XI. Radial limit of functions with finite Dirichlet integral

We expose results from the article *Ensembles exceptionnels* by Beurling in [Beur] devoted to the study of functions $f(\theta)$ on the unit circle T whose harmonic extensions H_f to D have a finite Dirichlet integral. For such functions we shall prove that H_f has radial limits outside a set whose capacitiy is zero. A real-valued functions $f(\theta)$ on the unit circle T has a Fourier series:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos n\theta + \sum_{n=1}^{\infty} b_n \cdot \sin n\theta$$

We say that f belongs to the class $\mathcal{D}$ if

$$(*) \qquad \sum_{n=1}^{\infty} n(a_n^2 + b_n^2) < \infty$$

When the constant term $a_0 = 0$ the sum in (*) is denoted by $D(f)$ and is called the Dirichlet norm. Denote by $\mathcal{E}_f$ the set of all θ where the partial sums of the Fourier series of f does not converge.

0.1 Theorem. *For each $f \in \mathcal{D}$ the outer capacity of $\mathcal{E}_f$ is zero.*

Remark. Recall from XXX that if $E \subset T$ then its outer capacity is defined by

$$\text{Cap}^*(E) = \inf_{E \subset U} \text{Cap}(U)$$

with the infimum taken over open neighborhoods of E.

The proof of Theorem 0.1 has two essential ingredients. First, given some $f \in \mathcal{D}$ with constant term $a_0 = 0$ we obtain the harmonic function $f(r, \theta)$ defined in the open disc by

$$f(r, \theta) = \sum_{n=1}^{\infty} r^n (a_n \cdot \cos n\theta + b_n \cdot \sin n\theta)$$

We construct partial derivatives with respect to r and obtain:

$$(1) \qquad f'_r(r, \theta) = \sum_{n=1}^{\infty} n \ \cdot r^{n-1} (a_n \cdot \cos n\theta + b_n \cdot \sin n\theta)$$

Define the function $F(r, \theta)$ in D by

$$(2) \qquad F(r, \theta) = \int_0^r |f'_s(s, \theta)| \cdot ds$$

Thus, for each θ we integrate the absolute value of (1) along a ray from the origin. For every fixed θ $r \mapsto F(r, \theta)$ is non-decreasing and hence there exists a limit

$$(3) \qquad \lim_{r \to 1} F(r, \theta) = F^*(\theta)$$

The limit value can be finite or $+\infty$. It is clear that if (3) is finite then there exists the radial limit

$$(4) \qquad \lim_{r \to 1} f(r, \theta) = f^*(\theta)$$

Remark. For every θ such that the radial limit (4) exists, it follows that Fourier's partial sums converge to $f^*(\theta)$. In fact, this follows from Abel's theorem in [Series] since the inclusion $f \in \mathcal{D}$ entails that a_n and b_n both are small ordo of $\frac{1}{n}$. Hence we have:

Lemma *For every $\rho > 0$ one has the inclusion*

$$\mathcal{E}_f \subset \{F^*(\theta) > \rho\}$$

We conclude that Theorem 0.1 follows if the capacity of $\{F^* > \rho\}$ tends to zero as $\rho \to +\infty$. This follows from the result below.

0.2 Theorem. *Let $f \in \mathcal{D}$ where $a_0 = 0$ and $D(f) = 1$. Then*

$$\mathrm{Cap}(\{F^* > \rho\}) \leq e^{-\rho^2}$$

hold for every $\rho > 0$.

The essential step to get Theorem 0.2 relies upon the following inequality:

0.3 Theorem. *For each $f \in \mathcal{D}$ with $a_0 = 0$ one has $F^* \in \mathcal{D}$ and*

$$D(F^*) \leq D(f)$$

Once this is proved we can deduce Theorem 0.2. This is done in § 2 after we have proved Theorem 0.3 in § 1. Before we proceed to § 1 we shall need a result about logarithmic potentials. Let μ be a probability measure on T, i.e a non-negative Riesz measure of total mass one and put:

$$U_\mu(z) = \int \log \frac{1}{|z - \zeta|} \cdot d\mu(\zeta)$$

This is a harmonic function in $\{|z| < 1\}$ and passing to its radial limits as $r \to 1$ the energy integral is defined by:

$$(*) \qquad J(\mu) = \lim_{r \to 1} \int U_\mu(r, \theta) \cdot d\mu(\theta) = \int U_\mu(\theta) \cdot d\mu(\theta)$$

One says that μ has finite energy when (*) is finite. Assume that μ has finite energy. Using polar coordinates in D we have a series expansion:

$$U_\mu(r, \theta) = \sum \frac{r^n}{n}(h_n \cos n\theta + k_n \sin n\theta)$$

where $\{h_n\}$ and $\{k_n\}$ are real numbers. The energy integral $J(\mu)$ becomes the limit of the following expression as $r \to 1$:

$$(1) \qquad \int U_\mu(r, \phi) \cdot d\mu(\phi) = \iint \log \frac{1}{|1 - re^{i(\phi - \theta)}|} d\mu(\phi) \cdot d\mu(\theta)$$

