Fundamental solutions to second order Elliptic operators.

Introduction. Elliptic partial differential operators appear frequently and one often employs fundamental solutions. The subsequent material is devoted to contruct fundamental solutions with best possible regularity conditions for second order elliptic operators with variable coefficients in $\mathbf{R}^3$. For PDE-operators with constant coefficients one employs Fouriers' inversion fourmula and we refer to Chapter X in vol.2 in Hörmander's text-book series on linear partial differential operators for a detailed account about constructions of fundamental solutions with optimal regularity In the case of second order elliptic operstors with constant coefficients the best funfsrmtnal solutions were alreafy found by Newton in pioneering eork from hids famous text-books published in 1666. Passing to the case of variable coefficients one profits Newton's constructions and following Carleman we will show that one can obtain fundamental solutions to second order elliptic operators with variable coefficientd in a canonical fashion. In contrast to the case of constant coefficients the subsequent constructions do not use the Fourier transform. Insetad one finds fundamental solutions by solving integral equations of the Neumann-Fredholm type. Let us remark that one does not need any concepts from distribution theory since we will find fundamental solutions are locally integrable and in such situations the notion of fundamental solutions were well understood at an early stage after pioneering work by Weyl and Zeilon prior to 1925. In his article Carleman refers to *Grundlösungen* and their requested properties are derived via Green's formula.

Remark. For students interested in PDE-theory the material below offers an instructive lesson and suggests further investigations. We restrict the study to $\mathbf{R}^3$ and remark only that similar constructions can be performed when $n \geq 4$ starting from Newton's potential $|x-\xi|^{-n+2}$. Here it would be interesting to clarify the precise estimates when $n \geq 4$ and establish similar inequalities as in the Main Theorem. It is also tempting to try to extend the whole constructions to elliptic operators of order ≥ 3. Recall that for elliptic operators with constant coefficiebts having an even order $2m$ with $m \geq 2$ there exist fuyndsrmtnal solutions constructed by Fritz John which like Newton's are found in a can onical fashion and therefore should be useful to treat the case of vartiable coefficiebts. This appears to be a "profitable research problem" for ph.d-students. Let us also remark that one does not assume that the elliptic operators are symmetric, i.e. both the constructions as well as estimates for the fundamental solutions do not rely upon symmetry conditions. Recall finally that fundamental solutions are used to construct various Greens' functions and here a priori estiamates are valuable We illustrate this in § xx where we expose some further results by Carleman concerned with asymptotic distributions of eigenvalues to elliptic boundary value problems.

An asymptotic formula for the spectrum. Let $n = 3$ and consider a second order PDE-operator

$$L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x)\cdot\frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p} + a_0(x)$$

The a-functions are real-valued and defined in a neighborhood of the closure of a bounded domain Ω in $\mathbf{R}^3$ with a C^1-boundary. Here one has the symmetry $a_{pq} = a_{qp}$, and $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x \in \Omega$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under these conditions, a result which goes back to work by Neumann and Poincaré, gives a positive constant κ_0 such that if $\kappa \geq \kappa_0$ then the inhomogeneous equation

$$L(u) - \kappa^2 \cdot u = f \quad : f \in L^2(\Omega)$$

has a unique solution u which is a C^2-function in Ω and extends to the closure where it is zero on $\partial\Omega$. Moreover, there exists some κ_0 and for each $\kappa \geq \kappa_0$ a Green's function $G(x, y; \kappa)$ such that

$$\text{(i)} \qquad (L-\kappa^2)\Big(\frac{1}{4\pi}\cdot\int_\Omega G(x,y;\kappa)\, f(y)\, dy\Big) = -f(x) \quad : f \in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

(ii) $$f \mapsto -\frac{1}{4\pi} \cdot \int_\Omega G(x, y; \kappa)\, f(y)\, dy$$

is Neumann's resolvent to the densely defined operator $L - \kappa^2$ on the Hilbert space $L^2(\Omega)$. Next, one seeks pairs (u_n, λ_n) where u_n are L^2-functions in Ω which extend to be zero on $\partial\Omega$ and satisfy

$$L(u_n) + \lambda_n \cdot u_n = 0$$

It turns out that the set of eigenvalues is discrete and moreover their real parts tend to $+\infty$. They are arranged with non-decreasing absolute values and in § xx we prove that there exist positive constants C and c such that

$$|\mathfrak{Im}(\lambda_n)|^2 \leq C \cdot (\mathfrak{Re}(\lambda_n) + c)$$

hold for every n. Next, the elliptic hypothesis means that the determinant function

$$D(x) = \det(a_{p,q}(x))$$

is positive in Ω. With these notations one has

Theorem. *The following limit formula holds:*

(*) $$\lim_{n\to\infty} \frac{\mathfrak{Re}(\lambda_n)}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2} \cdot \int_\Omega \frac{1}{\sqrt{D(x)}}\, dx$$

