Chapter III. Complex analytic functions

Contents

Introduction. Expressing a complex number by $z = x + iy$ we identify $\mathbf{C}$ with $\mathbf{R}^2$. Consider a complex valued function $f(z) = u(x,y) + iv(x,y)$ where $\mathfrak{Re}(f) = u$ is the real part and $\mathfrak{Im}(f) = v$ the imaginary part. Let f be defined in a domain $\Omega \in \mathcal{D}(C^1)$ which extends continuously to the closure $\bar{\Omega}$ where u and v are C^1-functions in Ω, i.e. the four partial derivatives u_x, u_y, v_x, v_y exist as continuous functions in Ω. Now there exists complex line integral

(*)
$$\int_{\partial\Omega} f(z) \cdot dz = \int_{\partial\Omega} udx - vdy + i \cdot \int_{\partial\Omega} udy + vdx$$

Theorem 2.4 in Chapter II shows that both integrals are zero if the pair $u, v)$ satisfies the following two differential equations in Ω:

(**)
$$u_x = v_y \quad : \quad u_y = -v_x$$

We refer to (**) as the *Cauchy-Riemann equations* and (u,v) is called a CR-pair when (**) holds and then we say that $f = u + iv$ is an analytic function of the complex variable z. In § 6 we establish *Cauchy's formula*:

(***)
$$f(z_0) = \frac{1}{2\pi i} \cdot \int_{\partial\Omega} \frac{f(z)dz}{z - z_0} \quad \text{for all } z_0 \in \Omega\,.$$

We will use (***) to prove that when (u,v) is a CR-pair in some open set Ω, then they are not only of class C^1 but have continuous derivatives of any order, i.e. both u and v are C^∞-functions.

A notation. Let Ω be an open set in $\mathbf{C}$. The family of C^1-functions $f = u + iv$ for which (u,v) is a CR-pair in Ω is denoted by $\mathcal{O}(\Omega)$. We refer to this as the class of analytic functions in Ω and remark that one also refers to the class of *holomorphic functions*, i.e. the notion of complex analytic functions and holomorphic functions are the same.

Next follows a brief account of the material in this chapter.

0.1 Complex derivatives. Consider a C^1-function $f(z) = u(x,y) + iv(x,y)$ defined in some open set Ω. If $z_0 \in \Omega$ there exist complex difference quotients

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \quad : \quad z_0 \in \Omega$$

We say that f has a *complex derivative* at z_0 if these complex difference quotients have a limit as $\Delta z = \Delta x + i\Delta y \to 0$. During the passage to the limit Δx and Δy are not constrained, i.e. it is only required that both Δ-numbers tend to zero. For example, we can allow that Δx tends much faster to zero than Δy, or vice versa. In § 4 we show that the existence of a complex derivative at a point $z_0 = x_0 + iy_0$ is *equivalent* to the condition that $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$ hold. Hence the C^1-function $f(z)$ in Ω has a complex derivative everywhere if and only if (u, v) is a CR-pair.

In § 5 we relax the regularity hypothesis that f from the start is of class C^1. More precisely, assume only that $f(z)$ is continuous in Ω and that the pointwise defined complex derivative exists for every $z_0 \in \Omega$. *Goursat's theorem* shows that f is automatically of class C^1 and hence (u, v) is a CR-pair and they even become C^∞-functions in Ω.

0.2 Schwarz' reflection. Let $f(z)$ be analytic in an open rectangle

$$\square_+ = \{(x,y) \quad : 0 < y < b \quad \text{and} \quad -A < x < A\}$$

A reflection in the real x-axis gives the rectangle

$$\square_- = \{(x,y) \quad : \ -b < y < 0 \quad \text{and} \quad -A < x < A\}$$

In $\square_-$ we define the complex valued function

$$g(z) = \bar{f}(\bar{z}) = u(x,-y) - iv(x,-y)$$

One verifies easily that $g(z)$ becomes analytic in $\square_-$. Suppose now that $f(z)$ extends continuously to the x-axis, i.e.

$$\lim_{\epsilon \to 0} f(x + i\epsilon) = f_*(x)$$

exists uniformly with respect to x. If $f_*(x)$ is *real-valued* then f and g attain the same boundary values on the real x-interval. In § 5 we prove that under this assumption the pair f, g not only match each other as continuous functions on the x-interval but there exists an analytic function $F(z)$ defined in the whole open square

$$\square = \{(x,y) \quad : \ -b < y < b \quad \text{and} \quad -A < x < A\}$$

such that $F = f$ in $\square_+$ and $F = g$ in $\square_-$. This result is due to Hermann Schwarz who applied the reflection principle to establish many other results, such as the existence of a conformal map from the unit disc to a domain bordered by a piecewise linear Jordan curve.

0.3 A more general reflection. Let $\square_+$ and $\square_-$ be as in (0.2). Consider a pair $f \in \mathcal{O}(\square_+)$ and $g \in \mathcal{O}(\square_-)$. Then the following extension of Schwarz's result holds:

0.4 Theorem. *Assume that*

$$\lim_{\epsilon \to 0} \int_{-A}^{A} \big| f(x + i\epsilon) - g(x - i\epsilon) \big| \cdot dx = 0$$

Then there exists $F \in \mathcal{O}(\square)$ which gives an analytic extension of the pair (f, g).

This result is due to Carleman in [Car]. The proof relies upon properties of subharmonic functions and is given in the Appendix about Distributions. Notice that no growth conditions on f and g are imposed from the start in the theorem above.

0.5 An inequality by Schwarz. Let D be the unit disc $|z| < 1$ and f an analytic function in D whose maximumn norm is at most one, i.e.

(i) $$|f(z)| \leq 1 \quad : z \in D$$

Consider a bounded and connected subset Ω of the upper half-disc D_+ whose boundary intersects the real axis in some interval (a,b). Now $\partial\Omega$ is the union of $[a,b]$ and the remaining portion of the boundary is denoted by Γ which is a subset of D_+ and the points a and b belong to the closure of Γ. Put

$$\Omega_* = \{z \,:\, \bar{z} \in \Omega\}$$

and assume that $\Omega \cup \Omega_* \cup (a,b)$ is an open subset of D. Under this condition one has:

$$(*) \qquad \max_{a \leq x \leq b} |f(x)| \leq \sqrt{|f|_\Gamma}$$

To prove this Schwarz considered the function

$$F(z) = f(z) \cdot \bar{f}(\bar{z})$$

which to begin with is analytic in $\Omega \cup \Omega_*$. The reflection principle entails that F extends to an analytic set in the open set $U = \Omega \cup \Omega_* \cup (a,b)$. Here we notice that the boundary $\partial U = \Gamma \cup \Gamma_*$. From (i) it is clear that

$$|F|_{\partial U} \leq |f|_\Gamma$$

In § xx we establish the maximum principle for analytic functions which entails that when $a \leq x \leq b$ then

$$|f(x)|^2 = |F(x)| \leq |f|_\Gamma$$

and (*) follows when we take the square root.

0.6 The $\bar{\partial}$-operator. The Cauchy-Riemann equations can be expressed by a a single first order differential equation. Namely, introduce the differential operator

$$\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$$

If $f = u + iv$ is a complex-valued function it is clear that (u,v) is a CR-pair if and only if f satisfies the homogeneous $\bar{\partial}$-equation:

$$\bar{\partial}(f) = \frac{1}{2}[f_x + if_y) = 0$$

An important result is the *Pompieu formula* which shows how to solve the *inhomogeneous* $\bar{\partial}$-equation

$$\bar{\partial}(f) = g$$

when g is a continuous function with compact suppoprt. See § 8 for details.

0.7 Normal families. Cauchy's integral formula is used to show that an analytic function f defined in an open disc $D_R(z_0) = \{|z - z_0| < R\}$ is represented by a power series

$$(*) \qquad \sum c_n(z - z_0)^n$$

whose radius of convergence is $\geq R$. In this way $\mathcal{O}(D_R(z_0))$ is identified with the set of power series whose radius of convergence is $\geq R$. Moreover, the coefficients $\{c_n\}$ are determined by the formula

$$c_k = \frac{1}{k!} \cdot f^{(k)}(z_0) \quad : \; k = 0, 1, 2, \ldots$$

Using power series we establish results due to Montel concerned with the topology on $\mathcal{O}(\Omega)$. Of special importance is the following result:

Let $\{f_\nu\}$ be a sequence of analytic functions in a domain Ω whose maximum norms are uniformly bounded, i.e. there is a constant M such that $|f_\nu(z)| \leq M$ hold for all $z \in \Omega$ and every ν. Then the sequence contains at least one subsequence $\{g_k = f_{\nu_k}\}$ which converges uniformly to an analytic function g_* in every relatively compact subset of Ω. Moreover, if there exists an integer N such that every g_k has at most N zeros counted with multiplicity in Ω, then the same hold for the limit function g_*, unless it is identically zero.

This result will be used frequently later on. For example in the study of conformal mappings in Chapter VI.

0.8 Laurent series. In § 10 we study analytic functions defined in domains $\{r < |z| < R\}$. Here the boundary consists of the inner circle $|z| = r$ and the outer circle $|z| = R$. Let $f(z)$ be analytic in such an annulus which extends to a continuous function on the boundary. Then we can apply Cauchy's formula from (***) and obtain a series representation of f where one part of the series is an expansion with *negative* powers of z.

0.9 Conformal properties Let $f(z)$ be an analytic function with non-zero complex derivative defined in some domain Ω. We can regard f as a map from the complex z-plane into another complex plane and put

$$\zeta = f(z)$$

where $\zeta = \xi + i\eta$. Consider some point $z_0 \in \Omega$ and with $f = u + iv$ we have $\xi = u(x, y)$ and $\eta = v(x, y)$. The complex-valued function f is now identified with a vector-valued function from the real (x, y)-space to the real (ξ, η)-space whose *Jacobian* is the 2×2-matrix

$$J = \binom{u_x \quad v_x}{u_y \quad v_y}$$

The Cauchy-Riemann equations imply that the two column vectors are orthogonal, i.e.

$$u_x v_x + u_y v_y = 0$$

holds at every point in Ω. We have also the determinant formula:

$$(*) \qquad \det(J) = u_x v_y - u_y v_x = u_x^2 + u_y^2 = |f'(z)|^2$$

By a wellknown result in Calculus this implies that the vector valued map is infinitesmally a rotation times a dilation with the factor $|f'(z)|^2$ at every point $z \in \Omega$. This implies that the map is locally conformal. Namely, let $z_0 \in \Omega$ and consider a pair of C^1-curves γ_1, γ_2 which pass z_0 and let α be the angle between them. Then the angle between the image curves $f(\gamma_1)$ and $f(\gamma_2)$ is equal to α. This means that infinitesmal angles are preserved and is expressed by saying that f yields a conformal map.

Remark on quasi-conformal mappings. In the theory about quasi-conformal mappings many "magical phenomena" occur. See the text-books [Ahl] by Ahlfors and [L-V] by Lehto and Virtanen for the quasi-conformal theory. Let us recall that a differentiable function $f = u + iv$ which locally is an orientation preserving homeomorphism is quasi-conformal of order $\leq K$ for some number $K \geq 1$ if the first order derivative of f satisfy:

$$\big|\bar{\partial}(f)\big| \leq \frac{K-1}{K+1} \cdot \big|\partial(f)\big|$$

When $K = 1$ this means that $\bar{\partial}(f) = 0$, i.e. f is complex analytic. Consider as an example the linear function $f(z) = z + \frac{\bar{z}}{2}$. Here we require that $2(K-1) \geq K + 1$, i.e. take $K = 3$. Notice that the linear map:

$$(x, y) \mapsto (3x/2, y/2)$$

sends small circles centered at the origin to small ellipses. So the geometry is more involved when quasi-confomal mappings are studied.

