

XIII. Uniform approximation by Blaschke products

Introduction. A bounded analytic function $f(z)$ in the unit disc D is an *inner function* if the boundary function on T has constant absolute value almost everywhere. The set of inner functions is denoted by $\mathcal{I}(D)$. The first result about inner functions is due to O. Frostman in [x] :

0.1 Theorem. *Let $w(z) \in \mathcal{I}(D)$. For every $\epsilon > 0$ there exists a real number c and a Blaschke product $B(z)$ such that*

$$\max_{z \in D} |w(z) - e^{ic} B(z)| < \epsilon$$

The next result is due to D. Marshall in [Mar] and asserts that the convex hull of Blaschke products is dense in the unit ball of $H^\infty(D)$.

0.2 Theorem. *Let $f(z) \in H^\infty(D)$ with $|f|_D \leq 1$. For each $\epsilon > 0$ there exists a finite family of Blaschke functions B_1, \dots, B_N and an N -tuple of positive real numbers a_1, \dots, a_N with $\sum a_\nu = 1$ such that*

$$|f(z) - \sum a_\nu \cdot B_\nu(z)|_D < \epsilon$$

Remark. Frostman's theorem reduces Theorem 0.2 to the assertion that the convex hull of inner functions is dense in $H^\infty(T)$. The first step in the proof of Theorem 0.2 is to use an approximation theorem which takes place in the Banach space $L^\infty(T)$. Notice that if w is an inner function then $\frac{1}{w} \in L^\infty(T)$. We can also take quotients $\frac{w_1}{w_2}$ and get new $L^\infty(T)$ -functions. With this in mind the following result is due to Douglas and Rudin in [xx]

0.3 Theorem *The convex hull formed by functions $\frac{w_1}{w_2}$ from pairs in $\mathcal{I}(D)$ is dense in $L^\infty(T)$.*

We prove this result in section 5. Now we consider some $f \in H^\infty(T)$ with norm $|f|_\infty = 1$. Theorem 0.3 gives for each $\epsilon > 0$ an N -tuple of unimodular functions $\{\frac{w_1^\nu}{w_2^\nu}\}$ formed by inner functions such that

$$(*) \quad \left| f - \sum a_\nu \cdot \frac{w_1^\nu}{w_2^\nu} \right|_\infty < \epsilon \quad : \quad a_1 + \dots + a_N = 1 : a_\nu \geq 0$$

However, (*) does not give Theorem 0.2 since the individual quotients $\frac{w_1^\nu}{w_2^\nu}$ are not analytic in D . To proceed from (*) Marshall considered the inner function

$$\mathcal{J} = w_2^1 \cdots w_2^N$$

Then we can write

$$(**) \quad \sum a_\nu \cdot \frac{w_1^\nu}{w_2^\nu} = \sum a_\nu \cdot \frac{W_\nu}{\mathcal{J}} \quad : \quad W_\nu = w_1^\nu \cdot \prod_{j \neq \nu} w_2^j$$

The next step in Marshall's proof is to apply a theorem by Nevanlinna which implies that (*) gives the existence of another inner function W_* such that the function

$$(***) \quad g = \sum a_\nu \cdot \frac{W_\nu}{\mathcal{J}} + \epsilon \cdot \frac{W_*}{\mathcal{J}} \quad \text{belongs to } H^\infty(T)$$

It follows that $|f - g| < 2\epsilon$ and to finish the proof of Theorem 0.2 there remains only to approximate the special $H^\infty(T)$ -function g -function by a convex combination of inner functions. This step in the proof of Theorem 0.2 is given under the heading *Marshall's Lemma* in section 4.

Remark. Before we enter the proofs of Theorem 0.1 and 0.2 we expose some preliminaries about inner functions and Blaschke products based upon results from Chapter XX about the Jensen-Nevanlinna class of analytic functions in D .

