k)}| \leq |\mathcal{D}_n^{(p-1)}|^{-1} \cdot C \cdot (\rho - \epsilon)^{-(p-1)n} \cdot H_n \tag{8}$$

Again we apply the lower bound in (§§ XX) to get upper estimates for $|\mathcal{D}_n^{(p-1)}|^{-1}$ and together with (§§ XX) we obtain the following:

7.3 Lemma *To each $\epsilon > 0$ there is a constant C_ϵ such that*

$$|\delta_n^{(k)}| \leq C_\epsilon \cdot \big(\frac{\rho + \epsilon}{\rho_* - \epsilon}\big)^n \quad : \quad 1 \leq k \leq p$$

7.4 The polynomial $Q(z)$. Lemma 7.3 and (2) entail that the sequence $\{A_n^{(k)} : n = 1, 2, \ldots\}$ converges for every k and we denote this limit by $A_*^{(k)}$. Here we notice that Lemma 7.2 also entails that

$$|A_*^{(k)} - A_n^{(k)}| = O\big(\frac{\rho + \epsilon}{\rho_* - \epsilon}\big)^n \tag{i}$$

Now we consider the sequence

$$b_n = c_{n+p} + A_*^{(1)} \cdot c_{n+p-1} + \ldots A_*^{(p)} \cdot c_n \tag{ii}$$

The equation (1) with $j = 0$ gives

$$b_n = c_{n+p} + (A_*^{(1)} - A_n^{(1)}) \cdot c_{n+p-1} + \ldots + (A_*^{(p)} - A_n^{(p)}) \cdot c_n \tag{iii}$$

We know already that $|c_n| \leq C \cdot (\rho - \epsilon)^{-n}$ hold for some constant C and together with (i) above we get

7.5 Lemma. *For every $\epsilon > 0$ there exists a constant C such that*

$$|b_n| \leq C \cdot \big(\frac{1 + \epsilon}{\rho_*}\big)^n$$

Finally, consider the polynomial

$$Q(z) = z^p + A_*^{(1)} \cdot z^{p-1} + \ldots A_*^{(p)}$$

Lemma 7.5 and the trivial version of Hadamard's formula for the radius of convergence implies that $Q(z) \cdot f(z)$ is analytic in the disc $\{|z| < \rho_*\}$ which finishes the proof of Theorem 7.1.

8. On positive definite quadratic forms

In many situations one is asking when a given a bi-linear form is positive definite. We prove a result of this nature which has a geometric interpretation. Let $m \geq 2$ and denote m-vectors in $\mathbf{R}^m$ with capital letters, i.e. $X = (x_1, \ldots, x_m)$. Let $N \geq 2$ be some positive integer and $X_1, \ldots, X_N$ an N-tuple of real m-vectors. To each pair $j \neq k$ we set

$$b_{ij} = ||X_j|| + X_k|| - ||X_j - X_k||$$

where $||\cdot||$ is the usual euclidian length in $\mathbf{R}^m$. We get the symmetric $N \times N$-matrix with elements $\{b_{ij}\}$ and the associated quadratic form

$$H(\xi_1, \ldots, \xi_N) = \sum\sum b_{ij} \cdot \xi_i \cdot \xi_j$$

8.1 Theorem. *If the X-vectors are all different then H is positive definite.*

The proof relies upon a useful formula to express the length of a vector in $\mathbf{R}^m$.

8.2 Lemma There exists a constant C_m such that for every m-vector X one has

$$||X|| = C_m \cdot \int_{\mathbf{R}^m} \frac{1 - \cos\langle X, Y\rangle}{||Y||^{m+1}} \cdot dY$$

Proof. We use polar coordinates and denote by dA the area measure on the unit sphere S^{m-1} and $\omega = (\omega_1, \ldots, \omega_m)$ denote points on the unit sphere S^{m-1}. Notice that

$$\int_{S^{m-1}} (1 - \cos\langle X, \omega\rangle) \cdot dA$$

only depend upon $||X||$. Hence it suffices to prove lemma 8.2 when $X = (R, \ldots, 0)$ where $R = ||X||$ and then the integral in (*) becomes:

$$\int_0^\infty \Big[\int_{S^{m-2}} (1 - \cos Rr\omega_1) \cdot dA_{m-1} \Big] \cdot \frac{dr}{r^2}$$

where dA_{m-1} is the area measure on S^{m-2}. Set

$$B(R, \omega_1) = \int_0^\infty (1 - \cos Rr\omega_1) \cdot \frac{dr}{r^2}$$

for each $-1 < \omega_1 < 1$. The variable substitution $r \to s/R$ gives

$$B(R, \omega_1) = R \cdot \int_0^\infty \frac{1 - \cos s\omega_1}{s^2} \cdot ds = R \cdot B_*(\omega_1)$$

With these notations the integral in (*) becomes

$$R \cdot \int_{S^{m-2}} B_*(\omega_1) \cdot dA_{m-2} \tag{1}$$

Hence Lemma 8. follows when C_m^{-1} is equal to (1).

Proof of Theorem 8.1. For a given pair i, j the addition formula for the cosine-function gives:

$$1 - \cos\langle X_i, Y\rangle + 1 - \cos\langle X_j, Y\rangle + \cos\langle (X_i - X_j), Y\rangle =$$

$$(1 - \cos\langle X_i, Y\rangle) \cdot (1 - \cos\langle X_i, Y\rangle) + \sin\langle X_i, Y\rangle \cdot \sin\langle X_j, Y\rangle \tag{1}$$

It follows that the matrix element b_{ij} is given by

$$C_m \cdot \int_{\mathbf{R}^m} \frac{(1 - \cos\langle X_i, Y\rangle) \cdot (1 - \cos\langle X_j, Y\rangle) + \sin\langle X_i, Y\rangle \cdot \sin\langle X_j, Y\rangle}{||Y||^{m+1}} \cdot dY$$

From this we see that

$$H(\xi) = C_m \cdot \int_{\mathbf{R}^m} \left([\sum (\xi_k \cdot (1 - \cos\langle X_k, Y\rangle)\,]^2 + [\sum (\xi_k \cdot (\sin\langle X_k, Y\rangle)\,]^2\right) \cdot \frac{dY}{||Y||^{m+1}}$$

This shows that H is positive definite as requested.

8.3 Exercise. Prove more generally that for every $1 < p < 2$ a similar result as above holds when the elements of the matrix are:

$$b_{ij} = ||X_j||^p + X_k||^p - ||X_j - X_k||^p$$

Hint. Employ a similar formula as in (*) where a new constant $C_{p,m}$ appears and $||Y||^{m+1}$ is replaced by $||Y||^{m+p}$.

8.4 A class of Hermitian matrices. *Let $z_1, \ldots, z_N$ be an n-tuple of distinct and non-zero complex numbers. Set*

$$b_{ij} = \{\frac{z_i}{z_j}\}$$

Then the matrix $B = \{b_{ij}\}$ is Hermitian and positive definite.

Again the proof is left as an exercise to the reader.

8.5 Remark. Theorem 8.1 has several applications. For example, Beurling used it to prove the existence of certain spectral measures which arise in ergodic processes. Another application from [Beurling: Notes Uppsala 1935] goes as follows: Let f and g be a pair of continuous and absolutely integrable functions on the real line. Define the function on the real t-line by

$$\phi(t) = \int_{-\infty}^{\infty} [f(t+s) - g(s)| \cdot ds$$

8.6 Theorem. *There exists a measure μ on the real ξ-line whose total variation is $\leq 2\sqrt{||f||_1 \cdot ||g||_1}$ such that*

$$\phi(t) = ||f||_1 + ||g||_1 + \int_{-\infty}^{\infty} e^{i\xi t} \cdot d\mu(\xi)$$

The reader is invited to try to prove this theorem using Theorem 8.1 and the observation that the a similar result as above holds for L^2-functions f and g, i.e. this time we set

$$\psi(t) = \int_{-\infty}^{\infty} [f(t+s) - g(s)|^2 \cdot ds$$

and one shows that there exists a measure γ whose total variation is $\leq 2\sqrt{||f||_2 \cdot ||g||_2}$ and

$$\psi(t) = ||f||_2 + ||g||_2 + \int_{-\infty}^{\infty} e^{i\xi t} \cdot d\gamma(\xi)$$

9. An application to integral equations.

Let $k(x,y)$ be a continuous function on the unit square $\{0 \leq x, y \leq 1\}$ and $f(x)$ is continuous on $[0,1]$. Assume that the maximum norms of k and f both are < 1. By induction over n starting with $f_0(x) = f(x)$ we get a sequence $\{f_n\}$ where

$$f_n(x) = \int_0^1 k(x,y) \cdot f_{n-1}(y) \cdot dy \quad : \quad n \geq 1$$

The hypothesis entails that each f_n has maximum norm < 1 and hence there exists a power series:

$$u_\lambda(x) = \sum_{n=0}^{\infty} f_n(x) \cdot \lambda^n$$

which converges when $|\lambda| < 1$ and yields a continuous function $u_\lambda(x)$ defined on $[0,1]$. Less obvious is the following:

9.1 Theorem. *The function $\lambda \mapsto u_\lambda(x)$ with values in the Banach space $B = C^0[0,1]$ extends to a meromorphic B-valued function in the whole λ-plane.*

This result is due to Carelan in[Carleman] and is based upon Hadamard's determinant calculus. Notice that we have only assumed that the maximum norm of k is < 1 while the kernel in general can be non-symmetric, i.e. $k(x,y) = k(y,x)$ need not hold. To establish Theorem xx we use Hadamard's theorem, i.e for each $0 \leq x \leq 1$ we take $\{f_n(x)\}$ and obtain for a given $p \geq 2$ the matrix $C_n^{(p)}(x)$ as above. With these notations we shall prove

Proposition. *For every $p \geq 2$ and $0 \leq x \leq 1$ onr has the inequality*

$$\big|\det(C_n^{(p)}(x))\big| \leq (p\,!)^{-n} \cdot \big(p^{\frac{p}{2}}\big)^n \cdot \frac{p^p}{p\,!}$$

The inequality (*) entails that

$$\limsup_{n\to\infty} \big[\det(C_n^{(p)}(x))\big|\big)^{1/n} \leq \frac{p^{p/2}}{p\,!}$$

Passing to the p:th roots the reader may using Stirling's formula that

$$\lim_{p\to\infty} \frac{p^{1/2}}{p\,!}^{-1/p} = 0$$

Hence Hadamard's theorem gives Theorem 8.2.

8.4 Proof of Proposition 8.3. To prove this inequality we shall use some general determinant formulas of independent interest. Let $\phi_x), \ldots, \phi_p(x)$ and $\psi_x), \ldots, \psi_p(x)$ be a pair of p-tuples of continuous functions on $[0,1]$. For each point $(s_1, \ldots, s_p)$ in $[0,1]^p$ we put

$$D_\phi(s_1, \ldots, s_p) = \Phi(x_1, \ldots, x_p) = \det \begin{pmatrix} \phi_1(x_1) & \cdots & \phi_1(x_p) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \phi_p(x_1) & \cdots & \phi_1(x_p) \end{pmatrix} \quad :$$

In the same way we define $D_\psi(s_1, \ldots, s_p)$. We also get the $p \times p$-matrix with elements

$$a_{jk} = \int_0^1 \phi_j(x) \cdot \psi_k(x) \cdot dx$$

With these notations we have

Lemma. *One has the equality*

$$\det(a_{jk}) = \frac{1}{p\,!} \int_{[0,1]^p} D_\phi(s_1, \ldots, s_p) \cdot D_\psi(s_1, \ldots, s_p) \cdot ds_1 \cdots ds_p$$

Exercise. Prove this result using standard formulas for determinants.

The next step towards the proof of Proposition xx uses the following equality below. First, inductively we get the sequence $\{k^{(m)}(x)\}$ starting with $k = k^{(1)}$ we put:

$$k^{(m)}(x) = \int_0^1 k^{(m-1)}(x,s)\ddot{k}(s) \cdot ds \quad : \quad m \geq 2$$

Now the reader can verify that

$$f_{n+m}(x) = \int_0^1 k^{m)}(x,s) \cdot f_n(s) \cdot ds$$

hold for all pairs $m \geq 1$ and $n \geq 0$.

On $[0,1]^{p+1}$ we define the function

$$\mathcal{K}(x, s_1, \ldots, s_p) = \det(XXXX)$$

next, on $[0,1]^p$ we define for every pair n, p the function

$$\mathcal{F}_n^{(p)}(s_1, \ldots, s_p) = \det(XXXX)$$

Exercise. Use Lemma XX and (xx) above to show the two equalities:

$$\mathcal{F}_n^{(p)}(s_1, \ldots, s_p) = \frac{1}{p\,!} \cdot \cdot \int_{[0,1]^p} \mathbf{K}(s_1, \ldots, s_p, t_1, \ldots, t_p) \cdot \mathcal{F}_{n-1}^{(p)}(t_1, \ldots, t_p) \cdot dt_1 \cdots dt_p$$

$$\det(\mathcal{C}_n^{(p)}(x)) = \frac{1}{p\,!} \cdot \int_{[0,1]^p} \mathcal{F}_n^{(p)}(s_1, \ldots, s_p) \cdot \mathcal{K}(s_1, \ldots, s_p) \cdot ds_1 \cdots ds_p$$

NOW easy to finish ....

D.I On zeros of polynomials

Contents

Introduction.

The results below are foremost due to Laguerre and the subsequent material is based upon Chapter X in Volume 2 from [Polya and Szegö]. The interested reader should also consult the text-book [XX] which contains a wealth of results concerned with zeros of polynomials. For each $n \geq 1$ we denote by $\mathcal{P}_n$ the space of polynomials $p(z)$ of degree $\leq n$. Consider a pair of such polynomials

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n \quad \text{and} \quad q(z) = b_0 + b_1 z + \ldots + b_n z^n$$

Their convolution is defined by:

$$(*) \qquad f * g(z) = a_0 b_0 + a_1 b_1 z + \ldots + a_n b_n z^n$$

The binomial convolution. We can express polynomials in the form

$$p(z) = \sum_{\nu=0}^{\nu=n} \binom{n}{\nu} \cdot a_\nu \cdot z^\nu \quad \text{and} \quad q(z) = \sum_{\nu=0}^{\nu=n} \binom{n}{\nu} \cdot b_\nu \cdot z^\nu$$

Their binomial convolution is defined by:

$$(**) \qquad p *_b q = \sum_{\nu=0}^{\nu=n} \binom{n}{\nu} \cdot a_\nu b_\nu \cdot z^\nu$$

Next, if $p = \sum a_k z^k$ is a polynomial in $\mathcal{P}_n$ we set

$$(***) \qquad p^*(z) = \sum \binom{n}{k} \cdot a_k z^k$$

With the notations above we shall prove the following results.

0.1 Theorem. *Assume that the zeros of q are real and contained in $(-1, 0)$. For every open and convex set K which contains the origin and all the zeros of p, it follows that K contains the zeros of $p *_b q$.*

0.2 Theorem. *If all the roots of p are real so are the roots of p^*.*

0.3 Theorem. *Assume that the zeros of p are real and that the zeros of q are all real and strictly negative, Then the zeros of $p * q$ are real.*

The proofs are given in section D and E.

A. Preliminary results

A.1 A geometric lemma *Let D be a disc in the complex z-plane. If $\lambda \in \mathbf{C} \setminus D$ we put:*

$$\lambda^*(D) = \{\frac{1}{\lambda - z} \quad : \; z \in D\}$$

Then $\lambda^*(D)$ is an open half plane if λ belongs to the circular boundary of D and an open disc if λ is outside the closed disc $\bar{D}$. In particular $\lambda^*(D)$ is always an open convex set

A.2 Proposition. *Let D be an open disc and $\alpha_1, \ldots, \alpha_n$ some n-tuple of points in D. Then*

$$\frac{1}{n}\sum \frac{1}{\lambda - \alpha_\nu} \neq \frac{1}{\lambda - \zeta} \quad \text{hold for all pairs } \lambda, \zeta \in \mathbf{C} \setminus D$$

A.3 Exercise. Prove the two results above.

A.4 Newton's formula. Consider some $p \in \mathcal{P}_n$. Let $\alpha_1, \ldots, \alpha_n$ be the zeros where multiple zeros can occur and assume that the zeros all belong to some open disc D. Suppose that $\lambda \in \mathbf{C}\setminus D$. Newton's formula for the logarithmic derivative gives:

$$\frac{p'(\lambda)}{p(\lambda)} = \frac{1}{n}\sum \frac{1}{\lambda - \alpha_\nu}$$

The convexity of the set $\lambda^*(D)$ from A.1 gives the inclusion

$$(*) \qquad \frac{p'(\lambda)}{p(\lambda)} \in \lambda^*(D)$$

A.5 Exercise. Deduce from A.4 that if K is a convex set containing the zeros of $p(z)$ then the zeros of $p'(z)$ are also contained in K.

B. The Laguerre operators A_ζ^n

.

Let $\zeta \in \mathbf{C}$ and $n \geq 1$. Define the linear operator from $\mathcal{P}_n$ into $\mathcal{P}_{n-1}$ by

$$(*) \qquad A_\zeta^n(p)(z) = (\zeta - z)p'(z) + np(z)$$

We have for example

$$A_\zeta^n(z^k) = k\zeta z^{k-1} + (n-k)z^k \quad : 1 \leq k \leq n$$

B.1 Proposition. *Let ζ and η be two complex numbers and $n \geq 2$. Then*

$$(*) \qquad A_\eta^{n-1} \circ A_\zeta^n = A_\zeta^{n-1} \circ A_\eta^n$$

Proof. Let $p \in \mathcal{P}_n$. The left hand side becomes:

$$(\eta - z)\big[(\zeta - z)p' + np\big]' + (n-1)(\zeta - z)p' + np) =$$

$$(\eta - z)(\zeta - z)p'' + (n-1)(\eta - z)p' + (n-1)(\zeta - z)p' + (n-1)np$$

The last expression is symmetric in η and ζ and (*) follows.

Composed Laguerre operators. Let $\zeta_1, \ldots, \zeta_n$ be an arbitrary n-tuple of complex numbers. So they are not necessarily distinct. We get the composed operator from $\mathcal{P}_n$ into $\mathbf{C}$ defined by:

$$A_{\zeta_1,\ldots,\zeta_n} = A_{\zeta_1}^1 \circ A_{\zeta_2}^2 \circ \ldots \circ A_{\zeta_n}^n$$

Proposition B.1 gives

$$A_{\zeta_{k-1}}^{k-1} \circ A_{\zeta_k}^k = A_{\zeta_k}^{k-1} \circ A_{\zeta_{k-1}}^k$$

for each $k \geq 2$. Since every permutation of an n-tuple can be achieved as the composition where one makes an interchange of a pair $(k-1, k)$ for some k, it follows that the n-fold composed operator $A_{\zeta_1,\ldots,\zeta_n}$ is symmetric with respect to the n-tuple, i.e. we get the same operator after an arbitrary permutation of $\zeta_1, \ldots, \zeta_n$. Next, for every k we have the the elementary symmetric polynomial of degree k:

$$\Sigma_k = \sum \zeta_{\nu_1} \cdots \zeta_{\nu_k}$$

where the sum extends over all k-tuples $1 \leq \nu_1 < \ldots < \nu_k \leq n$.

B.2 Proposition. *For each $1 \leq k \leq$ one has the equality*

$$(*) \qquad \binom{n}{k} \cdot A_{\zeta_1,\ldots,\zeta_n}(x^k) = n\,! \cdot \Sigma_k(\zeta)$$

Proof. Notice that (*) is equivalent with

(i) $$A_{\zeta_1,\ldots,\zeta_n}(x^k) = (n-k)! \cdot k! \cdot \Sigma_k(\zeta)$$

We prove (i) by an induction over n. Denote by Σ_k^* the elementary symmetric polynomials in $\zeta_1, \ldots, \zeta_{n-1}$. It is clear that

(ii) $$\Sigma_k = \zeta_n \cdot \Sigma_{k-1}^* + \Sigma_k^*$$

Next, we have:

(iii) $$A_{\zeta_1,\ldots,\zeta_n}(x^k) = A_{\zeta_1,\ldots,\zeta_{n-1}}(k\zeta_n x^{k-1} + (n-k)x^k)$$

By the induction over n the right hand side becomes

$$k\zeta_n \cdot (k-1)! \cdot (n-1-k)! \cdot \Sigma_{k-1}^* + (n-k)k! \cdot (n-1-k)! \cdot \Sigma_k^* =$$
$$k! \cdot (n-k)! \cdot \left[\zeta_n \cdot \Sigma_{k-1}^* + \Sigma_k^*\right] = k! \cdot (n-k)! \cdot \Sigma_k$$

B.3 The Laguerre form Consider a polynomial of degree n expressed in the form:

(*) $$q(x) = \binom{n}{0} \cdot b_0 + \binom{n}{1} \cdot b_1 x + \ldots + \binom{n}{n} \cdot b_n x^n$$

Let $\zeta_1, \ldots \zeta_n$ be the zeros of q which gives

$$q(x) = b_n \cdot \prod (x - \zeta_\nu) = b_n \cdot \sum_{k=0}^{k=n} (-1)^k \cdot \Sigma_k(\zeta) \cdot x^{n-k}$$

It follows that

(i) $$\binom{n}{k} b_{n-k} = b_n \cdot (-1)^k \cdot \Sigma_k(\zeta)$$

By additivity over k we see that (i) and Proposition B.2 together with the equalities $\binom{n}{k} = \binom{n}{n-k}$ give:

B.4 Proposition. *With q as above and $p(x) = \sum \binom{n}{k} a_k z^k$ we have*

$$\frac{1}{n\,!} \cdot b_n \cdot A_{\zeta_1,\ldots,\zeta_n}(p) = \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \cdot a_k \cdot b_{n-k}$$

This result suggests the following:

B.5 Definition. *For each pair of polynomials p and q in $\mathcal{P}_n$ we set*

$$\mathrm{Lag}(p,q) = \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \cdot a_k \cdot b_{n-k}$$

where p and q are expressed as in (*) above.

B.6 Remark. Notice that when p and q are interchanged we have

$$\mathrm{Lag}(q,p) = (-1)^n \cdot \mathrm{Lag}(p,g)$$

Next, with $q(x)$ as in (*) we have

$$\partial^\nu q(0) = \frac{n\,!}{(n-\nu!} \cdot b_\nu$$

A similar formula holds for p and now the reader can verify that

$$\mathrm{Lag}(p,q) = \frac{1}{n\,!} \sum_{k=0}^{k=n} (-1)^k \cdot \partial^k f(0) \cdot \partial^{n-k} g(0)$$

C. Properties of the Laguerre form

The result below is crucial in the study of Laguerre forms.

C.1 Lemma *Let $p(z) \in \mathcal{P}_n$ have all zeros in some open disc D. For every $\zeta \in \mathbf{C} \setminus D$ the zeros of the polynomial $A_\zeta(p)(z)$ belong to D.*

Proof. By definition

$$A_\zeta(p)(z) = (\zeta - z)p'(z) + np(z)$$

Suppose this polynomial has a zero α which does not belong to D. It follows that

$$\text{(i)} \qquad 0 = (\zeta - \alpha)p'(\alpha) + np(\alpha) \implies \frac{p'(\alpha)}{p(\alpha)} = \frac{n}{\alpha - \zeta}$$

Let $z_1, \ldots, z_n$ be the zeros of p where eventual multiple roots are repeated. Then (i) gives:

$$\text{(ii)} \qquad \frac{p'(\alpha)}{p(\alpha)} = \sum \frac{1}{\alpha - z_\nu}$$

It would follow that

$$\frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} \frac{1}{\alpha - z_\nu} = \frac{1}{\alpha - \zeta}$$

This contradicts Proposition A.1 and Lemma C.1 follows.

C.2 Proposition. *Let p and q be two polynomials of degree n such that $Lag(p, g) = 0$. Then, if D is an open disc which contains the zeros of p, it follows that q has at least one zero in D.*

Proof. Consider first the case $n = 1$ and let ζ_1 be the zero of q while $p(z) = \alpha z - \beta$. The hypothesis entails that

$$0 = A_{\zeta_1}(p) = (\zeta_1 - z)\alpha + \alpha z - \beta = \zeta_1 \cdot \alpha - \beta$$

It follows that ζ_1 is equal to the zero of p and hence it belongs to D. Next, let $n \geq 2$ and suppose that the zeros $\zeta_1, \ldots, \zeta_n$ of q all belong to $\mathbf{C} \setminus D$. By Lemma C.1 the zeros of $A_{\zeta_n}(p)$ belong to D and we can continue until the zero of the linear polynomial $\rho = A_{\zeta_2} \circ \ldots A_{\zeta_n}(p)$ belongs to D. Now ζ_1 is outside D and we get a contraidction from the linear case above since the hypothesis entails that $A_{\zeta_1}(\rho) = 0$.

D. Proof of Theorem 0.1

For a given complex number $\lambda \neq 0$ we set

$$\text{(1)} \qquad q^*(z) = z^n \cdot q(-\frac{\lambda}{z})$$

This gives a new polynomial of degree n. The construction of q^* and Definition B.7 give

$$\operatorname{Lag}(p, q^*) = (-1)^n \cdot \sum \binom{n}{k} \cdot a_k \cdot b_k \cdot \lambda^k$$

The right hand side is the evaluates the polynomial $p *_b q$ at λ which gives the implication

$$\text{(1)} \qquad p *_b q(\lambda) = \operatorname{Lag}(p, g^*) = 0$$

Let λ be a zero $p *_b q$ and consider an open disc D which contains the origin and all the zeros of p. Now (1) gives $\operatorname{Lag}(p, q^*) = 0$ and hence Proposition C.2 gives a point $z_* \in D$ such that $q^*(z_*) = 0$. The inclusion $q^{-1}(0) \subset (-1, 0)$ implies that

$$-1 < \frac{\lambda}{z^*} < 0$$

Hence

$$\text{(2)} \qquad \lambda = az_* \quad \text{for some} \quad 0 < a < 1$$

Since the origin belongs to the convex disc D it follows that $\lambda \in D$. Since this inclusion holds for every open disc D as above elementary geometry shows that λ belongs to the convex set K which finishes the proof of Theorem 0.1.

E. Proof of Theorem 0.2

Let the real roots of p be contained in an interval $(-a, b)$ where a and b both are > 0. For every polynomial q with zeros contained $(-1, 0)$ Theorem 0.1 gives the inclusion

$$p *_b q^{-1}(0) \subset (-a, b) \tag{1}$$

To profit upon (1) we shall use a special q-polynomial. Put

$$L_n(z) = \partial^n((1+z)^n(z-1)^n) \tag{2}$$

Leibniz's rule gives

$$L_n(z) = \sum_{k=0}^{k=n} \binom{n}{k} \cdot \partial^k((1+z)^n) \cdot \partial^{n-k}((z-1)^n) =$$

$$n\,! \cdot \sum_{k=0}^{k=n} \binom{n}{k}^2 \cdot (1+z)^{n-k} \cdot (z-1)^k$$

As explained in XXx the zeros of L_n belong to $(-1, 1)$. Now we define the polynomial

$$Q_n(z) = (z-1)^n \cdot L_n(\frac{z+1}{z-1}) \implies$$

$$Q_n(z) = 2^n \cdot n\,! \cdot \sum_{k=0}^{k=n} \binom{n}{k}^2 \cdot z^k \tag{3}$$

Let c_* be the largest zero of L_n so that $0 < c_* < 1$. Then we see that the zeros of Q_n are real and < 0 where the smallest zero is given by the negative number $-c^*$ where

$$c^* = \frac{1+c_*}{1-c_*}$$

If $c > c*$ then the zeros of the polynomial $q(z) = Q_n(cz)$ are contained in $(-1, 0)$. The construction of the polynomial p^* in Theorem 0.2 and (3) give

$$p^*(cz) = C \cdot p *_b q(z) \tag{4}$$

for some constant C. Theorem 0.1 implies that the zeros of $p^*(cz)$ are contained in the real interval $(-a, b)$ which implies that the zeros of $p^*(z)$ are contained in $(-ca, cb)$. Here cc^* can be arbitrarily small so actually we conclude that the zeros of p^* are contained in $-c^*a, c^*b)$.

E.1 Exercise. Deduce Theorem 0.3 from Theorem 0.1 and 0.2.

F. Legendre polynomials.

For each $n \geq 1$ an inner product is defined in $\mathcal{P}_n$ by

$$\langle q, p \rangle = \int_{-1}^{1} q(x)p(x) \cdot dx$$

Now $1, x, \ldots, x^n$ is a basis in $\mathcal{P}_n$. where $1, x, \ldots, x^{n-1}$ generate a subspace whose co-dimension is one which gives:

F.1 Proposition. *There exists a unique polynomial* $Q_n(x) = x^n + q_{n-1}x^{n-1} + \ldots + q_0$ *such that*

$$\int_{-1}^{1} x^\nu \cdot Q_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

To find $Q_n(x)$ we consider the polynomial $(1-x^2)^n$ whose derivative of order n belongs to $\mathcal{P}_n$ and partial integrations give:

$$\int_{-1}^{1} x^\nu \cdot \partial^n((x^2-1)^n)) \cdot dx = 0 \leq \nu \leq n-1$$

Notice that the leading coefficient of x^n becomes

$$c_n = 2n(2n-1)\cdots(n+1)$$

Hence we have

$$(*) \qquad Q_n(x) = \frac{1}{c_n} \cdot \partial^n((x^2-1)^n)$$

F.2 Definition. *The Legendre polynomial of degree n is given by*

$$L_n(x) = k_n \cdot \partial^n((x^2-1)^n)$$

where the constant k_n is chosen so that $L_n(1) = 1$.

Since L_n is equal to Q_n up to a constant we have

$$\int_{-1}^{1} x^\nu \cdot L_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

From this we conclude that

$$(**) \qquad \int_{-1}^{1} x^\nu \cdot L_n(x) \cdot L_m(x) dx = 0 \quad n \neq m$$

Thus, $\{L_n\}$ is an orthogonal family with respect to the inner product defined by the integral over $[-1, 1]$.

F.3 A generating function. Let w be a new variable and set

$$\phi(x, w) = 1 - 2xw + w^2$$

Notice that $\phi \neq 0$ when $-1 \leq x \leq 1$ and $|w| < 1$. Keeping $-1 \leq x \leq 1$ fixed we have the function

$$w \mapsto \frac{1}{\sqrt{1-2xw+w^2}}$$

Recall that when $|\zeta| < 1$ one has the Newton series

$$\frac{1}{\sqrt{1-\zeta}} = \sum g_n \cdot \zeta^n \quad \text{where} \quad g_n = \frac{3 \cdot 5 \cdots (2n-1)}{2^n}$$

It follows that

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum g_n (2xw - w^2)^2$$

With x kept fixed the series is expanded into w-powers, i.e. set

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum \rho_n(x) \cdot w^n$$

It is easily seen that as x varies then $\rho_n(x)$ is a polynomial of degree n. Moreover, we notice that the coefficient of x^n in $\rho_n(x)$ is equal to

$$g_n \cdot 2^n$$

Next, if $x = 1$ we have

$$\frac{1}{\sqrt{1-2w+w^2}} = \frac{1}{1-w} = \sum w^n$$

From this we conclude that

$$\rho_n(1) = 1 \quad \text{for all} \quad n \geq 0$$

With these notations one has:

F.4 Theorem. *One has $\rho_n(x) = L_n(x)$ for each n, i.e.*

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum L_n(x) \cdot w^n$$

holds when $-1 \leq x \leq 1$ and $|w| < 1$.

Exercise. Prove this result.

F.5 The trigonometric polynomial $L_n(\cos\theta)$

.

With x replaced by $\cos\theta$ we notice that

$$1 - 2\cos\theta\cdot w + w^2 = (1-e^{i\theta}w)(1-e^{-i\theta}w) \implies$$

$$\frac{1}{\sqrt{1-2\cos(\theta)w+w^2}} = \frac{1}{\sqrt{1-1-e^{i\theta}w)}}\cdot\frac{1}{\sqrt{1-e^{-i\theta}w)}}$$

The last product becomes

$$\sum\sum g_m e^{im\theta}w^m\cdot g_\nu e^{-i\nu\theta}w^\nu$$

Collecting w powers the double sum becomes

$$\sum\gamma_n(\theta)\cdot w^n \quad \text{where} \qquad \gamma_n(\theta) = \sum_{m+\nu=n} g_m g_\nu e^{i(m-\nu)\theta}$$

Since $\{g_m\}$ are real numbers we see that $\gamma_n(\theta)$ is equal to

$$g_n\cdot g_0\cos(\nu\theta) + g_{n-1}g_1\cos((n-2)\theta) + \ldots + g_1 g_{n-1}\cos((1-n)\theta) + g_0 g_n\cos(-n\theta)$$

By Theorem F.4 the last sum represents $L_n(\cos\theta)$. We have for example

$$L_3(\cos(\theta) = 2g_3\cdot\cos(3\theta) + 2g_2 g_1\cdot\cos(\theta)$$

where we used that $g_0 = 1$.

F.6 An inequality for $|L_n(x)|$. Since the g-numbers are all ≥ 0 we obtain

$$|L_n(\cos(\theta)| \leq g_n g_0 + g_{n-1}g_1 + \ldots + g_1 g_{n-1} + g_0 g_n = P_n(1) \quad : \quad 0\leq\theta\leq 2\pi$$

Hence we have proved

F.7 Theorem. *For each n one has*

$$|L_n(x)| \leq 1 \quad : \quad -1 \leq x \leq 1$$

Next, we study the values when $x > 1$.

F.8 Theorem. *For each $x > 1$ one has* $1 < L_1(x) < L_2(x) < \ldots$.
Proof. Put

$$\psi(x,w) = 1 + \sum_{n=1}^{\infty}[L_n(x) - L_{n-1}(x)]\cdot w^n$$

By Theorem F.4 this is equal to

$$(*) \qquad \frac{1-w}{\sqrt{1-2xw+w^2}}$$

With $x > 1$ we set $x = 1+\xi$ and notice that

$$1 - 2xw + w^2 = (1-w)^2 - 2\xi w$$

Hence (*) becomes

$$(**) \qquad \frac{1}{\sqrt{1-\frac{2\xi w}{1-w^2}}} = \sum g_n\cdot\frac{(2\xi w)^n}{(1-w^2)^n} = \sum g_n\cdot(2\xi)^n\cdot\frac{w^n}{(1-w^2)^n}$$

Next, for each $n \geq 1$ we notice that the power series of $\frac{w^n}{(1-w^2)^n}$ has positive coefficients. Since $g_n(2\xi)^n > 0$ also hold we conclude that (**) is of the form

$$1 + \sum_{n=1}^{\infty} q_n(\xi)\cdot w^n \quad \text{where} \quad q_n(\xi) > 0$$

Finally, since

$$L_n(1+\xi) - L_{n-1}(1+\xi) = q_n(\xi)$$

we get Theorem F.8

F.9 An L^2-inequality.

Let $n \geq 1$ and denote by $\mathcal{P}_n[1]$ the family of real-valued polynomials $Q(x)$ of degree $\leq n$ for which

$$\int_{-1}^{1} Q^2(x) \cdot dx = 1$$

Then we seek the number

$$\rho(n) = \max_{Q \in \mathcal{P}_n[1]} |Q|_\infty$$

where $|Q|_\infty$ is the maximum norm over $[-1, 1]$. To find this ρ-number we use the orthonormal basis from (xx) and write

$$Q(x) = t_0 \cdot L_0^*(x) + \ldots + t_n \cdot L_n^*(x)$$

Since $Q \in \mathcal{P}_n[1]$ we have $t_0^2 + \ldots + t_n^2 = 1$. Recall also that

$$L_\nu^*(x) = \sqrt{\frac{2\nu + 1}{2}} \cdot L_\nu(x)$$

Given $-1 \leq x_0 \leq 1$ the Cauchy-Schwarz inequality gives

$$Q^2(x_0) \leq \sum_{\nu=0}^{\nu=n} \frac{2\nu + 1}{2} \cdot |L_\nu(x_0)| \leq \sum_{\nu=0}^{\nu=n} \frac{2\nu + 1}{2}$$

where the last inequality follows since the maximum norm of each L_ν is one. Finally, we notice that

$$\sum_{\nu=0}^{\nu=n} \frac{2\nu + 1}{2} = \frac{(1+n)^2}{2}$$

We conclude that

$$(*) \qquad |Q(x_0)| \leq \frac{n+1}{\sqrt{2}}$$

F.10 Exercise. Show that (*) is sharp and find also the extremal polynomial whose L^2-norm is one while the maximum norm is $\rho(n)$.

I:E. Fourier series

Contents

Introduction. Section A contains basic material about Fourier series where we first introduce the kernels of Dini and Fejer. At the end of section A we construct the Jackson kernel which is used to get good approximations by trigonometric polynomials of a given periodic function f, where the rate of approximation is controlled by the modulos of continuity of f. Sections B-C are devoted to results about extremal polynomials. A complex version appears in section 4 where Theorem D.4 relates the transfinite diameter of compact subsets of **C** with Tchebysheff polynomials.

The study of Fourier series has a long history since they were invented by Fourier around 1810 to solve the heat-and the wave-equations. An obstacle when Fourier's method is applied is that even if $f(\theta)$ is a 2π-periodic and continuous function, it is not always true that the partial Fourier sums of f converge at a every point. Examples where pointwise convergence were discovered at an early stage and a classic example lead to the Gibb's phenomenon. For more than a century the question was open if pointwise convergence at least holds almost everywhere for an arbitrary continuous function. The affirmative answer that almost everywhere convergence holds was proved by Carleson in 1965 and constitutes one of the greatest achievements ever in analysis. Apart from the result, Carleson's proof is perhaps more important and the methods have been adapted in many other areas such as in the study of martingales in probability theory. So the Fourier series is an important and deep subject which serves as an essential tool many other areas in mathematics such as analytic number theory. Section F is devoted to a result by Carleman which proves convergence in the mean of partial Fourier sums. From a statistical point of view this result confirms the convergence of Fourier's partial sums. For example, one consequence if the material in section F gives the following:

Let f be a 2π-periodic and continuous function and $\{s_\nu\}$ are Fourier's partial sums. Then the average deviation of these partial sums from f is small. More precisely we have

$$\lim_{n\to\infty} \frac{1}{n+1}\cdot\sum_{\nu=0}^{\nu=n} ||s_\nu - f|| = 0$$

where $\{||s_\nu - f||\}$ denote maximum norms over $[0,2\pi]$.

A: Dini's and Fejer's kernels

We consider complex-valued and continuous functions $f(\theta)$ defined in the interval $[0,2\pi]$ which satisfiy $f(0) = f(2\pi)$. To each such function f and every integer n we set

$$\widehat{f}(n) = \frac{1}{2\pi}\cdot\int_0^{2\pi} e^{-in\phi} f(\phi)\cdot d\phi$$

We refer to $\{\widehat{f}(n)\}$ as the Fourier coefficients of f. Next, if $N \geq 1$ we set

$$S_N(\theta) = \sum_{n=-N}^{n=N} \hat{f}(n) \cdot e^{in\theta}$$

We refer to S_N as Fourier's partial sum function of degree N.

A.1.The Dini kernel. If $N \geq 1$ we set

$$D_N(\theta) = \frac{1}{2\pi} \sum_{n=-N}^{n=N} e^{in\theta}$$

A.2 Proposition. *One has the formula*

$$D_N(\theta) = \frac{\sin((N+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}$$

Proof. We have

$$\sum_{n=-N}^{n=N} e^{in\theta} = e^{-iN\theta} \cdot \frac{1}{2\pi} \sum_{n=0}^{n=2N} e^{in\theta} = e^{-iN\theta} \cdot \frac{e^{i(2N+1)\theta} - 1}{e^{i\theta} - 1}$$

Proposition A.2 follows after multiplication with $e^{i\theta/2}$ and the two equalities

$$e^{i(N+1/2)\theta} - e^{i(N+1/2)\theta} = 2i \cdot \sin((N+1/2)\theta)$$

$$e^{i\theta/2} - e^{-i\theta/2} = 2i \cdot \sin(\theta/2)$$

A.3 Exercise. Show that

$$S_N(\theta) = \int_0^{2\pi} D_N(\theta - \phi) \cdot f(\phi) \cdot d\phi = \int_0^{2\pi} D_N(\phi) \cdot f(\theta + \phi) \cdot d\phi$$

A.4 The Fejer kernel. Given f as above and $N \geq 1$ we set

$$F_N(\theta) = \frac{S_0(\theta) + \ldots + S_N(\theta)}{N+1} \implies$$

$$F_N(\theta) = \int_0^{2\pi} \mathcal{F}_N(\phi) \cdot f(\theta + \phi) \cdot d\phi \quad \text{where} \quad \mathcal{F}_N(\phi) = D_0(\phi) \ldots + D_N(\phi)$$

A.5 Proposition *One has the formula*

$$F_N(\theta) = \frac{1}{N+1} \cdot \frac{1 - \cos((N+1)\theta)}{2 \cdot \sin^2(\frac{\theta}{2})}$$

Proof. To each $\nu \geq 0$ we have

$$\sin((\nu + 1/2)\theta) = \mathfrak{Im}\big[e^{i(\nu+1/2)\theta)}\big]$$

It follows that $F_N(\theta)$ is equal to the imaginary part of

$$\frac{e^{i\theta/2}}{\sin(\theta/2)} \cdot \sum_{\nu=0}^{\nu=N} e^{i\nu\theta} = \frac{e^{i\theta/2}}{\sin(\theta/2)} \cdot \frac{e^{i(N+1)\theta} - 1}{e^{i\theta - 1}} = \frac{e^{i(N+1)\theta} - 1}{\sin(\theta/2)} \cdot \frac{1}{e^{i\theta/2} - e^{-i\theta/2}} =$$

$$\frac{e^{i(N+1)\theta} - 1}{2i \cdot \sin^2(\theta/2)} = i \cdot \frac{1 - e^{i(N+1)\theta}}{2 \cdot \sin^2(\theta/2)}$$

Finally, the imaginary part of the last term is equal to

$$\frac{1 - \cos((N+1)\theta)}{2 \cdot \sin^2(\theta/2)}$$

which proves Proposition A.5.

A.6 A limit formula. When f is given we set

$$\mathcal{F}_N(\theta) = \int_0^{2\pi} F_N(\phi) \cdot f(\theta+\phi) \cdot d\phi$$

If $a > 0$ and $a \leq \theta \leq 2\pi - a$ the sine-function $\sin^2(\theta/2)$ is bounded below, i.e.

$$\sin^2(\theta/2) \geq \sin^2(a/2)$$

So if M is the maximum norm of $|f(\theta)|$ over $[0, 2\pi]$ it follows that

$$\int_a^{2\pi - a} F_N(\phi) \cdot f(\theta+\phi) \cdot d\phi \leq$$

$$\frac{M}{(N+1) \cdot \sin^2(a/2)} \int_a^{2\pi - a} (1 - \cos(N\phi)) \cdot d\phi \leq \frac{2M}{(N+1) \cdot \sin^2(a/2)}$$

A.7 Exercise. Given some θ_0 and $a > 0$ we set

$$\omega_f(a) = \max_{|\theta - \theta_0| \leq a} |f(\theta) - f(\theta_0)|$$

Show that

$$|\mathcal{F}_N(\theta_0) - f(\theta_0)| \leq \frac{2M}{(N+1) \cdot \sin^2(a/2)} + \omega_f(a) \quad \text{for all} \quad 0 < a < \pi$$

Finally, use the *uniform continuity* of the function f over the interval $[0, 2\pi]$ to conclude that the sequence $\{\mathcal{F}_N\}$ converges uniformly to f over the interval $[0, 2\pi]$.

A.8 The case when f is real-valued. Above we used the complex Fourier series. When f is real-valued one often employs the real Fourier series which takes the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos kx + \sum_{k=1}^{\infty} b_k \cdot \sin kx$$

Here $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx$ and when $k \geq 1$ one has

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos kx \cdot dx$$

with similar formulas for the sine-coefficients $\{b_k\}$. With these notations Fourier's partial sum functions become

$$S_n(f) = \frac{a_0}{2} + \sum_{k=1}^{k=n} a_k \cdot \cos kx + \sum_{k=1}^{k=n} b_k \cdot \sin kx$$

The Fejer sum is given by

$$\mathcal{F}_n(f) = \frac{S_0(f) + S_1(f) + \ldots + S_n(f)}{n}$$

A.9 The Jackson kernel. The Fejer kernel leads to uniform approximations but is not so robust with respect to the modulos of continuity. More precisely, for a given continuous and periodic function f we have seen that the uniform continuity entails that

$$\lim_{n \to \infty} ||f - \mathcal{F}_n(f)|| = 0$$

One is led to ask if there exists a constant C which is independent of both f and of n such that

$$(*) \qquad ||f - \mathcal{F}_n(f)|| \leq C \cdot \omega_f(\frac{1}{n})$$

Examples show that (*) does not hold. To obtain a uniform constant C one must include an extra factor. More precisely there exists an absolute constant C such that

$$(**) \qquad ||f - \mathcal{F}_n(f)|| \leq C \cdot \omega_f(\frac{1}{n}) \cdot \big(1 + \log^+ \frac{1}{\omega_f(\frac{1}{n})}\big)$$

But this is weaker than (*) since the $\log^+$-factor increases when $\omega_f(n)$ tends to zero for large n.

A sharper method to approximate periodic functions was introduced by Durhan Jackson in his thesis *Über die Genauigkeit der Annährerung stegiger funktionen durch ganze rationala funktionen* from Göttingen in 1911. Namely, to each 2π-periodic and continuous function $f(x)$ on the real line and every $n \geq 1$ we set

$$\mathcal{J}_n^f(x) = \frac{3}{2\pi} \cdot \int_{-\infty}^{\infty} f(x + \frac{2t}{n}) \cdot (\frac{\sin t}{t})^2 \cdot dt$$

A.10 Theorem. *For each n the function $\mathcal{J}_n^f(x)$ is a trigonometric polynomial of degree $2n-1$ at most. Moreover one has the inequality*

$$\max_x |f(x) - \mathcal{J}_n^f(x)| \leq (+\frac{6}{\pi}) \cdot \omega_f(\frac{1}{n})$$

The proof will use some preliminary facts.

Lemma. *One has the equality*

$$\int_{-\infty}^{\infty} (\frac{\sin t}{t})^4 \cdot dt = \frac{2\pi}{3}$$

Lemma. *One has the inequality*

$$\frac{3}{2\pi} \int_{-\infty}^{\infty} 81 + 2|t|) \cdot (\frac{\sin t}{t})^4 \cdot dt \leq 1 + \frac{6}{\pi}$$

Lemma. *For every 2π-periodic function $f(x)$ and every positive integer n the function*

$$x \mapsto \int_0^{\pi} f(x + 2t) \cdot (\frac{\sin nt}{\sin t})^4 \cdot dt$$

is a trigonometric polynomial of degree $2n-1$ at most.

Proof of Theorem A.10. The variable substitution $t \to nt$ gives

$$\mathcal{J}_n^f(x) = \frac{3}{2\pi} \cdot \int_{-\infty}^{\infty} f(x + 2t) \cdot (\frac{\sin nt}{t})^4 \cdot dt$$

Since f is 2π-periodic it follows that (1) is equal to

$$\frac{3}{2\pi} \cdot \int_0^{\pi} \sum_{k=-\infty}^{\infty} \frac{\sin^4(nt)}{(k\pi + t)^4} \cdot f(t) \cdot dt$$

Next, the function of the complex variable z:

$$\frac{1}{\sin^4 z} - \sum_{k=-\infty}^{\infty} \frac{1}{(k\pi + z)^4}$$

is entire and the reader may conclude that it is identically zero. See also §§ in Chapter XX for details. Hence (1) is equal to

$$\frac{3}{2\pi} \cdot \int_0^{\pi} f(x + 2t) \cdot (\frac{\sin nt}{\sin t})^4 \cdot dt$$

By Lemma XX this yields a trigonometric function of degree $\leq 2n-1$ which proves the first part of Theorem A.10. To prove the inequality (*) we use Lemma A.xx which gives

$$\mathcal{J}_n^f(x) - f(x) = \mathcal{J}_n^f(x) = \frac{3}{2\pi} \cdot \int_{-\infty}^{\infty} [f(x + \frac{2t}{n}) - f(x)] \cdot (\frac{\sin t}{t})^4 \cdot dt$$

Now

$$|f(x + \frac{2t}{n}) - f(x)| \leq \omega_f(\frac{2t}{n}) \leq (2|t| + 1) \cdot \omega_f(n)$$

where the last equality follows from Lemma XX. Hence we get the following estimate for the maximum norm

$$\max_x |\mathcal{J}_n^f(x) - f(x)| \leq \omega_f(\frac{2t}{n}) \cdot \frac{3}{2\pi} \cdot \int_{-\infty}^{\infty} (2|t| + 1) \cdot \big(\frac{\sin t}{t}\big)^4 \cdot dt$$

Finally, by Lemma XX the last factor is majorized by $1 + \frac{6}{\pi}$ which proves (*) in Theorem XX.

A.11 A lower bound for polynomial approximation. Denote by $\mathcal{T}_n$ the linear space of trigonometric polynomials of degree $\leq n$. For a periodic and continuous function f we put

$$\rho_f(n) = \min_{T \in \mathcal{T}_n} ||f - T||$$

where the right hand side denotes the maximum norm. We shall establish a lower bound for the ρ-numbers when certain sign-conditions hold for Fourier coefficients. In general, let f be a periodic function and for each positive integer n we find $T \in \mathcal{T}_n$ such that $||f - T|| = \rho_f(n)$. Keeping n fixed we set $\rho = \rho_n(f)$. It is obvious that the Fejer kernels do not increase maximum norms. Hence we get

(i)
$$||\mathcal{F}_k(f) - \mathcal{F}_k(T)|| \leq \rho$$

for every positive integer k. Apply this with $k = n$ and $k = n + p$ where p is another positive integer. The equation from Exercise XX gives

(ii)
$$T = \frac{(n + p) \cdot \mathcal{F}_{n+p}(T) - n \cdot \mathcal{F}_n(T)}{p}$$

Since (i) hold for n, $n + p$ and $||f - T|| \leq \rho$, the triangle inequality gives

(iii)
$$||f - \frac{(n + p) \cdot \mathcal{F}_{n+p}(f) - n \cdot \mathcal{F}_n(f)}{p}|| \leq 2 \cdot \frac{n + p}{p} \cdot \rho_f(n)$$

Next, by the formula (§ xx) it follows that (iii) gives

$$||f - \frac{S_n(f) + \cdots S_{n+p-1}(f)}{p}|| \leq 2 \cdot \frac{n + p}{p} \cdot \rho_f(n)$$

In particular we take $p = n$ and get the inequality

(*)
$$||f - \frac{S_n(f) + \cdots S_{2n-1}(f)}{n}|| \leq \frac{4}{n} \cdot \rho_f(n)$$

A.12 A special case. Assume that $f(x)$ is an even function on $[-\pi, \pi]$ which gives a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos kx$$

A.12 Proposition *Let f be even as above and assume that $a_k \leq 0$ for every $k \geq 1$. Then the following inequality holds for every $n \geq 1$:*

$$f(0) - \frac{S_n(f)(0) + \cdots S_{2n-1}(f)(0)}{n} \leq -\sum_{k=2n}^{\infty} a_k$$

The easy verification is left to the reader. Taking the maximum norm over $[-\pi, \pi]$ it follows from (*) that

(**)
$$\rho_f(n) \geq \frac{n}{4} \cdot \sum_{k=2n}^{\infty} |a_k|$$

under the sign condition on the Fourier coefficients above. Notice that (**) means that one has a lower bound for polynomial approximations of f.

A.13 The function $f(x) = \sin|x|$ It is obvious that

$$\omega_f(\frac{1}{n}) = \frac{1}{n}$$

Next, the periodic function $f(x($ is even and hence we only get a cosine-series. For each positive integer m we have:

$$a_k = \frac{2}{\pi}\int_0^\pi \sin x \cdot \cos kx \cdot dx$$

To evaluate these integrals we use the trigonometric formula

$$sin\,(k+1)x - \sin(k-1)x = 2\sin x \cdot \cos kx$$

Now the reader can verify that $a_\nu = 0$ when ν is odd while

$$a_{2k} = -\frac{4}{\pi} \cdot \frac{1}{2k^2 - 1}$$

Hence the requested sign conditions hold and (**) entails that

$$\rho_f(n) \geq \frac{n}{\pi} \cdot \sum_{k=n}^\infty \frac{1}{2k^2 - 1}$$

it is easily seen that the right hand side is $\geq \frac{C}{n}$ for a constant C which is independent of n. So this specific example shows that Theorem XX is sharp up to a multiple with a fixed constant.

B. Legendre polynomials.

If $n \geq 1$ we denote by $\mathcal{P}_n$ the linear space of real-valued polynomials of degree $\leq n$. A bilinear form is defined by

$$\langle q, p\rangle = \int_{-1}^1 q(x)p(x) \cdot dx$$

Since $1, x, \ldots, x^{n-1}$ generate a subspace of co-dimension one in $\mathcal{P}_n$ we get:

B.1 Proposition. *There exists a unique* $Q_n(x) = x^n + q_{n-1}x^{n-1} + \ldots + q_0$ *such that*

$$\int_{-1}^1 x^\nu \cdot Q_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

To find $Q_n(x)$. we consider the polynomial $(1-x^2)^n$ which vanishes up to order n at the end-points 1 and -1. Its the derivative of order n gives a polynomial of degree n and partial integrations show that

$$\int_{-1}^1 x^\nu \cdot \partial^n((x^2-1)^n)) \cdot dx = 0 \leq \nu \leq n-1$$

The leading coefficient of x^n in $\partial^n((x^2-1)^n))$ becomes

$$c_n = 2n(2n-1)\cdots(n+1)$$

Hence we have

$$Q_n(x) = \frac{1}{c_n} \cdot \partial^n((x^2-1)^n)$$

B.2 Definition. *The Legendre polynomial of degree n is given by*

$$P_n(x) = k_n \cdot \partial^n((x^2-1)^n)$$

where the constant k_n *is determined so that* $P_n(1) = 1$.

Since P_n is equal to Q_n up to a constant we still have

$$\int_{-1}^{1} x^\nu \cdot P_n(x) \cdot dx = 0 \leq \nu \leq n-1$$

From this we conclude that

$$\int_{-1}^{1} x^\nu \cdot P_n(x) \cdot P_m x)dx = 0 \quad n \neq m$$

Thus, $\{P_n\}$ is an orthogonal family with respect to the inner product defined by the integral over $[-1,1]$.

B.3 A generating function. Let w be a new variable and set

$$\phi(x,w) = 1 - 2xw + w^2$$

Notice that $\phi \neq 0$ when $-1 \leq x \leq 1$ and $|w/ < 1$. Keeping $-1 \leq x \leq 1$ fixed we have the function

$$w \mapsto \frac{1}{\sqrt{1-2xw+w^2}}$$

Recall that when $|\zeta| < 1$ one has the Newton series

$$\frac{1}{\sqrt{1-\zeta}} = \sum g_n \cdot \zeta^n \quad \text{where} \quad g_n = \frac{3 \cdot 5 \cdots (2n-1)}{2^n}$$

It follows that

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum g_n (2xw - w^2)^2$$

With x kept fixed the series is expanded into w-powers, i.e. set

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum \rho_n(x) \cdot w^n$$

It is easily seen that as x varies then $\rho_n(x)$ is a polynomial of degree n. Moreover, we notice that the coefficient of x^n in $\rho_n(x)$ is equal to

$$g_n \cdot 2^n$$

Next, if $x = 1$ we have

$$\frac{1}{\sqrt{1-2w+w^2}} = \frac{1}{1-w} = \sum w^n$$

From this we conclude that

$$\rho_n(1) = 1 \quad \text{for all} \quad n \geq 0$$

B.4 Theorem. *One has the equality $\rho_n(x) = P_n(x)$ for each n, i.e.*

$$\frac{1}{\sqrt{1-2xw+w^2}} = \sum P_n(x) \cdot w^n$$

holds when $-1 \leq x \leq 1$ and $|w| < 1$.

B.5 Exercise. Prove this result.

B.6 The series for $P_n(\cos\theta)$. With x replaed by $\cos\theta$ we notice that

$$1 - 2\cos\theta \cdot w + w^2 = (1 - e^{i\theta}w)(1 - e^{-i\theta}w)$$

It follows that

$$\frac{1}{\sqrt{1-2\cos(\theta)w+w^2}} = \frac{1}{\sqrt{1-1-e^{i\theta}w)}} \cdot \frac{1}{\sqrt{1-e^{-i\theta}w)}}$$

The last product becomes

$$\sum\sum g_m e^{im\theta} w^m \cdot g_\nu e^{-i\nu\theta} w^\nu$$

Collecting w powers the double sum becomes

$$\sum \gamma_n(\theta)\cdot w^n \quad \gamma_n(\theta)=\sum_{m+\nu=n} g_m g_\nu e^{i(m-\nu)\theta}$$

By Theorem B.4 the last sum represents $P_n(\cos(\theta))$. One has for example

$$P_3(\cos(\theta)=2g_3\cdot\cos(3\theta)+2g_2g_1\cdot\cos(\theta)$$

where we used that $g_0=1$.

B.7 An inequality for $|P(x)|$. Since the g-numbers are ≥ 0 we obtain

$$|P_n(\cos(\theta)|\leq g_ng_0+g_{n-1}g_1+\ldots+g_1g_{n-1}+g_0g_n=P_n(1) \quad : \quad 0\leq\theta\leq 2\pi$$

Hence we have proved

B.8 Theorem. *For each n one has*

$$|P_n(x)|\leq 1 \quad : \quad -1\leq x\leq 1$$

Next, we study the values when $x>1$. Here one has

B.9 Theorem. *For each $x>1$ one has*

$$1<P_1(x)<P_2(x)<\ldots$$

Proof. Let us put

$$\psi(x.w)=1+\sum_{n=1}^{\infty}[P_n(x)-P_{n-1}(x)]\cdot w^n$$

By Theorem B.4 this is equal to

(*)
$$\frac{1-w}{\sqrt{1-2xw+w^2}}$$

With $x>1$ we set $x=1+\xi$ and notice that

$$1-2xw+w^2=(1-w)^2-2\xi w$$

Hence (*) becomes

(**)
$$\frac{1}{\sqrt{1-\frac{2\xi w}{1-w^2}}}=\sum g_n\cdot\frac{(2\xi w)^n}{(1-w^2)^n}=\sum g_n\cdot(2\xi)^n\cdot\frac{w^n}{(1-w^2)^n}$$

Next, for each $n\geq 1$ we notice that the power series of $\frac{w^n}{(1-w^2)^n}$ has positive coefficients. Since $g_n(2\xi)^n>0$ also hold we conclude that (**) is of the form

$$1+\sum_{n=1}^{\infty}q_n(\xi)\cdot w^n \quad \text{where} \quad q_n(\xi)>0$$

Finally, Theorem B.9 follows since

$$P_n(1+\xi)-P_{n-1}(1+\xi)=q_n(\xi)$$

B.10 An L^2-inequality.

Let $n\geq 1$ and denote by $\mathcal{P}_n[1]$ the space of real-valued polynomials $Q(x)$ of degree $\leq n$ for which $\int_{-1}^1 Q(x)^2\cdot dx=1$ and set

$$\rho(n)=\max_{Q\in\mathcal{P}-n[1]}|Q|_\infty$$

where $|Q|_\infty$ is the maximum norm over $[-1,1]$. To find $\rho(n)$ we use the orthonormal basis $\{P_k^*\}$ and write

$$Q(x)=t_0\cdot P_0^*(x)+\ldots+t_n\cdot P_n^*(x)$$

Since $Q\in\mathcal{P}_n[1]$ we have $t_0^2+\ldots+t_n^2=1$. Recall also that

$$P_\nu^*(x)=\sqrt{\frac{2\nu+1}{2}}\cdot P_\nu(x)$$

Given $-1 \leq x_0 \leq 1$ the Cauchy-Schwarz inequality gives

$$Q(x_0)^2 \leq \sum_{\nu=0}^{\nu=n} \frac{2\nu+1}{2} \cdot |P_\nu(x_0)| \leq \sum_{\nu=0}^{\nu=n} \frac{2\nu+1}{2}$$

where the last inequality follows since the maximum norm of each P_ν is ≤ 1. Finally, we notice that

$$\sum_{\nu=0}^{\nu=n} \frac{2\nu+1}{2} = \frac{(1-n)^2}{2}$$

We conclude that

$$|Q(x_0)| \leq \frac{n+1}{\sqrt{2}}$$

B.11 The case of equality. To have equality above we take $x_0 = 1$ and

$$t_\nu = \alpha \cdot P_\nu^*(1) \quad : \quad \nu \geq 0$$

C. The space $\mathcal{T}_n$

Let $n \geq 1$ be a positive integer. A real-valued trigonometric polynomial of degree $\leq n$ is given by

$$g(\theta) = a_0 + a_1 \cos\theta + \ldots + a_n \cos n\theta + b_1 \sin\theta + \ldots b_n \sin n\theta$$

Here $a_0, \ldots, a_n, b_1, \ldots, b_n$ are real numbers. The space of such functions is denoted by $\mathcal{T}_n$ which is a vector space over $\mathbf{R}$ of dimension $2n+1$. We can write

$$\cos kx = \frac{1}{2}[e^{ikx} + e^{-ikx}] \quad \text{and} \quad \sin kx = \frac{1}{2i}[e^{ikx} - e^{-ikx}] \quad : \ k \geq 1$$

It follows that there exist complex numbers $c_0, \ldots, c_{2n}$ such that

$$g(\theta) = e^{-in\theta} \cdot [c_0 + c_1 e^{i\theta} + \ldots + c_{2n} e^{i2n\theta}]$$

Exercise. Show that

$$c_\nu + c_{2n-\nu} = 2a_\nu \quad \text{and} \quad c_\nu - c_{2n-\nu} = 2b_\nu \implies$$

$$c_{2n-\nu} = \bar{c}_\nu \quad 0 \leq \nu \leq n$$

C.1 The polynomial $G(z)$. Given $g(\theta)$ as above we set

$$G(z) = c_0 + c_1 z + \ldots + c_{2n} z^{2n}$$

Then we see that

$$e^{-in\theta} \cdot G(e^{i\theta}) = g(\theta)$$

C.2 Exercise. Set

$$\bar{G}(z) = \bar{c}_0 + c - 1z + \ldots + \bar{c}_{2n} z^{2n}$$

and show that

$$(*) \qquad z^{2n} G(1/z) = \bar{G}(z)$$

Use this to show that if $0 \neq z_0$ is a zero of $G(z)$ then $\frac{1}{\bar{z}_0}$ is also a zero of $G(z)$.

C.3 The case when $g \geq 0$. Assume that the g-function is non-negative. Let

$$0 \leq \theta_1 < \ldots < \theta_\mu < 2\pi$$

be the zeros on the half-open interval $[0, 2\pi)$. Since $g \geq 0$ every such zero has a multiplicity given by an *even* integer. Consider also the polynomial $G(z)$. From Exercise C.2 it follows that $\{e^{i\theta_\nu}\}$

are complex zeros of $G(z)$ with multiplicities given by even integers. Next, if ζ is a zero where $\zeta \neq 0$ and $|\zeta| \neq 1$, then (*) in C.2 implies that $\frac{1}{\zeta}$ also is a zero and hence $G(z)$ has a factorisation

$$G(z) = c_{2n} \cdot \prod_{\nu=1}^{\nu=\mu} (z - e^{i\theta_\nu})^{2k_\nu} \cdot \prod_{j=1}^{j=m} (z - \zeta_j)(z - \frac{1}{\bar{\zeta}_j}) \cdot z^{2r} \quad \text{where} \quad 2\mu + 2m + 2r = 2n$$

Here $0 < |\zeta_j| < 1$ hold for each j and it may occur that multiple zeros appear, i.e. the ζ-roots need not be distinct and the integer r may be zero or positive.

C.4 The h-polynomial. Let $\delta = \sqrt{|\zeta_1| \cdots |\zeta_m|}$ and put

$$h(z) = c_{2n}\dot{\delta} \cdot \prod_{\nu=1}^{\nu=\mu} (z - e^{i\theta_\nu})^{k_\nu} \cdot \prod_{j=1}^{j=m} (z - \zeta_j) \cdot z^r$$

C.5 Proposition. *One has the equality*

$$|h(e^{i\theta})|^2 = g(\theta)$$

Proof. With $z = e^{i\theta}$ and $0 < |\zeta| < 1$ one has

$$(e^{i\theta} - \zeta)(e^{i\theta} - \frac{1}{\bar{\zeta}}) = (e^{i\theta} - \zeta) \cdot (\bar{\zeta} - e^{-i\theta}) \cdot e^{i\theta} \cdot \frac{1}{\bar{\zeta}}$$

Passing to absolute values it follows that

$$\big|(e^{i\theta} - \zeta)(e^{i\theta} - \frac{1}{\bar{\zeta}})\big| = \big|e^{i\theta} - \zeta\big|^2 \cdot \frac{1}{|\zeta|}$$

Apply this to every root ζ_ν and take the product which gives Proposition C.5.

C.6 Application. Let $g \geq 0$ be as above and assume that the constant coefficient $a_0 = 1$. This means that

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} g(\theta) \cdot d\theta$$

With $h(z) = d_0 + d_1 z + \ldots + d_n z^n$ we get

$$1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} h(e^{i\theta})|^2 \cdot d\theta = |d_0|^2 + \ldots + |d_n|^2$$

Notice that

(i) $$|d_n|^2 = |c_{2n}| \cdot \delta \quad \text{and} \quad |d_0|^2 = |c_{2n} \cdot \delta| \cdot \prod |\zeta_j|^2 = |c_{2n}| \cdot \frac{1}{\delta}$$

From this we see that

(iii) $$|c_{2n}| \cdot (\delta + \frac{1}{\delta}) = |d_0|^2 + d_n|^2 \leq 1$$

Here $0 < \delta < 1$ and therefore $\delta + \frac{1}{\delta} \geq 2$ which together with (iii) gives

$$|c_{2n}| \leq \frac{1}{2}$$

At the same time we recall that

$$c_{2n} = \frac{a_n + ib_n}{2}$$

Hence we have proved the inequality:

(*) $$|a_n + ib_n| \leq 1$$

Summing up we have proved the following:

C.7 Theorem. *Let $g(\theta)$ be non-negative in $\mathcal{T}_n$ with constant term $a_0 = 1$. Then*

$$|a_n + ib_n| \leq 1$$

C.8 An application. Let $n \geq 1$ and consider the space of all monic polynomials

$$P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0$$

where $\{c_\nu\}$ are real- To each such polynomial we can consider the maximum norm over the interval $[-1,1]$. Then one has

C.9 Theorem. *For each $P \in \mathcal{P}_n^*$ one has the inequality*

$$\max_{-1 \leq x \leq 1} |P(x)| \geq 2^{-n+1}$$

Proof. Consider some $P \in \mathcal{P}_n^*$ and define the trigonometric polynomial

$$g(\theta) = (\cos\theta)^n + c_{n-1}(\cos\theta)^{n-1} + \ldots + c_0$$

So here $P(\cos\theta) = g(\theta)$ and Theorem C.9 follows if we have proved that

$$2^{-n+1} \geq \max_{0 \leq \theta \leq 2\pi} |g(\theta))| \tag{1}$$

To prove this we set $M = \max_{0 \leq \theta \leq 2\pi} |g(\theta))|$. Next, we can write

$$g(\theta) = a_0 + a_1 \cos\theta \ldots + a_n \cos n\theta$$

Moreover, since

$$(\cos\theta)^n = 2^{-n} \cdot [e^{i\theta} + e^{-\theta}]^n$$

we get

$$a_n = 2^n$$

Now we shall apply Theorem C.8. For this purpose we construct non-negative trigonometric polynomials. First we define

$$g^*(\theta) = \frac{M - g(\theta)}{M - a_0}$$

Then $g^* \geq 0$ and its constant term is 1. We have also

$$g^*(\theta) = 1 - \frac{1}{M - a_0} \cdot \sum_{\nu=1}^{\nu=n} a_\nu \cos\nu\theta$$

Hence Theorem C.7 gives

$$\frac{1}{|M - a_0|} \cdot |a_n| \leq 1 \implies |M - a_0| \geq 2^{-n+1} \tag{1}$$

Next, we have also the function

$$g_*(\theta) = \frac{M + g(\theta)}{M + a_0}$$

In the same way as above we obtain:

$$|M + a_0| \geq 2^{-n+1} \tag{2}$$

Finally, (1) and (2) give

$$M \geq 2^{-n+1}$$

which proves Theorem C.9

D. Tchebysheff polynomials.

The inequality in Theorem C.9 is sharp. To see this we shall construct a special polynomial $T_n(x)$ of degree n. Namely, with $n \geq 1$ we can write

$$\cos n\theta = 2^{n-1} \cdot (\cos\theta)^n + c_{n-1} \cdot (\cos\theta)^{n-1} + \ldots + c_0$$

Set

$$T_n(x) = 2^{n-1}x^n + c_{n-1} \cdot x^{n-1} + \ldots + c_0$$

Hence

$$T_n(\cos\theta) = \cos n\theta$$

We conclude that the polynomial

$$p_n(x) = 2^{-n+1} \cdot T_n(x)$$

belongs to $\mathcal{P}_n^*$ and its maximum norm is 2^{-n+1}. This proves that the inequality in Theorem 10 is sharp.

D.1 Zeros of T_n. Set

$$\theta_\nu = \frac{\nu\pi}{n} + \frac{\pi}{2n}$$

It is clear that $\theta_1, \ldots, \theta_n$ are zeros of $\cos n\theta$. Hence the zeros of $T_n(x)$ are:

$$x_\nu = \cos\theta_\nu$$

Notice that

$$-1 < x_n < \ldots < x_1 < 1$$

Since $T_n(x)$ is a polynomial of degree n it follows that $\{x_\nu\}$ give all zeros and we have

$$T_n(x) = 2^{n-1} \cdot \prod (x - x_\nu)$$

D.2 Exercise. Show that

$$T_n'(x_\nu) \cdot \sqrt{1 - x_\nu^2} = n$$

hold for every zero of $T_n(x)$.

D.3 An interpolation formula. Since $x_1, \ldots, x_n$ are distinct it follows that if $p(x) \in \mathcal{P}_{n-1}$ is a polynomial of degree $\leq n - 1$ then

$$p(x) = \sum_{\nu 01}^{\nu=n} p(x_\nu) \cdot \frac{1}{T'(x_\nu)} \cdot \frac{T(x)}{x - x_\nu}$$

By the exercise above we get

$$p(x) = \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} (-1)^{\nu-1} p(x_\nu) \cdot \sqrt{1 - x_\nu^2} \cdot \frac{T(x)}{x - x_\nu}$$

We shall use the interpolation formula above to prove

D.4 Theorem *Let $p(x) \in \mathcal{P}_{n-1}$ satisfy*

(1)
$$\max_{-1 \leq x \leq 1} \sqrt{1 - x^2} \cdot |p(x)| \leq 1$$

Then it follows that

(2)
$$\max_{-1 \leq x \leq 1} |p(x)| \leq n$$

Proof. First, consider the case when

(*)
$$-\cos\frac{\pi}{2n} \leq x \leq \cos\frac{\pi}{2n}$$

Then we have

$$\sqrt{1 - x^2} \geq \sqrt{1 - \cos^2\frac{\pi}{2n}} = \sin\frac{\pi}{2n}$$

Next, recall the inequality $\sin x \geq \frac{2}{\pi} \cdot x$. It follows that

$$\sqrt{1 - x^2} \geq \frac{1}{n}$$

So when (1) holds in the theorem we have

$$|p(x)| = \frac{1}{\sqrt{1 - x^2}} \cdot \sqrt{1 - x^2} \cdot |p(x)| \leq \frac{1}{\sqrt{1 - x^2}} \leq n$$

Hence the required inequality in Theorem D.4 holds when x satisfies (*) above. Next, suppose that

(**) $$x_1 \leq x \leq 1$$

On this interval $T_n(x) \geq 0$ and from the interpolation formula xx and the triangle inequality we have

$$|p(x) \leq \frac{1}{n}\sum_{\nu=1}^{\nu=n} \sqrt{1-x_\nu^2}\cdot|p(x_\nu)|\cdot\frac{T(x)}{x-x_\nu} \leq \frac{1}{n}\sum_{\nu=1}^{nu=n}\frac{T(x)}{x-x_\nu}$$

Next, the sum

$$\frac{T(x)}{x-x_\nu} = T_n'(x) = n\cdot U_{n-1}(x)$$

So when (**) holds we have

(***) $$|p(x)| \leq |U_{n-1}(x)|$$

By xx the maximum normmof U_{n-1} over $[-1,1]$ is n and hence (***) gives

$$|p(x)| \leq n$$

In the same way one proves htat

$$-1 \leq x \leq x_n \implies |p(x)| \leq n$$

Together with the upper bound in the case (xx) we get Theorem D.4.

D.5 Bernstein's inequality.

Let $g(\theta) \in \mathcal{T}_n$. The derivative $g'(\theta)$ is another trigonometric polynomial and we have

Theorem. *For each $g \in \mathcal{T}_n$ one has*

$$\max_{0\leq\theta\leq 2\pi}|g'(\theta)| \leq n\cdot\max_{0\leq\theta\leq 2\pi}|g(\theta)|$$

Before we prove this result we establish an inequality for certain trigonometric polynomials. Namely, consider a real-valued sine-polynomial

$$S(\theta) = c_1\sin(\theta) + \ldots + c_n\sin(n\theta)$$

Now $\theta \mapsto \frac{S(\theta)}{\sin\theta}$ is an even function of θ and therefore one has

$$\frac{S(\theta)}{\sin\theta} = a_0 + a_1\cos\theta + \ldots + a_{n-1}(\cos\theta)^{-n-1}$$

Consider the polynomial

$$p(x) = a_0 + a_1x + \ldots + a_{n-1}x^{n-1}$$

Then e see that:

$$|p(\cos\theta)| = \frac{|S(\theta)|}{\sqrt{1-\cos^2\theta}}$$

Using this we apply Theorem D.4 to the polynomial $p(x)$ and conclude

D.6 Theorem. *Let $S(\theta) = c_1 sin(\theta) + c_n sin(n\theta)$ be a sine-polynomial as above. Then*

$$\max_{0\leq\theta\leq 2\pi}\frac{|S(\theta)|}{\sin\theta} \leq n\cdot\max_{0\leq\theta\leq 2\pi}|S(\theta)|$$

D.7 Proof of Bernstein's theorem. Fix an arbitrary $0 \leq \theta - 0 < 2\pi$. Set

$$S(\theta) = g(\theta_0+\theta) - g(\theta_0-\theta)$$

We notice that $S(\theta$is an odd polynomial of θ and $S(0) = 0$, It follows that $S(\theta)$ is a sine-polynomial as above of degree $\leq n$. Notice also that

$$\max_{0\leq\theta\leq 2\pi}|S(\theta)| \leq 2\cdot\max_{0\leq\theta\leq 2\pi}|g(\theta)|\max_{0\leq\theta\leq 2\pi}|g(\theta)|$$

Theorem D.6 applied to $S(\theta)$ gives

$$\text{(i)} \qquad \left|\frac{g(\theta_0+\theta)-g(\theta_0-\theta)}{\sin\theta}\right| \leq 2n \cdot \max_{0\leq\theta\leq 2\pi} |g(\theta)|$$

Next, in the left hand side we can take the limit as $\theta| \to 0$ and notice that

$$2 \cdot g'(\theta_0) = \lim_{\theta\to 0} \frac{g(\theta_0+\theta)-g(\theta_0-\theta)}{\sin\theta}$$

Hence (i) gives

$$|g'(\theta_0)| \leq n \cdot \max_{0\leq\theta\leq 2\pi} |g(\theta)|$$

Finally, since θ_0 was arbitrary we get Bernstein's theorem.

E. Fejers sums and Gibb's phenomenon.

Several remarkable inequalities for trigonometric polynomials were established by Fejer in [Fejer] where a central issue is to find trigonometric polynomials expressed by a sine series which are ≥ 0 on the interval $[0, \pi]$. Consider as an example is the sine-series

$$S_n(\theta) = \sum_{k=1}^{k=n} \frac{\sin k\theta}{k}$$

E.1 Theorem. *For every $n \geq 1$ one has the inequality*

$$0 < S_n(\theta) \leq 1 + \frac{\pi}{2} \quad : \quad 0 < \theta < \pi$$

The upper bound was proved by in [Fej] and Fejer conjectured that $S_n(\theta)$ stays positive on $(0, \pi)$. This was confirmed in articles by Jackson in [xx] and Cronwall in [xx]. The series (*) has a connection with Gibb's phenomenon and Theorem E.1 can be illustrated by drawing graphs of the S-polynomials where the situation when $\theta = \pi - \delta$ for small positive δ has special interest. Since $\cos \pi = -1$ the positivity entails that

$$(*) \qquad \sum_{k=2}^{n} (-1)^k \cdot \frac{\sin k\delta}{k} \geq \sin \delta \text{hold for every} \quad n \geq 2 \quad \text{and small} \quad \delta > 0$$

Exercise. Prove Theorem E.1 or consult the literature. It is also instructive to confirm (*) by numerical experiments with a computer.

E.2 Mehler's integral formula. In XX we introduced the Legendre polynomials. It turns out that the sum

$$\mathcal{P}_n(x) = \sum_{\nu=1}^{\nu=n} P_\nu(x)$$

is strictly positive for each $-1 < x < 1$. The proof of this positivity follows from Theorem E.1 and the following integral formula which is due to Mehler:

$$(*) \qquad \mathcal{P}_n(\cos\theta) = \frac{2}{\pi} \cdot \int_0^\pi \frac{\sin(n + \frac{1}{2})\phi \cdot d\phi}{\sqrt{2\cos\theta - 2\cos\phi}}$$

The positivity $\mathcal{P}_n(x) > 0$ on the real interval $(-1, 1)$ was generalized by Szegö in [Szegö] who proved that

$$\sum_{k=0}^{k=n} P_k(\cos\theta) \cdot z^k \neq 0$$

hold for all $0 < \theta < \pi$ and every complex z with $|z| \leq 1$. Here Szegö's inequality with $z = 1$ corresponds to the previous positivity.

F. Convergence of arithmetical means

Let $f(x)$ be a real-valued and square integrable function on $(-\pi, \pi)$, i.e.

$$\int_{-\pi}^{\pi} |f(x)|^2 \cdot dx < \infty$$

We say that f has a determined value A at $x = 0$ if the following two conditions hold:

$$\text{(i)} \qquad \lim_{\delta \to 0} \frac{1}{\delta} \cdot \int_0^\delta |f(x) + f(-x) - 2f(0)| \cdot dx = 0$$

$$\text{(ii)} \qquad \int_0^\delta |f(x) + f(-x) - 2f(0)|^2 \cdot dx \leq C \cdot \delta \quad \text{holds for some constant} \quad C$$

Remark. In the same way we can impose this condition at every point $-\pi < x_0 < \pi$. But to simplify the subsequent notations we take $x = 0$. If $x = 0$ is a Lebesgue point for f and $f(0)$ the Lebesgue value we have (i). Hence Lebegue's Theorem shows that (i) holds almost everywhere when $x = 0$ is replaced by other points x_0. We leave it to the reader to show that the second condition also is valid almost everywhere when f is square integrable. Notice also that (i-ii) hold at all points when f is a continuous function. Next, expand f in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

Examples show that f can have a determined value at $x = 0$ and yet the Fourier series at $x = 0$ is divergent, i.e. the series

$$\sum a_k$$

may diverge. In fact, this may even occur when f is continuous. But in any case we can consider the partial sums

$$s_n(0) = \frac{a_0}{2} + a_1 + \ldots + a_n + b_1 + \ldots + b_n$$

The result below is proved in [Carleman] and shows that $\{s_n\}$ are close to the determined value for many n-values.

F.1 Theorem. *Assume that f has a determined value $f(0)$ at $x = 0$. Then the following hold for every positive integer k*

$$\lim_{n\to\infty} \frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} |s_\nu - A|^k = 0$$

Remark. Carleson's famous theorem asserts that $\{s_n(x)\}$ converge to $f(x)$ almost everywhere when $f \in L^2$. Whenever pointwise convergence holds the limit formulas above are obvious. However, it is not clear that the pointwise convergence exists at *every point* where f has a determined value. So "ugly points" may appear in a null-set where pointwise convergence fails and here Carleman's result offers a substitute.

The case when $f \in \mathbf{BMO}(T)$. If f has bounded mean oscillation then results from § XX in section XX from Special Topics show that Carleman's two conditions (i-ii) hold at every Lebesgue point of f. So this gives a good control for averaged Fourier series of f.

Proof of Theorem F.1. Set $A = f(0)$ and

$$\phi(x) = f(x) + f(-x) - 2A$$

Applying Dini's kernel we have

$$s_n - A = \int_0^\pi \frac{\sin(n+1/2)x}{\sin x/2} \cdot \phi(x) \cdot dx$$

Trigonometric formulas show that the integral can be expressed by three terms where we have chosen some $0 < \delta < \pi$:

$$\alpha_n = \frac{1}{\pi} \cdot \int_0^\delta \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx$$

$$\beta_n = \frac{1}{\pi} \cdot \int_\delta^\pi \sin nx \cdot \cot x/2 \cdot \phi(x) \cdot dx$$

$$\gamma_n = \frac{1}{\pi} \cdot \int_0^\pi \cos nx \cdot \phi(x) \cdot dx$$

By Hölder's inequality it suffices to prove Theorem F.1 when $k = 2p$ is an even integer. Minkowski's inequality gives

$$(1) \qquad \Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq \Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p}$$

Denote by $o(\delta)$ small ordo and $O(\delta)$ is big ordo. When $\delta \to 0$ we shall establish the following:

(i) $$\big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\big]^{1/2p} = n^{1+1/2p)} \cdot o(\delta)$$

(ii) $$\big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\big]^{1/2p} \leq K \cdot p \cdot \delta^{-1/2p}$$

(iii) $$\big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\big]^{1/2p} = O(1)$$

In (ii) K is an absolute constant which is independent of p, n and δ. Let us first see why (i-iii) give Theorem F.1. For $O(1)$ in (iii) and the absolute constant K we take a common constant K^* and write $o(\delta) = \epsilon(\delta) \cdot \delta$ where $\epsilon(\delta) \to 0$. With these notations one has:

(*) $$\big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\big]^{1/2p} \leq n^{1+1/2p} \cdot \delta \cdot \epsilon(\delta) + K^* p \cdot \delta^{-1/2p} + K^*$$

Next, let $\rho > 0$ and choose b so large that

$$pK^* b^{-1/2p} < \rho/3$$

Take $\delta = b/n$ and with n large it follows that $\epsilon(\delta)$ is so small that

$$b \cdot \epsilon(b/n) < \rho/3$$

Then right hand side in (*) is majorized by

$$\frac{2\rho}{3} \cdot n^{1/2p} + K^*$$

When n is large we also have

$$K^* \leq \frac{\rho}{3} \cdot n^{1/2p}$$

Hence the left hand side in (*) is majorized by $\rho \cdot \cdot n^{1/2p}$ for all sufficiently large n. Since $\rho > 0$ was arbitrary we get Theorem F.1 when the power is raised by $2p$.

Proof of (i-iii)

The proofs of (i) and (iii) are easy and left as an exercise while the more involved inequality (ii) requires several steps. We are given ϕ and for each $0 < s < \pi$ we define the function $\phi_s(x)$ by

$$\phi_s(x) = \phi(x) \quad : \quad 0 < x < s$$

and extend it to an odd function, i.e. $\phi_s(-x) = -\phi_s(x)$ and finally $\phi_s(x) = 0$ when $|x| > s$. This odd function has a sine series

(1) $$\phi_s(x) = \sum_{\nu=1}^{\infty} c_\nu(s) \cdot \sin x$$

Let us also introduce the functions

(2) $$\rho(s) = \int_0^s |\phi(x)| \cdot dx \quad \text{and} \quad \Theta(s) = \int_0^s |\phi(x)|^2 \cdot dx$$

The first major step towards the proof of (ii) is the following:

Lemma. *One has the inequality*

$$\sum_{|nu=1}^{\infty} |c_\nu(s)|^{2p} \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot \rho(s)^{2p-2}$$

Proof. We employ convolutions and define inductively a sequence of functions $\{\phi_{n,s}(x)\}$ where $\phi_{1,s}(x) = \phi_s(x)$ and

$$\phi_{n+1,s}(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi_{n,s}(y)\phi_s(x+y)\cdot dy$$

Since convolution yield products of the Fourier coefficients and $2p$ is an even integer we have the standard formula:

$$\sum_{\nu=1}^{\infty} c_n(s)^{2p} = \phi_{2p,s}(0) \tag{1}$$

Next, using the Cauchy-Schwarz inequality the reader may verify that

$$|\phi_{2,s}(x)| \leq \frac{2}{\pi}\cdot\Theta(x)$$

This entails that

$$\phi_{3,s}(x) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |\phi_{2,s}(y)|\cdot|\phi_s(x+y)|\cdot dy \leq \frac{2}{\pi^2}\cdot\Theta(s)\cdot\int_{-\pi}^{\pi} |\phi_s(x+y)|\cdot dy = (\frac{2}{\pi})^2\cdot\Theta(s)\cdot\rho(s)$$

Proceeding in this way it follows by an induction that

$$\phi_{2p,s}(x) \leq (\frac{2}{\pi})^{2p-1}\cdot\Theta(s)\cdot(\rho(s))^{2p-2}$$

This holds in particular when $x = 0$ and then (1) above gives Lemma 1.

A formula for the β-numbers. We have by definition

$$\beta_\nu = \frac{2}{\pi}\int_{\delta}^{\pi} \sin\,\nu x\cdot\frac{1}{2}\cot(\frac{x}{2})\cdot\phi(x)\cdot dx$$

An integration by parts and the construction of the Fourier coefficients $\{c_\nu(s)\}$ which applies with $s = \delta$ give:

$$\beta_\nu = -\frac{1}{2}\cdot\,\cot\delta/2\cdot c_\nu(\delta) + +\frac{1}{4}\int_{\delta}^{\pi} c_\nu(x)\cdot\operatorname{cosec}^2(\frac{x}{2})\cdot dx \tag{*}$$

Now we profit upon Minkowski's inequality. Let q be the conjugate of $2p$, i.e $\frac{1}{q}+\frac{1}{2p} = 1$ and choose $\{\xi_\nu\}$ to be the sequence in ℓ^q of unit norm such that

$$|\sum \xi_\nu\cdot\beta_\nu| = ||\beta_\bullet||_{2p}$$

where the last term is the left hand side in (ii). At the same time (*) above and the triangle inequality give

$$||\beta_\bullet||_{2p} \leq -\frac{1}{2}\cdot\,\cot(\delta/2)\cdot\sum |c_\nu(\delta)|\cdot|\xi_\nu| + \frac{1}{4}\int_{\delta}^{\pi} \operatorname{cosec}^2(\frac{x}{2})\cdot\sum |c_\nu(x)\cdot\xi_\nu|\cdot dx \leq$$

$$\frac{1}{2}\cdot\,\cot(\delta/2)\cdot||c_\bullet(\delta)||_{2p} + \frac{1}{4}\int_{\delta}^{\pi} \operatorname{cosec}^2(\frac{x}{2})\cdot||c_\bullet(x)||_{2p}\cdot dx \tag{**}$$

At this stage we apply Lemma 1 and the assumption which give a constant K such that

$$\Theta(s) \leq K \quad\text{and}\quad \rho(s) \leq K\cdot s$$

The last estimate actually is weaker than the hypothesis but it will be sufficient to get the requested estimate of the ℓ^{2p}-norm in (ii). Lemma 1 gives a constant K_1 such that

$$||c_\bullet(\delta)||_{2p} \leq K_1\cdot\delta^{1-1/p}$$

At the same time we have a constant K_2 such that

$$\cot(\delta/2) \leq \frac{K_2}{\delta}$$

The product in the first term from (**) is therefore majorized by $K_1K_2 \cdot \delta^{-1/2p}$ as requested in (ii). For the second term we use Lemma 1 which first gives

$$||c_\bullet(x)||_p \leq K \cdot x^{-1/2p}$$

At this stage we leave it to the reader to verify that we get a constant K so that

$$\int_\delta^\pi x^{-1/2p} \cdot \operatorname{cosec}^2(\frac{x}{2}) \cdot dx \leq K \cdot \delta^{-1/2p}$$

which finishes the proof of (ii).

Chapter II: Stokes Theorem

0. Introduction.

The results in section A already contain the essential ingredients for integral formulas which appear in analytic function theory. However, the proofs in section A require certain regularity assumptions. Therefore we give another proof of Stokes Theorem under relaxed regularity conditions in Section 2 where Theorem 10 is a major result. Line integrals are expressed in two ways, either via differential 1-forms or by integrals taken with respect to the arc-length measure on oriented curves. Section 3 as well as section 5 describes these to alternative ways to express Stokes Thoerem. A notable point is that Theorem 10 in Section 2 gives more integral formulas where two functions appear. This is exposed in section 4 and leads to integral formulas where the second order differential operator given by the Laplacian appears. An example which later will be used to study harmonic, and more generally subharmonic functions, goes as follows: Let $u(x,y)$ be a C^2-function defined in some open disc of radius R centered at the origin in $\mathbf{R}^2$. Then, for each $0 < s < R$ one has the equality

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(s,\theta)\cdot d\theta + \iint_{D(s)} \log(\frac{s}{\sqrt{x^2+y^2}})\cdot \Delta u(x,y)\cdot dxdy \tag{1}$$

where $D(s)$ is the disc of radius s centered at the origin. Using this another integration when $0 < s < r < R$ gives

$$u(0) = \frac{1}{\pi r^2}\cdot \iint_{D(r)} u(x,y)\cdot dxdy - \int_0^r K(r,s)\cdot \big[\int_0^{2\pi} \Delta(u)(s,\theta)\cdot d\theta\big]\cdot ds \tag{2}$$

where $K_r(s)$ is a kernel function defined for pairs $0 < s \leq r$ by:

$$K(r,s) = s^3\cdot \int_1^{\frac{r}{s}} u\cdot \text{Log}(u)\cdot du \tag{3}$$

In (2) the first double integral in the right hand side is the mean-value of u over the disc $D(r)$. The second double integral describes the difference between this mean-value and the value of u at the center of the disc. If the function $\Delta(u)$ is non-negative in the whole disc $D(R)$ we get the mean-value inequality

$$u(0) \leq \frac{1}{\pi r^2}\cdot \iint_{D(r)} u(x,y)\cdot dxdy$$

for every $0 < r < R$. In Chapter V:B this gives the starting point for a study of subharmonic functions.

Another integral formula. Some special integral formulas are proved using polar coordinates rather than via Stokes theorem. Let (r,θ) be the polar coordinates so that

$$x = r\cdot\cos\theta \quad\text{and}\quad y = r\cdot\sin\theta$$

Looking at the first order partial derivatives outside the origin we have

$$\partial_r = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y \quad \text{and} \quad \frac{1}{r} \cdot \partial_\theta = -\sin\theta \cdot \partial_x + \cos\theta \cdot \partial_y$$

It follows that

$$\partial_x = \cos\theta \cdot \partial_r - \frac{1}{r} \cdot \sin\theta \quad \text{and} \quad \partial_y = \sin\theta \cdot \partial_r + \frac{1}{r} \cdot \cos\theta$$

Next, consider the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$. We leave it to the reader to verify that

$$\Delta = \partial_r^2 + \frac{1}{r} \cdot \partial_r + \frac{1}{r^2} \cdot \partial_\theta^2$$

Next, consider a C^2-function $F(x, y)$ defined in the unit disc $D = \{x^2 + y^2 < 1\}$ with $F(0, 0) = 0$.

Theorem. *The following equality holds for every* $0 < r < 1$*:*

$$(*) \qquad \int_0^{2\pi} \big[r^2 \cdot (\partial_r(F))^2 - (\partial_\theta(F))^2 \big] \cdot d\theta = 2 \cdot \int_0^r \big[\int_0^{2\pi} s^2 \partial_s(F) \cdot \Delta(F) \cdot d\theta \big] \dot{d}s$$

To prove this via Stokes or Green's theorem would be cumbersome. Instead on employs a Fourier series expansion, i.e. we can write

$$F(r, \theta) = \sum_{k=1}^{\infty} a_k(r) \cdot \cos\, k\theta + b_k(r) \cdot \sin\, k\theta$$

where $\{a_k(r)\}$ and $\{b_k(r)\}$ are two sequences of functions which only depend on r.

Exercise. Use familiar vanishing results for integrals of sine-and cosine-functions to conclude that it suffices to prove (*) in the special case when $F = a_k(r) \cdot \cos\, k\theta$ or $F = b_k(r) \cdot \sin\, k\theta$ for a single integer k. Next, consider the case when

$$F = a(r) \cdot \cos\, k\theta$$

for some positive integer k. Using the polar formula for Δ the right hand side in (*) becomes

$$\int_0^r \big[\int_0^{2\pi} s^2 \cdot a'(s) \cdot (a''(s) + \frac{1}{s} \cdot a'(s) - \frac{k^2}{s^2} a(s)\big] \cdot \cos^2 k\theta \cdot d\theta \,\big] \cdot ds$$

Notice that partial integration gives

$$\int_0^r s^2 \cdot a'(s) \cdot (a''(s) + \frac{1}{s} \cdot a'(s) - \frac{k^2}{s^2} a(s)\big] \cdot ds = \frac{r^2 a'(r)^2 - k^2 a(r)^2}{2}$$

It follows that the right hand side in (*) is equal to

$$(1) \qquad \big[r^2 a'(r)^2 - k^2 a(r)^2 \,\big] \cdot \int_0^{2\pi} \cos^2\, k\theta \cdot d\theta$$

At this stage the reader can confirm that (1) is equal to the right hand side in (*).

Remark. The integral formula in Theorem 0.0 can be used to study subharmonic functions, .e function f for which $\Delta(F) \geq 0$. For example, if we in addition assume that $\partial_r(F) \geq 0$ holds in the disc, then the right hand side in (*) is ≥ 0 which gives the inequality

$$(**) \qquad \int_0^{2\pi} \partial_\theta(F)^2 \cdot d\theta \leq r^2 \cdot \int_0^{2\pi} \partial_r(F)^2 \cdot d\theta \quad \text{for every} \quad 0 < r < 1$$

0:A. Calculus in $\mathbf{R}^2$

A parametrised curve γ in $\mathbf{R}^2$ is defined by a vector-valued function

(*) $$\gamma: \quad t \mapsto (x(t), y(t)) \quad : \quad 0 \leq t \leq T$$

Here $[0,T]$ is the interval of the parameter t. The curve is of class C^2 if $x(t)$ and $y(t)$ are both of class C^2. Notice that we do not assume that (*) is 1-1 so in general the curve can have self-intersections. We construct line integrals as follows: Let $u(x,y)$ and $v(x,y)$ be a pair of continuous functions and set

(1) $$\int_\gamma u \cdot dx = \int_0^T u(x(t), y(t)) \cdot x'(t) \cdot dt$$

(2) $$\int_\gamma v \cdot dy = \int_0^T v(x(t), y(t)) \cdot y'(t) \cdot dt$$

We refer to (1) as a line integral in the x-direction and (2) as a line integral in the y-direction. The notation $\int_\gamma$ is consolidated by the fact that if $s \mapsto t(s)$ is some strictly increasing C^1-function from an interval $0 \leq s \leq S$ onto $[0,T]$ then the new s-parametrisation does not change the integrals. In fact, a change of variables gives:

$$\int_0^T u(x(t), y(t) \cdot x'(t) \cdot dt = \int_0^S u(x(t(s)), y(t(s)) \cdot \frac{d}{ds}(x(t(s)) \cdot ds$$

where we used the differential equality

$$\frac{d}{ds}(x(t(s)) = x'(t(s)) \cdot t'(s)$$

However, the orientation is essential. One moves from the initial point $\gamma(0) = P$ to the end-point $\gamma(T) = Q$ where an arrow along the curve is used to indicate the direction during the integration and up to a sign the line integrals depend upon the chosen orientation.

A.1 Homotopy. Consider a family of C^2-curves $\{\gamma_s : 0 \leq s \leq 1\}$ where each single γ_s is parametrised over an interval $[0,T]$ as above and the curves have common end-points $P = (x_*, y_*)$ and $Q = (x^*, y^*)$. So when $t \mapsto (x(t,s), y(t,s))$ is a parametrisation of γ_s, then

(1) $$x(0,s) = x_* \quad \text{and} \quad x(T,s) = x^* \quad \text{for all} \quad 0 \leq s \leq 1$$

and similarly for the y-function. No further assumption is imposed, i.e. the γ curves in the family need not be simple and it may occur that $P = Q$ or that $P \neq Q$. Concerning the functions $x(t,s)$ and $y(t,s)$ we impose the condition that both are of class C^2 as functions of the two variabes. In particular the mixed second order derivatives are equal.

(2) $$x''_{ts} = x''_{st} \quad \text{and} \quad y''_{ts} = y''_{st}$$

A.2 Theorem. *Let $u(x,y)$ and $v(x,y)$ be C^1-functions. Then one has the two equalities*

(*) $$\int_{\gamma_1} u \cdot dx - \int_{\gamma_0} u \cdot dx = \iint_\square u'_y \cdot (y'_s \cdot x'_t - y'_t \cdot x'_s) \cdot dsdt$$

(**) $$\int_{\gamma_1} v \cdot dy - \int_{\gamma_0} v \cdot dy = \iint_\square v'_x \cdot (x'_s \cdot y'_t - x'_t \cdot y'_s) \cdot dsdt$$

Proof. We prove (*) while (**) is left to the reader since the proof is the same when x and y are interchanged. The fundamental theorem of calculus gives

$$\int_{\gamma_1} u \cdot dx - \int_{\gamma_0} u \cdot dx = \int_0^1 \frac{d}{ds}\Big(\int_0^T u \cdot x'_t \cdot dt\Big) \cdot ds =$$

(i) $$\iint_\square \big[u'_x \cdot x'_s x'_t + u'_y \cdot y'_s x'_t + u \cdot x''_{st}\big] \cdot dsdt$$

Next, consider for each *fixed* $0 \leq s \leq 1$ the integral

$$\text{(ii)} \qquad \int_0^T \frac{d}{dt}(u(x(t,s),y(t,s)) \cdot x_s'(t,s)) \cdot dt$$

By (1) the functions $s \mapsto x(0,s)$ and $s \mapsto x(T,s)$ are constant and therefore

$$\text{(iii)} \qquad x_s'(0,s) = x_s'(T,s) = 0 \quad \text{for each} \quad 0 \leq s \leq 1$$

Hence the integral (ii) is zero for every s At the same time we can differentiate the function under the integral sign and conclude that

$$\text{(iv)} \qquad \int_0^T (u_x' \cdot x_t' \cdot x_s' + u_y' \cdot y_t' \cdot x_s' + u \cdot x_{ts}'') \cdot dt = 0$$

Since (iv) holds for every s we get a vanishing double integral

$$\text{(v)} \qquad \iint_\square (u_x' \cdot x_t' \cdot x_s' + u_y' \cdot y_t' \cdot x_s' + u \cdot x_{ts}'') \cdot dsdt = 0$$

Finally, since $x(t,s)$ is assumed to be a C^2-function the mixed derivatives x_{ts}'' and x_{st}'' are equal. Subtracting the zero integral in (v) from (i) we conclude that (i) is equal to the double integral

$$\iint_\square [u_y' \cdot (y_s' \cdot x_t' - x_t' \cdot y_s') \cdot dsdt$$

which gives (*) in Theorem A.2.

A.3 Application. If the partial derivatives u_y' and v_x' are equal, then the reversed signs for $y_s'x_t' - y_t'x_s'$ which appear in (*) and (**) give the equality

$$\text{(***)} \qquad \int_{\gamma_1} (u \cdot dx + v \cdot dy) = \int_{\gamma_0} (u \cdot dx + v \cdot dy)$$

A.4 Remark. Given the pair (u,v) one refers to $u \cdot dx + v \cdot dy = 0$ as a differential 1-form. By definition it is closed if and only if

$$u_y' = v_x'$$

So Theorem A.2 shows that the line integral of a closed 1-form is not changed under a homotopy deformation where the end-points are kept fixed.

The case of closed curves. The same proof as in Theorem A.2 shows that if $\{\gamma_s\}$ is a homotopic family of closed curves and $u_y' = v_x'$ then we also have the equality (***) above.

A.5 Stokes Theorem. Consider a map Φ from a rectangle

$$\square = \{0 \leq t \leq T\} \times \{0 \leq s \leq 1\}$$

into $\mathbf{R}^2$. We write $\Phi(t,s) = (x(t,s), y(t,s))$ and assume that $x(t,s)$ and $y(t,s)$ are C^2.-functions. The *Jacobian* of Φ becomes

$$\mathcal{J}_\Phi = y_t' \cdot x_s' - x_t' \cdot y_s'$$

So if $u(x,y)$ is some C^1-function then the double integral

$$\text{(1)} \qquad \iint_{\Phi(\square)} u_y' \cdot dxdy = \iint_\square u_y'(x(t,s), y(t,s) \cdot [y_t' \cdot x_s' - x_t' \cdot y_s']\dot{d}sdt$$

By Theorem A.2 the double integral is a difference of two line integrals taken over $s = 0$ and $s = 1$ respectively. Suppose now that Φ is 1-1 so that $\square$ is mapped to a Jordan domain Ω whose boundary $\partial\Omega$ is a simple Jordan curve Γ. Here it may occur that Γ has corner points at the images of the four corner points of $\square$. But in any case the situation is sufficiently regular in order that we get a well defined line integral

$$\int_\Gamma u \cdot dx$$

Inspecting the sign of $y'_s x'_t - y'_t x'_s$ which appears in the Jacobian $\mathcal{J}_\Phi$ respectively in (*) from Theorem A.2 we conclude that one has a minus sign in the equation below:

$$\iint_\Omega u'_y \cdot dxdy = -\int_{\partial\Omega} u \cdot dx \tag{2}$$

Here the simple closed Jordan curve $\partial\Omega$ is oriented in the positive direction, i.e. one moves counter-clockwise as this curve encloses Ω. The reader should illustrate this by drawing a figure. In similar fashion one derives the formula

$$\iint_\Omega v'_x \cdot dxdy = \int_{\partial\Omega} v \cdot dy \tag{3}$$

A.5 Remark. The results above are in principle all we need to move directly to Chapter 3 and study analytic functions where the Cauchy-Riemann equations are used to ensure that we are in "favourable situations" such as (***) above. But for the reader's convenience we shall repeat certain arguments and give another proof in section 2 which has the merit that regularity conditions can be relaxed.

1. Some physical explanations.

Introduction. The material in this section is not necessary in the sequel. It is included to give a perspective upon Stokes Theorem and we also discuss results in dimension 3. The first lessons about line integrals, area integrals and volume integrals go back to Archimedes. The beginner should first of all understand how Archimedes computed the volume of a pyramid Δ. Start in the (x, y)-plane from a bounded and convex domain U bordered by a piecewise linear boundary ∂U with some finite set of corner points $\{p_k = (x_k, y_k)\}$. They are arranged so that a line ℓ_K joints p_k to p_{k+1} when $1 \leq k \leq N-1$ and finally a line ℓ_N joins p_N with p_1. Next, consider a point $p^* = (x_0, y_0, z_0)$ where $z_0 > 0$. We obtain the pyramid Ω by joining p^* to each corner point. It is clear that $\partial\Omega$ consists of has N many planar pieces plus lines when a pair intersect and a number of corner points. The reader may illustrate this by a figure when U is a square. With the notations above one has Arkimedes' formula:

$$(*) \qquad \operatorname{Vol}\Omega = \frac{z_0}{3} \cdot \operatorname{Area}(U)$$

Remark. The proof relies upon the fundamental fact that under dilation expressed by some $s > 0$ areas change with the scale factor s^2 and volumes by s^3. Then (*) follows when we regard portions of Ω where the z-coordinate is restricted to small intervals $\mathfrak{k}N \leq z \leq \mathfrak{k} + \imath N$ for some large integer k. Taking a limit as $N \to \infty$ the scale principle above gives

$$\operatorname{Vol}\Omega = \operatorname{Area}(U) \cdot \int_0^{z_0} (z_0 - z)^2 \cdot dz = \frac{z_0}{3} \cdot \operatorname{Area}(U)$$

Concerning the formula for the area of U it can be computed after a triangulation. After René Descartes introduced coordinates around 1640, geometry and algebra were put together and led to the analytic geometry which is the cornerstone for modern analysis. For example, with U given as above the cartesian coordinates of its corner points determine the area by the elegant formula:

$$(**) \qquad \operatorname{Area}(U) = \frac{1}{2} \sum_{k=0}^{N-1} (y_{k+1} - y_k) \cdot (x_{k+1} - x_k)$$

In XX we shall see how (**) is derived from the Stokes formula and we remark that in the area formula (**) one does not need to assume that U is convex. A more involved case as compared with a pyramid arises when we consider a bounded open set Ω where $\partial\Omega$ consists of N many planar sets $U_1, \ldots, U_N$ where the intersection between two such sets is a line segment. Several U-sets may also intersect at corner points on the boundary. Now we have the *area measure* dA_ν on U_ν and the *outer normal* vector $\mathbf{n}_\nu$ to U_ν, i.e. the unit vector which is $\perp$ to U_ν and points out from Ω. If $\mathbf{n}_\nu(x)$ denotes the x-component of this unit vector one has the formula

$$(***) \qquad \operatorname{Vol}(\Omega) = \sum_{\nu=1}^{N} \iint_{U_\nu} x \cdot \mathbf{n}_\nu(x) \cdot dA_\nu$$

Above we can replace x by y or z using the components $\mathbf{n}_\nu(y)$ r $\mathbf{n}_\nu(z)$ when we compute area integrals over the sides of $\partial\Omega$. Hence there are three ways to compute the volume of Ω. The proof of (***) relies upon Stokes formula in dimension 3.

The principle of Arkimedes. I turns out that (***) can be derived from the study of floating bodies in a liquid. An experience which every child less than five years has discovered, is that a body which is gently placed in water eventually comes to a position at rest when there are no waves or streams. The shape of the body can be highly irregular. Imagine a piece of a broken tree with several branches where some of them may be above the waterline in the floating position. The intersection with the free water line $z = 0$ and the boundary of the tree need not even be connected. In XX below we explain how the principle of Arkimedes confirms the validity of (***) even for domains Ω with a highly irregular boundary. So the formula (***) is therefore a Law of Nature rather than a theorem derived in mathematics.

Stokes Theorem. Let us describe how Newton and his contemporaries Boyle and Hooke were aware of a *conceptual proof* of Stokes formula in three dimensions. Newton attributes the basic ideas below to René Déscartes in his work *Principia* from 1687 where he argues as follows to confirm Stokes theorem:

A bounded connected domain Ω is given in $\mathbf{R}^3$. The boundary may consist of several pairwise disjoint closed surfaces of class C^1 at least, i.e. sufficiently regular in order that we can refer to surface area measure and the outer normal along every component of $\partial\Omega$. Inside Ω a large number of small particles - think of small balls - are moving. They have equal mass and when they impinge with each other the impact is elastic. At a point $p = (x, y, z)$ the "mean neighbor velocity" of balls close to p is a vector valued function $v(p) = (f(p), g(p), h(p))$, i.e. $f(p)$ is the velocity in the x-direction and so on. The interior of Ω is divided into small pairwise disjoint cubes $\{\square_\alpha\}$ with sides parallell to the coordinate axes. The effect of all impacts from balls inside one cube $\square_\alpha$ which hit the boundary of $\square_\alpha$ during a small time interval is a force vector which approximately will be

$$F_\alpha = \rho \cdot (f_x, g_y, h_z) \cdot \text{vol}(\square_\alpha)$$

Here f_x, g_y, h_z are the partial derivatives inside $\square_\alpha$, ρ a constant density of mass and $\text{vol}(\square)_\alpha$ the volume of the square. The boundary of each cube is some hard material so that via the force vectors F_α, each cube "pushes" - or alternatively gets a push - from some of its six many neighbor cubes with which it has a common side. For example, if $f_x > 0$ in a given cube $\square_\alpha$ then $\square_\alpha$ tends to push on the cube next to the right. Along the boundary the impact is expressed by the area integral

$$(*) \qquad \int_{\partial\Omega} \rho \cdot (fn_x + gn_y + hn_z)dA \quad : \quad dA = \text{ area measure}$$

where (n_x, n_y, n_z) the outer normal. Since the balls cannot escape the container Ω the *principle of reaction forces* implies that we must have the equality

$$(**) \qquad \sum F_\alpha = \int_{\partial\Omega} \rho \cdot (fn_x + gn_y + hn_z)dA$$

The sum to the left approximates the volume integral of the function $\rho \cdot (f_x + g_z + h_z)$. Dividing out ρ we get the equality

$$(***) \qquad \int\int\int (f_x + g_y + h_z)dxdydz = \int_{\partial\Omega} (fn_x + gn_y + hn_z)dA$$

Remark. In (***) one may encounter any triple of functions which proves Stokes formula for bounded domains in $\mathbf{R}^3$, expressed by the three equalities:

$$\iiint f_x \cdot dxdydz = \iint_{\partial\Omega} (fn_x \cdot dA$$
$$\iiint g_y \cdot dxdydz = \iint_{\partial\Omega} gn_y \cdot dA$$
$$\iiint h_z \cdot dxdydz = \iint_{\partial\Omega} hn_z \cdot dA\,,$$

Personally I find Newton's proof convincing. Advancements in mathematics rely upon ideas as above.

The formal proof of Stokes formula. In Section 2 we give a "formal" proof of Stokes Theorem for domains in $\mathbf{R^2}$ which are piecewise smooth. It starts by regarding graphic domains where Stokes formula is reduced to the fundamental theorem of calculus expressing a function as the integral of its derivative. After this we use partitions of the unity to get Stokes Theorem for general domains with a differentiable boundary. At the end we prove a more general result where corner points or even more irregular pieces appear on the boundary. See Theorem 2.6. In Section 3 we describe how Stokes Theorem is expressed by differential forms and Section 4 describes some consequences of Stokes Theorem which in particular lead to Green's formula and the end of his section we discuss the Dirichlet problem.

1.1. The principle of Archimedes

Consider a 3-dimensional body K placed in $\mathbf{R}^3$ where (x, y, z) are the coordinates and z is vertical so that the force of gravity is $-g \cdot e_z$. The body has some distribution of mass which need not have a constant density. Imagine a ship where the density is large in the machine room and considerably lighter in the lounge bar. In any case, the body has a specific weight $0 < s < 1$, i.e. if V is the volume then the mass of K is $s \cdot V$. Now K is gently put into the water by a five year old child from in the middle of a reasonably large lake at a time when there are no winds and hence no waves. The five year old child predicts correctly that K will float and after a short time even come to rest. The problem is to determine the floating position and *prove* it using basic laws of mechanics only, i.e. force of gravity and the law of momentum in statics. This problem was considered and solved by the the genius Archimedes. His studies about floating bodies were reconsidered by X. Stevin around 1600 who like Galilei in Pisa also performed experiments in Amsterdam to show that velocities of falling bodes are independent of their specific weight or total mass. Stevin is considered as the creator of modern statics. Let $\mathfrak{o}$ be the center of mass in K. Notice that $\mathfrak{o}$ need not be contained in K. A typical example is an oil-platform. Let K_* be the portion of K below the free waterline determined by the equation $z = 0$. In general K_* can be a disconnected set. But connected components of K_* are bounded by a surface below the free waterline and some area domain in the plane $z = 0$ which serves as a "roof" for this component. The reader should illustrate this by a figure. Now there exists the point $\mathfrak{o}^*$ which corresponds to the center of mass which is determined when K_* has a uniform density of mass. With these notations Archimedes stated that when the body has a resting floating position, then the mass M of the body is equal to the mass of water which would fill the portion of K below the free waterline. So with $0 < s < 1$ it means that

$$\mathrm{vol}(K_*) = s \cdot M$$

Moreover, the vector $\mathfrak{o} - \mathfrak{o}^*$ is $\perp$ to the horizontal water line, i.e. parallell to the direction where the force of gravity acts. Finally, these two conditions are both necessary and sufficient for a floating position at rest. That the two conditions are *necessary* seem likely while the *sufficiency* is more subtle. The reason is that there may exist *several floating positions* where the two conditions above hold. For example, let K be a solid cube with $\mathfrak{o}$ placed in the center and constant density of mass. imagine that K has a uniform distribution of mass and some specific weight $0 < s < 1$. The reader should discover that there exist several positions where Arhimedes' conditions hold. Take for example $s = 0.4$ and draw figures to find different solutions where $\mathfrak{o} - \mathfrak{o}^*$ is vertical. This leads to the question of *stability.* Stability conditions were found by Christian Huyghens. But his rather subtle stability analysis goes beyond the scope of these notes. Let us only mention that when Ω is a square the floating position where the free waterline is parallel to two of the sides of Ω is stable if and only $0 < s < 2 - \sqrt{3}$ or $\sqrt{3} - 1 < s < 1$. So we have an unstable equilibrium when $2 - \sqrt{3} < s < \sqrt{3} - 1$. Here stable floating positions occur when Ω is "tilted". For example when $s = \frac{1}{2}$ we get a stable position where the sides have the angle $\pi/4$ to the free waterline.

1.2 Proof of Archimedes' theorem. To begin with we must understand why the body can float at rest. This amounts to determine the *forces of lifting* on the part of K below the waterline. Following Stevin - and the later refinement by Huyghens - the force of lifting is obtained as follows. From the inside close to a point $p \in \partial K_*$ placed at some distance h below the waterline one makes a small circular hole of radius ϵ. A cylinder of equal radius is pressed a small bit ℓ outside the surface of the submarine. The effect is that a volume of water equal to $\pi \ell \epsilon^2$ is lifted to the free waterline. This requires a work equal to $gh \cdot \pi \ell \epsilon^2$. Then, if P is the force of *pressure* on the submarine close to p the work to push the cylinder as above is equal to $P \cdot \ell$. At the same time the area removed from the surface of the submarine is $\pi \epsilon^2$. The result is that the infinitesmal lifting force becomes

$$F = gh$$

Next, the force of pressure at p from the outside water on the surface of the submarine is parallell to the normal $\mathfrak{n}$ of ∂K_* where the reader by the aid of a figure realizes that one uses the *inward normal*. Thus, if $\mathfrak{n}$ denotes the *outer normal* to ∂K_* the discussion gives:

1.3 Proposition. *The total lifting force on the floating body is*

$$-g \cdot \int_{\partial K_*} h(p) \cdot \mathfrak{n}(p) d\sigma$$

where $d\sigma$ is the area measure on ∂K_*.

Notice that $h < 0$ below the free waterline so the integral above has a positive value. Archimedes' first principle asserts that the area integral above is equal to g times the volume of K_*. This reflects Stokes Theorem when we regard the z-component of the normal vector $\mathfrak{n}$. More precisely, since $z = 0$ on the free waterline, the principle of Archimedes gives the equality

$$\iint_{K_*} dxdydz = \int_{\partial K_*} z \cdot \mathfrak{n}_z \cdot d\sigma$$

where $h = -z$ above since we were below the free waterline.

1.4 A vanishing result. Since K comes to rest it cannot behave like a moving fish, i.e. the two horisontal components of the total lifting force must be zero. This means that we also must have

$$\int_{K_*} z \cdot \mathfrak{n}_x \cdot d\sigma = \int_{K_*} z \cdot \mathfrak{n}_y \cdot d\sigma = 0 \tag{1}$$

Again we shall learn that these two area integrals are zero by Stokes formula. Hence we have consolidated the first principle of Archimedes. Conversley, this principle already predicts the integration formulas since the shape of K_* is arbitrary.

Proof that $\mathfrak{o} - \mathfrak{o}^*$ is vertical. To show this we analyze the force of momentum. We may assume that coordinates are chosen so that $\mathfrak{o} = (0, b)$ for some b on the y-axis. The *Law of Momentum* gives

$$\mathcal{M} = g \cdot \int_{\partial K_*} (x, y - b) \times (-y\mathfrak{n}(x, y)) \cdot ds \tag{2}$$

Here the minus sign for y appears since the force of pressure was found via the distance h from a point below the water line up to $y = 0$. In (2) we decompose the vector $\mathbf{n}$ and expanding the vector product it follows that:

$$\mathcal{M} = g \cdot \int_{\partial K_*} -xy \cdot \mathfrak{n}_y(x, y)) \cdot ds - \int_{\partial K_*} y(y - b) \cdot \mathfrak{n}_x(x, y) \cdot ds$$

Now Stokes formula entails that

$$\cdot \int_{\partial K_*} y \cdot \mathfrak{n}_x(x, y) \cdot ds = \int_{\partial K_*} y^2 \cdot \mathfrak{n}_x(x, y) \cdot ds = 0 \implies$$

$$\mathcal{M} = -g \cdot \int_{\partial K_*} xy \cdot \mathfrak{n}_y(x, y)) \cdot ds \tag{3}$$

By Stokes formula (3) is equal to the area integral

$$-g \int_{K_*} xdxdy \tag{*}$$

This integral must be zero when the body is at rest. This means precisely that the x-component of $\mathfrak{o}^*$ is zero and Archimedes' second assertion follows.

1.5 Curvature and arc-length.

Arc length measure of curves in the (x, y)-plane and the curvature occur frequently in complex analysis. Following Huyghens we show how to determine the curvature by dynamical model. Let a plane curve be defined by an equation $y = y(x)$ where $y''(x) > 0$ for $x > 0$ and $y(0) = y'(0) = 0$. Up to translation and rotation this is a general situation. So we have a convex curve which we follow as x increases. To find an expression of curvature Huyghen's considered a particle of unit mass which can slide on the curve, say on the side just above the curve. No gravity occurs, i.e. imagine that a vertical wall is placed along the curve which prevents the particle to leave the curve. At time zero it has velocity v. No friction forces are present. This means that the force acting on the particle at each moment is directed along the normal to the plane curve. Let t be the time variable. So we have a time dependent function $t \mapsto (x(t), y(x(t))$. At a moment t we denote by $\rho(t)$ the reaction force on the particle, i,e. the force which keeps the particle to move on along the curve. Our assumptions imply that $\rho(t)$ is normal to the curve and directed upwards in the y-direction. Regarding a figure the reader discovers that the components are given by:

$$\rho_x(t) = \rho(t) \cdot \frac{-y'(x(t))}{\sqrt{1 + y'(x(t))^2}} \quad : \ \rho_y(t) = \rho(t) \cdot \frac{1}{\sqrt{1 + y'(x(t))^2}}$$

Newton's Law that "force=mass times acceleration", which Huyghens of course was well aware of in 1670 when he described the present situation, gives:

$$\ddot{x} = \rho(t) \frac{-y'(x(t))}{\sqrt{1 + y'(x(t))^2}} \quad \text{and} \quad \ddot{y} = \rho(t) \cdot \frac{1}{\sqrt{1 + y'(x(t))^2}}$$

Let us now notice that $\dot{y} = y'(x(t)) \cdot \dot{x}$. Hence

$$\dot{x} \cdot \ddot{x} = \rho(t) \frac{-\dot{y}}{\sqrt{1 + y'(x(t))^2}} = -\ddot{\dot{y}}y$$

It follows that $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$ which means that $v^2 = \dot{x}^2 + \dot{y}^2$ is constant. This proves the preservation of kinetic energy which of course must be valid since no other forces than the normal pressure acts on the particle. So the particle moves with constant speed.

1.6 Determination of ρ. To find ρ we start with $\dot{y} = y'(x(t)) \cdot \dot{x}$ and by taking the time derivative once more it follows that

$$\ddot{y} = y''(x(t)) \cdot \dot{x}^2 + y'(x(t)) \cdot \ddot{x}$$

Inserting the two formulas above for $\ddot{x}$ and $\ddot{y}$ it follows that

$$\sqrt{1 + y'(x(t))^2} \cdot \rho(t) = y''(x(t)) \cdot \dot{x}^2$$

Now we also have $v^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 + (1 + y'(x(t))^2 \cdot \dot{x}^2$. We conclude that

$$\rho(t) = \frac{y''(x(t))}{[1 + y'(x(t))^2]^{\frac{3}{2}}} \cdot v^2$$

This is gives the formula for the *centrifugal force* which appears in real life whenever a non-linear motion takes place. The term

$$\mathfrak{c}(x) = \frac{y''(x)}{[1 + y'(x)^2]^{\frac{3}{2}}}$$

is the *geometric curvature* of the plane curve. Huyghen's conclusion was that the centrifugal force is the quotient of v^2 with the curvature expressed as above. Having attained this one may give a geometric description of $\mathfrak{c}(x)$. Namely, $\frac{1}{\mathfrak{c}(x)}$ is the radius of a circle placed along the normal to the curve passing through x which has *best contact* with the curve at the point $(x, y(x))$. This geometric description of the curvature could of course have been given from the start. But the dynamical consideration gives a better insight and is extremely important in mechanics. Moreover, Huyghens clarified why the geometric description must be valid by computing the centrifugal force when a particle is constrained to move along a circular wall of some radius R. Namely, in this case the centrifugal force is constant during the motion and given by

(*)
$$\mathcal{C} = \frac{v^2}{R}$$

1.7 Huyghen's proof in the circular case. His proof is extremely elegant. To begin with he regards a particle which moves along a *regular polygon* with N corners inscribed in the circle of radius R. It moves with constant velocity v and hits the circle N times during one full turn. The *impact force* each time the particle of unit mass hits the circle is given by:

(**)
$$2 \cdot \sin\frac{\pi}{N} \cdot v$$

Here the reader should draw a figure and use that the corners of the polygon give rise to N many triangles with angle $2\pi/N$ at the center and discover that the sudden direction of the velocity vector is changed by $2\pi/N$ at every impact. Then (**) follows by decomposing the reaction force and use the definition of the sine-function. Next, the total length of the polygon is $N \cdot 2 \cdot \sin\frac{\pi}{N}$. Hence the time T to perform a full circular turn is

$$T = \frac{N \cdot 2 \cdot \sin\frac{\pi}{N}}{v}$$

Finally, we have N many instants when impact takes place. So after one circular turn we get

$$F_{\text{imp}} = N \cdot 2 \cdot \sin\frac{\pi}{N} \cdot v$$

Hence

$$\frac{F_{\text{imp}}}{T} = \frac{v^2}{R}$$

This is a remarkable equality, i.e. the left hand side expresses the effect of impact while the particle impinges the circle at the corner points and the effect of force per unit time does not depend on N. When $N \to \infty$ we get the "continuous formula" expressed by (*) above.

2. Stokes Theorem in $\mathbf{R}^2$

Introduction *Stokes Theorem* is often proved in an "intuitive fashion" where figures illustrate how one divides a domain into simpler so that repeated double integrals can be used. We shall give a proof without such artificial constructions. In the long run this will be essential since the cutting of domains tends to be quite involved when the number of its boundary components increases. Perhaps this section appears to be "overkilling" for the beginner. But my opinion is that it belongs to the foundation for complex analysis and a *strict proof* of Stokes Theorem is both important and instructive to learn. Moreover, to establish Stokes formula by a cutting process in dimension ≥ 3 is more or less hopeless. For any $n \geq 3$ one can give a proof of Stokes Theorem using the same procedure as in the 2-dimensional case.

2.A. The case of graphic domains

The FCT = fundamental theorem of calculus - asserts that if $g(x)$ is a function whose derivative exists as a continuous function, then

$$g(x) = g(a) + \int_a^x g'(t)dt$$

Next, consider $\mathbf{R}^2$ where (x, y) are the coordinates. A real valued function $f(x, y)$ is of class C^1 when the two partial derivatives f_x and f_y both exist as continuous functions. Consider a C^1-function $\phi(x)$ which depends on x only and is defined on a closed interval $0 \leq x \leq A$. Assume that $\phi(x) > 0$ which gives the open set

$$\Omega = \{(x, y) \colon\ 0 < x < A \quad 0 < y < \phi(x)\}$$

Now we have the double integrals

$$(*) \qquad \iint_\Omega f_x \cdot dxdy \quad : \quad \iint_\Omega f_y \cdot dxdy$$

We shall express these by certain line integrals. The second double integral is easy to handle since it is a repeated integral:

$$(1) \qquad \iint_\Omega f_y \cdot dxdy = \int_0^A [\int_0^{\phi(x)} f_y(x, y)dy]\, dx = \int_0^A [f(x, \phi(x) - f(x, 0)]dx$$

Later we explain the intrinsic nature of this formula. Let us turn to the double integral in the left hand side of (*). Here we cannot find a repeated integral when horizontal lines $\{x = a\}$ cut the domain Ω so that the sets $\Omega \cap \{x < a\}$ and $\Omega \cap \{x > a\}$ have several connected components. The reader should illustrate this by drawing some figures using an "ugly" ϕ-function. However, we can express the double integral as a sum of line integrals !

1. Construction of line integrals.

Put

$$J(f) = \iint_\Omega f_x \cdot dxdy$$

Consider the following function ψ of a single variable

$$\psi(x) = \int_0^{\phi(x)} f(x, y)dy \implies$$

$$\psi'(x) = f(x, \phi(x))\phi'(x) + \int_0^{\phi(x)} f_x(x, y)dy$$

The FTC gives

$$\psi(A) - \psi(0) = \int_0^A \psi'(x)dx = \int_0^A f(x, \phi(x)\phi'(x) + \int_0^A [\int_0^{\phi(x)} f_x(x, y)dy\,] \cdot dy$$

The last term is $J(f)$ and hence we get

$$(*) \qquad J(f) = \int_0^{\phi(A)} f(A,y)dy - \int_0^{\phi(0)} f(0,y)dy \; - \int_0^A f(x,\phi(x)\phi'(x)dx$$

In (*) three line integrals appear. The first is taken along the vertical line $x = A$, the second along $x = 0$ in the negative direction and the last along the curve $y = \phi(x)$. Now we explain their geometric meaning.

First, the vertical line $L_+ = \{x = A \quad 0 \leq y \leq \phi(A)\}$ is part of the boundary $\partial\Omega$. On this line the *arc-length measure* is equal to dy and *the outward normal* along L_+ is parallell to the x-axis so its component $n_x = 1$. Hence we can write

$$(1) \qquad \int_0^{\phi(A)} f(A,y)dy = \int_{L_+} fn_x \cdot ds$$

Second, consider $L_- = \{x = 0 \quad 0 \leq y \leq \phi(0)\}$. Again dy is the arc-length measure while the outer normal is directed in the negative x-direction, i.e. $n_x = -1$. Hence:

$$(2) \qquad -\int_0^{\phi(0)} f(0,y)dy = \int_{L_-} fn_x \cdot dy$$

Third. On the curve $\Gamma = \{y = \phi(x)\}$ the arc-length is $ds = \sqrt{1 + \phi'(x)^2} \cdot dx$ and the x-component of the outer normal is

$$n_x = \frac{-\phi'(x)}{\sqrt{1 + \phi'(x)^2}}$$

Here the minus sign becomes clear by inspecting a figure of the curve $y = \phi(x)$ where you discover that $n_x > 0$ if $\phi' < 0$ and vice versa ! Hence we obtain

$$(3) \qquad -\int_0^A f(x,\phi(x)\phi'(x)dx = \int_\Gamma fn_x ds$$

where the first minus sign reflects the sign-rule for n_x along the boundary curve Γ. Putting all this together we have proved the equality

$$(**) \qquad J(f) = \int_\Gamma fn_x ds + \int_{L_+} fn_x ds + \int_{L_-} fn_x ds$$

No "ackward signs" occur in the line integrals above since arc-length measure is always defined and the outer normal along the boundary of an open set is clarified by a picture. There remains the portion of $\partial\Omega$ defined by $I = \{0 \leq x \leq A \quad y = 0\}$. Here the outer normal is in the negative y-direction and hence $n_x = 0$. Therefore we only add a zero term by $\int_I fn_x ds$ and arrive at

2. Theorem. *One has the equality*

$$\iint_\Omega f_x \cdot dxdy = \int_{\partial\Omega} fn_x \cdot ds$$

3. An area formula. Let $\phi(x)$ be a piecewise linear function which is positive when $0 < x < A$ while $\phi(0) = \phi(A) = 0$. So we have corner points

$$p_\nu = (x_\nu, y_\nu) \quad 0 = x_0 < x_1 \ldots < x_N = A$$

while the $y_1, \ldots, y_{N-1}$ is any sequence of positive numbers. Apply Theorem 2 with $f(x,y) = x$. The double integral is the area of Ω, i.e. the domain bounded by the piecewise linear curve Γ and the interval $[0, A]$ on the x-axis. Regarding a figure we see that

$$n_x = \frac{y_\nu - y_{\nu+1}}{\sqrt{(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2}} \; : \; ds = \sqrt{(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2} \cdot dx$$

on each linear piece of Γ. After a summation the line integral becomes

$$\sum_{\nu=0}^{\nu=N-1}(x_{\nu+1}^2-x_\nu^2)(y_\nu-y_{\nu+1})$$

This yields a quite remarkable area formula. Consider the simplest case where just one corner point appears at $(1/2,H)$ with $A=1$. So here Ω is a triangle with area equal to $H/2$. The formula gives on the other hand

$$-1/4\cdot -H+(1-1/4)H=H/2$$

as it should be. The reader should continue to test the area formula in more complicated situations where the picewise linear curve has several negative and positive slopes and does not bound a convex set.

4. Expression of the first double integral Consider the component n_y of the outer normal to Ω. On the piece $I=\{0\leq x\leq A:\quad y=0\}$ we see that $n_y=-1$. On the curve $y=\phi(x)$ a picture shows that

$$n_y=\frac{1}{\sqrt{1+\phi'(x)^2}}$$

At the same time $dx=\frac{1}{\sqrt{1+\phi'(x)^2}}ds$ and hence $n_y ds=dx$ holds on the curve. Finally, the outer normal is in the x-direction on the two vertical lines of $\partial\Omega$. Hence we have proved:

5. Theorem. *One has*

$$\iint_\Omega f_y\cdot dxdy=\int_{\partial\Omega} fn_y\cdot ds$$

Thus, we have a similar formula as in Theorem 2 for the y-coordinate. There remains to extend these two formulas to general domains which even may have several disjoint closed boundary curves. But first we resume our special case a bit further. Consider an open cube $\square$ in $\mathbf{R}^2$ where we after a translation may assume that it is centered at the origin. Let $\psi(x,y)$ be a C^1-function whose gradient is $\neq 0$ at the origin and to make a choice we assume that $\psi_y(0,0)<0$. The *implicit function theorem* gives a C^1-function $\phi(x)$ and a positive function $h(x,y)$ such that

$$\psi(x,y)=(\phi(x)-y)h(x,y)$$

Shrinking $\square$ if necessary we may assume that $h>0$ in the whole of $\square$. Put

$$\Omega=\square\cap\{\psi>0\}=\square\cap\{y<\phi(x)\}$$

The previous results show that if f is a C^1-function with a compact support in $\square$, then the two FCT-formulas hold for Ω. We refer to Ω as a *graphic domain.* Next, the validity of the FCT-theorem is obviously invariant under a *linear change of coordinates.* Hence we can start with a cube whose sides are not parallell to the coordinate axis and use some direction of the gradient of ψ when the implicit function theorem is used to obtain a graphic domain. With this kept in mind we begin the proof in the general case.

6. The general case

First we give

7. Definition. *A bounded open and connected subset Ω of $\mathbf{R}^2$ has a C^1-boundary if $\partial\Omega$ is the disjoint union of a finite family of simple and closed curves $\Gamma_1,\ldots,\Gamma_k$ each of which are of class C^1.*

Notation. The class of domains from Definition 7 is denoted by $\mathcal{D}(C^1)$.

8. Remark about the arc-length. A simple closed curve Γ of class C^1 is the image of a vector-valued function

$$t\mapsto\gamma(t)=(x(t),y(t))\quad 0\leq t\leq T$$

which is 1-1 except that $\gamma(0) = \gamma(T)$. Moreover, the functions $x(t)$ and $y(t)$ are both of class C^1 and $\gamma(t)$ is "moving" which means that $\dot{\gamma}(t) \neq 0$ for all t, or equivalently $\dot{x}^2(t) + \dot{y}^2(t) > 0$ for all $0 \leq t \leq T$. The curve Γ can be parametrized in several ways. Among those one has the *parametrization by arc-length.* In this case we use s as parameter and then

$$(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 = 1 \quad : 0 \leq s \leq L$$

where L is the total arc-length of Γ. The arc-length measure along Γ is denoted by ds.

9. The normal to Γ. Let Γ be a C^1-curve. To each point $p \in \Gamma$ we find a unit vector $n(p)$ which is normal to Γ. Given a parametrization by arc-length we have:

$$n = (n_x, n_y) \quad : \quad n_x = \frac{dy}{ds} \quad \text{and} \quad n_y = -\frac{dx}{ds}$$

However, the *sign* of the normal depends on the chosen *orientation* of Γ. For example, let Γ be a circle of radius R centered at the origin. Its arc-lengtth is $2\pi R$ and it has the parametrisation

$$s \mapsto (R\cos(s), R\sin(s)) \quad 0 \leq s \leq 2\pi$$

Given this orientation we see that

$$n_x = \cos(s) \quad n_y = \sin(s)$$

Now we can announce the fundamental result called the FTC in dimension 2.

10. Theorem. *Let $\Omega \in \mathcal{D}(C^1)$. For each function $f(x,y)$ of class C^1 one has*

$$\iint_\Omega f_x \cdot dxdy = \int_{\partial\Omega} f n_x \cdot ds \quad : \quad \iint_\Omega f_y \cdot dxdy = \int_{\partial\Omega} f n_y \cdot ds$$

where n is the outward normal to Ω along each boundary curve.

Proof. The theorem asserts that the formula holds for every C^1-function. Following the method from the proof in the introduction by Déscartes and Newton we shall *decompose* a given C^1-function f and represent it by a sum of C^1-functions where each function of the sum has a small support which enable us to apply the previous result for *graphic domains.* To achieve such a decomposition we first consider some boundary point $p \in \partial\Omega$. It belongs to some boundary curve Γ. Let L be the tangent line to Γ passing through p. Then we construct a small square $\square(p)$ centered at p and with two sides parallel to L while the other are $\perp$ to L. Here Γ is locally defined by an equation $\phi(x,y) = 0$ close to p where ϕ is chosen so that

$$\Omega \cap \square(p) = \{\phi > 0\}$$

Regarding a picture - which the reader should draw - it is clear that $\Omega \cap \square(p)$ is a graphic domain if the sides of $\square(p)$ are sufficiently small. By the *Heine-Borel Lemma* we can cover the compact boundary $\partial\Omega$ by a finite set of such squares, say $\square(p_1), \ldots, \square(p_N)$. Notice that $p_1, \ldots, p_N$ are boundary points chosen from different boundary curves but the cubes are so small that

$$\square(p_\nu) \cap \square(p_j) = \emptyset \quad : \quad p_\nu, p_j \text{ belong to different boundary curves}$$

Next, consider the complementary set

$$K = \Omega \setminus \cup\square(p_\nu)$$

This becomes a compact subset of Ω. By the *Heine-Borel Lemma* it can be covered by a finite set of open cubes $W_1, \ldots, W_m$ chosen so small that each closure $\bar{W}_i$ stays inside Ω.

Partition of the unity. We have the finite family of cubes $\{\square(p_\nu), W_j\}$. By the result in XXX there exist C^1-functions $g_1, \ldots, g_N, h_1, \ldots, h_m$ such that

$$\sum g_\nu + \sum h_i = 1 \quad : \quad \text{Supp}(g_\nu) \subset \square(p_\nu) \quad \text{Supp}(h_i) \subset W_i$$

where the first equality holds in some open neighborhood U of $\bar{\Omega}$.

Final part of the proof. We treat the first formula expressing the double integral of f_x by line integral. Replacing x by y is proved in the same way. Given a C^1-function f it is expressed by a sum:

$$f = \sum f g_\nu + \sum f h_i$$

Since the sum of partial derivatives $\partial_x(\sum g_\nu + \sum h_i)$ and $\partial_y(\sum g_\nu + \sum h_i)$ both vanish we get

$$\iint_\Omega f_x \cdot dxdy = \sum \iint_\Omega (fg_\nu)_x \cdot dxdy + \sum \iint_\Omega (fh_i)_x \cdot dxdy$$

and similarly

$$\int_{\partial\Omega} fn_x \cdot ds = \sum \int_{\partial\Omega} fg_\nu n_x \cdot ds + \sum \int_{\partial\Omega} fh_i n_x \cdot ds$$

Hence it suffices to prove Theorem 10 for each term separately, i.e. for the functions fg_ν or fh_i. Here fg_ν has compact support in $\square(p_\nu)$ and the graphic case applies, i.e. the required formula in Theorem 10 holds in this case. Next, consider fh_i. This function has compact support in Ω so no boundary terms, i.e. no line integral appears. But at the same time XX shows that

$$\iint_\Omega (fh_i)_x \cdot dxdy = 0$$

So these double integrals give no contribution and Theorem 10 is proved.

11. Boundary with corner points

Consider a connected and bounded open set Ω. Its boundary is a compact set. A point $p \in \partial\Omega$ is called regular if there exists a small disc D centered at p and a C^1-function ϕ in D whose gradient vector is $\neq 0$ in D such that

$$\Omega \cap D = \{\phi < 0\} \quad : \quad \phi(p) = 0$$

It is obvious that the set of regular points is an open subset of $\partial\Omega$ to be denoted by $\text{reg}(\partial\Omega)$. We assume that the set is non-empty, i.e. we ignore to consider open sets with a very ugly boundary. On the regular part of $\partial\Omega$ the arc-length measure ds and the outer normal are defined. Next, put

$$\sigma_\Omega = \partial\Omega \setminus \text{reg}(\partial\Omega)$$

So this is a compact set and we shall impose a condition on its size.

12. Federer's conditions We say that σ_Ω satisfies Federer's condition if its two projections on the x-line respectively the y-inner are *null sets* in the sense of Lebesgue. Thus, the condition is that to every $\epsilon > 0$ there exists a finite family of disjoint open intervals $J_\nu = (a_\nu, b_\nu)$ on the x-line such that

$$\sum (b_\nu - a_\nu) < \epsilon \quad : \quad (x,y) \in \sigma_\Omega \implies x \in \cup J_\nu$$

and with a similar condition for the y-interval.

In addition to Federer's condition we assume that the *total arc-length* of the regular part is finite, i.e. that

$$\int_{\text{reg}(\partial\Omega)} ds < \infty$$

Now we can announce the FCT with corners:

13. Theorem *Assume that $\partial\Omega$ satisfies Federer's condition and that its regular part has finite arc-length. Then*

$$\iint_\Omega f_y dxdy = \int_{\text{reg}(\partial\Omega)} fn_y ds$$

and similarly for x.

Proof. Let $\epsilon > 0$ and choose intervals J_ν of total length $< \epsilon$ to satisfy Federer's condition for x-coordinates of points in σ_Ω. Choose a C^1-function $g(x)$ where $0 \leq g \leq 1$ and $g = 1$ outside $\cup J_\nu$ while $g(x) = 0$ in a neighbourhood of the compact set

$$\{x\colon \exists\, y\colon (x, y) \in \sigma_\Omega\}$$

Set

$$h = gf$$

By the choice of g the support of h avoids σ_Ω. So by repeating the proof of Theorem 10 for regular boundaries we obtain

$$\text{(1)} \qquad \iint_\Omega h_y dxdy = \int_{\text{reg}(\partial\Omega)} hn_y ds$$

Here $h_y = g(x) f_y$ and hence

$$\text{(2)} \qquad \iint_\Omega f_y dxdy - \iint_\Omega h_y dxdy = \iint_\Omega (1 - g(x)) f_y(x, y) dxdy$$

The last double integral is estimated as follows. By the choice of g we have $1 - g = 0$ outside a union of intervals of length $\leq \epsilon$. So if M is the maximum norm of f_y taken over $\bar{\Omega}$ and L is the maximum of two $y-$-coordinates for points in Ω with the same x-coordinate, then we get

$$\text{(3)} \qquad |\iint_\Omega (1 - g(x)) f_y(x, y) dxdy| \leq ML\epsilon$$

14. Estimate of $\int_{\text{reg}(\partial\Omega)} f(1-g) n_y ds$. Here we need a more delicate argument where the reader - as always when it comes to a more involved proof in analysis - should make suitable pictures to discover the geometry. Let $\delta > 0$ and consider the subset $W(\delta)$ of $\text{reg}(\partial\Omega)$ where $|n_y| \geq \delta$. On this set we notice that the arc-length is majorised by $|dx|$:

$$\text{(i)} \qquad |n_y| \geq \delta| \implies ds \leq \frac{1}{\delta}|dx|$$

Moreover, the projection $(x, y) \to x$ restricted to $W(\delta)$ has *discrete fibers*, i.e. it is locally 1-1 as you see by drawing a figure with a small curve passing through any point in $W(\delta)$. However, the set $W(\delta)$ need not be compact so we must perform another reduction. Namely, by hypothesis the total length of $\text{reg}(\partial\Omega)$ is finite. So with $\epsilon > 0$ we can find a *compact* set $K \subset \text{reg}(\partial\Omega)$ such that

$$\text{(ii)} \qquad \int_{\text{reg}(\partial\Omega)\setminus K} ds < \epsilon$$

Next, restrict the projection map π defined by $(x, y) \to x$ to $W(\delta) \cap K$. Since this set is *compact* and the projection is locally 1-1, *Heine-Borel Lemma* gives an integer M_δ such that the inverse fibers

$$\text{(iii)} \qquad \pi^{-1} \cap W(\delta) \cap K$$

contain at most M_δ points for every x. Notice that M_δ depends on δ but not upon ϵ. Using the above we obtain

$$\text{(*)} \qquad \int_{W(\delta)\cap K} (1 - g) ds \leq \frac{1}{\delta} \int_{W(\delta)\cap K} (1 - g) d|x| \leq \frac{M_\delta}{\delta} \cdot \int (1 - g) dx \leq \frac{M_\delta}{\delta} \cdot \epsilon$$

where the last inequality follows from the choice of g in XX above. We have also to other estimates. Let $|f[_K$ be the maximum norm of f over K. On On $K \setminus W(\delta)$ we have $|n_y| \leq \delta$ and hence

$$\text{(**)} \qquad |\int_{K\setminus W(\delta)} f(1 - g) n_y \cdot ds| \leq \delta \cdot \int_{K\setminus W(\delta)} f(1 - g) ds \leq \delta \cdot |f|_K \cdot \int_{\partial\Omega_{\text{reg}}} ds$$

Next, using (iii) above we have

$$\text{(***)} \qquad |\int_{[\text{reg}(\partial\Omega)\setminus K]} f(1 - g) ds| \leq |f|_{\partial\Omega} \cdot \int_{[\text{reg}(\partial\Omega)\setminus K]} ds \leq |f|_{\partial\Omega} \cdot \epsilon$$

Finally, notice that

(v) $$\text{reg}(\partial\Omega) = [W(\delta) \cap K] \cup [\text{reg}(\partial\Omega) \setminus K] \cup [K \setminus W(\delta)]$$

Putting all this together we obtain the inequality

(vi) $$|\int_{\text{reg}(\partial\Omega)} f(1-g)n_y ds| \leq \frac{M_\delta \epsilon}{\delta} + A \cdot |f|_K \cdot \delta + |f|_{\partial\Omega} \cdot \epsilon$$

Here (vi) hold for all pairs $\delta, \epsilon)$. To finish the proof of Theorem 13 we choose an arbitrary small$\delta > 0$ and after ϵ is chosen so small that we first have $\epsilon \leq \delta$ and also

$$\frac{M_\delta \epsilon}{\delta} < \delta$$

Then the left hand side is majorised by

$$(1 + A \cdot |f|_{\partial\Omega} + |f|_{\partial\Omega}) \cdot \delta$$

Since $\delta > 0$ is arbitrary we get Theorem 13.

3. Line integrals via differentials

In the previous section arc-length and the outer normal were used to construct line integrals. One may also introduce the differentials dx and dy. The construction of the outer normal shows that

(*) $$n_x \cdot ds = dy \quad n_y \cdot ds = -dx$$

Hence one can express Stokes Theorem in the form

$$\iint_\Omega f_x \cdot dxdy = \int_{\partial\Omega} f dy \quad : \quad \iint_\Omega f_y \cdot dxdy = -\int_{\partial\Omega} f dx$$

A Warning. When Stokes Theorem is expressed in this way one must be careful with the orientation. The *rule of thumbs* is used whenever $\partial\Omega$ borders an open set. Personally I prefer to express line integrals by $n_x ds$ or $n_y ds$ since the geometric picture becomes transparent. Historically this was also the case. For example, Gauss used the arc-length measure when he developed mathematical calculus which was needed for various calculations in electromagnetic fields.

However, differentials have an advantage when calculus is performed on *manifolds* rather than the euclidian plane $\mathbf{R}^2$, since here arc-length and normal derivatives are not even defined until the manifold has been equipped with a metric. Thus, complex analysis on *Riemann surfaces*, i.e. 1-dimensional complex manifolds uses differential forms.

3.1 Transformation laws. Let (ξ, η) be the coordinate functions in another copy of $\mathbf{R^2}$. Let Ω be a domain in the (x, y)-plane. Consider a 1-1- map

(1) $$Q\colon\ (x, y) \to (\phi(x, y), \psi(x, y)) \quad : \xi = \phi(x, y) \quad \eta = \psi(x, y)$$

defined in some neighborhood of $\bar{\Omega}$. Here ϕ and ψ are C^1-functions and we get the domain $Q(\Omega)$ in the (ξ, η)-space

Now we have

(2) $$d\xi = \phi_x \cdot dx + \phi_y \cdot dy \quad : \ d\eta = \psi_x \cdot dx + \psi_y \cdot dy \quad :$$

Next, the *Jacobian* of the Q-map is defined by

(3) $$\mathcal{J} = \phi_x \cdot \psi_y - \phi_x \cdot \psi_y$$

The Q-map preserves orientation when $\mathcal{J} > 0$ and from now on this is assumed. The Jacobian changes area which is expressed by

(3) $$d\xi d\eta = J \cdot dxdy$$

Now we take some C^1-function $g(\xi, \eta)$ defined in the $(\xi, \eta$-space. In the (x, y)-space we get the function

(4) $$g_*(x, y) = g(\phi(x, y), \psi(x, y))$$

Let us now study the result when Stokes formula is applied. Put $\Omega^* = Q(\Omega)$. We have the obvious equality:

(5) $$\int_{\partial\Omega^*} g d\xi = \int_{\partial\Omega} g_* \cdot (\phi_x dx + \phi_y dy)$$

Stokes formula applied to the right hand side gives

(6) $$\iint_\Omega -(g_*\phi_x)_y + (g_*\phi_y)_x \cdot dxdy$$

Here we get a cancellation since the mixed derivatives ϕ_{xy} and ϕ_{yx} are equal. Hence (6) becomes:

(7) $$\iint_\Omega \left[-(g_*)_y \cdot \phi_x + (g_*)_x \cdot \phi_y\right] \cdot dxdy$$

If Stokes formula is applied to the left hand side in (5) we get the area integral

(8) $$\iint_{\Omega^*} -g_\eta \cdot d\xi d\eta$$

Why is (7)=(8). The equality follows using transformation rules for partial derivatives. Namely, from (4) we get

(9) $$(g_*)_x = \phi_x \cdot g_\xi + \psi_x \cdot g_\eta \quad : \ (g_*)_y = \phi_y \cdot g_\xi + \psi_y \cdot g_\eta$$

Here we can solve out g_η and find

(10) $$(\phi_x \psi_y - \phi_y \psi_x) \cdot g_\eta = (g_*)_y \cdot \phi_x - (g_*)_x \cdot \phi_y$$

In (10) we discover the Jacobian as a factor for g_η. Hence the rule for area transformation in (3) and the two minus signs in (7) and (8) show that (7)=(8).

3.2 The pull-back of differential forms. The efficient way to analyze transforms which can be extended to maps between manifolds goes as follows: Let

(1) $$Q\colon\ (\xi, \eta) \mapsto (x, y) \quad : \ x = \phi(\xi, \eta) \quad y = \psi(\xi, \eta)$$

The differential 1-forms dx and dy have inverse images defined by

(2) $$(dx)^* = \phi_\xi \cdot d\xi + \phi_\eta \cdot d\eta \quad : \quad (dy)^* = \psi_\xi \cdot d\xi + \psi_\eta \cdot d\eta$$

More generally, to each pair of C^1-functions $A(x,y), B(x,y)$ we get the 1-form $\alpha = A(x,y)\dot{d}x + B(x,y) \cdot dy$. Its pull-back is

(3) $$\alpha^* = A^*(\xi, \eta) \cdot (dx)^* + B^*(\xi, \eta) \cdot (dy)^*$$

where $A^*(\xi, \eta) = A(\phi(x,y), \psi(x,y))$ and similarly for B^*.

3.3 Exterior differentials. If $\alpha = A \cdot dx + B \cdot dy$ is a 1-form its exterior differential is defined as

(4) $$d\alpha = -A_y \cdot dx \wedge dy + B_x \cdot \wedge dy$$

The minus sign in front of A_y is compatible with Stokes formula is expressed in (3.0),i.e. we get

(5) $$\iint_\Omega d\alpha = \int_\Omega \alpha$$

The formula in (5) summarizes the whole content of Stokes formula. It becomes especaily useful because of the following fundamental fact.

3.4 Theorem. *The pull-back of differential forms commutes with exterior differential.*

3.5 Remark. Theorem 3.4 asserts that if we start from α and get α^* then the 2-form $d(\alpha^*)$ in the $\xi, \eta)$-space is equal to the pull-back of the 2-form $d\alpha$. The reader should verify this or consult some text-book in calculus. Passing to Stokes formula the result is that if α is a 1-form in the (x,y)-space then one has equality for the area integrals

$$\iint_{\Omega} d\alpha = \iint_{\Omega^*} d\alpha^* \tag{6}$$

3.6 Currents. Above we recalled classic notions which after easily extend to manifolds in dimension 2 and by some extra linear algebra where one introduces differential forms of higher degree to manifolds in any dimension. However, the classic approach has a drawback since an equality like (6) assumes that one has a 1-1 map from the (ξ,η)-plane into the (x,y)-plane. The *modern procedure* is to use distribution theory. For example, consider a map

$$Q\colon (\xi,\eta) \mapsto (\phi(x,y),\psi(x,y))$$

where ϕ and ψ are C^∞-functions. but the map Q need not be 1-1. Let γ be a Jordan arc in the (ξ,η)-space. For example, an interval on some circle or a line segment. The image set $Q(\gamma)$ can be a curve with self-intersections and so on. A typical case is that for points $p \in Q(\gamma)$ the inverse set $Q^{-1}(p) \cap \gamma$ is a finite set of points on γ but the number may change as p moves in $Q(\gamma)$. So one should regard the image of γ under the Q-map as a *current* acting as a linear form on 1-forms in the (x,y)-space by the rule

$$\alpha \mapsto \int_\gamma \alpha^* \tag{1}$$

The point is that the pull-back α^* is defined even if Q is not 1-1, i.e. it is given by

$$A^*(\xi,\eta)\cdot[\phi^*_x \cdot d\xi + \phi^*_x \cdot d\eta] + B^*(\xi,\eta)\cdot[\psi^*_x \cdot d\xi + \psi^*_x \cdot d\eta] \tag{2}$$

The current defined by (1) is denoted by $Q_*(\gamma)$ and called the direct image of the integration current defined by γ in the (ξ,η)-space. Let us remark that it is essential that we prescribe an orientation on γ when we construct its direct image current. The current $Q_*(\gamma)$ has distribution coefficients when we specialize the 1-form α. That is, there exists a map

$$A(x,y) \in C^\infty(\mathbf{R^2}) \mapsto \int_\gamma A^*(\xi,\eta)\cdot(dx)^*$$

Similarly, $(dy)^*$ yields a distribution.

3.7 Stokes formula in higher dimension For readers who already are a bit familiar with differential geometry, the FCT in any dimension goes as follows: Let X be an oriented manifold of some dimension $n \geq 2$ and of class C^2 at least. Let V be a locally closed and oriented submanifold of some dimension $1 \leq k \leq n-1$. Assume that $\bar{V}$ is compact and that the boundary ∂V satisfies Federer's condition, i.e. it contains an open part $\text{reg}(\partial V)$ which is an oriented $k-1$-dimensional manifold whose $(k-1)$-dimensional volume is finite, and the $k-1$-dimensional Haussdorff measure of $\partial V \setminus \text{reg}(\partial V)$ is zero. Then the following hold for every differential $(k-1)$-form α of class C^1 defined in some open neighborhood of $\bar{V}$:

$$\int_V d\alpha = \int_{\text{reg}(\partial V)} \alpha$$

An example Let $n \geq 3$ and $1 \leq k \leq n-1$. Suppose that $P_1(x),\ldots,P_k(x)$ is a k-tuple of real valued polynomials of n variables such that the set where the $k \times k$-matrix whose elements are

$$\partial P_i/\partial x_\nu(x) \quad : \quad 1 \leq i,\nu \leq k$$

is invertible in some non-empty open set U of $\mathbf{R^n}$, i.e. the polynomial defined by the determinant of this matrix is not identically zero. Then we obtain a locally closed submanifold of $\mathbf{R^n}$ defined by

$$W = \{x\colon\ P_1(x) = \ldots = P_k(x) = 0\} \cap U$$

Next, let $Q_(x), \ldots, Q_m(x)$ be some m-tuple of polynomials and put

$$\mathfrak{Q} = \{x\colon\ Q_\nu(x) < 0 : 1 \leq \nu \leq m\}$$

Assume also that the open set $\mathfrak{Q}$ is bounded in $\mathbf{R^n}$, i.e. contained in some open ball with sufficiently large radius centered at the origin. Then the general Stokes Theorem holds for the locally closed k-dimensional submanifold $V = W \cap \mathfrak{Q}$. The proof that Federer's conditions hold follows from a result about *semi-algebraic sets* due to Tarski and Seidenberg. The reader may consult the appendix in Hörmander's text-book [Hö:1] or his more recent text-book series [Hö] for an account about semi-algebraic sets which verify Federer's conditions. We remark that the resulting boundary integrals may become quite involved since no extra conditions are imposed upon the Q-functions so the boundary ∂V may have "corners" which in the case $n \geq 3$ of course are hard to vizualise.

4. Green's formula and Dirichlet's problem

Let $\Omega \in \mathcal{D}(C^1)$ and f is a function of class C^2 which means that f_x and f_y are of class C^1. Stokes Theorem applied to f_x and f_y give

$$\iint_\Omega f_{xx} dxdy = \int_{\partial\Omega} f_x n_x ds \quad : \quad \iint_\Omega f_{yy} dxdy = \int_{\partial\Omega} f_y n_y ds$$

Adding the two equalities we obtain

$$\iint_\Omega (f_{xx} + f_{yy}) dxdy = \int_{\partial\Omega} (f_x n_x + f_y n_y) ds$$

Here $f_x n_x + f_y n_y$ is the *directional derivative* of f along the outer normal n which we denote by f_n. Next, $f_{xx} + f_{yy}$ is the *Laplacian* of f and is denoted by $\Delta(f)$. Hence we have proved

4.1 Theorem *Let $\Omega \in \mathcal{D}(C^1)$. For each f of class C^2 we have*

$$\iint_\Omega \Delta(f)\, dxdy = \int_{\partial\Omega} f_n \, ds$$

4.2 Remark. Stokes Theorem applied to the functions f_x and f_y gives the two formulas

$$\iint_\Omega f_{yx} dxdy = \int_{\partial\Omega} f_x n_y ds \quad : \quad \iint_\Omega f_{xy} dxdy = \int_{\partial\Omega} f_y n_x ds$$

When f is of class C^2, the mixed derivatives f_{yx} and f_{xy} are equal. Hence we obtain the following equality for line integrals

$$\int_{\partial\Omega} f_x n_y ds = \int_{\partial\Omega} f_y n_x ds$$

Notice that this equality is obvious using differentials to express the line integrals, i.e. since $dx = -n_y ds$ and $dy = n_x ds$ the equality above is expressed by

$$\int_{\partial\Omega} f_x dx + f_y dy = 0$$

The vanishing of this line integral follows trivally since each boundary curve of $\partial\Omega$ is closed. However, the two equalities above have a non-trivial consequence. Namely, let f be of class C^2 and consider its *gradient vector*

$$\nabla(f) = (f_x, f_y) = f_x \cdot e_x + f_y \cdot e_y$$

where e_x and e_y are the euclidian basis vectors in $\mathbf{R^2}$. Subtracting the two equalities above we get

$$\int_{\partial\Omega} (f_x n_y - f_y n_x) ds = 0$$

Here $f_x n_y - f_x n_x$ is equal to the *vector product* $\nabla \times n$. Hence we have proved

4.3 Theorem *Let $\Omega \in \mathcal{D}(C^1)$. For each f of class C^2 we have*

$$\int_{\partial\Omega} \nabla(f) \times n \cdot ds = 0$$

4.4 Mean value integrals Consider the case when Ω is an open disc. Without loss of generality we may assume its center is at the origin and let R be the radius. Denote the disc by D_R. Here ∂D_R is parametrized by

$$\theta \mapsto R(\cos(\theta), \sin(\theta)) \quad : \quad 0 \leq \theta \leq 2\pi$$

Let f be a C^2-function defined in some open neighbourhood of the closed disc $\bar{D}_R$. Put

$$M_f(R) = \frac{1}{2\pi} \int_0^{2\pi} f(R \cdot \cos\theta, R \cdot \sin\theta)\, d\theta$$

As R varies we can take the derivative and rules of differentiation yield the following equality for each $0 < r < R$:

$$\frac{d}{dr}(M_f(r)) = \frac{1}{2\pi}\int_0^{2\pi} [f_x(r\cos\theta, r\sin\theta)\cos(\theta) + f_y(r\cos\theta, r\sin\theta)\sin(\theta)]\, d\theta$$

Since $rd\theta = ds$ and $n = (\cos\theta, \sin\theta)$ we can express the equation by

$$\frac{d}{dr}(M_f(r)) = \frac{1}{2\pi r}\int_{\partial D_r} f_n ds$$

Hence Theorem 4.1 gives:

4.5 Theorem *For each $0 < r < R$ one has*

$$\int_{D_r} \Delta(f)dxdy = 2\pi r \cdot \frac{d}{dr}(M_f(r))$$

In particular, suppose that f is a *harmonic function* which means that it satisfies the Laplace equation, i.e. $\Delta(f) = 0$. Then the left hand side is zero above and hence the mean-value function $M_f(r)$ is constant. By continuity at the origin we see that

$$\lim_{\epsilon\to 0} M_f(\epsilon) = f(0,0)$$

Hence we get

4.6 Theorem *Let f be a harmonic function in disc D_R. Then*

$$f(0,0) = M_f(r) \quad : \quad 0 < r < R$$

Staying with harmonic functions we also notice that Theorem 4.1 gives

4.7 Theorem. *Let $\Omega \in \mathcal{D}(C^1)$. For each C^1-function h which is harmonic in Ω one has*

$$\int_{\partial\Omega} h_n ds = 0$$

4.8 Formulas with two functions. Let $\Omega \in \mathcal{D}(C^1)$ and f, g is a pair of C^2-functions. Applying Theorem 2.4 to $f_x g$ and $f_y g$ gives after a summation

$$\iint_\Omega [\Delta(f)\cdot g + f_x g_x + f_y g_y]\, dxdy = \int_{\partial\Omega} f_n \cdot g \cdot ds$$

Here $f_x g_x + f_y g_y$ is the *inner product* of the gradient vectors of f and g. Since this term is symmetric for the pair, we obtain the following when the same formula a above is applied with f and g interchanged:

4.9 Theorem *For each pair of C^2-functions f, g one has:*

$$\iint_\Omega \big[\Delta(f)\cdot g - f\cdot\Delta(g)\big]\, dxdy = \int_{\partial\Omega} \big[f_n \cdot g - f g_n\big]\cdot ds$$

4.10 Application. Let $\Omega = D_R$ be a disc centered at the origin. Let f be harmonic in D_R. Given a point $(a, b) \in D_R$ we define the g-function

$$g(x, y) = \text{Log}(\sqrt{(x-a)^2 + (y-b)^2})$$

An easy computation which is left to the reader shows that g is a harmonic function in $\mathbf{R^2}$ outsidee the point (a, b). With $\epsilon > 0$ small we remove the open disc $D_\epsilon(a, b)$ and apply Green's formula to the domain $\Omega_\epsilon = D_R \setminus \bar{D}_\epsilon(a, b)$. Since both f and g are harmonic in Ω_ϵ we obtain:

$$\int_{\partial\Omega_\epsilon} f_n g\, ds = \int_{\partial\Omega_\epsilon} f g_n\, ds$$

This equality yields an interesting formula when $\epsilon \to 0$. First, $\partial\Omega_\epsilon = \partial\Omega \cup \partial D_\epsilon$. For the line integrals over D_ϵ the following two limit formulas hold:

$$(*) \qquad \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} f_n g \, ds = 0 \quad : \quad \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} f g_n \, ds = -2\pi f(a,b)$$

where the last minus sign appears since ∂D_ϵ is a boundary component of Ω_ϵ so that the outward normal points into the disc D_ϵ.

4.11 Exercise. Prove the last limit formula in (*). The hint is: We may assume that (a,b) is the origin and since $\frac{d}{dr}\text{Log}\, r = \frac{1}{r}$ we find that the outer normal $g_n = -\frac{1}{\epsilon}$. At the same time $ds = \epsilon d\theta$ and now the reader verifies the second limit formula.

Using these two limit formulas we obtain

$$(**) \qquad 2\pi f(a,b) = \int_{\partial D_R} f g_n \, ds - \int_{\partial D_R} f_n g \, ds$$

Hence the value of the harmonic function f at (a,b) can be expressed by sum of two line integrals on ∂D_R where f and f_n appear. Later on we shall find *Poisson's formula* where the value of f is expressed by a line integral where only f appears.

4.12 The Dirichlet problem

The formula 4.8 with two functions was the starting point when Dirichlet around 1840 posed the following problem:

$$\text{Given } h \in C^0(\partial\Omega) \text{ find } f \in C^0(\bar{\Omega}) \quad : \quad f|\partial\Omega = h \quad \Delta(f)|\Omega = 0$$

To solve this Dirichlet considered the following *variational problem*

$$V(f) = \iint_\Omega (f_x^2 + f_y^2) \cdot dxdy \quad : \quad f|\partial\Omega = h$$

Now one seeks

$$V_* = \min_f V(f)$$

If $V(f) = V_*$ one has:

$$\text{(i)} \qquad \lim_{\epsilon \to 0} \frac{V(f + \epsilon g) - V(f)}{\epsilon} = 0 \quad : \quad \forall\, g \text{ such that } g|\partial\Omega = 0$$

Now we notice that

$$V(f + \epsilon g) - V(f) = 2\epsilon \iint (f_x g_x + f_y g_y) dxdy + \epsilon^2 \iint (g_x^2 + g_y^2) dxdy =$$

$$-2\epsilon \iint \Delta(f) g \, dxdy + \epsilon^2 \iint (g_x^2 + g_y^2) dxdy$$

Passing to the limit when $\epsilon \to 0$ we conclude that if f minimizes V then

$$\text{(ii)} \qquad \iint \Delta(f) g \, dxdy = 0$$

Since this hold for all g vanishing on $\partial\Omega$ it follows that $\Delta(f) = 0$ in Ω. Hence f solves Dirichlet's problem.

4.13 An obstacle. Dirichlet's solution is correct but his proof became "shaky" when Weierstrass discovered that there exist variational problems of a similar nature as above which *fail* to have an extremal solution. In 1923 O. Perron gave a rigorous proof using subharmonic functions which answers the question when Dirichlet's problem has a solution for every continuous boundary function.

4.14 Theorem. *Let Ω be a bounded open set such that for every point $a \in \partial\Omega$ the connected component of the set $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the singleton set $\{a\}$. Then each $h \in C^0(\partial\Omega)$ has a unique harmonic extension to $\bar{\Omega}$.*

Remark. We prove this in Chapter V. Notice that Theorem 4.14 applies when Ω is of class $\mathcal{D}(C^1)$.

4.15 Probabilistic solution. Theorem 4.14 can be proved by *probabilistic considerations*. Let us describe this under the assumption that $\partial\Omega$ is "nice". Pick some arc γ from the boundary, i.e. γ is a simple closed curve contained in one of the closed boundary curves Γ. Now one seeks a harmonic function f in Ω such that its boundary value is equal to 1 at interior points of γ and zero on $\partial\Omega \setminus \gamma$. So the boundary values of f are determined except at the the two end-points of the closed C^1-curve γ.

4.16 The harmonic measure. Let $p \in \Omega$. Starting from p we consider the *Brownian motion*, i.e. perform a 2-dimensional random walk which may be approximated by regarding small consequtive steps of length δ which moves the particle with probability 1/4 in each direction - i.e. changing x or y by + or -δ. With probability one the discrete random walk eventually crosses the boundary $\partial\Omega$. Some of these cross γ and this gives a number $0 < \pi_\gamma(p) < 1$ which is the probability for a random walk to cross γ. To be precise, one gets this number in the limit when $\delta \to 0$. Next, we use the fact that a function is harmonic if and only if it satisfies a local *mean-value condition*, i.e. a function f is harmonic in Ω if and only if its value at point $q \in \Omega$ is equal to its mean-value over small discs centered at q. From this it follows easily that the function

$$p \mapsto \pi_\gamma(p)$$

is harmonic in Ω and yields the solution to Dirichlet's problem with boundary values as above. Finally, since this is achieved for boundary arc γ one can deduce Theorem 4.14 by approximating a continuous function h on $\partial\Omega$ with functions which are piecewise constant

Remark The probability expressed by $\pi_\gamma(p)$ plays an important role later on since it is equal to the *harmonic measure* defined as the value at p of the harmonic function in Ω with boundary value zero on $\partial\Omega \setminus \gamma$ and equal to one on γ. This probabilistic interpretation of the harmonic measure gives also an intuitive feeling for the harmonic measure.

A notable point is that the probabilistic solution makes it possible to use *Monte Carlo simulations* in order to obtain good approximative solutions to Dirichlet's problem which in general have no "analytic solutions". For example, presence of several boundary components of $\partial\Omega$ does not in principle cause any problem when a Monte Carlo simulation is used. A drawback is that Monte Carlo simulations tend to be rather time consuming since one must repeat random walks several times over small grids. In an impressive *Examensarbete by Oskar Sandberg* (2003) at the Mathematics Department in Stockholm, a quite rapid Monte Carlo simulation was developed . Here one takes larger random steps in each simulation using the *mean-value property* for harmonic functions. The interested reader may consult Sandberg's work for further details where numerical solutions to the Dirichlet problem in dimension ≥ 3 also are obtained by Monte Carlo simulations. See also the section about the material about the *Brownian motion* in Chapter XX for further comments.

4.18 The Dirichlet problem in a half-space.

Consider the half space in $\mathbf{R}^2$ defined by $U = \{(x,y)\colon y > 0\}$. So here ∂U is the x-axis. We construct harmonic a class of harmonic functions in U by the following procedure:

4.19 Definition *To each pair of real numbers $a < b$ we let $H_{a,b}(x,y)$ be the function in U whose value at (x,y) is the angle at this point in the triangle with corners at $(a,0),(b,0),(x,y)$.*

The reader should draw a figure which explains why $0 < H_{a,b}(x,y) < \pi$ for all $(x,y) \in U$. Moreover one has the limit formulas

$$\lim_{y\to 0} H_{a,b}(x,y)=\pi \quad a<x<b \quad : \quad \lim_{y\to 0} H_{a,b}(x,y)=\pi \quad x<a \quad b<x$$

Finally we also have

$$\lim_{x^2+y^2\to\infty} H_{a,b}(x,y)=0$$

Less obvious is that H is a *harmonic* function in U. So let us give:

Proof Recall that the sum of the angles of a triangle is π. Given the poins $a<b$ a figure shows that the angle $H_{a,b}(x,y)$ is equal to $\beta-\alpha$ where α is the angle between the vector from a to (x,y) and the positive x-axis, and similarly β is the angle between the vector from b to (x,y) and the positive x-axis. Since the sum of harmonic functions is again harmonic, i suffices to show that the functions $\alpha(x,y)$ and $\beta(x,y)$ are harmonic. Now it is clear - again by a figure - that

$$\operatorname{tg}(\alpha)=\frac{y}{x-a} \implies \alpha(x,y)=\operatorname{arctg}\big(\frac{y}{x-a}\big)$$

When $y>0$ and $x=a$ the α-function is $\pi/2$. Outside this vertical line we can take irs derivatives. Using the wellknown formula for the derivative of the arctg-function the reader may verify that $\Delta(\alpha)=0$.

4.20 Remark See also XX where we give an alternative proof that H is harmonic using complex valued Log-functions. The $H_{a,b}$-functions can be used to solve the Dirichlet problem since the two limit formulas after Definition 4.19 settle the case when the boundary function is the characteristic function of an interval. To proceed further we need to study the H-functions when $b-a\to 0$. Given $(x,y)\in U$ and some a on the x-axis we consider for a small positive Δ the triangle with corners at $(x,y),(a,0),(a+\Delta,0)$. Let α be the angle at (x,y) which therefore gets small with Δ. By wellknown results about the cosine- and the sine-function - especially that $\frac{\sin\alpha}{\alpha}\to 1$ as $\alpha\to 0$, the reader can easily verify that:

(*)
$$\lim_{\Delta\to 0}\frac{\alpha}{\Delta}=\frac{y}{(x-a)^2+y^2}$$

Armed this result we solve the Dirichlet problem when $f(x)$ is a continuous function which vanishes outside a bounded interval $[-A,A]$. To keep variables distinct we use ξ as the coordinate on the x-axis. Let $\{\xi_\nu\}$ be a finite and strictly increasing sequence where the differences $\xi_{\nu+1}-\xi_\nu$ are small. Here $\xi_0=-A$ and $\xi_N=A$ for some $A>0$. Define the function

(i)
$$G_N(x,y)=\sum f(\xi_\nu)\cdot H_{\xi_\nu,\xi_\nu+1}(x,y)$$

Since f is continuous the limit formulas for the H-functions above show that

(ii)
$$\lim_{y\to 0} G_N(a,y)\simeq \pi\cdot f(a)$$

for every real a where this approximative equality becomes more and more accurate as the maximum of the differences $\xi_{\nu+1}-\xi_\nu$ tends to zero. At the same time the limit formulas for the H-functions imply that the G_N-fuction is approximated by the *Riemann sum*

$$\sum f(\xi_\nu)\cdot(\xi_{\nu+1}-\xi_\nu)\cdot\frac{y}{(x-\xi_\nu)^2+y^2}$$

Next, by the construction of the Riemann integral of f over $[-A,A]$ we also have

(iii)
$$G_N(x,y)\simeq\int_{-A}^{A}\frac{y}{(x-\xi)^2+y^2}f(\xi)d\xi$$

Passing to a limit where we take refined ξ-partitions, the sequence $\{G_N(x,y)$ converges to a limit function $G(x,y)$ which is harmonic in the upper half-plane and

$$\lim_{y\to 0} G(a,y)=\pi\cdot f(a) \quad : \quad -A<a<A$$

If a is outside the interval $[-A,A]$ we see that $\lim_{y\to 0} G(a,y)=0$. Hence we have arrived at Poisson's solution:

4.21 Theorem *Let $f(\xi)$ be a continuous function on the real x-axis which is zero outside a bounded interval. Then the function*

$$G(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2+y^2}\dot{f}(\xi)d\xi$$

is harmonic in the upper half plane and $G(\xi,0) = f(\xi)$ holds on the boundary.

5. Exact versus closed 1-forms

In this section a domain Ω in $\mathcal{D}(C^1)$ is given. It is connected and $\partial\Omega$ has p many boundary curves for some $p \geq 1$. The functions and other objects below are defined in Ω and when integrals are taken over the boundary it is assumed that the functions have been extended to the closure of Ω in order that boundary integrals are defined. A differential 1-form is given by:

$$W = f(x,y) \cdot dx + g(x,y) \cdot dy \tag{1}$$

Here f and g are supposed to be of class C^1 at least. The 1-form is closed if

$$f'_y = g'_x \tag{2}$$

Suppose there exists a C^2-function $U(x,y)$ such that

$$U'_x = f \quad \text{and} \quad U'_y = g \tag{3}$$

Since the mixed second order derivatives U''_{xy} and U''_{yx} are equal we see that (3) gives (2). When U exists we say that the 1-form W is exact and U is called the potential function of W.

Remark. In mechanics one refers to a 1-form W as a field of forces, i.e. to every point one assigns the force vector $F = (f,g)$. If (3) holds we have the equality $\nabla(U) = F$ and one says that F is a potential field where U is its potential function. Notice that U is determined up to a constant.

Let Γ be C^1-curve with end-points $A = (x_0,y_0)$ and $B = (x_1,y_1)$. For every 1-form W we get the line integral

$$\int_\Gamma W = \int_0^T \big[f(x(t),y(t)) \cdot \dot{x} + g(x(t),y(t)) \cdot \dot{y}\big] \cdot dt \tag{i}$$

If W is exact with a potential function U we notice that

$$\frac{d}{dt}(U(x(t),y(t)) = f(x(t),y(t)) \cdot \dot{x} + g(x(t),y(t)) \cdot \dot{y}$$

Hence (i) is equal to $U(B)-U(A)$. In other words, when W is exact then the line integral along a curve Γ only depends upon the two end-points and it is expressed by the difference $U(B)-U(A)$.

5.1 Exercise. Let W be a closed 1-form and assume that the line integral along every curve Γ only depends on the end-points. Show that W is exact.

5.2 Non-exact 1-forms. The standard example of a closed but non-exact 1-form occurs when Ω is an annulus $r^2 < x^2+y^2 < R^2$ and we take

$$W = \frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y2}$$

5.3 Starshaped domains. We say that Ω is star-shaped with respect to the origin if the line from 0 to every point $p \in \Omega$ is contained in Ω. In this case every closed 1-form is exact. To see this we define the function $U(x,y)$ in Ω by

$$U(x,y) = \int_0^1 \big[x \cdot f(tx,ty) + y \cdot g(tx,ty)\big] \cdot dt$$

Now we get

$$U'_x = \int_0^1 f(tx,ty) + \int_0^1 \big[tx \cdot f'_x(tx,ty) + ty \cdot g'_x(tx,ty)\big] \cdot dt \tag{1}$$

Next, $g'_x = f'_y$ is assumed and we notice that

$$\frac{d}{dt}(f(tx,ty)) = x \cdot f'_x(tx,ty) + y \cdot f'_y(tx,ty)$$

It follows that the last integral in (1) becomes

$$\int_0^1 t \cdot \frac{d}{dt}(f(tx,ty)) \cdot dt = f(x,y) - \int_0^1 f(tx,ty)) \cdot dt$$

The last integral is cancelled via (1) and hence $U'_x = f$. In the same way one shows that $U'_y = g$ and hence the 1-form is exact.

5.4 The deformation theorem. Let A and B be two given points in Ω. Consider a family of curves $\{\Gamma_s\}$ where each Γ_s has A and B as endpoints and we are given a vector valued function

(*) $$\rho\colon\ (s,t) \mapsto (x(s,t), y(s,t) \quad : 0 \leq s \leq 1 \quad \text{and} \quad 0 \leq t \leq T$$

Here $t \mapsto (x(s,t), y(s,t))$ is the parametrization of Γ_s. With these notations we have

5.5 Theorem. *For every closed 1-form W the function below is a constant*

(1) $$s \mapsto \int_{\Gamma_s} W$$

Remark. We have essentially proved this in the introduction, i.e. see Theorem A.X. But we give another proof below which involves less calculations and at the same time illustrates the efficiency when the calculus of differential forms is used.

Proof. To begin with, we can restrict s to some interval $[0, s_*]$ for every $0 < s_* \leq 1$ and therefore it suffices to prove that (1) takes the same value of Γ_1 and Γ_0. To achieve this we first consider an arbitrary C^1-function f and the 1-form $f \cdot dx$. We shall now find an expression of the difference

(3) $$\int_{\Gamma_1} f \cdot dx - \int_{\Gamma_0} f \cdot dx$$

For this purpose we consider the ρ-map in (*) and construct the inverse function

$$f^*(s,t) = f(x(s,t), y(s,t))$$

We have also the inverse 1-form

$$\rho^*(dx) = x'_s \cdot ds + x'_t \cdot dt$$

Put $\square = \{(s,t) \quad : \{0 \leq s \leq 1\} \cap \{0 \leq t \leq T\}\}$. Since the end-points of the curves $\{\Gamma_s\}$ are equal it follows that (3) is equal to the boundary integral

(4) $$\int_{\partial\square} f^* \cdot \rho^*(dx)$$

Stokes theorem applied to $\square$ implies that (4) is equal to

(5) $$\iint_\square df^* \cdot \rho^*(dx) = \iint_\square \rho^*(f \cdot dx) = \iint_\square \rho^*(f'_y \cdot dy \wedge \cdot dx)$$

Let us remark that the last integral becomes

(6) $$\iint_\square (f'_y)^* \cdot (y'_s \cdot x'_t - y'_t \cdot x'_s) \cdot ds \wedge dt$$

If we instead start with a 1-form $g \cdot dy$ then the same calculation gives

(7) $$\int_{\Gamma_1} g \cdot dy - \int_{\Gamma_0} g \cdot dy = \iint_\square \rho^*(g'_x \cdot dx \wedge \cdot dy)$$

Finally, using the equality $dy \wedge dx = dx \wedge dy$ we obtain the following equality which is valid for an *arbitrary* 1-form $W = f \cdot dx + g \cdot dy$ which need not be closed:

$$(**) \qquad \int_{\Gamma_1} W - \int_{\Gamma_0} W = \iint_{\square} \rho^*((g'_x - f'_y) \cdot dx \wedge \cdot dy)$$

In the special case when $f'_y = g'_x$ the last integral is zero and hence $\int_{\Gamma_1} W = \int_{\Gamma_0} W$ holds, i.e. Theorem 5.3 is a special case of (**).

5.6 When is a 1-form exact. If $p = 1$, i.e. if Ω is a Jordan domain then topology learns that every closed curve in Ω can be deformed to a point. Hence Theorem 5.3 implies that if W is a closed 1-form then its line integral over every closed curve is zero. It follows from Exercise 5.1 that W is exact. If $p > 1$ topology learns that there exist $p - 1$ many closed curves $\Gamma_1, \ldots, \Gamma_{p-1}$ which give a basis for the homology of the multiple connected domain. If W is a closed 1-form we assign the period numbers

$$(*) \qquad \int_{\Gamma_k} W \quad : \; 1 \leq k \leq p - 1$$

Now W is exact if and only if all the period numbers are zero. Suppose $W_1, \ldots, W_{p-1}$ is some $(p-1)$-tuple of closed 1-forms such that the $(p-1) \times (p-1)$-matrix with elements

$$a_{ik} = \int_{\Gamma_k} W_i$$

is invertible. Then, for every closed 1-form W we can find $c_1, \ldots, c_{p-1}$ such that the periods of $W - (c_1 W_1 + \ldots + c_{p-1} W_{p-1})$ vanish and hence there exists a potential function U such that

$$W = c_1 W_1 + \ldots + c_{p-1} W_{p-1} + dU$$

Remark. In Chapter 5 we shall find special $(p-1)$-tuples of closed 1-forms for which the period matrix is invertible.

6. An integral formula for the Laplace operator.

The formula in Theorem 4.9 will be applied to a special situation. Consider a C^2-function $f(x,y)$ defined in the open disc $D(R)$ of radius R centered at the origin. Let $0<r<R$ and choose also a small $\epsilon>0$ and consider the annulus $\Omega=\{\epsilon^2<x^2+y^2<r^2\}$. In Ω we have the function

$$g_r(x,y)=\text{Log}(\frac{r}{\sqrt{x^2+y^2}}) \tag{1}$$

We have already seen that g is harmonic in Ω. The boundary $\partial\Omega$ consists of the circle $x^2+y^2=\epsilon^2$ and the circe of radius r and we leave it to the reader to verify that Theorem 4.9 applied to f and g give the formula

$$\iint_\Omega \Delta(f)(x,y)\cdot g_r(x,y)\cdot dxdy=\int_0^{2\pi} f(r,\theta)\cdot d\theta-\int_0^{2\pi} f(\epsilon,\theta)\cdot d\theta \tag{2}$$

Keeping r fixed while $\epsilon\to 0$ the continuity of f at the origin gives:

$$f(0)=\frac{1}{2\pi}\cdot\int_0^{2\pi} f(r,\theta)\cdot d\theta-\frac{1}{2\pi}\cdot\iint_{D(r)} \Delta(f)(x,y)\cdot g(x,y)\cdot dxdy \tag{3}$$

With $0<r_*<R$ we can apply (3) to every $0<r<r^*$ and after an integration using polar coordinates we obtain:

$$f(0)=\frac{1}{\pi\cdot r_*^2}\cdot\iint_{D(r_*)} f(x,y)\cdot dxdy-\frac{1}{2\pi}\cdot\int_0^{r_*}\big[\iint_{D(r)} \Delta(f)(x,y)\cdot g_r(x,y)\cdot dxdy\big]\cdot dr$$

The last double integral is evaluated via polar coordinates and the result is the formula from the introduction, i.e. the reader may verify that the double integral becomes

$$\int_0^{r_*} K(r^*,s)\cdot\big[\int_0^{2\pi}\Delta(f)(s,\theta)\cdot d\theta\big]\cdot ds \tag{i}$$

where we for each pair $0<s\leq r^*$ have:

$$K(r^*,s)=s^3\cdot\int_1^{\frac{r}{s}} u\cdot\text{Log}(u)\cdot du \tag{ii}$$

Chapter III. Complex analytic functions

Contents

Introduction. Expressing a complex number by $z = x + iy$ we identify $\mathbf{C}$ with $\mathbf{R}^2$ and apply results from Chapter II to study complex-valued functions defined on $\mathbf{C}$. In Section 1 we define *complex line integrals.* Next, consider a complex valued function $f(z) = u(x, y) + iv(x, y)$ where $\mathfrak{Re}(f) = u$ is the real part and $\mathfrak{Im}(f) = v$ the imaginary part of f. Let f be defined in a domain $\Omega \in \mathcal{D}(C^1)$ which extends continuously to the closure $\bar{\Omega}$. We also assume that u and v are C^1-functions in Ω, i.e. the four partial derivatives u_x, u_y, v_x, v_y exist as continuous functions in Ω.

Under this hypothesis Stokes formula gives a condition which ensures that the complex line integral of f over the boundary of $\Omega \in \mathcal{D}(C^1)$ vanishes. Namely, the complex line integral

$$(*) \qquad \int_{\partial\Omega} f(z) \cdot dz = \int_{\partial\Omega} udx - vdy + i \cdot \int_{\partial\Omega} udy + vdx$$

Now Theorem 2.4 in Chapter II shows that both the real and the imaginary parts above are zero if the following two differential equations hold in the open set Ω:

$$(**) \qquad u_x = v_y \quad : \quad u_y = -v_x$$

We refer to (0.1) as the *Cauchy-Riemann equations* and (u, v) is called a CR-pair in Ω when (0.1) holds. Using (*) we can establish *Cauchy's formula*:

$$(***) \qquad f(z_0) = \frac{1}{2\pi i} \cdot \int_{\partial\Omega} \frac{f(z)dz}{z - z_0} \quad \text{for all } z_0 \in \Omega\,.$$

This is done in XX and applies in particular to the case when Ω is an open disc and will be used in Section 6 to show that when (u, v) is a CR-pair in some open set Ω, then they are not only of class C^1 but have continuous derivatives of any order, i.e. both u and v are C^∞-functions.

A notation. Let Ω be an arbitrary open set in $\mathbf{C}$. The family of C^1-functions $f = u + iv$ for which (u, v) is a CR-pair in Ω is denoted by $\mathcal{O}(\Omega)$. We refer to this as the class of analytic functions in Ω and remark that one sometimes refers to the class of *holomorphic functions*, i.e. the notion of complex analytic functions and holomorphic functions are the same.

0.1 Complex derivatives. Consider a C^1-function $f(z) = u(x,y) + iv(x,y)$ defined in some open set Ω. If $z_0 \in \Omega$ there exist complex difference quotients

$$(*) \qquad \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad : \quad z_0 \in \Omega\,.$$

We say that f has a *complex derivative* at z_0 if these complex difference quotients have a limit as $\Delta z = \Delta x + i\Delta y \to 0$. During this passage to the limit Δx and Δy are not constrained, i.e. it is only required that both Δ-numbers tend to zero. For example, we can allow that Δx tends much faster to zero than Δy, or vice versa. Another case is that $\Delta x = \Delta y$ during the limit. In Section 4 we show that the existence of a complex derivative at a point $z_0 = x_0 + iy_0$ is *equivalent* to the condition that $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$ hold. The conclusion is that a C^1-function $f(z)$ in Ω has a complex derivative everywhere if and only if (u,v) is a CR-pair.

In Section 5 we relax the regularity hypothesis that f from the start is of class C^1. More precisely, assume only that $f(z)$ is continuous in Ω and that the pointwise defined complex derivative exists for every $z_0 \in \Omega$. Then *Goursat's theorem* shows that f is automatically of class C^1 and hence (u,v) is a CR-pair and they even become C^∞-functions in Ω.

0.2 Schwarz' reflection. Let $f(z)$ be analytic in an open rectangle

$$\square_+ = \{(x,y) \quad : 0 < y < b \quad \text{and} \quad -A < x < A\}$$

A reflection in the real x-axis gives the rectangle

$$\square_- = \{(x,y) \quad : \quad -b < y < 0 \quad \text{and} \quad -A < x < A\}$$

In $\square_-$ we define the complex valued function

$$g(z) = \bar{f}(\bar{z}) = u(x,-y) - iv(x,-y)$$

One verifies easily that $g(z)$ becomes analytic in $\square_-$. Suppose now that $f(z)$ extends continuously to the x-axis, i.e.

$$\lim_{\epsilon \to 0} f(x + i\epsilon) = f_*(x)$$

exists uniformly with respect to x. If $f_*(x)$ is *real-valued* then f and g attain the same boundary values on the real x-interval. In XXX we prove that under this assumption the pair f, g not only match each other as continuous functions on the x-interval but there exists an analytic function $F(z)$ defined in the whole open square

$$\square = \{(x,y) \quad : \quad -b < y < b \quad \text{and} \quad -A < x < A\}$$

such that $F = f$ in $\square_+$ and $F = g$ in $\square_-$. This result is due to Hermann Schwarz who applied the reflection principle to establish many other results, such as the existence of a conformal map from the unit disc to a domain bordered by a piecewise linear Jordan curve.

0.3 An inequality by Schwarz. Let D be the unit disc $|z| < 1$ and f is a bounded analytic function in D with maximum norm $|f|_D = 1$. Consider a bounded and connected subset Ω of the upper half-disc D_+ whose boundary intersects the real axis in some interval (a,b). Now $\partial\Omega$ is the union of $[a,b]$ and the remaining portion Γ given by $\partial\Omega \cap D_+$. So here the two end-points a and b belong to the closure of Γ. With these notations we have:

$$(*) \qquad \max_{a \le x \le b} |f(x)| \le \sqrt{|f|_\Gamma}$$

To prove this Schwarz introduced the domain $\Omega_* = \{z \,:\, \bar{z} \in \Omega\}$. Now

$$F(z) = f(z) \cdot \bar{f}(\bar{z})$$

is analytic in $\Omega \cup \Omega_*$ and its boundary is $\Gamma \cup \Gamma_*$. Since $|f|_D = 1$ the maximum norm of F over $\Gamma \cup \Gamma_*$ is $\le |f|_\Gamma$. At the same time $F(x) = |f(x)|^2$ holds when $a \le x \le b$ and (*) follows from the maximum principle for analytic functions applied to F and $\Omega \cup \Omega_*$.

0.4 A more general reflection. Let $\square_+$ and $\square_-$ be as in (0.2). Consider a pair $f \in \mathcal{O}(\square_+)$ and $g \in \mathcal{O}(\square_-)$ where we assume that $f(x+iy)$ extends to a continuous function on the half-open rectangle $\{-A \leq x \leq A\} \times \{0 < y < b\}$ and similarly for g. We ask for conditions in order that they are analytic continuations of each other across the real x-axis. A sufficient condition for goes as follows

0.5 Theorem. *Assume that*

$$\lim_{\epsilon \to 0} \int_{-A}^{A} \big|f(x+i\epsilon) - g(x-i\epsilon)\big| \cdot dx = 0$$

Then there exists $F \in \mathcal{O}(\square)$ which gives an analytic extension of the pair (f, g).

The result above is due to Carleman in [Car]. In contrast to the other results in this section the proof relies upon properties of subharmonic functions and the proof appears in the Appendix about Distributions. Notice that we have not imposed growth conditions on f an g from the start.

0.6 The $\bar{\partial}$-operator. The Cauchy-Riemann equations can be put into a single first order differential equation. Namely, introduce the differential operator

$$\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

If $f = u + iv$ is a complex-valued function it is easily seen that (u, v) is a CR-pair if and only if f satisfies the homogenous $\bar{\partial}$-equation:

$$\bar{\partial}(f) = \frac{1}{2}[f_x + if_y) = 0$$

In Section 8 we establish the *Pompieu formula* which shows how to solve an *inhomogeneous* $\bar{\partial}$-equation.

0.7 Normal families. In Section 6 the Cauchy formula is used to show that an analytic function f defined in an open disc $D_R(z_0) = \{|z - z_0| < R\}$ is represented by a power series

$$(*) \qquad \sum c_n(z - z_0)^n$$

whose radius of convergence is $\geq R$. In this way $\mathcal{O}(D_R(z_0))$ is identified with the set of power series whose radius of convergence is $\geq R$. Moreover, the coefficients $\{c_n\}$ are determined by the formula

$$c_k = \frac{1}{k!} \cdot f^{(k)}(z_0) \quad : \ k = 0, 1, 2, \ldots$$

Using power series we establish results concerned with the topology on $\mathcal{O}(\Omega)$ where the original results are due to Montel. Of special importance is the following result:

Let $\{f_\nu\}$ be a sequence of analytic functions in a domain Ω whose maximum norms are uniformly bounded, i.e. there is a constant M such that $|f_\nu(z)| \leq M$ hold for all $z \in \Omega$ and every ν. Then the sequence contains at least one subsequence $\{g_k = f_{\nu_k}\}$ such that the g-sequence converges uniformly to an analytic function g_* in every relatively compact subset of Ω. Moreover, if the zero set of every g_k is empty then g_* has no zeros unless it is identically zero.

This result will be used frequently later on. For example in the proof of *Riemann's Mapping Theorem.*

0.8 Laurent series. In section 10 we study analytic functions defined in domains of the form $\{r < |z| < R\}$. Here the boundary consists of two circles, the inner circle $|z| = r$ and the outer circle $|z| = R$. Let $f(z)$ be analytic in such an annulus which extends to a continuous function on the boundary. Then we can apply Cauchy's formula from (**) above find a series representation of f where one part of the series is an expansion with *negative* powers of z.

0.9 Conformal properties Let $f(z)$ be an analytic function with non-zero complex derivative defined in some domain Ω. Consider f as a map from the complex z-plane into another complex plane and put

$$\zeta = f(z)$$

where $\zeta = \xi + i\eta$. Consider some point $z_0 \in \Omega$ and with $f = u + iv$ we have $\xi = u(x, y)$ and $\eta = v(x, y)$. The complex-valued function f can be identified with a vector-valued function from the real (x, y)-space to the real (ξ, η)-space whose *Jacobian* by definition is the 2×2-matrix

$$J = \binom{u_x \quad v_x}{u_y \quad v_y}$$

The Cauchy-Riemann equations imply that the two column vectors are orthogonal, i.e. $u_x v_x + u_y v_y = 0$ holds at every point in Ω. We have also the determinant formula:

$$(*) \qquad \det(J) = u_x v_y - u_y v_x = u_x^2 + u_y^2 = |f'(z)|^2$$

By a wellknown result in Calculus this implies that the vector valued map is infinitesmally a rotation times a dilation with the factor $|f'(z)|^2$ at every point $z \in \Omega$. This implies that the map is locally conformal. Namely, let $z_0 \in \Omega$ and consider a pair of C^1-curves γ_1, γ_2 which pass z_0 and let α be the angle between them. Then the angle between the image curves $f(\gamma_1)$ and $f(\gamma_2)$ is also equal to α. This geometric property is expressed by saying that f yields a conformal map since infinitesmal angles are preserved.

Remark on quasi-conformal mappings. We shall not discuss the theory about quasi-conformal mappings where many "magical phenomena" occur. See the text-books [Ahl] by Ahlfors and [L-V] by Lehto and Virtanen for the quasi-conformal theory. Let us only recall that a differentiable complex-valued function $f = u + iv$ which locally is an orientation preserving homeomorphism is quasi-conformal of order $\leq K$ for some number $K \geq 1$ if the first order derivative of f satisfy:

$$\left|\bar{\partial}(f)\right| \leq \frac{K-1}{K+1} \cdot \left|\partial(f)\right|$$

When $K = 1$ this means that $\bar{\partial}(f) = 0$, i.e. f is complex analytic. Consider as an example the linear function $f(z) = z + \frac{\bar{z}}{2}$. Here we require that $2(K-1) \geq K+1$, i.e. we can take $K = 3$. Notice that the linear map:

$$(x, y) \mapsto (3x/2, y/2)$$

sends small circles centered at the origin to small ellipses. So the geometry is more involved when quasi-confomal mappings are studied.

0.10 An area formula. Let Ω be a domain in $\mathcal{D}(C^1)$. So $\partial\Omega$ consists of simple and closed boundary curves $\gamma_1, \ldots, \gamma_p$. Let $f(z) \in \mathcal{O}(\Omega)$ and suppose it extends to a continuous function on $\bar{\Omega}$. Moreover, we assume that f is bijective on $\bar{\Omega}$ and that the image domain $f(\Omega)$ also belongs to $\mathcal{D}(C^1)$. This image domain is then bordered by a p-tuple of disjoint boundary curves $f(\gamma_1), \ldots, f(\gamma_p)$. We write as usual $f = u + iv$. To each $1 \leq k \leq p$ we consider a parametrisation by arc-length along γ_k and evaluate the line integral

$$(*) \qquad J(k) = \int_0^{L(\gamma_k)} u(z_k(s)) \cdot \frac{dv(z_k(s))}{ds} \cdot ds$$

where $\gamma_k \colon s \mapsto z_k(s)$ and $L(\gamma_k)$ is the arc-length of γ_k. With these notations Stokes Theorem from Chapter XX gives the following area formula:

0.11 Theorem *The area of $f(\Omega)$ is equal to $J(1) + \ldots + J(p)$.*

Remark. Since the absolute value of $f'(z)$ changes the area measure we have also the equality:

$$(**) \qquad \text{Area}[f(\Omega)] = \iint_\Omega |f'(z)|^2 \cdot dxdy$$

In Section XX we establish a third area formula using complex line integrals along the boundary curves of Ω:

$$(**) \qquad \text{Area}[f(\Omega)] = \int_{\partial\Omega} \bar{f}(z) \cdot f'(z) \cdot dz$$

The fact that one disposes three area formulas is quite useful.