0.10 Area formulas. Let Ω be a domain in $\mathcal{D}(C^1)$ where $\partial\Omega$ consists of simple and closed boundary curves $\gamma_1, \ldots, \gamma_p$. Let $f(z) \in \mathcal{O}(\Omega)$ and suppose it extends to a continuous function on $\bar{\Omega}$. Assume also that f is bijective on $\bar{\Omega}$ and that the image domain $f(\Omega)$ also belongs to $\mathcal{D}(C^1)$. Then $f(\Omega)$ is bordered by a p-tuple of disjoint boundary curves $f(\gamma_1), \ldots, f(\gamma_p)$. Write $f = u + iv$ and to each $1 \leq k \leq p$ we consider a parametrisation by arc-length along γ_k and evaluate the line integral

$$(*) \qquad J(k) = \int_0^{\ell(\gamma_k)} u(z_k(s)) \cdot \frac{dv(z_k(s))}{ds} \cdot ds$$

where $\gamma_k \colon s \mapsto z_k(s)$ and $\ell(\gamma_k)$ is the arc-length of γ_k. With these notations Stokes Theorem from Chapter II gives the following area formula:

0.11 Theorem *The area of $f(\Omega)$ is equal to $J(1)+\ldots+J(p)$.*

Remark. Since the absolute value of $f'(z)$ changes the area measure we have also the equality:

$$(**)\qquad \text{Area}[f(\Omega)] = \iint_\Omega |f'(z)|^2 \cdot dxdy$$

In Section XX we establish a third area formula using complex line integrals along the boundary curves of Ω:

$$(***)\qquad \text{Area}[f(\Omega)] = \int_{\partial\Omega} \bar{f}(z)\cdot f'(z)\cdot dz$$

The fact that one disposes these three area formulas is quite useful.

0.12 A local limit formula. If $f(z)$ is analytic in a bounded domain it suffices to know its values on a small portion of the boundary. Following the introduction in Carleman's book [Quasianalytic] we shall illuminate this in a special case where formulas are quite explicit. Here is the set-up. Let $0<\alpha<1/2$ be a real number and let ℓ_* be the ray along the non-negativel real axis ℓ_* and the second ray $\ell^* = \{re^{i\alpha} : r \geq 0\}$. Let Γ be a Jordan arc with end-points $A \in \ell_*$ and $B \in \ell^*$ while the remaining part of Γ is in the interior of the open sector bordered by the two rays. So here A is a positive real number and $B = b\cdot e^{i\alpha}$ for some $b>0$. Let Ω be the Jordan domain bordered by Γ and the straight lines $0A$ and $0B$ which intersect at the origin. Consider a point $\zeta = r\cdot e^{i\alpha/2}$ which belongs to Ω. The Jordan arc Γ is assumed to be rectifiable so that complex line integrals along Γ are defined. Let us now consider an analytic function f in Ω which extends to a continuous function on the closure $\bar{\Omega}$.

0.13 Theorem. *The value of f at ζ is obtained by the limit formula*

$$(*)\qquad f(\zeta) = \lim_{\sigma\to\infty} \frac{e^{-\sigma}}{2\pi i}\cdot \int_\Gamma \frac{f(z)\cdot e^{\sigma\left(\frac{z}{\zeta}\right)^{\frac{1}{\alpha}}}}{z-\zeta}\cdot dz$$

Proof. For each positive real number σ we have the function

$$(1)\qquad F_\sigma(z) = f(z)\cdot e^{\sigma\left(\frac{z}{\zeta}\right)^{\frac{1}{\alpha}}}$$

When $0 \leq s \leq A$ we have

$$\left(\frac{s}{\zeta}\right)^{\frac{1}{\alpha}} = \left(\frac{s}{r}\right)^{\frac{1}{\alpha}}\cdot e^{-\pi i/2} = -i\cdot\left(\frac{s}{r}\right)^{\frac{1}{\alpha}}$$

Since exponentials of purely imaginary numbers have absolute value we get

$$|F_\sigma(s)| = |f(s)|$$

On the ray OB we find a similar formula. Hence $F_\sigma(z)$ is bounded on OA and OB. We also notice that

$$F_\sigma(\zeta) = e^\sigma\cdot f(\zeta)$$

Cauchy's integral formula applied to F_σ gives

$$f(\zeta) = e^{-\sigma}\cdot F_\sigma(\zeta) = e^{-\sigma}\cdot\frac{1}{2\pi i}\cdot\int_{\partial\Omega}\frac{F_\sigma(z)}{z-\zeta}\cdot dz$$

The last line integral is the sum over Γ and the two line integrals along $0A$ and $0B$. Since F_σ is bounded on the line segments and $e^{-\sigma}\to 0$ as $\sigma\to+\infty$ we conclude that

$$f(z) = \lim_{\sigma\to\infty}\frac{e^{-\sigma}}{2\pi i}\cdot\int_\Gamma\frac{F_\sigma(z)}{z-\zeta}\cdot dz$$

By the construction of the F-functions this entails the limit formula (*).

Remark. If f from the start is defined in some domain U which is starshaped with respect to the origin then we pick $0\neq\zeta\in U$ and after a rotation we may assume that ζ is real and positive. With a very small α we can consider the rays from the origin where $\arg(z)$ is α or $-\alpha$ and by a picture the reader can see that $f(\zeta)$ via Theorem 0.13 is expressed by a limit where the integral is taken over a small portion of ∂U.

1. Complex Line Integrals

Consider a complex valued C^1-function of a real t-variable:

$$t \mapsto z(t) = x(t) + iy(t) \quad : \quad 0 \le t \le T$$

The C^1-condition means that both $x(t)$ and $y(t)$ are continuously differentiable functions of t. The t-derivative becomes:

$$\dot{z}(t) = \dot{x}(t) + i \cdot \dot{y}(t)$$

If $f = u + iv$ is a complex valued continuous function we get the line integral

$$\int_0^T f(z(t))\dot{z}(t)dt = \int_0^T [u(x(t), y(t)) + iv(x(t), y(t))](\dot{x}(t) + i \cdot \dot{y}(t)) \cdot dt$$

or expressed in a more abbreviated form:

$$(0.1) \qquad \int_0^T f(z)\dot{z} \cdot dt = \int_0^T [u(x,y) + iv(x,y)](\dot{x} + i\dot{y}) \cdot dt$$

The right hand side is a sum of line integrals with respect to x and y respectively. By the general result in XX, (0.1) does not depend on the chosen parametrization of the oriented image Γ. So we can therefore write (0.1) as

$$(0.2) \qquad \int_\Gamma f dz = \int_\Gamma u + iv)(dx + idy)$$

When the right hand side is decomposed into its real and imaginary parts we get:

$$(0.3) \qquad \int_\Gamma f dz = \int_\Gamma udx - vdy + i \cdot \int_\Gamma udy + ivdx$$

We refer to (0.2) as the complex line integral along Γ. Recall that the *choice of orientation* is essential, i.e. if the orientation on Γ is opposite the line integral changes sign.

1.1 Complex line integrals as Riemann sums. Consider as above a curve Γ with end points a and b. Following the orientation we choose a finite sequence of points $a = z_0, z_1, \ldots, z_N = b$ where each $z_\nu \in \Gamma$ and take the Riemann sum

$$\text{(i)} \qquad \sum_{\nu=0}^{N-1} f(z_\nu)(z_{\nu+1} - z_\nu)$$

When $\max |z_{\nu+1} - z_\nu| \to 0$ these sums converge to the line integral of f along Γ. To see this we use the hypothesis that Γ has a C^1-parametrisation, say $t \mapsto z(t)$. Now $z_\nu = z(t_\nu)$ with $0 = t_0 < t_1 < \ldots < t_N = T$. Since f is continuous the function $t \mapsto f(z(t))$ is continuous. From the previous definition of the line integral we have

$$\text{(ii)} \qquad \int_\Gamma f \cdot dz = \int_0^T f(z(t)) \cdot \dot{z}(t)dt$$

Here (ii) is approximated just as in Calculus by a Riemann sum:

$$\text{(iii)} \qquad \sum f(z(t_\nu)) \cdot \dot{z}(t_\nu) \cdot (t_{\nu+1} - t_\nu)$$

The continuity of the t-derivative $t \mapsto \dot{z}(t)$ give accurate approximations

$$\dot{z}(t_\nu) \cdot (t_{\nu+1} - t_\nu) \simeq z_{\nu+1} - z_\nu$$

for every ν. Hence the Riemann sums in (iii) converge to (ii) when

$$\max\{(t_{\nu+1} - t_\nu)_{\nu=1}^N\} \quad \text{tends to zero}$$

1.2 Integration on rectifiable curves. The approximative sums in (1) appear in the construction of Riemann-Stieltjes integrals. Since the complex integral is decomposed into a real and an imaginary part we can therefore use the result from XX where we proved the existence of integrals of the Riemann-Stieltjes type. More precisely, if Γ has a parametrisation $t \mapsto z(t) = x(t) + iy(t)$

where both $x(t)$ and $y(t)$ are continuous functions with bounded variation, then we can define Stieltjes' line integral

$$\int_\Gamma f(z)\cdot dz$$

where we only have to assume that $f(z)$ is a bounded Borel function. See Measure Appendix for details about this construction of general line integrals.

1.3 The case when Γ is a circle. Let $R>0$ and Γ is the circle $|z|=R$ equipped with its usual positive orientation. Since $z\neq 0$ on Γ we can divide a function f with z and get the line integral

$$\text{(i)} \qquad \int_\Gamma \frac{f(z)dz}{z}$$

Using the parametrisation $\theta\mapsto Re^{i\theta}$ this line integral becomes

$$\text{(ii)} \qquad \int_0^{2\pi} \frac{f(Re^{i\theta})\cdot iRe^{i\theta}d\theta}{Re^{i\theta}} = i\cdot\int_0^{2\pi} f(Re^{i\theta})d\theta$$

This formula plays a crucial role later on when we derive Cauchy's formula and develop residue calculus. Under the sole assumption that f is a continuous function defined in some open disc centered at the origin, we use the equality above when $R=\epsilon$ and $\epsilon\to 0$. Namely, since f is continuous at the origin we get the limit formula:

$$\text{(iii)} \qquad f(0)=\frac{1}{2\pi}\cdot\lim_{\epsilon\to 0}\int_0^{2\pi} f(\epsilon e^{i\theta})d\theta = \frac{1}{2\pi i}\cdot\lim_{\epsilon\to 0}\int_{|z|=\epsilon}\frac{f(z)dz}{z}$$

The origin can be replaced by any other point z_0. Since the limit formula above is so important for the subsequent residue calculus we state it separately:

1.4 Theorem *Let $f(z)$ be a continuous function in some open set Ω. Then*

$$f(z_0)=\frac{1}{2\pi i}\cdot\lim_{\epsilon\to 0}\int_{|z-z_0|=\epsilon}\frac{f(z)dz}{z-z_0} \quad : \quad z_0\in\Omega$$

§ 2. The Cauchy-Riemann equations

Consider a complex-valued function $f(z)=f(x+iy)=u(x,y)+iv(x,y)$. We assume that f is of class C^1 and let $\Omega\in\mathcal{D}(C^1)$. From § 1 we have

$$\int_{\partial\Omega} f dz = \int_{\partial\Omega} udx - vdy + i\int_{\partial\Omega} udy + vdx$$

Apply Theorem 2.4 in Chapter I to each term in the right hand side. This gives:

$$\text{(i)} \qquad \int_{\partial\Omega} f dz = \iint_\Omega (-(u_y+v_x)\cdot dxdy + i\cdot\iint_\Omega (u_x-v_y)\cdot dxdy$$

The right hand side is zero if the real and the imaginary part vanish which obviously follows if the following equations hold in the whole of Ω:

$$(*) \qquad u_x(x,y)=v_y(x,y) \quad : \quad u_y(x,y)=-v_x(x,y) \quad : \quad (x,y)\in\Omega$$

Hence we have proved the following

2.1 Theorem *Let $f=u+iv$ be a complex-valued C^1- function such that the pair (u,v) satisfies (*). Then*

$$\int_{\partial\Omega} f dz = 0 \quad : \quad \Omega\in\mathcal{D}(C^1)$$

Theorem 2.1 suggests the following

2.2 Definition *A pair of real-valued C^1-functions (u,v) satisfying $u_x = v_y$ and $u_y = -v_x$ is called a Cauchy-Riemann pair and in this case $f = u + iv$ is called an analytic function.*

§ 3. The $\bar{\partial}$-operator

The Cauchy-Riemann equations can be described by a single first order differential operator. Namely, set

$$\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y) \tag{i}$$

If $f = u + iv$ is a C^1-function we get:

$$\bar{\partial}(f) = \frac{1}{2}[\partial_x(f) + i\partial_y(f)] = \frac{1}{2}[u_x + iv_x + iu_y - v_y] = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$

We conclude that $\bar{\partial}(f) = 0$ if and only if (u,v) is a CR-pair. One refers to $\bar{\partial}$ as the *Cauchy-Riemann operator* since it determines when (u,v) becomes a CR-pair.