1. Blaschke products and inner functions

A sequence $\{z_n\}$ of non-complex numbers in D arranged so that $0 < |z_1| \leq |z_2| \leq \dots$ satisfies Blaschke's condition if

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$$

Blaschke's theorem asserts that the infinite product

$$(1) \quad B(z) = \prod_{n=1}^{\infty} \frac{z - z_n}{1 - \bar{z}_n z} \cdot \frac{\bar{z}_n}{z_n}$$

converges in D and yields an analytic function such that

$$\lim_{r \rightarrow 1} |B(re^{i\theta})| = 1 \quad \text{holds almost everywhere } 0 \leq \theta \leq 2\pi$$

In particular every Blaschke product belongs to $\mathcal{I}(D)$. Next, let μ be a singular non-negative Riesz measure on the unit circle, i.e. μ carries all mass on a null set in the sense of Lebesgue. Then there exists the analytic function in D defined by

$$(2) \quad G_{-\mu}(z) = \exp \left[-\frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{e^{i\theta} - z}{e^{i\theta} + z} \cdot d\mu(\theta) \right]$$

The Factorisation Theorem from XX gives:

1.1. Theorem *Every $w \in \mathcal{I}(D)$ is a unique product*

$$w(z) = e^{ic} \cdot B(z) \cdot G_{-\mu}(z) \quad : \quad c = \text{a real number}$$

where $B(z)$ is the Blaschke product formed by the zeros of w in D and μ a singular and non-negative measure.

We will also need Theorem XXX from XXX which characterizes when an inner function is a Blaschke product.

1.2 Theorem. *An inner function $w(z)$ is of the form $e^c \cdot B(z)$ if and only if*

$$\limsup_{r \rightarrow 1} \int_0^{2\pi} \log |w(re^{i\theta})| \cdot d\theta = 0$$

2. Proof of Frostman's theorem

Let $0 < \rho < 1$ and γ is a complex number with $|\gamma| \leq 1$. Then one has the equality

$$(*) \quad \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{\gamma - \rho e^{i\theta}}{1 - \rho e^{-i\theta} \gamma} \right| \cdot d\theta = \max(\rho, \log |\gamma|)$$

The verification is left to the reader. Now we consider some $w \in \mathcal{I}(D)$. Apply (*) with $\gamma = w(re^{it})$ for pairs $0 < r < 1$ and $0 \leq t \leq 2\pi$. Integration with respect to t gives the equality:

$$(*) \quad \frac{1}{2\pi} \int_0^{2\pi} \left[\int_0^{2\pi} \log \left| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \right| \cdot d\theta \right] dt = \int_0^{2\pi} \max[\rho, \log |w(re^{it})|] \cdot dt$$

Since $w \in \mathcal{I}(D)$ we have

$$(i) \quad \lim_{r \rightarrow 1} \log |w(re^{it})| = 0 \quad : \quad \text{for almost all } t$$

Keeping $0 < \rho < 1$ fixed the function $t \mapsto \max[\rho, \log |w(re^{it})|]$ is bounded so by *dominated convergence* under the integral sign we have:

$$(ii) \quad \lim_{r \rightarrow 1} \int_0^{2\pi} \max[\rho, \log |w(re^{it})|] \cdot dt = 0$$

Replace this limit by the double integral which comes from the equality (*) and apply *Fubini's theorem* to interchange the order of integration. Hence (ii) gives:

$$(iii) \quad \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} \left[\int_{t=0}^{t=2\pi} \log \left| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \right| \cdot dt \right] d\theta = 0$$

Now we use that the integrand

$$(iv) \quad \log \left| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \right| \leq 0 \quad : \quad 0 \leq \theta, t \leq 2\pi$$

Then (iii) and *Fatou's theorem* gives:

$$(v) \quad \limsup_{r \rightarrow 1} \int_0^{2\pi} \log \left| \frac{w(re^{it}) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(re^{it})} \right| \cdot dt = 0 \quad : \text{almost everywhere for } \theta$$

At this stage the proof is almost finished. Namely, we notice that the functions

$$F_\theta(z) = \frac{w(z) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} w(z)}$$

belong to $\mathcal{I}(D)$ for every θ . Moreover, (v) gives a null set \mathcal{N} in T such that

$$(vi) \quad \limsup_{r \rightarrow 1} \int_0^{2\pi} \log |F_\theta(re^{it})| dt = 0 \quad : \quad \theta \in T \setminus \mathcal{N}$$

Finally, for every $\theta \in T \setminus \mathcal{N}$, Theorem 1.2 implies that $F_\theta(z) = e^{ic_\theta} B_\theta(z)$ for some Blaschke product B_θ and a constant c_θ . With such a choice of θ we have

$$(vii) \quad |w(z) - e^{ic} \cdot B_\theta(z)| = \left| \frac{\rho e^{i\theta} - \rho e^{-i\theta} (w(z))^2}{1 - \rho e^{-i\theta} w(z)} \right| \leq \frac{2\rho}{1 - \rho}$$

Here we can choose any $\rho < 1$. So with $\epsilon > 0$ we choose ρ so small that $\frac{2\rho}{1-\rho} < \epsilon$ and Frostman's Theorem follows.

