Sobolev inequalities

Let $n \geq 2$ abnd $F(x)$ is a funvtion with compact support contaimed in a ball $\{|x| \leq K\}$ fopr some $K > 0$ whose partial derivtives $\{\partial_j(F)\}$ belong to $L^1(\mathbf{R}^n)$. Let ω denote points on the unit sphere S^{n-1}. For each fixed ω we set

$$F_\omega(x) = \int_0^\infty \frac{\partial}{\partial r}(F(x - r\omega) \cdot b(r)\, dr$$

A psrtial integration shows that the right hand side becomes

$$F(x) + \int_0^\infty F(x - r\omega) \cdot b'(r)\, dr$$

Now $b'(r) \neq 0$ only occurs if $2K < r < 3K$ so if $|x| \leq K$ it is clear that the last integral is zero because F vanishes when $|x| > K$. Hence $F_\omega(x) = F(x)$ when $|x| \leq K$. This implies that the L^q-norm of F is majorized by the L^q-norm of F_ω for every $q \geq 1$ and every $\omega \in S^{n-1}$. Since L^q-norms satisfy the triangle inequality we conclude that if $a(\omega)$ is some non-negaitve function on S^{n-1} such that

$$\int_{S^{n-1}} a(\omega)\, d\omega = 1$$

then every L^q-norm of F is majorized by that of

$$F_a(x) = \int_{S^{n-1}} \int_0^\infty \int_0^\infty \frac{\partial}{\partial r}(F(x - r\omega) \cdot b(r) a(\omega)\, dr d\omega$$

Notice that

$$\frac{\partial}{\partial r}(F(x - r\omega) = \sum_{j=1}^{j=n} \omega_j \cdot \frac{\partial}{\partial x_j}(F(x - rw)$$

Define the functions $h_1, \ldots, h_n$ in $\mathbf{R}^n$ by

$$h_j(r\omega) = \frac{b(r) \cdot \omega_j \cdot a(\omega)}{r^{n-1}}$$

Since $dx = r^{n-1} d\omega$ holds when we pass to polar coordinates in $\mathbf{R}^n$, it follows from (xx) that

$$F_a(x) = \sum \int_{\mathbf{R}^n} \partial_j(F(x - y)) \cdot h_j(y)\, dy$$

The individual h-functions satisfy

$$|h_j(r\omega)| \leq \frac{b(r) a(\omega)}{r^{n-1}} \leq C \cdot \frac{b(r)}{r^{n-1}}$$

where C is the maximum norm of a. So if σ_{n-1} denotes the $n-1$-dimensional volume of S^{n-1} and $s > 1$ it follows that

$$\int_{\mathbf{R}^n} |h_j(x)|^s\, dx \leq \sigma_{n-1} \int_0^\infty \frac{b(r)^s}{r^{(n-1)(s-1)}}$$

The last integral is convergent provided that

$$(n-1)(s-1) < 1 \implies 1 \leq s < \frac{n}{n-1}$$

Conclusion. If $p \geq 1$ and each $\partial_j(F)$ belongs to $L^p(\mathbf{R}^n)$ then Hölder's inequality entails that F_a belongs to L^{p_*} when

$$(*) \qquad \frac{1}{p_*} > \frac{1}{p} - \frac{1}{n}$$

The case of equality. The inequality (*) holds for every $p \geq 1$, i.r. even if $p = 1$. To get (*) in the critical case when equality holds one must appeal to the Calderon-Zygmund inequality and use the rather spcial properties of the h-functions above. More precisely, one should choose $a(\omega)$ so that not only (xx) above holds, but also

$$\int_{S^{n-1}} \omega_j \cdot a(\omega)\, d\omega = 0 \quad : \quad 1 \leq j \leq n$$

The fact that (xx) entails that Theorem xx also holds in the critical case when $\frac{1}{p_*} = \frac{1}{p} - \frac{1}{n}$ follows by general facts about convolution operators. More precsiely, (xx) entails that convolution by the h-functions satisfy a certain weak-type estimate in the critical case when one takes $p = 1$ and after Thorin's interpolation theorem is applied. We leave this to the reader who may consult text-books for details. See in particular [Stein-Fourier analysis] and the reader may also consult Chapter XIV: § 4 in [Dunford-Schwarz] for a further discussion of Sobolev inequalities.

Passage to higher order derivatives. By repeated use of Theorem XX it follows that if $F(x)$ has bounded support and $k \geq 2$ is an integer such that the partial derivatives $\frac{\partial^\alpha}{\partial x^\alpha}(F)$ belong to L^p for some $p > 1$, then

$$F \in L^{p_*}(\mathbf{R}^n) \quad \text{where} \quad \frac{1}{p_*} = \frac{1}{p} - \frac{k}{n}$$

Finally, if it happens that $\frac{1}{p} - \frac{1}{n} < 0$ one cann establish a continuity result which goes as follows:

Consider a bounded open set Ω in $\mathbf{R}^n$ with a smooth boundary, i.e. of class C^∞. Let $p \geq 1$ and k is a positive integer which yields the largest integer m such that

$$\frac{1}{m} < k - \frac{n}{p}$$

Then the folloewing hold:

Theorem. *Let $F(x)$ be a function in Ω whose partial derivatives up to order k belong to $L^p(\Omega)$. Then every derivative of order $\leq m$ exists and is even a continuous function defined on the closure of Ω.*

Sobolev inequalities

Theorem. *Let $p > 1$ and assume that each $\partial_j(F)$ belongs to $L^p(\mathbf{R}^n)$. Then it follows that $F \in L^{p_*}(\mathbf{R}^n)$ where*

$$\frac{1}{p_*} = \frac{1}{p} - \frac{1}{n}$$

The proof relies upon an interesting construction. With K given as above we cohoose some C^∞-function $b(r)$ on the real line where $b(r) = 1$ if $0 \leq r \leq 2K$ and zero if $r > 3K$. Let ω denote points on the unit sphere S^{n-1}. For each fixed ω we set

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Next, we notice that

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Let us now define the functions $h_1, \ldots, h_n$ in $\mathbf{R}^n$ by

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