3.1 Example Let $f = \bar{z} = x - iy$ be the conjugate function. Here $\bar{\partial}(\bar{z}) = 1$ and hence the pair $u = x$ and $v = -y$ is not CR. On the other hand, let $m \geq 1$ and consider $f(z) = z^m$. Since $\bar{\partial}$ is a first order differential operator, Leibniz' s rule from Calculus gives

$$\bar{\partial}(z^m) = mz^{m-1}\bar{\partial}(z)$$

Next, we have

$$\bar{\partial}(x + iy) = \frac{1}{2}[1 + i^2] = \frac{1}{2}[1 - 1] = 0$$

So with $z^m = u + iv$ it follows that (u,v) is a CR-pair. Take as an example the case $m = 3$ where $u = x^3 - 3x \cdot y^2$ and $v = 3x^2 \cdot y - y^3$.

3.2 Remark If f, g is a a pair of C^1-functions then Leibniz's rule for the first order differential operator gives

$$\bar{\partial}(fg) = f\bar{\partial}(g) + g\bar{\partial}(f)$$

It follows that if both f and g are analytic so is fg. Hence the class of analytic functions is stable under products. Next, let $f_1, \ldots, f_m$ a finite set of analytic functions. Put

$$\phi = \bar{z} \cdot f_1(z) + \ldots + \bar{z}^m \cdot f_m(z)$$

Then ϕ cannot be analytic unless every f_ν is zero. To see this one proceeds by induction over m. Namely, for each $m \geq 2$ we get the m:th order differential operator $\bar{\partial}^m$ which satisfies

$$\bar{\partial}^m(\bar{z}^\nu) = 0 \quad : \nu < m \quad : \quad \bar{\partial}^m(\bar{z}^m) = m$$

Using this the reader may verify the assertion about the ϕ-function.

3.3 Analytic polynomials. Since $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, every polynomial in the two variables (x,y) can be expressed as a polynomial in z and $\bar{z}$. Let $m \geq 1$ and denote by $H_{\text{Pol}}(m)$ the space of homogeneous polynomial of degree m with complex coefficients. Such a polynomial can be uniquely written in the form

$$P(z, \bar{z}) = \sum_{\nu=0}^{\nu=m} c_\nu \cdot z^{m-\nu}\bar{z}^\nu \quad : \quad c_0, \ldots, c_m \text{ complex constants}$$

The observation in 3.2 shows that P is analytic if and only if the sole term is z^m. Hence $H_{\text{Pol}}(m)$ is 1-dimensional complex vector space generated by the monomial z^m. This shows that the class of analytic functions is quite sparse.

§ 4. The Complex Derivative

Let $f(z) = u(x,y) + iv(x,y)$ be a complex-valued function of class C^1 in a domain Ω. Given a point $z_0 = x_0 + iy_0$ and a small complex number $\Delta z = \Delta x + i\Delta y$ we regard the difference

$$f(z_0 + \Delta z) - f(z_0) = u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)$$

Keeping Δx and Δy fixed for a while we have the function

$$\phi(s) = u(x_0 + s\Delta x, y_0 + s\Delta y) + iv(x_0 + s\Delta x, y_0 + s\Delta y) \quad : \quad 0 \leq s \leq 1$$

Rolle's mean value theorem gives a pair $0 < \theta_1, \theta_2 < 1$ such that the sum

$$\Delta x \cdot u'_x(x_0 + \theta_1\Delta x, y_0 + \theta_1\Delta y) + \Delta y \cdot u'_y(x_0 + \theta_1\Delta x, y_0 + \theta_1\Delta y) +$$

(1) $$i[\Delta x \cdot v'_x(x_0 + \theta_2\Delta x, y_0 + \theta_2\Delta y) + \Delta y \cdot u'_y(x_0 + \theta_2\Delta x, y_0 + \theta_2\Delta y)]$$

is equal to $\phi(1) - \phi(0) = f(z_0 + \Delta z) - f(z_0)$. In this sum the four first order partial derivatives are evaluated at points close to (x_0, y_0) when

$$|\Delta z| = \sqrt{(\Delta x^2 + (\Delta y)^2} \to 0$$

The continuity of the partial derivatives and the imply that the sum from (1) becomes

(2) $$\Delta x \cdot u'_x(x_0, y_0) + \Delta y \cdot u'_y(x_0, y_0) + i\Delta x \cdot v'_x(x_0, y_0) + i\Delta y \cdot v'_y(x_0, y_0) + \text{small ordo}(|\Delta z|)$$

The remainder term $o(|\Delta z|)$ comes from the continuity of first order derivatives. If we *assume* that u, v is a Cauchy-Riemann pair we can replace v'_y with u'_x and u'_y with $-v'_x$. Then (2) becomes

(3) $$(\Delta x + i\Delta y)u'_x + i(\Delta x + i\Delta y)v'_x + \text{small ordo}(\delta)$$

Here $\Delta z = \Delta x + i\Delta y$ appears as a common factor. The small ordo-term gives the limit formula:

(4) $$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u'_x(x_0, y_0) + +iv'_x(x_0, y_0)$$

So when (u, v) is a CR-pair the complex difference quotients $\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ have a limit as $\Delta z \to 0$. The limit is called the *complex derivative* of f at the point z_0. Put

(*) $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

A converse. Assume that $f = u + iv$ has a complex derivative at $z_0 = x_0 + iy_0$. We can approach this point in two ways - along the x axis or along the y-axis. With $\Delta z = \Delta x$ we get from the definition of partial derivatives

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_o)$$

If we instead take $\Delta z = i\Delta y$ we have

$$f'(z_0) = \lim_{i\Delta y \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{i}u_y(x_0, y_0) + v_y(x_0, y_o)$$

Identifying the real and the imaginary parts of the two expressions for the complex derivative $f'(z_0)$ we recover the Cauchy-Riemann equations. Hence we have proved the following

4.1 Theorem. *A C^1-function $f = u + iv$ defined in a domain Ω has a complex derivative at every point if and only if (u, v) is a CR-pair.*

4.2 The space $\mathcal{O}(\Omega)$. Let Ω be an open subset of $\mathbf{C}$. The class of analytic functions in Ω is denoted by $\mathcal{O}(\Omega)$. From the result in 2.7 this gives a subalgebra of all complex valued C^1-functions in Ω.

§ 5. Morera's and Goursat's theorems.

Let $f(z)$ be a continuous and complex-valued function defined in an open square $\square = \{-A < x, y < A\}$. To every point $p = (a, b) \in \square$ we get the rectangle Γ with corners at the origin, $(a, 0), (a, b), (0, b)$. Suppose that $\int_\Gamma f(z)dz = 0$ for every such rectangle. Define a function $F(z)$ by

$$F(x+iy) = \int_0^x f(t,0)dt + \int_0^y f(a,s)ids$$

It is obvious that $F'_y = if$. Next, the hypothesis on f implies that we also have

$$F(x+iy) = -\int_0^y f(0,s)ids - \int_0^x f(t,y)dt$$

From this we see that $F'_x = -f$. Hence $F'_x = F_y$ and since F also is a C^1-function this implies that $F(z)$ is analytic by the result in 2.5. Next, if we *knew* that F is of class C^2 we can take the mixed second order derivatives and obtain

$$-f'_y = F''_{yx} = F''_{xy} = if'_x$$

This gives $f'_x = if'_y$ and 2.5 proves that f is analytic. in the next section we show that F actually is of class C^2 and hence it follows that f is analytic. Of course, allowing more rectangles we can conclude:

5.1 Theorem. *Let $f(z)$ be continuous in an open set Ω. Assume that $\int_\Gamma f(z)dz = 0$ for every rectangle inside Ω with sides parallell to the coordinate axes. Then f is analytic in Ω.*

5.2 Some estimates. We begin with some preliminary results which will be used to prove Theorem 5.3 below. A rectangle Γ gives four smaller rectangles $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ which arise when we join the parallell sides with lines from their opposed mid-points. Now γ_1, γ_2 has one side in common and so on. By drawing a figure and keeping in mind how the orientation is chosen along each small rectangle we see that:

$$\int_\Gamma f(z)dz = \sum_{\nu=1}^{\nu=4} f(z)dz$$

This process can be continued. Suppose now that $f(z)$ is a bounded function which is no essential restriction since we otherwise may replace Ω by a smaller set. if $|f(z)| \leq M$ in Ω the construction of a complex line integral gives

$$|\int_\Gamma f(z)dz| \leq \ell(\Gamma) \cdot M \quad : \ell(\Gamma) = \text{sum of lengths of the sides}$$

Suppose now that we have a rectangle Γ where $\int_\Gamma f(z) \neq 0$. Let us put

$$|\int_\Gamma f(z)| = A \cdot |\ell(\Gamma)$$

where A now is > 0. Dividing Γ in four smaller portions we see that the equality above gives

$$\max_\nu |\int_{\gamma_\nu} f(z)dz| \geq \frac{1}{4} \cdot |\int_\Gamma f(z)dz| = \frac{A}{4} \cdot \ell(\Gamma)$$

At the same time $\ell(\gamma_\nu) = \frac{1}{2} \cdot \ell(\Gamma)$. Hence we find at least one small γ-rectangle where

$$\int_{\gamma_\nu} f(z)dz| \geq \frac{A}{2} \cdot \ell(\gamma_\nu)$$

Starting from one such γ-rectangle it is decomposed in four pieces and so on. In this way we obtain a nested descreasing sequence of rectangles $\{\gamma_\nu\} : \nu = 1, 2, \ldots\}$ such that

$$\int_{\gamma_\nu} f(z)dz| \geq \frac{A}{2^\nu} \cdot \ell(\gamma_\nu) \quad : \nu \geq 1 \quad : \ell(\gamma_\nu) = 2^{-\nu}\ell(\Gamma)$$

At this stage we are prepared to prove:

5.3 Goursat's theorem. *Let $f(z)$ be a continuous function in Ω and assume it has a complex derivative at every point. Then f is analytic.*

Proof. It suffices to work locally and we may take $\Omega = \square$ as above. If f fails to be analytic we find Γ and $A > 0$ where

$$|\int_\Gamma f(z)dz\,| = A \cdot \ell(\Gamma) \cdot M$$

Then there exists a nested sequence $\{\gamma_\nu\}$ and since there sides tend to zero, it follows by Bolzano's theorem that there is a limit point $z_0 = \cap\,\gamma_\nu$. By assumption f has a complex derivative at z_0. It means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon|z - z_0| \quad : \quad |z - z_0| < \delta$$

If ν is sufficiently large then γ_ν is contained in the disc $D_\delta(z_0)$. Next, we notice that

$$\int_{\gamma_\nu} [f(z_0) + (z - z_0)f'(z_0)]dz = 0$$

Next, when $z \in \gamma_\nu$ we notice that $|z - z_0| \leq \ell(\gamma_\nu)/2$. Hence by the vanishing in XX and the inequality XX above, the triangle inequality gives

$$|\int_{\gamma_\nu} f(z)dz\,| \leq \frac{1}{2}\epsilon\,\ell(\gamma_\nu)^2 = \frac{1}{2} \cdot \epsilon \cdot 4^{-\nu}\ell(\Gamma)$$

At the same time, during the construction of the nested sequence we have

$$\int_{\gamma_\nu} f(z)dz| \geq \frac{A}{2^\nu} \cdot \ell(\gamma_\nu) = A \cdot 4^{-\nu} \cdot \ell(\Gamma)$$

Now we get a contradiction since we take ϵ arbitrary small , i.e. it would even be sufficient to take $\epsilon < 2A$ and then the contradiction will follow as soon as ν is so large that $\gamma_\nu \subset D_\delta(z_0)$.