3. A theorem by Nevanlinna

Theorem 3.1 below was proved by R. Nevanlinna in his article [Nev:xx.] from 1919. On the unit circle T we have the Banach space $L^\infty(T)$ which contains the closed subspace $H_0^\infty(T)$ of boundary values from bounded analytic functions $h(z)$ in D which vanish at $z = 0$. Recall from XX that a bounded Lebesgue measurable function $f(e^{i\theta})$ belongs to $H_0^\infty(T)$ if and only if

$$(0.1) \quad \int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) d\theta = 0 \quad : \quad n = 0, 1, 2, \dots$$

Since $H_0^\infty(T)$ is a closed subset of $L^\infty(T)$ there exists the Banach space

$$(0.2) \quad \mathcal{B} = \frac{L^\infty(T)}{H_0^\infty(T)}$$

For each $F \in L^\infty(T)$ we set:

$$(0.3) \quad \mathbf{nev}(F) = \min \|F - h\|_\infty \quad : \quad h \in H_0^\infty(D)$$

We refer to $\mathbf{nev}(F)$ as the Nevanlinna norm of F . Since $\mathbf{nev}(F)$ is the norm in the quotient space \mathcal{B} it is trivial that

$$(*) \quad \mathbf{nev}(F) \leq |F|_\infty$$

When strict inequality holds in (*) one has the following:

3.1 Theorem *Let $F \in L^\infty(T)$ be such that $\mathbf{nev}(F) < |F|_\infty$. Then there exists $h^* \in H^\infty(T)$ such that*

$$|F(e^{i\theta}) - h^*(e^{i\theta})| = |F|_\infty \quad : \quad \text{almost everywhere on } T$$

Proof. By a change of scale, i.e. replacing F by F times the inverse of $|F|_\infty$ we may assume that $|F|_\infty = 1$. Set

$$(i) \quad \mathcal{H}_F = \{h \in H^\infty(T) : |F - h| \leq 1\}$$

The triangle inequality gives $|h|_\infty \leq 2$ for every $h \in \mathcal{H}_F$. So we have a uniform bound and hence \mathcal{H}_F is a normal family of analytic functions in the unit disc D . By the general result from Ch. 3-XX there exists $h^* \in \mathcal{H}_F$ such that

$$(ii) \quad h^*(0) = \max_{h \in \mathcal{H}_F} |h(0)|$$

Notice that $h^*(0) > 0$. In fact, since $\mathbf{nev}(F) < 1$ is assumed there exists to begin with some $g \in H_0^\infty(T)$ with $|F - g|_\infty \leq 1 - \delta$ for some $\delta > 0$. Hence \mathcal{H}_F contains the functions $g(z) + \delta$ which is $\neq 0$ at the origin.

Sublemma. One has the equality

$$|F - h^* - \phi|_\infty \geq 1 \quad : \quad \phi \in H_0^\infty(T)$$

Proof. Suppose there exists $\phi \in H_0^\infty(T)$ with $|F - h^* - \phi|_\infty = 1 - \delta$ for some $\delta > 0$. If $a > 0$ it follows that

$$|F - (1 + a)h^* - \phi| \leq 1 - \delta + a|h^*|_\infty \leq 1 - \delta + 2a$$

So $h_1 = (1 + \delta/2)h^* + \phi$ belongs to \mathcal{H}_F . Here $h_1(0) > h^*(0)$ which contradicts the maximality of $h^*(0)$ and the Sublemma follows.

