B. The Kovalevsky gyroscope

The article *Sur le probleme de la rotatiation d'un corps solide autotur d'un point fixe* was printed in Acta Mathematica 1889. A work for which Kovalevsky received the *Bordin Prize* on Christmas Eve 1888 in Paris. Here follows the first section of the article in English translation.

The problem of the rotation of a rigid body around a fixed point with gravity force g reduces, as is wellknown, to solve the following system of differential equations:

$$
\begin{aligned}
A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma') &:& \quad \frac{d\gamma}{dt} = r\gamma' - q\gamma'' \\
B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma'') &:& \quad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma \\
C\frac{dr}{dt} &= (A-B)rpq + Mg(x_0\gamma' - y_0\gamma) &:& \quad \frac{d\gamma''}{dt} = q\gamma - p\gamma'
\end{aligned}
$$

The constants $A, B, C, M, g, x_0, y_0, z_0$ denote the following: A, B, C are the positive eigenvalues of the intertia operator of the body, i.e. they correspond to lengths of the principal axis when the body is regarded as an ellipsoid. M is the mass of the body, g the force of gravity and x_0, y_0, z_0 the coordinates of the center of mass in the coordinate system whose origin is the fixed point with principal axis determined as above. Fuinally, the six quantities $p, q, r, \gamma, \gamma', \gamma''$ are time-dependent functions where $\omega = (p, q, r)$ is Euler's angular velocity and $(\gamma, \gamma', \gamma'')$ are the coordinates of the vector e_z taken in the body space. which extend to complex analytic functions whose singularities are at most poles.

Until now one has only found two cases where the equations of motion can be integrated and thus solved by quadrature:

The case of Poisson (or Euler): $x_0 = y_0 = y_0$.

The case of Lagrange: $A = B$: $x_0 = y_0 = 0$.

In these two cases the solution is found after integration by theta functions where the six quantities $p, q, r, \gamma, \gamma', \gamma''$ are time-dependent functions which extend to analytic functions whose singularities are at most poles.

Kovalevsky studied the power series solutions to the differential system above and found constraints on their complex Laurent series expansions which led to the conclusion that the constants A, B, C and the position of the center of mass, must be special in order find a fourth integral as as in the two special cases above.

After five pages of calculations an example occurs in section 2 from [ibid] which gives a rigid body whose equations of motion can be solved by quadrature for every initial position. More precisely, in the Kovalevsky gyroscope one has:

(*) $$A = B = 2 \quad C = 1 \quad x_0 = 1 \quad y_0 = z_0 = 0$$

The equality $A = B$ means that the body has a plane of symmetry and the center of mass is placed in this plane. But this center of mass does not belong to a principal axis so the gyroscope fails to satisfy the assumption which had been treated earlier by Lagrange. When (*) holds Kovalevsky found a *fourth integral* of the differential system expressed by a polynomial of the 4:th degree of the variables $p, q, r, \gamma, \gamma', \gamma''$. Once this is achieved there remains a considerable work to express the solution. In [ibid] more than ten pages are devoted to the study of *ultra-elliptic functions* which appear in the time-dependents solutions using the fourth integral.

The use of complex analysis. To find the fourth integral of the system when (*) holds Kovalevsky employed analytic function theory. Let us describe how one derives the fourth integral. We prefer to use other notations for p, q, r, i.e. Euler's angular velocities are denoted by

$$\omega_1 = p \quad : \quad \omega_2 = q \quad : \quad \omega_3 = r$$

Similarly, lower indices are used for the γ-functions:

$$\gamma_1 = \gamma \quad : \quad \gamma_2 = \gamma' \quad : \quad \gamma_3 = \gamma''$$

When A, B, C satisfy (*) and $gM = 1$ we obtain a system of six differential equations:

$$\begin{aligned} 2\dot{\omega}_1 &= \omega_2\omega_3 \\ 2\dot{\omega}_2 &= -\omega_1\omega_3 - \gamma_3 \\ \dot{\omega}_3 &= \gamma_2 \end{aligned}$$

$$\begin{aligned} \dot{\gamma}_1 &= \gamma_2\omega_3 - \gamma_3\omega_2 \\ \dot{\gamma}_2 &= -\gamma_1\omega_3 + \gamma_3\omega_1 \\ \dot{\gamma}_3 &= -\gamma_1\omega_2 - \gamma_2\omega_1 \end{aligned}$$

Invariant integrals. Since γ is a unit vector one has

(1) $$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

Next, since the sum of kinetic and potential energy is constant one has:

(2) $$\omega_1^2 + \omega_2^2 + \frac{\omega_3^3}{2} + \gamma_1 = E$$

Finally, the time derivative of the angular momentum $\mathfrak{M}$ is $\perp$ to e_z which gives the third algebraic equation:

(3) $$2\omega_1\gamma_1 + 2\omega_2\gamma_2 + \omega_3\gamma_3 = F$$

where F is a constant.