5.4 A result by Carleman. Analytic functions are also characterized by a local mean value condition.

5.5 Theorem *Let f be a continuous function in Ω such that* $\int_{\partial D} f(z)dz = 0$ for ever disc $D \subset \Omega$. *Then $f \in \mathcal{O}(\Omega)$.*

Proof. Let $r > 0$ be small and put

$$\Omega_r = \{z \in \Omega \,:\, \text{dist}(z, \partial\Omega) > r\}$$

To each $z \in \Omega_r$ we define the mean value

$$F_r(z) = \iint_{|\zeta - z| \leq r} f(z)dxdy = \int_0^r \int_0^{2\pi} f(z + se^{i\theta})sdsd\theta$$

Now we prove that $F_r \in (\Omega_r)$. To see this we consider its partial derivatives with respect to x and y. With $z = x + iy$ and Δx small, $F(x + \Delta x + iy)$ is the area integral over a disc centered at $(x + \delta x, y)$. Drawing a figure for computing area integrals the reader should discover that we obtain

$$F_x(z) = \int_0^{2\pi} \cos(\theta)f(z + re^{i\theta}d\theta \quad \text{and} \quad F_y(z) = \int_0^{2\pi} \sin(\theta)f(z + re^{i\theta}d\theta$$

It follows that

$$F_x + iF_y = \int_0^{2\pi} e^{i\theta}f(z + re^{i\theta}d\theta = 0$$

where the last equality follows from the mean value assumption. Hence F satisfies the $\bar{\partial}$-quation from XX and is therefore analytic. Now this holds for any $r > 0$ and the continuity of f gives:

$$f(z) = \lim_{r \to 0} \frac{1}{\pi r^2} \cdot F_r(z)$$

Hence f is the limit of a sequence of analytic functions and therefore analytic by the result to be proved in XXX.

§ 6. Cauchy's integral formula

6.1 The local residue. Let us repeat the result which led to Theorem 1.2 once more since it plays such a fundamental role. Let $z_0 \in \mathbf{C}$ and let $g(z)$ be a continuous function defined in some open disc of radious r centered at z_0. To each $0 < \epsilon < r$ we set

$$R_\epsilon(g) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{g(z)dz}{z-z_0}$$

Using polar coordinates we get

$$R_\epsilon(g) = \int_0^{2\pi i} \frac{g(z_0+\epsilon e^{i\theta} id\theta}{e^{i\theta}} = \int_0^{2\pi} \int_0^{2\pi} g(z_0+\epsilon e^{i\theta} d\theta$$

By continuity of g at z_0 it follows that

(*) $$\lim_{\epsilon \to 0} R_\epsilon(g) = g(z_0)$$

This local limit formula will now be applied below when $g(z)$ is an analytic function.

6.2 Cauchy's formula Let $f(z) \in \mathcal{O}(\bar{\Omega})$ where $\Omega \in \mathcal{D}(C^1)$. Let $z_0 \in \Omega$ and with $\epsilon > 0$ is small we remove the open disc of radius ϵ centered at z_0. Put

$$\Omega_\epsilon = \Omega \setminus \{|z-z_0| \leq \epsilon\}$$

Now $\partial\Omega_\epsilon = \partial\Omega \cup \partial D_\epsilon$ where $D_\epsilon = \{z - z_0| < \epsilon\}$. We get the function

$$g(z) = \frac{f(z)}{z-z_0} \in \mathcal{O}(\bar{\Omega}_\epsilon)$$

Applying Theorem 2.1 we get

$$\int_{\partial\Omega} \frac{f(z)dz}{z-z_0} = \int_{\partial D_\epsilon} \frac{f(z)dz}{z-z_0}$$

Here ϵ can be arbitrarily small. The limit formula in (*) from 4.1 shows that the last term is equal to $2\pi i f(z_0)$. Hence we have proved

6.3 Theorem. *Let $f \in \mathcal{O}(\bar{\Omega})$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int \int_{\partial\Omega} \frac{f(z)dz}{z-z_0} \quad : \quad z_0 \in \Omega$$

6.4 Expressions for derivatives Cauchys formula represents $f(z)$ inside Ω in the same way as in 3.3. Hence we can take derivative of any order, i.e. for each $m \geq 1$ we get

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int \int_{\partial\Omega} \frac{f(z)dz}{(z-z_0)^{m+1}} \quad : \quad z_0 \in \Omega$$

This proves in particular that when f is analytic, then it has complex derivatives. Moreover, $f^{(m)}(z)$ yield now analytic functions in Ω. In XXX we study power series representations and obtain even more information about regularity of f as well as of its real and imaginary parts. Notice that the conclusion above applies to the function $F(z)$ from section 5 and hence Theorem 6.3 above finishes the proof of *Morera's Theorem.*

6.5 The case when Ω is a disc Let $\Omega = D_R$ be the disc of radius R centered at the origin and $f(z)$ is an analytic function defined in a neighborhood of the closed disc $\bar{D}_R$. Using the parametrisation $\theta \mapsto Re^{i\theta}$ which gives $dz = iRe^{i\theta}d\theta$. Cauchy's formula gives:

(*) $$f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{f(Re^{i\theta}) \cdot Re^{i\theta} d\theta}{Re^{i\theta} - z} \quad : \quad z \in D_R$$

Remark. Now (*) yields a series representation of f. First division with $Re^{i\theta}$ gives

(i) $$f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{f(Re^{i\theta}) \cdot d\theta}{1 - \frac{z}{R} \cdot e^{-i\theta}}$$

Since $|z| < R$ we can expand the denominator in a convergent geometric series:

(ii) $$\frac{1}{1 - \frac{z}{R} \cdot e^{-i\theta}} = \sum_{\nu=0}^{\infty} R^{-\nu} e^{-i\nu\theta} \cdot z^{\nu}$$

Put

(iii) $$c_\nu = \frac{1}{2\pi} \cdot R^{-\nu} \int_0^{2\pi} f(Re^{i\theta}) \cdot e^{-i\nu\theta} d\theta \quad : \quad \nu = 0, 1, \ldots$$

Then we obtain the series representation

(iv) $$f(z) = \sum_{\nu=0}^{\infty} c_\nu \cdot z^\nu$$

Remark. The convergence of this series when $|z| < R$ is clear. For if M is the maximum of $|f(Re^{i\theta})|$ as $0 \leq \theta \leq 2\pi$ we see that

(*) $$|c_\nu| \leq M \cdot R^{-\nu}$$

The coefficients c_ν correspond to *Fourier series coefficients* of the periodic function of θ defined by

$$\theta \mapsto f(Re^{i\theta})$$

The interplay between Fourier series and analytic functions will be discussed in XXX.

6.6 Exercise. Consider a power series $f(z) = \sum c_\nu \cdot z^\nu$ as above where (*) holds for a pair M and R. Now we also get a convergent power series

$$g(z) = \sum_{\nu=1}^{\infty} \nu \cdot c_\nu \cdot z^{\nu-1}$$

It turns out that g is the derivative of f. To show this we consider a point z_0 in the disc D_R. With a small Δz we have the difference quotient:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Now the reader should verify the inequality:

$$\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - g(z_0) \right| \leq |\Delta z| \cdot \sum_{\nu=2}^{\infty} |c_\nu| \cdot$$

§ 7. Complex differentials

1. Calculus with differential forms. With $z = x + iy$ we get the differential 1-form

$$dz = dx + idy$$

Similarly, $\bar{z} = x - iy$ gives $d\bar{z} = dx - idy$. Hence we obtain

$$dx = \frac{1}{2}(dz + d\bar{z}) \quad : dy = \frac{1}{2i}(dz - d\bar{z})$$

So every differential 1-form $Adx + Bdy$ where A, B say are complex-valued C^1-functions can be expressed by dz and $d\bar{z}$:

$$Adx + Bdy = \frac{1}{2}(A - iB)dz + \frac{1}{2}(iA + B)d\bar{z}$$

2. Exterior derivatives. If $Adx + Bdy$ is a 1-form its exterior derivative is the 2-form

$$d(Adx + Bdy) = (A_y - B_x) \cdot dx \wedge dy$$

With these notations Stokes formula for a domain $\Omega \in \mathcal{D}(C^1)$ can be expressed by

$$\iint_\Omega d(Adx + Bdy) = \int_{\partial\Omega} Adx + Bdy$$

In the left hand side one integrates a function times the 2-form $dx \wedge dy$ over Ω which is the same thing as taking the usual area integral. Thus, the left hand side in Stokes formula becomes

$$\iint_\Omega (-A_y + B_x)dxdy$$

Regarding A and B separately we recover the formula in XXX.

3. Complex expressions. Consider now a 1-form expressed as fdz where f is a complex-valued C^1-function. We get

(*) $$d(fdz) = df \wedge dz = \bar{\partial}(f)d\bar{z} \wedge dz \quad : \quad \bar{\partial}(f) = \frac{1}{2}(f_x + if_y)$$

It is both illumminating and essential to confirm (*) using differentials expressed by dx and dy. To begin with we have

(i) $$dx \wedge dx = dy \wedge dy = 0 \quad : \quad dy \wedge dx = -dx \wedge dx$$

From this we get

(ii) $$d\bar{z} \wedge dz = 2idx \wedge dy \quad : \quad dz \wedge d\bar{z} = -2idx \wedge dy$$

Hence (*) is equivalent to

(**) $$d(fdx + ifdy) = (-f_y + if_x)dx \wedge dy = \frac{1}{2}(f_x + if_y) \cdot 2idx \wedge dy$$

The reader may discover that (**) holds. Next, let us start with a 1-form $gd\bar{z}$. Then we get

(***) $$d(gd\bar{z}) = \partial(g)dz \wedge d\bar{z} \quad : \quad \partial(g) = \bar{\partial}(f) = \frac{1}{2}(g_x - if_y)$$

4. Stokes formula in complex form. Using (*) and (***) together with (ii) above the real version of Stokes formula from II:XX gives the following two formulas:

(*) $$\iint d(fdz) = \iint \bar{\partial}(f)d\bar{z} \wedge dz = \int_{\partial\Omega} fdz$$

(**) $$\iint d(gd\bar{z}) = \iint \partial(g)dz \wedge d\bar{z} = \int_{\partial\Omega} gd\bar{z}$$

5. Example. Let $\Omega \in \mathcal{D}(C^1)$ and let $f(z)$ is an analytic function in Ω which extends to a C^1-function on its closure. With $\epsilon > 0$ we get the C^1-function

$$u_\epsilon = \operatorname{Log}|f|^2 + \epsilon$$

We see that

(i) $$\partial(u_\epsilon) = \frac{\bar{f} \cdot f'}{|f|^2 + \epsilon}$$

where f' is the complex derivative of the analytic function f. Stokes formula applied to the 1-form $\partial(u_\epsilon)dz$ gives:

(ii) $$\iint_\Omega \bar{\partial}(\frac{\bar{f} \cdot f'}{|f|^2 + \epsilon}) \cdot d\bar{z} \wedge dz = \int_{\partial\Omega} \frac{\bar{f} \cdot f'}{|f|^2 + \epsilon} \cdot dz$$

A computation left to the reader shows that the left hand side becomes

$$2i \cdot \iint_\Omega \frac{\epsilon \cdot |f'|^2}{(|f|^2 + \epsilon)^2} dxdy$$

Assume that f has no zeros on $\partial\Omega$. Then we get the limit formula

(iii) $$\lim_{\epsilon \to 0} \int_{\partial\Omega} \frac{\bar{f} \cdot f'}{|f|^2 + \epsilon} \cdot dz = \int_{\partial\Omega} \frac{f' dz}{f}$$

Hence the left hand side in (i) has a limit, i.e. there exists

(iv) $$\lim_{\epsilon \to 0} 2i \cdot \iint_\Omega \frac{\epsilon \cdot |f'|^2}{(|f|^2 + \epsilon)^2} dxdy$$

