20. A Non-Linear PDE-equation

Introduction. We expose Carleman's article *Über eine nichtlineare Randwertaufgabe bei der Gleichung* $\Delta u = 0$ (Mathematisches Zeitschrift vol. 9 (1921). Here is the equation to be considered: Let Ω be a bounded domain in $\mathbf{R}^3$ with C^1-boundary and $\mathbf{R}^+$ the non-negative real line where u is the coordinate. Let $F(u,p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$(0,1) \qquad u \mapsto F(u),p)$$

is strictly increasing for every $p \in \partial\Omega$ and that $F(0,p) \geq 0$. Moreover,

$$(0.2) \qquad \lim_{u \to \infty} F(u,p) = +\infty$$

holds uniformly with respect to p. For a given point $Q_* \in \Omega$ we seek a function $u(x)$ which is harmonic in $\Omega \setminus \{Q_*\}$ and at Q_* it is locally $\frac{1}{|x-Q_*|}$ plus a harmonic function and on $\partial\Omega$ the inner normal derivative $\partial u/\partial n$ satisfies the equation

$$(*) \qquad \frac{\partial u}{\partial n}(p) = F(u(p),p) \quad : p \in \partial\Omega$$

Finally u extends to a continuous function on $\partial\Omega$.

Theorem 1. *For each F as above the boundary value problem has a unique solution.*

Remark. The subsequent proof teaches how to handle of non-linear boundary value problems. The strategy in Carleman's proof is to consider the family of boundary value problems where we for each $0 \leq h \leq 1$ seek u_h to satisfy

$$(**) \qquad \frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p),p) \quad : p \in \partial\Omega$$

where u_h has the same pole as u above. Starting with $h = 0$ one has the linear Neumann problem

$$\frac{\partial u_0}{\partial n}(p) = u_0(p)$$

This equation has a unique solution given by

$$u_0 = G + \phi$$

where G is Green's function with a pole at Q_* and ϕ is the harmonic function in Ω satisfying the boundary equation

$$\text{(i)} \qquad \frac{\partial \phi}{\partial n}(p) + \frac{\partial G}{\partial n}(p) = \phi$$

Now G is a super-harmonic function in Ω and it is welknown that $\frac{\partial G}{\partial n}$ is a continuous and positive function on $\partial\Omega$ which gives a pair of positive constants $0 < \gamma_* < \gamma^*$ such that

$$\text{(ii)} \qquad \gamma_* \leq \frac{\partial G}{\partial n}(p) \leq \gamma^* \quad : \quad p \in \partial\Omega$$

If ϕ attains its maximum at some $p^* \in \partial\Omega$ its inner normal derivative at p^* must be ≤ 0 and hence (i-ii) and the maximum principle for harmonic functions entails that

$$\max_{p \in \bar{\Omega}} \phi(p) \leq \gamma^*$$

In a similar fashion one proves that

$$\min_{p \in \bar{\Omega}} \phi(p) \geq \gamma_*$$

Next, one reduces the proof of Theorem 1 to the case when F is a real-analytic function of u. This is easy and proved in § below. If F is real-analytic the subsequent proof will show that there exists $\epsilon > 0$ such that if $0 \leq h_0 < 1$ and a solution u_{h_0} to (**) has been found, then there exist solutions $\{u_h\}$ to (**) for all $h_0 < h < h_0 + \epsilon$ expressed by a convergent power series

$$u_h = u_{h_0} + \sum_{\nu 01}^{\infty} (h - h_0)^\nu \cdot u_\nu$$

where $\{u_\nu\}$ is a sequence of harmonic functions are found by solving linear boundary value problems. Starting with the solution u_0 it will follow that there exist solutions u_h for all $0 \leq h \leq 1$ and gives the requested solution in Theorem 1 when $h = 1$.

A.0. Proof of uniqueness.

Suppose that u_1 and u_2 are two solutions to the equation (*). Then $u_2 - u_1$. is harmonic in Ω and if $u_1 \neq u_2$ we may assume that the maximum of $u_2 - u_1$ is > 0. The maximum is attained at some $p_* \in \partial\Omega$ and the strict maximum principle for harmonic functions gives:

(i) $$u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*)$$

for all $x \in \Omega$. With $v = u_2 - u_1$ we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p), p) - F(u_1(p), p)$$

Now (0.1) and (*) entail that $\frac{\partial v}{\partial n}(p_*) > 0$ and since we have taken an inner normal derivative this violates (i) which proves the uniqueness.

A.1 Montone properties.

Let F_1 and F_2 be two functions which both satisfy (0.1) and (0.2) where

$$F_1(u,p) \leq F_2(u,p)$$

hold for all $(u,p) \in \mathbf{R}^+ \times \partial\Omega$. If u_1, respectively u_2 solve (*) for F_1 and F_2 it follows that $u_2(q) \leq u_1(q)$ for all $q \in \Omega$. To see this we set $v = u_2 - u_1$ which is harmonic in Ω. If $p \in \partial\Omega$ we get

(i) $$\frac{\partial v}{\partial n}(p) = F_2(u_2(p), p) - F_1(u_1(p), p) \geq 0$$

Suppose that the maximum of v is > 0 and let the maximum be attained at some point p_*. Since (i) is an inner normal it follows that we must have $0 = \frac{\partial v}{\partial n}(p)$ which would entail that

$$F_2(u_2(p_*)p_*) > F_2(u_1(p_*), p_*) \geq F_1(u_1(p_*), p_*) \implies$$

and this contradicts the strict inequality $u_2(p_*) > u_1(p_*)$ since we have an increasing function in (0.1).