0.12 A local limit formula.

If $f(z)$ is analytic in a bounded domain it suffices to know its values on a small portion of the boundary, i.e. analyticity gives a quite strong uniqueness principle. Following the introduction in Carleman's book [Quasianalytic] we shall illuminate this in a special case where formulas are quite explicit. Here is the set-up. Let $0 < \alpha < 1/2$ be a real number and consider the two rays given by the non-negativel real axis ℓ_* and the ray $\ell^* = \{re^{i\alpha} : r \geq 0\}$. Let Γ be a Jordan arc with end-points $A \in \ell_*$ and $B \in \ell^*$ while the remaining part of Γ is in the interior of the open sector bordered by the two rays. So here A is a positive real number and $B = b \cdot e^{i\alpha}$ for some $b > 0$. Let Ω be the Jordan domain bordered by Γ and the straight lines $0A$ and $0B$ which intersect at the origin. Consider a point $\zeta = r \cdot e^{i\alpha/2}$ which belongs to Ω. The Jordan arc Γ is assumed to be rectifiable so that complex line integrals along Γ are defined. Suppose now that f is analytic in Ω and extends to a continuous function on the closure $\bar{\Omega}$. With these notations one has

0.13 Theorem. *The value of f at ζ is obtained by the limit formula*

$$f(\zeta) = \lim_{\sigma \to \infty} \frac{e^{-\sigma}}{2\pi i} \cdot \int_\Gamma \frac{f(z) \cdot e^{\sigma \left(\frac{z}{\zeta}\right)^{\frac{1}{\alpha}}}}{z - \zeta} \cdot dz \tag{*}$$

To prove (*) we consider for a given positive real number σ the function

$$F_\sigma(z) = f(z) \cdot e^{\sigma \left(\frac{z}{\zeta}\right)^{\frac{1}{\alpha}}} \tag{1}$$

When $0 \leq s \leq A$ we have

$$\left(\frac{s}{\zeta}\right)^{\frac{1}{\alpha}} = \left(\frac{s}{r}\right)^{\frac{1}{\alpha}} \cdot e^{-\pi i/2} = -i \cdot \left(\frac{s}{r}\right)^{\frac{1}{\alpha}}$$

Since exponentials of purely imaginary numbers have absolute value we get

$$|F(s)| = |f(s)|$$

On the ray OB we find a similar formula. Hence $F(z)$ is bounded on OA and OB. We also notice that

$$F(\zeta) = e^\sigma \cdot f(\zeta)$$

Cauchy's integral formula applied to F gives

$$f(\zeta) = e^{-\sigma} \cdot F(\zeta) = e^{-\sigma} \cdot \frac{1}{2\pi i} \cdot \int_{\partial\Omega} \frac{F(z)}{z - \zeta} \cdot dz$$

The last line integral is the sum over Γ and the two line integrals along $0A$ and $0B$. Since F is bounded on the line segments and $e^{-\sigma} \to 0$ as $\sigma \to +\infty$ we conclude that

$$f(z) = \lim_{\sigma \to +\infty} \frac{e^{-\sigma}}{2\pi i} \cdot \int_\Gamma \frac{F(z)}{z - \zeta} \cdot dz$$

By the construction of F this is precisely the limit formula (*).

Remark. If f from the start is defined in some domain U which is starshaped with respect to the origin then we pick $0 \neq \zeta \in U$ and after a rotation we may assume that ζ is real and positive. With a very small α we can consider the rays from the origin where $\arg(z)$ is α or $-\alpha$ and by a picture the reader can see that $f(\zeta)$ via Theorem 0.13 is expressed by a limit where the integral is taken over a small portion of ∂U.

1. Complex Line Integrals

Consider a complex valued C^1-function of a real t-variable:

$$t \mapsto z(t) = x(t) + iy(t) \quad : \quad 0 \leq t \leq T$$

The C^1-condition means that both $x(t)$ and $y(t)$ are continuously differentiable functions of t. The t-derivative becomes:

$$\dot{z}(t) = \dot{x}(t) + i \cdot \dot{y}(t)$$

If $f = u + iv$ is a complex valued continuous function we get the line integral

$$\int_0^T f(z(t))\dot{z}(t)dt = \int_0^T [u(x(t), y(t)) + iv(x(t), y(t))](\dot{x}(t) + i \cdot \dot{y}(t)) \cdot dt$$

or expressed in a more abbreviated form:

$$\int_0^T f(z)\dot{z} \cdot dt = \int_0^T [u(x, y) + iv(x, y)](\dot{x} + i\dot{y}) \cdot dt \tag{0.1}$$

The right hand side is a sum of line integrals with respect to x and y respectively. By the general result in XX, (0.1) does not depend on the chosen parametrization of the oriented image Γ. So we can therefore write (0.1) as

$$\int_\Gamma f dz = \int_\Gamma u + iv)(dx + idy) \tag{0.2}$$

When the right hand side is decomposed into its real and imaginary parts we get:

$$\int_\Gamma f dz = \int_\Gamma udx - vdy + i \cdot \int_\Gamma udy + ivdx \tag{0.3}$$

We refer to (0.2) as the complex line integral along Γ. Recall that the *choice of orientation* is essential, i.e. if the orientation on Γ is opposite the line integral changes sign.

1.1 Complex line integrals as Riemann sums. Consider as above a curve Γ with end points a and b. Following the orientation we choose a finite sequence of points $a = z_0, z_1, \ldots, z_N = b$ where each $z_\nu \in \Gamma$ and take the Riemann sum

$$\sum_{\nu=0}^{N-1} f(z_\nu)(z_{\nu+1} - z_\nu) \tag{i}$$

When $\max |z_{\nu+1} - z_\nu| \to 0$ these sums converge to the line integral of f along Γ. To see this we use the hypothesis that Γ has a C^1-parametrisation, say $t \mapsto z(t)$. Now $z_\nu = z(t_\nu)$ with $0 = t_0 < t_1 < \ldots < t_N = T$. Since f is continuous the function $t \mapsto f(z(t))$ is continuous. From the previous definition of the line integral we have

$$\int_\Gamma f \cdot dz = \int_0^T f(z(t)) \cdot \dot{z}(t)dt \tag{ii}$$

Here (ii) is approximated just as in Calculus by a Riemann sum:

$$\sum f(z(t_\nu)) \cdot \dot{z}(t_\nu) \cdot (t_{\nu+1} - t_\nu) \tag{iii}$$

The continuity of the t-derivative $t \mapsto \dot{z}(t)$ give accurate approximations

$$\dot{z}(t_\nu) \cdot (t_{\nu+1} - t_\nu) \simeq z_{\nu+1} - z_\nu$$

for every ν. Hence the Riemann sums in (iii) converge to (ii) when

$$\max\{(t_{\nu+1} - t_\nu)_{\nu=1}^N\} \quad \text{tends to zero}$$

1.2 Integration on rectifiable curves. The approximative sums in (1) appear in the construction of Riemann-Stieltjes integrals. Since the complex integral is decomposed into a real and an imaginary part we can therefore use the result from XX where we proved the existence of integrals of the Riemann-Stieltjes type. More precisely, if Γ has a parametrisation $t \mapsto z(t) = x(t) + iy(t)$

where both $x(t)$ and $y(t)$ are continuous functions with bounded variation, then we can define Stieltjes' line integral

$$\int_\Gamma f(z)\cdot dz$$

where we only have to assume that $f(z)$ is a bounded Borel function. See Measure Appendix for details about this construction of general line integrals.

1.3 The case when Γ is a circle. Let $R>0$ and Γ is the circle $|z|=R$ equipped with its usual positive orientation. Since $z\neq 0$ on Γ we can divide a function f with z and get the line integral

(i)
$$\int_\Gamma \frac{f(z)dz}{z}$$

Using the parametrisation $\theta\mapsto Re^{i\theta}$ this line integral becomes

(ii)
$$\int_0^{2\pi}\frac{f(Re^{i\theta})\cdot iRe^{i\theta}d\theta}{Re^{i\theta}}=i\cdot\int_0^{2\pi} f(Re^{i\theta})d\theta$$

This formula plays a crucial role later on when we derive Cauchy's formula and develop residue calculus. Under the sole assumption that f is a continuous function defined in some open disc centered at the origin, we use the equality above when $R=\epsilon$ and $\epsilon\to 0$. Namely, since f is continuous at the origin we get the limit formula:

(iii)
$$f(0)=\frac{1}{2\pi}\cdot\lim_{\epsilon\to 0}\int_0^{2\pi} f(\epsilon e^{i\theta})d\theta=\frac{1}{2\pi i}\cdot\lim_{\epsilon\to 0}\int_{|z|=\epsilon}\frac{f(z)dz}{z}$$

The origin can be replaced by any other point z_0. Since the limit formula above is so important for the subsequent residue calculus we state it separately:

1.4 Theorem *Let $f(z)$ be a continuous function in some open set Ω. Then*

$$f(z_0)=\frac{1}{2\pi i}\cdot\lim_{\epsilon\to 0}\int_{|z-z_0|=\epsilon}\frac{f(z)dz}{z-z_0}\quad : \quad z_0\in\Omega$$

2. The Cauchy-Riemann equations

Consider a complex-valued function $f(z)=f(x+iy)=u(x,y)+iv(x,y)$. We assume that f is of class C^1 and let $\Omega\in\mathcal{D}(C^1)$. From section 1 we have

$$\int_{\partial\Omega} fdz=\int_{\partial\Omega} udx-vdy+i\int_{\partial\Omega} udy+vdx$$

Apply Theorem 2.4 in Chapter I to each term in the right hand side. This gives:

(i)
$$\int_{\partial\Omega} fdz=\iint_\Omega(-(u_y+v_x)\cdot dxdy+i\cdot\iint_\Omega(u_x-v_y)\cdot dxdy$$

The right hand side is zero if the real and the imaginary part vanish which obviously follows if the following equations hold in the whole of Ω:

(*)
$$u_x(x,y)=v_y(x,y)\quad : \quad u_y(x,y)=-v_x(x,y)\quad : \quad (x,y)\in\Omega$$

Hence we have proved the following

2.1 Theorem *Let $f=u+iv$ be a complex-valued C^1- function such that the pair (u,v) satisfies (*). Then*

$$\int_{\partial\Omega} fdz=0\quad : \quad \Omega\in\mathcal{D}(C^1)$$

Theorem 2.1 suggests the following

2.2 Definition *A pair of real-valued C^1-functions (u,v) satisfying $u_x = v_y$ and $u_y = -v_x$ is called a Cauchy-Riemann pair and in this case $f = u + iv$ is called an analytic function.*

3. The $\bar{\partial}$-operator

The Cauchy-Riemann equations can be described by a single first order differential operator. Namely, set

(i) $$\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

If $f = u + iv$ is a C^1-function we get:

$$\bar{\partial}(f) = \frac{1}{2}[\partial_x(f) + i\partial_y(f)] = \frac{1}{2}[u_x + iv_x + iu_y - v_y] = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$

We conclude that $\bar{\partial}(f) = 0$ if and only if (u,v) is a CR-pair. One refers to $\bar{\partial}$ as the *Cauchy-Riemann operator* since it determines when (u,v) becomes a CR-pair.

3.1 Example Let $f = \bar{z} = x - iy$ be the conjugate function. Here $\bar{\partial}(\bar{z}) = 1$ and hence the pair $u = x$ and $v = -y$ is not CR. On the other hand, let $m \geq 1$ and consider $f(z) = z^m$. Since $\bar{\partial}$ is a first order differential operator, Leibniz' s rule from Calculus gives

$$\bar{\partial}(z^m) = mz^{m-1}\bar{\partial}(z)$$

Next, we have

$$\bar{\partial}(x + iy) = \frac{1}{2}[1 + i^2] = \frac{1}{2}[1 - 1] = 0$$

So with $z^m = u + iv$ it follows that (u,v) is a CR-pair. Take as an example the case $m = 3$ where $u = x^3 - 3x \cdot y^2$ and $v = 3x^2 \cdot y - y^3$.

3.2 Remark If f, g is a a pair of C^1-functions then Leibniz's rule for the first order differential operator gives

$$\bar{\partial}(fg) = f\bar{\partial}(g) + g\bar{\partial}(f)$$

It follows that if both f and g are analytic so is fg. Hence the class of analytic functions is stable under products. Next, let $f_1, \ldots, f_m$ a finite set of analytic functions. Put

$$\phi = \bar{z} \cdot f_1(z) + \ldots + \bar{z}^m \cdot f_m(z)$$

Then ϕ cannot be analytic unless every f_ν is zero. To see this one proceeds by induction over m. Namely, for each $m \geq 2$ we get the m:th order differential operator $\bar{\partial}^m$ which satisfies

$$\bar{\partial}^m(\bar{z}^\nu) = 0 \quad : \nu < m \quad : \quad \bar{\partial}^m(\bar{z}^m) = m$$

Using this the reader may verify the assertion about the ϕ-function.

3.3 Analytic polynomials. Since $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, every polynomial in the two variables (x,y) can be expressed as a polynomial in z and $\bar{z}$. Let $m \geq 1$ and denote by $H_{\text{Pol}}(m)$ the space of homogeneous polynomial of degree m with complex coefficients. Such a polynomial can be uniquely written in the form

$$P(z,\bar{z}) = \sum_{\nu=0}^{\nu=m} c_\nu \cdot z^{m-\nu}\bar{z}^\nu \quad : \quad c_0, \ldots, c_m \text{ complex constants}$$

The observation in 3.2 shows that P is analytic if and only if the sole term is z^m. Hence $H_{\text{Pol}}(m)$ is 1-dimensional complex vector space generated by the monomial z^m. This shows that the class of analytic functions is quite sparse.

4. The Complex Derivative

Let $f(z) = u(x,y) + iv(x,y)$ be a complex-valued function of class C^1 in a domain Ω. Given a point $z_0 = x_0 + iy_0$ and a small complex number $\Delta z = \Delta x + i\Delta y$ we regard the difference

$$f(z_0 + \Delta z) - f(z_0) = u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)$$

Keeping Δx and Δy fixed for a while we have the function

$$\phi(s) = u(x_0 + s\Delta x, y_0 + s\Delta y) + iv(x_0 + s\Delta x, y_0 + s\Delta y) \quad : \quad 0 \leq s \leq 1$$

Rolle's mean value theorem gives a pair $0 < \theta_1, \theta_2 < 1$ such that the sum

$$\Delta x \cdot u'_x(x_0 + \theta_1\Delta x, y_0 + \theta_1\Delta y) + \Delta y \cdot u'_y(x_0 + \theta_1\Delta x, y_0 + \theta_1\Delta y) +$$

$$i[\Delta x \cdot v'_x(x_0 + \theta_2\Delta x, y_0 + \theta_2\Delta y) + \Delta y \cdot u'_y(x_0 + \theta_2\Delta x, y_0 + \theta_2\Delta y)] \tag{1}$$

is equal to $\phi(1) - \phi(0) = f(z_0 + \Delta z) - f(z_0)$. In this sum the four first order partial derivatives are evaluated at points close to (x_0, y_0) when

$$|\Delta z| = \sqrt{(\Delta x^2 + (\Delta y)^2} \to 0$$

The continuity of the partial derivatives and the imply that the sum from (1) becomes

$$\Delta x \cdot u'_x(x_0, y_0) + \Delta y \cdot u'_y(x_0, y_0) + i\Delta x \cdot v'_x(x_0, y_0) + i\Delta y \cdot v'_y(x_0, y_0) + \text{small ordo}(|\Delta z|) \tag{2}$$

The remainder term $o(|\Delta z|)$ comes from the continuity of first order derivatives. If we *assume* that u, v is a Cauchy-Riemann pair we can replace v'_y with u'_x and u'_y with $-v'_x$. Then (2) becomes

$$(\Delta x + i\Delta y)u'_x + i(\Delta x + i\Delta y)v'_x + \text{small ordo}(\delta) \tag{3}$$

Here $\Delta z = \Delta x + i\Delta y$ appears as a common factor. The small ordo-term gives the limit formula:

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u'_x(x_0, y_0) + +iv'_x(x_0, y_0) \tag{4}$$

So when (u, v) is a CR-pair the complex difference quotients $\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ have a limit as $\Delta z \to 0$. The limit is called the *complex derivative* of f at the point z_0. Put

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{*}$$

A converse. Assume that $f = u + iv$ has a complex derivative at $z_0 = x_0 + iy_0$. We can approach this point in two ways - along the x axis or along the y-axis. With $\Delta z = \Delta x$ we get from the definition of partial derivatives

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_o)$$

If we instead take $\Delta z = i\Delta y$ we have

$$f'(z_0) = \lim_{i\Delta y \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{i}u_y(x_0, y_0) + v_y(x_0, y_o)$$

Identifying the real and the imaginary parts of the two expressions for the complex derivative $f'(z_0)$ we recover the Cauchy-Riemann equations. Hence we have proved the following

4.1 Theorem. *A C^1-function $f = u + iv$ defined in a domain Ω has a complex derivative at every point if and only if (u, v) is a CR-pair.*

4.2 The space $\mathcal{O}(\Omega)$. Let Ω be an open subset of $\mathbf{C}$. The class of analytic functions in Ω is denoted by $\mathcal{O}(\Omega)$. From the result in 2.7 this gives a subalgebra of all complex valued C^1-functions in Ω.

5. Morera's and Goursat's theorems.

Let $f(z)$ be a continuous and complex-valued function defined in an open square $\square = \{-A < x, y < A\}$. To every point $p = (a, b) \in \square$ we get the rectangle Γ with corners at the origin,

$(a,0),(a,b),(0,b)$. Suppose that $\int_\Gamma f(z)dz=0$ for every such rectangle. Define a function $F(z)$ by

$$F(x+iy)=\int_0^x f(t,0)dt+\int_0^y f(a,s)ids$$

It is obvious that $F'_y=if$. Next, the hypothesis on f implies that we also have

$$F(x+iy)=-\int_0^y f(0,s)ids-\int_0^x f(t,y)dt$$

From this we see that $F'_x=-f$. Hence $F'_x=F_y$ and since F also is a C^1-function this implies that $F(z)$ is analytic by the result in 2.5. Next, if we *knew* that F is of class C^2 we can take the mixed second order derivatives and obtain

$$-f'_y=F''_{yx}=F''_{xy}=if'_x$$

This gives $f'_x=if'_y$ and 2.5 proves that f is analytic. in the next section we show that F actually is of class C^2 and hence it follows that f is analytic. Of course, allowing more rectangles we can conclude:

5.1 Theorem. *Let $f(z)$ be continuous in an open set Ω. Assume that $\int_\Gamma f(z)dz=0$ for every rectangle inside Ω with sides parallell to the coordinate axes. Then f is analytic in Ω.*

5.2 Some estimates. We begin with some preliminary results which will be used to prove Theorem 5.3 below. A rectangle Γ gives four smaller rectangles $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ which arise when we join the parallell sides with lines from their opposed mid-points. Now γ_1,γ_2 has one side in common and so on. By drawing a figure and keeping in mind how the orientation is chosen along each small rectangle we see that:

$$\int_\Gamma f(z)dz=\sum_{\nu=1}^{\nu=4} f(z)dz$$

This process can be continued. Suppose now that $f(z)$ is a bounded function which is no essential restriction since we otherwise may replace Ω by a smaller set. if $|f(z)|\leq M$ in Ω the construction of a complex line integral gives

$$|\int_\Gamma f(z)dz\,|\leq \ell(\Gamma)\cdot M \quad : \ell(\Gamma)=\text{sum of lengths of the sides}$$

Suppose now that we have a rectangle Γ where $\int_\Gamma f(z)\neq 0$. Let us put

$$|\int_\Gamma f(z)|=A\cdot|\ell(\Gamma)$$

where A now is >0. Dividing Γ in four smaller portions we see that the equality above gives

$$\max_\nu |\int_{\gamma_\nu} f(z)dz|\geq \frac{1}{4}\cdot|\int_\Gamma f(z)dz|=\frac{A}{4}\cdot\ell(\Gamma)$$

At the same time $\ell(\gamma_\nu)=\frac{1}{2}\cdot\ell(\Gamma)$. Hence we find at least one small γ-rectangle where

$$\int_{\gamma_\nu} f(z)dz|\geq\frac{A}{2}\cdot\ell(\gamma_\nu)$$

Starting from one such γ-rectangle it is decomposed in four pieces and so on. In this way we obtain a nested descreasing sequence of rectangles $\{\gamma_\nu\}:\ \nu=1,2,\ldots\}$ such that

$$\int_{\gamma_\nu} f(z)dz|\geq\frac{A}{2^\nu}\cdot\ell(\gamma_\nu) \quad :\nu\geq 1 \quad : \ell(\gamma_\nu)=2^{-\nu}\ell(\Gamma)$$

At this stage we are prepared to prove:

5.3 Goursat's theorem. *Let $f(z)$ be a continuous function in Ω and assume it has a complex derivative at every point. Then f is analytic.*

Proof. It suffices to work locally and we may take $\Omega = \square$ as above. If f fails to be analytic we find Γ and $A > 0$ where

$$|\int_\Gamma f(z)dz\,| = A \cdot \ell(\Gamma) \cdot M$$

Then there exists a nested sequence $\{\gamma_\nu\}$ and since there sides tend to zero, it follows by Bolzano's theorem that there is a limit point $z_0 = \cap\,\gamma_\nu$. By assumption f has a complex derivative at z_0. It means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon|z - z_0| \quad : \quad |z - z_0| < \delta$$

If ν is sufficiently large then γ_ν is contained in the disc $D_\delta(z_0)$. Next, we notice that

$$\int_{\gamma_\nu} [f(z_0) + (z - z_0)f'(z_0)]dz = 0$$

Next, when $z \in \gamma_\nu$ we notice that $|z - z_0| \leq \ell(\gamma_\nu)/2$. Hence by the vanishing in XX and the inequality XX above, the triangle inequality gives

$$|\int_{\gamma_\nu} f(z)dz\,| \leq \frac{1}{2}\epsilon\,\ell(\gamma_\nu)^2 = \frac{1}{2} \cdot \epsilon \cdot 4^{-\nu}\ell(\Gamma)$$

At the same time, during the construction of the nested sequence we have

$$\int_{\gamma_\nu} f(z)dz| \geq \frac{A}{2^\nu} \cdot \ell(\gamma_\nu) = A \cdot 4^{-\nu} \cdot \ell(\Gamma)$$

Now we get a contradiction since we take ϵ arbitrary small , i.e. it would even be sufficient to take $\epsilon < 2A$ and then the contradiction will follow as soon as ν is so large that $\gamma_\nu \subset D_\delta(z_0)$.

5.4 A result by Carleman. As an alternative to Morreras Theorem we show that analytic functions are characterized by a local mean value condition.

5.5 Theorem *Let f be a continuous function in Ω such that* $\int_{\partial D} f(z)dz = 0$ for ever disc $D \subset \Omega$. *Then $f \in \mathcal{O}(\Omega)$.*

Proof. Let $r > 0$ be small and put

$$\Omega_r = \{z \in \Omega \,:\, \text{dist}(z, \partial\Omega) > r\}$$

To each $z \in \Omega_r$ we define the mean value

$$F_r(z) = \iint_{|\zeta - z| \leq r} f(z)dxdy = \int_0^r \int_0^{2\pi} f(z + se^{i\theta})sdsd\theta$$

Now we prove that $F_r \in (\Omega_r)$. To see this we consider its partial derivatives with respect to x and y. With $z = x + iy$ and Δx small, $F(x + \Delta x + iy)$ is the area integral over a disc centered at $(x + \delta x, y)$. Drawing a figure for computing area integrals the reader should discover that we obtain

$$F_x(z) = \int_0^{2\pi} \cos(\theta)f(z + re^{i|theta}d\theta$$

Similarly we get

$$F_y(z) = \int_0^{2\pi} \sin(\theta)f(z + re^{i\theta}d\theta$$

It follows that

$$F_x + iF_y = \int_0^{2\pi} e^{i\theta}f(z + re^{i\theta}d\theta = 0$$

where the last equality follows from the mean value assumption. Hence F satisfies the $\bar{\partial}$-quation from XX and is therefore analytic. Now this holds for any $r > 0$ and by the continuty of f we have

$$f(z) = \lim_{r \to 0} \frac{1}{\pi r^2} \cdot F_r(z)$$

Hence f is the limit of a sequence of analytic functions and therefore analytic by the result to be proved in XXX.

6. Cauchy's formula

6.1 The local residue Let us repeat the result which led to Theorem 1.2 once more since it plays such a fundamental role. Let $z_0 \in \mathbf{C}$ and let $g(z)$ be a continuous function defined in some open disc of radious r centered at z_0. To each $0 < \epsilon < r$ we set

$$R_\epsilon(g) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{g(z)dz}{z-z_0}$$

Using polar coordinates we get

$$R_\epsilon(g) = \int_0^{2\pi i} \frac{g(z_0 + \epsilon e^{i\theta} id\theta}{e^{i\theta}} = \int_0^{2\pi} \int_0^{2\pi} g(z_0 + \epsilon e^{i\theta} d\theta$$

By continuity of g at z_0 it follows that

(*) $$\lim_{\epsilon \to 0} R_\epsilon(g) = g(z_0)$$

This local limit formula will now be applied below when $g(z)$ is an analytic function.

6.2 Cauchy's formula Let $f(z) \in \mathcal{O}(\bar{\Omega})$ where $\Omega \in \mathcal{D}(C^1)$. Let $z_0 \in \Omega$ and with $\epsilon > 0$ is small we remove the open disc of radius ϵ centered at z_0. Put

$$\Omega_\epsilon = \Omega \setminus \{|z - z_0| \leq \epsilon\}$$

Now $\partial\Omega_\epsilon = \partial\Omega \cup \partial D_\epsilon$ where $D_\epsilon = \{z - z_0| < \epsilon\}$. We get the function

$$g(z) = \frac{f(z)}{z - z_0} \in \mathcal{O}(\bar{\Omega}_\epsilon)$$

Applying Theorem 2.1 we get

$$\int_{\partial\Omega} \frac{f(z)dz}{z - z_0} = \int_{\partial D_\epsilon} \frac{f(z)dz}{z - z_0}$$

Here ϵ can be arbitrarily small. The limit formula in (*) from 4.1 shows that the last term is equal to $2\pi i f(z_0)$. Hence we have proved

6.3 Theorem. *Let $f \in \mathcal{O}(\bar{\Omega})$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int \int_{\partial\Omega} \frac{f(z)dz}{z - z_0} \quad : \quad z_0 \in \Omega$$

6.4 Expressions for derivatives Cauchys formula represents $f(z)$ inside Ω in the same way as in 3.3. Hence we can take derivative of any order, i.e. for each $m \geq 1$ we get

$$f^{(m)}(z) = \frac{m\,!}{2\pi i} \int \int_{\partial\Omega} \frac{f(z)dz}{(z - z_0)^{m+1}} \quad : \quad z_0 \in \Omega$$

This proves in particular that when f is analytic, then it has complex derivatives. Moreover, $f^{(m)}(z)$ yield now analytic functions in Ω. In XXX we study power series representations and obtain even more information about regularity of f as well as of its real and imaginary parts. Notice that the conclusion above applies to the function $F(z)$ from section 5 and hence Theorem 6.3 above finishes the proof of *Morera's Theorem.*

6.5 The case when Ω is a disc Let $\Omega = D_R$ be the disc of radius R centered at the origin and $f(z)$ is an analytic function defined in a neighborhood of the closed disc $\bar{D}_R$. Using the parametrisation $\theta \mapsto Re^{i\theta}$ which gives $dz = iRe^{i\theta} d\theta$. Cauchy's formula gives:

(*) $$f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{f(Re^{i\theta}) \cdot Re^{i\theta} d\theta}{Re^{i\theta} - z} \quad : \quad z \in D_R$$

Remark. Now (*) yields a series representation of f. First division with $Re^{i\theta}$ gives

(i) $$f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{f(Re^{i\theta}) \cdot d\theta}{1 - \frac{z}{R} \cdot e^{-i\theta}}$$

Since $|z| < R$ we can expand the denominator in a convergent geometric series:

(ii) $$\frac{1}{1 - \frac{z}{R} \cdot e^{-i\theta}} = \sum_{\nu=0}^{\infty} R^{-\nu} e^{-i\nu\theta} \cdot z^{\nu}$$

Put

(iii) $$c_\nu = \frac{1}{2\pi} \cdot R^{-\nu} \int_0^{2\pi} f(Re^{i\theta}) \cdot e^{-i\nu\theta} d\theta \quad : \quad \nu = 0, 1, \ldots$$

Then we obtain the series representation

(iv) $$f(z) = \sum_{\nu=0}^{\infty} c_\nu \cdot z^\nu$$

Remark. The convergence of this series when $|z| < R$ is clear. For if M is the maximum of $|f(Re^{i\theta})|$ as $0 \leq \theta \leq 2\pi$ we see that

(*) $$|c_\nu| \leq M \cdot R^{-\nu}$$

The coefficients c_ν correspond to *Fourier series coefficients* of the periodic function of θ defined by

$$\theta \mapsto f(Re^{i\theta})$$

The interplay between Fourier series and analytic functions will be discussed in XXX.

6.6 Exercise. Consider a power series $f(z) = \sum c_\nu \cdot z^\nu$ as above where (*) holds for a pair M and R. Now we also get a convergent power series

$$g(z) = \sum_{\nu=1}^{\infty} \nu \cdot c_\nu \cdot z^{\nu-1}$$

It turns out that g is the derivative of f. To show this we consider a point z_0 in the disc D_R. With a small Δz we have the difference quotient:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Now the reader should verify the inequality:

$$\left|\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - g(z_0)\right| \leq |\Delta z| \cdot \sum_{\nu=2}^{\infty} |c_\nu| \cdot$$

7. Complex differentials

1. Calculus with differential forms. With $z = x + iy$ we get the differential 1-form

$$dz = dx + idy$$

Similarly, $\bar{z} = x - iy$ gives $d\bar{z} = dx - idy$. Hence we obtain

$$dx = \frac{1}{2}(dz + d\bar{z}) \quad : dy = \frac{1}{2i}(dz - d\bar{z})$$

So every differential 1-form $Adx + Bdy$ where A, B say are complex-valued C^1-functions can be expressed by dz and $d\bar{z}$:

$$Adx + Bdy = \frac{1}{2}(A - iB)dz + \frac{1}{2}(iA + B)d\bar{z}$$

2. Exterior derivatives. If $Adx + Bdy$ is a 1-form its exterior derivative is the 2-form

$$d(Adx + Bdy) = (A_y - B_x) \cdot dx \wedge dy$$

With these notations Stokes formula for a domain $\Omega \in \mathcal{D}(C^1)$ can be expressed by

$$\iint_\Omega d(Adx + Bdy) = \int_{\partial\Omega} Adx + Bdy$$

In the left hand side one integrates a function times the 2-form $dx \wedge dy$ over Ω which is the same thing as taking the usual area integral. Thus, the left hand side in Stokes formula becomes

$$\iint_\Omega (-A_y + B_x) dxdy$$

Regarding A and B separately we recover the formula in XXX.

3. Complex expressions. Consider now a 1-form expressed as fdz where f is a complex-valued C^1-function. We get

(*) $$d(fdz) = df \wedge dz = \bar{\partial}(f) d\bar{z} \wedge dz \quad : \quad \bar{\partial}(f) = \frac{1}{2}(f_x + if_y)$$

It is both illumminating and essential to confirm (*) using differentials expressed by dx and dy. To begin with we have

(i) $$dx \wedge dx = dy \wedge dy = 0 \quad : \quad dy \wedge dx = -dx \wedge dx$$

From this we get

(ii) $$d\bar{z} \wedge dz = 2idx \wedge dy \quad : \quad dz \wedge d\bar{z} = -2idx \wedge dy$$

Hence (*) is equivalent to

(**) $$d(fdx + ifdy) = (-f_y + if_x) dx \wedge dy = \frac{1}{2}(f_x + if_y) \cdot 2idx \wedge dy$$

The reader may discover that (**) holds. Next, let us start with a 1-form $gd\bar{z}$. Then we get

(***) $$d(gd\bar{z}) = \partial(g) dz \wedge d\bar{z} \quad : \quad \partial(g) = \bar{\partial}(f) = \frac{1}{2}(g_x - if_y)$$

4. Stokes formula in complex form. Using (*) and (***) together with (ii) above the real version of Stokes formula from II:XX gives the following two formulas:

(*) $$\iint d(fdz) = \iint \bar{\partial}(f) d\bar{z} \wedge dz = \int_{\partial\Omega} fdz$$

(**) $$\iint d(gd\bar{z}) = \iint \partial(g) dz \wedge d\bar{z} = \int_{\partial\Omega} gd\bar{z}$$

5. Example. Let $\Omega \in \mathcal{D}(C^1)$ and let $f(z)$ is an analytic function in Ω which extends to a C^1-function on its closure. With $\epsilon > 0$ we get the C^1-function

$$u_\epsilon = \mathrm{Log}\,|f|^2 + \epsilon$$

We see that

(i) $$\partial(u_\epsilon) = \frac{\bar{f} \cdot f'}{|f|^2 + \epsilon}$$

where f' is the complex derivative of the analytic function f. Stokes formula applied to the 1-form $\partial(u_\epsilon)dz$ gives:

(ii)
$$\iint_\Omega \bar\partial(\frac{\bar f\cdot f'}{|f|^2+\epsilon})\cdot d\bar z\wedge dz=\int_{\partial\Omega}\frac{\bar f\cdot f'}{|f|^2+\epsilon}\cdot dz$$

A computation left to the reader shows that the left hand side becomes

$$2i\cdot\iint_\Omega\frac{\epsilon\cdot|f'|^2}{(|f|^2+\epsilon)^2}dxdy$$

Assume that f has no zeros on $\partial\Omega$. Then we get the limit formula

(iii)
$$\lim_{\epsilon\to 0}\int_{\partial\Omega}\frac{\bar f\cdot f'}{|f|^2+\epsilon}\cdot dz=\int_{\partial\Omega}\frac{f'dz}{f}$$

Hence the left hand side in (i) has a limit, i.e. there exists

(iv)
$$\lim_{\epsilon\to 0}2i\cdot\iint_\Omega\frac{\epsilon\cdot|f'|^2}{(|f|^2+\epsilon)^2}dxdy$$

Since ϵ appears in the numerator this area integral tends to zero outside zeros of f. But when $f(\alpha)=$ for some $\alpha\in\Omega$ we get a contribution. Suppose for example that $\alpha=0$ and close to the origin $f(z)=az^m(1+b_1z+\ldots+)$ where $a\neq 0$ and $m\geq 1$. Taking an area integral over a small disc D_δ centered at the origin we see that

(iv)
$$\lim_{\epsilon\to 0}\iint_{D_\delta}\frac{\epsilon|f'|^2}{(|f|^2+\epsilon)^2})\cdot dxdy=\lim_{\epsilon\to 0}\iint_{D_\delta}\frac{\epsilon\cdot m^2|a|^2|z|^{2m-2}}{(|a|^2|z|^{2m}+\epsilon)^2})\cdot dxdy$$

Integrating in polar coordinates the last term becomes

(v)
$$\lim_{\epsilon\to 0}2\pi\cdot\int_0^\delta\frac{\epsilon\cdot m^2|a|^2r^{2m-2}}{(|a|^2r|^{2m}+\epsilon)^2})\cdot rdr$$

A easy computation which is left to the reader shows that this limit is $\pi\cdot m$. Performing this limit in small discs around each zero of f in Ω we conclude the following:

5.1 Theorem. *Let $\mathcal{N}_f(\Omega)$ denote the integer equal to the sum of zeros of f in Ω counted with their multiplicities, while $f\neq 0$ on $\partial\Omega$. Then we have the equality:*

$$\mathcal{N}_\Omega(t)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f'(z)}{f(z)}\cdot dz$$

Another example. Let f be as above and let g be some C^1-function in Ω which vanishes the zeros of f which we assume are all simple. With u_ϵ defined as in the previous example we apply Stokes formula to the 1-form $g\partial(u_\epsilon)dz$. Computing $d(g\partial(u_\epsilon)dz$ gives a sum of two area integrals

(i)
$$2i\cdot\iint_\Omega\frac{\epsilon\cdot g\cdot|f'|^2}{(|f|^2+\epsilon)^2}dxdy+2i\cdot\iint_\Omega\frac{\bar\partial(g)\bar f\cdot f'}{|f|^2+\epsilon}dxdy$$

The last double integral has a limit. The reason is that

(1)
$$\lim_{\epsilon\to 0}\frac{\bar f\cdot f'}{|f|^2+\epsilon}=\frac{f'}{f}$$

exists in the space of integrable functions, i.e. we use that $\frac{1}{f}$ is locally integrable when f has simple zeros. So the limit hen $\epsilon\to 0$ of the second double integral becomes

(2)
$$2i\cdot\iint_\Omega\frac{\bar\partial(g)\cdot f'}{f}dxdy$$

In the first integral ϵ is in the denominator so the area integral tends to zero outside zeros of f. To analyze the situation at a zero taken as the origin where we may take $f(z) = z$ since f by assumoption has simple zeros there remains to regard

$$\lim_{\epsilon \to 0} \cdot \iint_{D_\delta} \frac{\epsilon \cdot g}{(|z|^2 + \epsilon)^2} dxdy \quad : \quad g(0) = 0 \tag{3}$$

We leave it as an exercise to show that this limit is zero for every g vanish at the origin. The hint is to integrate in polar coordinates. Hence we have established

6. Theorem. *Assume that f has simple zeros in Ω and g is a C^1-function which vanishes at these zeros. Then*

$$2i \cdot \iint_\Omega \frac{\bar{\partial}(g) \cdot f'}{f} dxdy = \int_{\partial\Omega} \frac{g \cdot f'}{f} dz$$

7. An area formula. Recall that $dz = dx + idy$. It follows that

$$\bar{z} \cdot dz = i(xdy - ydx) + xdx + ydy$$

If Ω is a domain in $\mathcal{D}(C^1)$ we proved in Chapter XX the two area formulas:

$$\int_{\partial\Omega} x \cdot dy = -\int_{\partial\Omega} y \cdot dx = \text{Area}(\Omega)$$

At the same time the line integrals of xdx and ydy are zero. We conclude that

$$2i \cdot \text{Area}(\Omega) = \int_{\partial\Omega} \bar{z} \cdot dz \tag{*}$$

This area formula is often used. Consider for example an analytic function $f(z)$ in Ω which extends to a C^1-function to the closure and assume that the map $f : \bar{\Omega} \to \bar{U}$ is a homeomorphism where U is another domain in $\mathcal{D}(C^1)$. So here f yields a con formal map from Ω onto U and it sends each of the p many boundary curves to Ω onto boundary curves of U. With $w = f(z)$ as a new complex variable the area formula applied to the image domain gives:

$$2i \cdot \text{Area}(f(\Omega)) = \int_{\partial U} \bar{w} \cdot dw$$

Here the last line integral is equal to

$$\int_{\partial\Omega} \bar{f}(z) \cdot f'(z) \cdot dz$$

Hence we have found an elegant formula to express the area of the image domain $f(\Omega)$ via a line integral along $\partial\Omega$.

8. The Pompeiu formula.

Let $\Omega \in \mathcal{D}(C^1)$. Then the following hold for each C^1-function f defined on $\bar{\Omega}$:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)dz}{\zeta - z_0} - \frac{1}{\pi} \iint_\Omega \frac{\bar{\partial}(f)}{z - z_0} \cdot dxdy \quad : \quad z_0 \in \Omega \tag{*}$$

Proof. Let D_δ be a small disc centered at z_0. Regarding the 1-form $\frac{1}{z - z_0} \cdot f(z)dz$ in $\Omega \setminus D_\delta$ we get by Stokes Theorem

$$\int_{\partial\Omega_\delta} \frac{f(z)dz}{z - z_0} = \iint_{\Omega \setminus D_\delta} \frac{\bar{\partial}(f)}{z - z_0} \cdot d\bar{z} \wedge dz = 2i \cdot \iint_{\Omega \setminus D_\delta} \frac{\bar{\partial}(f)}{z - z_0} \cdot dxdy$$

Since the function $z \mapsto \frac{1}{z-z_0}$ stays locally integrable even close to z_0 the last double integral has a limit as $\delta \to 0$ which simply is the area integral over Ω. Next, $\partial\Omega_\delta = \partial\Omega \cup \partial D_\delta$ so the right hand side becomes

$$\int_{\partial\Omega} \frac{f(z)dz}{z-z_0} - \int_{\partial D_\delta} \frac{f(z)dz}{z-z_0}$$

where the minus sign in front of the last integral appears since D_δ is a removed disc from Ω which means that its usual positive orientation is reversed while Stokes formula was applied. Putting all in order and dividing with $\frac{1}{2\pi i}$ we have therefore established the equality:

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)dz}{z-z_0} - \frac{1}{\pi}\iint_\Omega \frac{\bar\partial(f)}{z-z_0}\cdot dxdy = \lim_{\delta\to 0} \frac{1}{2\pi i}\int_{\partial D_\delta} \frac{f(z)dz}{z-z_0}$$

But the last limit is $f(z_0)$ by the residue formula in XX and hence Pompieu's formula is proved.

8.1 inhomogeneous $\bar\partial$-equations. The Pompeiu formula is used to solve $\bar\partial$-equations. Let $f(z)$ be a complex valued C^1- function with compact support in $\mathbf{C}$. Set

(1)
$$g(z) = \iint \frac{f(\zeta)d\xi d\eta}{\zeta - z}$$

After a change of variables we see that

(2)
$$g(z) = \iint_{\mathbf{C}} \frac{f(\zeta+z)\cdot d\xi d\eta}{\zeta}$$

Hence we obtain

(3)
$$\bar\partial(g) = \iint_{\mathbf{C}} \frac{\bar\partial(f)(\zeta+z)\cdot d\xi d\eta}{\zeta} = \iint_{\mathbf{C}} \frac{\bar\partial(f)(\zeta)\cdot d\xi d\eta}{\zeta - z}$$

Now Pompieiu's formula shows that

(**)
$$\bar\partial(g) = f$$

Remark. The solution $g(z)$ is not unique since one can add any analytic function $h(z)$ to g and still have (**). In many situations one wants a solution to (**) with certain prescribed bounds on g. In Section XX devoted to the Corona Theorem we shall encounter such a case where one first uses the existence of the Pompeiu solution and after finds h to get a solution with good bounds.

8.2 The equation $\bar\partial(f) = a\cdot f + b\cdot\bar f$.

Let $a(x,y)$ and $b(x,y)$ be a pair of continuous and complex-valued function defined in some open and connected domain Ω. We suppose that $f(z)$ is a solution to the differential equation above. The Pompeiu formula entails that if $z_0 \in \Omega$ and $R > 0$ is chosen so that the disc D_R centered at z_0 is contained in Ω, then the following holds when $|z - z_0| < R$:

(*)
$$f(z) = \frac{1}{2\pi i}\cdot\int_{\partial D_R} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_{D_R} [af + b\bar f]\cdot\frac{d\xi d\eta}{\zeta - z}$$

We shall use these locally defined integral equations to prove a uniqueness theorem. Let us say that a continuous function $g(z)$ in Ω is flat at a point z_0 if

$$\lim_{z\to z_0} \frac{g(z)}{[z-z_0)^n} = 0$$

holds for every positive integer n.

8.3 Theorem. *Let f be a continuous solution to the differential equation $\bar\partial(f) = af + b\bar f$ in Ω. Then f cannot be flat at any point in Ω unless f is identically zero.*

Proof. Suppose that f is flat at some $z_0 \in \Omega$. After a translation we may assume that z_0 is the origin. For each integer $n \geq 1$ we set:

$$f_n(z) = \frac{f(z)}{z^n}$$

Then we see that f_n satisfies the equation

$$\bar{\partial}(f_n) = af_n + b \cdot \big(\frac{\bar{z}}{z}\big)^n \cdot f_n$$

Using the Pompeiu formula it follows that f_n satisfies the integral equation

$$(*) \qquad f_n(z) = \frac{1}{2\pi} \cdot \int_{\partial D_R} \frac{f_n(\zeta) \cdot d\zeta}{\zeta - z} + \frac{1}{\pi} \iint_{D_R} [af_n + b_n f_n] \cdot \frac{d\xi d\eta}{\zeta - z}$$

The triangle inequality gives:

$$(**) \qquad |f_n(z)| \leq \frac{1}{2\pi} \cdot \int_{\partial D_R} \frac{|f_n(\zeta)| \cdot |d\zeta|}{|\zeta - z|} + \frac{1}{\pi} \iint_{D_R} |af_n + b_n \bar{f}_n| \cdot \frac{d\xi d\eta}{|\zeta - z|}$$

Next, for each $z \in D_R$ we notice the inequality

$$(1) \qquad \frac{1}{2\pi} \cdot \iint_{D_R} \frac{dxdy}{|\zeta - z|} \leq 2R$$

Using (**) and (1) an integration with respect to z over the disc $|z - z_0| < R$ gives:

$$(2) \qquad \iint_{D_R} |f(z)| dxdy \leq 2R \cdot \int_{\partial D_R} |f_n(\zeta)| \cdot |d\zeta| + 4R \cdot \iint_{D_R} |af_n + b_n \bar{f}_n| \cdot d\xi d\eta$$

Let M be the maximum norm of $|a| + |b|$ over D_R. Changing the integration variables $\zeta = \xi + i\eta$ in the last integral above it follows that

$$(3) \qquad \iint_{D_R} |f_n(z)| \cdot dxdy \leq 2R \cdot \int_{\partial D_R} |f_n(\zeta)| \cdot |d\zeta| + 4MR \cdot \iint_{D_R} |f_n(z)| \cdot dxdy$$

Above (3) holds for every R such that the disc of radius R centered at z_0 stays in Ω. We can choose R so small that $4MR < 1$ and replacing f_n by $\frac{f}{z^n}$ we get

$$(4) \qquad \iint_{D_R} \frac{|f(z)|}{|z|^n} \cdot dxdy \leq \frac{2R}{1 - 4MR} \cdot R^{-n} \cdot \int_{\partial D_R} |f(\zeta)| \cdot |d\zeta|$$

This inequality holds for every $n \geq 1$. if f is not identically zero in the disc D_R we find some $z_* \in D_R$ and a small $\delta > 0$ such that $|z_*| + \delta = \rho < R$ and $f(z) \neq 0$ in the closed disc $\{|z - z_*| \leq \delta\}$. If m_* is the minimum of $|f(z)|$ in this disc we get from (4):

$$(5) \qquad \rho^{-n} m_* \cdot \pi\delta^2 \leq \frac{2R}{1 - 4MR} \cdot R^{-n} \cdot \int_{\partial D_R} |f(\zeta)| \cdot |d\zeta|$$

Since $\rho < R$ this inequality cannot hold for all n. Hence we have established a contradiction and conclude that f must be identically zero in the disc D_R. Finally, since the domain is connected we can continue from new flat points in neighborhoods where f is identically zero and conclude that $f = 0$ holds in the whole domain Ω.

Exercise. Use similar methods as in the proof above to show that if f is a solution to the differential equation in 8.2 which is not identically zero, then the set of zeros must be a *discrete* subset of Ω, i.e. there cannot exist $z_* \in \Omega$ and a sequence of distinct points $\{z_\nu\}$ such that $z_\nu \to z_*$ and $f(z_\nu) = 0$ for all ν.

9. Normal families.

Introduction. Cauchys integral formula for a disc D gives the existence of power series expansions for analytic functions $f(z)$. Moreover, we obtain bounds for higher order derivatives. For example, suppose that $f(z) \in \mathcal{O}(D_r)$ where D_r is centered at the origin and has radius r. Assume that f extends to a continuous function on the closed disc $\bar{D}_r$. If $|z| < r$ we have seen that:

$$(*) \qquad f'(z) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

Let $|z| \leq r/2$ and with $\zeta = re^{i\theta}$ during the integration, the triangle inequality gives

$$|f'(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{|f(re^{i\theta}| \cdot r \cdot d\theta}{(r - |z|)^2}$$

So if $M = \max |f(re^{i\theta})|$ and $|z| \leq r/2$ we get

$$\max_{|z| \leq r/2} |f'(z)| \leq \frac{4M}{r}$$

Thus, we obtain a uniform bound for the derivative via the maximum norm of f over a larger disc. From this one gets useful convergence principle.

0.1. Theorem. *Let Ω be an open set and $\{f_\nu \in \mathcal{O}(\Omega)\}$ a sequence with uniform bounded maximum norms, i.e. $|f_\nu|_\Omega \leq M$ hold for all ν and some constant M. Then there exists at least one subsequence $\{g_j = f_{\nu_j}\}$ such that $g_j \to g_*$ holds uniformly over compact subsets of Ω where the limit function g_* is analytic i Ω.*

The proof uses *Arzela's Theorem.* Namely, by the local estimate for first order derivatives above, the uniform bound for maximum norms implies that $\{f_\nu\}$ yields an *equi-continuous* family of continuous functions over every compact subset of Ω. Next, one exhausts Ω by some increasing sequence of compact subsets to get the theorem above. We leave the details as an exercise. Bu we shall give a more detailed account in the case when Ω is a disc where one discovers how to obtain convergent subsequences. In addition to Theorem 0.1 the following result should also mentioned.

0.2. Theorem. *Let Ω be an open set and $\{f_\nu \in \mathcal{O}(\Omega)\}$ a sequence which converges uniformly to a limit function g_* over each compact subset of Ω. Assume also that there exists a relatively compact subset Ω_0 of Ω and an integer $k \geq 1$, such that every f_ν has exactly k zeros in Ω_0 - as usual counted with multiplicity, while no zeros occur outside Ω_0. Then the limit function g_* is either identically zero or it has exactly k zeros in Ω counted with multiplicity and they all belong to the closure $\bar{\Omega}_0$.*

Remark. The proof uses *Rouche's theorem* which is given in XXX. So for the moment we leave out the proof of Theorem 0.2 as an exercise. Moreover, once we have a convergence $f_\nu \to g_*$, it follows that the sequence of complex derivatives also converges, i.e. one has the implication:

$$(*) \qquad f_\nu \to g_* \implies f'_\nu \to g'_*$$

This will be used in the proof of *Riemann's Mapping Theorem* in XX where one regards a convergent sequence $f_\nu \to g_*$ and assume that the derivatives f'_ν have no zeros. Then we conclude from the above that g_* is either identically a constant or that g'_* never has zeros.

1. Convergence in discs

Let $\delta_0 > 0$ and denote by D^* the open disc centered at the origin of radius $1 + \delta_0$. If $f \in \mathcal{O}(D^*)$ we obtain the the series representation from 4.5:

$$f(z) = \sum c_\nu z^\nu$$

Assume that f is bounded, i.e. there is a constant M such that $|f(z)| \leq M$ for all $z \in D^*$. For any $0 < \delta < \delta_0$ we then have

$$c_\nu = \frac{1}{2\pi} \cdot (1+\delta)^{-\nu} \int_0^{2\pi} f((1+\delta)e^{i\theta})e^{-i\nu\theta} d\theta$$

The triangle inequality gives

$$|c_\nu| \leq M \cdot (1+\delta)^{-\nu} \quad : \nu = 0, 1, 2, \ldots$$

These upper bounds for the coefficients of f depend on M only. Moreover, since the inequality holds for any $\delta < \delta_0$ we can take $\delta = \delta_0$ above and still have the estimates of c_ν. Suppose now that N is a positive integer and $\epsilon > 0$ some positive number such that

$$|c_\nu| \leq \epsilon \quad : 0 \leq \nu \leq N$$

Then, if $0 < r < 1$ we have

$$f(re^{i\theta}) = \sum_{\nu=0}^{\nu=N} c_\nu r^\nu \cdot e^{i\nu\theta} + \sum_{\nu=N+1}^{\infty} c_\nu r^\nu \cdot e^{i\nu\theta}$$

By the triangle inequality we can estimate both sums. The result is

$$f(re^{i\theta})| \leq \epsilon \cdot \sum_{\nu=0}^{\nu=N} r^\nu + M \cdot \sum_{\nu=N+1}^{\infty} \frac{r^\nu}{(1+\delta_0)^\nu}$$

Taking the whole sum over the two geometric series above we get

1.1 Proposition. *One has*

$$\max_\theta |f(re^{i\theta})| \leq \epsilon \cdot \frac{1}{1-r} + M \frac{r^{N+1}}{(1+\delta_0)^{N+1}} \cdot \frac{1}{1 - \frac{r}{1+\delta_0}}$$

1.2 Application. Let $\{f_k\}$ be a sequence in $\mathcal{O}(D^*)$ where the maximum norm of every f_k is $\leq M$. Each f_k has a series expansion $\sum c_\nu(k) z^\nu$. For each fixed ν we have a bounded sequence $\{c_\nu(1), c_\nu(2), \ldots\}$ of complex numbers. By the diagonal procedure we can find a subsequence $k_1 < k_2 < \ldots$ such that there exists

$$\lim_{\to\infty} c_\nu(k_j) = c^*_\nu \quad : \nu = 0, 1, \ldots$$

The uniform estimates for the $c_\nu(k)$-coefficents give

$$|c^*_\nu| \leq M \cdot (1+\delta_0)^{-\nu}$$

Hence there exists the analytic function in D^*:

$$g(z) = \sum c^*_\nu \cdot z^\nu$$

Renumber the f-functions by setting $g_j = f_{k_j}$. By the convergence we see that for any $\epsilon > 0$ and any $N \geq 1$ there exists some integer N^* such that

$$|c_\nu(k_j) - c^*_\nu| \leq \epsilon \quad : 0 \leq \nu \leq N \quad : j \geq N^*$$

Now we can apply Proposition 1.1 to estimate the maximum norm of $|g_j - g^*|$ over discs of radius $r < 1$. In particular we obtain

1.3 Proposition. *The sequence $\{g_j = f_{k_j}\}$ converges uniformly to the limit function g^* over each disc $|z| \leq r$ with $r < 1$.*

1.4 Convergene in the whole of D^* The rate of convergence for maximum norms over discs $|z| \leq r < 1$ in Proposition 1.3 are well controlled. By relaxing the estimates a bit it is still true that $g_j \to g^*$ holds uniformly in discs $|z| \leq r$ for *any* $r < 1 + \delta_0$. To see this one applies the "(ϵ, δ) - yoga". So let $\epsilon > 0$ and $r < 1 + \delta_0$ be given. We first find a large N so that

$$M \frac{r^{N+1}}{(1+\delta_0)^{N+1}} \cdot \frac{1}{1 - \frac{r}{1+\delta_0}} \leq \epsilon$$

Then, if $|z| \leq r$ it follows from Proposition xx and the triangle inequality:

$$|g_j(z) - g^*(z)| \leq \sum_{\nu=0}^{\nu=N} |c_\nu(k_j) - c^*_\nu| \cdot r^\nu + 2\epsilon \quad : \ j \geq N^*$$

But now $\lim_{j\to\infty} c_\nu(k_j) \to c^*_\nu$ hold for each $0 \leq \nu \leq N$ so we find some large $N^{**} > N^*$ such that

$$\sum_{\nu=0}^{\nu=N} |c_\nu(k_j) - c^*_\nu| \cdot r^\nu \leq \epsilon \cdot \frac{1}{(1+\delta_0)^N}$$

Then we see that maximum norms of $g_j - g^*$ are $\leq 3\epsilon$ if $j \geq N^{**}$. This proves that uniform convergence holds over all discs inside D^*.

1.5 Remark. The conclusion is quite striking since we from the start only assume that the coefficients begin to converge and after conclude that one gets uniform convergence to the limit function g^* over every compact disc inside D^*.

10. Laurent series.

Let $R^* > 1$ and consider the open domain

$$\Omega = \{z \quad : \ 1 < |z| < R^*\}$$

We refer to Ω as an open annulus where $|z| = 1$ is the inner circle and $|z| = R^*$ is the outer circle. Let $f(z)$ be analytic in Ω. When $1 < |z| < R$ is fixed we can choose a pair r, R such that

$$1 < r < |z| < R < R^*$$

Cauchy's formula applies to the domain $r < |z| < R$ and gives the equality:

$$(*) \qquad f(z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=R} \frac{f(\zeta) \cdot d\zeta}{\zeta - z} - \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \frac{f(\zeta) \cdot d\zeta}{\zeta - z}$$

The idea is now to expand $\zeta - z$ in a geometric series. When $|\zeta| = R > |z|$ we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \sum_{\nu=0}^{\infty} \frac{z^\nu}{\zeta^\nu}$$

Let us put

$$a_\nu = \frac{1}{2\pi i} \cdot \int_{|\zeta|=R} \frac{f(\zeta) \cdot d\zeta}{\zeta^{\nu+1}} \quad : \quad \nu = 0, 1, 2, \ldots$$

Then the first integral in (*) becomes

$$(1) \qquad f^*(z) = \sum a_\nu \cdot z^\nu$$

Next, when $|\zeta| = r < |z|$ we have the series expansion

$$\frac{1}{\zeta - z} = -\frac{1}{z} \cdot \sum \frac{\zeta^\nu}{z^\nu}$$

Put

$$b_\nu = \frac{1}{2\pi i} \cdot \int_{|\zeta|=R} f(\zeta) \cdot \zeta^{\nu-1} \cdot d \quad : \quad \nu = 1, 2, \ldots$$

Taking the negative sign into the account in (*) we see that the second integral becomes

(2) $$f_*(z) = \sum_{\nu=1}^{\infty} \frac{b_\nu}{z^\nu}$$

1. Definition. *The function $f^*(z)$ is called the positive part of f and $f_*(z)$ the lower part of f.*

Remark. In the construction above R can be chosen arbitrarily close to R^*. With $R = R^* - \epsilon$ we get the finite maximum norm

$$||f||_{R_*-\epsilon} = \max_{\zeta|=R_*-\epsilon} |f(\zeta)|$$

The triangle inequality gives

$$|a_\nu| \leq \frac{||f||_{R^*-\epsilon}}{(R^* - \epsilon)^\nu}$$

Since ϵ can be made arbitrary small we conclude that the radiius of convergence for the series (1) is ≥ 1. Hence we have proved:

2. Proposition. *The positive part $f^*(z)$ is analytic in the disc $|z| < R^*$.*

In a similar way the reader may verify that the zseries for $f_* z)$ is convergent for all $|z| > 1$.

3. Proposition. *The inner part $f_*(z)$ is analytic in the exterior disc $|z| > 1$.*

4. The Laurent series. The analytic function $f(z)$ in the annulus can be written as a sum $f^*(z) + f_*(z)$. Here $f^* \in \mathcal{O}(D_{R^*})$ while f_* is analytic in the exterior disc $|z| > 1$. We can take the two series together and hence $f(z)$ is represented by

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu \cdot z^\nu + \sum_{\nu=1}^{\infty} \frac{b_\nu}{z^\nu}$$

This is called the Laurent series of $f(z)$.

5. The residue coefficient b_1. From the construction above we have

$$b_1 = \frac{1}{2\pi i} \int_{|\zeta|=R} f(\zeta) \cdot d\zeta$$

Notice that we can perform the integral over any circle, i.e. above we can take any $1 < R < R*$. The special role of b_1 is that $b_1 = 0$ holds if and only if $f(z)$ has a primitive analytic function in the annulus .

6. Exercise. Prove the assertion above, i.e. that there exists an analytic function $F(z)$ in the annulus such that $F'(z) = f(z)$ if and only if $b_1 = 0$.

7. Example. Consider the analytic function in the annulus $1 < |z| < R$ defined by

$$\phi(z) = \frac{1}{z+1} - \frac{1}{z-1}$$

Here $\phi(z)$ has the Laurent series expansion

$$\phi(z) = -2 \cdot (\frac{1}{z^2} + \frac{1}{z^4} + \ldots)$$

Hence there exists the primitive function

$$\Phi(z) = 2 \cdot \big(\frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \ldots\big)$$

Next, we notice that $\Phi(z)$ can be expressed by the complex Log-function

$$(*) \qquad \log\big(\frac{z+1}{z-1}\big)$$

Here some care must be taken. First it is clear that (*) is defined when $z = x$ is real with $x > 1$. So on the real interval $(1, +\infty)$ we have the equality

$$G(x) = \log\big(\frac{x+1}{x-1}\big)$$

8. Exercise. Let $R > 1$ and consider an analytic function

$$g(z) = \sum c_\nu \cdot z^\nu$$

in the open disc D_R. It can be restricted to the real line $[-1, 1]$. Set

$$K_g(z) = \int_{-1}^{1} \frac{g(z) - g(s)}{z - s} \cdot ds$$

The algebraic identity $z^\nu - s^\nu = (z - s)(z^{\nu-1} + \ldots + s^{\nu-1})$ gives

$$K_g(z) = 2c_1 + \sum_{\nu=2}^{\infty} c_\nu \cdot \int_{-1}^{1} \big[z^{\nu-1} + z^{\nu-2}s + \ldots + s^{\nu-1}\big] \cdot ds$$

Since we integrate over $[-1, 1]$ the integrals taken over odd s-powers are all zero and a computation gives:

$$(1) \qquad K_g(z) = 2c_1 + 2 \cdot \sum_{\nu=2}^{\infty} c_\nu \cdot \big[z^{\nu-1} + \frac{z^{\nu-3}}{3} + \frac{z^{\nu-5}}{5} + \ldots\big]$$

Notice that the sum in each bracket is finite. Show that $K_g(z)$ is analytic in the disc $|z| > R$ and using the Φ-function from Example 7 the reader should verify that $K_g(z)$ is equal to the positive part of the Laurent series defined by $g(z) \cdot \Phi(z)$ in $1 < |z| < R$.

11. An area formula.

Consider a Laurent series

$$f(z) = \sum_{-\infty}^{\infty} c_\nu \cdot z^\nu$$

which represents the analytic function defined in $\{R_* < |z| < R^*\}$. If $R_* < r < R^*$ we get the parametrised curve

$$(*) \qquad \theta \mapsto \theta \mapsto f(re^{i\theta})$$

Consider the situation where (*) is bijective so that the image is a closed Jordan curve J_r in the complex ζ-plane which borders a bounded Jordan domain Ω_r. We shall express the area of Ω_r with the the coefficients in Laurent series of f. The formula depends upon the orientation in the map (*) which may be either positive or negative.

1. Theorem. *One has has the equality*

$$\text{area}(\Omega_r) = \text{sign}(*) \cdot \pi \cdot \sum_{-\infty}^{\infty} n \cdot |c_n|^2 \cdot r^{2n}$$

where sign$(*)$ is +1 or -1 depending upon the orientation in (*).

2. Example. If $f(z) = z$ the orientation is positive and here Ω_r is the disc of radius r and the formula is okay since its area is $\pi \cdot r^2$. On the other hand, if $f(z) = \frac{1}{z}$ the orientation is negative and the sign-rule in Theorem 1 gives a correct formula.

emphProof of Theorem 1. Set $w = f(z)$ and suppose that (*) is positively oriented. The area formula from XX gives:

$$\text{area}(\Omega_r) = \int_{J_r} \bar{w} \cdot dw = \int_{|z|=r} \bar{f}(z) \cdot f'(z) \cdot dz$$

The right hand side is a double sum

$$\sum\sum \bar{c}_k \cdot m \cdot c_m \cdot \int_{|z|=r} \bar{z}^k \cdot z^{m-1} \cdot dz$$

extended over all pairs of integers. Since $\bar{z}^k = \frac{r^{2k}}{z^k}$ holds on $|z| = r$ we have

$$\int_{|z|=r} \bar{z}^k \cdot z^{m-1} \cdot dz = r^{2k} \cdot \int_{|z|=r} z^{-k} \cdot z^{m-1} \cdot dz$$

By Cauchy's residue formula the last integrals are zero for when $k \neq m$ and become $r^{2k} \cdot 2\pi i$ if $k = m$. Now we can read off the formula in Theorem 1. The proof when (*) has a negative orientation is the same after signs have been reversed.

3. A special case. Let

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} c_n \cdot z^n$$

where the positive series represents an analytic function in some disc $|z| < R$.Suppose that (*) holds for some $0 < r < R$ where the bijective map from (*) has a negative orientation which gives:

$$\text{area}(\Omega_r) = \pi \cdot \big[\frac{1}{r} - \sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot r^{2n}\big]$$

Since the area is a positive number we get the inequality

$$\sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot r^{2n} \leq \frac{1}{r} \tag{1}$$

4. Koebe's inequality. Let ϕ be an analytic function in the unit disc where $\phi(0) = 0$ and $\phi'(0) = 1$ and ϕ is a conformal map from D onto some simply connected domain in $\mathbf{C}$. Put

$$f(z) = \frac{1}{\phi(z)}$$

Now f is analytic in the punctured disc $0 < |z| < 1$ and since ϕ is a conformal map it follows that f maps circles $|z| = r$ onto closed Jordan curves where the orientation now is negative for each $0 < r < 1$. Moreover f has a Laurent series

$$\frac{1}{z} + \sum_{n=0}^{\infty} c_n \cdot z^n$$

and since the inequality (1) holds for every $r < 1$ we have

$$\sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot r^{2n} \leq 1 \tag{i}$$

At the same time the given ϕ-function has a series expansion

$$\phi(z) = z + d_2 z^2 + d_3 z^3 + \ldots$$

Let us assume that ϕ is an *odd* function which to begin with gives $d_2 = 0$ and

$$\frac{1}{\phi} = \frac{1}{z} \cdot \frac{1}{1 + d_3 z^2 + d_4 z^3 + \ldots}$$

From this we conclude that

$$c_1 = -d_3$$

and the inequality (i) gives

(ii) $$|d_3| \leq 1$$

Exercise. Use (i) above to show that equality holds in (ii) if and only if

$$\phi(z) = \frac{z}{1 - \lambda z^2}$$

for some λ whose absolute value is one.

Next, consider a conformal map ψ from D to a simply connected domain whose Taylor series at $z = 0$ is of the form

$$\psi(z) = z + a_2 z^2 + \ldots$$

5. Theorem. *One has the inequality* $|a_2| \leq 2$.

Proof. To begin with $\psi(z^2)$ is analytic in D and starts with z^2 which gives the analytic function

(i) $$\frac{\psi(z^2)}{z^2} = 1 + a_2 \cdot z^2 + \sum_{m \geq 2} b_m \cdot z^{2m}$$

Since $\psi(z) \neq 0$ when $z \neq 0$ the analytic function in (i) is also $\neq 0$ in D. Now the unit disc is simply connected and hence there exists an analytic square root function:

(ii) $$\phi(z) = z \cdot \sqrt{\frac{\psi(z^2)}{z^2}}$$

Sublemma The function ϕ is odd function and yields a conformal mapping, i.e. ϕ is 1-1 in D.

The proof of this Sublemma is left as an exercise. In particular we have a series expansion:

(iii) $$\phi(z) = z + d_3 \cdot z^3 + \ldots$$

Notice that the square root in (i) has a series expansion

(iv) $$1 + \frac{a_2}{2} \cdot z^2 + \text{higher order terms}$$

It follows that

(v) $$d_3 = \frac{a_2}{2}$$

Then (ii) from (4) above gives $|a_2 \leq 2$ as required.

6. Exercise. Show that the equality $|a_2| = 2$ holds in Theorem 5 if and only if ψ is of the form

$$\psi(z) = \frac{z}{(1 - \lambda z)^2} \quad \text{for some} \quad |\lambda| = 1$$

Show also that every such function indeed yields a conformal map from D onto a simply connected domain which becomes a *radial slit domain.* For example, take $\lambda = 1$ and show that the ψ-image of D becomes

$$\mathbf{C}) \setminus \big[-1/4, +\infty\big)$$

12. A theorem by Jentzsch

Introduction. Let $\sum c_n z^n$ b a convergent series in the unit disc D whose radius of convergence is one. To each $n \geq 1$ we get the Taylor polynomials

$$s_n(z) = c_0 + c_1 z + \ldots + c_n z^n$$

Denote by $\mathcal{N}(s_n)$ the zero set of s_n. It turns out that these zero sets always give cluster points on the unit circle.

12.1 Theorem. *For each point $e^{i\theta}$ on the unit circle there exists a strictly increasing sequence $1 \leq n_1 < n_2 < \ldots$ and points $z_k \in \mathcal{N}(s_{n_k})$ such that $\lim_{k\to\infty} z_k = e^{i\theta}$.*

Remark. This result was proved by X. Jentzsch in the article [Jen: Acta mathematica 1918]. To illustrate the theorem we consider the analytic function $f = \frac{1}{1-z}$. It has no zeros in D but the zeros of

$$s_n(z) = 1 + \ldots + z^n$$

are roots of unity which cluster on the whole unit circle as $n \to \infty$. The fact that a similar clustering of zeros occur for partial sums of a function of the form

$$f(z) = \sum z^{\nu_k}$$

where $1 \leq \nu_1 < \nu_2 < \ldots$ is an arbitrary increasing sequence of integers is less evident but has an affirmative answer by the theorem above.

Proof of Theorem 12.1

We argue by contradiction. Suppose that some point on T is not a cluster point for zeros of the s-functions. After a rotation we may assume that this point is $z = 1$. So now there exists $0 < \delta < 1$ and a positive integer k_* such that

$$(1) \qquad s_k(z) \neq 0 \quad : |z-1| < \delta \quad : k \geq k^*$$

There remains to derive a contradiction. Since $s_k \to f$ holds uniformly inside the unit disc and f is not reduced to a constant, it follows from XX and (1) that $f(z) \neq 0$ in the domain

$$\Omega = D \cap \{|z-1| < \delta\}$$

This domain is simply connected so we can find $h \in \mathcal{O}(\Omega)$ where

$$f(z) = e^{h(z)}$$

We also find a sequence $\{g_k \in \mathcal{O}(D_\delta(1))\}$ such that

$$s_k(z) = e^{g_k(z)} \quad : \quad z \in D_\delta(1)$$

Now $s_k(\xi) \to f(\xi)$ and we choose branches of $\{g_k\}$ so that

$$(2) \qquad \lim_{k\to\infty} g_k(\xi) = h(\xi)$$

Since $s_k(z) \to f(z)$ holds uniformly on compact subsets of Ω it follows that $g_k(z) \to h(z)$ also holds with uniform convergence over compact subsets of Ω. Next, consider the functions

$$(3) \qquad \phi_k(z) = e^{\frac{g_k(z)}{k}} - 1$$

The convergence $g_k \to h$ in Ω entails that

$$(4) \qquad \phi_k(z) \to 0$$

where this convergence is uniform over compact subsets of Ω. Next we have

Lemma. *The ϕ-functions are uniformly bounded in $D_\delta(1)$.*

Proof. First it is clear that there exists a constant A such that

$$|c_n| \leq A \cdot 2^n$$

With $\delta < 1$ and $|z| \leq 1 + \delta$ the triangle inequality gives

$$\text{(i)} \qquad |s_k(z)| \leq A \cdot 2^{k+1} \cdot (1+\delta)^k \quad : \quad k = 1, 2 \ldots$$

Since $e^{\mathfrak{Re}(g_k)(z)} = |s_k(z)|$ we see that (i) gives:

$$\text{(ii)} \qquad \mathfrak{Re}(g_k)(z) \leq (k+1) \cdot \log 2 + k(1+\delta) + \log^+ A$$

Dividing by k we see that $\{\frac{\mathfrak{Re}(g_k)(z)}{k}\}$ are uniformly bounded above when $z \in D_\delta(1)$ and Lemma 1 follows since

$$|\phi_k(z)| \leq e^{\frac{\mathfrak{Re}\, g_k(z)}{k}} + 1$$

Final part of the proof: The uniform bound in the Lemma and the convergence from (4) imply that $\{\phi_k(z)\}$ converges uniformly to zero in compact subsets of $D_\delta(1)$. In particular we can find some $0 < \epsilon < \delta$ such that

$$|\phi_k(1+\epsilon)| < \epsilon/2$$

hold for all large k. It follows that

$$(*) \qquad |s_k(1+\epsilon)| = |1 + \phi_k(1+\epsilon)|^k \leq (1+\epsilon/2)^k$$

holds when k large and then

$$|c_k(1+\epsilon)^k| = |s_k(1+\epsilon) - s_{k-1}(1+\epsilon)| \leq |s_k(1+\epsilon)| + |s_{k-1}(1+\epsilon)| \leq 2 \cdot (1+\epsilon/2)^k$$

It follows that

$$\limsup_{k\to\infty} |c_k|^{\frac{1}{k}} \leq \lim 2^{\frac{1}{k}} \cdot \frac{1+\epsilon/2}{1+\epsilon} = \frac{1+\epsilon/2}{1+\epsilon}$$

The last term is < 1 which contradicts that the radius of convergence of $\sum c_k z^k$ is one and this contradiction finishes the proof of Theorem 12.1

13. An inequality by Siegel

The result below was proved by C. Siegel in Math. Zeitschrift Bd. 10 page 175 (1921) Let $n \geq 2$ and $p(z)$ is a monic polynomial

$$p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

Let $\alpha_1, \ldots \alpha_n$ be the roots where eventual multiple roots are repeated.

Theorem.

$$\prod_{\nu=1}^{\nu=n} (1 + |\alpha_\nu|) \leq 2^n \cdot \sqrt{1 + |a_1|^2 + \ldots + |a_n|^2}$$

Proof. For every complex number the formula from XXX gives

$$(*) \qquad 1 + |z| \leq 2 \cdot \max(1, |z|) = 2 \cdot \exp\Big[\frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}|e^{i\theta} - z| d\theta\Big]$$

Apply this to each root of p and take the product. This gives

$$\prod_{\nu=1}^{\nu=n} (1 + |\alpha_\nu|) \leq 2^n \cdot \exp\Big[\frac{1}{2\pi} \cdot \int_0^{2\pi} \sum_{\nu=1}^{\nu=n} \operatorname{Log}\big|e^{i\theta} - \alpha_\nu\big| \cdot d\theta\Big] =$$

$$2^n \cdot \exp\Big[\frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}|p(e^{i\theta})|\, d\theta\Big]$$

To estimate the last term we employ Blascke's factorization of $p(z)$ in the unit disc $|z| < 1$:

$$p(e^{i\theta}) = B(e^{i\theta}) \cdot e^{\phi(e^{i\theta})} \quad : |B(e^{i\theta}| = 1 \quad : \phi(z) \in \mathcal{O}(D) \implies$$

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}|p(e^{i\theta})|\, d\theta = \frac{1}{2\pi} \cdot \int_0^{2\pi} \mathfrak{Re}(\phi(e^{i\theta}))d\theta = \mathfrak{Re}(\phi)(0)$$

where the last equality holds since $\mathfrak{Re}(\phi)$ is harmonic. Hence there only remains to show the inequality

$$(i) \qquad e^{\mathfrak{Re}(\phi)(0)} \leq \sqrt{1 + |a_1|^2 + \ldots + |a_n|^2}$$

To show this we use the Plancherel formula which gives

$$(ii) \qquad \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = 2\pi \cdot (1 + |a_1|^2 + \ldots + |a_n|^2)$$

In addition, we have

(iii) $$|p(e^{i\theta})|^2 = e^{2\cdot\mathfrak{Re}(\phi(e^{i\theta}))}$$

Now we can finish the proof as follows: Since $e^{\phi(z)}$ is analytic the mean value formula gives:

(iv) $$e^{\phi(0)} = \frac{1}{2\pi}\cdot\int_0^{2\pi} e^{\phi(e^{i\theta})}\cdot d\theta$$

Since the absolute value $|e^{\phi(z)}| = e^{\mathfrak{Re}(\phi)(z)}$ hold for all z, the triangle inequality gives

(v) $$e^{\mathfrak{Re}(\phi)(0)} \leq \frac{1}{2\pi}\cdot\int_0^{2\pi} e^{\mathfrak{Re}(\phi(e^{i\theta}))}\cdot d\theta$$

The Cauchy-Schwarz inequality applied to (v) gives:

(vi) $$e^{\mathfrak{Re}(\phi)(0)} \leq \frac{1}{\sqrt{2\pi}}\cdot\Big[\int_0^{2\pi} e^{2\mathfrak{Re}(\phi(e^{i\theta}))}\cdot d\theta\Big[^{\frac{1}{2}}$$

Now (ii) and (iii) give the requested inequality (i).

14. Zeros of product series.

Let $\{\lambda_k\}$ be a non-decreasing sequence of where $\lambda_1 \geq 2$ and

(1) $$\sum \lambda_k^{-(1+\epsilon)} < \infty \quad \text{hold for every} \quad \epsilon > 0$$

This gives (1) the analytic function $\phi(z)$ defined in $\mathfrak{Re}\, z > 1$ by

(*) $$\phi(z) = \prod (1 - \lambda_k^{-z})$$

The logarithmic derivative becomes

$$\frac{\phi'(z)}{\phi(z)} = -\sum \log \lambda_k \cdot \frac{\lambda_k^{-z}}{1 - \lambda_k^{-z}}$$

Expanding the denominator $1 - \lambda_k^{-z}$ into a geometric series we obtain:

$$\frac{\phi'(z)}{\phi(z)} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-nz}$$

If $z = 1 + \epsilon + iy$ for some $\epsilon > 0$ and a real y we set

$$\rho_\epsilon(n;y) = \sum_{k=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-n(1+\epsilon)} \cdot \lambda_k^{-iny}$$

A remarkable positive sum. Let $y_0 \neq 0$ and consider the sum

$$S_\epsilon(n) = 3\cdot\rho_\epsilon(n;0) + 4\cdot\rho_\epsilon(n;y_0) + \rho_\epsilon(n;2y_0)$$

Taking real parts we obtain

(i) $$\mathfrak{Re}\, S_\epsilon(n) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-n(1+\epsilon)} \cdot \big[3 + 4\cdot\mathfrak{Re}\,\lambda_k^{-iny_0} + \mathfrak{Re}\,\lambda_k^{-iny_0}\,\big]$$

In this double sum the terms inside the brackets become

$$3 + 4\cdot\cos\, n\cdot\lambda_k\cdot y_0 + \cos\, 2n\cdot\lambda_k\cdot y_0 = 2\big(1 + \cos\, n\cdot\lambda_k\cdot y_0\big)^2$$

where the last equality follows from the trigonometric formula $\cos 2a = 2\cos^2 - 1$.

Hence we have the formula

$$\mathfrak{Re}\big[\,3\cdot\frac{\phi'(1+\epsilon)}{\phi(1+\epsilon)}+4\cdot\frac{\phi'(1+\epsilon+iy_0)}{\phi(1+\epsilon+iy_0)}+\frac{\phi'(1+\epsilon+2iy_0)}{\phi(1+\epsilon+2iy_0)}\,\big]=$$

$$(*)\qquad -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\log\lambda_k\cdot\lambda_k^{-n(1+\epsilon)}\cdot 2\big(1+\cos\,n\cdot\lambda_k+\cdot y_0\big)^2$$

14.1 Conclusion. Keeping y_0 fixed we conclude that

$$\mathfrak{Re}\,3\cdot\frac{\phi'(1+\epsilon)}{\phi(1+\epsilon)}+4\cdot\frac{\phi'(1+\epsilon+iy_0)}{\phi(1+\epsilon+iy_0)}+\frac{\phi'(1+\epsilon+2iy_0)}{\phi(1+\epsilon+2iy_0)}<0$$

hold for all $\epsilon>0$.

14.2 Absence of asymptotic zeros. Let $y_0>0$ and assume that one has three limit expansions:

$$\phi(1+\epsilon)=\epsilon_*^{\alpha}\cdot\gamma_0(\epsilon)$$
$$\phi(1+\epsilon+iy_0))=\epsilon^{\beta}\cdot\gamma_1(\epsilon)$$
$$\phi(1+\epsilon+2iy_0))=\epsilon^{\alpha}\cdot\gamma_2(\epsilon)$$

where α_*,β,α are real constants and each γ-function is $\neq 0$ for small positive ϵ and together with its derivative satisfies

$$\lim_{\epsilon\to 0}\epsilon\cdot\frac{\gamma_j'(\epsilon)}{\gamma_j(\epsilon)}=0\quad:\quad 0\leq j\leq 2$$

Under the conditions above we see that the function

$$\epsilon\mapsto\mathfrak{Re}\big[\,3\cdot\frac{\phi'(1+\epsilon)}{\phi(1+\epsilon)}+4\cdot\frac{\phi'(1+\epsilon+iy_0)}{\phi(1+\epsilon+iy_0)}+\frac{\phi'(1+\epsilon+2iy_0)}{\phi(1+\epsilon+2iy_0)}\,\big]$$

has an asymptotic expansion of the form

$$\frac{3\cdot\mathfrak{Re}\,\alpha_*+4\cdot\mathfrak{Re}\,\beta+\mathfrak{Re}\,\alpha}{\epsilon}++\gamma^*(\epsilon)$$

where $\epsilon\cdot\gamma^*(\epsilon)\to 0$.

At the same time we have the inequality from 14.1 and checking signs we have the following result:

14.3 Theorem. *If $\phi(z)$ has the three asymptotic expansions above for some $y_0\neq 0$ then the following inequality must hold:*

$$3\cdot\mathfrak{Re}\,\alpha_*+4\cdot\mathfrak{Re}\,\beta+\mathfrak{Re}\,\alpha<0$$

14.4 Example. Theorem 14.3 applies in particular when $\phi(z)$ has a meromorphic extension across $\mathfrak{Re}(z)=1$ which give asymptotic expansions as above where α_*,β,α are integers. For example, suppose that ϕ has a simple pole at $z=1$ so that $\alpha_*=-1$. Theorem 14.3 implies that if ϕ has a zero at the $1+iy_0$ then it must have a pole at $1+2iy_0$. In particular we conclude that if ϕ has a simple pole at $z=1$ and extends to a meromorphic function across the line $\mathfrak{Re}\,z=0$ with no further poles, then it cannot have any zero on this line. This case will be applied to Riemann's ζ-function and used to prove the prime number theorem.

Remark. The method to employ positive cosine-functions to investigate zeros and poles of Dirichlet series was introduced by de Valle Poussin and occurs also frequently in work by Hardy, Littlewood and Titschmarsh. Above we discussed a case using the positivity of the function $a\mapsto 3+4\cos\,a+\cos 2a$. Other cosine-sums which produce functions which are everywhere ≥ 0 can be used to discover more involved relations about possible poles and zeros of Dirichlet series on the critical line of convergence.

Chapter 4: Multi-valued analytic functions

Introduction.

In section 1 we define winding numbers of curves in $\mathbf{R^2}$ where no complex variables appear. After this we study complex line integrals where Theorem 1.7 is the first main result in this chapter. The second main result is Theorem 2.1, referred to as the *argument principle*. It gives a gateway to find zeros of an analytic function since the counting function of its zeros is expressed by winding numbers arising from the image under the given function of closed boundary curves to a domain where we seek the number of zeros of f.

Section 3 starts with a construction due to Karl Weierstrass and leads to the total analytic continuation of germs of analytic functions. To grasp th construction of analytic continuations one can consider the *total sheaf space* $\widehat{\mathcal{O}}$ whose stalks are germs of analytic functions at points in $\mathbf{C}$. This topological manifold is locally homemorphic to small discs in $\mathbf{C}$ which express germs of multi-valued functions f. So if ρ denotes the local homemorphism from $\widehat{\mathcal{O}}$ onto $\mathbf{C}$, then the constructions by Weierstrass gives a 1-1 correspondence between connected open subsets of $\widehat{\mathcal{O}}$ whose ρ-image is equal to Ω), and the class of multi-valued analytic functions in Ω. Notice that in this correspondence one does not exclude functions which may have analytic continuation to larger sets. The constructions of analytic continuations are used to prove that certain classes of multi-valued analytic functions are *normal* in the sense of Montel. In section 4 we prove the *Monodromy Theorem* and describe multi-valued functions in a punctured disc. In Section 7 we construct the $\mathbf{p}^*$-function which will be used to solve the Dirichlet problem in Chapter 5. In Section 8 we prove result due to Eisenstein. The proof is instructive since it teaches how to manipulate with multiple-valued algebraic root functions. In Section 9 we recall the important *Spiegelungsprinzip* by Hermann Schwarz. It is applied in Section 10 to construct the *modular function* which is needed to establish the uniformisation theorem for domains in $\mathbf{C}$. Finally Section 11 gives a brief exposition about some of the contributions by Poincaré which has played a major role for the theory of Fuchsian groups and automorphic functions.

1. Angular variation and winding numbers

Let (x, y) be the coordinates in $\mathbf{R}^2$ and consider a vector-valued function of the real parameter t:

$$t \mapsto (x(t), y(t)) \quad : \quad 0 \leq t \leq T$$

We assume that the image does not contain the origin, i.e. $x^2(t) + y^2(t) > 0$ hold for each t and $x(t)$ and $y(t)$ are both continuously differentiable. Set

$$\dot{x}(t) = \frac{dx}{dt} \quad \text{and} \quad \dot{y}(t) = \frac{dy}{dt} \quad \text{and} \quad r(t) = \sqrt{x(t)^2 + y(t)^2}$$

1.1 Proposition. *There exists a unique continuous map $t \mapsto \phi(t)$ such that*

(*) $$x(t) = r(t) \cdot \cos\phi(t) \quad : \quad y(t) = r(t) \cdot \sin\phi(t) \quad : 0 \leq t \leq T$$

where the ϕ-function satisfies the initial condition:

$$x(0) = r(0) \cdot \cos\phi(0) \quad \text{and} \quad y(0) = r(0) \cdot \sin\phi(0$$

Proof. We solve the first order ODE-equation.

$$\dot{\phi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}$$

with initial condition $\phi(0)$ as above. There remains to show that (*) holds for all t. Let us for example verify the $x(t) = r(t) \cdot \cos\phi(t)$. It suffices to prove that

(1) $$\frac{d}{dt}(r(t) \cdot \cos\phi(t)) = \dot{x}$$

To prove this we notice that the left hand side becomes

(2) $$\dot{r} \cdot \cos(\phi) - r \cdot \sin(\phi)\dot{\phi} = \frac{\dot{r} \cdot x}{r} - y \cdot \dot{\phi}$$

Next, since $r = \sqrt{x^2 + y^2}$ we have

$$r\dot{r} = x\dot{x} + y\dot{y}$$

Hence (2) is equal to

$$\frac{x^2\dot{x} + xy\dot{y}}{r^2} - \frac{y(x\dot{y} - y\dot{x})}{r^2} = \frac{(x^2 + y^2)\dot{x}}{r^2} = \dot{x}$$

This proves Proposition 1.1

1.2 The angular variation. Since the sine and the cosine functions are 2π-periodic, the ϕ-function above is only uniquely determined up to integer multiples of 2π. A specific choice arises from the value $\phi(0)$. However, we get an *intrinsic number* by the difference

(**) $$\phi(T) - \phi(0)$$

This number is called the *angular variation* of the function $t \mapsto (x(t), y(t))$. If we choose another parametrization where $t = t(\tau)$ is non-decreasing and $0 \leq \tau \leq T^*$, then we start with the vector valued function $\tau \mapsto x(t(\tau), y(t(\tau))$ and find $\phi(\tau)$. Calculus shows that (**) is the same. Thus, the angular variation of an oriented parametrized C^1-curve is intrinsically defined.

A notation. The angular variation along a parametrized curve γ is denoted by $\mathfrak{a}(\gamma)$. If γ is a curve we can construct the curve γ^* with the opposite direction:

$$\gamma^*(t) = \gamma(T - t)$$

It is clear that

$$\mathfrak{a}(\gamma^*) = -\mathfrak{a}(\gamma)$$

In other words, up to a sign the angular variation is determined by the orientation of the curve.

1.3 The case of closed curves.

If $x(0) = x(T)$ and $y(0) = y(T)$ then the variation is an integer multiple of 2π. So if γ is any closed parametrized curve then $\mathfrak{a}(\gamma)$ is an integer multiple of 2π and we set

$$\mathfrak{w}(\gamma) = \frac{\mathfrak{a}(\gamma)}{2\pi}$$

We refer to $\mathfrak{w}(\gamma)$ as the *winding number* of γ. By the construction one has

$$\mathfrak{w}(\gamma) = \frac{1}{2\pi} \int_0^T \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \cdot dt \tag{1.1}$$

Example. Let m be a positive integer and

$$x(t) = \cos mt \quad : \quad y(t) = \sin mt$$

Notice that $x^2 + y^2 = 1$. It follows that

$$\frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \cdot dt = \cos mt \cdot m \cdot \cos mt - \sin mt \cdot (-m \cdot \sin mt) = m$$

Hence the winding number is equal to m.

1.4 Homotopy invariance. Consider a family of closed curves

$$\{\gamma_s\colon\ 0 \leq s \leq 1\} \quad : \gamma_s(0) = \gamma_s(T) : \quad 0 \leq s \leq 1$$

Let $t \mapsto (x_s(t), y_s(t))$ be the parametrization of γ_s. For each fixed s the curve $t \mapsto \gamma_s(t)$ has a winding number $\mathfrak{w}(\gamma_s)$. *Assume* that the two C^1-functions depend continuously upon s. Then $s \mapsto \mathfrak{w}(\gamma_s)$ is a continuous function. Since it is an integer-valued it must be a constant. Hence we have proved

1.5 Theorem *Let $\{\gamma_s\}$ be a homotopic family of closed curves Then they have the same winding number.*

Remark. In topology one refers to this by saying that the winding number is the same in each homotopy class of closed parametrized curves which surround the origin. A parametrized curve γ is defined by a map $t \mapsto \gamma(t)$ which need not be 1-1, i.e. we only assume that $\gamma(0) = \gamma(T)$. One may think of an insect which takes a walk on the horisontal (x, y)-plane starting at point p at time $t = 0$ and returns to p after a certain time interval T. During this walk the insect may cross an earlier path several times and even walk in the same path but in opposed direction for a while. The sole constraint is that the insect never attains the origin, i.e. $x(t)^2 + y(t)^2 > 0$ must hold in order to construct the winding number.

1.6 The case of non-closed curves. Let p and q be two points outside the origin. Consider two curves γ_1 and γ_2 where p is the common initial point and q the common end-point. Now we get the closed curve ρ defined by

$$\rho(s) = \gamma_1(2s) \quad : 0 \leq s \leq T/2 \text{and} \quad \rho(s) = \gamma_2(2T - 2s) \quad : T/2 \leq s \leq T$$

Here we find that

$$\mathfrak{w}(\rho) = \mathfrak{a}(\gamma_1) - \mathfrak{a}(\gamma_2)$$

Next, keeping p and q fixed we consider a continuous family of curves $\{\gamma_s\}$ where $\gamma_s(0) = p$ and $\gamma_s(T) = q$ for all $0 \leq s \leq 1$. To each s we get the two curves γ_0 and γ_s and construct the closed curve ρ as above. Theorem 1.5 implies that the difference

$$\mathfrak{a}(\gamma_0) - \mathfrak{a}(\gamma_s)$$

is a constant function of s. Since the difference obviously is zero when $s = 0$ we conclude that

$$\mathfrak{a}(\gamma_0) = \mathfrak{a}(\gamma_s) \quad : \ 0 \leq s \leq 1$$

Thus, the angular variation is constant in a homotopic family of curves which join a pair of points p and q.

1.7 Rouche's principle. Let γ_* be a parametrized closed curve. Suppose that γ is another closed curve such that

(i) $$|\gamma_*(t) - \gamma(t)| < |\gamma_*(t)| \quad : \quad 0 \leq t \leq T$$

To each $0 \leq s \leq 1$ we obtain the closed curve $\gamma_s(t) = s \cdot \gamma_* + (1-s)(\gamma_*(t) - \gamma(t))$ which by (i) also the surrounds the origin. This gives a homotopic family and Theorem 1.5 gives:

(*) $$\mathfrak{w}(\gamma_*) = \mathfrak{w}(\gamma)$$

Variation of vector-valued functions

Let γ be a parametrized C^1-curve. Here we do not exclude that $\gamma(t) = (0,0)$ for some values of t, i.e. γ is an arbitrary C^1-curve. Consider a pair of C^1-functions $u(x,y)$ and $v(x,y)$ defined in some neighborhood of the compact image set $\Gamma = \gamma([0,T])$. Assume that $u^2 + v^2 \neq 0$ on Γ. Then we get a curve γ^* which surrounds the origin defined by

(i) $$t \mapsto (u(\gamma(t)), v(\gamma(t))$$

Write $\gamma(t) = (x(t), y(t))$ and set

$$\xi(t) = u(x(t), y(t)) \quad : \quad \eta(t) = v(x(t), y(t))$$

Then we have

(ii) $$\mathfrak{a}(\gamma^*) = \int_0^T \frac{\xi\dot{\eta} - \eta\dot{\xi}}{\xi^2 + \eta^2} \cdot dt$$

Now $\dot{\xi} = u_x\ddot{x} + u_y\dot{y}$ and similarly for $\dot{\eta}$. So the last integral becomes

(*) $$\int_0^T \frac{u(v_x\dot{x} + v_y\dot{y}) - v(u_x\dot{x} + u_y\dot{y})}{u^2 + v^2} \cdot dt$$

This yields an integer which we refer to as the variation of the vector valued function (u,v) along the closed curve γ. We denote this integer by a subscript notation and write $\mathfrak{a}_{(u,v)}(\gamma)$. In the case when γ is a closed curve we define the winding number

$$\mathfrak{w}_{(u,v)}(\gamma) = \frac{1}{2\pi} \cdot \mathfrak{a}_{(u,v)}(\gamma)$$

Notice that this integer depends upon the pair (u,v) while γ is kept fixed.

1.8 The case of CR-pairs

Let γ be a curve and $f(z) = u + iv$ an analytic in a neighborhood of $\gamma(T)$ where we assume that $f(\gamma(t)) \neq 0$ for all t. Hence $u^2 + v^2 \neq 0$ on γ so we can define $\mathfrak{a}_{(u,v)}(\gamma)$. Now (u,v) satisfy the Cauchy-Riemann equations which enables us to express $\mathfrak{a}_{(u,v)}(\gamma)$ in an elegant way. Namely let $t \mapsto (x(t), y(t))$ be a parametrization of γ and write $z(t) = x(t) + iy(t))$. Then

$$\dot{z} = \dot{x} + i\dot{y}$$

Now we regard the function

(i) $$t \mapsto \mathfrak{Im}\,[\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t)\,]$$

Since the complex derivative $f'(z) = u_x + iv_x$ we obtain

(ii) $$\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t) = \frac{(u_x + iv_x)(u - iv)(\dot{x} + i \cdot \dot{y})}{u^2 + v^2}$$

The imaginary part becomes

(iii) $$\frac{u_xu\dot{y} - u_xv\dot{x} + v_xu\dot{x} + v_xv\dot{y}}{u^2 + v^2} = \frac{u(u_x\dot{y} + v_x\dot{x}) - v(u_x\dot{x} - v_x\dot{y})}{u^2 + v^2}$$

Next, we can apply the Cauchy-Riemann equations and replace u_x with v_y and $-v_x$ by u_y. Then we see that (iii) is equal to the integrand which appears in (*) in 1.7. Hence we have proved the following:.

1.9 Theorem *Let $f(z) = u + iv$ be holomorphic in a neighborhood of γ and set $\mathfrak{a}_f(\gamma) = \mathfrak{a}_{(u,v)}(\gamma)$. Then*

$$(*) \qquad \mathfrak{a}_f(\gamma) = \int_0^T \mathfrak{Im}\,\Big[\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t)\Big] \cdot dt$$

1.10 Remark By the construction of complex line integrals, the integral (*) above can be written as

$$\frac{1}{i} \cdot \int_\gamma \mathfrak{Im}\,\Big[\frac{f'(z)}{f(z)}\Big] \cdot dz$$

This complex notation is often used. When γ is a closed curve we get the winding number

$$\mathfrak{w}_f(\gamma) = \frac{1}{2\pi i} \cdot \int_\gamma \mathfrak{Im}\,\Big[\frac{f'(z)}{f(z)}\Big] \cdot dz$$

So this complex line integral always is an integer whenever $f(z)$ is analytic and $\neq 0$ in some open neighborhood of the compact set $\gamma([0,T])$.

1.11 Jordan's curve theorem

Let γ be a closed C^1-curve and set $\Gamma = \gamma([0,T])$. To each $a \in \mathbf{C} \setminus \Gamma$ the closed curve

$$t \mapsto \frac{1}{\gamma(t) - a}$$

surrounds the origin. Its winding number denoted by $\mathfrak{w}_a(\gamma)$. From (*) in 1.4 we see that this winding number is constant in every connected component if $\mathbf{C} \setminus \Gamma$.

1.12 The case when γ is 1-1 Assume that $\gamma(t)$ is 1-1 except for the common end-values. This means that the image curve $t \mapsto \gamma(t)$ is a *closed Jordan curve*. For each $a \in \mathbf{C} \setminus \Gamma$ we notice that $t \mapsto \gamma(t) - a$ is 1-1. In the equation from XX which determines the ϕ-function for a given a where we may take $\phi(0) = 0$ as initial value shows that $t \mapsto \gamma(t)$ is 1-1 on the open interval $(0,T)$. Hence $\phi(t)$ cannot be an integer multiple of 2π when $0 < t < T$. Starting with $\phi(0) = 0$ it follows that

$$-2\pi < \phi(t) < 2\pi \quad : \quad 0 < t < 2\pi$$

Hence $\phi(T)$ can only attain one of the values $-2\pi, 0, 2\pi$. The *Jordan curve theorem* tells us that the value zero is never attained. Moreover, the set of points a for which the winding number equals 1 is a conneced open set, called the Jordan domain bounded by Γ. The complementary set is also connected and here $\mathfrak{w}_a(\gamma) = 0$. This can be expressed by saying that the closed Jordan curve Γ divides $\mathbf{C}$ into two component. This topological result was proved by Camille Jordan in 1850 and it is is actually valid under the relaxed assumption that the γ-function is only continuous. In that case the proof of Jordan's Curve Theorem is more demanding. For a detailed proof of the continuous version of Jordan's Curve Theorem we refer [Newmann] where methods of algebraic topology are used. We remark that Jordan's theorem in the plane is subtle in view of a quite remarkable discovery in dimension 3 due to X. Alexander who constructed a *homeomorphic copy* of the unit sphere in R^3 where the analogue of Jordan's theorem is not valid. This goes beyond the scope of these notes. A recommended text-book in algebraic topology is Alexander's classic text-book [Al] which gives an excellent introduction to the subtle parts of the theory.

1.13 The case of a simple polygon. Let $p_1, \ldots, p_N$ be distinct points in $\mathbf{C}$ where $N \geq 3$. To each $1 \leq \nu \leq N-1$ we get a line segment $\ell_\nu = [p_\nu, p_{\nu+1}]$ and we also get the line segment $\ell_N = [p_N, p_1]$. Assume that they do not intersect.Then they give sides of a simple closed curve

Γ whose corner points are $p_1,\ldots,p_N$. The circle Γ is oriented where one travels in the positive direction from p_ν to $p_{\nu+1}$ when $1\leq\nu\leq N-1$ and makes the final positive travel from p_N to p_1. We can imagine a narrow channel $\mathcal{C}_+$ which surrounds Γ and from this one can "escape" to the point at infinity. For example at a corner point p_ν where $|p_\nu|$ is maximal the channel contains points of absolute value >1. From this picture it is clear the the *outer component* Ω_∞ of Γ is connected - and even simply connected if one adds the point at infinity. Rouche's principle shows that the winding number is zero for all points in the exterior component. If we instead construct a narrow channel $\mathcal{C}_*$ which moves "just inside" Γ then the channel itself is obviously connected. But their remains to see why the whole interior is connected and that the common winding number is equal to one. This, if Ω_* is the open complement of $\Gamma\cup\Omega_\infty$ we must first prove that Ω_* is connected. Since the narrow channel $\mathcal{C}_*$ is connected it suffices to show that when $p\in\Omega_*$ then there exists some curve γ from p which reaches $\mathcal{C}_*$. To obtain γ we consider a point $p^*\in\Gamma$ such that $|p-p^*|$ is the distance of p to Γ, i.e. we pick a point nearest to p. Now we draw the straight line L through p and p^* and by a picture the reader discovers that if we travel along L from p towards p^* then we reach $\mathcal{C}_*$ prior to the arrival at p^*. This proves that Ω_* is connected. The proof that the common winding number for points in Ω_* is equal to one is left as an *exercise* to the reader.

2. The argument principle

Let $\Omega \in \mathcal{D}(C^1)$ and $f(z)$ is an analytic function in Ω which extends to a C^1-function on its closure. Denote by $\mathcal{N}_\Omega(f)$ the number of zeros of f in Ω. We also assume that $f \neq 0$ on $\partial\Omega$.

2.1 Theorem. *Let $\Omega \in \mathcal{D}(C^1)$ and let $\Gamma_1, \ldots, \Gamma_k$ be its simple and closed boundary curves. Then*

$$N_\Omega(f) = \sum_{\nu=1}^{\nu=k} \mathfrak{w}_f(\Gamma_\nu)$$

Proof. By Theorem III.XX we have

$$N_\Omega(f) = \sum \frac{1}{2\pi i} \cdot \int_{\Gamma_\nu} \frac{f'(z)dz}{f(z)}$$

Since $N_\Omega(f)$ is an integer and hence a real nmber this gives

$$N_\Omega(f) = \sum \frac{1}{2\pi} \cdot \int_{\Gamma_\nu} \mathfrak{Im}\,[\frac{f'(z)dz}{f(z)}\,]$$

By Theorem 1.9 expressed in the complex notation each term of the sum above is equal to $\mathfrak{w}_f(\Gamma_\nu)$ and Theorem 2.1 follows.

2.2 Rouche's Theorem. *Let Ω and f be as above and let g be another holomorphic function in Ω which extends to be C^1 on the closure. If $|g| < |f|$ holds on $\partial\Omega$, it follows that*

$$\mathcal{N}_{f+g}(\Omega) = \mathcal{N}_f(\Omega)$$

Proof. Apply the result in 1.6.

2.3 An application to trigonometric series. Let $1 \leq m < n$ be a pair of positive integers. Consider a trigonometric polynomial

$$P(\theta) = \sum_{\nu=m}^{\nu=n} a_\nu \cos(\nu\theta) + b_\nu \sin(\nu\theta) \quad : \quad a_\nu, b_\nu \in \mathbf{R}$$

We assume that at least one of the coefficients a_m or b_m is $\neq 0$, and similarly at least one of the numbers a_n or b_n is $\neq 0$. Then one has

2.4 Theorem *P has at least $2m$ zeros on $[0, 2\pi]$ counted with multiplicity.*

Proof Consider the polynomial

$$Q(z) = (a_m - ib_m)z^m + \ldots + (a_n - ib_n)z^n$$

Notice that $\mathfrak{Re}(Q(e^{i\theta}) = P(\theta)$. The polynomial Q has a zero of multiplicity m at the origin. Consider some $r < 1$ chosen so that $Q \neq 0$ on the circle $T_r = \{|z| = r\}$. Since $Q(z)$ has at least m zeros counted with multiplicity in the disc D_r, it follows from Theorem 2.2 that

$$\mathfrak{w}_Q(T_r) \geq m$$

Regarding a picture the reader discovers that the curve $\theta \mapsto Q(re^{i\theta})$ must intersect the imaginary axis line at least $2m$ times which means that the function

$$\theta \mapsto \sum_{\nu=m}^{\nu=n} r^\nu \cdot a_\nu \cos(\nu\theta) + r^\nu \cdot b_\nu \sin(\nu\theta)$$

as at least $2m$ distinct zeros on $[0, 2\pi]$. Passing to the limit as $r \to 1$ we get Theorem 2.4.

2.5 A special estimate. Theorem 2.1 can be used to give upper bounds for the counting function $\mathcal{N}_\Omega(f)$. Suppose that Ω is a rectangle

$$\{z = x + iy \colon\ a < x < b \colon\ 0 < y < T\}$$

Here $\partial\Omega$ contains the vertical line $\ell = \{x = b : \ 0 < y < T\}$. The line integral along ℓ contributes to the evaluation of $\mathcal{N}_\Omega(f)$ by

$$\frac{1}{2\pi} \cdot \int_\ell \mathfrak{Im} \, | \frac{f'(z)dz}{f(z)}]$$

Now $dz = idy$ along ℓ and therefore the integral above is equal to

$$\frac{1}{2\pi} \cdot \int_0^T \mathfrak{Re}[\frac{f'(b+iy)}{f(b+iy)}] \cdot dy$$

Let us now assume that $\mathfrak{Re}\, f(b+iy) \geq c_0 > 0$ for all $0 \leq y \leq T$. Then there exists a single valued branch of the Log-function, i.e.

$$\operatorname{Log} f(b+iy) = \operatorname{Log}|f(b+iy)| + i \cdot \arg(f(b+iy)) \ : \ -\pi/2 < \arg(f(b+iy)) < \pi/2$$

Since $f'(z) = \frac{1}{i} \cdot \partial_y(f)$ it follows that

$$\frac{f'(b+iy)}{fb+iy)} = \frac{1}{i} \cdot [\partial_y(\operatorname{Log}|f(b+iy)|) + i \cdot \partial_y(\arg(f(b+iy)))]$$

Hence we obtain

2.6 Proposition. *One has the equality*

$$\mathfrak{Re}\, \frac{f'(b+iy)}{f(b+iy)} = \partial_y(\arg(f(b+iy)))$$

2.7 Remark. Proposition 2.6 gives therefore

$$(*) \qquad \frac{1}{2\pi} \cdot \int_\ell \mathfrak{Im} \, | \frac{f'(z)dz}{f(z)}] = \frac{1}{2\pi} \cdot \arg(f(b+iT))) - \arg(f(b)))$$

The right hand side is a real number in $(-1/4, 1/4)$ and hence we get a small contribution from the line integral in the left hand side when we regard whole line integral over $\partial\Omega$ which evaluates $\mathcal{N}_f(\Omega)$. This will be used to study the zeros of Riemann's ζ-function.

2.8 A local implicit function theorem. Let $m \geq 2$ and $g_2(z), \ldots, g_m(z)$ are analytic functions defined in an open disc D centered at $z = 0$ where $g_\nu(0) = 0$ for every ν. Let also $\phi(z)$ be another analytic function in D with $\phi(0) = 0$. Consider the equation

$$(*) \qquad y + g_2(z)y^2 + \ldots + g_m(z)y^m = \phi(z)$$

Thus, we seek $y(z)$ so that (*) holds. It turns out that there exists a unique analytic function $y(z)$ defined in some open disc D_* centered at $z = 0$ where $y(0) = 0$ and (*) hold for every $z \in D_*$. To prove this we set

$$P(y, z) = y + g_2(z)y^2 + \ldots + g_m(z)y^m$$

Now we can find $\delta > 0$ such that if $z \in D(\delta)$ then

$$\text{(i)} \qquad |\phi(z)| < |P(e^{i\theta}, z)| \quad \text{for all} \quad 0 \leq \theta \leq 2\pi$$

Next, let us put

$$P'_y(y, z) = 1 + 2g_2(z)y + \ldots + mg_m(z)y^{m-1}$$

From (i) there exists the integral

$$(1) \qquad \frac{1}{2\pi i} \cdot \int_{|y|=1} \frac{P'_y(y, z)}{P(y, z) - \phi(z)} \cdot dy \quad : \ z \in D(\delta)$$

By Rouche's Theorem this integer-valued function is constant when z varies in $D(\delta)$. When $z=0$ the integrand is $\frac{1}{y}$ and hence the constant integer is 1. This means that when $z\in D(\delta)$ is fixed, then the analytic function

$$y\mapsto P(y,z)-\phi(z)$$

has exactly one simple zero in $|y|<1$. Denote this zero by $y(z)$. The residue formula gives:

$$y(z)=\frac{1}{2\pi i}\cdot\int_{|y|=1}\frac{y\cdot P'_y(y,z)}{P(y,z)-\phi(z)}\cdot dy \tag{2}$$

It is clear that $y(z)$ is analytic in $D(\delta)$ and by the construction $P(y(z),z)=0$. Thus, $y(z)$ is the required solution.

2.9 The case of higher multiplicity. This time we consider the equation

$$P(z,y)=g_m(z)+g_1(z)y+g_2(z)y^2+\ldots+g_m(z)y^m=0$$

where $g_k(0)\neq 0$ for some $k\geq 2$ while $g_m(0)=\ldots=g_{k+1}(0)=0$. No special, assumption is imposed on $g_1(0),\ldots,g_{k-1}(0)$, i.e. some of these numbers may be $\neq 0$. Since $g_(k)\neq 0$in some disc around $z=0$ and we study a homogeneous equation we can divide out g_k and assume that it is 1 from the start. Let us then consider the polynomial

$$Q(y)=y^k+g_{k-1}(0)y^{k-1}+g_1(0)y+g_(0)$$

It has k zeros counted with multiplicity and we choose R so large that these zeros all belong to $|y|<R$. With R kept fixed it is clear that there exists $\delta_*>0$ such that

$$|z|<\delta_* \implies |g_m(z)y^m+\ldots+g_{k+1}(z)y^{m+1}|<|y^k+g_{k-1}(z)y^{k1}+\ldots+g_0(z)|$$

when $|y|=R$. Hence Rouche's theorem implies that $y\mapsto P(z,y)$ has m zeros in $|y|<R$ for each $|z|<\delta_*$. in general this m-tuple of zeros do not give rise to single-valued analytic functions in $|z|<\delta_*$. In XX we describe the multi-valued behaviour of these root functions in more detail.

2.10 Images of closed curves. Let γ be a closed Jordan curve which is parametrized by arc-length. So we have a map $s\to z(s)$ where $\gamma(0)=\gamma(L)$ and we assume that this function is C^1 with a non-zero derivative, i.e. with $z=x+iy$ the two functions $x(s)$ and $y(s)$ are both of class C^1 and $x'(s)^2+y'(s)^2>0$ hold when $0\leq s\leq L$. Suppose now that $f(z)$ is analytic in some open neighborhood of γ and that the absolute value $|f|$ is a constant $c>0$ on γ. We find a unique continuous function $\theta(s)$ such that

$$f(z(s))=c\cdot e^{i\theta(s)} \tag{i}$$

where $\theta(0)$ is determined by the equality $f(z(0))=c\cdot e^{i\theta(0)}$. Taking the derivative with respect to s we get

$$f'(z(s))\cdot z'(s)=c\cdot i\theta'(s)\cdot e^{i\theta(s)} \implies$$

$$\frac{f'}{f}\cdot z'(s)=i\cdot\theta'(s) \tag{i}$$

It follows that

$$\frac{1}{2\pi i}\cdot\int_\gamma\frac{f'\cdot dz}{f}=\frac{1}{2\pi}\int_0^{2\pi}\theta'(s)\cdot ds=\frac{\theta(L)-\theta(0)}{2\pi} \tag{*}$$

Let us now assume now that the left hand side is 1 which gives

$$\theta(L)=\theta(0)+2\pi \tag{ii}$$

In addition to this we assume that the complex derivative $f'\neq 0$ on γ. Then (i) shows that the s-derivative of the real-valued θ-function is always $\neq 0$ and since the value increases by (ii) we have $\theta'(s)>0$ for all s. From this we con clude

2.10 Theorem. *Assume that $f'\neq 0$ on γ and that the left hand side in (*) is one. Then f yields a bijective map from γ onto the circle of radius c centered at the origin.*

Next, with the assumptions in Theorem 2.10 we consider a complex number w of absolute value < 1. Since γ is a closed curve we know that

$$(1) \qquad \frac{1}{2\pi i}\cdot\int_\gamma \frac{f'\cdot dz}{f-w}$$

is an integer and just as in Rouche's theorem we conclude that (1) is equal to one for every $|w|<1$. The reader may also verify that

$$(2) \qquad |w|>1 \implies \frac{1}{2\pi i}\cdot\int_\gamma \frac{f'\cdot dz}{f-w}=0$$

Remark. Notice that (1-2) hold without the hypothesis that f extends to be analytic in the Jordan domain Ω bordered by γ. On the other hand, if we in addition assume that f from the start is analytic in a neighborhood of $\bar{\Omega}$ then (1) shows that $f(z)-w$ has exactly one zero in Ω for each $|w|<1$ which means that f yields a conformal map from Ω onto the disc $|w|<c$. This, to check when a given $f\in\mathcal{O}(\Omega)$ which extends to γ where $|f|=c$ and $f'\neq 0$, it suffices to check that the left hand side in (*) is equal to one in order that f yields a conformal map.

2.11 A conformal condition. Let γ and f be as above where f extends to be analytic in a neighborhood of γ.and we also assume that the curve γ is real-analytic, i.e. now $z(s)$ is a real-analytic function of s. W e can find a closed Jordan curve γ_* contained in the Jordan domain Ω which together with γ borders a doubly-connected domain U. Here γ_* can be close to γ as illustrated by figure XXX, Then one has

2.12 Theorem. *Assume that the restriction of f to U is 1-1. Then f yields a bijective map from γ onto the circle of radius c.*

Proof. We may assume that $\theta'(0)>0$ and Theorem xx follows if we prove that $\theta'(s)>0$ for all s. If $\theta'(s_0)=0$ for some $0<s_0<L$ and $z_0=z(s_0)$ we have $f'(z_0)=0$. Hence we have a Taylor series with

$$f(z_0+\zeta)=f(z_0)+c_2\zeta^2+c_3\zeta^3+\ldots$$

Let m be the smallest integer such that $c_m\neq 0$. We claim that $m=2$ must hold. To see this we use tha γ is of class C^1 which means that if $\epsilon>0$ is sn´mall then

$$\{|z-z_0|<\epsilon\}\cap U=V$$

is almost a small half-disc.

Exercise. Show that if $m\geq 3$ with $c_m\neq 0$ then the restriction of f to V cannot be 1-1.

We conclude that if $\theta'(s_0)=0$ then the second order derivative $f''(z_0)\neq 0$. At the same time we notice that another derivation in (ii) above at $s=s_0$ where $f'(z_0)=0$ holds

$$\frac{f''(z_0)}{f(z_0)}\cdot z'(s_0(^2)=i\cdot\theta''(s_0)$$

So whenever $\theta'(s_0)=0$ it follows that $\theta''(s_0)\neq 0$.

Exercise. Show that when $f|U$ is 1-1 then $\theta'(s)\neq 0$ for all s and hence $f|\gamma$ is bijective.

2.13 The case when the winding number is zero. We keep the assumption that γ is a real-analytic closed Jordan curve and f extends to an analytic function which is 1-1 in a domain U as above while $|f(z)|=c$ has constant absolute value along γ. But this time we suppose that

$$(*) \qquad \int_\gamma \frac{f'\cdot dz}{f}=0$$

So here $\theta(L)=\theta(0)$ and after a rotation we may assume that this common value is zero. while $\theta'(0)>0$. It follows that $\theta(s)$ takes a maximum >0 for some s^* where we therefore get $\theta'(s^*)$. So we always find the smallest $0<s_0<L$ where $\theta'(s_0)=0$. So $s\to\theta(s)$ is strictly increasing on $[0,s_0)$ and it is clear that the hypothesis that $f|U$ is 1-1 entails that the range $\theta[0,s_0]$ is an interval $[0,\theta_0|$ where $\theta_0<2\pi$, i.e. the θ-function has not made a full turn so the image set $f(\gamma[0,s_0|)$ is an interval on the circe $|w|=c$ where we write $w=f(z)$.

Exercise. Show by a similar reasoning as in XxX that $\theta''(s_0) \neq 0$ which means that $\theta(s)$ starts to decrease on some interval $s_0 \leq s \leq s_1$ until $\theta'(s_1) = 0$ which must occur for some $s_1 < L$ since $\theta'(L) \neq 0$ was assumed. Keeping this in mind the reader should supply a figure and verify the details of the following result:

2.14 Theorem. *f maps γ onto a proper circular interval of $|w| = c$ where it yields a double cover except at the two end-points of the interval given by $f(\theta(s_0)$ and $f(\theta(s_1))$ where the θ-function achieves its maxium respectively its minium.*

Example. Consider the analytic function

$$f(z) = z + \frac{1}{z}$$

When γ is the unit circle we have

$$f(e^{i\theta}) = 2 \cdot \cos\ \theta$$

This yields a double-cover onto the interval $[-2, 2]$. At the same time one easily verifies f restricts to a 1-1 map in the punctured disc $0 < |z| < 1$ and it is also 1-1 in the exterior disc $|z| > 1$. In fact, in the extended w-plane we have the simple connected domain $\Omega^* = \mathbf{C} \cup \infty \setminus [2, 2]$ and f is a conformal map from the union of the exterior disc $|z| > 1$ and $z = \infty$ onto Ω^*. The double cover of $[-2, 2]$ arises from this conformal map and Ω^* is an example of a so called slit-domain.

3. Multi-valued functions

Let Ω be an open connected subset of $\mathbf{C}$ and $D \subset \Omega$ is an open disc of some radius r centered at a point z_0. The material about power series in Chapter XX shows that $\mathcal{O}(D)$ is identified with convergent power series

$$\sum c_\nu (z - z_0)^\nu \quad : \quad \text{radius of convergenece } \geq r$$

So if $f \in \mathcal{O}(\Omega)$ its restriction to any disc $D \subset \Omega$ determines a convergent power series. These power series must be matching when two discs have a non-empty intersection. This observation is the starting point for a general construction due to Weierstrass.

3.1 Analytic continuation along paths Let $s \mapsto \gamma(s)$ be a continuous and complex valued function with values in Ω. We do not require that $\gamma(0) = \gamma(1)$ or that γ is 1-1. The points $p = \gamma(0)$ and $q = \gamma(1)$ are called the terminal points of γ. Let $f_0 \in \mathcal{O}(D_r(p))$ for some $r > 0$, i.e. f is analytic in a small disc centered at p. Consider a strictly increasing sequence $0 = s_0 < s_1 < \ldots < s_N = 1$ and to each $p_\nu = \gamma(s_\nu)$ we choose a small disc $D_{p_\nu}(r_\nu)$ such that:

$$D_{p_\nu}(r_\nu) \cap D_{p_{\nu+1}}(r_{\nu+1}) \neq \emptyset \qquad 0 \leq \nu \leq N-1$$

Assume that for each $1 \leq \nu \leq N$ exists $f_\nu \in \mathcal{O}(D_{r_\nu}(p_\nu))$ such that

$$f_\nu = f_{\nu+1} \text{ holds in } D_{p_\nu}(r_\nu) \cap D_{p_{\nu+1}}(r_{\nu+1})$$

After N many *direct analytic continuations over pairs of intersecting discs* we arrive at f_N which is analytic in an open disc centered at $\gamma(1)$. From the uniqeness of each direct analytic continuation, it follows that the locally defined analytic function f_N at $\gamma(1)$ is the same if we instead have chosen a *refined* partition of $[0,1]$. Since two coverings of γ via finite families of discs have a common refinement, we conclude that locally defined analytic function at the end-point is unique. Thus, the construction yields a map T_γ map which sends an analytic function f defined in a disc around $\gamma(0)$ to an analytic function $T_\gamma(f)$ defined is some disc centered at $\gamma(1)$. Of course, here T_γ is only defined on those f at $\gamma(0)$ which have an analytic continuation along γ in the sense of Weierstrass.

3.2 The class $M\mathcal{O}(\Omega)$. Let Ω be an open subset of $\mathbf{C}$. At each point $z \in \Omega$ we denote by $\mathcal{O}(z_0)$ the germs of analytic functions at z_0 and recall that this set is identified with power series $\sum c_\nu (z - z_0)^\nu$ which have some positive radius of convergence. In $\mathcal{O}(z_0)$ we can consider those germs which have analytic continuation along *every* curve in Ω whose initial point is z_0 while the end-point is arbitrary. This leads to:

3.3 Definition *A germ $f \in \mathcal{O}(z_0)$ generates a multi-valued analytic function in Ω if it can be extended in the sense of Weierstrass along every curve $\gamma \subset \Omega$ which has z_0 as initial point. The set of all these germs is denoted by $M\mathcal{O}(\Omega)(z_0)$.*

3.4 Remark. Notice that $M\mathcal{O}(\Omega)(z_0)$ contains those germs at z_0 which are induced by *single-valued* analytic functions in Ω. If $f \in M\mathcal{O}(\Omega)(z_0)$ and γ is a curve in Ω with z_0 as initial point and z_1 as end-point, then the germ $T_\gamma(f)$ at z_1 belongs to $M\mathcal{O}(\Omega)(z_1)$. This is obviuos since if γ_1 is a curve starting at z_1 with end-point at z_2, then f extends along the composed curve $\gamma_1 \circ \gamma_1$ and one has the composition formula:

$$(*) \qquad T_{\gamma_1}(T_\gamma(f)) = T_{\gamma \circ \gamma}(f)$$

3.5 The total sheaf space $\widehat{\mathcal{O}}$. We construct a big topological space $\widehat{\mathcal{O}}$ as follows: One has a map ρ from $\widehat{\mathcal{O}}$ onto $\mathbf{C}$. The inverse fiber $\rho^{-1}(z) = \mathcal{O}(z)$ for each $z \in \mathbf{C}$. An open neighborhood of a "point" $f \in \rho^{-1}(z_0)$ consists of a pair (f, D) where D is a small disc centered at z_0 such that the germ f extends to an analytic function in D. Then its induced germ at a point $z \in D$ belongs to $\rho^{-1}(z)$. The set of points in $\widehat{\mathcal{O}}$ obtained in this way yields the subset (f, D) and as D shrinks to z_0 they give by definition a fundamental system of open neighborhoods of the point

f in $\widehat{\mathcal{O}}$. With this topology on $\widehat{\mathcal{O}}$ the map ρ is a *local homeomorphism* and each inverse fiber $\rho^{-1}(z_0)$ appears as a *discrete* subset of $\widehat{\mathcal{O}}$.

Remark $\widehat{\mathcal{O}}$ is the first example of a sheaf which has led to the general construction of sheaves which is presented in elementary text on topology. The construction of the sheaf topology on $\widehat{\mathcal{O}}$ yields the following elegant description of multi-valued functions.

3.6 Proposition. *Let Ω be an open and connected subset of $\mathbf{C}$. Let $z_0 \in \Omega$ and $f \in M(\Omega)(z_0)$. Then f appears in the inverse fiber $\rho^{-1}(z_0)$ of an open and connected set $\mathcal{W}(f)$ of $\rho^{-1}(\Omega)$ called Weierstrass Analytische Gebilde of the germ of this multi-valued funtion. For each $z \in \Omega$ the set $\mathcal{W}(f) \cap \rho^{-1}(z)$ consists of all germs at z obtained by analytic continuation of f along some curve with end-point at z.*

Some notations. Let f be as above. If $z \in \Omega$ we denote by $W(f:z)$ the set of germs at z which arise via all analytic continuations of f. Thus, $W(f:z)$ is equal to $\mathcal{W}(f) \cap \rho^{-1}(z)$. In addition to this we can consider the set of values at z which are attained by these germs. So we have also the set

$$R_f(z) = \{T_\gamma(f)(z) \quad : \ T_\gamma(f) \in W(f:z)\}$$

Example Let $\Omega = \mathbf{C}$ minus the origin, i.e. the punctured complex plane. Then we have the multi-valued Log-function. At each point $z \in \Omega$ it has an infinite set of local branches which differ by integer multiples of $2\pi i$. The resulting connected set $\mathcal{W}(\text{Log}(z))$ can be regarded as a 2-dimensional connected manifold. In topology one learns that this is the *universal covering space* of Ω, so that $\mathcal{W}(\text{Log}(z))$ is a *simply connected* manifold.

3.7 Normal families.

Let Ω be some connected open set in $\mathbf{C}$. Let $x_0 \in \Omega$ and consider some germ $f \in M\mathcal{O}(\Omega)(x_0)$. We say that f yields a bounded multi-valued function if there exists a constant K such that

$$(*) \qquad \big|T_\gamma(f)(x)\big| \leq K$$

holds for all pairs x, γ where $x \in \Omega$ and γ is any curve from x_0 to x. Suppose that $\{f_\nu\}$ is a sequence of germs in $M\mathcal{O}(\Omega)(x_0)$ which are uniformly bounded, i.e. (*) above holds for some constant K and every ν. If we to begin with consider a small open disc D centered at x_0 we get the unique single-valued branches of each f_ν in $\mathcal{O}(D)$. This family in $\mathcal{O}(D)$ is normal by the results in XXX. Passing to a subsequence we may assume that there exists a limit function $g \in \mathcal{O}(D)$, i.e. shrinking D if necessary we may assume that

$$\text{(i)} \qquad \lim_{\nu \to \infty} ||f_\nu - g||_D \to 0$$

Next, if γ is a curve in which starts at x_0 and has some end-point x we cover γ with a finite number of open discs and each f_ν has its analytic continuation along γ by the Weierstrass procedure. From the material in XXX it is clear that during these analytic continuations the local series expansions of the sequence $\{f_\nu\}$ converge uniformly and as a result we find that g has an analytic extension along γ. Hence the germ of g at x_0 belongs to $M\mathcal{O}(\Omega)(x_0)$. Moreover, the uniform convergence "propagates". For example, if γ is a closed curve at x_0 we get the sequence $\{T_\gamma(f_\nu)\}$ after the analytic continuation along γ, and similarly $T_\gamma(g)$. Then

$$\text{(ii)} \qquad \lim_{\nu \to \infty} ||T_\gamma(f_\nu) - T_\gamma(g)||_D \to 0$$

holds for a small disc D centered at the end point of γ.

4. The Monodromy Theorem

Let $f \in M\mathcal{O}(\Omega)$. If $z_0 \in \Omega$ and γ is a curve starting at z_0 we obtain the germ $T_\gamma(f)$ at the end-point z_1 of γ. The analytic continuation is obtained by the Weierstrass procedure and since γ is a compact subset of Ω it can be covered by a finite set of discs $D_0, D_1, \ldots, D_N$ where $D_\nu \cap D_{\nu+1}$ are non-empty and the analytic continuation of f is achieved by succesive direct continuations of analytic functions $\{g_\nu \in \mathcal{O}(D_\nu)\}$ where $g_\nu = g_{\nu+1}$ holds in $D_\nu \cap D_{\nu+1}$. The discs are chosen so small that they are relatively compact in Ω. If γ_1 is another curve from z_0 to z_1 which stays so close to γ that the discs $D_0, \ldots, D_N$ again can be used to perform the analytic continuation of f along γ_1, then it is clear that $T_\gamma(f) = T_{\gamma_1}$. This observation gives:

4.1 Theorem *Let (z_0, z_1) be a pair in Ω and $\Gamma(s,t)$ a continuous map from the unit square in the (s,t)-space into Ω where*

$$\Gamma(s,0) = z_0 \quad : , \gamma(s,1) = z_1 \quad : \quad 0 \leq s \leq 1$$

Then, if $\{\gamma_s\}$ is the family of curves defined by $t \mapsto \Gamma(s,t)$, it follows that

$$T_{\gamma_s}(f) = T_{\gamma_0}(f) \quad : \quad 0 \leq s \leq 1$$

Remark This is called the monodromy theorem and can be expressed by saying that analytic continuation along a curve which joins a given pair of points only depends on the *homotopy* class of the curve, taken in the family of all curves which joint the two given points. Of course, when we deal with some multi-valued function in an open set Ω we are obliged to use curves inside Ω only.

4.2 The case of finite determination. Let $f \in M(\Omega)$. If $z_0 \in \Omega$ we get the set of germs $W(f : z_0)$ at z_0. This is a subset of $\mathcal{O}(z_0)$ and we can regard the complex vector space it generates. It is denoted by $\mathcal{H}_f(z_0)$. Suppose that this complex vector space has a finite dimension k. Then we can choose a k-tuple of germs $g_1, \ldots, g_k$ in $W(f : z_0)$ which give a basis of $\mathcal{H}_f(z_0)$. Thus, one has to begin with

$$\mathcal{H}_f(z_0) = Cg_1 + \ldots + Cg_k$$

Let z_1 be another point in Ω and fix some curve γ which joins z_0 and z_1. At z_1 we get the germs $T_\gamma(g_1), \ldots, T_\gamma(g_k)$. By the remark in XXX T_γ is a *bijective map* from $W(f : z_0)$ to $W(f : z_1)$. Moreover, if $\phi = c_1 g_1 + \ldots + c_k g_k$ belongs to $\mathcal{H}_f(z_0)$ we have

$$T_\gamma(\phi) = c_1 T_\gamma(g_1) + \ldots + c_k T_\gamma(g_k)$$

Hence the k-tuple $\{T_\gamma(g_\nu)\}$ generates the vector space $\mathcal{H}_f(z_1)$. Since we also can use the inverse map $T_{\gamma^{-1}}$ it follows that the k-tuple $\{T_\gamma(g_\nu)\}$ yields a basis of $\mathcal{H}_f(z_1)$. In particular the vector space $\mathcal{H}_f(z)$ have common dimension k as z varies in the connected open set Ω. *Summing up*, we can conclude the following:

4.3 Proposition *If $f \in M(\Omega)$ has finite determination the complex vector spaces $\mathcal{H}_f(z)$ have common dimension. Moreover, one gets a basis of these by starting at any point z_0 and choose some k-tuple of C-linearly germs $g_1, \ldots, g_k$ in $W(f : z_0)$. Then we obtain a basis in $\mathcal{H}_f(z)$ for any point $z \in \Omega$ by a k-tuple $\{T_\gamma(g_\nu)\}$ where γ is any curve which joins z_0 and z.*

4.4 The case of a punctured disc Let $\dot{D} = \{0 < |z| < R\}$ be a punctured disc centered at the origin. Consider some $f \in \mathcal{MO}(\dot{D})$ of finite determination and let k be its rank. In a punctured open disc every closed curve is homotopic to a closed circle parametrized by $\theta \mapsto re^{i\theta}$. Another way to express this is that the fundamental group $\pi_1(\dot{D})$ is isomorphic to the abelian group of integers. Thus, the multi-valuedness is determined by a sole T-operator which arises when we let γ be a circle surrounding the origin in the positive sense. Given $z_0 \in \dot{D}$ we consider the $\mathbf{C}$-linear operator

$$T_\gamma \colon \mathcal{H}_f(z_0) \mapsto \mathcal{H}_f(z_0)$$

By Jordan's decomposition theorem we can choose a basis in $\mathcal{H}_f(z_0)$ such that the matrix representing T_γ is of Jordan's normal form. This means that we have a direct sum

$$\mathcal{H}_f(z_0) = \oplus \mathcal{K}_\nu(z_0)$$

where $\{\mathcal{K}_\nu(z_0)\}$ are T_γ-invariant subspaces and the restriction of T_γ $\mathcal{K}_\nu(z_0)$ is represented by an elementary Jordan matrix $J(m,\lambda)$ for some complex number λ and $m \geq 1$. Given the pair m,λ we consider a local branch of the function

$$f(z) = z^\alpha \cdot [\operatorname{Log} z]^{m-1} \quad : e^{2\pi i\alpha} = \lambda$$

which for example is defined close to $z=1$ where $f(1)=0$. After one turn around the origin we get a new local branch of the form

$$f_1(z) = \lambda \cdot z^\alpha \cdot [\operatorname{Log} z + 2\pi i]^{m-1}$$

Continuing in this way m times we see that the local branches of f generate an m-dimensional complex vector space whose monodromy is determined by the matrix $J(m,\lambda)$. Using this fact it follows that if $f(z)$ is any local branch of a multi-valued function of finite determination, then it can be expressed as:

$$(*) \qquad f(z) = \sum_{\nu=1}^{k} \sum_j g_{\alpha_\nu,j}(z) \cdot z^{\alpha_\nu} \cdot [\operatorname{Log} z]^j$$

Here $0 \leq \mathfrak{Re}(\alpha_1) < \ldots < \mathfrak{Re}(\alpha_k) < 1$ and $\{j\}$ is a finite set of non-negative integers and the g-functions are *single-valued* in the punctured disc $\dot{D}$. Moreover these g-functions are uniquely determined provided a specific local branch of the Log-function is chosen. For example, when f is a local branch at some real point $0 < a < R$ where $\operatorname{Log} a$ is chosen to be real.

5. Homotopy and Covering spaces

Let X be a metric space, i.e. the topology is defined by some distance function. By a curve in X we mean a continuous map γ from the closed unit interval $[0,1]$ into X. In general γ need not be 1-1. The initial point is $\gamma(0)$ and the end point is $\gamma(1)$. If $\gamma(0)=\gamma(1)$ we say that γ is a *closed* curve. We say that X is *arcwise connected* if there to each pair of points x_0, x_1 exists some curve γ with $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$.

A notation. Given a point $x_0 \in X$ we denote by $\mathcal{C}(x_0)$ the family of all closed curves γ where $\gamma(0)=\gamma(1)=x_0$.

5.1 Definition. *A pair of closed curves γ_0 and γ_1 in $\mathcal{C}(x_0)$ are homotopic if there exists a continuos map Γ from the unit square $\square = \{(t,s) \quad : 0 \leq t, s \leq 1\}$ into X such that*

$$\Gamma(t,0) = \gamma_0(t) \text{ and } \Gamma(t,1) = \gamma_1(t) \quad \Gamma(0,s) = \Gamma(1,s) = x_0 \ : \ 0 \leq s \leq 1$$

It is clear that homotopy yields an equivalence relation on $\mathcal{C}(x_0)$. If $\gamma \in \mathcal{C}(x_0)$ then $\{\gamma\}$ denotes its homotopy class. Next, if γ_0 and γ_1 are two closed curves at x_0 we get a new closed curve γ_2 defined by

$$\gamma_2(t) = \gamma_1(2t) : \ 0 \leq t \leq \frac{1}{2} \text{ and } \gamma_2(t) = \gamma_2(2t-1) : \ \frac{1}{2} \leq t \leq 1$$

We refer to γ_2 as the composed curve and it is denoted by $\gamma_1 \circ \gamma_0$. One verifies easily that the homotopy class of γ_2 depends upon $\{\gamma_1\}$ and $\{\gamma_1\}$ only. In this way we obtain a composition law on the set of homotopy classes of closed curves at x_0 defined by

$$\{\gamma_1\} \cdot \{\gamma_0\} = \{\gamma_1 \circ \gamma_0\}$$

One verifies easily that this composition satisfies the associative law. A neutral element is the closed curve γ_* for which $\gamma_*(t) = x_0$ for every t. Finally, if $\gamma(t)$ is any closed curve at x_0 we get a new closed curve by reversing the direction, i.e. set

$$\gamma^{-1}(t) = \gamma(1-t)$$

Exercise. Show that the composed curve $\gamma^{-1} \circ \gamma$ is homotopic to γ_*.

5.2 The fundamental group. The construction of composed closed curves and the exercise above show that homotopy classes of closed curves at x_0 give elements of a group to be denoted by $\pi_1(X : x_0)$.

Remark. The group $\pi_1(X : x_0)$ is intrinsic in the sense that it does not depend upon the chosen point x_0. Namely, let x_1 be another point in X and fix a curve λ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$. Then we obtain a map from $\mathcal{C}(x_1)$ to $\mathcal{C}(x_0)$ defined by

(i) $$\gamma \mapsto \lambda^{-1} \circ \gamma \circ \lambda$$

One verifies that (i) sends homotopic curves to homotopic curves and by considering homotopy classes we obtain an isomorphism between $\pi_1(X : x_0)$ and $\pi_1(X : x_1)$. Hence there exists an intrinsically defined group denoted by $\pi_1(X)$. It is called the fundamental group of the metric space X. If $\pi_1(X)$ is reduced to a single element, i.e. when all closed curves in $\mathcal{C}(x_0)$ are homotopic we say that X is *simply connected*.

Exercise. Let x_0 and x_1 be two distinct points in X. Denote by $\mathcal{C}(x_0, x_1)$ the family of curves γ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Two such curves γ_0 and γ_1 are homotopic if there exists a continuos map Γ from the square $\square$ such that

$$\Gamma(t, 0) = \gamma_0(t) \text{ and } \Gamma(t, 1) = \gamma_1(t) \quad \Gamma(0, s) = x_0 \text{ and } \Gamma(1, s) = x_1 \;:\; 0 \le s \le 1$$

Show that a pair γ_0 and γ_1 are homotopic in $\mathcal{C}(x_0, x_1)$ if and only if the closed curve $\gamma_1^{-1} \circ \gamma_0$ is homotopic to γ^* in $\mathcal{C}(x_0)$ where

$$\gamma_1^{-1} = \gamma_1(1 - t)$$

In particular each pair of curves in $\mathcal{C}(x_0, x_1)$ are homotopic if X is simply connected.

5.3 Covering maps.

Let X and Y be two arcwise connected metric spaces. A continuous map ϕ from X onto Y is a local homeomorphism if the following hold: For each $y_0 \in Y$ there exists an open neighborhood U such that the inverse image $\phi^{-1}(U)$ is a union of pairwise disjoint open sets $\{U_\alpha^*\}$ and the restriction of ϕ to each U_α^* is a homeomorphism from this set onto U.

5.4 Lifting of curves. Let $\phi \colon X \to Y$ be a local homeomorphism where we assume that $\phi(X) = Y$. Let γ be a curve in Y defined by a continuous map $t \to \gamma(t)$ from the closed unit interval $[0, 1]$ into Y with some initial point $y_0 = \gamma(0)$ and some end-point $y_1 = \gamma(1)$. The case $y_0 = y_1$ is not excluded, i.e. γ may be a closed curve. Next, in X we chose a point x_0 such that $\phi(x_0) = y_0$. By assumption there exists an open neighborhood U of y_0 in X a unique open neighborhood U^* of x_0 such that $\phi \colon U^* \to U$ is a homeomorphism. Since $t \to \gamma(t)$is continuous there exists some $\delta > 0$ such that

(i) $$\gamma(t) \in U\,, \quad 0 \le t \le \delta$$

Then we get a *unique* curve γ^* in X defined for $0 \le t \le \delta$ such that

(ii) $$\phi(\gamma^*(t)) = \gamma(t)\,, \quad 0 \le t \le \delta \quad \text{and } \gamma^*(0) = x_0\,.$$

If this lifting process can continued for all $0 \le t \le 1$ we say that γ has a lifted curve γ^*. This means that there exists a curve $t \mapsto \gamma^*(t)$ from $[0, 1]$ into X such that

(*) $$\phi(\gamma^*(t)) = \gamma(t)\,, \quad 0 \le t \le 1 \quad \text{and } \gamma^*(0) = x_0\,.$$

Exercise. Show that the curve γ^* is unique if it exists. The hint is to use that ϕ is a local homeomorphism.

The whole discussion above leads to

5.5 Definition. *A local homeomorphism $\phi \colon X \to Y$ is called a covering map of $\mathcal{L}$-type if the following hold: For each pair of points $y_0 \in Y$ and $x_0 \in \phi^{-1}(0)$, every curve γ in Y with initial point y_0 and arbitrary end-point y can be lifted to a curve in X with initial point x_0.*

5.6 The case when X is simply connected. Assume this and let $\phi \colon X \to Y$ be a covering map of $\mathcal{L}$-type. Let $y_0 \in Y$ and choose some point $x_0 \in \phi^{-1}(y_0)$. Next, let γ be a closed curve in Y with $\gamma(0) = \gamma(1) = y_0$. By assumption there exists a unique lifted curve γ^* in X with

$\gamma^*(0) = x_0$. Suppose that $\gamma^*(1) = x_0$, i.e. the lifted curve is closed. Since X is simply connected it is homotopic to the trivial curve which stays at x_0, i.e. there exists a continuos map Γ^* from $\square$ into X such that

(i) $$\Gamma^*(t,0) = \gamma^*(t) \,\text{and}\, \Gamma^*(t,1) = x_0 \quad \Gamma^*(0,s) = \Gamma(1,s) = x_0 \;:\; 0 \leq s \leq 1$$

Now $\Gamma(t,s) = \phi(\Gamma^*(t,s)$ is a continuous map from $\square$ into Y and from (i) we see that Γ yields a homotopy between γ_0 and γ_1. Using this observation we arrive at:

5.7 Proposition. *Let γ_0 and γ_1 be two closed curves at y_0. Then they are homotopic if and only if $\gamma^*(1) = \gamma^*(1)$.*

Proof. We have already seen that if $\gamma^*(1) = \gamma^*(1)$ then the two curves are homotopic. Conversely, if they are homotopic we get a continus map $\Gamma(s,t)$ from $\square$ into Y and for each $0 \leq s \leq 1$ we have the closed curve $\gamma_s(t) = \Gamma(t,s)$ at y_0. Since the inverse fiber $\phi^1{}_1(y_0)$ by assumption is a discrete set in X, it follows by continuity and the unique path lifting that $s \mapsto \gamma_s^*(1)$ is constant and hence $\gamma_0^*(1) = \gamma_1^*(1)$.

5.8 Conclusion. Proposition 5.7 shows that homotopy classes of closed curves γ at y_0 are in a 1-1 correspondence with their end-points in X. Notice also that if x belongs to $\phi^{-1}(y_0)$ then the arc-wise connectivity of X gives a curve ρ where $\rho(0) = x_0$ and $\rho(1) = x$. Now $\gamma(t) = \phi(\rho(t))$ is a closed curve at y_0 and here $\gamma^*(t) = \rho(t)$ and hence x appears as an end-point for at least one closed curve at y_0. Identifying $\pi_1(Y)$ with homotopy classes of closed curves at y_0 we have therefore proved the following:

5.9 Theorem. *The map $\gamma \to \gamma^*(1)$ yields a bijective correspondence between the fundamental group $\pi_1(Y)$ and the inverse fiber $\phi^{-1}(y_0)$.*

Exercise. Let X and Z be two simply connected metric spaces. Suppose that $\phi\colon X \to Y$ and $\psi\colon Z \to Y$ are two covering maps which both belong to the class $\mathcal{L}$. Fix some $y_0 \in Y$. Choose $x_0 \in \phi^{-1}(y_0)$ and $z_0 \in \psi^{-1}(y_0)$. Next, let $y \in Y$ and consider some curve γ in Y with $\gamma(0) = y_0$ and $\gamma(1) = y$. Its unique lifted curve to X is denoted by γ^* and we get the end-point

$$\gamma^*(1) \in \phi^{-1}(y)$$

Similarly. we get a unique lifted curve γ^{**} in Z and the end-point

$$\gamma^{**}(1) \in \psi^{-1}(y)$$

From the above these two end-points only depend on the homotopy class of γ. Use this to conclude that we obtain a *unique and bijective* map from the discrete fiber $\phi^{-1}(y)$ to $\psi^{-1}(y)$. Moreover, as y varies in Y this gives a unique homeomorphism G from X onto Z with $G(x_0) = z_0$.

6. The uniformisation theorem.

Introduction. Let Ω be a connected open subset of $\mathbf{C}$. If the closed complement contains at least two points there exists a a covering map $f\colon\ D \to \Omega$ of $\mathcal{L}$-type given by an analytic function. This will be proved in Chapter 6. Here we take this existence for granted and analyze some consequences. More precisely, in Chapter VI we prove Riemann's mapping theorem for connected domains which asserts the following:

6.1 Theorem. *For every $z_0 \in \Omega$ there exists a unique analytic covering map f of $\mathcal{L}$-type where $f(0) = z_0$ and $f'(0)$ is real and positive.*

6.2 The multi-valued inverse to f. We take the theorem above for granted and discuss some consequences. Let f be a an analytic covering map as above. To distinguish the z-coordinate in Ω from D we let w be the complex coordinate in D. To begin with f yields a biholomorphic map from a small open disc D_* centered at the origin in D to a small open neighborhood U_0 of z_0. It gives the inverse analytic function $F(z)$ defined in U_0 such that

$$F(f(w)) = w \quad w \in D\,.$$

Next, let γ be a curve in Ω with $\gamma(0) = z_0$. Since f is of $\mathcal{L}$-type there exists a unique lifted curve γ^* in D with $\gamma^*(0) = 0$. Now the germ of F at z_0 can be continued analytically along γ where

(i) $$T_{\gamma(t)}(F(\gamma(t)) = \gamma^*(t) \quad : 0 \leq t \leq 1$$

Hence we get a multi-valued analytic function F in Ω. It gives an inverse to f in the following sense: Let $w \in D$ and consider the curve $t \mapsto t \cdot w$ in D. Now $t \mapsto f(t \cdot w)$ is a curve γ in Ω and by the construction (i) we have

(ii) $$T_{\gamma(t)}(F(f(t \cdot w)) = tw \quad : 0 \leq t \leq 1$$

We may express this by saying that the composed function $F \circ f$ is the identity on D.

6.3 Constructing single-valued functions. Consider the situation in Theorem 6.2, i.e. f is a covering map of $\mathcal{L}$-type from D onto Ω. Let $g(w)$ be some analytic function in D whose range $g(D) \subset \Omega$ and $g(0) = z_0$. We use F to construct a single-valued analytic function $F \circ g$ in D. Namely, let $w \in D$ which gives the curve γ_w parametrized by $t \mapsto g(t \cdot w)$ in Ω where $\gamma_w(0) = z_0$. We can continue F along this curve and when $t = 1$ we get the value

$$T_{\gamma_w(1)}(F(g(w))$$

It is clear that this gives an analytic function in D defined by

$$F \circ g(w) = T_{\gamma_w(1)}(F(g(w))$$

This construction can be performed for every $g \in \mathcal{O}(D)$ such that $g(0) = z_0$ and $g(D) \subset \Omega$. Hence we have proved the following:

6.4 Proposition. *Let $\mathcal{O}_*(D : \Omega)$ denote the family of analytic functions g in D where $g(0) = z_0$ and $g(D) \subset \Omega$. Then there exists a map from $\mathcal{O}_*(D : \Omega)$ into $\mathcal{O}(D)$ given by:*

$$g \mapsto F \circ g\,.$$

Here $F \circ g(0) = 0$ and the range $(F \circ g)(D) \subset D$.

6.5 Möbius transforms. Let f be a as in Theorem 6.1 and identify the fundamental group $\pi_1(\Omega)$ with homotopy classes of closed curves at z_0. Theorem xx gives a bijective map between elements in the group $\pi_1(\Omega)$ and the discrete subset $f^{-1}(z_0)$ of D. Let a be a point in this fiber, i.e. here $f(a) = z_0$. For each $0 \leq \theta < 2\pi$ we get an analytic function in D defined by

$$g(w) = f(e^{i\theta} \cdot \frac{w + a}{1 + \bar{a} \cdot w})$$

Here $g(0) = f(a) = z_0$ and the complex derivative at $w = 0$ becomes

$$g'(0) = f'(a) \cdot e^{i\theta} \cdot (1 - |a|^2)$$

We can choose θ so that $f'(a) \cdot e^{i\theta}$ is real and positive. With this choice of θ it follows from the uniqueness in Theorem 6.1 that $g = f$. Hence the function f satisfies

$$(*) \qquad f(w) = f(e^{i\theta} \cdot \frac{w+a}{1+\bar{a} \cdot w})$$

This means that f enjoys certain invariance properties. We return to a discussion at the end of section xx.

6.6 Inverse multi-valued functions.

Let ϕ be an analytic function defined in some open and connected subset U of $\mathbf{C}$. We assume that the derivative is $\neq 0$ at every point and get the open image domain $\Omega = \phi(U)$. Since ϕ is locally conformal it is in particular a local homeomorphism. We add the hypothesis that ϕ yields a covering map of $\mathcal{L}$-type. Consider some $\zeta_0 \in \Omega$ and put $x_0 = \phi(\zeta_0)$. We get a germ $f(x)$ of an analytic function at x_0 using the local inverse of ϕ, i.e. since $\phi'(\zeta_0) \neq 0$ there exists a small open disc $D_\delta(\zeta_0)$ such that

$$f(\phi(\zeta) = \zeta \quad : \quad |\zeta - \zeta_0| < \delta$$

In fact, we simply find the convergent power series

$$f(x) = \sum c_\nu (x - x_0)^\nu$$

where $c_0, c_1, \ldots$ are determined so that

$$\sum c_\nu \big(\phi(\zeta) - \phi(\zeta_0)\big)^\nu = \zeta$$

6.7 Proposition. *The germ f at x_0 extends to a multi-valued analytic function in U.*

Proof. Let γ be a curve in U having x_0 as initial point. The lifting lemma gives a unique curve γ^* in Ω. The required analytic continuation of f along γ now follows when we apply the Heine-Borel Lemma cover the compact set γ with a finite set of discs which are homemorphic images of discs in Ω whose consequtive union covers γ^*. Then we use that ϕ is everywhere analytic. The result is that the germ $T_\gamma(f)$ at the end-point $\zeta_1 = \gamma(1)$ satisfies

$$T_\gamma(f)(\phi(x)) = x$$

where x is close to the point $\phi(\gamma^*(1))$ in Ω. So in particular

$$T_\gamma(f)(\gamma(1)) = \gamma^*(1)$$

which clarifies how to determine values of the multi-valued analytic function.

6.8 Remark. It is instructive to consider some specific cases. Consider the entire function $\phi(\zeta) = e^\zeta$. With $\Omega = \mathbf{C}$ the image domain U is the punctured complex plane. If we take $x_0 = 1$ and $\zeta_0 = 0$ we find that f is the multi-valued Log-function where we start with the local branch at $x_0 = 1$ for which $\log 1 = 0$. Next, let us regard the polynomial $\phi(\zeta) = \zeta^2$. in order to get a covering we must exclude the origin to ensure that $\phi'(\zeta) \neq 0$. So if $\Omega = \mathbf{C} \setminus \{0\}$ we get a covering whose image set U also becomes the punctured complex plane. In this case the inverse fiber consists of two points and the function $f(z)$ is the multi-valued square-root of z.

6.9 Constructing single-valued functions.

Let Ω be a connected open set and consider some multi-valued analytic function F in Ω. Let U be some open and *simply connected* set. Consider some $h \in \mathcal{O}(U)$ whose image set $h(U)$ is contained in Ω. No further conditions on h are imposed, i.e. the inclusion $h(U) \subset \Omega$ may be strict and the derivative of h may have zeros. Using F we produce single valued analytic functions in U by the following procedure. Let us fix a point $\zeta_0 \in U$ and put $x_0 = h(\zeta_0)$. At x_0 we haver the family of local branches of F. Let f_* be one such local branch. Next, let γ be a curve in U where x_0 is the initial point and $x = \gamma(1)$ denotes the end-point. In Ω we get the image curve

(i) $$t \mapsto h(\gamma(t))$$

Now f_* has an analytic continuation along the curve in (i). When $t = 1$ we arrive at the endpoint $\gamma(1)$ which we denote by x. At x we can evaluate the local branch $T_\gamma(f_*)$. Next, let $\gamma_1(t)$ be another curve in U with the same end-point x as γ. By assumption U is simply connected which means that the curves γ and γ_1 are homotopic. It is clear that the homotopy in U implies that the two image curves obtained via (i) are homotopic in the curve family in Ω which joint x_0 and x. It follows that the image curves constructed via (i) are homotopic The monodromy theorem applied to F implies that

(ii) $$T_\gamma(f_*)(x) = T_{\gamma_1}(f_*)(x)$$

We conclude that (ii) gives an analytic function in U. Denote by $\mathcal{O}(U)_\Omega$ the family of analytic functions in U whose image is contained in Ω. With these notations the discussion above gives:

6.10 Proposition. For each point $\zeta_0 \in U$ there exists a map

$$\rho\colon\ \mathcal{O}(U)_\Omega \times M\mathcal{O}(\Omega)(x_0) \to \mathcal{O}(U)$$

where $x_0 = h(\zeta_0)$ and for a pair $h \in \mathcal{O}(U)_\Omega$ and $f_* \in M\mathcal{O}(\Omega)(x_0)$ the analytic function $\rho(h, f_*)$ satisfies

$$\rho(h, f_*)(\zeta) = T_\gamma(f_*)(h(\zeta)) \quad : \quad \zeta \in U$$

where γ is the h-image of any curve in U which joins ζ_0 with ζ.

Remark. Keeping h fixed we notice that the map $f_* \to \rho(h, f_*)$ is a **C**-algebra homomorphism from the complex **C**-algebra $M\mathcal{O}(\Omega)(x_0)$ into $\mathcal{O}(U)$.

7. The p^*-function.

We construct a special harmonic function which will be used to get solutions to the Dirichlet problem in XXX. Let Ω be an open and connected set in $\mathbf{C}$. Its closed complement has connected components. Let E be such a connected component. To each $a \in E$ we get the winding number $\mathfrak{w}_a(\gamma)$. If b is another point in E which is sufficiently close to a it is clear that

$$|\frac{1}{\gamma(t)-a} - \frac{1}{\gamma(t)-b}| < |\frac{1}{\gamma(t)-a}|$$

Rouche's theorem from 1.4 implies that $\mathfrak{w}_a(\gamma) = \mathfrak{w}_b(\gamma)$, i.e. for every closed curve γ in Ω, the winding number stays constant in each connected component of $\mathbf{C}\setminus\Omega$. This enable us to construct single valued Log-functions in Ω. Namely, let $a \in E$ where E is a connected componen in the complement of Ω. Consider $f = \text{Log}\,(z-a)$ and choose a single valued branch f_* at some point $z_0 \in \Omega$. If $\gamma \subset \Omega$ is a closed curve with initial point at z_0 the analytic continuation along γ of the Log-function gives:

$$T_\gamma(f_*) = f_* + 2\pi i \cdot \mathfrak{w}_a(\gamma) \tag{1}$$

Next, if b is another point in E we consider $g_* = \text{Log}(z-b)$ and obtain

$$T_\gamma(g_*) = g_* + 2\pi i \cdot \mathfrak{w}_b(\gamma) \tag{2}$$

Since $\mathfrak{w}_b(\gamma) = \mathfrak{w}_b(\gamma)$ it follows that

$$T_\gamma(f_*) - T_\gamma(g_*) = f_* - g_* \tag{3}$$

Hence the difference $\text{Log}(z-a) - \text{Log}(z-b)$ is a *single valued* analytic function in Ω. Taking is exponential we find $\Psi(z) \in \mathcal{O}(\Omega)$ such that

$$e^{\Psi(z)} = \frac{z-a}{z-b} \tag{4}$$

Since $a \neq b$ we see that $\Psi(z) \neq 0$ for all $z \in \Omega$. Next, we get the harmonic function defined in Ω by

$$p(z) = \mathfrak{Re}\big(\frac{1}{\Psi(z))}\big) = \frac{\mathfrak{Re}(\Psi(z)}{|\Psi(z)|^2} \tag{*}$$

Notice that $\mathfrak{Re}(\Psi(z)) = \text{Log}\,|z-a| - \text{Log}|z-b|$ and since $\text{Log}\,|z-a| \to -\infty$ as $z \to a$ we see from (*) that

$$\lim_{z\to a} p(z) = 0 \tag{**}$$

Notice also that $\Psi(z)$ exends to a continuous function on $\bar{\Omega} \setminus (a,b)$ and we can choose a small $\delta > 0$ such that

$$\text{Log}|z-a| - \text{Log}\,|z-b| < -1 \quad : \quad |z-a| \leq \delta \tag{ii}$$

Then (i) and (ii) give

7.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the single point a. Then there exists a harmonic function $p^*(z)$ in Ω for which*

$$\lim_{z\to a} p^*(z) = 0$$

and there exists $\delta > 0$ such that

$$\max_{\{|z-a|=r\}\,\cap\Omega} p^*(z) < 0 \quad : \quad z \in D_a(r) \cap \Omega$$

9. Extensions by reflection

Introduction. *Das Spiegelungsprinzip* is due to H. Schwartz. First we describe the standard case. Let $f(z)$ be analytic in the upper half plane $U_+ = \{\mathfrak{Im}\, z > 0\}$. Let $J(a,b) = \{a < x < b\}$ be an interval, on the real axis. Suppose that f extends to a continuous function to this open interval and takes real values. In the lower half-plane U_- we get the analytic function

$$\text{(i)} \qquad f_*(z) = \bar{f}(\bar{z})$$

By the result in XX the two functions are analytic continuations of each other over (a,b). This means that f has an analytic extension to the open set $\Omega = \mathbf{C} \setminus J$, where $J_* = (-\infty, a] \cup [b, +\infty)$ is the closed complement of (a,b) on the x-axis. Next, suppose that $e^{i\theta} f(z)$ extends to a real-valued function on (a,b) for some θ. After multiplication with $e^{-i\theta}$ we get an extension of f. That is, one has only to require that the argument of f is constant to obtain an analytic continuation. Suppose now that the argument of f is constant over a family of pairwise disjoint intervals $\{J(a_\nu, b_\nu)\}$. Then we get analytic continuations across each interval. In particular one has:

9.1 Theorem. *Let $a_1 < \ldots < a_N$ be a finite set of real numbers and assume that f extends to a continuous function on each of the intervals*

$$J_0 = (-\infty, a_1) \quad : \; J_\nu = (a_\nu, a_{\nu+1}) : 2 \leq \nu \leq N_1 \quad : \; J_N = (a_N, +\infty)$$

and on every such interval the argument of f is some constant. By successive reflections over there intervals we obtain a in general multi-valued analytic of f_*^ν defined in $\mathbf{C} \setminus (a_1, \ldots, a_N)$.

Example. In the upper half-plane U_+ we consider the analytic function

$$f(z) = \sqrt{z} \cdot \sqrt{1-z}$$

The single-valued branches of the root functions are chosen so that

$$\sqrt{z} = \sqrt{r} \cdot e^{i\theta/2} \quad : \quad \sqrt{z-1}) = \sqrt{1 + r^2 - 2r \cdot \cos\theta} \cdot e^{i\phi} \quad : z = re^{i\theta}$$

where $0 < \theta < \pi$ and ϕ is the outer angle of the triangle in figure xx. So here $0 < \phi < \pi$ holds. As we approach a point $0 < x < 1$ we get the boundary value

$$f(x) = \sqrt{x} \cdot i \cdot \sqrt{1-x}$$

Now get the analytic continuation f_*^1 across the interval $J_1 = (0,1)$ which becomes an analytic function defined in the lower half-plane U_- by

$$f_*^1(z) = -\bar{f}(\bar{z})$$

Notice that the minus-sign appears in order that $f(x) = f_*^1(x)$ holds for $0 < x < 1$. Suppose now that $x > 1$. Then we get

$$\lim_{y \to 0} f_*^1(x - iy) = \lim_{y \to 0} -\bar{f}(1(x + iy) = -\sqrt{x} \cdot \sqrt{x-1}$$

So f and f_*^1 do not agree on the real interval $(1, +\infty)$. At the same time f_*^1 can be continued analytically across $(1, +\infty$ and gives an analytic function f_+** defined in U_+ where we obtain

$$f_+^{**}(z) = -f(z) \quad : \quad z \in U_+$$

Another analytic continuation of f_*^1 takes place across $(-\infty, 0)$. When $x < 0$ we have

$$\lim_{y \to 0} f_*^1(x - iy) = \lim_{y \to 0} -f(1(x + iy) =$$

After f has been extended to the lower half-plane where we get an analytic function denoted by f_* we notice that f_* by the construction also has boundary values with a constant argument as we approach points on the real axis form below. So f_* also extends to the upper half-plane where we encounter a new analytic function $f^*(z)$. Next, we can continue f^* to the lower half-plane and so on. The result is that f extends to a multi-valued function in $\mathbf{C} \setminus (0,1)$.

9.2 The use of conformal maps. For local extensions there exists a general result. Let D be an open disc and γ a Jordan arc which joins two points on ∂D and separates $D \setminus \gamma$ into a pair of disjoint Jordan domains. Let f is analytic in one of the Jordan domains, say D^*. Assume also that f extends to a continuous and real-valued function on γ. Now there exists a conformal map from D^* to the upper half-plane and using this it follows that f extends analytically across γ. of course, the extension f_* will in general only exist in a small domain close to γ, i.e. this is governed via the conformal mapping. But here exists at least a locally defined analytic continuation across γ.

9.3 Boundary values on circles Let $f(z)$ be as above and suppose it extends continuously to γ where the absolute value is constant, say 1. Using a conformal map from D_* to the unit disc we may assume that γ is an interval of the unit disc D and f is analytic in a small region $U \subset D$ where γ appears as a relatively open subset of ∂U. By hypothesis $f(\gamma)$ is a subset of another unit circle and using a conformal map from the disc bordered by this unit circle we get the situation in 7.2. and conclude that f continues analytically across γ. More generally, using a locally defined conformal map there exists an analytic extension of f across γ if we only assume that the continuous boundary values of f on γ are contained in some locally defined *real-analytic curve.* Finally, by a two-fold application of conformal mappings we get the following quite general result:

9.4 Theorem. *Let $f(z)$ be analytic in a Jordan domain Ω and suppose that $\gamma$is an open arc of $\partial\Omega$ such that f extends continuosly from Ω to $\Omega \cup \gamma$ and the restriction $f|\gamma$has a range $f(\gamma)$ contained in a simple real-analytic curve γ^*. Then f extends analytically across γ, i.e. there exists an open and connected neighborhood U of γ such that the original f-function extends to the connected domain $\Omega \cup U$.*

Remark. Theorem 9.4 follows from the fact that if Ω_2 and Ω_2 are two Jordan domains whose boundaries both are *real analytic* closed Jordan curves, then a conformal map from Ω_1 to Ω_2 extends to a conformal map from an open neighborhood of $\bar{\Omega}_1$ to an open neighborhood of $\bar{\Omega}_2$.

10. The elliptic modular function

Introduction. We shall construct an analytic function $\phi(z)$ in the upper half-plane $U_+ = \mathfrak{Im}\, z > 0$ whose complex derivative of ϕ is everywhere $\neq 0$ and the image $\phi(U_+)$ is equal to the connected open set $\Omega = \mathbf{C} \setminus \{0,1\}$, i.e. the two points 0 and 1 are removed from the complex plane. So ϕ is locally conformal but not 1-1. Moreover the function is invariant under a group of Möbius maps which preserve U_+. To be precise, consider first the map

(i) $$z \mapsto \frac{z}{2z+1}$$

With $z = x + iy$ we have

$$\mathfrak{Im} x + iy2x + 2iy + 1 = \frac{y}{|2x+2iy+1|^2} > 0$$

So (i) maps U_+ ito itself. Moreover, the map is bijective for if we set $w = \frac{z}{2z+1}$ we can solve out

$$z = \frac{w}{1-2w}$$

which yields the inverse map to (i). Since (i) defines an analytic function it means that it gives a conformal map of U_+ onto itself. Another obvious conformal map from U_+ onto itself is given by the translation

(ii) $$z \mapsto z+1$$

Let $\mathcal{F}$ denote the group of conformal maps generated by (i) and (ii). We are going to construct the analytic function ϕ in such a way that it is {-invariant which means that

(iii) $$\phi(z) = \phi(z+1) = \phi(\frac{z}{2z+1}) \quad \text{hold for all} \quad z \in U_+$$

When (iii) holds one says that ϕ is an {-automorphic function. The subsequent construction of ϕ will show that not only is $\phi' \neq 0$ in U_+ but to each point $w \in U_+$ the inverse fiber $\phi^{-1}(w)$ is equal to an orbit under $\mathcal{F}$, i.e. if z_0 is chosen so that $\phi(z_0) = w$ then

(iv) $$\phi^{-1}(w) = \{\rho(z_0) \quad : \quad \rho \in \mathcal{F}\}$$

The construction of ϕ. Consider the simply connected domain:

$$V_0 = U_+ \,\cap\, |z - 1/2| > 1/2 \cap \{0 < \mathfrak{Re}\, z < 1\}$$

Here ∂V_0 consists of three pieces: The vertical half-line ℓ_0 on which $x = 0$ and $y > 0$, the half-line ℓ_1 on which $x = 1$ and $y > 0$, and finally the half-circle

$$T_0^+ = \{1/2 + 1/2 \cdot e^{i\theta}\} \quad : \quad 0 < \theta < \pi$$

Riemann's mapping theorem gives a conformal mapping ϕ_0 from V_0 onto U_+ such that ϕ yields bijective maps from ℓ_0 onto $(-\infty, 0)$ and T_0 onto $(0,1)$, and finally ℓ_1 onto $(1, +\infty)$. In particular $\phi_0(0) = 0$ and $\phi_0(1) = 1$ hold and as $z \to \infty$ in V_0 then $\phi_0(z) \to \infty$ in U_+. See figure 1 for an illustration of this conformal mapping. Following Schwarz we shall perform reflections to obtain an analytic function ϕ_* defined in the domain

$$V_* = \{0 < \mathfrak{Re}\, z < 1\} \cap \mathfrak{Im}\, z > 0\}$$

whose image is equal to $\mathbf{C} \setminus \{0,1\}$.

8.1 The first reflection. Consider the open set W_0 defined by

(1) $$W_0 = \{z \quad : \frac{\bar z}{2\bar z - 1} \in V_0\}$$

in W_0 a two-fold complex conjugation gives the analytic function $g_0(z)$ defined by

(2) $$g_0(z) = \bar\phi_0(\frac{\bar z}{2\bar z - 1})$$

Let us consider a point $z \in T_0^+$ which can be written as $1/2 + e^{i\theta}/2$. Now we have

$$\frac{1/2 + e^{-i\theta}/2}{1 + e^{-i\theta} - 1} = 1/2 + e^{i\theta}/2 \tag{3}$$

This shows that $g_0 = \phi_0$ on the half-circle T_0^+. So by Schwarz' reflection principle this pair of analytic functions extend to the union of V_0 and W_0. There remains to determine W_0. To see its geometric picture we first consider the piece of ∂V_0 where $z = iy$ for $y > 0$. This corresponds to points on ∂W_0 of the form

$$w(y) = \frac{-iy}{-2iy - 1} = \frac{iy}{2iy + 1}$$

Now we get

$$w(y) - 1/4 = \frac{1}{4} \cdot \frac{2iy - 1}{2iy + 1}$$

The right hand side has absolute value 1/4 so the resulting piece of ∂W_0 stays on the circle $|w - 1/4| = 1/4$. If we write

$$w(y) = 1/4 + 1/4e^{i\theta(y)/4}$$

then the reader may verify that $y \mapsto \theta(y)$ decreases from π to zero as y increases from 0 to $+\infty$. So the image of ℓ_0 is the upper half-circle T_{10}^+ of radius 1/4 and centered at 1/4. See figure XX for an illustration.

Exercise. Show by similar computations that when $z = 1 + iy$ with $y > 0$ then the w-image is the upper half-circle T_{11}^+ of radius 1/4 centered at 3/4. As a result W_0 is the domain illustrated by figure XX which is bordered by T_{10}^+, T_{11}^+ and T_0^+.

The second step. Put $V_1 = V_1 \cup T_0^+ \cup W_0$. Above we have constructed an analytic function ϕ_1 in V_1. Since $\phi_0(iy)$ and $\{phi_0(1 + iy$ takes real values for all $y > 0$, we see that ϕ_1 takes real values on the half-circles T_{10}^+ and T_{11}^+. This gives analytic extensions of ϕ_1 across each of these half-circles. To achieve this we first consider the domain

$$W_{10} = \{\bar{z} \quad : \quad \frac{\bar{z}}{4\bar{z} - 1} \in V_1\}$$

If $z = 1/4 + e^{i\theta}/4$ belongs to T_{10}^+ we obtain

$$\frac{1/4 + e^{-i\theta}/4}{e^{-i\theta}} = 1/4 + e^{i\theta}/4$$

So by reflection we obtain an analytic function g_{10} in W_{10} defined by

$$g_{10}(z) = \bar{\phi}_1(\frac{\bar{z}}{4\bar{z} - 1})$$

To determine the domain W_0 we regard the images of ℓ_0 and ℓ_1. if $z0iy$ with $y > 0$ we obtain

$$\frac{-iy}{-4iy - 1} - 1/8 = \frac{iy}{4iy + 1} - 1/8 = \frac{1}{8} \cdot \frac{4iy - 1}{4iy + 1}$$

Hence the image of ℓ_0 is the upper half-circle of radius 1/8 centered at 1/8. By a similar computation we see that the image of ℓ_1 is the half-circle of radius 1/8 centered at 3/8.

We also attain an extension across T_{11}^+. This time we put

$$W_{11} = \{\bar{z} \quad : \quad \frac{\bar{z}}{4\bar{z} - 3} \in V_1\}$$

If $z = 3/4 + e^{i\theta}/4$ we obtain

$$\frac{3/4 + e^{-i\theta}}{3 + e^{-i\theta} - 3} = 3/4 + e^{i\theta}/4$$

So again we obtain an analytic extension where $g_{11}(z)$ is defined in W_{11} by

$$g_{11}(z) = \phi_1(\frac{\bar{z}}{4\bar{z} - 3})$$

A computation which is left to the reader show that the image of ℓ_0 now becomes the upper half-circle centered at 5/8 with radius 1/8 and the image of ℓ_1 the half-circle centered at 7/8 with radius 1/8. The resulting domain after the second extension is illustrated by figure xx, i.e. it is bordered by ℓ_0, ℓ_1 and four upper half-circles each of radius 1/8 and in this domain we have constructed the analytic function ϕ_2.

At this stage it is clear how one proceeds to construct larger and larger domains $\{W_k\}$ and analytic functions ϕ_k via reflections over suitable half-circles. See figure XX. The result is an analytic function $\phi_*(z)$ defined in the half-strip

$$\square_+ = 0 < \mathfrak{Re}\, z < 1\} \cap \{\mathfrak{Im}\, z < 0\}$$

The construction shows that the derivative of the analytic function ϕ_* in $\square_+$ is everywhere $\neq 0$ and the range is $\mathbf{C} \setminus \{0,1\}$ Next, since $\{phi_*$ is real-valued on ℓ_0 and $|ell_1$ it extends by reflection over these two vertical lies and the reader may verify that we obtain an analytic function ϕ defined in the whole upper half-plane which is 1-periodic, i.e.

$$\phi(z+1) = \phi(z)$$

Invariance under $\mathcal{F}$. Let us prove the equality

$$\phi(z) = \phi(\frac{z}{2z+1})$$

By analyticity it suffices to check this on some curve in U_+. Consider for example ℓ_0. By the previous construction we have

$$\phi(\frac{iy}{2iy+1}) =$$

Check by previous !!!

8.8 The multi-valued inverse. Since ϕ is locally conformal we can construct a multi-valued inverse function to be denoted by $\mathfrak{m}$. Namely, set $\Omega = \mathbf{C} \setminus \{0,1\}$ and consider the point $\zeta_0 = i$. We first find the unique point $z_0 \in V_0$ such that $\phi(z_0) = i$. At ζ_0 we get a unique germ $\mathfrak{m}_0(\zeta) \in \mathcal{O}(\zeta_0)$ such that

$$\mathfrak{m}_0(\phi(z)) = z$$

hold for z close to i. Next, let γ be a curve in Ω which starts at i and has some end-point ζ_1. Since λ is locally conformal there exists a unique curve γ^* in U_+ such that

$$\phi(\gamma^*(t)) = \lambda(t) \quad : 0 \leq t \leq 1$$

As explained in xx there exists an analytic extension of $\mathfrak{m}_0$ along γ which locally produces inverses of the ϕ-function. The resulting multi-valued $\mathfrak{m}$-function gives the set of values $W(\mathfrak{m}, \zeta)$ for every $\zeta \in \Omega$ which is in a 1-1 correspondence with the inverse fiber $\lambda^{-1}(\zeta)$ and hence equal to an orbit under the group $\mathcal{F}$.

11. Poincare's theory of Fuchsian groups

The theory of Fuchsian groups was created by Poincaré. His two articles *Théorie des groupes fuchsiens* and *Memoire sur les fonctions fuchsiennes* were published 1882 in the first first volume of Acta Mathematica and the article *Memoire sur les groupes kleinéens* appeared in volume III. The last article is more advanced and we shall not discuss Kleinan groups here. Nor do we discuss the article *Memoire sur les fonctions zétafuchsiennes.* The connection to arithmetic was presented in a later article *Les fonctions fuchsiennes et l'Arithmétique* from 1887. One should also mention the article *Les fonctions fuchsiennes et l'équation* $\Delta(u) = e^u$ where Poincaré proved that this second order differential equation has a subharmonic solution with prescribed singularities on every closed Riemann surface attached to an algebraic equation. The last work started potential theoretic analysis on complex manifolds. Here we only discuss material from the first two cited articles.

Remark. Poincaré was inspired by earlier work, foremost by Bernhard Riemann, Hermann Schwarz and Karl Weierstrass. For example, he used the construction of multi-valued analytic extensions by Weierstrass which leads to the *Analytische Gebilde* of a multi-valued function f defined in some connected open subset Ω of $\mathbf{C}$. This *Analytische Gebilde* is a connected complex manifold X on which f becomes a single valued analytic function f^*. More precisely, there exists a locally biholomorphic map

$$\pi\colon\ X \mapsto \Omega$$

When $U \subset \Omega$ is simply connected the inverse image $\pi^{-1}(U)$ is a union of pairwise disjoint open sets U^*_γ where the single-calued analytic function f^* is determined by a branch T_γ of f, i.e. one has

$$T_\gamma(f)(\pi(x) = f^*(x) \quad : x \in U^*_\gamma$$

Major contributions are also due to Schwarz. In 1869 he used the reflection principle and calculus of variation to settle the Dirichlet problem and used this to prove the uniformisation theorem for connected domains bordered by p many real analytic and closed Jordan curves where p in general is ≥ 2. Of special interest is the multi-valued $\mathfrak{m}$-function defined in $\mathbf{C} \setminus \{0,1\}$ which is related to the elliptic integral of the first kind and hence to Jacobi's $\mathfrak{sn}$-function which appears in the equation of motion when a rigid body rotates around a fixed point.

A comment. The theory of Fuchsian functions was not restricted to analytic function theory. The main concern for Poincaré was to develop the theory of differential systems, both linear and non-linear. His research was also directed towards to the general theory about abelian functions and their integrals, inspired by Abel's pioneering work. Hundreds of text-books have appeared after Poincaré. Personally I find that his own and often quite personal presentation superseeds most text-books which individually only treat some fraction from the great visions by Poincaré. His original work offers therefore a good introduction for the student who enters studies about linear and non-linear differential systems in an algebraic context, together with function theory which leads to Fuchsian as well as Kleinian groups and there associated functions, See in particular the book *Analyse de ses travaux scientifiques* which contains a survey of his the scientific work. In several chapters Poincaré describes in his own words various research areas from the period between 1880 until 1907, which has the merit that it not only contains a summary of results but also explanations of the the main ideas and methods which led to the theories.

Of course there exists more recent advancement in function theory. Here one should foremost mention work by Lars Ahlfors. So in addition to the cited reference above I recommend text-books by Ahlfors, especially his book *Conformal Invariants* which contains material about the theory of extremal length which was created by Arne Beurling in the years 1942-1946. From a complex analytic point of view the discoveries by Ahlfors and Beurling have a wider scope and has led to many still unsolved problems in complex analysis. including the study of quasi-conformal mappings. In addition to this we refer to the excellent material in the text-book [A-S] by Ahlfors and Sario about Riemann surfaces.

Chapter 5.A Harmonic functions

Contents

Introduction.

The whole chapter is divided in two parts, where the second treats subharmonic functions. A strict logical procedure might have start with general results about subharmonic functions. However, we prefer to first study harmonic functions even though we are obliged to appeal to some facts from part B below. But the reader will hopefully accept this and whenever needed consult results in section B. A major result is the solution to the Dirichlet problem. Consider a bounded open and connected domain Ω in $\mathbf{C}$ such that for every boundary point $a \in \partial\Omega$ the connected component $\mathcal{C}(a)$ in $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the singleton set $\{a\}$. Under this assumption we shall prove that each continuous function ϕ on $\partial\Omega$ has a unique continuous extension to a harmonic function in Ω. The proof relies upon the existence of certain peak functions. In § xx from Chapter 4 we proved that if $a \in \partial\Omega$ and $\mathcal{C}(a) \neq \{a\}$ then there exists a harmonic function $p_a^*(z)$ in Ω such that

$$(*) \qquad \lim_{z \to a} p_a^*(z) = 0 \quad \text{and} \quad p_a^*(z) < 0 \quad : z \in \Omega$$

Next, if $\phi \in C^0(\partial\Omega)$ we have the Perron family $\mathcal{P}(\phi)$ from XXX and obtain the harmonic function $H_\phi(z)$ in Ω defined by

$$(**) \qquad H_\phi(z) = \max_{u \in \mathcal{P}(\phi)} u(z) \quad : \quad z \in \Omega$$

In Theorem 1.1 we show that for each boundary point a such that $\mathcal{C}(a) \neq \{a\}$ and every continuous boundary function ϕ one has the equality

$$(***) \qquad \lim_{z \to a} H_\phi(z) = \phi(a)$$

That (***) holds when $\mathcal{C}(a) \neq \{a\}$ was established by by Boulignad in [Boul] shortly after Perron's construction of the H_ϕ-functions. More precisely one considers the special boundary function function $B_a(z) = |z - a|$. *Boulignad's Lemma* asserts that if $a \in \partial\Omega$ is such that $\mathcal{C}(a) \neq \{a\}$, then

$$(****) \qquad \text{Lim}_{z \to a} H_{B_a}(z) = 0$$

In Section 1 we prove (****) and after we deduce (***) when $\mathcal{C}(a) \neq \{a\}$ while $\phi \in C^0(\partial\Omega)$ is arbitrary.

0.1 Harmonic measures. Let Ω be a bounded and connected domain where the Dirichlet problem can be solved. So to every $\phi \in C^0(\partial\Omega)$ we get its unique harmonic extension H_ϕ. Riesz's representation theorem gives for each $z \in \Omega$ a unique probability measure $\mathfrak{m}_z$ supported by $\partial\Omega$ such that

$$(*) \qquad H_\phi(z) = \int_{\partial\Omega} \phi(\zeta) \cdot d\mathfrak{m}_z(\zeta) \quad : \quad \phi \in C^0(\partial\Omega)$$

One refers to $\mathfrak{m}_z$ as the harmonic measure of z. When z varies in Ω it turns out that these measures are absolutely continuous with respect to each other. So if we fix some point $z_* \in \Omega$, then every $z \in \Omega$ gives a unique function $\rho_z \in L^1(\mathfrak{m}_{z_*})$ such that

$$\mathfrak{m}_z = \rho_z \cdot \mathfrak{m}_{z_*}$$

Since $C^0(\partial\Omega)$ is a dense subspace of $L^1(\mathfrak{m}_{z_*})$ we conclude that for every $f \in L^1(\mathfrak{m}_{z_*})$ there exists a harmonic function in Ω defined by

$$f^*(z) = \int \rho_z(\zeta) \cdot f(\zeta) \cdot d\mathfrak{m}_{z_*}(\zeta)$$

But in contrast to the special case when f is a continuous function on $\partial\Omega$ we cannot expect "nice limit values" of f^* as z approaches a boundary point.

Next, let g be a bounded Borel function defined on $\partial\Omega$. Recall from *Measure Appendix* that we can construct integrals of Borel functions with respect to an arbitrary Riesz measure. When $z \in \Omega$ we have the Riesz measure $\mathfrak{m}_z$ and set

$$G(z) = \int_{\partial\Omega} g(\zeta) \cdot d\mathfrak{m}_z(\zeta) \quad :$$

Then G is a bounded harmonic function in Ω. In particular we can take a Borel set E in $\partial\Omega$ and let $g = \chi_E$ be its characteristic function. This gives the harmonic function

$$\omega_E(z) = \int_E d\mathfrak{m}_z(\zeta)$$

One refers to ω_E as the harmonic measure function with respect to the E. In the case when E is a null-set with respect to $\mathfrak{m}_{z_*}$ the ω_E-function is identically zero.

0.2 Wiener's solution. For bounded open sets Ω where $\partial\Omega$ contains points a such that $\mathcal{C}(a)$ is reduced to $\{a\}$ it is not sure that the Dirichlet problem can be solved. However, there always exists a family of probability measures $\{\mathfrak{m}_z\}$ supported by $\partial\Omega$ which produce harmonic functions in Ω. More precisely, for every $f \in C^0(\partial\Omega)$ there exists a harmonic function W_f in Ω defined by

$$W_f(z) = \int f(\zeta) \cdot d\mathfrak{m}_z(\zeta)$$

Moreover Wiener found a necessary and sufficient geometric condition in order that an isolated boundary point a where $\{a\} = \mathcal{C}(a)$ is regular in the sense that

$$\lim_{z \to a} W_f(z) = f(a)$$

hold for every $f \in C^0(\partial\Omega)$. Wiener's constructions are given at the end of section 1.

0.2 Other topics. In Section 2 we construct the harmonic conjugate of a harmonic function and expose some results when $\Omega \in \mathcal{D}(C^1)$. Section 3 is devoted to the case when Ω is the unit disc where we confirm the solution to the Dirichlet problem using the Poisson kernel. Here we encounter some specific calculations. The Poisson kernel is expressed by the function

$$P(r, \theta - \phi) = \frac{1}{2\pi} \cdot \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)}$$

where $0 < r < 1$ while (ϕ, θ) are independent angular variables. When $U(\theta)$ is a continuous function on the circle its harmonic extension to D is given by

$$(1) \qquad u(r, \phi) = \int_0^{2\pi} P(r, \theta - \phi) \cdot U(\theta) \cdot d\theta$$

It is often convenient to rewrite the last integral to get

$$u(r,\phi)=\int_0^{2\pi} P(r,\theta)\cdot U(\theta+\phi)\cdot d\theta/tag2$$

Suppose for example that U is Lipschitz continuous, i.e. there is a constant C such that

$$|U(\theta_2)-U(\theta_1)|\leq C\cdot|\theta_1-\theta_2|$$

Then we use (2) and conclude that u restricts to a Lipschitz continuous function of the angular variable ϕ with norm $\leq C$ on every circle of radius $r<1$. In particular the angular derivatives

$$|\frac{\partial u}{\partial\phi}(r,\phi)|\leq C$$

holds in the whole disc D. To study radial derivatives we use the equation

$$\frac{\partial P(r,\theta)}{\partial r}=\frac{1}{2\pi}\cdot\frac{2\cos\theta+2r^2\cos\theta-4r}{(1+r^2-2r\cos\,\theta)^2}$$

With a fixed ϕ_* we consider the function

$$(*)\qquad r\mapsto\frac{\partial u}{\partial r}(r,\phi_*)$$

Now $\theta\mapsto U(\theta+\phi_*)$ has the same Lipschitz norm and we may therefore take $\phi_*=0$ and since the partial r-derivative of u is unchanged when we add a constant we can therefore replace $U(\theta)$ by $U(\theta)-U)0)$, i.e assume that $U(0)=0$. This entails that

$$|U(\theta)|\leq C\cdot|\theta|\quad:\quad-\pi<\theta<\pi$$

Now (*) is majorized in absolute value by the integral

$$\frac{C}{2\pi}\int_{-\pi}^{\pi}\frac{\bigl|2\cos\theta+2r^2\cos\theta-4r\bigr|}{(1+r^2-2r\cos\,\theta)^2}\cdot|\theta|\cdot d\theta$$

Exercise. Show that the last integral is majorized by an absolute constant which is independent of $0<r<1$ and conclude from the above that the following hold:

Theorem. *There exists an absolute constant C_* such that*

$$\max_{(r,\phi)\in D}\sqrt{\partial u/\partial r^2(r,\phi)+\partial u(\partial\phi^2(r,\phi)}\leq C_*\cdot|U|_{\text{Lip}}$$

where the last term is the Lipschitz norm of U over T.

Remark. If we remove a small disc centered at the origin the left hand side can be replaced by the length of the gradient vector $\nabla u=(u'_x,u'_y)$. For example, we find an absolute constant C^* such that

$$\max_{1/2\leq|z|<1}||\nabla(u)(z)||\leq C^*\cdot|U|_{\text{Lip}}$$

At the end of Section 3 we investigate properties of the harmonic conjugate where Theorem 9 is the main result. Section 4 is devoted to the mean-value condition. Here a major point is that the condition for a function to be harmonic can be expressed by mean-value properties. In section 5 we study harmonic measures and use these to prove some results due to Lindelöf concerning limits of bounded analytic functions. In section 6 we construct the Green's functions and the Neumann problem is treated in section 7. An example related to hydromechanics appears in Section 8. Section 9 presents a proof where Carleman for the first time employed differential inequalities to estimate harmonic functions Section 10 treats a result due to Nevanlinna about critical points for level curves of harmonic measure functions. The last two sections contain some supplementary facts about harmonic functions. For example, the estimates in section 12 will be used when Lindelöf indicators are studied for analytic functions in strip domains in Section XX from Special Topics.

1. The Dirichlet Problem.

Introduction. Let Ω be a bounded open set in $\mathbf{C}$. No connectivity assumptions are imposed, i.e. neither Ω or $\partial\Omega$ have to be connected. Given a continuous real-valued function ϕ on the compact boundary $\partial\Omega$ we seek a function H_ϕ which is harmonic in Ω and extends to a continuous function on the closure such that $H_\phi = \phi$ holds on $\partial\Omega$. A procedure to find H_ϕ was given by J. Perron who considered the family $\mathcal{P}(\phi)$ of subharmonic functions $u(z)$ in Ω satisfying

$$\limsup_{z\to w}\, u(z) \leq \phi(w) \quad : \; w \in \partial\Omega \tag{0.1}$$

In Theorem XX we prove that there a harmonic function H_ϕ in Ω given by:

$$H_\phi(z) = \max_{u\in\mathcal{P}(\phi)}\, u(z) \quad : \; z \in \Omega \tag{*}$$

There remains to analyze if H_ϕ solves the Dirichlet problem, i.e. if

$$\lim_{z\to a}\, H_\phi(z) = \phi(a) \quad : \; a \in \partial\Omega \tag{**}$$

If (**) holds for every boundary point it is clear that Perron's solution extends to a continuous function on the closure $\bar{\Omega}$ and solves the Dirichlet problem with the prescribed boundary function ϕ. It turns out that (**) holds under a geometric condition.

1.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of a in the closed complement $\mathbf{C}\setminus\Omega$ is not reduced to the singleton set $\{a\}$. Then (**) holds for every ϕ.*

Remark. The result above is stated for an individual boundary point. If the geometric condition holds for every $a \in \partial\Omega$ it means that the Dirichlet problem has a solution for every $\phi \in C^0(\partial\Omega)$. The proof of Theorem 1 relies upon the study of Perron's solution in the special case when $\phi(z) = |z-a|$. With a given we let $H_a(z)$ denote this Perron solution. Since the function $z \mapsto |z-a|$ is subharmonic in the whole of $\mathbf{C}$, the construction of H_a gives:

$$H_a(z) \geq |z-a| \quad : \quad z \in \Omega \tag{1}$$

The result below is due do J. Boulignad

1.2 Boulignad's Lemma. *Let $a \in \partial\Omega$ satisfy the condition in Theorem 1.1. Then*

$$\lim_{z\to a}\, H_a(z) = 0$$

Before we prove this Lemma we show why it settles the Dirichlet problem.

1.3 Proof of Theorem 1.1.

Let $a \in \partial\Omega$ be as in Theorem 1.1 and consider a continuous boundary function $\phi(z)$ with the Perron solution $H_\phi(z)$. If c is a constant it is clear that $H_{\phi-c} = H_\phi - c$. Replacing ϕ by $\phi(z)-\phi(a)$ we may therefore assume that $\phi(a) = 0$ and it remains to show that

$$\lim_{z\to a}\, H_\phi(z) = 0 \tag{*}$$

First we consider the limes superior and show that

$$\limsup_{z\to a}\, H_\phi(z) \leq 0 \tag{**}$$

To get (**) we put $M^* = \max_{z\in\partial\Omega}\, |\phi(z)|$. Let $\epsilon > 0$. The continuity of ϕ gives some $\delta > 0$ such that

$$\phi(z) \leq \epsilon \quad : \quad z \in \partial\Omega \cap D_a(\delta)$$

Define the harmonic function in Ω by

$$g^*(z) = \epsilon + \frac{M^*}{\delta} \cdot H_a(z)$$

Since $H_a(z) \geq |z-a|$ we have:

$$\liminf_{z \to b} g^*(z) \geq M^* \quad : \quad b \in \partial\Omega \setminus D_a(\delta)$$

At the same time $g^*(z) \geq \epsilon$ for every $z \in \Omega$ so if $u \in \mathcal{P}(\phi)$ we conclude that

$$\limsup_{z \to \xi} (u(z) - g^*(z)) \leq 0 \quad \text{for all boundary points } \xi \in \partial\Omega$$

The maximum principle for subharmonic functions gives $u(z) \leq g^*(z)$ for every $z \in \Omega$. Since $u \in \mathcal{P}(\phi)$ was arbitrary the construction of H_ϕ entails that $H_\phi \leq g^*$ holds in Ω. But then

$$\limsup_{z \to a} H_\phi(z) \leq \limsup_{z \to a} g^*(z) = \epsilon$$

where the last equality follows from Boulignad's Lemma. Since ϵ can be arbitrary small we get (**). There remains to show that

(***)
$$\liminf_{z \to a} H_\phi(z) \geq 0$$

To show this we put

$$g_*(z) = -\epsilon - \frac{M^*}{\delta} \cdot H_a(z)$$

It is clear that

$$\limsup_{z \to \xi} g_*(z) \leq \phi(\xi) \quad \text{for all boundary points } \xi \in \partial\Omega$$

Hence $g_* \in \mathcal{P}(\phi)$ which gives $g_* \leq H_\phi$. So now we have

$$\liminf_{z \to a} H_\phi(z) \geq \liminf_{z \to a} g_*(z) = -\epsilon$$

Since ϵ can be arbitrary small we get (***) and Theorem 1.1 is proved.

1.4 Proof of Boulignad's lemma.

By assumption the connected component of a is not reduced to a single point and hence Theorem XX from Chapter 4 gives the harmonic function $p_a^*(z)$ in Ω which satisfies:

(1)
$$\lim_{z \to a} p_a^*(z) = 0 \quad \text{and} \quad p_a^*(z) < 0 \quad : z \in \Omega$$

Next, let $\epsilon > 0$ which is kept fixed below. Since $a \in \partial\Omega$ we can find $0 < r \leq \epsilon$ such that the circle $|z - a| = r$ has a non-empty intersection with Ω. This intersection is an open subset of the circle $|z - a| = r$ denoted by Γ. Put

(2)
$$M = \max_{z \in \Omega} |z - a|$$

Now we can choose a compact subset Γ_* of Γ such that

(3)
$$\ell = \text{arc-length } (\Gamma \setminus \Gamma_*) \leq \frac{\epsilon}{M}$$

In the disc $D = \{|z - a| < r\}$ we find the harmonic function $V(z)$ whose boundary values on $|z - a| = r$ are zero outside the open set $\Gamma \setminus \Gamma_*$ while $V = M$ holds on $\Gamma \setminus \Gamma_*$. The mean-value formula for V entails that

(4)
$$V(a) = \ell \cdot M \leq \epsilon$$

Next, since Γ_* is a compact subset of Ω and $p_a^* < 0$ holds in Ω there exists $\delta > 0$ such that

(5)
$$p^*(z) \leq -\delta \quad : \quad z \in \Gamma_*$$

Set

(6)
$$B(z) = V(z) - \frac{M}{\delta} \cdot p^*(z) + \epsilon$$

This is a harmonic function in $\Omega \cap D$ and the construction of V together with (5) give

(7)
$$B(z) \geq M \quad : \quad z \in \Gamma$$

Final part of the proof. Consider the open set $U = \Omega \cap D$ where we get the subharmonic function

$$g = H_a - B$$

Since $|z-a| \leq \epsilon$ holds in the closed disc in $\bar{D}$ we have

(i) $$\limsup_{z \to w} H_a(z) \leq \epsilon \quad : \; w \in \bar{D} \cap \partial\Omega$$

Notice also that $H_a(z) \leq M$ holds in Ω which entails that

(ii) $$\limsup_{z \to w} H_a(z) \leq M \quad : \; w \in \Gamma$$

Next, we have the set-theoretic inclusion

(iii) $$\partial(D \cap \Omega) \subset \Gamma \cup (\bar{D} \cap \partial\Omega)$$

Hence (i-ii) and (7) give

(iv) $$\limsup_{z \to w} H_a(z) - B(z) \leq 0 \quad : \; w \in \partial(\Omega \cap D)$$

The maximum principle applied to the subharmonic function $H - B$ in $\Omega \cap D$ therefore gives

(v) $$H(z) \leq B(z) \quad : \quad z \in \Omega \cap D)$$

It follows that

(vi) $$\limsup_{z \to a} H_a(z) \leq \limsup_{z \to a} B(z) = V(a) + \epsilon + \limsup_{z \to a} p_a^*(z)$$

Notice that since a is an interior point of D the limes superior above is taken as $z \in \Omega$ tends to the boundary point a.Finally (1) and (4) show that the last term in (vi) is $\leq 2\epsilon$ and since $\epsilon > 0$ can be chosen arbitrary small we have proved that $\limsup_{z \to a} H_a(z) \leq 0$ which finishes the proof of Boulignad's lemma.

1.5 Wiener's solution.

Let Ω be a bounded open set. No further assumptions are imposed. For example, it may have infinitely many connected components. Consider the family $\mathcal{F}$ which consists of nested sequences of strictly increasing relatively compact subsets $\{\Omega_n\}$ such that $\cup \Omega_n = \Omega$ and Dirichlet's problem is solvable for each Ω_n. The nested property means that $\bar{\Omega}_n$ appears as a compact subset of Ω_{n+1} for every n. For example, if $N \geq 1$ we consider the family $\mathcal{D}_N$ of dyadic cubes whose sides have length 2^{-N}. We find the finite family $\mathcal{D}_N(\Omega)$ of cubes in this family whose closure stay in Ω. Their closed union is relatively compact in Ω and it is clear that Dirichlet's problem is solvable for the open set given by the union of the interior of this family. With increasing N this is an example of a nested sequence in $\mathcal{F}$.

1.6 Theorem. *For each $z \in \Omega$ and every nested sequence $\{\Omega_n$ the harmonic measures $\{\mathfrak{m}_z^{\Omega_n}\}$ converge weakly to a unique probability measure $\mathfrak{m}_z^*$ supported by $\partial\Omega$, and for each $f \in C^0(\partial\Omega)$ the function*

$$W_f(z) = \int f(\zeta) \cdot d\mathfrak{m}_z^*(\zeta)$$

is harmonic in Ω.

Proof. Let us first consider a ϕ be a continuous function ϕ on $\bar{\Omega}$ which is subharmonic in Ω. For every n its restriction to $\partial\Omega_n$ has a harmonic extension Φ_n to Ω_n. Next, let z be a point in Ω and start from some n_* such that $z \in \Omega_{n_*}$. To each $n \geq n_*$ we get the harmonic measure $\mathfrak{m}_z^{\Omega_n}$ and have

(1) $$\Phi_n(z) = \int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n}$$

for every $n \geq n_*$. Next, since ϕ is subharmonic and $\phi = \Phi_{n+1}$ holds on $\partial\Omega_{n+1}$, it follows that $\phi \leq \Phi_{n+1}$ in Ω_{n+1} and in particular it holds on $\partial\Omega_n$ which entails that

$$\Phi_n(z) = \int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \leq \int_{\partial\Omega_n} \Phi_{n+1} \cdot \mathfrak{m}_z^{\Omega_n} = \Phi_{n+1}(z) \tag{2}$$

where the last equality holds since the restriction of Φ_{n+1} to Ω_n is harmonic. Hence (1-2) entail that

$$\int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \leq \int_{\partial\Omega_{n+1}} \phi \cdot \mathfrak{m}_z^{\Omega_{n+1}} \quad \text{hold for every} \quad n \geq n_*$$

So we have a non-decreasing sequence of real numbers which in addition is bounded above because the maximum norm of ϕ on $\bar{\Omega}$ is finite. Hence there exists the limit

$$L(\phi) = \lim_{n\to\infty} \int_{\partial\Omega_n} \phi \cdot \mathfrak{m}_z^{\Omega_n} \tag{3}$$

Next, recall from XXX that a continuous function can be expressed as the difference of two subharmonic functions. We conclude that the limit in (3) exists for every $f \in C^0(\partial\Omega)$. By the Riesz representation Theorem this implies that weak-star limits from subsequences of $\{\mathfrak{m}_z^{\Omega_n}\}$ are equal and by the compactness of the set of probability measures on $\bar{\Omega}$ we obtain a unique weak limit $\mathfrak{m}_z^*$. Moreover, since the boundaries $\partial\Omega_n$ approach $\partial\Omega$ $\mathfrak{m}_z^*$ is supported by $\partial\Omega$.