Since ϵ appears in the numerator this area integral tends to zero outside zeros of f. But when $f(\alpha) =$ for some $\alpha \in \Omega$ we get a contribution. Suppose for example that $\alpha = 0$ and close to the origin $f(z) = az^m(1 + b_1 z + \ldots +)$ where $a \neq 0$ and $m \geq 1$. Taking an area integral over a small disc D_δ centered at the origin we see that

(iv) $$\lim_{\epsilon \to 0} \iint_{D_\delta} \frac{\epsilon |f'|^2}{(|f|^2 + \epsilon)^2}) \cdot dxdy = \lim_{\epsilon \to 0} \iint_{D_\delta} \frac{\epsilon \cdot m^2 |a|^2 |z|^{2m-2}}{(|a|^2 |z|^{2m} + \epsilon)^2}) \cdot dxdy$$

Integrating in polar coordinates the last term becomes

(v) $$\lim_{\epsilon \to 0} 2\pi \cdot \int_0^\delta \frac{\epsilon \cdot m^2 |a|^2 r^{2m-2}}{(|a|^2 r|^{2m} + \epsilon)^2}) \cdot rdr$$

A easy computation which is left to the reader shows that this limit is $\pi \cdot m$. Performing this limit in small discs around each zero of f in Ω we conclude the following:

5.1 Theorem. *Let $\mathcal{N}_f(\Omega)$ denote the integer equal to the sum of zeros of f in Ω counted with their multiplicities, while $f \neq 0$ on $\partial\Omega$. Then we have the equality:*

$$\mathcal{N}_\Omega(t) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} \cdot dz$$

Another example. Let f be as above and let g be some C^1-function in Ω which vanishes the zeros of f which we assume are all simple. With u_ϵ defined as in the previous example we apply Stokes formula to the 1-form $g\partial(u_\epsilon)dz$. Computing $d(g\partial(u_\epsilon)dz$ gives a sum of two area integrals

(i) $$2i \cdot \iint_\Omega \frac{\epsilon \cdot g \cdot |f'|^2}{(|f|^2 + \epsilon)^2} dxdy + 2i \cdot \iint_\Omega \frac{\bar{\partial}(g) \bar{f} \cdot f'}{|f|^2 + \epsilon} dxdy$$

The last double integral has a limit. The reason is that

$$\lim_{\epsilon\to 0} \frac{\bar{f}\cdot f'}{|f|^2+\epsilon} = \frac{f'}{f} \tag{1}$$

exists in the space of integrable functions, i.e. we use that $\frac{1}{f}$ is locally integrable when f has simple zeros. So the limit hen $\epsilon \to 0$ of the second double integral becomes

$$2i \cdot \iint_\Omega \frac{\bar{\partial}(g)\cdot f'}{f} dxdy \tag{2}$$

In the first integral ϵ is in the denominator so the area integral tends to zero outside zeros of f. To analyze the situation at a zero taken as the origin where we may take $f(z) = z$ since f by assumoption has simple zeros there remains to regard

$$\lim_{\epsilon\to 0} \cdot \iint_{D_\delta} \frac{\epsilon\cdot g}{(|z|^2+\epsilon)^2} dxdy \quad : \quad g(0) = 0 \tag{3}$$

We leave it as an exercise to show that this limit is zero for every g vanish at the origin. The hint is to integrate in polar coordinates. Hence we have established

5.2 Theorem. *Assume that f has simple zeros in Ω and g is a C^1-function which vanishes at these zeros. Then*

$$2i \cdot \iint_\Omega \frac{\bar{\partial}(g)\cdot f'}{f} dxdy = \int_{\partial\Omega} \frac{g\cdot f'}{f} dz$$

6. An area formula. Recall that $dz = dx + idy$. It follows that

$$\bar{z}\cdot dz = i(xdy - ydx) + xdx + ydy$$

If Ω is a domain in $\mathcal{D}(C^1)$ we proved in Chapter XX the two area formulas:

$$\int_{\partial\Omega} x\cdot dy = -\int_{\partial\Omega} y\cdot dx = \text{Area}(\Omega)$$

At the same time the line integrals of xdx and ydy are zero. We conclude that

$$2i\cdot \text{Area}(\Omega) = \int_{\partial\Omega} \bar{z}\cdot dz \tag{*}$$

This area formula is often used. Consider for example an analytic function $f(z)$ in Ω which extends to a C^1-function to the closure and assume that the map $f\colon \bar{\Omega} \to \bar{U}$ is a homeomorphism where U is another domain in $\mathcal{D}(C^1)$. So here f yields a con formal map from Ω onto U and it sends each of the p many boundary curves to Ω onto boundary curves of U. With $w = f(z)$ as a new complex variable the area formula applied to the image domain gives:

$$2i\cdot \text{Area}(f(\Omega)) = \int_{\partial U} \bar{w}\cdot dw$$

Here the last line integral is equal to

$$\int_{\partial\Omega} \bar{f}(z)\cdot f'(z)\cdot dz$$

Hence we have found an elegant formula to express the area of the image domain $f(\Omega)$ via a line integral along $\partial\Omega$.

§ 8. The equation $\bar{\partial}(f) = a \cdot f + b \cdot \bar{f}$.

Let $a(x,y)$ and $b(x,y)$ be a pair of continuous and complex-valued function defined in some open and connected domain Ω and suppose that $f(z)$ is a solution to the differential equation in (8.2). The Pompeiu formula entails that if $z_0 \in \Omega$ and $R > 0$ is chosen so that the disc D_R centered at z_0 is contained in Ω, then the following holds when $|z - z_0| < R$:

(*)
$$f(z) = \frac{1}{2\pi i} \cdot \int_{\partial D_R} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{\pi} \iint_{D_R} [af + b\bar{f}] \cdot \frac{d\xi d\eta}{\zeta - z}$$

These locally defined integral equations lead to a uniqueness theorem. Let us say that a continuous function $g(z)$ in Ω is flat at a point z_0 if

$$\lim_{z \to z_0} \frac{g(z)}{[z - z_0)^n} = 0 \quad \text{hold for every} \quad n \geq 1$$

7.1 Theorem. *Let f be a continuous solution to the differential equation $\bar{\partial}(f) = af + b\bar{f}$ in Ω. Then f cannot be flat at any point in Ω unless f is identically zero.*

Proof. Suppose that f is flat at some $z_0 \in \Omega$. After a translation we may assume that z_0 is the origin. For each integer $n \geq 1$ we set:

$$f_n(z) = \frac{f(z)}{z^n}$$

Then we see that f_n satisfies the equation

$$\bar{\partial}(f_n) = af_n + b \cdot \big(\frac{\bar{z}}{z}\big)^n \cdot f_n$$

The Pompeiu formula shows that f_n satisfies the integral equation

(*)
$$f_n(z) = \frac{1}{2\pi} \cdot \int_{\partial D_R} \frac{f_n(\zeta) \cdot d\zeta}{\zeta - z} + \frac{1}{\pi} \iint_{D_R} [af_n + b_n f_n] \cdot \frac{d\xi d\eta}{\zeta - z}$$

The triangle inequality gives:

(**)
$$|f_n(z)| \leq \frac{1}{2\pi} \cdot \int_{\partial D_R} \frac{|f_n(\zeta)| \cdot |d\zeta|}{|\zeta - z|} + \frac{1}{\pi} \iint_{D_R} |af_n + b_n \bar{f}_n| \cdot \frac{d\xi d\eta}{|\zeta - z|}$$

Next, for each $z \in D_R$ we notice the inequality

(1)
$$\frac{1}{2\pi} \cdot \iint_{D_R} \frac{dxdy}{|\zeta - z|} \leq 2R$$

Using (**) and (1) an integration with respect to z over the disc $|z - z_0| < R$ gives:

(2)
$$\iint_{D_R} |f(z)|dxdy \leq 2R \cdot \int_{\partial D_R} |f_n(\zeta)| \cdot |d\zeta| + 4R \cdot \iint_{D_R} |af_n + b_n \bar{f}_n| \cdot d\xi d\eta$$

Let M be the maximum norm of $|a| + |b|$ over D_R. Changing the integration variables $\zeta = \xi + i\eta$ in the last integral above it follows that

(3)
$$\iint_{D_R} |f_n(z)| \cdot dxdy \leq 2R \cdot \int_{\partial D_R} |f_n(\zeta)| \cdot |d\zeta| + 4MR \cdot \iint_{D_R} |f_n(z)| \cdot dxdy$$

Above (3) holds for every R such that the disc of radius R centered at z_0 stays in Ω. We can choose R so small that $4MR < 1$ and replacing f_n by $\frac{f}{z^n}$ we get

(4)
$$\iint_{D_R} \frac{|f(z)|}{|z|^n} \cdot dxdy \leq \frac{2R}{1 - 4MR} \cdot R^{-n} \cdot \int_{\partial D_R} |f(\zeta)| \cdot |d\zeta|$$

This inequality holds for every $n \geq 1$. if f is not identically zero in the disc D_R we find some $z_* \in D_R$ and a small $\delta > 0$ such that $|z_*|+\delta = \rho < R$ and $f(z) \neq 0$ in the closed disc $\{|z-z_*| \leq \delta\}$. If m_* is the minimum of $|f(z)|$ in this disc we get from (4):

$$\rho^{-n} m_* \cdot \pi\delta^2 \leq \frac{2R}{1-4MR} \cdot R^{-n} \cdot \int_{\partial D_R} |f(\zeta)| \cdot |d\zeta| \tag{5}$$

Since $\rho < R$ this inequality cannot hold for all n. Hence we have established a contradiction and conclude that f must be identically zero in the disc D_R. Finally, since the domain is connected we can continue from new flat points in neighborhoods where f is identically zero and conclude that $f = 0$ holds in the whole domain Ω.

Exercise. Use similar methods as in the proof above to show that if f is a solution to the differential equation in 8.2 which is not identically zero, then the set of zeros must be a *discrete* subset of Ω, i.e. there cannot exist $z_* \in \Omega$ and a sequence of distinct points $\{z_\nu\}$ such that $z_\nu \to z_*$ and $f(z_\nu) = 0$ for all ν.

§ 9. Normal families.

Introduction. Cauchys integral formula for a disc D gives the existence of power series expansions for analytic functions $f(z)$. Moreover, we obtain bounds for higher order derivatives. For example, suppose that $f(z) \in \mathcal{O}(D_r)$ where D_r is centered at the origin and has radius r. Assume that f extends to a continuous function on the closed disc $\bar{D}_r$. If $|z| < r$ we have seen that:

$$f'(z) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(\zeta)d\zeta}{(\zeta - z)^2} \tag{*}$$

Let $|z| \leq r/2$ and with $\zeta = re^{i\theta}$ during the integration, the triangle inequality gives

$$|f'(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{|f(re^{i\theta}| \cdot r \cdot d\theta}{(r - |z|)^2}$$

So if $M = \max\, |f(re^{i\theta})|$ and $|z| \leq r/2$ we get

$$\max_{|z| \leq r/2} |f'(z)| \leq \frac{4M}{r}$$

Thus, we obtain a uniform bound for the derivative via the maximum norm of f over a larger disc. From this one gets useful convergence principle.

9.1. Theorem. *Let Ω be an open set and $\{f_\nu \in \mathcal{O}(\Omega)\}$ a sequence with uniform bounded maximum norms, i.e. $|f_\nu|_\Omega \leq M$ hold for all ν and some constant M. Then there exists at least one subsequence $\{g_j = f_{\nu_j}\}$ such that $g_j \to g_*$ holds uniformly over compact subsets of Ω where the limit function g_* is analytic i Ω.*

The proof uses *Arzela's Theorem.* Namely, by the local estimate for first order derivatives above, the uniform bound for maximum norms implies that $\{f_\nu\}$ yields an *equi-continuous* family of continuous functions over every compact subset of Ω. Next, one exhausts Ω by some increasing sequence of compact subsets to get the theorem above. We leave the details as an exercise. Bu we shall give a more detailed account in the case when Ω is a disc where one discovers how to obtain convergent subsequences. In addition to Theorem 9.1 the following result is often used:

9.2. Theorem. *Let Ω be an open set and $\{f_\nu \in \mathcal{O}(\Omega)\}$ a sequence which converges uniformly to a limit function g_* over each compact subset of Ω. Assume also that there exists a relatively compact subset Ω_0 of Ω and an integer $k \geq 1$, such that every f_ν has exactly k zeros in Ω_0 - as usual counted with multiplicity, while no zeros occur outside Ω_0. Then the limit function g_* is either identically zero or it has exactly k zeros in Ω counted with multiplicity and they all belong to the closure $\bar{\Omega}_0$.*

Remark. The proof uses *Rouche's theorem* which is proved in Chapter IV. Once we have a convergence $f_\nu \to g_*$, it follows that the sequence of complex derivatives also converges, i.e. following implication holds:

$$(*) \qquad f_\nu \to g_* \implies f'_\nu \to g'_*$$

This will be used in the proof of *Riemann's Mapping Theorem* in XX where one regards a convergent sequence $f_\nu \to g_*$ and assume that the derivatives f'_ν have no zeros. For in this situation the discussion above shows that g_* is either identically a constant or else g'_* never has zeros.