A.2. A bound for the maximum norm. Let u be a solution to (*) and M_u denotes the maximum norm of its restriction to $\partial\Omega$. Choose $p^* \in \partial\Omega$ such that

(1) $$u(p^*) = M_u$$

Let G be the Green's function which has a pole at Q_* while $G = 0$ on $\partial\Omega$. Now

$$h = u - M_u - G$$

is a harmonic function in Ω. On the boundary we have $h \leq 0$ and $h(p^*) = 0$. So p^* is a maximum point for this harmonic function in the whole closed domain $\bar{\Omega}$. It follows that

$$\frac{\partial h}{\partial n}(p^*) \leq 0 \implies$$

$$F(u(p^*), p^*) = \frac{\partial u}{\partial n}(p^*) \leq \frac{\partial G}{\partial n}(p^*)$$

Set

$$A^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

Then we have

(*) $$F(M_u, p^*) \leq A^*$$

Hence the assumption (0.2) for F this gives a robust estimate for the maximum norm M_u. Next, let m_u be the minimum of u on $\partial\Omega$ and consider the harmonic function

$$h = u - m_u - G$$

This time $h \geq 0$ on $\partial\Omega$ and if $u(p_*) = m_u$ we have $h(p_*) = 0$ so here p_* is a minimum for h. It follows that

$$\frac{\partial h}{\partial n}(p_*) \geq 0 \implies F(u(p_*), p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$A_* = \min_{p\in\partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

(**) $$F(m_u, p^*) \geq A_*$$

Remark. Above $0 < A_* < A^*$ are constants which are independent of F. Hence the maximum norms of solutions $u = u_F$ are controlled if the F-functions stay in a family where (0.2) holds uniformly.

B. The linear equation.

Let $f(p)$ and $W(p)$ be a pair of continuous functions on the boundary $\partial\Omega$ where W is positive, i.e. $W(p) > 0$ for every boundary point. The classical Neumann theorem asserts that there exists a unique function U which is harmonic in Ω, extends to a continuous function on the closed domain and its inner normal derivative satisfies:

(1) $$\partial U/\partial n(p) = W(p) \cdot U(p) + f(p) \quad p \in \partial\Omega$$

The uniqueness is a consequence of Green's formula. For suppose that U_1 and U_2 are two solutions to (1) and set $v = U_1 - U_2$. Since v is harmonic in Ω it follows that:

$$\iiint_\Omega |\nabla(v)|^2 dxdydz + \iint_{\partial\Omega} v \cdot \partial v/\partial n \cdot dS = 0$$

Here $\partial v/\partial n = W(p)v$ and since $W(p) > 0$ holds on $\partial\Omega$ we conclude that v must be identically zero. For the unique solution to (1) some estimates hold. Namely, set

$$M_U = \max_p U(p) \quad \text{and} \quad m_U = \min_p U(p)$$

Since U is harmonic in Ω the the maximum and the minimum are taken on the boundary. If $U(p^*) = M_U$ for some $p^* \in \partial\Omega$ we have $\partial U/\partial n(p^*) \leq 0$. Set

$$W_* = \min_p W(p)$$

By assumption $W_* > 0$ and we get

$$M_U \cdot W(p^*) + f(p^*) = \partial U/\partial n(p^*) \leq 0 \implies M_U \leq \frac{|f|_{\partial\Omega}}{W_*}$$

where $|f|_{\partial\Omega}$ is the maximum norm of f on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial\Omega}}{W_*}$$

Hence the following inequality holds for the the maximum norm $|U|_{\partial\Omega}$:

(*) $$|U|_{\partial\Omega} \leq \frac{|f|_{\partial\Omega}}{W_*}$$

B.1 Estimates for first order derivatives. Let $p \in \partial\Omega$ and denote by N the inner normal at p. Since $\partial\Omega$ is of class C^1 a sufficiently small line segment from p along N stays in Ω. So at points $q = p + \ell \cdot N$ we can take the directional derivative of U along N_p This gives a function

$$\ell \mapsto \partial U/\partial N(p+\ell\cdot N)$$

Since the boundary is C^1 these functions are defined on a fixed interval $0 \leq \ell \leq \ell^*$ for all p. With these notations there exists a constant B such that

(**) $$\big|\partial U/\partial N(p+\ell\cdot N)\big| \leq B\cdot ||\partial U/\partial n||_{\partial\Omega} \quad : p\in\partial\Omega \;:\; 0\leq \ell\leq \ell^*$$

where the size of B is controlled by the maximum norm of f on $\partial\Omega$ and the positive constant W_* above.