At this stage we leave to the reader to show that if $\{U_n\}$ is another nested sequence then the weak star limit of every n some N such that $\{\mathfrak{m}_z^{U_n}\}$ is equal to the limit measure $\mathfrak{m}_z^*$ constructed via $\{\Omega_n\}$. Finally, the constructions above show that when z varies in Ω then

$$z \mapsto \int_{\partial\Omega} f \cdot \mathfrak{m}_z^*$$

is harmonic in Ω for every continuous boundary function f.

1.7 Regular boundary points. Above we constructed Wiener's harmonic function W_f. Let us now consider a point $a \in \partial\Omega$ such that $\{a\} = \mathcal{C}(a)$. We shall find a geometric condition order that a is regular for the Dirichlet problem which means that

$$\lim_{z|toa} W_f(z) = f(a)$$

hold for every $f \in C^0(\partial\Omega)$. Without loss of generality we can take a as the origin and for each $n \geq 1$ we consider the closed annulus

$$A_n = \{2^{-n} \leq |z| \leq 2^{-n+1}\}$$

Now $\{\partial\Omega \cap A_n\}$ are compact sets and to each we get the capacity $e^{-\gamma_n}$. As explained in XX this means that there exists a probability measure μ_n supported by $\partial\Omega \cap A_n$ such that the logarithmic potential

$$U_n(z) = \frac{1}{\gamma_n} \int \log \frac{1}{|z-\zeta|} \cdot d\mu_n(\zeta) = 1 \quad \text{for all points} \quad z \in \partial\Omega \cap A_n$$

1.8 Theorem. *The boundary point a is regular if and only if*

$$\sum_{n=1}^{\infty} \frac{n}{\gamma_n} = +\infty \tag{*}$$

Proof. Suppose first that (*) is convergent. Then we shall prove that a cannot be regular. We can take a as the origin and choose some $0 < \epsilon < 1/3$ which is kept fixed from now on. Next, (*) gives some N_* such that

$$\sum_{n=N_*}^{\infty} \frac{n}{\gamma_n} \leq \epsilon \tag{i}$$

Choose $F \in C^0(\partial\Omega)$ with $F(0) = 1$ and $F = 0$ on $\{|z| \geq 2^{-N_*+1} \cap \partial\Omega\}$ while the whole range $F(\bar{\partial}\Omega)$ is contained in $[0,1]$. Since the origin is assumed to be regular there exists $s > 0$ such that

(i) $$W_F(z) \geq 1 - \epsilon \quad \text{holds on} \quad D(s) \cap \Omega$$

where $D(s)$ is the disc of radius s centered at the origin.

The function V_r. When $r \to 0$ the discs $D(r)$ shrink to the singleton set $\{0\}$. So by the observation in XX the capacity expressed by $e^{-\gamma_r}$ tends to zero with r,i.e:

(iii) $$\lim_{r \to 0} \gamma_r \to +\infty$$

Let ν_r be the equilibrium distribution on $D(r) \cap \partial\Omega$ and put

$$V_r(z) = \frac{1}{\gamma_r} \cdot \int \log \frac{1}{|z - \zeta|} \cdot d\nu_r(\zeta)$$

So here $V(z) = 1$ on $D(r) \cap \partial\Omega$. From (iii) we can choose $r < s$ so small that

(iv) $$\max_{|z| \geq s} V_r(z) \leq \epsilon$$

Next, choose $M > N_*$ so large that $2^{-M} \leq r$ and set

$$U_M^*(z) = \sum_{n=N_*}^{n=M} \frac{1}{\gamma_n} \cdot U_n(z)$$

With this choice it is clear that

(v) $$V(z) + U_M^*(z) \geq 1 \quad \text{holds on} \quad D(2^{-N_*+1}) \cap \partial\Omega$$

Since $0 \leq F \leq 1$ and $F = 0$ on $\partial\Omega \setminus D(2^{-N_*+1})$ we have $F \leq V + U_M^*$ on the whole boundary of Ω. The minimum principle for super-harmonic functions gives the inequality

(vi) $$V + U_M^* \geq W_F \quad \text{in} \quad \Omega$$

Now (i) and (iv) entail that

(vi) $$\min_{z \in \Omega \cap D(s)} U_M^*(z) \geq 1 - 2\epsilon$$

At the same time we have $2^{-M} \leq s$ so the construction of U_M^* entails that $U_M^* = 1$ holds on $|z| \cap \partial\Omega$. Together with (vi) we obtain

(vii) $$\min_{|z|=s} U_M^*(z) \geq 1 - 2\epsilon$$

Now U_M^* restricts to a super-harmonic function on $D(s)$ so the minimum principle for super-harmonic functions gives

(viii) $$1 - 2\epsilon \leq U_M^*(0) = \sum_{n=N_*}^{n=M} \frac{1}{\gamma_n} \int \cdot \log \frac{1}{|\zeta|} \cdot d\mu_n(\zeta)$$

Finally, $|\zeta| \geq 2^{-n}$ holds on the support of μ_n and hence each integral is $\leq n \cdot \log 2$. So (viii) gives the inequality

$$1 - 2\epsilon \leq \sum_{n=N_*}^{n=M} \frac{n \cdot \log 2}{\gamma_n} < \log 2 \cdot \epsilon$$

This cannot hold since we have chosen $\epsilon < 1/3$. It means that (i) cannot hold and hence a is not a regular boundary point.

Proof of sufficiency.

Above we have proved that if a is regular then the series (*) in Theorem 1.8 must diverge. There remains to show that if (*) is divergent then a is regular. The proof of this converse is left as an exercise to the reader.

2. Harmonic conjugates

Let $H(x, y)$ be a harmonic function of class C^2 which means that

$$\Delta(H) = H_{xx} + H_{yy} = 0$$

Put

$$u = H_x \quad v = -H_y$$

Recall from Calculus that the mixed second order derivatives H_{xy} and H_{yx} are equal. Hence $u_y = -v_x$. Next, we have

$$u_x - v_y = H_{xx} + H_{yy} = 0$$

Hence (u, v) is a CR-pair and we can conclude:

2.1 Proposition. *Let H be harmonic in a domain Ω. Then*

$$f(z) = f(x + iy) = H_x - iH_y \in \mathcal{O}(\Omega)$$

2.2 The harmonic conjugate. Suppose that the analytic function $f(z)$ above has a primitive, i.e. there exists an analytic function $F(z) = U + iV$ whose complex derivative is f. We get

$$H_x - iH_y = U_x + iV_x = -iU_y + V_y \tag{1}$$

where we have used the two alternative expressions for the complex derivative $F'(z)$. Identifying real and imaginary parts in (1) give

$$H_x = V_y \quad \text{and} \quad H_y = -V_x \tag{2}$$

We refer to V as the harmonic conjugate of H. It is determined up to a constant. *Conversely*, assume that H has a harmonic conjugate V. Then we get analytic function $F = H + iV$ where

$$F' = H_x + iV_x = H_x - iH_y = f(z)$$

Hence we have proved

2.3 Theorem *A harmonic function H has a harmonic conjugate if and only if the analytic function $H_x - iH_y$ is the complex derivative of an analytic function.*

2.4 Existence of harmonic conjugates. Consider a domain $\Omega \in \mathcal{D}(C^1)$. Let H be harmonic in Ω which extends to a C^1-function up to the boundary. Put $f(z) = H_x - iH_y$. We seek a primitive function of f. For this purpose we fix a point $z_0 \in \Omega$ and if $z \in \Omega$ we choose a path γ from z_0 to z and construct the complex line integral of f along γ. Put

$$F_\gamma(z) = \int_\gamma f d\zeta \tag{i}$$

This gives an analytic function $F(z)$ in Ω provided that the integral above does not depend upon the particular path γ which joins z_0 and z. This independence holds if and only if the integral of f is zero along every *closed* curve inside Ω. Now $\partial\Omega$ consists of k disjoint closed curves $\Gamma_1, \ldots, \Gamma_k$ for some $k \geq 1$. By the *topological fact* from XX the f-intergral is zero along each closed curve in Ω if and only if

$$\int_{\Gamma_i} f(z)dz \quad : \quad 1 \leq i \leq k \tag{ii}$$

Let us express these complex line integrals in terms of derivatives of the H-function. We have

$$f dz = (H_x - iH_y)dz = H_x dx + H_y dy + i(H_x dy - H_y dx) \tag{iii}$$

Since Γ_i is closed we always have:

$$\int_{\Gamma_i} H_x dx + H_y dy = 0 \tag{iv}$$

Indeed, this holds by the observation in XXX. There remains to consider the line integrals

$$\int_{\Gamma_i} H_x dy - H_y dx = \int_{\Gamma_i} \big[H_x \cdot \mathbf{n}_x + H_y \cdot \mathbf{n}_y\big] \cdot ds = \int_{\Gamma_i} H_{\mathbf{n}} ds \tag{v}$$

Hence the results above give:

2.5 Proposition. *A harmonic function H which extends to a C^1-function on $\bar{\Omega}$ has a harmonic conjugate if and only if*

$$\int_{\Gamma_i} H_{\mathbf{n}} \cdot ds \quad : \quad 1 \leq i \leq k$$

where $H_{\mathbf{n}}$ is the normal derivative and ds the arc-length measure.

Remark. By Theorem Xx the line integral of $H_{\mathbf{n}}$ taken along the whole boundary is always zero, i.e. for every harmonic function H one has

$$\sum_{i=1}^{i=k} \int_{\Gamma_i} H_{\mathbf{n}} \cdot ds$$

So in order to check the conditions in Proposition 2.5 it suffices to regard $p-1$ many boundary curves.

2.6 The case $k=1$. If Ω only has one boundary curve Γ_1 the remark above shows that every harmonic function has a conjugate. Recall that Ω is a *Jordan domain* when its boundary consists of a single closed curve. Thus, in every Jordan domains there exist harmonic conjugates. If Ω is not a Jordan domain the situation is different, i.e. in general we cannot find harmonic conjugates.

2.7 Example. Consider an annulus $\Omega = \{r < |z| < R\}$. Here we have the harmonic function $H(z) = \log|z|$. On the boundary curve $|z| = R$ we notice that $H_n = \frac{1}{R}$ and since $ds = Rd\theta$ is the arc-length measure we get

$$\int_{\{|z|=R\}} H_{\mathbf{n}} ds = \int_0^{2\pi} d\theta = 2\pi$$

So here there does not exist a harmonic conjugate which reflects the fact that $\log z$ is multi-valued and hence prevents the analytic function $\frac{1}{z}$ to have a primitive in the annulus.

Exercise. We have $H = \frac{1}{2} \cdot \log(x^2 + y^2)$. Show that

$$H_x - iH_y = \frac{1}{z}$$

2.8 Exponential functions. Let $\Omega \in \mathcal{D}(C^1)$ where $\partial\Omega$ consists of p many pairwise disjoint and closed Jordan curves $\Gamma_1, \ldots, \Gamma_p$ and Γ_p is the outer curve. We assume that $p \geq 2$ and consider a harmonic function $H(z)$ which extends to a C^1-function on $\bar{\Omega}$. Put

$$a_j = \int_{\Gamma_j} H_{\mathbf{n}} \cdot ds \quad 1 \leq j \leq p-1\,.$$

Suppose there exist integers $m_1, \ldots, m_{p-1}$ such that

$$(*) \qquad a_j = 2\pi \cdot m_j \quad 1 \leq j \leq p$$

For each $1 \leq j \leq p-1$ we choose a point z_j in the interior of the Jordan domain bordered by Γ_j and set

$$(1) \qquad H^*(z) = H(z) - \sum_{j=1}^{j=p-1} m_j \cdot \log|z - z_j|$$

Then we see that

$$\int_{\Gamma_j} H^*_{\mathbf{n}} \cdot ds = 0 \quad 1 \leq j \leq p-1\,.$$

Proposition 2.5 shows that H^* has a harmonic conjugate V^* and there exists the analytic function

$$(2) \qquad f(z) = H^*(z) + iV^*(z)$$

It follows that

$$\left|e^{f(z)}\right| = e^{H^*(z)} = e^{H(z)} \cdot \left|(z-z_1)^{-m_1} \cdots (z-z_{p-1})^{-m_{p-1}}\right|$$

Hence, with $g(z) = (z-z_1)^{m_1} \cdots [z-z_{p-1})^{m_{p-1}} \cdot e^{f(z)}$ we have :

2.9 Theorem. *Let $H(z)$ be a harmonic function in Ω such that the periods $a_1, \ldots, a_{p-1}$ are integer multiples of 2π. Then there exists a zero-free analytic function $g(z) \in \mathcal{O}(\Omega)$ such that*

$$(**) \qquad |g(z)| = e^{H(z)}$$

Exercise. By the construction above the g-function extends to a C^1-function on the closure $\bar{\Omega}$ and the reader should verify that

$$\int_{\Gamma_j} H_{\mathbf{n}} \cdot ds = \mathfrak{Im}\big[\int_{\Gamma_j} \frac{g'(z)dz}{g(z)-w}\big]$$

hold for every boundary curve. Show also the converse to Theorem 2.9. Thus, assume that there exists an analytic function $g(z)$ with no zeros in Ω such that $|g(z)| = e^{H(z)}$ holds and deduce that the period numbers $\{a_j\}$ are integers times 2π.

2.10 The case when H is constant on boundary curves. Let H be as in Theorem 2.8 and suppose also that $H(z) = c_j$ is a constant on every boundary curve of Ω. Then $|g(z)| = e^{c_j}$ is constant on each Γ_j. Let w be a complex number whose absolute value is $\neq e^{c_j}$ for every j. Let $\mathcal{N}(g:w)$ be the number of zeros of $g(z)$ in Ω counted with multiplicity. The formula from Theorem XXX in Chapter 4 gives:

$$\mathcal{N}(g:w) = \mathfrak{Im}\,\big[\frac{1}{2\pi}\sum_{\nu=1}^{p} \int_{\Gamma_\nu} \frac{g'(z)dz}{g(z)-w}\big]$$

As explained in XX each single integral

$$\text{(ii)} \qquad \mathfrak{Im}\big[\frac{1}{2\pi} \cdot \int_{\Gamma_\nu} \frac{g'(z)dz}{g(z)-w}\big]$$

is equal to the winding number of the image curve $(g(z)-w) \circ \Gamma_\nu$. This individual winding number is zero if $|w| > e^{c_j}$ and if $|w| < e^{c_j}$ the stability of the winding number under a homotopic deformation identifies (ii) with the value when $w = 0$. Next, (**) in Theorem 2.9 gives the equality

$$g'(z) = g(z) \cdot (H_x - iH_y)$$

So with $w = 0$ it follows that (ii) becomes

$$\mathfrak{Im}\big[\frac{1}{2\pi} \cdot \int_{\Gamma_\nu} (H_x - iH_y) \cdot dz\big] = \frac{1}{2\pi} \cdot \int_{\Gamma_\nu} H_x \cdot dy + H_y \cdot dx = \frac{1}{2\pi} \cdot \int_{\Gamma_\nu} H_{\mathbf{n}} \cdot ds$$

Hence we have proved the following:

2.11 Theorem *Let w be a complex number such that $|w| \neq e^{c_j}$ for every j and denote by $w(*)$ the set of those j:s for which $|w| < e^{c_j}$. Then one has the equality*

$$\mathcal{N}(g:w) = \frac{1}{2\pi} \cdot \sum_{j \in w(*)} a_j$$

2.12 Example. Suppose that $a_p = 2\pi \cdot m$ for some positive integer m while $a_\nu = -2\pi\cdot$ for every $1 \leq \nu \leq p-1$. Suppose that w belongs to the open image set $g(\Omega)$ which means that the sum in Proposition 2.13 is a positive integer. Since $a_j \leq 0$ when $j \leq p-1$ it follows that the term from a_1 must be > 0 and hence $|w| < e^{c-1}$. Thus, $g(\Omega)$ is contained in the disc of radius e^{c_1}. Moreover, there cannot exist $2 \leq j \leq p-1$ with $e^{c_j} > e^{c-1}$ for then we can choose $|w| > e^{c_1}$ while $|w| < e^{c_j}$ and then the right hand side in Proposition 2.13 would be a negative integer.

3. Harmonic functions in a disc.

Above we studied general domains $\Omega \in \mathcal{D}(C^1)$. Here we specialize to the case when $\Omega = D$ is the unit disc centered at the origin. Already in Chapter 1 we encountered the Poisson kernel and described how to solve the Dirichlet problem. But let us resume this and give some alternative proofs. If f is analytic in a neighborhood of $\bar{D}$ we have:

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=1|}\frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta})\cdot d\theta}{1 - ze^{-i\theta}}$$

Since $z^n f(z)$ is analytic for each non-negative integer n we have

(i) $$\int_{|\zeta|=1}\zeta^n f(\zeta)d\zeta = i\cdot\int_0^{2\pi} e^{i(n+1)\theta}f(e^{i\theta})d\theta = 0$$

Consider the geometric series

$$\frac{\bar{z}e^{i\theta}}{1-\bar{z}e^{i\theta}} = \sum_{n=0}^{\infty}\bar{z}^{n+1}e^{i(n+1)\theta}$$

Since the series converges absolutely when $|z| < 1$, the vanishing (i) gives

(*) $$\int_0^{2\pi}\frac{\bar{z}e^{i\theta}\cdot f(e^{i\theta})\cdot d\theta}{1-\bar{z}e^{i\theta}} = 0$$

3.1 The Poisson kernel Let us add the zero (*) to the Cauchy integral and put

$$P(z,\theta) = \frac{1}{2\pi}\cdot\frac{1}{1-ze^{-i\theta}} + \frac{\bar{z}e^{i\theta}d\theta}{1-\bar{z}e^{i\theta}} = \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}$$

where the last equality follows since $|1-ze^{-i\theta}|^2 = (1-ze^{-i\theta})(1-\bar{z}e^{i\theta})$. We refer to $P(z,\theta)$ as the *Poisson kernel*. The results above give

3.2 Theorem *For each analytic function f in D which extends continuously to $\bar{D}$ one has*

$$f(z) = \int_0^{2\pi}P(z,\theta)f(e^{i\theta})d\theta \quad : \quad z \in D$$

Since the Poisson kernel is *real-valued* we can decompose $f = u + iv$ into its real and imaginary parts and obtain

$$u(z) = \int_0^{2\pi}P(z,\theta)u(e^{i\theta})d\theta \quad : \quad v(z) = \int_0^{2\pi}P(z,\theta)v(e^{i\theta})d\theta$$

3.3 Solution to Dirichlet's Problem Keeping θ fixed we have a function in D defined by $z \mapsto P(z,\theta)$. From the construction we have

$$2\pi\cdot P(z,\theta) = \frac{1}{1-ze^{-i\theta}} + \frac{\bar{z}e^{i\theta}d\theta}{1-\bar{z}e^{i\theta}}$$

The first term is an analytic function of z in D and since both the real and imaginary parts of an analytic function are harmonic, this complex valued function satisfies the Laplace equation. The second term is an *anti-analytic function*, i.e. analytic with respect to the conjugate complex variable. These functions also satisfy the Laplace equation and hence the function $z \to P(z,\theta)$ is harmonic in D. Now we solve the Dirichlet problem in D.

3.4 Theorem *Let $h \in C^0(T)$. Put*

$$H(z) = \int_0^{2\pi}P(z,\theta)h(e^{i\theta})\cdot d\theta$$

Then H solves the Dirichlet problem, i.e. it extends to a continuous function on T where it is equal to h.

Proof. We use that h is uniformly continuous and define its modulus of continuity:

$$\omega(\delta) = \max |h(\theta_1) - h(\theta_2)| \quad : \quad |\theta_1 - \theta_2| \leq \delta$$

For each $0 \leq \alpha \leq 2\pi$ we take a difference and obtain

$$2\pi \cdot [H(re^{i\alpha}) - h(e^{i\alpha})] = \int_0^{2\pi} \frac{(1-r^2)[h(e^{i(\theta+\alpha)} - h(e^{i\alpha})] \cdot d\theta}{1 + r^2 - 2r\cos\theta}$$

By the triangle inequality and the definition of the ω-function, the absolute value of the left hand side is majorized by

$$\int_0^{2\pi} \frac{(1-r^2) \cdot \omega(\theta) d\theta}{1 + r^2 - 2r\cos\theta}$$

With $\delta > 0$ given we split the θ-integral in two parts, where we integrate over $-\delta < \theta < \delta$ and in the second part $|\theta| \geq \delta$. Since the ω-function is non-decreasing, i.e. $\omega(\theta) \leq \omega(\delta)$ hold when $|\theta| \leq \delta$ we then majorize the integral above by

$$\omega(\delta) + M \cdot \int_{|\theta| \geq \delta} \frac{(1-r^2) d\theta}{1 + r^2 - 2r\cos\theta}$$

where M is the maximum norm of the ω-function. With $r = 1 - s$ and $0 < s \leq 1$ the last integral above is majorized by

$$2Ms \cdot \int_{|\theta| \geq \delta} \frac{d\theta}{(2-s)(1-\cos\theta)} \leq 2Ms \cdot \int_{|\theta| \geq \delta} \frac{d\theta}{1 - \cos\theta}$$

where the positive contribution of s^2 in the denominator is removed since this only strenghtens the inequality. Next, since $(1 - \cos\theta \geq \theta^2/2$ the lat integral is majorised by

$$\int_{|\theta| \geq \delta} \frac{2d\theta}{\theta^2} = 4\frac{\pi - \delta}{\delta} \leq \frac{4\pi}{\delta}$$

Hence (*) is majorized by

$$\omega(\delta) + \frac{8\pi M(1-r)}{\delta}$$

The required limit as $r \to 1$ follows since the ω-function tends to zero, i.e. given ϵ we choose δ so that $\omega(\delta) < \epsilon$ and after we choose $1 - r < \frac{1}{8\pi M}\delta\epsilon$ which majorizes (*) with 2ϵ.

3.5 Remark When the boundary function h has extra continuity, say that it is of class C^1 one can improve the rate of convergence. But we shall not to discuss detailed estimates concerned with the rate of convergence of

$$r \mapsto \max_\theta |\, H(re^{i\theta}) - h(e^{i\theta}) \,|$$

as $r \to 1$. One may also consider boundary functions for any L^1-function $h(\theta)$ on the unit circle we can apply the Poisson kernel and obtain a harmonic function $H(z)$ in the unit disc defined by

$$H(z) = \int_0^{2\pi} P(z, \theta) \cdot h(\theta) d\theta$$

The question arises if the H-function has *radial limit values*, i.e. if there exist limits of the form

$$\lim_{r \to 1} H(re^{i\theta}) = h(\theta)$$

One expects that the radial limit exists for all θ outside a null set, i.e. convergence holds almost everywhere in the sense of Lebesgue. This is indeed true. More precisely (*) holds at every Lebesgue point of the L^1-function h. We refer to 3.XX for details where we also study boundary values of H when we start from a Riesz measure on T.

3.6 Herglotz formula.

Given a real-valued continuos function u on the unit circle we define a function $U(z)$ in D by

(*)
$$U(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot u(e^{i\theta}) \cdot d\theta$$

It is clear that $U(z)$ is an analytic function in D, i.e. use that the geometric series in the denominator converges when $|z| < 1$. Next, multiply with $e^{-i\theta} - \bar{z}$ which is the complex conjugate of the denominator and notice that:

$$(e^{-i\theta} - \bar{z})(e^{i\theta} - z) = 1 - |z|^2 - 2i\mathfrak{Im}(ze^{-i\theta})$$

It follows that

(**)
$$U(z) = \frac{1}{2\pi} \int_0^{2\pi} [\frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot u(e^{i\theta}) d\theta + \frac{i}{2\pi} \int_0^{2\pi} [\frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta} - z|^2} \cdot u(e^{i\theta}) d\theta$$

Hence the real part of U is the Poisson integral of u, i.e. $\mathfrak{Re}(U)$ is the harmonic extension of u. Concerning the second integral which yields the imaginary part of U we know that it is harmonic since it is the harmonic conjugate to $\mathfrak{Re}(U)$. Starting with an analytic function $f(z) = u + iv$ and choosing $h = u|T$ as a boundary function, the whole consruction gives

3.7 Theorem. *Let $f(z)$ be analytic in D with continuous boundary values. Then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot \mathfrak{Re}(f(e^{i\theta})) \cdot d\theta + i \cdot \mathfrak{Im}(f(0)) \quad : \quad z \in D$$

Hence f is represented by a boundary integral expressed by its real part only, except for the value of its imaginary part at the origin.

3.8 Formula for a harmonic conjugate. Start with some real-valued continuous function u on the unit circle T. We find the unique analytic function $f(z)$ in D whose real part is u on T while $f(0)$ is real. Now $\mathfrak{Im}\, f$ is the harmonic conjugate of $\mathfrak{Re}\ f$ and (**) above gives:

(***)
$$\mathfrak{Im} f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta} - z|^2} \cdot u(e^{i\theta}) d\theta$$

3.9 The conjugate kernel $\mathcal{C}$. In the product set $D \times T$ we define the function

$$\mathcal{C}(z, \theta) = \frac{1}{2\pi} \cdot \frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta} - z|^2}$$

The $\mathcal{C}$-function is no longer positive and the integrals with respect to θ of its absolute value increase when $|z| \to 1$. Namely, let $z = r = 1 - s$ with $0 < s < 1$. Then

$$|\mathcal{C}(z, \theta)| = \frac{1 - s}{2\pi} \cdot \frac{|\sin \theta\,|}{[\, s^2 + 2(1 - s)(1 - \cos \theta)\,]^2}$$

So when s and θ both are close to zero the order of magnitude of $\mathcal{C}$ becomes

(1)
$$|\mathcal{C}(s, \theta)| \simeq \frac{|\theta|}{(s^2 + \theta^2)^2}$$

This explains why the map which sends a boundary function $u(e^{i\theta})$ into the boundary value function $v(e^{i\theta}$ of the harmonic conjugate of the harmonic extension of u is not so well-behaved. However, certain estimates are available. In particular the BMO-norm of the conjugate boundary value function v can be estimated by the maximum norm of u.

3.10 Theorem. *There exists an absolute constant C such that*

$$[v|_{\mathrm{BMO}} \leq C \cdot \max_{\theta} |u(e^{i\theta})|$$

The proof boils down to an exercise in one variable analysis. Namely, regard the following linear operators:

3.11 Definition. *To each $s > 0$ we define the operator T_s acting on real-valued functions $f(x)$ with support in $[-1, 1]$ by*

$$T_s(f)(y) = \int_{-1}^{1} \frac{x}{x^2 + s^2} \cdot f(x+y)dy$$

In the case when f is an even function we notice that $T_s(f)(0) = 0$. From the definition of functions in BMO and recalling that $\sin\theta \simeq \theta$ for small θ, the reader will have no difficulty to derive Theorem 3.10 from the following result:

3.12 Proposition. *There exists a constant C such that the following holds when f is even and has support in $[-1, 1]$:*

$$|\int \frac{1}{2h} \int_{-h}^{h} T_s(f)(y)dy| \leq |f|_\infty \quad : \forall \; 0 < s, h < 1$$

Proof. By partial integration the term above becomes

$$\frac{1}{2h} \int \mathrm{Log}\,[\frac{(\xi - h)^2 + s^2}{(\xi + h)^2 + s^2}] \cdot f(\xi)d\xi$$

Hence there only remains to show that there exists C so that

$$\int_{-1}^{1} |\mathrm{Log}\,[\,\frac{(\xi - h)^2 + s^2}{(\xi + h)^2 + s^2}]\,|d\xi \leq C \cdot h$$

Obviously C exists if there is a constant C^* such that

(i)
$$\int_0^1 \mathrm{Log}\,|\frac{(\xi + h)^2 + s^2}{\xi^2 + s^2}|d\xi \leq C^* \cdot h$$

Now we use that

$$\frac{\xi + h)^2 + s^2}{\xi^2 + s^2} = 1 + \frac{2h\xi + h^2}{\xi^2 + s^2} \leq 1 + \frac{2h^2 + \xi^2}{\xi^2} = 2 + 2\frac{h^2}{\xi^2}$$

So there remains only to show

(ii)
$$\int_0^1 \mathrm{Log}\,[2 + 2\frac{h^2}{\xi^2}\,] \cdot d\xi \leq C^* \cdot h \quad : \; 0 < h < 1$$

With $\xi = hu$ this follows since

(iii)
$$\int_0^\infty \mathrm{Log}\,[2 + 2\frac{1}{u^2}] \cdot du < \infty$$

In other words, we can take C^* as the value of this convergent integral.

3.13 A duality theorem.

Theorem 3.10 shows that the harmonic conjugation functor expressed via the boundary value function

$$u \mapsto \mathcal{C}(u) \quad : \; u(e^{i\theta}) \in L^\infty(T)$$

is a continuous linear operator from $L^\infty(T)$ into BMO. Next, consider the Hardy space $H^1(T)$ which consists of those $f \in L^1(T)$ which are boundary values of analytic functions in D, or equivalently those integrable functions on the unit circle for for which

$$\int_0^{2\pi} e^{in\theta}\cdot f(e^{i\theta})\cdot d\theta = 0 \quad : \; n=1,2,\ldots$$

The following result is due to C. Fefferman and E. Stein in [F-S]:

3.14 Theorem *There exists an absolute constant C such that*

$$|\int_0^{2\pi} \phi(e^{i\theta})\cdot f(e^{i\theta})\cdot d\theta\,| \leq C\cdot |\phi|_{\text{BMO}}\cdot |f|_1$$

hold for all pairs $f\in H^1(T)$ and ϕ in BMO.

For the proof of Theorem 3.14 and an account about Hardy spaces and BMO the reader is referred to text-books by E. Stein. See in particular (St:xxx].

3.15 The Riesz transform

Let $f(\theta)\in L^1(T)$. We get the Fourier coefficients

$$\hat{f}(n) = \int_0^{2\pi} e^{-in\theta} f(\theta)d\theta$$

A dense subspace of $L^1(T)$ consists of trigonometric polynomials, i.e. those functions on T with a finite Fourier series. Denote this subspace by $\mathcal{P}$. On this subspace we define the linear operator $\mathcal{R}$ by

(*) $$\mathcal{R}(e^{i\nu\theta}) = -e^{i\nu\theta} \;:\; \nu\leq -1 \quad : \mathcal{R}(e^{i\nu\theta}) = e^{i\nu\theta} \;:\; \nu\geq 1 \quad : \; \mathcal{R}(1)=0$$

This operator is closely related to the harmonic conjugation functor. To see this, let $n\geq 1$ and and put

$$u_n(\theta) = \frac{1}{2}[e^{in\theta}+e^{-in\theta}]$$

Its harmonic extension to D is $\mathfrak{Re}\,(z^n)$ and hence the boundary function of the harmonic conjugate is

$$v(\theta) = \frac{1}{2}[e^{in\theta}-e^{-in\theta}] \implies$$

(**) $$\mathcal{C}(u) = v = \mathcal{R}(u)$$

Taking linear sums we conclude that $\mathcal{R}$ via the associated Fourier series coincides with the harmonic conjugation functor. Passing to functions $f(\theta)$ which belong to $H^\infty(T)$ Theorem 3.10 therefore proves that $\mathcal{R}(f)$ has a bounded means oscillation. In other words, $\mathcal{R}$ gives a continuous linear operator from $H^\infty(T)$ to $\text{BMO}(T)$. We shall not continue this discussion since it is a topic in the theory about singular integral operators. See the text-books by E. Stein which presents this theory from many aspects including higher dimensional cases.

3.16 Poisson integrals of Riesz measures.

Let μ be a real Riesz measure on T. In general it is decomposed into $mu_c+\mu_s$ where μ_c is absolutely continuous and μ_s is singular. We obtain a harmonic function $H_\mu(z)$ in D defined by

$$H_\mu(z) = \frac{1}{2\pi}\int \frac{1-|z|^2}{|e^{i\theta}-z|^2}\cdot d\mu(\theta)$$

Exercise. Show the inequality below for each $r<1$:

(1) $$\int_0^{2\pi} |H(re^{i\phi}|\cdot d\phi \leq ||\mu|| \quad : \quad ||\mu||| \quad \text{is the total variation of} \quad \mu$$

Next, let $g(\theta)$ be a continuous function on T identified with the 2π-periodic θ-interval. Using the uniform continuity of g the reader may verify the limit formula

$$\lim_{r\to 1}\int_0^{2\pi} g(\theta)\cdot H(re^{i\theta}) = \int_0^{2\pi} g(\theta)\cdot d\mu(\theta) \tag{2}$$

Since (2) holds for every $g \in C^0(T)$ it means that the absolutely continuous measures $\{H(re^{i\theta})\}$ converge weakly to μ where Riesz' representation theorem identifies the dual space of $C^0(T)$ with Riesz measures on T.

A converse result. Let $H(z)$ be harmonic in D and assume that there is a constant C such that

$$\int_0^{2\pi} |H(re^{i\theta})|\cdot d\theta \leq C \quad \text{hold for all} \quad r<1 \tag{*}$$

3.17 Theorem. *When (*) holds there exists a unique Riesz measure μ on T such that $H = H_\mu$.*

Proof. Notice that (*) means that the family of absolutely continuous measures on T given by the family $\{\mu_r(\theta) = H(re^{i\theta})\}$ is bounded. So by the compactness in the weak topology of measures there exists an increasing sequence $\{r_n\}$ where $r_n \to 1$ and a Riesz measure μ such that

$$\lim_{n\to\infty}\int_0^{2\pi} g(\theta)\cdot H(r_ne^{i\theta})d\theta = \int_0^{2\pi} g(\theta)\cdot d\mu(\theta) \tag{i}$$

hold for every $g \in C^0(T)$. There remains to see why this gives $H = H_\mu$ in D. To prove this we fix $z \in D$ and if $r_n > |z|$ we have the Poisson formula

$$H(z) = \int_0^{2\pi} \frac{r_n^2 - |z|^2}{r_ne^{i\theta} - z|^2}\cdot H(r_ne^{i\theta})d\theta \tag{ii}$$

Keeping $z \in D$ fixed we set

$$g_n(\theta) = \frac{1}{2\pi}\cdot\frac{r_n^2-|z|^2}{|r_ne^{i\theta}-z|^2} \quad \text{and} \quad g_*(\theta) = \frac{1}{2\pi}\cdot\frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

It is clear that $g_n \to g_*$ holds uniformly over $0 \leq \theta \leq 2\pi$ as $n \to \infty$. So if $\epsilon > 0$ we can find $n*$ such that $n \geq n_*$ entails

$$\big|H(z) - \int_0^{2\pi} g_*(\theta)\cdot H(r_ne^{i\theta})d\theta\,\big| < \epsilon$$

Now

$$H_\mu(z) = \int_0^{2\pi} g_*(\theta)\cdot d\mu(\theta)$$

and at the same time (i) holds. So when $n \geq n*$ we see that

$$|H(z) - H_\mu(z)| < 2\epsilon$$

Finally, since $\epsilon > 0$ is arbitrary we get $H = H_\mu$ as required.

Remark. Theorem 3.17 gives a 1-1 correspondence between the space of Riesz measures on T and harmonic functions in D satisfying the L^1-condition (*).

3.18 Radial limits. Let μ be a Riesz measure on T which gives the harmonic function H_μ. it turns out that radial limits exist almost everywhere.

3.19 Theorem. *Write $\mu = \mu_c + \mu_s$. Then there exists a null set E on T such that*

$$\lim_{r\to 1} H_\mu(re^{i\theta}) = \mu_c(e^{i\theta})$$

holds when $\theta \in T \setminus E$ and θ is a Lebesgue point of the $L^1(T)$-function $\mu_c(e^{i\theta})$

Exercise. Prove this result. A hint concerning the singular part μ_s is as follows: Since μ_s is singular we know from XX that there exists a null set F on T such that

$$\lim_{\delta\to 0}\frac{1}{\delta}\cdot\int_{\theta-\delta}^{\theta+\delta} |d\mu_s(\theta)| = 0$$

for each $\theta \in T \setminus F$. Using the expression of the Poisson kernel the reader can verify that this gives:

$$\lim_{r \to 1} H_{\mu_s}(re^{i\theta}) = 0 \quad \text{for each} \quad \theta \in T \setminus F$$

3.20 A class of analytic functions. Let μ be a singular measure on T. In D we get the analytic function

$$U_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot d\mu(\theta)$$

We see that $\mathfrak{Re}\, U_\mu = H_\mu$ and taking the exponential we get the analytic function

$$\omega_\mu(z) = e^{U\mu(z)}$$

Here the absolute value

$$|\,\omega_\mu(z)| = e^{H_\mu(z)}$$

Theorem 3.19 entails that there is a null set F on T such that

(1) $$\lim_{r \to 1} |\omega_\mu(re^{i\theta})| = 1 \quad : \quad \theta \in T \setminus F$$

The question arises if there exist radial limits

$$\lim_{r \to 1} \omega_\mu(re^{i\theta}) = \omega_*(e^{i\theta})$$

outside a null set. This turns out to be true but the proof is more involved and relies upon the Brothers Riesz theorem which is established in Section X from Special Topics and a further study of analytic functions of the form $\omega_\mu(z)$ for singular Riesz measures is treated in XX from Special Topics.

4. The mean value property

Let H be a harmonic function defined in an open domain Ω. Let $z_0 \in \Omega$ and D is a disc centered at z_0 of some radius R with R chosen so that $\bar{D} \subset \Omega$. We already know the mean-value formula:

$$(*) \qquad H(z_0) = \frac{1}{2\pi} \int_0^{2\pi} H(z_0 + Re^{i\theta}) d\theta$$

It turns out that the converse is true. More precisely a real-valued and continuous function in Ω satisfying a local mean-value formula at every point in Ω is harmonic. Since the condition to be harmonic is local it suffices to prove this when Ω is a disc.

4.1 Theorem *Let $\bar{D}$ be a closed disc of some radius R centered at the origin. Let $h \in C^0(\bar{D})$ and assume that to every $z_0 \in D$ there exists some $0 < r \leq dist(z_0, \partial D)$ such that $h(z_0)$ equals its mean value over the circle $|z - z_0| = r$. Then h is harmonic in D.*

Proof. We solve the Dirichlet problem and find a harmonic function H in D which extends to a continuous function on $\bar{D}$ where $H = h$ on the circle $\{|z| = R\}$. Now we claim that $h = H$. Assume the contrary and suppose for example that there exists a point z_0 where $h(z_0) > H(z_0)$. Since a continuos function achieves its maximum on the compact set $\bar{D}$ we get the closed set K where $h - H$ takes its maximum. Here K must be inside the open disc since $h = H0$ on the boundary. Choose a point $z^* \in K$ such that $|z^*| \geq |z|$ for any other point $z \in K$. By hypothesis there exists a circle of some radius $r \leq R - |z^*|$ such that $h(z^*)$ is equal to its mean value over the circle $T = \{|z - z^*| = r\}$. Since the harmonic function H satisfies the mean value condition everywhere, it follows that

$$\text{(i)} \qquad h(z^*) - H(z^*) = \frac{1}{2\pi} \int_0^{2\pi} (h - H)(z^* + re^{i\theta}]) d\theta$$

But this gives a contradicton. For $H - h$ takes its maximum at z^* and by the choice of z^* we can find θ such that $z^* + re^{i\theta}$ is outside the set K and then the integral in (i) cannot be equal to $h(z^*) - H(z^*)$. This proves that $h \leq H$ holds in the whole disc. Since $-h$ also satisfies the local mean value condition we prove that $h \geq H$ in the same way and Theorem 3.1 follows.

4.2 Remark Notice that Theorem 4.1 implies that if h is *not* harmonic then there must exist some point $z_0 \in D$ such that

$$h(z_0) \neq \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta}) d\theta$$

hold for all $0 < r \leq R - |z_0|$. Notice also that this inequality means that $h(z_0)$ is either strictly less or strictly larger than all mean-values above.

4.3 A study of Log $|f|$. Let $\Omega \in \mathcal{D}(C^1)$ and f is an analytic function in Ω which extends to a continuos function on $\bar{\Omega}$. We also assume that $f \neq 0$ on $\partial\Omega$ but inside Ω it has some zeros $a_1, \ldots, a_k$ where the multiplicity of each zero can be any positive integer. Outside these zeros the function $\text{Log}\,|f|$ exists and it is harmonic in $\Omega \setminus (a_1, \ldots, a_k)$. Now one has the following useful inequality:

4.4 Propostion *Let h solve Dirichlet's problem for the boundary function Log $|f|$. Then*

$$\text{Log}\,|f(z)| \leq h(z) \quad : \quad z \in \Omega \setminus (a_1, \ldots, a_k)$$

Proof. Let $\epsilon > 0$ be small and let Ω_ϵ be the open set where the closed discs of radius ϵ around the zeros of f are removed from Ω. If m is the order of a zero of f at a_ν we have

$$\text{Log}(|f|)(z - a_\nu) = m \cdot \text{Log}|z - a_\nu| + \text{ a bounded function}$$

when z is close to a_ν. So on the boundary of one of the removed discs the harmonic function $\text{Log}(|f|)$ is negative and $\simeq$ to the negative number $m \cdot \text{Log}(\epsilon)$. For a sufficiently small ϵ it therefore follows that

$$\text{Log}(|f|)(z) \leq h(z) \quad : \quad z \in \cup \partial D_\epsilon(a_\nu)$$

At the same time equality holds on $\partial\Omega$. Hence the maximum principle for harmonic functions gives the inequality

$$\text{Log}(|f|)(z) \leq h(z) \quad : \quad z \in \Omega \setminus \cup \bar{D}_\epsilon(a_\nu)$$

Since we can take any small ϵ Proposition 2.4 follows.

4.5 A specific majorisation Let $f(z)$ be analytic in the half-plane $\mathfrak{Im}(z) > 0$. Assume also that f extends continuously to the real line where $y = \mathfrak{Im}(z) = 0$. Consider a finite sequence of real numbers $a_0 < a_1 < \ldots a_N$ where some a_ν may be negative. We get the pariwise disjoint intervals

$$J_0 = (-\infty, a_0) \quad J_\nu = (a_\nu, a_{\nu+1}) \colon \; 0 \leq \nu \leq N-1 \quad J_N = (a_n, +\infty)$$

Suppose there are constants $c_0, \ldots, c_N$ such that

$$\text{Log}\,|f(x)| \leq c_\nu \quad : \quad x \in J_\nu \quad : \quad 0 \leq \nu \leq N$$

In addition to this we assume that

$$(*) \qquad \lim_{|z| \to \infty} f(z) = 0$$

where z stay in the closed upper half-plane during the limit. In this case we can chose a specific harmonic majorant to $\text{Log}\,|f(x)|$. Namely, put

$$H(z) = \sum_{\nu=0}^{\nu=N} c_\nu \cdot H_{J_\nu}(z)$$

where $\{H_{J_\nu}\}$ are the harmonic measure functions of the J-intervals. Then we have

$$\text{Log}\,|f(x)| \leq H(x)$$

for real x.

Exercise. Show that (*) now entails the global inequality:

$$\text{Log}\,|f(z)| \leq H(z) \quad : \quad z \in U$$

5. Harmonic measures

Let Ω be a bounded connected and open set where Dirichlets problem can be solved. If $z \in \Omega$ we get a linear form on $C^0(\partial\Omega)$ defined by

$$\phi \mapsto H_\phi(z) \tag{1}$$

Thus, we evaluate the unique harmonic extension of the boundary function ϕ at the point z. If $\phi \geq 0$ then $H_\phi(z) \geq 0$ and when $\phi = 1$ is the identity we have $H_\phi = 1$. Riesz' Representation Formula gives a unique probability measure on $\partial\Omega$ denoted by $\mathfrak{m}_z$ such that

$$H_\phi(z) = \int_{\partial\Omega} \phi(\zeta) \cdot d\mathfrak{m}_z(\zeta) \tag{2}$$

When z varies in Ω these measures are absolutely continuous with respect to each other. To see this we fix some point p in Ω. Let q be another point and suppose that $\mathfrak{m}_q$ is not absolutely continuous with respect to $\mathfrak{m}_p$ which gives a closed subset E of $\partial\Omega$ where $\mathfrak{m}_p(E) = 0$ and $\mathfrak{m}_q(E) = a > 0$. Now we can choose a sequence of continuous boundary functions $\{\phi_n\}$ such that $0 \leq \phi_n \leq 1$ while $\phi_n = 1$ on E and

$$\lim_{n \to \infty} \int_{\partial\Omega} \phi_n \cdot d\mathfrak{m}_p = 0 \tag{1}$$

Here $\{H_n = H_{\phi_n}\}$ is a sequence of harmonic functions such that $0 \leq H_n \leq 1$ holds and $H_n(p) \to 0$. The minium principle for harmonic functions entails that $H_n \to 0$ holds on uniformly on every compact subset of Ω. In particular $H_n(q) \to 0$. But this gives a contradiction since the ϕ-functions give $H_n(q) \geq a$ for all n.

Öksendal's theorem. If may occur that the boundary $\partial\omega$ has a positive 2-dimensional Lebesgue measure. But the harmonic measures are always concentrated to null sets with respect to the Lebesgue measure in $\mathbf{R}^2$. Nasmely, one has

5.0 Theorem. *There exists a null-set E in $\partial\Omega$ such that $\mathfrak{m}_p(E) = 1$ for every $p \in \Omega$.*

We prove this result in 5.x below after some results concerning majorisation principles for harmonic measures have been established.

The principle of Gebietserweiterung. The inequality in (*) below is used in many situations. Let Ω and U be two domains where the Dirichlet problem is solvable for both. Suppose that $\Omega \subset U$ and put

$$\partial\Omega = \gamma \cup \gamma$$

where γ is a compact subset of ∂U while Γ is a subset of $\bar{\Omega}$. Consider also a compact subset K of ∂U such that $\bar{\Omega} \cap K = \emptyset$. Let z be a point in Ω which also may be regarded as a point in U. We get the harmonic measure $\mathfrak{m}_z^U(K)$, i.e. the value at z of the harmonic function h in U for which $h = 1$ on K while $h = 0$ on $\partial U \setminus K$. At the same time we have the harmonic measure $\mathfrak{m}_z^\Omega(\Gamma)$ which equals $H(z)$ for the harmonic function in Ω which is one on Γ and zero on $\gamma \setminus \Gamma$. Since $0 \leq h \leq 1$ holds in U it follows that $h \leq H$ on Γ and at the same time both H and h are zero on $\gamma \setminus \Gamma$. Hence $h \leq H$ holds on $\partial\Omega$ so the maximum principle for harmonic functions entails that

$$\mathfrak{m}_z^U(K) \leq \mathfrak{m}_z(\Gamma) \tag{*}$$

5.1. Conformal invariance. Let Ω and Ω^* be two bounded and connected open sets which both are of class $\mathcal{D}(C^1)$. Let $g\colon \Omega \to \Omega^*$ be a conformal map which extends to the boundary and gives a homeomorphism from $\bar{\Omega}$ to $\bar{\Omega}^*$. As explained in XX the g-function sends a harmonic function H in Ω^* to the harmonic function $H \circ g$ in Ω. Let $\Gamma_1, \ldots, \Gamma_p$ be the closed boundary curves to Ω. Their images $\{g(\Gamma_\nu)\}$ give the boundary curves of Ω^*. Let E be some Borel set in $\partial\Omega$ which gives the harmonic measure function

$$H_E(z) = \int_E d\mathfrak{m}_z(\zeta)$$

Similarly we get the harmonic measure function

$$H^*_{g(E)}(w) = \int_{g(E)} d\mathfrak{m}^*_w(\zeta)$$

where $\mathfrak{m}^*_w$ is the harmonic measure on $\partial\Omega^*$. Now we have the equality

$$H_E(z) = H^*_{g(E)} \circ g(z) \implies$$

$$(*) \qquad \int_E d\mathfrak{m}_z(\zeta) = \int_{g(E)} d\mathfrak{m}^*_{g(z)}(\zeta) \quad \text{holds for all points} \quad z \in \Omega$$

Since g restricts to a homeomorphism from $\partial\Omega$ into $\partial\Omega^*$ this gives a transformation rule for the harmonic measure. This invariance is often used and we shall meet such applications in the chapter about conformal mappings.

5.2 Harmonic measures in half-discs. Let $R > 1$ and consider the half-disc $D^+(R) = D_R \cap \{y > 0\}$. which is contained in the upper half-plane $\mathfrak{Im}(z) > 0$. Here there exists the harmonic function

$$u(z) = \mathfrak{Im}(\text{Log}(z-R)) - \mathfrak{Im}(\text{Log}(z+R)) = \arg(z-R) - \arg(z+R)$$

As already pointed out in XX we have

$$\lim_{y \to 0} u(x+iy) = \pi \quad : \quad -R < x < R$$

Moreover, by euclidian geometry we have

$$u(Re^{i\theta}) = \frac{\pi}{2} \quad : \; 0 < \theta < \pi$$

Set

$$(*) \qquad u^*(z) = \frac{2}{\pi} \cdot (u(z) - \frac{\pi}{2})$$

Then $u^*(x) = 1$ on the real interval $(-R, R)$ while $u^* = 0$ on the portion of $\partial D(R)^+$ where $|z| = R$. Hence, if $0 < a < R$ then the harmonic measure of the point ai with respect to the boundary pice $(-R, R)$ is equal to $u^*(ai)$.

Exercise. Show that

$$u^*(ai) = \frac{2}{\pi} \cdot \text{arctg}(\frac{R}{a})$$

We have also the harmonic measure with respect to the circular boundary where $|z| = R$ which is given by the function $1 - u^*$. hence, the harmonic measure at ai with respect to this circular portion is equal to

$$1 - \frac{2}{\pi} \cdot \text{arctg}(\frac{R}{a}) = \frac{2}{\pi} \cdot \int_{R/a} \frac{dt}{1+t^2}$$

The last term is majorised by

$$(*) \qquad \frac{2}{\pi} \cdot \int_{R/a} \frac{dt}{t^2} = \frac{2a}{\pi R}$$

In particular this harmonic measure behaves like $O(\frac{1}{R})$ when $R \to +\infty$ while a stays in a bounded interval.

5.3 Lindelöf's theorem.

Let D^+ be a half-disc where $|z| < R$ and $\mathfrak{Im}\, z > 0$. Let $f(z)$ be a bounded analytic function in D^+ with maximum norm ≤ 1. Consider also a Jordan arc γ which belongs to D^+ except for the endpoint $\gamma(1)$ which is taken as the origin. Suppose that f converges to zero along γ, i.e.

$$\lim_{s \to 1} f(\gamma(s)) = 0$$

Then Lindelöf proved that f has a non-tangential limit at $z = 0$. More precisely, for each $0 < \alpha < \pi/2$ one has

$$(*) \qquad \lim_{r\to 0}\,[\max_{\alpha\leq\theta\leq\pi-\alpha}\,|f(re^{i\theta})|\,] = 0$$

Proof. Consider the subharmonic function $u(z) = \log\,|f(z)|$. For each large $M > 0$ it follows from (1) that there exists $s_0 < 1$ such that

$$\text{(i)} \qquad u(\gamma(s)) \leq -M$$

for all $s_0 \leq s < 1$. Now we choose $0 < R_* < R$ such that the smallest s for which $|\gamma(s)| = R_*$ holds is $\geq s_0$. Denote this s-number with s_* So if D_*^+ is the half-disc where $|z| < R_*$ then

$$\text{(ii)} \qquad u(\gamma(s)) \leq -M$$

for all $s_* \leq s < 1$ and the half-open Jordan curve $\gamma[s_*, 1)$ stays in D_*. It follows from Jordan's theorem that this curve divides D_*^+ into a pair of disjoint Jordan domains Ω_1 and Ω_2. Moreover, as illustrated by a picture the real intervals $[-R_*, 0]$ and $[0, R_*]$ appear in the boundary of each Ω-domain. We may assume that

$$[-R_*, 0] \subset \partial\Omega_1$$

In addition $\partial\Omega_1$ consists of the Jordan arc $\gamma[s_*, 1]$ and an arc of the circle $|z| = R_*$ with end-points at $-R_*$ and $\gamma(s_*)$. Next, given $0 < \alpha < \pi$ we consider points $z \in \Omega_1$ such that:

$$\text{(iii)} \qquad z = re^{i\theta} \quad : \quad \alpha \leq \theta \leq \pi - \alpha \quad \text{and} \quad 0 < r < R_*/2$$

For each such point z we have the harmonic measure $\mathfrak{m}_z$ on $\partial\Omega_1$. Since $|f| \leq 1$ was assumed we have $u \leq 0$ and since u is subharmonic we get:

$$\text{(iv)} \qquad u(z) \leq -M \cdot \mathfrak{m}_z(\Omega_1 \cap \gamma)$$

At this stage we use the principle of *Gebietserweiterung*. Consider the whole half-disc D_*^+. Here $[0, R_*]$ is the portion of its boundary and since the Jordan arc γ separates the half disc where $[0, R_*] \subset \partial\Omega_2$ one has the inequality

$$\mathfrak{m}_z(\Omega_1 \cap \gamma) \geq \mathfrak{m}_z^*([0, R_*])$$

where $\mathfrak{m}_z^*$ is the harmonic measure for z in the half disc. It follows from (iv) that

$$u(z) \leq -M \cdot \mathfrak{m}_z^*(([0, R_*])/tagv$$

Finally, since z satisfies (1) it follows that

$$\text{(vi)} \qquad \mathfrak{m}_z^*(([0, R_*]) \geq \rho(\alpha)$$

where $\rho(\alpha)$ is a positive number which can be computed via a conformal mapping. See XX for details. So when z is as in (iii) we get the estimate

$$u(z) \leq -\rho(\alpha) \cdot M$$

Here $\rho(\alpha)$ depends on the fixed α-value only while M can be arbitrary large while R_* gets small and this obviously entails that we have the requested limit (*) in the sector $\alpha \leq \arg(z) \leq \pi - \alpha$. Finally, we proved that the limit when z stays in Ω_1. Reversing the role where one instead works with the interval $[-R_*, 0]$ we get again the limit in (*) and Lindelöf's theorem is proved.

Remark. The original proof by Lindelöf appears in [Lindelöf]. The idea to employ harmonic measures and estimate the subharmonic function $\log\,|f(z)|$ was introduced by Carleman. See page xxxx in Nevanlinna's book [Nev] for an account about the principle for harmonic measures under a change of domains.

5.4 Application to Laurent series. Let $f(z)$ be analytic in a punctured disc $0 < |z| < 1$. If $0 < r < 1/2$ and $|z| = r$ we find estimates for harmonic measures as follows. To each $0 < \epsilon < r$ we have the annulus $\epsilon < |z| < 1/2$ and put

$$\omega(z) = \frac{1}{\log\frac{1}{2\epsilon}} \cdot \log\frac{|z|}{\epsilon}$$

This yields a harmonic function and we conclude that

$$\mathfrak{m}_z(T_*) = \frac{1}{\log \frac{1}{2\epsilon}} \cdot \log \frac{2r}{2\epsilon} = 1 + \frac{\log 2r}{\log \frac{1}{2\epsilon}}$$

where T_* is the circle $|z| = 1/2$. It follows that the harmonic measure with respect to the inner circle $T_\epsilon = |z| = \epsilon$ becomes

(1) $$\mathfrak{m}_z(T_\epsilon) = 1 - \mathfrak{m}_z(T_*) = \frac{\log \frac{1}{2r}}{\log \frac{1}{2\epsilon}}$$

Suppose now that $f(z)$ is bounded in the punctured disc with a maximum norm M. At the same time we consider the maximum norm of $|f|$ on the circle $|z| = 1/2$ which we denote by M_*. Since $\log |f|$ is subharmonic it follows from (1) that we have

(2) $$\log |f(z)| \leq \log M_* + M \cdot \frac{\log \frac{1}{2r}}{\log \frac{1}{2\epsilon}}$$

This inequality holds for every $|epsilon > 0$ and since $\frac{1}{2\epsilon} \to +\infty$ we conclude that $\log |f(z)| \leq \log M_*$ which entails that

(3) $$|f(z)| \leq M_*$$

The point is that (3) does not depend upon M. Now we give an application:

5.5 Theorem. *Let f be an analytic function in the punctured disc $0 < |z| < 1$ where f and all is derivatives $f^{(n)}$ are bounded. Then f extends to an analytic function in the disc $|z| < 1$.*

Proof. Let M be the maximum norm of f in $0 < |z| < 1$. If $n \geq 1$ then Cauchy's inequality entails that

(i) $$|f^{(n)}(z)| \leq n\,! \cdot 2^n \cdot M$$

for each $|z| = 1/2$. Using (3) applied to the analytic function $f^{(n)}$ it follows that (i) holds for all $0 < |z| < 1/2$. Now we consider the series expansion at $z = 1/4$:

(ii) $$f(1/4 + z) = \sum \frac{f^{(n)}(1/4)}{n\,!} \cdot z^n$$

Since (i) hold with $z = 1/4$ we see that the series (ii) has a radius of convergence which is at least $1/2$ which implies that $f(z)$ extends beyond $z = 0$.

5.6 Proof of Öksendal's theorem.

We shall interpretate harmonic measures via the hitting probability for a Browninan motion. To begin with we consider the following situation. Let $\delta > 0$ and start a Brownian motion a point p with $|p| = \delta$. Suppose also that there is a closed subset Γ on a circle $|z| = r$ for some $\delta/2 \leq r < 3\delta/4$ whose linear measure is $(1 - \epsilon) \cdot 2\pi r$ for some $0 < \epsilon < 1$. This set is viewed as an obstacle and we seek the probability that the Brownian motion reaches some point on the circle $|z| = \delta/2$ before it has hit Γ which means that a Brownian path during passages through $|z| = r$ must stay in one of the open intervals on this circle which form the open complement of Γ. With this in mind one has

5.7 Proposition. *There exists an absolute constant C which is independent of both δ and ϵ such that the survival probalility is $\leq C \cdot \epsilon$.*

The result is intuitively clear and for a formal proof we refer to XXX. There remains to see why Propostion 5.7 gives Theorem 5.0. To achieve this we study Lebesgue points on the compact set $\partial\Omega$ where we assume that its 2-dimensional measure is > 0 since otherwise Theorem 5.0 is trivial. With C chosen as above we set

(1) $$\epsilon = \frac{1}{8C}$$

For each positive integer M we set we let K_M be the subset of $\partial\omega$ such that if $x \in K_M$ then

$$(2) \qquad 0 < \delta \leq 2^{-M} \implies \frac{|\partial\Omega \cap B_z(\delta)|_2}{\pi\delta^2} \geq 1 - \epsilon$$

Since almost every point in ω is a Lebesgue point one has

$$(3) \qquad \lim_{M\to\infty} |\Omega \setminus K_M|_2 = 0$$

Next, fix a point $p \in \Omega$ and consider the measure$\mathfrak{m}_p$. We must show that this Riesz measure is singular and from (3) it suffices to show that its restriction to K_M is singular for all M. Since the sets $\{K_M\}$ increase with M it suffices to show this when we start from M-values such that the distance of p to $\partial\omega$ is strictly greater than 2^{-M}. By the general result in MEASURE the restriction $\mathfrak{m}_p|K_M$ is singular if

$$(5) \qquad \lim_{N\to\infty} 4^N \cdot \mathfrak{m}_p(B_z(2^{-N})) = 0$$

for each $z_* \in K_M$. To prove (5) we consider integers $N \geq M+1$ and identify $\mathfrak{m}_p(B_{z_*}(2^{-N})$ with the probability that a Brownian path starting at p reaches the circle $|z - z_*| = 2^{-N}$ before it has hit $\partial\Omega$. In order that such a path survives it must pass the circles $T_k = |z - z_*| = 2^{-k}$ for every $M \leq k \leq N$. If the path has reached a point $p \in T_k$ the probability that it also reaches T_{k+1} can be estimated above by Proposition 5.x. Namely, for a given k we set

$$\rho_k = \max_r \frac{1}{2\pi r} \cdot \text{Length of} \quad \partial\Omega \cap T_r|$$

where the maximum is taken when $2^{-(k+1)} \leq r \leq 3 \cdot 2^{-k}/4$. It follows that the area:

$$|\Omega \cap B_{z_*}(2^{-k}|_2 \geq (1-\rho_k)\pi \cdot 2^{-2k} \cdot (9/16 - 1/4)$$

On the other hand, since $z_* \in K_M$ we also have

$$|\Omega \cap B_{z_*}(2^{-k})|_2 \leq \epsilon \cdot \pi \cdot 2^{-2k}$$

It follows that

$$1 - \rho_k \leq \frac{16}{5} \cdot \epsilon \implies \rho_k \geq 1 - \frac{16}{5} \cdot \epsilon$$

Hence we find a radius r in the interval above where the length of $\partial\Omega \cap T_r$ is at least $(1 - \frac{16}{5} \cdot \epsilon) \cdot 2\pi r$. This set is an obstacle and from Proposition together with the general principle in 5.1 it follows that the probability to move from p to a point on T_{k+1} without hitting $\partial\Omega$ is majorized by

$$C \cdot \frac{16}{5} \cdot \epsilon = \frac{1}{8}$$

where (1) above gives the last inequality. At this stage we easily get (5). For in order to move from a point p on the circle $|z - z_*| = 2^{-M}$ to a point on $|z - z_*| = 2^{-N}$ we must survive the passage for each k as above, i.e. we must survive a passage $N - M - 1$ many times. By (**) the probability for this survivial is estimated above by

$$\frac{1}{8^{N-M-1}} = 4^{-N} \cdot 8^{M+1} \cdot 2^{-N}$$

It follows that

$$4^N \cdot \mathfrak{m}_p(B_z(2^{-N})) \leq 8^{M+1} \cdot 2^{-N}$$

With M fixed the right hand side tends to zero with N and hence (5) follows which finishes the proof of Öksendal's result.

6. Green's functions.

Let Ω be a bounded and connected set of class $\mathcal{D}(C^1)$. Let $z_* \in \Omega$ which gives the continuous boundary function

(1) $$\zeta \to \text{Log}\,|\zeta - z_*|$$

In Ω we get the unique harmonic function whose boundary function is (1). It depends upon z_* and is denoted by $H(z_*; z)$ where z is the "active variable". Keeping z_* fixed the values of H inside Ω are recaptured by the harmonic measure, i.e. one has:

(2) $$H(z_*; z) = \int_{\partial\Omega} \log\,|\zeta - z_*|\, d\mathfrak{m}_z(\zeta) \quad : \; z \in \Omega$$

Keeping z fixed the measure $\mathfrak{m}_z$ can be weakly approximated by a sequence of discrete measures of the form $\rho = \sum p_k \cdot \delta_{\zeta_k}$ where $\sum p_k = 1$ and $\{\zeta_k\}$ is a finite subset of $\partial\Omega$. Next, since the functions

$$\zeta_* \mapsto \log\,|\zeta - z_*|$$

are harmonic in Ω for each $\zeta \in \partial\Omega$ the reader can verify that the functions

$$\zeta_* \mapsto H(z_*; z)$$

are harmonic for every z. It turns out that one has symmetry.

6.1. Theorem. *For each pair of points z_*, z in Ω one has:*

$$H(z_*, z) = H(z, z_*)$$

Proof. By the construction of the H-function we see that

(1) $$H(z_*, \zeta) = \log\,|\zeta - z_*|$$

holds for each $\zeta \in \partial\Omega$. To prove the symmetry we introduce Green's functions with logarithmic singularities. To each pair of distinct points in Ω we set:

(2) $$G(z_*; z) = H(z_*; z) - \text{Log}\,|z - z_*| \quad : \; z_* \neq z$$

Then (1) entails that

(3) $$G(z_*; \zeta) = 0 \quad : \; \zeta \in \partial\Omega$$

At the same time the function of z defined by

$$-G(z; z_*) = \text{Log}\,|z - z_*| - H(z_*; z)$$

is subharmonic in Ω and tends to $-\infty$ as $z \to z_*$. By (3) it vanishes on $\partial\Omega$ so by the maximum principle for subharmonic functions it is < 0 in $\Omega \setminus z_*$ and returning to G we get

(4) $$G(z_*; z) > 0 \quad : \; z \in \Omega \setminus z_*$$

Next, since $\text{Log}\,|z - z_*|$ is symmetric we have:

$$G(z_*; z) - G(z; z_*) = H(z_*; z) - H(z; z_*)$$

Keeping z_* fixed (3) and (4) show that the left hand side is < 0 when $z \in \partial\Omega$. Hence the maximum principle for harmonic functions gives

$$G(z_*; z) - G(z; z_*) \geq 0 \quad : \; z \in \Omega$$

Reversing the role the reader finds the opposite inequality and Theorem 6.1 follows.

6.2 Remark. Theorem 6.1 implies also that Green's function enjoys the symmetry property, i.e.

(*) $$G(z_*, ; z) = G(z; z_*)$$

6.3 Outer normal derivatives of G. It turns out that G recaptures harmonic measures. Keeping z_* fixed we know that $z \to G(z_*; z)$ is positive in Ω and tends to zero as $z \to \partial\Omega$. If $\Omega \in \mathcal{D}(C^1)$ we can apply Green's formula starting with an arbitrary harmonic function u in Ω which extends to a C^1-function on the boundary. More precisely, for a small $\epsilon > 0$ we remove the disc $D_\epsilon = |z - z_*| \leq \epsilon$ and with $\Omega_\epsilon = \Omega \setminus D_\epsilon$ we apply Green's formula to the pair u and $G(z_*, \zeta)$. This gives the equality

$$\int_{\partial\Omega_\epsilon} u(\zeta) \cdot \partial G/\partial \mathbf{n}(\zeta) \cdot ds(\zeta) = \int_{\partial D_\epsilon} G(z_*, \zeta) \cdot \partial u/\partial \mathbf{n}(\zeta) \cdot ds(\zeta)$$

where we used that $G(z_*, \zeta) = 0$ on $\partial\Omega$.

6.4 Exercise. Notice that we have taken outer normals with respect to the domain Ω_ϵ. Show that a passage to the limit a $\epsilon \to 0$ gives the equality

$$(*) \qquad u(z_*) = \frac{1}{2\pi} \cdot \int_{\partial\Omega} u(\zeta) \cdot \partial G/\partial \mathbf{n}_* \cdot ds(\zeta)$$

where we have taken the *inner normal* of $G(z_*, \zeta)$ along $\partial\Omega$. This choice is natural since $G > 0$ in Ω so that the inner normal is non-negative which means that we obtain a representation by a non-negative measure. So above we have found the equality

$$d\mathfrak{m}_{z_*} = \partial G/\partial \mathbf{n}_* \cdot ds$$

where ds is the arc-length measure on the boundary.

6.5 Example. Let $\Omega = D$ be the unit disc. In this case we have:

$$G(z, \zeta) = \text{Log}\, \frac{|1 - z \cdot \bar{\zeta}|}{|z - \zeta|}$$

With $z_* \in D$ kept fixed and $\zeta = e^{i\theta}$ is on the boundary circle T we see the inner normal derivative becomes

$$\frac{d}{dr}\big[(\text{Log}\, \frac{|re^{i\theta} - z_*|}{|1 - re^{-i\theta} \cdot z_*|}\big]$$

where this derivative is evaluated when $r = 1$. At the same time the arc-length measure is $d\theta$. The reader may now verify that the formula in the theorem corresponds to Poisson's formula for the harmonic extension of the boundary function $u(e^{i\theta})$.

6.6 Harmonic measures with respect to arcs. Let Ω as above be a domain in the class $\mathcal{D}(C^1)$ and let γ be a simple C^1-curve which appears as a compact subset of Ω. compact subset of Ω. The Dirichlet problem has a solution for the domain $\Omega \setminus \gamma$. Let H be harmonic in $\Omega \setminus \gamma$ with continuous boundary values. Along γ we recall that the outer normal derivative of H is taken from two sides while we apply Green's formula. See XXX for a detailed discussion. If $z_0 \in \Omega \setminus \gamma$ we apply Green's formula to $H(\zeta)$ and $G(z_0, \zeta)$ while a small disc centered at z_0 is removed. Since $G = 0$ on $\partial\Omega$ we obtain

$$\int_{\partial\Omega} H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds + \int_\gamma H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds + \int_{\partial D_\epsilon} H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds =$$
$$\int_\gamma G(z_0, \zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds + \int_{\partial D_\epsilon} G(z_0, \zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds$$

Passing to the limit as $\epsilon \to 0$ we get the equality

$$(1) \quad \int_{\partial\Omega} H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds + \int_\gamma H \cdot \partial G(z_0, \zeta)/\partial \mathbf{n} \cdot ds - 2\pi \cdot H(z_0) = \int_\gamma G(z_0, \zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds$$

Now $\zeta \mapsto G(z_0,\zeta)$ is harmonic in a neighborhood of γ and hence quite regular which implies that the outer normal derivatives along the two sides of γ cancel each other so the second integral above vanishes and there remains the equality

$$\int_{\partial\Omega} H \cdot \partial G(z_0,\zeta)/\partial \mathbf{n} \cdot ds - 2\pi \cdot H(z_0) = \int_\gamma G(z_0,\zeta) \cdot \partial H/\partial \mathbf{n} \cdot ds \tag{2}$$

In particular we can apply (*) when $H = 0$ on $\partial\omega$ and identically one on γ. This means that $H(z_0)$ evaluates the harmonic measure at $z_0 =$ with respect to the compact set γ inside the domain Ω. We set $H(z) = w_\gamma(z)$ where one should consider Ω as fixed while γ is a simple curve situated inside Ω. Since $w_\gamma = 0$ on $\partial\Omega$ we arrive at the formula

$$w_\gamma(z_0) = \frac{1}{2\pi} \cdot \int_\gamma G(z_0,\zeta) \cdot \partial w_\gamma/\partial \mathbf{n}_* \cdot ds \tag{3}$$

where we use the inner normal of w_γ. On γ we haver the non-negative measure defined by the density $\frac{1}{2\pi} \cdot \partial w_\gamma/\partial \mathbf{n}_* \cdot ds$. Let us denote it by σ_γ so we can write

$$w_\gamma(z_0) = \int_\gamma G(z_0,\zeta) \cdot d\sigma_\gamma(\zeta) \tag{*}$$

Remark. The point above is that (*) holds for every point $z_0 \in \Omega \setminus \gamma$ while the non-negative measure σ_γ is fixed.

6.7 Beurling's projection theorem.

The formula (*) in (6.6) has several applications. We shall give one which is due to Beurling. Let Ω be the unit disc D and consider an arc γ inside D which does not contain the origin. Keeping γ fixed we set $\sigma = \sigma_\gamma$. Now we have the circular projection

$$z \mapsto |z| \tag{1}$$

which sends γ onto some interval $[a,b]$ on the positive x-axis with $0 < a \leq b < 1$. The exceptional equality $a = b$ holds if γ happens an arc on the circle $|z| = a$. As illustrated by a figure the map (1) is in general not bijective. However, there always exists the push-forward measure denoted by σ^* which is supported by $[a,b]$ and

$$\int_a^b \phi(s) \cdot d\sigma^*(s) = \int_\gamma \phi(|\zeta|) \cdot d\sigma(\zeta) \tag{2}$$

holds for every $\phi \in C^0[a,b]$. We consider only the case $a < b$ and assume also the measure σ^* is absolutely continuous.

Exercise. Show that σ^* is absolutely continuous if the arc γ is real-analytic.

Next, consider the function

$$V(z) = \int_a^b G(z,s) \cdot d\sigma^*(s) \tag{3}$$

It is clear that $V(z)$ is a harmonic function in $D \setminus \gamma$ and $V = 0$ on the unit circle T. Since $z \mapsto g(z,\zeta)$ are super-harmonic for every ζ and $\sigma^* \geq 0$, it follows that $V(z)$ is super-harmonic in the whole disc D.

6.7.1 Proposition. *For each $a \leq s_* \leq b$ one has*

$$\liminf_{z \to s_*} V(z) \geq 1$$

where the lim inf *is taken as $z \in D \setminus [a,b]$ approach s_*.*

Proof. We have $s_* = |z_*|$ for some $z_* \in \gamma$ where $z_* = s_* \cdot e^{i\theta}$ for some θ. If $\liminf$ is taken along some sequence $\{z_k$ we put $z_k^* = |z_k| \cdot e^{i\theta}$ which implies that $z_k^* \to z_*$. Since $w_\gamma = 1$ on γ it follows that

$$\text{(i)} \qquad \lim \int G(z_k^*, \zeta) \cdot d\sigma(\zeta) = 1$$

Since $\sigma \geq 0$ the inequality for G in (xx) entails that the integrals above are majorized by

$$\text{(ii)} \qquad \int G(|z_k^*|, |\zeta|) \cdot d\sigma(\zeta) = \int G(|z_k|, s) \cdot d\sigma^*(s)$$

where the last equality used (2) above and the equalities $|z_k^*| = z_k|$ At the same time we have

$$\text{(iii)} \qquad V(z_k) = \int_a^b G(z_k, s) \cdot d\sigma^*(s)$$

Now we have the limit formulas

$$\lim z_k = \lim |z_k| = s_*$$

We have assumed that $d\sigma^*(s)$ is an absolutely continuous measure Moreover, it is clear the functions $s \mapsto G(z, s)$ are L^1-functions on $[a, b]$ for every $z \in D$ which in addition satisfy

$$\lim \int_a^b |G(z, s) - G(z', s)| \cdot d\sigma^*(s) = 0$$

when $|z| - z'| \to 0$. Since $|z_k - |z_k| \to 0$ we have

$$\lim V(z_k) = \lim V(|z_k|)$$

Finally, the majorization from (ii) and the limit formula (i) give Proposition 6.7.1.

6.7.2 An estimate for harmonic measures. The inequality in Proposition 6.7.1 implies that the V-function majorizes the harmonic measure function with respect to the interval $[a, b]$. Hence we get the inequality

$$\text{(i)} \qquad w_{[a,b]}(-[z|) \leq V(-|z|) = \int_a^b G(-|z|, s) \cdot d\sigma^*(s)$$

for each $z \in D$. By (2) the right hand side is equal to

$$\text{(ii)} \qquad \int_\gamma G(-|z|, |\zeta|) \cdot d\sigma(\zeta)$$

Now the inequality from (xxx) shows that (ii) is majorized by

$$\text{(iii)} \qquad \int_\gamma G(z, \zeta) \cdot d\sigma(\zeta) = w_\gamma(z)$$

Hence we arrive at the following

6.7.3 Theorem. *For every $z \in D$ one has the inequality*

$$w_{[a,b]}(-|z|) \leq w_\gamma(z)$$

where $[a, b]$ is the circular projection of γ.

Remark. The result above gives the starting point fora far-reaching study concerned with estimates of harmonic measures. We refer to Section 9 in chapter III in [Garnett-Marshall] for such results of this nature and mention that this section also contains several instructive exercises and further references to the literature.

7. The Neumann problem.

Let Ω be a domain in the class $\mathcal{D}(C^1)$ bordered by p many closed Jordan curves $\Gamma_1, \ldots, \Gamma_p$ where $p \geq 2$. Given a continuous function f on $\partial\Omega$ we seek a harmonic function v in Ω whose outer normal derivative satisfies

$$(*) \qquad \partial v/\partial \mathbf{n}(p) = f(p) \quad : \quad p \in \partial\Omega$$

Since

$$\int_{\partial\Omega} \partial v/\partial \mathbf{n} \cdot ds = 0$$

hold for every harmonic function a necessary condition in order that (*) can be solved is that

$$(**) \qquad \int_{\partial\Omega} f \cdot ds = 0$$

We are going to prove that (*) has a unique solution when f satisfies (**). Before we enter the proof we establish a mean-value inequality which is due to Schwarz. Namely, let v be harmonic in Ω and suppose it extends to a C^1-function on the boundary. Let γ be an arc of one boundary curve, say Γ_1, and suppose that

$$\text{(i)} \qquad \partial v/\partial \mathbf{n}(p) = 0 \text{ for all } p \in \gamma$$

If a and b are the two end-points of γ we can construct a closed Jordan curve J where γ appears as an arc and $J \setminus \gamma$ is a simple Jordan curve contained in Ω. As explained by a figure we may construct J so that a and b are two corner points and otherwise J is of class C^1. With this given one has the following result:

7.1 Schwarz integral formula. *Assume that (i) above holds. Then, for every point $p \in \gamma$ there exists a probability measure μ_p supported by $J \setminus \gamma$ such that*

$$v(p) = \int_{J \setminus \gamma} v(z) \cdot d\mu_p(z)$$

Proof. Riemann's mapping theorem gives a conformal map ϕ from the Jordan domain bordered by J onto the upper half-disc D^+ where $|w| < 1$ and $\mathfrak{Im}(w) > 0\}$, such that $\phi(\gamma)$ is the real interval $(-1, 1)$ while $\phi(J \setminus \gamma)$ is the half circle. Then $V = v \circ \phi^{-1}$ is harmonic in D^+ and with the complex coordinate $\zeta = \xi + i\eta$ in D^+ we see that (i) means that

$$\text{(ii)} \qquad \frac{\partial V}{\partial \eta}(\xi, 0) = 0 \quad : \quad -1 < \xi < 1\,.$$

Let U be the harmonic conjugate of V and consider the analytic function

$$F(\zeta) = U + iV$$

The Cauchy-Riemann equations and (ii) give $\frac{\partial U}{\partial \xi}(\xi, 0) = 0$. Hence U is a constant on this real interval and since we can choose this conjugate harmonic function up to a constant we may assume that $U(\xi, 0) = 0$. Hence Schwarz' reflection principle can be applied, i.e. we find an analytic function $\hat{F}(\zeta)$ defined in the unit disc $|\zeta| < 1$ with

$$\hat{F}(\zeta) = \bar{F}(\bar{\zeta}) \quad : \quad \mathfrak{Im}(\zeta) < 0$$

If $-1 < \xi < 1$ we represent $\hat{F}(\xi)$ by the Poisson integral over $|\zeta| = 1$ and since ξ is real we get

$$\hat{F}(\xi) = 2 \cdot \int_0^{\pi} P(\xi, e^{i\theta}) \cdot F(e^{i\theta}) \cdot d\theta$$

Taking the real part we can write

$$V(\xi) = 2 \cdot \int_0^{\pi} P(\xi, e^{i\theta}) \cdot V(e^{i\theta}) \cdot d\theta$$

Here $2\cdot\int_0^\pi P(\xi, e^{i\theta})d\theta = 1$. So $V(\xi)$ is a convex sum of values of V taken on the half-circle $\phi(J\setminus\gamma)$. Returning to the v-function this means that there exists a probability measure μ_ξ supported by $J\setminus\gamma$ such that

$$v(\phi^{-1}(\xi)) = \int_{J\setminus\gamma} v(z)\cdot d\mu_\xi(z)$$

Since $\phi^{-1}(\xi))$ corresponds to any preassigned point on γ we have proved Schwarz' integral formula.

7.2 A uniqueness result. Let v be harmonic in Ω and suppose that

$$\frac{\partial v}{\partial \mathbf{n}}(p) = 0 \text{ holds for every } p \in \partial\Omega$$

Then Schwarz' integral formula and the maximum principle for harmonic function implies that v is a constant. In fact, let M be the maximum of v over $\partial\Omega$ and choose $p\in\partial\Omega$ where $v(p)=M$. If v is not a constant we have seen in XX that $v(q)<M$ for every $q\in\Omega$. This strict inequality means that we cannot obtain a measure μ_p in the integral formula above. On the other hand we can find such a non-trivial formula when $\frac{\partial v}{\partial \mathbf{n}}$ is identically zero on $\partial\Omega$. Hence v must be reduced to a constant. This proves the uniqueness part for a solution to the equation (*). Their remains to prove the existence. Before this is done we establish another result of independent interest.

7.3 A non-singular matrix for harmonic measure functions. To each boundary Γ_i we have the harmonic function H_i in Ω with boundary value one on Γ_i while $H_i = 0$ on the remaining boundary curves. Let Γ_p be the outer curve and consider the functions $H_1,\ldots,H_{p-1}$. We get the $(p-1)\times(p-1)$-matrix A with elements

(i) $$a_{j,\nu} = \int_{\Gamma_\nu} \frac{\partial H_j}{\partial \mathbf{n}}\cdot ds \quad : 1\le j,\nu\le p-1$$

Proposition. *7.4 The matrix A is non-singular.*

Proof. Assume the contrary. This gives some $p-1$-tuple $c_1,\ldots,c_{p-1}$ which are not all zero such that

(ii) $$c_1\cdot a_{1,\nu}+\ldots+c_{p-1}\cdot a_{p-1,\nu} = 0 \quad : \quad 1\le\nu\le p-1$$

Set $V = c_1H_1+\ldots+c_{p-1}H_{p-1}$. Then (i) and (ii) give

(iii) $$\int_{\Gamma_\nu}\frac{\partial V}{\partial \mathbf{n}}\cdot ds = 0 \quad : \quad 1\le\nu\le p-1$$

From (xx) it follows that V has a harmonic conjugate U and we get the analytic function $F = U+iV$. Next, the *tangential* derivative of a harmonic measure function H_j is identically zero on every boundary curve. By the Cauchy- Riemann equations it follows that the normal derivative of U is identically zero on every boundary curve and then the uniqueness result in 7,2 above implies that U is a constant. But then V would also be reduced to a constant which obviously cannot occur when some $c_k\neq 0$. Hence the A-matrix is non-singular.

7.5 A consequence. Now we consider the whole p-tuple $H_1,\ldots,H_p$ and the $p\times p$-matrix B with elements

(i) $$b_{j\nu} = \int_{\Gamma_\nu}\frac{\partial H_j}{\partial \mathbf{n}}\cdot ds \quad : 1\le j,\nu\le p$$

By xx we have $b_{1\nu}+\ldots+b_{n\nu}=0$ for every ν and therefore the B-matrix must be singular, From Proposition xx its rank is $p-1$. More precisely, the column vectors when $1\le\nu\le p-1$ are linearly independent. Hence we can find complex numbers $\rho_1,\ldots,\rho_{p-1}$ such that

(ii) $$\int_{\Gamma_\nu} \frac{\partial H_p}{\partial \mathbf{n}} \cdot ds = \sum_{j=1}^{p-1} \rho_j \cdot \int_{\Gamma_\nu} \frac{\partial H_j}{\partial \mathbf{n}} \cdot ds \quad : \ 1 \leq \nu \leq p$$

7.6 Solution to the Neuman problem.

Since the equation (*) is linear it suffices by additivity to show that it has a solution when the function f is identically zero on $(p-1)$ many boundary curves and $\neq 0$ on one of them. Consider the case when $f = 0$ on the inner curves. Notice that (**) entails that

(1) $$\int_{\gamma_p} f \cdot ds = 0$$

where ds is the arc-length measure along γ_p. Fix a point z_* on γ_p and we move in the positive direction of this oriented curve to construct a primitive function with respect to s, i.e. we obtain a continuous function F on γ_p such that $F(z_*) = 0$ and its derivative with respect to s is equal to f on γ_p. .Next, we solve the Dirichlet problem and find G where $\Delta(G) = 0$ in Ω while $G = F$ on γ_p and $G = 0$ on the inner boundary curves. By 7.4 we can find constants $a_1, \ldots, a_{p-1}$ such that the function $g = G - (a_1 H_1 + \ldots + a_{p-1} H_{p-1})$ has a harmonic conjugate which we denote by v. Since $H_1, \ldots, H_{p-1}$ are constant functions on the boundary curves, the tangential g-derivative is equal to that of G along each boundary curve. From the construction of G this means that the tangential g-derivatives are zero on all inner curves and equal to f on the outer curve. Finally, if v is the conjugate harmonic function to g it follows that v is a solution to (*).

8. An example from Hydro mechanics.

Introduction. Consider a stationary motion of an ideal fluid in the plane which means that the velocity vector (u, v) at each point (x, y) is independent of the time. We also assume that the fluid is incompressible which means that whenever $U\Omega$ is some domain, then

$$\int_{\partial\Omega} (u \cdot \mathbf{n}_x + v \cdot \mathbf{n}_y) \cdot ds = 0$$

Stokes theorem implies that the area integral

$$\iint_{\Omega} (u_x + v_y) \cdot dxdy = 0$$

Since this holds for every small domain it follows that the velocity vector satisfies the equation

(*) $$u_x + v_y = 0$$

This is the continuity equation for an incompressible fluid. Next, one defines the *rotation* by

(i) $$\rho = \frac{1}{2}(v_x - u_y)$$

Assume that (*) holds in the whole plane which gives a function ψ such that

(ii) $$\frac{\partial\psi}{\partial x} = -v \quad \text{and} \quad \frac{\partial\psi}{\partial y} = u\,.$$

From (i-ii) we get

(iii) $$\Delta(\psi) + 2\rho = 0$$

An example. Consider the case when

$$\lim_{x^2+y^2\to+\infty} (u, v) = (0, 0)$$

and there exists a bounded Jordan domain Ω such that

$$\rho(x, y) = A = \text{a constant when} \quad (x, y) \in \Omega \quad \text{and } \rho = 0 \text{ in } \mathbf{C} \setminus \Omega$$

Then the ψ-function satisfies:

(1) $$\Delta(\psi) = 0 \text{ outside } \Omega \quad \text{and} \quad \Delta(\psi)(x, y) = -2A \,:\, (x, y) \in \Omega$$

(2) $$\lim_{x^2+y^2\to\infty} \big[\big(\frac{\partial\psi}{\partial x}\big)^2 + \big(\frac{\partial\psi}{\partial y}\big)^2\big] = 0$$

The Riesz representation formula for subharmonic functions gives:

8.1 Proposition. *The ψ-function is given by*

$$\psi(x,y) = \frac{A}{\pi} \iint_\Omega \log \frac{1}{|z-\zeta|} \cdot dxdy + C$$

where C is some constant.

Above we found a formula for the ψ-function. It turns out the domain Ω cannot be arbitratry. Namely, one has:

8.2 Theorem. *Under the assumptions above Ω is a disc.*

Proof. Since the motion is stationary it follows from the general physical principle by Helmholtz that the boundary curve $\partial\Omega$ must be a streamline. By (ii) above this means that the ψ-function is a constant C_0 on $\partial\Omega$, i.e. we have

$$(**) \qquad \frac{A}{\pi} \iint_\Omega \log \frac{1}{|z-\zeta|} \cdot d\xi d\eta = C_0 \quad \text{for all } z \in \partial\Omega$$

But then the general result below shows that Ω is a disc.

8.3 An isoperimetric result. Let $f(r)$ be a positive function defined on $r > 0$ which is strictly decreasing. Next, for an arbitrary Jordan domain Ω with a C^1-boundary we define the function

$$(*) \qquad F_\Omega(z) = \iint_\Omega f(|z-\zeta|) \cdot d\xi d\eta$$

With these notations Theorem 8.2 is a special case of the result below applied to the special function

$$f(r) = \text{Log}\,\frac{1}{r}$$

8.5 Theorem. *If Ω is a Jordan domain such that $F_\Omega(z)$ is a constant function on $\partial\Omega$ then Ω is a disc.*

Remark. We refer to XXX for the proof of this isoperimetric result. Notice that we can express Theorem 8.2 in physical terms as follows:

8.6 Theorem. *Every stationary motion of an incompressible fluid with constant rotational density and a simply connected (Querschnitt) is circular.*

9. A differential inequality

In this section we expose a result from Carleman's article *Sur les fonctions inverses des fonctions entières d'ordre fini* which was published in 1920 and gave the first example how the principle of harmonic majorisation can be applied in a fruitful manner. Here is the situation: A connected domain Ω is given. It is contained in $\mathfrak{Re}(z) > 0$ and for every $\xi > 0$ we assume that the set

$$\ell(\xi) = \{y \quad : \ (\xi, y) \in \Omega\}$$

is bounded. Hence $\ell(\xi)$ is some union of open intervals. We assume that that $\ell(\xi) \neq \emptyset$ for all ξ and the sum of the lengths of the intervals yields a positive function $h(\xi)$. Next, let $f(z)$ be an analytic function in Ω where

$$\max_{z \in \partial\Omega} |f(z)| \leq 1 \tag{1}$$

For every $\xi > 0$ we set

$$M(\xi) = \max_{z \in \ell(\xi)} |f(z)| \tag{2}$$

We shall assume that $M(\xi)$ tends to $+\infty$ as ξ increases. In particular there exists some ξ_* such that $M(\xi_*) > 1$. The maximum principle applied to f shows that the function

$$\xi \mapsto M(\xi)$$

is non-decreasing when $\xi \geq \xi_*$.

9.1 Theorem. *One has the inequality*

$$\log M(\xi) \geq \log M(\xi_*) \cdot \exp\left[\frac{4}{\pi} \int_{\xi_*}^{\xi} \frac{dt}{h(t)}\right] \quad \text{for all} \quad \xi > \xi_*$$

The proof requires several steps. Given $\xi > \xi_*$ we choose a point $p \in \ell(\xi)$ where $M(\xi) = |f(p)|$. Next, consider the subharmonic function

$$U(x + iy) = \log |f(x + iy)|$$

Let us also consider a consider some $\xi^* > \xi$ and let D be the connected subdomain of the set

$$\Omega \cap \{0 < \mathfrak{Re}\, z < \xi^*\}$$

which contains the point p. In D we have the Green's function G with a pole at p while $G(z)$ is harmonic in $D \setminus \{p\}$ and zero on ∂D. Next, in ∂D we have the portion

$$L(\xi_*) = \partial D \cap \ell(\xi_*)$$

With these notations we apply the general formula in XX to G and U. As pointed out in XX we get the following inequality

$$2\pi \cdot U(p) \leq \int_{L(\xi_*)} U \cdot \frac{\partial G}{\partial \mathbf{n}} \cdot ds \leq \log M(\xi_*) \cdot \int_{L(\xi_*)} \frac{\partial G}{\partial \mathbf{n}} \cdot ds \tag{*}$$

where we have used that $U \leq 0$ on $\partial D \setminus L(\xi)$. At this stage we use the principle of harmonic majorisation from XX. Namely, consider the half-plane $\mathfrak{Re}\, z < \xi_*$ and let G^* be the Green's function for this half-plane with a pole at p. Then the majorisation principle from XX gives:

$$\frac{\partial G}{\partial \mathbf{n}} \leq \frac{\partial G^*}{\partial \mathbf{n}} \quad \text{on} \quad L(\xi_*)$$

Next, with $p = (\xi, y_0)$ the explicit formula for G^* gives

$$\int_{L(\xi_*)} \frac{\partial G^*}{\partial \mathbf{n}} \cdot ds = \int_{L(\xi_*)} \frac{\xi_* - \xi}{(\xi_* - \xi)^2 + (y - y_0)^2} \cdot dy \tag{1}$$

Now (1) is $\leq$ than the integral taken over $\ell(\xi_*)$ and and it follows easily that (1) is majorised by

$$\frac{2}{\pi} \cdot \operatorname{arctg} \frac{h(\xi_*)}{2(\xi_* - \xi)} \tag{2}$$

Hence we have proved the inequality

$$\operatorname{Log} M(\xi) \leq \frac{2}{\pi} \cdot \operatorname{arctg} \frac{h(\xi_*)}{2(\xi_* - \xi)} \cdot \operatorname{Log} M(\xi) \quad \text{for all} \quad \xi > \xi_* \tag{3}$$

At this stage the proof is almost finished. The idea is to use (3) when $\xi = \xi_* + \delta$ and $\delta \to 0$. For when $\delta > 0$ is small we have

$$\frac{2}{\pi} \cdot \operatorname{arctg} \frac{h(\xi_*)}{2\delta} = 1 - \frac{4\delta}{\pi \cdot h(\xi)} + O(\delta^2)$$

Passing to the limit as $\delta \to 0$ we get the differential inequality

$$\frac{d}{d\xi} \big[\operatorname{Log} M(\xi)\big] \geq \frac{4}{\pi \cdot h(\xi)}$$

After an integration we get the required inequality in Theorem 9.1.

10. On harmonic measure functions.

We shall expose material from [§ 5: Chapter 1] in Nevanlinna's text-book [Nev]. Let Ω be a domain in $\mathcal{D}(C^1)$ with p boundary curves $\Gamma_1, \ldots, \Gamma_p$ where Γ_p is taken as the outer boundary curve. On each Γ_ν we consider a function ϕ_ν which takes the value $+1$ on a finite union of intervals and zero on the complementary intervals. So unless ϕ happens to be constant on a curve Γ-curves we get jumps at common boundary points of intervals where ϕ_ν is constant. The number of such points is obviously an even integer denoted by $2 \cdot k_\nu$ where $k_\nu = 0$ means that ϕ_ν is identically 1 or 0 on Γ_ν. Solving Dirichlet's problem we get a harmonic function $\omega(z)$ whose boundary function is ϕ. At jump points the boundary value is discontinuous. But in any case ω is defined in Ω where it takes values in the open interval $(0, 1)$ when we assume that the range of the ϕ-function contains both 1 and 0. In fact, the strict inequality

$$0 < \omega(z) < 1 \quad : z \in \Omega$$

follows from the maximum principle for harmonic functions. If $0 < \lambda < 1$ the level curve $\{\omega = \lambda\}$ is denoted by $S(\lambda)$. Since $\partial\Omega$ is of class C^1 it follows that $S(\lambda)$ contains all jump points, i.e. if $z_* \in \Gamma_\nu$ is a jump point for some ν then $S(\lambda)$ contains an arc which has z_* as end-point. This is proved using a locally defined conformal mapping and the special analysis which takes place when Ω is an open half-disc bordered by a real segment $-r \leq x \leq r$ and the upper half-circle $|z| = r$ where $y \geq 0$. When $\phi(x) = 0$ for $-r < x < 0$ and $\phi(x) = 1$ when $0 < x < r$ we describe the level curves $S(\lambda)$ in XX below which proves the assertion above.

The question arises about the behavior of the level curves $S(\lambda)$ in the interior of Ω. We are going to analyze the critical points for the ω-function, i.e. the set

$$\mathcal{C}_\omega = \{z = x + iy \quad : \nabla(\omega)(x, y) = 0\}$$

where $\nabla(\omega) = (\omega_x, \omega_y)$ is the gradient vector Consider the analytic function

$$g(z) = \omega_x - i \cdot \omega_y$$

Then $\mathcal{C}_\Omega$ is equal to the zeros of g in Ω. We shall prove the following:

10.1 Theorem. *The number of zeros of g counted with multiplicities is equal to*

$$k_1 + \ldots + k_p + p - 2$$

Examples. Before Theorem 10.1 is proved we consider the case $p = 1$ and Ω is the unit disc D. Here $k_1 = 1$ holds when $\phi = 1$ on some interval (θ_0, θ_1) and 0 on the complementary interval. So Theorem 10.1 implies that g has no zeros and hence each curve $S(\lambda)$ is smooth. In fact, as shown by a figure $S(\lambda)$ will be a simple curve with end-points at ${}^{i\theta_0}$ and $e^{i\theta_1}$. Next, consider the case $k_1 = 2$ which means that ϕ is 1 on two disjoint intervals. Suppose they are $\{-a < \theta < a\}$ and $\pi - b < \theta < \pi + b\}$ for some pair $0 < a, b < \pi/2$. In this case there exists a unique $0 < \lambda_* < 1$ where the level curve $S(\lambda_*)$ fails to be smooth. For symmetric reason the branch point of $S(\lambda_*)$ belongs to the real axis, i.e. at a point $-1 < x_* < 1$ where two pieces of $S(\lambda_*)$ intersect at a right angle by the general result about level curves for harmonic functions in XX.

Exercise. Use a computer to find numerical values for the point x_* as a function of a and b. At the same time one can make plots of level curves $S(\lambda)$ while λ varies. The special case $a = b$ can for example be illustrated.

More involved cases. Keeping $\Omega = D$ while k_1 increases give more complicated descriptions. Already the case $k_1 = 3$ is not obvious. Here one gets two critical points which depend on the positions of the three intervals where the ϕ-function is 1. A computer is needed for plots while Nevanlinna's theorem at least indicates what will occur during such a numerical investigation.

The case $p \geq 2$. Let Ω be an annulus $\{1 < |z| < R\}$ for some $R > 1$. Let $\phi = 0$ on $\{|z| = 1\}$, and on the outer circle we take some interval $-a < \theta < a$ where $\phi(Re^{i\theta}) = +1$ while $\phi = 0$ on the remaining part of $\{|z| = R\}$. Here $k_1 = 1$ and $p = 2$ so g has one simple zero which corresponds to a unique λ_* where $S(\lambda_*$ fails to be smooth. The reader is again invited to use a computer and to determine the zero of g and plot the critical level curve $S(\lambda_*)$.

Proof of Theorem 10.1

We use the argument principle from Theorem XX in Chapter XX. So we shall analyze the variation of the argument of g along the boundary curves. To begin with we study the angular variation outside the jump points. Let us first consider the special case when ϕ is constant on the outer curve γ_p where the constant can be 0 or 1. The harmonic function ϕ is > 0 in Ω since we assume that ϕ takes the value 1 on some interval on an inner curve. The maximum principle for harmonic functions entails that the outer normal derivative $\partial\Phi\partial n$ is > 0 if $\phi = 0$ on Γ_p and < 0 if $\phi = 1$ on Γ_p. it turns out that the sign of this outer normal derivative c does not affect the angular variation of the g-function along γ_p. More precisely we have

Lemma. One has the equality

$$\frac{1}{2\pi i} \cdot \int_{\Gamma_p} \frac{g'(z)}{g(z)} \cdot dz = -1$$

Proof. Let ds be the arc-length measure on Γ_p which means that the closed curve has a parametrisation $s \to z(s)$ where $0 \leq s \leq L$ and L is the total length of Γ_p. It is chosen in the positive sense which means that we have the winding number:

$$\text{(i)} \qquad \int_0^L \arg(\frac{dz}{ds}) \cdot ds = 2\pi$$

Next, locally along Γ_p we find the harmonic conjugate v of ϕ and set

$$G(z(s)) = \phi(z(s)) + i \cdot v(z(s)$$

Now $s \to \phi(z(s))$ is constant and the complex derivative $G' = g$ which after derivation with respect to s gives:

$$g(z(s)) \cdot z'(s) = i \cdot v * 8s)$$

Here $v'(s)$ is real-valued while $\arg(i) = \pi/2$ and we get the equality

$$\arg(g(z(s)) + \arg(\frac{dz}{ds}) = \pi/2$$

it follows from (i) that we have

$$\arg(g(z(L)) - \arg(g(z(0)) = -2\pi$$

which via the formula in XX from Chapter 4 proves Lemma 1.

Exercise. Show by a similar technique that the integral in Lemma 1 taken over an inner curve Γ_ν is equal to 2π when ϕ is constant on Γ_ν.

Next, suppose we have jumps on Γ_p a some points $p_1, p_2, \ldots, p_{2k-1}, p_{2k}$. for some $k \geq 1$. The points are ordered so that we move in the positive direction along the intervals $(p_\nu, p_{\nu+1})$ where $p_{2k+1} = p_1$. See figure XXX. Let $\epsilon > 0$ be a small number and remove the discs $D_\nu(\epsilon) = |z - p_\nu| < \epsilon$ which for a small ϵ are pairwise disjoint. on Γ_p we get the open intervals $J_\nu = \Gamma_\nu \cap D_\nu(\epsilon)$ where each J-interval contain the jump point p_ν. Set

$$\Gamma_\epsilon = \Gamma_p \setminus \cup J_\nu$$

Next, inside Ω we have the half-circles $D^*_\nu(\epsilon) = D_\nu(\epsilon) \cap \Omega$ and obtain the closed curve

$$\Gamma_p(\epsilon) = \Gamma_\epsilon \cup \partial D^*_\nu(\epsilon)$$

Exercise. Use that ϕ is constant on Γ_ϵ and the same argument as in the proof of Lemma 1 to show the limit formula

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{\Gamma_\epsilon} \frac{g'(z)}{g(z)} \cdot dz = -1$$

Local variation along the jumps. There remains to analyze the integrals

$$\frac{1}{2\pi i} \cdot \int_{\partial D_(\epsilon)} \frac{g'(z)}{g(z)} \cdot dz$$

when $1 \leq \nu \leq 2k$. Here one has

Lemma *For each ν one has the limit formula*

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{\partial D_(\epsilon)} \frac{g'(z)}{g(z)} \cdot dz = \pi$$

Exercise. Prove Lemma XX using the following hints. Using a locally defined conformal maps at each jump point the crucial situation arises when we consider the open half-disc D^+ where $\phi(z)$ is a harmonic function such that $\phi(x) = 1$ when $x < 0$ and 0 if $0 < x < 1$, or vice versa, while $\phi = 0$ on the half-circle $\{x^2 + y^2 = 1\}$ with $y > 0$. Now we consider the function

$$\omega(z) = \mathfrak{Re}(\frac{\log z}{i\pi}) \implies g(z) = \frac{1}{i\pi z}$$

Let z move from $-\epsilon$ to ϵ on the upper half-circle Then we see that the argument of g *increases* from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. i.e. the variation is $+\pi$. If we instead suppose that the ϕ-function is 0 when $-1 < x < 0$ and+1 when $x > 0$ we study the argument of

$$g_*(z) = -\frac{1}{i\pi z}$$

This time the variation starts from $-i$ and reaches $+i$ so the argument is again *increasing*, i.e the contribution from the jump point is always $+\pi$ which explains why we have the formula in Lemma 2.

11. Some estimates for harmonic functions.

Let u be harmonic in the unit disc D. We impose the condition that $\sqrt{|u|}$ is integrable on D and after a normalisation we suppose that

$$(*) \qquad \iint_D \sqrt{|u(x,y)|}\cdot dxdy = 1$$

We seek estimates of the maximum modulus function

$$M(r) = \max_\theta |u(r,\theta)|$$

where $re^{i\theta} = x + iy$ in D give the polar coordinates. Put

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{|u(r,\theta)|}\cdot d\theta$$

If $0 < r < 1$ we represent values of u on $|z| = r^2$ by a Poisson integral over $|z| = r$ and find that

$$M(r^2) \leq \frac{1+r}{1-r}\cdot\frac{1}{2\pi}\int\int_0^{2\pi} |u(r,\theta)|\cdot d\theta$$

The right hand side is estimate above by $\frac{2}{1-r}\cdot\sqrt{M(r)}\cdot I(r)$ which gives the inequality

$$M(r^2) \leq \frac{2}{1-r}\cdot\sqrt{M(r)}\cdot I(r) \implies$$

$$\text{(i)} \qquad \log\, M(r^2) \leq \log\,\frac{2}{1-r} + \frac{1}{2}\cdot\log M(r) + \log I(r)$$

Put

$$\mathcal{M}(A) = \int_A^1 \log\, M(s)\cdot\frac{ds}{s}$$

The substitution $s = r^2$ gives

$$\mathcal{M}(A) = 2\cdot\int_{\sqrt{A}}^1 \log\, M(r^2)\cdot\frac{dr}{r}$$

Using (i) we obtain

$$\text{(ii)} \qquad \mathcal{M}(A) \leq 2\cdot\int_{\sqrt{A}}^1 \log\,\frac{2}{1-r}\cdot\frac{dr}{r} + \int_{\sqrt{A})}^1 \log M(r)\cdot dr + 2\cdot\int_{\sqrt{A}}^1 \log I(r)\cdot\frac{dr}{r}$$

Put

$$C(A) = 2\cdot\int_{\sqrt{A}}^1 \log\,\frac{2}{1-r}\cdot\frac{dr}{r}$$

Then (ii) gives

$$\int_A^{\sqrt{A}} \log\, M(s)\cdot\frac{ds}{s} \leq C(A) + 2\cdot\int_{\sqrt{A}}^1 \log I(r)\cdot\frac{dr}{r}$$

Since the maximum modules function $M(r)$ is increasing the left hand side majorizes

$$\log M(A)\cdot\int_A^{\sqrt{A}}\frac{ds}{s} = \log M(A)\cdot\frac{1}{2}\cdot\log\,\frac{1}{A}$$

So for each $1/2 \leq A < 1$ we have

$$\text{(iii)} \qquad \log M(A)\cdot\frac{1}{2}\cdot\log\,\frac{1}{A} \leq C(A) + 2\cdot\int_{\sqrt{A}}^1 \log I(r)\cdot\frac{dr}{r}$$

The case $A \geq 1/2$. When this holds we have $r \geq \sqrt{A} \geq \sqrt{1/2}$ during the integration and hence the last integral in (iii) is estimated by

$$2\sqrt{2} \cdot \int_{\sqrt{A}}^1 \log I(r) \cdot dr$$

Let us also write $\sqrt{A} = 1 - s$ where we will study the situation as $s \to 0$.

Exercise. Show that there exists an absolute constant such that when $A \geq 1/2$ then

$$C(A) \leq C \cdot s$$

Use also that

$$s \mapsto \log \frac{1}{(1-s)^2}$$

starts a Taylor expansion with $2s$ and conclude that there is an absolute constant C such that

$$(*) \qquad \log M((1-s)^2) \leq C\big[1 + s^{-1} \cdot \int_{1-s}^1 \log I(r) \cdot dr\big]$$

Next, since the Log-function is concave we get the inequality

$$\frac{1}{s} \cdot \int_{1-s}^1 \log I(r) \cdot dr \leq \log \big[\frac{1}{s} \int_{1-s}^1 I(r) \cdot dr\big]$$

and the normalisation (*) gives

$$1 = 2\pi \cdot \iint_0^1 I(r) \cdot rdr$$

It follows that the integrals

$$\Phi(s) = \int_{1-s}^1 I(r) \cdot dr$$

converge to zero as $s \to 0$ and with $r = 1 - s$ we obtain

$$(**) \qquad \log M(r^2) \leq C[1 + \frac{\Phi(1-r)}{1-r}]$$

This estimate is rather weak and we are led to the following:

11.1 Question. What kind of upper bounds can we expect for $M(r)$ as $r \to 1$.

11.2 Interior estimates. Ignoring the eventual growth of $M(r)$ as $r \to 1$ we fix $r = 1/2$ and get the inequality

$$(1) \qquad M(1/4) \leq C[1 + 2\Phi(1/2)]$$

for an absolute constant C. Moreover, by the normalisation

$$1 = 2\pi \cdot \int_0^1 I(r) \cdot r \cdot dr$$

This gives

$$\Phi(1/2) \leq 2 \cdot \int_{1/2}^1 I(r) \cdot rdr \leq \frac{1}{\pi}$$

Hence (1) gives another absolute constant C^* such that

$$M(1/2) \leq C^*$$

In particular we can estimate $u(0)$ and the whole analysis above leads to

11.3 Theorem. *In the family of harmonic functions u where (*) holds there is an absolute constant c such that*

$$|u(0)| \leq C$$

12. A density result for harmonic functions.

Inside the unit disc D we consider the interval $[-a, a]$ on the real x-line for some $0 < a < 1$ which in general can be small. If $H(x, y)$ is harmonic in D we know that it is a real-analytic function and when H^* is the harmonic conjugate we get the analytic function $f = h + iH^*$ whose complex derivative becomes $f' = H_x - iH_y$. If both H and the partial derivative H_y are identically zero on $[|vvva, a]$ it follows that $f' = 0$ on this interval. So by analyticity f is a constant and since $H = 0$ on the interval, it follows that $H = 0$ in D. In other words, the pair of restricted functions H and H_y to $[-a, a]$ determine H in D. We can use this to get various density theorems. To each $-a \leq x \leq a$ we have the Poisson kernel

$$P_x(\theta) = \frac{1}{2\pi} \cdot \frac{1 - x^2}{1 + x^2 - 2x \cos \theta}$$

We also introduce a partial y-derivative and define

$$\partial_y(P_x)(\theta) = \frac{1}{2\pi} \cdot \frac{(1 - x^2) \cdot \cos \theta}{(1 + x^2 - 2x \cos \theta)^2}$$

It means that whenever $h(\theta)$ is a continuous function on the unit circle and $H(z)$ its harmonic extension to D, then

$$\partial_y(H)(x, 0) = \int_0^{2\pi} \partial_y(P_x)(\theta) \cdot h(\theta) \cdot d\theta$$

12.1 Proposition. *For each $0 < a < 1$ the linear space of functions on T generated by $\{P_x\}$ and $\partial_y(P_x)\}$ as $-a \leq x \leq a$ is dense in $L^2(T)$.*

Proof. If $h \in L^2$ is $\perp$ to this space it follows from the above that its harmonic extension is identically zero and since $h(\theta)$ is almost everywhere equal to the radial limit of its harmonic extension it follows that $h = 0$. This proves the requested density.

Let us now consider a point $z \in D$ which is outside the interval. Now $\theta \mapsto P_z(\theta)$ is an L^2-function. So to each $\epsilon > 0$ we can find an finite subsets $\{\alpha\nu\}$ and $\{\beta_\nu\}$ in $[-a, a]$ such that an **R**-linear combination

$$\sum c_j \cdot P_{x\alpha_j}(\theta) + + \sum d_k \cdot \partial_y(P_{\beta_k}(\theta) \tag{1}$$

whose distance in the L^2-norm to $P_z \leq \epsilon$. Applying Cauchy-Schwarz inequality it follows that for every harmonic function H in the unit disc whose boundary function $h(\theta)$ belongs to $L^2(T)($ one has

$$\big|H(p) - \sum c_j \cdot H(\alpha_j) - \sum d_k \cdot \partial_y(H)(\beta_k)\,\big| \leq \sqrt{\epsilon} \cdot ||h||_2 \tag{*}$$

Remark. The family of $L^2(T)$-functions used for the density in Proposition 12.1 is not orthogonal and it appears to be cumbersome to exhibit an orthonormal family in an efficient manner. So above we have just proved a density theorem while no specific control is given to the constants $\{c_\nu\}$ or $\{d_k\}$. which in addition depend on z while the ϵ-approximation in (*) is attained. But a merit is that (*) holds for all harmonic functions H with $h \in L^2(T)$. This can for example be used to study growth properties of harmonic functions in strip domains. See the section *Lindelöf functions* in Special Topics XXX for such an application.

On inner normal derivatives.

We shall study a local situation. Consider the domain

$$\Omega_A = \{0 < x < 1\} \cap \{x > A|y|\}$$

where $A > 0$. If A is small this domain is almost a half-disc while it shrinks as A increases. See figure XX. Le $u(x, y)$ be harmonic in Ω_A with continuous boundary values where we assume that

$u(0,0)=0$ and $u \geq 0$ in $\bar{\Omega}$ and not identically zero. This entails that $u>0$ in Ω and in particular we get the positive number $u(1/2)$. Now we can estimate $u(x,0)$ from below as $x \to 0$.

Exercise. Show that there exists a positive number A_* which depends on A only such that

$$u(x,0) \geq A_* \cdot u(1/2,0) \cdot x$$

hold for all $0 < x < 1/2$ and every harmonic function u as above. A hint is to employ the conformal map F from Ω_A onto the half-disc D_+ in the right half-plane where

$$F(z) = z^{\frac{1}{A}}$$

Then we can pass to the half-disc and seek B_* such that

$$U(x^{\frac{1}{A}},0) \geq B_* \cdot U(2^{-\frac{1}{A}},0) \cdot x^{\frac{1}{A}}$$

for non-negative harmonic functions in the half-disc which is achieved using a suitable version of Harnack's inequality which is left to the reader to analyze in detail.

An application. Let Ω be a connected domain where the origin is a boundary point which contain Ω_A for some $A>0$. Prove that there again exists a constant A_{**} which depends on A only such that (*) above holds for every harmonic function u in Ω which is >0 in Ω while $u(0,0)=0$. This entails in particular that

$$\liminf_{x \to 0} \frac{u(x)}{x} > 0$$

Remark. A special case occurs when $\partial\Omega$ is of class C^1 and working locally close to the boundary point $(0,0)$ the domain is defined by an equation $\{x > \rho(y)\}$ where $\rho(y)$ is a real-valued C^1-function with $\rho(0)=0$. In that case we get (xx) above and this strict inequality for C^1-domains is sometimes referred to as Hopf's lemma who established similar inequalities in higher dimensions. See also [Krantz-Green: page 440-444] which describes how Hopf's Lemma is used to establish certain smoothness results for conformal mappings.

Chapter 5.B. Subharmonic functions

Introduction.

The theory of subharmonic functions is foremost due to F. Riesz whose article [Ri] from 1926 contains the essential facts about subharmonic functions. We shall not discuss subharmonic functions in $\mathbf{R}^n$ when $n \geq 3$ which for example are treated in the text-book [Hayman]. When $n = 2$ we identify $\mathbf{R}^2$ with $\mathbf{C}$ where real-valued functions $u(x, y)$ become functions of the single complex variable $z = x + iy$. If $u(x, y)$ is a C^2-function we proved in XX that the Laplacian $\Delta(u)$ is a non-negative continuous function if and only if u satisfies the *local mean-value inequality.* In other words, for every point p there exists some $\delta > 0$ such that

$$(*) \qquad u(p) \leq \frac{1}{\pi r^2} \cdot \iint_{D_r(p)} u(x, y) \cdot dxdy \quad \text{for all} \quad 0 < r < \delta$$

where $D_r(p)$ is the disc of radius r centered at p. These local mean-value inequalities make sense for non-differentiable functions. Thus, let $u(x, y)$ be a real-valued and continuous function defined in some open set Ω. We say that u satisfies the local mean-value inequality if there for every $p \in \Omega$ exists some $\delta > 0$ with $\delta \leq \text{dist}(p, \partial\Omega)$ such that (*) above holds. In XX we will prove that this is equivalent to the condition that the Laplacian $\Delta(u)$ in the sense of distributions is a non-negative Riesz measure in Ω. When the continuous function u satisfies (*) we say that it is subharmonic. This class of functions is denoted by $\text{SH}_c(\Omega)$ where the prefix c indicates that we restrict the attention to continuous functions.

0.1 The majorant principle. The local mean value inequality for a function u in $\text{SH}_c(\Omega)$ gives the following majorisation principle. Consider a pair (U, h) where $U \subset \Omega$ is an open subset and h is a continuous function on the compact closure $\bar{U}$ and harmonic in U. Then one has the implication:

$$(**) \qquad u|\partial U \leq h|\partial U \implies u \leq h \quad \text{in the whole set } \bar{U}$$

The proof of (**) relies upon the mean-value equality for harmonic functions and is given in XXX below where we also prove the converse result, i.e. a continuous function u in Ω for which the majorisation (**) holds belongs to $\text{SH}_c(\Omega)$. Hence we have an equivalent condition for a continuous function to be subharmonic !

0.2 Logarithmic potentials. Let μ be a non-negative Riesz measures with compact support in $\mathbf{C}$. Since the function $\text{Log}\,|z|$ is locally integrable there exists the convolution integral:

$$(***) \qquad U_\mu(z) = \frac{1}{2\pi} \int \text{Log}\,|z - \zeta| \cdot d\mu(\zeta)$$

We refer to U_μ as the logarithmic potential of μ. Here $U_\mu(z)$ is locally integrable. Moreover, it can be attained as the pointwise limit of a decreasing sequence of C^∞-functions. Namely, when $\epsilon > 0$ we set

$$U_\mu^\epsilon(z) = \frac{1}{4\pi} \int \log\bigl(|z - \zeta|^2 + \epsilon\bigr) \cdot d\mu(\zeta)$$

The reader should verify that these functions are of class C^∞ and with z kept fixed we have

$$\frac{dU^\epsilon_\mu(z)}{d\epsilon} = \frac{1}{4\pi}\int \frac{1}{|z-\zeta|^2+\epsilon}\cdot d\mu(\zeta)$$

It follows that $\epsilon \to U^\epsilon_\mu\}$ is a pointwise decreasing sequence of C^∞-functions whose pointwise limit function is U_μ. Hence the L^1_{loc}-function U_μ has well-defined values everywhere and becomes upper semi-continuous. At some points it may have the value $-\infty$. Put

$$\text{Polar}(\mu) = \{z \quad : \ U_\mu(z) = -\infty\}$$

This polar set is the intersection of the open sets $\{U_\mu < -N\}$ taken over all positive integer N and is therefore a so called G_δ-set whose 2-dimensional Lebesgue measure is zero since U_μ is locally integrable. In most applications we are content with this fact. Actually the polar set of μ belongs to a more restricted family of sets than the family of all null sets. This is treated in Chapter XX devoted to complex potential theory.

0.3 The mean-value inequality of U_μ. With μ fixed we define for a given $r > 0$ the mean-value function

(1) $$M_{\mu,r}(z) = \frac{\pi}{r^2}\int_0^r\int_0^{2\pi} U_\mu(z+se^{i\theta})\cdot sds\cdot d\theta$$

This yields a continuous function of z and one has the equality

(2) $$M_{\mu,r}(z) - U_\mu(z) = \int \Phi_r(|z-\zeta|)\cdot d\mu(\zeta)$$

where the function $a \mapsto \Phi_r(a)$ is defined by

$$\Phi_r(a) = \log(r) - \log(a) + \frac{1}{2}\cdot(\frac{a^2}{r^2}-1) \quad : 0 < a < r \quad \text{and} \ \ \Phi_r(a) = 0 \ : \ \ a \geq r$$

Remark. It is easily seen that $\Phi_r(a) \geq 0$ and hence (1) implies that $U_\mu \leq M_{\mu,r}$ for every $r > 0$. The formula for $\Phi_r(a)$ above follows after a dilation of scale and the equality

(i) $$\frac{1}{\pi}\int_0^1\int_0^{2\pi} \log|se^{i\theta}-a|\cdot sds\cdot d\theta = \frac{|a|^2-1}{2} \quad : \quad |a| < 1$$

whose proof is left as an exercise to the reader.

Now we enlarge the class of sub-harmonic functions to include U_μ-functions which arise from measures non-negative μ.

0.4 Definition. *Let Ω be an open set in* $\mathbf{C}$. *Then $SH(\Omega)$ denotes the family of functions $u \in L^1_{loc}(\Omega)$ for which the distribution $\Delta(u)$ is a non-negative Riesz measure. We refer to $SH(\Omega)$ as the set of subharmonic functions in Ω.*

Remark. The condition for a locally integrable function to be subharmonic can be phrased in another way where we do not use distribution derivatives. Namely, let $u(x,y)$ be a function in $L^1_{\text{loc}}(\Omega)$. Denote by $\mathfrak{Leb}(u)$ the set of its Lebesgue points. If $p \in \mathfrak{Leb}(u)$ and $0 < r < \text{dist}(p,\partial\Omega)$ we consider the mean value

(1) $$M_r(p) = \frac{1}{\pi r^2}\cdot\iint_{D_r(p)} u(x,y)\cdot dxdy$$

We say that u satisfies the local mean value inequality if there to every Lebesugue point p exists some $\delta > 0$ such that

(2) $$u(p) \leq M_r(p) \quad : \quad 0 < r < \delta$$

In (XX) we prove that this is equivalent to the condition that u is subharmonic in the sense of Definition 0.4.

0.5 The Riesz representation formula. It turns out that every subharmonic function in the sense of Definition 0.4 is locally expressed by the logarithmic potential of a Riesz measure plus some harmonic function. More precisely, let u be subharmonic in an open set Ω and put $\mu = \Delta(u)$. Let Ω_0 be a relatively compact subset of Ω and denote by K its compact closure. Extending μ to be zero outside K we get the compactly supported measure μ_K and its logarithmic potential U_{μ_K}. In XXX we prove that there exists a harmonic function H in Ω_0 such that

$$(*) \qquad u = U_{\mu_K} + H$$

holds in Ω_0.

0.6 Examples of subharmonic functions. A class of subharmonic functions arises as follows: Let $f(z)$ be analytic in the open unit disc. Assume that $f(0) = 0$ and that $|f(z)| < 1$ for all $z \in D$. To each $0 < r < 1$ there exists the function $\mathcal{N}_r(w)$ in the disc $|w| < 1$, defined by

$$(***) \qquad \mathcal{N}_r(w) = \sum \log \frac{r}{|\zeta|} \quad : \text{sum taken over all } \zeta \in D_r \quad : \ f(\zeta) = w$$

Thus, in the sum above one repeats zeros of $f(z) - w$ with their multiplicity. In XXX we show that $\mathcal{N}_r(w)$ is a subharmonic function in the unit disc of the complex w-plane. This class of subharmonic functions was introduced by Nevanlinna when he developed the value distribution theory for meromorphic functions. A notable point is that one can extend the construction of these $\mathcal{N}$-functions to universal covering spaces of the image domains $f(D_r)$. The interested reader can consult the article [Lehto] by O. Lehto for applications of such subharmonic functions in value distribution theory of meromorphic functions. See also [Nev. page xx-xx] which describes the usefulness of his class of subharmonic functions.

Outline of the contents.

In § 1 we study log $|z|$ and show that its Laplacian in the sense of distributions is equal to 2π times the unit point mass at the origin. Next we consider logarithmic potentials. Denote by $\mathfrak{M}_+(\mathbf{C})$ the class of non-negative Riesz measures in $\mathbf{C}$ with compact support. To each such measure μ we define

$$(*) \qquad U_\mu(z) = \frac{1}{2\pi} \cdot \int \log |z - \zeta| \cdot d\mu(\zeta)$$

In Section 1 we show that U_μ is a subharmonic function and describe how U_μ is obtained via certain regularisations. Namely, to each $\epsilon > 0$ we define

$$(**) \qquad U_\mu^\epsilon(z) = \frac{1}{2\pi} \cdot \int \log \sqrt{|z - \zeta|^2 + \epsilon^2} \cdot d\mu(\zeta)$$

Here we get continuous functions and since $\mu \geq 0$ it is clear that this family is monotone, i.e.

$$\epsilon_1 < \epsilon_2 \implies U_\mu^{\epsilon_1}(z) \leq U_\mu^{\epsilon_2}(z)$$

From this we will deduce that one has a pointwise limit almost everywhere, i.e. the equality

$$(***) \qquad U_\mu(z) = \min_\epsilon U_\mu^\epsilon(z) = \quad \text{holds almost everywhere}$$

Since an L^1_{loc} is determined by its values outside a null set this means that $U_\mu(z)$ can be identified with the *upper semi-continuous* function defined by the last term in (***) which is given as a pointwise limit of a monotone sequence of continuous functions. At the end of section XX we study *Cauchy transforms*. Since $\frac{1}{z}$ belongs to $L^1_{\text{loc}}(\mathbf{C})$ we can define its convolution with every $\mu \in \mathfrak{M}_+(\mathbf{C})$. Thus, we set

$$\mathcal{C}_\mu(z) = \int \frac{1}{z - \zeta} \cdot d\mu(\zeta)$$

and refer to $\mathcal{C}_\mu$ as the Cauchy transform of μ.

We also establish formulas which relate U_μ and $\mathcal{C}_\mu$ to each other. More precisely, consider the *distribution derivative* $\partial(U_\mu) = \frac{\partial U_\mu}{\partial z}$. Then one has the equality

$$\partial(U_\mu) = \frac{1}{4} \cdot \mathcal{C}_\mu$$

We show that μ is recovered when the Laplace operator is applied to U_μ via the formula:

$$\Delta(U_\mu) = \mu$$

In Section 3 we study the general class of subharmonic functions. The main result is Theorem 2.X which shows that every subharmonic function u is given as a sum of the logarithmic potential of a non-negative measure μ and a harmonic function. Moreover, μ is equal to $\Delta(u)$ where the Laplacian of μ is defined in the sense of distributions. The proof of Riesz' formula requires several steps. One essential ingredient is the *elliptic property* of the Laplace operator which means that if μ is a measure which in the distribution sense satisfies $\Delta(\mu) = 0$ then μ is a "true" harmonic functions which therefore is a real-analytic function.

Further results. In Section 4 we study *Perron families* where Theorem 4.1 will be used to solve the Dirichlet problem in Chapter 5:A. Section deals with constructions of subharmonic functions which arise from harmonic functions and a technical result is proved in Section 6 which is a cornerstone for the main results in section 5.

1. The subharmonic Log-function

1.0 The function L_ϵ. To each $\epsilon > 0$ we get a function in $\mathbf{R}^2$ defined by:

$$F_\epsilon(x,y) = \text{Log}(x^2+y^2+\epsilon)$$

The partial derivatives with respect to x become:

$$\partial_x(F_\epsilon) = \frac{2x}{x^2+y^2+\epsilon} \quad : \quad \partial_x^2(F_\epsilon) = \frac{2}{x^2+y^2+\epsilon} - \frac{4x^2}{(x^2+y^2+\epsilon)^2}$$

and similarly for y-derivatives. A summation of the second order partial derivatives gives:

(i)
$$\Delta(F_\epsilon) = \frac{2\epsilon}{(x^2+y^2+\epsilon)^2}$$

The double integral over $\mathbf{R^2}$ becomes:

$$\iint \frac{2\epsilon}{(x^2+y^2+\epsilon)^2} = \int_0^\infty \int_0^{2\pi} \frac{2\epsilon}{(r^2+\epsilon)^2} \cdot r d\theta = 4\pi$$

So if we normalise F_ϵ and take

(ii)
$$L_\epsilon(x,y) = \frac{1}{2\pi} \cdot \text{Log}\,\sqrt{x^2+y^2+\epsilon}$$

then its double integral is equal to one and

(iii)
$$\Delta(L_\epsilon) = \frac{1}{2\pi}\frac{\epsilon}{(x^2+y^2+\epsilon)^2}$$

With $z = x+iy$ we can write

$$L_\epsilon(z) = \frac{1}{2\pi} \cdot \text{Log}\,|\sqrt{|z|^2+\epsilon}|$$

Outside the origin we get the limit formula:

(*)
$$\lim_{\epsilon\to 0} L_\epsilon(z) = \frac{1}{2\pi} \cdot \text{Log}\,|z|$$

Next, recall that $\partial = \frac{1}{2}(\partial_x - i\partial_y)$. A differentiation gives:

$$\partial(L_\epsilon) = \frac{1}{4\pi} \cdot \frac{x-iy}{x^2+y^2+\epsilon} = \frac{1}{4\pi} \cdot \frac{\bar{z}}{|z|^2+\epsilon}$$

So outside the origin we get the limit formula

(**)
$$\lim_{\epsilon\to 0} \partial(L_\epsilon)(z) = \frac{1}{4\pi z}$$

1.1 L_ϵ as distributions. The function L_ϵ defines a distribution, i.e. a linear functional on the space of test-functions in $\mathbf{R^2}$. Let $\phi \in C_0^\infty(\mathbf{R})^2$ be a test-function. Green's formula gives

(1)
$$\iint \Delta(L_\epsilon)\cdot\phi\cdot dxdy = \iint L_\epsilon \cdot \Delta(\phi)\cdot dxdy \quad : \quad \epsilon > 0$$

The left hand side has a limit as $\epsilon \to 0$. Namely, (iii) above gives:

(ii)
$$\lim_{\epsilon\to 0} \frac{1}{2\pi} \iint \frac{\epsilon}{(x^2+y^2+\epsilon)^2} \cdot \phi(x,y)dxdy = \phi(0)$$

The reader should confirm the limit formula (ii) by computing the integral in polar coordinates. In distribution theory this is expressed as follows:

1.2 Theorem. *The distribution densities $\Delta(L_\epsilon)$ converge to the unit point mass δ_0.*

Remark. Above we constructed a *regularisation* of the Dirac measure. Now we apply this in a more general set-up. Let μ be a Riesz measure in $\mathbf{R^2}$ with a compact support which defines a distribution by

$$\phi \mapsto \int \phi \cdot d\mu$$

To each $\epsilon > 0$ we construct the convolution

$$L_\epsilon * \mu(x,y) = \int L_\epsilon(x-t, y-s) \cdot d\mu(t,s)$$

Here $L_\epsilon * \mu$ are C^∞-functions and taking the Laplacian we get

$$\Delta(L_\epsilon * \mu)(x,y) = \frac{1}{2\pi} \int \frac{\epsilon}{(x-t)^2 + (y-s)^2 + \epsilon} \cdot d\mu(t,s)$$

Next, when $\phi \in C_0^\infty$ we perform integration with respect to (x,y) and obtain

$$\iint \Delta(L_\epsilon * \mu)(x,y) \cdot \phi(x,y)dxdy = \frac{1}{2\pi} \int [\iint \frac{\epsilon \cdot \phi(x,y)dxdy}{(x-t)^2 + (y-s)^2 + \epsilon}] \cdot d\mu(t,s)$$

Here the inner double intergal is a function of (s,t) which converges uniformly to ϕ as $\epsilon \to 0$. Hence a passage to the limit gives

$$\lim_{\epsilon \to 0} \iint \Delta(L_\epsilon * \mu)(x,y) \cdot \phi(x,y)dxdy = \int \phi \cdot d\mu$$

This is expressed by saying the the distribution densities $\Delta(L_\epsilon * \mu)$ converge to the distribution defined by μ. Next, recall from distribution theory that convolution commutes with differentiation. Hence we get

(*) $$\lim_{\epsilon \to 0} \Delta(L_\epsilon) * \mu) = \lim_{\epsilon \to 0} \Delta(L_\epsilon) * \mu) = \mu$$

where the last equality follows from Theorem 1.2.

1.3 The logarithmic potential. Recall from measure theory that one can define the convolution of a compactly supported Riesz measure μ with an L^1-functions. We apply this with the locally integrable function $\log \sqrt{x^2 + y^2}$ which in complex notation is written as $\log |z|$. Now there exists the convolution:

$$U_\mu(z) = \frac{1}{2\pi} \int \log |z - \zeta| \cdot d\mu(\zeta)$$

It is called the logarithmic potential of the Riesz measure μ and we notice that $U(z)$ is in $L^1_{\text{loc}}\mathbf{C})$. The limit formulas from Theorem 1.2 and (*) above yield

1.4 Theorem. *The Laplacian of U_μ taken in the distribution sense is equal to μ.*

Remark. To confirm Theorem 1.4 we consider a test-function g. Fubini's theorem gives the equality

$$\frac{1}{2\pi} \cdot \int \Delta\, g(z) \cdot] \int \log |z - \zeta| \cdot d\mu(\zeta)] \cdot dxdy = \frac{1}{2\pi} \cdot [\int \Delta\, g(z) \cdot \log |z - \zeta| \cdot dxdy] \cdot d\mu(\zeta)$$

By Theorem 1.2 the last term becomes

$$\int g(\zeta) \cdot d\mu(\zeta)$$

Hence the definition of distribution derivatives gives Theorem 1.4.

1.5 The Cauchy transform.

Starting from $U_\mu(z)$ we construct its distribution derivative with respect to the first order differential operator ∂. Using the limit formula (**) from 1.0 we obtain the following equality in the distribution sense:

(i) $$\partial(U_\mu(z) = \frac{1}{4\pi} \int \frac{d\mu(\zeta)}{z - \zeta}$$

Put

$$\mathcal{C}_\mu(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

Since this is a convolution of a measure with compact support and the locally integrable function $\frac{1}{z}$ it belongs to L^1_{loc}. We refer to $\mathcal{C}_\mu$ as the Cauchy transform of μ. Notice that $\mathcal{C}_\mu$ is an analytic function outside the support of μ. For example, its complex derivative becomes

$$\frac{d\mathcal{C}_\mu(z)}{dz} = -\int \frac{d\mu(\zeta)}{(z-\zeta)^2}$$

1.6 The equality $\bar{\partial}(\mathcal{C}_\mu) = \pi \cdot \mu$.

Recall from XX that the Laplacian Δ can be expressed as the product of the first order differential operators

$$\partial = \frac{1}{2}(\partial_x - i\partial_y) \quad : \quad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

More precisely we have $\Delta = 4 \cdot \bar{\partial}_z \cdot \partial_z$. Now we regard the logarithmic potential $U_\mu(z)$ from 1.3 and here (i) in 1.5 gives:

$$\partial(U_\mu) = \frac{1}{4\pi} \cdot \mathcal{C}_\mu$$

By Theorem 1 .2 we also have

$$\mu = \Delta(U_\mu) = 4 \cdot \bar{\partial}(\partial U)) = 4 \cdot \frac{1}{4\pi} \cdot \bar{\partial}(\mathcal{C}_\mu)$$

From this we get the equality (1.6) above. Since it is so important we state

1.7 Theorem. *One has the equality*

$$\bar{\partial}(\mathcal{C}_\mu) = \pi \cdot \mu$$

Example. Let $\mu = \delta_0$ be the unit mass at the origin. In this case $\mathcal{C}_\mu(z) = \frac{1}{z}$ and by the definition of distribution derivatives Theorem 1.7 means that

(i) $$g(0) = -\frac{1}{\pi} \int \frac{\bar{\partial}(g) dxdy}{z} \quad : \quad g \in C_0^\infty(\mathbf{C})$$

Recall from XX that $dz \wedge d\bar{z} = -2i \cdot dxdy$. So the minus sign is changed and the right and side becomes

(ii) $$\frac{1}{2\pi i} \int \frac{\bar{\partial}(g) \cdot dz \wedge d\bar{z}}{z} \quad : \quad g \in C_0^\infty(\mathbf{C})$$

Since $\frac{1}{z}$ is locally integrable this integral is equal to

(iii) $$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|>\epsilon} \frac{\bar{\partial}(g) \cdot dz \wedge d\bar{z}}{z} \quad : \quad g \in C_0^\infty(\mathbf{C})$$

Now we regard the differential 1-form $\alpha = \frac{g(z) \cdot dz}{z}$ and as explained in XX we have

(iv) $$d\alpha = \frac{\bar{\partial}(g) \cdot d\bar{z} \wedge dz}{z} = -\frac{\bar{\partial}(g) \cdot dz \wedge d\bar{z}}{z}$$

where we used the the exterior product of two 1-forms is anti-commutative. Next, the boundary of the *exterior* disc $|z| > \epsilon$ is $|z| = \epsilon$. When Stokes formula is applied the outer normal with respect to the exterior domain is minus the usual outer normal with respect to the open disc $|z| < \epsilon$. So when we apply Stokes Theorem to the differential 1-form α we see that (iii) above is equal to

(v) $$-\frac{1}{2\pi i} \cdot \lim_{\epsilon \to 0} \int_{|z|>\epsilon} d\alpha = \frac{1}{2\pi i} \cdot \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{g(z)dz}{z}$$

where the last line integrals over $|z| = \epsilon$ are taken in the *usual positive sense*, i.e. using $z = \epsilon e^{i\theta}$ which gives $\frac{dz}{z} = id\theta$, it it follows (v) that becomes

$$\frac{1}{2\pi}\cdot\lim_{\epsilon\to 0}\int_0^{2\pi} g(\epsilon\cdot e^{i\theta})\cdot d\theta = g(0)$$

This confirms Theorem 1.7 when $\mu = \delta_0$.

Remark. We included the computations above to avoid eventual confusion when the real version of Stokes Theorem is mixed with its complex counter part.

1.8 Approximation theorems

The fact that Theorem 1.7 holds for every Riesz measure has some remarkable consequences. Let K be a compact null set in $\mathbf{C}$, i.e. its 2-dimensional Lebesgue measure is zero. We have the Banach space $C^0(K)$ of continuous and complex valued functions on K. If $z \in \mathbf{C}\setminus K$ the rational function $\frac{1}{z-\zeta}$ restricts to a continuous function on K. Taking finite linear combinations we get functions

$$\zeta \mapsto \sum c_\nu \frac{1}{z_\nu - \zeta} \quad : z_1,\ldots,z_N \in \setminus K \quad : c_1,\ldots,c_N \in \mathbf{C}$$

Denote by $R(K)$ the closure of this linear subspace of $C^0(K)$. Thus, functions in $R(K)$ consist of continuous functions on K which can be uniformly approximated by rational functions with poles outside K.

1.9 Theorem. *For every compact null set K one has the equality $C^0(K) = R(K)$.*

Proof. Suppose that $R(K) \neq C^0(K$. Riesz' representation formula gives the existence of a non-zero measure μ supported by K such that $\mu \perp R(K)$. Consider the Cauchy transform

$$\mathcal{C}_\mu(z) = \int_K \frac{d\mu(\zeta)}{z-\zeta}$$

Since $\mu \perp R(K)$ it is identically zero outside K. Now K is a null set so the L^1_{loc}-function $\mathcal{C}_\mu$ is identically zero and so is its distribution derivative $\bar\partial(\mathcal{C}_\mu)$. But then we get a contradiction from Theorem 1.7 since this distribution derivative must recapture $\pi\cdot\mu$. which by assumption is non-zero.

Remark. The proof of Theorem 1.8 relies upon the Hahn-Banach theorem which gives the existence of a non-zero Riesz measure carried by K when $R(K) \neq C^0(K)$. The drawback of this proof is that it does not give any hint about how one actually approximates a given continuous function on K by rational functions having poles in the complement. So instead of the easy proof above which is based on an "argument by contradiction" one would like to have a *constructive proof* of Theorem 1.7, i.e. given a null-set K one may ask for some sort of algorithm to approximate every given continuous function on K. This point of view was put forward by E. Bishop in his book [Bish]. His objection to the proof above is not just a question of taste and philosophy. In fact, Erret Bishop is one of the most prominent analysists in complex function theory. It is therefore good to keep in mind that various theoretical results do not give the whole story if one really wants to apply them in more concrete situations.

1.10 Megelyan's swiss-cheese. Following [Merg] we describe the construction of a compact set K in C with empty interior where $R(K) \neq C^0(K)$. Take the unit disc D and remove a finite number of discs $D_1,\ldots,D_N$ inside D. They are chosen so that the closed discs $\{\bar D_\nu\}$ are pairwise disjoint and stay inside D. For each $1 \leq \nu \leq N$ we let μ_ν be the measure supported by ∂D_ν and given by $d\zeta$, i.e. if D_ν has a radius r_ν it is simply the measure whose Cauchy transform becomes

$$\mathcal{C}_\nu(z) = \int_0^{2\pi} \frac{ir_\nu e^{i\theta}}{z-(a_\nu + r_\nu)e^{i\theta}}\cdot d\theta$$

Notice that $\mathcal{C}_\nu(z) = 0$ outside $\bar D_\nu$. Indeed, this follows from the trivial observation from XXX. Next, on ∂D we get the measure μ^* defined by $d\zeta$ restricted to $|\zeta| = 1$. Let $\mathcal{C}^*(z)$ denote its Cauchy transform which now is identically zero outside D. In addition to this it has a compensating influence. For if $z_0 \in D_\nu$ for some ν, we have by Cauchys formula

$$\int_{\partial D_\nu} \frac{d\zeta}{z_0 - \zeta} = \int_{\partial D} \frac{d\zeta}{z_0 - \zeta}$$

Consider the measure

$$\rho_N = \mu^* - (\mu_1 + \ldots + \mu_N)$$

The previous observations show that $\mathcal{C}_{\rho_N}$ is zero in the set $\Omega_N = \cup D_\nu \cup |\zeta| > 1$. In other words, if

$$K_N = \bar{D} \setminus D_1 \cup \ldots \cup D_N$$

then this compact set is the support of the L^1-function $\mathcal{C}_{\rho_N}$.

At this stage we see how one should proceed to construct a swiss-cheese. Namely, inside D we construct a denumerable sequence of discs $D_1, D_2, \ldots$ so that the compact set

$$K = \bar{D} \setminus \cup D_\nu$$

has no interior points, i.e. just make sure that the center points of the discs $\{D_\nu\}$ appears as a dense subset of D. Moreover, let r_ν be the radius of D_ν and perform the construction so that

$$\sum r_\nu < \infty$$

The total variation of μ_ν becomes $2\pi \cdot r_\nu$ and hence we get a measure

$$\rho_* = \mu^* - \sum_{\nu=1}^{\infty} \mu_\nu$$

By the construction it is clear that $\rho_* \perp R(K)$ and hence $R(K) \neq C^0(K)$.

1.11 Wermer's example. In [We] appears a Jordan arc Γ whose 2-dimensional Lebesgue measure is positive. The the existence of such "fat" Jordan arcs goes back to work by Peano. The restriction of polynomial $P(z)$ to Γ gives a $\mathbf{C}$-subalgebra of $C^0(\Gamma)$. Let $P(\Gamma)$ be its uniform closure. Then

$$P(\Gamma) \neq C^0(\Gamma)$$

This inequality relies upon results about analytic capacity. The reader may consult the book [We] about uniform algebras for a detailed account about this example. See also the text-book [Ga] by T. Gamelin which is devoted to uniform algebras.

2. Subharmonic functions

A real-valued C^2-function $u(x,y)$ is called subharmonic if its Laplacian is a non-negative function. If Ω is an open subset of $\mathbf{C}$ we denote by $\mathrm{SH}^2(\Omega)$ the class of subharmonic C^2-functions defined in Ω. We begin to investigate this class. Let $u \in \mathrm{SH}^2(\Omega)$ and $\Omega_0 \in \mathcal{D}(C^1)$ is some relatively compact in Ω. Green's formula gives:

$$\iint_{\Omega_0} \Delta(u) dx dx = \int_{\partial\Omega_0} u_{\mathbf{n}} ds$$

Hence the subharmonicity entails that the integral of the outer normal derivative is ≥ 0 for any such domain Ω_0. Consider the case when $\Omega = D_R$ is a disc centered at the origin and $\Omega_0 = D_r$ for some $r < R$. In polar coordinates we get

$$\int_{\partial D_r} u_{\mathbf{n}} \cdot ds = \int_0^{2\pi} [\cos\theta \cdot u_x + \sin\theta \cdot u_y] \cdot r d\theta$$

Next, consider the mean-value function

$$M_u(r) = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(r,\theta) \cdot d\theta$$

Since $\frac{d}{dr}(u(r,\theta) = \cos\theta \cdot u_x + \sin\theta \cdot u_y$ we obtain

$$\frac{d}{dr}(M_u(r)) = \frac{1}{\pi r} \int_{\partial D_r} u_n ds = \frac{1}{2\pi r} \cdot \iint_{D_r} \Delta(u) dx dx$$

Hence the function $r \mapsto M_u(r)$ is non-decreasing and when $\Delta(u) > 0$ it is even strictly increasing. Since u is continuos we have

$$\lim_{r \to 0} M_u(r) = u(0,0)$$

It follows that u satisfies the mean-value inequality

$$u(0) \leq M_u(r) \quad : \quad 0 < r < R$$

2.1 Harmonic majorization. Let $u \in \mathrm{SH}^2(\Omega)$. Notice that if $\delta > 0$ and $u_\delta(x,y) = u(x,y) + \delta(x^2+y^2)$ then $\Delta(u_\delta) = \Delta(u) + 4\delta > 0$. Since $u_\delta \to u$ when $\delta \to 0$ we can always approximate a subharmonic function by a decreasing sequence of strictly subharmonic functions. Next, recall the following result from from *Calculus*. Let $f(x,y)$ be a C^2-function defined in some open subset of $\mathbf{R^2}$ which has a (not necessarily strict) maximum at some point (x_0,y_0) , i.e.

$$f(x,y) \leq f(x_0,y_0) \quad : \quad (x-x_0)^2 + (y-y_0)^2 < \epsilon$$

hold for some small ϵ. Then the *Hessian* of f at (x_0,y_0) must be negative semi-definite. In particular the trace $f_{xx} + f_{yy} = \Delta(f) \leq 0$. This elementary facts gives

2.2 Proposition. *Let u be a strictly subharmonic function of class C^2 defined in an open set Ω. Then u cannot have any local maximum in Ω. Thus, if U is a relatively compact subset of Ω then u takes its maximum on the boundary of U, i.e.*

$$\max_{\bar{U}} u = \max_{\partial U} u$$

Next, Proposition 2.2. together with the mean-value property of harmonic functions give the following:

2.3 Theorem. *Let $u \in SH^2(\Omega)$ and let h be a harmonic function in Ω. Then the following implication hold for every relatively compact open subset U of Ω;*

$$u \leq h \text{ on } \partial U \implies u \leq h \text{ on } U$$

Proof. Follows from Proposition 2.2 since $u - h$ is subharmonic. We refer to this as the principle of *harmonic majorization.*

2.4 Subharmonic functions in L^1_{loc}.

Now we relax the C^2-hypothesis. Let $u \in L^1_{\text{loc}}(\Omega)$ where u is real-valued. We get the distribution $\Delta(u)$ and impose the condition that it is equal to a non-negative Riesz measure μ. By the definition of distribution derivatives this means that

$$(*) \qquad \iint_\Omega \Delta(\phi) \cdot u dx dy = \iint_\Omega \phi \cdot d\mu \quad : \quad \phi \in C_0^\infty(\Omega)$$

2.5 Definition. *A function u in $L^1_{loc}(\Omega)$ is called subharmonic if the distribution $\Delta(u) \geq 0$. The class of subharmonic functions in Ω is denoted by SH(Ω).*

2.6 Regularisations.

Definition 2.5 is a bit abstract since it is not easy to discover the distribution $\Delta(u)$ when u is just assumed to be locally integrable. So we shall find other conditions in order that a function in L^1_{loc} is subharmonic. For this we use regularisations. In general, let $u \in \text{SH}(\Omega)$ where Ω is a bounded open set. If $\delta > 0$ we set

$$\Omega[-\delta] = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq \delta\}$$

Notice that $\Omega[-\delta]$ is a compact subset of Ω. For every test-function ϕ with compact support in the disc D_δ the convolution $\phi * u$ exists in $\Omega[-\delta]$. By the general formula from XXX we have

$$\Delta(\phi * u) = \phi * \Delta(u) = \phi * \mu$$

where μ by assumption is a non-negative Riesz measure.

2.7 The case when ϕ is radial. We shall use test-functions which depend on $x^2 + y^2$ only. Let us recall the construction. Start from a test-function $\phi_*(z)$ which is > 0 in $|z| < 1$ and has compact support in $|z| \leq 1$ and depends on $|z|$ only while

$$\iint_D \phi_*(z) dx dy = 1$$

Then, to every $0 < \delta < 1$ we get the test-function

$$\phi_\delta(z) = \frac{\phi_*(\frac{z}{\delta})}{\delta^2}$$

which has support in $\bar{D}_\delta$. Next, recall from XXX that for any L^1_{loc}-function f it follows that the convolution $\phi_\delta * f$ is a C^∞.function. We apply this with u above and conclude that $\phi_\delta * u$ is a C^∞-function defined in some open neighborhood of $\Omega[-\delta]$. By (*) in 2.6 we have

$$\text{(i)} \qquad \Delta(\phi_\delta * u) = \phi_\delta * \mu$$

Since both μ and ϕ_δ are ≤ 0, it follows that the convolution is ≥ 0. Hence the Laplacian of the C^∞-function $\phi_\delta * u$ is ≥ 0. So we can apply the results from the C^2-case. In particular $\phi_\delta * u$ satisfies the mean-value inequality

$$\phi_\delta * u(p) \leq \frac{1}{\pi r^2} \cdot \int_{D_r(p)} \phi_\delta(z - p) \cdot u(z) \cdot dx dy \quad : p \in \Omega[-2\delta] \quad : \quad 0 < r < \delta$$

If $p \in \Omega[-2\delta]$ is a Lebesgue point for u we can pass to the limit as $\delta \to 0$ and conclude that

$$(*) \qquad u(p) \leq \frac{1}{\pi r^2} \cdot \int_{D_r(p)} u(z) \cdot dx dy \quad : p \in \Omega[-2\delta] \quad : \quad 0 < r < \delta$$

This means that u satisfies the local mean-value inequality in $\Omega[-\delta]$. Since we can choose δ arbitrary small, it follows that u satisfies the local mean-value inequality in the whole of Ω, i.e. we have proved:

2.8 Theorem. *Let $u \in SH(\Omega)$. Then the following holds for each Lebesgue point of u in Ω:*

$$(*) \qquad u(p) \leq \frac{1}{\pi r^2} \cdot \int_{D_r(p)} u(z) \cdot dxdy \quad : \ 0 < r < \text{dist}(p, \partial\Omega)$$

Remark. Above we have recovered the definition of subharmonic functions from the introduction via Definition 2.5. The *converse* also holds, i.e. if we from start assume that the L^1_{Loc}-function u satisfies the local mean value inequality then it is subharmonic in the sense of Theorem 2.8.

2.9 Exercise. Prove the converse. The hint is that if ϕ_δ as above are radial test-functions and (*) is assumed, then the local mean value inequality hold for $\phi_\delta * u$. Here we have C^2-functions and by Green's formula one shows that $\delta(\phi_\delta * u) \geq 0$ follows. Finally one takes the limit as $\delta \to 0$ and the reader should now confirm that $\delta(u) \geq 0$ holds in the distribution sense.

Above we have constructed the convolutions $\phi_\delta * u$. It turns out that this family of functions is monotone.

2.10 Proposition. *Let $u \in SH(\Omega)$. Then the sequence of functions $\{\phi_\delta * u\}$ decrease, i.e.*

$$\phi_{\delta_1} * u(p) \leq \phi_{\delta_2} * u(p) \quad : \quad \delta_1 < \delta_2$$

Moreover, this decreasing sequence converges almost everywhere to the measurable function u.

2.11 Exercise. Give the details of proof. Show also that Proposition 2.11 implies u is almost everywhere equal to the pointwise limit if a monotone sequence of continuous functions and therefore we can always take u to be an upper semi-continuous function. The set where it becomes $-\infty$ is a null set. Thus, every subharmonic function enjoys similar properties as logarithmic potentials from Section 1.

3. Riesz representation formula

Let $u \in \mathrm{SH}(\Omega)$ where Ω is a bounded open set. Now $\Delta(u)$ exists as a distribution and we have by assumption

$$(*) \qquad \iint \Delta(\phi) \cdot u \cdot dxdy \geq 0 \quad : \phi \in C_0^\infty(\Omega)$$

We can take regularisations of u as in section 2, i.e. construct convolutions $\phi_k * u$ where $\{\phi_k\}$ is a sequence of non-negative test functions with smaller and smaller compact support in some disc $|x| \leq \delta$ while their integrals are one for every k. Let $g \in C_0^\infty(\Omega)$ with compact support in $\Omega[-\delta]$. Green's formula gives:

$$(**) \qquad \iint \Delta(g) \cdot \phi_k * u \, dxdy = \iint g \cdot \Delta(\phi_k * u) \cdot dxdy$$

Since u is subharmonic the functions $\Delta(\phi_k * u) \geq 0$ in $\Omega[-\delta]$. Hence they become non-negative measures. Let us fix a compact subset K in $\Omega[-\delta]$. For example, we can take

$$K = \text{closure of } \Omega[-2\delta]$$

Now we can regard the *total mass*

$$\rho_k = \iint_K \Delta(\phi_k * u) \cdot dxdy$$

3.1. An inequality. Since u is subharmonic the mean-value inequality from XX gives the inequality:

$$\phi_k * u(z) \leq u(z) \quad : \quad z \in \Omega[-2\delta]$$

From this we conclude that if $g \in C_0^\infty(\Omega)$ is non-negative and identically one on $\Omega[-2\delta]$ then (**) above gives

$$\text{(i)} \qquad \rho_k \leq \int \Delta(g) \cdot u \cdot dxdy \quad : \quad k = 1, 2, \ldots$$

Together with a general result about positive distributions in to be proved in XX below the inequalities (i) give:

3.2. Proposition. *There exists a constant C_K such that*

$$\rho_k \leq C_K \quad : , k = 1, 2, \ldots$$

The uniform bound in Proposition 3.2 gives the existence of a subsequence of the non-negative measures $\{\Delta(\phi_k * u)\}$ which converges weakly to non-negative Riesz measure μ in K. At the same time we notice that

$$\lim_{k \to \infty} \phi_k * u \to u$$

where the limit holds in L^1. Hence we have proved:

3.3. Proposition *For every test-function $g \in C_0^\infty(\Omega)$ with support contained in K one has:*

$$\iint \Delta(g) \cdot u dxdy = \int g d\mu$$

3.4. Constructing the Log-potential With $\delta > 0$ we construct a test-function χ satisfying

$$\chi = 1 \text{ in } \Omega[-3\delta] \quad : \chi \in C_0^\infty(\Omega[-\delta]$$

Next, we use the C^∞-functions L_ϵ and keepong $\delta > 0$ fixed we define the functions:

$$g_\epsilon = \chi \cdot L_\epsilon \quad : \quad 0 < \epsilon < \delta$$

Passing to the limit as $\epsilon \to 0$ while χ is kept fixed, Proposition 3 above and Theorem 1.4 imply that the function

$$w(z) = u(z) - \int \operatorname{Log}|\zeta - z| \cdot d\mu(\zeta) \quad : \; z \in \Omega[-3\delta]$$

will have a Laplacian in the *distribution sense* which is equal to zero in $\Omega[-3\delta]$.

3.5. Conclusions.

So are we have not proved anything definitive. But we have demonstrated that one should regard two problems. The first is to explain why the ρ-numbers stay bounded, i.e. to verify Proposition 3.2. The second is to show that if ϕ is some L^1_{loc}-function such that $\Delta(\phi) = 0$ holds in the distribution sense, then ϕ is *automatically* a nice function, i.e. at least C^2 and hence harmonic. If this has been achieved the results above show that the subharmonic function u is represented as a logarithmic potential of a measure plus a harmonic functions inside $\Omega[-3\delta]$. Since $\delta > 0$ can be made arbitrary small this gives a representation in any relatively compact subset of Ω. So there remains to establish two general results from distribution theory.

3.6. Positive distributions.

Consider an open square $\square = \{(x, y)\colon\; 0 < x, y < 1\}$. Let $\mathcal{L}$ be a linear form on $C_0^\infty(\square)$ and assume that there exists some integer $k \geq 0$ and a constant C such that

$$|L(g)| \leq C \cdot ||g||_k \quad : \; g \in C_0^\infty(\square) \text{ where } ||g||_k = \text{norm in } C^k(\square)$$

We say that L is positive if

$$g \geq 0 \implies L(g) \geq 0$$

3.7 Theorem. *Let L be defined and positive as above. Then, for every* $0 < r < 1$ *there is a constant* C_r *such that*

$$|L(g)| \leq C_r \cdot ||g||_0 \quad : \text{Supp}(g) \subset \square_r$$

Proof Given $r < 1$ we construct $\phi \in C_0^\infty(\square)$ where $\phi = 1$ on $\square_r$ and is non-negative. Now, if g ha support in $\square_r$ it follows that the function

$$||g||_0 \cdot \phi - g \geq 0$$

Since L is positive we get

$$L(g) \leq ||g||_0 \cdot L(\phi)$$

So we can take $C_r = L(\phi)$ and the Theorem 3.7 follows.

3.8. The elliptic property of Δ.

Let $w \in L^1_{\text{loc}}(\Omega)$ for some bounded open set. Assume that

$$\iint \Delta(g) \cdot w \cdot dxdy = 0 \quad : \; g \in C_0^\infty(\Omega)$$

3.9 Theorem. *Under the assumption above w is a harmonic function in Ω.*

Proof. We use similar regularisations as above. With $\delta > 0$ we choose the sequence $\{\phi_k\}$ and now $\phi_k * w \in C^\infty(\Omega[-\delta]$. Since convolution commutes with Δ, it follows that these functions are harmonic in $\Omega[-\delta]$. Moreover, since w b assumption has a finite L^1-norm over the relatively compact subset $\Omega[-\delta]$, the L^1-norms of $\{\phi_k * w\}$ are uniformly bounded in $\Omega[-2\delta]$, i.e. we have a constant C so that

$$\iint_{\Omega(-2\delta)} |\phi_k * w| \, dxdy \leq C$$

Poisson's formula implies that we get a uniform bound for the *maximum norms* in $\Omega[-3\delta]$, i.e. with another constant C_1 one has

$$\max |\phi_k * w(z)| \leq C_1 \quad : \; z \in \Omega[-3\delta]$$

At this stage we apply Montel's results for normal families of harmonic functions in XX. Passing to a subsequence if necessary, it follows that

$$\lim_{k \to \infty} \phi_k * w = G \text{ holds uniformly in compact subsets of } \Omega[-3\delta]$$

where the limit function G is harmonic. At the same time $w \in L^1_{\text{loc}}$. From Lebesgue theory we know that $\phi_k * w \to w$ and hence w must be equal to the "true" harmonic function G in $\Omega(-\delta)$. Since δ can be arbitrary small we conclude that w is a true harmonic function in the whole of Ω.

3.10 Remark. Above we gave a "pedestrian proof" which could have been given in a quicker way if one admits further results in distribution theory. Moreover, the elliptic property of Δ holds for distributions, i.e. if w is replaced by *any* distribution μ defined in some open set Ω where $\Delta(\mu) = 0$ holds in the sense of distributions, then μ is a "true" harmonic function. Thus can be shown by using regularisations as above. Namely, exactly as above $\phi_k * \mu \in C^\infty(\Omega[-\delta]$. Next, the distribution μ restricted to the relatively compact set $\Omega[-\delta]$ has a finite order k say. Using this one can proceed exactly as in the proof of Theorem 3.5, except that one has to be a bit more careful and take into the account growth of the derivatives of the ϕ-functions up to order k. We leave the details to the reader who also may consult text-books devoted to distribution theory which show that Theorem 3.9 holds with w replaced by a distribution.

3.11 Analytic expansions of harmonic functions

The elliptic character of Δ is made more precise by a result due to L. Ehrenpreis which shows how to express distributions via absolutely convergent integrals taken in $\mathbf{C}^2$ via the Fourier transform of μ. This result go beyond these notes since the proofs rely upon several complex variables. Proofs of Ehrenpreis' integral formulas appear in Chapter 8 of my book [Bj] and also in the later edition of [Hö-complex] as well as in [Hö:2 Chapter PDE]. But let as explain the result for harmonic functions $w(x, y)$ in the unit disc.

3.12 Integrals over harmonic exponentials Let ζ and w be two complex numbers and set:

$$\mathfrak{e}(x, y) = e^{i(x\zeta + y\eta)}$$

We see that $\Delta(\mathfrak{e}) = -(\zeta^2 + w^2) \cdot e$. Put

$$S = \{(\zeta, w) \in C^2 \quad : \; \zeta^2 + w^2 = 0\}$$

Points on this algebraic hypersurface in $\mathbf{C}^2$ produce harmonic $\mathbf{e}$-functions in the whole (x,y)-plane. It is therefore tempting to consider a complex-valued Riesz measure μ in the 4-dimensional real (ζ,w)-space with support in S and define the function

$$(*)\qquad U(x,y)=\int_S e^{i(x\zeta+yw)}\cdot d\mu(\zeta,w)$$

With $\zeta=\xi+i\eta$ and $w=u+iv$ we have

$$\text{(i)}\qquad |e^{i(x\zeta+yw)}|=e^{-(x\eta+yv)}$$

If $z=x+iy\in D$ so that $x^2+y^2<1$, the Cauchy-Schwartz inequality gives

$$\text{(ii)}\qquad |x\eta+yv|\leq\sqrt{\eta^2+v^2}$$

Assume that the mass distribution of μ satisfies

$$\text{(iii)}\qquad \int_S e^{\sqrt{\eta^2+v^2}}\cdot\,|d\mu(\zeta,w)|<\infty$$

Under this hypothesis we see from (i-iii) that the integral defining $U(x,y)$ in (*) converges for every point $(x,y)\in D$ and gives a harmonic function.

3.13 Extension to the complex Levi ball. From the real pair (x,y) we can take pass to complex numbers z_1,z_2 with $\mathfrak{Re}(z_1)=x$ and $\mathfrak{Re}(z_2)=y$. Let us then try to evaluate the integral

$$\mathcal{U}(z_1,z_2)=\int_S e^{i(z_1\zeta+z_2w)}\cdot d\mu(\zeta,w)$$

With $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ we get

$$|e^{i(z_1\zeta+z_2w)}|=e^{a_1\mathfrak{Re}(\zeta)-b_1\mathfrak{Im}(\zeta)+a_2\mathfrak{Re}(w)-b_2\mathfrak{Im}(w)}$$

Let us put

$$\mathcal{L}=\{(z_1,z_2)\quad:\ |z_1|+z_2|<1\}$$

This open subset of $\mathbf{C}^2$ is called the Levi ball. Notice that its intersection with the real subspace where $\mathfrak{Im}(z_1)=\mathfrak{Im}(z_2)$ is equal to the unit disc D in the (x,y)-plane. The Cauchy-Schwartz inequality gives

$$|a_1\mathfrak{Re}(\zeta)-b_1\mathfrak{Im}(\zeta)+a_2\mathfrak{Re}(w)-b_2\mathfrak{Im}(w)|\leq|z_1|\cdot\sqrt{\xi^2+\eta^2}+|z_2|\cdot\sqrt{u^2+v^2}$$

At the same time we stay on S so that $\zeta^2+w^2=0$. Regarding the real part this gives

$$\xi^2-\eta^2+u^2-v^2=0$$

At this stage the reader discovers the picture. More precisely we obtain

3.14 Lemma. *When* $(z_1, z_2) \in \mathcal{L}$ *one has the inequality*

$$|z_1| \cdot \sqrt{\xi^2 + \eta^2} + |z_2| \cdot \sqrt{u^2 + v^2} \leq \sqrt{\eta^2 + v^2} \quad : \; (\xi + i\eta, u + iv) \in S$$

The easy proof is left to the reader. Using this inequality and assuming that the integral (iii) above is finite, it follows that one has *absolutely convergent integrals*:

$$(**) \qquad \mathcal{U}(z_1, z_2) = \int_S e^{i(z_1\zeta + z_2 w)} \cdot d\mu(\zeta, w) \quad : \; (z_1, z_2) \in \mathcal{L}$$

Moreover, when $|z_1| + |z_2| < 1$, i.e. when we stay inside the open Levi ball we can take complex derivatives with respect to z_1 and z_2 to conclude that $\mathcal{U}$ is an analytic function in the open Levi ball. Its restriction to the real subspace is the harmonic function $U(x, y)$ which by (**) extends to an analytic function of two complex variables in the Levi ball.

3.15 Application. For each integer $N \geq 0$ we put

$$\mathcal{H}_N = \{u \in C^N(\bar{D}) \quad : \; u \text{ harmonic in } D\}$$

This is a Banach space where the norm $||u||_N$ can be taken as the sum of the maximum norm of its derivatives up to order N plus the maximum norm of u itself. With this notation the following result holds - i.e. a special case from the L^2-estimates from [Hör]. See also Chapter 4 in the text-book [Hör:xx.]

3.16 Theorem. *There exists a fixed integer* m^* *and for every* $N \geq 0$ *a constant* C_N *such that every* $u \in \mathcal{H}_N$ *can be represented inside* D *by an absolutely convergent integral*

$$u(x, y) = \int_S e^{i(x\zeta + yw)} d\mu(\zeta, w)$$

and the measure μ *satisfies*

$$\int_S [1 + \sqrt{\eta^2 + v^2}\,]^{m^* - N} \cdot e^{i\sqrt{\eta^2 + v^2}} \cdot |d\mu(\zeta, w)| \leq C_N \cdot ||u||_N$$

Remark. Hörmander's proof shows that the constants C_N have polynomial growth, i.e. there are constants A, B such that

$$C_N \leq A(1 + N)^B \quad : \; N = 1, 2, \ldots$$

3.17 Question. It would be interesting to determine the best possible m^* for which constants C_N as above exist for all N. For this question one need not insist that m^* is an integer.

3.18 Expansions by Hayman The analytic extension of a harmonic function $u(x, y)$ in D to the Levi ball can also be proved using the Poisson formula and expanding $u(x, y)$ into harmonic polynomials. This is done by Hayman in [xx] without any use complex analysis in several variables. Using Hayman's expansions it seems reasonable to get a good upper bound for m^* and perhaps even find the best possible choice.

4. Perron families.

Le Ω be a bounded open set in $\mathbf{C}$. No further assumptions are imposed, i.e. Ω need not be connected and its boundary can be "ugly". For example, it may have positive two-dimensional Lebesgue measure. On $\partial\Omega$ we have a function $\phi(x)$ which takes values between 0 and some $M > 0$. No other conditions are imposed, i.e. ϕ need not even be measurable. Given ϕ we denote by $\mathcal{P}(\phi)$ the family of all $u \in \mathrm{SH}^0(\Omega)$ for which

$$\limsup_{z \to w} u(z) \leq \phi(w) \quad : \quad w \in \partial\Omega$$

In Ω we get the function

$$H^*_\phi(z) = \max_{u \in \mathcal{P}(\phi)} u(z)$$

It is called Perron's maximal function of ϕ. With these notations one has

4.1 Theorem. *Perron's maximal function is harmonic in Ω.*

Proof. The constant functions 0 belons to $\mathcal{P}(\phi)$ and by the maximum principle we get $u(z) \leq M$ for every $u \in \mathcal{P}(\phi)$. It suffices to show that $H^*_\phi(z)$ is harmonic in a disc D inside Ω. When $u \in \mathcal{P}(\phi)$ its restriction to ∂D is a continuous function where we solve the Dirichlet problem and get the harmonic function u^* in D. The function defined as u^* in D and u in $\Omega \setminus D$ belongs to $\mathcal{P}(\phi)$. At the same time $u \leq u^*$ holds in D. We conclude from this that the values of H^*_ϕ inside D are obtained when we take u:s from the restricted class of $\mathcal{P}(\phi)$ which are harmonic and ≥ 0 in D. Denote this restricted class with $\mathcal{P}_*(\phi)$. So inside D we have

$$\text{(i)} \qquad H^*_\phi(z) = \max_{u \in \mathcal{P}_*(\phi)} u(z)$$

The harmonic functions in D which come from $\mathcal{P}_*(\phi)$ take values between 0 and M and hence their restrictions to D give a normal family of harmonic functions in D. See xx. Let a be the center of D. The normal family property yields a sequence $\{u_n\}$ in $\mathcal{P}_*(\phi)$ such that

$$\text{(ii)} \qquad H^*(a) = \lim_n u_n(a)$$

and $\{u_n\}$ converge uniformly to a harmonic function $U(z)$ in D. We claim that

$$\text{(iii)} \qquad H^*(z) = U(z) \quad : z \in D$$

For assume the contrary. To begin with, since $H^*(z)$ is the maximal function it is clear that $U(z) \leq H^*(z)$ holds in D. So if (iii) fails there exists

$$\text{(iv)} \qquad z_0 = a + re^{\theta_0} \in D \quad : \quad U(z_0) < H^*_\phi(z_0)$$

To see that (iv) cannot occur we regard the point z_0 and again use again the normal family to obtain a sequence $\{v_n\}$ in $\mathcal{P}_*(\phi)$ which converges uniformly to a harmonic function $V(z)$ in D where

$$\text{(v)} \qquad V(z_0) = H^*_\phi(z_0)$$

Now we can derive a contradiction from the supposed failure of (iii). Namely, in the whole of Ω we have the subharmonic function $w_n = \max(u_n, v_n)$ and taking its harmonic majorant inside D we get a new sequence $\{w^*_n\}$ in $\mathcal{P}^*_\phi$. Passing to a subsequence if necessary w^*_n converges to a limit function $W(z)$ which is harmonic in D and by construction it is $\geq$ both to U and to V. But from (ii) we have

$$\text{(vi)} \qquad U(a) = H^*_\phi(a) \implies U(a) = W(a)$$

At the same time $V \leq W$ and $V(z_0) = W(z_0) > U(z_0)$. Hence we get a strict inequality

$$\text{(vii)} \qquad \frac{1}{2\pi} \int_0^{2\pi} U(a + re^{i\theta}) \cdot d\theta < \frac{1}{2\pi} \int_0^{2\pi} W(a + re^{i\theta}) d\theta$$

But this contradicts the equality $U(a) = W(a)$ from (vi) since both terms in (vii) by the mean value equality express $U(a)$ and $W(a)$. Hence equality holds in (iii) and Theorem 4.1 is proved.

5. Maximum of several harmonic functions

Let $H_1, \ldots, H_k$ be a finite family of harmonic functions defined in some open set Ω. Put

$$u(x,y) = \max\{H_1(x,y), \ldots, H_k(x,y)\}$$

Since harmonic functions satisfy the mean-value condition it follows that the mean-value inequality holds for u and hence u is subharmonic. We are going to describe the non-negative measure $\Delta(u)$. To attain this we introduce the set

$$\Gamma = \bigcup_{i \neq \nu} \{H_i = H_\nu\}$$

Recall from XXX that the zero set of an arbitrary harmonic function consists of a union of smooth real analytic curves $\{\gamma_\alpha\}$ where each pair of these curves may interest in a discrete set and when it occurs this intersection is transveral. Since Γ is a finite union of zero sets of harmonic functions it enjoys the same description. There appears also the discrete set $\sigma(\Gamma)$ where at least two curves intersect. Now $\Gamma \setminus \sigma(\Gamma)$ is a disjoint union of smooth and connected real-analytic curves $\{\gamma_k\}$ called the *regular branches* of Γ. If Ω_0 is a relatively compact subset of Ω only finitely many regular branches intersect Ω_0.

Next, the open complement $\Omega \setminus \Gamma$ has connected components denoted by $\{\Omega_\alpha\}$. To every such component it is clear from the definition of u that there exists an integer $1 \leq i(\alpha) \leq k$ such that

$$u = H_{i(\alpha)} \quad \text{holds in} \quad \Omega_\alpha \tag{1}$$

Let us now consider a regular branch γ_k of Γ. It borders two connected components, say Ω_α and Ω_β. From (1) we get the pair $H_{i(\alpha)}$ and $H_{i(\beta)}$ where

$$H_{i(\alpha)} = H_{i(\beta)} \quad \text{holds on } \gamma_k$$

Since γ_k is a regular branch it follows that the two gradient vectors $\nabla(H_{i(\alpha)}$ and $\nabla)H_{i(\beta)}$ are not equal at any point on γ_k. Let $\mathfrak{n}_\alpha$ be the normal to γ_k which is directed into Ω_α. This means that

$$H_{i(\alpha)} > H_{i(\beta)} \quad \text{holds in} \quad \Omega_\alpha$$

and with this choice of $\mathfrak{n}_\alpha$ we have

$$\partial_{\mathfrak{n}_\alpha}(H_{i(\alpha)}) > \partial_{\mathfrak{n}_\alpha}(H_{i(\beta)}) \quad \text{on } \gamma_k$$

Thus, if ds is the arc-lengtth measure on γ_k we get the positive measure

$$\mu_{\gamma_k} = [\partial_{\mathfrak{n}_\alpha}(H_{i(\alpha)}) - \partial_{\mathfrak{n}_\alpha}(H_{i(\beta)})] \cdot ds \tag{*}$$

With these notations we have:

5.1 Proposition. *Along γ_k the measure $\Delta(u)$ is given by the positive density above.*

Exercise. Prove this result where the hint us to apply Stokes formula.

Proposition 5.1 gives us the following conclusive result:

5.2 Theorem. *The non-negative Riesz measure $\Delta(u)$ is equal to*

$$\sum \mu_{\gamma_k}$$

where the sum is taken over all regular branches of Γ.

Proof. By Propostion 5.1. there only remains to show that $\Delta(u)$ cannot contain a discrete part from discrete point masses in $\sigma(\Gamma)$. But this is clear for if $\Delta(u)$ contains a discrete measure $c \cdot \delta_p$ with $c \neq 0$ and $p \in \sigma(\Gamma)$ then the logarithimc potential of this point mass yields a discontinuous function while u from the start obviously is a continuos function.

5.3 Subharmonic configurations. Above we have clarified how a finite set of harmonic functions $H_1, \ldots, H_k$ yields a subharmonic maximum function. One may ask if this k-tuple can be used to construct other subharmonic functions u than the maximum function. Let us give

5.4 Definition. *A subharmonic configuration of $H_1, \ldots, H_k$ is a subharmonic function u in Ω whose Laplacian is supported by Γ and for every connected component Ω_α of of $\Omega \setminus \Gamma$ one has:*

$$(*) \qquad u = H_{i(\alpha)} \quad \text{for some} \quad 1 \leq i(\alpha) \leq k$$

Thus, when u is a subharmonic configuration then $\Omega \setminus \Gamma$ is covered by by a k-tuple of pairwise disjoint open sets $W_1, \ldots, W_k$ such that $u = H_i$ holds in W_i. Moreover, every W-set is a union of connected components of $\Omega \setminus \Gamma$.

An example. It turns out that there exist subharmonic configurations which which are not given by the maximum function. The following example is due to Borsea and Bögvad in [B-B]:

GIVE Example.

Local uniqueness.

The example above leads us to find conditions on the harmonic functions in order that they only admit the obvious subharmonic configuration. In the article [B-B-B] a local uniqueness result is proved which goes as follows: Let $p \in \Omega$ be a point such that k-tuple of gradient vectors $\{\nabla(H_i)(p)\}$ all are extreme points in the convex hull they generate in $\mathbf{R}^2$. Under this condition one has

5.5 Theorem. *Let u be a subharmonic configuration of $H_1, \ldots, H_k$ defined in some open neighborhood of p where all the H-functions are active, i.e. the closure of the open set where $u = H_i$ contains $\{p\}$ for each $1 \leq i \leq k$. Then*

$$u = \max(H_1, \ldots, H_k)$$

holds in a neighborhood of p.

5.6 A study of Cauchy transforms Let μ be a non-negative Riesz measure whose support is a compact null set K. Now we get the Cauchy transform

$$\mathcal{C}_\mu(z) = \int_K \frac{d\mu(\zeta)}{z - \zeta}$$

Let Ω be an open set which contains K and $g_1, \ldots, g_k$ is some k-tuple of holomorphic functions in Ω. We can impose the condition that for every connected component Ω_α of there exists $1 \leq i(\alpha) \leq k$ such that

$$(*) \qquad \mathcal{C}_\mu | \Omega_\alpha = g_{i(\alpha)}$$

Let us then consider the logarithmic potential

$$U_\mu(z) = \int_K \operatorname{Log}(|z - \zeta|) \cdot d\mu(\zeta)$$

It turns out that the subharmonic function U_μ is locally piecewise harmonic in Ω. To make this assertion more precise we work locally inside Ω and assume that Ω from the start is simply connected Then there exist primitive analytic functions $G_1, \ldots, G_k$ of the g-functions. The formulas from XXX show that (*) is equivalent to the condition that for every Ω_α there exists a constant c_α such that

$$(2) \qquad U_\mu(z) | \Omega_\alpha = \mathfrak{Re}(G_{i(\alpha)}) + c_\alpha$$

Here $\{H_i = \mathfrak{Re}(G_i)\}$ are harmonic functions in Ω. If the number of all constants $\{c_\alpha\}$ which appear above is finite it would follow that U_μ is piecewise harmonic with respect to a finite set of harmonic functions, i.e. given by the family $\{H_i\}$ plus eventual constants via (2) above. In this *favourable* case we get the similar result as in Theorem 5.2. In particular we conclude that the support of $\Delta(\mu)$ is a union of real-analytic γ-arcs. In [BV-B-B] the following affirmative result is proved without any initial assumption of the local finiteness of the c-constants.

5.7 Theorem. *When (*) holds above it follows that the logarthmic potential U_μ is locally piecewise harmonic and hence the support of $\Delta(\mu)$ consists of a locally finite union of real-analytic curves.*

Remark. The proof of this theorem is quite involved and we refer to [B-B-B] for details. The difficulty in the proof is to show that (*) implies that the number of c-constants from (**) is locally finite.

5.8 Cauchy transforms and algebraic functions. As above μ is a non-negative measure supported by a compact null set K in $\mathbf{C}$. The Cauchy transform $\mathcal{C}_\mu(z)$ is analytic in $\mathbf{C}\setminus K$. Suppose it satisfies an algebraic equation, i.e. there exists some $m \geq 1$ and polynomials $p_0(z), \ldots, p_m(z)$ such that

$$(*) \qquad p_m(z) \cdot \mathcal{C}_\mu^m(z) + \ldots p_1(z) \cdot \mathcal{C}_\mu(z) + p_0(z) = 0 \quad : \quad z \in \mathbf{C} \setminus K$$

Using Theorem 5.7 it is proved in [B-B-B] that (*) implies that K is a finite union of real-analytic curves which are related to roots of the algebraic equation

$$(**) \qquad p_m(z) \cdot y^m + \ldots p_1(z) \cdot y + p_0(z) = 0 \quad : \quad z \in \mathbf{C} \setminus K$$

5.9 Remark. The result in 5.8 is illustrated by examples from the article [Bergquist-Rullgård). Here asymptotic expansions for distributions of roots of eigenpolynomials which appear for a class of ODE-equations which extend the usual hypergeometric equation. The asymptotic distributions of roots are given by probability measures μ whose Cauchy transforms satisfy an algebraic equation of the form

$$\mathcal{C}_\mu^m(z) = \frac{1}{Q(z)}(*)$$

where $Q(z)$ is a monic polynomial of degree m with simple zeros $\alpha_1, \ldots, \alpha_k$. It is proved in [B-R] that there exists a *unique* probability measure μ with compact support whose Cauchy transform satisfies (*). Moreover, the support of μ is an analytic tree Γ, i.e. a connected set given by a finite union of real-analytic Jordan arcs which meet at some corner points. Moreover. $\mathbf{C} \setminus \Gamma$ is connected.

5.10 Question. It is not known if a similar result holds for a general algebraic equation. To begin with one may ask for which algebraic equations $p(z, y)$ there exists a probability measure μ whose Cauchy transform satisfies $p(z, \mathcal{C}_\mu(z)) = 0$. In addition there remains the question of uniqueness and also if the support of μ is an analytic tree. A specific family of algebraic equations which deserve to be investigated are of the form

$$(1) \qquad c_m y^m + \cdots c_2 y^2 + y = \frac{1}{z}$$

Here $m \geq 2$ and $c_m \neq 0$. The normalisation for the linear term y is necessary for the (eventual) existence of a probability measure μ such that $y = \mathcal{C}_\nu(z)$ satisfies (1). Let us remark that it is trivial to find a complex-valued and compactly supported measure μ whose Cauchy transform satisfies (1). Namely, at $z = \infty$ there exists the unique root function $\alpha(z)$ which satisfies (1) and has the series expansion

$$\alpha_*(z) = \frac{1}{z} + \sum_{k \geq 2} a_k z^{-k}$$

Next, set $p(y) = c_m y^m + \ldots + c_2 y^2 + y$ and let us assume that $f(z, y) = p(y) \cdot z$ is irreducible. Now α extends to a multiple-valued root function defined in $\mathbf{C} \setminus \Sigma$ where Σ is the union of $z = 0$ and the discriminant where multiple roots of the equation $p(y) - z = 0$ occur. We can choose some tree Γ whose set of corner points is Σ. Now the extended complex plane minus Γ is a simply connected domain Ω where we get a single-valued root function $\alpha \in \mathcal{O}(\Omega)$ which extends α_*. Now we find a complex-valued Riesz measure μ_Γ which is supported by Γ such that $\mathcal{C}_{\mu_\Gamma} = \alpha$ holds in Ω. Along a regular branch $\gamma \subset \Gamma$ we notice that μ is given by a density $(\alpha_k(z) - \alpha_j(z)) \cdot dz$ where $\alpha_k(z)$ and $\alpha_j(z)$ are two branches of the multiple valued root function which arrive from different sides of γ. So the problem is whether there exists some tree Γ such that the measure μ_Γ is real and positive. An (eventual) affirmative answer would be to solve a variational problem. The reason is that for any choice of Γ the Riesz measure μ_Γ has total variation ≥ 1 because when z approaches the point at infinity we have

$$\int_\Gamma \frac{d\mu_\Gamma(\zeta)}{z - \zeta} \simeq \frac{1}{z}$$

So in the family of compactly supported Riesz measures which represent the single-valued rot function α above we seek a measure with smallest total variation and expect that it is a probability measure, i.e. real-valued and non-negative. The next question is if such a probability measure is unique and if the support is a tree. Some affirmative answers to this problem have been found via numerical investigations of eigenpolynomials to ODE-equations in the article [Berqvist].

6. On zero sets of subharmonic functions.

Let Ω in $\mathbf{C}$ be a bounded open set and denote by $\mathrm{SH}_0(\Omega)$ the set of subharmonic functions in Ω whose Laplacian is a Riesz measure supported by a compact null set. Every such function v is locally a logarithmic potential of $\Delta(V)$ plus a harmonic function and can therefore be taken to be upper semi-continuous. Moreover, the distribution derivatives $\partial V/\partial x$ and $\partial V/\partial y$ belong to $L^1_{\mathrm{loc}}(\Omega)$. Before we announce Theorem XX below we introduce a geometric construction. If U is an open subset of Ω we construct its forward star-domain as follows: To each $\zeta \in U$ we find the largest $s(\zeta) > 0$ such that the line segment

$$\ell_\zeta(s(\zeta)) = \{\zeta + x :\ 0 \leq x < s(\zeta)\} \subset \Omega$$

Now we put

$$(*) \qquad \mathfrak{s}(U) = \bigcup_{\zeta \in U} \ell_\zeta(s(\zeta))$$

and refer to this open set as the forward star domain of U.

Theorem. *Let $V \in SH_0(\Omega)$ and put $K = Supp(\Delta(V))$. Suppose that $V = 0$ in an open subset U of $\Omega \setminus K$ and furthermore*

$$(*) \qquad \partial V/\partial x(z) < 0, \quad \text{holds in} \quad \Omega \setminus (K \cup U)\,.$$

Then $V = 0$ in $\mathfrak{s}(U)$.

Proof. It is clear that it suffices to show the following: Let $z_0 \in U$ and consider a horisontal line segment

$$\ell = \{z = z_0 + s :\ 0 \leq s \leq s_0\}$$

which is contained in Ω. Then, if $0 < \delta < \mathrm{dist}(\ell, \partial\Omega)$ and the open disc $D_\delta(z_0)$ of radius δ centered at z_0 is contained in U, it follows that V vanishes in the open set

$$(1) \qquad \{z :\ \mathrm{dist}(z, \ell) < \delta\}$$

Notice that (1) is a relatively compact subset of Ω. Consider the complex derivative

$$\partial V/\partial z = \frac{1}{2}(\partial V/\partial x - i\partial V/\partial y)$$

This yields a complex-valued and locally integrable function and since $V = 0$ in $D_\delta(z_0)$ it is clear that $V = 0$ in the open set from (1) if we prove that $\partial V/\partial z = 0$ holds almost everywhere in (1) To prove this we take some $\epsilon > 0$ and put

$$(2) \qquad \Psi_\epsilon(z) = \log\left(\partial V/\partial z - \epsilon\right)$$

where the single-valued branch of the complex Log-function is chosen so that

$$(3) \qquad \pi/2 < \mathfrak{Im}\,\Psi_\epsilon < 3\pi/2$$

Hence we can write

$$(4) \qquad \Psi_\epsilon(z) = \mathrm{Log}|\epsilon - \partial V/\partial z| + i\tau(z) \quad : \quad \pi/2 < \tau(z) < 3\pi/2$$

A regularisation. Choose a non-negative test-function ϕ with compact support in $|z| \leq \delta$ while $\phi(z) > 0$ if $|z| < \delta$ and $\iint \phi(z)dxdy = 1$. We construct the convolution $\sigma * \Psi_\epsilon$ which is defined in the subset of Ω whose points have distance $> \delta$ to $\partial\Omega$. Rules for first order derivations of a convolution give

$$\bar{\partial}/\bar{\partial}\bar{z}\big(\phi * \Psi_\epsilon\big) = \frac{\phi * \bar{\partial}\partial(V)}{\partial V/\partial z) - \epsilon} = \frac{1}{4} \cdot \frac{1}{\partial V/\partial z - \epsilon} \cdot \phi * \Delta(V)$$

Taking the real part we get

$$(5) \qquad \mathfrak{Re}(\bar{\partial}/\bar{\partial}\bar{z}(\phi * \Psi_\epsilon)) = \frac{\partial V/\partial x - \epsilon}{4|\epsilon - \partial V/\partial z|^2} \cdot \phi * \Delta(V)$$

To simplify notations we set

(6) $$\sigma(z) = \text{Log}|\epsilon - \partial V/\partial z|$$

The definition of the $\bar{\partial}$-derivative and the decomposition $\Psi_\epsilon = \sigma + i \cdot \tau$ together with the inequality (5) give

(7) $$\partial_x(\phi * \sigma) \leq \partial_y(\phi * \tau)$$

In the right hand side we use the partial y-derivative on ϕ, i.e. we use the general formula:

$$\partial_y(\phi * \tau) = \partial_y(\phi) * \tau$$

Since $\pi/2 \leq \tau \leq 3\pi/2$ the absolute value of this function is majorized by

(8) $$M = \frac{3\pi}{2} \cdot ||\partial_y(\phi)||_1$$

where $||\partial_y(\phi)||_1$ denotes the L^1-norm. Next, consider the function $s \mapsto \phi * \sigma(z_0 + s)$ where $0 \leq s \leq s_0$ whose s-derivative becomes $\partial_x(\phi * \sigma)(z + s)$. Hence (7-8) give:

$$\frac{d}{ds}(\phi * \sigma(z + s)) \leq M$$

(9) $$\implies \phi * \sigma(z_0 + s_0) \leq \phi * \sigma(z_0) + M \cdot s_0$$

From now on $\epsilon < 1$ so that $\log \epsilon < 0$. Since $V = 0$ in $D_\delta(z_0)$ we also have $\partial V/\partial z = 0$ in this disc and conclude that

(10) $$\phi * \sigma(z_0) = \log \epsilon$$

Next, we have

$$\sigma = \log \epsilon + \log |1 - \frac{\partial V/\partial z}{\epsilon}|$$

Put

(11) $$f_\epsilon = \log |1 - \frac{\partial V/\partial z}{\epsilon}|$$

Then (9-10) give the inequality

(12) $$\phi * f_\epsilon(z_0 + s_0) \leq M \cdot s_0$$

So from the construction of a convolution this means that

(13) $$\iint_{|\zeta| \leq \delta} f_\epsilon(z_0 + s_0 + \zeta) \cdot \phi(\zeta) \cdot d\xi d\eta \leq M \cdot s_0$$

Next, we can write

(14) $$f_\epsilon(z) = \frac{1}{2} \cdot \log \big((1 - \frac{\partial V/\partial x}{\epsilon})^2 + (\frac{\partial V/\partial y}{\epsilon})^2\big)$$

Since $\partial V/\partial x \leq 0$ holds almost everywhere we have $f \geq 0$ almost everywhere and at each point z where $\partial V/\partial z \neq 0$ we have

(15) $$\lim_{\epsilon \to 0} f_\epsilon(z) = +\infty$$

Since (13 holds for each $\epsilon > 0$ and $\phi > 0$ in the open disc $|\zeta| < \delta$ we conclude that $\partial V/\partial z = 0$ must hold almost everywhere in the disc $D_\delta(z_0 + s_0)$. Here we considered the largest s-value along the horisontal line ℓ. Of course, we get a similar conclusion for each $0 < s < s_0$ and hence $\partial V/\partial z = 0$ holds almost everywhere in the open set (1). At the same time $V = 0$ in $D_\delta(z_0)$ and we conclude that $V = 0$ in the open set from (1) as requested.

Remark. The method used in the proof above is due to Bergqvist-Rullgård in [Be-Ru] where the Theorem was proved under the assumption that the range of V is a finite set. But the essential idea to regard the complex log-function from (2) in the proof and employ regularisations already occur in [Be-Ru].

Chapter VI: Conformal mappings.

Introduction.

The central result is Riemann's mapping theorem which asserts that every simply connected open subset Ω of $\mathbf{C}$ which is not equal to the whole complex plane is conformal with the open unit disc D. More precisely, for every point $z_0 \in \Omega$ there exists a unique $f \in \mathcal{O}(\Omega)$ where $f(z_0) = 0$ and $f'(z_0)$ is real and positive such that

(*) $$f\colon \Omega \to D$$

is bijective and the complex derivative $f'(z) \neq 0$ for all $z \in \Omega$. The conformal map is the solution to a variational problem. Namely, consider the family $\mathcal{C}(\Omega, z_0)$ of all analytic functions g in Ω for which $g(\Omega) \subset D$ and $g(z_0) = 0$. Then one seeks

$$\max_{g \in \mathcal{C}(\Omega, z_0)} |g'(z_0)|$$

The extremal function to this variational problem gives the conformal map (*). This means that for every g-function as above one has the inequality

(**) $$|g'(z_0)| \leq f'(z_0)$$

and equality holds if and only if $g(z) = e^{i\theta} \cdot f(z)$ for some $0 \leq \theta \leq 2\pi$. The mapping theorem is actually not so general. The reason is that if Ω is bounded and simply connected then it can be exhausted by an increasing sequence of Jordan domains, i.e. by a sequence of closed Jordan curves $\{\Gamma_\nu\}$ which all are contained in Ω and the Jordan domains $\{\Omega_\nu\}$ bounded by these Γ-curves form an increasing sequence of relatively compact regions of Ω. If $z_0 \in \Omega$ is given we can start with some Γ_1 so that z_0 lies in its interior. Suppose we have found Riemann's mapping function f_ν from Ω_ν into D where $f_\nu(z_0) = 0$. Now $\{f_\nu\}$ is a normal family of analytic functions and there exists $f \in \mathcal{O}(\Omega)$ such that $\{f_\nu\}$ converges uniformly to f on every compact subset of Ω. It follows easily via an application of Rouche's theorem that f is the mapping function for Ω. Thus, Riemann's mapping theorem is essentially a result about conformal mappings from a Jordan domain to the unit disc. Moreover, during the exhaustion it suffices to consider Jordan curves which are piecewise linear. For such simply connected polygons Schwarz and Cristoffell independently gave a formula for the mapping function in 1868 which is exposed in a separate section.

Examples. Above we did not add the point at infinity which also appears in certain conformal mappings. An example is the function defined by

$$f(z) = z + \frac{1}{z}$$

When $z = e^{i\theta}$ moves on the unit circle we see that

$$f(e^{i\theta}) = 2 \cdot \cos\theta$$

whose range is the real interval $[-2,2]$ Here f has a simple pole at $z = 0$. But introducing the point at infinity we can regard f as an analytic map from the open unit disc D into $\Sigma \setminus [-2,2]$ where $\Sigma = \mathbf{C} \cup \infty$ and $f(0) = \infty$. It is easily seen that the mapping is one-to-one and $f(D) = \Sigma \setminus [-2,2]$. So this is an example of a conformal mapping Riemann's sense. The restriction of f to T yields a double cover onto $[-2,2]$ which reflects the geometric fact that this real segment as a boundary of the domain $\Sigma \setminus [-2,2]$ has two sides, i.e. one can approach a point $-2 < x < 2$ from the above with $z = x + i\epsilon$ and also from below by $z = x - i\epsilon$. As we shall see in Section 7 the conformal mapping is an example of a map from D onto a parallell slit region. The text-book [Nehari] contains many examples of conformal mappings and they occur frequently in classical hydro-mechanics where laws for ideal fluids without viscosity imply that streamlines are found via conformal mappings. The physically oriented reader should consult the eminent text-book *Hydromechnaics* by Horace Lamb for such examples.

Koebe's mapping function. The extension of Riemann'smapping theorem to multiply connected domains and more generally to Riemann surfaces leads to the uniformisation theorem where one foremost should mention contributions by Poincaré and Koebe. Here we shall not prove the uniformisation theorem in its full generality but restrict to the case of multiply connected domains. Koebe's proof of the uniformisation theorem employs a special conformal mapping defined by the function

$$k(z) = \frac{z}{(1-z)^2}$$

We leave it as an exercise to show that $k(z)$ is 1-1 on the open unit disc and hence it gives a conformal mapping from D onto a simply connected set which is unbounded since $k(x) \to +\infty$ as x is real and tends to 1. The Taylor series of k is special, i.e. we find that

$$k(z) = z + 2z^2 + 3z^3 + \ldots$$

A famous conjecture was posed by Bieberbach in 1916 which asserts that if

$$f(z) = z + \sum_{n=2}^{\infty} a_n \cdot z^n$$

yields a conformal mapping from D onto some simply connected domain, then the coefficients $\{a_n\}$ must satisfy

$$|a_n| \leq n \quad \text{for all} \quad n = 2, 3, \ldots$$

The affirmative answer to the Bieberbach conjecture was established by de Branges in 1985. The interested reader can consult [Bieberbachbook] for background and history how the proof was finally finished. In addition to Bieberbach's article [Bieb:Acta] we refer to Chapter XX in [Krantz] which gives a proof with all details.

Baernstein's inequality. Koebe's function enjoys other extremal properties. The following remarkable result was proved in [Acta:A. Baerenstein] in 1977:

Theorem. *For every conformal mapping function $f(z)$ where $f(0) = 0$ and $f'(0) = 1$ one has the inequality*

$$\int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta \leq \int_0^{2\pi} |k(re^{i\theta})| \cdot d\theta \leq \quad \text{for each} \quad 0 < r < 1$$

We shall not try to enter the proof which would require an extensive excursion into variational techniques in analytic function theory. The interested reader can consult the text-book *Univalent functions* by P.I. Duren which treats extremal problems and other interesting material in connection with conformal mappings. Concerning the mapping defined by k it is instructive to use a computer and plot closed Jordan curves which are images of circles $|z| = r$ when $0 < r < 1$.

0.1 The equilibrium potential on plane curves. Riemann's mapping theorem for Jordan domains was predicted in electric engineering. Namely, let Γ be a closed Jordan curve of class C^1. Then one seeks a positive density function μ on Γ such that the logarithmic potential

$$\int_\Gamma \log \frac{1}{|z-\zeta|} \cdot \mu(\zeta) \cdot |d\zeta|$$

is constant as z varies in Γ. The existence of μ was expected in electric engineering since μ corresponds to an equilibrium density for an electric field. It turns out that μ is found via the conformal mapping function f from the exterior domain bordered by Γ onto the exterior of the unit disc. The following is proved in Section 9:

$$(*) \qquad \mu(\zeta) = \frac{1}{|f'(\zeta)|}$$

0.2 Solution by Green's functions. Conformal mappings can be found via solutions to Dirichlet's problem. If Ω is a bounded Jordan domain and $z_0 \in \Omega$ there exists the unique harmonic function $H(z)$ in Ω with boundary function $\log|z - z_0|$. Since Ω is simply connected H has a unique harmonic conjugate V in Ω normalized so that $V(z_0) = 0$. Set

$$f(z) = (z - z_0)e^{-(H(z)+iV(z))}$$

Then f gives the conformal mapping from Ω onto D. We prove this result in Section XX.

