Stochastic processes in contionous time and their parabolic PDE:s

We consider a stiochastic differential equation of the form

$$(*) \qquad dX_t = b(t, X_t) \cdot dW$$

where $b(t, x)$ is a real-valued functions defined when $t \geq 0$ and x is real. For each $t > 0$ we denote by $f(t, x)$ be the frequency function of the stochastic variable X_t, and when $\Delta > 0$ g_Δ is the Gaussian kernel which gives the frequency function of a normal variable with mean-value zero and variance Δ. Now (*) means that

$$(0.1) \qquad X_{t+\Delta} = X_t + b(X_t, t) \cdot \mathcal{N}_\Delta + \text{small ordo}(\Delta)$$

More precisely, the small ordo term means that it is a stochastic variable whose mean value and variance both are small ordo of Δ. Introduce the characteristic function

$$\widehat{f}_t(\xi) = \int e^{ix\xi} \cdot f(t, x)\, dx$$

By assumption the stochastic variables $b(X_t, t)$ and $\mathcal{N}_\Delta$ are independent. So up to small ordo of Δ the charactersistic function of the right hand side becomes

$$\text{(ii)} \qquad \iint e^{i\xi(x + b(t,x) \cdot \sqrt{\Delta} y)} \cdot f(t, x) \cdot g_\Delta(y)\, dxdy$$

A Taylor expansion of $e^{i\xi(x+b(x,t)\cdot\sqrt{\Delta}y)}$ when Δ is small implies that (ii) up to small ordo of Δ is equal to

$$\text{(iii)} \quad \iint e^{ix\xi} \cdot (1 + i\xi b(t, x) \cdot \sqrt{\Delta} y - \xi^2/2 \cdot b(t, x)^2 \cdot \Delta \cdot y^2) \cdot f(t, x) \cdot g_\Delta(y)\, dxdy$$

Since $g_\Delta(y)$ has mean value zero and

$$\int \Delta \cdot y^2 g_\Delta(y)\, dy = 1$$

we see tahat (iii) - as usual up to small ordo Δ - becomes

$$\int e^{ix\xi}\, f(t, x)\, dx - \Delta \cdot \xi^2/2 \cdot \int e^{ix\xi}\, b(t, x)^2 \cdot f(t, x)\, dx$$

Passing to the limit as $\Delta \to 0$ and using Fourier's inversion formula we obtain

$$(**) \qquad \frac{\partial}{\partial t}(f(t, x)) = \frac{1}{2} \cdot \frac{\partial^2}{\partial x^2}(b(t, x)^2 \cdot f(t, x))$$

Remark. Above (**) is a parabolic partial differential equation. The deduction of (**) from (*) goes back to work by Fourier and Laplace. The study of equations such as (**) were performed by Hadamard, Gevrey and Holmgren around 1900, and later work which has led to many interesting and deep facts about parabolic equations are exposed in many text-books devoted by PDE-theory. An excellent introduction is the book by the eminent mathematician Petrovsky and let us remark that the interplay with random processes was recognised and used by him and his collaborators Khintchine and Kolmogorov. As a personal remark I want to add that I cannot see any novely in the article by Ito from 1950, except that it reproduces Fourier's inversion formula. As a graduate student I prefered to learn the presentation in text-books by the eminent Russian mathematicans above, while the so called "Ito-calculus" even today after my studies meany decades ago still appears for me as a nuisance which onlymtends to make matters more compicated than they are.

Of course one can extend the result above to vector-vaölued stochastic processes with independent increments and where the infinitesmal random change is caused by a vector-valued Guassian density which in general can have time dependent non-zero correöation terms. The resut is a non-linear secnod order parabolic system of PDE:s and a notable point is of course that every such system stems from a stochastic process. This probabilistic interpretation is often used and it was in this way a very refined studiy of the heat-equation was done by the three Russian mathematicians mentioned above in a famous joint article from 1930 which I recommend readers unterested in the subject to study in detail.

Green's function for domains in C.

Let Ω be a connected domain in **C** which belongs to the family $\mathcal{D}(C^1)$. Then Dirichlet's problem is solvable and Riesz's representation formula gives for each $p \in \Omega$ a unique probability measure μ_p on $\partial\Omega$ such that

$$\Phi(p) = \int_{\partial\Omega} \phi(q) d\mu_p(q)$$

for each $\phi \in C^0(\partial\Omega)$ where Φ is its unique harmonic extension to Ω. In particular, for each $p \in \Omega$ we solve the Dirichlet problem for boundary value functions

$$q \mapsto \log \frac{1}{|p-q|}$$

This gives the unique harmonic extension

$$H_p(q) = \frac{1}{2\pi} \cdot \int_{\partial\Omega} \log \frac{1}{|p-q|} \cdot d\mu_p(q) \tag{1}$$

Set

$$G(p,q) = \frac{1}{2\pi} \cdot \log \frac{1}{|p-q|} - H_p(q) \tag{2}$$

From the above the G-function is defined for all pairs (p,q) in Ω and for each $p \in \Omega$ we have

$$G(p,q) = 0 \quad : q \in \partial\Omega$$

We refer to $G(p,q)$ as Green's function. With p fixed we notice that the function

$$q \mapsto \frac{1}{2\pi} \cdot \log \frac{1}{|p-q|}$$

is super-harmonic in Ω, i.e. its negative is subharmonic. The minimum principle for superharmonic functions entails that $G(p,q) > 0$ for all pairs p,q in Ω. Next, recall that when p is fixed then the locally integrable function $q \mapsto \frac{1}{2\pi} \cdot \log |p-q|$ is a fundamental solution to the Laplcae operator.