Kovalevsky's fourth integral. There exists one more integral to the ODE-system. To obtain it we introduce the imaginary unit. The first two equations for the ω-system give:

(i) $$2(\dot{\omega}_1 + i\dot{\omega}_2) = -i\omega_3(\omega_1 + i\omega_2) - i\gamma_3$$

Next, the first two equations for the γ-system give:

(ii) $$\dot{\gamma}_1 + i\dot{\gamma}_2 = i\gamma_3(\omega_1 + i\omega_2) - i\omega_3(\gamma_1 + i\gamma_2)$$

Put

$$\phi = (\omega_1 + i\omega_2)^2 + (\gamma_1 + i\gamma_2)$$

It follows from (i-ii) that

$$\dot{\phi} = -(\omega_1 + i\omega_2) \cdot [i\omega_3(\omega_1 + i\omega_2) + i\gamma_3] + i\gamma_3(\omega_1 + i\omega_2) - i\omega_3(\gamma_1 + i\gamma_2)$$

A computation shows that the right hand side becomes $-i\omega_3 \cdot \phi$ and hence ϕ satisfies the differential equation

(iii) $$\dot{\phi} = -i\omega_3 \cdot \phi$$

Since the function $\omega_3(t)$ is real-valued this differential equation implies that the t-derivative of $\log(\phi(t))$ is purely imaginary. The equality

$$\mathfrak{Re}\ \log(\phi(t)) = \log\ |\phi(t)|$$

implies that the *absolute value* of ϕ is constant. Hence there exists a real constant k such that

(4) $$k^2 = |(\omega_1 + i\omega_2)^2 + \gamma_1 + i\gamma_2|^2 = (\omega_1^2 - \omega_2^2 + \gamma_1)^2 + (2\omega_1\omega_2 + \gamma_2)^2$$

Remark Above (4) gives the fourth integral which is used to solve the equations of motion by quadrature. In contrast to the case treated by Euler where the solution by quadrature is achieved by an elliptic integral of the first kind, the solution of the Kovalesvky gyroscope involves *hyper-elliptic integrals*. For details we refer to Kovalevsky's original article which in addition to the example of a special gyroscope contains very interesting calculations. Personally I think that every student should try to study original work. The material about hyperelliptic functions and their integrals in Kovalevsky's article offers an account which I personally find more transparent than more recent publications on these topics. The invariant integrals (1-3) will be established in the section devoted to rigid bodies.

Two years after Kovalevsky's disease, Liapounov proved that the example (*) is *unique*, i.e. it gives the sole example except the classical, where the equations of motion can be solved by quadrature. In an article published in Acta Mathematica 1892, Königsberg expressed the motion of Kovalevsky's gyroscope in Eulerian angles where the solution consists of generalised theta-functions. Königsberg's formula together with a computer provide a picture of the motion of the Kovalevsky gyroscope under arbitrary initial conditions. To avoid confusions we remark that there exist examples where the differential system for the motion can be integrated *provided* that special initial conditions are given. But these specific examples were not at stake in the far more difficult problem treated by Kovalesvky.

Another remark. Kovalevsky gave courses on many different subjects during her years in Stockholm. The interplay between equations derived from the "real world" of mechanics and complex integrals which lead to elliptic functions was put forward in these lectures. Let us illustrate this by an example which normally belongs to a course devoted to analytic function theory. Consider Weierstrass' $\mathfrak{p}$-function which is doubly meromorphic with respect to some lattice in $\mathbf{C}$ generated by two $\mathbf{R}$-linearly independent complex vectors ω_1 and ω_2. Set

$$e_1 = \frac{\omega_1}{2}\,, \quad e_2 = \frac{\omega_1}{2}\,, \quad e_3 = \frac{\omega_1 + \omega_2}{2}\,.$$