0.3 Beurling's mapping theorem. The formula (*) in (0.1) suggests that if g is a conformal map from the unit disc D onto a simply connected domain Ω then there is a close interplay between the absolute value of $|g'(z)|$ and values of $g(z)$ when $z \to \partial D$. A result about this correspondence was proved by Beurling in the article [Beurling] where the simply connected domain is not given in advance. Instead one starts from a positive and bounded continuous function $\Phi(w)$ defined in the whole complex plane such that $\text{Log}\,\frac{1}{|\Phi|}$ is subharmonic. Beurling proved that there exists a unique analytic function f in D with $f(0) = 0$ which yields a conformal map from D onto $f(D)$, and satisfies the limit formula:

$$(*) \qquad \lim_{r\to 1}\ \max{}_{0\le\theta\le 2\pi}\ \left[\left|f'(re^{i\theta})\right| - \Phi(f(re^{i\theta})\right] = 0$$

The proof is quite involved but it is rewarding to pursue since it gives an instructive lesson how problems in analytic function theory can be solved using calculus of variation.

0.4 The uniformisation theorem. Let Ω be a bounded and connected domain. But here we suppose that it is not simply connected. Then we cannot find a conformal map from Ω onto D. To compensate for this one constructs a *multi-valued* function on Ω which gives a substitute for Riemann's mapping theorem in the simply connected case. More precisely, let $z_0 \in \Omega$ be a given point. In Chapter 4 we defined the family $M\mathcal{O}(\Omega)(z_0)$ of germs of analytic functions at z_0 which extend to multi-valued functions on Ω. When f is such a germ and $z \in \Omega$ we denote by set $f^*(z)$ the set of values taken by all the local branches of f at z. We say that a germ $f \in M\mathcal{O}(\Omega)(z_0)$ gives a *multi-valued and locally conformal map* F from Ω onto D if the following hold:

$$\text{(i)} \qquad f^*(z_1) \cap f^*(z_2) = \emptyset \quad \text{for all pairs } z_1 \neq z_2 \text{ in } \Omega$$

$$\text{(ii)} \qquad \cup_{z\in\Omega}\, f^*(z) = D$$

In section 1.B we prove that for each point z_0 in Ω there exists a unique germ f as above which is normalised so that $f'(z_0)$ is real and positive and $f(z_0) = 0$. Moreover, f solves an extremal problem. Namely, for every $g \in M\mathcal{O}(\Omega)(z_0)$ such that $g(0) = 0$ and $\cup_{z\in\Omega}\, g^*(z) \subset D$ one has:

$$\text{(iii)} \qquad |g'(z_0)| \leq f'(z_0)$$

Remark. This result is called the *uniformisation theorem* for bounded and connected subsets of **C**. We shall say more about this in Section XX where the inverse function of f is constructed used to identify the fundamental group of Ω with a group of Möbius transforms on the unit disc.

0.5 Picard's theorem. In 1879 Picard found the affirmative answer to a question posed by Weierstrass. The result is that an entire function $f(z)$ whose range excludes two values must be a constant. Picard's proof used the modular function. A proof which does not use the modular function was found by Emile Borel in 1895. Inspired by Borel's methods, Shottky and Landau established extensions of Picard's theorem. For example, let a_0 and a_1 be two complex numbers where a_0 is $\neq 0$ and $\neq 1$. Then there exists a constant $C(a_0, a_1)$ which depends on this pair only such that the following hold:

If $f(z)$ is an analytic function in some disc D_R of radius R centered at the origin where the range $f(D_R)$ does not contain the two values 0 and 1 and the Taylor series at $z = 0$ starts with $a_0 + a_1 + \ldots$, i.e. $f(0) = a_0$ and $f'(0) = a_1$, then

$$R \leq C(a_0, a_1)\,.$$

Remark. Using the modular function we shall find the best possible constant $C(a_0, a_1)$ and various extensions of the Landau-Schottky theorem is discussed in Section 3.

0.6 Extension to the Boundary Let Ω be bounded and simply connected set and f is a conformal map from Ω onto the open unit disc D. With no conditions on $\partial\Omega$ one cannot expect a continuous extension of f. But for points p on $\partial\Omega$ which can be reached by a Jordan arc which except for p stays inside Ω there exist certain limits. Such boundary points are called *accessible* and we denote this set by $\mathcal{A}(\partial\Omega)$. We will prove that $\mathcal{A}(\partial\Omega)$ is dense in $\partial\Omega$. Next, for each $p \in \mathcal{A}(\partial\Omega)$ we denote by $\mathcal{J}(p)$ the family of Jordan arcs J where p is an end-point while the remaining part of J stays in Ω. So if $t \mapsto \gamma(t)$ defines J then $\gamma(t) \in \Omega$ when $0 \leq t < 1$ and $\gamma_J(1) = p$. The image of $J \setminus \{p\}$ under f is a half-open Jordan arc in D defined by:

$$\text{(i)} \qquad t \mapsto \gamma^*(t) \quad : \quad f(\gamma(t)) = \gamma^*(t) \quad : 0 \leq t < 1$$

With these notations one has:

0.6.1 Koebe's limit theorem. *For every $J \in \mathcal{J}(p)$ there exists the limit*

$$\lim_{t\to 1} \gamma^*(t))$$

The limit above is a point on the unit circle which depends on the pair (p, J). It is denoted by $\mathcal{K}(J, p)$ and called the Koebe limit attached to J. The second major result is due to Lindelöf.

0.6.2 Lindelöf's separation theorem. *Let $p \neq q$ be distinct points on $\partial\Omega$. Then*

$$\mathcal{K}(J, p) \neq \mathcal{K}(J', q) \quad : \quad J \in \mathcal{J}(p) \quad : J' \in \mathcal{J}(q)$$

Next, let $p \in \partial\Omega$ and consider some $J \in \mathcal{J}(p)$. Let $\{J_\nu \in \mathcal{J}(q_\nu)\}$ be a sequence of Jordan arcs with end points $\{q_\nu\}$.

0.6.3 Definition. *We say that the sequence $\{(J_\nu, q_\nu)\}$ converges to (J, p) if $q_\nu \to p$ and for every $\epsilon > 0$ there exists some ν^* such that whenever $\nu \geq \nu^*$ there exists a Jordan arc γ_ν which is contained in the disc $D_\epsilon(p)$ and has endpoints on J and J_ν.*

0.6.4 Koebe's Continuity Theorem. *If $(J_\nu, q_\nu) \to (J,p)$ it follows that*

$$\lim_{nu \to \infty} \mathcal{K}(J_\nu, q_\nu) = \mathcal{K}(J,p)$$

Next, consider a pair J_1, J_2 in $\mathcal{J}(p)$. The two Jordan arcs are *asymptotically linked* if there to every $\epsilon > 0$ exists a Jordan arc $\gamma_\epsilon \subset D_\epsilon(p)$ whose end-points belong to J_1 and J_2 respectively. Koebe's continuity theorem entails

0.6.5 Proposition. *Let J_1, J_2 be a pair in $\mathcal{J}(p)$ which are asymptotically linked. Then one has the equality:*

$$\mathcal{K}(J_1,p) = \mathcal{K}(J_2,p)$$

The results above show that if every pair of Jordan arcs in $\mathcal{J}(p)$ are asymptotically linked, then there exists a Koebe limit $\mathcal{K}_f(p)$ defined as the common value of $\mathcal{K}(J,p)$. Let $\mathcal{A}_*(\Omega)$ denote the set of such boundary points. Together with Lindelöf's separation theorem this yields an injective map from $\mathcal{A}_*(\Omega)$ into the unit circle T defined by

$$(*) \qquad p \mapsto \mathcal{K}_f(p)$$

0.6.6 The case of Jordan domains.

Suppose now that Ω is a Jordan domain. Here $\partial\Omega$ is a closed Jordan curve and a famous result due to Camille Jordan asserts that every boundary point is accessible. Moreover, a sharpended version of Camille Jordan's theorem was proved by von Schoenflies in [vScH].

0.6.7 Schoenflies' theorem. *Let Ω be a Jordan domain. For each boundary point p the family of all Jordan arcs is in $\mathcal{J}(p)$ are asymptotically linked and if $J \in \mathcal{J}(p)$ and $\{(J_\nu, q_\nu)\}$ is a sequence such that $q_\nu \to p$, then the sequence (J_ν, q_ν) converges to (J,p) in the sense of Definition 0.6.3.*

0.6.8 Conclusion. Koebe's Continuity Lemma and Schoenflies' result show that when Ω is a Jordan domain then the Koebe-Lindelöf map (*) above is a continuous and bijective map from $\partial\Omega$ into T. Since both $\partial\Omega$ and T are closed Jordan curves this implies that $\mathcal{K}^*$ must be surjective. Finally, a continuous and bijective map between two compact sets has a continuous inverse. Hence the mapping function f yields a homeomorphism from T onto $\partial\Omega$ and as one easily sees f actually extends to a continuous map from the closed unit disc D onto the closed Jordan domain $\bar{\Omega}$. This result for Jordan domains is sometimes attributed to Caratheodory but all essential steps in the proof rely upon the results above due to Jordan, von Schoenfliess, Koebe and Lindelöf. Caratheodory's contribution in this subject is foremost his elegant proof of the uniformisation theorem for arbitrary connected domains and certain topological constructions adapted to the Koebe-Lindelöf map (*) for an arbitrary simply connected domain.

0.6.9 Smooth boundary points. Let Ω be a Jordan domain and f a conformal map from D onto Ω. So now we have a homeomorphism

$$e^{i\theta} \mapsto f(e^{i\theta}) \quad : \quad 0 \leq \theta \leq 2\pi$$

The closed Jordan curve $\partial\Omega$ has a parametrisation $t \mapsto \gamma(t)$ which may have extra regularity. A result in the pointwise differentiable case was presented by Lindelöf at the Scandinavian Congress of Mathematics held at Institute Mittag-Leffler in 1916. It goes as follows:

Let $p \in \partial\Omega$ and assume that $\partial\Omega$ has a tangent line at p. We can take $p = 1$ and after a rotation in the complex w-plane also assume that the vertical line $\mathfrak{Re}(w) = 1$ is tangent to $\partial\Omega$ at p. This means that we can choose a parametrisation $\gamma(t)$ where we may translate t and assume that γ is defined on some interval $[-A, A]$ with $\gamma(-A) = \gamma(A)$ while $p = \gamma(0)$ and the differentiable assumption means that there exists some positive real number a such that

$$(*) \qquad \gamma(t) = 1 + iat + \text{small ordo}\,(t) \quad : \; t \to 0$$

Let $g(z)$ be the conformal mapping from D onto Ω normalised so that $g(1) = 1$. When θ is close to zero we get a real-valued function $\beta(\theta)$ such that

$$(**) \qquad g(e^{i\theta}) = \gamma(\beta(\theta))$$

We already know that the β-function is continuous. But when (*) is added Lindelöf proved the following result:

0.6.10 Theorem. *The β-function has an ordinary derivative at $\theta = 0$.*

Moreover, Lindelöf proved that f is locally conformal up to the boundary at $z = 1$:

0.6.11 Theorem. *Assume (*) above. Then there exists a positive constant B such that the following limits exist and are equal:*

$$\text{(i)} \qquad \lim_{\theta \to 0} \frac{f(e^{i\theta}) - 1}{i\theta} = B$$

$$\text{(ii)} \qquad \lim_{s \to 0} \frac{1 - f(1 - se^{i\alpha})}{s} = B \cdot e^{i\alpha} \quad : \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

0..6.12 Remark. Caratheodory proved later the conformal property holds up to the boundary in the more general case when p is a corner point, i.e. when (*) is replaced by the weaker assumption that the γ-function has one-sided derivatives as t decreases or increases to 0. For a full account of various results due to Lindelöf and Caratheodory we refer to Lindelöf's article in [XX]. Here we are content to give a proof of Theorem 0.6.10 in XXX. Regularity for higher order derivatives was studied by Painlavé in [Pain] (when???) who proved that if Ω is a Jordan domain whose boundary curve is of class C^∞, then a conformal map from Ω onto D extends to a C^∞-function from the closure of Ω onto the closed unit disc.

0.6.13 Properties of the inverse map f^{-1}.

Above we discussed the boundary behaviour of the conformal map from a bounded simply connected domain Ω onto the open unit disc. Let us instead consider the inverse function $\phi = f^{-1}$, i.e. now ϕ is a conformal map from D onto Ω. The Brothers Riesz theorem shows that ϕ has radial limits almost everywhere and. Actually the failure of radial limits is confined to a smaller set because the area integral

$$\iint_D |\phi'(z)|^2 \dot{d}xdy < \infty$$

We can therefore apply a result due to Beurling from XX from Special Topics which shows that radial limits exist outside a subset of T whose outer logarithmic capacity is zero. One can continue to analyze further regularity of ϕ. First we introduce the following:

0.6.14 Angular derivatives at the boundary. Consider some $e^{i\theta} \in T$ if ϕ where the radial limit exists:

$$(1) \qquad \lim_{r \to 1} \phi(re^{i\theta}) = p$$

here the limit is a point $p \in \partial\Omega$. Results about Fatou limits for bounded analytic functions imply that ϕ has a limit in every Fatou sector having $e^{i\theta}$ as a corner point. Consider the argument of the difference quotients

$$(2) \qquad z \mapsto \arg\left(\frac{\phi(e^{i\theta}) - \phi(z)}{e^{i\theta} - z}\right)$$

If (2) has a limit when $z \to 1$ in the sense of Fatou we say that ϕ is conformal at $e^{i\theta}$. For example, if $\theta = 0$ so that $e^{i\theta} = 1$ we require that for ever $0 < \delta < \pi/2$ there exists the limit

$$\lim_{s\to 0}\arg\big[\frac{\phi(1)-\phi(1-s\cdot e^{i\alpha})}{s\cdot e^{i\alpha}}\big] \tag{3}$$

uniformly with respect to α provided that $\pi/2-\delta \leq \alpha \leq \pi/2-\delta$. We can also consider the complex differences without introducing their arguments and say that ϕ has an *angular derivative* at $e^{i\theta}$ when

$$z \mapsto \frac{\phi(e^{i\theta})-\phi(z)}{e^{i\theta}-z} \tag{4}$$

has a Fatou limit as $z \to e^{i\theta}$. The limit is then called the angular derivative of ϕ and denoted by $\phi'(e^{i\theta})$. If $\phi'(e^{i\theta}) \neq 0$ it is clear that the limit in (2) also holds, i.e. the existence of a non-zero angular derivative implies that ϕ is conformal at $e^{i\theta}$. Results about the existence of angular derivatives are due to Ostrowski in his article [Ost] from 1937. Here it would take us to far to describe this in detail. Instead we refer the reader to the text-book [Marshall-Garnett where Chapter V.5 contains a detailed account of Ostrowski's results . Roughly speaking these results about angular derivatives and conformality at boundary points are expressed via geometric conditions of Ω and a boundary point p which to begin with can be reached from a radial limit (1) above. In [loc.cit] the reader also finds an exposition of more recent results which give conditions for the existence of angular derivatives using various subtle metric conditions which rely upon *extremal metrics*.

Remark. See also the article *Angular derivatives and Lipschitz majorants* by D.E. Marshall, available on-line at http:// math.washington.edu/- marshall/personal. html.

0.7 Comments on other sections.

In section 4 we prove Koebe's *"One Quarter Theorem"* together with the *Area Theorem* and Koebe's *Verzerrungssatz*. Section 5 is devoted to the construction of conformal maps from the unit disc onto convex polygons. In section 7 we construct conformal maps between multiple connected domains using certain harmonic functions and section 8 is devoted to the *Bergman kernel.*where we prove a result due to Carleman about the asymptotic behaviour of the kernel function with a sharp remained term for Jordan domains whose boundary curve is real-analytic. In Section 10 we prove a result due to Koebe about conformal mappings between domains which are bordered by circles. The final section 11 is devoted to Beurling's mapping theorem which was already described in 0.3.

I:A. Riemann's mapping theorem for simply connected domains

The family of simply connected domains were described by Hermann Schwartz in the article [Schwarz] from 1869.

The following are equivalent for an open subset Ω *of* $\mathbf{C}$:

(1) $$\mathbf{C} \setminus \Omega \text{ has no compact connected components}$$

(2) $$\text{For every closed curve } \gamma \text{ inside } \Omega \implies \mathfrak{w}_\gamma(a) = 0 \quad : a \in \mathbf{C} \setminus \Omega$$

(3) $$\text{Any } f \in \mathcal{O}(\Omega) \text{ has a primitive}$$

(4) $$\text{If } f \in \mathcal{O}(\Omega), \text{ has no zeros there exists } g \in \mathcal{O}(\Omega) \quad : f = e^g$$

(5) $$\text{If } f \in \mathcal{O}(\Omega), \text{ has no zeros there exists } h \in \mathcal{O}(\Omega) \quad : f = h^2$$

Remark. The reader should contemplate upon these conditions and prove they are equivalent. See also [Nah: page 151-153] which gives an instructive and detailed proof of the equivalence between (1-5) above. When Ω satisfies the equivalent conditions above we say that it is simply connected. The case $\Omega = \mathbf{C}$ is excluded. A punctured complex plane $\mathbf{C} \setminus \{a\}$ which arises when a single point is removed is obviously not simply connected. So when $\Omega \neq \mathbf{C}$ is simply connected then the boundary $\partial\Omega$ contains at least two points. This will be used below and we remark that the subsequent proof of Riemann's mapping theorem is due to Fejér and F. Riesz.

A.1 Reduction to bounded domains. Let $\Omega \neq \mathbf{C}$ be simply connected. Above we have seen that there exists two distinct points a and b in its closed complement. Consider the function

(i) $$w(z) = \sqrt{\frac{z-a}{z-b}}$$

From (5) in the list by Schwarz there exists a *single valued* branch of this root function which we denote by $w^*(z)$, i.e. here $w^* \in \mathcal{O}(\Omega)$. Let $z_0 \in \Omega$ and put $w^*(z_0) = c$. Since we have chosen a branch of the square root function it follow that $w^*(z)$ never attains the value $-c$. Moreover, since the complex derivative of w^* is $\neq 0$ there exists an open disc Δ centered at c which is disjoint from the disc $-\delta$ centered at $-c$, and an open neighborhood U of z_0 such that $w^* \colon U \to \delta$ is biholomorphic. It follows that the image $w^*(\Omega)$ has empty intersection with $-\Delta$. So if r is the radius of Δ we conclude that

$$|w^*(z) + c| \geq r \quad \text{for all} \quad z \in \Omega$$

We conclude that $\frac{1}{w(z)-c}$ yields a conformal map from Ω onto a bounded simply connected set U. Since the composition of two conformal maps is conformal there only remains only to prove Riemanns Mapping Theorem for U. So from now on we assume that Ω is bounded and proceed with the proof.

A.2 Proof of uniqueness. Let us prove that the mapping function is *unique* if it exists. For suppose that f and g are two conformal mappings from Ω onto D where $f(z_0) = g(z_0) = 0$ and both $f'(z_0)$ and $g'(z_0)$ are real and positive. Now there exists the inverse mapping function g^{-1} from D onto Ω and we set

$$\phi = f \circ g^{-1}$$

Then ϕ yields a conformal mapping of D onto itself where $\phi(0) = 0$. As explained in XX it follows that $\phi(z) = az$ for a constant a with $|a| = 1$. Here

$$a = \phi'(0) = \frac{f'(z_0)}{g'(z_0)}$$

Since both $f'(0)$ and $g'(0)$ are real and positive we get $a = 1$. So $\phi(z) = z$ is the identity map and $f = g$ follows.

A.3 Proof of Existence. Set

(i) $$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \quad : f(z_0) = 0 \quad f(\Omega) \subset D \quad : f \text{ is 1-1}\}$$

Thus, each $f \in \mathcal{F}$ gives a conformal map from Ω into some open subset of D. There remains to find some f such that $f(\Omega) = D$. To attain this we put

(ii) $$M = \max_{f \in \mathcal{F}} |f'(z_0)|$$

Here M is finite since there exists $r > 0$ such that the disc $D_r(z_0) \subset \Omega$ and Schwarz' inequality gives $|f'(z_0)| \leq \frac{1}{r}$ for each $f \in \mathcal{F}$. Next, by the Montel Theorem in XXX, the family $\mathcal{F}$ is *normal* in $\mathcal{O}(\Omega)$. Hence we can find $f \in \mathcal{F}$ such that $|f'(z_0)| = M$. Multiplying f with some $e^{i\theta}$ we may assume that $f'(z_0) = M$. There remains to show that $f(\Omega) = D$. Assume the contrary, i.e. suppose there exists $a \in D \setminus f(\Omega)$. Put

(iii) $$\phi(z) = \frac{f(z) - a}{1 - \bar{a}f(z)}$$

Since a Möbius transform is a conformal map on D, it follows that $\phi(\Omega) \subset D$. Moreover, $\phi \neq 0$ and by (4) in the list by Schwarz there exists an analytic function $F(z)$ in Ω such that

(iv) $$F(z) = \text{Log}[\frac{f(z) - a}{1 - \bar{a}f(z)}] \in \mathcal{O}(\Omega) \quad : \quad \mathfrak{Re}(F(z)) < 0 \quad z \in \Omega$$

It is clear that F yields a conformal mapping from Ω into an open subset of the left half-plane $\mathfrak{Re}(w) < 0$. Next, consider the function

(v) $$G(z) = \frac{F(z) - F(z_0)}{F(z) + \bar{F}(z_0)}$$

Since $F(z)$ and $F(z_0)$ belong to the same half-plane $\mathfrak{Re}(w) < 0$ we see that the absolute value of G is < 1 for every $z \in \Omega$ and conclude that G yields a conformal mapping from Ω into D. Now $G(z_0) = 0$ and by the maximal property of M we get a contradiction if we have proved the strict inequality:

(vi) $$|G'(z_0)| > M$$

To prove (vi) we notice that

(vii) $$G'(z_0) = \frac{F'(z_0)}{F(z_0) + \bar{F}(z_0)}$$

Next, the construction of the Log-function F gives

(vii) $$F(z_0) + \bar{F}(z_0) = 2\mathfrak{Re}\, F(z_0) = 2\,\text{Log}\,|a|$$

Moreover, since $f(z_0) = 0$ a derivation of the Log-function F gives

(viii) $$F'(z_0) = f'(z_0)(1 - |a|^2) = M(1 - |a|^2)$$

Putting this together we obtain

(ix) $$G'(z_0) = M \cdot \frac{1 - |a|^2}{2 \log |a|} \implies |G'(z_0)| = M \cdot \frac{1 - |a|^2}{2 \log \frac{1}{|a|}}$$

Here $|a| < 1$. A trivial verification which is left to the reader shows that

$$1 - |a|^2) > 2 \log \frac{1}{|a|} \quad : 0 < |a| < 1$$

Hence we have found $G \in \mathcal{F}$ with $|G'(z_0)| > M$. This is a contradiction and hence we must have $f(\Omega) = D$ which finishes the proof of Riemann's Mapping Theorem.

A.4 Other extremal properties.

Let Ω be simply connected and assume that $0 \in \Omega \subset D$. Let $f^* \colon \Omega \to D$ be the conformal mapping with $f^*(0) = 0$ and its derivative at 0 is real and positive. Then the following hold:

A.5 Theorem. *For every point $a \in \Omega$ one has:*

$$(*) \qquad [f^*(a)| = \max_f |f(a)| \quad : f \in \mathcal{O}(\Omega) \quad : f(0) = 0 \quad \text{and} \quad f(\Omega) \subset D$$

To prove (*) we use the inverse map $\phi\colon D \to \Omega$ which satisfies

$$\text{(i)} \qquad f^*(\phi(w) = w \quad : \quad w \in D$$

Next, consider some $f \in \mathcal{O}(\Omega)$ as above. Since $f(0) = 0$ and $\Omega \subset D$ we get $f \circ \phi \in \mathcal{O}(D)$. This analytic function in D is zero at the origin and has maximum norm ≤ 1. Hence Schwarz inequality gives:

$$\text{(ii)} \qquad |f(\phi(w))| \leq |w| \quad : w \in D$$

With $a \in \Omega$ we pick $w \in D$ so that $\phi(w) = a$ and then (i-ii) give

$$|f(a)| \leq |w| = |f^*(a)|$$

Since this hold for every f as above we get (*).

A.6 A result for Jordan domains. Assume that Ω is a Jordan domain bordered by a rectifiable closed Jordan curve Γ. Given $a \in \Omega$ we denote by $\mathcal{F}_a$ the family of analytic functions $g(z)$ in Ω such that $g(a) = 0$ and the complex derivative $g'(z)$ extends to a continuous function on Γ and

$$(*) \qquad \int_\Gamma |g'(z)| \cdot |dz = 2\pi$$

This means that the total length of the image curve $g(\Gamma)$ is 2π. Now one seeks

$$(**) \qquad \max_g |g'(a)| \quad : g \in \mathcal{F}_a$$

Theorem A.7 *The maximum in (**) is attained by the conformal map from Ω onto D which sends a to the origin.*

The proof is left as an exercise. The hint is to study the family $g \circ \phi$ where ϕ is the conformal map from D into Ω which sends the origin to a.

B. The mapping theorem for multiply connected sets.

Let Ω be an open and connected subset of $\mathbf{C} \setminus \{0, 1\}$. For each point $z_0 \in \Omega$ we have the family $M\mathcal{O}(\Omega)(z_0)$ of germs of analytic functions at z_0 which extend to multi-valued functions in Ω. If $f \in M\mathcal{O}(\Omega)(z_0)$ and $z \in \Omega$ we denote by $f^*(z)$ the set of values taken by all local branches of f at z. Let $\mathcal{C}_\Omega(z_0)$ be the family of germs $f \in M\mathcal{O}(\Omega)(z_0)$ such that $f(0) = 0$ and the following hold:

$$(*) \qquad \cup_{z \in \Omega} f^*(z) \subset D \quad \text{and} \quad f^*(z_1) \cap f^*(z_2) = \emptyset \quad : z_1 \neq z_2$$

The domain $\mathcal{D}_f$ and the inverse function W_f. In D we get the subset

$$\mathcal{D}_f = \cup_{z \in \Omega} f^*(z) \subset D \quad \text{and}$$

Notice that (*) implies that for every point $w_0 \in \mathcal{D}_f$ there exists a unique point $z(w_0) \in \Omega$ and a local branch $T_\gamma(f) \in \mathcal{O}(z(w_0))$ such that

$$(1) \qquad T_\gamma(f)(z(w_0)) = w_0$$

Here $T_\gamma(f)$ is analytic on some disc U centered at $z(w_0)$. If $z \in U$ the value $T_\gamma(f)(z)$ belongs to the set $f^*(z)$ and by the separation condition in (*) for the f^*-fibers it follows that $T_\gamma(f)$ is 1-1 in the disc. In particular its complex derivative is $\neq 0$ in U. From this we conclude that the map

$$w \to z(w)$$

constructed by (1) above yields an analytic function from $\mathcal{D}_f$ onto Ω. The resulting analytic function is denoted by W_f. From the construction we have seen that its complex derivative is everywhere $\neq 0$. But W_f is in general not 1-1, Namely, if $z \in \Omega$ then $f^*(z)$ may consist of several points. This gives an inverse fiber

$$W_f^{-1}(z) = \{w \in \mathcal{D}_f \quad : W_f(w) = z\} \tag{2}$$

which is a *discrete* subset of $\mathcal{D}_f$ because the W_f-function has a non-zero derivative at all points in $\mathcal{D}_f$. Now we shall find f in $\mathcal{C}_\Omega(z_0)$ such that $\mathcal{D}_f$ is the whole unit disc.

B.1 Theorem. *There exists a unique $f \in \mathcal{C}_\Omega(z_0)$ such that $f'(z_0)$ is real and positive and $\mathcal{D}_f = D$. Moreover one has the inequality:*

$$|g'(z_0)| \leq f'(z_0)$$

for all germs $g \in M\mathcal{O}(\Omega)(z_0)$ whose total image set $\mathcal{D}_g$ is contained in D.

Remark. Theorem B.1 is the uniformisation theorem for connected subsets of $\mathbf{C}$ whose complement contains at least two points. The proof below is due to Koebe and Caratheodory.

Proof of Theorem B.1

Since $\Omega \subset \mathbf{C} \setminus \{0, 1\}$ the modular function from XX restricts to a multi-valued function in Ω. After a translation and a Möbius transform we conclude that the family $M_*\mathcal{O}(\Omega(z_0))$ is non-empty. Next, the multi-valued version of Montel's theorem in XXX shows that family $\mathcal{C}_\Omega(z_0)$ is a normal. Hence the variational problem (*) in Theorem B.1 has a solution, i.e. there exists $f_* \in \mathcal{C}_\Omega(x_0)$ whose derivative at z_0 is real and positive and (*) form Theorem B.1 holds. There remains to prove that $D_f = D$. Assume the contrary, i.e. suppose there exists some $a \in D \setminus D_f$. Consider the germ in $\mathcal{O}(z_0)$:

$$\frac{f - a}{1 - \bar{a} \cdot f} \tag{1}$$

Since $D_f \subset D$ and $|a| < 1$ it is clear that that the germ (1) belongs to $\mathcal{C}_\Omega(z_0)$. Next, since we allow multi-valued functions we can proceed as in for the proof in the simply connected case and define the germ

$$F(z) = \log \frac{f - a}{1 - \bar{a} \cdot f} \tag{2}$$

This germ at z_0 extends to a multi-valued analytic function in Ω and since log-functions only add integer multiples of $2\pi i$ the assumption that the absolute values of local branches of f are < 1 implies that all local branches of F have real part < 0. Next, set

$$G = \frac{F - F(z_0)}{F + \bar{F}(z_0)} \tag{3}$$

From the above the germ G belongs to $\mathcal{C}_\Omega(z_0)$ and a repetition of the proof in 1.1 leads to a contradiction, i.e. one verifies that the complex derivative $|G' * (x_0)| > |f'(x_0)|$. This shows that $D_f = D$ must hold and the existence part in Theorem B.1 is proved.

Proof of uniqueness. Let f be the solution obtained above and consider a germ $g \in M\mathcal{O}(\Omega)(z_0)$ where $\mathcal{D}_g \subset D$. We must show that

$$|g'(z_0)| \leq f'(z_0) \tag{5}$$

To prove this we employ the single-valued analytic function W_f which was constructed in B.0. Since $\mathcal{D}_f = D$ we have $W_f \in \mathcal{O}(D)$. Now we consider the composed function

$$\phi = g \circ W_f$$

where we get:

$$\phi(0) = g(W_f(0)) = g(z_0) = 0 \tag{6}$$

Since $\mathcal{D}_g \subset D$ the germ ϕ extends to a multi-valued analytic function in the whole unit disc D. Now D is simply connected and hence ϕ is single valued. Since $\phi(0) = 0$ it follows from Schwarz' inequality that

(7) $$|\phi'(0)| = |g'(z_0)| \cdot |W_f'(0)| \leq 1$$

Finally, by the construction of W_f we have in particular

$$W_f(f(z)) = z$$

for points z close to z_0 which gives the equality

(8) $$W_f'(0) = \frac{1}{f'(z_0)}$$

Hence (7-8) give the requested inequality (5). At this stage the requested uniqueness of f is clear. For if g is such that $\mathcal{D}_g = D$ also holds hen we an reverse the role between f and g, i.e. using the existence of the function $W_g \in \mathcal{O}(D)$ and first conclude that the absolute values $|g'(z_0)|$ and $|f'(z_0)|$ are equal. Moreover, if both are real and positive we he reader may verify that $W_f \circ g$ and $W_g \circ f$ both give the identity map and conclude that $f = g$.

II. Boundary behaviour

First we study arbitrary bounded analytic functions and establish some results due to Lindelöf and Koebe. Let D be the open unit disc and T the unit circle. Let $\{\omega_\nu\}$ and $\{\omega_\nu^*\}$ be two sequences in D which converge to different points p and q on the unit circle T and $\{\gamma_\nu\}$ is a sequence of Jordan arcs which are contained in D and connect ω_ν with ω_ν^* for each ν. Moreover, we assume that:

(*)
$$\lim_{\nu\to\infty} \min_{z\in\gamma_\nu} |z| = 1$$

Thus, the joining Jordan arcs stay close to the unit circle T as ν increases.

2.1 Koebe's Lemma *Let f be a bounded analytic function in D such that the maximum norms of f on γ_ν tend to zero as $\nu \to \infty$. Then f must be identically zero.*

Proof. By continuity we may assume that the joining arcs γ_ν are polygons. After a rotation we may assume that $\omega_\nu \to 1$ and $\omega_\nu^* \to e^{i\theta^*}$ for some $0 < \theta^* < \pi$. Given ν we have $\omega_\nu = r_\nu e^{i\theta_\nu}$ and $\omega_\nu^* = r_\nu^* e^{i\theta_\nu^*}$. While drawing the Jordan arc γ_ν from ω_ν to ω_ν^* we encounter the last point $\xi_\nu \in \gamma_\nu$ whose argument is θ_ν and after the first point $\eta_\nu \in \gamma_\nu$ whose argument is θ_ν^*. Replace the pair ω_ν, ω_ν^* with ξ_ν, η_ν which then are joined by a simple polygon Γ_ν which except for its end-points stay in the circular sector where $0 < \arg(z) < \theta^*$. To each ν we therefore get a domain U_ν bordered by Γ_ν and the two rays from the origin to ξ_ν and η_ν. By the hypothesis the maximum norms

(i)
$$M_\nu = \max_{z\in\Gamma_\nu} |f(z)|$$

tend to zero as $\nu \to \infty$. At the same time the assumption in (*) above gives

$$\min_{z\in\Gamma_\nu} = \rho_\nu \quad \text{where } \rho_\nu \to 1$$

Since f is assumed to be bounded in D and without loss of generality we may assume that the maximum norm $|f|_D \leq 1$. Now we will finish the proof is using results from XXX applied to the subharmonic function $log\,|f|$. Namely, we have $\theta_\nu \to 0$ and $\theta_\nu^* \to \theta^*$ with $0 < \theta^* < \pi$ and may assume that $\rho_n u > 3/4$. So when ν is large we get

$$z_* = \frac{1}{2} \cdot e^{i\theta^*/2} \in U_\nu$$

Next, let U_ν^* be the circular sector bordered by the two rays which pass through ξ_ν and η_ν and put $\gamma_\nu^* = T \cap \partial U_v$. Let $\mathfrak{m}^*_{\gamma_\nu^*}(z_*)$ be the harmonic measure at z_* with respect to the boundary arc γ_ν^* of the circular sector. Then there is a fixed constant $a > 0$ such that

$$\mathfrak{m}^*_{\gamma_\nu^*}(z_*) \geq a \quad : \forall\, \nu$$

At the same time Carleman's majorant principle for harmonic measures gives:

$$\mathfrak{m}^*_{\gamma_\nu^*}(z_*) \leq \mathfrak{m}^*_{\Gamma_\nu^*}(z_*)$$

where the right and side is the harmonic measure at z_* in the domain U_ν. Regarding the subharmonic function log $|f|$ we obtain

$$\log |f(z_*)| \leq a \cdot \log(M_\nu)$$

By (i) the right hand side tends to $-\infty$ and hence $f(z_*) = 0$. We can achieve a similar vanishing for points in a small disc centered at z_*. So by analyticity f is identically zero in D and Koebe's Lemma is proved.

2.2. Lindelöf's Theorem. *Let J_1 and J_2 be two Jordan arcs in D with a common end-point $p \in T$. Let $f \in \mathcal{O}(D)$ be bounded and assume that it has a limit along both J_1 and J_2, Then the two limit values are equal. Moreover, if two Jordan arcs are disjoint then $f(z)$ converges to the common limit value when z tends to p inside the domain bordered by J_1 and J_2.*

Proof of Lindelöf's theorem

Let a be the limit of f along J_1 and b the limit along J_2. If the curves J_1 and J_2 intersect at some sequence of points $\{z_\nu\}$ which tends to p, then we immediately get $a = b$. So we may assume that J_1 and J_2 do not intersect in the open disc D, i.e. their sole common point is p. After a rotation we may take $p = 1$. When $\delta > 0$ we consider the line L defined by $\mathfrak{Re}(z) = 1 - \delta$ where $\delta > 0$ is small. On J_1 we find the last point ξ which intersects L and similarly the last point η on J_2. We get the domain G_δ bordered by the portions $\Lambda_1 \subset J_1$ and $\Lambda_2 \subset J_2$ where $\mathfrak{Re}(z) > 1 - \delta$ and the line segment which joins η and ξ on L. Put

$$F(z) = (f(z) - a)(f(z) - b)$$

By the assumption on f the maximum norm of F tends to zero on both Λ_1 and Λ_2. At the same time F is a bounded analytic in the whole disc D and put $M = |F|_D$. Now we apply the reflection construction by Schwartz from XX and conclude that the maximum norm

$$|F|_{\partial G_\delta} \leq \sqrt{M} \cdot \sqrt{|F|_{\Lambda_1} + |F|_{\Lambda_2}}$$

Hence F converges to zero in G_δ as $\delta \to 0$. By the construction of F this means that f must tends to a or to b uniformly in G_δ as $\delta \to 0$. But then it is obvious that $a = b$ and at the same time we have proved that f converges to this common number in the domain bordered by J_1 and J_2 as we approach p. This finishes the proof of Lindelöf's theorem.

2.3 Proof of Theorem 0.6.1

Let $f\colon \Omega \to D$ be a conformal map. Let $p \in \partial\Omega$ and consider some Jordan arc $J \in \mathcal{J}(p)$. Put $w = f(z)$ so that $w \in D$. Let J be defined by $t \mapsto \gamma(t)$. In D we get the image arc

(ii) $$t \mapsto f(\gamma(t)) \quad : \quad 0 \leq t < 1$$

Denote it by J^*. We must prove that J^* tends to a point $e^{i\theta} \in T$, i.e.

(iii) $$\lim_{t \to 1} f(\gamma(t)) = e^{i\theta}$$

Assume the contrary. Then we obtain two sequences of points in D:

(iii) $$q_\nu = f(\gamma(t_\nu)) \quad : \quad s_\nu = f(\gamma(\tau_\nu)) \quad : \quad t_1 < \tau_1 < t_2 < \tau_2 \ldots$$

where t_ν and τ_ν both tend to 1 and a pair of *distinct* points q^*, s^* on T such that

(iv) $$q_\nu \to q^* \quad : \quad s_\nu \to s^*$$

Next, to each ν we get the image curve in D given by

(v) $$\gamma_\nu^* = f(\gamma[t_\nu, \tau_\nu])$$

which joints q_ν with s_ν. Now we regard the inverse function $g = f^{-1}$ which is a bounded analytic function in D. Here

(vi) $$\lim_{t \to 1} |g(f(\gamma(t)) - p| = 0$$

With $h(z) = g(z) - p$ it follows that

(vii) $$\lim_{\nu \to \infty} \max_{z \in \gamma_\nu^*} |h(z)| = 0$$

At the same time we notice that since both g and f are conformal we must have

(viii) $$\lim_{\nu \to \infty} \min_{|z|} z \in \gamma_\nu^* = 1$$

Since $q^* \neq s^*$ in (iv) Koebe's Lemma applied to h would entails that $h = 0$. This is a contradiction and Theorem 0.6.1. follows.

2.4 Proof of Theorem 0.6.2

Suppose that $\mathcal{K}_{J_1}(f) = \mathcal{K}_{J_2}(f) = z^*$ for some $z^* \in T$. In D we get the two Jordan arcs $\{J_\nu^* = f(J_\nu)\}$ which both tend to z^*. If $g = f^{-1}$ is the inverse function the limit of $g(z)$ along J_1^* is p and the limit along J_2^* is q. Now $p \neq q$ was assumed which contradicts Lindelöf's Theorem and Theorem 0.6.2 follows.

2.5 Proof of 0.6.4-0.6.5.

Both results are easy consequences of Koebe's Lemma using the same device as in the proof of Theorem 0.6.1 above. So we leave the details of the proofs of the announced results from 0.6.4 and Proposition 0.6.5 as exercises to the reader.

2.6 Accessible points·

Let Ω be bounded and simply connected while no further assumptions are imposed. Let $\phi \colon D \mapsto \Omega$ be a conformal mapping where z is the complex coordinate in D while $\zeta = \phi(z)$ denote points in Ω. Consider a accessible point $p \in \mathcal{A}(\partial\Omega)$ which gives a half-open Jordan arc γ_* in D such that

(1)
$$\lim_{t \to 1} \phi(z(t)) = p$$

Lindelöf's theorem applies to the bounded analytic function ϕ and hence (1) entails that ϕ has a non-tangential limit at p_*. In particular there exists the radial limit

(2)
$$p = \lim_{r \to 1} \phi(re^{i\theta})$$

Conversely, let $e^{i\theta} \in T$ and assume that ϕ has a radial limit which yields a boundary point $p \in \partial\Omega$. Then it is clear that $p \in \mathcal{A}(\partial\Omega)$ and we have proved the following:

2.7 Theorem. *Let $\mathcal{R}(\phi)$ be the set of points on T where ϕ has a radial limit. Then $\mathcal{A}(\partial\Omega)$ is equal to the set of $p \in \partial\Omega$ such that*

$$p = \lim_{r \to 1} \phi(re^{i\theta}) \quad \text{for some} \quad e^{i\theta} \in \mathcal{R}(\phi)$$

Thus, $\mathcal{A}(\partial\Omega)$ is equal to the ϕ-image of all radial limit values. Recall from [Measure] that the subset of T where ϕ has a radial limit is dense Borel set. By Theorem 2.7 its image is equal to $\mathcal{A}(\partial\Omega)$ which to begin with implies that $\mathcal{A}(\partial\Omega)$ is a Borel subset of $\partial\Omega$.

2.8 Harmonic measures on $\partial\Omega$. When Ω is simply connected the condition in Theorem xx from Chapter V holds so Dirichlet's problem has a solution. Let $\phi \colon D \to \Omega$ be a conformal map where we let z be the variable in D and set $\zeta = \phi(z)$. Put $\zeta_* = \phi(0)$ and consider the harmonic measure $\mathfrak{m}_{z_*}$ on $\partial\Omega$. So when $f \in C^0(\partial\Omega)$ we have

(*)
$$H_f(\zeta_*) = \int f(\zeta) \cdot d\mathfrak{m}_{z_*}(\zeta)$$

if $r < 1$ the circle $|z| = r$ gives the image curve $\Gamma_r = \phi(|z| = r)$ which appears as a closed Jordan curve in Ω. When r is close to one the Jordan domain bounded by Γ_r contains z_* and we get the harmonic measure on Γ_r with respect to z_*. Restricting ϕ to the disc $|z| \leq r$ the transformation rule for harmonic measures under conformal mappings gives:

$$H_f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} H_f(\phi(re^{i\theta})) \cdot d\theta$$

Above $H_f \circ \phi$ is a bounded harmonic function in D. and has therefore radial limits almost everywhere which by dominated convergence gives:

$$H_f(\zeta_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} H_f(\phi(e^{i\theta})) \cdot d\theta$$

More precisely we have integrated the almost everywhere defined function $\theta \to H_f(\phi(e^{i\theta}))$ where $\phi(e^{i\theta})$ are points on $\mathcal{A}(\partial\Omega)$ Hence we have the equality

$$(**) \qquad H_f(\zeta_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(\phi(e^{i\theta})) \cdot d\theta$$

Above $f \in C^0(\partial\Omega)$ is arbitrary so (**) gives a linear functional

$$f \mapsto \frac{1}{2\pi} \cdot \int_0^{2\pi} f(\phi(e^{i\theta})) \cdot d\theta$$

which evaluates H_f at ζ_*. The uniqueness of the harmonic measure this entails that $\mathfrak{m}_{z_*}$ is the Riesz measure representing (**). We can interpretate this in a direct measure theoretic sense. Namely, let $\mathcal{R}(\phi)$ denote the set of points $e^{i\theta}$ where $r \mapsto \phi(re^{i\theta})$ has a radial limit. Then we get the Borel measurable function from $\mathcal{R}(\phi)$ to $\mathcal{A}(\partial\Omega)$ given by

$$(1) \qquad e^{i\theta} \mapsto \phi(e^{i\theta})$$

and the discussion above shows that $\mathfrak{m}_{z_*}$ is the push-forward of $\frac{1}{2\pi} \cdot d\theta$

Remark. The result above implies $\mathcal{A}(\partial\Omega)$ carries all mass of the harmonic measure, i.e.

$$\mathfrak{m}_{\zeta_*}(\mathcal{A}(\partial\Omega)) = 1$$

More generally, for every Borel subset E of $\mathcal{A}(\partial\Omega)$ we have the equality

$$\mathfrak{m}_{\zeta_*}(E) = \frac{1}{2\pi} \int_{\phi^{-1}(E)} d\theta$$

where $\phi^{-1}(E)$ is a Borel set in $\mathcal{R}(\phi)$. Notice also that the results by Koebe and Lindelöf imply that the map (1) is injective.

2.9 A further remark. If $f \in C^0(\partial\Omega)$ we consider its restriction to $\mathcal{A}(\partial\Omega)$ and obtain the function f^* on $\mathcal{R}(\phi)$:

$$(1) \qquad f^*(e^{i\theta}) = h(\phi(e^{i\theta}))$$

Here f^* is defined almost everywhere so via the Poisson kernel it has a unique harmonic extension H^* to D. The previous material shows that

$$H^*(z^*) = H_f(\phi(z))$$

hold when $z \in D$. This gives an injective linear map from $C^0(\Omega)$ to a space of bounded harmonic functions in D. The description of the resulting range of this linear map is unclear without further assumptions on $\partial\Omega$.

3. Picard's Theorem

Introduction In 1879 E. Picard gave the affirmative answer to a question posed by Weierstrass and proved that an entire function $f(z)$ which excludes two values must be a constant. Picard's original proof goes as follows: Suppose that $f(z)$ never takes the values 0 or 1. Consider the *modular function* $\mathfrak{w}(z)$ which exists as a multi-valued function in $\mathbf{C} \setminus \{0,1\}$ with values in the upper half plane U_+. The composed function $g(z) = \mathfrak{w}((f(z))$ becomes a multi-valued function defined in the whole of $\mathbf{C}$. Here $\mathbf{C}$ is simply connected and hence g is single valued. So g is a entire function with values in U_*. But then g is a constant which gives a contradiction, for then f would also be a constant function.

3.1 The Landau-Schottky Theorem

In 1904 Shottky discovered a surprising consequence of the proof by Picard which led to a refined version of Picard's Theorem in joint work with E. Landau. First we give

0.2 Definition *Let $f(z) = a_0 + a_1 z + \dots$ be an analytic function defined in a disc D_R of radius R centered at the origin. Then D_R is said to be $(0,1)$-excluding relative f if the range $f(D_R)$ does not contain the two points 0 and 1.*

0.3 Theorem *To each non-zero pair (a_0, a_1) there exists a constant $L(a_0, a_1)$ such that if D_R is $(0,1)$-excluding for some $f \in \mathcal{O}(D_R)$, then $R \leq L(a_0, a_1)$.*

Remark The point is of course that $L(a_0, a_1)$ does not depend on higher terms in the series expansion of f. The result above is due to Landau. A simple proof was later found by Caratheodory which leads also to a sharp estimate of $L(a_0, a_1)$ and goes as follows:

Proof of Theorem 0.3 Suppose that a disc D_R is $(0,1)$-excluding with respect to f. Let $\mathfrak{w}$ be the modular function. As explainedin XX we get the analytic function $g(z) = \mathfrak{w}(f(z))$ is in D_R where $g(D_R)$ is contained in the upper half plane U. Set

$$\ell(z) = \frac{g(z) - g(0)}{g(z) - \bar{g}(0)}$$

Since $\mathfrak{Im}(g(z)) > 0$ we see that $\ell(z)$ has absolute value < 1 in D_R. Here $\ell(0) = 0$ and Schwarz' Lemma gives:

$$|\ell(z)| \leq \frac{|z|}{R} \quad : \; z \in D_R\,.$$

It follows that the derivative at the origin has absolute value $\leq \frac{1}{R}$. Here

(i) $$\ell'(0) = \frac{g'(0)}{g(0) - \bar{g}(0))} \quad \text{which gives} \quad |\frac{g'(0)}{g(0) - \bar{g}(0))}| \leq \frac{1}{R}$$

Next, notice that

(ii) $$g'(0) = \mathfrak{w}'(a_0) \cdot a_1 \quad \text{and} \quad g(0) = \mathfrak{w}(a_0)$$

Hence (i-ii) give;

(iii) $$R \leq 2 \cdot |\frac{\mathfrak{Im}(\mathfrak{w}(a_0)}{\mathfrak{w}'(a_0) \cdot a_1}|$$

This proves the inequality

(*) $$L(a_0, a_1) \leq 2 \cdot |\frac{\mathfrak{Im}(\mathfrak{w}(a_0)}{\mathfrak{w}'(a_0) \cdot a_1}|$$

It turns out that (*) is sharp. To see this one constructs an Ahlfors' function associated to $\mathfrak{w}$. See Section XX for details.

0.4. The Schottky-Landau Theorem. Working a bit more using the $\mathfrak{w}$-function and various Green's functions, the following result was established by Landau and Schottky:

0.5 Theorem *For each pair (k, θ) where $k > 0$ and $0 < \theta < 1$ there exists a constant $S(k, \theta)$ such that if $R > 0$ and the open disc D_R is $(0, 1)$ excluding for some $f \in \mathcal{O}(D_R)$, one has*

$$\max_{|z| \leq \theta R} |f(z)| \leq S_1(k, \theta) \quad : \quad |f(0)| \leq k$$

Remark This result applies locally since R can be small and in contrast to the previous case we only assume that f is analytic in the disc D_R. Using this theorem one easily deduces the local version of Picard's Theorem, i.e. that an analytic function $f(z)$ with an isolated essential singularity at some point z_0 must take all values with at most one exception in arbitrary small punctured discs. For details of proof we refer to paragraph 2 in Ch.V from [Bieberbach] which contains further comments about upper bounds of the S-function. Of special interest is the asymptotic behavior as $k \to \infty$. For example, at the end of § 2 in [loc.cit], the following is proved:

0.6 Theorem *There exists an absolute constant* $\mathbf{B}$ *such that for any $R > 0$ and any $f \in \mathcal{O}(D_R)$ where D_R is (0,1)- excluding for f, one has*

$$\max_{|z| < \theta R} |f(z)| \leq \exp[\frac{\mathbf{B} \cdot \log\big(\,|a_0| + 2\,\big)}{1 - \theta} \quad : \quad 0 < \theta < 1 \quad : a_0 = f(0)$$

0.7 Remark. Landau' text.book [Landau] contains a historic account of the Picard's Theorem and also a proof of Theorem 0.5 which only uses elementary function theory, i.e avoiding the modular function. Let us also mention that Theorem 0.5 leads to certain a priori inequalities for conformal mappings. We announce one such result from Chapter 7 in [Landau]: Denote by $\mathcal{F}$ the family of all functions

$$f(z) = z + a_2 z^2 + \ldots$$

such that f gives a conformal map from the unit disc D onto some simply connected domain.

0.8 Theorem. *To each $0 < r < 1$ there exist constants $C_1(r)$ and $C_2(r)$ such that the following hold for every $f \in \mathcal{F}$:*

$$\max_{|z| \leq r} |f(z)| \leq C_1(r) \quad : \quad \frac{1}{C_2(r)} \leq \max_{|z| \leq r} |f'(z)| \leq C_2(r)$$

Since the image domain $f(D)$ need not be bounded this *a priori* inequality is quite remarkable.

1. The method by Ahlfors.

Landau's Theorem can be viewed as a special case of a more general problem where one starts from some open and connected set Ω in the complex plane such that $\mathbf{C} \setminus \Omega$ contains at least two points. Given Ω we consider an analytic function $f(z)$ defined in some open disc D_R centered at the origin such that $f(D_R) \subset \Omega$. At the origin we have the Taylor expansion

$$f(z) = a_0 + z + a_2 z^2 + \ldots \quad : \quad a_0 \in \Omega$$

For simplicity we have normalised the situation so that $f'(0) = 1$. Landau's theorem implies that there exists such an upper bound R^* which depends on a_0 and Ω only such that the analytic function $f(z)$ only can exist in a disc of radius $< R^*$. Ahlfors constructed certain subharmonic functions in Ω and used an extension of Schwartz inequality to obtain an upper bound for R^* which depends on the existence of a certain subharmonic function in Ω. We shall describe this result. Let w be the complex variable in Ω.

1.1 Definition. *Let $\gamma > 0$. Denote by $SH_\gamma(\Omega)$ the class of subharmonic C^2-functions U in Ω satisfying*

$$\Delta(U)(w) \geq e^{\gamma \cdot U(w)} \quad : \quad w \in \Omega$$

Let us assume that 0 and 1 stay outside Ω. In this case one can show that the class $\mathrm{SH}_\gamma(\Omega)$ is non-empy when γ is small enough. Notice that the constraint upon u is more restricted as γ increases and more relaxed when the open set Ω becomes smaller. Here follows Ahlfors' version of the Schwartz's inequality.

1.2 Theorem. *Let $U \in SH_\gamma(\Omega)$ and $f \in \mathcal{O}(D_R)$ for some $R > 0$. Then*

$$e^{\gamma \cdot U(f(z))} \cdot |f'(z)|^2 \leq \frac{8 \cdot R^2}{(R^2 - |z|^2)^2} \quad : \quad z \in D_R$$

Proof. In D we consider the function

(1) $$\Phi(z) = e^{\gamma \cdot U(f(z))} \cdot |f'(z)|^2 \cdot (R^2 - |z|^2)^2$$

Since $\Phi = 0$ on the boundary $|z| = R$ it takes its maximum at some $a \in D_R$, i.e.

$$\Phi(a) = \max_{z \in D_R} \Phi(z)$$

Then $\log \Phi = \phi$ also takes its maximum at a and since the maximum value of Φ is positive we have $f'(a) \neq 0$. Now

(2) $$\phi(z) = \gamma \cdot U(f(z)) + \log |f'(z)|^2 + 2 \cdot \log (R^2 - |z|^2)$$

Since ϕ takes a maximum at a ordinary Calculus gives $\Delta(\phi)(a) \leq 0$. Moreover, $|f'(z)|^2$ is harmonic in a neighborhood of a. Hence we have

(3) $$\gamma \cdot \Delta(U(f(z)) + 2 \cdot \Delta(\log (R^2 - |z|^2)) \leq 0 \quad \text{at the point} \quad z = a$$

Differential rules give:

(4) $$\Delta(U(f(a)) = \Delta U(f(a)) \cdot |f'(a)|^2$$

and an easy computation also gives that

(5) $$2 \cdot \Delta(\log (R^2 - |z|^2)) = -\frac{8R^2}{(R^2 - |z|^2)^2}$$

Since $U \in \mathrm{SH}_\gamma(\Omega)$ it follows from (3-5) that

(6) $$e^{\gamma \cdot U(f(a))} \cdot |f'(a)|^2 \cdot (R^2 - |a|^2)^2 \leq 8R^2$$

Here (6) is the maximum of $\Phi(z)$ over the whole disc D_R and hence we have proved the requested inequality

(7) $$e^{\gamma \cdot U(f(z))} \cdot |f'(z)|^2 \cdot (R^2 - |z|^2)^2 \leq 8 \cdot R^2 \quad : \quad z \in D_R$$

1.3 Upper bound for R^*. Let $f(z) = a_0 + z + a_2 z^2 + \ldots$ and suppose that $f(D_R) \subset \Omega$ holds for some R. With $z = 0$ in Theorem 1.2 we must have

$$e^{\gamma \cdot U(a_0)} \leq \frac{8}{R^2} \implies R \leq \sqrt{8} \cdot e^{-\gamma U(a_0)/2}$$

Thus, the existence of some U function in $\mathrm{SH}_\gamma(\Omega)$ gives

$$R^*(a_0, \Omega) \leq \sqrt{8} \cdot e^{-\gamma \dot{U}(a_0)/2}$$

1.4 Remark. Theorem 1.2 illustrates the importance to construct subharmonic functions with certain extremal properties. The article [Ahlfors] studies extremal metrics which in addition to analytic functions also apply to *quasi-conformal mappings.* In this way Ahlfors extended Picard's Theorem to a set-up where quasi-conformal mappings appear. For an account about Ahlfors' theory the reader may consult the presentation talk by Caratheodory from the IMU congress at Oslo in 1936 when L. Alhfors received the Fields Prize for his contributions.

4. Some geometric results.

We study geometric properties of maps defined by analytic functions. Let Γ be an interval on the circle $|z|=r$. Suppose that $f(z)$ is analytic in some neighborhood of Γ and that $f'(z)\neq 0$ when $z\in\Gamma$. So if $z_0\in\Gamma$ then f maps a small circle interval $\gamma\subset\Gamma$ centered at z_0 to a Jordan curve $f(\gamma)$. For each $z\in\gamma$ the curvature along γ at the point $f(z)$ is denoted by $\rho(z)$. With this notation one has:

4.1 Theorem *For each $z\in\gamma$ the following equality holds:*

$$\frac{1}{\rho(z)}=\frac{1+\mathfrak{Re}[z\cdot\frac{f''(z)}{f'(z)}]}{r|f'(z)|}$$

Proof. The image curve $f(\gamma)$ has the parametrisation $\theta\mapsto f(re^{i\theta})$ defined for some θ-interval. Let ds be the arc-length measure along this curve. By the general result in XXX we have

$$\text{(i)}\qquad \frac{ds}{d\theta}=r\cdot|f'(re^{i\theta})|$$

Next, the θ-parametrisation of $f(T_r)$ gives the complex tangent vector defined by

$$\text{(ii)}\qquad v(\theta)=\lim_{\Delta|\theta\to 0}\frac{f(re^{i\theta+i\Delta\theta}-f(re^{i\theta})}{\Delta\theta}=izf'(z)\quad:\quad z=r^{i\theta}$$

Let $\alpha(\theta)=\arg(v(\theta)$. The definition curvature gives:

$$\text{(iii)}\qquad \frac{1}{\rho}=\frac{d\alpha}{ds}=\frac{d\alpha}{d\theta}\cdot\frac{d\theta}{ds}$$

So by (i) here remains to show that

$$\text{(iv)}\qquad \frac{d\alpha}{d\theta}=1+\mathfrak{Re}[z\cdot\frac{f''(z)}{f'(z)}]$$

To show (iv) we notice that $z=re^{i\theta}$ gives

$$\alpha(\theta)=\arg(izf'(z))=\pi/2+\theta+\mathfrak{Im}[\mathrm{Log}(f'(re^{i\theta}))]$$

It follows that

$$\frac{d\alpha}{d\theta}=1+\mathfrak{Im}\bigl(ire^{i\theta}\cdot\frac{f''(re^{i\theta})}{f''(re^{i\theta})}\bigr)=1+\mathfrak{Re}\bigl(re^{i\theta}\cdot\frac{f''(re^{i\theta})}{f''(re^{i\theta})}\bigr)$$

Since $z=re^{i\theta}$ we have (iv) and Theorem 3.1 is proved.

4.2 Convexity of image curves. Let $f(z)$ be analytic in some open neighborhood of a circle $T_r=\{|z|=r\}$. Notice that we do not require that f extends to an analytic function in the disc D_r. But we assume that f is 1-1 on T_r which gives the closed Jordan curve $f(T_r)$. Recall from analytic geometry that this curve is *strictly convex* if and only if the curvature is everywhere >0. By Theorem 4.1 strict convexity therefore holds if and only if

$$\mathfrak{Re}\,z\cdot\frac{f''(z)}{f'(z)}>-1\quad:\quad z\in T_r$$

4.3 Example. Let a be a real number different from 1 and -1. Consider the function

$$f(z)=z+\frac{a}{z}$$

It is analytic in a neighborhood of the unit circle T. We obtain

$$z\cdot\frac{f''(z)}{f'(z)}=\frac{2a}{a-z^2}=\frac{2a(a-z^2)}{|2a-z^2|^2}$$

So with $z = e^{i\theta}$ the real part becomes

$$\frac{2a^2 - 2a \cdot \cos(\theta)}{|2a - e^{i\theta}|^2}$$

Hence we have strict convexity if and only if

$$2a^2 + |2a - e^{2i\theta}|^2 > 2a \cdot \cos(\theta) \quad : \quad 0 \leq \theta \leq 2\pi$$

Remark. Here is a good occasion to use a computer and plot the Jordan curves when a varies and discover when the are convex or not via the criterion above.

A bound for convexity. Let f give a conformal map from the unit disc D onto a simply connected domain. To each $0 < r < 1$ the image of $|z| < r$ is a Jordan domain $\Omega - [r]$.

4.4 Theorem. *Set $r_* = 2 - \sqrt{3}$. Then r_* is the largest number such that $\Omega_f[r]$ are convex for all conformal mappings f.*

Proof. Given f and some $z_0 \in D$ we put

$$\phi(z) = B \cdot f(\frac{z + z_0}{1 + \bar{z}_0 \cdot z})$$

where B is determined so that the derivative $\phi'(0) = 1$ which gives

$$B \cdot f'(z_0) \cdot (1 - |z_0|^2) = 1 \tag{1}$$

Now $\phi(z) = f(z_0) + z + a_2 \cdot z^2 + \ldots$ is a conformal map on D and Theorem XX in XXX gives

$$|a_2| \leq 2 \tag{2}$$

At the same time a computation gives

$$\phi'(z) = B \cdot f'(\frac{z + z_0}{1 + \bar{z}_0 \cdot z}) \cdot \frac{1 - |z_0|^2}{(1 + \bar{z}_0 \cdot z)^2} \implies$$

$$\phi''(0) = B \cdot f''(z_0) \cdot \left[1 - |z_0|^2\right]^2 - B \cdot f'(z_0) \cdot (1 - |z_0|^2) \cdot 2 \cdot \bar{z}_0 =$$

$$\frac{f''(z_0)}{f'(z_0)} \cdot (1 - |z_0|^2) - 2 \cdot \bar{z}_0 \tag{3}$$

Now (2) gives $|\phi''(0)| \leq 4$ and after a multiplication with z_0 we get the inequality

$$\left|z_0 \cdot \frac{f''(z_0)}{f'(z_0)} \cdot (1 - |z_0|^2) - 2 \cdot |z_0|^2\right| \leq 4 \cdot |z_0| \tag{4}$$

With $|z_0| = r < 1$ we see that (4) gives the inequality

$$\mathfrak{Re}\, z_0 \frac{f''(z_0)}{f'(z_0)} \geq \frac{-4r + 2r^2}{1 - r^2}$$

By (*) in 4.2 the image domain Ω_r is convex if the right hand side is > -1 Hence the critial value R_* is the smallest root of the equation $r^2 4r + 1 = 0$ which gives $r_* = 2 - \sqrt{3}$.

Exercise. Show that the bound r_* is sharp using the Koebe map from XX where we have the equality $|a_2| = 2$ for a normalised conformal map ϕ with $\phi'(0) = 1$.

4.5 Jensen's landscape surface.

Let $f(z) \in \mathcal{O}(\Omega)$ for some open set. Assume that $f \neq 0$ in Ω. To each $z \in \Omega$ we get the positive number $t = |f(z)|$. So with $z = x + iy$ we obtain a surface in the real (x, y, t)-space defined by

$$t = |f(x + iy)|$$

It is denoted by $\mathcal{J}_f$. To each $z = x + iy \in \Omega$ we get the point

$$p = (x, y, |f(x, y)|) \in \mathcal{J}_f$$

Let Π be the tangent plane to $\mathcal{J}_f$ at p. Let $\gamma(p)$ be the acute angle between Π and the (x, y)-plane. So here $0 < \gamma(p) < \frac{\pi}{2}$. With these notations one has;

$$\operatorname{tg}(\gamma(p)) = 2 \cdot |f(z)| \cdot |f'(z)|$$

Proof. Let n be the unit normal to Jensen's surface whose t-component becomes

$$n_t = \frac{1}{(\partial_x |f|)^2 + (\partial_y |f|)^2 + 1}$$

By elementary geometry we have

$$n_t \cos(\gamma(p)) = \frac{1}{(\partial_x |f|)^2 + (\partial_y |f|)^2 + 1}$$

At this stage the reader can finish the proof after calculating the partial derivatives of $|f|$ by expressing f as $u + iv$.

Koebe's One Quarter Theorem

The result below was established by Koebe in 1907. Let $f(z)$ be analytic in the unit disc D. Assume that $f(0) = 0$ and $f'(0) = 1$ and that f is 1-1, i.e. f gives a *conformal map* from D onto a domain Ω.

4.6 Theorem. *The image set $f(D)$ contains the open disc of radius $\frac{1}{4}$ centered at the origin, i.e. to any $|w| < 1/4$ there exists $z \in D$ so that $f(z) = w$.*

Proof. To begin with $f(D)$ certainly contains some open disc centered at the origin. This yields the existence of a positive number d defined by

$$d = \min |w[\quad : \quad w \in C \setminus f(D)$$

We find some $w^* C \setminus f(D)$ such that $|w^*| = d$. Next, since the image $f(D) = \Omega$ is *simply connected* the result in plane topology from XXX gives the existence of a simple curve Γ with a starting point at w^*, contained in $C \setminus f(D)$ and moving to the point at infinity. Removing Γ we also know from XXX that $C \setminus \Gamma$ is simply connected. This implies that there exists a *single-valued* branch of the root function $\sqrt{w - d}$ in $C \setminus \Gamma$ such that

$$\mathfrak{Re}(\sqrt{w - d}) > 0 \quad : \quad w \in C \setminus \Gamma$$

Set

$$g(w) = -4d \frac{\sqrt{w - d} - i\sqrt{d}}{\sqrt{w - d} + i\sqrt{d}}$$

Then we see that

$$g'(0) = 0 \quad : \quad |g(w)| < 4d \quad : w \in C \setminus \Gamma$$

Next, consider the composed function $H = g(f(z))$ which becomes analytic in D. Since $f(0 = 0$ and $f'(0) = 1$ we obtain

$$H'(0) = g'(0) f'(0) = 1 \quad : \quad |H(z)| < 4d \quad : z \in D$$

Since we also have $H(0) = g(0) = 0$, the last estimate and Schwarzs' inequality from XX give $|H'(0) \leq 4d$. Since we also have $H'(0) = 1$ we get $d \geq \frac{1}{4}$ as required.

4.7 The Area Theorem.

Consider a function

$$(*) \qquad w(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

where the negative Laurent series is convergent in the exterior disc $|z| > 1$. At the point at infinity the w-function has a simple pole since z appears in (*). Assume that (*) yields a conformal map from the exterior domain $D^* = \{|z| > 1\} \cup \{\infty\}$ onto a simply connected exterior domain which includes the point at infinity in the w-plane bordered by a closed Jordan arc Γ. Thus, the *bounded* Jordan domain bordered by Γ is outside the image of the map (*).

4.8 Theorem. *One has the inequality*

$$\sum_{n=1}^{\infty} n \cdot |a_n|^2 \leq 1$$

Proof. If $r > 1$ we denote by $\Gamma(r)$ the closed Jordan curve in the w-plane which is the image of the circle $|z| = r$. Let $J(r)$ be the area of the bounded Jordan domain in the w-plane which is bordered by $\Gamma(r)$. With $w = u + iv$ we recall from (xx) that Green's formula gives:

$$\text{(i)} \qquad J(r) = \int_0^{2\pi} u(re^{i\theta}) \cdot \frac{dv(re^{i\theta})}{d\theta} \cdot d\theta$$

The Cauchy Riemann equations imply that (i) is equal to:

$$\text{(ii)} \qquad \int_0^{2\pi} \frac{w(re^{i\theta}) + \bar{w}(re^{i\theta})}{2} \cdot \frac{w'(re^{i\theta}) - \bar{w}'(re^{i\theta})}{2i} \cdot d\theta$$

Now (ii) has the series expansion

$$\int_0^{2\pi} \Big[\frac{re^{i\theta} + re^{-i\theta}}{2} + \sum_{n=1}^{\infty} \frac{a_n \cdot re^{-in\theta}) + \bar{a}_n \cdot (re^{in\theta})}{2 \cdot r^n}\Big] \cdot$$
$$\Big[\frac{re^{i\theta} + re^{-i\theta}}{2} - \sum_{n=1}^{\infty} \frac{na_n \cdot re^{in\theta}) + n\bar{a}_n \cdot (re^{in\theta})}{2 \cdot r^n}\Big] \cdot d\theta$$

Since the integrals $\int_0^{2\pi} e^{ik\theta} \cdot d\theta = 0$ for all integers $k \neq 0$, a computation shows that the expression above becomes

$$\text{(iii)} \qquad \pi r^2 - \pi \cdot \sum_{n=1}^{\infty} \frac{n \cdot |a_n|^2}{r^{2n}}$$

Here (iii) holds for every $r > 1$ and since it expresses the non-negative area $J(r)$ it is non-negative. Passing to the limit as $r \to 1$ we get the inequality in the theorem.

4.9 An extremal problem. Consider a double connected domain Ω where $\partial\Omega$ consists of two disjoint and closed Jordan curves Γ_* and Γ^*. Here Γ^* is the outer curve which borders the unbounded component of $\mathbf{C} \setminus \bar{\Omega}$. By XXX there exists a unique number $0 < \ell < 1$ such that Ω is conformally equivalent to the annulus $\mathcal{A}(\ell) = \{0 < \ell < |z| < 1\}$ and we set $\ell = \ell(\Omega)$. Let Ω^* be the Jordan domain bounded by the outer curve Γ^* and Ω_* is the Jordan domain bounded by Γ_*. Set

$$\rho_* = \text{Area}(\Omega_*) \quad \text{and} \quad \rho^* = \text{Area}(\Omega^*)$$

So if Ω is equal to $\mathcal{A}(\ell)$ or a translate of this annulus we have

$$\ell^2 = \frac{\rho_*}{\rho^*}$$

4.10 Theorem. *For every doubly connected domain Ω which is not a translate of an annulus one has strict inequality*

$$\ell(\Omega)^2 > \frac{\rho_*}{\rho^*}$$

Proof. Set $\ell = \ell(\Omega)$ and let $F \colon \mathcal{A}(\ell) \to \Omega$ be a conformal map. Now $F(z)$ has a Laurent series expansion

$$\text{(1)} \qquad F(z) = \sum_{-\infty}^{\infty} c_n \cdot z^n$$

We get the area formulas

$$\rho_* = \frac{1}{\pi} \cdot \sum_{-\infty}^{\infty} n \cdot |c_n|^2 \cdot \ell^{2n} \quad \text{and} \quad \rho^* = \frac{1}{\pi} \cdot \sum_{-\infty}^{\infty} n \cdot |c_n|^2 \tag{2}$$

Since $\ell < 1$ we get :

$$\ell^2 \cdot \sum_{n=1}^{\infty} n \cdot |c_n|^2 \geq \sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot \ell^{2n} =$$

$$\pi \cdot \rho_* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2 \cdot \ell^{-2n} \geq \pi \cdot \rho_* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2 \tag{3}$$

Above strict inequality holds unless

$$c_n = 0 \quad \text{for all} \quad n \neq 1 \tag{4}$$

We have also the equality

$$\sum_{n=1}^{\infty} n \cdot |c_n|^2 = \pi \cdot \rho^* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2 \tag{5}$$

The inequality (3) and a division with $\sum_{n=1}^{\infty} n \cdot |c_n|^2$ gives

$$\ell^2 \geq \frac{\pi \cdot \rho_* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2}{\pi \cdot \rho^* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2} \tag{6}$$

Since $\rho^* - \rho_* > 0$ it is obvious that the last term is $\geq \frac{\rho_*}{\rho^*}$ and Theorem 4.10 follows.

Remark. Theorem 4.10 appears in Carleman's article *Über eine Minimalproblem der mathematischen Physik* from 1917 where an application of Theorem 4.10 to *Zylinderkondensatoren* is described.

5. Schwarz-Christofell maps

Introduction. Following original constructions by H. Schwartz and Christoffell we shall find a conformal mapping from the unit disc onto a convex polygon Π whose corner points are denoted by $w_1, \ldots, w_N$ where $N \geq 3$. Performing a translation if necessary we may assume that the origin is an interior point of Π. Here Π is placed in the complex w-plane. The corner points $w_1, \ldots, w_N$ are arranged so that the boundary has a positive orientation - i.e. anti-clockwise. See figure XXX. At each corner point w_k we get the two line segments $\ell_*(k)$ and $\ell^*(k)$ where $\ell_*(k)$ joins w_{k-1} with w_k and $\ell^*(k)$ joins w_k with w_{k+1}. In the case $k = 1$ then $\ell_*(1)$ joins w_N with w_1 and $\ell^*(N)$ joins w_N with w_1. At each corner point we have the interior angle β_k where $0 < \beta_k < \pi$. A wellknown formula from euclidian geometry gives

$$\sum \beta_k = (N-2)\pi$$

The outer angles are defined by:

$$\alpha_k = \pi - \beta_k \quad \Longrightarrow \quad \sum \alpha_k = 2\pi$$

See figure XXX for an illustration. Riemann's mapping theorem gives the unique analytic function $f(z)$ in D such that $f(0) = 0$ and $f'(0)$ is real and positive while f maps D conformally onto Π. Moreover, since Π is a Jordan domain we know that this conformal mapping. extends continuously up to the boundary and f yields a bi-continuous map from the unit circle T onto the boundary of Π. On T we get the points $z_1, \ldots, z_N$ which are mapped to the corner points of Π. Performing a rotation of Π in the w-plane we may assume that $z_1 = 1$ and if $2 \leq k \leq N$ we have

$$z_k = e^{i\theta_k} \quad \text{where} \quad 0 < \theta_2 < \ldots < \theta_N < 2\pi$$

When $k = 1$ we have $z_1 = 0$ so $\theta_1 = 0$. Put

$$a_k = \frac{\alpha_k}{\pi} \quad : \quad 1 \leq k \leq N$$

Hence (xx) entails that

$$\sum a_k = 2$$

To every $1 \leq k \leq n$ we get the analytic function in D defined by

(1) $$\phi_k(\zeta) = (1 - e^{-i\theta_k}\zeta)^{-a_k}$$

where single valued branches are chosen so that

$$-\pi/2 < \arg(\phi(\zeta)) < \pi/2$$

In particular $\phi_k(0) = 1$. With these notations one has:

5.1 Theorem. *The conformal map f from D onto Π is given by:*

$$f(z) = c_0 \cdot \int_0^z \frac{d\zeta}{(1 - e^{-i\theta_1}\zeta)^{\alpha_1} \cdots (1 - e^{-i\theta_N}\zeta)^{\alpha_N}} \quad : c_0 > 0 \text{ is a positive constant}$$

To prove this we analyze the function

$$\theta \mapsto f(e^{i\theta})$$

over θ-intervals which avoids the n-tuple $\{\theta_k\}$. When $\theta_k < \theta < \theta_{k+1}$ holds we have

$$f(e^{i\theta}) = c_0 \cdot \int_0^{e^{i\theta}} \frac{d\zeta}{(1 - e^{-i\theta_1}\zeta)^{a_1} \cdots (1 - e^{-i\theta_N}\zeta)^{a_N}}$$

It follows that the θ-derivative becomes

$$\frac{df}{d\theta} = c_0 \cdot i \cdot e^{i\theta} \cdot \frac{1}{\prod (1 - e^{-i\theta_k + i\theta})^{a_k}}$$

The argument of the right hand side becomes

(i) $$\frac{\pi}{2}+\theta-\sum a_k\cdot \arg\,(1-e^{-i\theta_k+i\theta})$$

As explained in XX: Chapter I we have

$$\arg\,(1-e^{-i\theta_k+i\theta})=\frac{\theta-\theta_k-\pi}{2}$$

for every k. Together with (*) above we conclude that

$$\arg\frac{df}{d\theta}=\frac{\pi}{2}+\theta+\sum a_k\cdot\frac{\theta_k+\pi-\theta}{2}=\frac{3\pi}{2}+\sum a_k\cdot\frac{\theta_k}{2}$$

where the last equality follows from XX. In particular the argument remains constant which means that f maps the circular arc $\{\theta_k<\theta<\theta_{k+1}\}$ onto a line segment. So this confirms that if f is s conformal mapping then it sends D onto a Jordan domain bounded by a piecewise linear boundary curve which we want to be equal to the given convex polygon Π.

Proof that f is a conformal mapping. To show this we investigate the conformal mapping $f_*(z)$ which is already predicted by Riemann's mapping theorem and we are going to show that the complex derivative of f_* is equal to the complex derivative

(1) $$f'(z)=c_0\cdot\phi_1(z)\cdot\phi_N(z)$$

To prove that $f_*'(z)$ is equal to the function in (1) we first consider a circular arc:

(1) $$\gamma_k=\{e^{i\theta}\quad:\theta_k<\theta<\theta_{k+1}\}$$

Now f_* maps this arc onto a line segment of the polygon Π. In particular $\arg(f_*')$ is constant on γ_k and then Schwarz' reflection principle implies that the complex derivative f_*' extends analytically across γ_k. In fact, set

(2) $$\beta_k=\arg f_*'(e^{i\theta})\quad:\quad e^{i\theta}\in\gamma_k$$

In the exterior disc $|z|>1$ we define the analytic function

(3) $$g_k(z)=e^{2i\beta_k}\cdot\bar{f}_*'(\frac{1}{\bar{z}})$$

When $e^{i\theta}\in\gamma_k$ we obtain

(4) $$\arg g_k(e^{i\theta})=2\beta_k\arg f_*'(e^{i\theta})$$

It follows that $g_k=f_*'$ on γ_k and hence g_k yields the analytic extension of f_*' across γ_k where g_k is analytic in the exterior disc $\{|z|>1\}$. Next, for any other arc γ_ℓ we can return to the unit disc by performing an analytic extension of g_k across ℓ_ν. Moreover, we see that the new analytic function in d must be a constant times f_*'. This process can be continued, and the conclusion is that f_*' is a single-valued branch of a multi-valued analytic function F defined in $\mathbf{C}\setminus(e^{i\theta_1},\ldots,e^{i\theta_N})$. Moreover, by the explicit analytic continuations over the γ-segments the rank of the multi-valued function F is equal to one and hence the general result in XX shows that the single-valued branch f_*' is given in the form

(5) $$f_*'(z)=\prod_{k=1}^{k=N}(1-e^{-i\theta_k}z)^{\rho_k}\cdot H(z)$$

where $\rho_1,\ldots,\rho_N$ are complex numbers and $H(z)$ is an entire function. From (4) above the jump of $\arg f_*'$ at each point $e^{i\theta_k}$, via the local study in XX gives the equalities:

$$\rho_k=-a_k$$

There remains to prove that the entire function $H(z)$ is constant. To see this we consider the analytic extension of f_* across some γ_k as above which leads to the analytic function g_k in the exterior domain. By the construction g_k is bounded in this exterior disc and from this we see that the entire function $H(z)$ is bounded and hence reduced to a constant by Liouville's theorem. Hence we have proved that there exists a constant c such that

$$f'_* = c \cdot f'$$

Since both f and f'_* are zero at the origin we have proved that f is a constant times f'_* which shows that f gives the requested conformal mapping from D onto Π.

5.4 A local study.

With the notations as above we analyze the behaviour of the mapping function $f_*(z)$ as $z \to z_k$ for every k. We already know that f_* extends to a multi-valued function in a punctured disc centered at z_k and there remains to consider the situation locally for each given z_k. After a rotation we may take $z_k = w_k = 1$ and at the corner point w_k we have some angle α. Put

$$a = \frac{\alpha}{\pi}$$

Consider the analytic function $g(z)$ in D defined by

(i) $$\phi(z) = (z-1)^{1-a}$$

When $\theta > 0$ is small the complex argument

(ii) $$\arg(e^{i\theta} - 1) \simeq \frac{\pi}{2}$$

At the same time, when θ is small and negative we have

(iii) $$\arg(e^{i\theta} - 1) \simeq -\frac{\pi}{2} \quad : \theta < 0$$

Hence we have two limit formulas:

(iv) $$\lim_{\theta \to 0_+} \arg \phi(e^{i\theta}) = (1-\alpha)\frac{\pi}{2} \quad : \quad \lim_{\theta \to 0_-} \arg \phi(e^{i\theta}) = (\alpha - 1)\frac{\pi}{2}$$

From figure XX the two limit formulas reflect the jump discontinuity of the argument for the mapping function f at the corner point. From this we conclude that $f(z)$ close to $z = 1$ has a fractional series expansion

(v) $$f(z) = 1 + [\,c_1(x-1) + c_2(x-1) + \ldots]^{1-\alpha}$$

Passing to the complex derivative we get

(vi) $$f'(z) = [\,c_1(x-1) + c_2(x-1) + \ldots]^{-\alpha}$$

It follows that

(iv) $$\frac{f''(z)}{f'(z)} = \frac{-\alpha}{z-1} + \sum_{\nu=0}^{\infty} d_\nu \cdot (z-1)^\nu$$

where the last sum is an analytic function in some disc centered at $z = 1$. So in (iv) above we encounter a simple pole whose residue is determined by the angle α. Applying this to every corner point of the given polygon Π we have proved:

5.5 Lemma *Under analytic continuation of $f(z)$ it follows that $\frac{f''(z)}{f'(z)}$ becomes a single valued meromorphic function of the form:*

$$\phi(z) = \sum \frac{-\alpha_\nu}{z_\nu - z} + H(z) \quad : \; H(z) \text{ entire function}$$

There remains only to see that the entire function is zero. To show this we first notice that the analytic continuation of f is achived by reflections in boundary arcs of T and therefore remains bounded. In particular the entire function H is bounded and hence a constant. Finally, this constant must be zero for otherwise

$$\operatorname{Log} f(z) = -\sum \alpha_\nu \cdot \operatorname{Log}(z_\nu - z) + bz \quad : \ b \neq 0$$

and taking the exponential we see that $f'(z)$ increases too fast.

5.2 Determination of the z_k-numbers. The given polygon Π gives the α-numbers in the Schwarz-Christoffell formula. There remains to determine the points $\{z_k = e^{i\theta_k}\}$ on the unit circle which are mapped to corner points of the polygon. This amounts to compute the integrand in Theiorem 5.1 to recover the lengths of the sides in Π. The determination of the N-tuple $z_1, \ldots, z_N$ when Π is prescribed is not easy. The reason is that one cannot solve these z_k-numbers step by step. One has to regard a system of N many *non-linear* equations where the z_k appear in a rather implicit way. However, computer programs offer algorithms which give numerical solutions for the z-numbers. See for example [XXX].

6. Privalov's theorem

Introduction. We shall prove a uniqueness result established by Privalov in 1917. Let $f(z) \in \mathcal{O}(D)$ where no conditions are imposed on $|f(z)|$, i.e. it may have arbitrary growth. There exist f which is not identically zero, and yet the radial limits are all zero, i.e.

$$\lim_{r \to 1} f(re^{i\theta}) = 0 \quad : \forall\, 0 \leq \theta \leq 2\pi$$

Remark. For the construction of such a function f we refer to Privalov's text-book [Pri]. See also [Bi:2.page 152-154] for the construction of an unbounded analytic function f in D which *never* has a radial limit.

To establish a uniqueness theorem for a general unbounded function one must therefore allow *non-tangential limits*. Following Privalov we describe how such non-tangential limit values should be defined.

The Jordan domain $\mathfrak{M}_E$ Let J be an interval in T with two end-points $\zeta_\nu = e^{i\theta_\nu}$ where $|\theta_1 - \theta_2| < \pi$. Then we obtain a curve linear triangle Δ_J constructed upon J as follows: Through ζ_1 we take the straight line ℓ_1 which has angle $\pi/4$ with T. Similarly we construct the line ℓ_2. The two lines intersect at a point $q \in D$. Then Δ_J is the domain bordered by J and the two line segments along the ℓ-lines from ζ_ν to q. The reader may illustrate this by a figure. Notice that the angle at q becomes $\pi/2$.

Next, let E be a closed subset of T. Now $T \setminus E$ is a disjoint union of open intervals J_ν. To each of them we get the domain Δ_ν. Put

$$\mathfrak{W}_E = D \setminus \cup \bar{\Delta}_\nu$$

We refer to $\mathfrak{W}_E$ as *Privalov's domain* attached to E. It is easily seen that it is a Jordan domain. In fact, to each $0 \leq \theta \leq 2\pi$ we consider the ray from the origin in the θ-direction and looking a a figure the reader discovers that there is a unique $0 < r(\theta) \leq 1$ such that $re^{i\theta} \in \partial\mathfrak{W}_E$ while $re^{i\theta} \in \mathfrak{W}_E$ when $0 \leq r < r(\theta)$. In this way we get a bi-continuous map from the periodic θ-interval onto $\partial\mathfrak{W}_E$. Notice also that $r(\theta) = 1$ precisely when $e^{i\theta} \in E$. By the construction the boundary of $\mathfrak{M}_E$ is piecewise linear inside D and where the total arc-length is $\sqrt{2} \cdot \sum |J_k\nu|$ as one sees from the construction of the Δ-triangles. When E has positive Lebesgue measure the remaining part of the simple and closed Jordan $\partial\mathfrak{W}_E$ has length $|E|$. Hence the boundary of $\mathfrak{M}_E$ is a *rectifiable* Jordan curve. Now we can announce Privalov's uniqueness theorem.

6.1 Theorem *Let $E \subset T$ be a closed set of positive measure. If $f \in \mathcal{O}(D)$ is such that*

$$\lim_{z \to E} f(z) = 0 \quad : \quad z \in \mathfrak{W}_E$$

Then f is identically zero.

Proof . By assumption we have a pointwise convergence to zero as $z \in \mathfrak{M}_E$ tends to E. By *Egoroff's theorem* in measure theory we can therefore find a closed subset E_* if E such that the limit is attained *uniformly* as $z \to E_*$ and at the same time E_* again has positive measure. Now we notice that $\mathfrak{W}_{E_*} \subset \mathfrak{W}_E$ and we restrict f to this Jordan domain. Let f_* denote this restricted function. By the uniform convergence it follows that f_* is a bounded analytic function in $\mathfrak{W}_{E_*}$. Next, consider a conformal map ϕ from D onto $\mathfrak{W}_{E_*}$ and put $g = f_* \circ g$. Then g is a bounded analytic function in D. Since E_* has positive Lebesgue measure and $\partial\mathfrak{W}_{E_*}$ is a rectifiable Jordan arc, it follows that the inverse set $\phi^{-1}(E_*)$ appears as a closed subset of T with positive Lebesgue measure. Hence g tends to zero on set of positive measure and is therefore identically zero, i.e. as a very special case of Fatou's result in XX. We conclude that $f = 0$ and Privalov's theorem is proved.

7. Maps between multiple connected domains.

Introduction. Let us begin the study with a doubly connected domain Ω whose outer boundary is the unit circle T and the inner boundary a closed Jordan curve γ contained in the unit disc. There exists the harmonic function ω with boundary value one on T and zero on γ, i.e. ω is the harmonic measure with respect to T. Set

$$b = \int_T \frac{\partial \omega}{\partial \mathbf{n}} \cdot ds$$

It is clear that $b > 0$ and we put:

$$\alpha = \frac{2\pi}{b}$$

The result in XXX shows that there exists an analytic function $f(z)$ in Ω defined by

$$f(z) = e^{\alpha \cdot \omega + iV}$$

where V is the locally defined harmonic conjugate of $\alpha \cdot \omega$. Since

$$|f(z)| = e^{\alpha \cdot \omega(z)}$$

and $\omega(z) = 0$ on γ we conclude that $|f| = 1$ on γ while $|f| = e^{\alpha}$ on T.

7.1. Proposition. *The function f maps from Ω conformally onto the annulus $1 < |z| < e^{\alpha}$.*

Proof. The maximum principle for analytic functions applied to f and $\frac{1}{f}$ give

$$1 < |f(z)| < e^{\alpha} \quad : \, z \in \Omega$$

Next, let w be a complex number in the annulus, i.e. $1 < |w| < e^{\alpha}$. By the general result in XX f is conformal if we prove the equality

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z) - w} \cdot dz = 1 \tag{1}$$

for every such w. To get (1) we use that $|f(z)| = 1 < |w|$ on the inner curve γ. So the result in XXX gives:

$$\int_{\gamma} \frac{f'(z)}{f(z) - w} \cdot dz = 0 \tag{2}$$

Next, on the unit circle T it follows from the general result in XX from Chapter 4 that

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z) - w} \cdot dz = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} \cdot dz = \tag{3}$$

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{dV}{d\theta}(e^{i\theta}) \cdot d\theta$$

Here V is the local harmonic conjugate to $\alpha\omega$ which means gives

$$\frac{dV}{d\theta} = \alpha \cdot \frac{\partial \omega}{\partial \mathbf{n}}$$

along T. Hence the choice of α shows that (3) has value one and together with (2) we conclude that (1) also has value one.

7.2 General case.

Consider a domain $\Omega \in \mathcal{D}(C^1)$ with p boundary curves $\gamma_1, \ldots, \gamma_p$ where $p \geq 3$. Let γ_p be the curve which borders the unbounded connected component of $\mathbf{C} \setminus \Omega$. To each $1 \leq j \leq p-1$ we have the harmonic function ω^j with boundary value 1 on γ_j and zero on the remaining boundary curves. This gives a $(p-1) \times (p-1)$-matrix $\mathbf{B}$ with elements

$$b_{\nu,j} = \int_{\gamma_\nu} \omega^j_{\mathbf{n}} \cdot ds \quad : \, 1 \leq j, \nu \leq p-1 \tag{1}$$

By the result in XXX this matrix is non-singular which gives a unique solution to the following system of inhomogeneous linear equations:

$$b_{11}\xi_1 + \ldots + b_{1,p-1} \cdot \xi_{p-1} = -2\pi$$
$$b_{\nu 1}\xi_1 + \ldots + b_{\nu,p-1} \cdot \xi_{p-1} = 0 \quad : 2 \leq \nu \leq p-1\,.$$

Set

$$H = \xi_1\omega_1 + \ldots + \xi_{p-1}\omega_{p-1}$$

This gives

(2) $$\int_{\gamma_1} H_{\mathbf{n}} \cdot ds = -2\pi \quad \text{and} \int_{\gamma_\nu} H_{\mathbf{n}} \cdot ds = 0 \quad : \; 2 \leq \nu \leq p-1\,.$$

Recall from XXX that the sum of the line integrals over all the boundary curves is zero and hence we have:

(*) $$\int_{\gamma_p} H_{\mathbf{n}} \cdot ds = 2\pi$$

Since the line integrals of $H_{\mathbf{n}}$ are integer multiples of 2π for every the boundary curve the locally defined harmonic conjugate V of H is determined up to integer multiples of 2π. Hence there exists the analytic function $f(z)$ in Ω defined by

$$f(z) = e^{H(z)+iV(z)}$$

Along each inner boundary curve γ_ν we get:

$$|f(z)| = e^{H(z)} = e^{\xi_\nu} \quad : \; 1 \leq \nu \leq p-1$$

and along the outer curve γ_p we have $|f(z)| = 1$. Moreover one has:

7.3 Theorem. *The function f yields a conformal map from Ω onto a domain Ω^*.*

Proof. The same reasoning as in the proof of Proposition 7.1 gives

$$\frac{1}{2\pi} \cdot \int_{\gamma_\nu} H_{\mathbf{n}} \cdot ds = \frac{1}{2\pi i} \int_{\gamma_\nu} \frac{f'(z)}{f(z)} \cdot dz \quad : \quad 1 \leq \nu \leq p$$

Moreover, from (2) we have

$$\int_{\gamma_\nu} \frac{f'(z)}{f(z)} \cdot dz = 0 \quad : \quad 2 \leq \nu \leq p-1$$

Since the absolute value of f along a curve γ_ν is the constant $e^{|xi_\nu}$ the general result in XX gives

(1) $$|w| \neq e^{\xi_\nu} \implies \int_{\gamma_\nu} \frac{f'(z)}{f(z)-w} \cdot dz = 0 \quad : \quad 2 \leq \nu \leq p-1$$

We have also

(2) $$\int_{\gamma_p} \frac{f'(z)}{f(z)-w} \cdot dz = 2\pi \quad \text{for each} \quad |w| < 1$$

(3) $$\int_{\gamma_1} \frac{f'(z)}{f(z)-w} \cdot dz = 0 \quad \text{for each} \quad |w| < e^{\xi_1}$$

while (2) is zero if $|w| > 1$ and similarly (3) is zero if $|w| > e^{\xi_1}$. Next, since f is a non-constant analytic function in Ω we find some w such that $f(z) = w$ has at least one solution where we may assume that $|w| \neq e^{\xi_\nu}$ for all ν and $|w| \neq 1$. Let $N(w)$ be the sum of zeros of $f(z) - w$ counted with multiplicities which by (1) give

$$2\pi i \cdot N(w) = \int_{\gamma_1} \frac{f'(z)}{f(z)-w} \cdot dz + \int_{\gamma_p} \frac{f'(z)}{f(z)-w} \cdot dz$$

From (2-3) we see that $N(w) \geq 1$ implies that $|w| < 1$ and $|w| > e^{\xi_1}$ and that the integer $N(w)$ cannot exceed one. Hence we have proved the inequality

$$e^{\xi_1} < 1$$

and we have also proved also that f restricts to a conformal map in Ω since $w \mapsto N(w)$ is the constant one.

7.4 The image domain $f(\Omega)$. Using (1-3) above one shows that f maps the outer curve γ_p onto the unit circle T and γ_1 onto the circle of radius e^{ξ_1}. Next, if $2 \leq \nu \leq p-1$ then f maps γ_ν onto an interval ℓ_ν of the circle $\{|z| = e^{\xi_\nu}\}$. See figure XX for an illustration of the image domain.

Exercise. Prove the assertions in 7.4. See also [page xx-xx] in [Ahlfors] for a discussion and a proof of (7.4).

Parallell slit regions Let Ω as above be a domain in the class $\mathcal{D})\mathbf{C}^1)$ with p many boundary curves. In Chapter V:A.6 we introduced the function $H(z,\zeta)$ defined in $\Omega \times \Omega$ where Theorem 6.1 in [ibid] shows the symmetry

$$H(z,\zeta) = H(\zeta,z)$$

for each pair of points in Ω. Moreover

$$H(z,\zeta) = \log|z-\zeta|$$

when $z \in \Omega$ and $\zeta \in \partial\Omega$. We can use H to construct special conformal mappings. Write $\zeta = \xi + i\eta$ and notice that the function

$$z \mapsto \frac{\partial}{\partial\xi} H(z,\zeta_0)$$

is a harmonic of the variable z in Ω. Keeping ζ_0 fixed we denote this function by $u(z)$. Next, to each inner boundary curve γ_ν we have the harmonic measure function ω_ν and as explained in Chapter V:A:xx we find real constants $a_1, \ldots, a_{p-1}$ such that if

$$U = u - (a_1 \cdot \omega_1 + \cdots + a_{p-1} \cdot \omega_{p-1})$$

then u has a single-valued harmonic conjugate V which gives the analytic function $f(z) = U + iV$ in Ω. Next, from (xx) we have the equality

$$u(z) = \mathfrak{Re}\,\frac{1}{z-\zeta_0} \quad : \quad z \in \partial\Omega$$

Let us put

$$p(z) = f(z) - \frac{1}{z-\zeta_0}$$

Then we see that $\mathfrak{Re}\,p$ is constant on each boundary curve.

Theorem. *The analytic function $p(z)$ yields a conformal mapping from Ω onto a a slit region whose complement consists of p vertical segments.*

Remark. Above p maps ζ_0 to the point at infinity. Its residue a ζ_0 is one and by this normalisation the vertical segments under a conformal map as above are determined except for a parallel translation.

Example.

8. The Bergman kernel.

Introduction. Let Ω be a bounded and connected open set in $\mathbf{C}$. Denote by $A^2(\Omega)$ the family of analytic functions $f(z)$ which are square integrable, i.e

$$(*)\qquad \iint_\Omega |f(z)|^2 \cdot dxdy < \infty$$

The space $A^2(\Omega)$ was introduced and studied by Stefan Bergman whose text-book [Bergman] used this Hilbert space to construct Riemann's mapping theorem via a kernel function. A merit of Bergman's approach is that the similar Hilbert spaces can be adapted in several variables, i.e. Bergman kernels exist when one regards square integrable analytic functions $f(z)$ defined in domains of $\mathbf{C}^n$ with $n \geq 2$. Here we only consider the 1-dimensional case. The starting point is the observation that if $f(z)$ is analytic in some open disc D_R of radius R centered at the origin, then the vanishing of the integrals

$$\int_0^{2\pi} e^{ik\theta} d\theta \quad : \quad k \neq 0$$

implies that with $f(z) = \sum a_n z^n$ one has the equality:

$$\iint_{D_R} |f(z)|^2 \cdot dxdy = \sum |a_n|^2 \cdot 2\pi \cdot \int_0^R r^{2n+1} dr = \sum |a_n|^2 \cdot 2\pi \cdot \frac{R^{2n+2}}{2n+2}$$

In particular

$$|f(0)|^2 = |a_0|^2 \leq \frac{1}{\pi R^2} \iint_{D_R} |f(z)|^2 \cdot dxdy$$

The square root of the double integral is by definition L^2-norm on $A^2(D_R)$. This gives:

$$(**)\qquad |f(0)| \leq \frac{1}{\sqrt{\pi} \cdot R} \cdot ||f||_2$$

Passing to a domain Ω we use that when $z_0 \in \Omega$ then the open disc of radius $\text{dist}(z_0, \partial\Omega)$ is contained in Ω. Since the area integral of $|f|^2$ taken over Ω is $\geq$ than that over $D_R(z_0)$ it follows that

$$(***)\qquad |f(z_0)| \leq \frac{1}{\sqrt{\pi} \cdot R} \cdot ||f||_2 \quad : \quad R = \text{dist}(z_0, \partial\Omega)$$

These local inequalities show that if $\{f_\nu\}$ is a Cauchy sequence with respect to the L^2-norm, then the maximum norms of $|f_\nu - f_j|$ over compact subsets of Ω tend to zero. Hence $\{f_\nu\}$ converges to an analytic function f_* in Ω and one easily verifies that $f_* \in A^2(\Omega)$ and that the L^2-norms of $f_\nu - f_*$ tend to zero. This proves that $A^2(\Omega)$ is a Hilbert space. Next, (***) shows that the linear functional on $f \mapsto f(z_0)$ is continuous on $A^2(\Omega)$ and the representation formula for elements in the dual of a Hilbert space gives a unique function $g(z) \in A^2(\Omega)$ such that

$$\text{(i)}\qquad f(z_0) = \iint_\Omega f(z)\bar{g}(z) \cdot dxdy \quad : \quad f \in A^2(\Omega)$$

To find an expression for the g-function we choose some orthonormal basis in $A^2(\Omega)$, i.e. a sequence $\{\phi_\nu(z)\}$ for which

$$\text{(ii)}\qquad \int_\Omega \phi_\nu(z)\bar{\phi}_j(z) \cdot dxdy = \text{Kronecker's delta function}$$

Now $g(z)$ has an expansion

$$\text{(iii)}\qquad g(z) = \sum c_k \cdot \phi_k(z)$$

Apply (i) with $f = \phi_k$. Then (ii) gives

$$\text{(iv)}\qquad \phi_k(z_0) = c_k$$

8.1. The kernel function.

Define the function $K(\zeta, z)$ on the product space $\Omega \times \Omega$ by

$$K(\zeta, z) = \sum \phi_\nu(\zeta) \cdot \bar{\phi}_\nu(z) \tag{1}$$

The previous results give

8.2 Theorem. *For every $f \in A^2(\Omega)$ and each $\zeta \in \Omega$ one has*

$$f(\zeta) = \iint f(z) \cdot K(\zeta, z) \cdot dxdy \tag{*}$$

Remark. The kernel function K is analytic in ζ for each fixed z while it is anti-analytic in z when ζ is fixed. A notable point is that the formula (*) does not depend on the chosen orthonormal basis in $A^2(\Omega)$, i.e. one has the equality

$$K(\zeta, z) = \sum \psi_\nu(\zeta) \cdot \bar{\psi}_\nu(z) \tag{*}$$

for *every* orthonormal basis $\{\psi_\nu\}$ in $A^2(\Omega)$.

8.3 Transformation laws. Let $F\colon U \to \Omega$ be an analytic map. That is, $F \in \mathcal{O}(U)$ and the image $F(U) \subset \Omega$. For the moment we do not assume that F is injective and it may occur that its derivative is zero at some points in U. If $g \in \mathcal{O}(\Omega)$ we get $g^* = g \circ F \in \mathcal{O}(U)$. Recall that the Jacobian of the F-map is $|F'(z)|^2$. So if $w = u + iv$ is the complex coordinate in Ω then

$$\iint_\Omega |g(w)|^2 \cdot dudv = \iint_U |g^*(z)|^2 \cdot |F'(z)|^2 \cdot dxdy \tag{i}$$

Hence there is a linear map from $A^2(\Omega)$ into $A^2(U)$ defined by:

$$\mathbf{T}\colon\ g(w) \mapsto F'(z) \cdot g^*(z)$$

Moreover, $\mathbf{T}$ is an isometry, i.e. the L^2-norms of g and of $\mathbf{T}(g)$ are the same. So if $\{\phi_\nu\}$ is an orthonormal basis in $A^2(\Omega)$ then $\{\mathbf{T}(\phi_\nu)\}$ is an orthonormal family in $A^2(U)$. The question arises when this orthonormal family is a basis in $A^2(U)$. From the construction of $\mathbf{T}$ via (ii) above we see that this holds if and only if the derivative $F'(z) \neq 0$ for all points in Ω and in addition F must be 1-1, i.e. $F(z)$ is a conformal map. Moreover, when F is conformal the Remark after Theorem 8.2 shows that the kernel function $K^*(\zeta, z)$ on U becomes:

$$K^*(\zeta, z) = F'(\zeta) \cdot \bar{F}'(z) \cdot K(F(\zeta), F(z)) \tag{**}$$

where K is the kernel function on Ω. We refer to (**) as the *transformation law* for Bergman's kernel function.

8.4 An extremal problem. Let Ω as be some bounded and connected domain. If $z_0 \in \Omega$ we put

$$\lambda^*(z_0) = \max_g |g(z_0)| \quad : \quad ||g||_2 = 1 \tag{i}$$

To find $\lambda^*(z_0)$ and a maximizing g-function we choose an orthonormal basis and write

$$g(z) = \sum c_\nu \cdot \phi_\nu(z)$$

Now $||g||_2 = 1$ means that $\sum |c_\nu|^2 = 1$ so

$$\lambda^*(z_0) = \max \left|\sum c_\nu \cdot \phi_\nu(z_0)\right| \quad : \quad \sum |c_\nu|^2 = 1$$

By the Cauchy-Schwartz inequality the right hand side is majorised by:

$$\sum |\phi_\nu(z_0)|^2 \quad : \quad \text{for all sequences } \{c_\nu\} : \sum |c_\nu|^2 = 1$$

Moreover, equality holds if and only if there is a complex number ρ such that

$$c_\nu = \rho \cdot \bar{\phi}_\nu(z_0) \quad : \quad \nu = 1, 2, \ldots$$

Now ρ must be chosen so that

$$1 = \rho^2 \sum |\phi(z_0)|^2 = \rho^2 \cdot K(z_0.z_0) \implies \rho = \frac{1}{\sqrt{K(z_0, z_0)}}$$

At the same time we get

$$\lambda^*(z_0) = \rho \cdot K(z_0, z_0)$$

Hence we have proved:

8.5 Theorem. *For each $z_0 \in \Omega$ one has the equality*

$$\lambda^*(z_0) = \sqrt{K(z_0, z_0)}$$

Moreover, the g-function which maximizes (i) above is given by

$$g(z) = \frac{1}{\sqrt{K(z_0, z_0)}} \cdot K(z, z_0)$$

8.6 The simply connected case. Let Ω be bounded and simply connected. If $a \in \Omega$ we find the conformal mapping function $f_a \colon \Omega \to D$ using the Kernel function. The result is

8.7 Theorem. *The conformal map f_a is given by*

$$f_a(z) = \sqrt{\frac{\pi}{K(a, a)}} \cdot \int_a^z K(z, a) dz$$

Exercise. Prove this result. The hint is to use the various extremal properties satisfied by the mapping function f_a.

8.8 Orthogonal polynomials. Let Ω be a bounded simply connected domain. The Gram-Schmidt construction gives a special orthonormal basis in $A^2(\Omega)$ given by a sequence of polynomials $\{P_n(z)\}$ where P_n has degree n and

$$\iint P_k \cdot \bar{P}_m \cdot dxdy = \text{Kronecker's delta function}$$

One expects that these polynomials are related to a mapping function. We shall consider the case when Ω is a Jordan domain whose boundary curve Γ is *real-analytic*. Let ϕ be the conformal map ϕ from the *exterior domain* $\Omega^* = \Sigma \setminus \bar{\Omega}$ onto the exterior disc $|z| > 1$. Here ϕ is normalised so that it maps the point at infinity into itself. The inverse conformal mapping function ψ is defined in $|z| > 1$ and has a series expansion

$$\psi(z) = \tau \cdot z + \tau_0 + \sum_{\nu=1}^{\infty} \tau_\nu \cdot \frac{1}{z^\nu} \tag{*}$$

where τ is a positive real number. The assumption that Γ is real-analytic gives some $\rho_1 < 1$ such that ψ extends to a conformal map from the exterior disc $|z| > \rho_1$ onto a domain whose compact complement is contained in Ω. It turns out that the polynomials $\{P_n\}$ are approximated by functions expressed by ϕ and is complex derivative on $\partial\Omega$.

8.9 Theorem. *There exists a constant C which depends upon Ω only such that to every $n \geq 1$ there is a function $\omega_n(z)$ defined in Ω^* and*

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \cdot \phi'(z) \cdot \phi(z)^n \cdot \big[1 + \omega_n(z)\big] \quad \text{holds on} \quad \partial\Omega$$

Finally the ω-functions satisfy

$$\max_{z \in \partial\Omega} \, |\omega_n(z)| \leq C \cdot \sqrt{n} \cdot \rho_1^n \quad : \quad n = 1, 2, \ldots$$

Remark. Theorem 8.10 stems from Faber's article *Über Tschebycheffsche Polynome* from 1920. The subsequent proof is taken from the article [Carleman]

Proof. Let $n \geq 2$ and consider the set of polynomials $Q(z)$ of degree n whose highest term is z^n. Keeping n fixed we set

$$I_*(n) = \min_Q I(Q) = \iint_\Omega |Q(z)|^2 \cdot dxdy$$

Given a polynomial Q as above we choose a primitive polynomial $R(z)$. So here

$$R(z) = \frac{z^{n+1}}{n+1} + b_n z^n + \ldots + b_0$$

1. Exercise. Use Green's formula to show that

$$I(Q) = \frac{1}{4} \int_{\partial\Omega} \partial_n(|R|^2) \cdot ds$$

where ds is the arc-length measure on $\partial\Omega$

Now we use the inverse conformal map $\psi(\zeta)$ and set

$$F(\zeta) = R(\psi(\zeta))$$

Then F is analytic in the exterior disc $|\zeta| > 1$ and (*) above Theorem 8.9 entails that F has the series expansion

$$\text{(i)} \qquad F(\zeta) = \tau^{n+1} \big[\frac{\zeta^{n+1}}{n+1} + A_n \zeta^n + \ldots + A_1 \zeta + A_0 + \sum_{nu=1}^{\infty} \alpha_\nu \cdot \zeta^{-\nu} \big]$$

2. Exercise. Use a variable substitution via ψ to show that the integral in Exercise 1 is equal to

$$\int_{|\zeta|=1} \frac{d}{dr} (|F(e^{i\theta})|^2 \cdot d\theta$$

and use the series expansion (i) to show that this integral is equal to

$$\pi \cdot \tau^{2n+2} \cdot \big[\frac{1}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 - \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \big]$$

An upper bound for I_*. The coefficients $A_1, \ldots, A_n$ are determined via Q. The reader may verify that there exists a polynomial $Q(z)$ with highest term z^n such that $A_1 = \ldots = A_n = 0$. It follows that

$$(*) \qquad I_* \leq \pi \cdot \tau^{2n+2} \cdot \big[\frac{1}{n+1} - \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \big] \leq \pi \cdot \tau^{2n+2} \cdot \frac{1}{n+1}$$

A lower bound for I_*. The upper bound (*) did not use that $\partial\Omega$ is real-analytic, i.e. it is valid for every Jordan domain whose boundary curve is of class C^1. To get a lower bound we choose some $\rho_1 < \rho < 1$ and by assumption ψ maps the exterior disc $|\zeta| > \rho$ conformally to an exterior domain $U^* = \Sigma \setminus \bar{U}$ where U is a relatively compact Jordan domain inside Ω. Let Q be a polynomial for which $I(Q) = I_*$. Now $\Omega \setminus \bar{U}$ is contained in Ω so we have

$$\text{(i)} \qquad I_* > \iint \Omega \setminus \bar{U}\, |Q(z)|^2 \cdot dxdx$$

3. Exercise. Show that the integral in (i) is equal to

$$\int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot d\theta - \int_{|\zeta|=\rho} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot \rho \cdot d\theta =$$

$$\text{(ii)} \qquad \pi \cdot \tau^{2n+2} \cdot [\frac{1-\rho^{2n+2}}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1)\,]$$

The last equality gives the lower bound

$$(**) \qquad I_* \geq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot (1-\rho^{2n+2})$$

Moreover the upper bound and (ii) give the inequality

$$\sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1) \leq \frac{\pi}{n+1} \cdot \rho^{2n+2}$$

Since $1-\rho^2 \leq 1-\rho^{2\nu}$ for every $\nu \geq 1$ it follows that

$$(***) \qquad \sum_{k=1}^{k=n} k \cdot |A_k|^2 + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \leq \frac{\pi}{(1-\rho^2) \cdot n+1} \cdot \rho^{2n+2}$$

Conclusion. Recall that $F(\zeta) = R(\psi(\zeta)$ and $R' = Q$. So after a derivation we get

$$F'(\zeta) = \psi'(\zeta) \cdot Q(\psi(\zeta))$$

Hence the series expansion of $F(\zeta)$ gives

$$\text{(i)} \qquad Q(\psi(\zeta)) = \frac{\tau^{n+1}}{\psi'(\zeta)} \cdot \big[\zeta^n + \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}\,\big]$$

where the equality holds for $|\zeta| > \rho$. Put

$$\omega^*(\zeta) = \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}$$

When $|\zeta| = 1$ the triangle inequality gives

$$\text{(ii)} \qquad |\omega^*(\zeta)| \leq \sum_{k=1}^{k=n} k \cdot |A_k| + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|$$

4. Exercise. Notice that (***) holds for every $\rho > \rho_1$ and use this together with suitable Cauchy-Schwarz inequalities to show that (i) above gives a constant C which is independent of n such that

$$|\omega^*(\zeta)| \leq C \cdot \sqrt{n} \cdot \rho_1^{n+1}$$

Final part of the proof. Since ψ is the inverse of ϕ we have

$$\psi'(\phi(z)) \cdot Q(\psi(\phi(z) = \frac{Q(z)}{\phi'(z)}$$

Then (iii) in the conclusion implies that if we define the function $\partial\Omega$ by

$$\omega_n(z) = \frac{\omega^*(\phi(z))}{\phi'(z)}$$

then we have

$$Q(z) = \tau^{n+1} \cdot \phi'(z) \cdot [\phi(z)^n + \omega_n(z)\,]$$

where Exercise 4 shows that $|\omega_n(z)|$ satisfies the estimate in Theorem 8.5. Finally, the Q-polynomial minimized th L^2-norm under the constraint that the leading term is z^n and for this variational problem he upper and the lower bounds in (*-**) above show that the minimum $I_*(n)$ satisfies

$$|I_*(n) - \frac{\pi}{n+1} \cdot \tau^{2n+2}| \leq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot \rho^{2n+2}$$

So if we normalise Q so that its L^2-norm is one and hence gives the polynomial $P_n(z)$ then the factor τ^{n+1} is replaced by $\frac{\sqrt{n+1}}{\sqrt{\pi}}$ which finishes the proof of Theorem 8.10.

9. The electric equilibrium potential.

Introduction. Let Γ be a closed Jordan circle of class C^1. We seek a positive density function $\mu(s)$ where ds is the arc-length measure on Γ such that the logarithmic potential from the introduction is constant on Γ. It turns out that μ is found from a conformal mapping. Namely, consider the *exterior domain* Ω^* which is bounded by Γ. So here the bounded Jordan domain D bordered by Γ is the open complement of $\bar{\Omega}^*$. Adding the point at infinity we get a simply connected set $\Omega_\infty = \Omega^* \cup \infty$. Riemann's mapping theorem gives a conformal map from Ω_∞ onto the exterior disc $|w| > 1$. Choose f so that it maps the point at infinity to the point at infinity . This means that there is some real number $a > 0$ and $f(z) \simeq az$ as $|z| \to \infty$. Now we prove that the function defined on Γ by

$$(*) \qquad z_* \mapsto \int_\Gamma \log\Big(\frac{1}{|z - z_*|}\Big) \cdot \frac{|dz|}{|f'(z)|}$$

is constant. To get this we set $w = f(z)$ and let $\phi(w)$ be the inverse function. So here ϕ is analytic in $|w| > 1$. Under the conformal map we have the arc-length formula

$$|dw| = \frac{|dz|}{|f'(z)|}$$

Hence (*) amounts to show that

$$(**) \qquad \theta_* \mapsto \int_0^{2\pi} \log\Big(\frac{1}{|\phi(e^{i\theta}) - \phi(e^{i\theta_*})|}\Big) \cdot d\theta$$

is a constant function of θ_*. Since the image set $\phi(T) = \Gamma$ we can define a function in the Jordan domain D by

$$H(z) = \int_0^{2\pi} \text{Log}\Big(\frac{1}{|\phi(e^{i\theta}) - z|}\Big) \cdot d\theta$$

This is a harmonic function in D given by the real part of the multi-valued function

$$z \mapsto \int_0^{2\pi} \text{Log}\Big(\frac{1}{\phi(e^{i\theta}) - z}\Big) \cdot d\theta$$

We avoid eventual multi-valuedness by taking derivatives, i.e. passing to the complex derivative we get

$$(1) \qquad H_x - iH_y = \int_0^{2\pi} \frac{1}{\phi(e^{i\theta}) - z} \cdot d\theta \quad : \quad z \in D$$

With $z \in D$ kept fixed, the right hand side can be written as a complex line integral:

$$\int_{|w|=1} \frac{1}{\phi(w) - z} \cdot \frac{dw}{iw}$$

Now the ϕ function is analytic in $|w| > 1$ and the ϕ-image of $|w| > 1$ is contained in Ω^* so $z \in D$ stays outside this set. By Cauchy's theorem we can therefore shift the integration to circle $|w| = R$ with $R > 1$ and hence (**) becomes:

$$(2) \qquad \int_{|w|=R} \frac{1}{\phi(w) - z} \cdot \frac{dw}{iw}$$

Next, we have $\phi(w) \simeq \frac{w}{a}$ when $|w| \to \infty$. It follows that (***) tends to zero when $R \to +\infty$. This proves that (**) is identically zero and hence the harmonic function $H(z)$ is a constant in D. Finally, $H(z)$ extends to a continuous function on the closed Jordan domain $\bar{D}$ and we conclude that the function from (**) is constant.

A special class of curves

.

Let Γ be a closed Jordan curve whose curvature ρ is $\neq 0$ at all points. One may ask when the equilibrium density μ_Γ is proportional to a fractional power of ρ, i.e. when there exists some $\beta > 0$ such that

$$(*) \qquad \mu_\Gamma^\beta(z) = \rho(z) \quad : \quad z \in \Gamma$$

To begin with one may notice that (*) holds when Γ is a circle. Conversely one can show that if (*) holds for some real number $\beta > 0$ which is not an integer ≥ 2, then Γ must be a circle. A proof is given in the article [xx] by Carleman. Here we shall present the non-trivial cases which appear in this article.

A preliminary construction. Let $m \geq 3$ be an integer. With $\beta = \frac{2}{m}$ one has Newton's formula for binomial series:

$$(1) \qquad (1+t)^\beta = 1 + \binom{\beta}{1} \cdot t + \binom{\beta}{2} \cdot t^2 + \ldots \quad : \quad |t| < 1$$

With $t = \frac{1}{z^m}$ where $|z| > 1$ we get

$$(1 + \frac{1}{z^m})^\beta = 1 + \binom{\beta}{1} \cdot \frac{1}{z^m} + \binom{\beta}{2} \cdot \frac{1}{z^{2m}} + \ldots$$

In the exterior disc $|z| > 1$ we now define the analytic function

$$(2) \qquad f(z) = z - \frac{1}{m-1}\binom{\beta}{1} \cdot \frac{1}{z^{m-1}} - \frac{1}{2m-1}\binom{\beta}{m} \cdot \frac{1}{z^{2m-1}} -$$

At the point at infinity the f-function has a simple pole while $f(z) - z$ extends analytically to ∞. With this in mind we have the following:

9.1 Theorem. *f gives a conformal map from the exterior unit disc to an exterior domain Ω which borders a closed Jordan curve Γ for which (*) holds with $\beta = \frac{2}{m}$.*

Remark. In particular this result asserts that f yields a bijective map from the unit circle $|z| = 1$ onto a simple and closed curve. By (2) above f maps ∞ into itself. The proof requires several steps and we begin with:

A. Proof that $f(z)$ is 1-1 outside unit disc. Consider a pair $z_1 \neq z_2$ in the exterior unit disc: Then we have

$$(1) \qquad \frac{f(z_1) - f(z_2)}{z_1 - z_2} = 1 + P \quad \text{where}$$

$$P = \frac{1}{(m-1)z_1 z_2} \cdot \binom{\beta}{1} \cdot [(\frac{1}{z_1})^{m-2} + (\frac{1}{z_1})^{m-3} \cdot \frac{1}{z_2} + \ldots + (\frac{1}{z_2})^{m-2}] +$$

$$\frac{1}{(2m-1)z_1 z_2} \cdot \binom{\beta}{2} \cdot [\frac{1}{z_1})^{2m-2} + (\frac{1}{z_1})^{2m-3} \cdot \frac{1}{z_2} + \ldots + \frac{1}{z_2})^{2m-2}] + \ldots$$

When $k \geq 3$ the k:th term in the sum defining P becomes

$$\frac{1}{(km-1)z_1 z_2} \cdot \binom{\beta}{k} \cdot [\frac{1}{z_1})^{km-2} + (\frac{1}{z_1})^{km-3} \cdot \frac{1}{z_2} + \ldots + \frac{1}{z_2})^{km-2}]$$

Since $m-1 \leq km-1$ when $k \geq 2$ we see that

$$(*) \qquad |P| \leq \binom{\beta}{1} + \binom{\beta}{2} + \ldots = 2^\beta - 1$$

where the last equality follows from Newton's formula. Now $2^\beta - 1 < 1$ so (1) implies that $f(z_1) \neq f(z_2)$.

Proof continued. Since $f(z) \simeq z$ when $|z|$ is large it follows from the above together with Rouche's theorem that f yields a conformal mapping. SAY that f is primitive of newton series and therefore extends to $|z| = 1$ and image nice jordan curve - also rectifiable. The required potential via Riemann's theorem. Remains to check the curvature of $f(T)$ which is found by general formula: Loewner in previous section.

10. Conformal maps of circular domains.

Introduction. In 1906 Koebe proved a result about conformal mappings between domains bordered by a finite set of circles. Let $p \geq 2$ and denote by $\mathcal{C}^*(p)$ the family of connected bounded domains Ω in $\mathbf{C}$ for which $\partial\Omega$ is the union of p many disjoint circles.

Theorem. *Let $f\colon \Omega \to U$ be a conformal map between two domains in $\mathcal{C}^*(p)$. Then $f(z)$ is a linear function, i.e. $f(z) = Az + B$ for some constants A and B.*

Koebe's original proof used reflections over the boundaries and results related to the uniformisation theorem. A more direct proof was given by Carleman in [Car] which we expose below. It teaches how to compute certain winding numbers in specific situations.

Let f be a mapping function as above. Let Ω be bordered by circles $C_1, \ldots, C_p$ where C_p is the outer circle. Similarly U is bordered by $C_1^*, \ldots, C_p^*$. Then f must map C_p to C_p^* and the remaining discs are arranged so that f maps C_ν onto C_ν^* for $1 \leq \nu \leq p-1$. Using a linear map of the outer discs we may assume from the start that $C_p = C_p^* = T$ where T is the unit circle. Moreover, after a suitable rotation we may also assume that the map $f\colon T \to T$ has at least 3 fixed points. There remains to show that when this holds, then $f(z) = z$ must be the identity map. To prove this we shall argue by a contradiction. Namely, if $f(z)$ is not the identity we get the non-constant function

$$(*) \qquad \phi(z) = f(z) - z$$

Notations. To each $1 \leq \nu \leq p$ we denote by $n^{(\nu)}$ the number of zeros of ϕ counted with multiplicities which belong to C_ν. We also set

$$J_\nu = \frac{1}{2\pi i} \cdot \int_{C_\nu} \frac{\phi'(z)}{\phi(z)} \cdot dz \quad : \quad 1 \leq \nu \leq p$$

Now we will derive a contradiction using residue formulas to compute these J-numbers. For each $1 \leq \nu \leq p-1$ one encounters nine different case which are listed below:

Separate cases.

Let $1 \leq \nu \leq p-1$ be given. Then J_ν is found by the equations below depending on the positions of the two circles C_ν and C_ν^*:

Case 1: $C_\nu = C_\nu^*$. Here

$$J_\nu = 1 + \frac{1}{2} \sum n_k^{(\nu)}$$

Case 2: C_ν and C_ν^* are exterior to each other, i.e. C_ν^* is outside the closed disc bordered by C_ν. Then

$$J_\nu = 0$$

Case 3: C_ν inside the open disc bordered by C_ν^* or conversely C_ν^* inside the open disc bordered by C_ν. Then

$$J_\nu = 1$$

Case 4: $C_\nu \cap C_\nu^*$ consists of two points P, Q, none of which is a zero of ϕ. Then

$$J_\nu = 0, 1, 2 \text{ i.e. one of these numbers are attained}$$

Case 5: $C_\nu \cap C_\nu^*$ consists of two points P, Q where one of these two points is a zero of ϕ of some multiplicity e. Then

$$J_\nu = \frac{1+e}{2} \text{ or equal to } \frac{3+e}{2}$$

Case 6: $C_\nu \cap C_\nu^*$ consists of two points P, Q where both are zeros of ϕ with multiplicity e and f. Then

$$J_\nu = 1 + \frac{e+f}{2}$$

Case 7: C_ν and C_ν^* has a a common tangential point P which is not a zero of ϕ. See Figure XX for this case. Then

$$J_\nu = 0 \text{ or equal to } 1$$

Case 8: $C_\nu \cap C_\nu^*$ is reduced to a single point P and are otherwise external to each other and P is a zero of ϕ with multiplicity e. Then

$$J_\nu = \frac{1+e}{2}$$

Case 9: $C_\nu \cap C_\nu^*$ is reduced to a single point P and this time C_ν is contained in the closed disc bordered by C_ν^* or vice versa and p is a zero of ϕ of multiplicity e. Then

$$J_\nu = \frac{k+e}{2} + \quad : \quad k = 1, 2, 3$$

Exercise. Verify the nine formulas above. A hint is that the conformal mapping f restricts to bijective map from C_ν onto C_ν^* and preserves orientation.

Conclusion. For each $1 \leq \nu \leq p$ we have found that J-number is ≥ 0. Next, on the outer circle T we have

$$J_1 = \frac{1}{2\pi i} \cdot \int_T \frac{\phi'(z)}{\phi(z)} \cdot dz = 1 - \frac{1}{2} \cdot N_T(\phi)$$

where $N_T(\phi)$ is the number of zeros of ϕ on the unit circle counted with multiplicities. By the hypothesis $N_T(\phi) \geq 3$ Hence $J_1 \leq -\frac{1}{2}$ is strictly negative. Now we get a contradiction. Namely, let $\mathcal{N}_\Omega(\phi)$ be the number of zeros of ϕ in the open domain ϕ. Then we have the general formula:

$$(*) \qquad \mathcal{N}_\Omega(\phi) = J_1 - (J_2 + \ldots + J_p)$$

But this is a contradiction, i.e. the left hand side is a non-negative integer and by the above the right hand side is < 0. Hence the ϕ-function must be identically zero which proves Koebe's theorem.

11. Beurling's conformal mapping theorem.

Introduction. Let D be the open unit disc $|z| < 1$. Denote by $\mathcal{C}$ the family of conformal maps $w = f(z)$ which map D onto some simply connected domain Ω_f which contains the origin and satisfy:

$$f(0) = 0 \quad \text{and} \quad f'(0) \text{ is real and positive}\,.$$

Riemann's mapping theorem asserts that for every simply connected subset Ω of $\mathbf{C}$ which is not equal to $\mathbf{C}$ there exists a unique $f \in \mathcal{C}$ such that $\Omega_f = \Omega$. We are going to construct a subfamily of $\mathcal{C}$. Consider a positive and bounded continuous function a function Φ defined in the whole complex w-plane.

0.1 Definition. *The set of all $f \in \mathcal{C}$ such that*

$$(*) \qquad \lim_{r \to 1} \max_{0 \leq \theta \leq 2\pi} \big[|f'(re^{i\theta})| - \Phi(f(re^{i\theta})) \big] = 0$$

is denoted by $\mathcal{C}_\Phi$.

Remark. Thus, when $f \in \mathcal{C}_\Phi$ then the difference of the absolute value $|f'(z)|$ and $\Phi(f(z))$ tends uniformly to zero as $|z| \to 1$. Let M be the upper bound of Φ. The maximum principle applied to the complex derivative $f'(z)$ gives

$$|f'(z)| \leq M \quad : \quad z \in D$$

Hence $f(z)$ is a continuous function in the open disc D whose Lipschitz norm is uniformly bounded by M. This implies that f extends to a continuous function in the closed disc, i.e. f belongs to the disc algebra $A(D)$. Notice also that (*) implies that the function $z \mapsto |f'(z)|$ extends to a continuous function on $\bar{D}$.

1. Theorem. *Assume that $Log\, \frac{1}{\Phi(w)}$ is subharmonic. Then $\mathcal{C}_\Phi$ contains a unique function f^*.*

Remark. In the special case when $\Phi(w) = \Phi(|w|)$ is a radial function we notice that for every $\rho > 0$ such that $\Phi(\rho) = \rho$ it follows that the function $f(z) = \rho \cdot z$ belongs to $\mathcal{C}_\Phi$. So for a radial Φ-function where different ρ-numbers exist one does not have uniqueness. The reader may verify that a radial function Φ for which $\Phi(\rho) = \rho$ has several solutions cannot satisfy the condition in Theorem 1. Next, let us give examples of Φ-functions which satisfy the condition in Theorem 1. Consider an arbitrary real-valued and non-negative L^1-function $\rho(t,s)$ which has compact support. Set

$$\Phi(w) = \exp\Big[\int \log \frac{1}{|w - t - is[} \cdot \rho(t,s) \cdot dtds\,\Big]$$

Here $\log \frac{1}{\Phi}$ is subharmonic and Theorem 1 asserts that there exists a unique simply connected domain Ω which contains the origin such that the normalised conformal mapping function $f\colon D \to \Omega$ satisfies

$$|f'(e^{i\theta})| = \Phi(f(e^{i\theta})) \quad : \quad 0 \leq \theta \leq 2\pi\,.$$

The proof of Theorem 1 relies upon some results where we only assume that the Φ-function is continuous and positive.

The family $\mathcal{A}_\Phi$. A conformal map $f(z)$ in $\mathcal{C}$ belongs to $\mathcal{A}_\Phi$ if

$$(1) \qquad \limsup_{|z| \to 1}\ |f'(z)| - \Phi(f(z)) \leq 0$$

Remark. By the definition of limes superior this means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f'(z)| \leq \Phi(f(z)) + \epsilon \quad : \text{for all } 1 - \delta < |z| < 1\,.$$

The maximal region $\Omega^*(\Phi)$. With Φ given we define a bounded open subset in the w-plane as follows:

$$\Omega^*(\Phi) = \cup f(D) \quad : \quad \text{union taken over all } f \in \mathcal{A}_\Phi \,.$$

With these notations we have

2. Theorem. *The maximal region $\Omega^*(\Phi)$ is simply connected. Moreover, there exists a unique conformal map with $f^*(D) = \Omega^*(\Phi)$ which in addition belongs to $\mathcal{C}_\Phi$.*

The family $\mathcal{B}_\Phi$. It consists of all $f \in \mathcal{C}$ such that

$$\text{Lim.inf.}_{\,|z|\to 1}\, |f'(z)| - \Phi(f(z)) \geq 0$$

To this family we assign minimal region

$$\Omega_*(\Phi) = \cap f(D) \quad \text{The intersection taken over all } f \in \mathcal{B}_\Phi$$

3. Theorem. *The set $\Omega_*(\Phi)$ is simply connected and the unique $f_* \in \mathcal{C}$ for which $f_*(D) = \Omega_*(\Phi)$ belongs to $\mathcal{C}_\Phi$.*

The constructions of the maximal and the minimal region give

$$\Omega_*(\Phi) \subset \Omega^*(\Phi) \tag{1}$$

In general this inclusion it is strict as seen by the example when Φ is radial. But when Log $\frac{1}{|\Phi|}$ is subharmonic the uniqueness in Theorem 1 asserts that one has the equality $\Omega_*(\Phi) = \Omega^*(\Phi)$.

Remark about the proofs. Following Beurling's article [Beur] we shall give the details of the proof of Theorem 2. Concerning Theorem 3 it is substantially harder and for this part of the proof we refer to [Beur: p. 127-130] for details, Before we enter the proof of Theorem 2 we show how Theorem 2 and 3 together give the uniqueness in Theorem 1.

A. Proof of Theorem 1.

Let Φ be as in Theorem 1. Admitting Theorem 2 and 3 we get the two simply connected domains $\Omega^*(\Phi)$ and $\Omega_*(\Phi)$. Keeping Φ fixed we set $\Omega^* = \Omega^*(\Phi)$ and $\Omega_* = \Omega_*(\Phi)$. Since $\Omega_* \subset \Omega^*$ Riemann's mapping theorem gives an inequality for the first order derivative at $z = 0$:

$$f_*'(0) \leq (f^*)'(0) \tag{i}$$

Next, we can write

$$\Phi(w) = e^{U(w)}$$

where $U(w)$ by assumption is super-harmonic. We can solve the Dirichlet problem with respect to the domain Ω^*. This gives the harmonic function U^* in Ω^* where

$$U^*(w) = U(w) \quad w \in \partial\Omega^* \,.$$

Similarly we find the harmonic function U_* in Ω_* such that

$$U_*(w) = U(w) \quad w \in \partial\Omega_* \,. \tag{*}$$

Next, since $f^* \in \mathcal{C}_\Phi$ we have the equality

$$\log|(f^*)'(z)| = U(f(z)) \quad |z| = 1 \tag{ii}$$

Now $\log|(f^*)'(z)|$ and $U^*(f(z))$ are harmonic in D and (ii) gives:

$$\log(f^*)'(0) = U^*(0)$$

In a similar way we find that

$$\log f_*'(0) = U_*(0)$$

Since U is super-harmonic in Ω^* and $\partial\Omega_*$ is a closed subset of $\bar{\Omega}^*$ we get:

$$U(w) \geq U^*(w) \quad w \in \partial\Omega_*$$

From (*) it therefore follows that $U_* \geq U^*$ holds in Ω_*. So in particular

$$\log f_*'(0) = U_*(0) \geq U^*(0) = \log (f^*)'(0)$$

Together with (i) we conclude that $f_*(0) = (f^*)'(0)$. Finally, the uniqueness in Riemann's mapping theorem gives $\Omega_* = \Omega^*$ and hence that $f_* = f^*$ which proves Theorem 1.

B. Proof of Theorem 2.

The first step in the proof is to construct a certain "union map" defined by a finite family $f_1, \ldots, f_n$ of functions $\mathcal{A}_\Phi$. Set

(*) $$S_\nu = f_\nu(D) \quad \text{and } S_* = \cup\, S_\nu$$

So above S_* is a union of Jordan domains which in general can intersect each other in a rather arbitrary fashion.

B.1 Definition. *The extended union denoted by $EU(S_*)$ is defined as follows: A point w belongs to the extended union if there exists some closed Jordan curve γ which contains w in its interior domain while $\gamma \subset S_*$.*

Exercise. Verify that the extended union is *simply connected*.

B.2 Lemma *Let f_* be the unique normalised conformal map from D onto the extended union above. Then $f_* \in \mathcal{A}_\Phi$.*

Proof. First we reduce the proof to the case when all the functions $f_1, \ldots, f_n$ extend to be analytic in a neighborhood of the closed disc $\bar{D}$. In fact, with $r < 1$ we set $f_\nu^r(z) = f_\nu(rz)$ and get the image domains $S_\nu[r] = f_\nu^r(D) = f_\nu(D_r)$. Put $S_*[r) = \cup\, S_\nu[r]$ and construct its extended union which we denote by $S_{**}[r]$. Next, let $\epsilon > 0$ and consider the new function $\Psi(w) = \Phi(w) + \epsilon$. Let $f_*[r]$ be the conformal map from D onto $S_{**}[r]$. If Lemma B.2 has been proved for the n-tuple $\{f_\nu^r\}$ it follows by continuity that $f_*[r]$ belongs $\mathcal{A}_\Psi$ if r is sufficently close to one. Passing to the limit we see that $f_* = \lim_{r\to 1} f_*[r]$ and we get $f_* \in \mathcal{A}_\Psi$. Since $\epsilon > 0$ is arbitrary we get $f_* \in \mathcal{A}_\Phi$ as required.

After this preliminary reduction we consider the case when each f-function extends analytically to a neighborhood of the closed disc $|z| \leq 1$. Then each S_ν is a closed real analytic Jordan curve and the boundary of S_* is a finite union of real analytic arcs and some corner points. In particular we find the outer boundary which is a piecewise analytic and closed Jordan curve Γ and the extended union is the Jordan domain bordered by Γ. It is also clear that Γ is the union of some connected arcs $\gamma_1, \ldots, \gamma_N$ and a finite set of corner points and for each $1 \leq k \leq N$ there exists $1 \leq \nu(k) \leq n$ such that

$$\gamma_k \subset \partial S_{\nu(k)}$$

Denote by $\{F_\nu = f_\nu^{-1}\}$ and $F = f_*^{-1}$ the inverse functions and put:

$$G = \text{Log}\,\frac{1}{|F|} \quad : G_\nu = \text{Log}\,\frac{1}{|F_\nu|} \quad : 1 \leq \nu \leq n\,.$$

With $1 \leq \nu \leq n$ kept fixed we notice that G_ν and G are super-harmonic functions in S_ν and the difference

$$H = G - G_\nu$$

is superharmonic in S_ν. Next, consider a point $p \in \partial S_\nu$. Then $|F_\nu(p)| = 1$ and hence $G_\nu(p) = 0$. At the same time p belongs to ∂S_* or the interior of S_* so $|F(p)| \leq 1$ and hence $G(p) \geq 0$. This shows that $H \geq 0$ on ∂S_ν and by the minimum principle for harmonic functions we obtain:

(i) $$H(q) \geq 0 \quad \text{for all } q \in S_\nu$$

Let us then consider some boundary arc γ_k where $\gamma \subset \partial S_\nu$, i.e. here $\nu = \nu(k)$. Now $H = 0$ on γ_k and since (i) holds it follows that the *outer normal derivative*:

(ii) $$\frac{\partial H}{\partial n}(p) \leq 0 \quad p \in \gamma_k$$

Since $|F| = |F_\nu| = 1$ holds on γ_k and the gradient of H is parallell to the normal we also get:

$$\frac{\partial G}{\partial n}(p) = -|F'(w)| \quad \text{and} \quad \frac{\partial G_\nu}{\partial n}(p) = -|F'_\nu(w)| \quad : w \in \gamma_k$$

Hence (ii) above gives

(iii) $$|F'(w)| \geq |F'_\nu(w)| \quad \text{when} \quad w \in \gamma_k$$

Next, since $f_\nu \in \mathcal{A}_\Phi$ we have

(iv) $$|f'_\nu(F_\nu(w))| \leq \Phi(w)$$

and since F_ν is the inverse of f_ν we get

$$1 = f'_\nu(F_\nu(w)) \cdot F'_\nu(w)$$

Hence (iv) entails

(v) $$|F'_\nu(w)| \geq \frac{1}{\Phi(w)}$$

We conclude from (iii) that

(vi) $$|F'(w)| \geq \frac{1}{\Phi(w)} \quad : \ w \in \gamma_k$$

This holds for all the sub-arcs $\gamma_1, \ldots, \gamma_n$ and hence we have proved the inequality

(*) $$|F'(w)| \geq \frac{1}{\Phi(w)} \quad \text{for all} \quad w \in \Gamma$$

except at a finite number of corner points. To settle the situation at corner points we notice that Poisson's formula applied to the harmonic function $\log |f'_*(z)|$ in the unit disc gives

(vii) $$\log |f'_*(z)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot \log |f'(e^{i\theta})| \cdot d\theta\,.$$

Next, since F is the inverse of f_* we have

$$|f'_*(z)| \cdot |F'(f(z))| = 1 \quad \text{for all } |z| = 1\,.$$

Hence (vi) gives

$$|f'(z)| \leq \Phi(f_*(z)) \quad \text{for all } |z| = 1\,.$$

With $\Phi = e^U$ we therefore get

$$\log |f'_*(z)| \leq U(f(z)) \quad \text{for all } |z| = 1\,.$$

From the Poisson integral (vii) it follows that

$$\log |f'_*(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot U(f_*(e^{i\theta})) \cdot d\theta\,. \quad z \in D$$

A passage to the limit. I addition to the obvious equi-continuity the passage to the limit requires some care which is exposed in [Beurling: Lemma 1, page 122]. Passing to the limit as $|z| \to 1$ the continuity of Φ implies that f_* belongs to $\mathcal{A}_\Phi$ which proves Lemma B.2.

B.3 The construction of f^*

By the uniform bound for Lipschitz norms the family $\mathcal{A}_\phi$ is equi-continuous. We can therefore find a denumerable dense subset $\{h_\nu\}$. It means that to every $f \in \mathcal{A}_\Phi$ and every $\epsilon > 0$ there exists some h_ν such that the maximum norm $|f - h_\nu|_D < \epsilon$. It follows that

$$\Omega^* = \cup\, h_\nu(D)$$

Next, to every $n \geq 2$ we have the n-tuple $h_1, \ldots, h_n$ and by Lemma B. 2 we construct the function f_n where we have the inclusions

$$h_\nu(D) \subset f_n(D) \quad : 1 \leq \nu \leq n$$

Moreover, the image domains $\{f_n(D)\}$ increase with n. So (i) above gives

(*) $$\Omega^* = \cup\, f_n(D)$$

Next, $\{f_n\}$ is a normal family of analytic functions and since their image domains increase it follows there exists the limit function f^* which belongs to $\mathcal{C}$ and (*) above gives the equality $f^*(D) = \Omega^*$. There remains to prove that f^* also belongs to $\mathcal{C}_\Phi$. To get the inclusion

$$f^* \in \mathcal{C}_\Phi$$

we establish a relation between Φ and the maximal domain $\Omega^*(\Phi)$.

B.4 Proposition. *Let Ψ be a positive continuous function which is equal to Φ outside $\Omega^*(\Phi)$ while its restriction to $\Omega^*(\Phi)$ is arbitrary. Then one has the equality*

$$\Omega^*(\Phi) = \Omega^*(\Psi)$$

Proof. The assumption gives

(i) $$\Psi(w) = \Phi(w) \quad \text{for all } w \in \partial\Omega^*(\Phi)$$

It follows that $f^* \in \mathcal{A}_\Psi$. Hence the equality $f^*(D) = \Omega^*(\Phi)$ and the construction of $\Omega^*(\Psi)$ gives the inclusion

(ii) $$\Omega^*(\Phi) \subset \Omega^*(\Psi)$$

Next, let h^* be the mapping function associated to Ψ. and by the construction of the maximal region $\Omega^*(\Phi$ we get:

(iv) $$\Omega^*(\Psi) = h^*(D) \subset \Omega^*(\Phi)$$

Hence (iii) and (iv) give the requested equality

(v) $$\Omega^*(\Psi) = \Omega^*(\Phi)$$

B.5 A special choice of Ψ

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Keeping Φ fixed we put $\Omega^*(\Phi) = \Omega^*$ to simplify the notations. We have the U-function such that

(vi) $$\Phi(w) = e^{-U(w)}$$

Now $U(w)$ is a continuous function on $\partial\Omega^*$ and solving the Dirichlet problem we obtain the function $U_*(w)$ where $U_* = U$ outside Ω^*, and in Ω^* the function U_* is the harmonic extension of the boundary function U restricted to $\partial\Omega^*$. Set

$$\Psi(w) = e^{U_*(w)}$$

Proposition B.4 gives

(i) $$f^* \in \mathcal{A}_\Psi$$

Next, consider the function in D defined by:

$$V(z) = \log \Big|\frac{df^*(z)}{dz}\Big| - U_*(f(z))$$

From (i) it follows that $V(z)$ either is identically zero in D or everywhere < 0. It is also clear that if $V = 0$ then $f^* \in \mathcal{C}_\Phi$ as required. So there remains only to prove:

B.6 Lemma. *The function $V(z)$ is identically zero in D.*

Proof. Assume the contrary. So now

(i) $$\Big|\frac{df^*(z)}{dz}\Big| < e^{U_*(z)} \quad \text{for all } z \in D\,.$$

Let $F(w)$ be the inverse of f so that :

$$F'(f^*(z)) \cdot \frac{df^*(z)}{dz} = 1 \quad z \in D$$

Then (i) gives:

(ii) $$\big|F'(w)\big| > e^{-U_*(w)} \quad w \in \Omega^*$$

Next, let $V(w)$ be the harmonic conjugate of $U_*(w)$ normalised so that $V(0) = 0$ and set

$$H(w) = \int_0^w e^{-U_*(\zeta)+iV(\zeta)} \cdot d\zeta\,.$$

Then (ii) gives

(iii) $$\big|H'(w)\big| < \big|F'(w)\big| \quad w \in \Omega^*$$

Since $H(w((0)) = F(w(0)) = 0$ it follows that

(iv) $$\inf_{w \in \Omega^*} \frac{|H(w)|}{|F(w)|} = r_0 < 1$$

Since $|F(w)| < 1$ in Ω^* while $|F(w)| \to 1$ as w approaches $\partial\Omega^*$ we see that (iv) entails the domain

(v) $$R_0 = \{w \in \Omega^* \quad : \ |H(w)| < r_0\}$$

has at least one boundary point w_* which also belongs to $\partial\Omega^*$.

Next, the function $H(w)$ is analytic in R_0 and its derivative is everywhere $\neq 0$ while $|H(w)| = 1$ on ∂R_0. It follows that H gives a conformal map from R_0 onto the disc $|z| < r_0$. Let $h(z)$ be the inverse of this conformal mapping. Now we get the analytic function in D defined by

$$g(z) = h(r_0 z)$$

Next, let $|z| = 1$ and put $w = h(r_0 z)$. Then

(v) $$|g'(z)| = r_0 \cdot |h'(r_0 z)| = r_0 \cdot \frac{1}{|H'(g(z))|} = r_0 \cdot e^{U_*(g(z))} = r_0 \cdot \Psi(g(z)) < \Psi(g(z))$$

At the same time we have a common boundary point

$$w_* \in \partial\Omega^* \cap \partial g(D)$$

Since the g-function extends to a continuous function on $|z| \leq 1$ there exists a point $e^{i\theta}$ such that

$$g(e^{i\theta}) = w_*$$

Now we use that $r_0 < 1$ above. The continuity of the Ψ gives some $\epsilon > 0$ such that for any complex number a which belongs to the disc $|a - 1| < \epsilon$, it follows that the function

$$z \mapsto a \cdot g(z)$$

belongs to $\mathcal{A}_\Psi$. Finally, by Proposition B.4 the maximal region for the Ψ-function is equal to Ω^* and we conclude that

$$ag(e^{i\theta}) = aw_* \in \Omega^* \quad |a - 1| < \epsilon$$

This would mean that w_* is an *interior point* of Ω^* which contradicts that $w_* \in \partial\Omega_*$. Hence Lemma B.6 is proved.

C. Proof of Theorem 3.

First we have a companion to Lemma B.2. Namely, let $g_1, \ldots, g_n$ be a finite set in $\mathcal{B}_\Phi$. Set $S_\nu = g_\nu(D)$. Following [Beur: page 123] we give

C.1 Definition. *The reduced intersection of the family $\{S_\nu\}$ is defined as the set of these points w which can be joined with the origin by a Jordan arc γ contained in the intersection $\cap S_\nu$. The resulting domain is denoted by $RI\{S_\nu\}$.*

C.2 Proposition. *The domain $RI\{S_\nu\}$ is simply connected and if $g \in \mathcal{C}$ is the normalised conformal mapping onto this domain, then $g \in \mathcal{B}_\Phi$.*

The proof of this result can be carried out in a similar way as in the proof of Lemma B.2 so we leave out the details. Next, starting from a dense sequence $\{g_\nu\}$ in $\mathcal{B}_\Phi$ we find for each n the function $f_n \in \mathcal{B}_\Phi$ where

$$f_n(D) = RI\{S_\nu\} \quad : \, S_\nu = g_\nu(D) : \, 1 \leq \nu \leq n \, .$$

Here the simply connected domains $\{f_n(D)\}$ decrease and there exists the limit function $f_* \in \mathcal{C}$ where

$$f_*(D) = \Omega_*$$

Moreover, $f \in \mathcal{B}_\Phi$. There remains to prove

C.3 Proposition. One has $f_* \in \mathcal{C}_\Phi$.

Remark. Proposition C.3 requires a quite involved proof and is given in [Beur: page 127-130]. We shall not try to present all the details and just sketch the strategy in the proof. Put

$$m = \inf_{g \in \mathcal{B}_\Phi} \, g'(0)$$

Now f^* belongs to $\mathcal{B}_\Phi$ because we have the trivial inclusion $\mathcal{C}_\Phi \subset \mathcal{B}_\Phi$ and using this one proves that

$$m \geq \min_{w \in \Omega^*} \, \Phi(w)$$

Next, using Proposition C.2 above, Beurling employs a normal family and proves that

$$f_*'(0) = m$$

Thus, f_* is a solution to an extremal problem. This is used in the final part of Beurling's proof to establish the inclusion $f_* \in \mathcal{C}_\Phi$. Let us remark that this part of the proof relies upon some subtle set-theoretic constructions where the family of regions of the *Schoenfliess' type* are introduced in [Beur: page 121]. The whole analysis involves topological investigations of independent interest.

Chapter 7. Residue calculus.

Contents

Introduction

The material in 0.A - 0.J describe general methods which often appear in residue calculus. A more extensive discussion about zeros of polynomials with real coefficients appears in section 0.M and the second part listed by sections A-M are devoted to specific examples which illustrate formulas in residue calculus. The literature is extensive and numerous text-books offer examples and exercises in residue calculus. her we try to focus upon various cases of a slightly more involved character where multi-valued functions often appear. So for more standard examples we refer to other text-books. The complex log-function and fractional powers z^α where α is not an integer appear often as integrands and here one must either pursue an integration where analytic continuation takes place, i.e. only in certain favourable cases it is possible to choose single-valued branches. A typical case is when $\sqrt{1-x^2}$ appears in the integrand and integration is over the real interval

$[-1,1]$. Here one often start from the single-valued analytic function $g(z)$ defined in $\mathbf{C} \setminus [-1,1]$ by

$$g(z) = z \cdot \sqrt{1 - z^{-2}}$$

To be precise, consider the extended complex plane where the point at infinity is added and then $\Omega = \mathbf{C} \cup \infty \setminus [-1,1]$ is simply connected which implies that there exists the single-valued branch of the square root of $1 - z^{-2}$, i.e. we have the analytic function

$$g_*(z) = \sqrt{1 - z^{-2}}$$

defined in the whole of Ω where g_* has a simple zero at the point at infinity. If $z = iy$ is purely imaginary then

$$g_*(iy) = iy \cdot \sqrt{1 + y^{-2}}$$

This implies that

$$g(iy) = i \cdot \sqrt{1 + y^2} \quad \text{when} \quad y > 0$$

while $g(iy) = -i\sqrt{1 + y^2}$ when $y < 0$. The change of sign is crucial when reside calculus is employed. For example, using this we can perform residue calculus to express the real integral

(*)
$$J = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}$$

Namely, we employ the g-function above and consider the complex line integral over the closed curve γ which consists of the two line segments from $-1 + i\epsilon$ to $1 + i\epsilon$ respectively $-i\epsilon$ to $1 - i\epsilon$ together with two small half-circles. See figure XX. Using the change of sign and passing to the limit as $\epsilon \to 0$ it follows that

$$2J = i \cdot \int_{|z|=R} \frac{dz}{g(z)}$$

where we have taken a line integral over circles of radius $R > 1$. Passing to the limit as $R \to +\infty$ one easily verifies that the last integral becomes 2π and hence we have

(**)
$$J = \pi$$

Let us remark that the last equality can be proved directly since we have

(*)
$$2J = \int_{0}^{1} \frac{dx}{\sqrt{1 - x^2}}$$

and the last integral is computed via the variable substitution $x \to \sin\theta$ which gives (**). So this example just serves to illustrate a method. In more involved cases residue calculus is necessary to attain exact formulas.

Exercise. Let $0 < a < 1$ and consider the integral

$$J = \int_{0}^{1} \frac{dx}{(1-x)^a \cdot x^{1-a}}$$

Here we use the existence of a single valued branch of the analytic function $g(z) = z \cdot (1 - \frac{1}{z})^a$ in $\mathbf{C} \setminus [0,1]$. Again we have

$$\int_{|z|=R} \frac{dz}{g(z)} = 2\pi i$$

Next, since we have $(-1)^a = e^{\pi i a}$ we can employ a similar line integral as in the previous example to conclude that

$$J = \frac{\pi}{\sin(\pi a)}$$

Next, let $m \geq 3$ be an integer and put

$$J_m = \int_{0}^{1} \frac{dx}{1 - x^m)^{1/m}}$$

The substitution $x \mapsto t^{1/m}$ gives

$$J_m = \frac{1}{m}\int_0^1 \frac{dt}{t^{1/m-1}\cdot(1-t)^{1/m}} = \frac{\pi}{m\cdot\sin(\pi/m)}$$

Notice that the last term converges to 1 as $m \to +\infty$. The reader should discover that this limit is a consequence of Neper's limit formula for e.

Example from mechanics. Historically many residue formulas were established via studies of particle systems in classical mechanics. Here one encounters most of the relevant cases which are used in reside calculus and at the same time a concrete physical meaning can be given to solutions. The interested reader should consult the excellent text-book on Hydromechanics by Horace Lamb which gives many instructive examples with a physical background where residue calculus is employed during the computation of constants. A fundamental function appears in the study a simple pendelum. Consider a particule p of unit mass attached at the end-point of a rigid bar of some length ℓ whose other end is suspended at a fixed point while the bar and p oscillates in a vertical plane where gravity is the sole external force. The system has one degree of freedom expressed by the angle θ between the bar and the vertical line which is directed downwards. The kinetic energy of the one-point system becomes

$$T = \frac{\ell^2}{2}\cdot\dot{\theta}^2$$

The equation of motion becomes

$$\ell^2\ddot{\theta} = -g\ell\cdot\sin\theta$$

With initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = v > 0$ the time-dependent θ-function satisfies the equation

$$\dot{\theta}^2 = \frac{2g}{\ell}\cdot(\cos\theta - 1) + v^2$$

We assume that v us not to large, i.e.

$$v^2 < \frac{2g}{\ell}$$

Then there exists $0 < \theta^* < \pi/2$ such that

$$\cos\theta^* = 1 - \frac{\ell v^2}{2g}$$

Now $t \mapsto \theta(t)$ oscillates between $-\theta^*$ and θ^*. The time for a quarter of the whole period, i.e. the time to reach θ^* becomes

$$T = \int_0^{\theta^*} \frac{d\theta}{\sqrt{\frac{2g}{\ell}\cdot(\cos\theta - 1) + v^2}}$$

This formula shows that the determination of exact T-values with varying intial velocity v boils down to study the function

$$\theta^* \mapsto \int_0^{\theta^*} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta^*}}$$

Admitting the inverse arccos-function as "elementary" and using the substitution $\cos\theta \to x$ we are led to consider integrals of the form

$$J(a) = \int_a^1 \frac{dx}{\sqrt{(1-x^2)\cdot(x-a)}}$$

where $0 < a < 1$. This J-function is contained in a class of functions which a an early stage were studied by Legendre and Jacobi. Let us also remark that not only the numerical value of $J(a)$ as $0 < a < 1$ is of interest here. It turns out that this real-analytic function defined on $(0,1)$ extends to a multi-valued analytic function in $\mathbf{C} \setminus \{0,1\}$ where it satisfies a Fuchsian differential equation which gives a further motivation for including $J(a)$ in a class of "elementary functions".

Example by Huyghens. Let p be a particle of unit mass which moves on the horizontal (x,y)-plane where no friction is present and gravity does not affect the motion. The particle slides on an infinite bar suspended at the origin which can rotate and the bar has no mass. So we have a particle systems with two degrees of freedom where the position of the mass point is given in polar coordinates (r,θ). Here θ is the angle between the bar and the positive x-axis. At time $t=0$ we suppose the initial conditions are

$$\theta(0)=0 \quad : \dot{\theta}(0)=\omega \quad : r(0)=A \quad : \dot{r}(0)=0$$

where ω and A are positive. in this situation we have Kepler's identity

(1) $$r^2\cdot\dot{\theta}=A^2\omega$$

We also get the differential equation

(2) $$\dot{r}^2+\frac{A^4\omega^2}{r^2}=A^2\omega^2$$

Exercise. Express θ as a function of r and use this to prove that the increasing time dependent function $\theta(t)$ has a limit as $t\to+\infty$. More precisely a calculation gives the formula

(3) $$\lim_{t\to\infty}\theta(t)=\int_1^\infty\frac{ds}{s\cdot\sqrt{s-1}}$$

At the time when Huyghens, Newton and Wallis performed calculations they used series expansions to show that (3) is equal to π. The reader is invited to prove this using residue calculus. Thus, as time increases the bar moves from the position along the x-axis to positions which come closer to the positive $y$axis. Notice that the limit formula is independent of the pair ω and A. The reader should contemplate upon this by reflecting over daily life experience of the centrifugal force.

Fourier transforms. One is often interested to get formulas where parameters occur. Typical cases arise in the study of Fourier transforms. Here is an example. On the real x line we have the function $\frac{1}{x-i}$. It is not integrable but defines a tempered distribution μ so we get the Fourier transform which is defined when $\xi\neq 0$ by the integral

(1) $$\widehat{\mu}(\xi)=\int_{-\infty}^\infty\frac{e^{-i\xi x}}{x-i}\cdot dx$$

If $\xi<0$ and $z=x+iy$ with $y>0$ we have $|e^{-i\xi z}=e^{\xi y}$ which decreases when $y\to+\infty$. Using this fact the reader can verify that

$$\widehat{\mu}(\xi)=2\pi i\cdot e^{\xi}$$

holds when $\xi<0$ and reversing the sign verify that $\widehat{\mu}(\xi)=0$ when $\xi>0$. On the reader ξ-line we have the tempered distribution defined by e^{ξ} when $\xi<0$ and zero if $\xi\geq 0$. Its inverse Fourier transform becomes

$$\frac{1}{2\pi}\cdot 2\pi i\int_{-\infty}^0 e^{ix\xi}\cdot e^{\xi}\cdot d\xi=\frac{i}{ix+1}=\frac{1}{x-i}$$

This confirms the calculations using residues via Fourier's inversion formula.

Fourier transforms on $\mathbf{R}^+$. Let $0<a<1$ and β is a complex number outside the non-negative real line. When $\zeta=\alpha+is$ is complex we set

(*) $$J(\zeta)=\int_0^\infty x^\zeta\cdot\frac{x^a}{x-\beta}\cdot\frac{dx}{x}$$

Sincc $|x^{\alpha+is}|=x^\alpha$ when x is real and positive we see that (*) converges when

(1) $$a<\alpha<1-a$$

When (1) holds we can apply residue calculus. After the reader has become familiar with this calculus it is an easy exercise to verify the equation:

$$J\cdot\left[1-e^{2\pi i(\zeta+a-1)}\right]=2\pi i\cdot\beta^{\zeta+a-1}$$

Specific examples. Let $\beta=b$ where $b>0$ which gives

$$\beta^{\zeta+a-1} = b^{\zeta+a-1} \cdot e^{\pi i(\zeta+a-1)}$$

where we used that $-1 = e^{\pi i}$. Suppose also that $\zeta = \alpha$ is real which gives

$$J \cdot \left[1 - e^{2\pi i(\alpha+a-1)}\right] = 2\pi i \cdot b^{\alpha+a-1} \cdot e^{\pi i(\zeta+a-1)}$$

Using the formula for the complex sine-function the reader may verify that

$$J = \pi \cdot \frac{b^{\alpha+a-1}}{\sin \pi(1-a-\alpha)}$$

Since $1 - a - \alpha > 0$ is assumed the formula shows that J is real and positive which it should be since the choice gives

$$J = \int_0^\infty \frac{x^{\alpha+a}}{x+b} \cdot \frac{dx}{x}$$

where the integrand is real and positive. Next, consider the case where $\zeta = is$ is purely imaginary and $\beta = b$ with $b > 0$ while $a = 1/2$. The general formula (xx) gives

$$J \cdot [1 - e^{-2\pi s} \cdot e^{-2\pi i/2}] = 2\pi i \cdot b^{is-1/2} \cdot e^{\pi s} \cdot e^{-\pi i/2}$$

Since $e^{-\pi i} = 1$ and $e^{-\pi/2} = i$ while $i^2 = -1$ it follows that

$$J(1 + e^{-2\pi s}) = e^{-\pi s} \cdot 2\pi \cdot b^{is-1/2}$$

Introducing the complex cosine-function we get the formula

$$\int_0^\infty x^{is} \cdot \frac{\sqrt{x}}{x+b} \cdot \frac{dx}{x} = \pi \cdot b^{is-1/2} \frac{1}{\cos(\pi is)}$$

Remark. The last equation yields a formula for the Fourier transform of the L^1-function $\frac{\sqrt{x}}{x+b}$ on the multiplicative line $\mathbf{R}^+$ where $\frac{dx}{x}$ is the Haar measure. Replace is by the complex variable ζ and set

$$J(\zeta) = \int_0^\infty x^\zeta \cdot \frac{\sqrt{x}}{x+b} \cdot \frac{dx}{x}$$

Then the computations above show that

$$J(\zeta) = \frac{\pi \cdot b^{\zeta-1/2}}{\cos(\pi\zeta)}$$

Notice that the right hand side is an analytic function in the strip domain $1/2 < \mathfrak{Re}\zeta < 1/2$ while we encounter poles when $\zeta = 1/2$ or $-1/2$ whose appearance is clear from the integral which defines $J(\zeta)$ because we get divergent integrals in these two cases. At the same time (xx) gives a meaning to the integral (xx) for al complex ζ, i.e. the result is a globally defined meromorphic functions with simple poles at the zeros of the complex cosine-function. This illustrates the usefulness of residue calculus since it was needed to get the precise formula (xx) above.

Principal values. Let us study with a real integral of the form

(*)
$$J(a) = \int_0^1 \frac{dx}{x-a}$$

where $0 < a < 1$. The principal value is defined by:

(**)
$$\lim_{\epsilon \to 0} \Big[\int_0^{a-\epsilon} \frac{dx}{x-a} + \int_{a+\epsilon}^1 \frac{dx}{x-a}\Big]$$

When $0 < \epsilon < a$ and $a + \epsilon < 1$ we can evaluate both integrals and get:

$$\int_0^{a-\epsilon} \frac{dx}{x-a} = -\log\epsilon + \log a \quad : \quad \int_{a+\epsilon}^1 \frac{dx}{x-a} = \log\epsilon - \log(1-a)$$

where it is not even necessary to perform a limit since (**) takes the same value for all $0 < \epsilon < a$. In particular we get the formula

$$(***) \qquad J(a) = \log \frac{1-a}{a}$$

The construction (**) can be understood by complex integrals. Namely, for any real number $a > 0$ there exists the complex log-function

$$\log(z-a)$$

with a single valued branch in the upper half-plane $\mathfrak{Im}(z) > 0$ whose complex derivative is $\frac{1}{z-a}$. For $\epsilon > 0$ we can take the line integral on the horizontal line from $i\epsilon$ to $1 + i\epsilon$ which gives:

$$(1) \qquad \int_0^1 \frac{dx}{x - a + i\epsilon} = \log 1 - a + i\epsilon - \log(-a + i\epsilon)$$

Passing to the limit as $\epsilon \to 0$ the right hand side becomes

$$(2) \qquad \log(1-a) - \log a - \pi i = \log \frac{1-a}{a} - \pi i$$

To clarify this limit formula we rewrite the left hand side in (1) which amounts to compute

$$(3) \qquad \int_0^1 \frac{(x - a - i\epsilon) \cdot dx}{(x-a)^2 + \epsilon^2}$$

Separating real and imaginary parts it follows from (1) that one has the two limit formulas:

$$(4) \qquad \lim_{\epsilon \to 0} \int_0^1 \frac{(x-a) \cdot dx}{(x-a)^2 + \epsilon^2} = \log \frac{1-a}{a} \quad \text{and} \quad \lim_{\epsilon \to 0} \int_0^1 \frac{\epsilon \cdot dx}{(x-a)^2 + \epsilon^2} = \pi$$

Exercise. Clarify why the first formula in (4) agrees with the previously defined principal value integral. Prove also the second formula using the arctan-function.

Another example. Consider the integral:

$$(1) \qquad J = \int_0^\infty \frac{1}{x} \cdot \log\left(\frac{|x+1|}{|x-1|}\right) \cdot dx$$

The reader may verify that the integrand in (1) is a continuous function whose value at $x = 0$ is equal to 2 and when $|x| \to \infty$ the integrand decays as x^{-2}. So we have an absolutely convergent integral. Residue calculus is used to compute the integral. The idea is to consider the function

$$g(z) = \frac{1}{z} \cdot \log \frac{z+1}{z-1}$$

which is analytic in the upper half-plane.

Exercise. For a large R and a small $\epsilon > 0$ we take the complex line integral of g along the closed curve Γ which consists of the real interval $[\epsilon, R]$, the quarter circle $\{z = Re^{i\theta}$ where $0 \leq \theta \leq \pi/2$ and the imaginary interval $[i\epsilon, iR]$ and finally the small quarter circle of radius ϵ. Since $g(z)$ is analytic the complex line integral over Γ is zero. The reader may verify that

$$(2) \qquad \lim_{R \to \infty} \int_0^{\pi/2} g(Re^{i\theta}) \cdot iRe^{i\theta} \cdot d\theta = 0$$

Next, the integral along the imaginary line $\epsilon \leq y \leq R$ where the line integral taken in the opposite direction becomes

$$-\int_\epsilon^R \log\left(\frac{iy+1}{iy-1}\right) \cdot \frac{dy}{y}$$

Since $|iy + 1| = |iy - 1|$ this integral is purely imaginary. Regarding the real part and using (2) above the reader should verify that:

$$J = \lim_{R,\epsilon} \int_\epsilon^R \frac{1}{x} \cdot \log\Big(\frac{|x+1|}{|x-1|} \cdot dx = \lim_{\epsilon \to 0} \mathfrak{Re} \int_0^{\pi/2} g(\epsilon e^{i\theta}) \cdot i\epsilon e^{i\theta} \cdot d\theta$$

Notice that

$$i\epsilon e^{i\theta} \cdot g(\epsilon e^{i\theta}) = i \cdot \log \frac{\epsilon e^{i\theta}+1}{\epsilon e^{i\theta}-1} = i \cdot \log\, 1 + \epsilon e^{i\theta}) - i \cdot \log(-1 + \epsilon e^{i\theta}) =$$

Now

$$\lim_{\epsilon \to 0} \log\,(-1 + e^{i\theta}) = \pi \cdot i \quad \text{and} \quad \lim_{\epsilon \to 0} \log\, \epsilon(1 + e^{i\theta} - 1) = 0$$

It follows that

$$\lim_{\epsilon \to 0} \int_0^{\pi/2} g(\epsilon e^{i\theta}) \cdot i\epsilon e^{i\theta} = \pi^2/2$$

Hence the integral from (1) has the value:

$$J = \frac{\pi^2}{2}$$

0:A Four examples of residue calculcus.

0.1 Example. Let $P(z)$ and Q be two polynomials where $\deg(P) \geq \deg(Q)+1$ and $-1 < a < 0$ is a real number. Assume that P has no zeros on the non-negative real axis and set:

$$J = \int_0^\infty \frac{x^a \cdot Q(x)}{P(x)} \cdot dx$$

To find J we consider the function $g(z) = \frac{z^a \cdot Q(z)}{P(z)}$ which is multi-valued outside the origin. The trick is to integrate g over a contour starting from $x = \epsilon$ until $x = R$ is reached, followed by an integral taken over the circle $|z| = R$ and after one returns from R to ϵ on the x-axis and finish by an integral over the circle $|z| = \epsilon$ which is performed clock-wise, i.e. in the negative direction. Passing to the limit as $R \to +\infty$ and $\epsilon \to 0$ one uses the multi-valued behaviour of z^α and get

$$(*) \qquad (1 - e^{2\pi i a}) \cdot J = 2\pi i \cdot \sum \mathfrak{res}(g : \alpha_\nu)$$

where the sum is taken over the zeros of P.

0.1.1 Exercise. Take $Q = 1$ and $P = z - i$ above. Then we have

$$(1) \qquad (1 - e^{2\pi i a}) \cdot J = 2\pi i \cdot i^a = 2\pi i \cdot e^{\pi i a/2} \implies J = -\pi e^{-\pi i a/2} \cdot \frac{1}{\sin \pi a}$$

Write $1 - e^{2\pi i a} = (1 - e^{\pi i a})(1 + e^{\pi i a})$ and show that we get

$$J = \frac{2\pi i}{1 + e^{\pi i a}} \cdot \frac{1}{-2i \sin(\pi a/2)} = -\frac{\pi}{(1 + e^{\pi i a}) \cdot \sin(\pi a/2)}$$

Consider in particular the case $a = -\epsilon$ with a positive ϵ. Since $\frac{1}{z-i} = \frac{z+i}{z^2+1}$ and the sine-function is odd we get

$$J = \int_0^\infty \frac{x^{-\epsilon}(x+i)}{1+x^2} \cdot dx = \frac{\pi}{(e^{\pi i \epsilon} + 1) \cdot \sin \pi\epsilon/2}$$

where we used that the sine-function is odd to change the minus sign from (1). Separating the real and imaginary part the reader may verify the formula

$$\int_0^\infty \frac{x^{1-\epsilon}}{1+x^2} \cdot dx = \pi \cdot \frac{1 + \cos(\pi\epsilon)}{(2 + 2\cos(\pi\epsilon) \cdot \sin(\pi\epsilon/2)}$$

To check this formula we consider a limit as $\epsilon \to 1$. Since $\cos \pi = 0$ the reader may verify that the limit in the right hand side becomes $\frac{\pi}{2}$ which is okay since we know from calculus that

$$\lim_{\epsilon \to 0} \int_0^\infty \frac{x^{1-\epsilon}}{1+x^2} \cdot dx = \int_0^\infty \frac{1}{1+x^2} \cdot dx = \frac{\pi}{2}$$

0.2 Example. Here we assume that $\deg(P) \geq \deg(Q) + 2$ and consider the integral

$$J = \int_0^\infty \frac{Q(x)}{P(x)} \cdot dx$$

To overcome the lack of a multi-valued integrand we use the complex log-function $\log z$ and define

$$g(z) = \frac{Q(z)}{P(z)} \cdot \log z$$

Perform an integral of g over the same contour as above After one turn around $|z| = R$ $\log z$ has changed its branch with the constant $2\pi i$ and since the "home run" is the integral from R back to the origin we get:

$$(*) \qquad J = -\sum \mathfrak{res}(g : \alpha_\nu)$$

For example, take $Q = 1$ and $P(z) = (z+a)(z+b)$ where $a > b > 0$ are positive real numbers. The right hand side in (*) becomes

$$-[\frac{\log a + \pi i}{b - a} + \frac{\log b + \pi i}{a - b}] = \frac{\log(a/b)}{a - b}$$

Notice that the right hand side is positive as it should. It is always good to confirm a general formula. Take $b = 1$ while $a = 1 + \epsilon$ and pass to the limit as $\epsilon \to 0$ which gives:

$$\int_0^\infty \frac{dx}{(x+1)^2} = \lim_{\epsilon \to 0} \frac{\log(1+\epsilon)}{\epsilon} = 1$$

as it should. Next, consider the integral

$$J = \int_0^\infty \frac{dx}{1+x^3}$$

The polynomial $1 + z^3$ has simple roots $j_1 = e^{\pi/3}, j_2 = 1, j_3 = e^{5\pi/3}$. The formula (*) above gives

$$J = -\frac{1}{3} \cdot \sum \frac{\log j_\nu}{j_\nu^2} = \frac{1}{3} \cdot \sum j_\nu \cdot \log j_\nu$$

where the last equality holds since $j_\nu^3 = 1$ for each ν. The reader should check that the last sum becomes

$$\frac{i}{3}(j_1 \cdot \pi/3 + j_2 \cdot \pi + j_3 \cdot 5\pi/3) = \frac{i}{3} \cdot (i\sqrt{3}/2 \cdot \pi/3 - i\sqrt{3}/2 \cdot 5\pi/3) = \frac{2\pi}{3 \cdot \sqrt{3}}$$

This gives a positive real number as it should since we started to integrate a positive function over $[0, +\infty)$.

0.3 Example. Let $a > 0$ be real and put

$$J = \int_{-1}^1 \frac{dx}{(1+ax^2)\sqrt{1-x^2}} \tag{0.1}$$

To compute this integral one uses the analytic function defined in the complement of the real interval $-1 \leq x \leq 1$ defined by

$$g(z) = z \cdot \sqrt{1 - z^{-2}}$$

In Xx we explain why $g(z)$ is a single-valued analytic function in $\mathbf{C} \setminus [1, 1]$. Moreover, regarding the series expansion at ∞ where $\sqrt{1 - z^{-2}}$ only give even negative z-powers, it follows that g is an odd function, i.e. $g(-z) = -g(z)$. The reader should verify that

$$y > 0 \implies g(iy) = i \cdot \sqrt{1 + y^2} \tag{1}$$

Moreover we have the limit formula

$$\lim_{\epsilon \to 0} g(x + i\epsilon) = \cdot \sqrt{1 - x^2} \quad : \quad 1 < x < 1 \tag{2}$$

where $\epsilon > 0$ during the limit. When the real interval $(-1, 1)$ is approached from the opposite side the reader should verify that:

$$\lim_{\epsilon \to 0} g(x - i\epsilon) = -i \cdot \sqrt{1 - x^2} \quad : \quad 1 < x < 1 \tag{3}$$

Now we consider the function

$$f(z) = \frac{1}{(1 + az^2) \cdot g(z)}$$

It has to simple poles when $1 + az^2 = 0$, i.e. we find purely imaginary poles at plus and minus $i \cdot a^{-1/2}$. Notice also that

$$\lim_{R \to \infty} \int_{|z|=R} f(z) \cdot dz = 0 \tag{4}$$

We use (4) and apply residue calculus while we integrate over closed curves around $[_1, 1]$ as illustrated by figure XX. Using (23) we find that

$$\frac{2}{i} \cdot J = 2\pi \cdot i \cdot \frac{2}{2a\alpha \cdot g(\alpha)}$$

where $\alpha = i \cdot a^{-1/2}$ and we used that g is odd when the two residues are added. Finally, using (1) the reader can verify that

$$J = \frac{\pi}{\sqrt{1+a}}$$

It is always good to check if the answer is reasonable. Consider for example the case when $a \to 0$ which after a limit would give

$$\pi = \int_1^1 \frac{dx}{\sqrt{1-x^2}}$$

and this is okay since one easily shows that the last equality holds.

Exercise. Let $P(z)$ be a polynomial of degree ≥ 1 with simple zeros outside $[-1, 1]$. Set

$$J = \int_{-1}^1 \frac{dx}{P(x) \cdot \sqrt{1-x^2}} \tag{0.1}$$

Show that

$$J = -\pi \cdot \sum \frac{1}{P'(\alpha_k) \cdot g(\alpha_k)} \tag{*}$$

Notice that P may have degree one. For example, take $P(z) = z - i$. Then the integral becomes

$$-\pi \cdot \frac{1}{g(i)} = -\frac{\pi}{\sqrt{2} \cdot i} = \frac{\pi \cdot i}{\sqrt{2}}$$

0.4 Example. Consider the integral

$$J = \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x - iA} \cdot dx \tag{0.4}$$

where $0 < a < 1$ and $A > 0$ is real. To find J one considers the meromorphic function $g(z) = \frac{e^{az}}{e^z - iA}$ which has simple poles when $e^z = iA$. Consider the complex line integral taken over the boundary of a rectangle

$$\square = \{-R \leq x \leq R\} \times \{0 \leq y \leq 2\pi\}$$

The reader should verify that $e^z - iA$ has a simple zero at the point $\log A + i\pi/2$ which therefore gives a simple pole of $g(z)$. We have also $g(x + 2\pi i) = e^{2\pi i a} \cdot g(x)$ when x is real. When $R \to +\infty$ we get a limit where residue calculus gives:

$$(1 - e^{2\pi i a}) \cdot J = 2\pi i \cdot \mathfrak{res}\big(g(z) : \log A + i\pi/2\big) = 2\pi i \cdot \frac{A^a \cdot i^a}{iA} = 2\pi A^{a-1} \cdot e^{a\pi i/2} \tag{*}$$

0:B Summation formulas.

Various sums are often found using the meromorphic function:

$$g(z) = \frac{\cos \pi z}{\sin \pi z} \tag{*}$$

It has simple poles at all integers with the common residue $\frac{1}{\pi}$. Consider a pair of polynomials P, Q where $\deg P \geq 2 + \deg(Q)$ and $P(n) \neq 0$ hold at all integers. Then residue calculus gives the summation formula

$$\frac{1}{\pi} \cdot \sum_{-\infty}^{\infty} \frac{Q(k)}{P(k)} = \sum \mathfrak{res}(g(z) \cdot \frac{Q(z)}{P(z)} : \alpha_\nu) \tag{**}$$

where the right hand is the sum of residues over all zeros of P. As an illustration we take $P(z) = z^2 + 1$ which has simple zeros at i and $-i$. Since

$$g(z) = i \cdot \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}$$

an computation shows that the right hand side in (*) becomes $\frac{e^\pi+e^{-\pi}}{e^\pi-e^{-\pi}}$. Hence

$$\sum_{-\infty}^{\infty}\frac{1}{1+k^2}=\pi\cdot\frac{e^\pi+e^{-\pi}}{e^\pi-e^{-\pi}}$$

0.B.1 Exercise. Let α be a complex number which is not an integer. Show that

$$\sum_{k=-\infty}^{\infty}\frac{1}{k+\alpha)^2}=\frac{\pi^2}{\sin^2\pi\alpha}$$

0.B.2 Exercise. Certain summation formulas can also be established directly without residue calculus. Consider the meromorphic function

$$g_*(z)=\frac{\pi}{\sin\pi z}$$

It has simple poles at all integers and we can write out an infinite sum of rational functions which will match these poles. Namely, consider

$$g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}(\frac{1}{z-2n}+\frac{1}{z+2n})-\sum_{n=0}^{\infty}(\frac{1}{z-2n-1}+\frac{1}{z+2n+1})$$

It is easily seen that g_*-g has no poles and the reader should verify that this entire is bounded and hence a constant and finally that this constant is zero. We can express g via a series where the convergence for z-values outside the set of integers is expressed more directly, i.e. one has

$$(***)\qquad \frac{\pi}{\sin\pi z}=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-4n^2}-\sum_{n=0}^{\infty}\frac{2z}{z^2-(2n+1)^2}$$

For example, with $z=1/4$ one gets

$$\sqrt{2}\cdot\pi=4-\sum_{n=1}^{\infty}\frac{8}{48n^2-1}+\sum_{n=0}^{\infty}\frac{8}{48(2n+1)^2-1}$$

The right hand side is an infinite sum of rational numbers while the trancendental number π appears in the left hand side. So the formula is quite remarkable.

0.B.3 Wallis product formula. There exists a meromorphic function with simple poles at all integers defined by the series

$$g_*(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-j^2}$$

At the same time we have the function from (*) which also has simple poles at the integers with residues $\frac{1}{\pi}$.

0.B.4 Exercise. Show that

$$\text{(i)}\qquad \frac{\cos\pi z}{\sin\pi z}=\frac{1}{\pi}\cdot g_*(z)$$

Next, use that the derivative of $\sin\pi z$ is equal to $\cos\pi z$ and deduce the product formula

$$\text{(ii)}\qquad \sin\pi z=\pi z\cdot\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})$$

Next, take $z=1/2$ in (ii). If $N\geq 2$ we consider partial products in the right hand side which gives the limit formula

$$\text{(iii)}\qquad \lim_{N\to\infty}\prod_{n=1}^{n=N}(1-\frac{1}{4\cdot n^2})=\frac{2}{\pi}$$

0.B.5 Exercise. Rewrite the product and show that (iii) entails Wallis' limit formula:

$$\sqrt{\frac{\pi}{2}}=\lim_{N\to\infty}\frac{2\cdot 4\cdots 2N}{1\cdot 3\cdot 5\cdots(2N-1)}\cdot\frac{1}{\sqrt{2N+1}}$$

0.C Asymptotic expansions.

Residue calculus is often used to find asymptotic formulas. We describe a result of this nature. Let $\{\lambda_\nu\}$ be a strictly increasing sequence of positive real numbers and $\{a_\nu\}$ some sequence of positive real numbers. Assume that there exists some positive number r_* such that

$$f(x) = \sum_{\nu=1}^{\infty} \frac{a_\nu}{\lambda_\nu + x}$$

is convergent for all $x > r_*$. To each $x > 0$ we denote by $\omega(x)$ the largest integer ν such that $\lambda_\nu < x$.

0.C.1 Theorem. *Suppose that the following limit formula holds for some* $0 < \alpha < 1$ *and a constant* A*:*

$$\lim_{x \to +\infty} x^{-\alpha} \cdot f(x) = A \tag{1}$$

Then it follows that

$$\lim_{x \to +\infty} x^{1-\alpha} \cdot \sum_{\nu=1}^{\nu=\omega(x)} a_\nu = \frac{A}{\pi} \cdot \frac{\sin \pi\alpha}{1-\alpha} \tag{2}$$

The proof requires Fourier analysis and Wiener's general Tauberian theorem. So here more advanced methods are needed but residue calculus is used to compute the value of this limit.

0.D Ugly examples.

There are situations where an integral cannot be expressed in an elementary fashion even if it is defined by elementary functions. For example, consider the integral

$$\int_0^1 \frac{e^x}{1+x} \cdot dx \tag{1}$$

With $z = x + iy$ the function $g(z) = \frac{e^z}{z+1}$ is analytic in the half space $\mathfrak{Re}\, z > -1$. The line integral of g along rectangles $\{0 \leq x \leq 1\} \times 0 \leq y \leq R\}$ is zero and after a passage to the limit when $R \to +\infty$ we see that (1) is equal to

$$\int_0^\infty \frac{e^{is}}{1+is} \cdot ids - \int_0^\infty \frac{e^{1+is}}{2+is} \cdot ids \tag{2}$$

The conclusion is that (1) can be calculated by an "exact formula" if we can handle integrals such as

$$J(a) = \int_0^\infty \frac{e^{is}}{s - ia} \cdot ds \quad : a > 0 \tag{3}$$

Here one encounters an annoying fact. If we instead consider the integral

$$J^*(a) = \int_{-\infty}^\infty \frac{e^{is}}{s - ia} \cdot ds \quad : a > 0 \tag{4}$$

then there is no problem to compute it. In fact, we shall learn that the value of (4) is found by ordinary residue calculus and becomes $2\pi i \cdot e^{-a}$, obtained from a residue at $z = ia$ when we consider the function $g(z) = \frac{e^{iz}}{z-ia}$ in the upper half-plane where it has a simple pole at $z = ia$. But (3) cannot be found in this simple fashion. After the substitution $s \mapsto a\xi$ we see that (3) is equal to

$$J(a) = \int_0^\infty \frac{e^{ia\xi}}{\xi - i} \cdot d\xi \tag{5}$$

Apart from the factor $\frac{1}{2\pi}$ this is an inverse Fourier transform of the tempered distribution on the real ξ-line which is supported by $\{\xi \geq 0\}$ given by the density $\frac{1}{\xi - i}$ on $\xi > 0$. This illustrates a close interplay between Fourier transforms and the calculations of various integrals. Let a be replaced by x to indicate that $J(x)$ is a function of x which is defined on $x > 0$ but becomes a tempered distribution on the real x-line via Fourier's inversion formula. It follows for example that the distribution J satisfies the differential equation

$$\partial_x(J) + J = 2\pi i \cdot H^* \tag{6}$$

where H^* is the inverse Fourier transform of the Heaviside distribtution on the ξ-line which is 1 if $\xi \geq 0$ and zero on $\{\xi < 0\}$. In XX we return to a study of the J-distribution and get a certain formula for the evaluation of the integral in (4).

Let us now turn to "nice" situations and begin with some general formulas which are used in residue calculus.

0:E Fractional decompostion

The vanishing below holds for every pair of polynomials p, q if $\deg(p) \geq \deg(q) + 2$:

$$\lim_{R\to\infty} \int_{|z|=R} \frac{q(z)\cdot dz}{p(z)} = 0 \tag{*}$$

A second useful formula is the fractional decomposition:

$$\frac{1}{p(z)} = \sum_{k=1}^{k=n} \frac{c_k}{z-\alpha_k} \quad \text{where} \quad c_k = \frac{1}{p'(\alpha_k)}$$

where

$$p(z) = \prod_{k=1}^{k=n} (z-\alpha_k) \quad \text{has simple zeros}$$

0.E.1 Exercise. Show that (*) applied with $q(z) = 1$ gives:

$$c_1 + \ldots + c_n = 0 \tag{1}$$

Next, let $1 \leq \nu \leq n-1$ and show that one has the fractional decomposition

$$\frac{z^\nu}{p(z)} = \sum_{k=1}^{k=n} \frac{\alpha_k^\nu}{p'(\alpha_k)} \cdot \frac{1}{z-\alpha_k} \tag{2}$$

0:F Computing local residues.

When multiple zeros occur local calculations are needed to find residues. The typical case is as follows: We have an analytic function $f(z)$ defined in disc centered at $\{z = 0\}$ and with a zero of order $k \geq 2$ at the origin. Now

$$\frac{1}{f(z)} = c_k z^{-k} + \ldots + c_1 z^{-1} + d_0 + d_1 z + \ldots$$

Here c_1 is the residue coefficient. In practice an expansion

$$f = bz^k(1 - (b_1 z + b_2 z^2 + \ldots))$$

is known from the start. To find c_1 therefore amounts to find the coefficient of z^{k-1} in the power series

$$\frac{1}{1-(b_1 z + b_2 z^2 + \ldots)} = 1 + w_1 z + w_2 z^2 + \ldots \tag{*}$$

In (*) we can take $\{b_\nu\}$ to be arbitrary and seek for algebraic expressions of the w-numbers. This leads to every integer $k \geq 1$ to a certain polynomial R_k of the b-variables. We see for example that

$$b_1 = w_1 \quad : \quad b_1^2 + b_2 = w_2 \quad : \quad b_1^3 + b_1 b_2 + b_3 = w_3 \tag{1}$$

0.F.1 Exercise. Show that for every $k \geq 1$ there exists a polynomial of the form

$$R_k(b_\bullet) = \sum \rho_{i_1 \ldots i_m} b_1^{i_1} \cdot b_m^{i_m} \tag{**}$$

where $1 \leq m \leq k$ holds in each term and

$$i_1 + 2i_2 + \ldots + ki_k = k$$

hold for every k-tuple of the non-negative i-numbers. Use also (**) to continue the computations in (1) above for higher k-values. One has for example

$$b_1^4 + 3b_1^2 b_2 + b_1 b_3 + b_2^2 + b_4 = w_4$$

Employ a computer to extend the result to get exact formulas for a set of positive integers, say up to $k = 50$. Notice that all the ρ-coefficients in (**) are positive integers.

0.F.2 Exercise. Let $g(z)$ be a meromorphic function with a pole of order k at $z = 0$. Then $z^k \cdot g(z)$ is holomorphic at the origin. Show that the residue of g given by the coefficient c_1 of z^{-1} in the Laurent expansion is given by

(1) $$\frac{1}{(k-1)!} \cdot \partial^{k-1}(z^k \cdot g)(0)$$

Take for example $g(z) = \frac{1}{\sin^3 z}$ which has a triple pole a $z = 0$. We write

$$\sin z = z(1 - z^2/3! + z^4/5! - \ldots))z \cdot \rho(z)$$

By (1) the residue becomes

$$\frac{1}{2}\partial^2(\frac{1}{\rho(z)} = -\frac{1}{2} \cdot \partial(\frac{\rho'(z)}{\rho^2(z)})$$

Now the reader can verify that the residue becomes $\frac{1}{6}$.

0.G Line integrals of multi-valued functions.

Let Ω be a connected domain in $\mathbf{C}$ and f_* is a germ of a multi-valued analytic function at some point $z_* \in \Omega$. Let γ be a curve which starts at z_* and stays in Ω. The end-point z^* of γ can be equal, to z_*, i.e. we do not exclude the case when γ is closed. Now f_* can be extended in the sense of Weierstrass along γ and using its analytic continuation along γ the line integral

(1) $$\int_\gamma f \cdot dz$$

is defined.

Exercise. Us the monodromy theorem to show that if γ and γ^* are curves starting at z_* with the same end-point z^* and homotopic in this family of such curves, then the integral (1) taken over γ or γ^* are equal.

Some examples. Consider the multi-valued function $f = \sqrt{z}$ in the punctured complex plane. At $z = 1$ the branch is chosen so that $f(1) = 1$. Let γ be the closed curve given by the unit circle oriented in counter-clockwise direction. Then we obtain

$$\int_\gamma f \cdot dz = \int_0^{2\pi} e^{i\theta/2} \cdot ie^\theta d\theta = \frac{i}{2} \cdot e^{3i\theta/2}|_0^\pi = \frac{i}{2}[-i-1] = \frac{1-i}{2}$$

Next, remove the to points -1 and $+1$ from $\mathbf{C}$ which gives the domain $\Omega = \mathbf{C} \setminus \{-1, 1\}$. In Ω we consider the multi-valued function

$$f(z) = \sqrt{1+z} \cdot (1-z)^a \quad \text{where} \quad 0 < a < 1$$

where the local branch f_* at the origin is chosen so that $f_*(0) = 1$. Let γ_1 be the closed curve at the origin which follows the circle $\{z - 1| = 1\}$ and is oriented in the counter-clockwise direction. Similarly, γ_2 is the closed curve which now follows the circle $\{|z+1| = 1\}$ in the counter clock-wise direction. We have also the closed curves γ_1^{-1} and γ_2^{-1} with reversed orientation. Now we get the composed closed curve

(1) $$\gamma = \gamma_2^{-1} \circ \gamma_1^{-1} \circ \gamma_2 \circ \gamma_1$$

Along γ_1 we have $z = 1 + e^{i\theta}$ and the line integral becomes

(i) $$J_1 = \int_0^{2\pi} \sqrt{2 + e^{i\theta}} \cdot e^{ia\theta} \cdot ie^{i\theta} \cdot d\theta$$

Next, when the integral over γ_2 is computed we have performed an analytic continuation of f along γ_1 which means that we have a new local branch of f at $z = 0$ which takes the value $e^{2\pi ia}$. So along γ_2 we get the contribution

$$J_2 = e^{2\pi ia} \cdot \int_0^{2\pi} e^{ia\theta} \cdot \sqrt{-2 - e^{i\theta}} \cdot ie^{i\theta} \cdot d\theta =$$

(ii) $$i \cdot e^{2\pi ia} \cdot \int_0^{2\pi} e^{ia\theta} \cdot \sqrt{2 + e^{i\theta}} \cdot ie^{i\theta} \cdot d\theta = i \cdot e^{2\pi ia} \cdot J_1$$

For the integral along γ_1^{-1} we start with a local branch where $f(0) = -e^{2\pi ia}$ and since it is performed in the clockwise direction the contribution becomes

$$J_3 = e^{2\pi ia} \cdot J_1$$

Finally, the local branch of f at $z = 0$ when we start integration along γ_2^{-2} is minus one and we see the contribution of the last line integral becomes

$$J_4 = e^{-2\pi ia} J_2 = i \cdot J_1$$

From this we conclude that the line integral of f taken over γ is equal to

(*) $$(1 + i)(1 + e^{2\pi\alpha}) \cdot J_1$$

The reader may verify that $J_1 \neq 0$ and hence the line integral of f over the closed curve γ is non-zero. The exercise above therefore shows that γ cannot be homotopic to the trivial curve which stays at the origin in Ω. This means that the image $\{\gamma\}$ in the fundamental group $\pi_1(\Omega)$ is non-zero, i.e. the homotopy classes $\{\gamma_1\}$ and $\{\gamma_2\}$ do not commute in this group.

Remark. The example above shows how one can establish results in topology using complex line integrals. The precise result is that the fundamental group $\pi_1(\Omega)$ is a free group generated by the homotopy classes of γ_1 and γ_2. That this indeed holds can be proved integrating multi-valued functions of the form along composed closed γ-curves.

$$f(z) = (z - 1)^a \cdot (z + 1)^b$$

where a, b can be arbitrary pairs of complex numbers.

0.G.2 Exercise. Let $0 < a < 1$ be a real number. Consider the multi-valued function $f(z) = z^a \cdot \log z$ defined outside the two points 0 and 1. Let $R > 1$ and at $z = R$ we choose the local branch f_* where $f_*(R) = R^a \cdot \log R$ is real and positive. Calculate the line integral

$$\int_\gamma f \cdot dz$$

where γ is the circle $\{|z| = R\}$ oriented in the counter-clock wise sense.

0.G.3 Example. Let $a > 0$ be real and consider the integral:

(*) $$J = \int_0^1 \frac{dx}{(1 + ax^2)\sqrt{1 - x^2}}$$

To evaluate this integral we use the fact that there exists a *single-valued* analytic function $\sqrt{1 - z^2}$ in $\mathbf{C} \setminus [0, 1]$. Choose a closed contour formed by a the line segment where $y = \epsilon$ and $0 \leq x \leq 1$ plus a small half circle around 1 and return along the line $y = -\epsilon$ while x moves from 1 to zero and finish with a small half-cirlce from $-i\epsilon$ to $i\epsilon$. Notice that the line integral over large circles $|z| = R$ of $\frac{1}{(1+az^2)\sqrt{1-z^2}}$ tend to zero. Now Cauchy's residue formula gives

$$(1 - i) \cdot J = 2\pi i \cdot \mathfrak{res}((1 + az^2)\sqrt{1 - z^2} : \frac{i}{a}) + 2\pi i \cdot \mathfrak{res}((1 + az^2)\sqrt{1 - z^2} : \frac{-i}{a})$$

Here a computation gives the equality:

$$J = \frac{\pi}{2\sqrt{1+a}}$$

0.H Solving a differential equation.

Line integrals of multi-valued functions are also used in other situations. Here follows an example from chapter VII in the text-book [Cartan]. Let $n \geq 2$ and consider the differential equation

$$(*) \qquad (a_n z + b_n) \cdot y^{(n)}(z) + \ldots + (a_1 z + b_1) \cdot y'(z) + (a_0 z + b_0) \cdot y(z) = 0$$

Here $\{a_k\}$ and $\{b_k\}$ are complex constants with $a_n \neq 0$. Define the two polynomials

$$(1) \qquad A(z) = \sum a_k z^k \quad \text{and} \quad B(z) = \sum b_k z^k$$

Assume that the zeros of A are simple and denote them by $c_1, \ldots, c_n$. Under this assumption the set of entire functions which solve (*) is a complex vector space of dimension $(n-1)$. To find these solutions we use the fractional decomposition and write

$$(2) \qquad \frac{B(z)}{A(z)} = \alpha + \frac{\alpha_1}{z - c_1} + \ldots + \frac{\alpha_n}{z - c_n}$$

Next, define the function

$$(3) \qquad U(z) = e^{\alpha z} \cdot \prod (z - c_k)^{\alpha_k}$$

Since $\{\alpha_k\}$ in general are not integers this U-function is multi-valued. Outside the zeros of A we notice that one has the equality

$$(4) \qquad \frac{U'(z)}{U(z)} = \frac{B(z)}{A(z)}$$

This will be used to construct solutions to (*). Namely, fix a point $z_0 \in \mathbf{C} \setminus \{c_k\}$ and in a small disc centered at z_0 we choose a local branch of U. Next, let γ be a closed curve which stays in $\mathbf{C} \setminus \{c_k\}$ and has z_0 as a common start and end-point. For each complex number z we can devaluate the line integral and get a function

$$(5) \qquad f(z) = \int_\gamma e^{z\zeta} \cdot \frac{U(\zeta)}{A(\zeta)} \cdot d\zeta$$

It is clear that f is an entire function of z and each complex derivative is given by:

$$f^{(k)}(z) = \int_\gamma e^{z\zeta} \cdot \zeta^k \cdot \frac{U(\zeta)}{A(\zeta)} \cdot d\zeta$$

So the construction of the polynomials A and B show that f is a solution to the differential equation (*) if

$$(**) \qquad \int_\gamma e^{z\zeta} \cdot [z \cdot A(\zeta) + B(\zeta)] \cdot \frac{U}{A}(\zeta) \cdot d\zeta = 0$$

where the equality holds for all z.