Exercise. Use the above to show that when g is a function in $C^2(\bar{\Omega})$, then

$$g(p) = \int_{\partial\Omega} \frac{\partial G(p,q)}{\partial n_*} \cdot g(q) \cdot ds(q) + \iint_{\Omega} G(p,q) \cdot \Delta(g)(q)\, dq \tag{*}$$

Here we have taken the inner normal derivative of $q \mapsto G(p,q)$ along $\partial\Omega$ and ds refers to the arc-length measure. Finally, the second term is an area integral over Ω. In the specia. case when g is harmonic in Ω we obtain

$$g(p) = \int_{\partial\Omega} \frac{\partial G(p,q)}{\partial n_*} \cdot g(q) \cdot ds(q) \quad : \ p \in \Omega$$

Since (*) hold for every such g-function and C^2-functions are dense in the space of continuous functions, we conclude:

0.1 Theorem. *For each $p \in \Omega$ one has the equality*

$$\mu_p = \frac{\partial G(p,q)}{\partial n_*} \cdot ds(q)$$

Remark. In particular μ_p is absolutely continuous with respect to the arc-length measure.

0.2 The Δ-operator on $L^2(\Omega)$. We have the complex Holbert space of square-integrable functions in Ω. Here $C_0^\infty(\Omega)$ appears as a dense subspace and Stokes theorem gives

$$\iint_{\Omega} \Delta(g) \cdot \bar{f}\, dq = \iint_{\Omega} g \cdot \overline{\Delta(f)}\, dq$$

for all pairs of test-functions. It means that Δ yields a densely defined and symmetric linear operator on $L^(\Omega)$. By definition the adjoint operator Δ^* has a domain of definition which consists of L^2-functions ϕ for which there exists a constant $C(\phi)$ such that

$$|\iint_\Omega \Delta(g)\cdot\overline{\phi}\,dq| \leq C(\phi)\cdot ||g||_2 \tag{0.2.1}$$

hold for all test-functions g, where the last term is the L^2-norm of g in $L^2(\Omega)$. Concerning the densely defined operatopr Δ its domain of definition consists of L^2–functions ϕ in Ω such that $\Delta(\phi)$ in the sense of distribution theory is locally integrable, and moreover its L^2-integral taken over Ω is finite. By general facts in distribution theory the graph of Δ is closed. Less obvious is the following:

0.2.2 Theorem. *The closed and densely defined operator* Δ *is self-adjoint, i.e. we have the equality*

$$\Delta = \Delta^*$$

Remark. The proof amounts to show that if $\phi \in L^2(\Omega)$ satisfies (0.2.1) then it belongs to $\mathcal{D}(\Delta)$, i.e. (0.2.1) implies that $\Delta(\phi)$ taken in the distribution sense belongs to $L^2(\Omega)$.

0.2.3 How to prove Theorem 0.2.2. The crucial point is to regard the linear operator on $L^2(\Omega)$ defined by

$$\mathcal{G}_\phi(p) = \iint_\Omega G(p,q)\cdot\phi(q)\,dq$$

Exercise. Show that $\mathcal{G}$ is a bounded operator and its range is equal $\mathcal{D}(\Delta)$. More precisely

$$\Delta\circ\mathcal{G}_\phi = -\phi \quad : \phi \in L^2(\Omega) \tag{0.2.4}$$

Finally, use this to prove Theorem 0.2.2

The equation (0.2.4) means that $\mathcal{G}$ is Neumann's resolvent of $-\Delta$. So if $\phi \in \mathcal{D}(\Delta)$ we also have

$$\Delta(\mathcal{G}_\phi) = -\phi$$

0.2.4 Eigenfunctions. The densely defined self-adjoint operator δ on $L^2(\Omega)$ has a discrete spectrum which consists of a non-decreasing sequence of positive real numbers $\{0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$. The associate eigenfunctions $\{\phi_n\}$ can be chosen so that they give an orthonormal basis for $L^2(\Omega)$. So here

$$\Delta(\phi_n) + \lambda\cdot\phi_n = 0 \tag{i}$$

This entails that

$$\phi_n(p) = \lambda_n\cdot\int G(p,q)\cdot\phi_n(q)\,dq$$

The symmetry of G entails that each ϕ_n extends to a continuous function on the closurse of Ω where $\phi_n = 0$ on the boundary. Next, we have an expansion

$$G(p,q) = \sum c_n\phi_n(p)\cdot\phi_n(q)$$

From (i) we see that

$$c_n = \frac{1}{\lambda_n} \tag{ii}$$

A hyperbolic equation.

Introduction. We shall expose a result due to Friedrichs from the article *xxx.* Here we shall consider hyperbolic equations of one space variable while the general case is treated in § xx. The boundary value equation in dimension one is as follows: Let x, s be coordinates in $\mathbf{R}^2$ and consider the rectangle

$$\square = \{(x,y)\colon\ 0 \leq x \leq \pi\colon\ 0 \leq s \leq s^*\}$$

where $s^* > 0$. A continuous and real-valued function $g(x,s)$ in $\square$ is x-periodic if

$$g(0,s) = g(\pi,s) \quad : 0 \leq s \leq s^*$$

More generally, if $k \geq 1$ and $g(x,s)$ belongs to $C^k(\square)$ then it is x-periodic if