Now there exists Jacobi's elliptic function

$$\mathfrak{sn}(z) = \frac{\sqrt{e_1 - e_2}}{\sqrt{\mathfrak{p}(z) - e_3}}$$

Jacobi's inversion formula asserts that

$$z = \frac{1}{\sqrt{e_2 - e_3}} \cdot \int_0^{\mathfrak{sn}(z)} \frac{dz}{\sqrt{(1-z^2)(1-\chi^2 z^2)}}$$

This inversion formula is a consequence of conservation laws applied to a two-particle system describing the motion of two mass-points which both perform a periodic motion in a simple pendelum. The addition formula for the $\mathfrak{p}$-function, and more generally for any elliptic function can also be derived via laws of classical mechanics. So students interested in analytic function theory or algebraic geometry devoted to projective curves in characteristic zero, should keep in mind that results which express properties of rational functions on such curves have a natural interpretation via classical mechanics and may therefore be considered as a Law of Nature rather than a mathematical discovery.

2. Rigid Bodies

In a rigid body K distances between points remain constant under motion, even when a sudden force acts on K by an impact. This implies that when K is placed in $\mathbf{R^3}$, the position of an arbitrary point $q \in K$ can be determined via six cordinates. To see this we choose a point $p \in K$ and three other points q_1, q_2, q_3 in K such that the euclidian distances satisfy

(1) $$|q_\nu - p| = 1 \quad \text{and} \quad |q_\nu - q_j| = \sqrt{2} \quad : \quad \nu \neq j$$

If necessary we adjoin such points withous mass and rigid bars without mass connecting them to p. Pythagoras' theorem implies this that the three unit vectors $\{q_\nu - p\}$ are pairwise orthogonal and we get the body space where p is the origin and these three vectors is an orthonormal basis. Denote this orthonormal space by $\mathcal{V}_K$. Next, when K is placed in the euclidian space $\mathbf{R}^3$ the postions of p and $\{q_\nu\}$ give four points denoted by p^*, q_1^*, q_2^*, q_3^*. Since distances are preserved (1) hold for the (*)-marked points, and Pythagoras's theorem entails that the vectors $\{\xi_\nu^* = q_\nu^* - p^*\}$ are pairwise orthogonal unit vectors. Next, let $q \in K$ be an arbitrary point. In the body space we get coordinates a_1, a_2, a_3 such that

(1) $$q = -p = a_1(q_1 - p) + a_2(q_2 - p) + a_3(q_3 - p)$$

Put $\xi_k = q_k^* - p^*$ for each k. Passing to the (*) marked points we write

(2) $$q^* - p^* = b_1 \cdot \xi_1^* + b_2 \cdot \xi_2^* + b_3 \cdot \xi_3^*$$

Now the reader can verify that the b-coordinates are equal to the a-coordinates, i.e.

(3) $$b_k = a_k$$

hold for each k.

Conclusion. For each $q \in K$ the position of q^* in $\mathbf{R}^3$ is determined by the positions of p^* and the three orthogonal unit vectors $\{q_\nu^* - p^*\}$. Together with (3) it follows that there exists an orthogonal 3×3-matrix S such that

(*) $$q^* - p^* = S(q - p)$$

hold for every $q \in K$ where S is regarded as a linear map from the orthogonal body space into the euclidian space $\mathbf{R}^3$. Recall that the determinant of an orthogonal matrix is +1 or -1. The set of orthogonal matrices with determinant +1 is a group denoted by S0(3). Under a continuous motion of K in $\mathbf{R}^3$ the sign of the Smatrix is unchanged and one can always start with the orthonormal basis in the body space so that the S-matrix in (*) has determinant +1 which means that it preserves orientation. At the same time p can be placed at an arbitrary point p^* in $\mathbf{R}^3$. It follows that the *configuration space* for K is given by the product space $\mathbf{R}^3 \times \mathrm{SO}(3)$. We conclude that a rigid body has *six degrees of freedom* when it moves in $\mathbf{R}^3$. Let us now consider a time-dependent motion of K. So when t is the time variable the vector valued function $p^*(t)$ describes the motion of p and we also have the SO(3)-valued function $t \mapsto S_t$ where

(**) $$q^*(t) = p^*(t) + S_t(q - p) \quad 1 \leq \nu \leq 3 \quad : \quad q \in K$$

2.1 Remark. Above the construction started from a chosen point $p \in K$. Consider another point $\rho \in K$ which gives

$$q^*(t) = \rho^*(t) + (p^*(t) - \rho^*(t)) + S_t(q - p) = \rho^*(t) - S_t(\rho - p) + S_t(q - p) = \rho^*(t) + S_t(q - \rho)$$

This shows that the rotation matrix S_t is the same when the body space is centered at ρ.