9.3 Convergence in discs Let $\delta_0 > 0$ and denote by D^* the open disc centered at the origin of radius $1+\delta_0$. If $f \in \mathcal{O}(D^*)$ we obtain the the series representation from 4.5:

$$f(z) = \sum c_\nu z^\nu$$

Assume that f is bounded, i.e. there is a constant M such that $|f(z)| \leq M$ for all $z \in D^*$. For any $0 < \delta < \delta_0$ we then have

$$c_\nu = \frac{1}{2\pi} \cdot (1+\delta)^{-\nu} \int_0^{2\pi} f((1+\delta)e^{i\theta})e^{-i\nu\theta} d\theta$$

The triangle inequality gives

$$|c_\nu| \leq M \cdot (1+\delta)^{-\nu} \quad : \nu = 0, 1, 2, \ldots$$

These upper bounds for the coefficients of f depend on M only. The inequality holds for every $\delta < \delta_0$ and with $\delta = \delta_0$ we still have the estimates of $\{c_\nu\}$. Suppose now that N is a positive integer and $\epsilon > 0$ some positive number such that

$$|c_\nu| \leq \epsilon \quad : 0 \leq \nu \leq N$$

Then, if $0 < r < 1$ we have

$$f(re^{i\theta}) = \sum_{\nu=0}^{\nu=N} c_\nu r^\nu \cdot e^{i\nu\theta} + \sum_{\nu=N+1}^{\infty} c_\nu r^\nu \cdot e^{i\nu\theta}$$

By the triangle inequality we can estimate both sums. The result is

$$f(re^{i\theta})| \leq \epsilon \cdot \sum_{\nu=0}^{\nu=N} r^\nu + M \cdot \sum_{\nu=N+1}^{\infty} \frac{r^\nu}{(1+\delta_0)^\nu}$$

Taking the whole sum over the two geometric series above we get

9.4 Proposition. *One has*

$$\max_\theta |f(re^{i\theta})| \leq \epsilon \cdot \frac{1}{1-r} + M \frac{r^{N+1}}{(1+\delta_0)^{N+1}} \cdot \frac{1}{1-\frac{r}{1+\delta_0}}$$

9.5 Application. Let $\{f_k\}$ be a sequence in $\mathcal{O}(D^*)$ where the maximum norm of every f_k is $\leq M$. Each f_k has a series expansion $\sum c_\nu(k)z^\nu$. For each fixed ν we have a bounded sequence $\{c_\nu(1), c_\nu(2), \ldots\}$ of complex numbers. By the diagonal procedure we can find a subsequence $k_1 < k_2 < \ldots$ such that there exists

$$\lim_{j\to\infty} c_\nu(k_j) = c^*_\nu \quad : \nu = 0, 1, \ldots$$

The uniform estimates for the $c_\nu(k)$-coefficents give

$$|c^*_\nu| \leq M \cdot (1+\delta_0)^{-\nu}$$

Hence there exists the analytic function in D^*:

$$g(z) = \sum c^*_\nu \cdot z^\nu$$

Renumber the f-functions by setting $g_j = f_{k_j}$. The convergence entails that for each pair $\epsilon > 0$ and $N \geq 1$, there exists some integer N^* such that

$$|c_\nu(k_j) - c^*_\nu| \leq \epsilon \quad : 0 \leq \nu \leq N \quad : j \geq N^*$$

Now we can apply Proposition 9.4 to estimate the maximum norm of $|g_j - g^*|$ over discs of radius $r < 1$. In particular we obtain

9.6 Proposition. *The sequence $\{g_j = f_{k_j}\}$ converges uniformly to the limit function g^* over each disc $|z| \leq r$ with $r < 1$.*

9.7 Convergene in the whole of D^*. The rate of convergence for maximum norms over discs $|z| \leq r < 1$ in Proposition 9.6 are well controlled. By relaxing the estimates a bit it is still true that $g_j \to g^*$ holds uniformly in discs $|z| \leq r$ for *any* $r < 1 + \delta_0$. To see this one applies the "(ϵ, δ) - yoga". So let $\epsilon > 0$ and $r < 1 + \delta_0$ be given. We first find a large N so that

$$M \frac{r^{N+1}}{(1+\delta_0)^{N+1}} \cdot \frac{1}{1 - \frac{r}{1+\delta_0}} \leq \epsilon$$

Then, if $|z| \leq r$,Proposition 9.4 and the triangle inequality yield:

$$|g_j(z) - g^*(z)| \leq \sum_{\nu=0}^{\nu=N} |c_\nu(k_j) - c^*_\nu| \cdot r^\nu + 2\epsilon \quad : j \geq N^*$$

Since $\lim_{j\to\infty} c_\nu(k_j) \to c^*_\nu$ hold for each $0 \leq \nu \leq N$ we find some large $N^{**} > N^*$ such that

$$\sum_{\nu=0}^{\nu=N} |c_\nu(k_j) - c^*_\nu| \cdot r^\nu \leq \epsilon \cdot \frac{1}{(1+\delta_0)^N}$$

Then we see that maximum norms of $g_j - g^*$ are $\leq 3\epsilon$ if $j \geq N^{**}$. This proves that uniform convergence holds over all discs inside D^*.

9.8 Remark. The conclusion above is quite striking since we from the start only assume that the coefficients begin to converge and after conclude that one gets uniform convergence to the limit function g^* over every compact disc inside D^*.

§ 10. Laurent series.

Let $R^* > 1$ and consider the open domain

$$\Omega = \{z \quad : 1 < |z| < R^*\}$$

We refer to Ω as an open annulus where $|z| = 1$ is the inner circle and $|z| = R^*$ is the outer circle. Let $f(z)$ be analytic in Ω. When $1 < |z| < R$ is fixed we can choose a pair r, R such that

$$1 < r < |z| < R < R^*$$

Cauchy's formula applies to the domain $r < |z| < R$ and gives the equality:

$$(*) \qquad f(z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=R} \frac{f(\zeta) \cdot d\zeta}{\zeta - z} - \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \frac{f(\zeta) \cdot d\zeta}{\zeta - z}$$

The idea is now to expand $\zeta - z$ in a geometric series. When $|\zeta| = R > |z|$ we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \sum_{\nu=0}^{\infty} \frac{z^\nu}{\zeta^\nu}$$

Let us put

$$a_\nu = \frac{1}{2\pi i} \cdot \int_{|\zeta|=R} \frac{f(\zeta) \cdot d\zeta}{\zeta^{\nu+1}} \quad : \quad \nu = 0, 1, 2, \ldots$$

Then the first integral in (*) becomes

$$(1) \qquad f^*(z) = \sum a_\nu \cdot z^\nu$$

Next, when $|\zeta| = r < |z|$ we have the series expansion

$$\frac{1}{\zeta - z} = -\frac{1}{z} \cdot \sum \frac{\zeta^\nu}{z^\nu}$$

Put

$$b_\nu = \frac{1}{2\pi i} \cdot \int_{|\zeta|=R} f(\zeta) \cdot \zeta^{\nu-1} \cdot d \quad : \quad \nu = 1, 2, \ldots$$

Taking the negative sign into the account in (*) we see that the second integral becomes

$$(2) \qquad f_*(z) = \sum_{\nu=1}^{\infty} \frac{b_\nu}{z^\nu}$$

1. Definition. *The function $f^*(z)$ is called the positive part of f and $f_*(z)$ the lower part of f.*

Remark. In the construction above R can be chosen arbitrarily close to R^*. With $R = R^* - \epsilon$ we get the finite maximum norm

$$||f||_{R_*-\epsilon} = \max_{\zeta|=R_*-\epsilon} |f(\zeta)|$$

The triangle inequality gives

$$|a_\nu| \leq \frac{||f||_{R^*-\epsilon}}{(R^* - \epsilon)^\nu}$$

Since ϵ can be made arbitrary small we conclude that the radiius of convergence for the series (1) is ≥ 1. Hence we have proved:

2. Proposition. *The positive part $f^*(z)$ is analytic in the disc $|z| < R^*$.*

In a similar way the reader may verify the following:

3. Proposition. *The inner part $f_*(z)$ is analytic in the exterior disc $|z| > 1$.*

4. The Laurent series. The analytic function $f(z)$ in the annulus can be written as a sum $f^*(z) + f_*(z)$. Here $f^* \in \mathcal{O}(D_{R^*})$ while f_* is analytic in the exterior disc $|z| > 1$. We can take the two series together and hence $f(z)$ is represented by

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu \cdot z^\nu + \sum_{\nu=1}^{\infty} \frac{b_\nu}{z^\nu}$$

This is called the Laurent series of $f(z)$.

5. The residue coefficient b_1. From the construction above we have

$$b_1 = \frac{1}{2\pi i} \int_{|\zeta|=R} f(\zeta) \cdot d\zeta$$

Notice that we can perform the integral over any circle, i.e. above we can take any $1 < R < R*$. The special role of b_1 is that $b_1 = 0$ holds if and only if $f(z)$ has a primitive analytic function in the annulus .

6. Exercise. Prove the assertion above, i.e. that there exists an analytic function $F(z)$ in the annulus such that $F'(z) = f(z)$ if and only if $b_1 = 0$.

7. Example. Consider the analytic function in the annulus $1 < |z| < R$ defined by

$$\phi(z) = \frac{1}{z+1} - \frac{1}{z-1}$$

Here $\phi(z)$ has the Laurent series expansion

$$\phi(z) = -2 \cdot (\frac{1}{z^2} + \frac{1}{z^4} + \ldots)$$

Hence there exists the primitive function

$$\Phi(z) = 2 \cdot (\frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \ldots)$$

Next, we notice that $\Phi(z)$ can be expressed by the complex Log-function

$$\log (\frac{z+1}{z-1}) \tag{*}$$

Here some care must be taken. First it is clear that (*) is defined when $z = x$ is real with $x > 1$. So on the real interval $(1, +\infty)$ we have the equality

$$G(x) = \log (\frac{x+1}{x-1})$$

8. Exercise. Let $R > 1$ and consider an analytic function

$$g(z) = \sum c_\nu \cdot z^\nu$$

in the open disc D_R. It can be restricted to the real line $[-1, 1]$. Set

$$K_g(z) = \int_{-1}^{1} \frac{g(z) - g(s)}{z - s} \cdot ds$$

The algebraic identity $z^\nu - s^\nu = (z - s)(z^{\nu-1} + \ldots + s^{\nu-1})$ gives

$$K_g(z) = 2c_1 + \sum_{\nu=2}^{\infty} c_\nu \cdot \int_{-1}^{1} \left[z^{\nu-1} + z^{\nu-2}s + \ldots + s^{\nu-1}\right] \cdot ds$$

Since we integrate over $[-1, 1]$ the integrals taken over odd s-powers are all zero and a computation gives:

$$K_g(z) = 2c_1 + 2 \cdot \sum_{\nu=2}^{\infty} c_\nu \cdot \left[z^{\nu-1} + \frac{z^{\nu-3}}{3} + \frac{z^{\nu-5}}{5} + \ldots\right] \tag{1}$$

Notice that the sum in each bracket is finite. Show that $K_g(z)$ is analytic in the disc $|z| > R$ and using the Φ-function from Example 7 the reader should verify that $K_g(z)$ is equal to the positive part of the Laurent series defined by $g(z) \cdot \Phi(z)$ in $1 < |z| < R$.