Proof continued. Put $G = F - h^*$. The Sublemma means that the norm of the G -image in the Banach space \mathcal{B} is at least 1. At the same time the $L^\infty(T)$ -norm of G is one since $h^* \in \mathcal{H}_F$. It follows that the \mathcal{B} -norm of G is 1. Next, by the duality between $H_0^\infty(T)$ and $H^1(T)$ from XX G yields a linear functional on the Hardy space $H^1(T)$ of norm one. Hence there exists a sequence $\{\phi_n\}$ in $H^1(T)$ with L^1 -norms equal to one such that:

$$(1) \quad \lim_{n \rightarrow \infty} \int_0^{2\pi} G \cdot \phi_n \cdot d\theta = 1$$

Put $c_n = \phi_n(0)$. Since $h^* \in H^\infty(T)$ we notice that

$$(2) \quad \frac{1}{2\pi} \cdot \int_0^{2\pi} h^* \cdot \phi_n \cdot d\theta = h^*(0) \cdot c_n$$

Sublemma 2. There exists some positive constant a such that

$$|c_n| \geq a \quad : \quad n = 1, 2, \dots$$

Proof. Assume the contrary. If $c_n \rightarrow 0$ then (1) and (2) give

$$(3) \quad \lim_{n \rightarrow \infty} \frac{1}{2\pi} \cdot \int_0^{2\pi} F \cdot \phi_n \cdot d\theta = 1$$

But this is impossible since $\mathbf{nev}(F) < 1$. Indeed, this gives the existence of some $\psi \in H^\infty(T)$ with $|F - \psi|_\infty = 1 - \delta$ for some $\delta > 0$. At the same time $c_n \rightarrow 0$ implies that

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi} \cdot \int_0^{2\pi} \psi \cdot \phi_n \cdot d\theta = 0$$

Now we get a contradiction, i.e. (3) cannot hold since

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} |F - \psi| \cdot \phi_n \cdot d\theta \leq |F - \psi|_\infty \cdot |\phi_n|_1 = 1 - \delta$$

Proof continued. Now we may assume that there is some $a > 0$ such that

$$(*) \quad |\phi_n(0)| \geq a \quad : \quad n = 1, 2, \dots$$

At the same time (1) above holds and we also know that $|G|_\infty = 1$. Using this we will show that

$$(**) \quad |G(e^{i\theta})| = 1 \quad : \text{ holds almost everywhere}$$

To prove (**) we use Jensen's inequality from XX which by (*) for every n gives:

$$(***) \quad 2\pi \cdot \text{Log } |a| \leq \int_0^{2\pi} \text{Log } |\phi_n(e^{i\theta})| \cdot d\theta$$

Suppose that (**) does not hold. This gives the existence of some $\rho < 1$ and a set E of positive Lesbesgue measure such that the maximum norm $|G|_E \leq \rho$. Since $|G|_\infty = 1$ it follows that

$$\int_0^{2\pi} G \cdot \phi_n \cdot d\theta \leq \rho \cdot \int_E |\phi_n| \cdot d\theta + \int_{T \setminus E} |\phi_n| \cdot d\theta$$

By (1) above the left hand side tends to one and since the L^1 -norms of ϕ_n are all equal to one, we conclude that

$$(***) \quad \lim_{n \rightarrow \infty} \int_E |\phi_n| \cdot d\theta = 0$$

But this contradicts (1) by the Nevanlinna-Jensen theory in XX. Namely, (****) first entails that

$$(1) \quad \lim_{n \rightarrow \infty} \int_E \text{Log } |\phi_n| \cdot d\theta = -\infty$$

At the same time ϕ_n have finite L^1 -norms which by the material from XXX gives a constant C which is independent of n so that

$$\int_0^{2\pi} \text{Log}^+ |\phi_n| \cdot d\theta \leq C$$

Then we see that (1) violates (***) and hence (**) must hold which finishes the proof of Nevanlinna's theorem.

Remark. Notice that the proof also shows how to find h^* , i.e. it is the extremal function in the family \mathcal{H}_F which maximizes the value at $z = 0$.