2.2 Euler's Angular velocity

Consider a motion of the rigid body whose time dependent functions are of class C^2 at least. We can take time derivatives of the nine elements in the matrix S_t. This yields for each t the 3×3-matrix $\dot{S}_t$ which again is a linear map from $\mathcal{V}_K$ into $\mathbf{R^3}$. Let us also regard the *inverse* linear map S_t^{-1}. Since S_t is orthogonal this inverse is the adjoint matrix S_t^*. Now $S_t^* \circ \dot{S}_t$ is a linear map from $\mathcal{V}_K$ into itself.

2.2.1 Proposition. *The matrices $S_t^* \circ \dot{S}_t$ are anti-symmetric for all t.*

Proof. Since S_t is orthogonal $t \mapsto \langle S_t(q), S_t(p)\rangle$ is a constant function of t for all pairs p, q in $\mathcal{V}_K$. Hence the time derivative is zero which gives

$$0 = \langle \dot{S}_t(q), S_t(p)\rangle + \langle S_t(q), \dot{S}_t(p)\rangle = \langle S_t^* \circ \dot{S}_t(q), p\rangle + \langle q, S_t^* \circ \dot{S}_t(p)\rangle$$

and the requested anti-symmetry follows.

Recall from linear algebra that every anti-symmetric matrix is expressed by a vector product. Thus, for each t there exists a unique vector $\omega(t) \in \mathcal{V}_K$ such that

$$S_t^* \circ \dot{S}_t(q) = \omega(t) \times q \quad : \; q \in \mathcal{V}_K$$

From (**) above Remark 2.1 it follows that the time derivatives of points $q \in K$ satisfy:

$$\dot{q}^*(t) = \dot{p}^*(t) + \dot{S}_t(q) \tag{2.2.2}$$

Since $S_t \circ S_t^*$ is the identity we obtain

$$\dot{q}^*(t) = \dot{p}^*(t) + S_t \circ S_t^* \circ \dot{S}_t(q) = \dot{p}^*(t) + S_t(\omega(t) \times q) \tag{2.2.3}$$

One refers to $\omega(t)$ as *Euler's angular velocity* which by the constructions above is a vector-valued on the body space.

2.3 Kinetic energy and momentum

Let K be a rigid body which consists of a finite set of mass points $p_1, \ldots, p_N$. One can imagine that they are joined by rigid bars with zero mass. Put

$$\mathfrak{o} = \frac{1}{M} \sum_{\nu=1}^{\nu=N} m_\nu \cdot p_\nu \quad \text{where} \quad M = \sum m_\nu \text{ is the total mass}$$

One refers to $\mathfrak{o}$ as the the center of mass, or simply the mass-point of K. When K moves in $\mathbf{R^3}$ the velocities varies between individual points because of rotation. Therefore it is no sufficient to regard a single point, such as the center of mass to obtain the kinetic energy. Let us choose $\mathfrak{o}$ as the origin in the body space $\mathcal{V}_K$. Under a motion we get time dependent rotation matrices S_t and for every mass point $p_\nu \in K$ one has

$$\dot{p}_\nu^*(t) = \dot{o}^*(t) + S_t(\omega_t \times p_\nu) \tag{2.3.1}$$

It follows that the kinetic energy becomes

$$T = \frac{1}{2} \cdot \sum m_\nu \cdot \langle \dot{p}_\nu^*, \dot{p}_\nu^*\rangle =$$

$$\frac{1}{2} \cdot \sum m_\nu \langle \dot{p}_\nu^*, \mathfrak{o}^*\rangle + \frac{1}{2} \cdot \sum m_\nu \cdot \langle \dot{p}_\nu^*, S_t(\omega_t \times p_\nu)\rangle \tag{2.3.2}$$

The first sum above is equal to $\frac{1}{2} \cdot M \cdot \langle \dot{o}^*, \dot{o}^*\rangle$. Expanding the second term the reader can verify that it becomes:

$$\frac{1}{2} \cdot \sum m_\nu \cdot \langle S_t(\omega_t \times p_\nu), S_t(\omega_t \times p_\nu)\rangle \tag{2.3.3}$$

Since the orthogonal matrix S_t preserves inner products it follows that (2.3.3) is equal to

$$\frac{1}{2} \cdot \sum m_\nu \cdot \langle \omega_t \times p_\nu, \omega_t \times p_\nu \rangle \tag{2.3.4}$$

This suggests that we introduce a linear operator on $\mathcal{V}_K$.