A partial integration. Since $\partial_\zeta(e^{z\zeta}) = z \cdot e^{z\zeta}$ it follows that

$$(6) \qquad \int_\gamma e^{z\zeta} \cdot z \cdot U(\zeta) \cdot d\zeta = e^{z\zeta} \cdot U(\zeta)|_{\gamma_*}^{\gamma^*} - \int_\gamma e^{z\zeta} \cdot U'(\zeta) \cdot d\zeta$$

At the same time (4) above gives the equality $U' = \frac{BU}{A}$ and we conclude that (**) holds if and only if

$$(7) \qquad e^{z\zeta} \cdot U(\zeta)|_{\gamma_*}^{\gamma^*} = 0$$

By the construction of the line integral where the multi-valued function U appears, (7) means precisely that the after analytic continuation along γ one has the equality

$$(***) \qquad T_\gamma(U)(z_0) = U(z_0)$$

If we want that the f-function in (5) is not identically zero we must choose closed curves γ which are not trivial, i.e. homotopic to the constant curve at z_0, and at the same time (***) should hold. To achieve this we consider for each $1 \leq k \leq n$ a simple closed curve γ_k at z_0 whose winding number with respect to c_k is equal to one, while the winding number with respect to the remaining c-rots are zero. This means that the homotopy classes of $\gamma_1, \ldots, \gamma_n$ generate the free group $\pi_1(\mathbf{C} \setminus \{c_k\})$. Notice also that

$$T_{\gamma_k}(U)(z_0) = e^{2\pi i \cdot \alpha_k} \cdot U(z_0) \tag{8}$$

hold for every k. To satisfy (***) we introduce the following $(n-1)$-tuple of closed curves:

$$\gamma_k^* = \gamma_k^{-1} \circ \gamma_1^{-1} \circ \gamma_k \circ \gamma_1 \quad : \; 1 \leq k \leq n \tag{9}$$

With this choice we have

$$T_{\gamma_k^*}(U)(z_0) = e^{-2\pi i \cdot \alpha_k} \cdot e^{-2\pi i \cdot \alpha_1} \cdot e^{2\pi i \cdot \alpha_k} \cdot e^{2\pi i \cdot \alpha_1} \cdot U(z_0) = U(c_0) \tag{10}$$

Hence the differential equation is solved by the functions

$$f_k(z) = \int_{\gamma_k^*} e^{z\zeta} \cdot \frac{U(\zeta)}{A(\zeta)} \cdot d\zeta \quad : 2 \leq k \leq n \tag{11}$$

There remains to prove that the functions above are $\mathbf{C}$-linearly independent and give a basis for the entire solutions to (*). The fact that the complex vector space of entire solutions to (*) has dimension $n-1$ at most can be proved in several ways. One is to apply results from $\mathcal{D}$-module theory. See (xx). Less obvious that $f_2, \ldots, f_n$ are $\mathbf{C}$-linearly independent. To see this we suppose that one has a relation $q_2 f_2(z) + \ldots + q_n f_n(z) = 0$ where $\{q_k\}$ are complex constants. This is an identity for all z and by expanding $e^{z\zeta}$ it would follow that:

$$\sum_{k=2}^{k=n} q_k \cdot \int_{\gamma_k^*} \zeta^m \cdot \frac{U(\zeta)}{A(\zeta)} \cdot d\zeta = 0 \quad \text{for all} \quad m = 0, 1, \ldots \tag{12}$$

Now the homotopy classes of the γ^*-curves are different in the fundamental group and using this one can show that (12) implies that all the q-numbers are zero which gives the requested $\mathbf{C}$-linear independence of the f-functions.

0:I. More involved integrals.

Even though standard residue calculus settles a quite extensive family of integrals there remain integrals where the evaluation is more cumbersome and eventually force us to employ numerical calculations. Consider for example the integral

$$J(a)=\int_0^1\frac{dx}{x^a\cdot(1+x)^a} \tag{*}$$

where $0<a<1$. We find a series for the solution using the expansion

$$(1+x)^{-a}=\sum_{n=0}^\infty c_n(a)\cdot x^n\quad : 0<x<1\implies$$

$$J(a)=\sum_{n=0}^\infty c_n(a)\cdot\frac{1}{n+1-a} \tag{**}$$

Recall also that

$$c_n(a)=(-1)^n\cdot\frac{a(a+1)\cdots(a+n-1)}{n!}\quad : n=1,2,\ldots$$

So above we have an explicit series and it is a matter of taste if one includes the J-function which evaluates the integral among the "elementary functions". The J-integral above is related to integrals of the form

$$I(a)=\int_{-\infty}^\infty\frac{dx}{(1+x^2)^a} \tag{***}$$

which exist when a is real and $>1/2$. To solve (***) it is tempting to consider the multi-valued analytic function $g(z)=(1+z^2)^{-a}$. If $R>1$ we get the simply connected domain Ω_R which is the upper disc $\{|z|<R\}$ where $\mathfrak{Im}(z)>0$ and the imaginary interval $[0,i]$ is removed. In this domain there exists a single-valued branch of the g-function which admits a factorisation

$$g(z)=(z+i)^{-a}\cdot(z-i)^{-a}$$

The whole line integral

$$\int_{\partial\Omega_R}g(z)\cdot dz=0$$

On the portion of $\partial\Omega_R$ given by the half-circle of radius R we get a vanishing integral as $R\to+\infty$. On the portion on the real x-axis we must take into the account that the restriction of g to the negative real axis has changed. More precisely we have performed an analytic continuation of $(z-i)^{-a}=e^{-a\log(z-i)}$ and the effect is that

$$g(x)=e^{-\pi ai}\cdot\frac{1}{1+x^2}\quad : x<0$$

0.I.1 Exercise. Conclude from the above that

$$(1+e^{-\pi ai})\cdot\frac{I(a)}{2}=\int_\Gamma g(z)\cdot dz \tag{1}$$

where Γ is the contour give by two copies of the imaginary interval $[0,i]$. As illustrated by a picture the portion of the complex line integral of $g(z)\cdot dz$ on the "positive side" is taken in the negative direction and therefore contributes with the term:

$$-\int_0^1\frac{idy}{(1-y^2)^a} \tag{2}$$

On the "negative side" the g-function has a new branch and taking the negative direction into the account the contribution to the complex line integral $\int_\Gamma g(z)\cdot dz$ becomes

$$e^{-\pi ai}\cdot\int_0^1\frac{idy}{(1-y^2)^a} \tag{3}$$

Conclusion. From (1-3) above we obtain

$$(1+e^{-\pi ai})\cdot\frac{I(a)}{2}=(1-e^{-\pi ai})\cdot i\cdot\int_0^1\frac{dy}{(1-y^2)^a}$$

Multiply both sides with $e^{\pi ia/2}$. Then the reader can conclude that

$$\text{(*)}\qquad \cos(\pi a/2)\cdot I(a)=2\cdot\sin(\pi a/2)\cdot\int_0^1\frac{dy}{(1-y^2)^a}$$

0.I.2 Exercise. To each $0<a<1$ we have the integral

$$L(a)=\int_0^1\frac{dy}{(1-y^2)^a}$$

By the substitution $y\mapsto\cos\theta$ the integral becomes

$$\int_0^1(\cos\theta)^{(1-a}\cdot d\theta$$

From a numerical point of view this is a robust integral since the integrand is a continuous function and the reader should check numerical values with a computer as $0<a<1$ varies.

0.J Abel integrals and equations

Residue calculus can be used to solve integral equations. Given $0<a<1$ we consider the equation

$$\text{(*)}\qquad \int_0^1\frac{\phi(y)}{|x-y|^a}\cdot dy=f(x)$$

Here $f(x)$ is a given function which at least is continuous and one seeks ϕ. A complete solution was given by Carleman in an article entitled *Über die Abelsche Integralgleichung mit konstanter Integrationsgrenzen* from 1922. To begin with f yields the function

$$\text{(1)}\qquad F(t)=-\frac{1}{\pi}\cos\frac{a\pi}{2}\cdot\int_0^1\frac{1}{s-t}\cdot f(s)\big[(s(1-s)\big]^{\frac{a-1}{a}}\cdot ds$$

The unique solution $\phi(x)$ is found given via a complex contour integral where Γ_x for each $0<x<1$ is a simple closed curve whose intersection with the positive real axis is the singleton set $\{x\}$ and one has;

$$\text{(2)}\qquad \phi(x)=\frac{1}{2\pi i}\int_{\Gamma_x}\frac{1}{(t-x)^{1-a}}\cdot\big[t(t-1)\big]^{\frac{1-a}{2}}\cdot F(t)dt$$

0.J.1 Experiment with a computer. Another integral equation is

$$\text{(**)}\qquad \int_0^1\log|x-y|\cdot\phi(y)\cdot dy=f(x)$$

Just as above it was shown in Carleman's article that the ϕ-solution is unique and determined by the formula:

$$\frac{1}{\pi^2}\cdot\frac{1}{\sqrt{x(1-x)}}\cdot\int_0^1\frac{f'(s)\sqrt{s(1-s)}}{s-x}\cdot ds-\frac{1}{2\pi^2\log 2\cdot\sqrt{x(1-x)}}\cdot\int_0^1\frac{f(s)}{\sqrt{s(1-s)}}\cdot ds$$

It is instructive to check this solution formula with the aid of a computer, i.e. choose various f-functions where it suffices to take polynomials of relatively small degree and try after perform the numerical study so when the (approximated) ϕ-function found by the formula above is inserted in (**), then f is approximately recaptured the right hand side. In Chapter XX we shall prove the inversion formula above using results about distributions and boundary values of analytic functions.

0.K Location of zeros of polynomials.

In the article [xx] from 1876, Eduard Routh applied Cauchy's residue calculus to analyze positions of roots to polynomials. For applications to dynamical systems the main concern is to determine zeros in a half-plane such as $\mathfrak{Re}(z) > 0$. We shall discuss some examples. Consider a polynomial

$$P(z) = z^{2m} + c_{2m-1}z^{2m-1} + \ldots + c_1 z + c_0$$

whose coefficients are real numbers. We assume also that P has no zeros in the imaginay axis and the zeros of

$$y \mapsto \mathfrak{Re}(P(iy))$$

are simple. When $|y|$ is large we have

$$P(iy) \simeq (-1)^m \cdot y^{2m}$$

Exercise. Assume that m is even and $\mathfrak{Re}(P(iy))$ has at least one zero. Show that there exists a positive integer k and a strictly increasing sequence

$$\alpha_1 < \beta_1 < \ldots < \alpha_k < \beta_k$$

where $\{i\alpha_\nu\}$ and $\{i\beta_\nu\}$ are the zeros of $\mathfrak{Re}(P)$ on the imaginary axis. By assumption $\mathfrak{Im}(P) \neq 0$ at these simple zeros of the real part. With these notations special case of Routh's formula gives:

K.1 Theorem. *The number of zeros of P counted with multiplicity in the right half-plane is equal to*

$$m - \frac{1}{2}\sum_{\nu=1}^{\nu=k} \big[\text{sign}(\mathfrak{Im}\, P(i\alpha_\nu) - \text{sign}(\mathfrak{Im}\, P(i\beta_\nu)\big]$$

Proof. We shall use the argument principle and study the the function

$$y \mapsto \arg P(iy)$$

as y decreases from $+\infty$ to $-\infty$ on the real y-line. By an induction over k the reader can verify that

$$\lim_{R\to\infty} [\arg P(iR) - \arg P(-iR)] = -\pi \cdot \sum_{\nu=0=}^{\nu=k} [\text{sign}\, \mathfrak{Im}\, P(i\alpha_\nu) - \text{sign}\mathfrak{Im}\, P(i\beta_\nu)]$$

At the same time the argument of P along a half-circle $z = Re^{i\theta}$ with $-\pi/2 < \theta < \pi/2$ increases by the term $\simeq 2m\pi$ when R is large. Now Routh's theorem follows from the argument principle in Chapter IV.

K.2 Example. Consider the case $m = 2$ and a polynomial of the form

$$P(z) = z^4 + 2Az^2 + Bz + Cz^3 - 1$$

where A, B, C are real. We get

$$\mathfrak{Re}P(iy) = y^4 - 2Ay^2 - 1 = (y^2 - A)^2 - A^2 - 1$$

Here two real roots appear via the equation

$$y^2 = A + \sqrt{A^2 + 1}$$

More precisely, we get two roots $-\rho$ and ρ where

$$\rho = \sqrt{\sqrt{A^2 + 1} + A}$$

At the same time

$$\mathfrak{Im}\, P(iy) = By - Cy^3$$

With the notations in Theorem K.1 we have $\alpha_1 = -\rho$ and $\beta_1 = \rho$. Here the difference

$$\text{sign}(B \cdot -\rho + C\rho^3 - \text{sign}(B \cdot \rho - C\rho^3 = 2 \cdot \text{sign}(C\rho^3 - B\rho)$$

Taking the minus sign into the account in Routh's formula we conclude that P has one zero in the right half-plane if

(i) $$C\rho^2 > B$$

while it has 3 zeros in this half-plane when

(ii) $$C\rho^2 < B$$

where $\rho = A + \sqrt{A^2+1}$. Notice that when P is restricted to the real axis then it is < 0 when $|x|$ is small and > 0 when $|x|$ is large so P has always at least one real zero on $x > 0$ and one on $x < 0$ while (i-ii) determine the real part of the two remaining zeros which are conjugate since the coefficients of P are real.

The case when P has odd degree. Consider the case of a cubic polynomial

$$P(z) = z^3 + a_2z^2 + a_1z + a_0$$

where each a-number is real and positive. Now

$$P(iy) = -iy^3 - a_2y^2 + ia_1y + a_0$$

We assume in addition that P has no zeros on the imaginary axis. The real part has two zeros $\rho > 0$ and $-\rho$ where

$$\rho = \sqrt{\frac{a_0}{a_2}}$$

Let us pursue the variation of $\arg P(iy)$ while y descreases from $+\infty$ to $-\infty$. On the positive real axis $P(x)$ is real and positive and it follows that

$$\lim_{R\to\infty} \arg P(iR) = 3\pi/2$$

Next, the real part of $P(iy)$ is < 0 as long as $y > \rho$ and when $y = \rho$ the imaginary part becomes

$$\rho(a_1 - \rho^2)$$

Suppose that this term is > 0. Then a figure shows that the argument has decreased from $3\pi/2$ to $\pi/2$. Next, while $-\rho < y < \rho$ the real part is > 0 so $P(iy)$ moves in the right half-plane and when $y = -\rho$ the imaginary part gets a reversed sign to (1) which means that the argument now has decreased from $\pi/2$ to $-\pi/2$. So up to $y = -\rho$ the decrease of the argument is -2π. Finally, when $y < -\rho$ then the real part is again < 0 while $\mathfrak{Im}\, P(-i\rho) < 0$ and after we see that the imaginary part increases when $y \to -\infty$ which means that the argument of P continues to decrease and the total effect is that

$$\lim_{R\to\infty} [\arg P(iR) - \arg P(-iR)] = -3\pi$$

At the same time $P(Re^{i\theta} \simeq R^3e^8i\theta$ when $R >> 1$ so the argument increases by 3π along the half-circle of radius R in the right half-plane. From this we conclude that P has no zeros in half-discs $\{|z| < R \cap \mathfrak{Re}\, z > 0\}$ which mens that the zero of the cubic polynomial are confined to the left half-plane. hence we have proved

Theorem. *Let $P(z)$ be a cubic polynomial where $\{a_k\}$ are real and positive and P has no zeros on the imaginary axis. Then all roots belong to the left half-plane if*

$$a_1a_2 > a_0$$

Exercise. Show that if $a_1a_2 < a_0$ then P has exactly one root in the right half-plane.

Some other examples. Consider a polynomial of degree 3:

$$P(z) = z^3 + Az - 1$$

where $A > 0$. Hence the derivative $P'(x) = 3x^2 + A > 0$ on the real x-axis so $P(x)$ has at most one real zero and since $P(0) = -1$ this zero ρ must be > 0. The two remaining roots appear in a conjugate pair $\xi, \bar{\xi}$ and since z^2 is missing in P we have

$$\xi + \bar{\xi} + \rho = 0$$

Since $\rho > 0$ we conclude that $\mathfrak{Re}\, \xi < 0$, i.e, P has one root in the right half-plane. It is instructive to check this via the argument principle. On the imaginary axis we notice that $\mathfrak{Re}\, P(iy) = -1$ is constant so $y \mapsto P(iy)$ moves in the left half-plane and when $R >> 0$ we see that

$$\text{arg}P(iR) \simeq -\pi/2$$

A figure shows that the argument of $P(iy)$ decreases from $-\pi/2$ to $-3\pi/2$. At the same time the argument increases by the factor 3π as we move on a circle from $-iR$ to iR. The total variation of the argument along a large half-circle becomes $3\pi - \pi = 2\pi$ which reflects the fact that P has one root in the right half-plane.

Another case. Consider a polynomial of the form

$$P(z) = z^3 + Az + 1$$

where A is real. Here $y \to P(iy)$ moves in the right half-plane and we have

$$\arg P(-iR) \simeq \pi/2 \quad \text{and} \quad \arg P(iR) \simeq -\pi/2$$

So the variation of the argument as y moves from R to $-R$ increases by the factor π. From this we can conclude that P has two zeros in the right half-plane. Notice that $P(0) = B > 0$ which implies that $P(x)$ has a real zero ρ on the negative axis. The two remaining roots appear in a conjugate pair $\xi, \bar{\xi}$ and since z^2 is missing we have

$$2\xi + \rho = 0$$

which implies that $\mathfrak{Re}\, \rho > 0$ in accordance with the previous verification via the argument principle.

The case $P(z) = z^5 + z + 1$**.** Here we get

$$P(iy) = i(y^5 + y) + 1$$

The variation of $\text{arg}(P)(iy)$ as y moves from R to $-R$ is now π while that along the half-circle is 5π. The conclusion is that $P(z)$ has two roots in the right half-plane. Notice that in this example

$$P'(x) = 5x^4 + 1 > 0$$

So $x \mapsto P(x)$ is strictly increasing on the real x-line where it has a simple zero $x_* < 0$ since $P(0) = 1 > 0$. We have seen that two complex roots α and $\bar{\alpha}$ appear with a common real part > 0 while the two other complex roots β and $\bar{\beta}$ have a negative real part. The reader should find numerical values for these complex roots and confirm the assertion that two roots appear in the right half-plane. Let us remark that the polynomial above appears in Abel's article [Abel] where he demonstrated that his specific algebraic equation of degree five cannot be solved by roots and radicals.

K.3 The Hurwitz-Routh theorem. Consider a polynomial

$$P(z) = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

whose coefficients are real numbers. We assign the 4×4-matrix

$$A = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 1 & a_2 & a_4 \end{pmatrix}$$

The matrix has four principal minors. The first is just a_1, the second $a_1 a_2 - a_3$ and the third

$$a_4(a_1(a_2 a_3 - a_1 a_4) - a_3^2)$$

The last minor is $\det(A)$. With these notations the Hurwitz-Routh theorem for polynomials of degree 4 asserts that the roots of $P(z)$ all belong to the left half-plane if and only if the four minors above are all > 0. A similar criterion holds for polynomials of arbitrary high degree. More precisely, consider a polynomial

$$P(z) = z^n + a_1 z^{n-1} + \ldots a_{n-1} z + a_n$$

with real coefficients. The necessary and sufficient condition in order that all roots belong to the half-plane $\mathfrak{Re}(z) < 0$ is expressed by minors of an $n \times n$-matrix A with elements

$$(*) \qquad \alpha_{ik} = a_{2k-i}$$

where $a_0 = 01$ while $a_\nu = 0$ when $\nu < 0$ or $\nu > n$. For example, with $k = n$ we get a non-zero element in the n:th column if and only if $2n - i \leq n$ which means that only $i = n$ is possible, i.e. $\alpha_{nn} = 1$ while $\alpha_{in} = 0$ when $1 \leq i \leq n - 1$. The Hurwitz-Routh theorem asserts that the roots of P belong to the right half-plane if and only if all principal minors of the A-matrix are positive. For a proof we refer to Chapter 11 in [xx] which in addition to this criterion expressed by signs of minors contains w ealth of other results and also an extensive historic account where one major contributions in addition to those of Routh and Hurwitz are due to Sturm. Let us also mention that instead of using the argument principle which involves complex computations, one can consider the so called Cauchy index which arises when one pursues the real-valued function

$$\frac{\mathfrak{Re}\, P(iy)}{\mathfrak{Im}\, P(iy}$$

where jumps at zeros of the imaginary part leads to sign-chains and makes it possible to apply the rule of Descartes for zeros of real-valued functions. here a very efficient algorithm was discovered and developed by Sturm which can be used to determine the number of zeros of a polynomial with real coefficients in the right and the left half-plane. The interested reader should consult chapter 10 in the excellent text-book [XX] for further details and which in addition gives a very complete account of the extensive theory dealing with positions of zeros of polynomials which in general can have complex coefficients.

Special Integrals.

A. The integral of e^{-x^2}

.

Recall that one can use a trick in calculus to evaluate:

$$J = \int_0^\infty e^{-x^2} \cdot dx \tag{1}$$

Namely, use polar coordinates in the first quadrant which gives

$$J^2 = \int_0^{\pi/2} [\int_0^\infty r \cdot e^{-r^2} dr] d\theta = \pi/4$$

Hence we get $J = \sqrt{\pi}/2$. If we instead take the substitution $x^2 \to t$ one gets:

$$J = \frac{1}{2} \cdot \int_0^\infty t^{-1/2} \cdot e^{-t} \cdot dt$$

In general we consider an integral of the form

$$J_a = \int_0^\infty t^{-a} \cdot e^{-t} \quad : \quad 0 < a < 1 \tag{1}$$

In (1) we recognize the Γ-function and conclude that:

$$J_a = \Gamma(1-a) \tag{*}$$

This is admitted as an "analytic formula". More precisely one should include the Γ-functions in the family of "elementary functions". Of course a computer is needed for numerical values as a varies.

B. Integrals of rational functions.

Let $P(z)$ be a polynomial of degree $n \geq 2$ and assume that it has no real zeros which implies that the integral below exists:

$$J = \int_{-\infty}^\infty \frac{dx}{P(x)} \tag{1}$$

Consider the case when the roots of $P(z)$ are simple. Newton's formula gives:

$$\frac{1}{P(z)} = \sum_{k=1}^{k=n} \frac{1}{P'(\alpha_k)} \cdot \frac{1}{z - \alpha_k} \tag{2}$$

where the sum extends over the roots $\alpha_1, \ldots, \alpha_n$. The absolute convergence of (1) implies that we have a limit as we integrate over $-R \leq x \leq R$.

Exercise. Show that

$$J = 2\pi i \cdot \sum^{*} \frac{1}{P'(\alpha_k)} = -2\pi i \cdot \sum_{*} \frac{1}{P'(\alpha_k)}$$

where the sum is taken over the zero of P in the upper, respectively the lower half-plane. Conclude that one always has

$$\sum_{k=1}^{k=n} \frac{1}{P'(\alpha_k)} = 0 \tag{*}$$

Hint. Let $\alpha = a + ib$ be a complex number with $b \neq 0$. If $b > 0$ there exists a single-valued branch of $\log(z - \alpha)$ along the real axis where

$$-\pi < \arg(x - \alpha) < 0 \tag{1}$$

while $-\alpha$ moves in the lower half-plane. It follows that

$$\int_{-R}^{R} \frac{dx}{x-\alpha} = \log(R-\alpha) - \log(-R-\alpha) = \log \frac{|R-\alpha|}{(R+\alpha|} + i \cdot (\arg(R-\alpha) - \arg(-R-\alpha))$$

From (1) the reader may verify that

$$\lim_{R\to\infty} \arg(R-\alpha) = 0 \quad \text{and} \quad \lim_{R\to\infty} \arg(-R-\alpha) = -\pi$$

At the same time we notice that

$$\lim_{R\to\infty} \log \frac{|R-\alpha|}{(R+\alpha|} = 0$$

It follows that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x-\alpha} = \pi i$$

In the case $\alpha = a + ib$ where $b < 0$ the reader can verify that the limit integral instead takes the value $-\pi i$.

C. The integral $\int_0^\infty \frac{dx}{P(x)}$

Assume as above that the zeros of P are simple and non-real. Removing the half-line $0 \leq x < \infty$ from the complex plane we obtain a single value branch of $\log z$ whose imaginary part stays in $(0, 2\pi)$. Now we apply residue calculus to the function

$$g(z) = \log z \cdot \frac{1}{P(z)}$$

Exercise. Explain how to choose suitable contours and use that $\log z$ after one positive turn around the origin changes the branch by adding $2\pi i$, to conclude that

$$(*) \qquad -2\pi i \cdot \int_0^\infty \frac{dx}{P(x)} = 2\pi i \cdot \sum \mathfrak{res}_{z=\alpha_k} (\log z \cdot \frac{1}{P(z)})$$

where the residue sum is taken over all zeros of P.

C.1 Example. Consider first the case $P(z) = z^2 + 1$. Now

$$\frac{1}{xz^2+1} = \frac{1}{2i} \cdot [\frac{1}{z-i} - \frac{1}{z+i}]$$

It follows tha the residue sum becomes

$$\frac{1}{2i} \cdot (\log i - \log(-i)) = \frac{1}{2i} \cdot (\pi i/2 - 3\pi i/2) = -\pi/2$$

Taking the minus sign into the account in (*) we conclude that

$$\int_0^\infty \frac{dx}{1+x^2} = \pi/2$$

which is a wellknown formula from calculus.

Exercise. Use the formula (*) above to calculate

$$\int_0^\infty \frac{dx}{1+x^3}$$

D. The integrals $\int_{-\infty}^{\infty} \frac{e^{iax}\cdot dx}{P(x)}$

Assume that P has no real zeros and of degree ≥ 2. When a is a real number it is clear that the integral above exists. Let us consider the case $a > 0$. The entire analytic function e^{iaz} is small in the upper half-plane since

$$|e^{ia(x+iy)}| = e^{-ay}$$

Assume that the complex roots of P are simple and consider its fractional decomposition. Then the integral in (D) becomes

$$\lim_{R\to\infty} \sum \frac{1}{P'(\alpha_k)} \cdot \int_{-R}^{R} \frac{e^{iax}\cdot dx}{x-\alpha_k} \tag{1}$$

D.1 Exercise. Show that (1) is equal to

$$2\pi i \cdot \sum^{*} \frac{e^{ia\cdot\alpha_k}}{P'(\alpha_k)} \tag{2}$$

where $\sum^*$ extends over those k for which $\mathfrak{Im}(\alpha_k) > 0$.

D.2 Example. Consider the case $P(x) = x^2 + 1$. Then $\alpha_1 = i$ is the sole simple root in the upper half-plane and we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}\cdot dx}{1+x^2} = 2\pi i \cdot c_1 \cdot e^{-a} = \pi \cdot e^{-a}$$

D.4 Exercise. Let $a < 0$ and assume that the zeros $\{\alpha_k\}$ o $P(z)$ which belong to the lower half-plane are all simple. Show that in this case the integral (1) becomes

$$-2\pi i \cdot \sum_{*} \frac{e^{ia\cdot\alpha_k}}{P'(\alpha_k)}$$

where the sum extends over zeros with $\mathfrak{Im}(\alpha_k) < 0$. The reader should explain why a minus sign occurs by the aid of a figure and the orientation of the continuous which is used when the residue formula is applied.

E. Principal value integrals.

Outside $x = 0$ we have the odd function $\frac{1}{x}$ which entails that

$$\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} = 0$$

From this it is clear that the limit below exists for every C^1-function $g(x)$ on $[0,1]$:

$$\lim_{\epsilon\to 0} \int_{\epsilon<|x|<1} \frac{g(x)\cdot dx}{x}$$

In general, let $P(x)$ be a polynomial of some degree $k \geq$ whose zeros are all real and simple. Using principal values we can define the integral

$$\text{PV} \int_{-\infty}^{\infty} \frac{dx}{P(x)} \tag{1}$$

To compute (1) we consider the function $g(z) = \frac{1}{P(z)}$ which is analytic in the upper half-plane and integrated along a large half-circle of radius R, and along the real axis where small intervals around each real root are replaced by half-circles of radius ϵ.

E.1 Exercise. Draw a figure to illustrate the contour integral of $g(z)$ which is used above. Next, the complex line integral of g along this contour is zero. The definition of the principal value integral shows that (1) is equal to the limit

$$(*)\qquad \sum \int_0^\pi \epsilon \cdot \frac{e^{i\theta} \cdot id\theta}{P(a_k + \epsilon \cdot e^{i\theta})} = \pi \cdot \sum \frac{1}{P'(a_k)}$$

F. The integral $J = \int_0^\infty \frac{T(\sin x)}{P(x)} \cdot dx$

Let $P(x)$ be a polynomial of degree $N \geq 2$ whose zeros are all real and simple. Set

$$T(x) = \sum_{\nu=1}^{\nu=m} c_\nu \cdot \sin(\nu x)$$

where $\{c_\nu\}$ are real and assume that $T(a_k) = 0$ holds for each real zero of P. To compute the integral in (F) we consider the analytic function

$$(1)\qquad g(z) = \sum c_\nu \cdot e^{i\nu z} \cdot \frac{1}{P(z)}$$

Take a complex line integral over a contour which consists of a large half-circle $\{|z| = R\}$ in the upper half-plane and on $-R \leq x \leq R$ we replace small intervals around the zeros $\{a_k\}$ by small half-circles. If $\Gamma_{R,\epsilon}$ denotes the contour we get

$$(2)\qquad 0 = \int_{\Gamma_{R,\epsilon}} g(z) \cdot dz$$

Next, in the upper half plane the imaginary part of g is small which implies that

$$(3)\qquad \lim_{R \to +\infty} \int_0^\pi g(Re^{i\theta}) \cdot iRe^{i\theta} \cdot d\theta = 0$$

F.1 Exercise. Conclude from the above that one has the formula

$$(1)\qquad \int_{-\infty}^\infty \frac{T(\sin x)}{P(x)} \cdot dx = \lim_{\epsilon \to 0} \mathfrak{Im}\,[\sum_{k=1}^{k=N} \int_0^\pi g(a_k + \epsilon \cdot e^{i\theta}) \cdot i\epsilon \cdot e^{i\theta} \cdot d\theta\,]$$

Show also that for each real zero a_K one has the formula

$$(2)\qquad \lim_{\epsilon \to 0} \mathfrak{Im}\,[\int_0^\pi g(a_k + \epsilon \cdot e^{i\theta}) \cdot i\epsilon \cdot e^{i\theta} \cdot d\theta\,] = \mathfrak{Im}\big[\sum c_\nu e^{i\nu a_k} \cdot \frac{1}{P'(a_k)} \cdot \pi i\big]$$

Finally, since $\{c_\nu\}$ are real and $P'(a_k)$ also is real we see that the last term becomes

$$(3)\qquad \frac{\pi}{P'(a_k)} \cdot \sum c_\nu \cdot \cos(\nu a_k)$$

Hence we have established the formula

$$(*)\qquad \int_{-\infty}^\infty \frac{T(\sin x)}{P(x)} \cdot dx = \sum_{k=1}^{k=N} \frac{\pi}{P'(a_k)} \cdot T(\cos a_k)$$

F.2 Exercise. Above we assumed that the degree of P is ≥ 2. Show that the same formula as in (*) holds if P is linear, i.e is of the from $x - a_1$ for some real a_1. In particular consider the case $a_1 = 0$ and the sine-function $\sin x$. Then the general formula (*) gives

$$\int_{-\infty}^\infty \frac{\sin x}{x} \cdot dx = \pi$$

G. Adding complex zeros to P

Above the zeros of P were real and simple. Suppose now that P has real coefficients but extra non-real roots occur which therefore appear in conjugate pairs. In this case the calculation are the same as in (F) except that we also get residues from zeros of P in the upper half-plane. More precisely, the line integrals of g over $\Gamma_{R,\epsilon}$ from (F:1) are no longer zero. With R large and ϵ small the contour $\Gamma_{R,\epsilon}$ contains all zeros of $P(z)$ i the open upper half-plane. Let us assume that these zeros are simple and denote them by $\beta_1,\ldots,\beta_M$. So here $\mathfrak{Im}(\beta_\nu)>0$ for each ν. Each β-root gives a residue

$$2\pi i\cdot\frac{1}{P'(\beta_j)}\cdot\sum c_\nu e^{i\nu\cdot\beta_j} \tag{1}$$

Here we shall take the imaginary part to get a contribution. The result is that in the formula (*) from (F) one adds a term in the right hand side which becomes

$$2\pi\sum_{j=1}^{j=M}\mathfrak{Re}[\frac{1}{P'(\beta_j)}\cdot\sum c_\nu e^{i\nu\cdot\beta_j}] \tag{**}$$

G.1 Example. Consider the case $P(x)=x(x^2+1)$ and $T(x)=\sin x$. We get the root $\beta_1=i$ and notice that

$$\frac{1}{P'(\beta_1)}\cdot e^{i\cdot\beta_1}=\frac{1}{2i^2}\cdot e^{-1}=-\frac{1}{2e}$$

The conclusion is that

$$\int_{-\infty}^{\infty}\frac{\sin x\cdot dx}{x(1+x^2)}=\pi-\frac{\pi}{e} \tag{1}$$

H. The integral $\int_0^\infty\frac{x^a}{1+x^2}\cdot dx$

.

Let $0<a<1$. To find the integral above use the function z^a which under analytic continuation in the upper half-plane reaches the negative real axis where we have

$$(-x)^a=x\cdot e^{\pi ia}\quad:\quad x>0 \tag{1}$$

At the same time $\frac{1}{1+z^2}$ has a simple pole at $z=i$. So if J is the value of the integral in (H:1) we obtain:

$$J(1-e^{\pi ia})=2\pi i\cdot\mathfrak{res}(\frac{z^a}{1+z^2}:i)$$

At $z=i$ we use that $i=e^{\pi i/2}$ so that $z^a=e^{a\pi i/2}$ and find that the right hand side above becomes:

$$2\pi i\cdot\frac{e^{a\pi i/2}}{2i}=\pi\cdot e^{a\pi i/2}\implies$$

$$J=\pi\cdot\frac{e^{a\pi i/2}}{1-e^{\pi ia})}=2\cdot\pi\cdot\sin(\pi a/2) \tag{*}$$

H.1 Exercise. Use the methods from XX and explain a formula for the integral

$$\int_0^\infty\frac{x^a\cdot dx}{P(x)}$$

when P is a polynomial of degree ≥ 2 and without real zeros.

H.2 The case when P has negative real zeros. If this occurs one uses another method. The strategy is to take a complex integral of $g(z)=\frac{z^a}{P(z)}$ which starts with the interval $[\epsilon,R]$ where ϵ is small and R is large. Then one takes the circle $|z|=R$ and after one turn one integrates back from $x=R$ to $x=\epsilon$. Finally a small integral is taken over $|z|=\epsilon$ and the reader should illustrate the whole construction by a figure.

H.3 Exercise. If $\Gamma_{\epsilon,R}$ is the contour described above then it borders a simply connected domain where a single-valued branch of z^a exists. Then we can apply the residue formula and the reader should verify that

$$(1-e^{2\pi ai})\cdot J=2\pi i\sum\mathfrak{res}(g(z):\alpha_k)$$

where the sum is taken over all zeros of P.

H.4 Exercise Consider the case $P(x) = (x+1)^2$ which has a double zero at $x = -1$. Use the formula above to show that

$$\int_0^\infty \frac{x^a \cdot dx}{(x+1)^2} = \frac{\pi a}{\sin(\pi a)}$$

I. Use of the Log-function.

Consider the integral

$$J = \int_0^\infty \frac{dx}{1+x+x^5}$$

To compute it we consider the multi-valued analytic function

$$g(z) = \frac{\log z}{1+z+z^5}$$

Start the integration on the real x-axis from 0 to R and continue the complex line integral over the large circle $|z| = R$ and after one returns from R to $x = 0$ in the negative direction. While this is done we have a new branch of the log-function, i.e. it is now $\log x + 2\pi \cdot i$. Taking the negative direction into the account during the last integration along the non-negative x-axis it follows that

$$(*) \qquad 2\pi i \cdot J = -2\pi i \cdot \sum \mathfrak{res}\Big(\frac{\log z}{1+z+z^5}\Big)$$

Notice the minus sign above !

I.1 A simpler example. Suppose above that we instead take the polynomial $1 + z^2$. It has simple roots at i and $-i$. Now

$$\log i = \pi \cdot i/2 \quad \text{and} \quad \log -i = 3\pi \cdot i/2$$

The sum of residues therefore becomes

$$\frac{\pi \cdot i/2}{2i} + \frac{3\pi \cdot i/2}{-2i} = \frac{\pi}{2}$$

Thanks to the minus sign in (*) above we conclude that

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

This reflects a wellknown formula which can be established directly, i.e use that $\frac{1}{1+x^2}$ is the derivative of the arctg-function. But it is illuminating to see that the general procedure using the multi-valued log-function works.

J. Trigonometric integrals.

A trigonometric polynomial is of the form

$$P(\theta)=\sum c_k\cdot e^{ik\theta}$$

where the coefficients $\{c-k\}$ in general are complex numbers and the sum extends over a finite set of integers which may be both positive and negative. Consider a quotient of two such trigonometric polynomials

$$R(\theta)=\frac{P(\theta)}{Q(\theta)}$$

Assume that $Q(\theta)\neq 0$ for all $0\leq\theta\neq 2\pi$ and put:

(*)
$$J_R=\int_0^{2\pi}R(\theta)\cdot d\theta$$

To find (*) we use the substitutions $e^{ik\theta}\mapsto z^k$ and obtain:

(**)
$$J_P=\int_{|z|=1}\frac{P(z)}{Q(z)}\cdot\frac{dz}{iz}$$

One must not forget $\frac{1}{iz}$ which appears since

$$ie^{i\theta}\cdot d\theta=dz\implies d\theta=\frac{dz}{iz}$$

If M is a sufficiently large integer then

(1)
$$Q(z)=z^{-M}\cdot Q_*(z)\quad\text{and}\quad P(z)=z^{-M}\cdot P_*(z)$$

where P_* and Q_* are polynomials in z. Using (1) there remains to evaluate

(2)
$$\int_{|z|=1}\frac{P_*(z)}{Q_*(z)}\cdot\frac{dz}{iz}$$

Usually one picks residues in the open unit disc D. However, there are cases when Q_* has many zeros in D and then one can use residue calculus in the exterior disc. More precisely, choose a large positive number r so that $\{|z|<r\}$ contains all zeros of P_* and Q_*. Let $m(P_*)$ and $m(Q_*)$ be the degrees of the polynomials. If $m(P_*<Q_*$ the reader may verify that the integral (*) becomes

$$2\pi\cdot\sum\mathfrak{res}(R(\alpha_k)$$

with the sum taken over zeros of $Q*$ in the exterior disc $|z|>1$.

J.1 Experiment with a computer. It is instructive to perform a calculation via residues and at the same time compare the result by asking the computer for a numerical answer within a fraction of a second. Consider for example the integral

$$J=\int_0^{2\pi}\frac{d\theta}{1+a\cdot\cos\theta}$$

where a is a complex number with absolute value <1. Then

$$J=\frac{1}{i}\cdot\int_{|z|=1}\frac{2zdz}{2z+az^2+a}$$

We see that the quadratic polynomial has a simple zero in D. Let us denote it by α. Residue calculus gives the formula below which after can be solved numerically for different values of $0<a<1$.

$$J=2\pi\cdot\frac{\alpha}{1+a\alpha}$$

K. Summation formulas.

Consider the meromorphic function

$$g(z) = \frac{\cos \pi z}{\sin \pi z} \tag{1}$$

It has simple poles at all integers. Notice that g can be written as

$$i \cdot \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \tag{2}$$

K.1 Exercise. Show that there exists a constant A_1 such that the following holds for all integers N and every real number s:

$$\big|g((N + 1/2)i + is)\big| \leq A_1 \tag{*}$$

Show also that there exists a positive constant A_2 such that the following hold for eavery $z = x+iy$ where $|y| \geq 1$:

$$|g(x + iy) \leq A \tag{**}$$

Thus, the g-function is bounded when we stay away a bit from the real axis. Using (*) and (**) we can establish various summation formulas. In general, let p and q be two polynomials where we assume that $\deg(p) \geq \deg(q) + 1$ and that p has no real zeros. If N is a positive integer and $R >\geq 1$ we consider the rectangle

$$\square_{R,A} = \{-N - 1/2 < x < N + 1/2\} \times \{-R < y < R\}$$

Here N and R are chosen so that this rectangle contains all zeros $\{\alpha_\nu\}$ of p. Then Cauchy's residue formula is applied when we integrate $\frac{q}{p} \cdot g$ over the boundary of this rectangle. The result is

$$\frac{1}{2\pi i} \int_{\partial\square_{R,A}} \frac{q(z)}{p(z)} \cdot g(z) \cdot dz = \pi \cdot \sum_{k=-N}^{k=N} \frac{q(k)}{p(k)} + \sum \mathfrak{res}(\frac{q}{p} \cdot g : \alpha_\nu) \tag{***}$$

K.2 Exercise. Assume that $\deg(p) \geq \det(q) + 1$. Show that the line integrals over $\partial\square_{A,R}$ tend to zero when $A >> R >> 1$ and conclude that one has the general summation formula:

$$\pi \sum_{k=-\infty}^{k=\infty} \frac{q(k)}{p(k)} = -\mathfrak{res}(\frac{q}{p} \cdot g : \alpha_\nu) \tag{****}$$

Remark. Using fractional decomposition of $\frac{q}{p}$ the formula above boils down to the case when $q = 1$ and $p = z - \alpha$ for some non-real α. Consider as an example the case $p(z) = z - i$. Then the left hand side becomes

$$\pi[-\frac{1}{i} + \sum_{k=1}^{\infty} (\frac{1}{k-i} - \frac{1}{k+i})] = -\cot(i) = i \cdot \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}} \implies$$

$$\pi \cdot [1 + \sum_{k=1}^{\infty} \frac{2}{1+k^2}] = \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}$$

It is instructive to check the last formula numerically with a computer.

L. A Fourier integral.

We shall calculate an integral which is used in certain Tauberian theorems. The formula in (**) below is used to calculate certain Fourier transforms and gives rise to highly non-trivial limit formulas in Wiener's study of Tauberian theorems. The computations below illustrate that one is sometimes confronted with extra difficulties in order to handle singular log-functions. Our aim is to find a formula for the integral:

$$(*) \qquad J(s)=\int_0^\infty \frac{\log|1-x^2|}{x^2}\cdot x^{is}dt \quad : \quad s>0$$

Since the absolute value $|x^{is}|=1$ and $\log(1-x^2)\simeq -x^2$ when x is small we see that the integral converges. But it is not clear how to compute it via residue calculus. However, we shall see that this can be done after a number of steps. In the upper half-plane there exists an analytic function defined by:

$$(1) \qquad g(z)=\frac{\log(1-z^2)}{z^2}\cdot z^{is}$$

Here the single valued branch of $\log(1-z^2)$ is chosen so that its argument belongs to $(-\pi,0)$. We have also $z^{is}=e^{is\log z}$ where the single valued branch of $\log z$ has an argument in $(0,\pi)$ as usual. On the positive imaginary axis we get

$$(2) \qquad g(iy)=\frac{\log(1+y^2)}{y^2}\cdot(iy)^{is}=\frac{\log(1+y^2)}{y^2}\cdot y^{is}\cdot e^{-\pi s/2}$$

After these preparations we consider the complex line integral of the g-function over the closed curve given by the real interval $0\leq x\leq R$, the quarter-circle $\{z=Re^{i\theta}:\ 0\leq\theta\leq\pi/2\}$ and the imaginary interval from iR to 0. Along the real axis the argument of the log-function changes. More precisely we notice that the imaginary part of $\log(1-x^2)$ is zero when $0<x<1$ and is $-\pi$ if $x>1$. From this we obtain

$$(3) \qquad \lim_{R\to\infty}\int_0^R g(x)\cdot dx=\int_0^\infty \frac{\log|1-x^2|}{x^2}\cdot x^{is}\cdot dx-i\pi\cdot\int_1^\infty \frac{x^{is}\cdot dx}{x^2}$$

Next, with s real and positive the absolute value $|z^{is}|$ is bounded in the upper half-plane and the reader can verify that the line integral of g along the quarter circle tends to zero when $R\to+\infty$. There remains to consider the line integral along the imaginary axis which on the closed contour above is taken in the negative direction. Taking this sign into the account together with (2) and the vanishing of the complex line integral over the whole closed contour we see that (3) is equal to

$$(4) \qquad \int_0^\infty \frac{\log(1+y^2)}{(iy)^2}\cdot y^{is}\cdot idy$$

After a partial integration (4) becomes

$$(5) \qquad \frac{1}{is-1}\cdot i\cdot\int_0^\infty \frac{2y}{1+y^2}\cdot y^{is-1}\cdot dy=\frac{2}{s+i}\cdot\int_0^\infty \frac{y^{is}\cdot dy}{1+y^2}$$

To calculate the last integral we use that $(-1)^{is}=e^{\pi i\cdot is}=e^{-\pi s}$ and conclude that

$$(6) \qquad (1+e^{-\pi s})\cdot\int_0^\infty \frac{y^{is}\cdot dy}{1+y^2}=\int_{-\infty}^\infty \frac{y^{is}\cdot dy}{1+y^2}$$

The last integral is found by residue calculus and as a consequence the reader may verify that (4) becomes

$$(7) \qquad \frac{2}{s+i}\cdot\frac{1}{1+e^{-\pi s}}\cdot 2\pi i\cdot\frac{i^{is}}{2i}=\frac{2}{s+i}\cdot\frac{1}{1+e^{-\pi s}}\cdot\pi\cdot e^{-\pi s/2}$$

Conclusion. One has the equality

$$(**) \qquad \int_0^\infty \frac{\log|1-x^2|}{x^2}\cdot x^{is}\cdot dx=\frac{i\pi}{1-is}+\frac{2\pi}{s+i}\cdot\frac{1}{e^{\pi s}+e^{-\pi s}}$$

Notice that the right hand side is zero when $s = 0$ which gives

$$(**) \qquad \int_0^\infty \frac{\log|1-x^2|}{x^2} \cdot dx = 0$$

M. Multi-valued integrands.

A more involved study arises when the integrands are branches of multi-valued functions and one seeks values which depend upon parameters. Let us give an example.

$$(*)\qquad J(z)=\int_0^1 \frac{dt}{\sqrt{t(z-t)}}$$

When z is real and >1 we can evaluate the integral as in ordinary calculus. In the half-plane $\mathfrak{Re}(z)>1$ we see that $J(z)$ is an analytic function of z whose complex derivative becomes

$$(**)\qquad J'(z)=-\frac{1}{2}\cdot\int_0^1 \frac{dt}{\sqrt{t(z-t)^3}}$$

It turns out that $J(z)$ extends to an analytic function where the sole branch points are 0 and 1. To begin with we can choose a single valued branch of $\sqrt{z-t}$ when $\mathfrak{Im}(z)>0$ so that $J(z)$ is analytic in the upper half-plane. Less obvious is that J extends analytically across the open real interval $0<x<1$. One can prove this using a deformation of the contour which defines J, i.e. replace $[0,1]$ by curves in the complex t-plane which joint 0 and 1. Examples of deformation the contour during the analytic continuation of the J-function appear in the classic literature. It was for example used by Hermite and appears in many text-books devoted to algebraic functions. See for example the excellent material in Paul Appel's books which contains a wealth of examples related to integrals on algebraic curves and especially so called hyper-elliptic integrals.

Use of $\mathcal{D}$-module theory. A method which avoids the rather involved deformation of contours to achieve the analytic continuation goes back to Fuchs, and was later put forward in a much wider context in lectures by Pierre Deligne at Harvard University in 1967, inspired by deep studies by Nils Nilsson from the article [Nilsson-1965] which deals with integrals over algebraic chains in higher dimensions and leads to the notion of Nilsson class functions. Here we stay in dimension one and begin to seek a differential operator $Q(z,\partial_z)$ with polynomial coefficients which annihilates the J-function. Set $\nabla=z\partial_z$ which gives

$$-\nabla(J)=\frac{1}{2}\cdot\int_0^1 \frac{(z-t)dt}{\sqrt{t(z-t)^3}}+\frac{1}{2}\cdot\int_0^1 \frac{t\cdot dt}{\sqrt{t(z-t)^3}}=\frac{J}{2}+\frac{1}{2}\int_0^1 \frac{\sqrt{t}\cdot dt}{(z-t)^{\frac{3}{2}}}$$

In the last integral we perform a partial integration with respect to t and obtain

$$(*)\qquad \sqrt{t}\cdot\frac{1}{\sqrt{z-t}}|_0^1-\frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t\cdot(z-t)}}\implies \nabla(J)=-\frac{1}{\sqrt{z-1}}$$

It follows that J extends to a multi-valued function outside 0 and 1. Since

$$(z-1)\partial(z-1)^a=a(z-1)^a$$

for all a it follows from (*) that

$$[(z-1)\partial+1/2]\cdot\nabla(J)=0\implies$$

$$(*)\qquad (z-1)z\cdot\partial^2(J)+\frac{3}{2}\cdot\nabla(J)-\partial(J)=0$$

Exercise Investigate the multi-valued behavior of J around 0 and 1. More precisely, J generates a Nilsson class function of rank 2 and as described in Chapter 4 this leads to the local monodromy expressed by 2×2-matrices at each of these ramification points.

L. Multi-valued Laplace integrals.

Let $k\geq 1$ and $a_1,\ldots,a_k$ is some k-tuple of distinct points in $\mathbf{C}$. Let us also consider another k-tuple of complex numbers $\lambda_1,\ldots,\lambda_k$. In as complex w-plane we consider the simple curve C_R which consists of the half circle $\{w=Re^{i\theta}\quad -\pi/2\leq\theta\leq\pi/2\}$ and the two horizontal lines

$$\ell*(R)=w=\{t-iR\quad -\infty<t\leq 0\}\quad\text{and}\quad \ell^*(R)=\{t+iR\quad -\infty<t\leq 0\}$$

Her R is chosen so large that the absolute values $|a_\nu| < R$ for each ν. In a neighborhood of $w = R$ we get local branches of the complex powers w^{λ_ν}, i.e. with $w = R + \zeta$ and ζ small one has

$$(R+\zeta)^{\lambda_\nu} = e^{\lambda_\nu \cdot \log(R+\zeta)}$$

and the branch is chosen so that the value when $\zeta = 0$ is the ordinary complex exponential $e^{\lambda_\nu \cdot \log(R}$ where $\log R$ is real. Now each function $w^{\lambda\nu}$ extends analytically along C_R and if z is another complex variable whose real part is < 0 we see that there exists a convergent integral

$$(*) \qquad J(z) = \int_{C_R} e^{zw} \cdot \prod (w - a_\nu)^{\lambda_\nu} \cdot dw$$

It is clear that this J-function is analytic in the half-plane $\mathfrak{Re}\, z > 0$. In general $J(z)$ does not extend to an entire function. Consider as an example the case $k = 1$ with $a_1 = 0$ and set $\lambda = \lambda_1$. So here

$$J(z) = \int_{C_R} e^{zw} \cdot w^\lambda \cdot dw$$

When $z = x$ is real and positive we can perform the variable substitution $w \mapsto u/x$ and get

$$J(x) = x^{-\lambda-1} \int_{C_{Rx}} e^u \cdot u^\lambda \cdot du$$

By Theorem 3.,2 in [Gamma] the integral above is found via the Γ-function, i.e. we obtain

$$J(x) = x^{-\lambda-1} \cdot \frac{2\pi i}{\Gamma(-\lambda)}$$

From this we conclude that $J(z)$ extends to the function $z^{-\lambda-1}$ times the constant $\frac{2\pi i}{\Gamma(-\lambda)}$. So unless λ is an integer we get a multi-valued J-function. For the general case (*) one has

Theorem. *The J-integral extends to $\mathbf{C} \setminus (a_1, \ldots, a_k)$ as an analytic function which in general is multi-valued with ramification points or poles at $a_1, \ldots, a_k$.*

To prove this result one finds a differential equation with polynomial coefficients satisfied by J, i.e. the efficient procedure is to use $\mathcal{D}$-module technique. More precisely, suppose we have found a differential operator

$$Q(w, \partial_w) = \sum q_j(w) \cdot \partial_w^j$$

in the Weyl algebra A_1 with respect to the w-variable such that

$$Q(\prod (w - a_\nu)^{\lambda_\nu}) = 0$$

Then we associate the differential operator Q^* in the z-variable given by

$$Q^*(z, \partial_z) = \sum q_j(-\partial_z) \cdot z^j$$

Exercise. Show that we get $Q^*(J(z)) = 0$. Then hint is that a partial integration gives

$$\text{(i)} \qquad zJ(z) = -\int_{C_R} e^{zw} \cdot \partial_w \big(\prod (w - a_\nu)^{\lambda_\nu}\,\big) \cdot dw$$

At the same time we notice that

$$\text{(ii)} \qquad \partial_z(J) = \int_{C_R} e^{zw} \cdot w \cdot \prod (w - a_\nu)^{\lambda_\nu}\,) \cdot dw$$

A computation will show that the differential operator Q^* has order k and that the polynomial coefficient in front of ∂_z^k is given by $\prod(z - a_\nu)$ from which Theorem XX follows.

Remark. The computations above can be reversed and lead to integral representations of functions which are solution to a differential equation defined by $Q^*(z, \partial_z)$ for a suitable Q^*. A classic case are the confluent hypergeometric functions which are solutions to certain differential equations. More precisely, let α and γ be a pair of non-zero complex numbers. One seeks solutions $f(z)$ to the second order equation

$$z \cdot \partial^2(f) + (\gamma - z)\dot{\partial}(f) - \alpha \cdot f = 0$$

So here

$$Q^* = z \cdot \partial^2 + (\gamma - z)\partial - \alpha$$

w hich in the non-commutative Weyl algebra it can be written in the form

$$\partial^2 \cdot z - 2\partial + \gamma\partial - \partial \cdot z + 1 - \alpha$$

Hence Q^* is associated to the differential operator

$$Q(w, \partial_w) = w^2\partial_w + 2w - \gamma w + w\partial_w + 1 - \alpha$$

Using this one can investigate the asymptotic behaviour of solutions to (xx).

Chapter VIII The Gamma function and Riemann's ζ-function

Contents

Introduction. The results in this chapter stem from early work by Euler and later studies by Gauss and Riemann. The Γ-function is defined and analyzed in Section I. A major result is that $\frac{1}{\Gamma(z)}$ is an entire function which is not obvious from its direct definition. Another important result in Section 1 is the integral formula for the Γ-function in I.3. Section II is devoted to Riemann's ζ-function which to begin with is defined by the Dirichlet series

(*) $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad : \quad \mathfrak{Re}\, s > 1$$

Euler proved that the ζ-function extends to the whole complex s-plane with a single pole at $s = 1$ whose reside is one which gives the entire function

$$\zeta(s) - \frac{1}{s-1}$$

A consequence is that the zeta-function has no zeros on the line $\mathfrak{Re}\, s = 1$, i.e.

(1) $$\zeta(1 + it) \neq 0 \quad \text{for all real} \quad t \neq 0$$

Indeed, this follows from the general result due to de Valle Poussin in § XX. From (1) we shall deduce the Prime Number Theorem in Section 4. We proceed to discuss further properties of the ζ-function which were announced by Riemann. The first major result is the functional equation:

(1) $$\zeta(1 - s) = \frac{2}{(2\pi)^s} \cdot \cos \frac{\pi}{2}\, s \cdot \Gamma(s) \cdot \zeta(s)$$

Here the Γ-function appears. It will be studied in Section 1 where we prove that it has simple poles at $(0, -1, -2, \ldots)$ and hence the functional equation implies that

(2) $$\zeta(-2n) = 0 \quad : \quad n = 1, 2, \ldots$$

The *Riemann hypothesis* states that all other zeros stay in the critical line $\mathfrak{Re}\, s = 1/2$. This conjecture is still open and considered as the utmost challenging open question in mathematics. Extensive use of computers confirm the hypothesis up to a high degree but there remains to give a proof. At the end of this introduction we give further comments about this famous hypothesis.

Some results by Beurling. The zeta-function appears in a class of Dirichlet series which arises as follows: Let $\mathcal{F}$ be the family of all non-decreasing increasing sequence of positive numbers $\lambda_1 \leq \lambda_2 \leq \ldots$ for which there exists some $\delta > 0$ such that

(i) $$\lambda_n \geq \delta \cdot n$$

hold for every n where δ can depend on the given sequence in the family. To every sequence in $\mathcal{F}$ we associate the Dirichlet series

$$\text{(ii)} \qquad \Lambda(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

It is clear that this gives an analytic function in the half-plane $\mathfrak{Re}\, s > 1$. Moreover, from (i) general facts about entire functions of exponential type imply that the Hadamard product

$$\text{(iii)} \qquad f(z) = \prod (1 + \frac{z^2}{\lambda_n^2})$$

yields an entire function in the class $\mathcal{E}$ from § xx in Special Topics. In XXX we prove that $\Lambda(s)$ is recovered from $f(z)$ by the formula

$$\text{(iv)} \qquad \Lambda(s) = s \cdot \sin \frac{\pi s}{2} \cdot \frac{1}{\pi} \cdot \int_0^{\infty} \log f(x) \cdot \frac{dx}{x^{s+1}}$$

The special role of the zeta-function is seen from the formula

$$\text{(v)} \qquad f(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$$

Indeed, it follows when we take $\lambda_n = n$ to construct the Hadamard product (iii) above. In § xx we shall use the formula above and some further equations to establish the following:

0.1 Theorem. *The zeta-functions extends to a meromorphic function in the complex s-plane with a simple pole at $s = 1$ of residue one. Moreover, there is a constant C such that it satisfies the growth condition*

$$\max_{|s|=r} |\zeta(s)| \leq C \cdot \frac{\Gamma(r)}{(2\pi)^r} \quad \text{for every} \quad r \geq 2$$

0.2 An extremal property of $\zeta(s)$. In a lecture at Harvard University in 1949, Beurling proved that the zeta-function has a remarkable position in a class of functions defined by Dirichlet series. More precisely, for any positive number k we denote by $\mathcal{C}_k$ the class of series $\Lambda(s)$ from (ii) above with the properties:

$$\text{(a)} \qquad \Lambda(s) - \frac{1}{s-1} \quad \text{is entire}$$

$$\text{(b)} \qquad \Lambda(-2n) = 0 \quad \text{for all positive integers}$$

Finally, there exists a constant C such that

$$\text{(c)} \qquad \max_{|s|=r} |\Lambda(s)| \leq C \cdot \frac{\Gamma(r)}{(2\pi k)^r} \quad \text{hold for a constant} \quad C \quad \text{and all} \quad r \geq 2$$

Notice that the class $\mathcal{C}_k$ becomes more restrictive as k increases. In section 5 we prove that for every $k > 1/2$ the sole functions in the class $\mathcal{C}_k$ are given by a **C**-linear combination of the two functions

$$\zeta(s) \quad \text{and} \quad (2^s - 1)\zeta(s)$$

The gap when $1/2 < k < 1$ illustrates the special role of the ζ-function. In Section 7 we prove another result by Beurling where Theorem 7.1 gives a *necessary and sufficient condition* for the validity of the Riemann hypothesis expressed by a certain L^2-closure on the interval $(0, 1)$ generated by a specific family of functions. Theorem 7.1 is based upon a closure theorem in Section 6 whose proof illustrates the efficiency of mixing functional analysis with analytic function theory. Let us remark that the study of distributions of primes and the Riemann hypothesis was one of the main issues in Beurling's research. His first extensive article on this subject from 1937 is entitled *Analyse de loi asymptotique de la distribution des nombres premier generalisés.* Even though this work does not settle the Riemann hypothesis the reader will find a number of interesting results concerned with Dirichlet series.

0.3. The distribution of prime numbers

A motivation to consider the ζ-function is Euler's product formula:

$$(*) \qquad \zeta(s) = \prod \frac{1}{1-p^{-s}} : \text{ product over all prime numbers } \geq 2$$

Indeed, (*) follows since every integer $n \geq 2$ can be factorised in a unique way as a product of of prime numbers. Let us introduce the counting function:

$$\mathcal{N}(x) = \text{number of primes } \leq [x] \quad : [x] = \text{least integer } \leq x$$

Thus $\mathcal{N}(x)$ is the primitive of the discrete measure supported by $[2, +\infty)$ which assigns a unit point mass at every prime. An integration by parts for this counting function gives:

$$\text{(i)} \qquad \log\ \zeta(s) = s \cdot \int_2^\infty \frac{\mathcal{N}(x) \cdot dx}{(x^s - 1)x} \quad : s \text{ real and } > 1$$

Insight about the increasing function $\mathcal{N}(x)$ arises if we have shown that the ζ-function extends beyond $s = 1$ as a meromorphic function of a complex variable with a simple pole at $s = 1$. Granted this we can write

$$\text{(ii)} \qquad \zeta(s) = \frac{1}{s-1} + g(s) \quad : g(s) \text{ analytic in a disc } |s-1| < \delta$$

With $s = 1 + \epsilon$ we get the following limit formula:

$$(**) \qquad \lim_{\epsilon \to 0} \frac{1}{\mathrm{Log}\,[\frac{1}{\epsilon}]} \cdot \int_2^\infty \frac{\mathcal{N}(x) \cdot dx}{x^{2+\epsilon}} = 1$$

The Prime Number Theorem. To begin with we notice that

$$\lim_{\epsilon \to 0} \frac{1}{\log \epsilon} [\frac{1}{\epsilon}] \cdot \int_2^\infty \frac{x \cdot dx}{x^{2+\epsilon} \cdot \log x} = 1$$

Indeed, this follows by the variable substitution $x \mapsto e^t$ and the observation that

$$\lim_{\epsilon \to 0} \frac{1}{\mathrm{Log}\,[\frac{1}{\epsilon}]} \cdot \int_1^\infty e^{-\epsilon t} \cdot \frac{dt}{t} = 1$$

In view of (**) above is therefore no surprise that the following limit formula holds for $\mathcal{N}$.

0.3.1 Theorem. *One has*

$$\lim_{x \to \infty} \frac{\mathrm{Log}\,(x) \cdot \mathcal{N}(x)}{x} = 1$$

Remark. This limit formula was known by heuristic considerations long before Riemann's study of the ζ-function. Using elementary arithmetic one gets the prime number theorem under the *extra hypothesis* that $\mathcal{N}(x)$ has a "regular growth". The reader may consult the book [Co-Rob] *What is Mathematics* where the prime number theorem is derived under such an extra hypothesis. But presumably both Riemann and Gauss understood that $\mathcal{N}(x)$ does *not* increase in a regular way. Hence a solid proof of the prime number theorem was requested and it was established by de Valle Poussin in 18XX(?) and is exposed in section 4. Considerable work has been devoted to find small remainder terms in the Prime Number Theorem, i.e. to write

$$\mathcal{N}(x) = \frac{x}{\mathrm{Log}\, x} + \text{smaller terms}$$

We shall not discuss this but refer to the extensive literature about such limit formulas with remainder terms, where refined versions often rely upon the Riemann hypothesis. That is, *granted that this hypothesis* is valid one can establish various limit formulas with small remainder terms.

0.4. The Riemann Hypothesis.

The conjecture by Riemann in [Rie] states that the zeros of $\zeta(s)$ in the critical strip $0 < \mathfrak{Re}(s) < 1$ belong to the line $\mathfrak{Re}(s) = 1/2$. This line is special since the functional equation yields an entire function $\xi(s)$ defined by

$$(*) \qquad \xi(s) = \frac{s(s-1)}{2} \cdot \Gamma(\frac{s}{2}) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s)$$

Moreover, the ξ-function satisfies

$$\xi(s) = \xi(1-s) \quad : \quad \xi(s) = \bar{\xi}(\bar{s})$$

From this it follows that the function

$$t \mapsto \Gamma(\frac{1/2+it}{2}) \cdot \pi^{-\frac{1/2+it}{2}} \cdot \zeta(\frac{1}{2}+it)$$

is real-valued. Here $\Gamma(\frac{1/2+it}{2}) \cdot \pi^{-\frac{1/2+it}{2}} \neq 0$ for all real t and we can define the function

$$(**) \qquad \theta(t) = \arg\big[\Gamma(\frac{1/2+it}{2}) \cdot \pi^{-\frac{1/2+it}{2}}\big] \quad : \quad 0 \leq t \leq \infty$$

where we take $\theta(0) = 0$. So now we have the real-valued function

$$(***) \qquad X(t) = e^{i\theta(t)} \cdot \zeta(\frac{1}{2}+it)$$

Hence zeros of ζ on the critical line $\mathfrak{Re}(s) = \frac{1}{2}$ correspond to zeros of this real valued function. It is not difficult to conclude that there must occur infinitely many zeros on the critical line. Namely, consider the two integrals

$$J(T) = \int_0^T X(t) \cdot dt \quad : \quad J^*(T) = \int_0^T |X(t)| \cdot dt$$

If J^* increases faster than $J(T)$ then the real valued function $X(t)$ must change sign and hence produce zeros on the critical line. This is indeed true since one can show that

$$(****) \qquad \lim_{T \to \infty} \frac{J(T)}{T} = 0 \quad : \quad \lim_{T \to \infty} \frac{J^*(T)}{T} = 1$$

From this it follows that the ζ-function has an infinity of zeros on the critical line.

0.4.1 The Riemann-Siegel formula. In [Sie] it is proved that as $t \to +\infty$ one has the asymptotic limit formula:

$$X(t) = \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{\cos\theta(t) - t \cdot \log n}{\sqrt{n}} + O(t^{-\frac{1}{2}})$$

Starting from this Gram and Backlund established numerical results. Namely, the results above one derives the asymptotic formula:

$$\theta(t) = \frac{t}{2} \cdot \big[\text{Log}\frac{t}{2\pi} - \frac{1}{2}\big] - \frac{\pi}{8} + O(\frac{1}{t})$$

Then one regards the increasing sequence of real numbers $\{t_\nu\}$ for which

$$\theta(t_\nu) = (\nu - 1) \cdot \pi \quad : \quad \theta'(t_\nu) > 0$$

Using the Riemann-Siegel formula one has

$$(1) \qquad \zeta(\frac{1}{2} + it_\nu) = 1 + \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{\cos(t_\nu \cdot \log n)}{\sqrt{n}} + O(t_\nu^{-\frac{1}{2}})$$

This formula suggests that $\zeta(\frac{1}{2} + it_\nu)$ in general is positive and that $X(t_{\nu-1})$ and $X(t_\nu)$ will have different sign which therefore gives a zero for the ζ-function in the interval $(t_{\nu-1} - t_\nu)$. The *Law of Gram* asserts that all zeros of $X(t)$ should appear in this fashion, i.e. one zero is produced in $(t_{\nu-1} - t_\nu)$ for every ν. Of course, this "law" was presented as an asymptotic formula only. A "weak asymptotic law" was confirmed in work by Hutchinson and Titchmarsh. But the situation is not so easy. For consider the actual zeros on the critical line:

$$0 < \gamma_1 < \gamma_2 < \ldots \quad : \quad \zeta(\frac{1}{2} + i\gamma_\nu) = 0$$

In an article from 1942, Atle Selberg proved that the γ-sequence increases in a certain *irregular* fashion.

0.4.2 Theorem. *There exists an absolute constant $0 < C_* < 1$ such that for every positive integer r one has*

$$\limsup_{n\to\infty} \frac{\gamma_{n+r} - \gamma_n}{2\pi r} \cdot \operatorname{Log} \gamma_n > 1 + C_* \quad \text{and} \quad \liminf_{n\to\infty} \frac{\gamma_{n+r} - \gamma_n}{2\pi r} \cdot \operatorname{Log} \gamma_n < 1 - C_*$$

Let us finish by a citation from A. Selberg's lecture on the Zeta Function and the Riemann Hypothesis at the Scandinavian Congress in mathematics held at Copenhagen in 1946, where he gave some comments about the eventual validity of the Riemann Hypothesis:

In spite of the numerical evidence by which it is supported there are still reasons to regard the Riemann Hypothesis with suspicion. For in the range covered by calculations. the exceptions from Gram's Law are few and they are of the simplest kind, farther out more severe departures from Gram's law must occur and it seems likely that the irregularities in the variation of $\zeta(s)$ which should be necessary for producing zeros outside $\mathfrak{Re}(s) = \frac{1}{2}$, should be far more remote than the first exceptions from Gram's Law.

Note. Atle Selberg (19xx-2007) received the Fields medal at the IMU-congress in 1950 for his outstanding contribution in number theory and deep studies of the ζ-function.

0.4.3 Hardy's inversion formula. Another contribution in the study of zeros on the critical line was achieved by Hardy in [Har]. His method was to regard the function

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} \quad : \ x > 0$$

The construction of the Γ function gives the equality

$$(*) \qquad \frac{1}{n^s}\Gamma(\frac{s}{2}) \cdot \pi^{-\frac{s}{2}} = \int_0^\infty e^{-n^2\pi x} \cdot x^{\frac{s}{2}} \cdot \frac{dx}{x}$$

for each positive integer n. A summation over n gives:

$$\zeta(s) \cdot \Gamma(\frac{s}{2}) \cdot \pi^{-\frac{s}{2}} = \int_0^\infty \omega(x) \cdot x^{\frac{s}{2}} \cdot \frac{dx}{x} \quad : \mathfrak{Re}(s) > 1$$

Using Fourier's inversion formula and a shift of certain complex line integrals, Hardy established the following result:

0.4.4 Theorem. *When $\mathfrak{Re}\, x > 0$ one has the equality*

$$\omega(x) = \frac{1}{2\sqrt{x}} + \frac{1}{4\pi} \cdot \int_{-\infty}^{\infty} \zeta(1/2 + it) \cdot \Gamma(1/4 + it/2)\pi^{-it/2-1/4} x^{-1/4-it} \cdot dt$$

Remark. The right hand side yields a nicely convergent integral. The reason is that one has an exponential decay for the Γ-function, i.e. there are constants A and k such that

$$|\Gamma(1/4 + it/2)| \leq A \cdot e^{-k|t|} \quad : \ -\infty < t < \infty$$

At the same time the ζ-function does not increase too fast. In fact, in xx we show that there is constant B such that

$$|\zeta(1/2 + it)| \leq B \cdot t^2 \quad : \ t \geq 1$$

We refer to Hardy's original work how the inversion formula is used to produce zeros of the zeta-function on the critical line. See also [Tit] devoted Riemann's ζ-function by E.C. Titchmarsch. Here methods and results from analytic function theory and Fourier analysis are used to the "bitter end" in the search for a positive answer to the Riemann Hypothesis.

Final remark. The literature about the Riemann hypothesis is extensive and there exist alternative conjectures, some of them even more general than Riemann's. We shall not enter into a discussion about this. But let also recall that André Weil solved the Riemann hypothesis in characteristic p. See [We]. This gives some support for the "optimistic point of view point " that Riemann's hypothesis is true. Until an answer is found the Riemann Hypothesis remains as an outstanding open problem in mathematics.

1. The Gamma function

The Γ-function has from the start a simple definition:

$$(*) \qquad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad : \mathfrak{Re}(z) > 0$$

With $z = x + iy$ the absolute value $|t^{z-1}| = t^{x-1}$ when t is real and positive. Moreover, $t^\alpha \cdot \log(t)$ is locally integrable on intervals $(0, t_*)$ with $t_* > 0$ and we have the exponential decay from e^{-t}. Hence (*) converges when $\mathfrak{Re}(z) > 0$ and gives a holomorphic function with the complex derivative

$$(0.1) \qquad \Gamma'(z) = \int_0^\infty e^{-t} \cdot \operatorname{Log}(t) \cdot t^{z-1} dt$$

Partial integration gives:

$$\Gamma(z) = e^{-t} \cdot \frac{t^z}{z} |_0^\infty + \frac{1}{z} \int_0^\infty e^{-t} t^z \, dt = \frac{1}{z} \cdot \Gamma(1+z)$$

Hence we have the equality

$$(0.2) \qquad z\Gamma(z) = \Gamma(z+1) \quad : \quad \mathfrak{Re}(z) > 0$$

Replacing z by $z + 1$ and so on we obtain

$$(0.3) \qquad z(z+1) \cdots (z+m)\Gamma(z) = \Gamma(z+m+1) \quad : \quad m = 0, 1, 2, \ldots$$

Since (0.3) hold for all non-negative integers, $\Gamma(z)$ extends to a meromorphic function defined in the whole complex plane and for every positive integer m one has:

$$(0.4) \qquad \Gamma(z) = \frac{1}{z(z+1)\ldots(z+m)} \cdot \Gamma(z+m) \quad : \quad \mathfrak{Re}(z) > -m$$

Here (0.4) shows that the poles of $\Gamma(z)$ are contained in the set of non-negative integers. We shall later prove that a simple pole exists for every such integer. Moreover, we will show that $\frac{1}{\Gamma(z)}]$ is an entire function and establish the functional equation:

$$(**) \qquad \frac{1}{\Gamma(z) \cdot \Gamma(1-z)} = \frac{\sin \pi z}{\pi}$$

1. The Gauss representation.

Consider the Hadamard product:

$$(1.1) \qquad H(z) = \prod_{m=1}^{\infty} (1 + \frac{z}{m}) e^{-\frac{z}{m}}$$

Here $H(z)$ is entire with simple zeros at negative integers and (0.4) gives the entire function

$$(1.2) \qquad zH(z) \cdot \Gamma(z) \in \mathcal{O}(\mathbf{C})$$

1.3 Theorem *Let γ be the Euler constant defined by*

$$\gamma = Lim_{n \to \infty} 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \operatorname{Log} n$$

Then one has

$$(**) \qquad zH(z) \cdot \Gamma(z) = e^{-\gamma z}$$

Proof. When $n \geq 2$ we consider the partial product

$$H_n(z) = \prod_{m=1}^{m=n} (1 + \frac{z}{m}) e^{-\frac{z}{m}}$$

A computation gives the identity

$$\frac{z(z+1)\cdots(z+n)}{n\,!\cdot n^z} = e^{z\cdot[1+\frac{1}{2}+\ldots+\frac{1}{n}-\operatorname{Log} n]}\cdot z\cdot H_n(z)$$

If $\mathfrak{Re}(z)>0$ the right hand side converges to the limit $e^{\gamma z}\cdot z\cdot H(z)$. Hence there exists the entire limit function

(i) $$G(z)=\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n\,!\cdot n^z}$$

It is clear that (**) holds in Theorem 1.3 if we have proved:

(ii) $$G(z)\cdot\Gamma(z)=1$$

To prove (ii) we regard the meromorphic function $\mathcal{G}=\frac{1}{G}$ so that

(iii) $$\mathcal{G}(z)=\lim_{n\to\infty}\frac{n\,!\cdot n^z}{z(z+1)\cdots(z+n)}$$

Let us put

(iv) $$\mathcal{G}_n(z)=\frac{n\,!\cdot n^z}{z(z+1)\cdots(z+n)}$$

Then we have

$$\mathcal{G}_n(z+1)=\frac{n\,!\cdot n^{1+z}}{(z+1)(z+2)\cdots(z+n+1)}=$$

$$\frac{n\,!\cdot n^z}{z(z+1)\cdots(z+n)}\cdot\frac{nz}{n+1+z}=z\mathcal{G}_n(z)\cdot\frac{n}{n+z+1}$$

The last quotient tends to z and we already know that $\mathcal{G}_n(z)\to\mathcal{G}(z)$. We conclude that the $\mathcal{G}$-function satisfies

(v) $$\mathcal{G}(z+1)=z\mathcal{G}(z)$$

Hence $\mathcal{G}$ satisfies the same functional equation as the Γ-function and there remains only to show that $\Gamma=\mathcal{G}$. Since we have two meromorphic functions it suffices that they are equal on the positive real axis. To show this we first regard complex derrivatives of $\operatorname{Log}\mathcal{G}$ which is holomorphic in the right half-plane. Since the limit in (iii) defines $\mathcal{G}$ in $\mathfrak{Re}(z)>0$ we have

(vi) $$\operatorname{Log}\mathcal{G}(z)=\lim\,[z\cdot\operatorname{Log}(n)+\sum_{\nu=1}^{\nu=n}\operatorname{Log}\nu-\sum_{\nu=0}^{\nu=n}\operatorname{Log}(z+\nu)\,]$$

In the right hand side we take the second order derivative which becomes

$$\sum_{\nu=0}^{\nu=n}\frac{1}{z+\nu)^2}$$

Passing to limit as $n\to\infty$ it follows that

(vii) $$\frac{d^2\operatorname{Log}\mathcal{G}(z)}{dz^2}=\lim_{n\to\infty}-\sum_{\nu=0}^{n}\frac{1}{(z+\nu)^2}=-\sum_{\nu=0}^{\infty}\frac{1}{(z+\nu)^2}$$

Let us now regard the function defined for $x>0$ by

(vii) $$\delta(x)=\operatorname{Log}\Gamma(x)-\operatorname{Log}\mathcal{G}(x)$$

If we prove that $\delta(x)=0$ for all $x>0$ then (ii) follows by analyticity. To show that $\delta(x)=0$ we first notice that (v) gives:

(ix) $$\delta(x+1)=\delta(x)\quad:\quad x>0$$

Next, we shall regard the second derivative of δ. First, put

(ix) $$\psi(x)=d^2\operatorname{Log}\Gamma(x)dx^2=\frac{\Gamma(x)\Gamma''(x)-\Gamma'(x)^2}{\Gamma'(x)^2}$$

Next we have

(x) $$\Gamma'(x) = \int_0^\infty e^{-t}\cdot \operatorname{Log} t\cdot t^{x-1}\cdot dt \quad : \quad \Gamma''(x) = \int_0^\infty e^{-t}(\operatorname{Log} t)^2\cdot t^{x-1}\cdot dt$$

From these two expressions we see that the Cauchy Schwarz inequality gives

(xi) $$\Gamma(x)\Gamma''(x) - \Gamma'(x)^2 \geq 0$$

At the same time (vii) shows that the second order derivative of $\operatorname{Log}\mathcal{G}(x)$ is < 0. We conclude that $\delta''(x) \geq 0$, i.e. the δ-function is strictly convex when $x > 0$. Next, the periodicity remains valid for the first order derivative, i.e.

(xii) $$\delta'(x+1) = \delta'(x) \quad : \quad x > 0$$

Finally, δ is convex the derivative $\delta'(x)$ is a non-increasing function and we notice that every non-increasing and 1-peridoic function on $x > 0$ is a constant. Hence $\delta'(x)$ is a constant which gives

(xiii) $$\delta(x) = ax + b \quad : a, b \text{ real constants}$$

Since $\delta(x+1) = \delta(x)$ it follows that $a = 0$. Hence $\delta(x) = b$ is a constant But $b = 0$ since it is clear that the functions Γ and $\mathcal{G}$ are equal at all positive integers. This proves that $\delta(x)$ is identically zero on $x > 0$ and the proof of Theorem 1.3 is finished.

1.4 Remark. The limit of products which defined $\mathcal{G}(z)$ was considered by Gauss. So one refers to $\mathcal{G}$ as the *Gauss representation* of the Γ-function. Theorem 1.3 is due to Weierstrass. The proof above using the δ-function was discovered by Erhard Schmidt. His method was later extended by Emil Artin who established a remarkable uniqueness property of the Γ-function in an article from 1931. More precisely he proved

1.5 Artin's Theorem *Let $f(z)$ be an entire function with $f(0) = 1$ satisfying:*

$$f(z+1) = f(z) \quad : : \quad \frac{d^2 \operatorname{Log} f(x)}{dx^2} \geq 0 \quad : x > 0$$

Then $f(z) = \frac{1}{z\cdot\Gamma(z)}$.

We refer [Artin] for details of proof and a further discussions about the Γ-function.

2. A functional equation.

Consider the product

(2.1) $$\Gamma(z)\cdot\Gamma(1-z)$$

This is a meromorphic function with simple poles at all integers. Next, we have the entire function $\sin \pi z$ with simple zeros at all integers. Hence the function

$$F(z) = \Gamma(z)\cdot\Gamma(1-z)\cdot\sin\pi z$$

has no poles and is therefore entire. It turns out that this function is constant.

2.1 Theorem. One has the equality

$$\frac{1}{\Gamma(z)\cdot\Gamma(1-z)} = \frac{\sin\pi z}{\pi}$$

Proof. The Gauss representation from (iii) in the proof of Theorem 1.3 gives

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\cdots(z+n)}{n!\cdot n^z} = \lim_{n\to\infty} \frac{z(z+1)\cdot(\frac{z}{2}+1))\cdots(\frac{z}{n}+1)}{n^z}$$

Similarly we obtain

$$\frac{1}{\Gamma(1-z)} = \lim_{n\to\infty} \frac{(1-z)\cdot(1-\frac{z}{2})\cdots(1-\frac{z}{n})}{n^{-z}}\cdot\frac{n+1-z}{n}$$

Since $\lim_{n\to\infty} \frac{n+1-z}{n} = 1$ we conclude that

$$\frac{1}{\Gamma(z)\cdot\Gamma(1-z)} = \lim_{n\to\infty} z\cdot(1-z^2)\cdot(1-(\frac{z}{2})^2)\cdots(1-(\frac{z}{n})^2)$$

But the last term is the Hadarmard product for $\frac{\sin \pi z}{\pi}$ and Theorem 2.1 follows.

3. The integral formula.

Let z be a non-zero complex number such that $\mathfrak{Re}\, z < 1$. Keeping z fixed we define the complex powers s^{-z} for all s in $\mathbf{C}\setminus(-\infty, 0]$. More precisely, when the negative real axis is removed we have a unique polar form

(i) $$s = re^{i\theta} \quad -\pi < \theta < \pi$$

Then we can write

(ii) $$s^{-z} = r^{-z}\cdot e^{-i\theta z}$$

For every fixed z the function $s\mapsto s^{-z}$ is analytic in the simply connected domain $\Omega = \mathbf{C}\setminus(-\infty, 0]$ where it is equal to $e^{-z\cdot\log s}$ and (i) determines the value of $\log s$. Multiplying with e^s we get

(1) $$f(s) = s^{-z}\cdot e^s \in \mathcal{O}(\Omega)$$

With $s = \sigma + i\tau$ we get

(2) $$|f(\sigma+i\tau)| = |r|^{-x}\cdot e^{-\theta y}\cdot e^{\sigma} \quad : \quad z = x+iy \quad :$$

Hence $f(s)$ has exponential decay in the right half-plane $\mathfrak{Re}(s) < 0$. We profit upon this to construct two absolutely convergent integrals. Given $\epsilon > 0$ we consider the two half-lines

(3) $$\ell^*(\epsilon) = \{s = \sigma + i\epsilon \quad : \sigma \leq 0\} \quad : \quad \ell_*(\epsilon) = \{s = \sigma - i\epsilon \quad : \sigma \leq 0\}$$

Let $T_+(\epsilon) = \{s = \epsilon e^{i\theta} : \ -\pi/2 \leq \theta < \pi/2\}$ and put:

(4) $$\gamma(\epsilon) = \ell^*(\epsilon)\cup T_+(\epsilon)\cup \ell_*(\epsilon)$$

We choose the *negative* orientation along $\gamma(\epsilon)$ which means that we first integrate along $\ell^*(\epsilon)$ as σ increases from $-\infty$ to 0 and and so on. This gives:

(5) $$\int_{\gamma(\epsilon)} f(s)ds = \int_{-\infty}^{0} f(\sigma+i\epsilon)d\sigma - i\epsilon\cdot\int_{-\pi/2}^{\pi/2} f(\epsilon\cdot e^{i\theta})e^{i\theta}\cdot d\theta - \int_{-\infty}^{0} f(\sigma - i\epsilon)d\sigma$$

With $\mathfrak{Re}(z) = x < 1$ we see from (2) that

(6) $$\lim_{\epsilon\to 0} \epsilon\cdot\int_{-\pi/2}^{\pi/2} f(\epsilon\cdot e^{i\theta})e^{i\theta}\cdot d\theta = 0$$

Next, when $\sigma < 0$ we see that (ii) and (1) give

(6) $$\lim_{\epsilon\to 0} f(\sigma+i\epsilon) - f(\sigma - i\epsilon) = |\sigma|^{-x-iy}[e^{i\pi\cdot z} - e^{-i\pi\cdot z}]$$

Hence we have proved

3.1 Proposition. *One has*

$$\lim_{\epsilon\to 0}\int_{\gamma(\epsilon)} f(s)ds = 2i\cdot\sin(\pi z)\cdot\int_{-\infty}^{0} |\sigma|^{-x-iy}\cdot e^{\sigma}d\sigma$$

Divide by $2\pi i$ and make the variable substitution $t = -\sigma$. This gives

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{\gamma(\epsilon)} f(s)ds = \frac{\sin(\pi z)}{\pi} \cdot \int_0^\infty t^{-x-iy} \cdot e^{-t} dt \tag{7}$$

The last integral is $\Gamma(1-z)$. Together with Theorem 2.1 we conclude the following:

3.2 Theorem. *When* $\mathfrak{Re}(z) < 1$ *one has the equality*

$$\frac{1}{\Gamma(z)} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{\gamma(\epsilon)} e^s \cdot s^{-z} ds$$

where $\gamma(\epsilon)$ *has the negative orientation as described in (4).*

3.3 Remark. In the right hand side we can change the contour γ_ϵ where convergence holds as long as the real part of s tends to $-\infty$ along the end-tails. For example, the integral representation holds for every $\epsilon > 0$, i.e. even for *large* ϵ. This flexible manner to represent $\frac{1}{\Gamma(z)}$ is used in many formulas where Γ-functions appear. An example are the integral formulas due to Barnes for hypergeometric functions.

II. Riemann's ζ-function

Introduction. The zeta-function is defined by the series:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\quad:\quad \mathfrak{Re}\, s>1 \tag{0.1}$$

It is clear that $\zeta(s)$ is an analytic function in the half-space $\mathfrak{Re}(s)>1$ whose complex derivative is found by termwise differentiation, i.e.

$$\zeta'(s)=-\sum_{n=1}^{\infty}\frac{\mathrm{Log}(n)}{n^s}\quad:\quad \mathfrak{Re}\, s>1 \tag{0.1}$$

In the subsequent sections we establish some classical results about the ζ-functions concerned with growth properties and distribution of its zeros.

1. The meromorphic extension.

The fact that $\zeta(s)$ has a meromorphic extension with a simple pole a $s=1$ goes back to Euler and is presented below.

1.1 Euler's summation formula. When $x\geq 1$ we let $[x]$ denote the largest integer which is $\leq x$. Set

$$P(x)=[x]-x+\frac{1}{2}\,,\quad x\geq 1$$

The differential dP is the counting function at positive integers. So if $\mathfrak{Re}\, s>1$ we have

$$\zeta(s)-\int_1^\infty\frac{dx}{x^s}=\int_1^\infty\frac{dP(x)}{x^s}=\frac{P(x)}{x^s}\Big|_1^\infty-\int_1^\infty\frac{P(x)}{x^{s+1}}\cdot dx$$

Since $P(1)=\frac{1}{2}$ and $\int_1^\infty\frac{dx}{x^s}=\frac{1}{s-1}$ we obtain

1.2 Euler's integral formula. One has

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+s\cdot\int_1^\infty\frac{P(x)}{x^{s+1}}\cdot dx$$

Remark. Since the function $P(x)$ is bounded the last integral extends to an analytic function in the half plane $\mathfrak{Re}\, s>0$. So Euler's integral formula shows that the zeta-function extends to a meromorphic function in $\mathfrak{Re}\, s>0$ with a simple pole at $s=1$.

1.3 Further integral formulas. Notice that $P(x)$ is 1-periodic:

$$P(x+1)=P(x)\quad x\geq 1\,.$$

Moreover, as explained in XX it has the Fourier series expansion

$$P(x)=\sum_{n=1}^{\infty}\frac{\sin(2n\pi x)}{n\pi}$$

The primitive function becomes

$$P_1(x)=-\sum_{n=1}^{\infty}\frac{\cos(2n\pi x)}{2n^2\pi^2}$$

A partial integration gives

$$s\cdot\int_1^\infty\frac{P(x)}{x^{s+1}}\cdot dx=s\cdot\frac{P_1(x)}{x^{s+1}}\Big|_1^\infty-s(s+1)\int_1^\infty\frac{P_1(x)}{x^{s+2}}\cdot dx$$

Next, one has the summation formula

$$P_1(1) = \sum_{n=1}^{\infty} \frac{1}{2n^2\pi^2} = \frac{1}{12}$$

It follows that

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - s(s+1) \cdot \int_1^{\infty} \frac{P_1(x)}{x^{s+2}} \cdot dx$$

The last integral is analytic when $\mathfrak{Re}\, s > -1$ which gives a further meromorphic extension of $\zeta(s)$. Repeating the process by taking the primitive function of P_1 one shows that $\zeta(s)$ extends to the whole complex plane with a single and simple pole at $s = 1$. Let us also notice that the boundedness of $P_1(x)$ and the formula above gives

1.4. Proposition. *Let $0 < \delta < 1$. Then there exists a constant $C(\delta)$ such that*

$$|\zeta(-1+\delta+it)| \leq C(\delta) \cdot t^2\,, \quad \text{for all } |t| \geq 1$$

2. Riemann's functional equation.

The next result consolidates Euler's results from section 1.

2.1. Theorem *The ζ-function extends to a meromorphic function in the whole complex s-plane where it satisfies the functional equation*

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \cdot \cos(\frac{\pi s}{2}) \cdot \Gamma(s) \cdot \zeta(s) \quad : \ s \in \mathbf{C}$$

The proof requires several steps. To begin with we have the integral formula for the Γ-function in § XX:

(1)
$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \cdot \int_{\mathbf{L}_\epsilon} e^z \cdot z^{-s} dz$$

Notice that the role of z and s are interchanged in (1) above as compared to the formula in section 3 about the Γ-function. Since both sides in (1) are entire functions of s we can replace s by $1-s$ and write

(2)
$$\frac{1}{\Gamma(1-s)} = \frac{1}{2\pi i} \cdot \int_{\gamma(\epsilon)} e^z \cdot z^s \cdot \frac{dz}{z}$$

If n is a positive integer the variable substitution $z \mapsto nz$ gives

$$\frac{1}{\Gamma(1-s)} = \frac{1}{2\pi i} \cdot n^s \cdot \int_{L_\epsilon} e^{nz} \cdot z^s \cdot \frac{dz}{z} \implies$$

(3)
$$n^{-s} \cdot \frac{1}{\Gamma(1-s)} = \frac{1}{2\pi i} \cdot \int_{L_\epsilon} e^{nz} \cdot z^s \cdot \frac{dz}{z}$$

Next, notice that

$$\sum_{n=1}^{\infty} e^{nz} = \frac{e^z}{1-e^z} \quad : \mathfrak{Re}\, z < 0$$

Taking the sum over n in (3) we obtain

2.2 Proposition. *One has the equality*

$$\frac{\zeta(s)}{\Gamma(1-s)} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{L_\epsilon} \frac{e^z}{1-e^z} \cdot z^s \cdot \frac{dz}{z}$$

Proof of Theorem 2.1

Let $R = (2N+1)\pi$ where N is a positive integer. With $0 < \epsilon < 1$ we obtain a simply connected domain Ω bounded by the circle $|z| = R$ and the portion of of L_ϵ. See figure ! With s fixed we find a single valued branch of z^s in Ω and regard the analytic function

(ii) $$g_s(z) = \frac{e^z}{1 - e^z} \cdot z^{s-1}$$

In Ω we encounter poles when $e^z = 1$, i.e. when $z = 2\pi i\nu$ when $1 \leq \nu \leq N$. Residue calculus gives - see figure !!:

(iii) $$\int_{-\pi+\delta}^{\pi-\delta} g_s(Re^{i\theta}) \cdot d\theta - \int_{L_\epsilon} g_s(z) \cdot dz = 2\pi i \cdot \sum_{\nu=1}^{\nu=N} \mathfrak{res}\,[g_s(2\pi i\nu) + g_s(-2\pi i\nu)]$$

The residue sum is easily found, i.e. one has

Sublemma The right hand side in (iii) becomes:

(*) $$2 \cdot \sum_{\nu=1}^{\nu=N} \frac{1}{\nu^{1-s}} \cdot (2\pi)^{s-1} \cdot \cos\frac{\pi}{2}(s-1)$$

So far we have allowed any s. Let us now specify s to be real and negative. A straightforward calculation which is left to the reader gives:

(iv) $$\lim_{N\to\infty} \int_{-\pi+\delta}^{\pi-\delta} g_s(Re^{i\theta}) \cdot d\theta = 0 : \quad s \text{ real and } < 0$$

Next, the definition of the ζ-function gives

(**) $$\zeta(1-s) = \lim_{N\to\infty} \sum_{\nu=1}^{\nu=N} \frac{1}{\nu^{1-s}} \quad : \; s < 0$$

Hence (iii-iv) together with Proposition 2.2, (*) and (**) give:

Sublemma. *When s is real and < 0 one has the equality:*

$$2 \cdot (2\pi)^{s-1} \cdot \zeta(1-s) \cdot \cos\frac{\pi}{2}(s-1) = \frac{\zeta(s)}{\Gamma(1-s)}$$

Final part of the proof . Notice that

(1) $$\cos\frac{\pi}{2}(s-1) = \sin\frac{\pi s}{2} = \frac{\sin \pi s}{2 \cdot \cos\frac{\pi s}{2}}$$

Hence the Sublemma gives

$$\frac{\zeta(s)}{\Gamma(1-s)} = (2\pi)^{s-1} \cdot \zeta(1-s) \cdot \sin \pi s \cdot \frac{1}{\cos\frac{\pi s}{2}} \implies$$

(1) $$\cos\frac{\pi s}{2} \cdot \zeta(s) = (2\pi)^{s-1} \cdot \zeta(1-s) \cdot \sin \pi s \cdot \Gamma(1-s)$$

At this stage we recall from GAMMA that

(2) $$\sin \pi s \cdot \Gamma(1-s) = \frac{\pi}{\Gamma(s)}$$

So (1-2) give together:

(3) $$\cos\frac{\pi s}{2} \cdot \zeta(s) = \frac{1}{2} \cdot (2\pi)^s \cdot \zeta(1-s) \cdot \frac{1}{\Gamma(s)}$$

Expressing $\zeta(1-s)$ alone we get the requested formula in Theorem 2.1 where analyticity gives equality for all s.

3. The asymptotic formula for $\mathcal{N}(T)$

The functional equation and the fact that the Γ-function has simple poles at all non-negative integers show that $\zeta(-2m) = 0$ for every positive integer m. Thus, $\zeta^{-1}(0)$ contains $\{-2, -4, -6, \ldots\}$. Apart from this it turns out that the remaining zeros of $\zeta(s)$ belong to the strip

$$\mathcal{S} = \{0 < \mathfrak{Re}\, s < 1\} \tag{0.1}$$

To find zeros in $\mathcal{S}$ we notice that the ζ-function is real when s is real and > 1. By analytic continuation it follows that

$$\zeta(s) = \bar{\zeta}(\bar{s}) \tag{0.2}$$

hold for all s. Hence zeros of ζ appear in conjugate pairs in $\mathcal{S}$ and it suffices to study the counting function

$$\mathcal{N}(T) = \text{number of zeros of } \zeta(s) \quad : \; s \in \mathcal{S} \cap \{\mathfrak{Im}\, s > 0\} \tag{0.3}$$

3.1 Riemann's asymptotic formula. *There exists a constant C_0 such that the following hold when $T \geq 1$:*

$$\mathcal{N}(T) = \frac{1}{2\pi} T \cdot \operatorname{Log} T - \frac{1 + \operatorname{Log} 2\pi}{2\pi} \cdot T + \rho(T) \cdot \operatorname{Log} T \quad : \; |\rho(T)| \leq C_0$$

Remark. Riemann announced this asymptotic formula in [Rie]. It was later proved by von Mangoldt. In XXX we present the elegant proof due to Backlund.

Before we begin the proof of the asymptotic formula in 3.1. from the introduction we draw some conclusions from Riemann's functional equation. Let $n \geq 1$ be a positive integer and put $s = 2n + 1$. Here the cosine function has a simple zero, i.e. $\cos \pi n + \frac{\pi}{2} = 0$ and at the same time $\zeta(2n+1)$ and $\Gamma(2n+1)$ are real and positive. Hence Theorem 2.2 implies that the ζ-function has a simple zero at $1 - s = -2n$. So we have proved:

3.2 Proposition. *The ζ-function has simple zeros at all even negative integers.*

3.3 The entire ξ-function. Let us define the function

$$\xi(s) = \frac{s(s-1)}{2} \cdot \Gamma(\frac{s}{2}) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s)$$

3.4 Proposition. *The function $\xi(s)$ is entire and satisfies*

$$\xi(s) = \xi(1-s) \quad : \; \xi(s) = \bar{\xi}(\bar{s})$$

Proof. That $\xi(s)$ is entire is clear from the construction since the zeros of the ζ-function at even and negative integers compensate the simple poles of the Γ-function at negative integers. Moreover, the factor $s(s-1)$ takes care of the simple pole of $\zeta(s)$ at $s = 1$ and the pole of the Γ-function at $s = 0$. Next, the equality $\xi(s) = \bar{\xi}(\bar{s})$ follows since the same conjugation property hold for the three factors defining $\xi(s)$. Finally, the equality $\xi(s) = \xi(1-s)$ follows from Riemann's functional equation.

3.5. Zeros in the critical strip.

The construction of the ξ-function in 3.3. and the fact that $\Gamma(\frac{s}{2})$ has no zeros in the critical strip give:

3.6 Proposition. *The zero sets of ζ and ξ in the critical strip are equal.*

Now we shall count zeros of ξ. For this purpose we consider the rectangle

$$\square_T = \{s = \sigma + it \quad : \; -1/2 < \sigma < 3/2 : \; -T < t < T\} \quad : \; T \geq 1$$

Notice that $\Box_T$ is symmetric around $\mathfrak{Re}(s) = 1/2$. We choose T so that no zeros occur when $\mathfrak{Im}(s) = T$. Then we have

$$(*) \qquad 2 \cdot \mathcal{N}(T) = \frac{1}{2\pi i} \int_{\partial \Box_T} \frac{\xi'(s)ds}{\xi(s)}$$

To estimate the line integral in (*) we recall that

$$\text{(i)} \qquad \xi(s) = \xi(1-s) \quad : \xi(s) = \bar{\xi}(\bar{s})$$

These two equalities show that the line integral over $\partial\Box_T$ is *four times* the line integral over the the "quarter part" given by the union of the two lines

$$\text{(ii)} \qquad \ell_*(T) = \{2 + it \quad : \ 0 < t < T\} \quad : \ \ell^*(T) = \{\sigma + iT \quad : \ 1/2 < \sigma < 2\}$$

Next, we know from the start that the line integral is the real number $2 \cdot \mathcal{N}(T)$. Taking the factor 4 into the account we get: we have

3.7 Lemma. One has

$$\mathcal{N}(T) = \frac{1}{\pi i} \cdot \int_{\ell_*} \frac{\xi'(s)ds}{\xi(s)} + \frac{1}{\pi i} \cdot \int_{\ell^*} \frac{\xi'(s)ds}{\xi(s)}$$

Following Backlund we decompose the logarithmic derivative of $\xi(s)$ which gives a sum of five terms:

$$\text{(iii)} \qquad \cdot \frac{\xi'(s)ds}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} + \frac{1}{2} \cdot \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} - \frac{\operatorname{Log} \pi}{2} + \frac{\zeta'(s)}{\zeta(s)}$$

There remains to study the line integrals of each of these separate terms. The reader can verify that the contribution from the four first terms give the sum of the four first terms which appear in Riemann's asymptotic formula. There remains to investigate the contribution of:

$$(*) \qquad \frac{1}{\pi i} \cdot \int_{\ell_*(T)} \frac{\zeta'(s)ds}{\zeta(s)} + \frac{1}{\pi i} \cdot \int_{\ell^*(T)} \frac{\zeta'(s)ds}{\zeta(s)}$$

The required estimate for the remainder term follows if we have proved

3.8 Lemma. *There exists a constant C such that the absolute value of (*) is bounded above by $C \cdot Log(T)$ for every $T \geq e$.*

3.9 The use of Jensen's formula. To prove Lemma 3.8 we first consider the the line integral over the vertical line $\ell_*(T)$. To pursue the logarithmic derivative of ζ along ℓ_* where it by assumption is $\neq 0$ we choose choose a branch of its Log-function and write

$$(1) \qquad \operatorname{Log} \zeta(2+it) = \operatorname{Log} |\zeta(2+it)| + i\phi(t)$$

where $\phi(t)$ is a real valued argument function. Now we get

$$(2) \qquad \frac{\zeta'(2+it)}{\zeta(2+it)} = \frac{1}{i}\frac{d}{dt}\,[\operatorname{Log} \zeta(2+it)\,] = \frac{1}{i}\frac{d}{dt}\,[|\operatorname{Log} \zeta(2+it)|\,] + \frac{d\phi}{dt}$$

Along ℓ_* we have $ds = idt$. Since $\mathcal{N}(T)$ is real our sole concern is to study:

$$(3) \qquad \mathfrak{Re}\,[\frac{1}{\pi i} \cdot \int_{\ell^*} \frac{\zeta'(s)ds}{\zeta(s)}\,] = \frac{1}{\pi} \int_0^T \frac{d\phi}{dt} \cdot dt$$

To control the last integral we use the following

Sublemma. One has

$$\mathfrak{Re}[\,\zeta(2+it)\,] > \frac{1}{3} \quad : \; t > 0$$

Proof. Since $n^{it} = e^{it\log n}$ we have $\mathfrak{Re}\, n^{it} = \cos(t \cdot \log n)$. It follows that

$$\mathfrak{Re}[\,\zeta(2+it)\,] = 1 + \sum_{n=2}^{\infty} \frac{\cos(t \cdot \log n)}{n^2} \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} = 2 - \frac{\pi^2}{6} > \frac{1}{3}\,.$$

The Sublemma implies that $\zeta(2+it)$ belongs to the right half-plane when $0 \leq t \leq T$ and hence the argument of the ϕ-function satisfies:

$$-\pi/2 < \phi(t) < \pi/2 \quad : \text{ along } \ell_*(T)$$

In particular we get

$$-1 < \frac{1}{\pi} \int_0^T \frac{d\phi}{dt} \cdot dt < 1$$

Thus, the contribution along $\ell_*(T)$ has absolute value < 1 for all T and is therefore harmless for the asymptotic estimate in Lemma 3.8.

3.10 The line integral over $\ell^*(T)$. Along $\ell^*(T)$ we have $ds = d\sigma$. So this time our concern is to estimate

(1) $$\frac{1}{\pi} \cdot \int_{1/2}^{2} \mathfrak{Im}\,[\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)}\,] \cdot d\sigma$$

By the result in XXX this amounts to find an upper bound for the zeros of the function

(2) $$\sigma \mapsto \mathfrak{Re}(\zeta(\sigma+iT) \quad : 1/2 < \sigma < 2$$

By the conjugate equality $\zeta(s) = \bar{\zeta}(\bar{s})$ this amounts to consider zeros of the function

(3) $$\sigma \mapsto \zeta(\sigma+iT) + \zeta(\sigma-iT) \quad : \; 1/2 < \sigma < 2$$

We seek an upper bound of zeros for large T. So from now on we assume that

$$T \geq \frac{7}{2}$$

To obtain an upper bound for the zeros in (3) we consider the following analytic function defined by a new complex variable w:

(4) $$\phi_T(w) = \zeta(w+iT) + \zeta(w-iT) \quad : \; |w-3| \leq 5/2$$

Let $\frac{1}{2} < \sigma_1 \leq \ldots \leq \sigma_m < 2$ be the m-tuple which yield all the zeros counted with multiplicities of (3). They also give zeros of the analytic function ϕ in the disc $|w-3| \leq 5/2$ and by the general inequality from XXXX we have

(5) $$\sum_{\nu=1}^{m} \log\,[\frac{5}{2\sigma_\nu}\,] + \log|\phi_T(3)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \log\big|\phi_T(3 + \frac{5}{2}e^{i\theta})\,\big| \cdot d\theta$$

By XX we have $|\phi_T(3)| \geq 1/3$ and since $1/2 \leq \sigma_\nu \leq 2$ hold we get

(6) $$m \cdot \log\frac{5}{4} \leq \log 3 + \max_{|w-3|=5/2} \log|\phi_T(w)|$$

Finally, Proposition xx gives the constant $C(1)$ such that

(7) $$\phi_T(w)| \leq 2 \cdot C(1) \cdot T^2 \quad : \quad T \geq 7/2$$

Passing to $\operatorname{Log}|\phi_T(w)|$ we conclude that the m-number which counts the zeros is bounded above by an absolute constant times $\log T$ for all $T \geq 7/2$ which finishes the proof of Lemma 3.8.

4. The prime number theorem

The counting function for prime numbers is defined by

(i) $$\Pi(x) = \text{number of primes} \quad \leq x$$

This is an increasing jump function where we for example have $\Pi(8) = \Pi(9) = \Pi(10) = 4$ and $\Pi(11) = 5$.

4.0 Exercise. Show that prime numbers are sufficiently sparse in order that

$$\lim_{x\to\infty} \frac{\Pi(x)}{x} = 0$$

4.1 Theorem *There exists the limit formula*

(*) $$\lim_{x\to\infty} \frac{\log x \cdot \Pi(x)}{x} = 1$$

To prove this we introduce the function defined for $x > 0$ by

$$\gamma(x) = \sum_{p\leq x} \log p$$

where the sum as indicated extends over all prime numbers $\leq x$. This means that γ is a non-decreasing jump-function. Next, let $d\Pi$ be the discrete measure which assigns a unit point mass at every prime number. A partial integration gives the equation:

(i) $$\gamma(x) = \int_2^x \log x \cdot d\Pi(x) = \log x \cdot \Pi(x) - \int_2^x \frac{\Pi(x)}{x} \cdot dx$$

In XX we show the elementary fact that

(ii) $$\lim_{x\to\infty} \frac{\Pi(x)}{x} = 0$$

By Exercise 4.0 the last term tends to zero and hence Theorem 4.1 follows if we have proved that

(**) $$\lim_{x\to\infty} \frac{\gamma(x)}{x} = 1$$

To obtain (***) the first step is the following:

Lemma 4.2. *The function* $\frac{\gamma(x)}{x}$ *is bounded.*

Proof. The idea is to use that if $N \geq 2$ is a positive integer then all terms in the binomial expansion of $(1+1)^{2N}$ are integers. In particular we have the integer

(i) $$\xi_N = \frac{(2N)\,!}{N\,! \cdot N\,!}$$

Let $q_1, \ldots, q_m$ be the distinct primes in $[N+1, 2N-1]$. By (i) each q_ν is a prime divisor of ξ_N. So we have trivially

$$q_1 \cdots q_m \leq \xi_N$$

Taking the logarithm the definition of the γ-function gives

(ii) $$\gamma(2N) - \psi(N) \leq \log\, \xi_N \leq N \cdot \log\, 2$$

where the reader may confirm the last inequality by a trivial calculation. Now we perform the usual trick using 2-powers, i.e. given $K \geq 2$ we apply (ii) with $N = 2^k \quad : 1 \leq k \leq K-1$. After a summation over k we get

$$\gamma(2^K - \psi(2) \leq \log\, 2 \cdot [1 + \ldots + 2^{K-1}] \leq \log\, 2 \cdot 2^K$$

Since this hold for all K and the γ-function is increasing we see that Lemma 4.2 holds.

Next, Lemma 4.2 and the general result in § XX give (**) if we have proved that the integral below exists:

$$(***) \qquad \int_2^\infty \frac{\gamma(x)-x}{x^2}\cdot dx$$

4.3 The Φ-function. To establish (***) we introduce the function

$$\Phi(z)=\sum \log p\cdot p^{-z}$$

where z is a complex variable. To begin with $\Phi(z)$ is defined in the half-plane $\mathfrak{Re}(z)>1$. Since $d\gamma$ is the discrete measure which assigns the mass $\log p$ at every prime number we have:

$$(1) \qquad \Phi(z)=\int_1^\infty x^{-z}\cdot d\gamma(x)=z\cdot\int_1^\infty x^{-z-1}\cdot\gamma(x)\cdot dx$$

With $z=1+\zeta$ and $\mathfrak{Re}(\zeta)>0$ we can write (1) as

$$(2) \qquad \frac{1}{1+\zeta}\Phi(1+\zeta)=\int_1^\infty x^{-\zeta-2}\cdot\gamma(x)\cdot dx$$

The substitution $x\to e^t$ identifies the last integral by

$$(3) \qquad \int_0^\infty e^{-\zeta t}\cdot e^{-t}\cdot\gamma(e^t)\cdot dy$$

Next, let us introduce the function

$$f(t)=e^{-t}\gamma(e^t)-1$$

Lemma 4.2 shows that f is a bounded function and the substitution $x\to e^t$ shows that the integral (***) converges if there exists the limit

$$(****) \qquad \lim_{T\to\infty}\int_0^T f(t)\cdot dt$$

Hence there only remains to prove (****). For this purpose we introduce the Laplace transform

$$(4) \qquad F(z)=\int_0^\infty e^{-zt}\cdot f(t)dt$$

Using the equality

$$\int_0^\infty e^{-zt}\cdot dt=\frac{1}{z}$$

we see that (2-3) above give

$$(5) \qquad \frac{1}{1+z}\cdot\Phi(1+z)=F(z)-\frac{1}{z}$$

At this stage we apply the result from XX applied to the bounded function $f(t)$ which shows that the requested lim (****) follows if $F(\zeta)$ extends to an analytic function in some open set Ω which contains the closed half-space $\mathfrak{Re}(z)\geq 0$. Using (5) this amounts to show the result below which by the previous results give the prime number theorem.

4.4 Lemma *The function* $\frac{1}{1+z}\cdot\Phi(1+z)+\frac{1}{z}$ *extends to be analytic in an open set which contains* $\mathfrak{Re}(z)\geq 0$.

Proof. When $\mathfrak{Re}\,z<1$ Euler's product formula gives

$$\log\zeta(z)=\sum\log(1-p^{-z})$$

Passing to the logarithmic derivative we get

$$-\frac{\zeta'(z)}{\zeta(z)}=\sum\frac{\log p}{p^z-1}$$

The last sum is rewritten as

$$\sum \frac{\log p}{p^z - 1} = \sum \frac{\log p(p^z - 1 + 1)}{p^z \cdot (p^z - 1)} =$$
$$\sum \frac{\log p}{p^z} + \sum \frac{\log p}{p^z \cdot (p^z - 1)}$$

So the construction of Φ gives the equality

$$\Phi(z) = -\frac{\zeta'(z)}{\zeta(z)} - \sum \frac{\log p}{p^z \cdot (p^z - 1)}$$

With $z = 1 + w$ we can write

$$\Phi(1+w) - \frac{\zeta'(1+w)}{\zeta(1+w)} = -\sum \frac{\log p}{p^{1+w} \cdot (p^{1+w} - 1)} \tag{1}$$

It is clear the right hand side is an analytic function of w in the half-plane $\mathfrak{Re}(w) > -1/2$. Next, recall that the ζ-function has a simple pole at $z = 1$ while $\zeta(1 + it) \neq 0$ for all real t. This means that

$$-\frac{\zeta'(1+w)}{\zeta(1+w)} = -\frac{1}{w} + H(w) \tag{2}$$

where H extends to be analytic in an open set Ω containing $\mathfrak{Re}(w) \geq 0$. Hence (1-2) imply that $\Phi(1 + w) + \frac{1}{w}$ extends analytically to Ω and since we only require that Ω is some open set containing $\mathfrak{Re}(w) \geq 0$ the same is true for the function $\frac{1}{1+z} \cdot \Phi(1 + z) + \frac{1}{z}$ which proves Lemma 4.4.

5. A uniqueness result for the ζ-function

Introduction. The situation was described in the introduction where we defined a class $\mathcal{D}_k$ of Dirichlet series for every $k > 0$ and announced Theorem 0.1. The proof of this result requires several steps which we briefly describe before the details of the proof start. To begin with we will show that a Dirichlet series $\phi(s)$ in the family $\mathcal{D}_k$ gives an even and entire function of exponential type defined by an Hadamard product:

$$f(z) = \prod (1 + \frac{z^2}{\lambda_n^2})$$

Using Phragmén-Lindelöf inequalities together with properties of the Γ-function and Mellin's inversion formula, we shall prove that for each $\epsilon > 0$ there exists a constant C_ϵ such that

$$(*) \qquad \big|f(x) - a|x|^p \cdot e^{\pi|x|}\big| \leq C_\epsilon \cdot e^{\pi(1-2k+\epsilon \cdot |x|}$$

hold for all real x. Here $a = e^{2\phi(0)}$ and $p = 2\phi'(0)$ are determined by ϕ. When $k > 1/2$ we can choose ϵ small so that $1 - k + \epsilon = -\delta$ for some $\delta > 0$ which means that $f(x) - a|x|^p \cdot e^{\pi|x|}$ has exponential decay as $|x| \to +\infty$. From this we shall deduce that $f(z)$ is of a special form and after deduce Theorem 0.1 via an inversion formula for Dirichlet series. Before the proof of Theorem 0.1 we establish a uniqueness result for entire functions in the class $\mathcal{E}$.

A. A uniqueness result in $\mathcal{E}$

.

Let a and δ be positive real numbers and p some real number. Consider an entire function $f(z)$ of exponential type for which there exists a constant C such that

$$(*) \qquad \big|f(x) - a|x|^p \cdot e^{\pi|x|}\big| \leq Ce^{-\delta|x|} \quad \text{hold for all} \quad |x| \geq 1$$

Notice that (*) can be satisfied for suitable a and p by the entire functions

$$(1) \qquad f_1(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z} \quad : \quad f_2(z) = \frac{e^{\pi z} + e^{-\pi z}}{2}$$

A.1 Theorem. *Every $f \in \mathcal{E}$ which satisfes (*) is equal to $c_1 \cdot f_1 + c_2 \cdot f_2$ for a pair of constants c_1, c_2.*

Proof. To begin we f must be even. For if $g(z) = f(z) - f(-z)$ then (*) gives

$$|g(x)| \leq 2C \cdot e^{-\delta|x|}$$

Now we recall from XXX that en entire function of exponential type with exponential decay on the real axis must be identically zero. From now on we work in the right half-plane and after we use that f us even. In $\mathfrak{Re}(z) > 1$ we have the entire function

$$(i) \qquad h(z) = f(z) - az^p \cdot e^{\pi z}$$

where the branch of z^p is taken so that $x^p > 0$ when $z = x$ is real and > 1. Consider first the domain

$$\Omega = \{z = x + iy \quad y > 0 \quad \text{and} \quad x > 1\}$$

Since $f \in \mathcal{E}$ there is a constant A such that $e^{-A|z|} \cdot f(z)$ is bounded. It follows that we have constants C and B such that

$$(ii) \qquad |h(1 + iy)| \leq C \cdot e^{By}$$

for all $y > 0$. At the same time (*) gives

$$(iii) \qquad |h(x)| \leq C \cdot e^{-\delta x}$$

The Phragmen-Lindelöf theorem applied to the quarter planer Ω therefore gives a constant C such that

$$(iv) \qquad |h(x + iy)| \leq Ce^{Ay - \delta x}$$

for all $x + iy \in \Omega$. In exactly the same way one proves (iv) with y replaced by $-y$ in the quarter-plane where $x > 0$ and $y < 0$. Let us then consider the strip domain

$$S = \{x + iy \quad |y| \leq 1 \quad \text{and} \quad x > 1\}$$

Then we see that there is a constant such that

$$h(x + iy)| \leq C \cdot e^{-\delta x} \quad : \quad x + iy \in S$$

If $n \geq 1$ we consider the complex derivative $h^{(n)}$ and with $x > 2$ Cauchy's formula gives

$$h^{(n)}(x) =$$

From this we conclude that there exists a constant C_n such that

$$|h^{(n)}(x)| \leq C_n \cdot e^{-\delta x}$$

for all $x \geq 2$. At this stage the proof is almost finished. For consider the second order differential operator

$$L = x^2 \partial_x^2 - 2p \cdot x\partial_x - \pi^2 x^2 + p(p+1)$$

The two functions $|x|^p e^{\pi x}$ and $|x|^p e^{-\pi x}$ are obviously solutions to the homogeneous equation $L = 0$ when $x \neq 0$, i.e. on the two half-lines $x > 0$ or $x < 0$. it follows that when f is restricted to the real $x > 0$ or $x < 0$ then

$$L(f) = L(h)$$

Now L also yields the holomorphic differential operator where ∂_x is replaced by ∂_z and then $g = L(f)$ us en entire function exponential type. Now (xx) and the estimates (xx for $n = 0, 1, 2$ give a constant C such that

$$|g(x| \leq C(1 + x^2) \cdot e^{-\delta |x|}$$

for all real x. But then the $\mathcal{E}$-function g is identically zero by the result in XXX. Hence $L(f) = 0$ holds when $x \neq 0$ and by wellknown result in ODE-theory it follows that its restriction to $x > 0$ and to $x < 0$ are given by linear combinations of the two functions in (xx). From this one easily deduces Theorem A.1 .

B. Dirichlet series and their transforms.

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be a non-decreasing sequence of positive real numbers. Assume that there exists some $\delta < 0$ such that

(i) $$\lambda_n \geq \delta \cdot n \quad \text{hold for all} \quad n$$

It follows that the Dirichlet series

(1) $$\phi(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

is analytic in the half-space $\mathfrak{Re}\, s > 1$. Moreover, from (i) and the results by Hadamard and Lindelöf in §§ XX there exists the entire function

(2) $$f(z) = \prod \big(1 + \frac{z^2}{\lambda_n^2}\big)$$

of exponential type, i.e. $f \in \mathcal{E}$. We are going to derive properties of ϕ via f, and vice versa. To attain this we shall use the following:

B.1 Inversion formula. *When* $0 < \mathfrak{Re}\, s < 2$ *and* $a < 0$ *is real the following equality holds:*

(1) $$\int_0^\infty \log\big(1 + \frac{x^2}{a^2}\big) \cdot \frac{dx}{x^{s+1}} = \frac{1}{a^s} \cdot \frac{\pi}{\cdot \sin \frac{\pi s}{2}}$$

The proof of this equality is left as an exercise for the reader.

Exercise. Apply (1) with $a = \lambda_n$ and show that a summation over n gives:

(2) $$\int_0^\infty \log f(x) \cdot \frac{dx}{x^{s+1}} = \frac{\pi \cdot \phi(s)}{s \sin \frac{\pi s}{2}} \quad : \quad \mathfrak{Re}\, s > 1$$

B.2 A condition on f. Suppose there exist real constants a, p, k where $k > 1/2$ such that

$$f(x) = ax^p e^{\pi x} + O(e^{\pi(1+\epsilon-2k)x}$$

hold for each $\epsilon > 0$ as $x \to +\infty$. In other words, for every $\epsilon > 0$ there exists a constant C_ϵ such that

$$(*) \qquad |f(x) - ax^p e^{\pi x}| \leq C_\epsilon \cdot e^{\pi(1+\epsilon-2k)x} \quad \text{for all} \quad x \geq 1$$

Exercise. Show that (*) entails that

$$(**) \qquad \log f(x) = \log a + p \log x + \pi \cdot x + O(e^{\pi(1+\epsilon-2k)x} \quad \text{for all} \quad x \geq 1$$

here $\epsilon > 0$ as above can be arbitrary small.

Next, set

$$\psi(s) = \int_0^\infty \log\, f(x) \cdot \frac{dx}{x^{s+1}}$$

Notice that f is an even entire function where the Hadamard product gives $f(0) = 1$ which implies that

$$\log f(x) \simeq x^2$$

when x is close to zero. Therefore it is the behaviour of $f(x)$ when x is large which determines when the ψ-function is nice in the half-space $\mathfrak{Re}\, s < 2$. Notice

$$\int_0^\infty [\log a + p \log x + \pi \cdot x] \cdot \frac{dx}{x^{s+1}} = \frac{\log a}{s} + \frac{p}{s^2} + \frac{\pi}{s-1}$$

Moreover, we have $k > 1/2$ so when ϵ is small it follows that $1 + \epsilon - 2k = -\delta$ for some $\delta > 0$. Here the function

$$g_\delta(s) = \int_0^\infty e^{-\pi\delta x} \cdot \frac{dx}{x^{s+1}}$$

is analytic in the half-space $\mathfrak{Re}\, s < 2$.

Exercise. Use the definition of the γ-function to conclude that

$$|g_\delta(-\sigma + it)| \leq \frac{\Gamma(\sigma)}{(\pi\delta)^\sigma}$$

hold for every $\sigma > 0$.

Let us now add the assumption that

$$(3) \qquad \phi(-2n) = 0 \quad \text{hold for every positive integer}$$

When (3) holds we see that the right hand side in (2) is a meromorphic function whose poles in the half-plane $\mathfrak{Re} s < 2$ are confined to $s = 0$ and $s = 1$. Denote this function with $\psi(s)$. Next, by assumption ϕ also belongs $\mathcal{D}_k$ for some $k > 1/2$ and using the growth condition (xx) from § XX we shall estimate certain L^1-integrals.

B.3 Proposition. *There exists a constant C such that*

$$\int_{-\infty}^\infty |\psi(-\sigma + it)| \cdot dt \leq C \cdot \frac{\sigma^3 \cdot \Gamma(\sigma)}{(2\pi k)^\sigma} \quad : \quad \sigma \geq 2$$

Proof. Consider the function

$$\psi_*(s) = (2\pi k)^s \cdot \Gamma(2-s)$$

From (xxx) we have a constant C such that

$$\text{(i)} \qquad \phi(3/2 + it)| \leq C \quad : \quad -\infty < t < +\infty$$

It follows that

$$\text{(ii)} \qquad \Big|\frac{\psi(3/2+it)}{\psi_*(3/2+it)}\Big| \leq \frac{C\pi}{|3/2+it|}\cdot\frac{1}{2\pi k)^{3/2}}\cdot\frac{1}{\sin\left(\pi(3/4+it/2)\cdot\Gamma(1/2-it)\right.}$$

The complex sine-function increases along this vertical line,i.e. there is a constant $c > 0$ such that

$$\text{(iii)} \qquad |\sin\left(\pi(3/4+it/2)\right| \geq c\cdot e^{\pi|t|/2}$$

At the same time the result in (xxx) gives the lower bound

$$\text{(iv)} \qquad |\Gamma(1/2-it)| \geq \sqrt{\pi}\cdot e^{-\pi|t|/2}$$

From (iii-iv) we conclude that the function $\frac{\psi}{\psi_*}$ is bounded on the line $\mathfrak{Re}(s) = 3/2$.

Sublemma 1. The function $\frac{\psi}{\psi_*}$ *is a bounded function in the domain*

$$\Omega = \{\mathfrak{Re}(s) < 3/2\} \cap \{|s| > 2\}$$

Proof. Follows easily via the Phragmén-Lindelöf theorem and the bound above on $\mathfrak{Re}(s) = 3/2$.

Next, Sublemma 1 gives a constant C such that

$$\text{(v)} \qquad |\psi(s)| \leq C\cdot\left|(2\pi k)^s\cdot\Gamma(2-s)\right| \quad : \quad s\in\Omega$$

We shall also need an inequality for the Γ-function which asserts that there exists a constant C such that

$$\text{(vi)} \qquad \int_{-\infty}^{\infty}\left|\,\Gamma(\sigma+2+it)\,\right|\cdot dt \leq C\cdot\sigma^3\cdot\Gamma(\sigma) \quad : \quad \sigma\geq 2$$

The verification of (vi) is left to the reader. Armed with the results above we prove the following.

B.4 Proposition. *There exists a constant* C *such that*

$$\int_{-\infty}^{\infty}\left|\,\psi(-\sigma+it)\,\right|\cdot dt \leq C\cdot\frac{\sigma^3\cdot\Gamma(\sigma)}{(2\pi k)^\sigma} \quad : \quad \sigma\geq 2$$

B.5 Mellin's inversion formula. The integrability in Proposition B.4 enable us to apply the Fourier-Mellin inversion formula via (**) from XX. This gives

$$\log f(x) = \frac{1}{2\pi i}\cdot\int_{c-i\infty}^{c+i\infty}\psi(s)\cdot x^s\cdot ds \quad : \quad 1 < c < 2$$

Using Proposition B.4 we can shift the contour the left and perform integrals over lines $\mathfrak{Re}\, s = -c$ where $c > 0$. During such a shift we pass the poles of ψ which appear at $s = 0$ and $s = 1$ with resides described in XXX above. From this the reader can deduce the integral formula:

$$(*) \qquad \log f(x) - \pi x - 2\phi(0)\cdot\log x - 2\phi'(0) = \frac{1}{2\pi i}\cdot\int_{-c-i\infty}^{-c+i\infty}\psi(s)\cdot x^s\cdot ds \quad \text{for all} \quad c > 0$$

B.6 A clever estimate. To profit upon (*) we shall adapt the c-values when x are real and large. More precisely, with $x\geq 2$ we take $c = x$ and notice that

$$|x^{(-x+it)}| = x^{-x}$$

Then Proposition B.4 and the triangle inequality show that the absolute value of the right hand side integral in (*) is majorized by

$$(**) \qquad 2\pi\cdot x^{-x}\cdot C\cdot\frac{x^3\cdot\Gamma(x)}{(2\pi k)^x} \quad : \quad x\geq 2$$

B.7 Exercise. Recall that $\Gamma(N) = N!$ for positive integers. Use this and Stirling's formula to conclude that for every $\epsilon > 0$ there is a constant C_ϵ such that (**) is majorized by

$$C_\epsilon \cdot e^{-2\pi(k-\epsilon)x}$$

B.8 Consequences. With $x \geq 2$, $p = 2\phi(0)$ and $a = 2\phi'(0)$ we obtain from above:

$$\big|\log f(x) - \pi x - p \cdot \log x - a\big| \leq C_\epsilon \cdot e^{-2\pi(k-\epsilon)x}$$

B.9 Exercise. Deduce from this estimate that there is a constant C_ϵ^* such that

$$|f(x) - e^a \cdot x^p \cdot e^{\pi x}| \leq C_\epsilon^* \cdot e^{\pi(1-2k+2\epsilon)\cdot x} \quad : \quad x \geq 2$$

6. A theorem on functions defined by a semi-group

Let $f(x)$ be a function in $L^2(0,1)$ which in general is complex-valued. To each $0 < a < 1$ we put

$$f_a(x) = f(ax)$$

We only consider $f_a(x)$ as a function on $(0,1)$. which again is defined on $(0,1)$ and we also impose the following:

Condition. For each $\delta > 0$ one has

$$(*) \qquad \int_0^\delta |f(x)| \cdot dx > 0$$

Denote by $\mathcal{C}_f$ the linear space generated by $\{f_a\}$ as $0 < a < 1$. Thus, a function in $\mathcal{C}_f$ is expressed as a finite sum

$$\sum c_k \cdot f_{a_k}(x)$$

where $\{c_k\}$ are complex numbers and $0 < a_1 < \ldots < a_n < 1$ some finite tuple of points in $(0,1)$. Next, consider some real number $1 < p < 2$. The inclusion $L^2(0,1) \subset L^p(0,1)$ identifies $\mathcal{C}_f$ with a subspace of $L^p(0,1)$. and we denote by $\mathcal{C}_f(p)$ the closure of $\mathcal{C}_f$ in the Banach space $L^p(0,1)$.

0.1 The function $F(s)$. It is defined for complex numbers s such that $\mathfrak{Re}(s) > 1/2$ by:

$$(1) \qquad F(s) = \int_0^1 f(x) \cdot x^{s-1} \cdot ds$$

To see that (1) is defined we notice that if $\sigma = \mathfrak{Re}(s) > 1/2$, then the Cauchy-Schwarz inequality gives

$$(2) \qquad |F(\sigma + it)| \leq \sqrt{\int_0^1 |f(x)|^2 \cdot dx} \cdot \sqrt{\int_0^1 |x|^{2\sigma-2} \cdot dx} = ||f||_2 \cdot \sqrt{\frac{1}{2\sigma - 1}}$$

6.1 Theorem *If there exists some $1 < p < 1$ such that $\mathcal{C}_f(p)$ is a proper subspace of $L^p[0,1]$, then $F(s)$ extends to a meromorphic function in the whole complex s-plane whose poles are confined to the open half-plane $\mathfrak{Re}(s) < 1/2$. Moreover, for every pole λ the function $x^{-\lambda}$ belongs to $\mathcal{C}_f(p)$.*

Proof. Recall that $L^q(0,1)$ is the dual of $L^p(0,1)$ where $\frac{1}{q} = 1 - \frac{1}{p}$. The assumption that $\mathcal{C}_f(p) \neq L^p(0,1)$ gives a non-zero $k(x) \in L^q(0,1)$ such that

$$(1) \qquad \int_0^1 k(x) f(ax) \cdot dx = 0 \quad : \quad 0 < a < 1$$

To the k-function we associate the transform

$$(2) \qquad K(s) = \int_0^1 k(x) \cdot x^{-s} \cdot dx$$

Hölder's inequality implies that $K(s)$ is analytic in the half-plane $\mathfrak{Re} s < \frac{1}{p}$. Next, we define a function $g(\xi)$ for every real $\xi > 1$ by

$$g(\xi) = \int_0^1 k(x) \cdot f(\xi x) \cdot dx$$

Hölder's inequality gives

$$|g(\xi)| \leq \Big[\int |k(x)|^q \cdot dx\Big]^{\frac{1}{q}} \cdot \Big[\int_0^{1/\xi} |f(\xi x)|^p \cdot dx\Big]^{\frac{1}{p}}$$

With $\xi > 1$ we notice that the last factor after a variable substitution is equal to

$$||f||_p \cdot |\xi|^{-1/p}$$

Since the L^p-norm of f is majorized by its L^2-norm we conclude that

$$(3) \qquad |g(\xi) \leq ||k||_q \cdot ||f||_p \cdot \xi^{-1/p} \quad : \quad \xi > 1$$

Next, put

$$G(s)=\int_1^\infty g(\xi)\cdot\xi^{s-1}\cdot d\xi \tag{4}$$

From (3) it follows that $G(s)$ is analytic in the half-space $\mathfrak{Re}\,s<1/p$. Let us now consider some s in the strip domain:

$$1/2<\mathfrak{Re}\,s<1/p$$

By variable substitutions of double integrals the reader can verify the equality

$$G(s)=F(s)\cdot K(s) \tag{*}$$

Conclusion. Recall the domains where F,G,K are analytic. Togeher with (*) this entails that F extends to a meromorphic function in the whole s-plane. In particular eventual singularities along $\mathfrak{Re}s=1/2$ are caused by poles. At the same time we have the inequality (2) from 0.1 which shows that no poles appear during the continuation across this line. We conclude that F either is an entire function or else it has a non-empty set of poles where each pole λ has real part $<1/2$. At this stage we are prepared to finish the proof of Theorem XX.

The case when a pole occurs. Suppose that F as a pole at some λ with a real part $<1/2$. Since G is analytic in $\mathfrak{Re}(s)<1/2$ the equality (*) implies that λ is a zero of K. Now we notice that the presence of the pole of F at λ is *independent* of the chosen L^q-function k which is $\perp$ to $\mathcal{C}_f$. Hence the following implication holds:

$$k\perp\mathcal{C}_f(p)\implies K(\lambda)=\int_0^1 k(x)x^{-\lambda}\cdot dx=0$$

The Hahn-Banach theorem entails that the L^p-function $x^{-\lambda}$ belongs to $/\mathcal{C}_h(p)$ which proves the last claim in Theorem 6.1

Existence of at least one pole. There remains to prove that F has at least one pole. We prove this by a contradiction, i.e. suppose that F is an entire function and consider a real number $1/2<\alpha<1/p$. The construction of F shows that its restriction to the half-space $\mathfrak{Re}\,s\geq\alpha$ is bounded an it is also clear that

$$\lim_{\sigma\to+\infty}F(\sigma+it)=0 \tag{i}$$

Next, in the half-space $\mathfrak{Re}\,s\leq\alpha$ we know that

$$F=\frac{G}{K}$$

where G and K both are bounded and at the same time their quotient is analytic in this half-space. Moreover their constructions imply that

$$\lim_{\sigma\to-\infty}G(\sigma+it)=0$$

and similarly for the K-function. Now the result by F. and R. Nevanlinna from XX gives some $M>0$ and real number c such that

$$F(\sigma+it)|\leq M\cdot e^{c(\sigma-\alpha)}\quad\text{holds when}\quad\sigma\leq\alpha \tag{iii}$$

Above two cases can occur. If $c\geq 0$ then we notice that the entire function F is bounded and (i) implies that $F=0$. But this is impossible since it entails that $f=0$.

The case $c<0$**.** When this holds we set $a=e^c$ so that $0<a<1$ and define

$$F_1(s)=\int_0^1 f(ax)x^{s-1}\cdot ds \tag{iv}$$

Here a variable substitution gives

$$F_1(s)=a^s\big(F(s)-\int_a^1 f(x)x^{s-1}\cdot ds\big) \tag{v}$$

It follows that the entire function $F_1(s)$ is bounded so by Liouville's theorem it is identically zero. Si by (iv) the the transform of the function $f_a(x) = f(ax)$ is identically zero. This means precisely that f vanishes on the interval $[0, a]$. But this was excluded by condition (*) above Theorem 6.1. which finishes the whole proof.

7. Beurlings criterion for the Riemann hypothesis

Let $\rho(x)$ denote the 1-periodic function on the real x-line where $\rho(x) = |x|$ of $-1/2 < x < 1/2$. To each $0 < \theta < 1$ we get the function

$$\rho_\theta(x) = \rho(\theta/x)$$

whose restriction to $(0,1)$ gives a non-negative function with range in $[-1/2, 1/2]$ with jump-discontinuites at the discrete set of x-values where θ/x is an integer. Denote by $\mathcal{D}$ the linear space of functions on $(0,1)$ of the form

$$f(x) = \sum c_\nu \cdot \rho(\theta_\nu/x)$$

where $0 < \theta_1 < \ldots < \theta_N < 1$ is a finite set and $\{c_\nu\}$ complex numbers such that

$$\sum c_\nu \cdot \theta_\nu = 0$$

7.1 Theorem. *The Riemann hypothesis is valid if and only if the identity function 1 belongs to the closure of $\mathcal{D}$ in $L^2(0,1)$.*

The proof will use the following formula:

7.2 Proposition. *For each $0 < \theta < 1$ one has the equality*

$$\int_0^1 \rho(\theta/x)x^{s-1} \cdot dx = \frac{\theta}{s-1} - \frac{\theta^s \cdot \zeta(s)}{s} \quad \text{when} \quad \mathfrak{Res} > 1$$

Proof. The variable substitutions $x \to \theta \cdot y$ and $y \to 1/u$ identifies the left hand side with

$$\theta^s \cdot \int_0^{1\theta} \rho(1/y) \cdot y^{s-1} \cdot dy =$$

$$\text{(i)} \qquad \theta^s \cdot \int_\theta^\infty \rho(u) \cdot u^{-s-1} \cdot du$$

To evaluate (i) we first consider the integral

$$\text{(ii)} \qquad \int_1^\infty \rho(u) \cdot u^{-s-1} \cdot du = \sum_{n=1}^\infty \int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du$$

where the last equation used the periodicity of ρ. An integration by parts gives for each $n \geq 1$:

$$\int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du = -\frac{1}{s}(n+1)^{-s} + \frac{1}{s}\int_0^1 \frac{du}{(n+u)^s}$$

After a summation over n we see that (ii) becomes

$$-\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s}\int_1^\infty u^{-s} \cdot du = -\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s(s-1)} = -\frac{\zeta(s)}{s} + \frac{1}{s-1}$$

It follows that the left hand side in (*) is equal to

$$\theta^s \cdot [\int_\theta^1 u \cdot u^{-s-1} \cdot du - \frac{\zeta(s)}{s} + \frac{1}{s-1}] =$$

$$\theta^s \cdot [\frac{\theta^{-s+1} - 1}{s-1} - \frac{\zeta(s)}{s} + \frac{1}{s-1}] = \frac{\theta}{s-1} - \frac{\theta^s \cdot \zeta(s)}{s}$$

Now we are prepared to begin the proof of Theorem 7.1

2. The case when 1 *in the* L^2*-closure of* $\mathcal{D}$.

When this holds we must show that $\zeta(s)$ has no zeros in the half-plane $\mathfrak{Re}\, s > 1/2$ which gives the Riemann hypothesis. To prove this we choose some small $\epsilon > 0$ and by assumption there exists $f \in \mathcal{D}$ such that the L^2-norm of $1 + f$ is $< \epsilon$. Since $\sum c_\nu \cdot \theta_\nu = 0$, Proposition 7.2 gives:

$$\int_0^1 (1 + f(x)) \cdot x^{s-1} \cdot dx = \frac{1}{s} - \frac{\zeta(s)}{s} \cdot \sum c_\nu \cdot \theta_\nu^s$$

With $s = \sigma + it$ and $\sigma > 1/2$ we have x^{s-1} in L^2 and Cauchy-Schwarz inequality gives:

$$\Big| \int_0^1 (1 + f(x)) \cdot x^{s-1} \cdot dx \,\Big| \leq ||f||_2 \cdot \sqrt{\int_0^1 x^{2\sigma-2} \cdot dx}$$

The last factor is equal to $\frac{1}{\sqrt{2\sigma-1}}$ and hence we obtain

$$\Big|\frac{1}{s} - \frac{\zeta(s)}{s} \cdot \sum c_\nu \cdot \theta_\nu^s\Big| \leq \epsilon \cdot \frac{1}{\sqrt{2\sigma - 1}} \quad : \quad \sigma > 1/2$$

If $\zeta(s_*) = 0$ holds for some $s_* = \sigma_* + it_*$ with $\sigma_* > 1/2$, then the left hand side is reduced to $\frac{1}{|\zeta_*|}$. Since we can find f as above for every $\epsilon > 0$ it would follow that

$$\frac{1}{|\zeta_*|} \leq \epsilon \cdot \frac{1}{\sqrt{2\sigma_* - 1}} \quad \text{for every} \quad \epsilon > 0$$

But this is clearly not possible and hence we have proved that if 1 belongs to the L^2-closure of $\mathcal{D}$ then the Riemann-Hypothesis holds.

3. Proof of necessity.

There remains to show that if 1 is outside the L^2-closure of $\mathcal{D}$ then the ζ-function has a zero in the half-plane $\mathfrak{Re}\, s > 1/2$. To show this we introduce a family of linear operators $\{T_a\}$ as follows: If $0 < a < 1$ and $g(x)$ is a function on $(0,1)$ we set

$$T_a(g)(x) = g(x/a) \quad : \quad 0 < x < a$$

while $T_a(g) = 0$ when $x \geq a$.

Exercise. Show that each T_a maps $\mathcal{D}$ into itself and one has the inequality

$$||T_a(f)||_2 \leq ||f||_2$$

Next, since 1 is outside the L^2-closure of $\mathcal{D}$ its orthogonal complement in the Hilbert space is $\neq 0$. Hence there exists a non-zero $g \in L^2(0,1)$ such that

$$(*) \qquad \int_0^1 f(x) \cdot g(x) \cdot dx = 0 \quad : \quad f \in \mathcal{D}$$

Since $\mathcal{D}$ is invariant under the T-operators it follows that if $0 < a < 1$ then we also have

$$(1) \qquad 0 = \int_0^a f(x/a) \cdot g(x) \cdot dx = a \cdot \int_0^1 f(x) \cdot g(ax) \cdot dx$$

At this stage we apply Theorem 6.1. To begin with we show that the g-function satisfies (*) in Theorem 6.1. For suppose that $g = 0$ on some interval $(0, a)$ with $a > 0$. Choose some b where

$$a < b < \min(1, 2a)$$

Now $\mathcal{D}$ contains the function $f(x) = b\rho(x/a) - a\rho(x/b)$. The reader may verify that $f(x) = 0$ for $x > b$ and is equal to the constant a on (a, b). With (1) applied to f we therefore get

$$\int_a^b g(x) \cdot dx = 0$$

This means that the primitive function

$$G(x) = \int_0^x g(u) \cdot du$$

has a vanishing derivative on the interval (a, b). The derivative is also zero on $(0, a)$ where $g = 0$. We conclude that $G = 0$ on the interval $(0, b)$ so the L^2-function g is almost everywhere a fixed constant on this interval. But this constant is zero since $g = 0$ on (a, b). Hence we have shown that $g = 0$ on the whole interval $(0, b)$. We can repeat this with a replaced by b and conclude that g also is zero on the interval

$$0 < x < \min(1, 2b) = \min(1, 4a)$$

After a finite number of steps $2^m a \geq 1$ and hence g would be identically zero on $(0, 1)$ which is not the case. Hence Theorem 6.1 applies to g and gives some λ_* with $\mathfrak{Re}\,\lambda_* < 1/2$ such that $x^{-\lambda_*}$ belongs to $\mathcal{C}_g(p)$ for every $p < 2$. Next, for each $\theta > 0$ we get the $\mathcal{D}$-function

$$x \mapsto \rho(1/x) - \frac{1}{\theta} \cdot \rho(\theta/x)$$

Since (1) holds for all $0 < a < 1$ it follows that

$$\text{(2)} \qquad \int_0^1 [\rho(1/x) - \frac{1}{\theta} \cdot \rho(\theta/x)] \cdot x^{-\lambda_*} \cdot dx = 0$$

Put $s_* = 1 - \lambda_*$. The formula in Proposition 7.2 shows that the vanishing in (2) gives

$$\frac{\theta^{s_*} - 1}{s_*} \cdot \zeta(s_*) = 0$$

This hold for every $0 < \theta < 1$ and we can choose θ so that $\theta^{s_*} - 1 \neq 0$ which gives the requested zero $\zeta(s_*) = 0$ where $\mathfrak{Re}(s_*) = 1 - \mathfrak{Re}(\lambda_*) > 1/2$.

1. Distributions and boundary values of analytic functions

Contents

Introduction.

The general notion of distributions and their basic properties was put forward by Laurent Schwartz, in 1945 and fully presented in his book *Théorie des distributions* from 1951. As pointed out by Lars Gårding in Chapter 12 from [History], Schwartz' broad attack, his radical use of infinitely differentiable functions and his conviction that distributions would be useful almost everywhere made the difference compared to earlier work where various special cases had adopted the idea of distributions but never in full generality. In section 1 we construct the Fourier transform of temperated distributions and derive Fourier's inversion formula for these. The proofs rely upon properties of the Schwartz class which consists of rapidly decreasing C^∞-functions which enable us to define the Fourier transform on a much wider class than integrable functions. Let us give some examples which illustrate the notion of distributions.

Let $\psi(x)$ be a real-valued and integrable function on the unit interval. Denote by $C_0^\infty(0,1)$ the space of test-functions with compact support in the open interval $(0,1)$. Suppose there exists a constant K such that

$$(*) \qquad \int_0^1 \phi'(x) \cdot \psi(x) \cdot dx \leq K \cdot \int_0^1 |\phi(x)| \cdot dx$$

hold for every $\phi \in C_0^\infty(0,1)$. Now $C_0^\infty(0,1)$ is a dense subspace of $L^1[0,1]$ whose dual is is$L^\infty[0,1]$. Hence (*) gives a unique $q(x) \in L^\infty[0,1]$ such that

$$(1) \qquad \int_0^1 \phi'(x) \cdot \psi(x) \cdot dx = \int_0^1 q(x) \cdot \phi(x) \cdot dx \quad \text{for all} \quad \phi \in C_*^\infty(0,1)$$

Let $Q(x) = \int_0^x q(s) \cdot ds$ be the primitive function. A partial integration identifies the right hand side in (1) with

$$(2) \qquad -\int_0^1 Q(x) \cdot \phi'(x) \cdot dx$$

Next, the set of first order derivatives produced by C_0^∞-functions generate a closed hyperplane in $L^1[0,1]$ which consists of all integrable functions on $[0,1]$ with mean-value zero. Hence (1) and (32) imply that the L^1-function $\psi - Q$ is reduced to a constant. The conclusion is that after ψ has been changed on a null-set, it is equal to an absolutely continuous function. Moreover, its derivative exists almost everywhere by Riesz' theorem in §§ Measure and given by the bounded and measurable function $q(x)$ which was initially found via a duality argument.

Remark. The description of ψ which was derived via the a priori inequality (*) was known long before the notion of distributions was introduced and we saw that the essential steps to describe ψ rely upon measure theory. However, when more complicated situations occur we shall see that the general notion of distributions becomes very useful and help us to clarify many results.

A result by Beurling. In distribution theory various limits are taken in a weak sense. For example, on the real line we consider the family of complex-valued Riesz measures of finite total variation. Let $\{\mu_n\}$ be a sequence of such measures for which there exists a constant M such that $||\mu_n|| \leq M$ hold for every n. The sequence is said to converge weakly to a Riesz measure μ when a pointwise limit hold for the Fourier transforms, i.e. if

$$(1) \qquad \lim_{n\to\infty} \widehat{\mu}_n(\xi) = \widehat{\mu}(\xi)$$

holds pointwise. Thanks to the uniform bound on $\{||\mu_n||\}$ one easily shows that (1) is equivalent to the condition that

$$\lim_{n\to\infty} \int f(x) \cdot d\mu_n(x) = \int f(x) \cdot d\mu(x)$$

hold for every continuous function $f(x)$ with a compact support. Next, let $\phi(x)$ be a bounded and uniformly continuous function on the real x-line. But we do not assume that it has compact support. However, the integrals below exist since $\{\mu_n\}$ have finite total variation:

$$(2) \qquad \int \phi(x) \cdot d\mu_n(x)$$

The question arises if the weak convergence entails that

$$(*) \qquad \lim_{n\to\infty} \int \phi(x) \cdot d\mu_n(x) = \int \phi(x) \cdot d\mu(x)$$

Let us remark that a given Riesz measure μ with finite total variation can be obtained as the weak limit via many different sequences $\{\mu_n\}$ whose total variations have a uniform bound. So the *consistent limit formula* above is not evident. Beurling proved that (*) holds for every weakly convergent sequence $\{\mu_n\}$ if and only if ϕ can be uniformly approximated on the whole x-line by functions of the form

$$\psi(x) = \int e^{i\xi x} \cdot d\gamma(\xi)$$

where γ is a Riesz measure on the ξ-line with a finite total variation. We prove this in Section 17 which teaches a lesson about topological considerations related to distribution theory and as we shall see the actual proof requires considerable work where solutions to certain variational problems appear.

Boundary values of analytic functions. This is a major issue in this chapter. The basic constructions appear in 2.1 and are used to obtain boundary values of analytic functions $f(x+iy)$

with a moderate growth as $y \to 0$. Theorem 2.7 gives a uniqueness properties when boundary values of analytic functions are taken from the upper, respectively the lower half plane. Section 3 describes a more general procedure to get Fourier's inversion formula which was presented by Carleman's lectures at Institute Mittag Leffler in 1935. Here Theorem 3.5 leads to the Fourier-Carleman transform which can be used to establish uniqueness results for tempered distributions. Section 4 is devoted to the *Paley-Wiener theorem* and Section 5 treats Runge's approximation theorem and results about the inhomogeneous $\bar{\partial}$-equation. Section 6 extends Fourier's inversion formula to non-tempered situations based upon constructions presented by Carleman in 1935 and leads to the space of hyperfunctions describing the dual to the space of real-analytic functions.

Section 7 contains one of Carleman's most valuabe discoveries which gives a sharp inequality concerned with the growth of derivatives of differentiable functions which vanish up to a high order the end-points of a bounded interval. This fundamental result clarifies to what extend it is possible to produce cut-off functions with good smoothing properties. In distribution theory this result is used to clarify when a distribution μ is locally expressed as the boundary value of an analytic function from defined above or below the real x-line which is described via the notion of analytic wave front sets.

Sections 8-15 contain scattered material which foremost deal with certain bounds and specific formulas related to the Fourier transform. Section 18 gives a proof of Lindeberg's precise version of the Central Limit Theorem.

0. Examples related to distribution theory.

The reader may turn directly to the subsection entitled *The origin of distribution theory*. But we start with with a fairly extensive discussion to illustrate how the notion of distributions appears naturally in many situations.

0.1 The Laplace operator. Let $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ be the Laplace operator in $\mathbf{R}^3$ where x, y, z are the coordinates. For each $\rho > 0$ we define the kernel function:

$$A_\rho(p,q) = \frac{2}{\rho} - \frac{[p-q|}{\rho^2} - \frac{1}{|p-q|}$$

where p and q are points in $\mathbf{R}^3$. The singularity of the last term is not too bad for if p is fixed then $q \mapsto \frac{1}{|p-q|}$ is locally square integrable as a function of q. Let D be some bounded open domain in $\mathbf{R}^3$ and ϕ a square integrable function over D. Suppose that u is a continuous function on the closure $\bar{D}$ such that for each point $p \in D$ and every $\rho < \text{dist}(p, \partial D)$ one has the equality

$$u(p) = \int_{B_p(\rho)} \frac{u(q)}{|p-q|} \cdot dq + \int_{B_p(\rho)} A_\rho(p,q) \cdot \phi(q) \cdot dq \tag{1}$$

Here $dq = dxdydz$ is the Lebesgue measure and $B_p(\rho)$ the open ball of radius ρ centered at p. The last integral in (1) is defined since $A_\rho(p,q) \cdot \phi(q)$ belongs to $L^1(B_p(\rho))$ by the Cauchy-Schwarz inequality. If v is a C^2-function in D then Green's formula gives the equality

$$v(p) = \int_{B_p(\rho)} \frac{v(q)}{|p-q|} \cdot dq + \int_{B_p(\rho)} A_\rho(p,q) \cdot \Delta(v)(q) \cdot dq \tag{2}$$

From this we conclude that (1) gives the equality below in the sense of distributions:

$$\Delta(u) = \phi \tag{3}$$

Before distributions were introduced, one used the integral formula (1) to express a solution u to the inhomogeneous equation (3) when the ϕ-function is given in $L^2(D)$. In other words, the inhomogeneous equation (3) is solved via the integral equation (1). Of course it requires a rather involved analysis to show that this integral equation has at least locally a solution u and to determine regularity properties of u under the sole assumption that ϕ is an L^2-function. Let us recall that pioneering work by Fredholm demonstrated that a quite extensive class of linear PDE-equations can be solved via integral equations.

0.2 Kernels in analytic function theory. Riemann posed the problem to find an analytic function $f(z)$ in a bounded domain Ω where the real and the imaginary part of f restrict to linearly dependent functions on the boundary. More generally, Hilbert treated the problem when $\mathcal{C}(z) = \{c_{pq}(z)\}$ is an $n \times n$-matrix of continuous complex valued function defined on $\partial\Omega$ and asked for a pair of n-tuples $\{f_p(z)\}$ and $\{g_q(z)\}$ where the f-functions are meromorphic in Ω with a finite set of poles and the g-functions are meromorphic in the exterior domain with a finite number of poles, such that

$$f_p(z) = \sum_{q=1}^{q=n} c_{pq}(z) \cdot g_q(z)$$

hold on $\partial\Omega$ for every $1 \leq p \leq n$. The system above also appears in work by Plemelj devoted to the problem of finding systems of linear differential equations whose solutions have a prescribed monodromy. A more general account was later given by Hasemann and one should also mention work by Uhler who used integral equations to extend the theory about zeta-functions of Fuchsian type. Cases where singular kernels appear lead to equations where regularisations in the spirit of general distribution theory are used and we remark that when smoothness of $\partial\Omega$ is relaxed many open problems remain to be analyzed in more detail.

0.3 The Pompieu formula. Let $f(z)$ be a continuous complex-valued function in a domain Ω which belongs to the class $\mathcal{D}(\mathbf{C}^1)$. In XX we established the Pompieu formula under the assumption that f is a C^1-function. This regularity assumption can be relaxed. Namely, assume only that f is continuous and that the *distribution derivative* $\bar{\partial}(f)$ belongs to $L^1_{\text{loc}}(\Omega)$ which means that there exists an L^1_{loc}-function ϕ in Ω such that the following equality holds for area integrals:

$$\iint \bar{\partial}(g) \cdot f \cdot dxdy = -\iint g \cdot \phi \cdot dxdy \tag{1}$$

for every test-function $g(x,y) \in C_0^\infty(\Omega)$. When $z \in \Omega$ is given we consider test-functions

$$g_\epsilon(\zeta) = \frac{\bar{\zeta} - \bar{z}}{|\zeta - z|^2 + \epsilon} \cdot \rho$$

where $\rho \in C_0^\infty(\Omega)$ is identical 1 in the compact set of points in Ω with distance $\geq \epsilon$ to $\partial\Omega$. Passing to the limit as $\epsilon \to 0$ the same use of Stokes Theorem as in XX from Chapter 3 gives the formula:

$$f(z) = \frac{1}{\pi} \iint_\Omega \frac{f(\zeta) \cdot \bar{\partial}(\rho)}{z - \zeta} \cdot d\xi d\eta + \frac{1}{\pi} \iint_\Omega \frac{\rho(\zeta) \cdot \phi(\zeta)}{z - \zeta} \cdot d\xi d\eta \tag{2}$$

The area integrals in the right hand side are both defined and yield continuous functions of z as it should since f was assumed to be continuous from the start. The Pompieu formula can be extended a bit further. Namely, assume only that f from the start is a *bounded* Borel function. It can be identified with an L^1_{loc}-function which therefore has a distribution derivative and if we again assume that $\bar{\partial}(f) = \phi$ for some L^1_{loc}-function ϕ then (2) still holds. To be precise, we get a pointwise equality at every *Lebesgue point* of f.

0.4 The elliptic property of $\bar{\partial}$. Let $f(x,y) \in L^1_{\text{Loc}}(\Omega)$ where Ω belongs to $\mathcal{D}(C^1)$. Suppose that

$$\iint f \cdot \bar{\partial}(g) \cdot dxdy = 0 \tag{*}$$

hold for all text-functions $g(x,y)$ with compact support in Ω. Then one says that $\bar{\partial}(f) = 0$ holds in the distribution sense. Here $\phi = 0$ holds in the Pompieu formula using (2) where the ρ-functions can be identically one on arbitrary large compact subsets o Ω we conclude that f belongs to $\mathcal{O}(\Omega)$. The remarkable fact is of course that (*) entails that the L^1_{loc}-function f *automatically is analytic* in Ω. If necessary one has only to change its values on a null-set which is harmless since two L^1_{loc}-functions are considered to be equal if they coincide outside a null set. A similar result is valid for the Laplacian, i.e. let f again be in $L^1_{\text{loc}}(\Omega)$ and suppose that

$$\iiint f \cdot \Delta(g) \cdot dxdy = 0 \tag{**}$$

holds for every test-function g in Ω. Then f is a harmonic function in Ω which in particular entails that it is a real-analytic function of the two real variables x and y.

Remark. In a broader context the two results above hold because the differential operators $\bar{\partial}$ and Δ are *elliptic*. Actually the result above was established in the chapter devoted to harmonic functions. In XXX we proved that if $g(x,y)$ is a test function then

$$\bar{\partial}(g)(z) = \frac{1}{2\pi i} \iint \frac{g(z + \zeta)}{\zeta} \cdot d\xi d\eta$$

Expressed by distribution theory this means that the $\bar{\partial}$-derivative of the locally integrable function $\frac{1}{z}$ is equal to $\pi \cdot \delta_0$ where δ_0 is the the Dirac distribution at the origin.

Exercise. Show that

$$-\iint \bar{\partial}(g) \cdot \frac{1}{z} \cdot dxdxy = \pi \cdot g(0)$$

hold for every test function $g(x,y)$ which by the construction of distribution derivatives means that

$$\bar{\partial}(\frac{1}{z}) = \pi \cdot \delta_0 \tag{*}$$

It follows that the L^1-density function $\frac{1}{\pi \cdot z}$ is a *fundamental solution* to the $\bar{\partial}$-operator. For the Laplace operator we find a fundamental solution via the formula

$$(**) \qquad \Delta(g)(z) = \frac{1}{\pi} \cdot \iint \operatorname{Log}|z-\zeta| \cdot g(z+\zeta) \cdot d\xi d\eta \quad : \ g(x,y) \in C_0^\infty(\mathbf{C})$$

0.5 Riesz measures and positive harmonic functions. Consider a non-negative harmonic function $u(x,y)$ defined in the open unit disc. The mean-value property gives

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \cdot d\theta \quad \text{for all} \quad 0 < r < 1$$

In general the u-function is unbounded and examples show that the radial limits $\lim_{r \to 1} u(re^{i\theta})$ can be quite irregular. It may also occur that these radial limits exist and are zero for all θ-angles outside a null set on the periodic interval $[0, 2\pi]$. However, a consistent and unique limit exists in the distribution sense. Namely, there exists a unique non-negative Riesz measure μ on the unit circle whose total mass is $u(0)$ and

$$(*) \qquad \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \cdot u(re^{i\theta}) \cdot d\theta = \int_0^{2\pi} g(\theta) \cdot d\mu(\theta)$$

hold for every 2π-periodic and continuous g-function. Thus, we can refer to a "boundary value" of u expressed by the equation (*). Moreover, by Herglotz' formula the map which sends a positive harmonic function u to the boundary measure μ is bijective. In other words, there is a *one-to-one correspondence between the family of positive harmonic functions in D and positive Riesz measures on T.*

0.6 Cauchy problems.

A basic boundary value problem for a second order PDE-equation is to find a function $u(x,y)$ which for $x > 0$ satisfies some PDE-equation, and at $x = 0$ the two boundary conditions:

$$(*) \qquad u(0,y) = \phi(y) \quad \text{and} \quad \frac{\partial u}{\partial x}(0,y) = \psi(y)$$

Here ϕ and ψ are given functions of the real y-variable and when $x > 0$ the u-function solves a PDE-equation of the *Riemann type*:

$$(**) \qquad \frac{\partial^2 u}{\partial x^2} = F\Big(\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u, x, y\Big)$$

Above F is supposed to be a real valued polynomial of seven variables or more generally some analytic series. If ϕ and ψ both are analytic functions of the y-variable the Cauchy-Kovalevsky theorem gives existence and uniqueness of a solution $u(x,y)$ defined in some open interval $0 \leq x < \delta$ when the analytic Cauchy data is given on some y-interval. The study of this boundary value problem when ϕ and ψ no longer are analytic leads to a much more difficult theory. That one should expect this was demonstrated by examples in the article [xx] by Sophie Kovalevsky from her dissertation in 1874. See also the article [HS] by Harold Shapiro for an account about the history of the Cauchy-Kovalevsky theorem.

Conditions on the pair ϕ, ψ in order that the Cauchy problem has a solution were investigated around 1900 by Gevrey, Hadamard and Holmgren. In his article [XX] from 1902 Hadamard considered the case when F is linear and elliptic. For example, let F be the Laplace operator. So here we seek a harmonic function $u(x,y)$ defined in some open rectangle $\{0 < x < \delta\} \cap \{a < y < b\}$ which satisfies (*). Using Schwarz reflection principle and the fundamental solution of Δ expressed by $\operatorname{Log}|z|$, Hadamard proved that the boundary value problem has a solution u if and only if the function defined for $a < y < b$ by

$$y \mapsto \phi(y) + \frac{1}{\pi} \int_a^b \log \frac{1}{|y-s|} \cdot \psi(s) \cdot ds$$

is real-analytic on the interval (a,b). If the PDE-equation is of *hyperbolic type*. Hadamard proved that the boundary value problem has a solution for an arbitrary pair (ϕ,ψ). The reader can consult the text-book [Petrovksy] on PDE-theory by Petrowsky for studies of PDE-equations related to the Cauchy problem with special emphasis on the hyperbolic case which includes studies in dimension ≥ 3.

Now we consider the *heat equation*. So here u satisfies $\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial y}$ when $x>0$. In his article *Sur l'extension de la méthode d'integration de Riemann* from 1904, Holmgren showed that the boundary value problem for the heat operator has solutions if and only if ϕ and ψ are related to each other by an equation

$$\psi(y)=-\frac{1}{\sqrt{\pi}}\cdot\int_0^y\frac{\phi'(u)}{\sqrt{y-u}}\cdot du+g((y)$$

where the g-function must be C^∞ and the higher order derivatives satisfy the inequalities

$$\left|g^{(n)}(y)\right|\leq M\cdot K^n\cdot(2n+1)!$$

for some pair of constants M and K. These estimates on the higher order derivatives are sufficiently relaxed in order that there exists g-functions with arbitrary small compact supports, i.e. the class is not quasi-analytic. We mention this to illustrate that in many applications one should not restrict the attention to the analytic case. On the other hand the discussion above shows that analytic functions play a central role. The use of distributions have clarified many questions concerned with well-posedness of boundary value problems. It would lead us to far to discuss this. Volume II in Lars Hörmander's text-book series [Hörmander I-IV] contains a wealth of results and here the reader will find an extensive list of references, covering all essential work dealing with linear PDE-theory.

0.7 Cauchy transforms and the logarithmic potential

Let μ be a Riesz measure on the unit interval $\{0\leq t\leq 1\}$. With $z=x+iy$ we get the Cauchy transform

$$\mathcal{C}_\mu(z)=\int_0^1\frac{d\mu(t)}{z-t}\tag{*}$$

We shall learn that there exist the two boundary value distributions

$$\mathfrak{b}^*\mathcal{C}_\mu=\lim_{\epsilon\to 0}\mathcal{C}_\mu(x+i\epsilon)\quad\text{and}\quad\mathfrak{b}_*\mathcal{C}_\mu=\lim_{\epsilon\to 0}\mathcal{C}_\mu(x-i\epsilon)\tag{1}$$

For example, when $f(x)$ is a differentiable function on the real x-line with compact support on some interval $[-a,a]$ then

$$\mathfrak{b}^*\mathcal{C}_\mu(f)=\lim_{\epsilon\to 0}\int_0^1\Big[\int_{-a}^a\frac{f(x)\cdot dx}{x-t+i\epsilon}\Big]\cdot d\mu(t)\tag{2}$$

The fact that this limit exists is not evident and will be demonstrated in XX. Next, let us notice the equality

$$\mathcal{C}_\mu(x+i\epsilon)-\mathcal{C}_\mu(x-i\epsilon)=-2i\cdot\int_0^1\frac{\epsilon}{(x-t)^2+\epsilon^2}\cdot d\mu(t)\tag{3}$$

When $\epsilon>0$ the right hand side is a function of x which we denote by $\rho_\epsilon(x)$. If $g(x)$ is a test-function on the real x-line we get:

$$\int\rho_\epsilon(x)\cdot g(x)\cdot dx=2i\cdot\int_0^1\Big[\int\frac{\epsilon}{(x-t)^2+\epsilon^2}\cdot g(x)dx\Big]\cdot d\mu(t)\tag{4}$$

The limit of the inner integral is found for each t since g is a test-function. More precisely the reader can verify that

$$\lim_{\epsilon\to 0}\int\frac{\epsilon}{(x-t)^2+\epsilon^2}\cdot g(x)dx=\pi\cdot g(t)\tag{5}$$

where the convergence even holds uniformly with respect to t. Hence

$$\lim_{\epsilon\to 0}\int \rho_\epsilon(x)\cdot g(x)\cdot dx = 2\pi i\cdot\int g(t)\cdot d\mu(t) \tag{6}$$

In terms of distributions this gives the equality

$$\mathfrak{b}^*\mathcal{C}_\mu - \mathfrak{b}_*\mathcal{C}_\mu = 2\pi i\cdot\mu \tag{**}$$

Hence μ is expressed as a difference of two distributions which arise via boundary values of analytic functions. Now we consider the sum

$$\mathcal{C}_\mu(x+i\epsilon)+\mathcal{C}_\mu(x-i\epsilon) = \int_0^1 \frac{2(x-t)}{(x-t)^2+\epsilon^2}\cdot d\mu(t) \tag{7}$$

To get a formula for the limit in (7) we introduce the function

$$F(z) = \int_0^1 \log(z-t)\cdot d\mu(t) \tag{8}$$

Here F is defined outside the real interval $[0,1]$ as a multi-valued analytic function. But its complex derivative is single-valued and we have

$$F'(z) = \mathcal{C}_\mu(z) \tag{9}$$

At the same time we can choose single-valued branches of $\log(z-t)$ in the the half-planes $\mathfrak{Im}(z)>0$ and $\mathfrak{Im}(z)<0$. Let us notice that if $0<t<1$ is fixed then one has the limit formula

$$\lim_{\epsilon\to 0}\,[\log(x+i\epsilon-t)+\log(x-i\epsilon-t)) = 2\cdot\log[x-t|+\pi\cdot i$$

From this we deduce the limit formula:

$$\lim_{\epsilon\to 0} F(x+i\epsilon)+F(x-i\epsilon) = 2\cdot\int_0^1 \log|x-t|\cdot d\mu(t)\cdot dt + i\pi\cdot\int_0^1 d\mu(t) \tag{10}$$

In XX we shall learn that the passage to boundary value distributions commute with derivations. So (9) and (10) give the following equality of distributions on the real x-line:

$$\frac{1}{2}\big[\mathbf{b}^*(\mathcal{C}_\mu)+\mathbf{b}_*(\mathcal{C}_\mu)\big] = \frac{d}{dx}(f(x)) \tag{***}$$

where the right hand side is the distribution derivative of the L^1-function $f(x)$ defined by:

$$f(x) = \int_0^1 \log|x-t|\cdot d\mu(t) \tag{11}$$

0.7.1 An inversion formula. Above we announced formulas which are expressed by distributions. However, one cannot deduce more precise results by a mere appeal of notions from distribution theory. For example, suppose that $\mu = g\cdot dt$ where $g(t)$ is an L^1-function on $[0,1]$. We expect that the f-function in (11) has more regularity than a mere L^1-function and we also ask for an inversion formula which recaptures the g-function from f. For this purpose we consider the linear operator from $L^1[0,1]$ into itself defined by

$$T_g(x) = \int_0^1 \log|x-t|\cdot g(t)\cdot dt$$

It is easily seen that the T-operator is injective. In Section 14 we shall establish exhibit an inversion formula which recaptures g from T_g and in this way also find a description of its range.

0.8 The Fourier transform.

The fact that one can define the Fourier transform of a distribution is very useful. An example occurs in Section 9 where we consider the integral equation

$$f*\phi(x) = \int_0^\infty f(x-y)\phi(y)dy = 0$$

Here f is a function in $L^1(\mathbf{R})$ and we seek solutions ϕ in the space $L^\infty(\mathbf{R})$, i.e. bounded and Lebesgue measurable functions on the real line. When the zeros of the Fourier transform $\widehat{f}(\xi)$ is a discrete set on the real ξ-line we shall find all ϕ-solutions. More precisely, every such solution is the limit of functions given by finite sums of the exponential functions $\{e_\alpha(y) = e^{i\alpha\cdot x}\}$ where $\{\alpha\}$ are the zeros of $\widehat{f}$. Even if this result is expected the systematic use of distribution fascilates the proof. One can also reverse the consideration and start from some $\phi \in L^\infty(\mathbf{R})$ and seek the set of all $f \in L^1(\mathbf{R})$ such that $f * \phi = 0$. This leads to the problem of *spectral synthesis* which is treated in Section 10, and again the notion of distributions is helpful to analyze this problem.

0.8.1 Plancherel's theorem. The construction of Fourier transforms of tempered distributions gives in particular the existence of Fourier transforms for functions in $L^2(\mathbf{R})$. This was actually achieved by Plancherel before the general notion of distributions had appeared. More precisely, for every $A > 0$ one defines the operator $\mathcal{P}_A$ which sends a square integrable function $f(x)$ to

$$(*) \qquad \mathcal{P}_A(f)(\xi) = \int_{-A}^{A} e^{-ix\xi} \cdot f(x) \cdot fx$$

Plancherel proved that there exists an L^2-function $g(\xi)$ on the real ξ-line such that

$$(**) \qquad \lim_{A\to\infty} ||g - \mathcal{P}_A(f)||_2 = 0$$

In the context of distribution theory, g is the Fourier transform of the tempered distribution on the x-line expressed by the L^2-density $f(x)$ and we set $g = \widehat{f}$.

0.8.2 Parseval's formula. By $f \mapsto \widehat{f}$ one gets a linear isomorphism beween the L^2-spaces on the real x- respectively the real ξ-line. More precisely, introducing the inner product on these complex Hilbert spaces one has the equality

$$(***) \qquad \langle\, f, g \,\rangle = \frac{1}{2\pi} \cdot \langle\, \widehat{f}, \widehat{g} \,\rangle \quad \text{for all pairs} \quad f, g \in L^2(\mathbf{R})$$

Remark. In the literature one sometimes normalises the Fourier transform of L^2-functions to attain an isometry, i.e. one employs $\frac{1}{\sqrt{2\pi}} \cdot \widehat{f}$ rather than $\widehat{f}$. However, to fit everything with the general construction of Fourier transforms of tempered distributions we prefer to define $\widehat{f}$ without this normalizing factor and then $\frac{1}{2\pi}$ appears in the Parseval formula.

0.8.3 Comment to Plancherel's theorem. The Fourier transform of the characteristic function for the interval $[-A, A]$ becomes

$$(1) \qquad \rho_A(\xi) = 2 \cdot \frac{\sin(A\xi)}{\xi}$$

Parseval's formula via a convolution gives

$$(2) \qquad \mathcal{P}_A(f)(\xi) = \frac{1}{\pi} \cdot \int \rho_A(\eta) \cdot \widehat{f}(\xi + \eta) \cdot d\eta$$

The limit in (**) with $g = \widehat{f}$ can be established using the oscillatory behaviour of the ρ_A-function as $A \to +\infty$ and the continuity under translations of L^2-functions, i.e. one uses that:

$$(3) \qquad \lim_{\delta\to 0} \int \left|\widehat{f}(\xi + \delta) - \widehat{f}(\xi)\right|^2 \cdot d\xi = 0$$

Remark. In harmonic analysis one investigates the rate of convergence in (**) as $A \to +\infty$ via the behaviour of (3) as $\delta \to 0$. Such results are not covered by the mere passage to weak limits of distributions. An example is the *Central Limit Theorem* in probability theory. Here one can easily establish the existence of a limit expressed in a distribution theoretic context. But further analysis is needed to attain a more precise information about rate of convergence. This occurs in Lindeberg's Central Limit Theorem where on allows "fat tails" during the passage to the limit of sums of independent random variables. So the reader should be aware of the fact that when one refers to limits in spaces of distributions they are often taken in a weak sense, while more precise limit formulas require additional work. The next example illustrates this.

0.8.4 A pointwise limit formula. Above the Fourier transform was considered on L^2-functions and inversion formulas expressed in terms of L^2-norms. When $f(x)$ from the start is a continuous function which vanishes outside an interval one gets pointwise limit formulas provided that the Dini condition holds below. Let us recall this classical result. For simplicity we assume that $f(x)$ is an even and continuous funtion which is zero outside $[-1,1]$ and impose:

Dini's condition. *It holds at $x=0$ when*

(*) $$\int_0^1 \frac{|f(x)|}{x}\cdot dx<\infty$$

From now on (*) is assumed. Since f is even we have:

$$\widehat{f}(\xi)=2\cdot\int_0^1 \cos(x\xi)\cdot f(x)\cdot dx$$

With $A>0$ we set

(1) $$\gamma(A)=\frac{1}{2\pi}\int_{-A}^{A}\widehat{f}(\xi)\cdot d\xi$$

Our aim is to show that Dini's condition implies that

(**) $$\lim_{A\to\infty}\gamma(A)=f(0)$$

To prove (**) we first evaluate (1) which gives

(2) $$\gamma(A)=\frac{2}{\pi}\int_0^1 \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx$$

Next, we have the limit formula

(3) $$\lim_{A\to\infty}\frac{2}{\pi}\int_0^1 \frac{\sin(Ax)}{x}\cdot dx=\frac{2}{\pi}\int_0^A \frac{\sin(t)}{t}\cdot dt=1$$

So in order to get $\gamma(A)\to f(0)$ we can replace f by $f(x)-f(0)$, i.e. it suffices to show that $\gamma(A)\to 0$ when $f(0)=0$ is assumed. To obtain this we fix some $0<\delta<1$ and put

(4) $$\gamma_\delta(A)=\frac{2}{\pi}\int_0^\delta \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx$$

Since $|\sin(Ax)|\leq 1$ the triangle inequality gives

(5) $$\gamma_\delta(A)\leq\int_0^\delta \frac{|f(x)|}{x}\cdot dx<\infty$$

for all A and every $\delta>0$. Dini's condition implies that the right hand side tends to zero as $\delta\to 0$. Next, we set

(6) $$\gamma^\delta(A)=\frac{2}{\pi}\int_\delta^1 \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx$$

Here $\frac{f(x)}{x}$ is continuous on $[\delta,1]$ and has therefore a finite modulus of continuity, i.e. we get the function

(7) $$\omega_\delta(r)=\max_{\delta\leq x_1,x_2\leq 1}\left|\frac{f(x_1)}{x_1}-\frac{f(x_2)}{x_2}\right|\quad\text{where}\quad |x_1-x_2|\leq r$$

With these notations one has the inequality:

(***) $$\gamma^\delta(A)\leq\frac{8\pi}{\pi}\cdot\omega_\delta(\frac{2\pi}{A})$$

The verification is left to the reader as an exercise. We remark only that the extra factor 4 replacing 2 by 8 comes form $4 = \int_0^{2\pi} |\sin(t)| \cdot dt$. Hence we have

$$(8) \qquad \gamma(A) \leq \gamma_\delta(A) + \frac{8\pi}{\pi} \cdot \omega_\delta(\frac{2\pi}{A})$$

This holds for all pairs δ and A and now we conclude that Dini's condition indeed gives the limit formula in (**).

Remark. Above $x = 0$. More generally we can impose Dini's condition for f at an arbitrary point a, i.e. for every a we set

$$D_f(a) = \int \frac{|f(x) - f(a)|}{|x - a|} \cdot dx$$

The results above show that whenever $D(a) < \infty$ one has a pointwise limit

$$(1) \qquad f(a) = \lim_{A \to \infty} \frac{1}{2\pi} \cdot \int_{-A}^{A} e^{ia\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

An example when this occurs is when $f(x)$ is Hölder continuous of some order > 0.

0.8.5 Carleson's Theorem. With no other assumption than continuity on f the question about the pointwise limits in (*) was unclear for more than a century. In 1965 Carleson proved that pointwise limit in (*) holds almost everywhere for an arbitrary continuous function $f(x)$ with a compact support. This famous result from [Carleson] goes far beyond the scope of this book. In the sequel we define the Fourier transform of distributions where the resulting inversion formula is taken in the distribution theoretic sense which means that one can *ignore* precise pointwise limits and so on. In (1) from the Remark above we are content to assert that the right hand side as functions of a converge to f in the L^2-norm on the real a-line.

0.8.6 A theorem by Carleman and Hardy. A affirmative result about pointwise convergence goes as follows: We are given some L^1-function $u(x)$ which is even and zero outside $[-1, 1]$ and of class C^2 when $x \neq 0$. Moreover, we assume that there exists a constant C such that

$$(*) \qquad |u''(x)| \leq \frac{C}{x^2} \quad : \quad x \neq 0$$

Since u is an L^1-function we can construct the Fourier series

$$F_u(x) = \sum_{n=0}^{\infty} A_n \cdot \cos nx \quad \text{where} \quad A_n = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\xi) \cdot u(\xi) \cdot d\xi$$

Since u is a C^2-function when $x \neq 0$ the series converges uniformly to u on every interval $[\delta, 1]$ when $0 < \delta < 1$. There remains to analyze the situation at $x = 0$. The following result is due to Carleman and Hardy and will be proved in Section 13:

0.8.7 Theorem. *The series $\sum A_n$ converges if and only if there exists the limit*

$$\lim_{x \to 0} u(x) = S_*$$

and in this case S_* is the series sum of $\sum A_n$.

0.8.8 The central limit theorem.

Consider a family of independent random variables $\chi_1, \ldots, \chi_N$, where each individual variable has mean-value zero and a finite variance σ_ν. This yields the sum variable

$$(*) \qquad S_N = \frac{\chi_1 + \ldots + \chi_N}{\sqrt{\sigma_1^2 + \ldots + \sigma_N^2}}$$

which has been normalised so that its variance is one. In 1812 Simon Laplace stated that $\{S_N\}$ converges to the normal distribution whose frequency function is $\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$, under the hypothesis that each individual variable yields a *relatively small contribution.* The last condition is achieved if there exists a constant M such that

$$\sigma_\nu \leq M \tag{1}$$

hold for all ν. It turns out that one needs to assume a bit more than (1) to get the convergence of $\{S_N\}$ to the normal distribution. The conclusive result was established in 1920 by Lindeberg who proved the convergence with the sole extra hypothesis that tails of second order moments admit a certain weak estimate. Lindeberg's theorem is proved in Section XX. The crucial step in the proof is the inequality (**) below. To motivate its relevance we consider two probability measures μ and ν on the real line expressing distributions of two random variables. For a given pair of real numbers $a < b$ and small $\delta > 0$ there exists a non-negative C^2-function $g_\delta(x)$ which is identically one on $[a, b]$ and vanishes outside $[a-\delta, b+\delta]$. Elementary calculus shows that g_δ can be constructed so that the maximum norm of its second derivative is bounded by $\frac{2}{\delta^2}$. At the same time we have the Fourier transforms $\widehat{\mu}$ and $\widehat{\nu}$. With these notations, Parseval's formula gives equality

$$2\pi \cdot \int g_\delta(x) \cdot [d\mu(x) - d\nu(x)] = \int \widehat{g}_\delta(\xi) \cdot (\widehat{\mu}(\xi) - \widehat{\nu}(\xi)) \cdot d\xi$$

Next, if $|\xi| \geq 1$ we have the inequality

$$|\widehat{g}_\delta(\xi)| \leq \frac{1}{\xi^2} \cdot \int |g''_\delta(x)| \cdot dx \leq \frac{4}{\delta \cdot \xi^2}$$

where the last inequality follow since the support of g''_δ is contained in a union of two δ-intervals while is maximum norm is $\leq \frac{2}{\delta^2}$. At the same time the maximum norm of $\widehat{\mu} - \widehat{\nu}$ is bounded by 2. Moreover, the L^2-norm of $\widehat{g}_\delta$ is $\sqrt{2\pi}$ times the L^2-norm of g_δ and the latter is bounded by $\sqrt{b-a+2\delta}$. So if $A \geq 1$ the triangle inequality together with the Cauchy-Schwarz inequality show that the absolute value of the right hand side in (**) is majorised by

$$\sqrt{(2\pi(b-a+2\delta)} \cdot [\int_{-A}^{A} |\widehat{\mu}(\xi) - \widehat{\nu}(\xi)|^2 \cdot d\xi]^{\frac{1}{2}} + 2 \cdot \frac{4}{\delta} \cdot 2 \cdot \int_A^\infty \frac{d\xi}{\xi^2} \tag{**}$$

Notice that the last term is $\frac{16}{A \cdot \delta}$ which for a given $\delta > 0$ can be made arbitrary small when $A >> 1$ and from this one can derive limit formulas when one has an L^2-convergence of Fourier transforms of probability measures over arbitrary large ξ-intervals. A variant of (**) is to get rid of the factor $\sqrt{(2\pi(b-a+2\delta)}$ where we use that the maximum norm of the g_δ-function is one and get the majorisation

$$\int_{-A}^{A} |\widehat{\mu}(\xi) - \widehat{\nu}(\xi)| \cdot d\xi + \frac{16}{\delta \cdot A} \tag{***}$$

Example. Let $\{\mu_N\}$ be the sequence of probability measures expressing distributions of the sum variables $\{S_N\}$ in (*) above. From (***) it follows that the Central Limit Formula holds if

$$\lim_{N \to \infty} \int_{-A}^{A} |\widehat{\mu}_N(\xi) - e^{-\xi^2/2}| \cdot d\xi = 0 \tag{1}$$

holds for every $A \geq 1$. The case proved by de Moivre in 1733 was when S_N is the sum of N many Bernoulli variables which with probability 1/2 take the values $+- \frac{1}{\sqrt{N}}$. Here

$$\widehat{\mu}_N(\xi) = \left[\cos(\frac{\xi}{\sqrt{N}})\right]^N \tag{2}$$

Regarding the Taylor series of the cosine function at $\xi = 0$ we get (**) from Neper's limit formula

$$\lim_{N \to \infty} (1 - \frac{\xi^2}{2N})^N = e^{-\xi^2/2} \tag{3}$$

0.9 Dirac's δ-function

A basic object in distribution theory is Dirac's δ-function which for $n \geq 1$ is the unit point mass at the origin of $\mathbf{R}^n$. So let us discuss this central object in distribution theory. We restrict the discussion to the case $n = 2$ and mention only that similar results hold verbatim in dimension ≥ 3. To begin with δ_0 evaluates test functions $\phi(x, y)$ in $\mathbf{R}^2$ at the origin:

$$\delta_0(\phi) = \phi(0)$$

A far more valuable fact is that δ_0 is embedded in an analytic family which leads to its *plane wave decomposition.* With $z = x + iy$ we consider a parametrised family of functions:

$$G_\lambda(z) = \frac{2 \cdot |z|^\lambda}{2\pi \cdot \Gamma(\lambda/2 + 1)} \quad : \quad z \in \mathbf{C} \quad : \lambda \in \mathbf{C}$$

where Γ is the usual Gamma-function and recall that $\frac{1}{\Gamma(\lambda)}$ is an entire function in the complex λ-plane.

0.9.1 The analytic continuation of G_λ. If $\mathfrak{Re}\,\lambda > 0$ it is clear that $G_\lambda(z)$ is a continuous function in $\mathbf{C}$ which by integration yields a distribution on the underlying real (x, y)-space. Moreover, this distribution valued function is analytic as λ varies in the right half plane where the complex derivative becomes:

$$\frac{d}{d\lambda}\big(G_\lambda(z)\big) = \operatorname{Log}|z| \cdot G_\lambda(z) - \frac{2 \cdot |z|^\lambda \cdot \Gamma'(\lambda/2 + 1)}{2\pi \cdot \Gamma(\lambda/2 + 1)^2}$$

It turns out that this distribution-valued function extends to the whole complex λ-plane. In XX we prove:

0.9.2 Theorem *The distribution valued function G_λ extends to an entire function and at $\lambda = -2$ one has the equality $G_{-2} = \delta_0$.*

We prove this in XXX and remark that it relies upon Taylor expansions of C^∞-functions of the two real variables x and y. The fact that G_λ is an entire function becomes useful when we write out the integral formula:

$$(*) \qquad G_\lambda(x + iy) = \frac{2}{2\pi \cdot \Gamma(\lambda/2 + 1)} \cdot \int_0^{2\pi} \big|\cos(\theta) \cdot x + \sin(\theta) \cdot y\big|^\lambda \cdot d\theta$$

Here (*) together with the Theorem 0.4.2 constitute the plane wave decomposition of δ_0 in dimension two.

0.9.3 Fundamental solutions. Using (*) and Theorem 0.9.2 one constructs a fundamental solution to elliptic PDE-operators with constant coefficients of the form:

$$L(\partial) = \sum_{j+k \leq 2m} c_{j,k} \cdot \partial_x^j \cdot \partial_y^k$$

Here m is a positive integers and the c-coefficients are complex numbers. The elliptic property means that the leading polynomial

$$L^*(\theta) = \sum_{j+k=2m} c_{j,k} \cdot \cos^j(\theta) \cdot \sin^k(\theta) \neq 0 \quad \text{for all} \quad 0 \leq \theta \leq 2\pi$$

Assume that $L(\partial)$ is an elliptic operator and let us construct a distribution μ_L such that $L(\mu_L) = \delta_0$. To find μ_L we shall construct a certain distribution-valued entire function

$$(1) \qquad \lambda \mapsto \mu_\lambda$$

and the fundamental solution will be the constant term μ_{-2} at $\lambda = -2$. To attain (1) we consider for each $0 \leq \theta \leq 2\pi$ the ordinary differential operator of the single real variable s defined by:

$$L_\theta\Big(\frac{d}{ds}\Big) = \sum_{j+k \leq 2m} c_{j,k} \cdot \cos^j(\theta) \cdot \sin^k(\theta) \cdot \Big(\frac{d}{ds}\Big)^{j+k}$$

It has order $2m$ and the elliptic property of $L(\partial)$ means that when we write

$$L_\theta(\frac{d}{ds}) = c_{2m}(\theta) \cdot (\frac{d}{ds})^{2m} + \ldots + c_0(\theta)$$

then $c_{2m}(\theta) \neq 0$ for all θ. By elementary ODE-theory this non-vanishing gives for each $0 \leq \theta \leq 2\pi$ and every $\lambda \in \mathbf{C}$ a unique C^∞-function $v_{\lambda,\theta}(s)$ on the real s-line which satisfies $(\frac{d}{ds})^{2m}(v_{\lambda,\theta})(0) = 1$ and

$$L_\theta(\frac{d}{ds})(v_{\lambda,\theta}) = 0 \quad \text{and} \quad (\frac{d}{ds})^j(v_{\lambda,\theta})(0) = 0 \ : \ 0 \leq j \leq 2m-1$$

Moreover, classical formulas for solutions to inhomogeneous ODE-equations imply that the map:

$$\lambda \mapsto v_{\lambda,\theta}$$

is an entire function of λ with values in the space of C^∞-functions on the s-line, and at the same time $\theta \mapsto v_{\lambda,\theta}(s)$ is a real-analytic function of θ for every pair (λ, s). Define the functions

$$V_{\lambda,\theta}(z) = v_{\lambda,\theta}\big(\cos(\theta) \cdot x + \sin(\theta) \cdot y\big)$$

In XXX we show that

$$(1) \qquad L(\partial)\big(V_{\lambda,\theta}\big)(z) = \frac{1}{\Gamma(\lambda/2+1)} \cdot \big|\cos(\theta) \cdot x + \sin(\theta) \cdot y\big|^\lambda$$

Next, let us put

$$u_\lambda(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} V_{\lambda,\theta}(z) \cdot d\theta$$

Now $\lambda \to u_\lambda$ is an entire distribution-valued function. Using (1) and the plane wave decomposition (*) one verifies that when $\lambda = -2$ then u_{-2} gives a fundamental solution to $L(\partial)$.

0.9.4 On distribution solutions The discussions above may give the impression that calculus with distributions enable us to avoid all kinds of technical problems. However, this is not the case when one begins to ask for *distribution solutions* of a differential equation. Consider as an example the first order differential equation on the real x-line

$$(*) \qquad x \cdot \partial(f) - af = 0$$

where a is a constant which may be a complex number. If $x > 0$ the solution is x^a times a constant, i.e. we have a 1-dimensional solution space. On the negative half-line $x < 0$ we also get a 1-dimensional solution space using a complex power, i.e. this time the solution is $e^{\pi i a} \cdot |x|^a$. The question arises what occurs at $x = 0$ and if we can find distributions μ on the whole line which satisfy the homogeneous equation (*). It turns out that the distribution solutions to (*) defined on the whole real line $\mathbf{R}$ is a 2-dimensional vector space. To prove this one construct distributions from boundary values of analytic functions of $z = x + iy$ defined in the upper- respectively the lower half-plane. More geneally, consider a differential operator of some order $\geq m$ with polynomial coefficients:

$$P(x, \partial) = q_m(x) \cdot \partial^m + \ldots + q_1(x) \cdot \partial + q_0(x)$$

When the leading polynomial $q_m(x)$ has real zeros the problem is to find all distributions μ on the real line which solve the homogeneous equation $P(\mu) = 0$. This turns to be a technically involved problem where the results are not easy to express. See XX for further details.

A. The origin of distributions.

Let us cite an excerpt from Jean Dieudonne's article *300 years of analyticity* which was presented at the Symposium on the Occasion of the *Proof of Bieberbach's conjecture*. See [XX].

Since Cauchy and Weierstrass, the central fact in complex analysis has been the one-to-one correspondence

$$\{c_n\}_{n\geq 0} \mapsto \sum_{n=0}^{\infty} c_n z^n$$

between sequences of complex numbers which do not increase too fast and functions holomorphic in a neighborhood of 0. When you turn to Fourier series, you immediately meet the same kind of correspondence

$$\{c_n\}_{n\in Z} \mapsto \sum_{-\infty}^{\infty} c_n e^{in\theta}$$

between families of coefficients and sums of trigonometric series, which has been one of the most unsatisfactory and thickest jungles of classical analysis. A situation as satisfactory as in the analytic case has only been achieved by substituting distributions in place of functions. More precisely, there is a one-to-one correspondence above when, on the left hand side, only families of *polynomial growth* are considered, that is, families such that

$$|c_n| \leq C(1+|n|)^k \quad : \text{for some} \quad k>0 \quad \text{and some constant } C$$

and the right hand side is replaced by *any periodic distribution* T on $\mathbf{R}$. The beauty of this correspondence is that it is *stable* for derivative, primitive, and convolution; so Euler was perfectly justified in taking derivatives of Fourier series and considering them again as Fourier series !

Assuming that the sequence $\{c_n\}$ has polynomial growth the right hand side is a distribution T which can be split into a sum of two periodic distributions T_1+T_2, corresponding to families $\{c_n\}$ having $c_n=0$ for all $n<0$ (resp. $n\geq 0$), and which have holomorphic extensions

$$f_1(z)=\sum_{n=0}^{\infty} \quad \text{for} \quad |z|<1 \quad : \quad f_2(z)=\sum_{-\infty}^{-1} \quad \text{for} \quad |z|<1$$

and $f_1(re^{i\theta})$ (resp. $f_2(r^{-1}e^{i\theta})$), defined for $0<r<1$, *tends* indeed to T_1 (resp. T_2) when $r\to 1$ for the *weak topology* of distributions.

Remark. Dieudonne's concise decription of distributions on the unit circle is treated in many text-books and easy to understand. Namely, let $C^\infty(T)$ be the linear space of infinitely differentiable functions on the unit circle T, or equivalently 2π-period functions on the real line. Partial integration shows that Fourier coefficients have rapid decay, i.e. with

$$\widehat{f}(n)=\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})dx \quad : f\in C^\infty(T) \quad : \quad n \text{ any integer}$$

it follows that

$$n^k\cdot|\hat{f}(n)|\leq \max_{0\leq\theta\leq 2\pi}|f^{(k)}(e^{i\theta})| \quad : \quad k=0,1,2,\ldots$$

The topology on $C^\infty(T)$ is defined by the metric

$$d(f,g)=\sum_{k=0}^{\infty} 2^{-k}\frac{|f-g|_k}{1+|f-g|_k}$$

In this way $C^\infty(T)$ is a Frechet space and for every continuous linear form L there exists some integer k and a constant C so that

$$[L(f)|\leq C\cdot|f|_k \quad : \quad f\in C^\infty(T)$$

From this one gets the assertion by Dieudonné which identifies distributions on T and sequences $\{c_n\}$ with polynomial growth, i.e. any such sequence yields a distribution L defined by

$$L(f) = \sum_{-\infty}^{+\infty} c_n \cdot \widehat{f}(n)$$

This enable us to define boundary values of analytic functions $g(z)$ in the open unit disc which satisfy the growth condition

$$(1) \qquad |g(z)| \leq C \cdot (1 - |z|^{-m}$$

for some $m \geq 2$. Namely, (1) implies that the coefficients $\{c_n\}$ in the series expansion $g(z) = \sum c_n \cdot z^n$ have polynomial growth and therefore define a distribution on the unit circle as above.

A.1 Example. An illustration to the remarks by Dieudonné about the usefulness of expressing equations goes as follows. For every function $f(x)$ in the Schwartz class $\mathcal{S}$ on the real x-line there exists a limit of the integrals

$$(1) \qquad \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\sin(x + i\epsilon)}{\cos(x + i\epsilon)} \cdot f(x) \cdot dx$$

where the limit is taken as $\epsilon > 0$ decrease to zero. With $z = x + iy$ we consider the function in the upper half-plane

$$\frac{\sin(z)}{\cos(z)} = \frac{1}{i} \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} =$$

$$(2) \qquad i \cdot \frac{1 - e^{2iz}}{1 + e^{2iz}} = i + 2i \cdot \sum_{k=1}^{\infty} (-1)^k \cdot e^{2ikz}$$

The last sum converges as long as $z = x + i\epsilon$ with $\epsilon > 0$. Then, let $\phi(\xi)$ be a Schwartz function on the real ξ-line and set

$$f(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot \phi(\xi) \cdot d\xi$$

So here $\phi = \widehat{f}$ holds by Fourier's inversion formula. Now (2) implies that the limit in (1) is equal to

$$(*) \qquad i \cdot \phi(0) + 2i \cdot \sum_{k=1}^{\infty} \phi(-2k)$$

B. A bird's view upon distributions on the real line.

A substitute for $C^\infty(T)$ was introduced by Laurent Schwartz who defined the class $\mathcal{S}$ of C^∞-functions on the real line which together with all their derivatives have a rapid decay, i.e. when $f \in \mathcal{S}$ there exists for any pair of positive integers N, M a constant $C_{N,M}$ such that

$$|f^{(N)}(x)| \leq C_{N,M} \cdot (1 + |x|^{-M} \quad : \quad x \in \mathbf{R}$$

The usefulness of this class is that the Fourier transform

$$f \mapsto \widehat{f}(\xi) = \int e^{-ix\xi} f(x) dx$$

yields a 1-1 map from $\mathcal{S}$ to the corresponding $\mathcal{S}$-class on the real ξ-line. Moreover, one has *Fourier's inversion formula*:

$$f(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \widehat{f}(\xi) d\xi \quad : \quad \text{for all} \quad f \in \mathcal{S}$$

We review this in Section 1 and after we discuss the space $\mathcal{S}^*$ of *tempered distributions* which consist of continuous linear forms on $\mathcal{S}$. The construction of the topology on $\mathcal{S}$ and the Riesz representation formula imply that for every $\gamma \in \mathcal{S}^*$ there exists some integer $N \geq 0$ and a Riesz measure μ_N such that:

$$(*) \qquad \gamma(f) = \int_{-\infty}^{\infty} f^{(N)}(x) \cdot d\mu_N(x)$$

Moreover, the Riesz measure μ_N has temperate growth which means that there exists some $m \geq 0$ such that

$$(1) \qquad \int_{-\infty}^{\infty} (1 + |x|)^{-m} \cdot \big|d\mu_N(x)\big| < \infty$$

Let us remark that the given distribution can be represented by different pairs (N, μ_N) as N changes. For example, δ_0 is also represented by

$$f \mapsto \int_{-\infty}^{0} f'(x) \cdot dx$$

B.1 Boundary values of analytic functions. Using (*) every tempered distribution is recaptured as a sum of boundary values of analytic functions defined in the upper, resp. the lower half-plane. To obtain this one first extends Fourier's inversion formula to tempered distributions which implies that every $\gamma \in \mathcal{S}$ is the inverse Fourier transform of a tempered distribution on the real ξ-line with a similar representation as in (*) above. Suppose now that μ is a Riesz measure on the real ξ-line such that

$$(1) \qquad \int_{-\infty}^{\infty} (1 + |\xi|)^{-m} \cdot d\big|\mu\big|(\xi) < \infty$$

holds for some integer $m \geq 0$. If $z = x + iy$ belongs to the upper half-plane we set

$$G_+(z) = \int_0^{\infty} e^{iz \cdot \xi} \cdot d\mu(\xi) = \int_0^{\infty} e^{ix \cdot \xi - y \cdot \xi} \cdot d\mu(\xi)$$

This integral is absolutely convergent since $\xi \to e^{-y\xi}$ tends to zero more rapidly than any power of ξ as $\xi \to \infty$ when $y > 0$ is fixed. Hence $G_+(z)$ is an analytic function in $\mathfrak{Im}(z) > 0$. In a similar fashion we get the analytic function $G_-(z)$in the lower half-plane defined by

$$G_-(z) = \int_{-\infty}^{0} e^{iz \cdot \xi} \cdot d\mu(\xi)$$

In Section XX we prove that these two G-functions have boundary values on the real x-line given by tempered distributions. More precisely, there exists $\gamma_+ \in \mathcal{S}^*$ defined by

$$\gamma_+(f) = \lim_{y \to 0} \int_0^{\infty} G_+(x + iy) \cdot f(x) \cdot dx \quad : \quad f(x) \in \mathcal{S}$$

In a similar fashion we obtain the tempered distribution γ_- defined by

$$\gamma_-(f) = \lim_{y \to 0} \int_0^{\infty} G_-(x - iy) \cdot f(x) \cdot dx \quad : \quad f(x) \in \mathcal{S}$$

These constructions together with Fourier's inversion formula enable us to express every tempered distribution as a sum of tempered distributions which arise via boundary values of a pair of G-functions as above. Conversely, let $g(z)$ be an arbitrary bounded analytic function in the upper half-plane U_+. Then there exists the tempered distribution $\mathbf{b}g$ defined by

$$\mathbf{b}_g(f) = \lim_{y \to 0} \int_0^{\infty} g(x + iy) \cdot f(x) \cdot dx \quad : \quad f \in \mathcal{S}$$

Indeed, this follows since the bounded analytic function $g(z)$ has Fatou limits almost everywhere on the real x-line and if $g(x) = \lim_{y \to 0} g(x + iy)$ is the limit function then the tempered distribution

$\mathbf{b}g$ is defined by integrating $\mathcal{S}$-functions with respect to the L^1_{loc}-density $g(x) \cdot dx$. In Section XX we prove that an analytic function $G(z)$ in the upper half-plane has a boundary value given by a tempered distribution if and only if $G(z) = P(\partial)(g)(z)$ where $g(z)$ is a bounded analytic function in U_+ and $P(\partial) = \sum c_\nu \cdot \partial^\nu$ is a differential operator with constant coefficients. Using this together with Fourier's inversion formula one arrives at the following conclusive result:

B.2 Theorem. *Every tempered distribution on the real x-line can be expressed by a sum*

$$p(\partial)\mathbf{b}_{g_+} + q(\partial)\mathbf{b}_{g_-} + P(x)$$

where $P(x)$ is a polynomial and the g-functions are bounded analytic functions in U_+ and U_- while $p(\partial)$ and $q(\partial)$ are two ∂-polynomials.