$$\text{(i)} \qquad \partial_x^\nu(g(0,s)) = \partial_x^\nu(g(\pi,s)$$

hold for each $0 \leq \nu \leq k$. In particular we can consider real-valued C^∞-functions on $\square$ for which (i) hold for every $\nu \geq 0$. Let $a(x,s)$ and $b(x,s)$ be a pair real-valued C^∞-functions on $\square$ which are periodic in x. Consider the PDE-operator

$$P = \partial_s - a \cdot \partial_x - b$$

The boundary value problem. Let $p \geq 1$ and $f(x)$ is a periodic function on $[0,\pi]$ which is p-times continuously differentiable. We seek $F(x,s) \in C^p(\square)$ which is x-periodic and satisfies $P(F) = 0$ in $\square$ and the initial condition

$$F(x,0) = f(x)$$

We are going to prove that this boundary value equation has a unique solution F for every f. Notice that the regularity is expressed by p, i.e one has a specific boundary value problem for each positive integer p. The proof requires several steps and is not finished unitl § 4. A crucial result of independent interest occurs ion § 3 where we encounter certain densely defined linear operators on Hilbert spsces of the Sobolev type.

§ 1. Differential inequalities.

Let $M(s)$ be a non-negative real-valued continuous function on a closed interval $[0,s^*]$. To each $0 \leq s < s^*$ we set

$$d_M^+(s) = \limsup_{\Delta s \to 0} \frac{M(s+\Delta s) - M(s)}{\Delta s}$$

where Δs are positive during the limit.

1.1 Proposition. *Let B be a real number such that $d_M^+(s) \leq B \cdot M(s)$ holds in $[0,s^*)$. Then*

$$M(s) \leq M(0) \cdot e^{Bs} \quad : 0 < s \leq s^*$$

The proof of this result is left as an exercise. The hint is to consider the function $N(s) = M(s)e^{-Bs}$ and show that $d_N^+(s) \leq 0$ for all s. Notice that B is an arbitrary real number, i.e. it may also be < 0. More generally, let $k(s)$ be a non-decreasing continuous function with $k(0) = 0$. suppose that

$$d_M^+(s) \leq B \cdot M(s) + k(s) \quad : 0 \leq s < s^*$$

Now the reader may verify that

$$\text{(1.1.1)} \qquad M(s) \leq M(0) \cdot e^{Bs} + \int_0^s k(t)\,dt$$

Next, consider a product set $\Box = [0, \pi] \times [0, s^*]$ where $0 \leq x \leq \pi$. A C^1-function g is periodic with respect to x if g and the partial derivatives $\partial_s(g)$ and $\partial_x(g)$ are periodic in x, i.e.

$$g(0, s) = g(\pi, s) \quad : 0 \leq s \leq s^*$$

and similarly for $\partial_x(g)$ and $\partial_s(g)$.

1.2 Theorem. *Let g be a periodic C^1-function which satisfies the PDE-equation*

$$(*) \qquad \partial_s(g) = a \cdot \partial_x(g) + b \cdot g$$

in $\Box$ where a and b are x-periodic real-valued continuous functions on $\Box$.. Set

$$M_g(s) = \max_x |g(x, s)| \quad : \ B = \max_{x,s} |b(x, s)|$$

Then one has the inequality

$$M_g(s) \leq M_g(0) \cdot e^{Bs}$$

Proof. Consider some $0 < s < s^*$ and let $\epsilon > 0$. Put

$$m^*(s) = \{x : \ g(x, s) = M_g(s)\}$$

The continuity of g entails that the function $M_g(s)$ is continuous and the sets $m^*(s)$ are compact. If $x^* \in m^*(s)$ the periodicity of the C^1-function $x \mapsto g(x, s)$ entails that $\partial_x(x^*, s) = 0$ and (*) gives

$$\partial_s(g)(x, s) = b(x, s)g(x, s) \quad : x \in m^*(s)$$

Next, let $\epsilon > 0$. We find an open neighborhood U of $m^*(s)$ such that

$$|\partial_x(g)(x, s)| \leq \epsilon \quad : x \in U$$

Now there exists $\delta > 0$ such that

$$|g(x, s)| \leq M_g(s) - 2\delta \quad : x \in [0, \pi] \setminus U$$

Continuity gives some $\rho > 0$ such that if $0 < \Delta s < \rho$ then the inequalities below hold:

$$\text{(i)} \quad |g(x, s + \Delta s)| \leq M_g(s) - \delta \quad : x \in [0, \pi] \setminus U \quad : \ M_g(s + \Delta s) > M_g(s) - \delta$$

$$\text{(ii)} \qquad M_g(s + \Delta s) \leq M_g(s) + \epsilon \quad : \ |\partial_x(g)(x, s + \Delta s)| \leq 2\epsilon \quad : x \in m^*(s)$$

If $0 < \Delta s < \rho$ we see that (i) gives $x \in m^*(s + \Delta s) \subset U$ and for such x-values Rolle's mean-value theorem and the PDE-equation give

$$M_g(x, s + \Delta s) - g(x, s) = \Delta s \cdot \partial_s(g(x, s + \theta \cdot \Delta s) =$$

$$\text{(iii)} \qquad \Delta s \cdot \big[a(x, s + \Delta s) \cdot \partial_x(g)(x + \theta \cdot \Delta s) + b(x, s + \Delta s) \cdot g(x, s + \theta \cdot \Delta s)\big]$$

Let A be the maximum norm of $|a(x, s)|$ taken over $\Box$. Since $|g(x, s)| \leq M_g(s)$ the triangle inequality and (iii) give

$$M_g(s + \Delta s) \leq M_g(s) + \Delta s[\cdot A \cdot 2\epsilon + B \cdot M(s + \theta \cdot \Delta s)]$$

Since the function $s \mapsto M_g(s)$ is continuous it follows that

$$\limsup_{\Delta s \to 0} \frac{M_g(s + \Delta s) - M_g(s)}{\Delta s} \leq A \cdot 2\epsilon + BM_g(s)$$

Above ϵ can be arbitrary small and hence

$$d^+(s) \leq B \cdot M_g(s)$$

Then Proposition 1.1 gives (*) in the theorem.