2.3.5 Definition *The central operator of inertia is the linear operator defined on* $\mathcal{V}_K$ *by*

$$q \mapsto \sum m_\nu \cdot p_\nu \times (q \times p_\nu)$$

It is denoted by $\mathcal{M}_{\mathfrak{o}}$.

Recall that the vector product satisfies the following for each pair of vectors u, v in the orthonormal space $\mathcal{V}_K$:

$$\langle u, (u \times v) \times u \rangle = ||u \times v||^2$$

Applied to (2.3.4) above we get:

2.3.6 Theorem *The kinetic energy is expressed by the equation*

$$T = \frac{1}{2} M \cdot |\dot{o}^*|^2 + \frac{1}{2} \langle \omega_t, \mathcal{M}_{\mathfrak{o}}(\omega_t) \rangle$$

One refers to $\frac{1}{2}\langle \omega_t, \mathcal{M}_{\mathfrak{o}}(\omega_t) \rangle$ *as the rotational kinetic energy and it is denoted by* T_{rot}.

Change of center. It is sometimes easier to pursue the motion with respect to another point in K than the mass point. For this purpose we give

2.3.7 Definition *The inertia operator in a body space centered at a point* $p \in K$ *is defined by*

$$q \mapsto \sum m_\nu \cdot (p_\nu - p) \times [(q - p) \times (p_\nu - p)]$$

It is denoted by $\mathcal{M}_p$.

Exercise. Show the equality below for each pair of points p, q in K:

$$\mathcal{M}_p(q) = \mathcal{M}_{\mathfrak{o}}(q - p) + M \cdot (p \times (q \times p)) \tag{2.3.8}$$

and conclude that ine has the equation

$$T = \frac{1}{2} M \cdot ||\dot{p}^*||^2 + \frac{1}{2} \langle \omega_t, \mathcal{M}_p(\omega_t) \rangle \tag{2.3.9}$$

2.4 Angular momentum

When K is in motion we define for each time value t the vector

$$(2.4.1)\qquad \mathfrak{M}_{\mathfrak{o}}(t) = \sum m_\nu \cdot (p^*_\nu(t) - \mathfrak{o}^*(t)) \times \dot{p}^*_\nu$$

Exercise. Show the equality

$$(2.4.2)\qquad \mathfrak{M}_{\mathfrak{o}}(t) = S_t(\mathcal{M}_o(\omega_t))$$

Show also that the time derivative of the vector valued function $\mathfrak{M}_{\mathfrak{o}}$ becomes:

$$(2.4.3)\qquad \frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = S_t\left[\,\mathcal{M}_o(\dot{\omega}_t)) + \omega_t \times \mathcal{M}_o(\omega_t)\,\right]$$

One refers to (**) as Euler's identity for the time derivative of angular momentum.

The time derivative of $\mathfrak{M}_{\mathfrak{o}}$. Since the vector product is anti-commutative the reader can check that (2.4.1) gives the equation

$$(2.4.4)\qquad \frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = \sum m_\nu \cdot (p^*_\nu(t) - \mathfrak{o}^*(t)) \times \ddot{p}^*_\nu$$

2.5 Equations of motion

Now we study the effect of fores acting on a rigid body during its motion. At a given time monent we suppose that force vectors $\{F_\nu\}$ act on the mass-points $\{p_\nu\}$. Then (2.4.4) and Newtons formula give:

$$(2.5.1)\qquad \frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = \sum (p^*_\nu(t) - \mathfrak{o}^*(t)) \times F_\nu$$

External and inner forces. During a motion inner forces keep the body rigid. They appear in pairs f_{ij} and $-f_{ij}$ where f_{ij} acts on the mass-point p_j while the opposed force vector $-f_{ij} = f_{ji}$ acts on p_i. Apart from these there exist *external forces* denoted by F^{ext}_ν. Since the vector product is anti-commutative, the reader can check that the total effect of inner forces disappears in (2.5.1) and hence one has:

$$(2.5.2)\qquad \frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = \sum (p^*_\nu(t) - \mathfrak{o}^*(t)) \times F^{\text{ext}}_\nu$$