§ 11. An area formula.

Consider a Laurent series

$$f(z) = \sum_{-\infty}^{\infty} c_\nu \cdot z^\nu$$

which represents the analytic function defined in $\{R_* < |z| < R^*\}$. If $R_* < r < R^*$ we get the parametrised curve

$$(*) \qquad \theta \mapsto \theta \mapsto f(re^{i\theta})$$

Consider the situation where (*) is bijective so that the image is a closed Jordan curve J_r in the complex ζ-plane which borders a bounded Jordan domain Ω_r. We shall express the area of Ω_r with the the coefficients in Laurent series of f. The formula depends upon the orientation in the map (*) which may be either positive or negative.

1. Theorem. *One has has the equality*

$$\text{area}(\Omega_r) = \text{sign}(*) \cdot \pi \cdot \sum_{-\infty}^{\infty} n \cdot |c_n|^2 \cdot r^{2n}$$

where sign$(*)$ is +1 or -1 depending upon the orientation in (*).

2. Example. If $f(z) = z$ the orientation is positive and here Ω_r is the disc of radius r and the formula is okay since its area is $\pi \cdot r^2$. On the other hand, if $f(z) = \frac{1}{z}$ the orientation is negative and the sign-rule in Theorem 1 gives a correct formula.

emphProof of Theorem 1. Set $w = f(z)$ and suppose that (*) is positively oriented. The area formula from XX gives:

$$\text{area}(\Omega_r) = \int_{J_r} \bar{w} \cdot dw = \int_{|z|=r} \bar{f}(z) \cdot f'(z) \cdot dz$$

The right hand side is a double sum

$$\sum \sum \bar{c}_k \cdot m \cdot c_m \cdot \int_{|z|=r} \bar{z}^k \cdot z^{m-1} \cdot dz$$

extended over all pairs of integers. Since $\bar{z}^k = \frac{r^{2k}}{z^k}$ holds on $|z| = r$ we have

$$\int_{|z|=r} \bar{z}^k \cdot z^{m-1} \cdot dz = r^{2k} \cdot \int_{|z|=r} z^{-k} \cdot z^{m-1} \cdot dz$$

By Cauchy's residue formula the last integrals are zero for when $k \neq m$ and become $r^{2k} \cdot 2\pi i$ if $k = m$. Now we can read off the formula in Theorem 1. The proof when (*) has a negative orientation is the same after signs have been reversed.

3. A special case. Let

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} c_n \cdot z^n$$

where the positive series represents an analytic function in some disc $|z| < R$.Suppose that (*) holds for some $0 < r < R$ where the bijective map from (*) has a negative orientation which gives:

$$\text{area}(\Omega_r) = \pi \cdot \Big[\frac{1}{r} - \sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot r^{2n}\Big]$$

Since the area is a positive number we get the inequality

$$(1) \qquad \sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot r^{2n} \leq \frac{1}{r}$$

4. Koebe's inequality. Let ϕ be an analytic function in the unit disc where $\phi(0) = 0$ and $\phi'(0) = 1$ and ϕ is a conformal map from D onto some simply connected domain in $\mathbf{C}$. Put

$$f(z) = \frac{1}{\phi(z)}$$

Now f is analytic in the punctured disc $0 < |z| < 1$ and since ϕ is a conformal map it follows that f maps circles $|z| = r$ onto closed Jordan curves where the orientation now is negative for each $0 < r < 1$. Moreover f has a Laurent series

$$\frac{1}{z} + \sum_{n=0}^{\infty} c_n \cdot z^n$$

and since the inequality (1) holds for every $r < 1$ we have

(i) $$\sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot r^{2n} \leq 1$$

At the same time the given ϕ-function has a series expansion

$$\phi(z) = z + d_2 z^2 + d_3 z^3 + \dots$$

Let us assume that ϕ is an *odd* function which to begin with gives $d_2 = 0$ and

$$\frac{1}{\phi} = \frac{1}{z} \cdot \frac{1}{1 + d_3 z^2 + d_4 z^3 + \dots}$$

From this we conclude that

$$c_1 = -d_3$$

and the inequality (i) gives

(ii) $$|d_3| \leq 1$$

Exercise. Use (i) above to show that equality holds in (ii) if and only if

$$\phi(z) = \frac{z}{1 - \lambda z^2}$$

for some λ whose absolute value is one.

Next, consider a conformal map ψ from D to a simply connected domain whose Taylor series at $z = 0$ is of the form

$$\psi(z) = z + a_2 z^2 + \dots$$

5. Theorem. *One has the inequality* $|a_2| \leq 2$.

Proof. To begin with $\psi(z^2)$ is analytic in D and starts with z^2 which gives the analytic function

(i) $$\frac{\psi(z^2)}{z^2} = 1 + a_2 \cdot z^2 + \sum_{m \geq 2} b_m \cdot z^{2m}$$

Since $\psi(z) \neq 0$ when $z \neq 0$ the analytic function in (i) is also $\neq 0$ in D. Now the unit disc is simply connected and hence there exists an analytic square root function:

(ii) $$\phi(z) = z \cdot \sqrt{\frac{\psi(z^2)}{z^2}}$$

Sublemma The function ϕ is odd function and yields a conformal mapping, i.e. ϕ is 1-1 in D.

The proof of this Sublemma is left as an exercise. In particular we have a series expansion:

(iii) $$\phi(z) = z + d_3 \cdot z^3 + \dots$$

Notice that the square root in (i) has a series expansion

(iv) $$1 + \frac{a_2}{2} \cdot z^2 + \text{higher order terms}$$

It follows that

(v) $$d_3 = \frac{a_2}{2}$$

Then (ii) from (4) above gives $|a_2 \leq 2$ as required.

6. Exercise. Show that the equality $|a_2| = 2$ holds in Theorem 5 if and only if ψ is of the form

$$\psi(z) = \frac{z}{(1-\lambda z)^2} \quad \text{for some} \quad |\lambda| = 1$$

Show also that every such function indeed yields a conformal map from D onto a simply connected domain which becomes a *radial slit domain*. For example, take $\lambda = 1$ and show that the ψ-image of D becomes

$$\mathbf{C}) \setminus \big[-1/4, +\infty\big)$$

§ 12. A theorem by Jentzsch

Introduction. Let $\sum c_n z^n$ b a convergent series in the unit disc D whose radius of convergence is one. To each $n \geq 1$ we get the Taylor polynomials

$$s_n(z) = c_0 + c_1 z + \ldots + c_n z^n$$

Denote by $\mathcal{N}(s_n)$ the zero set of s_n.

12.1 Theorem. *For each point $e^{i\theta}$ on the unit circle there exists a strictly increasing sequence $1 \leq n_1 < n_2 < \ldots$ and points $z_k \in \mathcal{N}(s_{n_k})$ such that $\lim_{k\to\infty} z_k = e^{i\theta}$.*

Remark. This result was proved by X. Jentzsch in the article [Jen: Acta mathematica 1918]. To illustrate the theorem we consider the analytic function $f = \frac{1}{1-z}$. It has no zeros in D but the zeros of

$$s_n(z) = 1 + \ldots + z^n$$

are roots of unity which cluster on the whole unit circle as $n \to \infty$. The fact that a similar clustering of zeros occur for partial sums of a function of the form

$$f(z) = \sum z^{\nu_k}$$

where $1 \leq \nu_1 < \nu_2 < \ldots$ is an arbitrary increasing sequence of integers is less evident but has an affirmative answer by the theorem above.

Proof of Theorem 12.1

We argue by contradiction. Suppose that some point on T is not a cluster point for zeros of the s-functions. After a rotation we may assume that this point is $z = 1$. So now there exists $0 < \delta < 1$ and a positive integer k_* such that

(1) $$s_k(z) \neq 0 \quad : |z-1| < \delta \quad : k \geq k^*$$

There remains to derive a contradiction. Since $s_k \to f$ holds uniformly inside the unit disc and f is not reduced to a constant, it follows from XX and (1) that $f(z) \neq 0$ in the domain

$$\Omega = D \cap \{|z-1| < \delta\}$$

Since Ω is simply connected there exists $h \in \mathcal{O}(\Omega)$ where

$$f(z) = e^{h(z)}$$

We also find a sequence $\{g_k \in \mathcal{O}(D_\delta(1))\}$ such that

$$s_k(z) = e^{g_k(z)} \quad : \quad z \in D_\delta(1)$$

Cnosider the fixed point $\xi = 1 - \delta/8$. Now $s_k(\xi) \to f(\xi)$ and we can theefore choose branches of $\{g_k\}$ so that

$$\lim_{k \to \infty} g_k(\xi) = h(\xi) \tag{2}$$

Since $s_k(z) \to f(z)$ holds uniformly on compact subsets of Ω it follows that $g_k(z) \to h(z)$ holds with uniform convergence over compact subsets of Ω. Next, consider the functions

$$\phi_k(z) = e^{\frac{g_k(z)}{k}} - 1 \tag{3}$$

The convergence $g_k \to h$ in Ω entails that

$$\phi_k(z) \to 0 \tag{4}$$

where this convergence is uniform over compact subsets of Ω. Next we have

Lemma. *The ϕ-functions are uniformly bounded in $D_\delta(1)$.*

Proof. First it is clear that there exists a constant A such that

$$|c_n| \le A \cdot 2^n$$

With $\delta < 1$ and $|z| \le 1 + \delta$ the triangle inequality gives

$$|s_k(z)| \le A \cdot 2^{k+1} \cdot (1+\delta)^k \quad : \quad k = 1, 2 \dots \tag{i}$$

Since $e^{\mathfrak{Re}(g_k)(z)} = |s_k(z)|$ we see that (i) gives:

$$\mathfrak{Re}(g_k)(z) \le (k+1) \cdot \log 2 + k(1+\delta) + \log^+ A \tag{ii}$$

Dividing by k we see that $\{\frac{\mathfrak{Re}(g_k)(z)}{k}\}$ are uniformly bounded above when $z \in D_\delta(1)$ and Lemma 1 follows since

$$|\phi_k(z)| \le e^{\frac{\mathfrak{Re}\, g_k(z)}{k}} + 1$$

Final part of the proof: The uniform bound in the Lemma and the convergence from (4) imply that $\{\phi_k(z)\}$ converges uniformly to zero in compact subsets of $D_\delta(1)$. In particular we can find some $0 < \epsilon < \delta$ such that

$$|\phi_k(1+\epsilon)| < \epsilon/2$$

hold for all large k. It follows that

$$|s_k(1+\epsilon)| = |1 + \phi_k(1+\epsilon)|^k \le (1+\epsilon/2)^k \tag{*}$$

holds when k large and then

$$|c_k(1+\epsilon)^k| = |s_k(1+\epsilon) - s_{k-1}(1+\epsilon)| \le |s_k(1+\epsilon)| + |s_{k-1}(1+\epsilon)| \le 2 \cdot (1+\epsilon/2)^k \implies$$

$$\limsup_{k \to \infty} |c_k|^{\frac{1}{k}} \le \lim 2^{\frac{1}{k}} \cdot \frac{1+\epsilon/2}{1+\epsilon} = \frac{1+\epsilon/2}{1+\epsilon}$$

The last term is < 1 which contradicts that the radius of convergence of $\sum c_k z^k$ is one and hence Theorem 12.1 is proved.

§ 13. An inequality by Siegel

The result below was proved by C. Siegel in Math. Zeitschrift Bd. 10 page 175 (1921) Let $n \geq 2$ and $p(z)$ is a monic polynomial

$$p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

Let $\alpha_1, \ldots \alpha_n$ be the roots where eventual multiple roots are repeated.