1.3 L^2-inequalities. Let $g(x,s)$ be a C^1-function satisfying (*) in Theorem 1.2. Set

$$J_g(s) = \int_0^\pi g^2(x,s)\,dx$$

Taking the s-derivative we obtain with respect to s and (*) give

$$\frac{dJ_g}{ds} = 2 \cdot \int_0^\pi g \cdot \partial_s(g)\,ds = 2 \cdot \int_0^\pi (a\partial_x(g) \cdot \partial g + b \cdot g)\,dx$$

The periodicity of g with respect to x gives $\int_0^\pi \partial_x(ag^2)\,dx = 0$. This entails that the right hand side becomes

$$\int_0^\pi (-\partial_x(a) + b) \cdot g^2\,dx$$

So if K is the maximum norm of $-\partial_x(a) + b$ over $\square$ it follows that

$$\frac{dJ_g}{ds}(s) \leq K \cdot J_g(s)$$

Hence Theorem 1.2 gives

$$(1.3.1) \qquad \int_0^\pi g^2(x,s)\,dx \leq e^{Ks} \cdot \int_0^\pi g^2(x,0)\,dx \quad : 0 < s \leq s^*$$

Integration with respect to s entails that

$$(1.3.2) \qquad \iint_\square g^2(x,s)\,dxds \leq \int_0^{s^*} e^{Ks}\,ds \cdot \int_0^\pi g^2(x,0)\,dx$$

Thus, the L^2-integral of $x \to g(x,0)$ majorizes both the area integral and each slice integral when $0 < s \leq s^*$.

§ 2. A boundary value equation

Let $a(x,s)$ and $b(x,s)$ be real-valued C^∞-functions on $\square$ which are periodic in x. Consider the PDE-operator

$$P = \partial_s - a \cdot \partial_x - b$$

Given a periodic C^1-function $f(x)$ on $[0,\pi]$ we seek a periodic C^1-function $g(x,s)$ in $\square$ which satisfies $P(g) = 0$ and the initial condition

$$g(x,0) = f(x)$$

2.1 Theorem. *For every positive integer p and each periodic $f \in C^p[0,\pi]$ there exists a unique periodic $g \in C^p(\square)$ where $P(g) = 0$ and $g(x,0) = f(x)$.*

The uniqueness follows from the results in § 1. For if g and h are solutions in Theorem 2.1 then $\phi = g - h$ satisfies $P(\phi) = 0$. Here $\phi(x,0) = 0$ which gives $\phi = 0$ in $\square$ via (1.3.2). The proof of existence requires several steps and employs Hilbert space methods. So first we introduce certain Hilbert spaces.

2.2 The space $\mathcal{H}^{(k)}$. To each integer $k \geq 2$ the complex Hilbert space $\mathcal{H}^{(k)}$ defined is the completion of complex-valued C^k-functions on $\square$ which are periodic with respect to x. A wellknown Sobolev inequality entails that every function in $\mathcal{H}^{(2)}$ is continuous, and more generally one has the inclusion

$$\mathcal{H}^{(k)} \subset C^{k-2}(\square) \quad : k \geq 3$$

It is also clear that the first order PDE-operator P maps $\mathcal{H}^{(k+1)}$ into $\mathcal{H}^{(k)}$.

Next, on the periodic x-interval $[0,\pi]$ we have the Hilbert space $\mathcal{H}^k[0,\pi]$.

2.3 Definition. *For each $k \geq 2$ we denote by $\mathcal{D}_k(P)$ the family of functions $f(x) \in \mathcal{H}^k[0,\pi]$ for which there exists some $F(x,s) \in \mathcal{H}^{(k)}$ such that*

$$(*) \qquad P(F) = 0 \quad : F(x,0) = f(x)$$

The results in § 1 show that F is uniquely determined by (*). Moreover. there exists a constant C which only depends upon the C^∞-functions a and b and the given integer k such that

$$(2.3.1) \qquad ||F||_k \leq C \cdot ||f||_k$$

where we have taken norms in $\mathcal{H}^{(k)}$ and $\mathcal{H}^k[0,\pi]$ respectively. Moreover, the last inequality in (1.3.2) shows that C can be chosen such that we also have

$$(2.3.3) \qquad ||f^*||_k \leq C \cdot ||f||_k$$

where $f^*(x) = F(x,s^*)$.

2.4 A density principle Above we introduced the space $\mathcal{D}_k(P)$. It turns out that if it is dense in $\mathcal{H}^k[0,\pi]$ then one has the equality

$$(2.4.1) \qquad \mathcal{D}_k(P) = \mathcal{H}^k[0,\pi]$$

Proof. Suppose that $\mathcal{D}_k(P)$ is dense. So if $f \in \mathcal{H}^k[0,\pi]$ there exists a sequence $\{f_n\}$ in $\mathcal{D}_k(P)$ where $||f_n - g||_k \to 0$. By (2.2.2) we have

$$||F_n - F_m||_k \leq C||f_n - f_m||_k$$

Hence $\{F_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}^{(k)}$ and converges to a limit F. Since each $P(F_n) = 0$ it follows that $P(F) = 0$ and it is clear that the continuous boundary value function $F(x,0)$ is equal to $f(x)$ which entails that f belongs to $\mathcal{D}_k(P)$.