Example In $\mathbf{R^3}$ we denote the vertical direction by z. So here gravity yields an external force whose strength is g. At every mass-point p_ν we have

$$F^{\text{ext}}_\nu = -m_\nu \cdot g \cdot e_z$$

So when gravity is the sole external force acting on the rigid body one has:

$$(2.5.3)\qquad \frac{d}{dt}(\mathfrak{M}_{\mathfrak{o}}) = -g \cdot \sum m_\nu \cdot (p^*_\nu(t) - \mathfrak{o}^*(t)) \times e_z = 0$$

Notice that the vector in the right hand side is $\perp$ to e_z. It follows that the function

$$t \mapsto \langle \mathfrak{M}_{\mathfrak{o}}(t), e_z \rangle$$

is constant. Notice also that (2.4.x) gives

$$\langle \mathfrak{M}_{\mathfrak{o}}(t), e_z \rangle = \langle S_t(\mathcal{M}_o)(\omega_t), e_z \rangle = \langle \mathcal{M}_o)(\omega_t), S^*_t(e_z) \rangle$$

Hence we have proved the following:

2.5.4 Theorem. *When gravity is the sole external force on a body there exists a constant C such that*

$$\langle \mathcal{M}_{\mathfrak{o}}(\omega_t), S_t^*(e_z)\rangle = C$$

2.6. Rotation around the mass-point

We shall study a rigid body K whose mass-point remains fixed during the motion. Recall that the motion is described by matrices $\{S_t\}$ which map a body space $\mathcal{V}_K$ centered at $\mathfrak{o}$ into $\mathbf{R^3}$. In $\mathbf{R^3}$ the z-axis is vertical where gravity acts in the negative z-direction and we assume that this is the sole external force acting on K. Since the mass-point is fixed it means that no external forces appear during the rotation. Now we shall describe the equations of motion. For this purpose we consider the central angular momentum which from previous results becomes:

$$\mathfrak{M}_{\mathfrak{o}}(t) = S_t(\mathcal{M}(\omega(t))$$

By 2.5.2 the time-derivative is zero and applied to the right hand side Leibniz' rule gives

$$\dot{S}_t(\mathcal{M}_{\mathfrak{o}}(\omega(t)) + S_t(\mathcal{M}_{\mathfrak{o}}(\dot{\omega}(t)) = 0$$

Applying S_t^* on both sides it follows that

$$(*) \qquad \omega(t) \times \mathcal{M}_{\mathfrak{o}}(\omega(t)) + \mathcal{M}_{\mathfrak{o}}(\dot{\omega}(t)) = 0$$

The system (*) are called the the *Euler-Lagrange equations* for the body. To solve (*) we will choose a suitable *orthonormal basis* in $\mathcal{V}_K$ adapted to the linear operator $\mathcal{M}$.

2.6.1 Principal axes. Recall from § xx that $\mathcal{M}_{\mathfrak{o}}$ is a symmetric linear operator on $\mathcal{V}_K$. The spectral theorem for symmetric matrices gives an orthogonal basis e_1, e_2, e_3 which diagonalizes $\mathcal{M}_{\mathfrak{o}}$ and there exist constants A_1, A_2, A_3 such that

$$\mathcal{M}(e_i) = A_i e_i \quad 1 \leq i \leq 3$$

Above each $A_i > 0$ unless K is a linear body, i.e. where all the mass is concentrated to a single line. We ignore to discuss this special case and express the vectors $\omega(t)$ in this basis:

$$\omega(t) = \omega_1(t)e_1 + \omega_2(t)e_2 + \omega_3(t)e_3$$

The $e-$basis is chosen so that $e_1 \times e_2 = e_3$ and so on, i.e. it is positively oriented with respect to the vector product. Rules for vector products show that the system (*) becomes:

$$\begin{aligned} A_1 \cdot \dot{\omega}_1 + (A_3 - A_2)\omega_2\omega_3 &= 0 \\ A_2 \cdot \dot{\omega}_2 + (A_1 - A_3)\omega_1\omega_3 &= 0 \\ A_3 \cdot \dot{\omega}_3 + (A_2 - A_1)\omega_1\omega_2 &= 0 \end{aligned}$$

This first order system of differential equations has a unique vector-valued solution $\omega(t)$ when initial values are given a time $t = 0$.