C. Proof of Theorem 1

It suffices to prove the theorem when $F(u,p)$ is an analytic function with respect to u. For if we have an arbitrary F-function satisfying (0.1) and (0.2), then F can be uniformly approximated by a sequence $\{F_n\}$ of analytic functions. See §§ below for an explicit approximation when a continuous function F is given. If $\{u_n\}$ are the unique solutions to $\{F_n\}$ the estimates in (B) show that there exists a limit function $\lim_{n\to\infty} u_n = u$ where u solves (*) for the given F-function. So now we can assume that $u \mapsto F(u,p)$ is a real-analytic function on the positive real axis for each $p\in\partial\Omega$ where local power series converge uniformly with respect to p and there remains to prove the existence of a solution to the PDE in (*) above Theorem 1.

C.1 The succesive solutions $\{u_h\}$. To each real number $0\leq h\leq 1$ we seek a solution u_h where

(1) $$\frac{\partial u_h}{\partial n}(p) = h\cdot F(u_h,p) + (1-h)\cdot u_h(p)$$

With $h=0$ we get a solution as explained in the introduction. Next, let $0\leq h_0<1$ and suppose we have found the solution u_{h_0} in (1) above. Set $u_0 = u_{h_0}$ and with $h = h_0+\alpha$ for some small $\alpha>0$ we shall find u_h by a series

(3) $$u_h = u_0 + \sum_{\nu=1}^{\infty} \alpha^\nu\cdot u_\nu$$

The pole at Q_* occurs already in u_0 so $u_1,u_2,\ldots$ are harmonic functions in Ω. There remains to determine this sequence so that u_h yields a solution to (1). We will show that this can be achieved when α is sufficiently small. To begin with the results from (B) give positive constants $0<c_1<c_2$ such that

(4) $$0<c_1\leq u_0(p)\leq c_2 \quad : p\in\partial\Omega$$

Next, the analyticity of F with respect to u enables us to write:

(5) $$F(u_h(p),p) = F(u_0(p) + \sum_{k=1}^{\infty} c_k(p)\cdot \big[\sum_{\nu=1}^{\infty}\alpha^\nu u_\nu(p)\big]^k$$

where $\{c_k(p)\}$ are continuous functions on $\partial\Omega$ which appear in an expansion

(6) $$F(u_0(p)+\xi,p) = F(u_0(p),p) + \sum_{k=1}^{\infty} c_k(p)\cdot\xi^k$$

Here (4) and the hypothesis on F entail that the radius of convergence has a uniform bound below, i.e. there exist positive constants $\rho>0$ and C such that which are independent of h such that

(7) $$\max_{p\in\partial\Omega} |c_k(p)| \leq C\cdot\rho^k \quad : \quad k=0,1,\ldots$$

Moreover, the hypothesis (0.2) from the introduction gives a posotove constant C_* whgich also is independent of h such that

(8) $$\min_{p\in\partial\Omega} |c_1(p)| \geq C_*$$

Next, (1) is solved where the harmonic functions $\{u_\nu\}$ which are determined inductively while α-powers are identified. The linear α-term gives the equation

$$\frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1 - h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p) \tag{9}$$

For u_2 we find that

$$\frac{\partial u_2}{\partial n} = \big(1 - h_0 + h_0 c_1(p)\big)u_2 - u_1 + c_1(p)u_1 + c_2(p)u_1^2 \tag{10}$$

In general, for $\nu \geq 3$ one has

$$\frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \ldots, u_{\nu-1}, p) \tag{11}$$

where $\{R_\nu\}$ are polynomials in the preeceding u-functions whose coefficients are determined via the c-functions above. Notice that (8) gives a positive constant C_* which again is independent of h such that

$$\min_{p \in \partial\Omega} 1 - h_0 + h_0 \cdot c_1(p) \geq C_* \tag{12}$$

Next, the equations in (11) can be expressed as follows:

$$\frac{\partial u_m}{\partial n} = \big(1 - h_0 + h_0 \cdot c_1(p)\big) \cdot u_\nu(p) + \alpha \cdot \Big\{ \sum_{k=1}^{\infty} c_k(p) \big[\sum \alpha^\nu u_\nu(p)\big]^k \Big\}_{m-1} \tag{13}$$

where the index $\nu - 1$ indicates that one takes out the coefficient of α^{m-1} when the double sum inside the bracket is expanded as a series in α. Next, using (12) and the estimates for the inhomogeneous linear equation in § B we have a constant C^* which again is independent of h such that

$$\max_{p \in \Omega} |u_m(p)| \leq \alpha \cdot \max_{p \in \Omega} \Big| \Big\{ \sum_{k=1}^{\infty} c_k(p) \big[\sum \alpha^\nu u_\nu(p)\big]^k \Big\}_{m-1} \Big| \leq$$

$$C \cdot \alpha \cdot \sum_{k=1}^{\infty} \rho^k \cdot \max_{p \in \Omega} \Big| \Big\{ \big[\sum \alpha^\nu u_\nu(p)\big]^k \Big\}_{m-1} \Big| \tag{14}$$

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