2.5 The operators S_k. Each $f \in \mathcal{D}_k(P)$ gives the function $f^*(x) = F(x,s^*)$ in $\mathcal{H}^k[0,\pi]$ and set

$$S_k(f) = f^*(x)$$

So the domain of definition of S_k is equal to $\mathcal{D}_k(P)$ and (2.3.3) gives a constant M_k such that

$$||S_k(f)|| \leq M_k \cdot ||f||_k \quad : f \in \mathcal{D}_k(P)$$

where M_k by the above depends on the integer k and the given PDE-operator P.

2.6 Proposition. *For each k there exists some $\alpha(k) < 0$ such that for every $0 < \alpha < \alpha(k)$ the range of the operator $E - \alpha \cdot S_k$ contains all periodic C^∞-functions on $[0,\pi]$.*

2.7 The density of $\mathcal{D}_k(P)$. We prove Proposition 2.6 in § xx and proceed to that it gives the density of $\mathcal{D}_k(P)$. For if $\mathcal{D}_k(P)$ fails to be dense there exists a non-zero $f_0 \in \mathcal{D}_k(P)$ which is $\perp$ to $\mathcal{D}_k(P)$. In Proposition 2.6 we choose $0 < \alpha \leq \alpha(k)$ so small that

$$\text{(i)} \qquad \alpha < M_k/2$$

Since periodic C^∞-functions are dense in $\mathcal{H}^k[0,\pi]$, Proposition 2.6 gives a sequence $\{h_n\}$ in $\mathcal{D}_k(P)$ such that

$$\text{(ii)} \qquad \lim_{n\to\infty} ||h_n - \alpha \cdot S_k(h_n) - f_0||_k \to 0$$

It follows that

$$\text{(iii)} \qquad \langle f_0, f_0 \rangle = 1 = \lim \langle f_0, h_n - \alpha \cdot S_k(h_n) \rangle = -\alpha \cdot \lim \langle f_0, S_k(h_n) \rangle$$

Next, the triangle inequality and (ii) give

$$\text{(iv)} \qquad ||h_n||_k \leq 1 + \alpha \cdot ||(S_k(h_n)|| \leq 1 + 1/2 \cdot ||h_n|| \implies ||h_n||_k \leq 2$$

Funally, by the Cauchy-Schwarz inequality the absolute value in the right hand side of (iii) is majorized by

$$\alpha \cdot M_K \cdot 2 < 1$$

which contradicts (iii). Hence the orthogonal complement of $\mathcal{D}_k(P)$ is zero which proves the requested density.

Together with (2.4) we get the following conclusive result:

2.8 Theorem. *For each $k \geq 2$ and $f(x) \in \mathcal{H}^k[0,\pi]$ there exists a unique function $F(x,s) \in \mathcal{H}^{(k)}$ such that (*) holds in Definition 2.3.*

§ 3. A class of inhomogeneous PDE-equations.

Before Theorem 3.1 is announced we introduce some notations. Put

$$\square = \{0 \leq x \leq \pi\} \times \{0 \leq s \leq 2\pi\}$$

In this section we shall consider doubly periodic functions $g(x,s)$ on $\square$, i.e.

$$g(\pi,s) = g(0,s) \quad : g(x,0) = g(x,2\pi)$$

For each non-negative integer k we denote by $C^k(\square)$ the space of k-times doubly periodic continuously differentiable functions. If $g \in C^k(\square)$ we set

$$||g||^2_{(k)} = \sum_{j,\nu} \int_\square \Big|\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}(x,s)\Big|^2 \, dxds$$

with the double sum extended pairs $j+\nu \leq k$. This gives the complex Hilbert space $\mathcal{H}^{(k)}$ after a completion of $C^k(\square)$ with respect to the norm above. Recall from § xx that every function $g \in \mathcal{H}^{(2)}$ is automatically continuous and doubly periodic on the closed square. More generally, if $k \geq 3$ each $g \in \mathcal{H}^{(k)}$ has continuous and doubly periodic derivatives up to order $k-2$. Next, consider a first order PDE-operator

$$P = \partial_s - a(x,s)\partial_x - b(x,s)$$

where a and b are real-valued doubly periodic C^∞-functions. It is clear that P maps $\mathcal{H}^{(k)}$ into $\mathcal{H}^{(k+1)}$ for every $k \geq 2$. Keeping $k \geq 2$ fixed we set

$$\mathcal{D}_k(P) = \{g \in \mathcal{H}^{(k)} : \; P(g) \in \mathcal{H}^{(k)}\} \tag{3.1}$$