2.6.2. Exercise. Show that the differential system above give positive constants T and E such that

$$\text{(i)} \qquad A_1 \cdot \dot{\omega}_1^2 + A_2 \cdot \dot{\omega}_2^2 + A_3 \cdot \dot{\omega}_3^2 = 2T$$

$$\text{(ii)} \qquad A_1^2 \cdot \dot{\omega}_1^2 + A_2^2 \cdot \dot{\omega}_2^2 + A_3^3 \cdot \dot{\omega}_3^2 = E$$

2.6.3 Solution by quadrature. Consider the case when $A_3 < A_2 < A_1$. From (i-ii) we can eliminate ω_2^2 and ω_3^2 and from this the reader can deduce that ω_1 satisfies a differential equation of the form

$$\dot{\omega}_1 = + \text{or } - C \cdot \sqrt{\alpha^2 - \omega_1^2} \cdot \sqrt{\beta^2 - \omega_1^2}$$

where C, α, β are constants which in addition to the body-constants A_1, A_2, A_3 depend upon T and E. Except for very special case one has $\alpha \neq \beta$ and with $0 < \alpha < \beta$ it follows that $\omega_1(t)$ is a periodic function of t which oscillates between $-\alpha$ and α. The time intervals for a full oscillation is expressed by an elliptic integral. More precisely, if T_1 is the full period one has the equation

$$T_1 = 4C \cdot \int_0^\alpha \frac{dw}{\sqrt{\alpha^2 - \omega_1^2} \cdot \sqrt{\beta^2 - \omega_1^2}}$$

The functions $\omega_3(t)$ and $\omega_3(t)$ are also periodic with certain periods T_2 and T_3. Except for special cases the three periods are different and the whole rotation around the mass-point is "chaotic".

2.6.4 Rotation outside the mass-point

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Consider the case when K rotates around a point $p \neq \mathfrak{o}$ while gravity is the sole external force. Herte we let the body space be centered at p. The linear operator $\mathcal{M}_p$. is symmetric so by the spectral theorem we can choose an orthonormal basis in $\mathcal{V}_K$ where $\mathcal{M}_p$ is diagonal. Consider the angular momentum

$$\mathfrak{M}_p(t) = S_t(\mathcal{M}_p(\omega(t))$$

2.6.5 Exercise. Show that the time derivative

$$\text{(1)} \qquad \frac{d}{dt}(\mathfrak{M}_p(t) = M \cdot g \cdot o^*(t) \times e_z \implies$$

$$S_t\big[\omega(t) \times \mathcal{M}(\omega(t)) + \mathcal{M}(\dot{\omega}(t))\big] = -M \cdot g \cdot o^*(t) \times e_z$$

Applying S_t^* to both sides one gets

$$\text{(2)} \qquad \omega(t) \times \mathcal{M}_p(\omega(t)) + \mathcal{M}_p(\dot{\omega}(t)) = -M \cdot g \cdot \mathfrak{o} \times S_t^*(e_z)$$

In a basis where $\mathcal{M}_p(e_1) = A$, $\mathcal{M}_p(e_2) = B$ and $\mathcal{M}_p(e_3) = C$ we get a system of differential equations which was presented in the introduction from Kovalevsky's article. Notice that the Euler's equations no longer are homogenous since the function

$$t \mapsto S_t^*(e_z)$$

with values in the body space appears above. So here one encounters a system with six time-dependent functions, where three of them express time derivatives of the components of the vectors $S_t^*(e_z)$.

3. Eulerian angles

Given two angles $0 < \theta < \pi$ and $0 \leq \phi \leq 2\pi$ we put

$$e_3 = \cos\theta \cdot e_z + \sin\theta(\cos\phi \cdot e_x + \sin\phi \cdot e_y)$$

To this vector we associate the unit vectors

$$\xi_2 = -\sin\theta \cdot e_z + \cos\theta\,[\cos\phi \cdot e_x + \sin\phi \cdot e_y] \quad : \quad \xi_1 = \sin\phi \cdot e_x - \cos\phi \cdot e_y$$