Example. In the upper half-plane we choose the branch of $\log(z)$ which takes real values on $x > 0$. In the lower half-plane we take $g_(z)$ to be the branch of $\text{Log}(z)$ which also is real on $x > 0$. Then $g_+ - g_-$ is supported by $x \leq 0$ and the reader may verify that it is equal to the constant density $2\pi \cdot i$ on $x \leq 0$.

B.3 Parseval's formula. In XX we show that Fourier's inversion formula gives the equality:

$$\int |f(x)|^2 dx = \frac{1}{2\pi} \int |\widehat{f}(\xi)|^2 d\xi \quad : f \in \mathcal{S} \tag{1}$$

Next, on the real x-line we have the Hilbert space $L^2(\mathbf{R})$. Since $\mathcal{S}$ is a dense subspace the construction of the Fourier transform for tempered distributions will show that if $f(x) \in L^2(\mathbf{R})$, then its Fourier transform is an L^2-function on the ξ-line and the equality (1) holds. Suppose now that $f(x) \in L^2$ and that the Fourier transform $\widehat{f}(\xi)$ is supported by $\xi \geq 0$. In this case we can get an inverse transform defined in the upper half-plane by

$$F(z) = \frac{1}{2\pi} \cdot \int_0^\infty e^{iz\xi} \cdot \widehat{f}(\xi) \cdot d\xi \quad : \quad y > 0 \tag{2}$$

Fourier' inversion formula and the existence of Fatou limits gives the limit formula:

$$\lim_{y \to 0} F(x + iy) = f(x) \quad \text{for almost every} \quad x$$

At this stage we encounter an example where complex analysis enable us to get a result which goes beyond real variable methods. Namely, the L^2-function $f(x)$ whose Fourier transform by assumption is supported by $\xi \geq 0$ cannot be too small in average. More precisely, Carleman's formula from [XX] gives:

$$\int_{-\infty}^\infty \text{Log}^+ \frac{1}{|f(x)|} \cdot \frac{dx}{1 + x^2} < \infty \tag{*}$$

In XXX we prove a converse result from Carleman's article [Car:xx].

B.4 Theorem. *Let $f(x) \in L^2(\mathbf{R})$ be such that (*) holds. Then there exists an L^2-function $g(\xi)$ supported by $\xi \geq 0$ such that*

$$|f(x)| = \int_0^\infty e^{ix\xi} \cdot g(\xi) \cdot d\xi$$

B.5 Spectral gaps and Cauchy-Fourier formulas The support of a tempered distributions will be defined in Section XX. Suppose now that γ is a tempered distribution whose support has gaps, i.e. the complement is a union of disjoint open intervals $\{(a_\nu, b_\nu)\}$. We have a decomposition from Theorem 0.2. Set $G_+(\zeta) = p(\partial)(g_+)$ and $G_-(\zeta) = q(\partial)(g_-)$. The definition of the two boundary value distributions of $\mathbf{b}G_+$ and $\mathbf{b}G_-$ gives:

$$\gamma(\phi) = \lim_{\epsilon \to 0} \int_{-\infty}^\infty \big[G_+(x + i\epsilon) - G_-(x - i\epsilon)\big] \cdot \phi(x) + \int_{-\infty}^\infty P(x) \cdot \phi(x) \cdot dx$$

for every $\phi \in \mathcal{S}$. For each open interval (a_ν, b_ν) we shall learn that G_+ and G_* extend each other across this interval on the real x-axis. The conclusion is that there exists an analytic function $G^*(z)$ defined in the connected set $\mathbf{C} \setminus \cup [a_\nu, b_\nu]$ such that $G^* = G_+$ in U_+ and $G^* = G_-$ in U_-. Next, by Fourier's inversion formula a dense subset of $\mathcal{S}$-functions on the x-line are inverse Fourier transforms of test-functions on the ξ-line, i.e.

$$(*) \qquad \phi(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{ix\xi} \cdot \widehat{\phi}(\xi) \cdot d\xi \quad \text{where} \quad \widehat{\phi} \in C_0^\infty$$

When this holds $\phi(x)$ extends to an entire function $\phi(z)$. Using Cauchy's integral formula we prove in XX that for every ϕ as above one has the equality

$$(*) \qquad \gamma(\phi) = \int P(x) \cdot \phi(x) \cdot dx + \lim_{R \to \infty} \frac{1}{2\pi i} \cdot \int_{|z|=R} G^*(z) \cdot \phi(z) \cdot dz$$

where the limit is taken over those R which belong to $\cup\,(a_\nu, b_\nu)$. This formula will be applied in XX to establish certain uniqueness results for tempered distributions whose support have gaps.

B.6 A specific example. The definition of distributions is easy to grasp and some basic results are trivial consequences of basic measure theory. But even so, the beginner is often confronted with difficulites in order to manipulate distributions. Let us discuss an example. The Heaviside distribution H_+ is defined by

$$(1) \qquad \phi \mapsto \int_0^\infty \phi(x) \cdot dx$$

By definition the distribution derivative $\partial_x(H_+)$ is defined by

$$(2) \qquad \phi \mapsto -\int_0^\infty \phi'(x) \cdot dx = \phi(0)$$

Hence we have the equality $\partial_x(H_+) = \delta_0$. Next, it is obvious that the Fourier transform $\widehat{\delta}_0$ is the identity function on the real ξ-line. The *interchange rules* under the Fourier transform which we describe in Section 1 imply that

$$i\xi \cdot \widehat{H}_+ = 1$$

It follows that when $\xi \neq 0$ then $\widehat{H}_+$ is given by the density function $\frac{1}{i\xi}$. But there remains to determine this distribution on the whole real ξ-line. An obstacle is that the Fourier transform of H_+ is not directly defined since the function $x \mapsto e^{-ix\xi}$ is not integrable on $[0, +\infty)$. One way to overcome this is to restrict the integration and for every $A > 0$ define

$$(1) \qquad \widehat{h}_A(\xi) = \int_0^A e^{-ix\xi} \cdot dx = \frac{1 - e^{-iA\xi}}{i\xi}$$

If $g(\xi)$ is a test-function on the ξ-line whose support does not contain the origin then elementary Fourier analysis teaches that

$$(2) \qquad \lim_{A \to \infty} \int \frac{e^{-iA\xi}}{i\xi} \cdot g(\xi) d\xi = 0$$

From this we conclude that a passage to the limit as $A \to +\infty$ indeed implies that $\widehat{H}_+$ is equal to $\frac{1}{i\xi}$ outside $\xi = 0$. Next, if $g(\xi)$ is a test-function such that $g(0) = 0$ it is divisible by ξ and it follows again that (2) holds. Let us also consider a test-function g which is identically one on $[-1, 1]$. While we integrate outside $[-1, 1]$ we get a limit. There remains to analyze the limit

$$(3) \qquad \lim_{A \to \infty} \int_{-1}^{1} \frac{e^{-iA\xi}}{i\xi} \cdot d\xi$$

Here we can use residue calculus. Namely, with $\zeta = \xi + i\eta$ we have the entire function $\frac{1 - e^{-iA\zeta}}{i\zeta}$. It follows that (3) is equal to the line integral over the lower half circle:

$$(4) \qquad \int_\pi^{2\pi} \left(1 - e^{-iA \cdot e^{i\theta}}\right) \cdot d\theta$$

Now the absolute value $|e^{-iA\cdot e^{i\theta}}| = e^{A\sin\theta}$. Since the sine-function is ≤ 0 on $[\pi, 2\pi]$ we see that the limit as $A \to +\infty$ becomes π. Next, consider the boundary value distribution defined by $\frac{1}{i\zeta}$ from the lower half-plane which is defined by

$$g \mapsto \lim_{\epsilon \to 0} \frac{1}{i} \cdot \int \frac{g(\xi) - i\epsilon \cdot g'(\xi)}{\xi - i \cdot \epsilon} \cdot d\xi \tag{5}$$

Let μ denote this distribution. It is clear that $\mu = \frac{1}{i\xi}$ when $\xi \neq 0$. Next, let g as above be identically one on $[-1, 1]$. The contribution for the integral over $[-1, 1]$ for a given $\epsilon > 0$ becomes:

$$\frac{1}{i} \cdot \int_{-1}^{1} \frac{1}{\xi - i \cdot \epsilon} \cdot d\xi = \frac{1}{i} \cdot \cdot \log\big(\frac{1 - i\epsilon}{-1 - i\epsilon}\big) \tag{6}$$

Calculating the argument for the complex Log-function the reader can verify that the limit in (6) is equal to π as $\epsilon \to 0$. Using this we conclude that μ is equal to $\widehat{H}_+$, i.e. the Fourier transform of H_+ is the boundary value distribution of an analytic function in the lower half plane.

1 Tempered distributions on the real line.

The Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions on the real line is equipped with a topology defined by the sequence of norms $\{\rho_k\}_0^\infty$ where

$$\rho_k(f) = \max_{x \in \mathbf{R}} \, (1+|x|)^k \cdot \sum_{\nu=0}^{\nu=k} |f^{(\nu)}(x)|$$

To this sequence of ρ-norms one associates the distance function on $\mathcal{S}$ defined by

$$d(f,g) = \sum_{k=1}^\infty 2^{-k} \cdot \frac{\rho_k(f-g)}{1+\rho_k(f-g)}$$

One verifies easily that this metric above is complete, i.e. $\mathcal{S}$ is a Frechet space.

1.1 Exercise. Prove that there exists a constant C such that the following hold for each pair of integers $0 \leq \nu < k$ and ever $f \in \mathcal{S}$.

$$\max_x \, [1+|x|)^\nu \cdot |f^{(\nu)}(x)| \leq C^{k-\nu} \cdot \max_x \, [1+|x|)^k \cdot |f^{(k)}(x)|$$

1.2 An isomorphism. We have the bicontinuous map from the unit circle with the point 1 removed onto the real x-axis defined by

$$e^{i\theta} \mapsto i \cdot \frac{e^{i\theta}+1}{e^{i\theta}-1} \quad : 0 < \theta < 2\pi$$

If $f(x) \in \mathcal{S}$ we define the function

$$f_*(\theta) = f(i \cdot \frac{e^{i\theta}+1}{e^{i\theta}-1})$$

The rapid decay of f as $|x| \to +\infty$ implies that f_* extends to a C^∞-function on the whole unit circle which is flat at 1, i.e. the derivatives $f^\nu(1) = 0$ for all ν, or equivalently

$$\lim_{\theta \to 0} \frac{f(e^{i\theta})}{(e^{i\theta}-1)^n} = 0$$

for every positive integer n. Denote by $C_*^\infty(T)$ the class of C^∞-functions on T which are flat at $\theta = 0$. The discussion above shows that one has an isomorphism

$$\mathcal{S} \simeq C_*^\infty(T)$$

which also is topological, i.e. an isomorphism of Frechet spaces.

1.3 The dual space $\mathcal{S}^*$. Exercise 1.1 shows that to every continuous linear functional γ on $\mathcal{S}$ there exists some pair of integers $N, M \geq 0$ and a Riesz measure μ such that

$$(*) \qquad \gamma(f) = \int f^{(N)}(x) \cdot d\mu_N(x) \quad \text{where} \quad \int [1+|x|)^{-M} \cdot |d\mu_N(x)| < \infty$$

Remark. The integer N and the associated Riesz measure μ are not uniquely determined by γ. For example, let δ_0 be the Dirac distribution defined by $\delta_0(f) = f(0)$. It can also be defined by

$$\delta_0(f) = \int_0^\infty f'(x) \cdot dx$$

So with $N = 1$ we see that μ_1 is the Lebesgue measure concentrated to $x \geq 0$. A more involved case appears when we consider the distribution γ defined by a principal value:

$$\gamma(f) = \lim_{\epsilon \to 0} \text{PV} \int_{-\infty}^\infty \frac{f(x) \cdot dx}{x}$$

Here the reader may verify that

$$\gamma(f) = \int_{-\infty}^\infty \log|x| \cdot f'(x) \cdot dx$$

Thus, with $N = 1$ we get the locally integrable function $\log|x|$ and notice that we can take $M = 2$ above, i.e. the integral

$$\int \big|\log|x|\,\big| \cdot (1+|x|)^{-2}dx < \infty$$

1.4 The Fourier transform on $\mathcal{S}$.

If $f \in \mathcal{S}$ its Fourier transform is defined by

$$\widehat{f}(\xi) = \int e^{-ix\xi} \cdot f(x) \cdot dx$$

Since $f(x)$ is rapidly decreasing we can differentiate with respect to ξ and obtain

$$\partial_\xi^k(\hat{f}) = \int (-ix)^k \cdot e^{-ix\xi} \cdot f(x) \cdot dx \quad : \quad k \geq 1$$

Next, partial integration with respect to x gives

$$i \cdot \xi \cdot \hat{f}(\xi) = \int e^{-ix\xi} \cdot f'(x) \cdot dx$$

Let $\mathcal{F}$ denote the Fourier operator from $\mathcal{S}$ on the x-line to the corresponding $\mathcal{S}$-space on the ξ-line. Then we can express the formulas above as follows:

1.4.1 Proposition. *The following two interchange formulas hold:*

(ii) $$-i\partial_\xi \circ \mathcal{F} = \mathcal{F} \circ x \quad : \quad i\xi \circ \mathcal{F} = \mathcal{F} \circ \partial_x$$

1.4.2 Fourier's inversion formula. *Let $f(x) \in \mathcal{S}$ and set*

$$F(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

Then one has the equality

(*) $$f(x) = F(x)$$

Proof. First we establish the equality when $x = 0$. Notice that $f \mapsto F(0)$ is a linear functional on $\mathcal{S}$. Next, a function $f \in \mathcal{S}$ such that $f(0) = 0$ can be divided by x, i.e. $f = x \cdot \phi(x)$ with $\phi \in \mathcal{S}$. When this holds we have

(i) $$\widehat{f} = -\partial_\xi(\widehat{\phi})$$

The Fundamental Theorem of Calculus gives

$$\int_{-\infty}^{\infty} \partial_\xi(g) \cdot d\xi = 0$$

for all $g(\xi) \in \mathcal{S}$. Applied to $\widehat{\phi}$ and using (i) we conclude that

$$f(0) = 0 \implies F(0) = 0$$

But then the linear functional on the vector space $\mathcal{S}$ defined by $f \mapsto f(0)$ must be a constant times the functional $f \mapsto f(0)$. Hence there exists a constant c such that

$$f(0) = c \cdot \int \widehat{f}(\xi) \cdot d\xi$$

There remains to determine c. For this purpose we choose the special function

$$f(x) = e^{-x^2/2}$$

A verification which is left to the reader yields

$$\widehat{f}(\xi) = 2\pi \cdot e^{-\xi^2/2}$$

From this we deduce that $c = \frac{1}{2\pi}$.

The general case. With a fixed real number a and $f \in \mathcal{S}$ we set

$$f_a(x) = f(x+a)$$

It follows that

$$f(a) = f_a(0) = \frac{1}{2\pi} \int \widehat{f}_a(\xi) \cdot d\xi$$

Next, notice that a variable substitution gives:

$$\hat{f}_a(\xi) = \int f(x+a) \cdot e^{-ix\xi} \cdot dx = e^{ia\xi} \int f(x) \cdot e^{-ix\xi} \cdot dx = e^{ia\xi} \cdot \hat{f}(\xi)$$

From this we get the equality

$$f(a) = \frac{1}{2\pi} \int e^{ia\xi} \widehat{f}_a(\xi) \cdot d\xi$$

Since a is an arbitrary real number we have proved Fourier's inversion formula.

Exercise. If $n \geq 2$ we define the Schwarz class of rapidly decreasing C^∞-functions in $\mathbf{R}^n$. The Fourier transform is defined by

$$\widehat{f}(\xi) = \int e^{-i\langle \xi, x\rangle} \cdot f(x)dx$$

where the integration now is over $\mathbf{R}^n$. Fourier's inversion formula in dimension $n \geq 2$ amounts to show that

$$f(0) = \frac{1}{(2\pi)^n} \cdot \int \widehat{f}(\xi) \cdot d\xi$$

This formula can be proved via the fundamental theorem of calculus by an induction over n. Let us give the details when $n = 2$ where (x, y) are the coordinates in $\mathbf{R}^2$. Let $f(x, y)$ be given in $\mathcal{S}(\mathbf{R}^2)$. Define the partial Fourier transform

$$f^*(\xi, y) = \int e^{-ix\xi} f(x, y)dx$$

With ξ kept fixed we notice that the Fourier transform of the function $y \mapsto f^*(\xi, y)$ is equal to $\hat{f}(\xi, \eta)$. The 1-dimensional case applied to the y-variable gives for every ξ:

(i) $$f^*(\xi, 0) = \frac{1}{2\pi} \int \widehat{f}(\xi, \eta) \cdot d\eta$$

Next, the 1-variable case is also applied to the x-variable which gives

(ii) $$f(0,0) = \frac{1}{2\pi} \int f^*(\xi, 0)d\xi$$

Now (i-ii) give the required formula

$$f(0,0) = \frac{1}{(2\pi)^2} \iint \widehat{f}(\xi, \eta) \cdot d\xi d\eta$$

1.5 The Fourier transform of tempered distributions.

Let $\gamma \in \mathcal{S}^*$ be given. Since the Fourier transform on $\mathcal{S}$ is bijective and bi-continuous with respect to the Frechet metric there exists a unique tempered distribution $\widehat{\gamma}$ on the real ξ-line defined on functions $g(\xi) \in \mathcal{S}$ by:

$$(*) \qquad \widehat{\gamma}(g) = \gamma(g_*) \quad : \quad g_*(x) = \int e^{-ix\xi} g(\xi) d\xi \quad :$$

Remark. Let $f(x) \in \mathcal{S}$ and denote by γ_f the tempered distribution defined by the density $f(x)dx$. Then (*) gives

$$\widehat{\gamma}_f(g) = \iint f(x) \cdot e^{-ix\xi} g(\xi) \cdot dx d\xi = \int \hat{f}(\xi) \cdot g(\xi) \cdot d\xi$$

Thus, under the inclusion $\mathcal{S} \cdot dx \subset \mathcal{S}^*$, the construction of the Fourier transform of functions in $\mathcal{S}$ extend to tempered distributions via (*).

Examples. Let a be a real number and δ_a the unit point mass at a. This is a tempered distribution on the real x-line. By definition

$$\widehat{\delta}_a(g) = g_*(a) = \int e^{-ia\xi} \cdot g(\xi) \cdot d\xi$$

Hence the Fourier transform is the exponential density function $e^{-ia\xi} \cdot d\xi$.

1.5.1 The Fourier transform of H_+. On the real x-line we have the Heaviside distribution H_+ defined by the density 1 when $x \geq 0$ and zero if $x < 0$. To find its Fourier transform we shall perform certain limits. To begin with, for every large real number N we have the distribution on the x-line defined by

$$\mu_N(f) = \int_0^N f(x) \cdot dx$$

Its Fourier transform becomes

$$\widehat{\mu}_N(\xi) = \int_0^N e^{-ix\xi} \cdot dx = \frac{1 - e^{-iN\xi}}{i\xi}$$

It is clear that

$$\lim_{N \to \infty} \mu_N(f) = H_+(f)$$

hold for every $f \in \mathcal{S}$. From this it follows that $\{\widehat{\mu}_N\}$ converges weakly, i.e. for each $g(\xi) \in \mathcal{S}$ there exists

$$\text{(ii)} \qquad \lim_{N \to \infty} \int_{-\infty}^{\infty} \frac{1 - e^{-iN\xi}}{i\xi} \cdot g(\xi) \cdot d\xi$$

and by Remark 1.5.0 the limit is equal to $\widehat{H}_+(g)$. We can also find $\widehat{H}_+$ by other limit formulas. Namely, if $\epsilon > 0$ we define the distribution γ_ϵ on the x-line by

$$\gamma_\epsilon(f) = \int_{-\infty}^0 e^{\epsilon x} f(x) dx$$

Here we find that

$$\widehat{\gamma}_\epsilon(\xi) = \int_{-\infty}^0 e^{\epsilon x - i\xi x} dx = \frac{1}{\epsilon - i\xi} = \frac{i}{\xi + i\epsilon}$$

This gives the limit formula

$$\text{(ii)} \qquad \widehat{H}_+(g) = \lim_{\epsilon \to 0} i \cdot \int \frac{g(\xi) \cdot d\xi}{\xi + i\epsilon}$$

Hence the tempered distribution $\widehat{H}_+$ can be found by different limit formulas. Both (i) and (ii) are useful, though (ii) is the most frequent and we remark that the limit in (ii) means that one has to consider a so called principal value in order to overcome the obstacle that the function $\frac{1}{\xi}$ is not locally integrable close to zero. We refer to § xx for a detailed account about principal value integrals and how they give rise to distributions.

1.6 Tempered distributions in $\mathbf{R}^2$

The 1-dimensional case extends more or less verbatim to the case $n = 2$. Here one has the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions $g(x, y)$ of the two real variables x and y. The dual $\mathcal{S}^*$ consists of tempered distributions and every such distribution γ is represented as a finite sum

$$\gamma(f) = \iint_{\mathbf{R}^2} g^{(\alpha)}(x, y) \cdot d\mu_\alpha(x, y)$$

where $\{\mu_\alpha\}$ is a finite family of Riesz measures and α are multi-indices which yield higher order derivatives of g. Finally, there exists an integer N such that

$$\iint_{\mathbf{R}^2} (1 + x^2 + y^2)^N \cdot |d\mu_\alpha(x, y)| < \infty$$

hold for every α. Fourier's inversion formula was already established in 1.4.1 and leads to the construction of Fourier transform of tempered distributions. Let us give some examples.

Identifying $\mathbf{R}^2$ with $\mathbf{C}$ we have the first order differential operators $\partial = \frac{1}{2}[\partial_x - i\partial_y)$ and $\bar{\partial} = \frac{1}{2}[\partial_x + i\partial_y)$. Next, the locally integrable function $g = \frac{1}{z}$ yields a tempered distribution. The calculus for Fourier transforms give

(i) $$\partial_\xi(\hat{g}) = -i \cdot \widehat{xg}(\xi, \eta) \quad \text{and} \quad \partial_\eta(\hat{g}) = -i \cdot \widehat{yg}(\xi, \eta)$$

In the (ξ, η)-space we have the complex variable $\zeta = \xi + i\eta$ and $\bar{\partial}_\zeta = \frac{1}{2}(\partial_\xi + i\partial_\eta)$. We can express (i) by the equation:

(ii) $$i \cdot \bar{\partial}_\zeta(\hat{g}) = \text{The Fourier transform of} \quad (x + iy)g = 1$$

Now we notice that the Fourier transform of the identity function 1 in the (x, y)-space becomes

$$\widehat{1} = \delta_0$$

where δ_0 is the Dirac measure at the origin in the (ξ, η)-space. Hence (ii) means that

$$\bar{\partial}_\zeta(\widehat{g}) = \frac{1}{i} \cdot \delta_0$$

in particular $\widehat{g}$ is satisfies the Cacuhy-Riemann equation outside the origin so we expect that it is given by a holomorphic density in the punctured complex ζ-plane. This is indeed true and the precise formula is:

1.6.1 Theorem. *The Fourier transform of* $\frac{1}{x+iy}$ *is equal to* $\frac{2\pi i}{\xi + i\eta}$.

Exercise. Prove the formula above. See also § xx for further comments where the fact that g from the start is a homogenous distribution can be used to obtain Theorem 1.6.1.

2. Boundary values of analytic functions

Introduction. We are going to construct boundary values of analytic functions $f(z)$ defined in an open rectangle $\{-a < x < a \quad : 0 < y < b\}$. One says that f has moderate - or temperate - growth when the real axis is approached if there exists some integer $N \geq 0$ and a constant C such that

$$(*) \qquad |f(x+iy)| \leq C \cdot y^{-N}$$

When (*) holds it turns out that f has a boundary value given by a distribution $\mathbf{b}(f)$ defined on the interval (a, b) of the real x-axis. Moreover, we will show that the map $f \mapsto \mathbf{b}(f)$ commutes with derivations, i.e. if $\partial_z = d/dz$ while ∂_x is the derivation on the real x-axis then

$$\mathbf{b}(\partial_z(f)) = \partial_x(\mathbf{b}(f))$$

where the right hand side is a distribution derivative.

2.1 The construction. Let $f(z)$ be analytic in a rectangle

$$\square = \{(x+iy\colon \quad 0 < x < A \quad : 0 < y < B\}$$

where A, B are positive constants and f satisfies the growth condition

$$|f(x+iy)| \leq C \cdot y^{-N} \quad : \quad x+iy \in \square$$

where C is a constant and N some non-negative integer. Under this condition f has a boundary value as $y \to 0$ expressed by a distribution acting on test-functions $g(x)$ with compact support in $(0, A)$. To achieve this we extend a test-function $g(x)$ in such a way that $\bar{\partial}$-derivatives are small when $y \to 0$.

2.2 Small $\bar{\partial}$-extensions Recall that $\bar{\partial} = \frac{1}{2}[\partial_x + i\partial_y]$. Given a positive integer N and some $g(x) \in C_0^\infty(0, A)$ we construct a function $G_N(x+iy)$ as follows:

$$G_N(x+iy) = g(x) + \sum_{\nu=1}^{\nu=N} i^\nu \cdot \frac{g^{(\nu)}(x) \cdot y^\nu}{\nu\,!}$$

It follows that one has the equality

$$2 \cdot \bar{\partial}(G_N)(x+iy) =$$

$$(*) \qquad \sum_{\nu=0}^{N} i^\nu \cdot \frac{g^{(\nu+1}(x) \cdot y^\nu}{\nu\,!} \sum_{\nu=1}^{N} i^{\nu+1} \cdot \frac{g^{(\nu}(x) \cdot y^{\nu-1}}{(\nu-1)\,!} = i^N \cdot \frac{g^{(N+1)}(x) \cdot y^N}{N\,!}$$

2.3 The distribution $\mathfrak{b}(f)$. Let $f(z)$ satisfy the growth condition above. Given $g \in C_0^\infty(0, A)$ we construct G_N. With $0 < \epsilon < b < B$ we apply Stokes formula and obtain:

$$\int_0^A G_N(x+i\epsilon)f(x+i\epsilon)dx =$$

$$\int_0^A G_N(x+ib)f(x+ib)dx + 2i \cdot \int_0^A \int_\epsilon^b \bar{\partial}(G_N)(x+iy)f(x+iy)dxdy$$

The growth condition on f and (*) imply that the absolute value of the double integral is majorized by

$$(**) \qquad \frac{Cb}{N\,!} \cdot \int_0^A |g^{(N+1)}(x)| \cdot dx$$

Since this holds for any $\epsilon > 0$ we can pass to the limit $\epsilon \to 0$ while the absolutely integrable double integral is computed. Hence we have proved

2.4 Proposition *There exists the limit*

$$\lim_{\epsilon \to 0} \int_0^A G_N(x+i\epsilon)f(x+i\epsilon)dx$$

Moreover, the limit is equal to

$$\int_0^A G_N(x+ib)f(x+ib)dx + 2i \cdot \int_0^A \int_0^b \bar{\partial}(G_N)(x+iy)f(x+iy)dxdy \quad : \quad 0 < b < B$$

where the absolute value of the double integral is majorized by (**) above.

2.5 Definition. *The limit integrals above yield a linear functional on test-function which gives a distribution of order $\leq N+1$. It is denoted by $\mathfrak{b}(f)$ and called the boundary value distribution of f.*

2.6 Use of primitive functions. Starting with $f \in \mathcal{O}(\square)$ we construct primitive functions which behave better as we approach the real x-axis. For example, fix a point $p = ia$ with $a > 0$ and set

$$F(z) = \int_{ia}^z f(\zeta)d\zeta$$

If $|f(x+iy)| \leq C \cdot y^{-N}$ for some $N \geq 2$ we get $|F(i+iy) \leq C_1 \cdot y^{-N+1}$ for another constant C_1. In the case $N = 1$ we get $[F(x+iy)| \leq C_1 \cdot \text{Log}\,\frac{1}{|y|}$. So by choosing N sufficiently large and taking the N:th order primitive F_N of f it has even continuous boundary values and $\mathfrak{b}(F_N)$ is just the density function $F_N(x)$. Then one can take distribution derivatives on the real x-line and get

$$\mathfrak{b}(f) = \frac{d^N}{dx^N}(\mathfrak{b}(F_N(x))$$

So this is an alternative procedure to define $\mathfrak{b}(f)$ without small $\bar{\partial}$-extensions. Both methods have their advantage depending on the situation at hand.

2.7 Example. Let $f(z) = \log z$ where the single valued branch is chosen in $\mathfrak{Im}(z) > 0$ so that the argument is between 0 and π. Then $\mathfrak{b}(f)$ is the distribution defined by the density $\log x$ when $x > 0$ and if $x < 0$ by

$$\log |x| + \pi \cdot i$$

The complex derivative of f is $\frac{1}{z}$. Here one finds that $\mathfrak{b}(\frac{1}{z})$ is the distribution defined by

$$g \mapsto \lim_{\epsilon| \to 0} \int \frac{g(x)dx}{x + i\epsilon} = \int \frac{(g(x) - g(0)) \cdot dx}{x} + \pi i \cdot g(0) \tag{1}$$

Above it is an instructive exercise to take the distribution derivative of $\mathfrak{b}(\log z)$ and verify that it is equal to the distribution (1).

2.7 The reflection principle. Let

$$\square_- = \{(x + iy\colon \quad 0 < x < A \quad : \quad -B < y < 0\}$$

be an opposed rectangle in the lower half-plane and let $h \in \mathcal{O}(\square_-)$ satisfy the moderate growth condition. We construct $\mathfrak{b}(h)$ in the same way as above. Then one has

2.8 Theorem. *Let $f \in \mathcal{O}(\square)$ and $h \in \mathcal{O}(\square_-)$ be a pair such that $\mathfrak{b}(f) = \mathfrak{b}(h)$ holds as distributions. Then they are analytic continuations of each other, i.e. there exists an analytic function Φ defined in $\{-B < y < B\colon \ -A < x < A\}$ such that $\Phi = f$ in $\square$ and $\Phi = h$ in $\square_-$.*

Proof. We choose a large N so that th N:th order primitive functions F_N and H_N both extend continuously to the real x-axis. The equality $\mathfrak{b}(f) = \mathfrak{b}(h)$ entails that

$$\frac{d^N}{dx^N}(\mathfrak{b}(F_N) - \mathfrak{b}(H_N) = 0$$

Now a distribution on the real x-line whose N:th order derivative is zero is a polynomial $p(x)$ of degree $\leq N - 1$. So the pair of continuous functions $F(x)$ and $H(x)$ satisfy

$$H(x) = F(x) + p(x)$$

Hence the analytic functions $F(z) + p(z)$ and $H(z)$ have a common continuous boundary value function so by the Schwarz reflection principle they are analytic continuations of each other. Let $G(z)$ be the resulting analytic function defined in the open domain where $-B < y < B$ now holds. Its N:th order complex derivative is also analytic in this domain and equal to f in $\square_+$ and to h in $\square_-$. This proves Theorem 2.8 with $\Phi = G^{(N)}$.

2.9 Extension to several variables. With two real variables (x_1, x_2) one considers the 2-dimensional complex space (z_1, z_2) with $z_k = x_k + iy_k$. In the real y-space we consider an open truncated cone:

$$\mathcal{K} = \{(y_1, y_2) \quad y_2 > M \cdot |y_1|\} \cap \{0 < y_1^2 + y_2^2 < \delta^2\}$$

where δ, M are positive constants. Put

$$\square = \{(x_1, x_2 \quad -1 < x_1, x_2 < 1\}$$

In $\mathbf{C}^2$ we get the open set $\mathcal{T} = \square + i \cdot \mathcal{K}$. called a truncated tube. Let $f(z_1, z_2)$ be a bounded analytic function in $\mathcal{T}$. To each point $(y_1^*, y_2^*) \in \mathcal{K}$ the 1-dimensional results show that if $g(x_1, x_2)$ is a test-function in $\square$ then there exists a limit

$$\lim_{\epsilon \to 0} \int_\square g(x_1, x_2) \cdot f(x_1 + i\epsilon y_1^*, x_2 + i\epsilon y_2^*) \cdot dx_1 dx_2 \tag{*}$$

Moreover, (*) is independent of the point $y^* = (y_1^*, y_2^*)$ and we obtain a distribution in $\square$ denoted by $\mathbf{b}f$. In the case above where f was bounded the resulting distribution has order zero, i.e. expressed by a Riesz measure μ_f in $\square$. The reader may notice the close interplay to the 1-dimensional case and we remark that the existence of a limit distribution can be established by regarding Fatou limits. Moreover, using the Brothers Riesz theorem one proves that the measure μ_f is absolutely continuous with respect to the Lebesgue measure in the 2-dimensional (x_1, x_2)-space.

Next, the assumption that f is bounded can be relaxed, i.e. assume onloy that there is some integer $m \geq 1$ such that

$$|f(x_1 + iy_1, x_2 + iy_2)| \leq C.(y_1^2 + y_2^2)^{-m/2}$$

hold for some constant C_m and all points in the tube. To get the distribution $\mathbf{b}f$ one employs small $\bar{\partial}$-extension of test-functions g which are constructed as in the case $n = 1$. Thus, with $N \geq 1$ we set

$$G_N(x+iy) = \sum i^{k+j} \cdot \frac{\partial^{k+j}(g)}{\partial^k x_1 \partial^j x_2}(x) \cdot \frac{y_1^k \cdot y_2^j}{k! \cdot j!}$$

Using Stokes Theorem one verifies that if $f(z)$ has temperate growth then there exists a distribution $\mathbf{b}_f$ defined on test-functions $g(x)$ by

(**) $$\mathbf{b}_f(g) = \lim_{\epsilon \to 0} \int_\Box G_m(x_1, x_2) \cdot f(x_1 + i\epsilon y_1^*, x_2 + i\epsilon y_2^*) \cdot dx_1 dx_2$$

where the limit does not depend upon the chosen point $y^* \in \mathcal{K}$. The conclusion is that if $\mathcal{O}(\mathcal{T})_{\text{temp}}$ denotes the space of all analytic functions $f(z)$ in the tube satisfying (**) above for some integer m, then $f \mapsto \mathbf{b}_f$ yields a map to a space of distributions in $\Box$.

2.9 Analytic wave front sets. The range of the map above can be described using the notion of analytic wave front sets of distributions. It would bring us too far to give the details of the construction of $\mathrm{WF}_A(\mu)$ for a general distribution in $\mathbf{R}^2$. Apart from the original article [Hö:xx] the reader may consult Chapter X in [Hörmander] or the material in [Björk:xx: page. xxx-xx] for an account. about analytic wave front sets using cut-off functions and certain decay properties of Fourier transform.

2.10 The Schwarz reflection in two variables. Let $\mathcal{K}^*$ be the opposed cone defined by

$$K^* = \{(y_1, y_2) \quad y_2 < -M \cdot |y_1|\} \cap \{0 < y_1^2 + y_2^2 < \delta 2\}$$

If $\phi(z)$ is a tempered analytic function in the corresponding truncated tube domain T^*) we obtain the distribution ϕ_g. Suppose we have an equality of distributions:

(1) $$\mathbf{b}_\phi = \mathbf{b}_f$$

Then there exists an analytic function $\Psi(z)$ defined in a complex neighborhood of the real $\Box$ whose restriction to $\mathcal{T}$ is f while $\Psi|\mathcal{T}^* = g$. A proof can either be established by scrutinizing the 1-dimensional case carefully. But the efficient method is to study the Fourier transforms of the two distributions $\mathbf{b}_f$ and $\mathbf{b}_\phi$ where one shows that (1) entails that we obtain a common distribution whose analytic wave front set is empty and therefore defined by a real-analytic density on $\Box$ which extends to be holomorphic in a small complex neighborhood and therefore yields the complex analytic extension of the two given functions $f(z)$ and $\phi(z)$. Details about this procedure which holds for any dimension is given in Chapter XX from [Björk].

3. Fourier-Carleman transforms

Introduction. In this section we describe how the ordinary Fourier transform is obtained from a pair of analytic functions defined in the upper - resp. the lower half plane. The idea is that a Fourier transform

$$\widehat{g}(\xi) = \int e^{-ix\xi} \cdot g(x)dx$$

becomes a sum when we integrate over $(-\infty, 0)$ respectively $(0, +\infty)$. For each of these "halves" of the Fourier transform we get analytic functions $G_+(\zeta$ and $G_-(\zeta)$ in the upper, resp. the lower half plane of the complex ζ-plane where $\zeta = \xi + i\eta$. After one can take their boundary values. This construction is especially useful when the support of the Fourier transform has gaps, i.e. when its complement consists of many open intervals.

3.1 The functions G_+ and G_-. Consider a continuous complex-valued function $g(x)$ defined on the real x-line which is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |g(x)|dx < \infty$$

We obtain analytic functions $G_+(\zeta)$ and $G_-(\zeta)$ defined in the upper, respectively the lower half-plane of the complex ζ-plane where $\zeta = \xi + i\eta$.

$$(*) \qquad G_+(\zeta) = \int_{-\infty}^{0} g(x)e^{-i\zeta x}dx \quad : \quad G_-(\zeta) = \int_{0}^{\infty} g(x)e^{-i\zeta x}dx$$

With $\zeta = \xi + i\eta$ we have $|e^{-i\zeta x}| = e^{\eta x}$. This number is < 1 when $x < 0$ and $\eta > 0$, and vice versa. We conclude that $G_+(\zeta)$ is analytic in $\mathfrak{Im}(\zeta) > 0$ while $G_-(\zeta)$ is analytic in $\mathfrak{Im}(\zeta) < 0$. Since $|g|$ is integrable we see that G_+ extends continuously to the closed upper half plane where

$$G_+(\xi) = \int_{-\infty}^{0} g(x)e^{-i\xi x}dx$$

Similarly G_- extends to $\mathfrak{Im}(\zeta) \leq 0$. Notice that the sum

$$G_+(\xi) + G_-(\xi) = \int_{-\infty}^{\infty} g(x)e^{-i\xi x}dx = \widehat{g}(\xi) = \text{ The usual Fourier transform}$$

3.2 The case when $\widehat{g}$ has compact support. Suppose this holds. Hence we have two intervals $(-\infty, a)$ and $(b, +\infty)$ and some family of bounded interval $\{(\alpha_\nu, \beta_\nu)\}$ whose union is the open complement of $\text{Supp}(\widehat{g})$. On each such interval $G_+(\xi) + G_-(\xi)$ is identically zero. Hence we get

3.3 Theorem *Put $\Omega = \mathbf{C} \setminus Supp(\widehat{g})$. Then there exists a analytic function $\mathcal{G} \in \mathcal{O}(\Omega)$ such that $\mathcal{G} = G_+$ in the upper half plane and $\mathcal{G} = -G_-$ in the lower half plane.*

Consider some $R > 0$ which is chosen so large that the open disc D_R centered at the origin in the ζ-space contains the compact set $\text{Supp}(\widehat{g})$. For each real x the function $e^{ix\zeta}\mathcal{G}(\zeta)$ is again analytic in Ω. In particular we can consider the line integral

$$\text{(i)} \qquad J = \int_{|\zeta|=R} e^{ix\zeta} \cdot \mathcal{G}(\zeta) \cdot d\zeta$$

Keeping orientations in mind, the construction of $\mathcal{G}$ shows that

$$J = \int_{-R}^{R} e^{ix\xi}G_+(\xi)d\xi + \int_{-R}^{R} e^{ix\xi}G_-(\xi)d\xi = \int_{-R}^{R} e^{ix\xi} \cdot \widehat{g}(\xi) \cdot d\xi$$

The last integral appears in *Fourier's inversion formula.* Hence we arrive at the following

3.4 Theorem *Let $g(x)$ belong to $L^1(\mathbf{R})$ and assume that its Fourier transform has compact support. Then*

$$g(x) = \frac{1}{2\pi} \cdot \int_{|\zeta|=R} e^{ix\zeta} \cdot G(\zeta) \cdot d\zeta$$

for all R such that $Supp(\widehat{g}) \subset [-R, R]$.

3.5 Remark. The condition that $\widehat{g}$ has compact support is restrictive since it implies that $g(x)$ extends to an entire function in the complex z-plane. In order to relax this we consider the case where we only assume that the complement of $\text{Supp}(\widehat{g})$ contains some open intervals both on positive and the negative real ξ-axis. In fact, with $\Omega = C \setminus \text{Supp}(\widehat{g})$ it is still holds that $\mathcal{G} \in \mathcal{O}(\Omega)$ and if $R > 0$ is an positive number such that R and $-R$ are outside the support of $\hat{g}$, then we have the equality

$$(*) \qquad J(R) = \frac{1}{2\pi} \cdot \int_{|\zeta|=R} e^{ix\zeta} \mathcal{G}(\zeta) \cdot d\zeta = \frac{1}{2\pi} \cdot \int_{-R}^{R} e^{ix\xi} \cdot \widehat{g}(\xi) d\xi$$

If we *assume* that $\widehat{g}$ is absolutely integrable, then the last term almost recaptures $g(x)$ since Fourier's inversion formula gives

$$g(x) = \lim_{R \to \infty} \frac{1}{2\pi} \cdot \int_{-R}^{R} e^{ix\xi} \cdot \widehat{g}(\xi) d\xi$$

Hence, when $\widehat{g} \in L^1(\mathbf{R})$ and "gaps" occur in the support of $\widehat{g}$, i.e. if there exists some sequence $\{R_\nu\}$ where R_ν and $-R_\nu$ both are outside $\text{Supp}(\widehat{g})$, then one has the limit formula:

$$(**) \qquad g(x) = \lim_{\nu \to \infty} \frac{1}{2\pi} \cdot \int_{|\zeta|=R_\nu} e^{ix\zeta} \cdot \mathcal{G}(\zeta) \cdot d\zeta$$

3.6 Further extensions

Above we assumed that $g(x)$ was absolutely integrable which implies that $\widehat{g}$ is a bounded and continuous function. Suppose now that $g(x)$ is a continuous function such that

$$\int_{-\infty}^{\infty} \frac{|g(x)|}{1 + |x|^N} \cdot x < \infty$$

holds for some positive integer N. We can still define the two analytic functions G_+ and G_-. Consider the behaviour of G_+ as ζ approaches the real ξ-line. We have by definition

$$G_+(\xi + i\eta) = \int_{-\infty}^{0} g(x)^{-i\xi x} e^{\eta x} dx$$

Taking absolute values we get for $\eta > 0$:

$$|G_+(\xi + i\eta)| \le \int_{-\infty}^{0} \frac{|g(x)|}{(1 + |x|)^N} \cdot (1 + |x|)^N \cdot e^{\eta x} dx$$

One variable calculus shows that if $\alpha > 0$ and then the function

$$t \mapsto (1 + t)^N e^{-\alpha t} : t \ge 0$$

takes its maximum when $1 + t = \frac{N}{\alpha}$ so the maximum value over $[0, +\infty)$ is therefore $\le \frac{N^N}{\alpha^N}$. Apply this with $\eta > 0$ and $x < 0$ above. Then we see that there exists a constant C such that

$$(*) \qquad |G_+(\xi+i\eta)| \leq \frac{C}{\eta^N} \cdot \int_{-\infty}^{\infty} \frac{|g(x)|dx}{1+|x|^N}$$

Hence G_+ has temperate growth as $\eta \to 0$ so its boundary value distribution $\mathfrak{b}(G_+)$ exists. Similarly we find the boundary value distribution $\mathfrak{b}(G_-)$. The Fourier transform of $g(x)$, regarded as a *tempered distribution* is equal to $\mathfrak{b}(G_+) + \mathfrak{b}(G_-)$. Again, if $\text{Supp}(\widehat{g})$ has gaps we can proceed as in 3.5 and construct the complex line integral

$$J(R) = \frac{1}{2\pi} \cdot \int_{|\zeta|=R} e^{ix\zeta} \mathcal{G}(\zeta) \cdot d\zeta$$

for those values of R such that $-R$ and R are outside the support of $\hat{g}$. If this can be achieved for large R-values one recaptures g exactly as in (**) from 3.5. when $\hat{g}$ is in L^1.

3.7 Use of Fourier's inversion formula.

Above we regarded the Fourier transform. Using Fourier's inversion formula we can use a similar device to recapture a function on the x-line. Consider the following situation: Let $f(x)$ be a function in the Schwartz class. Assume that its Fourier transform $\widehat{f}(\xi)$ vanishes on some open interval $a < \xi < b$.

Set $c = \frac{a+b}{2}$ and $g(x) = e^{ixc} \cdot f(x)$. Then we get

$$\widehat{g}(\xi) = \widehat{f}(\xi + c)$$

Here $\widehat{g}$ is zero on an interval centered at $\xi = 0$ and we may therefore assume from the start that $\widehat{f}$ is zero on some interval $(-A, A)$. Now we define two analytic functions:

$$F_+(x+iy) = \frac{1}{2\pi} \cdot \int_A^\infty e^{(x+iy)i\xi} \cdot \widehat{f}(\xi) \cdot d\xi \quad : \quad F_-(x+iy) = -\frac{1}{2\pi} \cdot \int_{-\infty}^{-A} e^{(x+iy)i\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

When $y = 0$ we see that

$$(*) \qquad F_+(x) - F_-(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{ix\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

By Fourier's inversion formula the last integral is $f(x)$, i.e. here we used that $\widehat{f} = 0$ on $(-A, A)$ so we can ignore the integral over this interval. Hence $f(x)$ is represented as a difference of two analytic functions defined in the upper and the lower half-plane respectively. Let us also notice that we have the estimates:

$$(1) \qquad \big|F_+(x+iy)\big| \leq \int_A^\infty e^{-y\xi} \cdot |\widehat{f}(\xi)| \cdot d\xi \leq e^{-Ay} \cdot \int_A^\infty |\widehat{f}(\xi)| \cdot d\xi$$

$$(2) \qquad |F_-(x+iy)| \leq e^{-A|y|} \cdot \int_{-\infty}^{-A} |\widehat{f}(\xi)| \cdot d\xi$$

Suppose now that $f(x)$ also is zero on some interval, say $a < x < b$. This means that the two analytic functions $F_+(z)$ and $F_-(z)$ agree on this interval and by the Schwarz reflection principle they are analytic continuations of each other. Hence, we get the following:

3.8 Proposition. *Assume that Supp(f) is a proper subset of* $\mathbf{R}$ *and consider the open complement*

$$U = \cup\,(a_\nu, b_\nu)$$

where $\{(a_\nu, b_\nu)\}$ is a family of disjoint open intervals. Then there exists an analytic function $\mathcal{F}(z)$ defined in the connected set $\mathbf{C} \setminus$ Supp(f) where

$$\mathcal{F}(z) = F_+(z) \quad : z \in U_+ \quad : \quad \mathcal{F}(z) = F_-(z) \quad : z \in U_*$$

Remark. We refer to $\mathcal{F}$ as the *inverse Fourier-Carleman transform* of $\widehat{f}(\xi)$.

3.9 A local estimate. Consider an open interval (a_ν, b_ν) in U. Set

$$r = \frac{b_\nu - a_\nu}{2} \quad : \quad c = \frac{a_\nu + b_\nu}{2}$$

Hence the open disc $D_r(c)$ stays in the open set $\Omega = \mathbf{C} \setminus \text{Supp}(f)$. Next, when $0 < \phi < \pi$ we have

$$\mathcal{F}(c + re^{i\phi}) = \frac{1}{2\pi} \cdot \int_A^\infty e^{(c+r\cos\phi)i\xi - r\sin\phi\cdot\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

Since $\left|e^{(c+r\cos\phi)i\xi}\right| = 1$ the triangle inequality gives

$$|\mathcal{F}(c + re^{i\phi})| \leq \frac{1}{2\pi} \cdot e^{-rA\cdot\sin\phi} \cdot \int_A^\infty \left|\widehat{f}(\xi)\right| \cdot d\xi$$

When $-\pi \leq \phi \leq 0$ we get a similar estimate where we now use that $\mathcal{F} = F_-$. Introducing the L^1-norm of $\widehat{f}$ we conclude

3.10 Proposition. *One has the inequality*

$$\mathcal{F}(c + re^{i\phi}\Big| \leq \frac{||\widehat{f}||_1}{2\pi} \cdot e^{-Ar\cdot|\sin\phi|} \quad : \quad 0 \leq \phi \leq 2\pi$$

3.11 The subharmonic function $U = \mathbf{Log}\,|\mathcal{F}|$. Replacing f by $c \cdot f$ with some constant c we assume that $|\text{Log}\,\frac{||\widehat{f}||_1}{2\pi}| \leq 1$. Then Proposition 3.10 gives the inequality

$$U(c + re^{i\phi}) \leq -Ar \cdot \sin|\phi| \quad : \quad -\pi \leq \phi \leq \pi$$

Since U is subharmonic we can apply Harnack's inequality from XX and conclude

3.12 Proposition. *One has the inequality*

$$U(x) \leq -\frac{Ar}{2\pi}\cdot \quad : \quad c - \frac{r}{2} \leq x \leq c + \frac{r}{2}$$

3.13 A vanishing theorem·

The previous results give a vanishing theorem. Namely, let f be as above. We get the subharmonic function U and notice that in addition to the inequality in Proposition 3.12 which is valid for every open interval of the x-axis outside the support of f, we also have the estimate from Proposition 3.10 which gives

$$(*) \qquad U(x + iy) \leq -A|y|$$

where we for simplicity assumed that $||\widehat{f}||_1 \leq 1$. Now we can apply the general result from XX and conclude that if f is not identically zero then

$$(**) \qquad \sum (b_\nu - a_\nu) \cdot \int_{a_\nu}^{b_\nu} \frac{dx}{1 + x^2} < \infty$$

Conversely this gives a uniqueness theorem. Suppose for example that there is an infinite sequence of interval outside Supp(f). So here the mid-points form an increasing sequence $0 < c_1 < c_2 \ldots$ and the length of each interval is $\ell_\nu = (b_\nu - a_\nu)$. With these notations we get

3.14 Vanishing theorem. *One has the implication:*

$$\sum \frac{\ell^2}{c_\nu^2} = +\infty \implies f = 0$$

Remark. The result above was proved by A. Beurling in his lectures from Stanford University in 1961. The proof above is due to Mikael Benedicks and we refer to [Benedicks] for further gap-theorems which are derived using the Fourier-Carleman transform.

4. The Paley-Wiener theorem

We shall give a description of all distributions on the real x-line with compact support.

4.1 Theorem. *Let $H(\zeta)$ be an entire function for which there exist B and some integer m such that*

$$(1) \qquad |H(\zeta)| \leq (1+|\zeta|)^m \cdot e^{B|\mathfrak{Im}(\zeta)|} \quad :$$

Then $H = \hat{\mu}$ for a distribution supported by $[-B, B]$.

Proof. If $g(x)$ is a test-function on the real x-line we get its entire Fourier-Laplace transform $\widehat{g}(\zeta)$. Since g in particular belongs to $\mathcal{S}$ it follows that the restriction of $\widehat{g}$ to the real ξ-axis is rapidly decreasing. Hence we can define a linear form μ on $C_0^\infty(\mathbf{R})$ by:

$$(1) \qquad \mu(g) = \int_{-\infty}^{\infty} \widehat{g}(-\xi) \cdot H(\xi) \cdot d\xi$$

Next, we apply Cauchy's vanishing theorem applied to the entire function $\hat{g}(-\zeta) \cdot H(\zeta)$ and rectangles in the complex ζ-space defined by

$$\square_{R:\rho} = \{(x,y) \quad : \quad -R < x < R \quad \text{and} \quad 0 < y < \rho\}$$

where (R, ρ) is a pair of positive numbers. The decay of $\widehat{g}$ from XX and the assumption on H give:

$$\lim_{R \to \infty} \int_0^\rho \widehat{g}(-R - is) \cdot H(R + is) \cdot ds = 0$$

A similar vanishing holds for the line integral along $\mathfrak{Re}(\zeta) = -R$. It follows that

$$(2) \qquad \mu(g) = \lim_{R \to \infty} \int_{-R}^{R} \widehat{g}(-\xi - i\rho) \cdot H(\xi + i\rho) \cdot d\xi$$

This holds for every $\rho > 0$. Next, assume that the support of g is contained in $[B^*, +\infty)$ for some $B^* > B$. With the integer m given the result in XX gives a constant C which depends on m and g only such that

$$(3) \qquad |\widehat{g}(\xi + i\rho)| \leq C \cdot (1 + |\xi| + \rho)^{-m-2} \cdot e^{-B^*\rho}$$

hold for every ξ. It follows that the absolute value of $\mu(g)$ is estimated by

$$(5) \qquad C \cdot e^{(B-B^*)\rho} \cdot \int (1 + |\xi| + \rho)^{-2} \cdot d\xi$$

Here we can choose ρ arbitrarily large and conclude that $\mu(g) = 0$. in the same way the reader can verify that $\mu(g) = 0$ when the support of g is contained in $(-\infty, -A]$ for some $A > 0$. The conclusion is that μ is a distribution on the x-line whose support is contained in $[-A, B]$. Finally, by the construction of the Fourier transform of a distribution we see that $\widehat{\mu} = XX \cdot H(\xi)$.

4.3 A relaxed condition on H. Let $H(\zeta)$ be an entire function. Suppose that the following two growth conditions hold for every real ξ and every $\zeta \in \mathbf{C}$:

$$(*) \qquad |H(\xi) \leq (1+|\xi|)^m \quad \text{and} \quad |H(\zeta) \leq e^{B|\zeta|}$$

Exercise. Prove that (*) implies that $H = \widehat{\mu}$ for a distribution supported by $[0, B]\pounds$. The hint is to establish a groweth for $H(\zeta)$ as in Theorem XX. To achieve this one studies the subharmonic function $U(\xi, \eta) = \text{Log}^+ |H(\xi + i\eta)|$. In particular, with $\zeta_0 = \xi_0 + is_0$ where $s_0 > 0$ we take a large semi-disc

$$D^+(R) = \{(\xi, \eta) \quad : \eta > 0 \quad \text{and} \quad (\xi - \xi_0)^2 + \eta^2 < R^2\}$$

Now $U(\xi, \eta)$ is $\leq B \cdot (R + |\xi_0|$ on the circular boundary part while

$$U(\xi, 0) \leq m \cdot \text{Log}\, 1 + |\xi - \xi_0|$$

on the interval $(-R+\xi_0, R+\xi_0)$. At this stage the reader should finish the proof using properties of the harmonic measure in a half-disc which has been at several occasions. Of course, the estimate of $[H(\zeta)|$ when ζ belongs to the lower half-plane is proved in a similar way.

4.4 A division problem Let μ and γ be a pair of distribtutions with compact support. Assume that the quotient

$$\frac{\hat{\gamma}}{\hat{\mu}} \in \mathcal{O}(\mathbf{C})$$

Since $\hat{\mu}$ and $\hat{\gamma}$ both are entire functions of exponential type the division theorem by Lindelöf and Nevanlinna in XXX implies that the quotient is of exponential type, i.e. there exists some B and a constant C so that

$$\frac{|\hat{\gamma}(\zeta)|}{|\hat{\mu}(\zeta)|} \leq Ce^{B|\zeta|}$$

However, the polynomial growth which is required in Theorem 4.2 on the real ξ-line fails in general. See XX below for an example. To compensate for this failure, A. Beurling and P. Malliavin made an intensive study of functions in the Carleman class in [B-M]. In these notes we must refrain from a further discussion of these more advanced reults. The interested reader may also consult the article [Malliavin:xx].

5. Runge's theorem

To each compact subset K of $\mathbf{C}$ we consider the algebra $\mathcal{O}(K)$ of germs of analytic functions on K. We can take the restriction to K of every $f \in \mathcal{O}(K)$ to K. This gives a subalgebra of $C^0(K)$ whose uniform closure is denoted by $\mathcal{H}(K)$. Let us now consider some open set Ω which contains K as a relatively compact subset. Each $g \in \mathcal{O}(\Omega)$ can be restricted to K and these restrictions give a subalgebra of $C^0(K)$ whose uniform closure is denoted by $\mathcal{H}_\Omega(K)$. We have the obvious inclusion

$$\mathcal{H}_\Omega(K) \subset \mathcal{H}(K) \tag{0.1}$$

The question arises when equality holds. It turns out that the criterion is expressed by plane topology. Before we announce the result we notice that $\Omega \setminus K$ is an open set and hence a disjoint union of connected components denoted by $\{U_\alpha\}$. Consider a single connected component U_α. Suppose it is a *relatively compact* subset of Ω, i.e. the closure $\bar{U}_\alpha$ has empty intersection with $\partial\Omega$. If this holds it is clear that

$$\partial U_\alpha \setminus K \tag{0.2}$$

Since U_α also is connected, the inclusion (0.2) means precisely that U_α is a connected component of $\mathbf{C} \setminus K$. Moreover, notice that since $U_\alpha \subset \Omega$ one "digs a hole" in Ω when $\bar{U}_\alpha$ is removed. At the same time the maximum principle for analytic functions in Ω give

$$|f|_{\bar{U}_\alpha} = |f|_{\partial U_\alpha} \quad : \; f \in \mathcal{O}(\Omega) \tag{0.3}$$

On the other hand, if $z_0 \in U_\alpha$ we notice that the function

$$f(z) = \frac{1}{z - z_0} \in \mathcal{H}(K) \tag{0.4}$$

This meromorphic function has a pole at z_0 and it is therefore not likely that its restriction to K can be uniformly approximated by functions from $\mathcal{H}_\Omega(K)$ since they are analytic in U_α and obey the maximum principle (0.3).

From the observations above the result below is no surprise.

5.1 Theorem. *The equality (0.1) holds if and only if $\bar{U}_\alpha \cap \partial\Omega \neq \emptyset$ for every connected component U_α of $\Omega \setminus K$.*

Proof. Suppose first the inclusion (0.1) is strict. Then we are going to show the existence of a connected component V_* of $\Omega \setminus K$ such that $\partial V_* \subset K$ and therefore $\bar{V}_* \cap \partial\Omega = \emptyset$. To obtain V_* we apply the Riesz representation formula and the Hahn-Banach theorem which give a complex Riesz measure μ supported by K such that

$$\mu \perp \mathcal{H}_\Omega(K) \quad : \quad \exists\, g \in \mathcal{H}(K) : \int_K g \cdot d\mu \neq 0 \tag{1}$$

By density we find $f \in \mathcal{O}(K)$ so that f_K approximates g so close that $\int f_K \cdot d\mu \neq 0$. By the result in XXX this gives the existence of a point $z_* \in \mathbf{C} \setminus K$ such that

$$\int_K \frac{d\mu(\zeta)}{\zeta - z_*} \neq 0 \tag{2}$$

Moreover, as explained in XX the point z_* must belong to a bounded connected component of $\mathbf{C} \setminus K$ which we denote by V_*. Now we shall prove

Sublemma One has the inclusion

$$V \subset \Omega$$

Proof. Suppose the inclusion fails and pick a point

$$a \in V \setminus \Omega \tag{3}$$

Now the functions $\frac{1}{(z-a)^m}$ belong to $\mathcal{O}(\Omega)$ for all $m \geq 1$. Hence (1) gives

$$\int_K \frac{d\mu(\zeta)}{(\zeta - a)^m} = 0 \quad : \; m = 1, 2, \ldots \tag{4}$$

Next, in V_* we have an analytic function $\phi(z)$ defined by

$$z \mapsto \int_K \frac{d\mu(\zeta)}{(\zeta - z)} = 0$$

Now (4) means that the series expansion of ϕ at $z = a$ is identically zero. Hence ϕ must be zero in the connected open set. In particular $\phi(z_*) \neq 0$ which contradicts (2) above. This proves the Sublemma.

Proof continued. Since V_* is a connected component of $\mathbf{C} \setminus K$ we have the inclusion

$$\partial V_* \subset K$$

At the same time $V \subset \Omega$ holds by the Sublemma. But then it is clear that ∂V_* cannot intersect $\partial\Omega$. This finishes one half of Theorem 5.1. There remains to prove the implication

$$\mathcal{H}_\Omega(K) = \mathcal{H}(K) \implies \bar{U}_\alpha \cap \partial\Omega \neq \emptyset \quad : \; \forall \; \alpha \tag{5}$$

If this implication fails we find some U_α such that $\bar{U}_\alpha \cap \partial\Omega = \emptyset$ and we have seen above that this gives

$$\partial U_\alpha \subset K$$

To get a contradiction we pick $z_0 \in U_\alpha$ and notice that the function

$$f(z) = \frac{1}{z - z_0} \in \mathcal{O}(K) \tag{6}$$

Since we assume the equality $\mathcal{H}_\Omega(K) = \mathcal{H}(K)$ thee exists a sequence $\{g_n\}$ in $\mathcal{O}(\Omega$ which converges uniformly to f on K. hence

$$\lim_{n \to \infty} (z - z_0)(g(z) = 1 \quad : \text{holds uniformly on } \; K \tag{7}$$

But this is impossible. Namely, we have $\bar{U} \subset \Omega$ and $\partial U \subset K$. At the same time we have the functions

$$h_n(z) = 1 - (z - z_0) g_n(z) \in \mathcal{O}(\Omega) \tag{8}$$

The maximum principle for analytic functions apply to every h_n in the open set U_α. This gives in particular

$$1 = |h(z_0)| \leq \max_{z \in \partial U_\alpha} |h_n(z)| \leq \max_{z \in K} |h_n(z)| \tag{9}$$

This gives a contradiction since (7) entails that the right hand side in (9) tends to zero when n increases.

5.2 The inhomogeneous $\bar{\partial}$-equation.

Let Ω be a bounded and connected open set. Then one has

5.3 Theorem. *For every $g \in C^\infty(\Omega)$ there exists $\phi \in C^\infty(\Omega)$ such that*

$$\bar{\partial}(\phi) = g \tag{1}$$

Proof. By the topological result in XX there exists an increasing sequence of compact subsets $\{K_\nu\}$ such that for every ν one has:

(1) $$\partial U \cap \partial \Omega \neq \emptyset \quad : \ \forall \text{ connected components of } U \subset \mathbf{C} \setminus K_\nu$$

Next, Theorem XX gives for each ν a C^∞-function ϕ_ν which solves

(2) $$\bar{\partial}(\phi_\nu) = g \quad : \text{ in a neighborhood of } K_\nu$$

It follows that $\phi_{\nu+1} - \phi_\nu \in \mathcal{O}(K_\nu)$. Now (1) and Runge's Theorem give the existence of some $h_\nu \in \mathcal{O}(\Omega)$ such that the maximum norm

(3) $$|\phi_{\nu+1} - \phi_\nu - h_\nu|_{K_\nu} \leq 2^{-\nu}$$

Now we get a function ϕ_* which for every p can be written as

(4) $$\phi_* = \phi_p + \sum_{\nu=p}^{\infty} (\phi_{\nu+1} - \phi_\nu - h_\nu) - (h_1 + \ldots + h_{p-1})$$

where the reader can check by a direct calculation that the right hand sum does not depend upon p. By (3) the infinite sum in (4) converges uniformly over compact subsets of Ω for any $p \geq 1$. Moreover, with p fixed we have

(5) $$\phi_{\nu+1} - \phi_\nu - h_\nu \in \mathcal{O}(K_p) \quad : \ \nu \geq p$$

Next, since $h_1, \ldots, h_{p-1}$ are analytic in Ω, we conclude that

(6) $$\phi_* - \phi_p \in \mathcal{O}(K_p) \quad : \ p = 1, 2, \ldots$$

Since the increasing sequence $\{K_p\}$ exhaust Ω it first follows that $\phi_* \in C^\infty(\Omega)$ and then (6) and (2) show that $\bar{\partial}(\phi) = g$ holds in Ω.

6. The generalised Fourier transform.

Introduction. The book *L'Integrale de Fourier et questions qui s'y rattachent* was published in 1944 by Institute Mittag-Leffler. It is based upon lectures which Torsten Carleman gave at the institute in 1935. In the introduction he writes: *C'est avant tous les travaux fondamentaux de M. Wiener et Paley qui ont attiré mon attention.* The book *Fourier transforms on the complex domain* by Raymond Paley and Norbert Wiener was published the year before. We are going to expose material from Chapter II in [Car] which contains results about analytic extensions across intervals on the real line and culminates in the construction of a generalised Fourier transform which goes beyond the ordinary Fourier transform for tempered distributions on the real line. This generalised Fourier transform is useful when analytic function theory is applied to study singular integral equations with non-temperate solutions. An example is the Wiener-Hopf equation where one seeks eigenfunctions $f(x)$ to the integral equation

$$(*) \qquad f(x) = \int_0^\infty K(x-y)f(y)dy$$

In many physical applications the kernel K has exponential decay and one seeks eigenfunctions f which are allowed to increase exponentially. The major result about solutions (*) appear in Theorem XVI on page 56 in [Pa-Wi] based upon the original article [Ho-Wi]. This inspired the construction presented in Section 1 below. Let us remark that the generalised inversion formula yields hyperfunctions. For comments about Carleman's constructions and its relation to more recent studies of hyperfunctions we refer to the article [Kis] by Christer Kiselman.

6.1 Preliminary constructions

Let U^* denote the upper half-plane in $\mathbf{C}$. Given a pair of positive numbers a, b we denote by $\mathcal{O}_{a,b}(U^*)$ the family of analytic functions $f(z)$ in U^* such that for every $0 < \theta_0 < \pi/2$ there exists a constant $A(\theta_0)$ and

$$(*) \qquad |f(re^{i\theta})| \leq A(\theta_0) \cdot \big(r^a + \frac{1}{r^b}\big) \quad : \quad \theta_0 < \theta < \pi - \theta_0$$

Remark. No condition is imposed on the A-function as $\theta_0 \to 0$. In particular $f(z)$ need not have tempered growth as one approaches the real line. In the same way we define the family $\mathcal{O}_{a,b}(U_*)$ of analytic functions defined in the lower half-plane U_* satisfying similar estimates as above.

Example. Let $f(z)$ be the ordinary Fourier-Laplace transform of a tempered distribution μ on the real t-line supported by the half-line $t \leq 0$. Recall that this gives an integer N and a constant C such that

$$|f(x+iy)| \leq C \cdot y^{-N} \quad : \quad y > 0$$

Here we can take $a = b = N$ and $A(\theta) = \frac{C}{\sin(\theta)}$ to get $f(z)$ in $\mathcal{O}_{a,b}(U^*)$.

Let us return to the general case. Consider some $f \in \mathcal{O}_{a,b}(U^*)$. If $b \geq 1$ we choose a positive integer m so that $b < 1 + m$. For each $0 < \theta < \pi$ we consider the half space

$$U^*_\theta = \{z = re^{i\phi} \quad : -\pi - \theta < \phi < -\theta\}$$

The choice of m and (*) give an analytic function in $U^*_\theta(z)$ defined by:

$$\text{(i)} \qquad f_\theta(z) = \frac{i}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-izre^{i\theta}} \cdot r^m \cdot e^{im\theta} \cdot f(re^{i\theta})e^{i\theta} \cdot dr$$

Cauchy's theorem applied to the analytic function $f(z)$ shows that these f_θ-functions are glued together as we rotate the angle θ in the open interval $(0, \pi)$. Notice also that

$$\text{(ii)} \qquad \cup_{0<\theta<\pi} U^*(\theta) = \mathbf{C} \setminus [0, +\infty)$$

Hence we obtain an analytic function $f^*(z)$ defined in $\mathbf{C} \setminus [0,+\infty)$ such that

(iii) $$f^*\big| U^*_\theta = f_\theta \quad : \quad 0 < \theta < \pi$$

If we instead start with an analytic function $g(z)$ in the class $\mathcal{O}_{a,b}(U_*)$ we obtain for each $0 < \theta < \pi$ an analytic function

$$g_\theta(z) = \frac{i}{\sqrt{2\pi}} \cdot \int_0^\infty e^{izre^{i\theta}} \cdot r^m \cdot e^{im\theta} \cdot g(re^{-i\theta})e^{-i\theta} \cdot dr \quad : \quad 0 < \theta < \pi$$

which now is defined in the half-plane

$$U_*(\theta) = \{z = re^{i\phi} \quad : -\pi + \theta < \phi < \theta\}$$

These g_θ-functions are again glued together and we get an analytic function $g_*(z)$ defined in $\mathbf{C} \setminus (-\infty, 0]$ which satisfies:

(iii) $$g_*\big| U_*(\theta) = g_\theta \quad : \quad 0 < \theta < \pi$$

The $\mathcal{S}$-transform. Consider a pair $f \in \mathcal{O}_{a,b}(U^*)$ and $g \in \mathcal{O}_{a,b}(U_*)$. We get the functions f^* and g_*. Here $g_* - f^*$ is analytic outside the real axis. So this difference can be restricted to both the half-planes. We use this to give

6.2 Definition. *To every pair f, g as above we set*

$$\mathcal{S}^*(z) = g_*(z) - f^*(z) \quad : \quad \mathfrak{Im}(z) > 0$$
$$\mathcal{S}_*(z) = g_*(z) - f^*(z) \quad : \quad \mathfrak{Im}(z) < 0$$

Remark. From the construction of f^* and g_* it is easily seen that $g_* - f^*$ restricts to a function in $\mathcal{O}_{a,b}(U)$ when U is the upper or the lower half-plane. Hence $\mathcal{S}$ is a map from $O_{a,b}(U^*) \times \mathcal{O}_{a,b}(U_*)$ into itself.

The reflection operator. If $\phi \in \mathcal{O}(U^*)$ we get the analytic function in the lower half-plane defined by

$$T(\phi)(z) = \bar{\phi}(\bar{z})$$

In the same way T sends an analytic function defined in U_* to an analytic function defined in U^*. Now we can define the composed operator $T \circ \mathcal{S}$. By the construction it gives a pair of analytic functions defined by

$$(T \circ \mathcal{S})^* = \bar{\mathcal{S}}_*(\bar{z}) \quad : \quad \mathfrak{Im}(z) > 0$$
$$(T \circ \mathcal{S})_* = \bar{\mathcal{S}}^*(\bar{z}) \quad : \quad \mathfrak{Im}(z) < 0$$

With the notations as above one has the following result which extends the standard inversion formula for tempered distributions. In the theorem below $\simeq$ means that two functions differ by a polynomial in z.

6.3 The inversion Theorem. *For each pair (f, g) in $\mathcal{O}_{a,b}(U^*) \times \mathcal{O}_{a,b}(U^*)$ one has*

$$T \circ \mathcal{S} \circ T \circ \mathcal{S}(f) \simeq f \quad : \quad T \circ \mathcal{S} \circ T \circ \mathcal{S}(g) \simeq g$$

where $\simeq$ means that the differences are polynomials in z.

Remark. Theorem 6.3 is the assertion from p. 49 in [Car]. For details of the proof we refer to [loc.cit. p. 50-52] where the main burden are some results concerning analytic extensions across a real interval which are presented in the first section of Chapter II from [Car]. Since these results have independent interest we proceed to discuss material from from the first section in Chapter II and once this has been done we leave it to the reader to find the proof of Theorem 6.3 or consult Carleman's proof.

6.4 Some analytic extensions.

Let D be the unit disc centered at the origin. We get the two half-discs

$$D^* = D \cap \mathfrak{Im}(z) > 0 \quad \text{and} \quad D_* = D \cap \mathfrak{Im}(z) < 0$$

Consider a pair $f^* \in \mathcal{O}(D^*)$ and $f_* \in \mathcal{O}(D_*)$. With these notations one has

6.5 Theorem. *Assume that*

(*) $$\lim_{y \to 0} f^*(x+iy) - f_*(x-iy) = 0$$

holds uniformly with respect to x. Then there exists $F \in \mathcal{O}(D)$ such that $F|D^* = f^*$ and $F|D_* = f_*$.

Remark. No special assumptions are imposed on the two functions except for (*). For example, it is from the start not assumed that they have moderate growth as we apprach the real x-line.

Proof. In D^* we get the analytic function

(i) $$G(z) = f^*(z) - \bar{f}_*(\bar{z})$$

Write $G = U + iV$. We notice that (*) gives

(ii) $$\lim_{y \to 0} U(x,y) = 0$$

Hence the harmonic function U in D^* converges uniformly to zero on the part of ∂D^* defined by $y = 0$. If $\delta > 0$ is small we restrict U to the upper half-disc $D^*(\delta)$ of radius $1 - \delta$. Now (ii) implies that when G is expressed by the Poisson kernel of $D^*(\delta)$ then the boundary integral is only taken over the upper half-circle. It follows by the analyticity of the kernel function for $D^*(\delta)$ that $G(x,y)$ extends to a real analytic function across the the real interval $-1 + \delta < x < 1 - \delta$. The same holds for the derivatives $\partial G/\partial x$ and $\partial G/\partial y$. Next, the Cauchy Riemann equations show that the complex derivative of $F(z)$ extends analytically across the real interval and then the *classic reflection principle* by Hermann Schwarz finishes the proof.

6.6 Another continuation. We expose the proof of Théorème 3 in [Car: p. 40]. Put $\square = \{(x,y) \quad -1 < x < 1 \quad \text{and} \quad 0 < y < 1\}$ and consider some $F(z) \in \mathcal{O}(\square)$. For the moment no extra hypothesis is imposed on F. With some small $\ell > 0$ we consider the half-disc $D_+(\ell) = \{|\zeta| < \ell \cap \mathfrak{Im}(\zeta) > 0\}$. Next, let $z_0 = x_0 + iy_0$ where $-1/2 < x_0 < 1/2$ and $0 < y_0 < 1 - \ell$. Then we get an analytic function

(i) $$G_\zeta(z) = F(z+\zeta) - F(z)$$

which is defined in some neighborhood of z_0. It has a series expansion:

$$G_\zeta(z) = F(z+\zeta) - F(z) = \sum P_\nu(\zeta)(z-z_0)^\nu \quad \text{where :}$$

(*) $$P_\nu(\zeta) = \frac{1}{\nu!} \cdot [F^{(\nu)}(z_0+\zeta) - F^{(\nu)}(z_0)]$$

Keeping z_0 fixed we let $\rho(\zeta)$ be the radius of convergence for this series. Hadamard's formula gives:

(**) $$\log \frac{1}{\rho(\zeta)} = \limsup_{\nu \to \infty} \frac{\log |P_\nu(\zeta)|}{\nu}$$

Suppose we knew that

(***) $$\rho(\zeta) \geq y_0 \quad : \quad \zeta \in D_+(\ell)$$

Then we can pick $\zeta = \frac{iy_0}{2}$ and conclude that the function

(ii) $$z \mapsto F(z + \frac{iy_0}{2}) - F(z)$$

is analytic in the disc $|z - z_0| < y_0$. At the same time the function $z \mapsto F(z + \frac{iy_0}{2})$ is analytic when $\mathfrak{Im}(z) > -\frac{y_0}{2}$ and hence $F(z)$ extends as an analytic function across a small interval on the real

x-axis centered at x_0. So if we can establish (***) for every $-1/2 < x_0 < 1/2$ it follows that $F(z)$ extends analytically across the real interval $-1(2 < x < 1/2$. There remains to find a condition on F in order that (***) holds. Notice that it suffices to get (***) for sufficiently small y_0 if we seek some analytic extension of F across the real x-line. To obtain (***) Carleman imposed the following:

6.7 Hypothesis on F. *There exists a pair $\ell > 0$ and $\delta > 0$ such that if ξ is real with if $|\xi| < \ell$ then $z \mapsto G_\xi(z)$ extends to an analytic function in the domain where $|z| < 1/2$ and $\mathfrak{Im}(y) > -\delta$.*

It is clear that this hypothesis implies that if y_0 is sufficiently small then there exists a constant k such that

(1) $$|P_\nu(\zeta) \leq k^\nu \quad : \quad \zeta \in D_+(\ell) \quad : \nu = 1, 2, \ldots$$

Moreover, we see from a figure that the hypothesis also implies that

(2) $$\rho(\zeta) \geq y_0 \quad : \quad |\zeta| = \ell \quad : \mathfrak{Im}(\zeta) \geq 0$$

It is also trivial that

(3) $$\rho(\zeta) \geq y_0 \quad : |\zeta| = \ell \quad : \mathfrak{Im}(\zeta) = 0$$

Proof of (***). We use that the functions $\zeta \mapsto \text{Log}\,|P_\nu(\zeta)|$ are subharmonic in $D_+(\ell)$ for every ν. So if G is Green's function for $D_+(\ell)$ we have the inequality

(i) $$\log\,|P_\nu(\zeta)| \leq \frac{1}{2\pi} \int_{\partial D_*(\ell)} \frac{\partial G(\zeta, w)}{\partial n_w} \cdot \frac{\log\,|P_\nu(w)|}{\nu} \cdot |dw|$$

Now (1) above entails that

(ii) $$\frac{\log\,|P_\nu(w)}{\nu} \leq k \quad : w \in \partial D_+(\ell)$$

At the same time (2-3) and Hadamard's formula give

(iii) $$\limsup_{\nu \to \infty} \frac{\log\,|P_\nu(w)}{\nu} \leq \log\,\frac{1}{y_0} \quad : w \in \partial D_+(\ell)$$

Thanks to (ii) we can apply Lebesgue's dominated convergence theorem when we pass to the Lim.Sup in (i) and hence (iii) gives

(iv) $$\limsup_{\nu \to \infty} \frac{\log\,|P_\nu(\zeta)}{\nu} \leq \log\,\frac{1}{y_0} \quad : \zeta \in D_+(\ell)$$

Now we apply Hadamard's formula for points in $D_+(\ell)$ and (***) follows.

Remark. The continuation found above can be applied to relax the assumption in Theorem 6.5. For example, here exists an analytic extension for a pair f^*, f_* under the less restrictive condition that

$$\lim_{y \to 0} \int_a^b \big[f^*(x+iy) - f_*(x-iy)\big] \cdot dx = 0$$

This follows when 6.6 is applied to the primitive functions of the pair. For further results of analytic extensions across a boundary we refer to Beurling's article [Beurling].

7. Carleman's inequality

Introduction. In the article [Ca:1] from 1923, Carleman proved a result about differentiable functions on the real line which confirms the general philosophy that in order for a polynomial $P(x)$ *of any degree* $\geq n$ to have multiple roots of some order n at two points, say 0 and 1, while it does not vanish identically, the maximum norms of its derivatives up to order n *cannot be too small.* Theorem 1 below gives a conclusive answer to this problem. The proof is instructive since it shows that it is often useful to move into the complex domain to solve a problem on the real line. The key point is to use *subharmonic majorisations.* In 1923 this was a pioneering idea which after has become a standard tool in analysis. See § 2 in Chapter 4 in R. Nevanlinna's book [Nev[for a further discussion and examples which illustrate *Carleman's Prinzip der Gebietserweiterung.*

Now we begin to announce Carleman's theorem. Let $[0,1]$ be the closed unit interval on the real t-line. To each integer $n \geq 1$ we denote by S_n the class of n-times differentiable functions and non-negative functions on $[0,1]$ satisfying

$$f^{(\nu)}(0) = f^{(\nu)}(1) = 0 \quad : \quad 0 \leq \nu \leq n \quad : \quad \int_0^1 f^2(t)dt = 1 \quad : \ f(t) \geq 0$$

Thus, we regard non-negative functions which are "flat up to order n" at the end points. Notice that S_n contains all polynomials of the form

$$t^n(1-t)^n \cdot Q(t) \ : \ Q(t) \text{ any polynomial} \geq 0 \quad \text{where} \quad \int_0^1 t^n(1-t)^n Q(t)dt = 1$$

Since the degree of $Q(t)$ can be arbitrarily large the set of such polynomials is dense in S_n. Next, to each $f \in S_n$ we introduce the p:th roots of the L^2-norms of its derivatives of order $1 \leq p \leq n$:

$$\beta_p(f) = [\int_0^1 [f^{(p)}(t)]^2 dt\ ]^{\frac{1}{2p}} \quad : \quad 1 \leq p \leq n$$

With these notations Carleman's inequality asserts:

Theorem 1. *For every n and each $f \in S_n$ one has*

$$\sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \leq 2e\pi \cdot (1 + \frac{1}{4\pi^2 e^2 - 1})$$

Remark. The absolute constant in the right hand side appears as a consequence of the subsequent proof where several majorisations appear, such as the Cauchy-Schwarz inequality and a harmonic majorant to a certain subharmonic function. The best constant $\mathcal{C}_*$ which would give

$$\text{(i)} \qquad \sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \leq \mathcal{C}^*$$

for all n and every $f \in \mathcal{S}_n$ is not known. Let us remark that the general inequality above is sharp in the sense that there exists a constant C^* such that for every n one can find $f \in \mathcal{S}_n$ for which

$$\text{(ii)} \qquad \sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \geq C_*$$

See Chapter 1 in [Hö:xx] for the construction of such functions. Before we enter the proof if Theorem 1 we cite an excerpt from Emile Borel's comments to [Ca]:

La demonstration donnée par M. Carleman de l'ènoncé que j'avais induit du théorème de Denjoy est remarkquable par sa profondeur et par sa simplicité. Il serait toutefois désirable d'arriver à donner une demonstration sinon algébrique, du moins ne faisant appel qu'aux variables réelles du théorème auquel M. Carlemnan vient d'áttacher son nom. Ce théorème de Carleman me parait en effet devoir être considéré, avec le théorème de Denjoy, comme l'un des théorèmes *fondamentaux de la théorie des functions indéfiniment dérivables de variables réelles. Il serait encore plus intèressant de complêter les théorèmes de Denjoy et de Carleman pour une étude asymptotique aussi précise que possible des séries de toute terme géneral quand $n \to \infty$.*

Proof of Theorem 1.

Let $n \geq 1$ and $f \in S_n$. Keeping f fixed we put $\beta_p = \beta_p(f)$ to simplify notations. The result in XXX below show that the β-numbers are non-decreasing. i.e. we have

$$1 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_{n+1} \tag{*}$$

Define the complex Laplace transform

$$\Phi(z) = \int_0^1 e^{-zt} f(t)dt$$

Since f is n-flat at the end-ponts, integration by parts p times gives:

$$\Phi(z) = z^{-p} \int_0^1 e^{-zt} \cdot \partial^p(f^2)(t)dt \quad : \quad 1 \leq p \leq n+1$$

where $\partial^p(f^2)$ is the derivative of order p of f^2. Now we study the absolute value of Φ on the vertical line $\mathfrak{Re}(z) = -1$. To each $1 \leq p \leq n+1$ we have

$$\partial^p(f^2) = \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot f^{(\nu)} \cdot f^{(p-\nu)} \tag{1}$$

Since $|e^{t-iyt}| = e^t$ for all y, the triangle inequality gives

$$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \int_0^1 e^t \cdot |f^{(\nu)}(t)| \cdot |f^{(p-\nu)}(t)| \cdot dt \tag{2}$$

Now we use that $e^t \leq e$ on $[0,1]$ and apply the Cauchy-Schwarz inequality which gives by the definition of the β-numbers give:

$$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu} \tag{3}$$

From (*) it follows that $\beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu} \leq \beta_p^p$ for each ν and since $\sum_{\nu=0}^{\nu=p} \binom{p}{\nu} = 2^p$ we obtain

$$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot 2^p \cdot \beta_p^p \tag{4}$$

Passing to the logarithm we get

$$\log |\Phi(-1+iy)| \leq 1 + p \cdot \log \frac{2\beta_p}{|-1+iy|} \tag{5}$$

here (5) holds when $1 \leq p \leq n+1$. Notice that the assumption that $\beta_0 = 1$ also gives

$$\log |\Phi(-1+iy)| \leq 1 \tag{6}$$

The ω-function. To each $1 \leq p \leq n+1$ we find a positve number y_p such that

$$|-1+iy_p| = 2e\beta_p$$

Now we define a function $\omega(y)$ where $\omega(y) = 0$ when $y < y_1$ and

$$\omega(y) = p \quad : \quad y_p \leq y < y_{p+1}$$

and finally $\omega(y) = n+1$ when $y \geq y_{n+1}$. Friom this we see that (5-6) give the inequality

$$\log |\phi(-1+iy)| \leq 1-\omega(y) \tag{7}$$

for all $-\infty < y < +\infty$.

Harmonic majorisation. With $1-\omega(y)$ as boundary function in the half-plane $\mathfrak{Re}(z) > -1$ we construct the harmonic extension $H(z)$ where Poisson's formula in a half-plane in particular gives:

$$H(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\omega(y)}{1+y^2} \cdot dy$$

Since the function $\log|\Phi(z)|$ is subharmonic in this half-plane it follows from (7) that

$$0 = \log|\Phi(0)| \leq H(0)$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{\omega(y)}{1+y^2} \cdot dy \leq \pi \tag{8}$$

Now $\omega(y) = 0$ when $y \leq y_1$ and we see that (8) gives the inequality

$$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy \leq \frac{y_1^2}{1+y_1^2} \cdot \pi \tag{9}$$

Next, the construction of the ω-function obviously gives the equality

$$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy = \frac{1}{y_1} + \ldots + \frac{1}{y_{n+1}} \tag{10}$$

Next, the construction of the y_p-numbers entail that $y_p \leq 2e\beta_p$ and hence (9) and (10) give

$$\frac{1}{\beta_1} + \ldots + \frac{1}{\beta_{n+1}} \leq 2e\pi \cdot \frac{1}{1+\frac{1}{y_1^2}} \tag{11}$$

Finally, we have $1 + y_1^2 = 4e^2\beta_1^2$ and by XX we also have $\beta_1 \geq \pi$ which entails that

$$\frac{1}{1+\frac{1}{y_1^2}} \leq 1 + \frac{1}{4\pi^2e^2 - 1} \tag{12}$$

Now (11-12) give the requested inequality in Theorem 1.

Appendix

The proof used an elementary result which we recall here. Namely, that the β-sequence increases when $f \in \mathcal{S}_n$. To see this, let $1 \leq p \leq n-1$ and after a partial integration

$$\beta_p^2 = \int_0^1 f^{(p-1)}(t)f^{(p+1)}(t)dt$$

The Cauchy-Schwarz inequality gives

$$\beta_p^2 \leq \beta_{p-1}\beta_{p+1}$$

By an induction over p it follows that $\beta_1 \leq \ldots \leq \beta_n$ provided that we prove the inequality

$$\beta_0^2 = \int_0^1 f^2(t)dt \leq \beta_1^2 \tag{*}$$

This inequality follows easily by regarding Fourier's development of f into a sine series $\sum a_\nu \cdot \sin(\nu\pi t)$.

8. Carleman's inequality for inverse Fourier transforms in $L^2(\mathbf{R}^+)$.

Introduction. By Parsevel's theorem the Fourier transform sends L^2-functions on the ξ-line to L^2-functions on the x-line. We shall consider the class of non-negative L^2-functions $\phi(x)$ such that there exists an L^2-function $F(\xi)$ supported by the half-line $\xi \geq 0$ and

$$\phi(x) = \Big| \int_0^\infty e^{ix\xi} \cdot F(\xi) \cdot d\xi \Big| \tag{*}$$

Theorem 1 below gives a necessary condition in order that (*) holds. The result below is due to Carleman in [Carleman]. Apart from applications for to the study of quasi-analytic functions this theorem has several other consequences which are put forward by Paley and Wiener in [Pe-Wi].

Theorem 1. *An L^2-function $\phi(x)$ satisfies (*) if and only if*

$$\int_{-\infty}^\infty \text{Log}^+\big[\frac{1}{\phi(x)}\big] \cdot \frac{dx}{1+x^2} < \infty \tag{*}$$

Remark. Theorem 1 means that $\phi(x)$ in the average cannot be too small when (*) holds. Let us also point out the inequality (**) below. Namely, suppose that $F(\xi)$ satisfies the weighted mean-value equality

$$\int_0^\infty F(\xi) \cdot e^{-\xi}\, d\xi = 1 \tag{i}$$

The proof of Theorem 1 will show that when $\phi(x)$ is given by (*) then one has the inequality

$$\int_{-\infty}^\infty \text{Log}^+\big[\frac{1}{\phi(x)}\big] \cdot \frac{dx}{1+x^2} \leq \int_{-\infty}^\infty \frac{\phi(x)^2}{1+x^2} \cdot dx \tag{**}$$

Proof of Theorem 1

First we prove the suffiency. So let let $\phi(x)$ be a non-negative L^2-function where the integral in Theorem 1 is finite. Then there exists the harmonic extension of log $\phi(x)$ to the upper half-plane;

$$\lambda(x+iy) = \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\log\ \phi(t)}{(x-t)^2+y^2} \cdot dt \quad : \ y > 0 \tag{1}$$

Let $\mu(z)$ be the conjugate harmonic function of λ and set

$$h(z) = e^{\lambda(z)+i\mu(z)} \tag{2}$$

By Fatou's theorem there exists for almost every x the limit

$$\lim_{y\to 0} \lambda(x+iy) = \log\ \phi(x) \tag{3}$$

Or, equivalently

$$\lim_{y\to 0} |h(x+iy)| = \phi(x) \tag{4}$$

Next, from (1) and the fact that the geometric mean value of positive numbers cannot exceed their arithmetic mean value, one has

$$\big|h(x+iy]\big| = e^{\lambda(x+iy)} \leq \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\phi(t)}{(x-t)^2+y^2} \cdot dt \quad : \ y > 0 \tag{5}$$

Then (5) the Schwartz inequality gives:

$$\int_{-\infty}^{\infty} \bigl|h(x+iy)\bigr|^2 \leq \int_{-\infty}^{\infty} \bigl|\phi(x)\bigr|^2 \quad : \ y > 0 \tag{6}$$

Since $h(z)$ is analytic in the upper half-plane it follows from (6) and Cauchy's formula that if $\xi < 0$, then the integrals

$$J(y) = \int_{-\infty}^{\infty} h(x+iy) \cdot e^{-ix\xi + y\xi} \cdot dx \quad : \ y > 0 \tag{7}$$

are independent of y. Passing to the limit as $y \to \infty$ and using the uniform upper bounds on the L^2-norms of the functions $h_y(x) \mapsto h(x+iy)$, it follows that $J(y)$ vanishes identically. So the Fourier transforms of $h_y(x)$ are supported by $\xi \geq 0$ for all $y > 0$. Passing to the limit as $y \to 0$ the same holds for the Fourier transform of $h(x)$. Finally (4) gives

$$\phi(x) = |h(x)| \tag{8}$$

Since $\widehat{h}(\xi)$ is an L^2-function by Parseval's theorem we conclude that $\phi(x)$ is of the requested form (*).

Necessity. Now we are given the formula

$$\phi(x) = \lim_{N \to \infty} \frac{1}{2\pi} \cdot \int_0^N e^{ix\xi} \cdot F(\xi) \cdot d\xi \tag{9}$$

In the upper half plane we get the analytic function

$$\psi(x+iy) = \frac{1}{2\pi} \cdot \int_0^{\infty} e^{ix\xi - y\xi} \cdot F(\xi) \cdot d\xi \tag{10}$$

let us also assume that $F(\xi)$ satisfies (i) in the remark after Theorem 1 which means that

$$\psi(i) = 1$$

Next, consider the conformal map from the upper half-plane into the unit disc where w is the complex variable, defined by

$$w = \frac{z-i}{z+i}$$

Then $\phi(x)$ corresponds to a function $\Phi(e^{is})$ on the unit circe $|w| = 1$ which gives the equality

$$\int_{-\pi}^{\pi} |\Phi(e^{is})|^2 \cdot ds = 2 \cdot \int_{-\infty}^{\infty} \frac{|\phi(x)|^2}{1+x^2} \cdot dx \tag{11}$$

Similarly let $\Psi(w)$ be the analytic function in $|w| < 1$ which corresponds to $\psi(z)$. From (10-11) it follows that $\Psi(w)$ is the Poisson extension of Φ, i.e.

$$\Psi(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |w|^2}{|e^{is} - w|^2} \cdot \Phi(e^{is}) \cdot ds \tag{12}$$

If $0 < r < 1$ it follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Log}^+ |\Psi(re^{is}| \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \bigl|\Psi(re^{is}\bigr|^2 \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{is}|^2 \cdot ds \tag{13}$$

By (12) we have

$$\lim_{r \to 1} \Psi(re^{is}) = \Phi(e^{is}) \quad : \text{almost everywhere} \quad 0 \leq s \leq 2\pi \tag{14}$$

Next, since $\psi(i) = 1$ we have $\Psi(0) = 1$ which gives the inequality

$$\text{(15)} \qquad \int_{-\pi}^{\pi} \text{Log}^+ \frac{1}{|\Psi(re^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \text{Log}^+ |\Psi(re^{is}| \cdot ds \quad : 0 < r < 1$$

By (13-15) a passage to the limit as $r \to 1$ gives

$$\text{(16)} \qquad \int_{-\pi}^{\pi} \text{Log}^+ \frac{1}{|\Phi(e^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \left| \Phi(e^{is} \right|^2 \cdot ds$$

Returning to the real x-line we get the inequality (**) in the Remark 1 which at the same time finishes the proof of Theorem 1.

9. An integral equation.

Let $f(x)$ be an L^1-function on the real x-line such that the zeros of $\widehat{f}$ is a discrete subset $\{\alpha_\nu\}$ of $\mathbf{R}$, enumerated so that their absolute values are non-decreasing. In general this sequence is infinite. Consider the set of all L^∞-functions $\phi(x)$ which satisfy the convolution equation

$$(*) \qquad f * \phi(x) = \int f(x-y)\phi(y)dy = 0 \quad : \quad -\infty < x < +\infty$$

It is clear that the exponential function $e^{-i\alpha_\nu x}$ is a solution to (*) for every zero of $\widehat{f}$. More generally any exponential polynomial

$$P(x) = \sum c_\nu \cdot e^{-i\alpha_\nu x}$$

solves (*). It turns out that these exponential polynomials is a dense subset of all L^∞-functions.

Theorem. *To every L^∞-solution ϕ there exists a sequence of exponential solutions $\{P_N(x)\}$ such that $\lim_{N\to\infty} P_N(x) = \phi(x)$ holds almost everywhere and here $\{P_N\}$ can be chosen so that their maximum norms over the whole x-line satisfy*

$$||P_N||_\infty \leq ||\phi||_\infty$$

The crucial point in the subsequent proof is to use the local invertibility for L^1-functions whose Fourier transforms are $\neq 0$ over open intervals on the ξ-line. To begin with the L^∞-function ϕ which solves (*) is a tempered distribution on the real x-axis. Since ϕ is not integrable we cannot find its Fourier transform in an explicit way. But the following holds:

9.1 Proposition. *The support of $\hat{\phi}$ is contained in the discrete set $\{\alpha_\nu\}$.*

Proof. Let $J = (a, b)$ be an open interval on the ξ-axis where $\hat{f}(\xi)$ has no zeros on the closed interval $[a, b]$. The result in XX about the Banach algebra $L^1(\mathbf{R}$ gives an L^1-function g such that

$$\hat{g}(\xi) \cdot \hat{f}(\xi) = 1 \quad : \quad a \leq \xi \leq b$$

Next, consider some $\psi \in \mathcal{S}$ whose Fourier transform $\hat{\psi}$ has support contained in $[a, b]$. Now we can define the convolution $\psi * g * f$ in L^1 and its Fourier transform becomes

$$\hat{\psi} \cdot \hat{g} \cdot \hat{f} = \hat{\psi}$$

Since $f * \phi = 0$ it follows that $\psi * g * f * \phi = 0$ and hence also $\psi * \phi = 0$. Now ϕ belongs to $\mathcal{S}$ and the space of tempered distributions is a module over $\mathcal{S}$.So in the space of tempered distributions on the ξ-line we can define the product $\hat{\psi} \cdot \hat{\phi}$. and the vanishing of the convolution $\psi * \phi$ on the x-line entails that $\hat{\psi} \cdot \phi = 0$. Above $\hat{\psi}(\xi)$ can be taken as an arbitrary test-function in $C_0^\infty(a, b)$. It follows that the support of the distribution $\hat{\phi}$ does not intersect the open interval (a, b). Since (a, b) can be an arbitrary interval in the complement of the discrete set of zeros we get Proposition 9.1.

Proposition 9.1 shows that $\hat{\phi}$ is a sum of Dirac distributions at points from the discrete set $\{\alpha_\nu\}$. Let us show that every such Dirac distribution has order zero, i.e. of the form $c \cdot \delta_{\alpha_\nu}$ for some constant c. To prove this we consider some α_ν and choose $\psi(x) \in \mathcal{S}$ such that its Fourier transform a $\hat{\psi}(\xi)$ is a test-function which is identically one in a small neighborhood of α_ν while its support does not contain any other zero. Now the distribution

$$\text{(i)} \qquad \hat{\psi} \cdot \hat{\phi} = \rho_\nu$$

where ρ_ν is the Dirac distribution determined by $\hat{\phi}$ at α_ν. At the same time (i) is the Fourier transform of $\psi * \phi$ which is a bounded function on the x-line, Namely, by the the L^1-norm of ψ times the sup-norm of ϕ. Hence the inverse Fourier transform of the Dirac distribution ρ_ν is a bounded function on the x-line. But this can only occur if ρ_ν has order zero.

Summing up we have proved the following:

9.2 Proposition. *The Fourier transform $\hat{\phi}$ is given by:*

$$\hat{\phi} = \sum a_\nu \cdot \delta_{\alpha_\nu}$$

where $\{a_\nu\}$ is some sequence of complex numbers.

Remark. By Fouriers inversion the a-numbers are determined via the equations

$$\sum a_\nu \cdot \hat{g}(\alpha_\nu) = \frac{1}{2\pi} \int \phi(x) \cdot g(-x)dx \quad : g \in \mathcal{S}$$

The right hand side is defined for every L^1-function. So if the maximum norm $||\phi||_\infty = 1$ then the series

$$\sum a_\nu \cdot \hat{g}(\alpha_\nu)$$

converges for every L^1-function g and the absolute value of the sum is $\leq ||g||_1$.

9.3 Approximation by exponential solutions. At this stage we can finish the proof of Theorem 1. First, fix some non-negative function $g \in \mathcal{S}$ such that $\hat{g}(\xi)$ is a test-function and $\hat{g}(0) = 1$. If $N \geq 2$ we set $g_N(x) = Ng(Nx)$ which gives

$$\hat{g}_N(\xi) = \hat{g}(\xi/N)$$

Next, the Fourier transform of the convolution $g_N * \phi$ is equal to

$$\hat{g}_N(\xi) \cdot \hat{\phi}(\xi)$$

Since $\hat{g}_N$ has compact support the left hand side is a finite sum

$$\sum \hat{g}_N(\alpha_\nu) \cdot a_\nu \cdot \delta_{\alpha_\nu}$$

This gives the exponential polynomial

$$P_N(x) = (g_N * \phi)(x) = \frac{1}{2\pi} \cdot \sum \hat{g}_N(\alpha_\nu) \cdot a_\nu \cdot e^{i\alpha_\nu x}$$

Finally, by Lebesgue's theorem it is clear that

$$\lim_{N \to \infty} P_N(x) = \phi(x)$$

holds almost everywhere. For the maximum norms we get

$$||P_N||_\infty \leq |g||_1 \cdot ||\phi||_\infty$$

and Theorem 9.1 follows since since $||g||_1 = \widehat{g}(0) = 1$.

10. Spectral synthesis.

The subsequent material comes from the article [Beur] first presented by Beurling in 1938 at the Scandinavian Congress in Helsinki. Let $\phi(x)$ be an L^∞-function. In the convolution algebra $L^1(\mathbf{R})$ we get the ideal:

$$J_\phi = \{f \in L^1(\mathbf{R} \quad : \; f * \phi = 0\}$$

Notice that for any pair $g \in L^1(\mathbf{R})$ and $\psi \in L^\infty(\mathbf{R})$ the convolution

$$g * \psi(x) = \int g(x-y) \cdot \psi(y) \cdot dy$$

yields a bounded and continuous function whose maximum norm is majorised by $||g||_1 \cdot ||\psi||_\infty$. Next, from XX there exists the Banach algebra $B = \mathbf{C} \cdot \delta_0 + L^1(\mathbf{R})$ whose maximal ideal space is the real ξ-line plus a point at ∞ which corresponds to the multiplicative functional on B which vanishes on the ideal $L^1(\mathbf{R})$. Following [Beu] we give

10.1 Definition. *The set of common zeros of Fourier transforms of functions in J_ϕ is denoted by $\sigma(\phi)$ and called the spectrum of ϕ.*

Let us first notice that the spectrum is non-empty, unless $\phi = 0$. For if $\sigma(\phi) = \emptyset$ then the result from XXX shows that the $J_\phi = L^1(\mathbf{R})$ and since $L^\infty(\mathbf{R})$ is the dual space we get $\phi = 0$.

Exercise. Show that $\sigma(\phi)$ is equal to the support of the temperate distribution $\widehat{\phi}$. The hint is to use the fact that the inverse Fourier transform of every test-funtion on the ξ-line is an L^1-function on the x-line.

10.2 The spectral synthesis problem. *Let $f \in L^1(\mathbf{R})$ be such that $\widehat{f}(\xi) = 0$ on $\sigma(\phi)$. Does this imply that $f \in J_\phi$.*

In general the answer is negative, i.e. there exist ϕ-functions for which the spectral synthesis fails. See XXX (??). But if $\sigma(\phi)$ satisfies a certain topological condition then the answer is affirmative. Recall that a closed set K without interior points on the real ξ-line is *perfect* if the closure of $K \setminus \{p\}$ is equal to K for every $p \in K$. So if F is a closed set with empty interior which does not contain any perfect subset, then F must contain at least one isolated point. This last property was used in [Beu] to establish the following:

10.3 Theorem. *Let $\phi \in L^\infty(\mathbf{R})$ be such that the boundary of $\sigma(\phi)$ does not contain any perfect subset. Then spectral synthesis holds for the ideal J_ϕ.*

Before we enter the proof of Theorem 10.3 we establish a result of independent interest.

10.4 Theorem. *Consider a pair $\phi \in L^\infty(\mathbf{R})$ and $h \in L^1(\mathbf{R})$. Let a be a real number such that the Fourier transform $\hat{h}(a) = 0$. Then the Fourier transform of $h * \phi$ cannot be a constant times the Dirac measure δ_a unless $h * \phi$ is identically zero.*

Proof. Replacing h by the translated function $h_a(x) = e^{-iax} \cdot h(x)$ we may take $a = 0$. So now $\hat{h}(0) = 0$ and we also assume that the Fourier transform of $h * \phi$ is $c_0 \cdot \delta_0$ for some constant c_0. There remains to show that c_0 must be zero, i.e that the convolution $h * \phi = 0$. To show this we use the result in XX which gives a sequence $\{g_N\}$ in $\mathcal{S}$ such that $\hat{g}_N(0) = 0$ for all N and at the same time:

(i) $$\lim_{N \to \infty} ||g_N * h - h||_1 = 0$$

Next, since every $g_N \in \mathcal{S}$ we get

$$\widehat{g_N * h * \phi} = \widehat{g}_N \cdot \widehat{h * \phi} = \widehat{g}_N \cdot c_0 \cdot \delta_0 = 0$$

where the last equality holds since $\widehat{g}_N(0) = 0$. So the convolutions $g_N * h * \phi = 0$ for all N. Finally, since ϕ is i L^∞ we conclude that (i) implies that $h * \phi = 0$ as required.

Proof of Theorem 10.2. Let f be an L^1-function such that $\widehat{f} = 0$ on $\sigma(\phi)$. We must prove that $f * \phi = 0$. Assume the contrary which gives the L^∞-function $\psi = f * \phi$ which is not identically zero. Now we are going to derive a contradiction. First it is obvious that one has the inclusion

$$\sigma(\psi) \subset \sigma(\phi) \tag{1}$$

Indeed, this follows since the commutative law for convolutions imply that $h \in J_\phi$ gives $h*(f*\phi) = f * (h * \phi) = 0$. Hence $J_\phi \subset J_\psi$ and (1) follows. Now we shall improve (1) and establish:

$$\sigma(\psi) \subset \partial(\sigma(\phi)) \tag{2}$$

To prove this may assume that the interior of $\sigma(\phi) \neq \emptyset$ and let $(a, b) \subset \sigma(\phi)$ be an open interval. To every $\rho(\xi) \in C_0^\infty(a, b)$ the inverse Fourier transform is an L^1-function which we denote by ρ_*. Since $\hat{f} = 0$ on $\sigma(\phi)$ we get $\rho \cdot \widehat{f} = 0$ which entails $\rho_* * f = 0$ and then

$$0 = (\rho_* * f) * \phi = \rho_* * (f * \phi) = \rho_* * \psi \implies \rho \cdot \widehat{\psi} = 0$$

where we notice that the last product is defined since $\hat{\psi}$ is a tempered distribution on the ξ-line while ρ is a test-function so we can use that $\mathcal{S}^*$ is a module over the space of test-functions. Since $\rho \in C_0^\infty(a, b)$ was arbitrary we conclude that the support of $\hat{\psi}$ does not intersect (a, b), and since (a, b) was an arbitrary interval of the interior of $\sigma(\phi)$ we get (2) from Exercise 1.

Next, (2) and the hypothesis on the topology of $\partial\sigma(\phi)$ imply that $\sigma(\psi)$ contains at least one isolated point a. Now we choose a test-function $\rho(\xi)$ which is identically one in a small neighborhood of a while the support of ρ has empty intersection with the rest of $\sigma(\psi)$. Then the support of the tempered distribution $\rho \cdot \hat{\psi}$ is reduced to $\{a\}$. Moreover, let g be the inverse Fourier transform of ρ so that $\rho \cdot \hat{\psi}$ is the Fourier transform of $g * \psi$. here $g * \psi$ is a bounded continuous function and exactly as in XX we conclude that the Dirac distribution defined by $\rho \cdot \hat{\psi}$ can only be a constant times δ_a. This means that

$$g * \psi(x) = c_0 \cdot e^{aix}$$

Let us then consider the L^1-function $h = g * f$. Since $\hat{f} = 0$ holds on $\sigma(\phi)$ we have $\hat{h}(a) = 0$. At the same time:

$$h * \phi(x) = c_0 \cdot e^{aix} \quad \text{where} \quad c_0 \neq 0 \tag{*}$$

But this contradicts the result in Theorem 2 and hence Theorem 1 is proved.

11. On inhomogeneous $\bar{\partial}$-equations.

Recall that $L^1(\mathbf{R}^2)$ is a convolution algebra, i.e. for a pair of L^1-functions $f(x,y)$ and $g(x,y)$ one defines

$$f * g(x,y) = \iint f(x-s, y-t)g(s,t)dsdt \tag{1}$$

The convolution is commutative and satisfies the associative law. Next, given some L^1-function $f(x,y)$ there exists the *distribution derivatives* $\partial_x(f)$ and $\partial_y(f)$. We shall impose the condition that these distribution derivatives are bounded measurable functions, i.e. both belong to $L^\infty(\mathbf{R}^2)$.

Exercise. Recall that $L^\infty(\mathbf{R}^2)$ is the dual space of $L^1(\mathbf{R}^2)$. Use this together with the definition of distribution derivatives and conclude that $\partial_x(f) \in L^\infty(\mathbf{R}^2)$ if and only if there to every compact set K in $\mathbf{R}^2$ exists a constant C_K such that

$$|\iint f(x,y) \cdot \partial_x(g(x,y) \cdot dxdy \leq C_K \cdot ||g||_1$$

for every test-function $g(x,y)$ whose support is contained in K.

Next, let f be a complex valued function L^1-function with compact support. With $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ we get the distribution

$$\bar{\partial}(f) = \frac{1}{2}\big(\partial_x(f) + i\partial_y(f)\big)$$

11.1 Theorem. *The inclusion $\bar{\partial}(f) \in L^\infty(\mathbf{R}^2)$ implies that f is a continuous function whose modules of continuity is bounded by $C \cdot \delta \cdot Log\frac{1}{\delta}$ for some constant C which only depends on the size of the support of f.*

Proof To prove this we use the L^1_{loc} -function $\frac{1}{z}$. Without loss of generality we can assume that f has compact support. Put

$$g = \frac{1}{z} * f$$

So g is a convolution of two L^1-functions. Recall that

$$\bar{\partial}(\frac{1}{z}) = 2\pi i \cdot \delta_0 \tag{i}$$

From (i) and the hypothesis that $\bar{\partial}(f) \in L^\infty(\mathbf{R}^2)$ we get the following:

Sublemma. One has the equality:

$$2\pi i f = \frac{1}{z} * \bar{\partial}(f)$$

where the right hand side is the convolution of $\frac{1}{z}$ and the L^∞-function $\bar{\partial}(f)$ and the equality holds in L^1.

Proof continued. Set $g = \bar{\partial}(f)$. Given a pair of points z_1, z_2 in the complex plane we get

$$|f(z_2) - f(z_1)| \leq \frac{1}{2\pi} \cdot \big|\iint [\frac{1}{z-z_1} - \frac{1}{z-z_2}] \cdot g(z)dxdy\,\big| \leq$$

$$\frac{||g||_\infty}{2\pi} \cdot \iint_K |\frac{1}{z-z_1} - \frac{1}{z-z_2}| \cdot dxdy$$

where K is the compact support of f. There remains only to estimate the double integral. Suppose for example that K is contained in the disc D_R of radius R centered at the origin. Given a pair of distinct points z_1, z_2 in this disc we notice that

$$\frac{1}{z-z_1} - \frac{1}{z-z_2} = \frac{z_2 - z_1}{(z-z_1)(z-z_2)}$$

With $\delta = |z_1 - z_2|$ Theorem 11.1 follows if we find a constant C such that

$$(*) \qquad \iint_{D_R} \frac{dxdy}{|z-z_1| \cdot |(z-z_2|} \leq C \cdot \text{Log}\,\frac{1}{\delta}$$

The verification of (*) is left as an exercise to the reader.

12 Some integral equations.

Introduction. The study of integral equations is a very extensive subject and has inspired both Fourier analysis and analytic function theory. here we review some material from [Paley-Wiener] which in addition to the examples below contain several other examples where analytic function theory is used to solve integral equations.

A. Planck's equation.

Following [Paley-Wiener. page 40] we recall the physical background for the integral equation in (*) below. By Planck's law the radiation per unit volume in a black cavity at temperature T in a state of steady equilibrium and of frequency between ν and $\nu + d\nu$ is given by

(1) $$\frac{8\pi \cdot h\nu^3}{c^3(e^{\frac{h\nu}{kT}} - 1} \cdot d\nu$$

Here h is Planck's constant and c is the velocity of light, and k is the "gas constant" reckoned for one molecule. This suggests that the radiation from a source in approximative local equilibrium but consisting of a mixture of black bodies of different temperatures, will have a distribution as a function of ν given by

(2) $$\nu^3 \cdot \int_0^\infty \frac{\phi(T) \cdot dT}{e^{\frac{h\nu}{kT}} - 1}.$$

where $\phi(T)$ represents the amount of radiation coming from black bodies at temperature T. If then, we have an observable radiation with frequency distribution $\psi(\nu)$, the problem of resolving this into its constituent black-body radiations is equivalent to the solution of the equation:

(*) $$\psi(\nu) = \nu^3 \cdot \int_0^\infty \frac{\phi(T) \cdot dT}{e^{\frac{h\nu}{kT}} - 1}.$$

One refers to (*) as Planck's integral equation. Following [Paley-Wiener] we show how one can find ϕ for a given ψ-function. Set

(i) $$\mu = \frac{h}{kT} \quad : \quad \phi(T) \cdot dT = \Phi(\mu) \cdot d\mu \quad : \quad \frac{\psi(\nu)}{\nu^2} = \Psi(\nu)$$

Then (*) assumes the form:

(ii) $$\Psi(\nu) = \int_0^\infty \Phi(\mu) \cdot \frac{\mu\nu}{e^{\mu\nu} - 1} \cdot d\mu$$

Above Ψ and Φ are both defined on $\mathbf{R}^+$. Recall that on this multiplicative group with coordinate t the invariant Haar measure is $\frac{dt}{t}$. We impose the L^2-condition on Ψ by

(iii) $$\int_0^\infty |\Psi(\nu)|^2 \cdot \frac{d\nu}{\nu} < \infty$$

Under this condition we shall find the solution $\Phi(\mu)$ which turns out to be an L^2-function on the multiplicative μ-line, i.e.

(iv) $$\int_0^\infty |\Phi(\mu)|^2 \cdot \frac{d\mu}{\mu} < \infty$$

Solution. The idea is to express (ii) in terms of Fourier transforms of Ψ and Φ. Given $\Phi \in L^2(\mathbf{R}^+)$ we define its normalised Fourier transform

(1) $$\widehat{\Phi}(u) = \frac{1}{\sqrt{2\pi}} \cdot \lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} \Phi(\mu) \cdot \mu^{(iu-1/2)} \cdot d\mu$$

At the same time the given L^2-function Ψ yields the L^2-function defined on the real v-line by

$$(1)\qquad \widehat{\Psi}(v)=\frac{1}{\sqrt{2\pi}}\cdot\lim_{\epsilon\to 0}\int_\epsilon^{1/\epsilon}\Psi(\nu)\cdot\nu^{(iv-1/2)}\cdot d\nu$$

With these notations the requested Φ-function is found via Fourier's inversion formula and the equation below:

$$(3)\qquad \Gamma(\frac{3}{2}-i\nu)\cdot\zeta(\frac{3}{2}-i\nu)\cdot\widehat{\Phi}(\nu)=\widehat{\Psi}(v)$$

Proof. To prove (3) one uses the equality

$$(A)\qquad \int_0^\infty\frac{x^{1/2+iv}\cdot dx}{e^x-1}=\Gamma(\frac{3}{2}+i\nu)\cdot\zeta(\frac{3}{2}+i\nu)$$

Exercise. Verify (A) and use Parseval's formula to recover (3). If necessary, consult page 40-41 in [Paley-Wiener].

B. The Laplace equation

Consider the integral equation

$$(*)\qquad g(u)=\int_0^\infty e^{-ux}\cdot f(x)\cdot dx$$

where $g(u)$ is defined when $u>0$. We assume that the L^2-integral

$$(1)\qquad \int_0^\infty|f^2(x)|\cdot dx<\infty$$

Then there exists the Laplace transform defined in the right half-space $u>0$ where $u+iv$ are complex coordinates and

$$(2)\qquad F(u+iv)=\int_0^\infty e^{-(u+iv)x}\cdot f(x)\cdot dx$$

Plancherel's theorem shows that $F(iv)$ belongs to L^2 and Fourier's inversion formula recaptures $f(x)$ via the restriction of F to the imaginary axis. However, in the equation (*) we are given $g(u)=F(u)$ when $u>0$. Now f is found by the Laplace inversion formula:

B.1 Theorem. *When (1) holds one has the inversion formula*

$$f(x)=\lim_{A\to\infty}\frac{1}{2\pi x}\int_0^\infty\rho_A(t)\cdot F(\frac{t}{x})\cdot\frac{dt}{\sqrt{t}}\quad\text{where}\quad\rho_A(t)=\int_{-A}^{A}\frac{t^{i\xi}\cdot d\xi}{\Gamma(i\xi+\frac{1}{2})}$$

Remark. This formula means that if we are given a function $g(u)$ on $u>0$ which is the restriction to this half-line of an analytic function $G(u+iv)$ in the complex half-space $u>0$ and the boundary value function $v\mapsto G(u+iv)$ belongs to L^2, then we obtain a solution $f(x)$ in L^2 expressed by the Laplace formula where $F(\frac{t}{x})=g(\frac{t}{x})$ for all pairs $t,x>0$.

Exercise. Prove the inversion formula of Laplace.

B.2 Widder's solutions. In the article *The inversion of the Laplace integral and the related moment problem*, D.V Widder found an exceedingly simple method to solve (*). Consider first a bounded and continuous function $f(x)$ defined on $x>0$. Then it is clear that the g-function in (*) is infinitely differentiable on $u>0$ and for every $n\geq 1$ we have

$$(-1)^n\cdot g^{(n)}(u)=\int_0^\infty x^n\cdot e^{-ux}\cdot f(x)dx$$

With $x > 0$ fixed we therefore get the following equality for every $n \geq 1$:

$$\frac{(-1)^n}{n!} \cdot g^{(n)}(\frac{n}{x}) \cdot (\frac{n}{x})^{n+1} = \frac{\int_0^\infty \xi^n \cdot e^{-n\xi/x} \cdot f(\xi)dx}{\int_0^\infty \xi^n \cdot e^{-n\xi/x} \cdot dx} \tag{1}$$

Exercise. Use the assumption that $f(x)$ is a bounded and continuous function on $x > 0$ to show that (1) gives the limit formula:

$$f(x) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \cdot g^{(n)}(\frac{n}{x}) \cdot (\frac{n}{x})^{n+1}$$

The hint is to use the functions $\psi_n(x) = \frac{1}{n!} \cdot x^n \cdot e^{-x}$ which are all ≥ 0 and the integral over $(0, +\infty)$ is one. Here $\psi_n(x)$ takes its maximum when $x = n$. Now we consider the functions

$$\rho_n(x) = \frac{\psi_n(\frac{x}{n})}{n}$$

Then the reader can verify that if $p(x)$ is an arbitrary bounded and continuous function on $x > 0$, then

$$p(x) = \lim_{n \to \infty} \frac{1}{n} \cdot \int_0^\infty \psi_n(\frac{x}{n}) \cdot p(x) \cdot dx \quad \text{hold for every} \quad x > 0$$

13. The Carleman-Hardy theorem.

We are given the situation in Theorem 0.3.7. The proof relies upon inequalities about differentiable functions.

Lemma. *Let $\psi(x)$ be a C^1-function defined for $x>0$ such that*

$$\lim_{x\to 0}\psi(x)=0 \quad \text{and} \quad |\psi'(x)|\leq \frac{C}{x} \quad : x>0$$

holds for some constant C. Then it follows that

(*)
$$\lim_{x\to 0} x\cdot\psi'(x)=0$$

The proof is left as an exercise to the reader. See also the article *Contributions to the Arithmetic Theory of Series* by Hardy and Littlewood for the proof and further extension to limit formulas for higher order derivatives. Next we establish a result due to Landau from the article *Einige ungleichungen für zweimal differentierbare Funktionen.*

1. Proposition. *Let $\psi(x)$ be a C^3-function defined on $x>0$ such that*

(i)
$$\lim_{x\to 0}\frac{\psi(x)}{x^2}=0 \quad \text{and} \quad |\psi'''(x)|\leq \frac{C}{x}$$

hold for some constant C. Then it follows that

(ii)
$$\lim_{x\to 0}\psi''(x)=0$$

Proof. Let $x>0$ and set $\xi=\zeta\cdot x$ where $0<\zeta<1/2$. Keeping these numbers fixed, Taylor's formula gives

$$\psi(x+\xi)+\psi(x-\xi)-2\psi(x)=\xi^2\psi''(x)+\frac{\xi^3}{6}\cdot[\psi'''(x+\theta_1\xi)-\psi'''(x-\theta_2\xi)] \quad : \ 0<\theta_1,\theta_2<1$$

The triangle inequality gives

$$|\psi''(x)|\leq$$

(2)
$$\frac{1}{\xi^2}\cdot[|\psi(x+\xi)|+|\psi(x-\xi)|+2|\psi(x)|]+\frac{\xi^3}{6}\cdot\big[\big(|\psi'''(x+\theta_1\xi)|+|\psi'''(x-\theta_2\xi)|\big)$$

By the second condition in (i) the last term above is majorised by

(3)
$$\frac{C}{6}\cdot\xi\cdot(\frac{1}{x+\theta_1\xi}+\frac{1}{x-\theta-2\xi}]\leq\frac{C}{6}\cdot\frac{2\zeta}{1-\zeta}$$

Given $\epsilon>0$ we can choose ζ so small that (3) is $<\epsilon/2$. Next, keeping ζ fixed the first term in (2) above is majorised by

(4)
$$\frac{1}{\zeta^2x^2}\cdot\big[(1+\zeta)^2\cdot o(x^2)+(1-\zeta)^2\cdot o(x^2)+2\cdot o(x^2)\big]$$

where the small ordo terms follows from the first condition in (i). Now we see that (ii) in Proposition 1 follows from the inequality (2) above.

Proof of Theorem 0.3.7 Assume first that $\sum A_n$ converges and define the following two functions when $x > 0$:

(i) $$U(x) = \frac{1}{2}A_0 \cdot x^2 + \sum_{n=1}^{\infty} \frac{A_n}{n^2} \cdot (1 - \cos(nx))$$

(ii) $$V(x) = \int_0^x \big[\int_0^y u(s) \cdot ds\big] \cdot dy$$

In (i) $U(x)$ is the associated Riemann function of u. Since u is of class C^2 when $x > 0$ it is clear that $U''(x) = V''(x)$ when $x > 0$ and we conclude that $U(x) - V(x)$ is a linear function $C + Dx$ on $(0, +\infty)$. Next, since u by assumption is an L^1-function we see that $V(x) = o(x)$ and we also have $U(x) = o(x^2)$ by a classical result known as Riemann's Lemma. It follows that $C = D = 0$, i.e. the functions U and V are identical which gives

$$V(x) = o(x^2)$$

Next, notice that

(ii) $$x > 0 \implies V'''(x) = u'(x)$$

Now $|u''(x)| \leq \frac{C}{x^2}$ was assumed in 0.3.6 which obviosly gives another constant C^* such that $|u'(x)| \leq \frac{C^*}{x}$ so (ii) that V satisfies the Landau conditions in Proposition 1 which gives:

(iii) $$\lim_{x \to 0} V''(x) = 0$$

Finally, since $V''(x) = u(x)$ holds when $x > 0$ we conclude that $\lim_{x \to 0} u(x) = 0$. This proves one half of Theorem 0.3.7.

The case $\lim_{x \to 0} u(x) = 0$. We must show that $\sum A_n$ converges. By assumption $|u''(x)| \leq \frac{C}{x^2}$ which gives a constant C^* such that $|u'(x)| \leq \frac{C^*}{x}$ and then Lemma 1 gives

(i) $$u'(x) = 0(x)$$

Next, we use a result by Lebesgue from his book *Lecons des series trigonmétriquees* which asserts that the series $\sum A_n$ converges if

(ii) $$\lim_{\epsilon \to 0} \int_\epsilon^1 \frac{|u(x + \epsilon) - u(x)|}{x} \cdot dx = 0$$

To get (ii) we use Rolle's theorem and write

(iii) $$u(x + \epsilon) - u(x) = \epsilon \cdot u'(x + \theta \cdot \epsilon)$$

Now it is clear that (i) and (iii) give (ii) which finishes the proof that $\sum A_n$ is convergent.

14. The log-potential and an integral equation

Let μ be a Riesz measure on the closed unit interval $0 \leq t \leq 1$. We get the logarithmic potential and the Cauchy transform:

$$(*) \qquad U_\mu(z) = \int_0^1 \log |z-t| \cdot d\mu(t) \quad \text{and} \quad \mathcal{C}_\mu(z) = \int_0^1 \frac{d\mu(t)}{z-t}$$

Restrict U_μ to the real interval $0 \leq x \leq 1$ and set

$$(1) \qquad T_\mu(x) = \int_0^1 \log |x-t| \cdot d\mu(t)$$

Since the function $t \mapsto \log |t|$ is locally integrable it follows that $T_\mu(x)$ is an L^1-function on $[0,1]$.

0.1. Some boundary values. In the upper half-plane $\mathfrak{Im}(z) > 0$ we choose a single-valued branch of $\log(z-t)$ when $0 \leq t \leq 1$ and get the analytic function

$$(i) \qquad F_+(z) = \int_0^1 \log (z-t) \cdot d\mu(t)$$

Similarly we get an analytic function $F_-(z)$ in the lower half-plane and take their boundary values as distributions on the real x-line. Here

$$\mathbf{b}F_+(x) + \mathbf{b}F_-(x) = \lim_{\epsilon \to 0} \int_0^1 [\log (x-t+i\epsilon) + \log (x-t-i\epsilon)] \cdot d\mu(t) =$$

$$(1) \qquad \lim_{\epsilon \to 0} \int_0^1 [\log ((x-t)^2 + \epsilon^2) \cdot d\mu(t) = 2 \cdot T_\mu(x)$$

Next, let $\mathcal{C}_\mu^+$ be the restriction of the Cauchy transform to the upper half-plane, and similarly $\mathcal{C}_\mu^-$ is the restriction to the lower half-plane. Now the complex derivative $\partial_z(F_+) = \mathcal{C}_\mu^+$ and similarly in the lower half-plane and since the passage to boundary values commutes with differentiation as explained in XX we get the equality

$$(2) \qquad 2 \cdot \partial_x(T_\mu) = \mathbf{b}(\mathcal{C}_\mu^+) + \mathbf{b}(\mathcal{C}_\mu^+)$$

where we have restricted the boundary value distributions of the two $\mathcal{C}_\mu$ parts to $0 < x < 1$. The distribution in (2) is given by

$$(3) \qquad \lim_{\epsilon \to 0} \int_0^1 [\frac{1}{x-t+i\epsilon} + \frac{1}{x-t-i\epsilon}] \cdot d\mu(t) = \lim_{\epsilon \to 0} \int_0^1 [\frac{x-t}{(x-t)^2+\epsilon^2} \cdot d\mu(t)$$

where the limit is defined as a principal value.

0.2 A Fourier transform. The T-operator is like a convolution. Using the variable substitution $(x,t) \mapsto (x-t, x+t)$ we can describe the behaviour of the Fourier transform

$$\widehat{T}_\mu(\xi) = \iint_\square e^{-ix\xi} \cdot \log |x-t| \cdot d\mu(t)$$

0.3 Exercise. Verify the formula

$$\text{(i)} \qquad \int_{-1}^{1} e^{-iu\xi} \cdot \log|u| \cdot du = 2 \cdot \int_{-1}^{1} \cos u\xi \cdot \log u \cdot du = -\frac{2}{\xi} \int_0^{\xi} \frac{\sin u}{u} \cdot du$$

Use this to conclcude that

$$\widehat{T}_\mu(\xi) = -2 \cdot \frac{\widehat{\mu}(\xi)}{\xi} + \rho(\xi) \cdot \widehat{\mu}(\xi)$$

where the ρ-function decays at infinity like ξ^{-2}, i.e. there is an absolute constant C for which

$$|\rho(\xi)| \leq C \cdot \xi^{-2} \quad : \quad |\xi| \geq 1$$

0.4 Remark. Above μ has compact support so $\widehat{\mu}(\xi)$ is the restriction to the real ξ-line of the entire Fourier-Laplace transform. In particular $\widehat{\mu}$ is a bounded function so (1) entails that the Fourier transform of the L^1-function $T_\mu(x)$ at least decays like $\frac{1}{\xi}$ as $\xi \to +\infty$. Parseval's formula gives the inclusion

$$T_\mu \in L^2[0,1]$$

However, the continuity of T_μ is not clear since the decay is not sufficient to ensure that $\widehat{T}_\mu(\xi)$ is absolutely integrable on the real ξ-line.

0.5 Exercise. Restrict the T-operator to absolutely continuous measures, i.e now $\mu = g(t) \cdot dt$ for some L^1-function. Use the material above to show that

$$g \mapsto T_g$$

is an injective operator from $L^1[0,1]$ into itself whose range is a dense subset of $L^1[0,1]$.

After the preliminary results above we turn to the main issue in this section.

1. The Abel-Carleman inversion formula.

With $g \in L^1[0,1]$ we set $f = T_g$ and our aim is to express g via f. It turns out that there exists an inversion formula which is similar to the classic inversion established by Abel in a remarkable note from 1824 where he expressed the potential function $U(x)$ in a conservative force field on the real line. Adopting Abel's formula to the present case an inversion formula was established in our present context by Carleman in in the article *Ablesche Integralgleichungen mit konstanten Integrationsgrenzen.* In order that the inversion formula in Theorem1 below is valid one needs some extra regularity on f. This is o be expected since we already have seen that the range of T is a proper subspace of $L^1[0,1]$. Before we announce the result we insert a preliminary discussion.

If $h(x)$ is an L^1-function on $[0,1]$ there exists the distribution $P(h)$ given via a principal value limit as in (3) from (0.1), i.e. it is given by

$$(*) \qquad P_h(x) = \lim_{\epsilon \to 0} \int_0^1 \frac{x-t}{(x-t)^2 + \epsilon^2} \cdot h(t)$$

Here P_h yields an L^1-function for a dense subspace of L^1. For example, it suffices that h is continuously differentiable. Denote by $\mathcal{P}$ the set of all h-functions in $L^1[0,1]$ such that P_h also belongs to L^1. Consider the integrable function $\frac{1}{\sqrt{x(1-x)}}$ and denote by $\mathcal{P}_*$ the set of L^1-functions h such that

$$\frac{1}{\sqrt{x(1-x)}} \cdot h \in \mathcal{P}$$

1. Theorem. *Let $f(x)$ be an absolutely continuous function on $[0,1]$ such that its L^1-derivative $f'(x)$ belongs to $\mathcal{P}_*$. Then f belongs to the range of T and one has the inversion formula*

$$T^{-1}f(x) = \frac{1}{\pi^2} \cdot \frac{1}{\sqrt{x(1-x)}} \cdot P_h(x) - \frac{K}{\sqrt{x(1-x)}} \cdot \int_0^1 \frac{f(s) \cdot ds}{\sqrt{s(1-s)}}$$

where K is a constant given by

$$K = -\frac{1}{2\pi^2 \cdot \log 2}$$

Remark. The inversion formula gives a description of the range of the T-operator, i.e. the necessary and sufficient condition for f to be in the range is that it is absolutely continuos and its L^1-derivative belongs to $\mathcal{P}_*$. The proof of Theorem 1.1 relies upon the general formulas in section 0 above together with an integral equality which goes back to Legendre.

1.2 Legendre's Lemma. *The function*

$$x \mapsto \int_0^1 \log|x-t| \cdot \frac{1}{\sqrt{t}(1-t)} \cdot dt$$

is a constant function whose value is given by $-2\pi \cdot \log 2$.

1.3 Exercise. Prove Legendre's Lemma.

Proof of Theorem 1.1

In $\mathbf{C} \setminus [0,1]$ there exists the analytic function $\sqrt{z(z-1)}$ and we set

$$G(z) = \sqrt{z(z-1)} \cdot \mathcal{C}_g(z)$$

Notice that one has

$$\lim_{\epsilon \to 0} \sqrt{z+i\epsilon)(z+i\epsilon-1)} = i \cdot \sqrt{x(1-x)} \quad : \quad 0 < x < 1$$

From this we conclude that

$$\mathbf{b}(G_+) = i \cdot \sqrt{x(1-x)} \cdot \mathbf{b}\mathcal{C}_g^+$$

In the same way we find that

$$\mathbf{b}(G_-) = -i \cdot \sqrt{x(1-x)} \cdot \mathbf{b}\mathcal{C}_g^-$$

It follows that

$$\mathbf{b}(G_+) + \mathbf{b}(G_-) = 2i\sqrt{x(1-x)} \cdot (\mathcal{C}_g^+ - \mathcal{C}_g^-)$$

Using (2.0) we obtain

$$2 \cdot \partial_x(T_g) = 2i\sqrt{x(1-x)} \cdot \cdot g$$

TO CONTINUE AND FINISH

15. An L^1-inequality for inverse Fourier transforms.

Introduction. Theorem 1 below is due to Beurling in [Beurling]. Let $g(t)$ be a function defined on $t \geq 0$ and assume that the inverse Fourier transform of $tg(t)$ is integrable, i.e. the function defined on the x-axis by

$$f(x) = \int_0^\infty e^{itx} \cdot tg(t) \cdot dt \tag{*}$$

belongs to $L^1(\mathbf{R})$.

15.1 Theorem. When $f \in L^1(\mathbf{R})$ it follows that $g(t)$ is integrable and one has the inequality

$$\int_0^\infty |g(t)| \cdot dt \leq \frac{1}{2} \int_{-\infty}^\infty |f(x)| \cdot dx$$

Proof. Since (*) is taken over $t \geq 0$ it follows that $f(x)$ is the boundary value function of the analytic function defined in $\mathfrak{Im}(z) > 0$ by

$$f(z) = \int_0^\infty e^{itz} \cdot tg(t) \cdot dt \tag{1}$$

Here f belongs to the Hardy space $H^1(\mathbf{R})$ which is defined and studied in section XX from Special Topics. Let us first prove the inequality in Theorem 15.1 under the assumption that $f(z)$ is zero-free in the upper half-plane and consider the normalised situation where the L^1-integral of $|f(x)|$ is one. Then we construct the complex square root of $f(z)$ and obtain the analytic function $F(z)$ for which $F^2 = f$. Here $|F(x)|^2) = |f(x)|$ which means that F belongs to the Hardy space $H^2(\mathbf{R})$. Plancherel's theorem gives a function $h(t)$ on $t \geq 0$ where

$$F(z) = \int_0^\infty e^{itz} \cdot h(t)dt \tag{2}$$

Moreover, Parseval's equality gives

$$1 = \int |F^2(x)| \cdot dx = 2\pi \cdot \int_0^\infty |h^2(t)| \cdot dt$$

The Fourier transform of the convolution $h * h$ is equal to $F^2(x) = f(x)$ which gives

$$\cdot g(t) = \int_0^t h(t-s)h(s) \cdot ds \tag{3}$$

This gives the inequality

$$|t \cdot g(t)| \leq G(t) = \int_0^t |h(t-s)| \cdot |h(s)| \cdot ds$$

Let us define the function

$$F_*(x) = \int_0^\infty e^{itz} \cdot |h(t)|dt \tag{4}$$

Parseval's formula applied to $/h/$ and F_* gives

$$\int |F^2(x)| \cdot dx = 2\pi \cdot \int_0^\infty |h^2(t)| \cdot dt$$

We conclude that the L^2-norm of F^* also is one and here

$$F_*^2(x) = \int_0^\infty e^{itz} G(t) \cdot dt$$

At this stage we use a result from XXX which shows that the function

$$\theta \mapsto \log \big[\int_0^\infty |F_*(re^{i\theta}]|^2 \cdot dr \big]$$

is a convex function of θ where $-\pi \leq \theta \leq 0$. We apply this when $\theta = \pi/2$ and the end-values 0 and π which gives

$$\int_0^\infty |F_*(iy)|^2 \cdot dy \leq \sqrt{\int_{-\infty}^0 |F_*(x)|^2 \cdot dx} \cdot \sqrt{\int_0^\infty |F_*(x)|^2 \cdot dx}$$

Notice that $x \mapsto |F_*(x)|$ is an even function of x which implies that the product above via (XX) is equal to one. Hence we have

$$\int_0^\infty \big[\int_0^\infty e^{-ty} \cdot G(t) \cdot dt\big] \cdot dy \leq \frac{1}{2}$$

integration by parts shows that the left hand side is equal to

$$\int_0^\infty \frac{G(t)}{t} \cdot dt$$

Finally, from (xx) we have $|g(t)| \leq \frac{G(t)}{t}$ and conclude that the L^1-norm of g is bounded by $\frac{1}{2}$.

Removing zeros. If f is not zero-free we let $B(z($ be the Blaschke product of its zeros nd write

$$f = B(z)\phi(z)$$

So here ϕ is zero-free and we do not change the L^1-norm on the x-line where $|B(x)| = 1$ holds almost everywhere. Notice that we can write

$$f = \phi\big(\frac{1+B}{2}\big)^2 + \phi\big(\frac{1-B}{2}\big)^2 = F_1^2 - F_2^2$$

where F_1 and F_2 as above are zero-free in the Hardy space H^2. Since we have

$$|\frac{1+B}{2}|^2 + |\frac{1-B}{2}|^2 \leq 1$$

it follows that

$$|F_1|^2 + |F_2|^2 \leq |\phi|$$

Using the previous zero-free case we read off the requested inequality in Theorem X.

16. On function with spectral gap

Introduction. A fore-runner to distribution theory appeared in work by Beurling around 1940 when he analyzed the effect of spectral gaps of functions f on the real x-line. A spectral gap occurs when the Fourier transform $\widehat{f}(\xi)$ vanishes on some interval on the ξ-line. We expose a result from a seminar by Beurling at at Uppsala University in March 1942 where spectral gaps yield a certain estimate,

16.1 Theorem. *Let $f(x)$ be a bounded and continuous function on the real x-line with a spectral gap on $(-1,1)$ and assume also that*

$$f(x+h)-f(x)\leq h$$

hold for all $h>0$ and every x. Then its maximum norm is at most π.

The proof relies uses the following entire function:

$$2\cdot H(z)=\big(2\sin\frac{z}{2}\big)^2\cdot\big[\sum_{n=1}^{\infty}\frac{1}{(z-2\pi n)^2}-\sum_{n=0}^{\infty}\frac{1}{(z+2\pi n)^2}+\frac{1}{\pi z}\big]$$

A. Exercise. Verify the identity

$$\frac{1}{\big(2\sin\frac{z}{2}\big)^2}=\sum_{-\infty}^{\infty}\frac{1}{(z-2\pi n)^2}$$

The hint is to consider the meromorphic function

$$\phi(z)=\frac{\cos z/2}{2\sin z/2}$$

It has simple poles at $\{2\pi n\}$ where n runs over all integers and we notice that

$$\psi(z)=\phi(z)-\sum_{-\infty}^{\infty}\frac{1}{z-2\pi n}$$

is entire. Hence the complex derivative is entire too. Since $\cos^2\, z/2+\sin^2\, z/2=1$ we see that

$$\psi(z)=-\frac{1}{\big(2\sin\frac{z}{2}\big)^2}+\sum_{-\infty}^{\infty}\frac{1}{(z-2\pi n)^2}$$

At the same time the reader may verify that the right hand side is bounded so this entire function must be identically zero which gives (*).

B. Exercise. Use (*) to show that if $x>0$ then

$$2H(x)=1-\big(2\sin\frac{x}{2}\big)^2\cdot\big[\sum_{n=1}^{\infty}\frac{2}{(x+2\pi n)^2}+\frac{1}{x^2}-\frac{1}{\pi x}\big]$$

The θ-function. It is defined for all real x by:

$$\theta(x)=\frac{1}{2}\cdot\operatorname{sign}(x)-H(x)$$

where $\operatorname{sign}(x)$ is -1 if $x<0$ and $+1$ if $x>0$.

16.2 Proposition. *The θ-function is everywhere ≥ 0 and*

$$(*)\qquad\int_{-\infty}^{\infty}\theta(x)\cdot dx=1$$

Proof. If $x > 0$ we see that Exercise B gives

$$\theta(x) = \frac{1}{2}\big(2\sin\frac{x}{2}\big)^2 \cdot \big[\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} - \frac{1}{\pi x}\big]$$

Next, notice the two inequalities

$$\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} \leq \int_0^{\infty} \frac{2dt}{(x+2\pi t)^2} = \frac{1}{\pi x} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} \geq \int_0^{\infty} \frac{2dt}{(x+2\pi t)^2} = \frac{1}{\pi x} \tag{2}$$

Here (2) entails that $\theta(x) \geq 0$ on $x > 0$ and (1) obviously implies that the integral

$$\int_0^{\infty} \theta(x) \cdot dx < \infty \tag{3}$$

We leave as an exercise to the reader to verify the similar result for $x < 0$, i.e. that $\theta(x) \geq 0$ hold for $x < 0$ and that its integral over $-\infty, 0)$ is finite. There remains to establish the equality (*) in Proposition 16.2. To show this we notice that the function

$$\text{sign}(x) + (2\sin\frac{x}{2})^2 \cdot \frac{1}{\pi x}$$

is odd so its integral over the real line is zero. Similarly we notice the equalities the integrals

$$\int_{-\infty} \frac{(\sin\frac{x}{2})^2}{(x-2\pi n)^2} \cdot dx = \int_{-\infty} \frac{(\sin\frac{x}{2})^2}{(x+2\pi n)^2} \cdot dx \quad \text{for every} \quad n \geq 1$$

From this the reader can verify the equality

$$\int_{-\infty}^{\infty} \theta(x) \cdot dx = \int_0^{\infty} \frac{(2\sin\frac{x}{2})^2}{x^2} \cdot dx$$

and residue calculus shows that the last integral is π.

Proof of Theorem 16.1

Notice that $\frac{1}{2}\text{sign}(x)$ is equal to $\mathcal{H} - \frac{1}{2}$ where $\mathcal{H}$ is the Heaviside function. Regarding f as a temperate distribution we have the identity

$$f = \frac{d}{dx} f * (\mathcal{H} - \frac{1}{2}) \tag{i}$$

Next, as explained in XXX the Fourier transform $\widehat{H}(\xi)$ is supported by $[-1,1]$ and since the support of $\widehat{f}$ is disjoint from $[-1,1]$ it follows that $f * H = 0$ and hence its distribution derivative also is zero. So from (i) we conclude that

$$f = \frac{d}{dx}(f * \theta) = 0$$

This means that we have the formal equality

$$f(x) = \int_{-\infty}^{\infty} \theta(x-y) \cdot f'(y) \cdot dy$$

By assumption $f'(y) \leq 1$ for all y and since $\theta \geq 0$ and has integral one we conclude that the right hand side is bounded above by

$$\int_{-\infty}^{\infty} \theta(x-y) \cdot dy = \pi$$

Hence we have proved the inequality

$$f(x) \leq \pi$$

for all x. To get $f(x) \geq -\pi$ we consider the function

$$g(x) = -f(-x)$$

It is again a bounded continuous function and the reader easily verifies that $g(x+h) - g(x) \leq h$ for all $h > 0$. Moreover, $\widehat{g}(\xi)$ is minus the complex conjugate of $\widehat{f}$ so g has the same spectral gap as f and now we have the upper bound $g(x) \leq \pi$ for all x which entails that $f(x) \geq -\pi$ hold for all x and we conclude that its maximum norm is bounded by π which finishes the proof of Theorem 16.1

17. A theorem by Beurling

Denote by C_* the space of complex-valued bounded and uniformly continuous functions on the real x-line which becomes a Banach space when we use the maximum norm taken over the whole x-line. A subspace arises as follows: On the real ξ-line we have the space of complex Riesz measures γ with a finite total variation and to each γ we get the function

$$\mathcal{F}_\gamma(x) = \int_{-\infty}^{\infty} e^{ix\xi} \cdot d\gamma(\xi)$$

It is clear that $\mathcal{F}_\mu$ belongs to C_*. Denote by $\mathcal{A}$ the subspace of C_* given by $\mathcal{F}_\mu$-functions as γ varies over all Riesz measures as above. We shall determine the closure of $\mathcal{A}$ in the Banach space C_*. Before we announce Theorem 17.1 below we recall the notion of weak-star limits in the space of Riesz measures on the x-line. Let $\{\mu_n\}$ be a bounded sequence of Riesz measures, i.e. there exists a constant such that

$$||\mu_n|| \leq M$$

hold for all n. The sequence $\{\mu_n\}$ converges weakly to zero if

$$\lim_{n\to\infty} \int e^{ix\xi} \cdot d\mu_n(x) = 0$$

holds for every ξ.

17.1 Theorem. A function $\psi \in C_*$ belongs to the closure of $\mathcal{A}$ if and only if

$$(*) \qquad \lim_{n\to\infty} \int \psi(x) \cdot d\mu_n(x) = 0$$

whenever $\{\mu_n\}$ is a sequence of Riesz measures which converges weakly to zero.

Let us first show the sufficiency, i.e. that we get the requested zero-limit if ψ belongs to the closure of $\mathcal{A}$. Since the total variations in a weakly convergent sequence of measures by definition is uniformly bounded it suffices to show that (*) holds when $\psi \in \mathcal{A}$. So let $\psi = \mathcal{F}_\gamma$ and a sequence $\{\mu_n\}$ converges weakly to zero. Since γ and μ_n both have a finite total variation it is clear that

$$\int \psi(x) \cdot d\mu_n(x) = \int \mathcal{F}_{\mu_n}(\xi) \cdot d\gamma(\xi)$$

Moreover $\{\mathcal{F}_{\mu_n}(\xi)\}$ is a sequence of uniformly bounded continuous functions on the real $|xi$-oine which by assumption converges pointwise to zero. Since the Riesz measure γ has a finite total variation it follows by the Borel-Riesz convergence result in[Measure] that (xx) tends to zero with n.

Proof of necessity.

There remains to show that if $\psi \in C_*$ is outside the closure of $\mathcal{A}$, then there exists a sequence $\{\mu_n\}$ which converges weakly to zero while $\int \psi \cdot d\mu_n$ stay away from zero. To attain this we shall consider a class of variational integrals and extract a certain sequence of measures which does the job.

The variational integrals $\mathcal{J}(g; q, b, a, s)$. Let a, b, s be positive numbers and $q > 2$. With p chosen so that

$$\frac{1}{p} + \frac{1}{q} = 1$$

we have the space $L^p[-a,a]$ where $[-a,a]$ is an interval on the ξ-line. To each function $g(\xi) \in L^p[-a,a]$ we get the function

$$\mathcal{F}_g(x) = \int e^{ix\xi} \cdot g(\xi) \cdot d\xi$$

This gives a continuous function which is restricted to $[-b,b]$ and we set

$$||\psi - \mathcal{F}_g||_q^b = \big[\int_{-b}^{b} [\psi(x) - \mathcal{F}_g(x)|^q \cdot dx\,\big]^{1/q}$$

where the upper index b indicates that we compute a L^q-norm on the bounded interval $[-b,b]$. To each $g \in L^p[-a,a]$ we set

$$(*) \qquad \mathcal{J}(g;q,b,a,s) = ||\psi - \mathcal{F}_g||_q^b + ||g||_p$$

where the last term is the L^p-norm of g taken over $[-a,a]$. Recall that the Banach space $L^p[-a,a]$ is strictly convex. using this one easily verifies:

17.2 Proposition. *The variational problem where $\mathcal{J}$ is minimized over g while a,b,s are fixed has a unique extremal solution.*

17.3 Exercise. Regarding infinitesmal variations via the classic device due to Euler and Lagrange, the reader can verify that there exists a unique extremal solution g which satisfies

$$(*) \qquad ||g||_p)^{1-p} \cdot \frac{|g(\xi)|^p}{g(\xi)} = M^{-1/p} \cdot \int_{-b}^{b} e^{i\xi x} \cdot \frac{|\psi(x) - \mathcal{F}_g(x)|^q}{|\psi(x) - \mathcal{F}_g(x)} \cdot dx$$

where we have put

$$M = \int_{b}^{b} |\psi - \mathcal{F}_g|^q \cdot dx$$

Using (*) we establish some useful formulas. Consider the absolutely continuous measure on the x-line defined by the density

$$d\mu = M^{-1/p} \cdot \frac{\psi(x) - \mathcal{F}_g(x)|^q}{|\psi(x) - \mathcal{F}_g(x)} \quad : \quad -b \leq x \leq b$$

This gives

$$\int_{-b}^{b} |d\mu(x)| = M^{-1/p} \cdot \int_{-b}^{b} |\psi(x) - \mathcal{F}_g(x)|^{q-1} \cdot dx$$

Hölder's inequality applied to the pair of functions $|\psi(x) - \mathcal{F}_g(x)|^{q-1}$ and 1 on $[-b,b]$ gives the inequality below for the total variation:

$$(*) \qquad ||\mu|| \leq (2b)^{1/q}$$

17.4 Lemma. The following two formulas hold:

$$\int \psi \cdot d\mu = J(g;q,b,a,s)$$

$$\int \mathcal{F}_g \cdot d\mu = s \cdot ||g||_p$$

Proof. To begin with we have

$$\int (\psi - \mathcal{F}_g) \cdot d\mu = M^{-1/p} \cdot \int_{b}^{b} |\psi - \mathcal{F}_g| \cdot dx = M^{1-1/p} \cdot M = M^{1/q}$$

Next Fubini's theorem gives

$$\int \mathcal{F}_g \cdot d\mu = \int_a^a [\int e^{ix\xi} \cdot \mu(x)] \cdot g(\xi) \cdot d\xi = s \cdot \int_a^a ||g||_p)^{1-p} \cdot \frac{|g(\xi)|^p}{g(\xi)} \cdot g(\xi) d\xi = s \cdot ||g||_p$$

This proves formula (2) and (1) follows via (i) and the equality

$$J(g; q, b, a, s) = M + s \cdot ||g||_p$$

17.4 Passage to limits. Following [Beurling] we now consider certain limits where we first let $q \to +\infty$ and after $b \to \infty$, and finally we use pairs $a = 2^m$ and $s = 2^{-m}$ where m will be large positive integers. To begin with, the measure μ depends on q, b, a, s and let us denote it by $\mu_q(b, a, s)$. The uniform bound (*) from Exercise 17.3 entails that while a, b, s are kept fixed, then there is a sequence $\{q_\nu\}$ which tends to $+\infty$ and give a weak limit measure

$$\mu_*(b, a, s) = \lim \mu_{q_\nu}(b, a, s)$$

The extremal g-functions depend on q and are denoted by g_q. Their L^q-norms remain bounded and passing to a subsequence we get an L^∞-function g_* on $[-a, a]$ where $g_{q_\nu} \to g_*$. Here g_* depends on b, a, s and is therefore indexed as $g_*(b, a, s)$. We have also a limit:

$$\lim \mathcal{J}(g_g; q_\nu; b, a, s) = \mathcal{J}_*(g_*(a, b, s))$$

Moreover

$$\mathcal{J}_*(g_*(a, b, s)) = \max_{-b \leq x \leq b} |\psi(x) - \mathcal{F}_{g_*(b,a,s)}| + s \cdot \int_{-a}^a |g_*(b, a, s)(\xi)| \cdot d\xi$$

At this stage we use the hypothesis that ψ does not belong to the closure of $\mathcal{A}$ which entails that with a and s fixed, then here is a constant $\rho > 0$ such that

$$\liminf_{b \to \infty} \max_{-b \leq x \leq b} |\psi(x) - \mathcal{F}_{g_*(b,a,s)}| \geq \rho$$

At this stage proof is easily FINISHED.

18. Lindeberg's central limit theorem.

Introduction The classical version of the CLT (central limit theorem) was proved by De Moivre in 1733 and asserts that as n tends to infinity, the standardized binomial distribution tends to the normal distribution. Thus, let $\mathbf{B}$ be the two point random variable which takes the values +1 or -1 with probability 1/2. If $n \geq 2$ we denote by $\mathbf{B}_1, \ldots, \mathbf{B}_n$ an n-tuple of independent two points variables. This gives the random variable

$$(*) \qquad \chi_n = \frac{\mathbf{B_1} + \ldots + \mathbf{B_n}}{\sqrt{n}}$$

De Moivre's proof that χ_n tends to the normal distribution was direct in the sense that characteristic functions were not used. To see the idea we let $n = 2N$ be a large even integer. If $k \geq 1$ we denote by $\rho_N(k)$ the probability that the number of heads minus the number of tails is $\leq 2k$ after $2N$ many trials. The binomial formula gives

$$\rho_N(k) = \frac{1}{2} + 2^{-2N} \cdot \sum_{\nu=0}^{\nu=N+k} \binom{2N}{\nu}$$

The normal distribution is given by the increasing function

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-t/2} \cdot dt$$

With $k = \sqrt{N} \cdot a$ one wants the limit formula

$$(1) \qquad \lim_{N \to \infty} \rho_N(\sqrt{N} \cdot a) = \mathcal{N}(2a)$$

for each $a > 0$. This can be proved directly using Wallis' limit formula for products of sine-functions which gives Stirling's formula with a remainder term and therefore a quite sharp estimate for the rate of convergence. For readers familiar with Swedish a proof with good upper bounds for the rate of convergence is presented by Carleman in his outstanding text-book from 1926 for first year studies on university level which actually covers the more general where the equally distributed variables have a finite distribution. See also the section *Residue Calculus* for a proof of Wallis' formula.

A more general version of the CLT was formulated by Simon Laplace in his classic treatise on probability from 1812. His assertion was that sums of independent random variables which are suitably scaled so that the partial sums χ_n are random variables with mean value zero and variances σ_n which converge to a number σ, implies that the sequence χ_n converges to a normal distribution with variance σ, under the *hypothesis* that the individual random variables defining the sum variables χ_n give a *relatively insignificant contribution.* A rigorous proof of the CLT in the spirit of Laplace was given by Liapunoff in 1901. His result goes as follows: Let $W_1, W_2, \ldots$ be a sequence of independent random variables, each with mean value zero and variance σ_ν. Assume that there exists a constant M such that the *moment of order 3* is $\leq M$ for each W_ν and that the limit

$$\lim_{n \to \infty} \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\sqrt{n}} = \sigma^2$$

exists. Then that partial sums variables defined by

$$\chi_n = \frac{W_1 + \ldots + W_n}{\sqrt{n}}$$

converge to the normal distribution with variance σ. The conclusive version of the CLT was proved by Lindeberg in 1920. He weakened the conditions in Liapunoff's result by relaxing the hypothesis about finite moments of order 3 and proved the following:

1. Theorem *Let $W_1, W_2, \ldots$ be independent random variables with mean values zero. A sufficient condition in order that the sum variables*

$$\chi_n = \frac{W_1 + \ldots + W_n}{\sqrt{n}}$$

converge to a normal distribution with variance σ is that the following three conditions hold:

$$\lim_{n\to\infty} \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\sqrt{n}} = \sigma^2 \tag{1}$$

(2) There exists a constant M such that $\sigma_\nu \leq M$ for all ν

$$\lim_{n\to\infty} \frac{\sum_{\nu=1}^{\nu=n} \int_{|x|>\delta\sqrt{n}} x^2 \cdot dW_\nu(x)}{n} = 0 \quad \text{hold for all} \quad \delta > 0 \tag{3}$$

Remark. Condition (3) means that the individual random variables do not have fat tails.

Proof of Theorem 1

Let $W_1, W_2, \ldots$ be a sequence of random variables satisfying the conditions above. Without essential loss of generality we may assume that each W_ν is a finite discrete random variable given by a finite sum $\sum p_j(\nu) \cdot \delta_{x_j(\nu)}$. Let $\mathcal{C}_\nu(\xi)$ be its characteristic function:

$$\mathcal{C}_\nu = \sum p_j(\nu) e^{ix_j(\nu)\xi} \tag{*}$$

Theorem 1 follows by the convergence criterion in XX if

$$\lim_{n\to\infty} \prod_{\nu=1}^{\nu=n} \mathcal{C}_\nu(\xi/\sqrt{n}) \to e^{-\xi^2\sigma^2/2}$$

holds uniformly when ξ varies in a compact interval $[-A, A]$. From now on $A > 0$ is kept fixed and we choose $\delta > 0$ such that $A \cdot \delta \leq 1$. For each positive integer n we set

$$\Psi_\nu(n) = \sum_{|x_j(\nu)|\geq\delta\sqrt{n}} p_j(\nu) x_j(\nu)^2 \tag{1}$$

Lindeberg's 3rd condition gives

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} \Psi_\nu(n) = 0 \tag{2}$$

To simplify notations, let Σ_* denote summation over those j where $|x_j(\nu)| < \delta\sqrt{n}$ and Σ^* is the summation when $|x_j(\nu)| \geq \delta\sqrt{n}$. For a given ν we obtain

$$\mathcal{C}_\nu(\xi/\sqrt{n}) = \sum_* p_j(\nu) e^{ix_j(\nu)\xi} + \sum^* p_j(\nu) e^{ix_j(\nu)\xi} \tag{3}$$

We study the two sums separately. and begin to stud σ_*. Since $|x_j(\nu)| \cdot \xi/\sqrt{n}| \leq \delta \cdot A \leq 1$ hold under Σ_*, the Taylor estimate in XX applied to each j gives:

$$\sum_* p_j(\nu) e^{ix_j(\nu)\xi} =$$

$$\sum_* p_j(\nu) + \frac{i\xi}{\sqrt{n}} \sum_* p_j(\nu) x_j(\nu) - \frac{\xi^2}{2n} \cdot \sum_* p_j(\nu) x_j^2(\nu) + \text{error}(\nu)$$

where

$$|\text{error}(\nu)| \leq \frac{|\xi|^3}{n\sqrt{n}} \cdot \sum_* p_j(\nu)|x_j(\nu)|^3 \leq \frac{A^3\delta\sqrt{n}}{n\sqrt{n}} \cdot \sum_* p_j(\nu) x_j(\nu)^2 \leq \frac{A^3 M\delta}{n} \tag{4}$$

Above M is the uniform upper bound of the variances of the random variables $\{W_\nu\}$. Next, we use that

$$(5)\qquad \sum_* p_j(\nu) = 1 - \sum^* p_j(\nu) \quad \text{and} \quad \sum_* p_j(\nu)x_j(\nu) = -\sum^* p_j(\nu)x_j(\nu)$$

where the last equality follows since the mean value of χ_ν is zero. Now (4-5) and the triangle inequality give:

$$\big|\mathcal{C}_\nu(\xi/\sqrt{n}) - 1 + \frac{\xi^2}{2n}\cdot\sum_* p_j(\nu)x_j^2(\nu)\big| \leq$$

$$(6)\qquad \sum^* p_j(\nu) + \frac{|\xi|}{\sqrt{n}}\cdot\sum^* p_j(\nu)|x_j(\nu)| + \frac{A^3M\delta}{n}$$

Next, since $|x_j(\nu)| \geq \delta\sqrt{n}$ holds under Σ^* we obtain

$$(7)\qquad \sum^* p_j(\nu) \leq \frac{1}{\delta^2\cdot n}\cdot\sum^* p_j(\nu)\cdot x_j^2(\nu) = \frac{\Psi_\nu}{\delta^2\cdot n}$$

In a similar way we see that

$$(8)\qquad \sum^* p_j(\nu)\cdot|x_j(\nu)| \leq \frac{1}{\delta\cdot\sqrt{n}}\cdot\sum^* p_j(\nu)\cdot x_j^2(\nu) = \frac{W_\nu}{\delta\cdot\sqrt{n}}$$

We have also

$$(9)\qquad \frac{\xi^2}{2n}\cdot\sum_* p_j(\nu)x_j^2(\nu) = \frac{\xi^2}{2n}\cdot\sigma_\nu^2 - \frac{\xi^2}{2n}\cdot\sum^* p_j(\nu)x_j^2(\nu) = \frac{\xi^2}{2n}\cdot\sigma_\nu^2 - \frac{\xi^2}{2n}\cdot W_\nu$$

From (7-9) we see that (6) give the result below below when $|\xi| \leq A$:

$$\big|\mathcal{C}_\nu(\xi/\sqrt{n}) = 1 - \frac{\xi^2}{2n}\cdot\sigma_\nu^2 + \rho_\nu(n) \quad \text{where}$$

$$|\rho_\nu(n)| \leq \frac{\xi^2}{2n}\cdot W_\nu + \frac{|\xi|}{\sqrt{n}}\cdot\frac{W_\nu}{\delta\cdot\sqrt{n}} + \frac{W_\nu}{\delta^2\cdot n} + \frac{A^3M\delta}{n} \leq \frac{W_\nu}{n}\big(\frac{A^2}{2} + \frac{A}{\delta} + \frac{1}{\delta^2}\big) + \frac{A^3M\delta}{n}$$

Next, by the uniform bound of variances we have $\sigma_\nu^2 \leq M$ and keeping δ fixed we can choose n so large that

$$(\mathrm{i})\qquad \frac{A^2M^2}{2n} + |\rho_\nu(n)| \leq \frac{1}{2}$$

At this stage we recall that for every complex number α with absolute value $\leq 1/2$ one has the inequality:

$$(\mathrm{ii})\qquad |\mathrm{Log}(1+\alpha) - \alpha| \leq |\alpha|^2$$

So when (i) holds we conclude that

$$\log(\mathcal{C}_\nu(\xi/\sqrt{n}) = -\frac{\xi^2}{2n}\cdot\sigma_\nu^2 + \rho_\nu(n) + \rho_\nu^*(n) \quad \text{where}$$

$$(*)\qquad |\rho_\nu^*(n)| \leq \big(\frac{A^2M^2}{n} + |\rho_\nu(n)|\big)^2$$

At this stage the proof is almost finished. Namely, using (*) we obtain

$$(**)\qquad \sum_{\nu=1}^{\nu=n} \log(\mathcal{C}_\nu(\xi/\sqrt{n}) = -\frac{\xi^2}{2}\cdot\frac{\sigma_1^2 + \ldots + \sigma_n^2}{n} + \sum_{\nu=1}^{\nu=n}[\rho_\nu(n) + \rho_\nu^*(n)]$$

We want to have a uniform limit:

$$\lim_{n\to\infty}\sum_{\nu=1}^{\nu=n}\log(\mathcal{C}_\nu(\xi/\sqrt{n}) = -\frac{\xi^2\cdot\sigma^2}{2}$$

when $[-A\leq\xi\leq A]$. To get this we consider some $\epsilon>0$ and choose δ so small that in addition to the previously imposed inequality $A\cdot\delta\leq 1$ we also have

$$A^3M\delta<\epsilon/2$$

With this choice of δ the previous estimate for the ρ-functions give:

$$\text{(i)}\qquad \sum_{\nu=1}^{\nu=n}|\rho_\nu(n)|\leq\frac{\epsilon}{2}+\big(\frac{A^2}{\delta}+\frac{A}{\delta}+\frac{1}{\delta^2}\big]\cdot\frac{W_1+\ldots+W_n}{n}$$

By Lindeberg's tail condition the last term tends to zero as $n\to\infty$ so we can find n so large that (i) is $<\epsilon$. The verification that (**) above also gives

$$\sum_{\nu=1}^{\nu=n}|\rho^*_\nu(n)|<\epsilon$$

if n is sufficiently large is left to the reader. Finally, since we also have that $\frac{\sigma_1^2+\ldots+\sigma_n^2}{n}\to\sigma^2$ and ϵ can be arbitrary small the requested uniform convergence follows.

19. Poisson's summation formula.

Let μ be a distribution on the ξ-line with a compact support contained in the open interval $(-\pi,\pi)$. Here $\mu = \widehat{\phi}$ where the tempered distribution ϕ on the x-line extends to an entire function of the complex variable z where $z = x+iy$. For example, if μ is a Riesz measure we have

(1) $$\phi(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\xi}\cdot d\mu(\xi)$$

It turns out that $\phi(x)$ is determined by its value taken on the set of integers. To see this we introduce the ordinary Fourier coefficients

(2) $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\xi}\cdot d\mu(\xi)$$

Now μ is recovered via Fourier's inversion formula on the periodic interval $(-\pi,\pi)$:

(3) $$\mu = \sum c_n\cdot e^{in\xi}$$

At the same time (1) gives the equalities

$$c_n = \phi(-n)$$

for all integers n. This means that we formally can write

$$\phi(x) = \sum_{n\in\mathbf{Z}} \phi(-n)\cdot\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} e^{i(x+n)\xi}\cdot d\xi$$

Evaluating the integrals we obtain

(*) $$\phi(x) = \sum_{n\in\mathbf{Z}} \phi(-n)\cdot\frac{\sin\,\pi(x+n)}{\pi}$$

Under the condition that μ is smooth, say a density given by a twice continuously differentiable function one has $\phi(-n) = O(|n|^{-2})$ and the series in the right hand side of (*) converges pointwise. Taking integrals we obtain:

$$\int_{-\infty}^{\infty}\phi(x)\cdot dx = \sum_{n\in\mathbf{Z}}\phi(-n)\cdot\int_{-\infty}^{\infty}\frac{\sin\,\pi(x+n)}{\pi}$$

Notice that

$$\int_{-\infty}^{\infty}\frac{\sin\,\pi(x+n)}{\pi} = 1 \quad\text{hold for every integer}\quad n$$

Hence we get the equality

(**) $$\int_{-\infty}^{\infty}\phi(x)\cdot dx = \sum_{n\in\mathbf{Z}}\phi(-n)$$

This is no surprise since Fourier's non-periodic inversion formula gives

$$\int_{-\infty}^{\infty}\phi(x)\cdot dx = \mu(0) = \sum c_n = \sum \phi(-n)$$

Remark. One refers to (**) as Poisson's summation formula which was established by Poisson round 1810 soon after Fourier had defined Fourier series in the periodic case.

19.1 Remark. If we relax the assumption and only assume that the distribution $\leq |mu$ has compact support on the closed interval $[-\pi,\pi]$ then ϕ need not be determined by its restriction to $\mathbf{Z}$. A counter-example is when we let μ be the difference of the Dirac measures at π and $-\pi$ in which case

$$\phi(x) = \frac{\sin\pi x}{i\pi}$$

and here the sine-function vanishes on the set of integers. A more refined question arises if we suppose that $\mu(\xi)$ is a continuous density where this continuous function is zero at the two end-points π and $-\pi$. Assume that and suppose also that the inverse Fourier transform ϕ vanishes on $\mathbf{Z}$. This means that we have the entire function

$$\psi(z) = \frac{\phi(z)}{\sin \pi z}$$

Since ψ is a quotient of two entire functions of exponential type it follows from Lindelöf's division theorem in § XX that ψ also belongs to the class $\mathcal{E}$. We can consider its restriction to the imaginary axis. With $z = iy$ and $y << 0$ we notice that the absolute value $|\sin(\pi iy)| \simeq \frac{e^{\pi|y|}}{2}$. At the same time the growth of $\phi(iy)$ is controlled by the integral

$$\int_0^\pi e^{|y|\cdot\xi} \cdot u(\xi) \cdot d\xi$$

When π belongs to the support of u the integral above can be quite large as $y \to -\infty$, But certain estimates are available. Suppose for example that

$$|u(\xi)| \leq C \cdot (\pi - \xi)$$

holds for some constant C as $\xi \to \pi$.

19.1 A class of smoothing functions. In numerical analysis one often employs ϕ-functions as above via inverse Fourier transforms of even functions $\mu(\xi)$ which are supported by $[-\pi, \pi]$ and have derivatives up to some order k. In particular we can take such μ-functions where $\mu(0) = 0$ while the derivatives

$$\partial_\xi^j(\mu)(0) = 0 \quad : \quad 1 \leq j \leq k$$

At the end-points π and $-\pi$ we impose the condition that both μ and all derivatives up to order k are zero. Since μ is even we get:

$$\phi(x) = \frac{1}{\pi} \int_0^\pi \cos(x\xi) \cdot \mu(\xi) \cdot d\xi$$

Next, when μ is of class C^k we get a constant C such that decay condition:

$$(1) \qquad |\phi(x)| \leq C \cdot (1 + |x|)^{-k}$$

holds on the x-line. But ϕ may have a rather slow decay. To see this we consider L^2-integrals. Parseval's equality gives for each $0 \leq p \leq k$ the equality:

$$\int_{-\infty}^\infty |x|^{2p} \cdot |\phi(x)|^2 \cdot dx = c_* \cdot \int_0^\pi |\mu^{(p)}(\xi)|^2 \cdot d\xi$$

In the right hand side we encounter L^2-integrals which appear in Carleman's inequality from Section XX. Let us put

$$\gamma_p(\phi) = \Big| \int_{-\infty}^\infty |x|^{2p} \cdot |\phi(x) \cdot dx\Big|^{\frac{1}{2p}} \quad : \quad 0 \leq p \leq k$$

Then Theorem 1 from § xx gives the inequality

$$(*) \qquad \sum_{p=1}^{p=k} \frac{1}{\gamma_p(\phi)} \leq C$$

for an absolute constant C. This means that it is not possible to obtain small L^2-integrals for all $1 \leq p \leq k$ while k increases. In other words, the *a priori inequality* (*) puts a constraint when one trues to exhibit ϕ-functions which have a good decay as $|x| \to \infty$ via (1) above and at the same time do not increase too much before this good decay begins to be uniformly effective.

Remark. One reason why it is of interest in numerical investigations to construct ϕ-functions as above stems from Poisson's formula which entails that via dilations where ϕ is replaced by $\delta \cdot \phi(x/\delta)$ for $\delta > 0$, it follows that certain discrete moment conditions hold up to order k which can be used to approximate functions $f(x)$ with bounded derivatives up to order k on the real

x-line from its values on a discrete grid given by integer multiples of some positive number. We shall not try to pursue these topics any further since it would lead to an extensive discussion related to numerical analysis. The interested reader can consult [Zahedi: Chapter 3] for some theoretically interesting material about approximations by delta functions which are adapted for delicate numerical investigations. As a final point we notice that Carleman's inequality shows that an "optimal choice" of μ-functions is to choose the cut-off functions by Denjoy in § XX.

The case $\mu(\xi) = \sin \xi$**.** Here we find that

$$\phi(x) = \frac{1}{\pi} \cdot \frac{\sin \pi(x+1)}{x+1} - \frac{1}{\pi} \cdot \frac{\sin \pi(x-1)}{x-1}$$

Here $\phi(n) = 0$ for all integers except $n = 1$ and $n = -1$. More generally the reader is invited to find ϕ when $\mu(\xi) = \sin^k \xi$ with $k \geq 2$ and investigate maximum norms of ϕ over the real x-line and L^2-integrals of its derivatives.

Appendix A: Measure theory

Contents

Introduction

There exist many text-books devoted to measure theory so it is not obvious why we should include an account at all. But it is motivated because measure theory appears frequently in the study of analytic functions. We shall foremost restrict the study of measures to the real line or in $\mathbf{R}^2$ where the Lebesgue measure in dimension one respectively two play a central role. In addition there exists the class of singular Riesz measures which concentrate their distributions of mass to null-sets in the sense of Lebesgue. A subset E of the real x-line is called a null-set if there to every $\epsilon > 0$ exists sequence $\{(a_\nu, b_\nu)\}$ of open interval whose union contains E while

$$\sum (b_\nu - a_\nu) < \epsilon$$

Properties which are valid outside a null-set are said to hold almost everywhere. Lebesuge theory was created by Henri Lebesgue and presented in his famous text-book *Lecons xxx* from 1904. It contains many important results such as the notion of points of density for arbitrary compact subsets E in $\mathbf{R}^n$ where $n \geq 1$. More precisely, a point $x \in E$ is a point of density in the sense of Lebesgue if

$$(*) \qquad \lim_{h \to 0} \frac{\mathrm{vol}_n(\square \setminus E)}{h^n} = 0$$

where $\square$ are open cubes which contain x and h is the the length of the sides in $\square$. This means that h^n is the n-dimensional volume of $\square$ while the numerator in (*) computes the n-dimensional volume of the open complement $\square \setminus E$. So Lebegue's density condition is that these volumes are small relative the volume of the cubes. A notable point is that the limit in (*) is taken over cubes under the sole condition that $x \in \square$, i.e. x need not be the center of the cube and the sides of $\square$ need not be parallell to the coordinate axis. A major result in Lebegue's theory is that (*) holds almost everywhere, i.e. for all $x \in E$ with the eventual exception of a null-set.

Remark. The definition of n-dimensional volumes of bounded and open subsets i $\mathbf{R}^n$ will be given later on and we remark that it is found in a constructive manner using coverings by dyadic cubes.

Derivatives for functions of one variable. Lebesgue also proved that if $f(x)$ is a continuous and monotone function, i.e. non-increasing of non-decreasing, then f has an ordinary derivative almost everywhere. We prove this in § XX and also expose a result due to Denjoy and Young which applies to an arbitrary real-valued continuous function $f(x)$ defined on some open interval (a, b). More precisely, for each $a < x < b$ we set

$$D^*(x) = \limsup_{h+k \to 0} \frac{f(x+h) - f(x-k)}{h+k} \quad \text{and} \quad D_*(x) = \liminf_{h+k \to 0} \frac{f(x+h) - f(x-k)}{h+k}$$

where h and k are positive when we pass to the limes superior. The Denjoy-Young result is:

Theorem. *Outside a (possibly empty) null-set E of (a,b) the following two possibilities occur for each $x \in (a,b) \setminus E$: Either there exists a common finite limit*

(i) $$D^*(x) = D_*(x)$$

Or else one has

$$D^*(x) = +\infty \quad \text{and} \quad D_*(x) = -\infty$$

Remark. As one may expect the proof of this theorem is rather involved. In fact, it constitutes one of the the hardest result in measure theory which otherwise is a relatively soft subject in the sense that most results are derived via set-theoretic considerations. However, the merit is the perfect organisation and the generality which can be applied in many different situations. To fully appreciate measure theory the reader should be familiar with Weierstrass' construction of a continuous function $f(x)$ which is nowhere differentiable. For this famous construction we refer to text-books in calculus. Let us remark that the specific example by Weierstrass is no accident. In fact, later investigations by Dini, Mazurkiewics and Banach demonstrated that in the family of all continuos functions defined on the unit interval $[0,1]$ the generic function fails to have derivatives at all points $0 \leq x \leq 1$. From this point of view the discoveries by Lebesgue were invaluable and serve nowadays as a veritable cornerstone in analysis.

Applications to analytic function theory. Let us first remark that measure theoretic notions often tend to be vague in the sense that results only asserts that some "nice property" holds almost everywhere, while the actual null-set on which the property fails is not described. To give an example of this nature we consider a continuous and real-valued function $u(\theta)$ on the unit circle, i.e. this function of θ is 2π-periodic. Now one defines a function $v(r,\phi)$ when $0 \leq r < 1$ and $0 \leq \phi \leq 2\pi$ by

$$v(r,\phi) = \frac{r}{\pi} \int_0^{2\pi} \frac{\sin(\theta - \phi) \cdot u(\theta)}{1 + r^2 - 2r\cos(\theta - \phi)} \cdot d\theta$$

We remark that v is the conjugate harmonic function in the unit disc of the harmonic extension of u via the Poisson kernel. The Brothers Riesz Theorem asserts that

(*) $$\lim_{r \to 1} v(r,\phi) = v^*(\phi)$$

exists almost everywhere, i.e. for all ϕ outside a null-set on $[0,2\pi]$. However, the precise description of this null-set in terms of the given continuous function u cannot be attained in general. So one must be content with the almost everywhere assertion. However, it is consolidated via Lebesgue integrals. Namely, the full content of the Brothers Riesz theorem is that the almost everywhere defined function $v^*(\theta)$ is integrable in the sense of Lebesgue and one has the limit formula

(**) $$\int_0^{2\pi} \left| v(re^{i\phi}) - v^*(\phi) \right| \cdot d\phi = 0$$

The discussion above shows that it is essential to become familiar with measure theory during the study of analytic functions. Let us now give some historic comments about measure theory.

A problem which has played an important role to develop general measure theory appeared in the article *Sur la distribution de l'ectrecité à la surface des conducteurs fermés et des conducteurs ouverts* from 1886 by G. Robin. He asked if every compact set E in $\mathbf{C}$ without interior points exists carries a distribution of mass whose logarithmic potential function is constant on E. Robin's problem led to the notion of sets of absolute harmonic measure zero and other thin sets. Modern integration theory started in work by Stieltjes from 1894 in connection with moment problems. Soon after Stieltjes' publication, Emile Borel realised that integrals can be defined in a robust way by decomposing the range of the function which is integrated, instead of decomposing the domain where integration takes place. Borel's method leads to abstract measure theory which is exposed in Section 2. A fundamental example is the Lebesgue measure in $\mathbf{R}^n$ which assigns the n-dimensional volume a^n to every cube whose sides have length $a > 0$. In many situations, especially in probability theory, one also needs to consider measures on the infinite product of $\mathbf{R}$. A general method to construct integrals over infinite product spaces is due to Daniell in [Dan]. It is logically transparent but has a disadvantage because one cannot so easily call upon existing theorems in Lebesgue theory from the finite dimensional case. To overcome this, Norbert Wiener used *binary expansions* which extends the Lebesgue measures to an infinite product of $\mathbf{R}$. This relies upon a measure preserving map from the the two-dimensional unit square $\square = \{(x, y)\colon\ 0 < x, y < 1\}$ onto the interval $(0, 1)$ which is constructed as follows: Every real number $0 < x < 1$ has a binary expansion:

$$x = \alpha_1/2 + \alpha_2/4 + \ldots \qquad \alpha_k = 0 \text{ or } 1$$

Similarly $0 < y < 1$ has a binary expansion

$$y = \beta_1/2 + \beta_2/4 + \ldots \qquad \beta_k = 0 \text{ or } 1$$

These expansions are unique when one prescribes that the expansion never terminates with a sequence where the number 1 appears. For example, one represents $1/4$ by $0, 1, 0, , 0 \ldots$ while the sequence $0, 0, 1, 1, 1, \ldots$ is not allowed. A pair of sequences as above is arranged into a single sequence which is used to express a real number $0 < t < 1$ whose binary series is:

$$t(x, y) = \alpha_1/2 + \beta_2/4 + \alpha_2/8 + \beta_2/16 + \ldots \tag{*}$$

This yields a bijective map from $\square$ onto $(0, 1)$ which is *measure preserving*, i.e. if $F(x, y)$ is a continuous function of (x, y) then

$$\iint_\square F(x, y) \cdot dxdy = \int_0^1 f(t) \cdot dt \quad \text{where} \quad f(t(x, y)) = F(x, y)$$

In this way Lebesgue theory in dimension two can be recaptured from the one-dimensional case. Similar measure preserving transformations from $\mathbf{R}^n$ into $\mathbf{R}$ exist for every $n \geq 2$ and once this is achieved it is not difficult to construct a measure preserving transformation from a denumerable product to the real t-line. Details of this construction appear in the text-book [Paley-Wiener].

In addition to pioneering work by Stieltjes, Borel and Lebesgue, important contributions are also due to F. Riesz. For each $n \geq 1$ he constructed a class of measures in $\mathbf{R}^n$ where Borel's abstract theory is adapted to the σ-algebra of Borel sets in $\mathbf{R}^n$, i.e. the smallest σ-algebra which contains all open and closed sets. One consequence of Riesz' theory goes as follows: The solution of the Dirichlet problem in the unit disc implies that there exists a 1-1 correspondence between the class of positve harmonic functions $H(x, y)$ in the open unit disc $D = \{x^2 + y^2 < 1\}$ normalised so that $H(0, 0) = 1$ and the class of probability measures supported by the unit circle $T = \{x^2 + y^2 = 1\}$. In this class there appear measures which are singular with respect to the angular Lebesgue measure on T. A more general construction due to Wiener works for every bounded open set Ω even if the Dirichlet problem cannot be solved for every continuous boundary function f in $C^0(\partial\Omega)$. More precisely, for each $z \in \Omega$, Wiener constructed a unique probability measure $\mathfrak{m}_z^\Omega$ supported by $\partial\Omega$ such that each f in $C^0(\partial\Omega)$ produces a unique harmonic function W_f defined in Ω by

$$W_f(z) = \int_{\partial\Omega} f(\zeta) \cdot \mathfrak{m}_z^\Omega(\zeta)$$

Wiener's result in proved in the chapter about harmonic and subharmonic functions. In Ergodic Theory measures are used frequently. Here the literature is extensive and current research active. However, We shall enter a discussion of this subject except for material about almost periodic functions which were created and developed by Harald Bohr in the beginning of 1920 and we also give an application of Bohr's theory in the section from *Special Topics* entitled *Almost periodic functions and additive number theory* which presents a result due to Beurling from seminars at Uppsala University in 1948.

Measures are also used to study algebraic number fields. The text-book [Weil] by Andre Weil employs measure theoretic results to establish reciprocity theorems and advanced results in class-field theory. Recall that Weil proved the existence of the *Haar measure* on every locally compact group, i.e. even those which are not abelian. See his book [Weil xx] for the proof. Another recommended text-book for the construction of Haar measures is [Measure theory] by Halmos. Let us finish this introduction by describing two deep results expressed via measure theoretic concepts whose proofs require "hard analysis".

Hitting probability for the Brownian motion. The following result is due to the late Björn Dahlberg:

Let U be a bounded open set in $\mathbf{R}^n$ defined by $\{F(x) < 0\}$ where $F(x)$ is a Lipschitz continuous function and $\partial U = \{F(x) = 0\}$. Then the class of null sets in ∂U for the Brownian motion which starts at some point $x_0 \in U$ is equal to the class of subsets of ∂U whose $(n-1)$-dimensional Hausdorff measure are zero.

Above $E \subset \partial U$ is a null set for the Brownian motion if the probability to hit E before the Brownian path reaches ∂U at points outside E is equal to zero. For this discovery Dahlberg was awarded the Salem Prize in 197?.

Carleson' convergence theorem in L^2. In 1965 Lennart Carleson proved the following:

Let $\{a_n\}$ be a sequence of complex numbers whose ℓ^2-norm is finite, i.e. $\sum |a_n|^2$ is finite. To every $N \geq 1$ one constructs Fourier's partial sum:

$$S_N(x) = \sum_{\nu=-N}^{\nu=N} a_n e^{inx}$$

Then $\lim_{N\to\infty} S_N(x)$ exists for all $0 \leq x \leq 2\pi$ except for a set of Lebesgue measure zero.

This result is one of the greatest achievements ever in mathematical analysis. In 2006 Carleson was awarded the Abel Prize for this deep theorem together with his contributions in other areas of mathematics. Already Carleson's Ph.D-thesis from 1950 at Uppsala University when he was only 22 years, contains deep results concerned with sets of uniqueness for boundary values taken by analytic functions in the unit disc where measure theoretic concepts are expressed in terms of capacities. So here one considers classes of sets which are more restricted compared to null-sets in the sense of Lebesgue. An example is a result due to Beurling which is exposed in Section XX from *special topics* and devoted to sets of uniqueness in the unit circle for boundary values of analytic functions $f(z)$ in the unit disc for which the Dirichlet integral

$$\iint_D |f'(z)|^2 \cdot dxdy < \infty$$

Arithmetical conditions also appear in more subtle problems. An example is the notion of Kronecker sets on the unit circle which was introduced in [Carleson]: Let E be a compact subset of the unit circle and consider the class $\mathcal{M}_*(E)$ of all real-valued Riesz measures μ supported by E and of total variation one. Put

$$\text{Kron}(E) = \min_{\mu\in\mathcal{M}_*(E)} \max_{n\geq 0} \Big| \int_E e^{in\theta} \cdot d\mu(\theta) \Big|$$

This yields a number in $[0,1]$. If E has positive Lebesgue measure we can choose a non-zero function $f \in L^1(T)$ in the class $\mathcal{M}_*(E)$ and then the sequence $\{\mu_N = e^{iN\theta} \cdot f(\theta)\}$ stay in the class while

$$(*) \qquad \max_{n\geq 0} \int_E e^{in\theta} \cdot d\mu_N(\theta) = 2\pi \cdot \max_{k\geq N} |\widehat{f}(-k)|$$

where $\{\widehat{f}(\nu)\}$ are the usual Fourier coefficients. The Riemann-Lebesgue theorem entails that (*) tends to zero as $N \to \infty$ and hence $\text{Kron}(E) = 0$ whenever E has positive Lebesgue measure. But if E is a null-set in the sense of Lebesgue then its Kronecker index may be positive and even equal to one. The positivity of this index depends upon arithmetic properties of E. Recall that a closed subset E if T is perfect if it contains no isolated points. While it is trivial to construct perfect null-sets it requires more effort to find closed and perfect sets E such that $\text{Kron}(E) = 1$. For such constructions we refer to work by Salem and the reader may also consult the text-book [Kahane] for studies of various thin sets where arithmetic properties of sets are studied. An example of "refined measure theoretic analysis" is the following result from [Carleson: Theorem 9, p. 349]:

Theorem. *Let E be a closed subset of T with positive Kronecker index. Then each continuous function $\phi(\theta)$ on E has a representation*

$$\phi(\theta) = \sum_{n \geq 0} a_n \cdot e^{in\theta}$$

where the sequence $\{a_n\}$ is absolutely convergent.

Outline of the contents

Section 1 is devoted to measure theory on the real line. The reader may postpone this section until the abstract theory has been studied. But the 1-dimensional study has the merit that proofs are more transparent compared to the abstract theory.

Section 2 treats abstract measure theory. Here one starts from a sample space Ω and a Boolean σ-algebra $\mathcal{B}$ of subsets. Given such a pair $(\Omega, \mathcal{B})$ one gets the class of σ-additive measures. For each specific measure μ there exists the family $\mathcal{N}_\mu$ of its null sets. Then one proceeds to study μ-measurable sets and μ-measurable functions which are identified when they agree outside a null set. The main results deal with integrals of μ-measurable functions. The reader should pay attention to the distinction betwen *convergence in measure* and *convergence almost everywhere.* An important result is Hahn's decomposition theorem which shows that a signed measure can be expressed as the difference of two non-negative measures. One consequence is the Radon-Nikodym theorem which is used to construct the singular part of one measure with respect to another. These results were established a few years after Borel's pioneering work which is fully presented in his text-book [Borel 1908].

A. Borel's construction of integrals. A major point in section 2 is Borel's construction of integrals of μ-measurable functions f. The fundamental idea is to use a partition of values taken by f. Then one can compute μ-measures of measurable inverse images of sets $\{a \leq f < b\}$ for all pairs of real numbers $a < b$ and after a limit one gets the Borel integral denoted by $\int f \cdot d\mu$.

B. Lebesgue theory. The major result in Section 3 is the existence of *Lebesgue points* which implies that Lebesgue measurable sets, and more generally measurable functions are realized in a concrete manner. For example, a bounded Lebesgue measurable function f is the pointwise limit outside a null set of a sequence of Lipschitz continuous functions defined by mean values of f over smaller and smaller cubes centered at the point where such a pointwise convergence takes place.

C. Riesz measures in $\mathbf{R^n}$. Section 4 is devoted to the construction of these measures in $\mathbf{R^n}$. A Riesz measure may have all its mass concentrated to a null set in Lebegue's sense and when it occurs one refers to a singular Riesz measure. Singular Riesz measures are used to construct *inner* analytic functions in the unit disc which illustrates the importance of measure theory in analytic function theory.

D. Special topics. In section 5 we prove Stokes Theorem for domains in $\mathbf{R}^n$ where one allows irregular points on the boundary. Minor sections treat special topics, such as the Hardy-Littlewood maximal function and the Rademacher functions. In section 8 is devoted to the construction of the Brownian motion. Here the main result is the almost everywhere continuity of individual Brownian paths.

0.1 Rational series and measure theory.

The subsequent material is independent of the general measure theory so the reader can return to this at a later occasion. We shall discuss series of the form:

$$(*)\qquad \sum \frac{A_\nu}{z-a_\nu}$$

Before general measure theory was developed such series were studied by Poincaré, Goursat and Pringheim who used them to construct functions via lacunary series. The general theory about the series (*) is foremost due Emile Borel who introduced a class of generalised analytic functions by extending Cauchy's concept *de fonction monogène.* A full account of Borel's work would lead us too far in these notes and the interested reader should consult his text-book [Borel] for details. A non-trivial situation arises when the sequence $\{a_\nu\}$ is everywhere dense in an open subset Ω of $\mathbf{C}$, or even in the whole complex plane.

A special case. Suppose there exists some $\gamma > 1/2$ such that

$$(1)\qquad \sum |A_\nu| \cdot \nu^\gamma < \infty$$

For a given $\ell > 0$ we consider the open discs $\{D_\nu\}$ of radius $\ell \cdot \nu^{-\gamma}$ centered at the points $\{a_\nu\}$. The 2-dimensional area of the domain given by the union of there discs is majorized by:

$$(2)\qquad \pi \cdot \ell^2 \cdot \sum \nu^{-2\gamma}$$

Let $C_\ell = \mathbf{C} \setminus \cup D_\nu$ be the closed complement. These sets increase as $\ell \to 0$ and (2) shows that the complement of $\cup C_\ell$ has planar measure zero, i.e. it is a null-set in the complex plane. Next, if $z \in C_\ell$ the series (*) is absolutely convergent since

$$\frac{1}{|z-a_\nu|} \leq \ell^{-1} \cdot \nu^\gamma$$

hold for each ν and (1) gives therefore the absolute convergence

$$\sum \frac{|A_\nu|}{|z-a_\nu|} \leq \ell^{-1}$$

When the sequence $\{a_\nu\}$ is everywhere dense the interior of C_ℓ is empty so the resulting function

$$(*)\qquad f(z) = \sum \frac{A_\nu}{z-a_\nu}$$

is defined on every closed set C_ℓ but it cannot be regarded as an analytic function. But f is the limit of meromorphic functions

$$(*)\qquad f_m(z) = \sum_{\nu=1}^{\nu=m} \frac{A_\nu}{z-a_\nu}$$

where the convergence holds uniformly on C_ℓ for every fixed ν. So one may expect some analyticity properties of f. Let us say that f is of quasi-analytic type if it cannot vanish identically on any Jordan arc γ situated in C_ℓ for some $\ell > 0$. The condition for f to be quasi-analytic depends on $\{A_\nu\}$. But even if $\{A_\nu\}$ decreases quite rapidly there may exist non-quasi-analytic series. Examples where

$$|A_\nu \leq e^{-n^\alpha} \quad : \quad 0 < \alpha < 1/2$$

are given in [Denjoy 1922]. However, if $\{A_\nu\}$ decreases sufficiently rapidly to zero then quasi-analyticity holds. The following sufficiency result was proved by Carleman in [Carleman 1922]:

Theorem. *Suppose there exists $\epsilon > 0$ such that*

$$|A_n| \leq e^{-(\gamma+\epsilon)\cdot n \cdot \log n} \quad : \quad n \geq 1$$

Then $f(z)$ is quasi-analytic in C_ℓ for every $\ell > 0$.

The proof employs analytic function theory and is given in § XX from Chapter III. Carleman's book [xxx] contains several examples including necessity conditions for certain classes of series in (*). The interested reader should also consult [Wolff] where J. Wolff constructs series as above when $\{a_\nu\}$ is everywhere dense and (*) converges at every point outside this denumerable set.

Another case. Now we consider the series (*) under the condition that

$$\sum_{\nu=1}^{\infty} \sqrt{|A_\nu|} < \infty \tag{**}$$

Let us take a dense set $\{a_\nu\}$ in some open set Ω. Consider an analytic Jordan curve Γ , i.e. the image under a bijective and real analytic function $t \mapsto z(t)$ from the unit interval $[0,1]$. When all the a-points are outside Γ we can define the partial sums

$$S_N^\Gamma(t) = \sum_{\nu=1}^{\nu=N} \frac{A_\nu}{z(t) - a_\nu} \tag{1}$$

We say that the infinite series (*) converges uniformly on Γ if $\{S_N^\Gamma(t)\}_1^\infty$ converges uniformly on $[0,1]$. Borel proved that when (**) is assumed then this uniform convergence holds on *almost every closed line segment* in the complex plane. To be precise, if we represent a line segment by its two end-points (z_0, z_1) then there exists a nullset $\mathcal{N}$ in $\mathbf{C}^2$ such that the series

$$t \mapsto \sum \frac{A_\nu}{tz_0 + (1-t)z_1 - a_\nu}$$

converges uniformly for $0 \leq t \leq 1$ whenever (z_0, z_1) is outside $\mathcal{N}$. Next, relax the condition of uniform convergence on $[0,1]$ to the weaker condition that $\{S_N^\Gamma(t)\}$ converge pointwise for all t outside a null set on $[0,1]$. With this relaxed notion of convergence Borel proved the following:

0.1.2 Theorem. *Assume that $\{A_\nu\}$ satisfies*

$$\sum_{\nu=1}^{\infty} |A_\nu|^{\frac{2}{3}} < \infty$$

Then the partial sum sequence $\{S_N^\Gamma(t)\}$ converges almost every along every real-analytic Jordan arc Γ in $\mathbf{C}$.

The results above illustrate that the study of series (*) is a rich subject with wide range of applications and many important problems remain unsolved. See [Borcea. et.al.] for an account.

I. Measure theory on the real line.

Introduction. The modern theory of integration started in 1894 when Stieltjes published the article *Recherches sur les fractions continues* which contains a wealth of new ideas; among others, a new conceptual integral which originally was used to study *moment problems.* He considered a sequence of non-negative real numbers $\{c_n\}_0^\infty$ and asked if there exists a non decreasing function $f(t)$ defined on the non-negative real axis $t \geq 0$ such that

$$(*) \qquad c_n = \int_0^\infty t^n \cdot df(t) \quad \text{hold for all } n = 0, 1, \ldots$$

To analyze this problem Stieltjes constructed integrals of the form:

$$\int_a^b g(x) \cdot df(x)$$

Here $f(x)$ is a continuous and non-decreasing function and $g(x)$ is a continuous function on the closed interval $[a, b]$. Stieltjes defined the integral (*) as the limit of sums:

$$(**) \qquad \sum g(x_\nu) \cdot \big[f(x_{\nu+1} - f(x_\nu)\big]$$

where $a = x_0 < \ldots < x_N = b$ and $\max(x_{\nu+1} - x_\nu)$ tends to zero. The *uniform continuity* of g on $[a, b]$ gives a limit exactly as for the ordinary Riemann integral. Namely, consider the modulus of continuity:

$$\omega_g(\delta) = \max_{x_1, x_2} |g(x_1) - g(x_2)| \quad : \text{maximum taken over pairs } |x_2 - x_1| \leq \delta$$

Put $M = f(b) - f(a)$. If $\delta > 0$ and $x_{\nu+1} - x_\nu \leq \delta$ hold for every ν in the partition which defines (**), then Stieltjes proved that this sum differs form the limit (*) by a quantity which is $\leq M \cdot \omega_g(\delta)$. Hence the integral by Stieltjes exists since uniform continuity of g entails that

$$\lim_{\delta \to 0} \omega_g(\delta) = 0$$

Remark. The beautiful discoveries by Stieltjes will nor be treated in these notes. We remark only that they lead to moment problems and that Stieltjes found a necessary and sufficient condition on the sequence $\{c_n\}$ in order that there exists a function f such that (*) above holds. For an account about moment problems we refer to the text-book [Japanese...].