To compute the right hand side we expand the complex Log-function:

$$\log \frac{1}{1 - re^{i(\phi - \theta)}} = \sum_{n=1}^{\infty} \frac{r^n}{n} \cdot e^{in(\phi - \theta)}$$

Taking real parts it follows that (1) is equal to

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cdot \cos n(\phi - \theta) \cdot d\mu(\phi) \cdot d\mu(\theta)$$

Now we use the trigonometric formula

$$\cos n(\phi - \theta) = \cos n\phi \cdot \cos n\theta + \sin n\phi \cdot \sin n\theta$$

Put

$$(2) \qquad h_n = \int \cos n\theta \cdot d\mu(\theta) \quad \text{and} \quad k_n = \int \sin n\theta \cdot d\mu(\theta)$$

Then we obtain

$$(3) \qquad J(\mu) = \sum \frac{1}{n}(h_n^2 + k_n^2)$$

Next, let $g(\theta) \in \mathcal{D}$ with Fourier coefficients $\{a_n\}$ and $\{b_n\}$ where $a_0 = 0$. Then we have

$$\int g \cdot d\mu = \sum a_n \cdot h_n + b_n \cdot k_n$$

and Cauchy-Schwarz inequality gives:

$$(4) \qquad [\int g \cdot d\mu]^2 \leq S(g) \cdot J(\mu)$$

From the above we obtain the following:

0.4 Theorem. *For each probability measure μ with finite energy and every function $g(\theta) \in \mathcal{D}$ which is lower semi-continuous one has the inequality*

$$\big[\int g(\theta) \cdot d\mu(\theta)\big]^2 \leq S(g) \cdot J(\mu)$$

Remark. Above the lower semi-continuity is imposed in order to ensure that the Borel integral of g with respect to μ is defined.

1. Proof of Theorem 0.3

To begin with one has

1.1 Lemma. *The function F is subharmonic in D.*

For each fixed $0 < \alpha < 1$ we define the function ϕ_α in d by

$$\phi_\alpha(x, y) = \frac{\partial}{\partial \alpha} f(\alpha x, \alpha y) = x \cdot f'_x(\alpha x, \alpha y) + y \cdot f'_y(\alpha x, \alpha y)$$

Now we notice that the function $f_\alpha(x, y) = f(\alpha x, \alpha y)$ is harmonic and (1) means that

$$\phi_\alpha = (x\partial_x + y\partial_y)(f_\alpha)$$

where $\mathfrak{e} = x\partial_x + y\partial_y$ is the Euler field. As explained in XX this first order operator satisfies the identity

$$\Delta \circ \mathfrak{e} = \Delta + \mathfrak{e} \cdot \Delta$$

in the ring of differential operators and then we conclude that ϕ_α is harmonic. Next, the absolute value of a harmonic function is subharmonic so $\{|\phi_\alpha|\}$ yield subharmonic functions and a change of variables gives:

$$F = \int_0^1 |\phi_\alpha| \cdot d\alpha$$

This shows that F is a Riemann integral of subharmonic functions which in compact subsets of D is uniformly approximated by finite sums

$$\frac{1}{N} \sum_{k=1}^{k=N} |\phi_{k/N}|$$

Lemma 1.1 follows since a convex sum of subharmonic functions again is subharmonic.

An inequality. Notice that the function $F(r, \theta)$ is continuous and its derivative with respect to r exists and equals $|f'_r(r, \theta)|$. But the partial derivative $\partial F/\partial\theta$ may have jump discontinuities along rays where the derivative f'_r has a zero. However, this cannot occur too often so when $0 < r < 1$ is fixed there exists the integral

$$I(r) = \int_0^{2\pi} (\frac{\partial F}{\partial \theta}(r, \theta))^2 \cdot d\theta$$

We have proved that F is subharmonic and from its definition it is clear that the partial derivative $\partial F/\partial r$ is non-negative. By the general result in Chapter V:B:xxx we obtain

1.2 Lemma. *The inequality below holds for each* $0 < r < 1$*:*

(*) $$I(r) \leq r^2 \cdot \int_0^{2\pi} (\frac{\partial F}{\partial r}(r, \theta))^2 \cdot d\theta$$

1.3 Dirichlet integrals. Let $f \in \mathcal{S}$ with $a_0 = 0$. We construct the Dirichlet integral

$$\text{Dir}(f) = \frac{1}{\pi} \cdot \iiint_D [(f'_x)^2 + (f'_y)^2] \cdot dxdy$$

Then one has the equality:

$$\text{(*)} \qquad \mathrm{Dir}(f) = D(f)$$

To see this we identify $f(r,\theta)$ with the real part of the analytic function

$$G(z) = \sum (a_n - i \cdot b_n) \cdot z^n$$

The Cauchy-Riemann equations give

$$\mathrm{Dir}(f) = \frac{1}{\pi} \cdot \iiint_D |G'(z)|^2 \cdot dxdy$$

Now the reader can verify that the double integral above is equal to $D(f)$. Notice that (*) identifies $\mathcal{D}$ with the space of real-valued functions on T whose harmonic extensions to D have a finite Dirichlet integral.