Notice that $\xi_1 \times \xi_2 = e_3$ and hence the triple ξ_1, ξ_2, e_3 is a positively oriented orthonormal frame. A general ON-frame (e_1, e_2, e_3) arises when we introduce another angular variable ψ and set:

$$e_1 = \cos\psi\,\xi_1 + \sin\psi\,\xi_2 \quad : \quad e_2 = \sin\psi\,\xi_1 + \cos\psi\,\xi_2$$

We refer to θ, ϕ, ψ as the Euler angles defining this ON-frame. Let us now consider a rigid body which rotates around the origin and let e_1, e_2, e_3 be some postively oriented ON-frame in the body space. We get the time dependent vectors in $\mathbf{R^3}$:

$$e^*_\nu(t) = S_t(e_\nu)$$

For each t they give an ON-frame in $\mathbf{R^3}$. We assume that $e^*_3(t)$ is not parallell to the z-axis. Then there exist time dependent functions $\theta(t)$ and $\phi(t)$ such that

$$e^*_3(t) = \cos\theta(t) \cdot e_z + \sin\theta(t)(\cos\phi(t) \cdot e_x + \sin\phi(t) \cdot e_x)$$

The remaining vectors $e^*_1(t)$ and $e^*_2(t)$ are determined by the two angular functions $\theta(t), \phi(t)$ and a ψ-function which corresponds to a rotation of K around e^*_3. The result is that the time dependent rotation matrix S_t is a function of the three angular variables. Now we can take time derivatives and in this way express Euler's angular velocity by the three angle functions and their time derivatives. A straightforward calculation which is left to the reader gives:

$$\begin{aligned} \omega_1 &= \dot\phi \cdot \sin\theta\sin\psi + \dot\theta\cos\psi \\ \omega_2 &= \dot\phi \cdot \sin\theta\cos\psi - \dot\theta\sin\psi \\ \omega_3 &= \dot\phi \cdot \cos\theta + \dot\psi \end{aligned}$$

The kinetic energy. Suppose that e_1, e_2, e_3 yield principal axes for the operator of inertia. Hence

$$T = \frac{1}{2}[A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2]$$

Inserting the equations above we express T as a function of the angle functions and their time derivatives.

3.1 The symmetric case. Assume that $A_1 = A_2$. Then a simple reduction yields:

$$T = \frac{1}{2}A_3(\dot\psi + \dot\phi\cos\theta)^2 + \frac{1}{2}A_1(\dot\theta^2 + \dot\phi^2\sin^2\theta)$$

Above the ψ-function only enters via its time derivative. Using this, Lagrange reduced the system to a single differential equation for θ. One refers to the symmetric case above as Lagrange's top.

3.2 The Kovalevsky's gyroscope in Eulerian angles

Here $A_1 = A_2 = 2$ and $A_3 = 1$ and the center of mass $\mathfrak{o}$ is placed at ae_1 for some $a > 0$. The kinetic energy becomes

$$T = \frac{1}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + (\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

To study the effect of the external gravity force we express the inner product

$$\langle e_1^*(t), e_z\rangle = -\cos\psi\langle\xi_1, e_z\rangle + \sin\psi\langle\xi_2, e_z\rangle = -\sin\psi\cdot\sin\theta$$

Preservation of energy gives a constant E such that

$$\frac{1}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + (\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - ag\cdot\sin\psi\cdot\sin\theta = E \tag{1}$$

Next, we consider the D'Alembert -Lagrange equations. Here ϕ is a cyclic variable which gives a constant C such that

$$\cos\theta\cdot(\dot{\psi} + \dot{\phi}\cos\theta) + 2\dot{\phi}\cdot\sin^2\theta = C \tag{2}$$

From this equation we can eliminate $\dot{\phi}$ so that the constant energy integral is a function of ψ, θ and its time derivatives. We have also the Lagrangean equation for the ψ-variable which yields

$$\frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta) = -ag\cdot\cos\psi\cdot\sin\theta \tag{3}$$

Above (1-3) is a system of ODE-equations which has a unique solution for any given initial condition. In the article [Acta 1892] Königsberg implemented Kovalevsky's fourth integral to express the solutions via integrals of hyperelliptic functions which means that one encounters generalised theta-functions which appear in the rich and beautiful theory which foremost is due to Abel, Jacobi, Hermite and Weierstrass. So at this point a "pure mathematical study is needed. Apart from this it is of course valuable to find numerical solutions to the system above which describes the motion of Kovalevsky's gyroscope in $\mathbf{R}^3$.