Exercise. Let f be a C^1-function,i.e. the derivative $f(x)$ exists as a continuous function. Show that (*) is equal to the ordinary Riemann integral

$$\int_a^b g(x) f'(x) \cdot dx$$

If g also is a C^1-function then partial integration gives

$$(1) \qquad \int_a^b g(x) f'(x) = g(b)f(b) - g(a)f(a) \int_a^b f(x) g'(x) dx$$

Verify that the partial integration formula (1) remains valid for Stieltjes' integrals, i.e. if f and g are continuous and non-decreasing on $[a, b]$ then partial summations give the equality:

$$\int_a^b f(x) \cdot dg(x) + \int_a^b f(x) \cdot dg(x) = f(b)g(b) - f(a)g(a)$$

0.1 Functions of bounded variation. A continuous function $f(x)$ on $[a,b]$ has a bounded variation if there exists a constant M such that

$$\sum_{\nu=0}^{\nu=N-1} \big|f(x_{\nu+1})-f(x_\nu)\big| \leq M$$

for all partitions $a=x_0<x_1<\ldots<x_N=b$. The Stieltjes integrals

$$\int_a^b g(x)\cdot df(x)$$

are again defined for every continuous function $g(x)$. To prove this one uses the following result:

0.2 Decomposition Lemma. *Let f have a bounded variation. Then there exists a unique pair of continuous functions (f_*,f^*) where f^* is non-decreasing and f_* is non-increasing such that*

$$f(x)=f^*(x)-f_*(x) \quad \text{where} \quad f^*(a)=f(a) \quad \text{and } f_*(a)=0$$

Exercise. Prove this result. The hint is to define $f^*(x)$ for every $a\leq x\leq b$ by

$$f^*(x)=f(a)+\max\sum f(\xi_\nu)-f(\eta_\nu)$$

where the maximum is taken over sequences $0\leq\eta_1<\xi_1<\eta_2<\ldots<\eta_N<\xi_N\leq x$.

0.3 Borel's construction

.

Stieltjes' integral (*) can be computed in another way using the variation of f over a certain family of subsets of $[a,b]$. This fundamental method was introduced by Emile Borel in 1895. Recall that if Ω is an arbitrary bounded and open subset of $\mathbf{R}$ then it is a unique union of pairwise disjoint open interval $\{(a_\nu,b_\nu)\}$. In general this family is infinite but it is at most denumerable. Let $f(x)$ be continuous and non-decreasing. The variation of f over Ω is defined by

$$\mathrm{Var}_f(\Omega)=\sum_{\nu=1}^\infty \big[f(b_\nu)-f(a_\nu)\big]$$

If E is a closed subset of $[a,b]$ we put:

(i) $$\mathrm{Var}_f(\Omega\cap E)=\mathrm{Var}_f(\Omega\cap E)-\mathrm{Var}_f(\Omega\setminus E)$$

Next, consider a continuous function g on $[a,b]$. For each ν we get the following subset of $[a,b]$:

$$S_g[a_\nu,b_\nu)=\{x \quad : \quad a_\nu\leq g(x)<b_\nu\}$$

Since the set $\{g<b_\nu\}$ is open and $\{g\geq a_\nu\}$ is closed, it follows from (i) that the variation of f is defined over these S_g-sets. Suppose that the range $g([a,b])$ is contained in the interval $[-A,A]$ for some $A>0$. For every sequence

$$-A\leq\xi_0<\xi_1<\ldots<\xi_N=A$$

we set

$$S_*(\xi_\bullet)=\sum\xi_\nu\cdot\mathrm{Var}_f(S_g[\xi_\nu,\xi_{\nu+1}) \quad \text{and} \quad S^*(\xi_\bullet)=\sum\xi_{\nu+1}\cdot\mathrm{Var}_f(S_g[\xi_\nu,\xi_{\nu+1})$$

When $M=f(b)-f(a)$ we see that

$$S^*(\xi_\bullet)-S_*(\xi_\bullet)\leq M\cdot\max_\nu\,(\xi_{\nu+1}-\xi_\nu)$$

It follows that there exists

(*) $$S=\lim S_*(\xi_\bullet)$$

where the limit is taken over arbitrary partitions of $[-A,A]$ such that the numbers

$$\max_\nu\ \xi_{\nu+1}-\xi_\nu\to 0$$

By Abel's summation formula the limit in (*) is equal to the previous Stieltjes integral, i.e. one has the equality:

$$(**) \qquad S = \int_a^b g(x) \cdot df(x)$$

This equality paves the way to constructions of integrals where the regularity of g can be relaxed. The reason is that the Borel limit above is more *robust* as compared with the construction by Stieljes. For let $A > 0$ and consider the class $\mathcal{C}_A$ of continuous functions $g(x)$ on $[a, b]$ whose range is contained in $[-A, A]$. To every ξ-sequence $-A = \xi_0 < \xi_1 < \ldots < \xi_N = A$ we set

$$\delta(\xi_\bullet) = \max_\nu \xi_{\nu+1} - \xi_\nu$$

Now the lower- resp. the over-sums S_* and S^* differ from the limit S in (*) by number majorised by $A \cdot \delta(\xi_\bullet)$. This hold for all g-functions in $\mathcal{C}_A$. So when Borel's sums are used, the rate of convergence depends upon $\delta(\xi_\bullet)$ and not upon the chosen g-function in $\mathcal{C}_A$, i.e. the rate of convergence in Borel's limit is *independent* of the modules of continuity of an individual g-function.

0.4 Borel functions.

The robust limit via Borel's procedure makes it possible to construct integrals where the g-functions no longer have to be continuous. In fact, there exists the integral

$$(***) \qquad \int_a^b g(x) df(x)$$

under the sole assumption that g is a *bounded and Borel measurable function.* This means that $g(x)$ is a function on $[a, b]$ with some bounded range $[-A, A]$ and for each pair of real numbers $\eta < \xi$ the set $\{\eta \leq g < \xi\}$ belongs to the Borel algebra $\mathfrak{B}$ which by definition is the smallest Boolean σ-algebra of subsets of $\mathbf{R}$ containing all half-open intervals. In order to define (***) for an arbitrary bounded Borel function g one first constructs the variation of f over an arbitrary Borel set. Once this is done the existence of limits which give (***) follow exactly as in the case when g is a continuous function. This will be done in 0.6 below.

0.5 Baire classes. The class $\mathcal{B}$ of all Borel measurable functions arises from increasing families of functions. First $\mathcal{B}_0$ is the family of continuous functions. Next, a function g is of the first Baire class if it is equal to the *pointwise limit* of some sequence of continuous functions. This gives the class $\mathcal{B}_1$. Next we get the class $\mathcal{B}_2$ which consists of pointwise limits of $\mathcal{B}_1$-functions. One proceeds by an induction over positive integers and arrive at the class

$$\mathcal{B}_\infty = \cup B_n$$

However, this does not stop the process via pointwise convergence. The reason is that there exist sequences $\{g_k\}$ of functions in B_w which have a pointwise limit function which does not belong to B_∞, i.e. the limit function does not belong to B_w for a given integer w. So in order to get a family of functions which is stable under pointwise limits one must continue and define $\mathcal{B}$-classes by an induction using *ordinal numbers.* We shall not pursue this any further but refer to the extensive literature about the construction of all Borel functions on the real line. See for example [XXX].

0.6 Monotone functions and their Riesz measures

.

Given an interval $[a, b]$ we consider a non-decreasing and continuous functions $f(x)$ on this interval. Now we have the σ-algebra $\mathfrak{B}[a, b]$ of Borel sets in $[a, b]$. Consider a denumerable family of disjoint half-open intervals $\{[a_\nu, b_\nu)\}$ on $[a, b]$. Then the positive series

$$(1) \qquad \sum f(b_\nu) - f(a_\nu)$$

converges. Next, since $\mathfrak{B}[a,b]$ is the smallest σ-algebra which contains half-open subintervals of $[a,b]$, the convergence in (1) and general set-theoretic arguments imply that the variation of f is defined over every Borel set S, i.e. we obtain a σ-additive measure μ_f on $\mathfrak{B}[a,b]$ where

$$\mu_f(S) = \mathrm{Var}_f(S)$$

holds for every Borel set S in $[a,b]$. More precisely,

$$S \mapsto \mu_f(S) \tag{2}$$

yields a σ-additive map.

A converse result. Suppose that we have a non-negative σ-additive map

$$S \mapsto \mu(S)$$

from $\mathfrak{B}[a,b]$ into the set of non-negative real numbers. Thus, if μ is normalised so that its mass on $[a,b]$ is one, then it is a probability measure on the sample space formed by $[a,b]$ and $\mathfrak{B}[a,b]$ in the sense of abstract measure theory. To μ we associate the non-decreasing function $f(x)$ defined by

$$f(x) = \mu([a,x]) \quad : a \leq x \leq b$$

Exercise. Prove the equality

$$\mu = \mu_f \tag{1}$$

under the assumption that μ has no *discrete point masses*, i.e. that the μ-mass is zero on every singleton set. An equivalent condition is that

$$\mu(\eta,\xi) = \mu([\eta,\xi]) \tag{2}$$

hold for all pairs $\leq \eta < \xi \leq b$. When μ satisfies this condition one says that μ has no atoms or refer to an atomless measure. Hence we can conclude:

0.6.1 Proposition. *There exists a 1-1 correspondence between the class of non-decreasing continuous functions on $[a,b]$ and the class of σ-additive and non-negative measurs on $[a,b]$ without atoms.*

0.6.2 Discrete measures and jump functions. Let $\{x_k\}$ be a sequence of points on $\mathbf{R}$ indexed by positive integers. No special assumption is imposed. For example, the sequence may be some enumeration of all rational numbers. If $\{p_k\}$ is a sequence of positive numbers such that the series $\sum p_k < +\infty$ then we get the jump-fuction

$$s(x) = \sum p_k \cdot H_{x_k}(x) \tag{1}$$

Above we introduced Heaviside functions, i.e. for every real number x_* we define $H_{x_*}(x) = 1$ if $x \geq x_*$ and 0 when $x < x_*$. To the s-function corresponds the discrete measure which assigns the mass p_k at every x_k.

This yields a 1-1 correspondence between non-decreasing jump functions and non-negative discrete measures. Let us return to the closed interval $[a,b]$ and consider some arbitrary σ-additive and non-negative measure μ defined on $\mathfrak{B}[a,b]$. It may have point masses. Since the total mass of μ over $[a,b]$ is finite the sum of all mass assigned to the atoms is finite, i.e. we get a discrete part given by

$$\mu_d = \sum p_k \cdot \delta_{x_k}$$

Here $\sum p_k < \infty$ and $\{x_k\}$ is at most a denumerable subset of $[a,b]$. The difference $\mu - \mu_d$ has no atoms and corresponds to a non-decreasing continuos function f as above. So here $\mu - \mu_d = \mu_f$ and one refers to μ_f as the continuous part of μ. It is clear that this decomposition of μ is unique.

0.7 Signed measures.

A continuous function f on $[a,b]$ has a bounded variation if there exists a constant C such that

$$\sum \big|f(\xi_{\nu+1}) - f(\xi_\nu)\big| \leq C$$

for all partitions $a = \xi_0 < \ldots < \xi_N = b$. The smallest number C for which this holds is called the total variation of f and is denoted by $V(f)$. A result due to Ascoli asserts the following:

0.8 Theorem. *Let f be a continuous function with bounded variation. Then there exists a unique pair of non-decreasing functions g, h such that*

$$f(x) = g(x) - h(x)$$

where $g(a) = f(a)$ and $h(a) = 0$.

Exercise. Prove this. Here is the hint is to construct the g-function. Put

$$\phi(x) = f(a) + \max \sum_{\nu 01}^{\nu=N} f(\xi_\nu - f(\eta_\nu)$$

where the maximum is taken over all sequences

$$a < \eta_1 < \xi_1 < \eta_2 < \ldots < \xi_{N-1} < \eta_N < \xi_N \leq x$$

Intuitively, we seek intervals in $[0,x]$ where f increases as much as possible, while eventual intervals where f decrease are omitted. The reader should verify that the ϕ-function is non-decreasing and continuous and that the function

$$x \mapsto f(x) - \phi(x)$$

is non-increasing. Now $g = f(a) + \phi(x)$ and $h = f(x) - \phi(x) - f(a)$. This proves the existence in Theorem 0.8 and the proof of uniqueness is left to the reader.

The signed measure μ_f. The decomposition above gives a signed measure μ_f defined by

$$\mu_f = \mu_g - \mu_h$$

In this decomposition there exists a Borel set S in $[a,b]$ such that

$$\mu_g([a,b]) = \mu_g(S) \quad \text{and} \quad \mu_h([a,b]) = \mu_h([a,b] \setminus S)$$

This means that μ_g and μ_h are orthogonal and one writes

$$\mu_g \perp \mu_h$$

Summing up we have the following

0.9 Theorem. *There exists a 1-1 correspondence between the family of signed Riesz measures μ on $[a,b]$ without atoms and the class of continuous functions $f(x)$ of bounded variation normalised with $f(a) = 0$.*

Let us remark that the results above are special cases of the general decomposition by Hahn which is proved in the section devoted to abstract measure theory.

0.10 Borel-Stieltjes integrals. The previous material shows that if f is a continuous function of bounded variation on $[a,b]$ then there exists an integral

$$(*) \qquad \int_a^b g(x) \cdot d\mu_f(x) = \int_a^b g(x) \cdot df(x)$$

for every bounded Borel function $g(x)$.

0.11 Weak limits of measures

Above we have seen that every signed Riesz measure μ on $[a,b]$ is a unique sum of an atomless measure ν_c and a discrete measure μ_d. It turns out that the class of discrete measures recapture all Riesz measures after suitable passage to the limit. Let us clarify this and begin with the case of non-negative measures. Consider a positive integer N and some N-tuple $a \leq \xi_1 < \ldots < \xi_N \leq b$

and a sequence $\{c_\nu \geq 0\}$ where $\sum c_\nu = 1$ To this we associate the discrete measure μ_N and obtain:

$$\int g \cdot d\mu_N = \sum c_\nu \cdot g(\xi_\nu) \quad \text{for every } g \in C^0[a,b]$$

Suppose now that there exists the limits:

$$(*) \qquad \lim_{N\to\infty} \int g \cdot d\mu_N \quad \text{for all} \quad g \in C^0[a,b]$$

When (*) holds we say that $\{\mu_N\}$ is weakly convergent. Now the following holds:

0.12 Theorem. *Let $\{\mu_N\}$ be weakly convergent. Then there exists a unique non-negative Riesz measure μ such that*

$$\lim_{N\to\infty} \int g \cdot d\mu_N = \int_a^b g \cdot d\mu$$

hold for all $g \in C^0[a,b]$.

Proof. The existence of limits mean that we have a linear map

$$g \mapsto \lim_{N\to\infty} \int g \cdot d\mu_N = \int_a^b g \cdot d\mu$$

on the vector space $C^0[a,b]$. Let us denote it by L. Notice that $g \geq 0$ gives $L(g) \geq 0$ and taking the identity function we have $L(1) = 1$. Now we construct a non-decreasing function $f(x)$ as follows: To each $a < x \leq b$ we consider the family $\mathcal{F}_x$ of continuos functions g such that $0 \leq g \leq 1$ and there exists some $\epsilon > 0$ with $g(y) = 0$ when $y \geq x - \epsilon$. In other words, g has a support given by a compact subset of the half-open interval $[a,x)$. Set

$$f(x) = \max_{g \in \mathcal{F}_x} L(g)$$

Here $f(a) = 0$ and it is obvious that f is non-decreasing. The reader should also verify that f is left continuous, i.e.

$$\lim_{\epsilon\to 0} f(x-\epsilon) = f(x)$$

hold for every $a < x \leq b$. There may exist a set $\{\eta_\nu\}$ where f is discontinuous. Here a jump at η_ν is given by a positive number

$$\rho_\nu = \lim_{\epsilon\to 0} f(\eta_\nu + \epsilon) - f(\eta_\nu)$$

To every such point we have the Heaviside function $H_\nu(x)$ and define the jump function

$$s(x) = \sum \rho_\nu \cdot H_\nu(x)$$

Then $f - s = f_*$ is continuous and we get the Riesz measure

$$\mu = \mu_d + \mu_{f_*}$$

where the discrete part

$$\mu_d = \sum \rho_\nu \cdot \delta_{\eta_\nu}$$

At this stage we leave it to the reader to verify the equality:

$$L(g) = \int_a^b g \cdot d\mu$$

for all continuous functions $g(x)$.

Remark. Theorem 0.12 is the Riesz representation theorem for non-negative and continuous linear functionals on $C^0[a,b]$. If X^* denotes the dual space of the Banach space $C^0[a,b]$, then the limit in Theorem 0.12 is of weak type, i.e. the sequence $\{\mu_N\}$ converge weakly to μ in X^*. Conversely let μ be a non-negative Riesz measure on $[a,b]$ of total mass one. We also assume that μ is continuous. So $\mu = \mu_f$ for some non-decreasing and continuous function f on $[a,b]$ with

$f(a) = 0$. In this situation we can approximate μ weakly by a sequence of discrete measures. Namely, for every positive integer $N \geq 2$ we define the discrete measure

$$\mu_N = \sum_{k=1}^{k=N} p_k \delta_{x_k}$$

where

$$p_k = f(a + \frac{k(b-a)}{N}) - f(a + \frac{(k-1)(b-a)}{N}) \quad \text{and} \quad x_k = a + \frac{k(b-a)}{N})$$

Exercise. Show that the sequence $\{\mu_N\}$ converges weakly to μ_f. The hint is to use uniform continuity of the g-functions in Theorem 0.12.

0.13 Cantor's function. Consider the unit interval $[0,1]$. Let $\mu_1 = \frac{1}{2}[\delta_0 + \delta_1]$. Next, we get

$$\mu_2 = \frac{1}{4}[\delta_0 + \delta_{1/3} + \delta_{2/3} + \delta_1]$$

$$\mu_3 = \frac{1}{8} \cdot [\delta_0 + \delta_{1/9} + \delta_{2/9} + \delta_{3/9} + [\delta_{6/9} + \delta_{7/9} + \delta_{8/9} + \delta_1]$$

and so on where the reader can recognize the inductive construction. Intuitively one removes one third of middle intervals at each step. For every integer $k \geq 4$ we get a discrete measure μ_k which assigns the point mass 2^{-k} at 2^k many points.

0.14 Exercise. Show that the sequence $\{\mu_k\}$ converges weakly to a limit measure μ_f where f is continuous and non-decreasing. The notable fact is that f is constant on many intervals. To begin with it is constant on the middle interval $(1(3, 2/3)$. It is also constant on the intervals $(1/9, 2/9)$ and $(7/9, 8/9)$. The total length on intervals where f is constant becomes

$$(*) \qquad \frac{1}{3} + 2 \cdot \frac{1}{3^2} + \ldots = \frac{1}{3} \cdot \sum_{\nu=0}^{\infty} (2/3)^\nu = 1$$

The support of the measure μ_f is the closed complement of the removed intervals above. The construction is remarkable since f has a vanishing derivative in each removed open interval and at the same time (*) holds while f increases from zero to $f(1) = 1$ so the function is not constant. Hence Cantor's function f violates the fundamental theorem of calculus, i.e. it is not recovered by an integral of its derivative. Moreover, there is defined intervals on the closed complement of the removed intervals above.

0.15 Absolutely continuous functions. Let f be a non-decreasing and continuous function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. To each $0 < \delta < 1$ we consider open sets which consist of pairwise disjoint open intervals $\{(a_\nu, b_\nu)\}$ where

$$(*) \qquad \sum (b_\nu - a_\nu) < \delta$$

No condition is imposed on the number of these intervals. With δ fixed we set:

$$V_f(\delta) = \max \sum f(b_\nu) - f(a_\nu)$$

where the maximum is taken over all finite families $\{(a_\nu, b_\nu)\}$ for which (*) hold.

0.16 Definition. *The function f is called absolutely continuous if*

$$\lim_{\delta \to 0} V_f(\delta) = 0$$

The space of such functions is denoted by $AC^0[0,1]$.

It turns out that if $f \in AC^0[0,1]$ then it is recaptured from its derivative $f'(x)$. This relies upon the theorem by F. Riesz in the next section which shows that f has an ordinary derivative outside a null set in the sense of Lebesgue. After this one constructs the Lesbegue integral of f' to get a primitive function defined by

$$F(x) = \int_0^x f'(t) \cdot dt$$

Here F is also absolutely continuous and the almost everywhere existence of Lebesgue points imply that the derivative of F is equal to that of f almost everywhere. There remains to consider the difference

$$g = f - F$$

Here g belongs to $AC^0[0,1]$ and its derivative is almost everywhere zero. From this one can show that g is identically zero and hence one has the equality

$$(*) \qquad f(x) = \int_0^x f'(t) \cdot dt$$

In other words, an absolutely continuous function is recovered by its derivative.

0.17 Fubini's theorem. Let $\{f_n\}$ be a sequence of continuous and non-decreasing functions defined on some interval $[a,b]$. We also assume that every f_n is non-negative and that the positive series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges for every x. The sum yields a function $s(x)$ and it is clear that it is non-decreasing. As we shall explain in more detail later on, it follows that $s(x)$ is a sum of a non-decreasing function $s_*(x)$ and a discrete jump-function $s_d(x)$. Moreover, $s(x)$ has a derivative almost everywhere and the same is true for each f_n by Lebesgue's theorem to be proved in the next section. Hence there exists a null-set E_0 such that every f_n and s have derivatives at points outside E_0. In this situation Fubini proved that the sum of derivatives converges, i.e. when x is outside E_0 then

$$s'(x) = \lim_{N \to \infty} \sum_{n=1}^{n=N} f'_n(x)$$

The proof of Fubini's theorem occurs at the end of the next section under the headline Exercises and Examples.

0.18 Points of density.

The 1-dimensional Lebesgue measure is first defined on bounded open sets U. Recall the fundamental fact from Calculus that U has an interval decomposition, i.e. it is the disjoint union of interval $\{(a_n, b_n)\}$. Its Lebesgue measure is defined by $\sum (b_n - a_n)$ and denoted by $|U|$. A subset S of the real line is a null-set if for every $\epsilon > 0$ there exists an open set U such that $S \subset U$ and $|U| < \epsilon$. One shows easily that every denumerable union of null-sets again is a null-set. Next, let E be a compact subset of $\mathbf{R}$. To each $\delta > 0$ we get the open set

$$E_\delta = \{x : \operatorname{dist}(x, E) < \delta\}$$

These sets decrease with δ and the Lebesgue measure of E is defined to be

$$\text{(i)} \qquad |E| = \lim_{\delta \to 0} |E_\delta|$$

Exercise. Show that a compact set E is a null-set if and only if (i) above is zero. The hint is that whenever $> u$ is an open set which contains E, then there exists $\delta > 0$ such that $E_\delta \subset u$.

Next, let E be compact and consider a point $x \in E$. It is called a point of density in E if:

$$(*) \qquad \lim_{h+k \to 0} \frac{\big|E \cap [x-k, x+h]\big|}{h+k} = 0$$

where h, k both tend to zero under the sole assumption that $h + k \to 0$.

0.19 Theorem. *Almost every $x \in E$ is a point of density.*

This result was established by Lebesgue. S simplification of the original proof was later found by Vitali using a general covering lemma which we announce below while the proof of Theorem 0.19 is postponed until 0.xx.

0.20 Vitali's covering theorem. Let F be an arbitrary bounded subset of $\mathbf{R}$. Let $\mathcal{V}$ be a family of open intervals with the properties: To every $x \in F$ and $\delta > 0$ there exists an interval $\omega \in \mathcal{V}$ such that $x \in \omega$ and $|\omega| < \delta$. In addition we assume that $\omega \subset (-A, A)$ hold for some $A > 0$ which is chosen so large that F is contained in a compact subset of $(-A, A)$. If $\omega = (a, b)$ is an interval we denote by $3 \cdot \omega$ the enlarged interval $(a - \ell, b + \ell)$ where $\ell = b - a$ is the length of ω so that $3 \cdot \omega$ has length 3ℓ. With this notation one has:

0.21 Theorem. *There exists a sequence of pairwise disjoint intervals $\{\omega_n\}$ in $\mathcal{V}$ which covers F in the sense that for every positive integer N one has the inclusion*

$$(*) \qquad F \subset \bar{\omega}_1 \cup \ldots \cup \bar{\omega}_N \bigcup_{n > N} 3 \cdot \omega_n$$

Proof. Put

$$\delta_1 = \sup_{\omega \in \mathcal{V}} |\omega|$$

Choose ω_1 with $|\omega_1| > 2\delta_1/3$. Let $\mathcal{V}_1$ be the subfamily of intervals $\omega \in \mathcal{V}$ which have empty intersection with the closed interval $\bar{\omega}_1$. Put

$$\delta_2 = \sup_{\omega \in \mathcal{V}_1} |\omega|$$

Choose $\omega_2 \in \mathcal{V}_1$ with $|\omega_2| > 2\delta_2/3$. in the next step $\mathcal{V}_2$ is the subfamily of intervals in $\mathcal{V}$ which have empty intersection with $\bar{\omega}_1 \cup \bar{\omega}_2$. Then we put

$$\delta_3 = \sup_{\omega \in \mathcal{V}_2} |\omega|$$

and choose $\omega_3 \in \mathcal{V}_2$ with $|\omega_3| > 2\delta_3/3$. We continue in this way and get a sequence $\{\omega_n\}$ where the intervals by construction are pairwise disjoint. At this stage we leave it to the reader to verify that the sequence $\{\omega_n\}$ satisfies (*) in Theorem 0.21 for every positive integer N.

0.22 Exercise Since $\{\omega_\nu\}$ is a sequence of disjoint intervals which all are contained in a bounded set, it follows that

$$\sum \delta_n < \infty$$

Use this convergence and the inclusions in (*) for every positive integer N to show that the outer measure of F is majorized by:

$$|F|^* \leq \sum \delta_n$$

0.23 Proof of Theorem 0.19. For each integer $n \geq 2$ we denote by $E(n)$ the subset of E which consists of points such that

$$\text{(i)} \qquad \liminf_{h+k \to 0} \frac{\big|E \cap (x-k, x+h)\big|}{h+k} \leq 1 - \frac{1}{n}$$

Let $\mathcal{L}(E)$ be the set of points of density in E. It is clear that

$$E \setminus \mathcal{L}(E) = \bigcup_{n \geq 2} E(n)$$

We conclude that $E \setminus \mathcal{L}(E)$ is a null-set if $E(n)$ are null-sets for every $n \geq 2$. To prove this for a given n we proceed as follows: For each $\epsilon > 0$ we can choose $\delta > 0$ such that

$$\text{(ii)} \qquad |E_\delta \setminus E| < \frac{\epsilon}{n}$$

If $x \in E(N)$ the definition of limes inferior means that we can choose arbitrary small open intervals ω which contain x while $\omega \subset E_\delta$ and at the same time

$$\text{(iii)} \qquad \frac{\big|E \cap \omega\big|}{|\omega|} \leq 1 - \frac{1}{n}$$

It is clear that the family of such intervals is a Vitali covering of $E(n)$. Hence we find a sequence $\{\omega_\nu\}$ from Theorem XX and by the remark in XX the outer measure

$$|E(n)|^* \leq \sum |\omega_\nu|$$

Next, for each fixed ν we have

$$|\omega_\nu| = |\omega_\nu \cap E| + |\omega_\nu \cap (E_\delta \setminus E)|$$

Since $\{\omega_\nu\}$ are disjoint and (iii) above hold for every ν it follows that

$$\sum |\omega_\nu| \leq (1 - \frac{1}{n}) \cdot \sum |\omega_\nu| + |E_\delta \setminus E|$$

Hence (ii) gives

$$\sum |\omega_\nu| \leq \epsilon$$

Since ϵ was arbitrary we conclude that $E(n)$ is a null-set and Lebegue's theorem is proved.

0.24 Exercise. Use Lebegue's theorem to show that if F is a set such that there exists a constant $c < 1$ for which

$$\big|F \cap (a,b)\big|^* \leq c(b-a)$$

hold for every interval (a,b), then F is a null-set.

1. Derivatives of functions

In the text-book *Théorie de l'intégration* from 1904, Lebesgue proved that a monotone function defined in a real interval has an ordinary derivative outside a null-set. For an arbitrary continuous function a general result about derivatives was discovered by M. Young and Denjoy which goes as follows: Let $f(x)$ be a real-valued continuous function defined on some interval (a,b). For each $a<x<b$ we set

$$D^*(x)=\limsup_{h+k\to 0}\frac{f(x+h)-f(x-k)}{h+k}$$

where h and k are positive when we pass to the limes superior. Similarly

$$D_*(x)=\liminf_{h+k\to 0}\frac{f(x+h)-f(x-k)}{h+k}$$

1.1 Theorem. *Outside a (possibly empty) null-set E of (a,b) the following two possibilities occur for each $x\in(a,b)\setminus E$: Either there exists a common finite limit*

$$(*)\qquad D^*(x)=D_*(x)$$

Or else one has

$$(**)\qquad D^*(x)=+\infty\quad\text{and}\quad D_*(x)=-\infty$$

Remark. Above the pair $h,k)$ tends to zero under the sole condition that $h+k\to 0$. We can take $k=0$ or $h=0$ and consider one-sided limits:

$$\text{(i)}\qquad D^+(x)=\limsup_{h\to 0}\frac{f(x+h)-f(x)}{h}\quad\text{and}\quad d^+(x)=\liminf_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\text{(ii)}\qquad D_+(x)=\limsup_{k\to 0}\frac{f(x)-f(x-k)}{k}\quad\text{and}\quad d_+(x)=\liminf_{k\to 0}\frac{f(x)-f(x-k)}{k}$$

With these notations it is clear that:

$$D_*(x)\le d^+(x)\le D^+(x)\le D^*(x)$$

So the equality $D_*(x)=D^*(x)$ entails that f has an ordinary right derivative and since we also have

$$D_*(x)\le d_+(x)\le D_+(x)\le D^*(x)$$

we conclude that if (*) holds in the theorem, then f has a ordinary derivative at x. If (**) occurs at a point x, then the graph of f close to x is steep but may also change sign in a small interval around such a point. Take for example $x=0$ and let

$$f(x)=\sqrt{x}\quad\text{when}\quad x>0\quad:\quad f(x)=\sqrt{-x}\quad\text{when}\quad x<0$$

With $k=0$ and $h>0$ we see that $D^*(0)=+\infty$ and with $h=0$ and $k>0$ we see that $D_*(=)=-\infty$. Next, recall from Calculus the famous construction by Weierstrass of a continuous function $f(x)$ which fails to have an ordinary derivative at every point in the interval (a,b). The Denjoy-Young theorem shows that such a continuous function has a "turbulent" graph where $D^*(x)=+\infty$ and $D_*(x)=-\infty$ hold for all x outside a null-set.

1.2 The case of monotone functions. If the continuous function f is non-increasing or non-decreasing, then that case (**) cannot occur. So Theorem 1.1 implies that a monotone continuous function has an ordinary derivative almost everywhere which was the original result by Lebesgue. The proof of Theorem 1.1 relies upon a useful construction which in the present context was invented by F. Riesz a few years after Lebegue's pioneering work. Together with a general covering result due to Vitali it leads to to a quite simple proof of lebegue's theorem and concerning Theorem 1.1. we shall see in § xx that the proof relies upon the monotone case and Lebesgue's result about points of density. The interested reader may consult his plenary talk at the IMU-congress in Zürich (1932) for a historic account about derivative of functions on the real line and t he subsequent proof follows Riesz' presentation in [ibid] closely.

1.3 Forward Riesz intervals. Let $g(x)$ be a real-valued and continuous function defined on some open interval (a,b), The forward Riesz set $\mathcal{F}_g$ consists of all points $a < x < b$ for which there exists some $x < y \leq b$ such that

$$(*) \qquad g(x) < g(y)$$

If g is non-decreasing then $\mathcal{F}_g = \emptyset$. Excluding this case, continuity entails that $\mathcal{F}_g$ is an open subset of (a,b) and hence a disjoint union of intervals

$$(1) \qquad \mathcal{F}_g = \cup\,(\alpha_\nu, \beta_\nu)$$

Each interval in (1) is called a *forward Riesz interval* of g. It may occur that some interval is of the form (α, b) i.e. b is a right end-point. Similarly a can be a left end-point. For example, if g from the start is strictly increasing then $\mathcal{F}_g = (a,b)$.

1.4 Proposition *For each forward Riesz interval* (α,β) *one has*

$$(*) \qquad g(\beta) = \max_{\alpha \leq x \leq \beta} g(x)$$

Proof. Assume the contrary. This gives some maximum point $\alpha \leq x^* < \beta$ for the g-function on the closed interval $[\alpha,\beta]$. Now $x^* \in \mathcal{F}_g$ which means that

$$\exists\, y > x^* \quad \text{and} \quad g(x^*) > g(y)$$

Since x^* is a maximum point over $[\alpha,\beta]$ we must have $y > \beta$. But then $\beta \in \mathcal{F}_g$ which is impossible since β was a boundary point of the open set $\mathcal{F}_g$.

1.5 Backward Riesz intervals Put

$$\mathcal{B}_+ g = \{a < x < b\colon\; \exists\, a < y < x \quad : \quad g(y) > g(x)\}$$

Again $\mathcal{B}_g$ is open and hence a disjoint union of open intervals (c_ν, d_ν). They are called backward Riesz intervals. By similar reasoning as above one shows that if (c,d) is a backward Riesz interval then

$$(**) \qquad g(c) = \max_{c \leq x \leq d} g(x)$$

1.6 A study of monotone functions.

Let $f(x)$ be a continuous and *non-decreasing* function on $[a,b]$. To each $a < x < b$ we set

$$(1) \qquad D^+(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

where limes superior is taken as $h > 0$ decrease to zero. The function $x \mapsto D^+(x)$ takes values in $[0, +\infty]$.

1.7 Proposition. *For each positive number* C *the following set-theoretic inclusion holds:*

$$\{D^+(x) > C\} \subset \mathcal{F}_g \quad \text{where} \quad g(x) = f(x) - Cx$$

Proof. Suppose that $D^+(x) > C$ for some $a < x < b$. The definition of limes superior gives some $y > x$ such that

$$\text{(i)} \qquad \frac{f(y) - f(x)}{y - x} > C$$

Then $g(y) - g(x) = f(y) - f(x) - C(y-x) > 0$ and hence $x \in \mathcal{F}_g$.

1.8 Proposition *For every $C > 0$ the outer Lebesgue measure of the set $\{D^+ > C\}$ satisfies the inequality*

$$\big|\{D^+ > C\}\big|^* \leq \frac{f(b) - f(a)}{C}$$

Proof. With $g(x) = f(x) - Cx$ as above one has an interval decomposition $\mathcal{F}_g = \cup\,(\alpha_\nu, \beta_\nu)$ and the inclusion from Proposition 1.4 gives

$$|\{D^+ > C\}|^* \leq \sum (\beta_\nu - \alpha_\nu) \tag{1}$$

Apply Proposition 1.4 to the forward Riesz intervals of g. This gives for every ν:

$$f(\alpha_\nu) - C \cdot \alpha_\nu \leq f(\beta_\nu) - C \cdot \beta_\nu$$

Rewriting the last inequality we get

$$C(\beta_\nu - \alpha_\nu) \leq f(\beta_\nu) - f(\alpha_\nu)$$

Taking the sum over all ν we obtain

$$C \cdot \sum (\beta_\nu - \alpha_\nu) \leq \sum f(\beta_\nu) - f(\alpha_\nu) \leq f(b) - f(a) \tag{2}$$

where the last inequality holds since f is non-decreasing. Now (1) and (2) give the required inequality in Proposition 1.8.

1.9 The d_+-function. To each $a < x < b$ we put

$$d_+(x) = \liminf_{k \to 0} \frac{f(x) - f(x-k)}{k}$$

Let $c > 0$ and set

$$h(x) = f(x) - c$$

A similar reasoning as in Proposition 1.4 gives the inclusion

$$\{d_+ < c\} \subset \mathcal{B}_h$$

where the right hand side is the backward Riesz set from 1.5.

1.10 Some inequalitites. Consider a pair $0 < c < C$ and the intersection

$$E = \{d_+ < c\} \cap \{D^+ > C\}$$

Now (1.9) gives the inclusion

$$E \subset \{D^+ > C\} \cap \mathcal{B}_h \tag{i}$$

Let $\{(\alpha_\nu, \beta_\nu)\}$ be the interval decomposition of the open set $\mathcal{B}_h$. For each ν we consider the restriction of $g(x) = f(x) - Cx$ to the interval (α_ν, β_ν) and Proposition 1.8 gives the inequality

$$|\{D^+ > C\} \cap (\alpha_\nu, \beta_\nu)\,|^* \leq \frac{f(\beta_\nu) - f(\alpha_\nu)}{C} \tag{ii}$$

Since (α_ν, β_ν) is a backward Riesz interval of $f(x) - c$ we have $f(\beta_\nu) - f(\alpha_\nu) \leq c(\beta_\nu - \alpha_\nu)$. Hence (i) gives:

$$|\{D^+ > C\} \cap (\alpha_\nu, \beta_\nu)\,|^* \leq \frac{c}{C} \cdot (\beta_\nu - \alpha_\nu) \tag{iii}$$

Since the backward Riesz intervals are disjoint a summation over ν and the inclusion (i) give:

$$|\{D^+ > C\} \cap \{d_+ < c\}|^* \leq \frac{c}{C}(b - a) \tag{*}$$

1.11 Proof of Lebesgue's theorem. The function f restricts to a non-decreasing function on an arbitrary open subinterval (a_*, b_*) of (a, b) and since both D^+ and d_+ are constructed by limits close to a point we get the same inequality as in (*) above, i.e. one has the inequality

$$|\{d_+ < c\} \cap \{D^+ > C\} \cap (a_*, b_*)|^* \leq \frac{c}{C} \cdot (b_* - a_*)$$

Now the criterion from §XX implies that $\{d_+ < c\} \cap \{D^+ > C\}$ is a null-set. Apply this for pairs $c = q < r = C$ where q, r are rational numbers. Since a denumerable union of null-sets is a null-set we conclude that the equality

$$\text{(i)} \qquad d_+(x) = D^+(x)$$

holds almost everywhere. In the same way one proves that the equality

$$\text{(i)} \qquad d^+(x) = D_+(x) \quad \text{holds almost everywhere}$$

Finally, it is obvious that when (i-ii) hold then f has an ordinary derivative which proves Lebegue's theorem that every monotone function has a derivative almost everywhere.

1.12 An extension of Lebesgue's theorem. Let f be a continuous function on the closed unit interval $[0, 1]$. Suppose that E is a measurable subset of $(0, 1)$ such that the restriction of f to E is non-decreasing. Removing an eventual zero set we also assume that $E = \mathcal{L}(E)$, i.e. every $x \in E$ is a point of density for E as explained in § XX. Using exactly the same methods as above it follows that there is a (possibly empty) null-set $S \subset E$, there exists a derivative at every $x \in E$ in the sense that

$$\text{(1)} \qquad \lim_{h+k \to 0} \frac{f(x+h) - f(x-k)}{h+k} = f'_E(x)$$

exists for each $x \in E \setminus S$ where the limit is restricted in the sense that $x + h$ and $x + k$ belong to E during the passage to $h + k \to 0$. But since x is a point of density (1) holds without this restriction, i.e. $f'_E(x)$ gives an ordinary derivative of f. Let us supply the details for this assertion. We may take $x = 0$ and replacing f by $f - f'_E(0)x - f(0)$ we can assume that $f'_E(0) = f(0) = 0$. Next, let $0 < \epsilon < 1/4$ which gives some $\delta > 0$ such that if $0 < x < \delta$ and $x \in E$ then

$$f(x) \leq \epsilon \cdot x$$

At the same time the density condition entails that if δ is small enough then

$$|E \cap (-x, x)| \geq 2x(1 - \epsilon) \quad : \quad 0 < x < \delta$$

If we consider some $0 < x < \delta/2$ we see that (xx) implies the interval $(x + 4\epsilon \cdot x, x)$ must intersect E and if $x^* \in E$ is in this interval we get

$$f(x) \leq f(x^*) \leq \epsilon \cdot x^* \leq \epsilon \cdot 2x$$

Since $\epsilon > 0$ this proves that $D^+(0) = 0$ and in the same way the reader can verify that the right derivative at $x = 0$ vanishes.

1.13 Proof of Theorem 1. For each non-negative integer $n = 0, 1, 2, \ldots$ and every rational number $r \in (a, b)$ we denote by $E_{n,r}$ the set of all $r < x < b$ such that

$$\frac{f(x) - f(\xi)}{x - \xi} > -n \quad : \quad r < \xi < x$$

Exercise. Show the set-theoretic inclusion

$$\{D_*(x) > -\infty\} \subset \bigcup E_{n,r}$$

where the union is taken over all $n \geq 0$ and every rational number $a < r < b$.

1.14 Proposition. *For each pair (n, r) the equality*

$$D^*(x) = D_*(x)$$

holds almost everywhere in the measurable set $E_{n,r}$.

Proof. Replacing the interval (a, b) by (r, b) and f by $f(x-r)+nx$ we can assume that $r = n = 0$ and now $E_{0,0} \subset (0, b-r)$ where the restriction of f to this measurable set is monotone, i.e.

$$0 < \xi < x \implies f(x) > f(\xi)$$

holds for every pair $\xi < x$ in $E_{0,0}$. To simplify notations we set $E = E_{0,0}$. Let E_* be the set of density for E as defined in XX and recall from XX that $E \setminus E_*$ is a null-set. Ignoring this null-set we consider the restriction of f to E_* which again is a non-decreasing function. The extended version from 1.12 of Lebegue's theorem applies and shows that after removing another null-set from E_* if necessary, then the limit below exists for each $x \in E_*$:

$$(*) \qquad D(x) = \lim_{h+k \to 0} \frac{f(x+h) - f(x-k)}{h+k}$$

In the same way one proves that if a null-set is removed from the set

$$D^*(x) = +\infty\}$$

then f has an ordinary derivative so that $D^*(x) = D_*(x)$. This finishes the proof of the Denjoy-Young theorem.

1.15 Examples and Exercises.

Above we have studied monotone continuous functions. There also exist non-decreasing jump functions which arise as follows: Let $\{\xi_n\}$ be a sequence of real numbers in $(0, 1)$. They are not ordered and may give a dense set. For example, we can take some enumeration of all rational numbers in $(0, 1)$. Next, let $\{\delta_n\}$ be a sequence of positive numbers such that $\sum \delta_n < \infty$. To each n we get the jump function $H_n(x)$ where

$$H_n(x) = 0 \quad : \quad x < \xi_n \quad \text{and} \quad H_n(x) = \delta_n \quad : \quad x \geq \xi_n$$

Now

$$s(x) = \sum H_n(x)$$

is a non-decreasing function which has jump-discontinuites at each ξ_n.

Exercise. Show that s is pointwise continuous at every x outside the set $\{\xi_n\}$, i.e. show that if $\epsilon > 0$ then there exists $\delta > 0$ such that

$$s(x+\delta) < s(x) + \epsilon \quad \text{and} \quad s(x-\delta) > s(x) - \epsilon$$

Less evident is the following:

Theorem. *$s(x)$ has an ordinary derivative which is equal to zero almost everywhere.*

Proof. Let $\alpha > 0$ and denote by E be the subset of $(0, 1)$ which consists of numbers $0 < x < 1$ such that

$$\limsup_{h+k \to 0} \frac{s(x+h) - s(x-k)}{h+k} > \alpha$$

It suffices to show that E is a null-set. To prove this we consider some $\epsilon > 0$ and choose N so large that

$$\text{(i)} \qquad \sum_{n>N} \delta_n < \alpha \cdot \epsilon$$

Set $s_*(x) = s(x) - (H_1(x) + \ldots + H_N(x))$. If E_* is the corresponding set in (x) with s replaced by s_* then E and E_* only differ by the finite set $\xi_1, \ldots, \xi_N$ so the measures of e and E_* are the same. Now we apply Vitali's covering theorem using s_* and obtain a a sequence of disjoint intervals $\{a_n, b_n)\}$ which yields a Vitali covering of E_* and at the same time

$$\frac{s_*(b_\nu) - s_*(a_\nu)}{b_\nu - a_\nu} \geq \alpha$$

It follows that

$$\text{(ii)} \qquad s_*(1) - s_*(0) \geq \alpha \cdot \sum (b_\nu - a_\nu)$$

At the same time (i) entails that $s_*(1) - s_*(0) \leq \alpha \cdot \epsilon$ and hence we have

$$|E|^* = |E_*|^* \leq \sum (b_\nu - a_\nu) \leq \epsilon$$

Since ϵ was arbitrary we get $|E|^* = 0$ as requested.

II. Abstract measure theory

Introduction.

The subsequent material is foremost due to Emile Borel. The proofs rely upon straightforward set-theoretic constructions. But the perfect organisation leads to a powerful theory whose merit is that the results can be applied in many different contexts. To get some intuition we present the theory using concepts from probability theory where the abstract theory foremost is used. A stochastic variable χ on a sample space equipped with a probability measure μ, is by definition μ-measurable function. The distribution function of χ is the non-decreasing function F_χ defined on the real x-line by

$$F_\chi(x) = \mu(\{\chi < x\})$$

In other words, for each real number x we consider the set $\{w \in \Omega : \chi(w) < x\}$ which in probability theory is called an event. This event has a μ-measure and as x varies we get the function $F_\chi(x)$. To avoid technical matters we will expose the Borel's theory on measure spaces where the measures have finite total variation. The passage to general measure spaces is straightforward. See XXX below.

1. General constructions

1.1 Sample spaces. A sample space consists of a pair $(\Omega, \mathcal{B})$ where Ω is a set and $\mathcal{B}$ a Boolean σ-algebra of subsets. Thus, if $\{A_\nu\}$ is a sequence of sets in $\mathcal{B}$ indexed by non-negative integers, then $\cap A_\nu$ and $\cup A_\nu$ stay in $\mathcal{B}$.

1.2 Probability measures. A probability measure μ is a map

$$\mu\colon \mathcal{B} \to [0,1] \quad : \quad A \to \mu(A) \text{ is } \sigma - \text{additive}$$

The last condition means that if $\{A_\nu\}$ is a sequence of pairwise disjoint sets in $\mathcal{B}$, then

$$\mu(\cup A\nu) = \sum \mu(A_\nu)$$

Finally, every probability measure is normalized, i.e. $\mu(\Omega) = 1$.

1.3 Nullsets. Let μ be a probability measure. A subset F of Ω is a *null set* with respect to μ if there to each $\epsilon > 0$ exists $A \in \mathcal{B}$ such that

$$F \subset A \quad \text{and} \quad \mu(A) < \epsilon$$

The class of null-sets is denoted by $\mathcal{N}_\mu$. The σ-additivity implies that $\mathcal{N}_\mu$ is stable under a denumarable union, i.e.

$$\{F_\nu\} \subset \mathcal{N}_\mu \implies \cup F_\nu \in \mathcal{N}_\mu$$

Notice that one does not require that a null-set belongs to $\mathcal{B}$.

1.4 μ-measurable sets. Let $B \subset \Omega$ be an arbitrary subset. Its outer - resp. inner μ-measure are defined by

$$\mu^*(B) = \min_{A \in \mathcal{B}} \mu(A) \quad : B \subset A \quad : \quad \mu_*(B) = \max_{A \in \mathcal{B}} \mu(A) \quad : \; A \subset B$$

If $\mu_*(B) = \mu^*(B)$ we say that B us μ-measurable and the common number is denoted by $\mu(B)$. Denote by $\mathfrak{M}_\mu$ the family if all μ-measurable subsets of Ω. It is clear that we have the inclusion

$$\mathcal{B} \subset \mathfrak{M}_\mu$$

Exercise. Show that a set B is μ-measurable if and only if there exists a set $B_* \subset B$ such that $B \setminus B_*$ is a nullset, i.e. belongs to $\mathcal{N}_\mu$. Conclude that $\mathfrak{M}_\mu$ is a Boolean σ-algebra and that the map

$$B \mapsto \mu(B) \quad \text{is} \quad \sigma - \text{additive on} \quad \mathfrak{M}_\mu$$

1.5 Measurable functions. A real valued function f defined on Ω is measurable if the inverse sets

$$f^{-1}(-\infty, a) \text{ and } f^{-1}(-\infty, a] \text{ both belong to } \mathcal{B} \quad : \quad a \text{ any real number}$$

The class of measurable functions on Ω is denoted by $\mathcal{M}_{\mathcal{B}}$. Notice that $\mathcal{M}_{\mathcal{B}}$ is defined without any specific reference to a probability measure.

1.6 μ-measurable functions. Let μ be a probability measure. A real-valued function f on Ω is μ-measurable if

$$f^{-1}(-\infty, a) \text{ and } f^{-1}(-\infty, a] \text{ both belong to } \mathfrak{M}_\mu \quad : \quad a \in \mathbf{R}$$

Denote this class by $\mathcal{M}_\mu$. Since $\mathfrak{M}_\mu$ contains $\mathcal{B}$ this gives a more relaxed condition, i.e. one has the inclusion

$$\mathcal{M}_{\mathcal{B}} \subset \mathcal{M}_\mu$$

1.7 Teorem. *Let $f \in \mathcal{M}_\mu$. Then there exists a null set $F \in \mathcal{N}_\mu$ and some $g \in \mathcal{M}_{\mathcal{B}}$ such that $f = g$ in $\Omega \setminus F$. Thus, after modifying a μ-measurable function on a null set with respect to μ it becomes "truly measurable".*

Proof. Let $\{q_\nu\}_1^\infty$ enumerate the set Q of rational numbers. The exercise after 1.4 shows that to each ν there exists a pair of disjoint sets $F_\nu \in \mathcal{B}$ and $G_\nu \in \mathcal{N}_\mu$ such that

$$f^{-1}(-\infty, q_\nu) = F_\nu \cup G_\nu$$

We also regard the inverse image of the singleton sets and find for every ν a disjoint pair $H_\nu \in \mathcal{B}$ and $S_\nu \in \mathcal{N}_\mu$

$$f^{-1}(\{q_\nu\}) = H_\nu \cup S_\nu$$

Set

$$G^* = \cup G_\nu \bigcup \cup S_\nu$$

Here $G^* \in \mathcal{N}_\mu$ which gives the existence of a decreasing sequence of sets $W_\nu \in \mathcal{B}$ such that $\mu(W_\nu) < 2^{-\nu}$ and $G^* \subset \cap W_\nu$. Now $W_* = \cap W_\nu$ belons to $\mathcal{B}$ and it is also a null-set for μ. Finally, define the function f_* by

$$f_*(\omega) = f(\omega) \quad : \omega \in \Omega \setminus W_* \quad \text{and} \quad f_*|W_* = 0$$

By this construction $f_* \in \mathcal{M}_{\mathcal{B}}$ and f_* differs from f on a null set.

1.8 Equivalence classes of μ-measurable functions. A pair f, g in $\mathcal{M}_\mu$ are called equivalent if they are equal outside a null set. Theorem 1.7 shows that every equivalence class can be represented by a $\mathcal{B}$-measurable function.

1.9 Elementary functions. If $B \in \mathcal{B}$ we get the characteristic function χ_B which is 1 on B and 0 in $\Omega \setminus B$. By an elementary function we mean a finite linear sum of characteristic functions, i.e. a function defined as

$$F = \sum a_k \cdot \chi_{B_k} \tag{*}$$

where $B_1, \ldots, B_N$ is a finite set of disjoint sets in $\mathcal{B}$ whose union is Ω and and $a_1, \ldots, a_N$ are real numbers. For example, with $B \in \mathcal{B}$ then χ_B is an elementary function where the a-number of B is one and the a-number of $\Omega \setminus B$ is zero. Notice that F is unchanged if we take a refined partition, i.e. if every B_k is a disjoint union of some $\mathcal{B}$-sets $\{C_{k\nu}\}$ and set

$$F = \sum \sum c_{k\nu} \cdot \chi_{C_{k\nu}} \quad : c_{k\nu} = a_k$$

Let $E(\mathcal{B})$ denote the set of all elementary functions. It is clear that this is a linear space of functions on Ω. Moreover, if F and G belong to $E(\mathcal{B})$ we can represent both as in (*) after a common refinement and then the product $G \cdot F$ is the elementary function defined by

$$G \cdot F = \sum b_k \cdot a_k \cdot \chi_{B_K}$$

where $F = a_k$ and $G = b_k$ hold on B_k for every k. Next, let F be a bounded $\mathcal{B}$-measurable function i.e. we assume that there exists a constant K such that $-K \leq F(\omega) \leq K$ hold for all $\omega \in \Omega$. Then F can be *uniformly approximated* by elementary functions. Namely, if $N \geq 1$ we set

$$B_N(\nu) = \{-K + \nu \cdot 2^{-N} \leq F < -K + (\nu+1) \cdot 2^{-N}\} \quad : 0 \leq \nu \leq 2^{2N} - 2$$

$$\text{and} \quad B_N(2^{2N} - 1) = \{K - 2^{-N} \leq F \leq K\}$$

We get the elementary function

$$F_* = \sum_{\nu=0}^{2^{2N}-1} (-K + \nu \cdot 2^{-2N}) \cdot \chi_{B_N(\nu)}$$

It is clear that

$$F_* \leq F \leq F_* + 2^{-N}$$

1.10 Convergene in μ-measure. Let μ be a probability measure. We say that a sequence $\{f_n\}$ in $\mathcal{M}_\mu$ converges in μ-measure to a limit function g in $\mathcal{M}_\mu$ if

$$\lim_{n \to \infty} \mu(\{|f_n - g| \geq \epsilon\}] = 0$$

hold for every $\epsilon > 0$. When it holds we write

$$(*) \qquad f_n \xrightarrow{\mu} g$$

1.11 Convergence almost everywhere. As above μ is a probability measure.

1.12 Definition *A sequence of μ-measurable functions $\{f_\nu\}$ is said to converge μ-almost everywhere to a limit function g if there exists a null set $F \in \mathcal{N}_\mu$ such that*

$$\lim_{\nu \to \infty} f_\nu(\omega) = g(\omega) \text{ hold when } \omega \in \Omega \setminus F$$

When it holds we write

$$f_n \xrightarrow{a.e} g$$

1.13 Remark. Almost everywhere convergence means that pointwise convergence holds outside a null set F for μ. In general the limit function g is only defined in $\Omega \setminus F$ but when we regard μ-measurable functions this is harmless and the reader should verify the following:

1.14 Proposition. *If g is an almost everywhere limit of some sequence $\{f_n\}$ from $\mathcal{M}_\nu$ then g also belongs to $\mathcal{M}_\mu$.*

1.15 An example. Convergence in measure need not imply convergence almost everywhere. Consider as an example the case when $\Omega = [0,1]$ and μ is the Lebesgue measure. Given a sequence of real numbers $\{0 < a_\nu < 1 - 1/\nu)\}_2^\infty$ we define a sequence of functions $\{f_\nu\}$ where

$$f_\nu(x) = 1 \quad a_\nu < x < a_\nu + 1/\nu$$

while $f_\nu(x) = 0$ holds outside this interval. It is clear that $\{f_\nu\}$ converges to the zero function in measure. On the other hand the reader should find a sequence $\{a_\nu\}$ as above such that for every $0 < x < 1$ there exists an *infinite* family of ν:s such that $a_\nu < x < a_\nu + 1/\nu$ and as a consequence we never have a pointwise convergence except at the two end-points $x = 0$ or $x = 1$. On the other hand we have a positive result if we allow subsequences.

1.16 Proposition. *Let $\{f_n\}$ be a sequence of measurable functions which converges in μ-measure to a μ-measurable function g. Then there exists a subsequence*

$$g_k = f_{\nu_k} \quad : \quad \nu_1 < \nu_2 < \ldots$$

such that the sequence $\{g_k\}$ converges almost everywhere to g.

Proof. Regarding the sequence $\{f_n - g\}$ it suffices to prove the result when $g = 0$. Now we first find an integer ν_1 such that

$$\nu \geq \nu_1 \implies \mu(\{|f_\nu \geq 2^{-1}\} \leq 2^{-1}$$

Next, we find $\nu_2 > \nu_1$ such that

$$\nu \geq \nu_2 \implies \mu(\{|f_\nu \geq 2^{-2}\} \leq 2^{-2}$$

Inductively we get a sequence $\nu_1 < \nu_2 \ldots$ where

$$\nu \geq \nu_k \implies \mu(\{|f_\nu \geq 2^{-k}\} \leq 2^{-k}$$

Then, the sequence $\{g_k = f_{\nu_k}\}$ converges almost everywhere to the zero function. To see this we set:

$$g_k^* = \max_{\nu \geq k} |g_\nu|$$

The inductive construction above gives

(i) $$\mu(\{g_k^* \geq 2^{-k}\} \leq \sum_{\nu \geq k} \mu(\{|g_\nu| \geq 2^{-k}\} \leq \sum_{\nu \geq k} 2^{-\nu} = 2^{-k+1}$$

Here $g_1^* \geq g_2^* \geq \ldots$ is a monotone sequence and (i) shows that

$$g_k^* \overset{a.e}{\to} 0$$

From this it is clear that $\{g_k\}$ also converges almost everywhere to zero.

1.17 Exercise Let $\{f_n\}$ be a sequence in $\mathcal{M}_\mu$ such that

$$\lim_{n,m} \mu(\{|f_n - f_m| \geq |\epsilon\}) = 0$$

hold for each $\epsilon > 0$, where (m, n) ted to ∞. This means that to every $\epsilon > 0$ there exists some integer N such that for every pair $n \geq N$ and $m \geq N$ one has

$$\mu(\{|f_n - f_m| \geq |\epsilon\}) < \epsilon$$

When it holds we say that $\{f_n\}$ is a Cauchy-sequence with respect to the μ-measure Prove by a similar method as in Proposition 1.16 that there exists a subsequence $\{g_k = f_{n_K}\}$ which converges almost everywhere to a limit function f_*. Show also that f_* is unique in $\mathcal{M}_\mu$ since the almost everywhere convergence of the subsequence entails that we have

$$f_n \overset{\mu}{\to} f_*$$

2. Integrals

In (1.9) we defined the family $E(\mathcal{B})$ of elementary functions. This is a vector space and to each elementary function

$$F = \sum a_k \cdot \chi_{B_k}$$

we assign the μ-integral

(*) $$\int F \cdot d\mu = \sum a_k \cdot \mu(B_k)$$

It is clear that (*) yields an additive map from $E(\mathcal{B})$ into the complex number, i.e. here we regard complex-valued functions. The maximum-norm $||F||_\infty$ is the maximum of $\{|a_k|\}$. Since μ is a probability measure we get:

(**) $$|\int F \cdot d\mu\,| \leq ||F||_\infty$$

2.1 Integrals of bounded functions. If $K > 0$ is a positive number we denote by $\mathcal{M}_\mu(K)$ the class of μ-measurable functions f such that the absolute value $|f|$ is $\leq K$ almost everywhere. So in the equivalence class we can take f to be a function in $\mathcal{M}_{\mathcal{B}}$ where this everywhere defined function has maximum norm K at most. By (xx) the bounded function f can be uniformly approximated by a sequence $\{F_n\}$ from $E(\mathcal{B})$. For each pair n, m the triangle inequality gives

$$(1) \qquad \Big|\int F_n \cdot d\mu - \int F_n \cdot d\mu\Big| \leq \int |F_n - F_m| \cdot d\mu \leq ||F_n - F_m||_\infty$$

Since $||F_n - f||_\infty \to 0$ it follows that (1) tends to zero as n and m increase. Hence the evaluated integrals $\{\int F_n \cdot d\mu\}$ is a Cauchy sequence of complex numbers and we conclude that there exists a limit

$$(2) \qquad \lim_{n\to\infty} \int F_n \cdot d\mu$$

Moreover, (**) shows that this limit is intrinsic, i.e. independent of the chosen sequence $\{F_n\}$ which approximates f uniformly. The limit (2) is called the μ-integral of f and is denoted by

$$(3) \qquad \int f \cdot d\mu$$

2.2 Absolutely integrable functions For a positive real number K we define the the truncation operator T_K from $\mathcal{M}_\mu$ to $\mathcal{M}_\mu(K)$ by:

$$T_K(f)(x) = f(x) : \ |f(x)| \leq K \quad \text{and} \quad T_K(f)(x) = 0 : \ |f(x)| > K$$

When $f \in \mathcal{M}_\mu$ is given we get bounded functions $\{T_K(f) : \ K > 0\}$. We can also take their absolute values. It is clear that

$$(1) \qquad K_1 < K_2 \implies |T_{K_1}(f)| \leq |T_{K_2}(f)|$$

From (1) it is clear that the μ-integrals of the non-negative functions $\{|T_K(f)|\}$ increase with K, i.e. one has

$$(2) \qquad K_1 < K_2 \implies \int |T_{K_1}(f)| \cdot d\mu \leq \int |T_{K_2}(f)| \cdot d\mu$$

We can impose the condition that the non-decreasing sequence (2) is bounded. This leads to

2.3 Definition. *A function $f \in \mathcal{M}_\mu$ is said to be absolutely integrable if there exists a constant C such that*

$$(*) \qquad \int |T_{K_1}(f)| \cdot d\mu \leq C \quad \text{for all} \quad K > 0$$

and when (*) holds we say that f belongs to $L^1(\mu)$.

2.4 Integrals of $L^1(\mu)$-functions. Let f belong to $L^1(\mu)$. From the construction of the T-operators it is clear that if $K_2 > K_1 > 0$ then

$$(\text{i}) \qquad \Big|\int T_{K_2}(f) \cdot d\mu - \int T_{K_1}(f) \cdot d\mu\Big| \leq \int |T_{K_2}(f)| \cdot d\mu - \int |T_{K_1}(f)| \cdot d\mu$$

Since $f \in L^1(\mu)$ the non-decreasing sequence $\{\int |T_{K_2}(f)| \cdot d\mu\}$ is bounded above and therefore convergent. Since a convergent sequence also is a Cauchy sequence, it follows from (i) that

$$\lim_{K_2, K_1} \int T_{K_2}(f) \cdot d\mu - \int T_{K_1}(f) \cdot d\mu = 0$$

as K_1 and K_2 tend to $+\infty$. Since every Cauchy sequence of complex numbers converge, we conclude that there exists a limit

$$(*) \qquad \lim_{K\to\infty} \int T_K(f) \cdot d\mu$$

This limit is denoted by $\int f \cdot d\mu$ and is called the μ-integral of f.

2.5 Exercise. Show that the integral on $L^1(d\mu)$ is additive, i.e.

$$\int f d\mu + \int g d\mu = \int (f+g) d\mu \quad : \quad f, g \in L^1(d\mu)$$

Hence the μ-integral defines a linear functional on the linear space of absolutely integrable functions. Show also that if $f \in L^1(\mu)$ then the absolute value $|f|$ belongs to $L^1(\mu)$.

2.6 The L^1-norm. The linear space $L^1(d\mu)$ becomes a normed space when we set

$$||f||_1 = \int |f[d\mu \quad : \quad f \in L^1(d\mu$$

2.7 Exercise. Show that $E(\mathcal{B})$ is a dense subspace of $L^1(\mu)$.

2.8 Convergence in the L^1-norm.. By definition a sequence $\{f_n\}$ converges in the L^1-norm to a limit function g in $L^1(\mu)$ when

$$\lim_{n\to\infty} ||f_n - g||_1 = 0 \tag{1}$$

If $\epsilon > 0$ we notice that

$$\mu(\{|f_n - g| \geq \epsilon\} \leq \epsilon^{-1} \cdot ||f_n - g||_1 \tag{2}$$

Even though $\epsilon > 0$ can be small, it is kept fixed in (2) and since (1) holds we conclude that

$$\lim_{n\to\infty} \mu(\{|f_n - g| \geq \epsilon\} = 0 \tag{3}$$

This means that $f_n \xrightarrow{\mu} g$, i.e. L^1-convergence implies convergence in μ-measure.

2.9 Example. The converse is not true. For example, let μ be the Lebesgue measure on $[0,1]$. To each $n \geq 1$ we define $f_n(x)$ to be n if $0 \leq x \leq \frac{1}{n}$ and otherwise zero. Then

$$\int_=^1 f_n(x) \cdot dx = 1$$

hold for all n. At the same time $f_n(x) \to 0$ for every $x > 0$, i.e. the sequence converges almost every where to zero and hence also in measure. But if we restrict the attention to bounded functions a converse result holds.

2.10 Proposition *Let $K > 0$ be fixed. Then a sequence $\{f_\nu\}$ in $\mathcal{M}_\mu(K)$ converges to a limit function g in the L^1-norm if and only if the sequence converges in measure to g.*

Proof. Consider a pair f, g in $\mathcal{M}_\mu(K)$. If $\epsilon > 0$ we get the measurable set

$$E_\epsilon(f,g) = \{|f - g| \geq \epsilon\}$$

Since the maximum norm $||f - g||_\infty \leq 2K$ it follows that

$$||f - g||_1 = \int_{E_\epsilon(f,g)} |f - g| d\mu + \int_{\Omega \setminus E_\epsilon(f,g)} |f - g| d\mu \leq 2K \cdot \mu(E_\epsilon(f,g)) + \epsilon \tag{i}$$

If $f_n \xrightarrow{\mu} g$ hold for some sequence $\{f - n\}$ in $\mathcal{M}_\mu(K)$ then

$$\lim_{n\to\infty} \mu(E_\epsilon(f,g)) = 0 \quad \text{hold for each} \quad \epsilon > 0$$

Hence (i) implies that $||f_n - g||_1|| \to 0$. Since we already proved that L^1-convergence entails convergence in measure we get the equivalence assertion in Proposition 2.10.

2.11 Lebesgue's dominated convergence theorem

Let $\{f_n\}$ be a sequence in $L^1(\mu)$ which converges in μ-measure to a limit function g. The example in 2.9 shows that convergence need not hold in the L^1-norm even if $f_n \xrightarrow{a.e} g$ is assumed. To compensate for this we impose a certain bound so that the situation is essentially the same as in Proposition 2.10.

2.12 Theorem *Let $\{f_n\}$ be a sequence in $L^1(\mu)$ where $f_n \xrightarrow{\mu} g$ holds. Assume in addition that there exists a non-negative $\phi \in L^1(\mu)$ such that*

(1) $$|f_n| \leq \phi$$

hold almost everywhere for each n. Then the limit function g belongs to $L^1(\mu)$ and

(2) $$\lim_{\nu \to \infty} ||f_n - g||_1 = 0$$

2.13 Exercise Prove Theorem 2.12. The hint is to apply a truncation operator to ϕ so that the L^1-norm of $\phi - T_K(\phi)$ is small and then use Theorem 2.11.

2.14 Remark. Recall that almost everywhere convergence implies convergence in measure. It follows that if $\{f_n\}$ is some $L^1(\mu)$-sequence and a dominating ϕ-function as above, then an almost everywhere convergene $f_n \xrightarrow{a.e} g$ gives the L^1-convergence. This version of Lebegue's dominated convergence theorem is weaker than the result in Theorem 2.12. However, in many applications one is content with the "almost everywhere"-version of Theorem 2.12.

2.15 A completeness theorem. Let $\{f_n\}$ be Cauchy sequence in the L^1-norm, i.e. assume that

(*) $$\lim_{n,m \to \infty} \int |f_n - f_m| \cdot d\mu = 0$$

2.16 Exercise. Prove that (*) gives the existence of a unique L^1-function f_* such that

$$\lim_{n,m \to \infty} \int |f_n - f_*| \cdot d\mu = 0$$

This means that the normed space $L^1(\mu)$ is complete, i.e. it is a Banach space.

3. Signed measures and the Radon-Nikodym theorem.

Let $(\Omega, \mathcal{B})$ be a sample space. Consider an additive real-valued map $\mu \colon \mathcal{B} \to \mathbf{R}$ which may take negative values. But we impose the condition that μ is bounded, i.e. there exists a constant C such that

(*) $$-C \leq \mu(A) \leq C \quad : \quad A \in \mathcal{B}$$

The uniform bound and additivity imply that if $\{A_\nu\}$ is any sequence of disjoint sets in $\mathcal{B}$, then the series

$$\sum |\mu(A_\nu)| < 2C$$

Hence the series $\sum \mu(A_\nu)$ is absolutely convergent. We say that μ is σ-additive if

$$\sum \mu(A_\nu) = \mu(\cup A_\nu)$$

hold for every sequence of disjoint sets in $\mathcal{B}$ and refer to μ as a *signed* measure. From now on all measures are σ-additive and bounded, i.e. (*) above holds for some constant $C = C_\mu$. One also says that μ has a finite total variation when (*) holds. If $\mu(A) \geq 0$ for all $A \in \lfloor$ we say that μ is a positive measure. Before we announce Hahn's decompostion theorem below we give

3.1 Definition *Two positive measures μ and ν are mutually singular if there exist nullsets $A \in \mathcal{N}_\mu$ and $B \in \mathcal{N}_\nu$ such that*

$$\mu(B) = \mu(\Omega) \quad : \quad \nu(A) = \nu(\Omega)$$

When this holds we write $\mu \perp \nu$.

3.2 Hahn's Theorem *Every signed measure μ has a unique decomposition*

$$\mu = \mu_+ - \mu_- \quad \text{where} \quad \mu_, \mu_- \quad \text{are both positive and} \quad \mu_+ \perp \mu_-$$

Proof. $A \in \mathcal{B}$ is called a μ-positive set if

$$(*) \qquad E \subset A \quad \text{and} \quad E \in \mathcal{B} \implies \mu(E) \geq 0$$

Denote this class by $P_+(\mu)$. Obviously the union of two μ-positive sets is again μ-positive and by σ-additivity there exists a $A^* \in P_+(\mu)$ such that

$$(1) \qquad \mu(A^*) = \max_{A \in P_+(\mu)} \mu(A)$$

Sublemma. One has $\mu(B) \leq 0$ for each $B \subset \Omega \setminus A^*$.

Proof of Sublemma. We argue by contradiction. Suppose there is some $B_0 \subset \Omega \setminus A^*$ with $\mu(B_0) = \delta > 0$. The maximality of $\mu(A^*)$ implies that B_0 does not belong to $P_+(\mu)$. Hence there exists some $\delta_1 > 0$ such that

$$\text{(i)} \qquad -\delta_1 = \min_{E \subset B_0} \mu(E)$$

Choose $E_1 \subset B_0$ with $\mu(E_1) \leq -\delta_1/2$ and set $B_1 = B \setminus E_1$. Now $\mu(B_1) \geq \delta + \delta_1/2$ and we get a negative number

$$-\delta_2 = \min_{E \subset B_1} \mu(E)$$

Then we choose $E_2 \subset B_1$ with $\mu(E_2) \leq -\delta_2/2$. Inductively we get a decreasing sequence of sets

$$B_\nu = B_0 \setminus (E_1 \cup \ldots \cup E_\nu)$$

where $\{E_\nu\}$ are disjoint. Moreover, we have a sequence $\{\delta_\nu\}$ of positive numbers where

$$\text{(ii)} \qquad -\delta_\nu = \min_{E \subset B_\nu} \mu(E) \quad \text{and} \quad \mu(E_{\nu+1}) \leq -\delta_\nu/2$$

Since μ is a signed measure there is a constant A such that

$$\text{(iii)} \qquad \mu(E_1) + \ldots + \mu(E_N) \geq -A \quad \text{for all} \quad N \geq 1$$

It follows that

$$\text{(iv)} \qquad \delta_1 + \ldots + \delta_N \leq 2A \quad \text{for all} \quad N \geq 1$$

Hence the positive series $\sum \delta_\nu$ is convergent which gives $\delta_\nu \to 0$ as $\nu \to +\infty$. Put

$$\text{(v)} \qquad B_* = \cap B_\nu$$

For each $\nu \geq 1$ the inclusion $B_* \subset B_N$ and the definition of δ_ν from (ii) give:

$$\text{(vi)} \qquad \min_{E \subset B_*} \mu(E) \geq -\delta_\nu$$

This hold for every ν and since $\delta_\nu \to 0$ the minimum in (vi) is ≥ 0 which means that B_* belongs to $P_+(\mu)$. At the same time the construction above gives

$$\text{(vii)} \qquad \mu(B_\nu \geq \delta_0 + \frac{1}{2}(\delta_1 + \ldots + \delta_\nu) \geq \delta_0$$

for every ν. By σ-additivity we have

$$\mu(B_*) = \lim_{\nu \to \infty} \mu(B_\nu)$$

and then (vii) entails that $\mu(B_*) > 0$. This contradicts the (1) since A^* and B_* are disjoint and the Sublemma is proved.

Final part of the proof .

The Sublemma gives two positive measures μ_+ and μ_- defined by by

$$\mu_+(E) = \mu(E \cap A^*) \quad : \quad \mu_-(E) = -\mu(E \cap (\Omega \setminus A^*)$$

We see that $\mu_+ \perp \mu_-$ and $\mu = \mu_+ - \mu_-$. This proves the existence of at least one Hahn-decompostion. The proof of *uniqueness* of such a decompositon is left to the reader.

3.3 Radon-Nikodym derivatives.

Let μ be a positive measure and consider a non-negative function $f \in L^1(d\mu)$. Theorem XX gives a positive measure defined by the σ-additive map

$$E \mapsto \int_E f \cdot d\mu \quad : E \in \mathcal{B}$$

Denote this positive measure by $f \cdot \mu$. If $E \in \mathcal{N}_\mu$ the construction of μ-integrals implies that $\int_E f \cdot d\mu = 0$. So one has the inclusion

(**) $$\mathcal{N}_\mu \subset \mathcal{N}_{f \cdot \mu}$$

In general, a positive measure ν is called *absolutely continuous* with respect to μ if one has the inclusion

$$\mathcal{N}_\mu \subset \mathcal{N}_\nu$$

It turns out that such positive measures are of the form $f \cdot \mu$ with $f \in L^1(\mu)$.

3.4 Theorem. *Let μ be a positive measure. Then every positive measure ν which is absolutely continuous with respect to μ is of the form $f \cdot \mu$ for a unique non-negative function $f \in L^1(d\mu)$.*

Proof. For each pair (k, N) where N is a positive integer and k a non-negative integer we consider the following two signed measures

(i) $$\nu - k2^{-N} \cdot \mu \quad \text{and} \quad (k+1)2^{-N} \cdot \mu - \nu$$

The Hahn decomposition applied to $\nu - k2^{-N} \cdot \mu$ gives a maximal set

(ii) $$S_N(k) \in P_+(\nu - k2^{-N} \cdot \mu)$$

and similarly we find a maximal set

(iii) $$T_N(k) \in P_+((k+1)2^{-N} \cdot \mu - \nu)$$

If N is fixed the measure $(\nu - k2^{-N}) - (\nu - (k+1)2^{-N}) = 2^{-N} \cdot \mu \geq 0$. This implies that

(iv) $$S_N(k+1) \subset S_N(k) \quad \text{for all} \quad k = 0, 1, \ldots$$

Moreover, since ν is absolutely continuous with respect to μ it is clear that:

(v) $$\cap_{k \geq 1} S_N(k) = \emptyset$$

Finally the reader may observe that

(vi) $$S_N(k) \setminus S_N(k+1) = S_N(k) \cap T_N(k)$$

The functions $\{f_N\}$. Armed with the results above we shall construct functions via he following procedure. Set

(1) $$W_N(k) = S_N(k) \cap T_N(k)$$

From (vi) it follows that $\{W_N(k)\}$ is a family of disjoint subsets of Ω whose union taken over all $k \geq 0$ is equal to Ω. Notice also that

(2) $$k2^{-N}\mu(E) \leq \nu(E) \leq (k+1)2^N\mu(E) \quad \text{for all} \quad E \subset W_N(k)$$

Next, for a given $N \geq 1$ we define the function

(3) $$f_N = \sum k2^{-N} \cdot \chi_{W_N(k)}$$

From (2) we conclude that

$$\text{(3)} \qquad \int_E f_N \cdot d\mu \leq \nu(E) \leq \int_E f_N \cdot d\mu + 2^{-N} \cdot \mu(E) \quad \text{for all} \quad E \in \mathcal{B}$$

Finally, the reader may verify that the sequence $\{f_N\}$ is non-decreasing and hence there exists a limit function

$$f_* = \lim_{N \to \infty} f_N$$

where the convergence holds almost everywhere. Here $f_* \in L^1(d\mu)$ and sung (3) above the passage to the limit shows that one has the equality

$$\nu = f \cdot d\mu$$

This finishes the proof of Theorem 3.4.

3.5 A general decomposition.

Let μ be a positive measure. For any other positive measure ν there exists a unique decomposition of ν into a sum of one measure which is singular with respect to μ, while the other term is given by an $L^1(d\mu)$-function. More precisely one has:

3.6 Theorem *Given a positive measure μ every other positive measure ν is of the form*

$$\nu = \nu_s + f d\mu \quad : \quad \nu_s \perp \mu \quad \text{and} \quad f \in L^1(d\mu)$$

Proof To find the singular part ν_s we put

(1) $$M = \max \nu(A) \quad : \quad A \in \mathcal{N}_\mu$$

By σ-additivity there exists some $A_* \in \mathcal{N}_\mu$ such that $\nu(A_*) = M$. Define the measure ν_* by:

(2) $$\nu_*(E) = \nu(A_* \cap E)$$

Here $\nu_* \perp \mu$. Put $\gamma = \nu - \nu_*$. The construction of ν_* gives

(3) $$A \in \mathcal{N}_\mu \implies \gamma(A) = 0$$

Then Theorem 3.5 gives $\gamma = f \cdot \mu$ for some $f \in L^1(\mu)$ and Theorem 3.6 follows.

III. Lebesgue Theory

Contents

Introduction.

In the euclidian space $\mathbf{R}^n$ it is natural to define the n-dimensional volume of a cube whose sides have length a to be a^n. After this one expects that it is possible to associate a non-negative number to every bounded open set and construct an additive measure which enjoys an invariance property, i.e. the n-dimensional measure of a bounded open set Ω should be the same after a translation and also under a rotation expressed by an orthogonal linear map. It turns out that this can be achieved in a unique way. But after one is confronted with a problem, i.e. how to extend the construction of n-dimensional volumes to a larger family than bounded open sets. More generally, one would like to find a Boolean σ-algebra of subsets of $\mathbf{R}^n$ which contain all bounded open sets and on which there exists a σ-additive measure which eventually can assign an infinite volume to "large unbounded sets".

A solution to this was achieved by Henri Lebesgue. The material in his book [Leb] from 1904 has found a wide range of applications and we shall expose the main results in Lebesgue's theory. From the abstract measure theory from Section II one can quickly attain the requested properties of the Lebesgue measure and after define the class of Lebesgue measurable functions and construct their integrals. However, the geometric picture is lost if one proceeds directly with abstract notions. Instead we shall use an effective procedure to construct the n-dimensional Lebesgue measure of a bounded open set Ω which will be denoted by $\text{vol}_n(\Omega)$. The method is to employ *dyadic grids*. To each positive integer N one has the family $\mathcal{D}_N$ of open cubes whose sides have length 2^{-N} and the corner points belong to the lattice $2^{-N}\cdot\mathbf{Z}^n$. In Section 1 we explain how dyadic grids enable us to construct $\text{vol}_n(F)$ where F is a bounded open set or a compact set. After one defines the *inner* respectively the *outer* measure of an arbitrary subset E as follows:

$$\text{inner measure}(E) = \max_{K\subset E} \text{vol}_n(K) \tag{1}$$

where the maximum is taken over compact subsets of E. Next

$$\text{outer measure}(E) = \min_{E\subset\Omega} \text{vol}_n(\Omega) \tag{2}$$

where the minimum is taken over open sets which contain E. If (1) and (2) are equal one says that E is measurable in the sense of Lebesgue and the common number is its measure. It may occur that the common number is $+\infty$. But if E is relatively compact then its measure is always finite. The class of all measurable subsets of $\mathbf{R}^n$ is denoted by $\mathcal{L}(\mathbf{R}^n)$. It contain a subclass called *null sets*. More precisely, a set E is called a null-set if its outer measure is zero. The family of null sets is denoted by $\mathcal{N}(\mathbf{R}^n)$. One says that two measurable sets E and F are equivalent if their Boolean difference

$$(E\setminus F)\cup(F\setminus E)\in\mathcal{N}(\mathbf{R}^n) \tag{3}$$

0.1 Lebesgue points.

Let E be measurable where we assume that $\text{vol}_n(E) > 0$ and $\bar{E}$ denotes its closure. To each point x_0 in the closed set $\bar{E}$ we denote by $\mathcal{S}_\delta(x_0)$ the family of all open cubes $\square$ whose sides have length $\leq \delta$ and contain x_0. Here is not required that x_0 is the center of the cube or that the sides are parallell to the coordinate axis. Following Lebesgue we set

$$(*) \qquad \lambda_\delta^E(x_0) = \min_{\square \in \mathcal{S}_\delta(x_0)} \frac{\text{vol}_n(E \cap \square)}{\text{vol}_n(\square)}$$

Since the cube-family $\{\mathcal{S}_\delta(x_0)\}$ decrease as $\delta \to 0$, $\delta \mapsto \lambda_\delta^E(x_0)$ increases as δ shrinks to zero and we obtain a limit

$$(**) \qquad \lambda^E(x_0) = \lim_{\delta \to 0} \lambda_\delta^E(x_0)$$

It is clear from (*) that this λ-number is between 0 and 1. The set of Lebesgue points is defined by

$$(***) \qquad \mathcal{L}(E) = \{x \in \bar{E} \quad : \lambda^E(x) = 1\}$$

It urns out that the Boolean difference below is a null set, i.e. its n-dimensional Lebesgue measure is zero, i.e. one has the inclusion:

$$(****) \qquad (E \setminus \mathcal{L}(E)) \cup (\mathcal{L}(E)) \setminus E) \in \mathcal{N}(\mathbf{R}^n)$$

0.2 Absence of phantoms. Let us remark that (****) already is non-trivial when E is an open or a closed set in $\mathbf{R}^n$. For example, if Ω is open the Boolean difference $\mathcal{L}(\Omega) \setminus \Omega$ appears as a subset of the closed boundary $\partial\Omega$. The fact that this is a null set means that Ω cannot "surround" - in a measure theoretic sense - too many of its boundary points. If F is a closed set we have the zero-set $F \setminus \mathcal{L}(F)$ and (****) means that F is "fat" from a measure theoretic point of view around almost every point in this closed set.

More generally, (****) implies that Lebesgue measurable sets and Lebesgue measurable functions, can be approximated in a rather explicit fashion. By definition a real-valued function $f(x)$ is measurable in the sense of Lebesgue if the inverse image sets $f^{-1}(-\infty, a)$ and $f^{-1}(-\infty, a]$ are measurable for every real number a. Functions whose absolute values attain large values too often are not so interesting, i.e. one studies foremost measurable functions which are *locally integrable.* In XX we construct the Lebesgue integral which gives the space $L^1_{\text{loc}}(\mathbf{R}^n)$. Using (****) we shall prove that a function $f \in L^1_{\text{loc}}(\mathbf{R}^n$ has almost everywhere a Lebesgue value defined as

$$\mathcal{L}_f(x) = \lim \frac{1}{\text{vol}_n(\square)} \cdot \int_\square f(x) \cdot dx$$

where the limit is taken over open cubes $\square$ which contain x and their volumes tend to zero. Moreover, the limit value is equal to $f(x)$ almost everywhere. Thanks to this the analysis becomes transparent. For example, given $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ we fix $\delta > 0$ and set:

$$F_\delta(x) = \frac{1}{\delta^n} \cdot \int_{\square_\delta(x)} f(\xi) \cdot d\xi$$

where $\square_\delta(x)$ is the cube centered at x and with sides of length δ. Lebegue's theorem implies that one has a pointwise convergence

$$\lim_{\delta \to 0} F_\delta(x) = f(x)$$

at every Lebesgue point of f. Identifying measurable functions which are equal almost everywhere we can agree that every L^1_{loc}-function is equal to its associated Lebesgue function $\mathcal{L}_f$, i.e one ignores "silly values". Concerning the F_δ-functions they are continuous functions. So outside a null set a given L^1_{loc}-function is the pointwise limit of a sequence of continuous functions.

0.3 Other results. Using abstract measure theory one then proceeds to prove results about integrals, such as Lebegue's dominated convergence theorem. These proofs are more or less verbatim to those which appear in the abstract theory. An important result is *Fubini's theorem* which shows how n-dimensional Lebesgue integrals can be achieved via repeated integrals just as in ordinary calculus where one takes Riemann integrals with continuos function as integrands.

0.4 Non-measurable sets . Above we introduced measurable sets. To show that there exist *non-measurable sets* one is obliged to use the axiom of choice. So these sets do not exist in "reality". But their eventual existence means that in specific situations it is sometimes necessary to show that a certain set is measurable.

1. Dyadic grids and Lebesgue points.

If N is a positive integer we obtain a family of pairwise disjoint open cubes of the form

$$\square(\nu_\bullet) = \{(x_1,\ldots,x_n) :\ \nu_k 2^{-N} < x_\nu < (\nu_k+1)2^{-N} :\ 1 \leq k \leq n\}$$

where $\nu_\bullet = (\nu_1,\ldots,\nu_n)$ runs over all n-tuples of integers. This family of cubes is denoted by $\mathcal{D}_N$. Notice that each $\square \in \mathcal{D}_N$ contains 2^n many cubes in $\mathcal{D}_{N+1}$.

1.1 Dyadic exhaustion of open sets. Let Ω be a bounded open set. To each $N \geq 1$ we get the family $\mathcal{D}_N(\Omega)$ of cubes from $\mathcal{D}_N$ which are contained in Ω. Let $\rho_N(\Omega)$ be the number of such cubes. The volume of the union becomes

$$w_N(\Omega) = 2^{-nN} \cdot \rho_N(\Omega)$$

If $\square \in \mathcal{D}_N(\Omega)$, then the 2^n many dyadic cubes from $\mathcal{D}_{N+1}$ which are contained in $\square$ all belong to $\mathcal{D}_{N+1}(\Omega)$. It follows that

$$\rho_{N+1}(\Omega) \geq 2^n \cdot \rho_N(\Omega)$$

From this we conclude that $w_N(\Omega) \leq w_{N+1}(\Omega)$. Hence $\{w_N(\Omega)\}$ is increasing. Since Ω is bounded this sequence is bounded above. For example, if Ω is contained in the unit cube $\{0 \leq x_\nu \leq 1\}$ then $w_N(\Omega) \leq 1$ for every N. Since every non-decreasing sequence of non-negative real numbers which is bounded above has a limit we obtain the number

$$\text{(*)} \qquad \text{vol}_n(\Omega) = \lim_{N\to\infty} w_N(\Omega)$$

The limit number is called the n-dimensional Lebesgue measure of the bounded open set.

1.2 The unique dyadic exhaustion. Let Ω be a bounded open set. We first seek cubes in $\mathcal{D}_1(\Omega)$. It may be empty but is otherwise a finite set of unit cubes. Then we consider the cubes in $\mathcal{D}_2(\Omega$ which are not contained in those from $\mathcal{D}_1(\Omega)$. We continue in this way, i.e. if $k \geq 3$ we take only those cubes in $\mathcal{D}_k(\Omega$ which are outside the union of all cubes from $\mathcal{D}_1(\Omega),\ldots,\mathcal{D}_{k-1}(\Omega)$. Let us denote this family by $\mathcal{D}^*_k(\Omega)$. In this way we obtain a *uniquely determined* sequence $\{\mathcal{D}^*_k(\Omega)\}$ where each non-empty family $\mathcal{D}^*_k(\Omega)$ is a finite subfamily of $\mathcal{D}_k$. Let $\rho^*_k(\Omega)$ be the number of cubes in $\mathcal{D}^*_k(\Omega)$. By this construction it is clear that

$$\Omega_N(\Omega) = \sum_{k=0}^{k=N} 2^{-nk} \cdot \rho^*_k(\Omega)$$

hold for every N. Passing to the limit we obtain

$$\text{vol}_n(\Omega) = \sum_{k=0}^{\infty} 2^{-nk} \cdot \rho^*_k(\Omega)$$

1.3 Remark. If $\square$ is an arbitrary cube whose sides are parallell to the coordinate axis and of common length a then its Lebesgue measure is a^n. This equality is clear from the construction above if $2^N \cdot a$ is an integer for some N and the corner points belong to the lattice $2^N \cdot \mathbf{Z}^n$. The fact that $\text{vol}_n(\Omega) = a^n$ for an arbitrary cube follows when the real number a is approximated by "dyadic integers" , i.e. for every $N \geq 1$ we find an integer $q_N$so that $q_N 2^N \leq a < (q_N + 1)2^{-N}$ and in a similar fashion we perform small translations of the corner points. Next, we may also consider open cubes whose sides are not parallel to the coordinate axis. Then we still have $\text{vol}_n(\Omega) = a^n$ where a is the common length of the sides. To prove this one studies a general *linear transformation* on $\mathbf{R}^n$. Thus, let A be a real $n \times n$-matrix whose determinant is $\neq 0$. By $x \mapsto y = A(x)$ we get a bijective map from $\mathbf{R}^n$ onto itself. Then one has the equality

$$(*) \qquad \text{vol}_n(A(\Omega)) = \big|\det(A)\big| \cdot \text{vol}_n(\Omega)$$

for every bounded open set Ω.

Exercise. Prove (*). The hint is that we first have the equality in the special case when the A-matrix permutes the coordinates, i.e. when

$$A(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

where $i \to \sigma(i)$ is a permutation. In addition to such linear maps we recall from Linear Algebra that every matrix with a non-zero determinant is the product of such permuting matrices and special matrices of the form

$$A(x_1, \ldots, x_n) = (y_1, x_2, \ldots, x_n)$$
$$\text{where} \quad y_1 = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \quad \text{and} \quad a_1 \neq 0$$

The proof that (*) holds for such linear transforms is left as an exercise.

1.4 Additivity. Let $\Omega_1, \ldots, \Omega_k$ be finite family of disjoint and bounded open sets. Since every open cube $\square$ is connected an inclusion $\square \subset \Omega_1 \cup \ldots \cup \Omega_k$ implies that $\square$ is contained in one of the cubes. From this we conclude that

$$\rho_N(\Omega_1 \cup \ldots \cup \Omega_2) = \rho_N(\Omega_1) + \ldots + \rho_N(\Omega_k)$$

hold for each N. Passing to the limit we get:

$$\text{vol}_n(\cup\, \Omega_\nu) = \sum_{\nu=1}^{\nu=k} \text{vol}_n(\Omega_\nu)$$

More generally, let $\{\Omega_\nu\}$ be a denumerable sequence of pairwise disjoint open sets such that the union is contained in a bounded set. Then we get convergent positive series and the reader can verify that

$$\text{vol}_n(\cup\, \Omega_\nu) = \sum_{\nu=1}^{\infty} \text{vol}_n(\Omega_\nu)$$

Thus, the volume is σ-additive on bounded open sets.

1.5 Interior approximation. Let Ω be a bounded open set. For each N we may also consider the family $\mathcal{D}_N^*$ of *closed* dyadic cubes. Let $\bar{\rho}_N(\Omega)$ be the number of closed cubes from $\mathcal{D}_N^*$ which are contained in Ω. Notice that every such closed cube appears as a *compact* subset of Ω and hence the finite union of these cubes is again compact subset of Ω. It is clear that the function

$$N \mapsto 2^{-nN} \cdot \bar{\rho}_N(\Omega)$$

is increasing. So the limit exists and we have

$$(1) \qquad \lim_{N \to \infty} 2^{-nN} \cdot \bar{\rho}_N(\Omega) \leq \text{vol}_n(\Omega)$$

It turns out that (1) is an equality. To prove this we notice that if $\square \in \mathcal{D}_N$ and k is another large positive integer, then almost every small cube from $\mathcal{D}_{N+k}$ which is contained in $\square$ has is closure contained in $\square$. More precisely, the reader may verify that if Ω is a bounded open set then the inequality below holds for for every pair of positive numbers N and k:

$$\text{(i)} \qquad \bar{\rho}_{N+k}(\Omega) \geq 2^{nk} \cdot \rho_N(\Omega) - 2n \cdot 2^{(n-1)k} \cdot \rho_N(\Omega) = 2^{nk}\rho_N(\Omega) \cdot (1 - 2n \cdot 2^{-k})$$

The point is that we can choose large k-numbers at the same time let $N \to \infty$. Namely, for each $\epsilon > 0$ we first choose k such that $2n \cdot 2^{-k} < \epsilon$. Then (i) and a passage to the limit obviously give:

$$\lim_{N\to\infty} 2^{-nN} \cdot \bar{\rho}_N(\Omega) \geq (1-\epsilon) \cdot \mathrm{vol}_n(\Omega)$$

Since ϵ is arbitrary we get equality in (1).

1.6 The measure of compact sets. Let E be a compact subset of $\mathbf{R}^n$. For each N we denote by $\rho_N^*(E)$ the number of cubes $\square \in \mathcal{D}_N$ such that the closure $\bar{\square}$ has a non-empty intersection with E. Next, let Ω be some bounded open set which contains E. If $\square \in \mathcal{D}_N$ and $\bar{\square} \subset \Omega$ then we either have that $\bar{\square} \cap E \neq \emptyset$ or $\bar{\square} \subset \Omega \setminus E$. This gives the equality:

$$\bar{\rho}_N(\Omega) = \bar{\rho}_N(\Omega \setminus E) + \rho_N^*(E)$$

Passing to the limit where we apply the interior approximation from 1.5 to the open sets Ω and $\Omega \setminus E$, it follows that

$$\lim_{N\to\infty} 2^{-nN} \cdot \rho_N^*(E) = \mathrm{vol}_n(\Omega) - \mathrm{vol}_n(\Omega \setminus E)$$

In particular the limit in the left hand side exists and we use it as a definition of $\mathrm{vol}_n(E)$ for a compact set E. The reader may also verify that the limit exists directly. Namely, if $\square \in \mathcal{D}_N$ has a closure whose closure has a non-empty intersection wit E then it can only occur that some of the smaller closed cubes from $\mathcal{D}_{N+1}$ which are contained $\square$ have empty intersection with E. This entails that the function

$$N \mapsto 2^{-nN} \cdot \rho_N^*(E)$$

is increasing and hence has a limit which by the above is the measure of E.

1.7 An outer approximation. Let E be a compact set. If $\delta > 0$ we obtain the open set

$$E_\delta = \{x\colon\ \mathrm{dist}(x, E) < \delta\}$$

With this notation one has

$$(*) \qquad \lim_{\delta\to 0} \mathrm{vol}_n(E_\delta) = \mathrm{vol}_n(E)$$

To prove this we take some $\epsilon > 0$ and find N so large that

$$2^{-nN} \cdot \rho_N^*(E) < \mathrm{vol}_n(E) + \epsilon$$

Next, with N fixed we notice that if $\square \in \mathcal{D}_N$ is such that $\bar{\square} \cap E = \emptyset$ then $\bar{\square} \cap E_\delta = \emptyset$ for a sufficiently small δ. Hence with N fixed we can find δ so small that

$$E_\delta \subset \cup \bar{\square}$$

where the union is taken over cubes in $\mathcal{D}_N$ whose closures intersect E. This gives

$$\mathrm{vol}_n(E_\delta) \leq 2^{-nN} \cdot \rho_N^*(E) < \mathrm{vol}_n(E) + \epsilon$$

Since ϵ is arbitrary we get (*).

1.8 Lebesgue points

Let E be a compact set. Examples show that E may have positive Lebesgue measure and yet its interior is empty. Suppose this holds and let $x_0 \in E$ be a given point. If $\square$ is an open cube which contains x_0 then $\square \setminus E$ is non-empty so we have the strict inequality

$$\mathrm{vol}_n(E \cap \square) = \mathrm{vol}_n(\square) = \mathrm{vol}_n(\square \setminus E)$$

In spite of this it turns out that

$$(*) \qquad \lim_{\square \to x_0} \frac{\mathrm{vol}_n(E \cap \square)}{\mathrm{vol}_n(\square)} = 1$$

can hold where the limit is taken over cubes which tend to the singleton set $\{x_0\}$. To be precise, (*) means that for every $\epsilon > 0$ there exists some integer M such that

$$x_0 \in \square \quad \text{and} \quad \mathrm{vol}_n(\square) < \frac{1}{M} \implies \mathrm{vol}_n(E \cap \square) > (1-\epsilon) \cdot \mathrm{vol}_n(\square)$$

where $\square$ are arbitrary open cubes. Notice that we do not require that x_0 is the center if $\square$ when we regard an inclusion $x_0 \in \square$.

1.9 Definition. *A point $x \in E$ is called a Lebesgue point if (*) holds. The set of Lebesgue points in E is denoted by $\mathcal{L}(E)$.*

It turns out that $\mathcal{L}(E)$ is so large that the complement $E \setminus \mathcal{L}(E)$ is a null set, i.e. one has:

1.10 Theorem. *Almost every point in E is a Lebesgue point.*

The proof relies upon a covering Lemma due to Vitali. So we first expose this result and prove Theorem 1.10 in 1.13 below.

1.11 Vitali coverings. Let A be a subset of $\mathbf{R}^n$ and $\mathcal{V} = \{\square_\alpha\}$ a family of open cubes. It is called a Vitali covering of A if the following hold:

For each point $a \in A$ and each $\epsilon > 0$ there exists some $\square_\alpha \in \mathcal{V}$ such that $a \in \square_\alpha$ and vol$_n(\square_\alpha) < \epsilon$.

Remark. The $\mathcal{V}$ -cubes may consist of a non-denumarable family, i.e. the indices α can be taken from any set.

1.12 Vitali's Theorem *Let $\mathcal{V}$ be a Vitali covering of a bounded set A. Then there exists a sequence of pairwise disjoint cubes $\square_1, \square_2, \ldots$ in $\mathcal{V}$ such that for every N one has the inclusion*

$$A \subset \bar{\square}_1, \cup \ldots, \cup \bar{\square}_N \bigcup_{\nu > N} 3 \cdot \square_\nu$$

where $\bar{\square}_1, \ldots, \bar{\square}_N$ denote the closure of the first N cubes and if $\nu > N$ then $3 \cdot \square_\nu$ is the expanded cube whose sides are three times larger than those of $\square_\nu$.

Proof. Pick a cube $\square_1 \in \mathcal{V}$ such that

$$|\square_1|_n > \frac{2}{3} |\square_\alpha|_n \quad : \quad \forall\, \square_\alpha \in \mathcal{V}$$

Next, let $\mathcal{V}_1$ be the subclass of $\mathcal{V}$-cubes which have empty intersection with the closed cube $\bar{\square}_1$. Then we pick $\square_2 \in \mathcal{V}_1$ such that

$$|\square_2|_n > \frac{2}{3} |\square_\alpha|_n \quad : \quad \forall\, \square_\alpha \in \mathcal{V}_1$$

We continue in this way and obtain a sequence of pairwise disjoint cubes $\square_1, \square_2, \ldots$ where one for each $k \geq 2$ has

$$|\square_{k+1}|_n > \frac{2}{3} \cdot |\square_\alpha|_n \quad : \quad \forall\, \square_\alpha \in \mathcal{V}_k$$

and $\mathcal{V}_k$ is the family of cubes which have empy intersection with $\bar{\square}_1 \cup \ldots \cup \bar{\square}_k$. It remains to show that this sequence gives the covering lemma. First, since the $\mathcal{V}$-cubes all stay inside a bounded set we have

$$(1) \qquad \sum_{\nu=1}^{\infty} |\square_\nu|_n < \infty$$

Next, let $a \in A$ and let N be some positive integer. If a already belongs to $\bar{\square}_1 \cup \ldots, \cup \bar{\square}_N$ we are done. If the inclusion fails the Vitali covering gives some $\square_\alpha \in \mathcal{V}$ which contains a and at the same time is so small that it has empty intersection with the union of the N first closed cubes. Now $|\square_\alpha|_n > 0$ and we claim that $\square_\alpha$ cannot be disjoint from $\bar{\square}_{N+1} \cup \ldots, \cup \bar{\square}_M$ for all $M \geq N+1$. For if it is disjoint from such a union up to some integer , the construction entails that

$$|\square_\alpha|_n \leq \frac{2}{3} |\square_M|_n$$

But this cannot hold when M is large since the convergence in (1) implies that $|\square_M|_n \to 0$ as M increases. Hence we can find a *smallest* $M \geq N+1$ such that

$$\square_\alpha \cap \bar{\square}_M \neq \emptyset$$

Since M is minimal $\square_\alpha$ is disjoint from the closed union of the first $M-1$-cubes and the construction of $\square_M$ entails

$$|\square_M|_n > \frac{2}{3} |\square_\alpha|_n$$

This gives obviously the inclusion $\square_\alpha \subset 3 \cdot \bar{\square}_M$ and finishes the proof of Vitali's covering lemma.

1.13 Proof of Lebesgue's theorem

Let $\epsilon > 0$ and $0 < \rho < 1$ be kept fixed for a while. When N is a positive integer we define the subset $U_N(E)$ of E which consists of points $x \in E$ for which there exist a cube $\square$ such that

$$\frac{\text{vol}_n(\square \cap E)}{\text{vol}_n(\square)} < \rho \quad : \quad \text{vol}_n(\square) < \frac{1}{N}$$

It is obvious that $U_N(E)$ is an relatively open subset of E. Moreover these sets decrease as N increases. Put

$$A = \cap_{N \geq 1} U_N(E)$$

Next, by 1.5 above there exists $\delta > 0$ such that

$$|A_\delta \setminus A| < \epsilon$$

Next, let $\mathcal{V}(\rho)$ be the family of cubes $\square$ for which

$$\frac{\text{vol}_n(\square \cap E)}{\text{vol}_n(\square)} < \rho \quad : \quad 3 \cdot \bar{\square} \subset A_\delta$$

It is clear that $\mathcal{V}(\rho)$ is a Vitali covering of A. The Covering lemma gives a sequence of pairwise disjoint cubes $\square_1, \square_2, \ldots$. so that

$$A \subset \bar{\square}_1 \cup \ldots \cup \bar{\square}_N \bigcup 3 \cdot \square_\nu \quad : \quad N \geq 1$$

Next, using the obvious the inequalities

$$\frac{\text{vol}_n(A \cap \square_\nu)}{\text{vol}_n(\square_\nu)} \leq \frac{\text{vol}_n(E \cap \square_\nu)}{\text{vol}_n(\square_\nu)} < \rho \quad : \quad 1 \leq \nu \leq N$$

it follows that

$$|A|_n < \rho \cdot \sum_{\nu=1}^{\nu=N} \text{vol}_n(\square_\nu) + 3^n \cdot \sum_{\nu > N} \text{vol}_n(\square_\nu)$$

Finally, since the cubes $\{\square_\nu\}$ are pairwise disjoint the series $\sum \mathrm{vol}(\square_\nu)$ converges. So if N is sufficently large we have

$$3^n \cdot \sum_{\nu>N} \mathrm{vol}_n(\square_\nu) < \epsilon$$

At the same time $\square_1, \ldots, \square_N$ are all contained in A_δ. Hence XX gives

$$|A|_n < \rho \cdot |A_\delta|_n + \epsilon < \rho \cdot |A|_n + \rho\epsilon + \epsilon = \leq \rho \cdot |A|_n + 2\epsilon$$

Here ϵ can be arbitrarily small and since $\rho < 1$ we conclude that A is a null set. Its construction dependson ρ so let us denote this null set with $A(\rho)$. As $\rho \to 1$ the condition in (x) becomes more relaxed so these null sets increase. We get in particular the null set

$$A^* = \cup_{\nu \geq 2} A(1 - \frac{1}{\nu})$$

At this stage the reader may verify that $E \setminus A^*$ is contained in $\mathcal{L}(E)$ and Lebesgue's theorem is proved.

Remark Above we found that $A(1 - \frac{1}{\nu})$ is a so called G_δ set, i.e. the intersection of a decreasing sequence of relatively open subsets of E. So A^* is the union of an increasing sequence of G_δ-subsets of E, i.e. it a set-theoretically "relatively nice" subset of E.

1.14 Example. Let $n = 1$ and consider a denumerable family of open intervals $\{(a_\nu, b_\nu)\}$ all of which are contained in $(0, 1)$. Suppose also that $\sum (b_\nu - a_\nu) = 1/2$. In addition these intervals can be chosen so that their union is dense in $(0, 1)$. We obtain the compact set

$$E = [0, 1] \setminus \cup (b_\nu - a_\nu)$$

It has no interior points and its Lebesgue measure is $1/2$. Suppose that $\xi \in \mathcal{L}(E)$ where $0 < \xi < 1$. Given a small $\delta > 0$ we put

$$\rho(\delta) = \sum_{\nu=1}^{\infty} |(b_\nu - a_\nu) \cap (\xi - \delta, \xi + \delta)|_n$$

By the definition of Lebesgue points it follows that

$$\lim_{\delta \to 0} \frac{\rho(\delta)}{\delta} = 0$$

The fact that this hold for all $\xi \in E$ outside a null set is remarkable since the chosen family of open intervals $\{(a_\nu, b_\nu)\}$ is quite general.

2. Measurable sets

Let A be a subset of $\mathbf{R}^n$. Let Ω denote open sets and E compact sets. The outer - respectively the inner measure - of A are defined by

$$|A|^* = \min \mathrm{vol}_n(\Omega) \quad : \; A \subset \Omega \quad : \quad |A|_* = \max \mathrm{vol}_n(E) \quad : \; E \subset A$$

2.1 Definition. *A set A is measurable in the sense of Lebesgue if $|A|^* = |A|_*$. When it holds the common number is denoted by $vol_n(A)$ and called the Lebesgue measure of A. The class of measurable sets is denoted by $\mathcal{M}(\mathbf{R^n})$.*

Remark. Let A be measurable. The definition of inner measure yields an increasing squence of compact subsets $\{F_\nu\}$ of A such that $\lim |F_\nu| \to |A|_*$. Since $|A| = |A|_*$ it follows that that if $F^* = \cup F_\nu$, then $A \setminus F^*$ is a null set. Similarly, using the equality $|A| = |A|^*$ we find that that there exists a descreasing sequence of open sets $\{\Omega_\nu\}$ containing A such that if $\Omega_* = \cap \Omega_\nu$, then $\Omega_* \setminus A$ is a null set. Thus, apart from a null set every measurable set can be taken as a G_δ-set, i.e. a denumerable intersection of open sets, or as a denumerable union of closed sets. In general, a pair of measurable sets A, B are called *equivalent* if the Boolean difference

$$(A \setminus B) \cup (B \setminus A)$$

is a null set. This is an equivalence relation on the class of measurable sets and whenever needed one may choose a special measurable set from such an equivalence class.
Example. If A is measurable we use that its outer measure is equal to $|A|_n$ and construct a decreasing sequence of open set $\{U_\nu\}$ which all contain A while $\text{vol}_n(U_\nu \setminus A) < \frac{1}{\nu}$ hold for every ν. With $U_* = \cap U_\nu$ it follows that $U_* \setminus A$ is a null set. So in the equivalence class of A we can always choose a G_δ-set, i.e. a set formed by the intersection of a denumerable sequence of open sets.

Lebesgue points of measurable sets

Let A be a measurable set. A point x_0 is called a Lebesgue point for A if

$$(*) \qquad \lim_{\square \to x_0} \frac{\text{vol}_n(A \cap \square)}{\text{vol}_n(\square)} = 1$$

holds where the limit is taken over cubes which tend to the singleton set $\{x_0\}$. The set of Lebesgue points is denoted by $\mathcal{L}(A)$. Notice that we do not require that x_0 belongs to A. However, if E is a closed set it is clear that a Lebesgue point of E must belong to E which means that our definition above extends that for closed sets in Definition 1.9. In particular we see that $\mathcal{L}(A)$ is contained in the clopsure of A. If Ω is an open set it is clear that $\mathcal{L}(\Omega) \subset \Omega$ and it may occur that the boundary $\partial\Omega$ contains some Lebesgue points, i.e. the set $\mathcal{L}(\Omega) \setminus \Omega$ can be non-empty.

2.2 Theorem. *For every measurable set A the Boolean difference of A and $\mathcal{L}(A)$ is a null set.*

Proof. First we prove that $A \setminus \mathcal{L}(A)$ is a null set. To show this we consider some $0 < \rho < 1$ and for each integer $N \geq 1$ we get the subset $U_N(A)$ which consists of all $x \in A$ for which there exists an open cube $\square$ such that

$$(1) \qquad \frac{\text{vol}_n(\square \cap A)}{\text{vol}_n(\square)} < \rho \quad : \quad \text{vol}_n(\square) < \frac{1}{N} \quad : x \in \square$$

It is clear that $U_N(A)$ is relatively open in A and hence measurable. Put

$$(2) \qquad A_*(\rho) = \cap U_N(A)$$

Then $A_*(\rho)$ is measurable and hence equal to its outer measure. So for any $\epsilon > 0$ there exists an *open set* Ω which contains $A_*(\rho)$ such that

$$(3) \qquad |\Omega \setminus A_*(\rho)| < \epsilon$$

Next, define the family $\mathcal{V}(\rho)$ of open cubes for which

$$(4) \qquad \frac{\text{vol}_n(\square \cap A)}{\text{vol}_n(\square)} < \rho \quad \text{and} \quad 3 \cdot \bar{\square} \subset \Omega$$

This yields a Vitali covering of $A_*(\rho)$ and we get a sequence of parirwise disjoint cubes $\square_1, \square_2, \ldots$ in $\mathcal{V}(\rho)$ such that

$$A_*(\rho) \subset \bar{\square}_1 \cup \ldots \cup \bar{\square}_N \bigcup 3 \cdot \square_\nu \quad : \quad N \geq 1$$

From this we conclude that

$$|A_*(\rho)| < \rho \sum_{\nu=1}^{\nu=N} \text{vol}_n(\square_\nu) + 3^n \cdot \sum_{\nu > N} \text{vol}_n(\square_\nu)$$

As in the proof for compact sets we choose N so that $3^n \cdot \sum_{\nu > N} \text{vol}_n(\square_\nu) < \epsilon$ and since each $\square_\nu \subset \Omega$ we obtain

$$|A_*(\rho)| < \rho \cdot |\Omega| + \epsilon < \rho \cdot |A_*(\rho)| + \rho\epsilon + \epsilon \leq \rho \cdot |A_*(\rho)| + 2\epsilon$$

where we used that $|\Omega \setminus A_*(\rho)| < \epsilon$. Hence

$$|A_*(\rho)| < \frac{2\epsilon}{1-\rho}$$

Since $\epsilon > 0$ can be made arbitrary small we conclude that $A_*(\rho)$ is a null set. Now we choose $\rho = 1 - \frac{1}{k}$ with $k = 1, 2, \ldots$ and get the null set

$$A_* = \cup A_*(1 - \frac{1}{N})$$

Funally, from the construction of the sets $A_*(\rho)$ it is clear that one has the inclusion:

$$A \setminus A_* \subset \mathcal{L}(A)$$

Hence $A \setminus \mathcal{L}(A)$ is contained in the null set A_*. To show that $\mathcal{L}(A) \setminus A$ is a null set we can work locally and consider a bounded and relatively open subset B of $\mathbf{R}^n \setminus A$. Now $B \setminus \mathcal{L}(B)$ is a null set and at the same time it is obvious that

$$\mathcal{L}(A) \cap B \subset B \subset \mathcal{L}(B)$$

From this we conclude that $\mathcal{L}(A) \setminus A$ also is a null set and Theorem 2.2 follows.

2.3 Remark. The result above means that we can choose a canonical set in every equivalence class of measurable sets. Namely, if A and B are two measurable sets whose Boolean difference is a null set then $\text{vol}_n(A \cap \square) = \text{vol}_n(B \cap \square)$ hold for every cube. It follows that we have the equality $\mathcal{L}(A) = \mathcal{L}(B)$. So in the given equivalence class we can choose this common set denoted by A_* where we now have the equality

$$(*) \qquad A_* = \mathcal{L}(A_*)$$

2.4 A criterion for null sets Let A be a measurable set. Suppose there exists some $0 < \rho < 1$ such that

$$|A \cap \square|_n \leq \rho \cdot |\square|_n$$

hold for a family of open cubes which is a Vitali covering of A. Then we see that A cannot have any Lebesgue point and hence it must be a null set.

3. Measurable functions and their integrals

A real-valued function f on $\mathbf{R^n}$ is measurable if the sets

$$\{x\colon\, f(x) < a\} \quad : \quad \{x\colon\, f(x) \leq a\}$$

are measurable for every real number a. Among these occur finite $\mathbf{R}$-linear sums of characteristic functions. That is, let $A_1, \ldots, A_m$ be a finite family of pairwise disjoint and measurable sets. Then the functions

$$\sum c_\nu \cdot \chi_{A_\nu} \quad : \quad c_1, \ldots, c_m \text{ in } \mathbf{R}$$

are measurable where we for each set A denote by χ_A its characteristic function. Let us now study bounded and measurable functions which for simplicity takes values in the interval $[-1, 1]$. Given such a function f we can approximate it from below and from above. When $N \geq 1$ we get the pairwise disjoint and measurable sets

$$A_\nu(N) = \{\nu 2^{-N} \leq f < (\nu+1)2^{-N}\}$$

Put

$$S_*(N) = \sum_{\nu=-2^N}^{\nu=2^N-1} \nu 2^{-N} \chi_{A_\nu(N)} \quad \text{and} \quad S^*_N = \sum_{\nu=-2^N}^{\nu=2^N-1} (\nu+1)2^{-N} \chi_{A_\nu(N)}$$

It follows that

$$S_*(N) \leq f \leq S^*(N) \quad : \quad S^*(N) - S_*(N) \leq 2^{-N}$$

Remark. Notice that $\{S_*(N)$ is an increasing sequence of functions, while $\{S^*(N)\}$ is decreasing. Hence we can approximate the bounded function f *uniformly* by characteristic functions from below or from above.

3.1 The Lebesgue integral. Let f be bounded and measurable. Consider also some bounded measurable set E which is kept fixed. To each $N \geq 1$ we define the integrals

$$\int_E S_*(N)(x)dx = \sum_{\nu=-2^N}^{\nu=2^N-1} \nu 2^{-N} \mathrm{vol}_n(A_\nu(N) \cap E)$$

$$\int_E S^*(N)(x)dx = \sum_{\nu=-2^N}^{\nu=2^N-1} (\nu+1) 2^{-N} \mathrm{vol}_n(A_\nu(N) \cap E)$$

It is clear that their difference is $\leq (b-a)2^{-N}\mathrm{vol}_n(E)$ and their exists the common limit

$$\lim_{N\to\infty} \int_E S_*(N)(x)dx = \int_E S^*(N)(x)dx$$

It is denoted by $\int_E f(x)dx$ and called the Lebesgue integral of f over the measurable set E.

Exercise Show that the integral is approximated by small partitions of the range of f. More precisely, let $-1 = a_0 < a_1 \ldots < a_N = 1$ and put

$$I(a_\bullet) = \sum a_\nu \cdot \chi_{\{a_\nu \leq f < a_{\nu+1}\}}$$

Then one has the inequality

$$|\int_E f(x)dx - I(a_\bullet)| \leq \mathrm{vol}_n(E) \cdot \max\,(a_{\nu+1} - a_\nu)$$

Above the rate of convergence these $I(a_\bullet)$-sums to the Lebesgue integral of f is *independent* of f as long as its range is confined to $[-1, 1]$. This means that the construction of the Lebegue inegral is "robust".

Exercise Show that the Lebesgue integral is additive in the sense that

$$\int_{E_1 \cup E_2} f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx \quad : E_1, E_2 \text{ disjoint bounded measurable sets}$$

Show also additivity with respect to f, i.e. that

$$\int_E (f_1 + f_2)dx = \int_E f_1 dx = \int_E f_2 dx$$

for a pair of bounded and measurable functions.

3.2 Lebesgue points. Let f be a bounded measurable function. A point x_0 is called a Lebesgue point of f if

$$(*) \qquad \lim_{\delta\to 0} \frac{1}{|\square|_n} \int_\square f(x) \cdot dx = f(x_0)$$

hold where the limit it is taken over cubes $\square$ which contain x_0 with measure $< \delta^n$. Thus, if $f = \chi_A$ for some measurable set A, then we encounter the previous notion of Lebesgue points for sets.

3.3 Theorem *Outside a null set a bounded measurable function has Lebesgue points.*

Proof To prove this we employ the functions $S^*(N)$ and $S_*(N)$ above. Each S-function is finite linear sum of characteristic functions and by Theorem 2.2 measurable sets have Lebesgue points almost every where. Since the union of a denumerable family of null sets is again a null set we see that there exists a null set F such that the following hold for every point $x_0 \in \mathbf{R}^n \setminus F$ and every $N \geq 1$:

$$\lim_{\delta\to 0}\frac{1}{|\square|_n}\int_\square S^*(N)(x)dx = S^*(N)(x_0) \quad \text{and} \quad \lim_{\delta\to 0}\frac{1}{|\square|_n}\int_\square S_*(N)(x)dx = S_*(N)(x_0)$$

Since $f \leq S^*(N)$ for all N the first limit formula entails that

$$\text{Lim.sup}_{\,\delta\to 0}\,\frac{1}{|\square|_n}\int_\square f(x)dx \leq f(x_0)$$

Similarly, the second limit formula gives

$$\text{Lim.inf}_{\,\delta\to 0}\,\frac{1}{|\square|_n}\int_\square f(x)dx \geq f(x_0)$$

Hence the equality (*) holds almost everywhere.

3.4 Approximating bounded measurable functions. The existence of Lebesgue points outside a null set enable us to approximate a bounded measurable function in a nice manner. Namely, given some positive number δ we define the function $F_\delta(x)$ by

$$F_\delta(x) = \delta^{-n}\cdot\int_{\square_\delta(x)} f(x)dx$$

where $\square_\delta(x)$ is the square centered at x with volume δ^n. The definition of Lebesgue points give

(*) $$\lim_{\delta\to 0} F_\delta(x) = f(x) \quad : \quad x_0 \in \mathcal{L}(f)$$

Concerning the F_δ-functions we notice that they are not only continuous but even *Lipschitz continuous.*

3.5 Exercise Show that there exists a constant C_n which depends on n only such that when$|f| \leq 1$ and $0 < \delta \leq 1$ then

$$|F_\delta(x) - F_\delta(y)| \leq C_n\cdot\delta^{-n}\cdot|x-y| \quad : \quad \text{for any pair } x, y$$

3.6 Remark The convergence in (*) is *pointwise*. Thus, every bounded measurable function outside a null set is a pointwise limit of Lipschitz continuous functions.Moreover, identifying a pair of measurable functions which are equal almost everywhere we may assume from the start that f is equal to its Lebesgue value outside a certain nullset $\mathcal{N}_f$. Then the pointwise limit in (*) holds at all points outside $\mathcal{N}_f$.

3.7 Approximation of measurable functions. We agree to identify a locally integrable and measurable function $f(x)$ with its Lebesgue values outside the null set $\mathcal{N}_f$. Then the pointwise limi (*) above holds. In addition to this we define the functions

(1) $$F^*_\delta(x) = \max_{0<\rho\leq\delta} F_\rho(x)$$

Since the maximum is taken over a family of Lipschitz continuous functions, it follows that F^*_δ is an *upper semi-continuous function* for every $\delta > 0$. As δ decreases the maximum is taken over smaller families of functions. So $\{F^*_\delta\}$ is a non-increasing sequence of functions. Moreover, we still have the point-wise convergence

$$\lim_{\delta\to 0} F^*_\delta(x) = f(x)$$

for all points outside the null set $\mathcal{N}_f$.

3.8 Egoroff-s Theorem.

Let f be a bounded measurable function defined on a compact set, say the unit cube $\square$ in R^n. In general f fails to be continuous. However, the *restriction* of f to a suitable set is continuous. To prove this we consider an arbitrary measurable and bounded function f defined in the unit cube $\square$. Let $\mathcal{L}(f)$ denote the set of its Lebesgue points. If $\epsilon > 0$ we can choose a compact set $E \subset \mathcal{L}(f)$ such that $|\square \setminus E| < \epsilon$. With $\{g_n = F_{1/n}\}$ we have a sequence of continuous functions on E which converges pointwise to f, i.e. at every point in E. Let us now construct a subset of E where this pointwise convergence is uniform. To each pair of positive integers ν, N we set

$$G_N(\nu) = \bigcup_{k,m>N} \{x \in E : \ |g_k(x) - g_m(x)| \geq \frac{1}{\nu}\}$$

Since $g_k \to f$ holds pointwise on E, the decreasing sequence $N \mapsto G_N(\nu)$ tends to the empty set. So with $\epsilon > 0$ fixed there exists for every $\nu \geq 1$ some $N(\nu)$ such that

$$\text{vol}_n(|G_{N(\nu)}(\nu)) < 2^{-\nu} \cdot \epsilon$$

Next, since $G_{N(\nu)}(\nu)$ is measurable we find an open set Ω_ν such that

$$G_{N(\nu)}(\nu) \subset \Omega_\nu \quad : \ |\Omega_\nu| < 2^{-\nu} \cdot \epsilon$$

Put $E_* = E \setminus \cup \Omega_\nu$. Then $|\square \setminus E_*| < 2\epsilon$ and from the construction of the G-sets we see that the sequence $\{g_k\}$ converges *uniformly* on E_* and hence the limit function f restricts to be continuos on E_*. Hence we have proved:

3.9 Theorem. *To every $\epsilon > 0$ there exists a compact set $E \subset \square$ such that the measure of $\square \setminus E$ is $< \epsilon$ and the restriction of f to E is a continuous function.*

4. Fubini's Theorem.

We study images of measurable sets under projections and consider first the case $n = 2$ where (x, y) are the coordinates in $\mathbf{R}^2$. Let A be an arbitrary bounded set. To each x we get the slice

$$A(x) = \{y : \ (x, y) \in A\} \tag{1}$$

If Ω is a bounded and open set it is clear that the slices $\Omega(x)$ are open or eventually empty for every x. Next, for an arbitrary set A we get the open set A_δ whose points have euclidian distance δ to A. It is obvious that

$$A(x)_\delta \subset A_\delta(x) \tag{2}$$

So if $|A(x)|^*$ denotes the outer measure of the set $A(x)$ on the real y-line then we get the inequality:

$$|A(x)|^* \leq \lim_{\delta \to 0} |A_\delta(x)| \tag{3}$$

Next, let E be a compact subset of A. Then $E(x)$ is a compact subset of $A(x)$ so this time we get the inequality

$$|E(x)| \leq |A(x)|_*$$

where the right hand side is the inner measure of $A(x)$.

4.1 The case when A is measurable. In this case we prove the following:

4.2 Theorem. *The sets $A(x)$ are measurable for almost every x and the function $x \mapsto |A(x)|$ is measurable. Finally one has the equality*

$$\int_0^1 |A(x)| \cdot dx = \mathrm{vol}_2(A)$$

Proof. The result is obvious if A is an open, closed or half-open cube whose sides are parallell to the coordinate axis. This observation together with the unique dyadic decomposition of a bounded open set Ω in $\mathbf{R}^2$ gives Theorem 4.2 for Ω. Next, if E is a compact set we choose some open neighborhood Ω and since Theorem 4.2 hold for Ω and $\Omega \setminus E$ it holds for E by additivity.

Now we consider a bounded measurable set A. Choose an increasing sequence of compact subsets $\{E_n\}$ such that $\mathrm{vol}_2(A \setminus E_n) \to 0$ and consider also since descreasing sequence of open sets $\{\Omega_n = A_{\frac{1}{n}}\}$. Since A is measurable we know that:

$$\lim_{n\to\infty} \mathrm{vol}_2(\Omega_n \setminus E_n) = 0 \tag{1}$$

To each n we set

$$g_n(x) = |\Omega_n(x)| - |E_n(x)| \tag{2}$$

It is clear that $\{g_n(x)\}$ is a non-increasing sequence of functions. Since we already proved that Theorem 4.2 hold for open and for compact sets, i.t follows that

$$\int_0^1 g_n(x) \cdot dx = \mathrm{vol}_2(\Omega_n \setminus E_n) \tag{3}$$

Since $\{g - n\}$ is non-decreasing and (1) holds we conclude that

$$\lim_{n\to\infty} g_n(x) = 0 \tag{4}$$

holds almost everywhere. Consider a point x where the limit is zero. For each n we have

$$|A(x)|_* \geq |E_n(x)| \quad \text{and} \quad |A(x)|^* \leq |\Omega_n(x)| \tag{5}$$

So when (4) holds we conclude that the inner and the outer measures of $A(x)$ are equal. At this stage we leave it to the reader to conclude that the function $x \mapsto |A(x)|$ is measurable and that the equality in Theorem 4.1 holds.

Remark. Let E be a compact subset of $\mathbf{R^2}$. In this case the reader may verify that the function

$$x \mapsto |E(x)|_1 \tag{1}$$

is *upper continuous.* Similarly, if Ω is a bounded open set, then the function

$$x \mapsto |\Omega(x)|_1 \tag{2}$$

is *lower continuous.*

4.3 Fubini's theorem for functions.

Consider a bounded and measurable function $f(x,y)$ which is zero outside the unit cube $\square$. As in 3.6 we agree to identify f with its Lebesgue values outside a null set $\mathcal{N}_f$. From (3.7) we have the sequence $\{F^*_\delta\}$ which converges almost everywhere to f. if x is fixed the functions $y \mapsto F^*_\delta(x,y)$ are upper semi-continuos in the single y-variable and hence the integrals are defined. So for each $\delta > 0$ we get a function

$$\phi_\delta(x) = \int_0^1 F^*_\delta(x,y) \cdot dy \tag{1}$$

(Exercise.) Show that the ϕ-functions are upper semi-continuous for all $\delta > 0$. Next, since $\{F^*_\delta\}$ is a non-decreasing sequence it follows that $\{\phi_\delta\}$ also is non-decreasing and then the integral values

$$\delta \mapsto \int_0^1 \phi_\delta(x) \cdot dx$$

decrease with δ and we obtain a limit value

$$(*) \qquad J_* = \lim_{\delta \to 0} \int_0^1 \Big[\int_0^1 F^*_\delta(x,y) \cdot dy\Big] \cdot dx$$

Reversing the role of x and y we also have the limit value

$$(**) \qquad J_{**} = \lim_{\delta \to 0} \int_0^1 \Big[\int_0^1 F^*_\delta(x,y) \cdot dx\Big] \cdot dy$$

With these notations one has the equality

$$J_* = J_{**}$$

It may be expressed by sating that the two-dimensional Lebesgue integral of f over $\square$ can be identified with either of the two iterated integrals

$$(***) \qquad \int_0^1 \Big[\int_0^1 f(x,y) \cdot dx\Big] \cdot dy \quad \text{or} \quad \int_0^1 \Big[\int_0^1 f(x,y) \cdot dy\Big] \cdot dx$$

However, some care must be taken. The point is that we start with the null set $\mathcal{N}_F$ where f has no precise value. But the nul set is defined and for each fixed x can consider the slice

$$\mathcal{N}_f(x) = \{0 \leq y \leq 1 \quad : (x,y) \in \mathcal{N}_f\}$$

Now almost every such slice is a null set. For those x where the slice is a null set the function

$$(1) \qquad y \mapsto f(x,y)$$

is defined almost everywhere on the interval $0 \leq y \leq 1$ and hence we can take is integral which eventually may be infinite. However, (1) belongs to the L^1-space of this unit interval of the y-variable. for almost every x. This means that the function

$$\phi(x) = \int_0^1 f(x,y) \cdot dy$$

is defined almost everywhere. Finally, Fubini's theorem means that this ϕ-function also is integrable and we have the equality

$$\int_0^1 \phi(x) \cdot dx = \iint_\square f(x,y) \cdot dxdy$$

IV. Riesz measures

Contents

Introduction

The Lebesgue measure is invariant under translations as well as rotations in $\mathbf{R}^n$. Thus, the volume of a cube depends only upon the length of its sides and not upon its position. We shall construct measures which do not enjoy this invariance. If f is a locally integrable function in $\mathbf{R}^n$ in the sense of Lebesgue then we obtain a measure which to every bounded Lebesgue measurable set E assigns the mass

$$\int_E f \cdot dx$$

We say that $f \cdot dx$ is absolutely continuous with respect to dx. The new class consists of *singular* Riesz measures μ whose mass is concentrated to a null set in the sense of Lebesgue. Among these occur discrete measures which attach masses at some sequence of points. Less evident is the description of singular Riesz measures μ where the μ-mass of every singleton set is zero. If $n = 1$ we described such measures in Section I via non-decreasing and continuos functions $f(x)$ on the real x-line whose derivative is zero outside a set with Lebegue measure zero. When $n \geq 2$ we have no clear "geometric picture" of a singular Riesz measure. They are of a more implicit character and in many cases one can just say that a Riesz measure μ represents a continuous linear form on the space of continuous functions. This follows from a result due to Friedrich Riesz which for example asserts that if K is a compact subset of $\mathbf{R}^n$ and $C^0(K)$ the Banach space of continuous and real-valued functions on K, then its dual space consists of singular Riesz measures supported by K. The *Riesz representation formula* will be discussed in XXX.

In section 1 we construct Riesz measures in $\mathbf{R}^n$ using dyadic grids which is similar to the construction of the Lebesgue measure, except that one now arrives at a family of σ-additive measures which assign masses to arbitrary Borel sets. In $\mathbf{R}^n$ there exists the algebra $\mathcal{B}(\mathbf{R}^n)$ of *Borel measurable functions.*. Using material from abstract measure theory we will show that if f is a bounded Borel function and μ some compactly supported Riesz measure then there exists an integral

$$(*) \qquad \int f \cdot d\mu$$

Their existence and properties constitute a major part in the theory about Riesz measures in $\mathbf{R}^n$.

1. Dyadic grids.

If $N \geq 0$ we obtain the family $\mathcal{D}_N$ of half-open cubes where we for each n-tuple $\nu_\bullet \in \mathbf{Z}^n$ assign the cube

$$\square_N(\nu_\bullet) = \{x\colon\ \nu 2^{-N} \leq x_\nu < (\nu+1)2^{-N}\}$$

It is clear that $\{\square_N(\nu_\bullet)\}$ is a disjoint covering of $\mathbf{R}^n$ for each fixed N. Next, let Ω be a bounded open set. For each $x \in \Omega$ there exists a unique smallest integer $N(x)$ such that x belongs to a cube $\square \in \mathcal{D}_{N(x)}$ with $\square \subset \Omega$. This cube is denoted by $\square(x)$. The reader may verify the implication

$$y \in \square(x) \cap \Omega \implies \square(y) = \square(x)$$

From this we see that the open set Ω is a unique union of disjoint dyadic cubes. In other words, there exists a denumerable sequence $\{x_\nu\}$ in Ω such that the cubes $\{\square(x_\nu)\}$ are disjoint and

$$\Omega = \cup \square(x_\nu) \tag{*}$$

We shall use this unique decomposition of open sets to construct a family of additive functions. Let $\mathcal{D}_*$ be the family of subsets of $\mathbf{R}^n$ given by a finite and disjoint union of cubes $\{\square_\alpha\}$ where each $\square_\alpha \in \mathcal{D}_{N(\alpha)}$ for some integer $N(\alpha) \geq 0$. Suppose that

$$\mu\colon \mathcal{D}_* \mapsto \mathbf{R}^+$$

is a non-negative and additive function. It means that

$$\mu(\cup\square_\alpha) = \sum \mu(\square_\alpha)$$

hold for every finite and disjoint family of dyadic half-open cubes. Above the μ-mass of every single cube is some non-negative real number.

1.1 The μ-measure on open sets. Consider and additive function μ on $\mathcal{D}_*$. Let Ω be a bounded open set. The unique decomposition (*) defines the number:

$$\mu(\Omega) = \sum \mu(\square(x_\nu)) \tag{**}$$

In the right hand side is a positive series. Since Ω is bounded the series is convergent. To see this it suffices to observe that Ω is contained in some finite union of cubes from the family $\mathcal{D}_0$. Then, since μ is non-negative and additive, it follows that each finite partial sum of the series in (**) is bounded by the sum of μ-masses over the K-tuple of cubes in $\mathcal{D}_0$. Hence the positive series (**) is convergent. Next, let $\{\Omega_\nu\}$ be a sequence of pairwise disjoint open sets and the union is bounded. The uniqueness of dyadic decompositions implies that

$$\mu(\cup\,\Omega_\nu) = \sum \mu(\Omega_\nu) \tag{1}$$

1.2 The μ-measure on compact sets. Let E be a compact subset of $\mathbf{R}^n$. If $N \geq 0$ we set

$$\rho_N(E) = \sum \mu(\square) \quad : \quad \square \in \mathcal{D}_N \quad \text{and} \quad \square \cap E \neq \emptyset$$

If $N \geq 1$ and some $\square \in \mathcal{D}_N$ has a non-empty intersection with E it may occcur that some of the half-open cubes in $\mathcal{D}_{N+1}$ which are contained in $\square$ have empty intersection with E. This observation shows that

$$N \mapsto \rho_N(E)$$

decreases with N and hence there exists the limit

$$\lim_{N\to\infty} \rho_N(E)$$

The limit is denoted by $\mu(E)$ and is called the μ-measure of the compact set.

1.3 Additivity. Let E be a compact set and Ω some bounded open set which contains E. Consider some cube $\square(x_\nu) \subset \Omega$ in the decomposition of Ω. Here two cases can occur : Either $\square(x_\nu) \cap E \neq \emptyset$ or else $\square(x_\nu) \subset \Omega \setminus E$. From this we conclude that the following equality holds for every integer $N \geq 0$:

$$\rho_N(\Omega) = \rho_N(\Omega \setminus E) + \rho_N(E)$$

where we have set

$$\rho_N(\Omega) = \sum \mu(\square) \quad : \quad \square \in \mathcal{D}_N \quad \text{and} \quad \square \subset \Omega$$

and similarly we define $\rho_N(\Omega \setminus E)$. Since the positive series (*) in 1.1 is convergent for the open sets Ω and $\Omega \setminus E$ a passage to the limit as $N \to \infty$ gives:

1.4 Proposition. *If E is a compact set and if $E \subset \Omega$ for some bounded open set, then*

$$\mu(E) + \mu(\Omega \setminus E) = \mu(\Omega)$$

1.5 Exercise. Let $\mathfrak{B}_*$ denote the Boolean algebra of subsets of $\mathbf{R}^n$ generated by all open and all compact subsets. Let μ be a non-negative and additive function on $\mathcal{D}_*$ which has a bounded support in the sense that $\mu(\square) \neq 0$ only occurs for a finite set of cubes in $\mathcal{D}_0$. Use the results above to show that μ extends to an additive map on $\mathfrak{B}_*$, i.e for each finite family of disjoint sets $\{F_k\}$ in $\mathfrak{B}_*$ it holds that

$$\mu(\cup F_\nu) = \sum \mu(F_\nu)$$

whenever F and G are disjoint and bounded sets in $\mathfrak{B}_*$.

1.6 σ-additivity.

Above we found an additive measure defined on bounded sets in $\mathfrak{B}_*$. Now there exists the σ-algebra $\mathfrak{B}$ of Borel sets which by definition this is the smallest σ-algebra containing $\mathfrak{B}_*$. In order that μ can be extended to a σ-additive measure defined on $\mathfrak{B}$ we must impose additional conditions. In general, let $\square \in \mathcal{D}_N$ be some half-open dyadic cube. If $0 < a < 1$ we obtain the closed cube $a\bar{\square}$ defined by

$$\nu_k 2^{-N} \leq x_k \leq (\nu_k + a)2^N \quad : 1 \leq k \leq n\}$$

By the construction in 1.2 we can assign the μ-measure to each of these compact cubes and since μ is non-negative the function

$$\text{(i)} \qquad a \mapsto \mu(a \cdot \bar{\square})$$

is increasing. Moreover we have

$$\text{(ii)} \qquad \mu(a \cdot \bar{\square}) \leq \mu(\square) \quad : \text{for each} \quad a < 1$$

Hence (i-i) suggest the following:

1.7 Definition. *We say that the additive measure μ is regular if*

$$\lim_{a \to 1} \mu(a\bar{\square}) = \mu(\square)$$

holds for every half-open dyadic cube.

1.8 Remark. This regularity condition is necessary if μ has some σ-additive extension since we have the empty intersection:

$$\cap_{a<1}\,(\square \setminus a\cdot\bar{\square}) = \emptyset$$

The case when μ is regular. Let μ satisfy the condition from Definition 1.7. If Ω is a bounded open set and $\epsilon > 0$ we find N such that

(i) $$\mu(\Omega) < \rho_N(\Omega) + \epsilon$$

Here $\rho_N(\Omega) = \sum \mu(\square_\nu)$ where the sum extends over a finite set of cubes in $\mathcal{D}_N$. The regularity condition gives some $a < 1$ such that

(ii) $$\sum \mu(\square_\nu) < \sum \mu(a\cdot\bar{\square}_\nu) + \epsilon$$

With $E = \cup\, a\cdot\bar{\square}_\nu$ it follows that

(iii) $$\mu(\Omega) < 2\epsilon + \mu(E)$$

Notice that E is a *compact* subset of Ω. Since $\epsilon > 0$ can be arbitrary small above we have therefore proved:

1.9 Proposition. *Let μ be regular, Then*

$$\mu(\Omega) = \max_K \mu(K)$$

hold for every bounded open set Ω where the maximum is taken over compact subsets of Ω.

1.10 Exercise. Let μ be regular and E is a compact set. To each $\delta > 0$ we set $E_\delta = \{x\colon\ \mathrm{dist}(x,E) < \delta\}$. Show that

$$\lim_{\delta\to 0} \mu(E_\delta) = \mu(E)$$

1.11 Extension to the Borel algebra. Denote by $\mathfrak{B}$ the σ-algebra generated open and compact subsets of $\mathbf{R}^n$. Let μ be a regular and bounded measure. Using the results above it follows that μ extends n a unique fashion to a σ-additive measure defined on all sets in $\mathfrak{B}$. So here

$$\mu(\cup F_\nu) = \sum \mu(F_\nu)$$

hold for every denumerable disjoint union of Borel sets.

1.12 Non-negative Riesz measures .

This class consists of non-negative and regular measures μ defined as above. We have seen that the σ-additivity follows from the regularity. Moreover the resulting σ-additive function on bounded Borel sets is fully determined by the values assigned to all dyadic cubes.

(*) $$\square_N(\nu\cdot) = \{x = (x_1,\ldots,x_n)\colon\ \nu_j 2^{-N} \leq x_j < (\nu_j+1)2^{-N}\}$$

Let us restrict the attention to the cube

$$\square_* = \{0 \leq x_j < 1\colon\ 1 \leq j \leq n\}$$

Then $\mathcal{D}_N$ contains 2^{Nn} many pairwise disjoint and half-open cubes whose union cover $\square_*$.

By an induction over N we assign a non-negative mass to these cubes in $\mathcal{D}_N$. The constraint is to preserve additivity. First $\mathcal{D}_0$ is a single cube to which we assign the unit mass. Next, in $\mathcal{D}_1$ we have 2^n cubes and we assign a non-negative mass to each of them so that the sum is one. Next, let $N \geq 1$ and suppose that a mass $m_N(\nu\cdot)$ is assigned to every cube $\square_N(\nu\cdot) \in \mathcal{D}_N$. Notice that a given cube $\square_N(\nu\cdot) \in \mathcal{D}_N$ is the union of 2^n many cubes from $\mathcal{D}_{N+1}$. To each of these we assign a mass so that the sum of these 2^n-many masses is equal to $m_N(\nu\cdot)$.

This inductive construction works for all N. Suppose we have performed such a construction over all positive integers N. Hence a mass $m_N(\nu\cdot)$ is assigned to $\square_N(\nu\cdot)$ for every N and each of the 2^{Nn} many n-tuples $\nu\cdot = (\nu_1, \ldots, \nu_n)$.

The regularity condition. Above we have seen how an inductive construction yields an additive measure μ. But it does not follow that it is regular, i.e. the condition from Definition 1.7 is not automatic. So here one encounters a set-theoretic problem, i.e. to describe conditions during the inductive construction of the assigned masses $\{m_N(\nu\cdot)\}$ in order that μ is regular. A necessary and sufficient criterium for this regularity is not known, i.e. this would mean that there exists a fully *constructive* method to achieve all Riesz measures. So the reader should be aware of the fact that the space of Riesz measures is an abstract object even though we have found that it appears as a subset of all additive measures constructed by the induction above.

2. Singular Riesz measures

By the Radon Nikodym theorem every Riesz measure μ whose total mass is finite is the unique sum of a singular part and an absolutely continuous measures defined as a density by some $L^1(\mathbf{R}^n)$-function. Consider for simplicity a measure μ supported by the unit cube, i.e. it is constructed via the dyadic grid on $\square$. Th result below gives the condition in order that μ is singular, i.e. its total mass taken over a set of Lebesgue measure.

Theorem The measure μ is singular if and only if the following hold: For each $\epsilon > 0$ there exists an integer N such that we can find a subfamily $\mathcal{F}_N = \{\square_N(\nu\cdot)\}$ of $\mathcal{D}_N$ where the number of cubes in $\mathcal{F}_N$ is $< \epsilon \cdot 2^{Nn}$ while

$$\sum \mu(\square_N(\nu\cdot)) > 1 - \epsilon \quad : \quad \sum \text{ taken over cubes } \square_N(\nu\cdot) \in \mathcal{F}_N$$

Remark. The proof is left as an exercise. The criterion for μ to be singular means roughly speaking that there exist enough cubes in $\mathcal{D}_N$ whose union gives a μ-mass which is considerably larger than its n-dimensional Lebesgue measure. On the other hand, if K is a fixed real number and a measure μ is constructed from a sequence where

$$\mu_N(\cdot\nu) \leq K \cdot 2^{-nN}$$

hold for all N and every $\nu\cdot$), then μ is absolutely continuous and given by $f \cdot dx$ where f is a bounded Lebesgue measurable function whose maximum norm is $\leq K$.

3. Signed measures

.

A signed measure μ supported by the unit cube arises by the inductive construction above when we allow that some of the numbers $m_N(\nu\cdot)$ are negative. In addition to additivity we impose the condition that

$$\sum_\nu |m_N(\nu\cdot)[= 1$$

hold for each N. In this case μ is again a σ-additive measure defined on the Borel algebra $\mathcal{B}$. Moreover, by the Hahn decomposition theorem from XX we can write μ as difference of two non-negative Riesz measures ν_1, ν_2 which are $\perp$ to each other.

4. μ-integrals.

Let μ be a non-negative measure on the unit cube constructed as above. Let $f(x)$ be a continuous function defined on the compact unit cube. Let $M = |f|_\square$ be the maximum norm of $|f|$ over $\square$. The integral $\int f d\mu$ is defined as the limit of the increasing sequence

$$\rho_N = \sum_{\nu=0}^{\nu=2^N-1} M \cdot \nu \cdot 2^{-N} \cdot \mu(\,\nu 2^{-N} \leq f < M \cdot (\nu+1)2^{-N}\,)$$

The convergence of $\{\rho_N\}$ is robust exactly as in the abstract theory from XX. In this way we obtain an additive map

$$f \mapsto \int f d\mu \tag{1}$$

With the terminology of functional analysis this means that the μ-integral yields a continuous linear form on the normed space of continuous functions on $\square$. The *converse* is fundamental, i.e. one has

5. Riesz' representation theorem.

Let $\mathcal{L}$ be a continuous linear functional on the Banach space $C^0(\square)$ of real-valued continuos functions on $\square$. Using a similar procedure as in the proof of the Hahn decomposition theorem for signed measures one shows that $\mathcal{L}$ can be written in a unique way as the difference of two non-negative linear forms. Let us then assume that that $\mathcal{L}(f) \geq 0$ for every non-negative continuous function. If 1 is the identity number we have the positive number $\mathcal{L}(1)$ and without loss of generality we assume that $\mathcal{L}(1) = 1$. Now we construct a regular measure μ as follows: Let $\square$ be a dyadic cube in $\mathcal{D}_N$ for some $N \geq 1$. To each $0 < a < 1$ we get the compact cube $a \cdot \square$ and set

$$\text{(i)} \qquad \mu_a(\square) = \min_f \mathcal{L}(f): \quad \text{where} \quad f|a \cdot \square = 1 \quad \text{and} \quad f \geq 0$$

It is clear that $a \mapsto \mu_a(\square)$ increases with a and we set

$$\text{(ii)} \qquad \mu(\square) = \lim_{a \to 1} \mu_a(\square)$$

If $\square_1$ and $\square_2$ are two different dyadic cubes in $\mathcal{D}_N$ we notice that the compact cubes $a \cdot \square_1$ and $a \cdot \square_2$ are disjoint. This observation and the possibility to construct continuous functions in (i) which vanish outside arbitrary small open neighborhoods of $a \cdot \square$, imply that (i) yields an additive function on the family of all dyadic cubes. Moreover, by the construction via (ii) it is automatically regular. and hence μ is a Riesz measure.

Next, let E be a closed subset of $\square$. One says that E is a null set with respect to $\mathcal{L}$ if

$$\min_f \mathcal{L}(f) = 0 \quad : \quad f|E = 1 \quad : f \geq 0$$

This makes sense because for every $\delta > 0$ one constructs a continuous function f which is 1 on E and has compact support in its δ-open neighborhood E_δ. Let us now *assume* that the boundaries of all dyadic cubes are null sets for $\mathcal{L}$. This hypothesis is not restrictive since we otherwise can use a translation of $\mathbf{R}^n$ by an n-tuple of numbers from the non-denumarable set of real numbers, i.e. after such a translation we get null sets and place the dyadic grid in the corresponding unit cube to construct measures as in 0.X. Under this assumption we obtain a non-negative Riesz measure by the following procedure: If $N \geq 1$ and $\square_N(\nu\cdot)$ is a dyadic cube we define:

$$\rho_N(\nu) = \max_f \mathcal{L}(f) \quad : 0 \leq f \leq 1 \text{ and has compact support in } \square_N(\nu\cdot)$$

One verifies easily that these ρ-numbers satisfy the additivity during the inductive construction as N increases and hence we obtain a Riesz measure μ. Moreover, it is clear from the whole construction that

$$\mathcal{L}(f) = \int f d\mu \quad : f \in C^0(\square)$$

Notice that this gives a canonical procedure to find μ when $\mathcal{L}$ is given.

The conclusion from this is the following result from the original work by F. Riesz:

Theorem. *Let Ω be an open set in $\mathbf{R}^n$ and $\mathcal{L}$ is a linear form on the space $C_0^0(\Omega)$ of continuous functions with compact support in Ω. Assume also that to every compact subset K of Ω there exists a constant C_K such that*

$$|\mathcal{L}(f)| \leq C_K \cdot |f|_K \quad : \quad \text{Supp}(f) \subset K$$

Then there exists a unique - in general signed - Riesz measure μ in Ω such that

$$\mathcal{L}(f) = \int f \cdot d\mu \quad : \quad f \in C_0^0(\Omega)$$

6. Borel measurable functions.

In many important applications one needs to integrate functions which are not necessarily continuous. This occurs for example when one studies the spectral decomposition of unbounded linear operators on Hilbert spaces. To attain extensions of μ-integrals with continuous functions as integrands we regard the unique smallest σ-algebra of subsets of $\mathbf{R}^n$ generated by open sets. It is denoted by $\mathcal{B}$ and sets in this σ-algebra are called Borel sets after Emile Borel who introduced then while he developed the abstract measure theory from Section X. Now every Riesz measure μ is σ-additive and hence we can assign a measure $\mu(B)$ to every Borel set B. Next, we consider $\mathcal{B}$-measurable functions f. It means that a real-valued function f is Borel measurable if the sets

$$(*) \qquad \{x \in \mathbf{R}^n \quad : \quad f(x) < t\} \in \mathcal{B} \quad : \text{for every real number } t$$

Using (*) one constructs integrals $\int f \cdot d\mu$ exactly as in the abstract theory from Section x.

7. Weak convergence.

Consider a sequence of non-negative measures $\{\mu_\nu\}$ where each measure has total mass one on $\square$. One says that the sequence converges weakly to a limit measure μ^* if

$$(*) \qquad \lim_{\nu \to \infty} \int f d\mu_\nu = \int f d\mu^* \quad : \quad f \in C^0(\square)$$

7.1 Remark. Suppose that the standard dyadic grid is a null set for all measures -$\{\mu_\nu\}$ and also for the limit measure μ^*. Then the *uniform* continuity of each $f \in C^0(\square)$ shows that weak convergence holds if and only if

$$\lim_{\nu \to \infty} \mu_\nu(\square_N(k\cdot)) = \mu^*(\square_N(k\cdot)) \quad : \; N \geq 1 \quad : \text{for every } k(\cdot) - \text{tuple}$$

Thus, weak convergence means precisely that the mass distributions during the inductive partition of $\square$ into smaller dyadic cubes give convergent sequences of the associated real 2^N-tuples for every $N \geq 1$. Next, recall Bolzano's theorem, i.e. that every infinite sequence of real numbers on a bounded interval has a convergent subsequence. Using this and the "diagonal procedure" the reader may verify the following fundamental fact:

7.2 Theorem. *Let $\{\mu_\nu\}$ be a sequence of non-negative measures on $\square$ - each with total mass one. Then there exists at least one subsequence which converges weakly.*

7.3 Weak density of discrete measures. Among Riesz measures occur finite sums of point-masses. Consider a sequence

$$\mu_j = \sum c_{j,k} \cdot x_k \quad : \; j = 1, 2, \ldots$$

where $\{x_1, x_2, \ldots\}$ is a denumarable sequence of points in the unit cube $\square$. At the same time each μ_j is a probability measure, i.e.

$$\sum_j c_{j,k} = 1 \quad : \; k = 1, 2, \ldots$$

The condition for the sequence to converge weakly is now easy to check. Namely, to every $N \geq 1$ and every cube $\square_N(\dot{\nu})$ in $\mathcal{D}_N$ we have

$$\mu_j(\square_N(\nu\cdot)) = \sum_* c_{j,k} \quad : \text{sum taken over those } k \text{ with } x_k \in \square_N(\nu\cdot)$$

So above weak convergence holds if and only if these numbers have a limit as $j \to \infty$ for every $N \geq 1$ and every cube in $\mathcal{D}_N$. Expanding each x_k in the binary system one can proceed to get an understanding when such a weak convergence takes place. Notice that if the sequence $\{x_k\}$ is given from the start then there may exist many doubly indexed c-sequences where we get a weak limit and moreover one may attain different limit measures depending on the chosen c-sequence. This occurs in particular when the sequence $\{x_k\}$ is everywhere dense. For example, if one has enumarated points in the unit cube whose coordinates are all rational numbers. In this situation every probability measure on $\square$ is the weak limit of a sequence $\{\mu_j\}$.

7.4 Exercise. Show that if μ is a Riesz measure with compact support then it can be approximaed weakly by a sequence of discrete measures.

8. Products and convolution of Riesz measures

Let n and m be two positive integers. Let μ and ν be Riesz measures in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. The product measure is trivial to define since the dyadic family of cubes in $\mathbf{R}^{n+m}$ is obtained by products of cubes from grids in $\mathbf{R}^n$ and $\mathbf{R}^m$. In this way we obtain the product measure $\mu \times \nu$ which satisfies

$$(\mu \times \nu)(\Omega \times U) = \mu(\Omega) \cdot \nu(U)$$

where $\Omega \subset \mathbf{R}^n$ and $U \subset \mathbf{R}^n$ are open sets.

8.1 Projections of measures. Let γ be a measure in $\mathbf{R}^{n+m}$ where the coordinates are (x, y) with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. We have the projection from $\mathbf{R}^{n+m}$ onto $\mathbf{R}^n$ defined by

$$\pi(x, y) = x$$

Assume that γ has a finite total mass. Then we obtain a measure $\pi_*(\gamma)$ in $\mathbf{R}^n$ defined on Borel sets F by

$$(*) \qquad \pi_*(\gamma)(F) = \gamma(F \times \mathbf{R}^m)$$

One refers to γ_* as he direct image measure.

8.2 Convolution. Let μ, ν be a pair of measures in $\mathbf{R}^n$. We get the product measure $\mu \times \nu$ in $\mathbf{R}^{2n}$ whose points are denoted by (x, y) where we regard μ as a measure in the n-dimensional x-space and ν as a measure in the y-space. In $\mathbf{R}^{2n}$ we can take new coordinates (t, s) where

$$t_\nu = x_\nu + y_\nu \quad : \quad s_\nu = x_\nu - y_\nu$$

Now $\mu \times \nu$ is a measure in the (t, s) space and we use the projection $\pi(t, s) = t$ which gives the measure

$$\pi_*(\mu \times \nu)$$

It is denoted by $\mu * \nu$ and called the convolution of μ and ν. If $f(t)$ is a continuous function with compact support in the t-space the constructions of $\pi_*(\mu \times \nu)$ and $\mu \times \nu$ give

$$(*) \qquad \int f(t) \cdot d(\mu * \nu) = \iint f(x + y) d\mu(x) d\nu(y)$$

Remark. The last integral is symmetric with respect to μ and ν. Hence the convolution satisfies the commutative law:

$$\mu * \nu = \nu * \mu$$

One can continue and construct the convolution of a triple of measures, and more generally the convolution of any number of measures in $\mathbf{R}^n$. For example, if μ, ν, γ is a triple of measures in $\mathbf{R}^n$ then we get the measure defined by

$$\int_{\mathbf{R}^n} f(t) \cdot d(\mu * \nu * \gamma) = \iiint f(x + y + z) \cdot d\mu(x) d\nu(y) d\gamma(z)$$

8.3 Exercise. Show that the convolution satisfies the *associative law.* So if $M(\mathbf{R}^n)$ denotes the space of Riesz measures with finite total variation, then convolution equips it with a commutative product, i.e. $M(\mathbf{R}^n)$ becomes a commutative algebra. The multiplicative unit is the point mass at the origin. One has also the inequality for norms:

$$||\mu * \nu|| \leq ||\mu|| \cdot ||\nu||$$

i.e. the total mass of a convolution is at most the product of the individual total masses.

Example. Let E and F be two compact sets in $\mathbf{R}^n$, each with a positive Lebesgue measure. We get the new compact set defined by

$$E + F = \{x + y \quad : \quad x \in E \quad : y \in F\}$$

Next, the two characteristic functions χ_E and χ_F define measures where

$$\int f \cdot d\chi_E = \int_E f \cdot dx \quad : \quad \int f \cdot d\chi_F = \int_F f \cdot dx$$

The construction of the convolution shows that

$$\chi_E * \chi_F = \chi_{E+F}$$

8.4 Exercise. Let μ be the discrete measure on the real line which has mass $1/2$ at $x = 0$ and at $x = 1$. Let $N \geq 2$ be an integer and consider the N-fold convolution μ^N. Show that this is a discrete measure supported by the integers $0, 1, \ldots, N$ and the mass at $0 \leq k \leq N$ is given by $2^{-N} \cdot \binom{N}{k}$.

8.5 Exercise. Let $n = 2$ and consider the Riesz measure μ supported by the unit circle $\{x^2 + y^2 = 1\}$ where it is given by the constant angular measure, i.e. if $f(x, y)$ is a continuous function then

$$\int f \cdot d\mu = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(\cos\theta, \sin\theta) \cdot d\theta$$

Now we take the convolution of μ with itself, i.e. we get the Riesz measure

$$\nu = \mu * \mu$$

Since μ is invariant under rotations we expect that ν also enjoys this invariance and given by

$$\int f \cdot d\nu = \int_0^2 \big[\int_0^{2\pi} f(r\cos\theta, r\sin\theta) \cdot d\theta\big] \cdot \phi(r) \cdot dr$$

where $\phi(r)$ is some non-negative function of r. We leave as an exercise to the reader to find this ϕ-function.

5. Stokes Theorem

Consider a bounded and connected open set Ω in $\mathbf{R}^\mathbf{n}$. Its boundary $\partial\Omega$ is a compact set but we do no assume that $\partial\Omega$ is connected. A boundary point p is said to be *regular* if there exists some open ball B centered at p and a real-valued C^1-function ϕ defined in B such that

$$\Omega \cap B = \{\phi < 0\} \quad : \partial\Omega \cap B = \{\phi = 0\} \quad : \nabla(\phi) \neq 0$$

It is clear that the set of regular boundary points is a relatively open subset of $\partial\Omega$. This set is denoted by $\mathfrak{reg}(\partial\Omega)$. Put

$$K = \partial\Omega \setminus \mathfrak{reg}(\partial\Omega) \tag{1}$$

We impose the condition that K has Hausdorff measure zero in dimension $n-1$. Next, recall from XX that there exists the *area measure* on $\mathfrak{reg}(\partial\Omega)$, to be denoted by dS. We shall assume that:

$$\int_{\mathrm{reg}(\partial\Omega)} dS < \infty \tag{*}$$

When (*) holds and $|K|_{n-1} = 0$ we say that Ω is a domain of the Federer type.

5.1 Theorem. *Let Ω be a domain of the Federer type and $f(x)$ a function of class C^1 defined in some open neighborhood of the closure $\bar{\Omega}$. Then*

$$\int_\Omega \partial f/\partial x_j \cdot dx = \int_{\mathrm{reg}(\partial\Omega)} f \cdot \mathfrak{n}_j \cdot dS \quad : \quad 1 \leq j \leq n$$

where $\mathfrak{n}_j$ is the x_j component of the outer normal $\mathfrak{n}$.

We refer to Theorem 5.1 as Federer's version of Stokes theorem for domains with corner points. The proof requires several steps and we begin with the ordinary case when the "ugly set" K is empty.

1.2 The case of graphic domains. Consider an open cube of the form

$$\square = \{x\colon -A \leq x_\nu < A\colon \nu = 1, \ldots, n\}$$

Put $x' = (x_1, \ldots, x_{n-1}$ and let $\phi(x')$ be a C^1 function defined on the cube

$$\square_* = \{-A < x_\nu < A\colon \nu = 1, \ldots, n-1\}$$

Here $-A < \phi(x') < A$ is assumed and we get the graphic domain

$$\Omega = \{(x', x_n)\colon -A < x_n < \phi(x') \quad : x' \in \square_*\} \tag{1}$$

Next, let $f(x)$ be a C^1-function with compact support in $\square$. Then we have:

$$\int_\Omega \partial f/\partial x_j \cdot dx = \int_{\partial\Omega)} f \cdot \mathfrak{n}_j \cdot dS \quad : \quad 1 \leq j \leq n \tag{*}$$

To prove (*) we first notice that $f(x) =$ when at least some x_ν has absolute value A for then $x \in \partial\square$. Next, consider the case $j = n$. Here the volume integral is the repeated integral

$$\int_{\square_*} \Big[\int_{-A}^{\phi(x')} \partial f(x', x_n)/\partial x_n \cdot dx_n\Big]\, dx' = \int_{\square_*} f(x', \phi(x')dx' \tag{1}$$

Recall from XX that $dx' = \mathfrak{n}_\mathfrak{n} \cdot dS$ holds on the part of $\partial\Omega$ defined by $x_n = \phi(x')$. Moreover, since f has compact support in $\square$ it vanishes on the remaining boundary of $\partial\Omega$ where at least some $|x_j| = A$. From this it is clear that (1) holds.

The case $1 \leq j \leq n-1$. We may assume that $j = 1$ to simplify the notations. Here we cannot use a repeated integral. Instead we define the function

$$\psi(x') = \int_{-A}^{\phi(x')} f(x', s) ds$$

Partial integration with respect to x_1 gives

$$\partial\psi/\partial x_1(x') = f(x', \phi(x')) \cdot \partial\phi/\partial x_1(x') + \int_{-A}^{\phi(x')} \partial f/\partial x_1(x', s) \cdot ds$$

Now we integrate over $\square_*$. Since the ψ-function is zero when $|x_1| = A$ we have trivially

$$\int_{\square_*} \psi(x') dx' = 0$$

In the right hand side we notice that integrating the second term yields the repeated integral

$$\int_{\square_*} \big[\int_{-A}^{\phi(x')} \partial f/\partial x_1(x', s) \cdot ds\big] \cdot dx' = \int_{\Omega} \partial f/\partial x_1(x) \cdot dx$$

Hence this volume integral is equal to

$$-\int_{\square_*} f(x', \phi(x')) \cdot \partial\phi/\partial x_1(x') \cdot dx'$$

Recall from XX that one has the equality

$$\mathfrak{n}_1 \cdot dS = -\partial\phi/\partial x_1(x') \cdot dx'$$

on the part of $\partial\Omega$ defined by $\{x_n - \phi(x') = 0\}$. Hence we have the required equality with $j = 1$ and the case $2 \leq j \leq n-1$ is treated in the same way. This finishes the proof of (*) above. 2.

1.3. Using partition of the unity Consider the situation in Theorem 1 where we impose the extra assumption that $f = 0$ in an open neighborhood U of the ugly set K. In this case the result in 1.2 gives Theorem 1 after a partition of the unity. Namely, first we have the compact set $W = \partial\Omega \setminus U$ which is contained in $\mathfrak{reg}(\partial\Omega)$. For each $p \in W$ there exists a cube $\square$ centered at p such that $\square \cap \Omega$ is a graphic domain. By Heine-Borel's Lemma we can cover W by a finite set of such cubes, say $\square_1, \ldots, \square_M$. Finally we can cover the compact set $\Omega \setminus U \cup \square_1 \cup \ldots \cup \square_M$ by a finite set of cubes $\square_{M+1}, \ldots, \square_{M+N}$ where the closure of these cubes are contained in Ω. Then we construct a C^∞-partition of the unity, i.e. we find a family $\{\phi_\nu \in C_0^\infty(\square_\nu)\}$ so that $\sum \phi_\nu = 1$ holds in a neighborhood of the support of f. Now Stokes formula from (1.2) hold for every function $f \cdot \phi_\nu$ and by adding the result we get Stokes formula for f.

1.4 How to avoid K

Now we consider the general case with no assumption on the support of f. We have to prove the integral formula in Theorem 1 for each x_j. Without loss of generality we can take $j = n$. Let $\pi(x', x_n) = x*$ be the projection to the $n-1$-dimensional x'-space. The assumption that $|K|_{n-1} = 0$ entails that the image set $\pi(K)$ is a null set in the $(n-1)$-dimensional x'-space. Let $\phi_1(x'), \phi_2(x'), \ldots$ be a sequence of test-functions in $\mathbf{R}^{n-1}$ which converge almost everywhere to 1 outside $\pi(K)$, while each of them has compact support in $\mathbf{R}^{n-1} \setminus K$ and $0 \leq \phi_\nu \leq 1$ hold for every ν.

By he result in (1.3) Stokes Formula holds for every function $\phi_\nu f$. Since the functions $\{\phi_n u(x')\}$ are independent of x_n we get

$$\int_{\Omega} \phi_\nu \cdot \partial_n(f) \cdot dx = \int_{\mathfrak{reg}(\partial\Omega)} \phi_\nu \cdot f \cdot \mathfrak{n}_n \cdot dS \tag{1}$$

Since $\phi_\nu \to 1$ holds almost everywhere in the x'-space, Lebesgue's dominated convergence theorem implies that the left hand side in (1) tends to the volume integral of $\partial_n(f)$. There remains to prove that the right hand side tends to the integral of $f \cdot \mathfrak{n}_n$ taken over the whole regular boundary.

Since f is a bounded function and (*) above Theorem 1 holds, the requested limit this follows if we have proved:

$$\text{(2)} \qquad \lim_{\nu\to\infty} \int_{\mathfrak{reg}\partial\Omega)} (1-\phi_\nu)\cdot \mathfrak{n}_n \cdot dS = 0$$

To prove (2) we let $\epsilon > 0$ and set

$$\Gamma_\epsilon = \{|\mathfrak{n}_n| \geq \epsilon\} \cap \mathfrak{reg}(\partial\Omega)$$

Put $M = \text{Area}\,[\mathfrak{reg}(\partial\Omega)]$ which by (*) above Theorem 1 is finite. Then (2) is majorized by

$$\text{(3)} \qquad M\cdot\epsilon + \int_{\Gamma_\epsilon} (1-\phi_\nu)\cdot \mathfrak{n}_n \cdot dS = 0$$

Next, on Γ_ϵ the area measure dS is majorized by $\frac{1}{\epsilon}\cdot dx'$. Therefore the sequence $g_\nu = 1-\phi_\nu$ regarded as functions on the measure space $(\Gamma_\epsilon\,;\,dS)$ converge almost everywhere to zero. Hence

$$\text{(4)} \qquad \lim_{\nu\to\infty} \int_{\Gamma_\epsilon} (1-\phi_\nu)\cdot \mathfrak{n}_n \cdot dS = 0$$

In particular we can take ν so large that (4) is $< \epsilon$. Since ϵ was arbitrary the limit formula (2) holds and the proof of Theorem 1 is finished.

6. The Hardy-Littlewood maximal function

.

Let f be a non-negative L^1-function on the real line supported by $x \geq 0$, i.e. $f(x) = 0$ when $x < 0$. Its forward maximal function is defined by

$$f^*(x) = \max_{h>0} \frac{1}{h}\int_x^{x+h} f(t)dt$$

It is clear that f^* is a measurable function. For example, it suffices to seek the maximum when h runs over positive rational numbers which we enumerate as $\{q_\nu\}$. For a fixed ν we have a continuous function

$$g_\nu(x) = \frac{1}{q_\nu}\int_x^{x+q_\nu} f(t)dt$$

Then $f^*(x)$ is the maximum of the denumerable family $\{g_\nu\}$ and hence measurable, i.e. it is even *lower semi-continuous.* So if $\lambda > 0$ we get the open set:

$$E(\lambda) = \{x :\ f^*(x) > \lambda\}$$

1. Theorem *One has the inequality*

$$\text{meas}\,E(\lambda) \leq \frac{1}{\lambda}\cdot \int_{E(\lambda)} f(x)dx$$

Proof. Given $\lambda > 0$ we set $g(x) = \frac{f(x)}{\lambda}$. Then we must show:

$$\text{(1)} \qquad \text{meas}(E_g) \leq \int_{E_g} g(x)dx \quad : \quad E_g = \{g^* > 1\}$$

To get (1) we consider the primitive function

$$\text{(2)} \qquad G(x) = \int_0^x g(t)dt$$

It is non-decreasing and continuous. Let $\{(a_\nu, b_\nu)\}$ be the disjoint intervals of E_g.

Sublemma. For each ν one has

$$G(b_\nu) - G(a_\nu) \geq b_\nu - a_\nu$$

Proof. Suppose strict inequality holds. By continuity we find $x > a_\nu$ with $x - a_\nu$ small so that

$$G(b_\nu) - G(x) < b_\nu - x$$

Next, since $x \in E_g$ there exists $\xi > x$ such that

$$G(\xi) - G(x) > \xi - x$$

Then $b_\nu < \xi$ must hold. On the other hand, since b_ν is a boundary point of E_g we also have

$$G(\xi) - G(b_\nu) \leq \xi - b_\nu$$

Now we get a contradiction since it would follows that

$$G(\xi) - G(x) = G(\xi) - G(b_\nu) + G(b_\nu) - G(x) < \xi - b_\nu + b_\nu - x = \xi - x$$

Proof continued. Since the intervals (a_ν, b_ν) are disjoint we get

$$\int_{E_g} g dt = \sum \int_{a_\nu}^{b_\nu} g dt = \sum G(b_\nu) - G(a_\nu)$$

By the Sublemma the last sum is $\geq \sum (b_\nu - a_\nu)$ which is equal to $\text{meas}(E_g)$ and Theorem 1 is proved.

An L^2-inequality. If x is a Lebesgue point of f we have the inequality $f^*(x) \geq f(x)$. Using Theorem 1 we shall now establish an inequality for L^2-norms.

6.2. Theorem *For every $f(x)$ on $x \geq 0$ which is square integrable one has the inequality*

$$||f^*||_2 \leq 2 \cdot ||f||_2$$

where $||\cdot||_2$ denotes the L^2-norm.

Proof. Since we claim an a priori inequality it suffices to prove the result when $f(x)$ is bounded and has compact support. Let N be a positive integer which eventually will be very large. Keeping N fixed we set

$$E(\nu) = \{\frac{\nu}{N} \leq f^* < \frac{\nu + 1}{N}\} \quad : \quad \nu = 0, 1, \ldots$$

Consider the two functions

$$f_-^* = \sum \frac{\nu}{N} \cdot \chi_{E(\nu)} \quad : \quad f_+^* = \sum \frac{\nu + 1}{N} \cdot \chi_{E(\nu)}$$

Sublemma For each $N \geq 1$ we have

$$\int_0^\infty (f_-^*(x))^2 dx \leq 2 \cdot \int_0^\infty f_+^*(x) \cdot f(x) dx$$

Proof of Sublemma Set $E_\nu = \{f^* > \frac{\nu}{N}\}$. The left hand side is equal to

$$\sum_{\nu \geq 0} \frac{\nu^2}{N^2} \cdot (\,|E_\nu| - |E_{\nu+1}|\,) = \sum_{\nu \geq 0} \frac{1}{N^2} \cdot |E_\nu| \cdot (\nu^2 - (\nu - 1)^2) \leq \sum_{\nu \geq 0} \frac{2\nu}{N^2} \cdot |E_\nu|$$

By Theorem 1 the last sum is majorized by

$$\sum_{\nu \geq 0} \frac{2}{N} \int_{E_\nu} f dt = \sum_{\nu \geq 0} \frac{2(\nu + 1)}{N} \int_{E(\nu)} f dt$$

By the construction of f_+^* the last sum is $2 \int f_+^*(x) \cdot f dx$ and the Sublemma is proved.

Proof continued. The Cauchy-Schwarz inequality for L^2-norms gives:

$$\int f_+^*(x) \cdot f dx \leq ||f_+^*||_2 \cdot ||f||_2$$

Hence the Sublemma gives

$$||f_-^*||_2^2 \leq 2 \cdot ||f_+^*||_2 \cdot ||f||_2$$

Theorem 1 follows since the L^2-norms of f_+^* and f_-^* both tend to $||f^*||_2$ when N increases.

Remark. The proof of the L^2-inequality is easily extended to give an L^p inequality for each $1 < p < 1$ where the constant 2 is replaced by $\frac{p}{p-1}$. In Theorem 1 we regarded the *forward* maximum function. Of course, one could also have defined the backward maximum function and more generally, without any restriction on the support of f its full maximal function

$$f^{**}(x) = \max_{\xi,\eta} \frac{1}{\xi+\eta} \int_{x-\eta}^{x+\xi} f(t)dt$$

with the maximum taken over all pairs of positive numbers ξ, η. Then Theorem 2 gives:

$$||f^{**}||_2 \leq 4 \cdot ||f||_2$$

7. Rademacher functions

For each positive integer N the interval $[0,1)$ can be decomposed into half-open intervals

$$\Delta_N(\nu) = [\nu 2^{-N}, \nu+1)2^{-N}) \quad : \quad 0 \leq \nu \leq 2^N - 1$$

Define the function $R_N(x)$ by

$$R_N(x) = (-1)^\nu \quad : \quad x \in \Delta_N(\nu)$$

This construction applies to each $N \geq 1$. The construction of the R-function gives

$$\int_{\Delta_N(\nu)} R_M(x)dx = 0 \quad : \quad 0 \leq \nu < 2^N - 1 \quad : M > N$$

In particular we get

$$\int_0^1 R_N(x) \cdot R_M(x)dx = 0 \quad : \quad M \neq N$$

Hence the R-functions is an orthogonal family in the Hilbert space $L^2[0,1]$. Next, let $\{\alpha_\nu\}$ be a sequence in ℓ^2, i.e. $\sum |\alpha_\nu|^2 < \infty$. Consider the partial sum functions

$$S_N(x) = \sum_{\nu=1}^{\nu=N} \alpha_\nu \cdot R_\nu(x)$$

We shall analyze the limit as $N \to \infty$. For this purpose we use expansions of real numbers in the binary system. That is, when $0 < x < 1$ we have a series expansion

$$x = \epsilon_1(x)2^{-1} + \epsilon_2(x)2^{-2} + \ldots \quad : \quad \epsilon_\nu(x) = 1 \text{ or } 0.$$

The expansion is unique unless $2^N x$ is an integer for some N. Ignoring this denumerable set we define for each x the partial sum of its binary expansion

$$\xi_N(x) = \sum_{\nu=1}^{\nu=N} \epsilon_\nu(x) \cdot 2^{-\nu}$$

So here $\xi_N(x) < x < \xi_. N(x) + 2^{-N}$ hold. With these notations one has the following for the given ℓ^2-sequence above: '

1. Proposition *For each x and $N \geq 1$ one has the equality*

$$S_N(x) = \sum_{\nu=1}^{\nu=N} (2\epsilon_\nu(x) - 1) \cdot \alpha_\nu$$

The straightforward proof is left to the reader. Now we announce

2. Theorem *For each sequence $\{\alpha_\nu\}$ in ℓ^2 there exists a null set E such that one has a pointwise limit*

$$\lim_{N\to\infty} S_N(x) = \sum_{\nu=1}^{\infty} (2\cdot\epsilon_\nu(x)-1)\cdot\alpha_\nu \quad : \quad x \text{ outside } E$$

Proof. Since the R-functions is an orthonormal family in $L^2[0,1]$ we have

$$\int_0^1 S_N(x)^2 dx = \sum_{\nu=1}^{\nu=N} \alpha_\nu^2 \quad : \quad N \geq 1$$

It follows that $\{S_N\}$ is a Cauchy sequence in $L^2[0,1]$ and hence it converges in the L^2-norm to a limit function $S(x)$. Let $\mathcal{L}(S)$ be the set of Lebesgue points for S. If $x \in \mathcal{L}(x)$ we have in particular

$$S(x) = \lim_{N\to\infty} \int_{\xi_N(x)}^{\xi_N(x)+2^{-N}} S(t)dt$$

From the construction of the partial sum functions it follows that $\lim S_N(x) = S(x)$, i.e. the pointwise limit exists at each Lebesgue point of S. Since $(0,1)\setminus\mathcal{L}(S)$ is a null set we get Theorem 2.

8. The Brownian motion

Introduction. Inspired by P.J. Daniell's work [DA] which fascilated previous results about infinite products of measure spaces, Norbert Wiener gave a rigorous construction of a stochastic process $\mathcal{W}$ in continuous time with the following properties: For each $t > 0$ one has a normally distributed random variable W_t whose variance is t and $\{W_t\}$ have independent increments, i.e if $s < t$ then the random variables $W_t - W_s$ and W_s are independent. Wiener employed Fourier series to construct such a process whose sample space is a *denumerable* product of $\mathbf{R}$. The book [Pa-Wi] contains an interesting historic account about the role of the Browninan motion outside the "abstract world of mathematics". Here we add some further comments.

Albert Einstein's first scientific paper was published in december 1900. It contains a statistical analysis of experimental data from phenomenon of capillarity and shows that Einstein already during the early period of his career was familiar with statistical arguments. Five years later it led to his article [XX] which justified the existence of atoms and made it possible to determine the Avogrado number. After Einstein's work many skilled experimental scientists used stochastic analysis based upon the Brownian motion. Among those who applied Einstein's recipe was Theo Svedberg who received the Nobel Prize in chemistry in 1925 (6 ?). The interested reader may also consult the book *Atoms* by Jean Perrin [Per] which gives a fascinating description of the nature of Brownian motion and ells about empirical experiments which predict that individual paths are non-differentiable.

From a mathematical point of view, Einstein's article [Ei] shows the usefulness of a probabilistic interpretation to solutions of partial differential equations which appear in diffusion processes of colloidal material. This inspired later work dealing with the Brownian motion and the heat-equation such as in the joint article [KKP] by Khintchine, Kolmogorv and Petrowsky from 1930. Here one finds very precise results about continuity properties of individual Brownian paths based upon studies of boundary value problems for the heat equation. Later Paul Levy found a more direct proof of the continuity properties of individual Brownian paths. We shall present his construction of the Brownian motion below and establish the Hölder continuity of individual paths. The material below is inspired from notes by Anders Martin-Löf who attributes the main constructions to P. Levy.

The Levy-Lööf construction.

We seek a stochastic process $\{X(t)\colon\ t \geq 0\}$ such that for every pair $0 \leq s < t$, $X(t) - X(s)$ is a normally distributed random variable with variance $t - s$ and independent of $X(s)$. In particular each $X(t)$ is normally distributed with variance t. Recall that two normally distributed variables are independent if and only if they are uncorrelated. Hence, if E denotes expected value, the condition that the process has independent increments is equivalent to the condition that

$$E(X(t) \cdot X(s)) = \min(s,t) \quad : \text{for every pair } \ s,t$$

Before we give the construction of a stochastic process satisfying the conditions above we make some observations if such a process is given.

1. Conditioned mean-values. Let $s < t < u$ and consider the mean value of $X(t)$ when $X(u)$ and $X(s)$ are given, i.e. we seek

$$E((X(t)\,|\,X(s), X(u)$$

Since the stochastic variables $X(s), X(t), X(u)$ are normally distributed it is wellknown that this conditioned mean value is a linear function of $X(s)$ and $X(u)$, i.e. there exists a pair of real numbers (a, b) such that:

$$\bar{X}(t) = E((X(t)\,|X(s), X(u)) = a \cdot X(s) + b \cdot X(u))$$

Since we are assuming independent increments, $X(t) - \bar{X}(t)$ is independent of both $X(s)$ and $X(u)$. From this a calculation which is left to the reader gives

(*)
$$a = \frac{u-t}{u-s} \quad : \quad b = \frac{t-s}{u-s}$$

which of course is what one should expect from the start. For example, if $t - s \to 0$ while u stays fixed, then $a \to 1$ which reflects that the conditioned mean value is close to $X(s)$. Inserting the values of a, b we now have a linear function

$$\bar{X}(t) = \frac{(u-s)X(s) + (t-s)X(u)}{u-s} \quad : \quad s < t < u$$

which coincides with $X(s)$, respectively $X(u)$ at the end points of the interval $[s, u]$.

2. The conditioned variance $V(t)$ With a and b chosen as in (*) we get:

$$V(t) = E((X(t) - \bar{X}(t))^2 \mid X(s), X(u)) =$$

$$E[(a(X(t) - X(s)) + b(X(t) - X(u))^2] = a^2(t-s) + b^2(u-t) = \frac{(u-s)(t-s)}{u-s}$$

Hence the graph of $V(t)$ is a parabola whose values at the end points s and u are zero. The maximum is attained at the mid-point which reflects that we obtain a maximal variance when random values taken by $X(s)$ and $X(u)$ affect $X(t)$ in equal parts.

3. The inductive construction. We shall construct the Brownian motion over the unit interval $0 \leq t \leq 1$. After one can of course continue to any time interval. Let $N \geq 1$ and put

$$t_\nu(N) = \nu \cdot 2^{-N} \quad : \quad 0 \leq \nu \leq 2^N$$

Using the previous remarks we obtain a stochastic process as follows: Let $\chi_1, \ldots, \chi_{2^N}$ be independent and normally distributed variables, each with mean value zero and variance 2^{-N}. Consider a time value

$$t = k2^{-N} + \delta \quad : \quad 0 \leq k \leq 2^N - 1 \quad : 0 \leq \delta \leq 2^{-N}$$

At this moment of time we define the random variable

(i)
$$X_N(t) = \sum_{\nu=0}^{\nu=k-1} \chi_\nu + (1-\delta)\chi_k + \delta\chi_{k+1}$$

With a sample space of dimension 2^N over the real line the outcome consists of the values taken by the independent and normally distributed χ-variables. This sample determines X_t for every time value $0 \leq t \leq 1$. Hence the outcome of each sample point consists of a piecewise linear curve $t \mapsto \gamma(t)$ defined on $[0, 1]$ with eventual corner points at when $t = k2^{-N}$, where $\gamma(k2^{-N})$ is the sum of the first k many χ-variables.

4. Passage to the limit. Above we constructed a stochastic process $\{X_N(t)\}$ whose sample space is of dimension 2^N. For the inductive construction we pass from stage N to $N+1$ allowing "white noise" during time intervals $[k2^{-N}, (k+1)2^{-N}]$. This means that we introduce 2^N many new random variables $g_1, \ldots, g_{2^N}$ which are independent of those which were used to get $\{X_N(t)\}$. Moreover, every g_k is normally distributed with variance 2^{-N-1}. Thus, at stage $N+1$ we have a process $t \mapsto X_{N+1}(t)$ where

$$X_{N+1}(k2^{-N} + \delta 2^{-N-1}) = X_N(k2^{-N}) + \delta g_k : \; 0 \leq \delta \leq 1$$
$$X_{N+1}((k+1/2)2^{-N} + \delta 2^{-N-1}) = X_N(k2^{-N}) + g_k + \delta g_{k+1} : \; 0 \leq \delta \leq 1$$

Inductively we get a sequence of processes $\{X_N(t)\}_{N \geq 1}$ where the sample spaces increase. Thus, at each stage "white noise" appears, expressed by a block of 2^N many g-variables. The whole sample space is a denumerable product of the real line on which we define the usual product of the

Lebesgue measure. The outcome from a sample point is a sequence of polygons $\gamma_1, \gamma_2, \ldots$. At a dyadic time value $t = k2^{-N}$ the polygons $\{\gamma_\nu\}$ pass the same point, There remains to investigate whether or not the sequence of these polygons converge to a continuous limit curve. It turns out that the convergence holds for almost every sample point which by the above is taken from the denumerable product of copies of the real line $\mathbf{R}$.

5. Continuity of Brownian paths. We shall study the effect of white noise during the passage from stage $N-1$ to the stage N. For this purpose we consider a family $\chi_1, \ldots, \chi_{2^N}$ of independent normally distributed with mean value zero and variance one. Define for each $\alpha > 0$:

$$\Pi_N(\alpha) = \text{Prob}\{\max : \ 2^{-N/2}|\chi_k| \geq \alpha : 1 \leq k \leq 2^N\}$$

Thus, if $g_k = 2^{-N/2}\chi_k$ are the random variables which cause white noise, then $\Pi_N(\alpha)$ is the probability that at least some g_k has absolute value $\geq \alpha$. By the previous discussion this means that we measure the probability for the *maximal distance* between the curves γ_{N-1} and γ_N over the whole time interval. So we want are upper bounds for the Π-numbers with α small as N increases. Here is the crucial inequality:

6. Proposition. *For all pairs N, α one has the inequality*

$$\Pi_N(\alpha) \leq \sqrt{\frac{2}{\pi}} \cdot \frac{2^{N/2}}{\alpha} \cdot e^{-2^N \cdot \alpha^2/2}$$

Proof. Consider first a single normal distribution with variance one. If $A > 0$ we have the tail probability

$$\frac{1}{\sqrt{2\pi}} \int_A^\infty e^{-x^2/2} dx \leq \frac{1}{\sqrt{2\pi}} \int_A^\infty \frac{x}{A} \cdot e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-A^2/2}}{A}$$

Put

$$A = 2^{N/2} \cdot \alpha \quad : \quad \xi = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{A} \cdot e^{-A^2/2}$$

Next, notice that $1 - \Pi_N(\alpha)$ is the probability that $|\chi_k| \leq A$ for every k. The tail inequality applied to every χ_k gives

$$\text{Prob}\{|\chi_k| \geq A\} \geq \sqrt{\frac{2}{\pi}} \cdot \frac{1}{A} \cdot e^{-A^2/2} = \xi$$

The elementary inequality $(1-t)^M \leq Mt$ for every $0 < t < 1$ therefore gives

$$\Pi_N(\alpha) \leq (1-\xi)^{2^N} \leq 2^N \cdot \xi = 2^N \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{A} \cdot e^{-A^2/2}$$

Since $A = 2^{N/2}\alpha$ the last inequality proves Proposition 6.

Specific choice of α. With $\alpha = 2^{-N/2}\beta$ we have

$$\Pi_N(\alpha) \leq \sqrt{\frac{2}{\pi}} \cdot 2^N \cdot \frac{1}{\beta} \cdot e^{-\beta^2/2} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\beta} \cdot e^{\,\text{Log}(2) \cdot N - \beta^2/2}$$

Let us choose

$$\beta = [2N\text{Log}(2)\,]^{\frac{1}{2}} \quad : \quad \theta > 1$$

Then the last term becomes

$$\frac{1}{\sqrt{\pi \cdot \text{Log}(2)}} \cdot 2^{N(1-\theta^2)}$$

So with this specific choice of α we have established the inequality

$$\Pi_N(\alpha) \leq \frac{1}{\sqrt{\pi \cdot \mathrm{Log}(2)}} \cdot 2^{N(1-\theta^2)} \quad : \quad \alpha = [2N\mathrm{Log}(2)\,]^{\frac{1}{2}} \cdot 2^{-N/2}$$

Remark. Keeping $\theta > 1$ fixed and choosing $\alpha(N) = [2N\mathrm{Log}(2)\,]^{\frac{1}{2}} \cdot 2^{-N/2}$ for each N, we get the convergence of the series

$$\sum_{N=1}^{\infty} \Pi_N(\alpha(N))$$

In the construction of the Brownian motion the "white noise" as we pass from period N to $N+1$ arises from 2^N many independent normally distributed variables, each with variance 2^{-N}. So $\Pi_N(\alpha(N))$ measures the maximal deviation over the whole time interval $[0,1]$ from a polygon at period N to the new polygon at period $N+1$. But notice that $\alpha(N)$ are not of size 2^{-N}, i.e. instead like the square root of 2^{-N}. So even though the convergent series above implies that a single Brownian path is almost surely continuous, the estimates above only ensure that the resulting curve is Hölder continuous of order $\sqrt{t}| \cdot \mathrm{Log}(\frac{1}{t})$. In other words, we have established the following:

7. Theorem. *A single Brownian path is almost surely Hölder continuous of order $\sqrt{t} \cdot Log(\frac{1}{t})$.*

Remark. The result above means that almost every Brownian path is Hölder continuous of order $\sqrt{t} \cdot \mathrm{Log}(\frac{1}{t})$ for *every* value of t. A relaxed condition would be to ask for some Hölder continuity which holds for almost all time values. This leads to another result due to Khintchine which asserts that almost every Brownian path will be Hölder continuous of order

$$(*) \qquad \sqrt{t} \cdot \mathrm{Log}(\mathrm{Log}(\frac{1}{t}))$$

for almost every time value. Notice that since we have taken a double Log-function we come closer to Hölder continuity of order $\frac{1}{2}$ in (*). The proof of (*) uses Khintchine's *Law of the iterated logarithm.* For the proof of (*) we refer to [Khi] and the interested reader may also consult his excellent text-book on statistical mechanics in [Khi] which offers a good introduction for students of mathematics to the world of physics without having prerequisite knowledge about physical laws.

Bachelier's work

To the historic account one must add the pioneering work by X.X Bachelier. His article *La Bourse* from 1900 studies probability densities attached to the Brownian motion with barriers. For example, given some $a > 0$ we can ask for the time when a Brownian first hits the barrier $x = a$. This yields a probability distribution which depends on a and was determined explicitly by Bachelier. His text-book on probability theory from 1910 is the first modern account dealing with stochastic processes on an advanced level. The Brownian motion can be used to investigate various probability distributions, apart from the "hitting probability" on a given arc on the boundary of a domain in $\mathbf{C}$ which correspond to a harmonic measure. At several occasions in these notes we expose this "tautology" between probabilistic interpretations via the Brownian motion and solutions to the heat equation and other boundary value problems which are obtained via solutions to the Dirichlet problem.

Appendix B: Functional analysis.

Contents.

Introduction.

The material is foremost related to analytic function theory where major results appear about the operational calculus in Section 7. We also give an account about the spectral theorem for unbounded self-adjoint operators on Hilbert spaces. Many topics are excluded, such as a general study of geometry in normed spaces and examples which exhibit dual spaces. A good reference which covers all essential facts in functional analysis are the text-books by Dunford and Schwartz. The general concept of a normed linear space was put forward by Banach in the monograph [Banach]. Of course the study of infinite dimensional systems of equations was done before, but never put in full generality. A notable exception is the class of Hilbert spaces whose abstract definition was given by Hilbert around 1904. Let us give an example of a "concrete" result which is due to F. Riesz and also established before the general notion of normed vector spaces became standard. We are given a doubly indexed sequence $\{c_{\nu.n}\}$ of complex numbers where (ν, n) are pairs of non-negative integers. We assume that there exists a strictly increasing sequence of positive numbers $\{A_n\}$ which tends to $+\infty$ such that

$$|c_{\nu.n}| \leq \frac{1}{A_n} \quad \text{hold for all pairs} \quad \nu, n$$

Under this assumption we study the inhomogeneous system of linear equations

$$(*) \qquad \sum_{n=1}^{\infty} c_{n,\nu} \cdot x_n = y_\nu$$

where $\{y_\nu\}$ is a bounded sequence of complex numbers. The equation is said to be solvable in ℓ^1 if there exists a sequence $\{x_n\}$ which is absolutely convergent, i.e. $\sum |x_n| < \infty$. The following result was proved by Riesz:

Theorem. *Let $\{c_{n,\nu}$ and $\{y_\nu\}$ be such that for every finite sequence $\lambda_0,\ldots,\lambda_r$ it holds that*

$$|\sum_{\nu=0}^{\nu=r}\lambda_\nu\cdot y_\nu|\leq\sup_{n\geq 0}|\sum_{\nu=0}^{\nu=r}\lambda_\nu\cdot c_{n\nu}|$$

Then (*) has an ℓ^1-solution such that $\sum|x_n|\leq 1$.

This result has a wide range of applications. The fact that it is stated for an denumerable set of linear equations does not impose severe restrictions, i.e. in a more abstract context problems of a linear nature are reduced to solve linear systems of equations. Bu such a eduction may require some extra work because the spaces in functional analysis can be quite huge, i.e. they can for example be non-separable. Let us give an example where Riesz' theorem is used after some initial constructions. on the unit circle T we let E be a compact subset. We have the Banach space $C^0(E)$ of complex-valued continuous functions on E. Measure theory teaches that the dual space consists of Riesz measures μ of finite total variation on E. let $M(E)$ denote this set of measures. To every $\mu\in M(E)$ and each non-negative integer we set

$$\widehat{\mu}(n)=\int_E e^{in\theta}\cdot d\mu(\theta)$$

We say that E is of the Carleson-Kronecker set if there exists $0<p(E)\leq 1$ such that

$$||\mu||\leq\frac{1}{p(E)}\cdot\sup_{n\geq 0}|\widehat{\mu}(n)|\quad\text{hold for all}\quad\mu\in M(E)$$

Theorem. *Let E be a Carleson-Kronecker set. Then, for each $\phi\in C^0(E)$ there exists an absolutely convergent sequence $\{x_n\}$ such that*

$$\phi(e^{i\theta})=\sum_{n\geq 0}x_n\cdot e^{in\theta}\quad\text{holds on}\quad E$$

Remark. We prove this theorem in § XX in *Special Topics* and remark only that the proof relies upon some initial constructions until Riesz' theorem can be applied which is caused by the non-separability of the space $M(E)$ and of course also due to the fact that the ℓ^1-sequence which appears in the theorem for a given ϕ is not unique. Now we begin to summarize the contents in this chapter.

A.0 Normed spaces. Section 1 studies normed vector spaces over the complex field **C** or the real field **R**. We explain how each norm is defined by a *convex* subset of V with special properties. If X is a normed vector space such that every Cauchy sequence with respect to the norm $||\cdot||$ converges to some vector in X one says that the norm is *complete* and refer to the pair $(X,||\cdot||)$ as a Banach space. The *Banach-Steinhaus theorem* asserts that if X is a Banach space equipped with the complete norm $||\cdot||^*$, then this norm is stronger than any other norm $||\cdot||$ on X, i.e. there exists a constant C such that

$$(*)\qquad ||x||\leq C\cdot||x||^*\quad:\quad x\in X$$

Thus, up to equivalence, a vector space can only be equipped with one complete norm. The proof of (*) is an immediate consequence of *Baire's category theorem.* The result has a wide range of applications. For example, the Banach-Steinhaus theorem gives the *Open Mapping Theorem* and the *Closed Graph Theorem* for linear operators from one Banach space into another.

A.1 Dual spaces. When X is a normed linear space one constructs the linear space X^* whose elements are continuous linear functionals on X. The *Hahn-Banach Theorem* identifies norms of vectors in X via evaluations by X^*-elements. More precisely, denote by S^* the unit sphere in X^*, i.e. linear functionals x^* on X of unit norm. Then one has the equality

$$\text{(i)}\qquad ||x||=\max_{x^*\in S^*}|x^*(x)|:\ \text{for all } x\in X\,.$$

The determination of X^* is often an important issue. See for example Section XXX where we determine the dual of the normed space $X = H^\infty(T)$ of bounded Lebesgue measurable functions on the unit circle which are boundary values of bounded analytic functions in the open disc D. A portion of its dual space is given by the quotient space:

$$\text{(ii)} \qquad Y = \frac{L^1(T)}{H_0^1(T)}$$

where $H_0^1(T)$ is the closed subspace of $L^1(T)$ whose functions are boundary values of analytic functions in D which vanish at $z = 0$. However, the dual X^* is considerably larger. In fact, we shall learn that $H^\infty(T)$ is an example of a commutative Banach algebra to which we can assign the maximal ideal space $\mathfrak{M}_X$, also called the Gelfand space. Its precise description is not clear a all. Via point evaluations in D it is first clear that D appears as a subset of $\mathfrak{M}_X$. A deep result, known as the Corona Theorem is due to Carleson and asserts that D is dense in $\mathfrak{M}_X$.

Reflexive spaces. Starting from a Banach space X we get X^* and then its dual $(X^*)^*$ to be denoted by X^{**} and called the bi-dual of X. There is a natural injective map $j\colon X \to X^{**}$ and (i) above shows that j is an isometry, i.e. the norms $||x||$ and $||j(x)||$ are equal. But in general j is nor surjective. In the case when j is surjective so that $X = X^{**}$ one says that X is reflexive. Hilbert spaces are examples of reflexive spaces as we shall explain later on.

A.3 Calculus on Banach spaces. The study of differentiable maps from $\mathbf{R}^n$ to $\mathbf{R}^m$ extends verbatim to maps from one Banach space X into another Banach space Y. In Section 7 we define the differential of a C^1-map $g\colon X \to Y$. Notice that g is not assumed to be a linear map. The differential of g at a point $x_0 \in X$ becomes a bounded linear operator from X into Y. This extends the construction of the Jacobian for a C^1-map from $\mathbf{R}^n$ into $\mathbf{R}^m$ expressed by an $m \times n$-matrix. More generally one constructs higher order differentials, and in this way one can refer to C^∞-maps from one Banach space into another. We review this at the end of Section 7. Baire's category theorem together with the Hahn-Banach theorem show that if K is any compact metric space and ϕ is a continuous function with values in a normed space X, then ϕ is *uniformly continuous*, i.e. to every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_K(p,q) \leq \delta \implies ||\phi(p) - \phi(q)|| \leq \epsilon$$

where d_K is the distance function on the metric space K and the right hand side the norm in X. We have also the notion of differentiable Banach spaces. By definition a Banach space X is differentiable at a point x if there exists a linear functional $\mathcal{D}_x$ on X such that

$$(*) \qquad \mathcal{D}_x(y) = ||x + \zeta \cdot y|| - ||x|| = \mathfrak{Re}\,(\zeta \cdot \mathcal{D}_x(y)) + \text{small ordo}(|\zeta|)$$

hold for every $y \in X$ where the limit ia taken over complex ζ which tend to zero. one says that X is differentiable if $\mathcal{D}_x$ exist for every $x \in X$. For differentiable Banach spaces there exists a calculus which is often used in optimizations. See X.X for further details.