Theorem.

$$\prod_{\nu=1}^{\nu=n} (1 + |\alpha_\nu|) \leq 2^n \cdot \sqrt{1 + |a_1|^2 + \ldots + |a_n|^2}$$

Proof. For every complex number the formula from XXX gives

$$(*) \qquad 1 + |z| \leq 2 \cdot \max(1, |z|) = 2 \cdot \exp\Big[\, \frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}|e^{i\theta} - z| d\theta \,\Big]$$

Apply this to each root of p and take the product. This gives

$$\prod_{\nu=1}^{\nu=n} (1 + |\alpha_\nu|) \leq 2^n \cdot \exp\Big[\, \frac{1}{2\pi} \cdot \int_0^{2\pi} \sum_{\nu=1}^{\nu=n} \operatorname{Log}|e^{i\theta} - \alpha_\nu \,| \cdot d\theta \,\Big] =$$

$$2^n \cdot \exp\Big[\, \frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}|p(e^{i\theta})|\, d\theta \,\Big]$$

To estimate the last term we employ Blascke's factorization of $p(z)$ in the unit disc $|z| < 1$:

$$p(e^{i\theta}) = B(e^{i\theta}) \cdot e^{\phi(e^{i\theta})} \quad : |B(e^{i\theta}| = 1 \quad : \phi(z) \in \mathcal{O}(D) \implies$$

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}|p(e^{i\theta})|\, d\theta = \frac{1}{2\pi} \cdot \int_0^{2\pi} \mathfrak{Re}(\phi(e^{i\theta}))d\theta = \mathfrak{Re}(\phi)(0)$$

where the last equality holds since $\mathfrak{Re}(\phi)$ is harmonic. Hence there only remains to show the inequality

$$\text{(i)} \qquad e^{\mathfrak{Re}(\phi)(0)} \leq \sqrt{1 + |a_1|^2 + \ldots + |a_n|^2}$$

To show this we use the Plancherel formula which gives

$$\text{(ii)} \qquad \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = 2\pi \cdot (1 + |a_1|^2 + \ldots + |a_n|^2)$$

In addition, we have

$$\text{(iii)} \qquad |p(e^{i\theta})|^2 = e^{2 \cdot \mathfrak{Re}(\phi(e^{i\theta}))}$$

Now we can finish the proof as follows: Since $e^{\phi(z)}$ is analytic the mean value formula gives:

$$\text{(iv)} \qquad e^{\phi(0)} = \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{\phi(e^{i\theta})} \cdot d\theta$$

Since the absolute value $|e^{\phi(z)}| = e^{\mathfrak{Re}(\phi)(z)}$ hold for all z, the triangle inequality gives

$$\text{(v)} \qquad e^{\mathfrak{Re}(\phi)(0)} \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{\mathfrak{Re}(\phi(e^{i\theta}))} \cdot d\theta$$

The Cauchy-Schwarz inequality applied to (v) gives:

$$\text{(vi)} \qquad e^{\mathfrak{Re}(\phi)(0)} \leq \frac{1}{\sqrt{2\pi}} \cdot \Big[\int_0^{2\pi} e^{2\mathfrak{Re}(\phi(e^{i\theta}))} \cdot d\theta \,\Big[^{\frac{1}{2}}$$

Now (ii) and (iii) give the requested inequality (i).

§ 14. Zeros of product series.

Let $\{\lambda_k\}$ be a non-decreasing sequence of positive real number where $\lambda_1 \geq 2$ and

$$\sum \lambda_k^{-(1+\epsilon)} < \infty \quad \text{hold for every} \quad \epsilon > 0 \tag{1}$$

This gives the analytic function $\phi(z)$ defined in $\mathfrak{Re}\, z > 1$ by

$$\phi(z) = \prod (1 - \lambda_k^{-z}) \tag{*}$$

The logarithmic derivative becomes

$$\frac{\phi'(z)}{\phi(z)} = -\sum \log \lambda_k \cdot \frac{\lambda_k^{-z}}{1 - \lambda_k^{-z}}$$

Expanding the denominator $1 - \lambda_k^{-z}$ into a geometric series we obtain:

$$\frac{\phi'(z)}{\phi(z)} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-nz}$$

If $z = 1 + \epsilon + iy$ for some $\epsilon > 0$ and a real y we set

$$\rho_\epsilon(n; y) = \sum_{k=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-n(1+\epsilon)} \cdot \lambda_k^{-iny}$$

14.1 A remarkable positive sum. Let $y_0 \neq 0$ and consider the sum

$$S_\epsilon(n) = 3 \cdot \rho_\epsilon(n; 0) + 4 \cdot \rho_\epsilon(n; y_0) + \rho_\epsilon(n; 2y_0)$$

Taking real parts we obtain

$$\mathfrak{Re}\, S_\epsilon(n) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-n(1+\epsilon)} \cdot \big[3 + 4 \cdot \mathfrak{Re}\, \lambda_k^{-iny_0} + \mathfrak{Re}\, \lambda_k^{-iny_0}\,\big] \tag{i}$$

In this double sum the terms inside the brackets become

$$3 + 4 \cdot \cos\, n \cdot \lambda_k \cdot y_0 + \cos\, 2n \cdot \lambda_k \cdot y_0 = 2\big(1 + \cos\, n \cdot \lambda_k \cdot y_0\big)^2$$

where the last equality follows from the trigonometric formula $\cos 2a = 2\cos^2 -1$. Hence we have the formula

$$\mathfrak{Re}\big[\, 3 \cdot \frac{\phi'(1+\epsilon)}{\phi(1+\epsilon)} + 4 \cdot \frac{\phi'(1+\epsilon+iy_0)}{\phi(1+\epsilon+iy_0)} + \frac{\phi'(1+\epsilon+2iy_0)}{\phi(1+\epsilon+2iy_0)}\,\big] =$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \log \lambda_k \cdot \lambda_k^{-n(1+\epsilon)} \cdot 2\big(1 + \cos\, n \cdot \lambda_k + \cdot y_0\big)^2 \tag{*}$$

14.2 Conclusion. Keeping y_0 fixed we get the following for each $\epsilon > 0$:

$$\mathfrak{Re}\big[\, 3 \cdot \frac{\phi'(1+\epsilon)}{\phi(1+\epsilon)} + 4 \cdot \frac{\phi'(1+\epsilon+iy_0)}{\phi(1+\epsilon+iy_0)} + \frac{\phi'(1+\epsilon+2iy_0)}{\phi(1+\epsilon+2iy_0)}\big] < 0$$

14.3 Absence of asymptotic zeros. Let $y_0 > 0$ and assume that one has three limit expansions:

$$\phi(1+\epsilon) = \epsilon^\alpha \cdot \gamma_0(\epsilon)$$

$$\phi(1+\epsilon+iy_0)) = \epsilon^\beta \cdot \gamma_1(\epsilon)$$

$$\phi(1+\epsilon+2iy_0)) = \epsilon^\kappa \cdot \gamma_2(\epsilon)$$

where α, β, κ are real constants and each γ-function is $\neq 0$ for small positive ϵ and together with their derivatives satisfy

$$\lim_{\epsilon \to 0} \epsilon \cdot \frac{\gamma_j'(\epsilon)}{\gamma_j(\epsilon)} = 0 \quad : \quad 0 \leq j \leq 2$$

Under the conditions above we see that the function

$$\epsilon \mapsto \mathfrak{Re}\big[\,3\cdot\frac{\phi'(1+\epsilon)}{\phi(1+\epsilon)}+4\cdot\frac{\phi'(1+\epsilon+iy_0)}{\phi(1+\epsilon+iy_0)}+\frac{\phi'(1+\epsilon+2iy_0)}{\phi(1+\epsilon+2iy_0)}\,\big]$$

has an asymptotic expansion of the form

$$\frac{3\cdot\mathfrak{Re}\,\alpha+4\cdot\mathfrak{Re}\,\beta+\mathfrak{Re}\,\kappa}{\epsilon}+\gamma^*(\epsilon)$$

where $\epsilon\cdot\gamma^*(\epsilon)\to 0$.

At the same time we have the inequality from 14.2 and checking signs we have the following result:

14.4 Theorem. *If $\phi(z)$ has the three asymptotic expansions above for some $y_0 \neq 0$ then the following inequality must hold:*

$$3\cdot\mathfrak{Re}\,\alpha+4\cdot\mathfrak{Re}\,\beta+\mathfrak{Re}\,\kappa<0$$

14.5 The case when ϕ has a meromorphic extension. When $\phi(z)$ has a meromorphic extension across $\mathfrak{Re}(z)=1$ the asymptotic expansions in (14.3) exist for every $y_0>0$ and α,β,κ are integers. Suppose also that ϕ has a simple pole at $z=1$ which gives $\alpha=-1$. If $\phi(1+iy_0)=$ for some $y_0>0$ the corresponding β-integer is ≥ 1 and Theorem 14.4 implies that $\kappa\leq -1$ which means that ϕ must have a pole at $1+2iy_0$. In particular we have the following:

14.6 Theorem *If ϕ has a simple pole at $z=1$ and extends to a meromorphic function to an open domain Ω which contains the closed half-space $\mathfrak{Re}\,z\geq 1$ with no further poles on $\mathfrak{Re}\,z=1$ then*

$$\phi(1+iy)\neq 0 \quad \text{for all} \quad y>0$$

Remark. The idea to employ positive cosine-functions to investigate zeros and poles of Dirichlet series was introduced by de Valle Poussin in his proof of the Prime Number Theorem. Above we used the positivity of the function $a\mapsto 3+4\cos a+\cos 2a$. Other cosine-sums which produce functions which are everywhere ≥ 0 can be used to obt¡ain more involved relations about poles and zeros of Dirichlet series on the critical line of convergence.

§ 15. Hadamard products.

We expose material from the article *Les series entières* (Acta Math. 1899) by Hadamard and start with some geometric considerations. Let U and V be two Jordan domains which both contain the origin. Put $\sigma = \partial U$ and $\gamma = \partial V$. The inversion map $z \mapsto z^{-1}$ maps every closed Jordan curve which does not pass the origin to a similar closed Jordan curve. In particular, each $z \neq 0$ yields a closed Jordan curve $\gamma(z)$ whose points are $\frac{z}{w}, : w \in \gamma\}$. Following Hadamard we give

15.1 Definition. *Denote by $\mathcal{H}(U,V)$ the set of z such that the closed Jordan curve $\gamma(z)$ is contained in U.*

It is clear that $\mathcal{H}(U,V)$ is an open set which contains a neighborhood of the origin since this by hypothesis is a common interior point of U and V.

15.2 Exercise. Show that the open set $\mathcal{H}(U,V)$ is connected and the equality

$$\mathcal{H}(U,V) = \mathcal{H}(V,U)$$

where the right hand side is constructed when we instead consider the closed Jordan curves $\sigma(z)$ and require that they are contained in V.

Next, let $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$ which extend continuously up to the boundaries of U and V respectively. If $z \in \mathcal{H}(U,V)$ we can take a complex line integral along γ and define

$$\psi(z) = \frac{1}{2\pi i} \int_\gamma f(\frac{z}{\zeta}) \cdot g(\zeta)\, d\zeta$$

15.3 Exercise. Show that $\psi(z)$ is analytic in $\mathcal{H}(U,V)$.

We refer to ψ as the Hadamard convolution of the pair f, g and write $\psi = f * g$. The equality in Exercise 15.2 gives $f * g = g * f$ which me ns that we also have

$$\psi(z) = \frac{1}{2\pi i} \int_\sigma g(\frac{z}{\zeta}) \cdot f(\zeta)\, d\zeta$$

Example. Let U and V be discs of radius R and r centered at the origin where $R \geq r$. In this case $\mathcal{H}(U,V)$ is the disc of radius rR and the reader can verify that the Taylor series of ψ becomes

$$\psi(z) = \sum a_n \cdot b_n \cdot z^n$$

where $f = \sum a_n z^n$ and $g = \sum b_n z^n$.

A study of poles. Suppose that f and g both are rational functions and let $\alpha_1, \ldots, \alpha_m$ be the poles of f and $\beta_1, \ldots, \beta_k$ the poles of g. We assume that both f and g are analytic in a neighborhood of the origin and taking their Taylor series we construct $\psi(z)$ as above. Then the following result was proved by Hadamard in ibid]:

15.4 Theorem. *The function $\psi(z)$ extends to a meromorphic function in the complex plane with poles contained in the set $\{\alpha_\nu \cdot \beta_j\}$.*

Exercise. Prove this theorem. The hint is to use Newton's frational decomposition of the rational functions f and g.