Since $C^\infty(\square)$ is dense in $\mathcal{H}^{(k)}$ this yields for each $k \geq 2$ a densely defined operator

$$P \colon \mathcal{D}_k(P) \to \mathcal{H}^{(k)} \tag{i}$$

In $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$ we get the graph

$$\Gamma_k = \{(g, P(g) \colon \; g \in \mathcal{D}_k(P)\}$$

Since P is a differential operator the general result in § xx entails that Γ_k is a closed subspace so the densely defined operator in (i) has a closed graph. Thus. for each $k \geq 2$ we have a densely defined linear operator and closed operator on $\mathcal{H}^{(k)}$ denoted by $\mathcal{T}_k$. So its domain of definition $\mathcal{D}(T_k) = \mathcal{D}_k$. Next, we consider the graph

$$\gamma_* = \{(g, P(g) \colon \; g \in C^\infty(\square)\} \tag{ii}$$

This is a subspace of Γ_k and denote by $\overline{\gamma}_k$ its closure taken in $\mathcal{H}^{(k)} \times \mathcal{H}^{(k)}$. So here

$$\overline{\gamma}_k \subset \Gamma_k$$

and this inclusion yields another densely defined linear operator denoted by T_k whose graph is $\overline{\gamma}_k$. So here $\mathcal{T}_k$ is an extension of T_k and we have an inclusion

$$\mathcal{D}(T_k) \subset \mathcal{D}(\mathcal{T}_k) \tag{iii}$$

in general is strict. Let E be the identity operator on $\mathcal{H}^{(k)}$. With these notations one has

3.2 Theorem. *For each integer $k \geq 2$ there exists a positive real number $\rho(k)$ such that $T_k - \lambda \cdot E$ is surjective on $\mathcal{H}^{(k)}$ for every $\lambda > \rho(k)$ and its kernel is zero.*

The proof requires several steps and is not finished until § 3.x. First we establish the following:

3.3 Proposition. *One has the equality $\mathcal{D}(T_k^*) = \mathcal{D}_k$ and there exists a bounded self-adjoint operator B_k on $\mathcal{H}^{(k)}$ such that*

$$T_k^* = -\mathcal{T}_k + B_k$$

Proof of Proposition 3.3 Keeping $k \geq 2$ fixed we set $\mathcal{H} = \mathcal{H}^{(k)}$. For each pair g, f in $\mathcal{H}$ their inner product is defined by

$$\langle f, g\rangle = \sum \int_{\square} \frac{\partial^{j+\nu} f}{\partial x^j \partial s^\nu}(x,s) \cdot \overline{\frac{\partial^{j+\nu} g}{\partial x^j \partial s^\nu}}(x,s)\, dxds$$

where the sum is taken when $j + \nu \leq k$. Introduce the differential operator

$$\Gamma = \sum_{j+\nu \leq k} (-1)^{j+\nu} \cdot \partial_x^{2j} \cdot \partial_s^{2\nu}$$

Partial integration gives

(i) $$\langle f, g\rangle = \int_{\square} f \cdot \Gamma(\bar{g})\, dxds = \int_{\square} \Gamma(f) \cdot \bar{g}\, dxds \quad : \ f, g \in C^\infty$$

Now we consider the operator $P = \partial_s - a \cdot \partial_x - b$ and get

(ii) $$\langle P(f), g\rangle = \int_{\square} P(f) \cdot \Gamma(\bar{g})\, dxds$$

Partial integration identifies (ii) with

(iii) $$-\int_{\square} f \cdot \big(\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma(\bar{g})\, dxds$$

1.1 Exercise. In (iii) appears the composed differential operator

$$\partial_s - \partial_x(a) - a \cdot \partial_x - b) \circ \Gamma$$

Show that in the ring of differential operators with C^∞-coefficients this differential operator can be written in the form

$$\Gamma \circ (\partial_s - a \cdot \partial_x - b) + Q(x, s, \partial_x, \partial_s)$$

where Q is a differential of order $\leq 2k$ with coefficients in $C^\infty(\square)$. Conclude from the above that

(1.1.1) $$\langle Pf, g\rangle = -\langle f, Pg\rangle + \int_{\square} f \cdot Q(\bar{g})\, dxds$$

1.2 Exercise. With Q as above we have a bilinear form which sends a pair f, g in $C^\infty(\square)$ to

(1.2.1) $$\int_{\square} f \cdot Q(\bar{g})\, dxds$$

Use partial integration and the Cauchy-Schwarz inequelity to show that there exists a conatant C which depends on Q only such that the absolute value of (1.2.1) is majorized by $C_Q \cdot ||f||_k \cdot ||g||_k$. Conclude that there exists a bounded linear operator B_k on $\mathcal{H}$ such that

(1.2.2) $$\langle f, B_k(g)\rangle = \int_{\square} f \cdot Q(\bar{g})\, dxds$$

1.3 Proof that B_k is self-adjoint From the above we have

(1.3.1) $$\langle Pf, g\rangle = -\langle f, Pg\rangle + \langle f, B_k(g)\rangle$$

Keeping f in $C^\infty(\Box)$ we notice that $\langle f, B_k(g)\rangle$ is defined for every $g \in \mathcal{H}$. From this the reader can check that (1.3.1) remains valid when g belongs to $\mathcal{D}(\mathcal{T}_k)$ which means that

$$\langle Pf, g\rangle = -\langle f, \mathcal{T}_k g\rangle + \langle f, B_k(g)\rangle \quad : f \in C^\infty(\Box) \tag{1.3.2}$$

Moreover, when both f and g belong to $C^\infty(\Box)$ we can reverse their positions in (*) which gives

$$\langle Pg, f\rangle = -\langle g, Pf\rangle + \langle g, B_k(f)\rangle \tag{1.3.3}$$

Since a and b are real-valued it is clear that

$$\langle Pg, f\rangle = -\langle f, Pg\rangle \tag{1.3.4}$$

It follows that

$$\langle f, B_k(g) = \langle g, B_k(f) \quad : f, g \in C^\infty(\Box) \tag{1.3.5}$$

Since this hold for all pairs of C^∞-functions and B_k is a bounded linear operator on $\mathcal{H}$ the density of $C^\infty(\Box)$ entails that B_k is a bounded self-adjoint operator on $\mathcal{H}$.