3.2 A consequence of Nevanlinna's Theorem. Let $g \in H^\infty(T)$. If \mathcal{J} is an inner function then $\frac{g}{\mathcal{J}}$ belongs to $L^\infty(T)$. But in general this quotient does not belong to $H^\infty(T)$. To compensate for this we study its deviation from $H^\infty(T)$ and obtain:

3.3 Theorem. Let $0 < \epsilon < |g|_\infty$ be such that there exists $f \in H^\infty(T)$ of norm one for which

$$(*) \quad \left| f - \frac{g}{\mathcal{J}} \right|_\infty = \epsilon$$

Then there exists an inner function w such that

$$(**) \quad \frac{\epsilon \cdot w + g}{\mathcal{J}} \in H^\infty(T)$$

Proof. Nevanlinna's Theorem gives some $h^* \in H^\infty(T)$ such that

$$\left| \frac{g}{\mathcal{J}} - h^* \right| = \epsilon \quad : \text{ almost everywhere}$$

Since the inner function \mathcal{J} has absolute value one almost everywhere, it follows that the analytic function $g - h^* \mathcal{J}$ has absolute value ϵ almost everywhere on T . This gives an inner function w such that

$$(iii) \quad \mathcal{J} \cdot h^* - g = \epsilon \cdot w$$

After division with \mathcal{J} we get (**) in Theorem 3.3

4. Marshall's Approximation Lemma

Consider a function $g \in H^\infty(T)$ expressed as

$$(*) \quad g = \sum_{k=1}^{k=N} a_k \cdot \frac{w_k}{J}$$

where w_1, \dots, w_N and J are inner functions while a_1, \dots, a_N are some real numbers. Here we do not assume that they are ≥ 0 or that the sum is one. But we assume that the maximum norm $|g|_\infty < 1$.

4.1 Proposition. *For every $\epsilon > 0$ there exists a convex sum of inner functions V such that*

$$|g - V|_\infty < \epsilon$$

Proof. For the inner functions $\{w_k\}$ and J we have the equalities below on T :

$$(i) \quad \frac{1}{w_k} = \bar{w}_k \quad : \quad \frac{1}{J} = \bar{J}$$

It follows that almost everywhere on T :

$$(ii) \quad \bar{g}(e^{i\theta}) = \sum_{k=1}^{k=N} a_k \cdot \frac{J(e^{i\theta})}{w_k(e^{i\theta})}$$

Next, $W = w_1 \cdots w_N$ is an inner function and (ii) implies that the product

$$(iii) \quad W \cdot \bar{g} \in H^\infty(T)$$

Now we shall use (iii) to get another expression for g . By assumption $|g|_\infty \leq 1 - \delta$ for some $\delta > 0$. Since $\int_0^{2\pi} e^{i\nu t} dt = 0$ for every $\nu \geq 1$ we have the equality below for every complex number λ of absolute value < 1 :

$$(iv) \quad g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\lambda e^{it} + g(z)}{1 + \lambda e^{it} \bar{g}(z)} \cdot dt$$

Next, for each $z \in D$ we can take $\lambda = W(z)$ and obtain

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{W(z)e^{it} + g(z)}{1 + e^{it} \cdot W(z) \cdot \bar{g}(z)} \cdot dt \quad : \quad z \in D$$

The family of functions

$$t \mapsto \Phi_t(z) = \frac{W(z)e^{it} + g(z)}{1 + e^{it} \cdot W(z) \cdot \bar{g}(z)}$$

is equi-continuous with respect to t since $|g|_\infty < 1 - \epsilon$ is assumed. So we can evaluate the integral which defines $g(z)$ by Riemann sums in a uniform manner, i.e. to every $\epsilon > 0$ there exists a finite set $0 \leq t_1 < \dots < t_M < 2\pi$ and positive numbers $\{b_\nu\}$ with $\sum b_\nu = 1$ such that

$$(v) \quad \left| g(z) - \sum b_\nu \cdot \frac{W(z)e^{it_\nu} + g(z)}{1 + e^{it_\nu} W(z) \cdot \bar{g}(z)} \right| < \epsilon$$

Next, since $W(z)$ is inner we notice that the functions

$$\phi_\nu(z) = \frac{W(z)e^{it_\nu} + g(z)}{1 + e^{it_\nu} W(z) \cdot \bar{g}(z)} \quad : \quad 1 \leq \nu \leq M$$

are all inner. Since $\epsilon > 0$ can be made arbitrary small it follows that $g(z)$ can be uniformly approximated by a convex combination of inner functions.