A.4 Analytic functions. Let X be a Banach space. Consider a power series with coefficients in X:

$$\text{(i)} \qquad f(z) = \sum_{\nu=0}^{\infty} b_\nu \cdot z^\nu \quad b_0, b_1, \ldots \text{ is a sequence in } X\,.$$

Let $R > 0$ and suppose there exists a constant C such that

$$\text{(ii)} \qquad ||b_\nu|| \leq C \cdot R^\nu \quad : \nu = 0, 1, \ldots$$

Then the series (i) converges when $|z| < R$ and $f(z)$ is called an X-valued analytic function in the open disc $|z| < R$. More generally, let Ω be an open set in $\mathbf{C}$. An X-valued function $f(z)$ is analytic if there to every $z_0 \in \Omega$ exists an open disc D centered at z_0 such that the restriction of f to D is represented by a convergent power series

$$f(z) = \sum b_\nu (z - z_0)^\nu$$

Using the dual space X^* one extends results about ordinary analytic functions to X-valued analytic functions. Namely, for each fixed $x^* \in X^*$ the complex valued function

$$z \mapsto x^*(f(z))$$

is analytic in Ω. By this one recovers the Cauchy formula. For example, let Ω be a domain in the class $\mathcal{D}(C^1)$ and let $f(z)$ be an analytic X-valued function in Ω which extends to a continuous X-valued function on $\bar{\Omega}$. If $z_0 \in \Omega$ there exists the complex line integral

$$\int_{\partial\Omega} \frac{f(z)dz}{z - z_0}$$

It is evaluated by sums just as for a Riemann integral of complex-valued functions. One simply replaces absolute values of complex valued functions by the norm on X in approximating sums which converge to the Riemann integral. From this we obtain Cauchy's formula

$$f(z_0) = \int_{\partial\Omega} \frac{f(z)dz}{z - z_0}\,.$$

A.5 Borel-Stieltjes integrals. Let μ be a Riesz measure on the unit interval $[0,1]$ and f an X-valued function, which to every $0 \leq t \leq 1$ assigns a vector $f(t)$ in X. Suppose that the X-norm $||f(t)|| \leq M$ hold for some constant M and every t. We say that f is Borel measurable if the complex-valued functions $t \mapsto x^*(f(t))$ are Borel functions on $[0,1]$ for every $x^* \in X^*$. Then there exists the integral

$$J(x^*) = \int_0^1 x^*(f(t))\dot{d}t$$

for every x^*. The boundedness of f implies that $x^* \mapsto J(x^*)$ is a continuous linear functional on X^* which means that there exists a vector $\xi(f)$ in the bi-dual X^{**} such that

(1) $$\xi(f)(x^*) = J(x^*) \quad : \quad x^* \in X^*$$

When X is reflexive the f-integral yields a vector in $\mu_f \in X$ which computes (1), i.e.

$$x^*(\mu_f) = \int_0^1 x^*(f(t))\dot{d}t \quad \text{hold for all} \quad x^* \in X$$

Keeping μ fixed this means that $f \mapsto \mu_f$ is a bounded linear operator from the Borel algebra of functions on $[0,1]$ to X. This map applies in particular if X is a Hilbert space since they are reflexive.

A.6 Operational calculus. Commuative Banach algebras are defined and studied in Section XX. If B is a semi-simple Banach algebra with a unit element e and $x \in B$, then the spectrum $\sigma(x)$ is a compact subset of $\mathbf{C}$. and we get the resolvent function

(i) $$\lambda \mapsto R_x(\lambda) = (\lambda \cdot e - x)^{-1} \quad : \quad \lambda \in \mathbf{C} \setminus \sigma(x)$$

If $\lambda_0 \in \mathbf{C} \setminus \sigma(x)$ there exists the *local Neumann series* which represents $R_x(\lambda)$ when λ stays in the open disc of radius $\text{dist}(\lambda_0, \sigma(x))$. It follows that $R_x(\lambda)$ is a B-valued analytic function of the complex variable λ defined in the open complement of $\sigma(x)$. Starting from this, Cauchy's formula is used to construct elements in B for every analytic function $f(\lambda)$ which is defined in some open neighborhood of $\sigma(x)$. More precisely, denote by $\mathcal{O}(\sigma(x))$ the algebra of germs of analytic functions on the compact set $\sigma(x)$. Then there exists an *algebra homomorphism* from $\mathcal{O}(\sigma(x))$ into X which sends $f \in \mathcal{O}(\sigma(x))$ into an element $f(x) \in X$. Moreover, the *Gelfand transform* of $f(x)$ is related to that of x by the formula

(*) $$\widehat{f}(x)(\xi) = f(\widehat{x}(\xi)) \quad : \quad \xi \in \mathfrak{M}_B$$

This general fact is used in many applications and here we shall foremost apply it to Banach algebra generated by s single bounded linear operator on a Hilbert space.

A.7 Hilbert spaces. An non-degenerate inner product on a complex vector space $\mathcal{H}$ is a complex valued function on the product set $\mathcal{H} \times \mathcal{H}$ which sends each pair (x, y) into a complex number denoted by $\langle x, y\rangle$ satisfying the following three conditions:

$$x \mapsto \langle x, y\rangle \text{ is a linear form on } \mathcal{H} \text{ for each fixed } y \in \mathcal{H} \tag{1}$$

$$\langle y, x\rangle = \text{ the complex conjugate of } \langle x, y\rangle \text{ for all pairs } x, y \in \mathcal{H} \tag{2}$$

$$\langle x, x\rangle > 0 \text{ for all } x \neq 0 \tag{3}$$

Here (1-3) imply that $\mathcal{H}$ is equipped with a norm defined by $||x|| = \sqrt{\langle x, x\rangle}$. If this norm is complete we say that $\mathcal{H}$ is a Hilbert space. A fundamental fact is that a Hilbert space is *self-dual.* This means that if γ is an element in the dual $\mathcal{H}^*$, then there exists a unique vector $y \in \mathcal{H}$ such that

$$\gamma(x) = \langle x, y\rangle \quad \text{for all } x \in \mathcal{H}\,.$$

We prove this in the section devoted to Hilbert spaces.

B. Hilbert's spectral theorem for bounded self-adjoint operators.

The theory about integral equations created by Fredholm led to Hilbert's result from 1904 which we begin to describe. Let $\mathcal{H}$ be a Hilbert space and denote by $L(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. Every $T \in L(\mathcal{H})$ has its operator norm

$$||T|| = \max_x ||T(x)|| \quad \text{maximum over vectors of norm } \leq 1$$

Next, let A be a bounded self-adjoint operator on a Hilbert space whose compact spectrum is denoted by $\sigma(A)$. The Operational Calculus in Section 7 will show that there exists an algebra isomorphism from the sup-norm algebra $C^0(\sigma(A))$ into a closed subalgebra $\mathcal{A}$ of $L(\mathcal{H})$, i.e. to every continuous function g on the compact spectrum $\sigma(A)$ one gets a bounded linear operator G and $g \mapsto G$ is an algebra isomorphism. Moreover, it is an isometry which means that

$$|g|_{\sigma(A)} = ||G|| \tag{*}$$

where the left hand side is the maximum of $|g|$ over $\sigma(A)$. Next, since A is self-adjoint its spectrum $\sigma(A)$ is a compact subset of the real line where we use t as the variable. If $g(t) = c_0 + c_1 t + \ldots + c_m t^m$ is a polynomial, the operational calculus shows that $G = E + c_1 A + \ldots + c_m A^m$ where E is the identity operator on $\mathcal{H}$. By Weierstrass' theorem the set of polynomials a dense subalgebra of $C^0(\sigma(A))$ and hence $\mathcal{A}$ is the closure in $L(\mathcal{H})$ of the algebra formed by all polynomials of A.

B.1 The spectral measure. The algebra isomorphism above gives a map from the product $\mathcal{H} \times \mathcal{H}$ to the space of Riesz measures on $\sigma(A)$ which to every pair (x, y) in $\mathcal{H}$ assigns a Riesz measure $\mu_{x,y}$ such that

$$\langle g(A)x, y\rangle = \int_{\sigma(A)} g(t) \cdot d\mu_{x,y}(t) \tag{**}$$

holds for every $g \in C^0(\sigma(A))$. The isometry (*) implies that the total variation of $\mu_{x,y}$ is bounded by $||x|| \cdot ||y||$ for every pair x, y. Now measure theory is applied to construct a larger subalgebra of $L(\mathcal{H})$. Namely, for every bounded Borel function $g(t)$ on $\sigma(A)$ the integrals in the sense of Borel and Stieltjes exists in the right hand side of (**) for each pair x, y in $\mathcal{H}$. In this way the g-function gives a bounded linear operator G such that

$$\langle G(x), y\rangle = \int_{\sigma(A)} g(t) \cdot d\mu_{x,y}(t) \quad \text{hold for all pairs} \quad x, y \tag{***}$$

This yields an algebra isomorphism from the algebra $\mathcal{B}^\infty(\sigma(A))$ of bounded Borel functions to a subalgebra of $L(\mathcal{H})$ denoted by $B(\mathcal{A})$. Again the map $g \mapsto G$ is an isometry and in this extended algebra we can construct an ample family of self-adjoint operators. Namely, for every Borel subset γ of $\sigma(A)$ we can take its characteristic function and get the bounded linear operator Γ. By the Operational Calculus the spectrum of Γ is equal to the closure of γ. Moreover, Γ is a self-adjoint operator and commutes with A. In particular we can consider partitions of $\sigma(A)$. Namely, choose

$M > 0$ so that $\sigma(A) \subset [-M, M]$ and M is outside $\sigma(A)$. With a large integer N we consider the half-open intervals

$$\gamma_\nu = \left[-M + \frac{\nu}{N} \cdot M, -M + \frac{\nu+1}{N} \cdot M\right] \quad : 0 \leq \nu \leq 2N-1$$

Then $\Gamma_0 + \ldots + \Gamma_{2N-1} = E$ and we also get the decomposition

(1) $$A = A_0 + \ldots + A_{2N-1} \quad : \; A_\nu = A\Gamma_\nu$$

Above $\{\Gamma_\nu\}$ gives a *resolution of the identity* where (1) means that A is a sum of self-adjoint operators where every individual operator has a spectrum confined to an interval of length $\leq \frac{1}{N}$. This resembles the finite dimensional case and constitutes Hilbert's Theorem for bounded self-adjoint operators.

C. Carleman's theorem for unbounded operators

In a note from May 1920 [Comptes rendus], Carleman indicated a procedure to handle unbounded self-adjoint operators expressed via integral kernels which do not satisfy the Fredholm conditions. The conclusive theory appeared in *Sur les équations singulières à noyau reèl et symmétrique* from 1923 published by Uppsala University. Here one starts with a linear operator A on $\mathcal{H}$ which only is *densely defined*. That is, the domain of definition $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$ while A is unbounded, i.e.

$$\sup_x ||A(x)|| = +\infty$$

with the supremum taken over the unit ball in $\mathcal{H}$. One says that A is *symmetric* if

(1) $$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \text{hold for all pairs} \quad x, y \in \mathcal{D}(A)$$

The adjoint A^*. Let A be a symmetric operator. Given a vector x_* in $\mathcal{H}$ we define a linear functional on $\mathcal{D}(A)$ by

$$x \mapsto \langle Ax, x_*\rangle$$

Suppose there exists a constant $C(x_*)$ such that

(1) $$|\langle Ax, x_*\rangle| \leq C(x_*) \cdot ||x|| \quad \text{hold for all pairs} \quad x, y \in \mathcal{D}(A)$$

Since $\mathcal{D}(A)$ is dense and $\mathcal{H}$ is self-dual this gives a unique vector y_* such that

(2) $$\langle Ax, x_*\rangle = \langle x, y_*\rangle \quad : \; x \in \mathcal{D}(A)$$

The set of vectors x_* for which $C(x_*)$ exists is a subspace of $\mathcal{H}$ which we denote by $\mathcal{D}^*$. From (2) we get the linear operator $x_* \mapsto y_*$. It is denoted by A^* and called the adjoint operator of A. So here $\mathcal{D}(A^*) = \mathcal{D}^*$. Next follow some exercises where the reader if necessary can consult XX for a more detailed account.

Exercise A. Show that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and that A^* extends A in the sense that $A^*(x) = A(x)$ for every $x \in \mathcal{D}(A)$.

Exercise B. Show that A^* has a closed graph, i.e. put

$$\Gamma(A^*) = \{(x, A^*x) \quad : x \in \mathcal{D}(A^*)\}$$

and verify that $\Gamma(A^*)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$.

Exercise C. On $\mathcal{D}(A^*)$ we define an inner product by

$$\{x, y\} = \langle x, y\rangle + \langle A^*x, A^*y\rangle$$

Use that $\Gamma(A^*)$ is closed to conclude that this inner product is complete and hence $\mathcal{D}(A^*)$ is a Hilbert space under this inner product.

C.1 The eigenspaces $\mathcal{D}_+$ and $\mathcal{D}_-$. Put

$$\mathcal{D}_+ = \{x \in \mathcal{D}(A^*) \quad : A^*(x) = ix\} \quad \text{and} \quad \mathcal{D}_- = \{x \in \mathcal{D}(A^*) \quad : A^*(x) = -ix\}$$

Since A^* has a closed graph it is obvious that these two subspaces of $\mathcal{H}$ are closed.

C.2 A direct sum decomposition. Recall the inclusion $\mathcal{D}(A) \subset \mathcal{D}(A^*)$. We can therefore construct the closure of $\mathcal{D}(A)$ under the norm defined by the complete inner product in Exercise C. Let $cl(\mathcal{D}(A))$ denote this closure. The following will be proved in Section 9:

C.3 Proposition. *One has the following orthogonal decomposition in the Hilbert space $\mathcal{D}(A^*)$:*

$$\mathcal{D}(A^*) = cl(\mathcal{D}(A)) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$$

C.4 The self-adjoint case. Following [Carleman] the symmetric operator A gives *Case I* if $\mathcal{D}_+$ and $\mathcal{D}_-$ both are zero spaces. Then Proposition C.3 gives the equality $\mathcal{D}(A^*) = cl(\mathcal{D}(A))$ and since A is symmetric it follows that A^* also is symmetric, i.e.

$$(*) \qquad \langle A^*x, y\rangle = \langle x, A^*y\rangle$$

holds for all pairs (x, y) in $\mathcal{D}(A^*)$. Starting from the symmetric and densely defined operator A^* we can construct its adjoint. But this time the process stops, i.e. one finds that $(A^*)^* = A^{**}$.

C.5 The bounded resolvent operator. Let A be a densely defined and self-adjoint operator. Thus, it is symmetric and one has the equality $\mathcal{D}(A) = \mathcal{D}(A^*)$. The extension of Hilbert's theorem for bounded self-adjoint operators relies upon the existence of a bounded resolvent.

C.6 Theorem. *There exists a bounded and normal operator S such that the range $S(\mathcal{H}) = \mathcal{D}(A)$ and*

$$(*) \qquad (i \cdot E + A)(S(x)) = x$$

hold for all $x \in \mathcal{H}$. Moreover, the spectrum $\sigma(S)$ is contained in the set

$$\Sigma = \Big\{\frac{1}{a+i} \quad : a \in \mathbf{R} \cup \{0\}\Big\}$$

Remark. We refer to Section 9 for the proof. Next, a bounded linear operator R on $\mathcal{H}$ is normal if it commutes with its adjoint R^*. Hilbert's theorem extends verbatim to the normal operator S above. Namely, if $\sigma(S)$ is the compact spectrum there exists an isometric algebra isomorphism from $C^0(\sigma(S))$ onto the closed subalgebra $\mathcal{S}$ of $L(\mathcal{H})$ generated by S and its adjoint S^*. Moreover, exactly as in the self-adjoint case we use this to construct a map from $\mathcal{H} \times \mathcal{H}$ into Riesz measures on $\sigma(S)$. The isometric algebra isomorphism extends to a map from $\mathcal{B}^\infty(\sigma(S))$ onto a closed subalgebra $B(\mathcal{S})$ of $L(\mathcal{H})$ where each operator in $B(\mathcal{S})$ is normal and commutes with S.

Theorem C.6 implies that the set Σ is a simple closed curve which contains $-i$ and the origin in the complex λ-plane and we can apply the operational calculus to the normal operator S. So for every positive integer N we get the bounded self-adjoint operator Γ_N on $\mathcal{H}$ obtained via the characteristic function of the set

$$(1) \qquad \gamma_N = \{\lambda \in \sigma(S) \quad : \mathfrak{Im}(\lambda) \geq \frac{1}{N}\}$$

Since γ_N does not contain $\lambda = 0$ there exists the bounded normal operator :

$$S_N = \int_{\gamma_N} \frac{1 - i\lambda}{\lambda} \cdot d\mathcal{S}$$

The equality (*) in Theorem C.6 gives

$$(*) \qquad A\Gamma_N = S_N$$

Moreover, from the equation which defines the set Σ it follows that the spectrum of S_N is *real.* Since every normal operator with a real spectrum is self-adjoint we conclude that S_N is so. Finally, the construction of the γ_N-sets implies that the sequence $\{\Gamma_N\}$ converges to the identity operator. More precisely, the kernels of these bounded self-adjoint operators decrease and the intersection

$$\bigcap_{N \geq 1} \mathrm{Ker}(\Gamma_N) = \{0\}$$

C.7 Conclusion. *The results imply that the sequence $\{S_N\}$ converges to A in the sense that*

$$A(x) = \lim_{N\to\infty} S_N(x) \quad \text{holds for all } \ x \in \mathcal{D}(A)$$

Moreover, $\mathcal{D}(A)$ is equal to the set of $x \in \mathcal{H}$ for which the limit of $\{S_N(x)\}$ exists.

This is Carleman's theorem for unbounded self-adjoint operators. *Case II* arises when we start from a symmetric operator A where at least one of the eigenspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ of A^* is non-zero was also considered in [Carleman] where it is proved that if they are finite dimensional and have the same dimension, then one can still construct a self-adjoint operator A_0^* attached to the given symmetric operator A and after apply the spectral theorem to A_0^* to investigate A.

D. Application to a dynamical system.

Using the spectral theorem a rigorous proof of the Ergodic Hypothesis in Statistical Mechanics was given by Carleman at seminars held at Institute Mittag-Leffler in May 1931. Here is the situation: There is given an n-tuple of C^1-functions $A_1(x), \ldots, A_n(x)$ where $x = (x_1, \ldots, x_n)$ are points in $\mathbf{R}^n$. Let t be a time variable and consider the differential system

$$\text{(1)} \qquad \frac{dx_k}{dt} = A_k(x_1(t), \ldots, x_n(t)) \quad : \quad 1 \leq k \leq n$$

Assume that there exists a compact hypersurface S in $\mathbf{R}^n$ such that if $p \in S$ and $\mathbf{x}_p(t)$ is the vector-valued solution to (1) with initial condition $\mathbf{x}_p(0) = p$, then $\mathbf{x}_p(t)$ stay in S for every t. The uniqueness for solutions to the differential systems above gives for every t a bijective map $p \mapsto \mathbf{x}_p(t)$ from S onto itself. It is denoted by $\mathcal{T}_t$ and we notice that

$$\mathcal{T}_s \circ \mathcal{T}_t = \mathcal{T}_{s+t}$$

In addition to this we assume that there exists an invariant measure σ on S for the $\mathcal{T}$-maps. In other words, a non-negative measure σ such that

$$\sigma(\mathcal{T}_t(A)) = \sigma(A)$$

hold for every σ-measurable set. For the applications it suffices to consider the case when σ is absolutely continuos, i.e. a positive continuous function times the area measure on S. So now we have the Hilbert space $L^2(\sigma)$ of complex-valued measurable functions U on S for which

$$\int_S |U(p)|^2 \cdot d\sigma(p) < \infty$$

next, on the Hilbert space $L^2(\sigma)$ there exists the following densely defined symmetric operator:

$$(*) \qquad U \mapsto i \cdot \sum_{\nu=1}^{\nu=n} A_\nu \cdot \frac{\partial U}{\partial x_\nu}$$

It is easy to verify that *Case 1* holds for this operator and hence we can apply the spectral theorem. In particular, to each a pair of L^2-functions U and V we consider the following mean-value integrals over time intervals $[0, T]$:

$$(*) \qquad J_T(U, V) = \frac{1}{T} \cdot \int_0^t \left[U(\mathcal{T}_t(p)) \cdot V(p) \cdot d\sigma(p) \right] \cdot dt$$

Using the spectral theorem from [Carleman 1923], the result below was proved in [Carleman 1931]:

Theorem. *Let $\{\omega_\nu\}$ be an orthonormal basis in $\mathcal{H}$. For each pair U, V in $L^2(\sigma)$ one has the equality*

$$\lim_{T\to\infty} J_T(U,V) = \sum_{\nu=1}^{\infty} \langle \omega_\nu, U\rangle \cdot \langle \omega_\nu, V\rangle$$

where

$$\langle \omega_\nu, U\rangle = \int_S \omega_\nu(p)\cdot U(p)\cdot d\sigma(p)$$

and similarly with U replaced with V.

Remark. Let $\mathcal{H}_*$ be the space of $L^2(\sigma)$-functions which are $\mathcal{T}$-invariant, i.e. L^2-functions ω satisfying

$$(2) \qquad \mathcal{T}_t(\omega) = \omega \quad \text{for all} \quad t$$

Here $\mathcal{H}_*$ is a closed subspace of $L^2(\sigma)$. A special case occurs when $\mathcal{H}_*$ is reduced to the one-dimensional space of constant functions. In this case the theorem above implies that almost every trajectory which comes from the differential system comes close to every point in S which confirms the ergodic condition.

E. Schrödinger's equation.

In 1923 quantum mechanics had not yet appeared so the studies in [Car] were concerned with singular integral equations, foremost inspired from previous work by Fredholm and Volterra. The creation of quantum mechanics led to new applications of the spectral theorem. The interested reader can consult the lecture held by Niels Bohr at the Scandianavian congress in mathematics held in Copenhagen 1925 where he speaks about the interplay between the new physics and pure mathematics. Bohr's lecture presumably inspired Carleman when he some years later resumed his work in [Car 1923]. Recall that the fundamental point in Schrödinger's theory is the hypothesis on energy levels which correspond to the possible orbits in Bohr's theory of atoms which are described by Bohr in his plenary talk when he received the Nobel Prize in physics 1923. Mathematically this means that one considers an equation

$$(*) \qquad \Delta\phi + 2m\cdot\big(E-U\big)\big(\frac{2\pi}{h}\big)^2\cdot\phi = 0$$

Here Δ is the Laplace operator in the 3-dimensional (x,y,z)-space, m the mass of a particle and h Planck's constant while $U(x,y,z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Let us cite an excerpt from Carlemans lectures in Paris at Institut Henri Poincaré held in 1931:

Dans ces dernières années l'ínteret de la question qui nous occupe a considérablement augmenté. Cést en effet in instrument mathématique indispensable pour development de la mechanique moderne crée par M.M. de Brogile, Heisenberg et Schrödinger. Etude de l'équation integrale:

$$\phi(x) = \lambda\cdot\int_a^b K(x,y)\phi(y)dy + f(x) \quad : \lambda\in\mathbf{C}\setminus\mathbf{R}$$

The theory from [Carleman :1923] applies to the following PDE-equations attached to a second order differential operator

$$(*) \qquad L = \Delta + c(x,y,z) \quad : \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

where $c(x,y,z)$ is a real-valued function. The L-operator is densely defined on test-functions u and obviously symmetric on this dense subspace of $L^2(\mathbf{R}^3)$. The major problem is to find conditions on the c-function in order that the favourable Case 1 occurs, i.e. when the closure of the graph of L yields a densely defined and self-adjoint operator. Conditions in order that Case 1 occur were investigated in [Car:xx] and led in particular to the following *sufficiency result*:

D.1 Theorem *Let $c(x,y,z)$ be a continuous and real-valued function such that there is a constant M for which*

$$\limsup_{x^2+y^2+z^2\to\infty} c(x,y,z) \leq M$$

Then the favourable Case 1 holds for the operator $\Delta + c(x,y,z)$.

A special case. Here one considers a potential function:

$$W(p) = \sum \frac{\alpha_k}{|p - q_k|} + \beta$$

where $\{q_k\}$ is a finite subset of $\mathbf{R}^3$ and the α-numbers and β are real and positive. With $c(x,y,z) = W$ we get the favourable case and hence this central case for Schrödinger equations is covered by Theorem D.1 above. We refer to [Carleman] for a detailed proof of Theorem D.1 which also describes how to attain solutions via a limit process where Neumann's boundary value problem is considered on an increasing sequence of balls in $\mathbf{R}^3$.

Remark. The literature about the Schrödinger equation and other equations which emerge from quantum mechanics is very extensive. Numerical solutions to the special equation considered above can be obtained of computers. But already the determination of some initial spectral values when W is a Newtonian potential and the number of mass-points is some finite number ≥ 3 is quite involved. For sources of quantum mechanics the reader should first of all consult the plenary talks by Heisenberg, Dirac and Shrödinger when they received the Nobel prize in physics. Apart from physical considerations the reader will find expositions where explanations are given in a mathematical framework. Actually Heisenberg was sole winner 1931 while Dirac and Shrödinger shared the prize in 1932. But they visited Stockholm together in December 1932. For the mathematician a gateway to understand the theory behind quantum mechanics is the book *Grundlagen der quantenmechanik* by von Neumann published in 1931. In addition to this the reader may consult the book *xxx* by Weitzman which offers an extensive account about quantum mechanics with more up-to-date physical applications.

1. Normed spaces.

A normed space over the complex field is a complex vector space X equipped with a norm $||\cdot||$ expressed by a map from X into $\mathbf{R}^+$ satisfying:

$$(*) \qquad ||x+y|| \leq ||x|| + ||y|| \quad \text{and} \quad ||\lambda \cdot x|| = |\lambda \cdot ||x|| \quad : \quad x, y \in X \quad : \lambda \in \mathbf{C}$$

Moreover $||x|| > 0$ holds for every $x \neq 0$. A norm gives a topology on X defined by the distance function

$$(**) \qquad d(x,y) = ||x-y||$$

1.1 Real versus complex norms. The real numbers appear as a subfield of $\mathbf{C}$. Hence every complex vector space has an underlying structure as a vector space over $\mathbf{R}$. A norm on a real vector space Y is a function $y \mapsto ||y||$ where (*) holds for real numbers λ. Next, let X be a complex vector space with a norm $||\cdot||$ satisfying (*) above. Since we can take $\lambda \in \mathbf{R}$ in (*) the complex norm induces a real norm on the underlying real vector space of X. Complex norms are more special than real norms. For example, consider the 1-dimensional complex vector space given by $\mathbf{C}$. When the point 1 has norm one there is no choice for the norm of any complex vector $z = a + ib$, i.e. its norm becomes the usual absolute value. On the other hand we can define many norms on the underlying real (x, y)-space. For example, we may take the norm defined by

$$\text{(i)} \qquad ||(x,y)|| = |x| + |y|$$

It fails to satisfy (*) under complex multiplication. For example, with $\lambda = e^{\pi i/4}$ we send $(1,0)$ to $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ whose norm from (i) becomes $\sqrt{2}$ while it should remain with norm one if (*) holds.

1.2 Convex sets. We shall work on real vector spaces for a while. Let Y be a real vector space. A subset K is convex if the line segment formed by a pair of points in K stay in K, i.e.

$$\text{(i)} \qquad y_1, y_2 \in K \implies s \cdot y_2 + (1-s) \cdot y_1 \in K \quad : \quad 0 \leq s \leq 1$$

Let $\mathbf{o}$ denote the origin in Y. Let K be a convex set which contains $\mathbf{o}$ and is symmetric with respect to $\mathbf{o}$:

$$y \in K \implies -y \in K$$

The symmetric convex set K is called *absorbing* if there to every $y \in Y$ exists some $t > 0$ such that $ty \in K$. Suppose that K is symmetric and absorbing. To every $s > 0$ we set

$$sK = \{sx : \; x \in K\}$$

Since $\mathbf{o} \in K$ and K is convex these sets increase with s and since K is absorbing we have:

$$\text{(ii)} \qquad \bigcup_{s>0} sK = Y$$

Next, we impose the condition that K does not contain any 1-dimensional subspace, i.e. whenever $y \neq 0$ is a non-zero vector there exists some large t^* such that $t^* \cdot y$ does not belong to K. The condition is equivalent with

$$\text{(iii)} \qquad \bigcap_{s>0} s \cdot K = \mathbf{o}$$

1.3 The norm ρ_K. Let K be convex and symmetric and assume that (ii-iii) hold. To each $y \neq 0$ we set

$$(*) \qquad \rho_K(y) = \min_{t>0} \frac{1}{t} \quad : \; t \cdot y \in K$$

Notice that if $y \in K$ then $t = 1$ is competing when we seek the minimum and hence $\rho_K(y) \leq 1$. On the other hand, if y is "far away" from K we need small t-values to get $t \cdot y \in K$ and therefore $\rho_K(y)$ is large. It is also clear that

$$\text{(i)} \qquad \rho_K(ay) = a \cdot \rho_K(y) \quad : \; a \text{ real and positive}$$

Finally , since K is symmetric we have $\rho_K(y) = \rho_K(-y)$ and hence (i) gives

$$\text{(ii)} \qquad \rho_K(ay) = |a| \cdot \rho_K(y) \quad : \; a \text{ any real number}$$

1.4 Proposition. *By (*) we get a norm which is called the K-norm defined by the convex set K.*

Proof. The verification of the triangle inequality:

$$\rho_K(y_1 + y_2) \leq \rho_K(y_1)\rho_K(y_2)$$

is left as an exercise. The hint is to use the convexity of K.

1.5 A converse. Let $|| \cdot ||$ be a norm on Y. Then we get a convex set

$$K^* = \{y \in Y \quad : ||y| \leq 1\}$$

It is clear that $\rho_{K^*}(y) = ||y||$ holds, i.e. the given norm is recaptured by the norm defined by K^*. We can also regard the set

$$K_* = y \in Y \quad : ||y| < 1\}$$

Here $K_* \subset K^*$ but he reader should notice that one has the equality

$$\rho_{K_*}(y) = \rho_{K^*}(y)$$

Thus, the two convex sets define the same norm even if the set-theoretic inclusion $K_* \subset K^*$ may be strict. In general, a pair of convex sets K_1, K_2 satisfying (i-ii) above are equivalent if they define the same norm. Starting from this norm we get K_* and K^* and then the reader may verify that

$$K_* \subset K_\nu \subset K^* \quad : \nu = 1, 2$$

Summing up we have described all norms on Y and they are in a 1-1 correspondence with equivalence classes in the family $\mathcal{K}$ which consists of all convex sets which are symmetric, absorbing and satisfy (iii) above, i.e. when $K \in \mathcal{K}$ then K does not contain any 1-dimensional subspace. For each specific norm on Y we can assign the largest convex set K^* from the corresponding equivalence class.

1.6 Equivalent norms. Two norms $|| \cdot ||_1$ and $|| \cdot ||_2$ are equivalent if there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \cdot ||y||_1 \leq ||y||_2 \leq C \cdot ||y||_1 \quad : \quad y \in Y \tag{0.6}$$

Notice that if the norms are defined by convex sets K_1 and K_2 respectively, then (0.6) means that there exists some $0 < t < 1$ such that

$$tK_1 \subset K_2 \subset t^{-1}K_1$$

The case $Y = \mathbf{R}^n$. If Y is finite dimensional all norms are equivalent. To see this we consider the euclidian basis $e_1, \ldots, e_n$. To begin with we get the *euclidian norm* which by definition measures the euclidian length from a vector y to the origin. It means that

$$||y||_e = \sqrt{\sum_{\nu=1}^{\nu=n} |a_\nu|^2} \quad : \quad y = a_1e_1 + \ldots + a_ne_n \tag{i}$$

The reader should verify that the norm satisfies the triangle inequality

$$||y_1 + y_2||_e \leq ||y_1||_e + ||y_2||_e$$

which amounts to verify the Cauchy-Schwartz inequality. In the euclidian norm the unit sphere S^{n-1} corresponds to vectors whose euclidian norm os one. We also define the norm $|| \cdot ||^*$ by

$$||y||^* = \sum_{\nu=1}^{\nu=n} |a_\nu| \quad : \quad y = a_1e_1 + \ldots + a_ne_n \tag{ii}$$

This norm is equivalent to the euclidian norm. More precisely the reader may verify the inequality

$$\frac{1}{\sqrt{n}} \cdot ||y||_e \leq ||y||^* \leq \sqrt{n} \cdot ||y||_e \tag{iii}$$

Next, let $||\cdot||$ be some arbitrary norm. Put

(iv) $$C = \max_{1\leq\nu\leq n} ||e_\nu||$$

Then (ii) and the triangle inequality for the norm $||\cdot||$ gives

(v) $$||y|| \leq C\cdot ||y||^*$$

By the equivalence (iii) the norm topology defined by $||\cdot||^*$ is the same as the usual euclidian topology in $Y = \mathbf{R}^n$. Next, notice that (v) implies that the sets

$$U_N = \{y \in Y \quad : \quad ||y|| < \frac{1}{N}\} \quad : \quad N = 1, 2, \ldots$$

are *open* sets when Y is equipped with its usual euclidian topology. Now $\{U_N\}$ is an increasing sequence of open sets and their union is obviously equal t oY. in particular this union covers the compact unit sphere S^{n-1}. This gives an integer N such that

$$S^{n-1} \subset U_N$$

This inclusion gives

$$||y||_e \leq N\cdot ||y||$$

Together with (iii) and (v) we conclude that $||\cdot||$is equivalent with $||\cdot||_e$. Hence we have proved

1.7 Theorem. *On a finite dimensional vector space all norms are equivalent.*

1.8 The complex case. If X is a complex vector space we obtain complex norms when we restrict the attention to convex sets K which not only are symmetric with respect to scalar multiplication with real numbers but is also invariant under i. To be precise, one requires that

$$\lambda\cdot K \subset K \quad : \quad \forall\,\lambda \in \mathbf{C} \quad : |\lambda| \leq 1$$

Here a similar result as in Theorem 1.7 holds for complex norms on $\mathbf{C}^n$, i.e. they are all equivalent.

1.9 Non-linear convexity.

Let $f(x)$ be a real-valued function in $\mathbf{R}^n$ of class C^2. To every point x we assign the hessian $H_f(x)$ which is the symmetric matrix whose elements are $\{\partial^2 f/\partial x_j \partial x_k\}$. The function is strictly convex if $H_f(x)$ is positive for all x, i.e. if the eigenvalues are all > 0. Assume in addition that

(1) $$\lim_{|x|\to+\infty} f(x) = +\infty$$

Under these conditions one has the results below due to Lagrange and Legendre:

1.10 Theorem. *The vector valued function*

$$x \mapsto \nabla_f(x)$$

is a C^1-differomorphism of $\mathbf{R}^n$ onto itself.

Next, with f still as above one defines the function below for each $y \in \mathbf{R}^n$:

(*) $$\mathcal{L}_f(y) = \max_x \, \langle x, y\rangle - f(x)$$

1.11 Theorem. *For each y the maximum in (*) is taken at a unique pointt $x^*(y)$ and one has the equality*

(**) $$y = \nabla_f(x^*(y)$$

Moreover, $\mathcal{L}$ is again strictly convex and one has the biduality formula:

(***) $$f = \mathcal{L}\circ\mathcal{L}$$

1.12 Exercise. Prove the two theorems above. Legendre's biduality means that $\mathcal{L}$ is a bijective map on the class of strictly convex functions which satisfy (1) and the composed operator $\mathcal{L} \circ \mathcal{L}$ is the identty.

1.13 Remark. Legendre used Theorem 1.11 to study extremal solutions in the calculus of variation where it was foremost used for problems in mechanics. Recall also the more involved construction of quantized contact transformations which were introduced in some special cases by Hamilton and later extended by Jacobi to solve various Hamiltonian systems. For this more advanced account related to convexity, the reader may consult the Chapter XX in the text-book on mechanics by Lev Landau, which is the first volume in his outstanding series of books devoted to theoretical physics.

1.14 On cones in $\mathbf{R}^n$. Here follow an exercise which helps the reader to grasp some geometry in $\mathbf{R}^n$. A subset Γ is a cone if $x \in \Gamma$ implies that the half-ray $\mathbf{R}^+ \cdot x \subset \Gamma$. We suppose that the cone is closed. In particular the origin is included and notice that Γ is determined by the compact subset $\Gamma_* = \Gamma \cap S^{n-1}$ where S^{n-1} is euclidian the unit sphere. We say that Γ is *fat* if Γ_* has a non-empty interior in the unit sphere and Γ is *proper* if $\Gamma_* \cap -\Gamma_* = \emptyset$, i.e. equivalently Γ does not contain any 1-dimensional subspace. next, the *dual cone* is defined by

$$\widehat{\Gamma} = \{x \quad : \langle x, \Gamma \rangle \leq 0\}$$

1.15 Exercise. Show that a cone Γ is proper if and only if $\widehat{\Gamma}$ is fat and show also that Γ is equal to the dual of $\widehat{\Gamma}$.

2. Banach spaces.

Let Y be a normed space over $\mathbf{C}$ or over $\mathbf{R}$. A sequence of vectors $\{y_n\}$ is called a Cauchy sequence if

$$(*) \qquad \lim_{n,m \to \infty} ||y_n - y_m|| = 0$$

We obtain a vector space $\widehat{Y}$ whose vectors are defined as equivalence classes of Cauchy sequences. The norm of a Cauchy sequence $\hat{y} = \{y_n\}$ is defined by

$$||\hat{y}|| = \lim_{n \to \infty} ||y_n||$$

One says that the norm on Y is complete if every Cauchy sequence converges, or equivalently $Y = \widehat{Y}$. A complete normed space is called a *Banach space* as an attribution to Stefan Banach whose pioneering article [Ban] introduced the general concept of normed vector spaces.

2.1 The Banach-Steinhaus theorem. *Let X be a Banach space equipped with the complete norm $|| \cdot ||^*$. Then for every other norm $|| \cdot ||$ there exists a constant C such that*

$$||x|| \leq C \cdot ||x||^* \quad : \quad x \in X$$

Remark. In particular we see that if $|| \cdot ||_1$ and $|| \cdot ||_2$ are two complete norms on the same vector space then they are equivalent in the sense that there exists a constant C such that

$$C^{-1} \cdot || \cdot ||_2 \leq || \cdot ||_1 \leq C \cdot || \cdot ||_2$$

The proof of Theorem 2.1 relies upon a result due to Baire which we recall below.

The Baire category theorem. Let X be a metric space whose metric d is complete, i.e. every Cauchy sequence with respect to the distance function d converges.

2.2 Theorem. *Let $\{F_n\}$ is an increasing sequence of closed subsets of X where each F_n has empty interior. Then the union $F^* = \cup F_n$ is meager, i.e. F^* does not contain any open set.*

Proof. Let $x_0 \in X$ and $\epsilon > 0$ be given. It suffices to show that $B_\epsilon(x_0)$ contains a point x_* outside F^* for every $\epsilon > 0$. To show this we first use that F_1 has empty interior which gives some $x_1 \in B_{\epsilon/2}(x_0) \setminus F_1$ and we choose $\delta_1 < \epsilon/2$ so that

(i) $$B_{\delta_1}(x_1) \cap F_1 = \emptyset$$

Now $B_{\delta_1/2}(x_1)$ is not contained in F_2 and we find a pair x_2 and $\delta_2 < \delta_1/2$ such that

(ii) $$B_{\delta_2}(x_2) \cap F_2 = \emptyset$$

We can continue in this way and to every n find a pair x_n, δ_n such that

(iii) $$B_{\delta_n}(x_n) \cap F_n = \emptyset \quad : \quad x_n \in B_{\delta_{n-1}}(x_{n-1}) \quad : \quad \delta_n < \delta_{n-1}/2$$

Since X by assumption is complete and $\{x_n\}$ by the construction is a Cauchy sequence there exists a limit $x_n \to x^*$. The rapid decrease of the δ-numbers gives $x^* \in B_\epsilon(x_0)$ and the inductive construction shows that x^* does not belong to the union F^*.

2.3 Proof of the Banach-Steinhaus theorem. Let X be a Banach space equipped with the complete norm $||\cdot||^*$ and let $||\cdot||$ be some other norm. To each positive integer N we put

$$F_N = \text{The closure of the set } \{x : \ ||x|| \leq N\} \text{ with respect to } ||\cdot||^* - \text{topology}$$

Obviously $\cup F_N = X$ and Baire's category theorem gives the existence of some $N \geq 1$, a point $x_0 \in X$ and some $\delta > 0$ such that the open ball

(i) $$B_\delta(x_0) = \{x : \ ||x - x_0||^* < \delta\} \subset F_N$$

Next, notice that F_N is convex and symmetric. So if $||x|| < \delta$ we get

$$x = \frac{x_0 + x}{2} + \frac{-x_0 + x}{2} \in F_N$$

Hence we get the implication:

(ii) $$||x|| \leq \delta \implies ||x||^* \leq N$$

But this means precisely that

$$||x|| \leq \frac{N}{\delta} \cdot ||x||^*$$

This finishes the proof of the Banach-Steinhaus theorem.

2.4 Separable Banach spaces. This is the class of Banach spaces which contain a denumerable and dense subset. Let Y be a separable Banach space and $\{y_n\}$ a dense subset indexed by positive integers $n = 1, 2, \ldots$. To every n we get the finite dimensional vector space Y_n generated by $y_1, \ldots, y_n$ and by the procedure in Linear algebra we can construct a basis in Y_n and when $Y_n \subset Y_{n+1}$ get a new basis vector. In this way one arives at a denumarable sequence of linearly independent vectors $e_1, e_2, \ldots$ such that the increasing sequence of subspaces $\{Y_n\}$ are all contained in the vector space

(i) $$Y_* = \oplus \mathbf{R} \cdot \mathbf{e_n}$$

By the construction Y_* is a dense subspace of Y. Of course, there are many ways to construct a denumerable sequence of linearly independent vectors which by (i) give a dense subspace of Y.

2.5 Schauder basis. One may ask if it is possible to choose a sequence $\{e_n\}$ as above such that every $y \in Y$ can be expanded in this basis as follows:

2.6 Definition. *A denumerable sequence $\{e_n\}$ of $\mathbf{R}$-linearly independent vectors is called a Schauder basis if there to each $y \in Y$ exists a unique sequence of real numbers $c_1(y), c_2(y), \ldots$ such that*

$$\lim_{N \to \infty} ||y - \sum_{n=1}^{n=N} c_\nu(y) \cdot e_\nu|| = 0$$

2.7 Per Enflo's example. The existence of a Schauder basis in every separable Banach space appears to be natural and Schauder constructed such a basis in several cases, such as the Banach space $\mathbf{C}^0[0,1]$ of continuous functions on the closed unit interval equipped with the maximum norm. For several decades the question of existence of a Schauder basis in *every* separable Banach space was open until Per Enflo at seminars in Stockholm University during the autumn in 1972 presented an example where a Schauder basis does not exist. Actually Enflo also gave a counter-example concerning compact operators. More precisely, to every $2 < p < \infty$ he constructed a closed subspace Y of the Banach space ℓ^p on which there exists a *compact* linear operator T which cannot be approximated in the operator norm by linear operators on Y with finite dimensional range. One verifies easily that the failure of such an approximation implies that Y cannot have a Schauder basis. So Enflo constructed a very "ugly" separable Banach space. For the detailed construction we refer to his article [En-Acta Mathematica]. Let us remark that the essential ingredient in Enflo's construction relies upon a study of Fourier series where the efficient tool is to employ *Rudin-Schapiro* polynomials which consist of trigonometric polynomials

$$(*) \qquad P_N(x) = \epsilon_0 + \epsilon_1 e^{ix} + \ldots + \epsilon_N \cdot e^{iNx}$$

where each ϵ_ν is +1 or -1. For any such sequence Plancherel's equality gives

$$\frac{1}{2\pi}\int_{=}^{2\pi} |P_n(x)|^2 \cdot dx = 2^{N+1}$$

This implies that the maximum norm of $|P(x)|$ is at least $2^{\frac{N+1}{2}}$. In [Ru-Sch] it is shown that there exists a fixed constant C such that to every $N \geq 1$ there exists at least one choice of signs of the $\epsilon_\bullet$-sequence so that

$$\max_{0\leq x\leq 2\pi} |P_N(x)| \leq C \cdot 2^{\frac{N+1}{2}}$$

A Remark. After Enflo's work [En] it became a veritable industry to *verify* that various "concrete" Banach spaces Y do have a Schauder basis and perhaps more important, enjoy the approximation property , i.e. that the class of linear operators on Y with finite dimensional range is dense in the linear space of all compact operators on Y. Fortunately most Banach spaces do have a Schauder basis. But the construction of a specific Schauder basis is often non-trivial. It requires for example considerable work to exhibit a Schauder basis in the Banach space $A(D)$ of continuous functions on the closed unit disc which are analytic in the interior.

3. Linear operators.

Let X and Y be two normed spaces and $T\colon\ X \to Y$ a linear operator. We say that T is continuous if there exists a constant C such that

$$||T(x)|| \leq C \cdot ||x||$$

where the norms on X respectively Y appear. Denote by $\mathcal{L}(X,Y)$ the set of all continuous linear operators from X into Y. This yields a vector space equipped with the norm:

$$(*) \qquad ||T|| = \max_{||x||=1} ||T(x)|| \quad : T \in \mathcal{L}(X,Y)$$

Above X and Y are not necessarily Banach spaces. But one verifies easily that if $\hat{X}$ and $\hat{Y}$ are their completitions, then every $T \in \mathcal{L}(X,Y)$ extends in a unique way to a continuous linear operator $\hat{T}$ from $\hat{X}$ into $\hat{Y}$. One refers to $\hat{T}$ as the completion of T. Let us also notice the following:

3.1 Proposition. *If Y is a Banach space then the norm on $\mathcal{L}(X,Y)$ is complete, i.e. this normed vector space is a Banach space.*

The easy verification is left to the reader.

3.2 The open mapping theorem. Let X and Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$. In X we get the subspace

$$\mathcal{N}(T) = \{x\colon\ T(x) = 0\}$$

Since T is continuous it is obvious that the kernel is a closed subspace of X. So by the general construction in XX we get the quotient space

$$\bar{X} = \frac{X}{\mathcal{N}(T)}$$

One verifies that T yields a linear operator $\bar{T}$ from $\bar{X}$ into Y which by the construction of the quotient norm on $\bar{X}$ has the same norm as T. Next, consider the image $T(X)$. It is obvious that

(i) $$T(X) = \bar{T}(\bar{X})$$

One says that T has *closed range* if the linear subspace $T(X)$ of Y is closed. Assume this holds. Then the complete norm on Y induces a complete norm on the closed subspace $T(X)$. In addition to this complete norm on $T(X)$ we have the norm defined by

$$||y||^* = ||\bar{x}|| \quad : \quad y = \bar{T}(\bar{x})$$

The Banach-Steinhaus theorem gives a constant C such that

$$||y||^* \leq C \cdot ||y||$$

This means that if $y \in T(X)$, then there exists $x \in X$ such that

(*) $$y = T(x) \quad : \quad ||x|| \leq C \cdot ||y||$$

Remark. One refers to (*) as the Open Mapping Theorem. The terminology is perhaps a bit confusing since (*) means that given a vector y in the closed range of T one can always find $x \in X$ such that $y = T(x)$ and at the same time choose x so that its norm in X does not exceed the constant C times $||y||$.

3.3 The closed graph theorem Let X and Y be Banach spaces. Consider a linear operator T from X into Y. In the product space $X \times Y$ we get the graph

$$\Gamma_T = \{(x, T(x)) \quad : x \in X\}$$

Now we can impose the condition that Γ_T is a closed subset of the Banach space $X \times Y$. Notice that

$$\mathcal{N}(T) = \{x\colon\ (x, 0) \in \Gamma_T\}$$

The hypothesis that Γ_T is a closed subset of $X \times Y$ obviously implies that $\mathcal{N}(T)$ is a closed subspace of X. Now we get the Banach space $X_* = \frac{X}{\mathcal{N}(T)}$ and obtain a *bijective* linear map:

(i) $$\mathfrak{i}\colon\ \bar{x} \mapsto (\bar{x}, T(\bar{x}))$$

from X_* into Γ_T. The induced complete norm on the closed graph Γ_T is defined by

(ii) $$||(\bar{x}, T(\bar{x})|| = ||\bar{x}|| + ||T(\bar{x})||$$

Theorem xx applies to $\mathfrak{i}$ and proves that the inverse map is continuos. This gives a constant C such that

(iii) $$||\bar{x}|| + ||T(\bar{x}]|| \leq C \cdot ||\bar{x}|| \implies ||T(\bar{x}]|| \leq C \cdot ||\bar{x}||$$

This implies that T has finite norm. Hence we have proved the following:

3.4 Theorem. *Let T be a linear operator from one Banach space X into another Banach space Y with a closed graph Γ_T. Then T is continuous.*

3.5 Densely defined operators.

Let $X_* \subset X$ be a dense subspace and $T\colon\ X_* \to Y$ a linear operator where Y is a Banach space. We get the linear subspace of $X \times Y$ defined by

$$\Gamma_T = \{(x, y)\colon \quad x \in X_*\ :\ y = T(x)\}$$

We can impose the condition that Γ_T is a closed subspace of $X \times Y$. When it holds we say that the densely defined operator T has a closed graph. Let us give

3.6 Example. Let $X = C^0_*[0,1]$ be Banach space whose elements are continuous functions $f(x)$ on the closed interval $[0,1]$ with $f(0) = 0$. The space $X_* = C^1_*[0,1]$ of continuously differentiable functions appears as a dense subspace of X. Next, let $Y = L^1[0,1]$. We get a linear map T from X_* into Y defined by

$$\text{(i)} \qquad T(f) = f' \quad : \quad f \in C^1_*[0,1]$$

In other words, we take the derivative $f'(x)$ which belongs to Y since it is a continuous function. Now T has a graph

$$\text{(ii)} \qquad \Gamma_T = \{(f, f') \quad : f \in C^1_*[0,1]\}$$

Here Γ_T is no a closed subspace of $X \times Y$. But we can construct its closure which yields a closed subspace denoted by Γ^*_T. By definition a pair (f, g) belongs to Γ^*_T if and only if

$$\exists \; \{f_n\} \in C^1_*[0,1] \quad : ||f - f_n|| \to 0 \quad : \int_0^1 |f'_n(t) - g(t)| \cdot dt = 0$$

The last limit means that the derivatives f'_n converge to an L^1-function g. Since $f_n(0) = 0$ are assumed we have

$$f_n(x) = \int_0^x f'_n(t \cdot dt \to \int_0^x g(t) \cdot dt$$

It follows that the continuous limit function f is equal to the primitive integral

$$\text{(iii)} \qquad f(x) = \int_0^x g(t) \cdot dt$$

3.7 Conclusion. The linear space Γ^*_T consists of pairs (f, g) with $g \in L^1[0, T)$ and f is the g-primitive defined by (iii). In this way we obtain a linear operator T^* with a closed graph. More precisely, T^* is defined on the linear subsapce of X given by functions $f(x)$ which are primitives of L^1-functions. This means by Lebesgue theory that the domain of definition of T^* consists of *absolutely continuous functions.* Thus, by enlarging the domain of definition the linear operator T is extended to a linear operator T^* whose graph is closed in $X \times Y$. One refers to T^* as a closed graph extension of T.

The example above is typical for many constructions where one starts with some densely defined linear operator T and finds an extension T^* whose graph is the closure of Γ_T. Notice that the choice of the target space Y affects the situation. As a further illustration, replace $L^1[0,1]$ with the Banach space $L^2[0,1]$ of square integrable functions on $[0,1]$. In this case we find a closed graph extension T^{**} whose domain of definition consists of continuous functions $f(x)$ which are primitives of L^2-functions. Since the inclusion $L^1[0,1] \subset L^2[0,1]$ is strict the domain of definition for T^{**} is a proper subspace of the linear space of all absolutely continuous functions. At the same time one gets a complete linear space given by

$$\mathcal{D}_{T^{**}} = \{f \in C^0_*[0,1] \quad : f(x) = \int_0^x g(t) \cdot dt \quad : g \in L^2[0,1]\}$$

This linear space is indeed complete when it is equipppped with the norm

$$||f|| = ||g||_2 = \sqrt{\int_0^1 |g(t)|^2 \cdot dt}$$

This is an example of a Sobolev space. Constructions as above are often used in PDE-theory where one in general starts from a differential operator

$$(*) \qquad P(x, \partial) = \sum p_\alpha(x) \cdot \partial^\alpha$$

Here $x = (x_1, \ldots, x_n)$ are coordinates in $\mathbf{R}^n$ and ∂^α denote the higher order differential operators expressed by products of the first order operators $\{\partial_\nu = \partial/\partial x_\nu\}$. The coefficients $p_\alpha(x)$ are in general only continuous functions defined in some open subset Ω of $\mathbf{R}^n$, though the case when p_α

are C^∞-functions is the most frequent. Depending upon the situation one takes various target spaces Y, for example the Hilbert space $L^2(\Omega)$ of functions which are square integrable over Ω. To begin with one restricts $P(x,\partial)$ to the linear space $C_0^\infty(\Omega)$ of test-functions in Ω and constructs the corresponding graph. Then one seeks for extensions of this linear operator to larger subspace of functions on Ω and in favourable cases there exists a densely defined linear operator with a closed graph. We cannot enter this in more detail since this is a subject within PDE-theory. Let us only mention that the us of "abstract functional analysis" in this context is quite useful in PDE-theory. A result of this nature is *Gårding's inequality* established by Lars Gårding in [Gå] and later extended to the so called sharp Gårding inequality by L. Hörmander in [Hö]. This illustrates the usefulness of functional analysis, though one must not forget that delicate parts in the proofs rely upon "hard analysis".

4. Hilbert spaces.

Introduction. First we recall some geometric facts in the finite dimensional case which later on clarify properties of Hilbert spaces in the infinite dimensional case. A result in euclidian geometry asserts that if A is some invertible $n \times n$-matrix whose elements are real numbers and we regard A as a linear map from $\mathbf{R}^n$ into itself, then the image of the euclidian unit sphere S^{n-1} is an ellipsoid $\mathcal{E}_A$, and conversely if $\mathcal{E}$ is an ellipsoid then there exists an invertible matrix A such that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$.

0.1 The case $n = 2$. Already this case is instructive and the reader is invited to contemplate upon the two-dimensional case and study specific examples. For example, let (x, y) be the coordinates in $\mathbf{R}^2$ and A the linear map

$$(x, y) \mapsto (x + y, y) \tag{0.1}$$

To get the image of the unit circle $x^2 + y^2 = 1$ we use polar coordinates and write $x = \cos\phi$ and $y = \sin\phi$. This gives the closed image curve

$$\phi \mapsto (\cos\phi + \sin\phi\,;\, \sin\phi) \quad : | \quad 0 \leq \phi \leq 2\pi \tag{i}$$

It is not obvious how to determine the principal axes of this ellipse. The gateway is to consider the *symmetric* 2×2-matrix $B = A^*A$. If u, v is a pair of vectors in $\mathbf{R}^2$ we have

$$\langle Bu, v\rangle = \langle Au, Av\rangle \tag{ii}$$

It follows that $\langle Bu, u\rangle > 0$ for all $u \neq 0$. By a wellknown result in elementary geometry it means that the symmetric matrix B is positive, i.e. the eigenvalues arising from zeros of the characteristic polynomial $\det(\lambda E_2 - B)$ are both positive. Moreover, the *spectral theorem* for symmetric matrices shows that there exists an orthonormal basis in $\mathbf{R}^2$ given by a pair of eigenvectors for B denoted by u_* and v_*. So here

$$B(u_*) = \lambda_1 \cdot u_* \quad : \quad B(v_*) = \lambda_2 \cdot v_*$$

Next, since (u_*, v_*) is an orthonormal basis in $\mathbf{R}^2$ points on the unit circle are of the form

$$\xi = \cos\phi \cdot u_* + \sin\phi \cdot v_*$$

Then we get

$$|A(\xi)|^2 = \langle A(\xi).A(\xi)\rangle = \langle B(\xi), \xi) = \cos^2\phi \cdot \lambda_1 + \sin^2\phi \cdot \lambda_2$$

From this we see that the ellipse $\mathcal{E}_A$ has u_* and v_* as principal axes. It is a circle if and only if $\lambda_1 = \lambda_2$. If $\lambda_1 > \lambda_2$ the largest principal axis has length $2\sqrt{\lambda_1}$ and the smallest has length $2\sqrt{\lambda_2}$. The reader should now compute the specific example (*) and find $\mathcal{E}_A$.

4.2 A Historic Remark. The fact that $\mathcal{E}_A$ is an ellipsoid was wellknown in the Ancient Greek mathematics when $n = 2$ and $n = 3$. Moreover, the geometric constructions by Appolonius can be used to determine $\mathcal{E}_A$ when the linear map A is given. After general matrices and their determinants were introduced, the spectral theorem for symmetric matrices was established by A. Cauchy around 1810 under the assumption that the eigenvalues are different. Later Weierstrass found the proof in the general case. Independently Gram-Schmidt and Weierstrass also gave a method to produce an orthonormal basis of eigenvectors for a given symmetric $n \times n$-matrix B. An eigenvector with largest eigenvalue is found when one studies the extremal problem

$$\max_x \langle Bx, x\rangle \quad : \quad ||x|| = 1 \tag{1}$$

If a unit vector x_* maximises (1) then it is an eigenvector, i.e.

$$Bx_* = a_1 x_*$$

holds for a real number a. In the next stage one takes the orthogonal complement $x_*^\perp$ and proceed to study the restricted extremal problem where x say in this orthogonal complement. Here we find a new eigenvector whose eigenvalue $a_2 \leq a_1$. After n steps we obtain an n-tuple of pairwise orthogonal eigenvectors to B. In the orthonormal basis given by this n-tuple the linear operator of B is represented by a diagonal matrix.

Singular values. *Mathematica* has implemented programs which for every invertible $n \times n$-matrix A determines the ellipsoid $\mathcal{E}_A$ numerically. This is presented under the headline *singular values for matrices.* In general the A-matrix is not symmetric but the spectral theorem is applied to the symmetric matrix A^*A which determines the ellipsoid $\mathcal{E}_A$ and whose principal axis are pairwise disjoint.

4.3 Rotating bodies. The spectral theorem in dimension $n = 3$ is best illustrated by regarding a rotating body. Consider a bounded 3-dimensional body K in which some distribution of mass is given. The body is placed i $\mathbf{R}^3$ where (x_1, x_2, x_3) are the coordinates and the distribution of mass is expressed by a positive function $\rho(x, y, z)$ defined in K. The *center of gravity* in K is the point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ where

(i) $$\bar{x}_\nu = \iiint_K x_\nu \cdot \rho(x_1, x_2, x_3) \cdot dx_1 dx_2 dx_3 \quad : 1 \leq \nu \leq 3$$

After a translation we may assume that the center of mass is the origin. Now we imagine that a rigid bar which stays on a line ℓ is attached to K with its two endpoints p and q, i.e. if γ is the unit vector in $\mathbf{R}^3$ which determines the line then

$$p = A \cdot \gamma \quad : \quad q = -A \cdot \gamma$$

where A is so large that p and q are outside K. The mechanical experiment is to rotate around ℓ with some constant angular velocity ω while the two points p and q are kept fixed. The question arises if such an imposed rotation of K around ℓ implies that external forces at p and q are needed to prevent these to points from moving. It turns out that there exist so called free axes where no such forces are needed, i.e. for certain directions of ℓ the body rotates nicely around the axis with constant angular velocity. The free axes are found from the spectral theorem. More precisely, one introduces the symmetric 3×3-matrix A whose elements are

(i) $$a_{pq} = \bar{x}_\nu = \iiint_K x_p \cdot x_q \cdot \rho(x_1, x_2, x_3) \cdot dx_1 dx_2 dx_3$$

Using the expression for the centrifugal force by C. Huyghen's one has the *Law of Momentum* which in the present case shows that the body has a free rotation along the lines which correspond to eigenvectors of the symmetric matrix A above. In view of the historic importance of this example we present the proof of this in a separate section even though some readers may refer to this as a subject in classical mechanics rather than linear algebra. Hence the spectral theorem was evident by via this mechanical experiment, i.e. just as Stokes Theorem the spectral theorem for symmetric matrices is rather a Law of Nature than a mathematical discovery.

4.4 Inner product norms

Let A be an invertible $n \times n$-matrix. The ellipsoid $\mathcal{E}_A$ defines a norm on $\mathbf{R}^n$ by the general construction in XX. This norm has a special property. For if $B = A^*A$ and x, y is a pair of n-vectors, then

(i) $$||x + y||^2 = \langle B(x + y), B(x + y) \rangle = ||x||^2 + ||y||^2 + 2 \cdot B(x, y)$$

It means that the map

(ii) $$(x, y) \mapsto ||x + y||^2 - ||x||^2 - ||y||^2$$

is linear both with respect to x and to y, i.e. it is a bilinear map given by

(iii) $$(x, y) \mapsto 2 \cdot B(x, y)$$

We leave as an exercise for the reader to prove that if K is a symmetric convex set in $\mathbf{R}^n$ defining the ρ_K-norm as in xx, then this norm satisfies the bi-linearity (ii) if and only if K is an ellipsoid and therefore equal to $\mathcal{E}_A$ for an invertible $n \times n$-matrix A. Following Hilbert we refer to a norm defined by some bilinear form $B(x, y)$ as an *inner product norm.* The spectral theorem asserts that there exists an orthonormal basis in $\mathbf{R}^n$ with respect to this norm.

4.5 The complex case. Consider a Hermitian matrix A, i.e an $n \times n$-matrix with complex elements satisfying

$$(*) \qquad a_{qp} = \bar{a}_{pq} \quad : \quad 1 \leq p, q \leq n$$

Consider the n-dimensional complex vector space $\mathbf{C}^n$ with the basis $e_1, \ldots, e_n$. An inner product is defined by

$$(**) \qquad \langle x, y \rangle = x_1 \bar{y}_1 + \ldots + x_n \bar{y}_n$$

where $x_\bullet = \sum x_\nu \cdot e_\nu$ and $y_\bullet = \sum y_\nu \cdot e_\nu$ is a pair of complex n-vectors. If A as above is a Hermitian matrix we obtain

$$(***) \qquad \langle Ax, y \rangle = \sum \sum a_{pq} x_q \cdot \bar{y}_p \sum \sum x_p \cdot \bar{a}_{qp} \bar{y}_q = \langle x, Ay \rangle$$

Let us consider the characteristic polynomial $\det(\lambda \cdot E_n - A)$. If λ is a root there exists a non-zero eigenvector x such that $Ax = \lambda \cdot x$. Now (***) entails that

$$\lambda \cdot ||x||^2 = \langle Ax, x \rangle = \langle x, Ax \rangle = \bar{\lambda} \cdot ||x||^2$$

It follows that λ is *real*, i.e. the roots of the characteristic polynomial of a Hermitian matrix are always real numbers. If all roots are > 0 one say that the Hermitiain matrix is *positive.*

4.6 Unitary matrices. An $n \times n$-matrix U is called unitary if

$$\langle Ux, Ux \rangle = \langle x, x \rangle$$

hold for all $x \in \mathbf{C}^n$. The spectral theorem for Hermitian matrices asserts that if A is Hermitian then there exists a unitary matrix U such that

$$UAU^* = \Lambda$$

where Λ is a diagonal matrix whose elements are real.

4.7 The passage to infinite dimension.

Around 1900 the need for a spectral theorem in infinite dimensions became urgent. In his article *Sur une nouvelle méthode pour la resolution du problème de Dirichlet* from 1900, Ivar Fredholm extended earlier construction by Volterra and showed the importance to study systems of linear equations in an infinite number of variables with certain bounds. For this purpose Fredholm constructed infinite families of pairwise orthogonal functions attached to a concrete inner product space. His procedure was to regard a sequence of matrices $A_1, A_2, \ldots$ where A_n is an $n \times n$-matrix and an infinite dimensional vector space

$$V = \mathbf{R}e_1 + \mathbf{R}e_2 + \ldots$$

To each $N \geq 1$ we get the finite dimensional subspace $V_N = \mathbf{R}e_1 + \ldots \mathbf{R}e_N$. Now A_N is regarded as a linear operator on V_N and we assume that the A-sequence is matching , i.e. if $M > N$ then the restriction of A_M to V_N is equal to A_N. This means that we take any infinite matrix A_∞ with elements $\{a_{ik}\}$ and here A_N is the $N \times N$-matrix which appears as an upper block with N^2-elements $a_{ik} : 1 \leq i, k \leq N$. To each N we get the ellipsoid $\mathcal{E}_N = \mathcal{E}_{A_N}$ on V_N where it defines a norm. As N increases the norms are matching and hence V is equipped with a norm which for every $N \geq 1$ restricts to the norm defined by $\mathcal{E}_N$ on the finite dimensional subspace V_N. Notice that the norm of any vector $\xi \in V$ is finite since ξ belongs to V_N for some N, i.e. by definition any vector in V is a finite $\mathbf{R}$-linear combination of the basis vectors $\{e_\nu\}$. Moreover, the norm on V satisfies the bilinear rule from (0.3), i.e. on $V \times V$ there exists a bilinear form B such that

$$(*) \qquad ||x + y||^2 - ||x||^2 - ||y||^2 = 2B(x, y) \quad : \quad x, y \in V$$

Remark and an Exercise. Certain inequalities for determinants due to Hadamard play an important role in Fredholm's work and since the Hadamard inequalities are used in many other situations we announce some of his results, leaving proofs as an exercise or consult the literature. An excellent source is the introduction to integral equations by the former professor at Harvard University Maxime Bochner [Cambridge University Press: 1914):

4.8 Two inequalities. Let $n \geq 2$ and $A = \{a_{ij}\}$ some $n \times n$-matrix whose elements are real numbers. Show that if

$$a_{i1}^2 + \ldots + a_{in}^2 = 1 \quad : \quad 1 \leq i \leq n$$

then the determinant of A has absolute value ≤ 1. Next, assume that there is a constant M such that the absolute values $|a_{ij|}| \leq M$ hold for all pairs i, j. Show that this gives

$$\big|\det(A)\big| \leq \sqrt{n^n} \cdot M^n$$

4.9 The Hilbert space $\mathcal{H}_V$. This is the completition of the normed space V. That is, exactly as when the field of rational numbers is completed to the real number system one regards Cauchy sequences for the norm of vectors in V and in this way we get a normed vector space denoted by $\mathcal{H}_V$ where the norm topology is complete. Under this process the bi-linearity is preserved, i.e. on $\mathcal{H}_V$ there exists a bilinear form $B_{\mathcal{H}}$ such that (*) above holds for pairs $x, y \in \mathcal{H}_V$. Following Hilbert we refer to $B_{\mathcal{H}}$ as the *inner product* attached to the norm. Having performed this construction starting from any infinite matrix A_∞ it is tempting to make a further abstraction. This is precisely what Hilbert did, i.e. he ignored the "source" of a matrix A_∞ and defined a complete normed vector space over $\mathbf{R}$ to be a real Hilbert space if the there exists a bilinear form B on $V \times V$ such that (*) holds.

Remark. If V is a "abstract" Hilbert space the restriction of the norm to any finite dimensional subspace W is determined by an ellipsoid and exactly as in linear algebra one constructs an orthonormal basis on W. Following the Gram-Schmidt construction it follows that there exists an orthonormal sequence $\{e_n\}$ in V. However, in order to be sure that it suffices to take a *denumerable* orthonormal basis it is necessary and sufficient that the normed space V is *separable.* Assuming this it follows that every $v \in V$ has a unique representation

(i) $$v = \sum c_n \cdot e_n \quad : \quad \sum |c_n|^2 = ||v||^2$$

The existence of an orthonormal family therefore means that every separable Hilbert space is isomorphic to the standard space ℓ^2 whose vectors are infinite sequences $\{c_n\}$ where the square sum $\sum c_n^2 < \infty$. So in order to prove general results about separable Hilbert spaces it is sufficient to regard ℓ^2. However, the abstract notion of a Hilbert space turns out to be very useful since inner products on specific linear spaces appear in many different situations. For example, in complex analysis an example occurs when we regard the space of analytic functions which are square integrable on a domain or whose boundary values are square integrable. Here is the inner product is given in advance but it can be a highly non-trivial affair to exhibit an orthonormal basis.

4.10 Linear operators on ℓ^2. A bounded linear operator T from the complex Hilbert space ℓ^2 into itself is described by an infinite matrix $\{a_{p,q}\}$ whose elements are complex numbers. Namely, for each $p \geq 1$ we set

(i) $$T(e_p) = \sum_{q=1}^{\infty} a_{pq} \cdot e_q$$

For each fixed p we get

(ii) $$||T(e_p)||^2 = \sum_{q=1}^{\infty} |a_{pq}[^2$$

Next, let $x = \sum \alpha_\nu \cdot e_\nu$ and $y = \sum beta_\nu \cdot e_\nu$ be two vectors in ℓ^2. Then we get

$$||x + y||^2 = \sum |\alpha_\nu + \beta_\nu|^2 \cdot e_\nu$$

For each ν we have the pair of complex numbers α_ν, β_ν and here we have the inequality

$$|\alpha_\nu + \beta_\nu|^2 \leq 2 \cdot |\alpha_\nu|^2 + 2 \cdot |\beta_\nu|^2$$

It follows that

$$\text{(iii)} \qquad ||x + y||^2 \leq 2 \cdot ||x||^2 + 2 \cdot ||y||^2$$

In (iii) equality holds if and only if the two vectors x and y are linearly dependent, i.e. if there exists some complex number λ such that $y = \lambda \cdot x$. Let us now return to the linear operator T. In (ii) we get an expression for he norm of the T-images of the orthonormal basis vectors. So when T is bounded with some operator norm M then the sum of the squared absolute values in each row of the matrix $A = \{a_{p,q}\}$ is $\leq M^2$. However, this condition along is not sufficient to guarantee that T is a bounded linear operator. For example, suppose that the row vectors in T are all equal to a given vector in ℓ^2, i.e. $a_{p,q} = \alpha_q$ hold for all pairs where $\sum, |\alpha_q|^2 = 1$. Then

$$T(e_1 + \ldots + e_N) = N \cdot v \quad : \quad v = \sum \alpha_q \cdot e_q$$

The norm in the right hand side is N while the norm of $e - 1 + \ldots + e_n$ is $\sqrt{N}$. Since $N >> \sqrt{N}$ when n increases this shows that T cannot be bounded. So the condition on the matrix A in order that T is bounded is more subtle. In fact, given a vector $x = \sum \alpha_\nu \cdot e_\nu$ as above with $||x|| = 1$ we have

$$(*) \qquad ||T(x)||^2 = \sum_{p=1}^{\infty} \sum_q \sum_k a_{p,q} \cdot \alpha_q \cdot \bar{a}_{pk} \cdot \bar{\alpha}_k$$

So we encounter a rather involved triple sum. Notice also that for each fixed p we get a *non-negative* term

$$\rho_p = \sum_q \sum_k a_{p,q} \cdot \alpha_q \cdot \bar{a}_{pk} \cdot \bar{\alpha}_k = |\sum_{q=1}^{\infty} a_{pq} \cdot \alpha_q|^2$$

Final remark. Thus, the description of the Banach space $L(\ell^2, \ell^2)$ of all bounded linear operators on ℓ^2 is not easy to grasp. In fact, no "comprehensible" description exists of this space.

4.11 General results on Hilbert spaces.

Let $\mathcal{H}$ for a while be a real Hilbert space. A fundamental result is that if K is a closed convex subset of $\mathcal{H}$ and if $\xi \in \mathcal{H} \setminus K$, then there exists a unique $k_* \in K$ such that

$$(*) \qquad \min_{k \in K} ||\xi - k|| = ||\xi - k_*||$$

The proof is easy. For let ρ be the minimal distance. We find a sequence $\{k_n\}$ in K such that $||\xi - k_n|| \to \rho$. Now we shall prove that $\{k_n\}$ is a Cauchy sequence. To show this we let $\epsilon > 0$ and find first N_* such that

$$\text{(i)} \qquad ||\xi - k_n|| < \rho + \epsilon \quad : \quad n \geq N_*$$

The convexity of K implies that if $n, n \geq N_*$ then $\frac{k_n + k_m}{2} \in K$. Hence we have

$$\text{(ii)} \qquad \rho^2 \leq ||\xi - \frac{k_n + k_m}{2}||^2 \implies 4\rho^2 \leq ||(\xi - k_n) + (\xi - k_m]||^2$$

By the identity (**) the right hand side is

$$\text{(iii)} \qquad 2||\xi - k_n||^2 + 2||\xi - k_m||^2 - ||k_n - k_m||^2$$

It follows from (i-iii) that

$$||k_n - k_m||^2 \leq 4(\rho + \epsilon)^2 - 4\rho^2 = 8\rho \cdot \epsilon + 4\epsilon^2$$

Since ϵ can be made arbitrary small we conclude that $\{k_n\}$ is a Cauchy sequence and hence there exists a limit $k_n \to K_*$ where $k_* \in K$ since K is closed. Finally, the uniqueness of k_* is a direct consequence of (XX).

4.12 The decomposition theorem. Let V be a closed subspace of H. Its orthogonal complement is defined by

(i) $$V^\perp = \{x \in H \quad : \langle x, V \rangle = 0\}$$

It is obvious that $V^\perp$ is a closed subspace of H and that $V \cap V^\perp = 0$. There remains to prove the equality

(ii) $$H = V \oplus V^\perp$$

To see this we take some $\xi \in H \setminus V$. Now V is a closed convex set so we find v_* such that

(iii) $$\rho = ||\xi - v_*|| = \min_{v \in V} ||\xi - v||$$

If we prove that $\xi - v_* \in V^\perp$ we get (ii). To show this we consider some $\eta \in V$. If $\epsilon > 0$ we have

$$\rho^2 \leq ||\xi - v_* + \epsilon \cdot \eta||^2 = ||\xi - v_*||^2 + \epsilon^2 \cdot ||\eta||^2 + \epsilon \langle \xi - v_*, \eta \rangle$$

Since $||\xi - v_*||^2 = \rho^2$ and $\epsilon > 0$ it follows that

$$\langle \xi - v_*, \eta \rangle + \epsilon \cdot ||\eta||^2 \geq 0$$

here ϵ can be arbitrary small and we conclude that $\langle \xi - v_*, \eta \rangle \geq 0$. Using $-\eta$ instead we get the opposed inequality and hence $\langle \xi - v_*, \eta \rangle = 0$ as required.

4.13 Complex Hilbert spaces. On a complex vector space similar results as above hold provided that we regard convex sets which are **C**-invariant. We leave details to the reader and refer to the literature for a more detailed account about general properties on Hilbert spaces. See for example the text-book [Hal] by P. Halmos - a former student to J. von Neumann - which in addition to theoretical results contains many interesting exercises.

4:B. Eigenvalues of matrices.

Using the Hermitian inner product on $\mathbf{C}^n$ we establish results about eigenvalues of an $n \times n$-matrices A with complex elements. The spectrum $\sigma(A)$ is the n-tuple of roots $\lambda_1, \ldots, \lambda_n$ of the characteristic polynomial $P_A(\lambda) = \det(\lambda \cdot E_n - A)$, where eventual multiple eigenvalues are repeated. We also have the Hermitian matrix A^*A. Recall from (*) that $\sigma(A^*A)$ consists of non-negative real numbers denoted by $\{\mu_k\}$ which are arranged so that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. In particular one has

(1) $$\mu_1 = \max_{|x|=1} \langle Ax, Ax \rangle$$

4:B.1 Polarisation. Let A be an arbitrary $n \times n$-matrix. Then there exists a unitary matrix U such that the matrix U^*AU is upper triangular. To prove this we first use the wellknown fact that there exists a basis $\xi_1, \ldots, \xi_n$ in $\mathbf{C}^n$ in which A is upper triangular, i.e.

$$A(\xi_k) = a_{1k}\xi_1 + \ldots a_{kk}\xi_k \quad : \ , 1 \leq k \leq n$$

The *Gram-Schmidt orthogonalisation* gives an orthonormal basis $e_1, \ldots, e_n$ where

$$\xi_k = c_{1k} \cdot e_1 + \ldots c_{kk} \cdot e_k \quad \text{for each} \quad 1 \leq k \leq n$$

Let U be the unitary matrix which sends the standard basis in $\mathbf{C}^n$ to the ξ-basis. Now the reader can verify that the linear operator U^*AU is represented by an upper triangular matrix in the ξ-basis.

A theorem by H. Weyl. Let $\{\lambda_k\}$ be the spectrum of A where the λ-sequence is chosen with non-increasing absolute values, i.e. $|\lambda_1| \geq \ldots \geq |\lambda_n|$. With these notations the following holds for an arbitrary $n \times n$-matrix A:

B.2 Theorem. *For every $1 \leq p \leq n$ one has the inequality*

$$|\lambda_1 \cdots \lambda_p| \leq \sqrt{\mu_1 \cdots \mu_p}$$

Before we begin the proof for a general p we consider the special case $p = 1$ and prove:

B.3 Proposition. *One has the inequality*

$$|\lambda_1| \leq \sqrt{\mu_1}$$

Proof. Since λ_1 is an eigenvalue there exists a vector x_* with $|x_*| = 1$ so that $A(x_*) = \lambda_1 \cdot x_*$. It follows from (1) above that

$$\mu_1 \geq \langle A(x_*), A(x_*) \rangle = |\lambda_1|^2$$

Remark. The inequality is in general strict. Consider the 2×2-matrix

$$A = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

where $0 < b < 1$ and $a \neq 0$ some complex number which gives

$$A^*A = \begin{pmatrix} 1 & a \\ a & a^2 + b^2 \end{pmatrix}$$

Here $\lambda_1 = 1$ and the eigenvector $x_* = e_1$ and we see that $\langle A(x_*), A(x_*) \rangle = 1 + |a|^2$.

Proof of Weyl's theorem. The proof employs a construction of independent interest. Let $e_1, \ldots, e_n$ be some orthonormal basis in $\mathbf{C}^n$. For every $p \geq 2$ we get the inner product space V^p whose vectors are

$$v = \sum c_{i_1, \ldots, i_p} \cdot e_{i_1} \wedge \ldots \wedge e_{i_p}$$

where the sum extends over p-tuples $1 \leq i_1 < \ldots < i_p$. This is an inner product space of dimension $\binom{n}{p}$ where $\{e_{i_1} \wedge \ldots \wedge e_{i_p}\}$ is an orthonormal basis. Next, consider a linear operator A on $\mathbf{C}^n$ which in the e-basis is represented by a matrix with elements

$$a_{ik} = \langle Ae_i, e_k\rangle$$

If $p \geq 1$ we define the linear operaror $A^{(p)}$ on $V^{(p)}$ by

$$A^{(p)}(e_{i_1} \wedge \ldots \wedge e_{i_p}) = A(e_{i_1}) \wedge \ldots \wedge A(e_{i_p}) =$$

$$\sum a_{j_1 i_1} \cdots a_{j_p i_p} \cdot e_{j_1} \wedge \ldots \wedge e_{j_p}$$

with the sum extended over all $1 \leq j_1 < \ldots < j_p$.

Exercise. Show that the eigenvalues of $A^{(p)}$ consists of the $\binom{n}{p}$-tuple given by the products

(*) $$\lambda_{i_1} \cdots \lambda_{i_m} \quad : \quad 1 \leq i_1 < \ldots < i_p \leq n$$

Hint. First, the eigenvalues are independent of the chosen orthonormal basis $e_1, \ldots, e_n$ since a change of this basis gives another orthonormal basis in $V^{(p)}$ which does not affect the eigenvalues of $A^{(p)}$. Next, using a Polarisation we may assume from the start that A is an upper triangular matrix and at this stage the reader can verify (*).

Final part of the proof. If $p \geq 2$ it is clear that one has the equality

(i) $$(A^{(p)})^* \cdot A^{(p)} = (A^* \cdot A)^{(p)}$$

At this stage the reader can apply the Exercise and finish the proof of Weyl's theorem.

B.4 An inequality by Pick. Let C be a skew-symmetric $n \times n$-matrix, i.e. here $C^* = -C$. Notice that it implies that the eigenvalues of C are pure imaginary. Denote by g the maximum of the absolute values of the matrix elements of C. With these notations we have

B.5 Theorem. *One has the inequality*

$$\max_{|x|=1} \big|\langle Ax, x\rangle\big| \leq g \cdot \cot(\frac{\pi}{2n}) \cdot \sqrt{n(n-1)/2}$$

Proof. Since g is unchanged if we permute the columns of the given A-matrix it suffices to prove (*) for a vector x of unit length such that

(1) $$\mathfrak{Im}(x_i \bar{x}_k - x_k \bar{x}_i) \geq 0 \quad : \quad 1 \leq i < k \leq n$$

It follows that

$$\langle Ax, x\rangle = \sum\sum a_{ik} x_i \bar{x}_k = \sum_{i<k} a_{ik} x_i \bar{x}_k + \sum_{i>k} a_{ik} x_i \bar{x}_k$$

where the last equality simply follows when i and k are interchanged in the last sum on the first line. Since A is skew-symmetric the last term becomes

$$\sum_{i<k} a_{ik}[x_i \cdot \bar{x}_k - \bar{x}_i \cdot x_k]$$

B.5 Results by A. Brauer.

Let A be an $n \times n$-matrix. To each $1 \leq k \leq n$ we set

$$r_k = \min\,[\sum_{j \neq k} |a_{jk}| : \sum_{j \neq k} |a_{kj}|\,]$$

B.6 Theorem. *Denote by C_k the closed disc of of radius r_k centered at the diagonal element a_{kk}. Then one has the inclusion:*

(*) $$\sigma(A) \subset C_1 \cup \ldots \cup C_n$$

Proof. Consider some eigenvalue λ so that $Ax = \lambda \cdot x$ for a non-zero eigenvector. It means that

$$\sum_{j=1}^{j=n} a_{j\nu} \cdot x_\nu = \lambda \cdot x_j \quad : \quad 1 \leq j \leq n$$

Choose k so that $|x_k| \geq |x_j|$ for all j. Now we have

$$(1) \qquad (\lambda - a_{kk}) \cdot x_k = \sum_{j \neq k} a_{j\nu} \cdot x_\nu \implies |\lambda - a_{kk}| \leq \sum_{j \neq k} |a_{kj}|$$

At the same time the adjoint A^* satisfies $A^*(x) = \bar{\lambda} \cdot x$ which gives

$$\sum_{j=1}^{j=n} \bar{a}_{\nu,j} \cdot x_\nu = \bar{\lambda} \cdot x_j \quad : \quad 1 \leq j \leq n$$

Exactly as above we get

$$(2) \qquad |\lambda - a_{kk}| = |\bar{\lambda} - \bar{a}_{kk}| \leq \sum_{j \neq k} |a_{jk}|$$

Hence (1-2) give the inclusion $\lambda \in C_k$.

B.7 Theorem. *Assume that the closed discs $C_1, \ldots, C_n$ are disjoint. Then the eigenvalues of A are simple and for every k there is a unique $\lambda_k \in C_k$.*

Proof. Let D be the diagonal matrix where $d_{kk} = a_{kk}$. For ever $0 < s < 1$ we consider the matrix

$$B_s = sA + (1-s)D$$

Here $b_{kk} = a_{kk}$ for every k and the associated discs of the B-matrix are $C_1(s), \ldots, C_b(s)$ where $C_k(s)$ is again centered at a_{kk} while the radius is $s \cdot r_k$. When $s \simeq 0$ the matrix $B \simeq D$ and then it is clear that the previous theorem implies that B_s has simple eigenvalues $\{\lambda_k(s)\}$ where $\lambda_k(s) \in C_k(s)$ for every k. Next, since the "large discs" $C_1, \ldots, C_n$ are disjoint, it follows by continuity that these inclusions holds for every s and with $s = 1$ we get the theorem.

Exercise. Assume that the elements of A are all real and the discs above are disjoint. Show that the eigenvalues of A are all real.

B.8 Results by Perron and Frobenius

Let $A = \{a_{pq}\}$ be a matrix where all elements are real and positive. Denote by Δ^n_+ the standard simplex of n-tuples $(x_1, \ldots, x_n)$ where $x_1 + \ldots + x_n = 1$ and every $x_k \geq 0$. The following result was originally established by Perron in [xx]:

B.9 Theorem. *There exists a unique $\mathbf{x}^* \in \Delta^n_+$ which is an eigenvector for A with an eigenvalue s^*. Moreover. $|\lambda| < s^*$ holds for every other eigenvalue.*

Remark. We refer to $\mathbf{x}^*$ as the Perron vector of A. The proof below is more or less verbatim to Perron's original proof except that we employ the more recent vocabulary of a spectral radius. In [Frob] a proof is given by an induction over n based upon a calculus with loc.cit] the following addendum to Theorem B.9 is proved.

B.10 Theorem. *Let A as above be a positive matrix which gives the eigenvalue s^*. For every complex $n \times n$-matrix $B = \{b_{pq}\}$ such that $|b_{pq}| \leq a_{pq}$ hold for all pairs p, q, it follows that every root of $P_B(\lambda)$ has absolute value $\leq s^*$ and equality holds if and only if $B = A$.*

Proof of Theorem B.9. First we establish the existence part. In $\mathbf{C}^n$ we have the norm defined by

$$\text{(i)} \qquad ||y|| = |y_1| + \ldots + |y_n|$$

Next, for each $\mathbf{x} \in \Delta^n_+$ we set

$$\phi(\mathbf{x}) = s \quad \text{where} \quad s = \max_{\xi > 0} \text{ such that } \quad \sum a_{pq} \cdot x_q \leq \xi \cdot x_p \quad : 1 \leq p \leq n$$

It is clear that ϕ is a continuous function on Δ_+^n and hence it takes is maximum at some point $\mathbf{x}^*$. Next, let $\lambda \in \Sigma(A)$ have a maximal absolute value and let $\mathbf{y}$ be an eigenvector of norm one. The triangle inequality gives

$$||A(\mathbf{y})|| = \sum_{p=1}^{p=n} \big|\sum_{q=1}^{q=n} a_{pq} \cdot y_q\big| \leq \sum_{p=1}^{p=n} \sum_{q=1}^{q=n} a_{pq} \cdot |y_q| \leq s^* \cdot ||y|| = s^*$$

Hence we have the inequality

$$\text{(ii)} \qquad s^* \geq |\lambda| \quad \text{for all} \quad \lambda \in \sigma(A)$$

On the other hand we notice that if $N \geq 2$, then the definition of s^* gives

$$A^N(\mathbf{x}^*) \leq (s^*)^N$$

It follows that

$$\text{(iii)} \qquad s^* \leq \lim_{N \to \infty} [||A^N||]^{\frac{1}{N}}$$

We conclude that equality holds in (ii) and that s^* must be an eigenvalue for A which gives an eigenvector $\mathbf{x}^* \in \Delta_+^n$.

The uniqueness.

There remains to prove that $\mathbf{x}^*$ is unique, or equivalently that the ϕ-function above attains its maximum at a unique point on Δ_+^n. Notice that this also is equivalent to the condition that s^* is a *simple root* of $P_A(\lambda)$. NOW prove it by the Frobenius method - to illustrate algebraic methods.

B.11 The case of probability matrices. Le A have positive elements and assume that the sum in every column is one. In this case $s^* = 1$ for with $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ we have

$$s^* = s^* \cdot \sum x_p^* = \sum \sum a_{pq} \cdot x_q^* = \sum x_q^* = 1$$

The components of the Perron vector $\mathbf{x}^*$ yields the probabilities to arrive at a station q after many independent motions in the associated stationary Markov chain where the A-matrix defines the transition probabilities.

Example. Let $n = 2$ and take $a_{11} = 3/4$ and $a_{21} = 1/4$, while $a_{12} = a_{22} = 1/2$. A computation gives $s^* = 2/3$ which in probabilistic terms means that the asymptotic probability to arrive at station 1 after many steps is $2/3$ while that of station 2 is $1/3$. Here we notice that the second eigenvalue is $s_* = 1/4$ and an associated eigenvector is $(1, -2)$.

5. Dual vector spaces

Let X be a normed space over the complex field. A continuous linear form on X is a **C**-linear map γ from X into **C** such that there exists a constant C with:

(iii) $$\max_{||x||=1} |\gamma(x)| \leq C$$

The set of these continuous linear forms is denoted by X^*. It is obvious that X^* is a vector space and that (iii) defines a norm on X^*. Moreover, since Cauchy-sequences of complex numbers converge it follows that X^* is a Banach space. Notice that this holds even if X from the start is not complete. One refers to X as the dual of X. Next, let Y be a subspace of X. Every $\gamma \in X^*$ can be restricted to Y and gives an element of Y^*, i.e. we have the restriction map

(i) $$\mathfrak{res}_Y : X^* \to Y^*$$

Since a restricted linear form cannot increase the norm on X one has the trivial inequality

$$||\mathfrak{res}_Y(\gamma)| \leq ||\gamma|| \quad : \quad \gamma \in X^*$$

The kernel of $\mathfrak{res}_Y$. The kernel is by definition the set of X^*-elements which are zero on Y. This is a closed subspace of X^* which can be identfied with the dual of a new normed space. Namely, consider the linear quotient space

$$Z = \frac{X}{Y}$$

Thus, elements in Z are images of vectors $x \in X$. Here two vectors x_1 and x_2 give the same vector in Z if and only if $x_2 - x_1 \in Y$. Let $\pi_Y(x)$ denote the image of $x \in X$. Now Z is is equipped with a norm defined by

$$||z|| = \min\, ||x|| \quad : \quad z = \pi_Y(x)$$

This gives a norm on Z and by he construction above one has a canonical isomorphism

$$Z^* \simeq \mathrm{Ker}(\mathfrak{res}_y)$$

Thus, the dual space Z^* can be identified with a closed subspace of X^*.

5.1 The Hahn-Banch Theorem. It asserts that every continuous linear form on a subspace Y of X has a *norm preserving extension* to a linear form on X. Thus, if $\gamma_* \in Y^*$ has some norm C, then there exists $\gamma \in X$ such that

$$\mathfrak{res}_Y(\gamma) = \gamma_*$$

One refers to γ^* as a norm-åreserving extension of γ.

5.2 Exercise. Consult a text-book for the proof or give alternatively the details using the following hint. Given the pair (Y, γ_*) we consider all pairs (Z, ρ) where $Y \subset Z \subset X$ and $\rho \in Z^*$ is such that its norm is C and $\rho|Y = \gamma_*$.Thus, ρ is a norm preserving extension of γ_* to Z. By *Zorn's Lemma* there exists a maximal pair (Z, ρ) in this family. There remains only to show that $Z = X$ for then ρ gives the required norm-preserving extension. To prove that $Z = X$ one argues by contradiction. Namely, suppose $Z \neq X$ and choose a vector $x_0 \in X \setminus Z$ of norm one. Next, if α is a complex number we get a linear form on $Z^* = Z + \mathbf{C} \cdot x_0$ defined by

$$\rho_\alpha(ax_0 + z) = a \cdot \alpha + \rho(z)$$

where $a \in \mathbf{C}$ and $z \in Z$ are arbitrary. The contradiction follows if we can find α so that the norm of ρ_α again is $\leq C$. It is clear that $\rho_\alpha|| \leq C$ holds if and only if

(*) $$|\alpha + \rho(z)| \leq C \cdot ||x_0 + z|| \quad \text{hold for all } z \in Z$$

At this stage the reader should be able to finish the proof.

5.3 A separation theorem. Above we studied complex normed spaces. We can also consider a normed space X over the real numbers in which case the dual X^* consists of bounded **R**-linear forms. Let us now consider some closed and convex subset K of X. Then, if $p \in X \setminus K$ is outside K there exists $x^* \in X^*$ which separates p from K in the sense that there is some $\delta > 0$ so that

$$x^*(p) \geq \delta + x^*(x) \quad \text{for all} \quad x \in K$$

An exact sequence. Let $Y \subset X$ as above be a subspace and $Z = \frac{X}{Y}$ the quotient space. The Hahn-Banach Theorem shows that there exists an exact sequence

$$0 \to Z^* \to X^* \to Y^* \to 0$$

Moreover, the restriction map $X^* \to Y^*$ sends the unit ball in X^* onto the unit ball of Y^*, Notice that this precisely is the assertion of the Hahn-Banach Theorem.

An example Let X, Y, Z be as above and consider some $\gamma \in X^*$. Now γ can be restricted to Y where we get the norm

$$A = ||\mathfrak{res}_Y(\gamma)||$$

By the Hahn-Banach theorem there exists some $\hat{\gamma} \in X^*$ of norm A whose image under $\mathfrak{res}_Y$ is the same as for γ. Identifying Z^* with a subspace of X^* this means that

$$\gamma - \hat{\gamma} \in Z^*$$

Let us give a specific example which will be applied to Hardy spaces. Here $X = L^1(T)$ is the normed space of integrable functions on the unit circle. Recall that the dual space $X^* = L^\infty(T)$. Next, we have the subspace $H^\infty(T)$ of X^* of those Lebesgue measurable and bounded functions on T which are boundary values to analytic functions in the unit disc D. We have also the subspace $Y = H_0^1(T)$ of L^1-functions which are boundary values of analytic functions which are zero at the origin. As explained in XXX expansions in Fourier series show that if $g \in L^\infty(T)$ then

$$\int_0^{2\pi} g \cdot h \cdot d\theta = 0 \quad \text{for all} \quad f \in H_0^1$$

holds if and only if $g \in H^\infty(T)$. This means precisely that

$$H^\infty(T) = \text{Ker}(\mathfrak{res}_Y) \quad : Y = H_0^1(T) \subset L^1(T) = X$$

Consider now some $g \in L^\infty(T)$ and define the constant C by:

$$C = \max_h |\int_0^{2\pi} g \cdot h \cdot d\theta \quad : \quad h \in H_0^1(T) \text{ and } ||h||_1 = 1$$

The general result above applies and gives the existence of some $h \in H^\infty(T)$ such that the L^∞-norm norm $||g - h||_\infty = C$. Thus, we have

$$g = h + f \tag{1}$$

where $h \in H^\infty$ and the L^∞-function f has norm C. Notice that $||g||_\infty \geq C$ holds here. The norm of h is not determined because one may have several decompositions in (1). However, in XX we shall find a specific decompositions of g in certain cases.

5.4 Weak Convergence.

Let X be a normed space. On the dual X^* one can define a topology where convergence only has to be pointwise. It means that a fundamental system of open neighborhood of the origin in the vector space X^* is given by

$$U(x_1, \dots, x_N; \epsilon) = \{\gamma \in X^* \quad : |\gamma(x_\nu)| < \epsilon \quad : x_1, \dots, x_N \text{ finite set}\} \tag{*}$$

Notice that each such U-set is a convex subset of X^*. Let Y be the finite dimensional subspace of X generated by $x_1, \dots, x_n$. Then it is clear that the kernel of $\mathfrak{res}_Y$) is contained in the U-set above.

If k is the dimension of the vector space generated by $x_1, \dots, x_n$ then Linear Algebra implies that the kernel of $\mathfrak{res}_Y$) has codimension k in X. So the U-set in (*) contains a subspace of X^* with finite codimension, i.e. roughly speaking the open U-set in X^* is quite large.

5.5 The case when X is separable. Suppose that a sequence $x_1, x_2, \ldots$ is a dense subset of X. Let $B(X^*)$ denote the unit ball in X^*, i.e. elements $\gamma \in X^*$ of norm ≤ 1. On this unit ball we define a metric by

$$d(\gamma_1, \gamma_2) = \sum_{n=1}^{\infty} 2^{-n} \cdot \big|\gamma_1(x_n) - \gamma_2(x_n)\big|$$

Exercise. Verify that the metric above gives the induced weak topology on $B(X^*)$ defined via the U-sets in (*).

5.6 Theorem. *The metric space $B(X^*)$ is compact.*

Proof. Let $\{\gamma_k\}$ be a sequence in $B(X^*)$. To every j we get the bounded sequence of complex numbers $\{\gamma_k(x_j)\}$. By the wellknown diagonal construction there exists a strictly increasing sequence $k_1 < k_2 < \ldots$ such that if $\rho_\nu = \gamma_{k_\nu}$ then

(*) $$\lim_{\nu \to \infty} \rho_\nu(x_j)$$

exists for every j. Since every ρ_j has norm ≤ 1 and $\{x_j\}$ is dense in X it follows that

$$\lim_{\nu \to \infty} \rho_\nu(x) \quad \text{exist for all } x \in X$$

This gives some $\rho \in X^*$ such that $\rho(x)$ is the limit value above for every x. It is clear that the norm of ρ is ≤ 1 and by the construction of the distance function on $B(X^*)$ we have:

$$\lim_{\nu \to \infty} d(\rho_\nu, \rho) = 0$$

This proves that the given γ-sequence contains a convergent subsequence. So by the definition of compact metric spaces Theorem 6.1 follows.

5.7 Weak hulls in X^*. Let X be separable and choose a denumerable and dense subset $\{x_n\}$. Examples show that in general the dual space X^* is no longer separable in its norm topology. However, there always exists a denumerable sequences $\{\gamma_k\}$ in X^* which is dense in the weak topology.

Exercise. For every $N \geq 1$ we let V_N be the finite dimensional space generated by $x_1, \ldots, x_N$. It has dimension N at most. Applying the Hahn-Banch theorem the reader finds a sequence $\gamma_1, \gamma_2, \ldots$ in X^* such that for every N the restricted linear forms

$$\gamma_\nu | V_N \quad 1 \leq \nu \leq N$$

generate the dual vector space V_N^*. Next, let Q be the field of rational numbers. Show that if Γ is the subset of X^* formed by all finite Q-linear combinations of the sequence $\{\gamma_\nu\}$ then this denumerable set is dense in X^* with respect to the weak topology.

Another exercise. Let X be a separable Banach space and let E be a subspace of X^*. We say that E point separating if there to every $0 \neq x \in X$ exists some $e \in E$ such that $e(x) \neq 0$. Show first that every such point-separating subspace of X^* is dense with respect to the weak topology. This is the easy part of the exercise. The second part is less obvious. Namely, put

$$B(E) = B(X^*) \cap E$$

Prove now that $B(E)$ is a dense $B(X^*)$. Thus, if $\gamma \in B(X^*)$ then there exists a sequence $\{e_k\}$ in $B(E)$ such that

$$lim_{k \to \infty} e_k(x) = \gamma(x)$$

hold for all $x \in X$.

5.8 Example from integration theory. An example of a separable Banach space is $X = L^1(\mathbf{R})$ whose elements are Lebesgue measurable functions $f(x)$ for which the L^1-norm

$$\int_{-\infty}^{\infty} |f(x)| \cdot dx < \infty$$

If $g(x)$ is a bounded continuous functions on $\mathbf{R}$, i.e. there is a constant M such that $|g(x)| \leq M$ for all x, then we get a linear functional on X defined by

$$g^*(f) = \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx < \infty$$

Let E be the linear space of all bounded and continuous functions. By the previous exercise it is a dense subspace of X^* with respect to the weak topology. Moreover, by the second part of the exercise it follows that if $\gamma \in X^*$ has norm one, then there exists a sequence of continuous functions $\{g_n\}$ of norm one at most such that $g_\nu \to \gamma$ holds weakly. Let us now find γ. For this purpose we define the functions

$$\text{(i)} \qquad G_n(x) = \int_0^x g_n(t) \cdot dt \quad : \quad x \geq 0$$

These primitive functions are continuous and enjoy a further property. Namely, since the maximum norm of every g-function is ≤ 1 we see that

$$\text{(ii)} \qquad |G_n(x) - G_n(x')| \leq |x - x'| \quad : \quad x, x' \geq 0$$

This is means that whenever $a > 0$ is fixed, then the sequence $\{G_n\}$ restricts to an *equi-continuous* family of functions on the compact interval $[0, a]$. Moreover, for each $0 < x \leq a$ since we can take $f \in L^1(\mathbf{R})$ to be the characteristic function on the interval $[0, x]$, the weak convergence of the g-sequence implies that there exists the limit

$$\text{(iii)} \qquad \lim_{n \to \infty} G_n(x) = G_*(x)$$

Next, the equi-continuity in (ii) enable us to apply the classic result due to C. Arzéla in his paper *Intorno alla continua della somma di infinite funzioni contionue* from 1883 and conclude that the point-wise limit in (iii) is uniform. Hence the limit function $G_*(x)$ is continuous on $[0, a]$ and it is clear that G_* also satisfies (ii), i.e. it is Lipschitz continuous of norm ≤ 1. Since the passae to the limit can be carried out for every $a > 0$ we conclude that G_* is defined on $[0, +\infty >)$. In the same way we find G_* on $(-\infty, 0]$. Next, by the result in [XX-measure] there exists the Radon-Nikodym derivative $G_*(x)$ which is a bounded measurable function $g_*(x)$ whose maximum norm is ≤ 1. So then

$$G_*(x) = \int_0^x g_*(t) \cdot dt = \lim_{n \to \infty} G_n(x) = \lim_{n \to \infty} \int_0^x g_n(t) \cdot dt$$

holds for all x. Since finite $\mathbf{C}$-linear sums of characteristic functions is dense in $L^1(\mathbf{R})$ we conclude that the limit functional γ is determined by the L^∞-function g_*. So this shows the equality

$$L^1(\mathbf{R})^* = L^\infty(\mathbf{R})$$

Remark. The result above is of course wellknown. But it is interesting to see how the last duality formula is derived from studies of the Lebesgue integral.

5.9 The weak topology on X. Let X be a Banach space. Here we do not assume that X is separable. A sequence $\{x_k\}$ in X converges weakly to a limit vector x if

$$\lim_{k \to \infty} x^*(x_k) = x^*(x) \quad \text{hold for all } x^* \in X^*$$

Exercise. Show that when $\{x_k\}$ is a weakly convergent sequence then it must be bounded, i.e. there exists a constant C such that

$$||x_k|| \leq C$$

hold for all k.

5.10 Weak versus strong convergence. A weakly convergent sequence need not be strongly convergent. An example is when $X = C^0[0, 1]$ is the Banach space of continuous functions on the closed unit interval. By the Riesz representation theorem the dual space X^* consists of Riesz measures. A sequence $\{x_n(t)\}$ of continuous functions converge weakly to zero if

$$\lim_{n\to\infty}\int_0^1 x_n(t)\cdot d\mu(t)=0$$

hold for every Riesz measure μ. By the result from [Measure] this holds if and only if the maximum norms of the x-functions are uniformly bounded and the sequence converges pointwise to zero. We can construct many such pointwise convergent sequences which fail to converge in the maximum norm. So in this example the weak convergence is *strictly weaker* than the topology defined by the maximum norm on X.

5.11 Remark. The example below was not so special. Namely, if X is an arbitrary infinite dimensional Banach space then the norm-topology is always strictly stronger than the weak topology. The proof is very easy for by the definition of the weak topology an equality with the norm topology implies that there exists a finite subset $x_1^*,\ldots,x_N^*$ of X^* and a constant C such that one has the implication

$$\max_\nu |x_\nu^*(x)|<C \implies ||x||<1 \quad \text{for all} \quad x\in X$$

But then it is clear that X^* as a complex vector space is generated by the n-tuple $x_1^*,\ldots,x_N^*$ and via the Hahn-Banach theorem it follows that X has dimension N at most.

5.12 Final remark. Much more could have been said about topologies on X and its dual. For example, we have not defined *reflexive* Banach spaces which are characterised by the condition that the natural map from X into its bi-dual X^{**} is surjective. Other results deal with various separation theorems. Here a major result asserts that if K is a closed and convex set in X then it is also closed with respect to the weak topology on X. We refer the reader to the very extensive literature which treats functional analysis. The outstanding reference for the foundations in functional analysis is the text-book Linear Operators Volume 1 by Dunford and Schwarz which covers all essential results in functional analysis. with elegant and very detailed proofs, including very many instructive exercises and an extensive list of references covering all discoveries before 1960.

6. Fredholm theory.

Introduction. We prove some general results about bounded operators which go back to Ivar Fredholm's article [Fredholm] about integral equations from 1901. Fredholm's work was restricted to special Banach spaces but the proofs in the general case are verbatim except for Theorem 6.12 which in this generality is attributed to Laurent Schwartz. Here is the set-up for this section. Let X and Y be two Banach spaces. Their dual spaces are denoted by X^* and Y^*.

6.1 Adjoint operators. Let $u\colon X \to Y$ be a bounded linear operator . The adjoint u^* is the linear operator from Y^* to X^* defined by

(1) $$u^*(y^*)\colon\, x \mapsto y^*(u(x)) \quad : \quad y^* \in Y \quad : x \in X$$

Exercise. Show the equality of operator norms:

$$||u|| = ||u^*||$$

The hint is to apply the Hahn-Banach theorem.

6.2 Operators with finite dimensional range. The range is the image space $u(X)$. Suppose the range is finite dimensional and let N be the dimension of the vector space $u(X)$. Then we can choose an N-tuple $x_1, \ldots, x_N$ in X such that the vectors $\{u(x_k)\}$ is a basis for $u(X)$. Notice that this implies that $x_1, \ldots, x_N$ are linearly independent in X. Hence we get the N-dimensional subspace of X :

$$V = \mathbf{C}x_1 + \ldots + \mathbf{C}x_N$$

Consider the u-kernel

$$N_u = \{x \quad : \; u(x) = 0\}$$

The reader should verify that

$$X = N_u \oplus V$$

Next, consider the adjoint operator u^*. In Y^* we can find an N-tuple $y_1^*, \ldots, y_N^*$ such that

$$j \neq k \implies y_j^*(u(x_k)) = 0 \quad \text{and} \quad y_j^*(u(x_j)) = 1$$

If N_{u^*} is the kernel of u^* the reader may verify that

$$Y^* = N_{u^*} \oplus \mathbf{C}y_1^* + \ldots + \mathbf{C}y_N^*$$

Conclude from this that the range of u^* also is an N-dimensional vector space.

6.3 The operator $\bar{u}$. Let $u\colon X \to Y$ which gives the closed null space N_u in X. Now $\frac{X}{N_u}$ is a new Banach space and we get the induced linear operator

$$\bar{u}\colon\, \frac{X}{N_u} \to Y$$

By construction $\bar{u}$ is an *injective* linear operator and it is clear that it has the same range as u, i.e. one has the equality

(2) $$u(X) = \bar{u}(\frac{X}{N_u})$$

6.4 The image of u^*. In the dual space X^* we get the subspace

(3) $$N_u^\perp = \{x^* \in X^* \quad : \; x^*(N_u) = 0\}$$

Next, let $y^* \in Y^*$ and consider the image $u^*(y^*)$. If $x \in N_u$ we have by (1)

$$u^*(y^*)(x) = y^*(u(x)) = 0$$

Hence we get the following inclusion:

(4) $$u^*(Y^*) \subset N_u^\perp$$

Next, recall the canonical isomorphism

$$[\frac{X}{N_u}]^* \simeq N_u^\perp \tag{5}$$

Now we consider the linear operator $\bar{u}$ whose adjoint $\bar{u}^*$ maps Y^* into the dual of $\frac{X}{N_u}$. Using the canonical isomorphism (5) this means that

$$\bar{u}^*\colon\ Y^* \mapsto N_u^\perp$$

At this stage the reader should verify the equality

$$\text{Im}(\bar{u}^*) = \text{Im}(u^*) \tag{6}$$

The closed range property

A bounded linear operator $u\colon X \to Y$ has closed range if $u(X)$ is a closed subspace of Y. Suppose this holds. By the constructions above the linear operator $\bar{u}$ is injective and its image space is $u(X)$. So when $u(X)$ is closed it follows that $\bar{u}\colon X \to u(X)$ is a bijective map between Banach spaces. The Open Mapping Theorem applies and implies that $\bar{u}$ is an invertible linear operator between $\frac{X}{N_u}$ and $u(X)$. Passing to its adjoint we get a bijective and bi-continuous map

$$\bar{u}^*\colon\ u(X)^* \to N_u^\perp \tag{i}$$

where $N_u^\perp$ is identified with the dual of $\frac{X}{N_u}$. Using the equality (6) above we conclude that the image space

$$u^*(Y^*) = N_u^\perp$$

Here $N_u^\perp$ is a closed subspace of X^* and hence we have proved:

6.5 Proposition. *Assume that u has closed range. Then u^* has closed range and one has the equality*

$$\text{Im}(u^*) = N_u^\perp$$

A converse result. Let $u\colon X \to Y$ be a bounded linear operator. But this time we do not assume that it has a closed range from the start. Instead we assume that the adjoint operator u^* has a closed range. By the equality (xxx) this implies that the injective linear operator $\bar{u}$ is such that its adjoint $\bar{u}^*$ has closed range. Using this the reader should verify the following converse to Proposition 6.5.

6.6 Proposition. *If u^* has closed range it follows that u also has closed range.*

6.7 Compact operators.

A linear operator $T\colon X \to Y$ is compact if the the image under T of the unit ball in X is relatively compact in Y. In other words, compactness means that if $\{x_k\}$ is an arbitrary sequence in the unit ball $B(X)$ then there exists a subsequence of $\{T(x_k)\}$ which converges to some $y \in Y$. Next, let $\{T_n\}$ be a sequence of compact operators which converge to another operator T, i.e.

$$\lim_{n\to\infty} ||T_n - T|| = 0$$

where we employ the operator norm on the Banach space $L(X,Y)$. Then the reader may verify that T also is a compact operator.

6.8 Operators with finite-dimensional range. If $T\colon X \to Y$ is such that $T(X)$ is a finite dimensional subspace of Y then it is easily seen that T is compact. Denote by $\mathcal{F}(X,Y)$ the linear space of operators from X to Y with finite dimensional range. So now $\mathcal{F}(X,Y)$ is a subspace of the linear space $\mathcal{C}(X,Y)$ of all compact operators.

6.9 Enflo's example. The question arises if $\mathcal{F}(X,Y)$ is a dense subspace of $\mathcal{C}(X,Y)$. This was an open problem for many decades until Per Enflo in a seminar at Stockholm University in 1972 constructed an example of a separable Banach space X and a compact operator $T \in \mathcal{C}(X,X)$ which cannot be approximated by operators from $\mathcal{F}(X,X)$. This example has led to a veritable industry where one seeks to determine "good pairs" of Banach spaces X and Y for which $\mathcal{F}(X,Y)$ is dense in $\mathcal{C}(X,Y)$. We shall not dwell upon this but remark only that for most of the "familiar" Banach spaces one has the density which therefore means that a compact operator can be approximated in the operator norm by operators having finite dimensional range.

Before we announce the result below we notice that if T belongs to $\mathcal{F}(X,Y)$ then it has closed range. Indeed, this follows since every finite dimensional subspace of Y is closed. Moreover, the reader should verify that the image space $T^*(Y^*)$ also is finite dimensional and hence the adjoint T^* is a compact operator. However, taking Enflo's example into the account this special case does not cover the result below.

6.10 Theorem. *Let T be compact. Then the adjoint T^* is also compact.*

Exercise. Prove this result.

6.11 Stable range.

Now we study compact pertubations of linear operator. The main result goes as follows:

6.12 Theorem. *Let $u\colon X \to Y$ be an injective operator with closed range and $T\colon X \to Y$ a compact operator. Then the kernel of $u+T$ is finite dimensional and $u+T$ has closed range.*

Proof. First we show that N_{u+T} is finite dimensional. By XX it suffices to show that the set

$$B = \{x \in N_{u+T} \quad : \ ||x| \leq 1\}$$

is compact. So let $\{x_n\}$ be a sequence in B. Since T is compact there is a subsequence $\{\xi_j = x_{n_j}\}$ such that lim $T\xi_j = y$. It follows that

$$u(\xi_j) = -T(\xi_j) \to y$$

Now u is injective and has closed range so by the Open Mapping Theorem it is bi-continuous. So from the Cauchy sequence $\{u(\xi_j)\}$ produced via the limit in (i),it follows that the sequence $\{\xi_j\}$ converges to a vector ξ^* in X. But then it is clear from (i) that $u(\xi_*) = -T(\xi_*$ and hence $\xi_* \in B$. This proves that B is compact.

The closedness of Im$(u+T)$. Since N_{u+T} is finite dimensional we have a direct sum decomposition

$$X = N_{u+T} \oplus X_*$$

Now $(u+T)(X) = (u+T)(X_*)$ so it suffices that the last image is closed and we can restrict both u and T to X_* where we notice that the restricted operator T_* again is compact. Hence we may assume that the operator $u+T$ is *injective*. Next, let y be in the closure of Im$(u+T)$. It means that there is a sequence ξ_n in X such that

(i) $$\lim\,(u+T)(x_n) = y$$

Suppose first that the norms of $\{x_n\}$ are unbounded. Passing to a subsequence if necessary we may assume that $||x_n|| \to \infty$. With $x_n^* = \frac{x_n}{||x_n||}$ it follows that

(ii) $$\lim\, u(x_n^*) + T(x_n^*) = 0$$

Now $\{x_n^*\}$ is bounded and since T is compact we can pass to another subsequence and assume that $T(x_n^*) \to y$ holds for some $y \in Y$. But then (x) entails that $u(x_n^*)$ also has a limit and since u is ionjective it follows as above that $\{x_n^*\}$ is convergent. It $x_n^* \to x_*$. Here $x^* \neq 0$ since $||x_n^*|| = 1$ for all n. We see that (xx) entails that $u(x_*) + T(x_*) = 0$ and this is contradiction since N_{u+T} is assumed to be the null space.

So in (i) we now have that the sequence $\{x_n\}$ is bounded. Since T is compact we can pass to a subsequence and assume that $T(x_n) \to \xi$ holds for some $\xi \in Y$. But then (i) entails that the sequence $\{u(x_n)\}$ converges to $y - \xi$. Now u is injective so the Open Mapping Theorem implies that ¨$\{x_n\}$ is a Cauchy sequence in X and hence converges to some x^*. Passing to the limit in (i) we then get

$$u(x_*) + T(x_*) = y$$

Hence y belongs to $\mathrm{Im}(u+T)$ and Theorem 6.12 is proved.

6.13 Fredholm operators.

Let $u\colon X \to Y$ be a bounded linear operator. It is called a Fredholm operator if the kernel and the cokernel of u both are finite dimensional. In particular $\frac{Y}{u(X)}$ is a finite dimensional space and therefore $u(X)$ is closed, i.e. every Fredholm operator has a closed range. When u is a Fredholm operator we set

$$\mathfrak{ind}(u) = \dim N_u - \dim\big[\frac{Y}{u(X)}\big]$$

6.14 Theorem. *Let u be a Fredholm operator and $T\colon X \to Y$ a compact operator. Then $u+T$ is Fredholm and one has the equality*

$$\mathfrak{ind}(u) = \mathfrak{ind}(u+T)$$

Remark. When T has finite dimensional range this is an easy exercise and the equality for the indices follows from the Fredholm index formula in Linear algebra.

Proof in the general case.

Since $u(X)$ has finite codimension there exists a closed complement in Y, i.e.

$$Y = u(X) \oplus W$$

where W is a finite dimensional. Let $\pi\colon Y \to u(X)$ be the projection. Now $\pi \circ T$ is a compact operator from X into $u(X)$. If $\epsilon > 0$ we get the induced linear operator Next, we have the bijective operator

$$\bar{u}\colon \frac{X}{N_u} \to u(X)$$

Moreover, since N_u is finite dimensional we have another direct sum

$$X = N_u \oplus X_*$$

where X_* now is a finite dimensional subspace of X. Then u restricts to a bijective linear operator

$$u_*\colon X_* \to u(X)$$

Next, we can restrict T to the subspace X_* which yields an operator T_* from X_* to Y. Then we regard the composed operator $\pi \circ T_*$. With these notations we obtain for every $\epsilon > 0$ a linear operator $S_\epsilon = u_* + \epsilon \cdot \pi \circ T_*$. Here

(*) $$S_\epsilon\colon\ X_* \to u(X)$$

Now u_* is an isomorphism. By the general result in XX it first follows that the null space of S_ϵ is zero if ϵ is small. Notice that this only uses that the operator T is bounded. Next, since T is compact it follows from XX that S_ϵ has a closed range. Next, since the adjoint S_ϵ^* is injective when ϵ is small, it follows from XX that S_ϵ is an isomorphism, i.e. this conclusion holds for sufficiently small ϵ. Finally, since W and N_u are finite dimensional it follows via Linear algebra that $u + \epsilon T$ is Fredholm and has the same index as u. Now the reader can finish the proof using a homotopy argument over ϵ.

7. Calculus on Banach spaces.

Let X and Y be two Banach spaces and $g\colon X \to Y$ some map. Here g is not assumed to be linear. But just as in calculus one can impose the condition that when $x_0 \in X$ is a given then the difference $g(x) - g(x_0)$ is approximated in a linear way as the norms of $x - x_0$ becomes small. This leads to:

7.1 Definition. *We say that g is differentiable at x_0 if there exists a linear operator $L \in \mathcal{L}(X,Y)$ such that*

$$(*) \qquad \lim_{||x-x_0||\to 0} \frac{||g(x)-g(x_0)-L(x-x_0)||}{||x-x_0||} = 0$$

Remark. One verifies easily that L is unique if it exists. It is denoted by $D_g(x_0)$ and called the differential of g at x_0. If g has a derivative everywhere we get a new function

$$\text{(i)} \qquad x \mapsto D_g(x)$$

with values in the Banach space $\mathcal{L}(X,Y)$. If D_g also has derivatives one says that g is twice differentiable and we get its second order differential defined by

$$D_g^2 = D_{D_g}$$

One continues in this way and for each $k \geq 1$ we get the class $C^k(X,Y)$ of k-times differentiable functions from X onto Y. Notice that the higher order differential maps have target manifolds which are iterated constructions of $\mathcal{L}(X,Y)$.

7.1.B Exercise. Let X be a Banach space and $g\colon X \to X$ a C^1-map such that D_g is the identity at the origin So the assumption is that

$$\lim_{||x||\to 0} \frac{||g(x)-x||}{||x||} = 0$$

Show that g is a local diffeomorphism, i.e. there exists some $\epsilon > 0$ such that g yields a bijective map from the open ball $B(\epsilon) = \{||x|| < \epsilon\}$ onto an open neighborhood U of the origin and $g^{-1}\colon U \to B(\epsilon)$ is a C^1map.

Remark. We shall not enter a more detailed discussion of the differential calculus of Banach-space valued functions but refer to the concise presentation of basic facts from Chapter 1 in Hörmander's text-book [Hö] where the reader also can find a proof of the exercise above.

7.2 Line integrals

Let $X = \mathbf{C}$ equipped with its usual norm given by absolute values of complex numbers. Let Y be a Banach space. Consider continuous maps g defined on some open set Ω in $\mathbf{C}$ with values in Y. Let $t \mapsto \gamma(t)$ be a parametrized C^1-curve whose image is a compact subset of Ω. Then we can define the line integral

$$(*) \qquad \int_\gamma g \cdot dz = \int_0^T g(\gamma(t)) \cdot \dot{\gamma}(t) \cdot dt$$

The evaluation is performed exactly as for ordinary Riemann integrals, Namely, one uses the fact that the Y-valued function

$$t \mapsto g(\gamma(t))$$

is uniformly continuous with respect to the norm on Y, i.e. the Bolzano-Weierstrass theorem gives:

$$\lim_{\epsilon\to 0} \max_{|t-t'|\leq\epsilon} ||g(t)-g(t')|| = 0$$

Then (*) is approximated by Riemann sums exactly as in ordinary Calculus.

7.3 Analytic functions.

Let $g(z)$ be a continuous map from the open set Ω into the Banach space Y. We say that $g(z)$ is analytic a point $z_0 \in \Omega$ if there exists some $\delta > 0$ and a convergent power series expansion

(*)
$$g(z) = g(z_0) + \sum (z - z_0)^\nu \cdot y_\nu : \qquad \sum ||y_\nu|| \cdot \delta^\nu < \infty$$

The last condition implies that the power series $\sum (z - z_0)^\nu \cdot y_\nu$ converges in the Banach space Y when $z \in D_\delta(z_0)$. Notice that if $\gamma \in Y^*$ then (*) gives an ordinary complex-valued analytic function

(**)
$$\gamma(g)(z) = \gamma(g(z_0) + \sum c_\nu \cdot (z - z_0)^\nu : \qquad c_\nu = \gamma(y_\nu)$$

Since elements y in Y are determined when we know $\gamma(y)$ for every $\gamma \in Y^*$ we see that (**) entails that the sequence $\{y_\nu\}$ in (*) is unique, i.e. Y-valued analytic functions have unique power series expansions. Moreover, using (**) the reader may verify the following Bamach-space version of Cauchy's theorem.

7.4 Theorem. *Let $\Omega \in \mathcal{D}^1(\mathbf{C})$ and $g(z)$ is an Y-valued function which is analytic in Ω and extends to a continuous function on $\bar{\Omega}$. Let $f(z)$ be an ordinary analytic function in Ω which extends continuously to $\bar{\Omega}$. Then*

$$f(z_0) \cdot g(z_0) = \int_{\partial\Omega} \frac{f(\zeta) g(\zeta) d\zeta}{\zeta - z_0} \quad : \quad z_0 \in \Omega$$

Similarly, with the assumptions as above on f and g we have the vanishing result

$$\int_{\partial\Omega} f(\zeta) \cdot g(\zeta) d\zeta = 0$$

Remark. The results above show that analytic function theory can be applied in a quite general context. In these notes we have illustrated this in a section devoted to an existence proof of a non-linear PDE-quation where the strategy of the proof is to reduce everything to solutions of linear PDE-equations and use convergent series expansions with values in a suitable Banach space.

7.5 Resolvent operators

Let A be a continuous linear operator on a Banach space X. In XX we defined the spectrum $\sigma(A)$ and proved that the resolvent function

(i)
$$R_A(z) = (z \cdot E - A)^{-1} \quad : \quad z \in \mathbf{C} \setminus \sigma(A)$$

is an analytic function, i.e. the local Neumann series from XX show that $R_A(z)$ is an analytic function with values in the Banach space $Y = \mathcal{L}(X, X)$. Let us now consider a connected bounded domain $\Omega \in \mathcal{D}^1(\mathbf{C})$ whose boundary $\partial\Omega$ is a union of smooth and closed Jordan curves $\Gamma_1, \ldots, \Gamma_p$. Let $f(z)$ be an analytic function in Ω which extends to a continuous function on $\bar{\Omega}$. We impose the condition

(ii)
$$\partial\Omega \cap \sigma(A) = \emptyset$$

Then we can construct the line integral

(*)
$$\int_{\partial\Omega} f(\zeta) \cdot R_A(\zeta) \cdot d\zeta$$

This yields an element of Y denoted by $f(A)$. Thus, if $\mathcal{A}(\Omega)$ is the space of analytic functions with continuous extension to $\bar{\Omega}$ then (*) gives a map

(**)
$$T_A \colon \mathcal{A}(\Omega) \to Y$$

Let us put

$$\delta = \min\{|z-\zeta| \quad : \zeta \in \partial\Omega : \ z \in \sigma(A)\}$$

By the result in XX there is a constant C which depends on A only such that the operator norms:

(***) $$||R_A(\zeta)|| \leq \frac{C}{\delta} \quad : \quad \zeta \in \partial\Omega$$

From (***) and the construction in (*) we conclude that the linear operators $T_A(f)$ have norms which are estimated by

$$||T_A(f)|| \leq \frac{C}{\delta} \cdot \ell(\partial\Omega) \cdot |f|_{\partial\Omega}$$

where $\ell(\partial\Omega)$ is the total arc-length of the boundary. Hence we have proved:

7.6 Theorem. *With Ω as above there exists a continuos linear map $f \mapsto T_A(f)$ from the Banach space $\mathcal{A}(\Omega)$ into Y and one has the norm inequality*

$$||T_A|| \leq \frac{C}{\delta} \cdot \ell(\partial\Omega)$$

The range of T_A. There remains to describe the range of the linear operator T_A and to discover further properties. Recall first that the resolvent operators $R_A(z)$ commute with A in the algebra of linear operators on X. Since $f(A)$ is obtained by a Riemann sum of resolvent operators, it follows that $f(A)$ commutes with A for every $f \in \mathcal{A}(\Omega)$. At the same time $\mathcal{A}(\Omega)$ is a *commutative Banach algebra*. It turns out that one has a multiplicative formula for T_A. More precisely one has:

7.7 Theorem. *T_A yields an algebra homomorphism from $\mathcal{A}(\Omega)$ into a commutative subalgebra of Y, i.e.*

$$T_A fg) = T_A(f) \cdot T_A(g) \quad : f, g \in \mathcal{A}(\Omega)$$

Proof of Theorem 7.7

The proof requires several steps. To begin with, in Y we get the closed subalgebra $\mathbf{A}$ generated by A and all the resolvent operators $R_A(z)$ as z moves outside $\sigma(A)$. Then $\mathbf{A}$ is a commutative Banach algebra whose Gelfand space is denoted by $\mathfrak{M}$. The first step towards the proof of Theorem 7.7. is:

7.8 Proposition *The Gelfand space $\mathfrak{M}$ can be identified with the compact set $\sigma(A)$.*

Proof. Let λ be a multiplicative linear functional on $\mathbf{A}$. By the definition of $\sigma(A)$ we must have

$$\lambda(A) = z_* \quad : z_* \in \sigma(A)$$

Now z_* determines λ. For if $R_A(z)$ is a resolvent operator we have

$$R_A(z) \cdot (z \cdot E - A) = E$$

where E is the identity in $\mathbf{A}$. Since λ is multiplicative this entails that

(i) $$1 = \lambda(R_A(z) \cdot (z - z_*) \implies \lambda(R_A(z) = \frac{1}{z - z_*}$$

Hence z_* determines λ. Conversely, if we take $z_* \in \sigma(A)$ then we *define* λ such that $\lambda(A) = z_*$ and (i) holds for every $z \in \mathbf{C} \setminus \sigma(AS)$ and the reader may verify that this yields a multiplicative functional.

Remark. Recall that $\mathfrak{M}$ is the maximal ideal space of $\mathbf{A}$. If $z_* \in \sigma(A)$ and regard the *non-closed* algebra $\mathbf{A}_*$ generated by A and its resolvent operators, then it is obvious that we get the maximal ideal

$$\mathfrak{m}_*(zE - AS) \cdot \mathbf{A}_*$$

Taking its closure in $\mathbf{A}$ we get a maximal ideal in this commutative Banach algebra which corresponds to the point in $\mathfrak{M})$ determined by z_*.

Final part in the proof of Theorem 7.7. In addition to the given domain Ω we construct a slightly smaller domain Ω_* which also is bordered by p many disjoint and closed Jordan curves $\Gamma_1^*, \ldots, \Gamma_p^*$ where each single Γ_ν^* is close to Γ_ν and $\partial\Omega^*$ stays so close to $\partial\Omega$ that does not intersect $\sigma(A)$. Let us then consider pair f, g in $\mathcal{A}(\Omega)$. Now $\partial\Omega \cup \partial\Omega_*$ border a small domain where all functions are analytic. By analyticity and line integrals over $\partial\Omega$ or $\partial\Omega_*$ are equal. In particular we get:

$$T_A(g) = \int_{\partial\Omega_*} g(\zeta_*) \cdot R_A(\zeta_*) \cdot d\zeta_*$$

where we use ζ_* as a variable to distinguish from the subsequent integration along $\partial\Omega$. To compute $T_A(f)$ we keep integration on $\partial\Omega$ and obtain

$$\text{(i)} \qquad T_A(f) \cdot T_A(g) = \int_{\partial\Omega_*} \int_{\partial\Omega} g(\zeta_*) \cdot f(\zeta) \cdot R_A(\zeta_*) \cdot R(\zeta) \cdot d\zeta_* d\zeta$$

Next we use the resolvent equation

$$\text{(ii)} \qquad R_A(\zeta_*) \cdot R(\zeta) = \frac{R(\zeta_*) - R(\zeta)}{\zeta - \zeta_*}$$

The double integral in (i) therefore becomes a sum of two integrals

$$C_1 = \int_{\partial\Omega_*} \int_{\partial\Omega} g(\zeta_*) \cdot f(\zeta) \frac{R(\zeta_*)}{\zeta - \zeta_*} \cdot d\zeta_* d\zeta$$

$$C_2 = -\int_{\partial\Omega_*} \int_{\partial\Omega} g(\zeta_*) \cdot f(\zeta) \frac{R(\zeta)}{\zeta - \zeta_*} \cdot d\zeta_* d\zeta$$

To find C_1 we first perform integration with respect to ζ. Since every ζ_* from the inner boundary $\partial\Omega_*$ belongs to the domain Ω Cauchy's formula applied to the analytic function f gives:

$$f(\zeta_*) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta) d\zeta}{\zeta - \zeta_*}$$

Inserting this in the double integral defining C_1 we get

$$\text{(iii)} \qquad C_1 = \frac{1}{2\pi i} \int_{\partial\Omega_*} \int_{\partial\Omega_*} f(\zeta_*) g(\zeta_*) \cdot R(\zeta_*) \cdot d\zeta_* = T_A(fg)$$

To evaluate C_2 we first perform integration along $\partial\Omega_*$, i.e. we regard:

$$\text{(iv)} \qquad \int_{\partial\Omega_*} \frac{g(\zeta_*)}{\zeta - \zeta_*} \cdot d\zeta_*$$

Now ζ stays *outside* the domain Ω_* and hence (iv) is zero by Cauchy's vanishing theorem. So $C_2 = 0$ and then (i) gives the equality in Theorem 7.7.

7.9 The sup-norm case. Assume now that $\mathbf{A}$ is a sup-norm algebra and put $K = \sigma(A)$. Consider an open set Ω which contains the compact set K. By the previous results there exists an algebra homomorphism

$$T_A \colon \mathcal{O}(\Omega) \to \mathbf{A}$$

Let $f \in A(\Omega$. The spectrum of the $\mathbf{A}$-element $T_A(f)$ is equal to $f(\sigma(A))$. Since $\mathbf{A}$ is assumed to be a sup-norm algebra it follows that

$$(*) \qquad \max_{z \in K} |f(z)| = ||T_A(f)||$$

Here Ω is an arbitrary open neighborhood of K. Since $\mathbf{A}$ is a Banach algebra we can therefore perform a limit as the open sets Ω shrink to K and obtain another algebra homomorphism as follows: First we have the sup-norm algebra $\mathcal{A}(K)$ which consists of continuous functions on

K which an be uniformly approximated b on K by analytic functions defined in small open neighborhoods. Then (*) implies that we have an algebra homomorphism

$$T_A\colon\, f \mapsto T_A(t) \quad : \quad f \in \mathcal{A}(K)$$

Moreover it is an isometry, i.e.

$$\max_{z\in K} |f(z)| = ||T_A(f)$$

In this way the Banach algebra **A** is identified with the sup-norm algebra $\mathcal{A}(K)$.

7.10 A special case. If K is "thin" one has the equality

$$\mathcal{A}(K) = C^0(K) \tag{*}$$

For example, Theorem XXX shows that if the 2-dimensional Lebesgue measure of K is zero then all continuous functions on K can be uniformly approximated by rational functions with poles outside K and then (*) holds. If we also assume that $\mathbf{C} \setminus K$ is connected then Runge's Theorem from XX shows that $C^0(K)$ is equal to the closure of analytic polynomials $P(z)$. Passing to $\mathfrak{A}$ this implies that polynomials in A generate a dense subalgebra of **A**.

7.11 Strictly convex Banach spaces.

A Banach space B is strictly convex if

$$||x+y|| < ||x|| + ||y||$$

for all pairs x, y except when $y = a \cdot x$ for some real and positive a. The notion of uniformly convex Banach spaces was introduced by Clark in 1936. Next, the Banach space is differentiable at a point x if there for every $y \in B$ exists a real number $\mathcal{D}_x(y)$ such that

$$\lim_{\zeta \to 0} ||x+ty|| - ||x|| = \mathfrak{Re}(\zeta) \cdot \mathcal{D}_x(y) + \text{small ordo}(|\zeta|)$$

Exercise. Verify that when B is differentiable at some x, then

$$y \mapsto \mathcal{D}_x(y)$$

is a linear functional on X whose norm is one.

7.12 Conjugate vectors. Let S be the unit sphere in X and S^* the unit sphere in X^*. A pair $x \in S$ and $x^* \in S^*$ are said to be conjugate if

$$x * (x) = 1$$

Exercise. Assume that B is uniformly convex and differentiable at each $x \in X$. Show that every $x \in S$ has a unique conjugate x^* and that the map $x \to x^*$ from S to S^* is bijective. The last assertion means that when $x^* \in S^*$ then there exists a unique $x \in X$ such that $x^*(x) = 1$. Show also that the conjugate map is still bijective when B is strictly convex and reflexive.

7.13 Duality maps. Assume that B is uniformly convex and differentiable at every non-zero vector. Let $\phi(r)$ be a strictly increasing and continuous function on $r \geq 0$ where $\phi(0) = 0$ and $\phi(r) \to +\infty$ when $r \to +\infty$. A map Φ from X into X^* is called an associated duality map if

$$\Phi(rx) = \phi(r) \cdot x^*$$

hold for all $x \in X$ and $r \geq 0$. Let X be as above and Φ a duality map associated to some ϕ-function. If C is a closed subspace in X we set:

$$C^\perp = \{\xi \in X^* \quad : \quad \xi(C) = 0\}$$

7.14 Theorem. *For each closed subspace C of X where $C \neq X$ is assumed and every pair of vectors $x \in X$ and $y \in X^*$ the intersection*

$$\Phi(C+x) \cap \{C^\perp + y\}$$

is non-empty and consists of a single point ξ.

7.15 Exercise. Prove this theorem. If necessary consult the article [B-L] by Beurling and Lorch for the proof and example of various dual maps which occur in L^p-spaces on measure spaces.

8. Bounded self-adjoint operators.

Introduction. Let $\mathcal{H}$ be a complex Hilbert space. A bounded linear operator S on $\mathcal{H}$ is called is called self-adjoint if

(*) $$\langle x, Sy\rangle = \text{the complex conjugate of } \langle Sx, y\rangle \quad : \quad x, y \in \mathcal{H}$$

If S is self-adjoint we have the equality of operator norms:

(1) $$||S||^2 = ||S^2||$$

To see this we notice that if $x \in \mathcal{H}$ has norm one then

(i) $$\langle Sx, Sx\rangle = \langle x, S^*Sx\rangle = \langle x, S^2x\rangle$$

By the Cauchy-Schwarz inequality the last term is $\leq ||x| \cdot ||S^2||$. Since (i) holds for every x of norm one we conclude that

$$||S||^2 \leq ||S^2||$$

Now (1) follows from the multiplicative inequality for operator norms. Next, by induction over n we get the equalities

$$||S||^{2n} = ||S^n||^2 \quad : \ n \geq 1$$

Taking the n:th root and passing to the limit the spectral radius formula gives

$$||S|| = \max_{z \in \sigma(S)} |z|$$

It follows that if **S**is the closed subalgebra of Y generated by S and the identity, then it becomes a sup-norm algebra, i.e. isometric to a closed subalgebra of $C^0(\sigma(S))$. We can say more because one has:

8.1 Theorem. *The spectrum of a bounded self-adjoint operator is a compact real interval.*

Proof. If $\mathfrak{Im}(\lambda \neq 0$ there cannot exists a non-zero vector x in $\mathcal{H}$ such that

$$Sx = \lambda \cdot x$$

For this would give

$$\lambda \cdot ||x||^2 = \langle Sx, x\rangle\langle x, Sx\rangle = \bar{\lambda} \cdot ||x||^2$$

which cannot hold when $\lambda \neq \bar{\lambda}$. So when $\mathfrak{Im}(\lambda) \neq 0$ we have an injective linear operator

$$T\colon\ x \to \lambda x - Sx$$

There remains to show that T also is surjective which means that $\lambda \cdot E - S$ is invertible and hence λ is outside $\sigma(S)$ as required. First we show that T has closed range. To obtain this we consider some x and set

$$y = \lambda x - Sx$$

It follows that

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 + \lambda \cdot \langle x, Sx\rangle + \bar{\lambda} \cdot \langle Sx, x\rangle$$

Since S is self-adjoint we get

$$\lambda \cdot \langle x, Sx\rangle + \bar{\lambda} \cdot \langle Sx, x\rangle = 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle Sx, x\rangle$$

Now $|\langle Sx, x\rangle| \leq ||Sx|| \cdot ||x||$ so the triangle inequality gives

$$||y||^2 \geq |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 - 2|\mathfrak{Re}(\lambda)|| \cdot ||Sx|| \cdot ||x||$$

From this the reader easily shows that

$$||y||^2 \geq \mathfrak{Im}(\lambda)^2 \cdot ||x||^2$$

So we have proved that

(*) $$|\lambda \cdot x - Sx|| \geq |\mathfrak{Im}(\lambda)| \cdot ||x||$$

This implies that T has a closed range. To prove surjectivity it suffices to show that the orthogonal complement of $T(\mathcal{H})$ is zero. To see this we suppose that y is a vector such that

$$\langle \lambda \cdot x - Sx, y \rangle = 0$$

for all x. Since S is self-adjoint it follows that

$$\langle \lambda \cdot x, y \rangle = \langle x, Sy \rangle$$

This holds for every x and therefore $Sy = \lambda \cdot y$. But we have already seen that this gives $y = 0$ and Theorem XX is proved.

A consequence. Theorem 8.1 together with the general result from 7.XX gives the following:

8.2 Theorem. *Let S be a self-adjoint operator. Then the closed subalgebra of $L(\mathcal{H}, \mathcal{H})$ generated by S is a sup-norm algebra which is isomorphic to $C^0(\sigma(S))$.*

8.3 Normal operators.

A bounded linear operator A is normal if it commutes with its adjoint A^*.

Exercise. Let A be a normal operator. Show that the operator A^*A is self-adjoint. The hint is to use the general equality:

$$B^*S^* = (SB)^*$$

for an arbitrary pair of linear operators.

Next, let A be normal and set $S = A^*A$ which is self-adjoint by the exercise. It follows that $S^2 = A^2(A^*)^2$ and the multiplicative inequality for operator norms gives:

$$||S^2|| \leq ||A^2|| \cdot ||(A^*)^2|| = ||A^2||^2 \tag{1}$$

where the last equality follows since the norm of an operator is equal to the norm of its adjoint. Next, since S is self-adjoint we have already proved that

$$||S^2|| = ||S||^2 = ||AA^*||^2 = ||A||^4 \tag{2}$$

where the last equality follows from the general identity

$$||T||^2 = ||T^*T||$$

when T is an arbitrary operator on $\mathcal{H}$. From (1-2) we conclude that

$$||A||^2 = ||A^2||$$

We can take higher powers and exactly as in XX the spectral radius formula gives the equality:

$$||A|| = \max_{z \in \sigma(A)} |z| \tag{*}$$

Since every polynomial in A again is a normal operator for which (*) holds we have proved the following:

8.4 Theorem *Let A be a normal operator. Then the closed subalgebra* $\mathbf{A}$ *generated by A in $L(\mathcal{H}, \mathcal{H})$ is a sup-norm algebra.*

Remark. The spectrum $\sigma(A)$ is some compact subset of $\mathbf{C}$. In general we cannot affirm that $\mathcal{A}(\sigma(A)) = C^0(\sigma(A))$. To overcome this we shall also use the adjoint operator A^* and consider the closed subalgebra of $L(\mathcal{H}, \mathcal{H})$ which is generated by A and A^*. Notice that every polynomial in A and A^* again is a normal operator and it is celar that if a sequenceof normal operators converge in the operator norm then the limit is again a normal perator. So if $\mathcal{B}$ is the closed subalgebra of $L(\mathcal{H}, \mathcal{H})$ then every operator in B is normal. We conclude as above that $\mathcal{B}$ is a sup-norm algebra. There remains to prove the following concluive result:

8.5 Theorem. *The sup-norm algebra $\mathcal{B}$ is via the Gelfand transform isomorphic with $C^0(\mathfrak{M}_{\mathcal{B}})$.*

Proof. If $S \in \mathcal{B}$ is self-adjoint then we know from the previous section that its Gelfand transform is real-valued. Next, let $Q \in \mathcal{B}$ be arbitrary. Now $S = Q + Q^*$ is self-adjoint. So if $p \in \mathfrak{M}_{\mathcal{B}}$ it first follows that $\hat{Q}(p) + \hat{Q}^*(p)$ is real, i.e. with $\hat{Q}(p) = a + ib$ we must have $\hat{Q}^* = a_1 - ib$ for some real number a_1. But now QQ^* is also sefl.-adjoint and hence $(a + ib)(a - 1 - ib)$ is real. This gives $a = a_1$ and hence we have proved that the Gelfand transform of Q^* is the complex conjugate function of $\hat{Q}$. Hence the Gelfand transforms of $\mathcal{B}$-elements is a self-adjoint algebra and the theorem follows from the general fact that a self-adjoint and point separating sup-norm algebra on a compact space X is equal to $C^0(X)$.

Remark. Since $\hat{A}^*$ is the complex conjugate function of $\hat{A}$ it follows that $\hat{A}$ alone separates points on $\mathfrak{M}_{\mathcal{B}}$. We conclude that the Gelfandspace of $\mathcal{B}$ can be identified with $\sigma(A)$.

8.6 Spectral measures.

Given $\mathcal{B}$ and $\sigma(A)$ as above we can construct certain Riesz measures on $\sigma(A)$. Namely, let x, y be a pair of vectors in $\mathcal{H}$. Now we get a linear functional on the Banach space $\mathcal{B}$ defined by

$$T \mapsto \langle Tx, y\rangle$$

The Riesz representation formula gives a *unique* Riesz measure $\mu_{x,y}$ on $\sigma(A)$ such that

$$(*) \qquad \langle Tx, y\rangle = \int \hat{T}(z) \cdot d\mu_{x,y}(z)$$

hold for every $T \in \mathcal{B}$. Since $\hat{A}(z) = z$ is the identity function we have in particular

$$\langle Ax, y\rangle = \int z \cdot d\mu_{x,y}(z)$$

Similarly we get

$$\langle A^*x, y\rangle = \int \bar{z} \cdot d\mu_{x,y}(z)$$

8.7 The operators $E(\delta)$. Notice that (*) implies that the map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures on $\sigma(A)$ is bi-linear. We have for example:

$$\mu_{x_1+x_2,y} = \mu_{x_1,y} + \mu_{x_2,y}$$

Moreover, since $\mathcal{B}$ is a sup-norm algebra it follows from (*) that one has the inequality

$$||\mu_{x,y}|| \leq \max, |\langle Tx, y\rangle|$$

Here $||\nu_{x,y}||$ is the total variation of the complex-valued Riesz measure and the maximum is taken over all $T \in \mathcal{B}$ with operator norm ≤ 1. It follows that

$$(*) \qquad ||\mu_{x,y}|| \leq ||x|| \cdot ||y||$$

Next, let δ be a Borel subset of $\sigma(A)$. Keeping y fixed in $\mathcal{H}$ we obtain a linear functional on $\mathcal{H}$ defined by

$$x \mapsto \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta)$$

By (*) it has norm $\leq ||y||$ and itis represented by a vector $E(\delta)x$ in $\mathcal{H}$. More precisely we have

$$\langle E(\delta)x, y\rangle = \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta)$$

Finally, using the additivity in (xx) once more we see that

$$x \mapsto E(\delta)(x)$$

is linear and hence we obtain the linear operator $E(\delta)$.

Exercise. Show that $E(\delta)$ commutes with all operators in $\mathcal{B}$ and that it is a self-adjoint projection, i.e.

$$E(\delta)^2 = E\delta) \quad \text{and} \quad E(\delta)^* = E(\delta)$$

Show also that the spectrum of this linear operator is contained in the closure of the Borel set δ. Finally, show that

$$E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$$

holds for every pair of Borel subsets and if we take $\delta = \sigma(A)$ we get the identity operator.

8.8 Resolution of the identity. The self-adjoint projection operators above enable us to decompose the identity on $\mathcal{H}$. Namely, if $\delta_1, \ldots, \delta_N$ is any finite family of disjoint Borel sets whose union is $\sigma(A)$ then

$$1 = E(\delta_1) + \ldots + E(\delta_N)$$

At the same tine we get a decomposition of the operator A, i.e.

$$A = A_1 + \ldots + A_N \quad \text{where} \quad A_k = E(\delta_k) \cdot A$$

For each k the spectrum $\sigma(A_k)$is contained in the closure of δ_k. So the normal operator is represented by a sum of normal operators where the individual operator has a small spectrum when the δ-partition is fine.

9. Unbounded self-adjoint operators.

First we prove some general results about densely defined linear operators. A linear operator T on $\mathcal{H}$ is densely defined if there exists a dense subspace $\mathcal{D}(T)$ on which T is defined, i.e. to every $x \in \mathcal{D}(T)$ we get an image vector Tx. For the moment no further assumption is imposed on T. In particular it may be unbounded, i.e. if $\Sigma(T) = \mathcal{D}(T) \cap \Sigma$ where Σ is the unit ball in $\mathcal{H}$ then it can occur that

$$\max_{x \in \Sigma(T)} ||Tx|| = +\infty$$

9.1 Constructions of graphs. The product $\mathcal{H} \times \mathcal{H}$ is a Hilbert space whose inner product is defined by

$$\langle (x,y),(x_1,y_1)\rangle = \langle x, x_1\rangle + \langle y, y_1\rangle$$

If T is densely defined we set

$$\Gamma(T) = \{(x, Tx) \quad : \, x \in \mathcal{D}(T)\}$$

This graph is a subspace of $\mathcal{H} \times \mathcal{H}$. Its closure consists of points (x_*, y_*) for which there exists a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that

$$\lim x_n = x_* \quad \text{and} \quad \lim Tx_n = y_*$$

It is an easy exercise to verify that we obtain a linear operator T_c whose graph is the closure of $\mathcal{D}(T)$. Thus, $\mathcal{D}(T_c)$ consists of all x_* for which a limit as above exists. So T_c is a extension of T in the sense that $\Gamma(T) \subset \Gamma(T_c)$. In this way the study of densely defined linear operator is essentially reduced to operators with a closed graph.

9.2 Inverse operators. Let T be a densely defined operator. We do not assume that it has a closed graph. We say that T is injective if $Tx \neq Ty$ when $x \neq y$ and both x and y belong to $\mathcal{D}(T)$. Assume this and suppose also that the range $T(\mathcal{D}(T))$ is a dense subspace of $\mathcal{H}$. Then we define the inverse operator T^{-1} where $\mathcal{D}(T^{-1}) = T(\mathcal{D}(T))$ and

$$Tx = y \implies T^{-1}y = x \quad : \, x \in \mathcal{D}(T)$$

9.3 A useful graph map. On $\mathcal{H} \times \mathcal{H}$ there exists the isometry defined by

$$\mathcal{A}_1(x,y) = (y,x)$$

The construction of T^{-1} gives the equality

$$(*) \qquad \mathcal{A}_1(\Gamma(T)) = \Gamma(T^{-1})$$

Exercise. Prove (*) and conclude that if T has a closed graph so has T^{-1}.

9.4 Adjoint operators. Let T be a densely defined operator. Given a vector $y \in \mathcal{H}$ we define a linear functional on $D(T)$ by

$$x \mapsto \langle Tx, y\rangle$$

Suppose there exists a constant $C(y)$ such that

$$\text{(i)} \qquad |\langle Tx, y\rangle| \leq C(y) \cdot ||x|| \quad \text{for all} \quad x \in \mathcal{D}(T)$$

This densely defined linear functional has a unique extension to $\mathcal{H}$ and since a Hilbert space is self-dual there exists a unique vector y^* such that

$$\text{(ii)} \qquad \langle Tx, y\rangle| = \langle x, y^*\rangle \quad \text{for all} \quad x \in \mathcal{D}(T)$$

The set of all y for which a constant $C(y)$ exists is a subspace of $\mathcal{H}$ which we for the moment denote by $\mathcal{H}_*$. It is clear that the map

$$y \mapsto y^*$$

gives a linear operator from $\mathcal{H}_*$ into $\mathcal{H}$. It is denoted by T^* and is called the adjoint of T. So here $\mathcal{D}(T^*) = \mathcal{H}_*$ holds.

9.5 Another graph equality. On $\mathcal{H} \times \mathcal{H}$ we have the isometry defined by

$$\mathcal{A}_2(x,y) = (y, -x)$$

Exercise. Let T be densely defined. Show that

$$(*) \qquad \Gamma(T^*) = \big[\mathcal{A}_2(\Gamma(T))\big]^\perp$$

In other words, the graph of T^* is the orthogonal complement of $\mathcal{A}_2(\Gamma(T))$. We remark that this equality holds in general, i.e. even if $\mathcal{D}(T^*)$ is not dense. Since the orthogonal complement of an arbirary subspace of a Hilbert space is closed, it follows from (*) that an adjoint operator T^* always has a closed graph.

Next, assume that T is such that T^* also is densely defined. Hence we can construct its inverse $(T^*)^{-1}$. We have also the operator T^{-1} and again we assume that it is densely defined which is equivalent to the condition that the range of T is a dense subspace of $\mathcal{H}$. Now we also get the adjoint operator $(T^{-1})^*$ and with these notations one has

9.6 Theorem. *One has the equality*

$$(T^{-1})^* = (T^*)^{-1}$$

Proof. We must prove that the two operators have the same graph, To get the equality we use the two $\mathcal{A}$-operators. First

$$(1) \qquad \Gamma((T^*)^{-1}) = \mathcal{A}_1(\Gamma(T^*)) = \mathcal{A}_1\big(\big[\mathcal{A}_2(\Gamma(T))\big]^\perp\big)$$

Since $\mathcal{A}_1$ is an isometry the last term is equal to

$$(2) \qquad \big[\mathcal{A}_1\big(\mathcal{A}_2(\Gamma(T))\big]^\perp$$

Next, we notice that the composed operator $\mathcal{A}_1 \circ \mathcal{A}_2 = -\mathcal{A}_2 \circ \mathcal{A}_1$ and while we regard images of subspoaces in $\mathcal{H} \times \mathcal{H}$ the sign does not matter. So (2) becomes

$$(3) \qquad \big[\mathcal{A}_2\big(\mathcal{A}_1(\Gamma(T))\big]^\perp = \big[\mathcal{A}_2\big(\Gamma(T^{-1})\big)\big]^\perp$$

Finally, by another application of (*) from the Exercise above we see that (3) is equal to $\Gamma\big((T^{-1})^*\big)$ and Theorem 9.6 follows.

9.7 Symmetric operators

A symmetric operator is a densely defined operator T such that

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

hold for each pair x, y in $\mathcal{D}(T)$. It is easily seen that the symmetry is preserved by the operator T_c whose graph is the closure of T. So without loss of generality we consider symmetric operators with a closed range. Next, the symmetry obviously implies that the adjoint T^* is an extension of T, i.e. one has the inclusion

$$(*) \qquad \Gamma(T) \subset \Gamma(T^*)$$

Now we shall find a condition in order that equality holds in (*).

The spaces $\mathcal{D}_+$ and $\mathcal{D}_-$. Let T be a symmetric operator and use its adjoint to define the following two eigenspaces:

$$\mathfrak{D}_+ = \{x \in \mathcal{D}(T^*) \quad : T^*x = ix\} \quad \text{and} \quad \mathfrak{D}_- = \{x \in \mathcal{D}(T^*) \quad : T^*x = -ix\}$$

With these notations one has

9.8 Theorem. *If* $\mathfrak{D}_+ = \mathfrak{D}_- = 0$ *it follows that* $T_c = T^*$

Remark. So when the two $\mathfrak{D}$-spaces are zero we obtain the natural self-adjoint extension of T given by its closure T_c. However, this is not the only case when T has a self-adjoint extension. Namely, the following more general existence result holds:

9.9 Theorem. *Assume that the two linear spaces $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are finite dimensional complex vector spaces of the same dimension. Then the symmetric operator T has a self-adjoint extension.*

We refer to section XX for the proof of this result. It is illustrated by an example below. But the reader who is content with the case in Theorem 9.8 can proceed directly to its proof.

9.10 Example. Let us give an example of a symmetric operator T which has a self-adjoint extension but not given by T_c. Let $\mathcal{H}$ be the Hilbert space $L^2[0,1]$, i.e. the elements are square-integrable functions on the unit interval $[0,1]$ where the coordinate is denoted by t. A dense subspace $\mathcal{H}_*$ consists of functions $f(t) \in C^1[0,1]$ such that $f(0) = f(1) = 0$. On this dense subspace we define the operator T by

$$T(f) = if'(t)$$

A partial integration gives

$$\langle T(f), g\rangle = i\int_0^1 f'(t)\cdot \bar{g}(t)\cdot dt = \int_0^1 \bar{g}'(t)\cdot f(t)dt = \langle f, T(g)\rangle$$

Hence T is symmetric. Next, an L^2-function h belongs to $\mathcal{D}(T^*)$ if and only if there exists a constant $C(h)$ such that

$$\Big|\int_0^1 if'(t)\cdot \bar{h}(t)dt\Big| \leq C(h)\cdot ||f||_2$$

hold for all in $f \in \mathcal{H}_*$. By elementary distribution theory this means that $\mathcal{D}(T^*)$ consists of all L^2-functions h for which the distribution derivative $\frac{dh}{dt}$ again belongs to L^2. Let us then consider the operator T^*. Notice that $\mathcal{D}(T^*)$ contains *all* C^1-functions f, i.e. with no constraint upon the end-values $f(0)$ and $f(1)$. For such pair f, g a partial integration gives

$$\langle T^*(f), g\rangle\rangle - \langle f, T^*(g)\rangle = i\cdot (f(1)\bar{g}(1) - f(0)\bar{g}(0))$$

Hence the left hand side can be $\neq 0$, i.e. choose for example $f(t) = g(t) = t$. Next, we notice that

$$\mathfrak{D}_+ = \{h \in L^2 \quad : \frac{dh}{dt} = h\}$$

This is a 1-dimensional vector space generated by the exponential function e^t. Similarly

$$\mathfrak{D}_+. = \{h \in L^2 \quad : \frac{dh}{dt} = -h\}$$

is 1-dimensional and generated by e^{-t}.

The self-adjoint extension of T. Let $\bar{T}$ be the closure of T. By XX it is again a symmetric operator. Next, consider the exponential function e^t. It belongs to $\mathcal{D}(T^*)$ and satisfies

$$T^*(e^t) = i\cdot e^t$$

Thus, e^t belongs to $\mathfrak{D}_+$. The reader may verify that e^t does not belong to $\mathcal{D}(\bar{T})$. So we get a new subspace of $\mathcal{H}$ generated $\mathcal{D}(\bar{T})$ and e^t. On this dense subspace we define the linear operator

$$S(f + ce^t) = \bar{T}(f) + ice^t$$

when $f \in \mathcal{D}(\bar{T})$ and c is a complex constant.

Exercise Prove that S is symmetric and that $S = S^*$, i.e. S gives a self-adjoint extension of T.

Proof of Theorem 9.8

Recall that $\Gamma(T^*)$ is a closed subspace of $\mathcal{H}\times\mathcal{H}$. It follows that $\mathcal{D}(T^*)$ is equipped with a complete inner product defined by

(1) $$\{x, y\} = \langle x, y\rangle + \langle T^*x, T^*y\rangle$$

defined for pairs x, y in $\mathcal{D}(T^*)$. Since T^* is an extension of T, the graph $\Gamma(T)$ appears as a closed subspace of $\Gamma(T^*)$ which via (1) is identified with a closed subspace of $\mathcal{D}(T^*)$. To prove the equality in Theorem 9.8 suffices to show that the orthogonal complement of $\mathcal{D}(T)$ is zero. Suppose that some $\xi \in \mathcal{D}(T^*)$ is $\perp$ to $\mathcal{D}(T)$. This means that

(i) $$\langle \xi, x\rangle + \langle T^*\xi, Tx\rangle = 0 \quad \text{for all} \quad x \in \mathcal{D}(T)$$

From this it is clear that $\xi \in \mathcal{D}(T)$ which gives

$$0 = \langle \xi, x\rangle + \langle T\xi, Tx\rangle = \langle \xi, x\rangle + \langle T^2\xi, x\rangle$$

where the last equality holds by the symmetry of T. Since $\mathcal{D}(T)$ is dense it follows that

(ii) $$0 = T^2(\xi) + \xi = (T + iE)(T - iE)(\xi) = 0$$

Now the hypothesis that $\mathcal{D}_+ = \mathcal{D}_- = 0$ give $\xi = 0$ and Theorem 9.8 is proved.

9.11 Resolvents of self-adjoint operators.

Let A be a densely defined self-adjoint operator. If $x \in \mathcal{D}(A)$ we get the vector $y = ix - Ax$. Then we obtain

$$||y||^2 = ||x||^2 + ||Ax||^2 - i\langle x, Ax\rangle - \langle Ax, ix\rangle$$

Here we notice that

$$-\langle Ax, ix\rangle = i\langle Ax, x\rangle = i\langle x, Ax\rangle$$

where the last equality holds since A is symmetric. We conclude that

(*) $$||ix - Ax||^2 = ||x||^2 + ||Ax||^2 \quad \text{when} \quad x \in \mathcal{D}(A)$$

9.12 Proposition. *By $x \mapsto ix - Ax$ we get a bijective linear map from $\mathcal{D}(A)$ onto $\mathcal{H}$.*

Proof. Let ρ denote this map. By XX above it is injective. To prove surjectivity we set $Y = \rho(\mathcal{D}(A))$. First we show that the orthogonal complement $Y^\perp = 0$ which means that Y is a dense subspace of $\mathcal{H}$. To see this we consider some vector $\xi \in \mathcal{H}$ such that

(i) $$\langle \xi, i \cdot x - Ax\rangle = 0 \quad \text{for all} \quad x \in \mathcal{D}(A)$$

This implies that the linear functional on $\mathcal{D}(A)$ defined by

$$x \mapsto \langle \xi, Ax\rangle = \langle \xi, i \cdot x\rangle$$

is bounded, i.e. we see that $C(\xi) \leq ||\xi||$. So by definition ξ belongs to $\mathcal{D}(A^*)$ and since $A = A^*$ we have $\xi \in \mathcal{D}(A)$. Then the symmetry of A and (i) give:

$$\langle A\xi, x\rangle = \langle \xi, i \cdot x\rangle = -i \cdot \langle \xi, x\rangle$$

This hold for all x in the dense space $\mathcal{D}(A)$ which gives $A(\xi) = -i \cdot \xi$. But this contradicts the result in XX and hence $Y^\perp = 0$. There remains to show that Y is closed. But this follows easily from (*) above. For if $\{x_n\}$ is a sequence in $\mathcal{D}(A)$ and $y_n = ix_n - A(x_n)$ converge to some y_* then (*) entails

$$||x_n - x_m||^2 \leq ||y_n - y_m||^2$$

for all pair n, m. Since $\{y_n\}$ by hypothesis is a convergent sequence it is a Cauchy sequence and hence $\{x_n\}$is also a Cauchy sequence. Therefore $x_n \to x_*$ hold for some $x_* \in \mathcal{H}$ and at this stage the reader may verify that x_* belongs to $\mathcal{D}(A)$ and that $y_* = \rho(x_*)$.

9.13 The operator R. Since the ρ-image is $\mathcal{H}$ we get a linear operator R defined on the whole Hilbert space such that

$$(iE - A) \circ R(x) = x \quad \text{for all} \quad x \in \mathcal{D}(A)$$

Moreover, by the inequality (*) it follows that R is a bounded linear operator whose operator norm is ≤ 1 and we notice that the range

(1) $$R(\mathcal{H}) = \mathcal{D}(A)$$

Next, from the proof of Proposition 9.12 it is clear that the densely defined operator $iE + A$ has a bounded inverse which we deonte by S. So here

$$(iE + A) \circ S(x) = x \quad \text{for all} \quad x \in \mathcal{D}(A)$$

9.14 An adjoint formula. Above R is the inverse of the densely defined operator $iE - A$. Since A is self-adjoint we have

$$(iE - A)^* = -iE - A = -(iE + A)$$

Now $-S$ is the inverse operator of $-(iE+A)$ and hence Theorem XX gives

$$(*) \qquad R^* = -S$$

Using this equality we can prove the following:

9.15 Proposition. *The operator R is normal.*

Exercise. Prove this result where the hint is to use the equality (*) above.

9.16 The spectrum of S. Above we have found the normal and bounded operator R. We also get the normal operator S and from now on we prefer to work with S instead of R and establish the following inclusion for the spectrum $\sigma(S)$:

9.17 Theorem. *The spectrum $\sigma(S)$ contains 0 and is otherwise contained in the set*

$$\Sigma = \{\frac{1}{a+i} \quad : \ a \in \mathbf{R}\}$$

Proof. Since S is the inverse of $iE + A$ it follows from XX that

$$\Gamma(S) = \{(ix + Ax, x) \quad : x \in \mathcal{D}(A)\}$$

So if λ is a non-zero complex number we get

$$(1) \qquad \Gamma(\lambda \cdot E - S) = \{(ix + Ax, -x + \lambda(ix + Ax)) \quad : x \in \mathcal{D}(A)\}$$

Suppose now that λ is outside the set Σ We must show that that $\lambda \cdot E - S$ is invertible. First we prove that the range of $\lambda \cdot E - S$ is dense. For otherwise the formula for its graph in (1) above gives the existence of a non-zero vector y such that

$$(2) \qquad \langle -x + \lambda(ix + Ax), y\rangle = 0 \quad : x \in \mathcal{D}(A)$$

Since $A = A^*$ and $\lambda \neq 0$ hold, it is clear that (2) implies that y belongs to $\mathcal{D}(A)$. Now $\langle Ax, y\rangle = \langle x, Ay\rangle$ hold for all $x \in \mathcal{D}(A)$ and hence (2) gives the equality

$$(3) \qquad \frac{1 - i\lambda}{\lambda}\langle x, y\rangle = \langle x, Ay\rangle$$

If we set $\mu = \frac{1-i\lambda}{\lambda}$ the density of $\mathcal{D}(A)$ implies that

$$(4) \qquad Ay = \bar{\mu} \cdot y$$

By the result in (xx) this is only possible if $\mu = a$ is real and this entails that $\lambda = \frac{1}{a+i}$ which contradicts the hypothesis that λ is outside Σ. Hence the range of $\lambda \cdot E - S$ is dense. To finish the proof we consider the vector:

$$(5) \qquad \xi(x) = -x + \lambda(ix + Ax) = \lambda \cdot (\frac{i\lambda - 1}{\lambda} - Ax)$$

Next, put

$$\frac{i\lambda - 1}{\lambda} = a + ib$$

Notice that $b \neq 0$ since λ is outside Σ. Now we get

$$(6) \qquad ||\xi(x)||^2 = |\lambda|^2 \cdot ||ibx + ax - Ax||^2$$

Next, $aE - A$ is self-adjoint which by XX gives the equality

$$||ibx + ax - Ax||^2 = b^2||x||^2 + ||ax - Ax||^2$$

Next, with the notations above we notice that $\lambda \cdot E - S = ix + Ax \mapsto \xi(x)$ So the required invertibility of $\lambda \cdot E - S$ follows if we can find a constant M such that

$$(*) \qquad ||x||^2 + ||Ax||^2 = ||x + Ax||^2 \leq M \cdot ||\xi(x)||^2$$

The existence of a constant M follows easily because we have already seen that

$$||\xi(x)||^2 = |\lambda|^2 \cdot \left[b^2||x||^2 + ||ax - Ax||^2\right]$$

This finishes the proof of Theorem 9.17.

10. Commutative Banch algebras

Contents

Introduction

Consider a complex Banach space B equipped with a commutative product such that the norm satisfies the multiplicative inequality

(*) $$||xy|| \leq ||x|| \cdot ||y|| \quad : \; x, y \in B$$

We also assume that B has a multiplicative unit element e where $ex = xe$ hold for all $x \in B$ and $||e|| = 1$. When this holds we refer to B as a commuative Banach algebra with a multiplicative unit. A $\mathbf{C}$-linear form λ on B is called multiplicative if:

(**) $$\lambda(xy) = \lambda(x) \cdot \lambda(y) \quad \text{for all pairs} \quad x, y \in B$$

When λ satisfies (**) and is not identically zero it is clear that $\lambda(e) = 1$ must hold.

0.1 Theorem. *Every multiplicative functional λ on B is automatically continuous, i.e. an element in the normed dual space B^* and its norm is equal to one.*

The proof in A.1 below uses analytic function theory via a study of certain Neumann series. The crucial point is that when $x \in B$ has a norm strictly less than one, then $e - x$ is invertible in B whose inverse is given by the B-valued power series

(1) $$(e - x)^{-1} = e + x + x^2 + \ldots$$

The spectral radius formula. Given $x \in B$ we can take its powers and for each n set

$$\rho_n(x) = ||x^n||^{\frac{1}{n}}$$

In XX we show that these ρ-numbers have a limit as $n \to \infty$, i.e. there exists

$$\rho(x) = \lim_{n \to \infty} \rho_n(x)$$

Using Hadamard's formula for the radius of convergence of power series we prove the following in XX:

0.2 Theorem. For each $x \in B$ one has the equality

$$\rho(x) = \max_{\lambda \in \mathcal{M}(B)} |\lambda(x)|$$

where $\mathcal{M}(B)$ denotes the set of all multiplicative functionals on B.

0.3 The Gelfand transform. Keeping an element $x \in B$ fixed we get the complex-valued function on $\mathcal{M}(B)$ defined by:

$$\lambda \mapsto \lambda(x)$$

The resulting function is denoted by $\widehat{x}$ and called the Gelfand transform. Since $\mathcal{M}(B)$ is a subset of the dual space B^* it is equipped with the weak-star topology which is called the Gelfand topology. By definition this is the weakest topology on $\mathcal{M}(B)$ for which every Gelfand transform

$\widehat{x}$ becomes a continuous function. In particular there exists an algebra homomorphism from B into the commutative algebra $C^0(\mathcal{M}(B))$:

$$(*) \qquad x \mapsto \widehat{x}$$

0.4 Semi-simple algebras. The spectral radius formula shows that $\widehat{x}$ is the zero function if and only if $\rho(x) = 0$. One says that the Banach algebra B is *semi-simple* if (*) is injective. An equivalent condition is that

$$0 \neq x \implies \rho(x) > 0$$

0.5 Uniform algebras. If B is semi-simple the Gelfand transform identifies B with a subalgebra of $C^0(\mathcal{M}(B))$. In general this subalgebra is not closed. The reason is that there can exist B-elements of norm one while the ρ-numbers can be arbitrarily small. If the equality below holds for every $x \in B$:

$$(*) \qquad ||x|| = \rho(x) = |\widehat{x}|_{\mathcal{M}(B)}$$

one says that B is a uniform algebra.

Remark. Multiplicative functionals on specific Banach algebras were used by Norbert Wiener and Arne Beurling where the focus was on Banach algebras which arise via the *Fourier transform.* Later Gelfand, Shilov and Raikow established the abstract theory which has the merit that it applies to quite general situations such as Banach algebras generated by linear operators on a normed space. Moreover, Shilov applied results from the theory of analytic functions in several complex to construct *joint spectra* of several elements in a commutative Banach algebra. See [Ge-Raikov-Shilov] for a study of commutative Banach algebras which include results about joint spectra. One should also mention the work by J. Taylor who used integral formulas in several complex variables to analyze the topology of Gelfand spaces which arise from the Banach algebra of Riesz measures with total bounded variation on the real line, and more generally on arbitrary locally compact abelian groups.

A. Neumann series and resolvents

Let B be a commutative Banach algebra whose identity element is denoted by e. The set of elements x whose norms have absolute value < 1 is denoted by $\mathfrak{B}$ and called the open unit ball in B.

A.1 Neumann series. Let us prove that $e-x$ is invertible for every $x \in \mathfrak{B}$. We have $||x|| = \delta < 1$ and the multiplicative inequality for the norm gives:

$$(1) \qquad ||x^n|| \leq ||x||^n = \delta^n \quad : \quad n = 1, 2, \ldots$$

If $N \geq 1$ we set:

$$(2) \qquad S_N(x) = e + x + \ldots + x^N$$

For each pair $M > N$ the triangle inequality for norms gives:

$$(3) \qquad ||S_M(x) - S_N(x)|| \leq ||x^{N+1}|| + \ldots + ||x^M|| \leq \delta^{N+1} + \ldots + \delta^M$$

It follows that

$$||S_M(x) - S_N(x)|| \leq \frac{\delta^{N+1}}{1-\delta} \quad : M > N \geq 1$$

Hence $\{S_N(x)\}$ is a Cauchy sequence and is therefore convergent in the Banach space. For each $N \geq 1$ we notice that

$$(e - x)S_N(x) = e - x^{N+1}$$

Since $x^{N+1} \to 0$ we conclude that if $S_*(x)$ is the limit of $\{S_N(x)\}$ then

$$(*) \qquad (e - x)S_*(x) = e$$

This proves that $e - x$ is an invertible element in B whose inverse is the convergent B-valued series

$$(**) \qquad S_*(x) = e + \sum_{k=1}^{\infty} x^k$$

We refer to (**) as the Neumann series of x. More generally, let $0 \neq x \in B$ and consider some λ such that $|\lambda| > ||x||$. Now $\lambda^{-1} \cdot x \in \mathfrak{B}$ and from (**) we conclude that $\lambda \cdot e - x = \lambda(e - \lambda^{-1} \cdot x)$ is invertible where

$$(***) \qquad (\lambda \cdot e - x)^{-1} = \lambda^{-1} \cdot \big[\, e + \sum_{k=1}^{\infty} \lambda^{-k} \cdot x^k \,\big]$$

Exercise. Deduce from (***) that one has the inequality

$$||(\lambda \cdot e - x)^{-1}|| \leq \frac{1}{|\lambda| - ||x||}$$

A.2. Local Neumann series expansions. To each $x \in B$ we define the set

$$(*) \qquad \gamma(x) = \{\lambda : \ e - x \quad \text{is invertible}\}$$

Let $\lambda_0 \in \gamma(x)$ and put

$$(1) \qquad \delta = ||(\lambda_0 \cdot e - x)^{-1}||$$

To each complex number λ we set

$$(2) \qquad y(\lambda) = (\lambda_0 - \lambda) \cdot (\lambda_0 \cdot e - x)^{-1}$$

If $|\lambda - \lambda_0| < \delta$ we see that $y(\lambda) \in \mathfrak{B}$ and hence $e - y(\lambda)$ is invertible with an inverse given by the Neumann series:

$$(3) \qquad (e - y(\lambda))^{-1} = e + \sum_{\nu=1}^{\infty} (\lambda_0 - \lambda)^{\nu} \cdot (\lambda_0 \cdot e - x)^{-\nu}$$

Next, for each complex number λ we notice that

$$(\lambda \cdot e - x) \cdot (\lambda_0 \cdot e - x)^{-1} =$$

$$\big[(\lambda_0 \cdot e - x) + (\lambda - \lambda_0) \cdot e\big](\lambda_0 \cdot e - x)^{-1} = e - y(\lambda) \implies$$

$$(4) \qquad (\lambda \cdot e - x) = (\lambda_0 \cdot e - x)^{-1} \cdot (e - y(\lambda))$$

So if $|\lambda - \lambda_0| < \delta$ it follows that $(\lambda \cdot e - x)$ is a product of two invertible elements and hence invertible. Moreover, the series expansion from (3) gives:

$$(**) \qquad (\lambda \cdot e - x)^{-1} = (\lambda_0 \cdot e - x) \cdot [\, e + \sum_{\nu=1}^{\infty} (\lambda_0 - \lambda)^{\nu} \cdot (\lambda_0 \cdot e - x)^{-\nu} \,]$$

We refer to (**) as a local Neumann series. The triangle inequality gives the norm inequality:

$$||(\lambda \cdot e - x)^{-1}|| \leq ||(\lambda_0 \cdot e - x)|| \cdot \big[1 + \sum_{\nu=1}^{\infty} |\lambda - \lambda_0|^{\nu} \cdot \delta^{\nu} \,\big] =$$

$$(***) \qquad ||(\lambda_0 \cdot e - x)|| \cdot \frac{1}{1 - |\lambda - \lambda_0| \cdot \delta}$$

A.3. The analytic function $R_x(\lambda)$. From the above we see that $\gamma(x)$ is an open subset of $\mathbf{C}$. Let us put:

$$R_x(\lambda) = (\lambda \cdot e - x)^{-1} \quad : \lambda \in \gamma(x)$$

The local Neumann series (**) shows that $\lambda \mapsto R(\lambda)$ is a B-valued analytic function in the open set $\gamma(x)$. We use this analyticity to prove:

A.4 Theorem. *The set* $\mathbf{C} \setminus \gamma(x) \neq \emptyset$.

Proof. If $\gamma(x)$ is the whole complex plane the function $R_x(\lambda)$ is entire. When $\lambda| > ||x||$ we have seen that the norm of $R_x(\lambda)$ is $\leq \frac{1}{|\lambda|-||x||}$ which tends to zero as $\lambda \to \infty$. So if ξ is an element in the dual space B^* then the entire function

$$\lambda \mapsto \xi(R_x(\lambda))$$

is bounded and tends to zero and hence identically zero by the Liouville theorem for entire functions. This would hold for every $\xi \in B^*$ which clearly is impossible and hence $\gamma(x)$ cannot be the whole complex plane.

A.5 Definition *The complement* $\mathbf{C} \setminus \gamma(x)$ *is denoted by* $\sigma(x)$ *and called the spectrum of* x.

A.5 Exercise. Let λ_* be a point in $\sigma_B(x)$. Show the following inequality for each $\lambda \in \gamma(x)$:

$$||(\lambda \cdot e - x)^{-1}|| \geq \frac{1}{|\lambda - \lambda_*|}$$

The hint is to use local Neumann series from A.2.

B. The Gelfand transform

Put

$$(*) \qquad \mathfrak{r}(x) = \max_{\lambda \in \sigma(x)} |\lambda|$$

We refer to $\mathfrak{r}(x)$ as the spectral radius of x since it is the radius of the smallest closed disc which contains $\sigma(x)$. The next result shows that the spectral radius is found via a limit of certain norms.

B.1 Theorem. *There exists the limit* $\lim_{n\to\infty} ||x^|| ^{\frac{1}{n}}$ *and it is equal to* $\mathfrak{r}(x)$.

Proof. Put

$$\xi(n) = ||x^n||^{\frac{1}{n}} \quad n \geq 1\,.$$

The multiplicative inequality for the norm gives

$$\log \xi(n+k) \leq \frac{n}{n+k} \cdot \log \xi(n) + \frac{k}{n+k} \cdot \log \xi(k) \quad \text{for all pairs } n, k \geq 1\,.$$

Using this convexity it is an easy exercise to verify that there exists the limit

$$\text{(i)} \qquad \lim_{n\to\infty} \xi(n) = \xi_*$$

There remains to prove the equality

$$\text{(ii)} \qquad \xi_* = \mathfrak{r}(x)\,.$$

To prove (ii) we use the Neumann series expansion for $R_x(\lambda)$. With $z = \frac{1}{\lambda}$ this gives the B-valued analytic function

$$g(z) = z \cdot e + \cdot \sum_{\nu=1}^{\infty} z^\nu \cdot x^\nu$$

which is analytic in the disc $|z| < \frac{1}{\mathfrak{rad}(x)}$. The general result about analytic functions in a Banach space from XX therefore implies that when $\epsilon > 0$ there exists a constant C_0 such that

$$||x^n|| \leq C_0 \cdot (\mathfrak{r}(x) + \epsilon)^n \quad n = 1, 2, \ldots \implies$$

$$\xi(n) \leq C_0^{\frac{1}{n}} \cdot (\mathfrak{r}(x) + \epsilon))$$

Since $C_0^{\frac{1}{\nu}} \to 1$ we conclude that

$$\limsup_{n\to\infty} \xi(n) \leq \mathfrak{r}(x) + \epsilon)$$

Since $\epsilon > 0$ is arbitrary and the limit (i) exists we get

$$\text{(iii)} \qquad \xi_* \leq \mathfrak{r}(x)$$

To prove the opposite inequality we use the definition of the spectral radius which to begin with shows that the B-valued analytic function $g(z)$ above cannot converge in a disc whose radius

$$r^* > \frac{1}{\mathfrak{r}(x)}$$

Hence Hadamard's limit formula for B-valued power series in XX gives

$$\limsup_{n\to\infty} \xi(n) \geq \mathfrak{r}(x) - \epsilon \quad \text{for every } \epsilon > 0\,.$$

Since the limit in (i) exists we conclude that $\xi_* \geq \mathfrak{r}(x)$ and together with (iii) above we have proved Theorem B.1.

B.2 The Gelfand space $\mathcal{M}_B$.

Let B be a commutative Banach algebra with a unit element e. As a commutative ring we can refer to its *maximal ideals*. Thus, a maximal ideal $\mathfrak{m}$ is $\neq B$ and not contained in any strictly larger ideal. The maximality means that every non-zero element in the quotient ring $\frac{B}{\mathfrak{m}}$ is invertible, i.e. this quotient ring is a *commutative field*. Since the maximal ideal $\mathfrak{m}$ cannot contain an invertible element it follows from A.1 that

$$\text{(i)} \qquad x \in \mathfrak{m} \implies ||e - x|| \geq 1$$

Hence the closure of $\mathfrak{m}$ in the Banach space is $\neq B$. So by maximality $\mathfrak{m}$ is a *closed subspace* of B and hence there exists the Banach space $\frac{B}{\mathfrak{m}}$. Moreover, the multiplication on B induces a product on this quotient space and in this way $\frac{B}{\mathfrak{m}}$ becomes a new Banach algebra. Since $\mathfrak{m}$ is maximal this Banach algebra cannot contain any non-trivial maximal ideal which means that when ξ is any non-zero element in $\frac{B}{\mathfrak{m}}$ then the principal ideal generated by ξ must be equal to $\frac{B}{\mathfrak{m}}$. In other words, every non-zero element in $\frac{B}{\mathfrak{m}}$ is *invertible*. Using this we get the following result.

B.3 Theorem. *The Banach algebra $\frac{B}{\mathfrak{m}} = \mathbf{C}$, i.e. it is reduced to the complex field.*

Proof. Let e denote the identity in $\frac{B}{\mathfrak{m}}$. Let ξ be an element in $\frac{B}{\mathfrak{m}}$ and suppose that

$$\text{(i)} \qquad \lambda \cdot e - \xi \neq 0 \quad \text{for all} \quad \lambda \in \mathbf{C}$$

Now all non-zero elements in $\frac{B}{\mathfrak{m}}$ are invertible so (i) would entail that the spectrum of ξ is empty which contradicts Theorem 3.1. We conclude that for each element $\xi \in \frac{B}{\mathfrak{m}}$ there exists a complex number λ such that $\lambda \cdot e = \xi$. It is clear that λ is unique and that this means precisely that $\frac{B}{\mathfrak{m}}$ is a 1-dimensional complex vector space generated by e.

B.4 The continuity of multiplicative functionals. Let $\lambda\colon B \to \mathbf{C}$ be a multiplicative functional. Since $\mathbf{C}$ is a field it follows that the λ-kerenl is a maximal ideal in B and hence closed. Recall from XX that every linear functional on a Banach space whose kernel is a closed subspace is automatically in the continuous dual B^*. This proves that every multiplicative functional is continuous and as a consequence its norm in B^* is equal to one.

B.5 The Gelfand transform. Denote by $\mathcal{M}_B$ the set of all maximal ideals in B. For each $\mathfrak{m} \in \mathcal{M}_B$ we have proved that $\frac{B}{\mathfrak{m}}$ is reduced to the complex field. This enable us to construct complex-valued functions on $\mathcal{M}_B$. Namely, to each element $x \in B$ we get a complex-valued function on $\mathcal{M}_B$ defined by:

$$\hat{x}(\mathfrak{m}) = \text{ the unique complex number for which } x - \hat{x}(\mathfrak{m}) \cdot e \in \mathfrak{m}$$

One refers to $\hat{x}$ as the Gelfand transform of x. Now we can equip $\mathcal{M}_B$ with the *weakest topology* such that the functions $\hat{x}$ become continuous.

B.6 Exercise. Show that with the topology as above it follows that $\mathcal{M}_B$ becomes a *compact Hausdorff space.*

B.7 Semi-simple algebras. The definition of $\sigma(x)$ shows that this compact set is equal to the range of $\hat{x}$, i.e. one has the equality

$$(*) \qquad \sigma(x) = \hat{x}(\mathcal{M}_B)$$

Hence Theorem 4.1 gives the equality:

$$(**) \qquad \lim_{n\to\infty} ||x^n||^{\frac{1}{n}} = \max_{\mathfrak{m}} \hat{x}(\mathfrak{m}) = |\hat{x}|_{\mathcal{M}_B}$$

where the right hand side is the maximum norm of the Gelfand transform. It may occur that the spectral radius is zero which by (**) means that the Gelfand transform $\hat{x}$ is identically zero. This eventual possibility leads to:

B.8 Definition. *A Banach algebra B is called semi-simple if $\mathfrak{r}(x) > 0$ for every non-zero element.*

B.9 Remark. So when B is semi-simple then the Gelfand map $x \mapsto \widehat{x}$ from B into $C^0(\mathcal{M}_B)$ is injective. In this way B is identified with a subalgebra of all continuous and complex-valued functions on the compact Hausdorff space $\mathcal{M}_B$. Moreover one has the inequality

$$(*) \qquad |\widehat{x}|_{\mathcal{M}_B} \leq ||x||$$

It is in general strict. When equality holds one says that B is a *uniform algebra*. In this case the Gelfand transform identifies B with a closed subalgebra of $C^0(\mathcal{M}_B)$. For an extensive study of uniform algebras we refer to the books [Gamelin] and [Wermer].

C. Examples of Banach algebras.

Below we illustrate the general theory by some examples which appear in applications. Let us start with:

1. Operator algebras. Let B be a Banach space and T is a bounded linear operator on B. Together with the identity operator we construct the subalgebra of $\mathcal{L}(B)$ expressed by polynomials in T and take the closure of this polynomial T-algebra in the Banach space $\mathcal{L}(B)$. In this way we obtain a Banach algebra $\mathcal{L}(T)$ So if $S \in \mathcal{L}(T)$ then $||S||$ is the operator norm taken in $\mathcal{L}(B)$. Here the Gelfand space of $\mathcal{L}(T)$ is identified with a compact subset of $\mathbf{C}$ which is the spectrum of T denoted by $\sigma(T)$. By definition $\sigma(T)$ consists of those complex numbers λ such that the operator $\lambda \cdot E - T$ fails to be invertible in $\mathcal{L}(T)$.

1.0 Permanent spectrum. Above $\sigma(T)$ refers to the spectrum in the Banach algebra $\mathcal{L}(T)$. But it can occur that $\lambda \cdot e - T$ is an invertible linear operator on B even when $\lambda \in \sigma(T)$. To see an example we let $B = C^0(T)$ be the Banach space of continuous functions on the unit circle. Let T be the linear operator on B defined by the multiplication with z, i.e. when $f(\theta)$ is some 2π-periodic function we set

$$T(f)(\theta) = e^{i\theta} \cdot f(e^{i\theta})$$

If λ belongs to the open unit disc we notice that for any polynomial $Q(\lambda)$ one has

$$|Q(\lambda)| \leq \max_{\theta} |Q(e^{i\theta})| = ||Q(T)||$$

It follows that the spectrum of T in $\mathcal{L}(T)$ is identified with the closed unit disc $\{|\lambda| \leq 1\}$. For example, $\lambda = 0$ belongs to this spectrum. On the other hand T is invertible as a linear operator on B where T^{-1} is the operator defined by

$$T^{-1}(f)(\theta) = e^{-i\theta} \cdot f(e^{i\theta})$$

So in this example the spectrum of T taken in the space of all continuous linear operators on B is reduced to the unit circle $\{|\lambda| = 1\}$.

In general, let B be a commutative Banach algebra which appears as a closed subalgebra of a larger Banach algebra B^*. If $x \in B$ we have its spectrum $\sigma_B(x)$ relative B and the spectrum $\sigma_{B^*}(x)$ relative the larger algebra. The following inclusion is obvious:

$$\sigma_{B^*}(x) \subset \sigma_B(x) \tag{1}$$

The example above shows that this inclusion in general is strict. However, one has the opposite inclusion

$$\partial(\sigma_B(x)) \subset \sigma_{B^*}(x) \tag{2}$$

In other words, if λ belongs to the boundary of $\sigma_B(x)$ then $\lambda \cdot e - x$ cannot be inverted in any larger Banach algebra. It means that λ is a permanent spectral value for x. The proof of (2) is given in XX using Neumann series.

2. Finitely generated Banach algebras. A Banach algebra B is finitely generated if there exists a finite subset $x_1, \ldots, x_k$ such that every B-element can be approximated in the norm by polynomials of this k-tuple. Since every multiplicative functional λ is continuous it is determined by its values on $x_1, \ldots, x_k$. It means that we have an injective map from $\mathcal{M}(B)$ into the k-dimensional complex vector space $\mathbf{C}^k$ defined by

$$\lambda \mapsto (\lambda(x_1), \ldots \lambda(x_k)) \tag{1}$$

Since the Gelfand topology is compact the image under (1) yields a compact subset of $\mathbf{C}^k$ denoted by $\sigma(x_\bullet)$. This construction was introduced by Shilov and one refers to $\sigma(x_\bullet)$ as the joint spectrum of the k-tuple $\{x_\nu\}$. It turns out that such joint spectra are special. More precisely, they are polynomially convex subsets of $\mathbf{C}^k$. Namely, let $z_1, \ldots, z_k$ be the complex coordinates in $\mathbf{C}^k$. If z_* is a point outside $\sigma(x_\bullet)$ there exists for every $\epsilon > 0$ some polynomial $Q[z_1, \ldots, z_k]$ such that $Q(z_*) = 1$ while the maximum norm of Q over $\sigma(x_\bullet)$ is ϵ. To see this one argues by a contradiction, i.e. if this fails there exists a constant M such that

$$|Q(z^*)| \leq M \cdot |Q|_{\sigma(x_\bullet)}$$

for all polynomials Q. Then the reader may verify that we obtain a multiplicative functional λ^* on B for which

$$\lambda^*(x_\nu) = z_\nu^* \quad : 1 \leq \nu \leq k$$

By definition this would entail that $z^* \in \sigma(x_\bullet)$.

Remark. Above we encounter a topic in several complex variables. In contrast to the case $n = 1$ it is not easy to describe conditions on a compact subset K of $\mathbf{C}^k$ in order that it is polynomially convex, which by definition means that whenever z^* is a point in $\mathbf{C}^k$ such that

$$|Q(z^*)| \leq |Q|_K$$

then $z^* \in K$.

3. Examples from harmonic analysis.

The measure algebra $M(\mathbf{R}^n)$. The elements are Riesz measures in $\mathbf{R}^n$ of finite total mass and the product defined by convolution. The identity is the Dirac measure at the origin. Set $B = M(\mathbf{R}^n)$. The Fourier transform identifies the n-dimensional ξ-space with a subset of $\mathcal{M}(B)$. In fact, this follows since the Fourier transform of a convolution $\mu * \nu$ is the product $\widehat{\mu}(\xi) \cdot \widehat{\nu}(\xi)$. In this way we have an embedding of $\mathbf{R}^n_\xi$ into $\mathcal{M}(B)$. However, the resulting subset is not dense in $\mathcal{M}(B)$. It means that there exist Riesz measures μ such that $|\widehat{\mu}(\xi)| \geq \delta > 0$ hold for all ξ, and yet μ is not invertible in B. An example of such a measure was discovered by Wiener and Pitt and one therefore refers to the *Wiener-Pitt phenomenon* in B. Further examples occur in [Gelfand et. all]. The idea is to construct Riesz measurs μ with independent powers, i.e. measures μ such that the norm of a μ-polynomial

$$c_0 \cdot \delta_0 + c_1 \cdot \mu + \ldots + c_k \cdot \mu^k$$

is roughly equal to $\sum |c_k|$ while $||\mu|| = 1$. In this way one can construct measurs μ for which the spectrum in B is the unit disc while the range of the Fourier transform is a real interval. Studies of $\mathcal{M}(B)$ occur in work by J. Taylor who established topological properties of $\mathcal{M}(B)$. The proofs rely upon several complex variables and we shall not try to expose material from Taylor's deep work. Let us only mention one result from Taylor's work in the case $n = 1$. Denote by $i(B)$ the multiplicative group of invertible measures in B where $B = \mathcal{M}(B)$ on the real line. If $\nu \in B$ we construct the exponential sum

$$e^\nu = \delta_0 + \sum_{k=1}^{\infty} \frac{\nu^k}{k!}$$

In this way e^B appears as a subgroup of $i(B)$. Taylor proved that the quotient group

$$\frac{i(B)}{e^B} \simeq \mathbf{Z}$$

where the right hand side is the additive group of integers. More precisely one finds an explicit invertible measure μ_* which does not belong to e^B and for any $\mu \in i(B)$ there exists a unique integer m and some $\nu \in B$ such that

$$(*) \qquad \mu = e^\nu * \mu_*^k$$

The measure μ_* is given by

$$\mu_* = \delta_0 + f$$

CONTINUE...

3.1 Wiener algebras. We can ask for subalgebras of $M(\mathbf{R}^n)$ where the Wiener-Pitt phenomenon does not occur, i.e. subalgebras B where the Fourier transform gives a dense embedding of $\mathbf{R}^n_\xi$ into $\mathcal{M}(B)$. A first example goes as follows: Let $n \geq 1$ and consider the Banach space $L^1(\mathbf{R}^n)$ where convolutions of L^1-functions is defined. Adding the unit point mass δ_0 at the origin we get the commutative Banach algebra

$$B = \mathbf{C} \cdot \delta_0 + L^1(\mathbf{R}^n)$$

Here the Fourier transform describes $\mathcal{M}(B)$. More precisely, if λ is a multiplicative functional on B whose restriction to $L^1(\mathbf{R}^n)$ is not identically zero, then one proves that there exists a unique point $\xi \in \mathbf{R}^n$ such that

$$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in L^1(\mathbf{R}^n)$$

In this way the n-dimensional ξ-space is identified with a subset of $\mathcal{M}(B)$. An extra point λ^* appears in $\mathcal{M}(B)$ where $\lambda^*(\delta_0) = 1$ while its restriction to $L^1(\mathbf{R}^n)$ vanishes. Hence the compact Gelfand space $\mathcal{M}(B)$ corresponds to the one-point compactification of the ξ-space. Here the continuity of Fourier transforms of L^1-functions correspond to the fact that their Gelfand transforms are continuous. An important consequence of this is that when $f(x) \in L^1(\mathbf{R}^n)$ is such that $\widehat{f}(\xi) \neq 1$ for every ξ, then the B-element $\delta_0 - f$ is invertible, i.e. there exists another L^1-function g such that

$$\delta_0 = (\delta_0 - f) * (\delta_0 + g) \implies f = g - f * g$$

The equality

$$(*) \qquad \mathcal{M}(B) = \mathbf{R}^n_\xi \cup \{\lambda^*\}$$

was originally put forward by Wiener prior to the general theory about Banach algebras. Another Banach algebra is $M_d(\mathbf{R})^{\mathbf{n}}$ whose elements are discrete measures with a finite total variation. Thus, the elements are measures

$$\mu = \sum c_\nu \cdot \delta(p_\nu)$$

where $\{p_\nu\}$ is a sequence of points in $\mathbf{R}^n$ and $\{c_\nu\}$ a sequence of complex numbers such that $\sum |c_\nu| < \infty$. Here the Gelfand space is more involved. To begin with the Fourier transform identifies $\mathbf{R}^n_\xi$ with a subset of $\mathcal{M}(B)$. But the compact space $\mathcal{M}(B)$ is considerably and given by a compact abelian group which is called the Bohr group after Harald Bohr whose studies of almost periodic functions led to the description of $\mathcal{M}(B)$. However one has the following result:

3.2 Bohr's Theorem. *The subset $\mathbf{R}^n_\xi$ is dense in $\mathcal{M}(B)$.*

Remark. See XX for an account about almost periodic functions which proves Bohr's theorem in the case $n = 1$.

3.3 Beurling's density theorem. Consider the Banach algebra B generated by $M_d(\mathbf{R}^n)$ and $L^1(\mathbf{R}^n)$. So its elements are measures of the form

$$\mu = \mu_d + f$$

where μ_d is discrete and f is absolutely continuous. Here the Fourier transform identifies $\mathbf{R}^n_\xi$ with an open subset of $\mathcal{M}(B)$. More precisely, a multiplicative functional λ on B belongs to the open set $\mathbf{R}^n_\xi$ if and only if $\lambda(f) \neq 0$ for at least some $f \in L^1(\mathbf{R}$. The remaining part $\mathcal{M}(B) \setminus \mathbf{R}^n_\xi$ is equal to the Bohr group above.. It means that when λ is an arbitrary multiplicative functional on B then there exists $\lambda_* \in \mathcal{M}(B)$ such that λ_* vanishes on $L^1(\mathbf{R}^n$ while $\lambda_*(\mu) = \lambda(\mu)$ for every discrete measure. The density of $\mathbf{R}^n_\xi$ follows via Bohr's theorem and the fact that Fourier transforms of L^1-functions tend to zero as $|\xi| \to +\infty$.

3.4 Varopoulos' density theorem. For each linear subspace Π of arbitrary dimension $1 \leq d \leq n$ we get the space $L^1(\Pi)$ of absolutely continuous measures supported by Π and of finite total mass. Thus, we identify $L^1(\Pi)$ with a subspace of $M(\mathbf{R}^n)$. We get the closed subalgebra of $M(\mathbf{R}^n)$ generated by all these L^1-spaces and the discrete measures. It is denoted by $\mathcal{V}(\mathbf{R}^n)$ and called the Varopoulos measure algebra in $\mathbf{R}^n$. In [Var] it is proved that the Fourier transform identifies $\mathbf{R}^n_\xi$ with a dense subset of $\mathcal{M}(\mathcal{V}(\mathbf{R}^n))$.

3.5 The extended $\mathcal{V}$-algebra. In $\mathbf{R}^n$ we can consider semi-analytic strata which consist of locally closed real-analytic submanifolds S whose closure $\bar{S}$ is compact and the relative boundary $\partial S = \bar{S} \setminus S$ is equal to the zero set of a real analytic function. On each such stratum we construct measures which are absolutely continuous with respect to the area measure of S. Here the dimension of S is between 1 and $n-1$ and now each measure in $L^1(S)$ is identified with a Riesz measure in $\mathbf{R}^n$ which happens to be supported by S, One can easily prove that every $\mu \in L^1(S)$ has a power which belongs to the Varopolulos algebra and from this deduce that if $\mathcal{V}^*$ is the closed subalgebra of $M(\mathbf{R}^n)$ generated by the family $\{L^1(S)\}$ and $V(\mathbf{R}^n))$ then one gets a new Wiener algebra.

3.6 Olofsson's example. Above real analytic strata were used to obtain $\mathcal{V}^*$. That real-analyticity is essentially necessary was proved by Olofsson in [Olof]. For example, he found a C^∞-function $\phi(x)$ on $[0, 1]$ such that if μ is the measure in $\mathbf{R}^2$ defined by

$$\mu(f) = \int_0^1 f(x, \phi(x)) \cdot dx$$

then μ has independent powers and it cannot belong to any Wiener subalgebra of $M(\mathbf{R}^n)$. Actually [Olofson] constructs examples as above on curves defined by C^∞-functions outside the Carleman class of quasi-anaytic functions.