1.4 The equality $\mathcal{D}(T_k^*) = \mathcal{D}_k$. The density of $C^\infty(\Box)$ in $\mathcal{H}$ entails that a function $g \in \mathcal{H}$ belongs to $\mathcal{D}(T_k^*)$ if and only if there exists a constant C such that

$$|\langle Pf, g\rangle| \leq C \cdot ||f|| \quad : f \in C^\infty(\Box) \tag{1.4.1}$$

Since B_k is a bounded operator, (1.3.2) gives the inclusion

$$\mathcal{D}_k \subset \mathcal{D}(T_k^*) \tag{1.3.3}$$

To prove the opposite inclusion we use that the Γ-operator is elliptic. If $g \in \mathcal{D}(T_k^*)$ we have from (i) in § 1.1:

$$\langle Pf, g\rangle = \langle f, T_k^* g\rangle = \int \Gamma(f) \cdot \overline{T_k^*(g)}\, dxds \quad : f \in C^\infty(\Box)$$

Similarly

$$\langle f, B_k(g)\rangle = \int \Gamma(f) \cdot \overline{B_k(g)}\, dxds$$

Treating $\mathcal{T}_k(g)$ as a distribution the equation (1.3.2) entails that the elliptic operator Γ annihilates $T_k^*(g) - \mathcal{T}_k(g) + B_k(g)$. Since both $T_k^*(g)$ and $B_k(g)$ belong to $\mathcal{H}$ this implies by the general result in § xx that $\mathcal{T}_k(g)$ belongs to $\mathcal{H}$ which proves the requested equality (1.4) and at the same time the operator equation

$$T_k^* = -\mathcal{T}_k(g) + B_k \tag{1.4.2}$$

3.4 An inequality.

Let $f \in C^\infty(\Box)$ and λ is a positive real number. Then

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f) - \lambda \cdot f||^2 =$$

$$||\mathcal{T}_k(f) - \frac{1}{2}B_k(f)||^2 + \lambda^2 \cdot ||f||^2 - \lambda\big(\langle \mathcal{T}_k(f) - \frac{1}{2}B_k(f), f\rangle + \langle f, \mathcal{T}_k(f) - \frac{1}{2}B_k(f)\rangle\big)$$

The last term is λ times

$$\langle \mathcal{T}_k(f), f\rangle + \langle f, \mathcal{T}_k(f)\rangle - \langle f, B_k f\rangle \tag{i}$$

where we used that B_k is symmetric. Now $T_k = \mathcal{T}_k$ holds on $C^\infty(\Box)$ and the definition of adjoint operators give

$$\langle \mathcal{T}_k(f), f\rangle = \langle f, T_k^*\rangle \tag{ii}$$

Then (1.4.2) implies that (i) is zero and hence we have proved

(iii) $$||T_k(f)-\frac{1}{2}B_k(f)-\lambda\cdot f||^2=\lambda^2\cdot ||f||^2+||T_k(f)-\frac{1}{2}B_k(f)||^2\geq \lambda^2\cdot ||f||^2$$

From (iii) and the triangle inequality for norms we obtain

(iv) $$||T_k(f)-\lambda\cdot f||\geq \lambda\cdot ||f||-\frac{1}{2}||B_k(f)||$$

Now B_k has a finite operator norm and if $\lambda\geq ||B_k||$ we see that

(v) $$||T_k(f)-\lambda\cdot f||\geq \frac{\lambda}{2}\cdot ||f||$$

Finally, since $C^\infty(\Box)$ is dense in $\mathcal{D}(T_k)$ it is clear that (v) gives

(3.4.1) $$||T_k(f)-\lambda\cdot f||\geq \frac{\lambda}{2}\cdot ||f|| \quad : f\in\mathcal{D}(T_k)$$

§ *3.5. Proof of Theorem 3.2*

Suppose we have found some $\lambda^*\geq\frac{1}{2}\cdot ||B||$ such that $T_k-\lambda$ has a dense range in $\mathcal{H}$ for every $\lambda\geq\lambda^*$. If this is so we fix $\lambda\geq\lambda^*$ and take some $g\in\mathcal{H}$. The hypothesis gives a sequence $\{f_n\in\mathcal{D}(T_k)$ such that

$$\lim_{n\to\infty}||T(f_n)-\lambda\cdot f_n-g||=0$$

In particular $\{||T_k(f_n)-\lambda\cdot f_n\}$ is a Cauchy sequence in $\mathcal{H}$ and (1.5.x) implies that $\{f_n\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$ and hence converges to a limit f_*. Since the operator T_k is closed we conclude that $f_*\in\mathcal{D}(T)$ and we get the equality

$$T(_kf_*)-\lambda\cdot f_*=g$$

Finally, since the graph of T is contained in T_1 we have the requested equation

$$P(f_*)-\lambda\dot{f}_*=g$$

Thus finishes the proof of Theorem 3.2 provided we have established the existence of λ_* above.