5. Proof of Theorem 0.3

Let $\mathcal{U}(T)$ be the unimodular functions on T , i.e. L^∞ -functions with absolute value one almost everywhere. Elementary geometry shows that the convex hull of $\mathcal{U}(T)$ is dense in the unit ball of $L^\infty(T)$, i.e. to every G with $|G|_\infty \leq 1$ there exist g_1, \dots, g_M in $\mathcal{U}(T)$ such that

$$(*) \quad |G - \sum a_\nu \cdot g_\nu|_\infty < \epsilon \quad : \quad a_1 + \dots + a_M = 1$$

Since products of inner functions are inner, Theorem 0.3 follows if we show that every $g \in \mathcal{U}(T)$ can be uniformly approximated by convex combinations of quotients of inner functions. Again, since products of inner functions again are inner, it suffices to achieve this approximation for a generating set in the multiplicative group of unimodular functions on T . Hence it suffices to consider g -functions which only take two values. After a rotation we may assume that they are $+1$ and -1 . So now we have a measurable set E where $g = 1$ on E and $g = -1$ on $T \setminus E$. Let K be some positive number. Solving the Dirichlet problem with a real valued function u which is K on E and 0 on $T \setminus E$ we take the bounded analytic function

$$f(z) = e^{-(u+iv)}$$

where v is the harmonic conjugate of u . Here the absolute value $|f| = e^{-K}$ on E and $|f| = 1$ on $T \setminus E$. To be precise, equality holds almost everywhere in the sense of Lebesgue. Next, let $\epsilon > 0$. Let \mathbf{C}^e denote the extended complex plane. By the general result from XX there exists for a suitable K a conformal map Φ from the annulus $1 < |z| < e$ to the doubly connected domain

$$\Omega = \mathbf{C}^e \setminus [-\epsilon, 0] \cup [\ell, \ell^*]$$

where ℓ can be made large and $\ell^* \leq \ell + \epsilon$. Here $\Phi(z_0) = \infty$ for some $e^{-K} < |z_0| < 1$. Now we put

$$\Psi = \frac{i + \Phi \circ f}{i - \Phi \circ f}$$

This yields a meromorphic in D with poles at the points $a \in D$ for which $f(a) = z_0$. On T the composed function $\Phi \circ f$ takes values in the union of the real intervals $[-\epsilon, 0]$ and $[\ell, \ell^*]$ which implies that $|\Psi| = 1$ holds almost everywhere on T . With ϵ small we see that $\Psi \simeq 1$ holds on E and when ℓ is large we have $\Psi \simeq -1$ on $T \setminus E$. So the Ψ -function approximates the given unimodular function uniformly up to a small number of order ϵ . Now Ψ may have some poles and we take the Blaschke product B for this so that $B \cdot \Psi$ is analytic in D and since $|B| = 1$ holds almost everywhere this yields an inner function denoted by ψ . Now

$$\Psi = \frac{\psi}{B}$$

is a quotient of inner functions which gives the requested approximation.

6. Proof of Theorem 0.2

Consider some $f \in H^\infty(T)$ of norm 1. Theorem 0.3 gives a convex combination of quotients of inner functions such that:

$$|f - \sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{w_1^\nu}{w_2^\nu}|_\infty < \epsilon$$

Set

$$\mathcal{J} = \prod w_\nu^2$$

This is an inner function and we also get the inner functions

$$(1) \quad W_\nu = \frac{w_1^\nu}{w_2^\nu} \cdot \mathcal{J} \implies |f - \sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{W_\nu}{\mathcal{J}}|_\infty < \epsilon$$

Theorem 3.3 applies to the pair f and $g = \sum a_\nu W_\nu$ and gives an inner function W_* such that

$$(2) \quad \sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{W_\nu}{\mathcal{J}} + \epsilon \cdot \frac{W_*}{\mathcal{J}} \in H^\infty(T)$$

Let g be this analytic function. From (1) and the triangle inequality we have

$$(3) \quad |f - g|_\infty < 2 \cdot \epsilon$$

Finally, with $N = k + 1$ we can apply Marshall's lemma to the function $\frac{1}{1+\epsilon} \cdot g$ which is expressed by a convex combination (*) from (4). This gives a convex sum V formed by inner functions such that

$$(4) \quad |V - g|_\infty < 2 \cdot \epsilon$$

Since $\epsilon > 0$ is arbitrary we get Theorem 0.1 via (3) and (4) above.