1.4 Exercise. Show that the Dirichlet integral of a function g of class C^2 in D also is given by the double integral

$$\text{(i)} \qquad \frac{1}{\pi} \cdot \int_0^1 \int_0^{2\pi} \big[r^2 \cdot (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2} \cdot (\frac{\partial g}{\partial \theta})^2 \big] \cdot r \cdot d\theta dr$$

Show also that if g is harmonic then

$$\text{(ii)} \qquad \mathrm{Dir}(g) = \frac{2}{\pi} \cdot \int_0^1 \int_0^{2\pi} (\frac{\partial g}{\partial r})^2 \cdot r \cdot d\theta dr$$

1.5 Proof of Theorem 0.3

Apply (i) in 1.4 with $g = F$ where the inequality in Lemma 1.2 and an integration with respect to r give

$$\text{(1)} \qquad \mathrm{Dir}(F) \leq \frac{2}{\pi} \cdot \int_0^1 \int_0^{2\pi} (\frac{\partial F}{\partial r})^2 \cdot r \cdot d\theta dr$$

Next, the construction of F gives the equality

$$(\frac{\partial F}{\partial r})^2 = (\frac{\partial f}{\partial r})^2$$

in the whole disc D. Then (1) and the equality (ii) applied to the harmonic function f give:

$$\text{(2)} \qquad \mathrm{Dir}(F) \leq \mathrm{Dir}(f) = D(f)$$

where the last equality used (*) in 1.3. Next, construct the harmonic extension of the boundary function $F^*(\theta)$ which we denote by H_F. Here we have the equations

$$\text{(3)} \qquad D(F^*) = D(H_F)$$

Next, recall that the Dirchlet integral is minimized when we take a harmonic extension which entails that

$$\text{(4)} \qquad \mathrm{Dir}(H_F) \leq \mathrm{Dir}(F)$$

Hence (2-4) give the requested inequality

$$D(F^*) \leq D(f)$$

2. Proof of Theorem 0.2

Let $\rho > 0$ and apply Theorem 0.4 to the function $g = F^*$ and the equilibrium distribution μ assigned to the set $E = \{F^* > \rho\}$. This gives

$$\rho^2 \leq \Big[\int F^* \cdot d\mu\Big]^2 \leq S(F^*) \cdot J(\mu) \tag{4}$$

Now $D(F^*) \leq D(f) = 1$ holds by Theorem 0.3 and hence we have:

$$\rho^2 \leq J(\mu) \tag{5}$$

Next, recall from XX that $J(\mu)$ is the the constant value $\gamma(E)$ of the potential function U_μ restricted to E. Hence (5) gives

$$e^{-\gamma(E)} \leq e^{-\rho^2} \tag{6}$$

By definition the left hand side is the capacity of E which proves Theorem 0.2.

An application

Let Ω be a simply connected domain which contains the origin in the complex ζ-plane and $\partial\Omega$ contains a relatively open set given by an interval ℓ situated on the line $\mathfrak{Re}\,\zeta = \rho$ for some $\rho > 0$. Consider the harmonic measure $\mathfrak{m}_0^\Omega(\ell)$. In other words, the value at the origin of the harmonic function in Ω which is 1 on ℓ and zero on $\partial\Omega \setminus \ell$. We shall find an upper bound for (*) in the family of simply connected domains which contain the origin and ℓ and at the same time has area π. To attain this we consider the conformal map ϕ from the unit disc onto Ω with $\phi(0) = 0$. The invariance of harmonic measures gives:

$$\mathfrak{m}_0^\Omega(\ell) = \mathfrak{m}_0^D(\alpha)$$

where α is the interval on T such that $\phi(\alpha) = \ell$. For an interval on the unit circle one has the equality

$$\text{Cap}(\alpha) = \sin \alpha/4$$

At the same time $\mathfrak{m}_0^D(\alpha) = \frac{\alpha}{2\pi}$ which entails that

$$\mathfrak{m}_0^\Omega(\ell) = \frac{2}{\pi} \text{ arcsin Cap}(\alpha) \tag{1}$$

There remains to estimate last term above. Put $u = \mathfrak{Re}\,\phi$. The inclusion $\ell \subset \mathfrak{Re}\zeta = \rho$ means that $u = \rho$ on ℓ. So when ϕ is considered in the class $\mathcal{S}$ we have the inclusion

$$\alpha \subset \{|\phi| > \rho - \epsilon\}$$

for each $\epsilon > 0$. Next, since the area of $\phi(D) = \pi$ we have $S(u) = 1$ and Theorem 0.2 gives

$$\text{Cap}(\alpha) \leq e^{-\rho^2}$$

Hence we have proved the general inequality

$$\mathfrak{m}_0^\Omega(\ell) \leq \frac{2}{\pi} \cdot \text{arcsin } e^{-\rho^2} \tag{**}$$

Remark. There exists a special simply connected domain Ω for which equality holds in (**). See [Frostman: p. 39] : Potential theory.