3.5.1 Density of the range. By the construction of adjoint operators the range of $T_k-\lambda\cdot E$ fails to be dense if and ony if $T_k^*-\lambda$ has a non-zero kernel. So assume that

(i) $$T_k^*(f)-\lambda\cdot f=0$$

for some $f\in\mathcal{D}(T_k^*)$ which is not identically zero. Notice that T_k sends real-valued functions into real-valued functions. So above we can assume that f is real-valued and also assume that f is normalised so that

$$\int_\Box f^2(x,s)\,dxds=1$$

By (**) the equation (xx) gives

(ii) $$\mathcal{T}_k(f)+\lambda\cdot f-B(f)=0$$

Let us then consider the function

$$V(s)=\int_0^\pi f^2(x,s)\,dx$$

Recall from § xx that the $\mathcal{H}$-function f is of class C^1. Now

(iii) $$\frac{1}{2}\cdot V'(s)=\int_0^\pi f\cdot\frac{\partial f}{\partial s}\,dx$$

By (ii) we have

$$\frac{\partial f}{\partial s} - a(x)\frac{\partial f}{\partial x} - b \cdot f = B(f) - \lambda \cdot f$$

Hence the right hand side in (iii) becomes

$$-\lambda \cdot V(s) + \int_0^\pi f(x,s) \cdot B(f)(x,s)\,dx + + \int_0^\pi a(x,s) \cdot f(x,s) \cdot \frac{\partial f}{\partial x}(x,s)\,dx$$

By partial integration the last term is equal to

$$-\frac{1}{2}\int_0^\pi \partial_x(a)(x,s) \cdot f^2(x,s)\,dx$$

Set

$$M = \frac{1}{2} \cdot \max_{(x,s)\in\Box} |\partial_x(a)(x,s)|$$

Then we get the inequality

$$\frac{1}{2} \cdot V'(s) \leq (M - \lambda) \cdot V(s) + \int_0^\pi f(x,s) \cdot B(f)(x,s)\,dx$$

Set

$$\Phi(s) = \int_0^\pi |f(x,s)| \cdot |B(f)(x,s)|\,dx$$

Since the L^2-norm of f is one the Cauchy-Schwarz inequality gives

$$\int_{-\pi}^\pi \Phi(s)\,ds \leq \sqrt{\int_\Box |B(f)(x,s)|^2\,dxds} \leq ||B(f)||$$

where the last equality follows since the squared integral of $B(f)$ is majorized by its squared norm in $\mathcal{H}$. When $\lambda > M$ it follows from (xx) that

$$(\lambda - M) \cdot V(s) + \frac{1}{2} \cdot V'(s) \leq \Phi(s)$$

Next, since f is double periodic we have $V(-\pi) = V(\pi)$ so after an integration (xx) gives

$$(\lambda - M) \cdot \int_\pi^\pi V(s)\,ds = \int_{-\pi}^\pi \Phi(s)\,ds \leq ||B(f)||$$

By (xx) we have $\int_\pi^\pi V(s)\,ds = 1$ which gives a contradiction if $\lambda > M + ||B(f)||$.

Remark. Set

$$\tau = \min_f ||B(f)||$$

with the minimum taken over funtions $f \in \mathcal{D}(T_0^*)$ whose L^2-integral is normalised by (xx). The proof has shown that the kernel of $T_0^* - \lambda$ is zero for all $\lambda > M + \tau$.

A special solution.

Let $f(x)$ be a periodic C^∞-function on $[0,\pi]$. Put

$$Q = a(x,s)\cdot\frac{\partial}{\partial x} + b(x,s)$$

Let $\eta(s)$ be a C^∞-function of s and m some positive integer If $\lambda > 0$ is a real number. we set

(i) $$g_\lambda(x,s) = \eta(s)\cdot f + \eta(s)\cdot\sum_{j=1}^{j=m}\frac{(s-\pi)^j}{j!}\cdot(Q-\lambda)^j(f) \quad : 0\le s\le\pi$$

We choose η to be a real-valued C^∞-function such that $\eta(s)=0$ when $s\le 1/4$ and -1 if $s\ge 1/2$. Hence $g_\lambda(x,s)=0$ in (i) when $0\le s\le 1/4$ and we extend the function to $[-\pi\le s\le\pi$ where $g_\lambda(x,-s)=g_\lambda(x,s)$ if $0\le s\le\pi$. So now g_λ is π-periodic with respect to s and vanishes when $|s|\le 1/4$.

Exercise. If $1/2\le s\le\pi$ we have $\eta(s)=1$. Use (i) to show that

$$(P+\lambda)(g_\lambda) = \frac{\partial g_\lambda}{\partial s} - (Q-\lambda)(g_\lambda) = \frac{(s-\pi)^m}{m!}\cdot(Q-\lambda)^{m+1}(f)$$

hold when $1/2\le s\le\pi$. At the same time $g_\lambda(s)=0$ when $0\le s\le 1/4$. So $(P+\lambda)(g)$ is a function whose derivatives with respect to s vasnish up to order m at $s=0$ and $s=\pi$ and is therefore doubly periodic of class C^m in $\square$. Now Theorem 2.2 applies. For a given $k\ge 2$ we choose a sufficently large m and find $h(x,s)$ so that

$$P(h) + \lambda\cdot h = (P+\lambda)(g_\lambda)(x,s)$$

where h is s-periodic, i.e.

$$h(x,0) = h(x,\pi)$$

Notice also that $g_\lambda(x,0)=0$ while $g_\lambda(x,\pi)=f(x)$. Set

$$g_*(x) = h - g_\lambda$$

Then $P(g_*)+\lambda\cdot g_*=0$ and

$$g_*(x,0) - g_*(x,\pi) = f(x)$$

Above we started with the C^∞-function. Given $k\ge 2$ we can take m sufficiently large during the constructions above so that g_* belongs to $\mathcal{H}^{(k)}(\square)$.