Remark. The formula above is due to Courant and Weyl when P is symmetric and was extended to non-symmetric operators during Carleman 's lectures at Institute Mittag-Leffler in 1935. Weyl and Courant used calculus of variation in the symmetric case while Carleman employed different methods which have the merit that the passage to the non-symmetric case does not cause any trouble. A crucial step during the proof of the theorem above is to construct a fundamental solution $\Phi(x, \xi; \kappa)$ to the PDE-operators $L - \kappa^2$ which done in § 1 while § 2 treats the asymptotic formula above. As pointed out by Carleman the methods in the proof give similar asymptotic formulas in other boundary value problems such as those considered by Neumann where one imposes boundary value conditions on outer normals. As an example we consider an elliptic operator of the form

$$L = \Delta + \sum_{p=1}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + a_0(x)$$

where Δ is the Laplace operator. Given a positive real-valued continuous function $\rho(x)$ on $\partial\Omega$ we obtain the Neumann-Poincaré operator $\mathcal{NP}$ which sends each $u \in C^0(\partial\Omega)$ to

$$\mathcal{NP}(u) = \frac{\partial u^*}{\partial \mathbf{n}_i} - \rho \cdot u$$

Here u^* is the Dirichlet extension of u to Ω which is equal to u on $\partial\Omega$ and satisfies $L(u) = 0$ in Ω, while $\frac{\partial u^*}{\partial \mathbf{n}_i}$ is the inner normal along the boundary. In the special case when $L = \Delta$ this boundary value problem has unique solutions, i.e. for every $f \in C^0(\partial\Omega)$ there exists a unique u such that

$$\mathcal{NP}(u) = f$$

This was proved by Poincaré in 1897 for domains in $\mathbf{R}^3$ whose boundaries are of class C^2 and the extension to domains with a C^1-boundary is also classic. Passing to general operators L as above which are not neccesarily symmetric one encounters spectral problems, i.e. above $\mathcal{NP}$ regared as a linear operator on the Banach space $C^0(\partial\Omega)$ is densely defined and one seeks its spectrum, i.e, complex numbers λ for which there exists a non-zero u such that

$$\mathcal{NP}(u) + \lambda \cdot u = 0$$

I do not know if there exists an analytic formula for these eigenvalues. Notice that a new feauture is that the ρ-function affects the spectrum.

Fundamental solutions.

In $\mathbf{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ we consider a second order PDE-operator

$$L = \sum_{p=1}^{p=3} \sum_{q=1}^{q=3} a_{pq}(x) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p(x) \frac{\partial}{\partial x_p} + a_0(x)$$

where a-functions are real-valued and one has the symmetry $a_{pq} = a_{qp}$. To ensure existence of a globally defined fundamental solutions we suppose the the following limit formulas hold as $(x, y, z) \to \infty$:

(0.0) $\qquad \lim a_\nu(x, y, z) = 0 \colon 0 \leq p \leq 3 \quad : \ \lim a_{pq}(x, y, z) =$ Kronecker's delta function

Thus, L approaches the Laplace operator as (x, y, z) tends to infinity. Moreover L is elliptic which means that the eigenvalues of the symmetric matrix with elements $\{a_{pq}(x)\}$ are positive for every x. Recall the notion of fundamental solutions. Consider the adjoint operator:

$$L^*(x, \partial_x) = P - 2 \cdot (\sum_{p=1}^{p=3} (\sum_{q=1}^{q=3} \frac{\partial a_{pq}}{\partial x_q}) \cdot \frac{\partial}{\partial x_p} - \sum_{p=1}^{p=3} \frac{\partial a_p}{\partial x_p} + 2 \cdot \sum \sum \frac{\partial^2 a_{pq}}{\partial x_p \partial x_q} \tag{0.1}$$

Partial integration gives the equation below for every pair of C^2-functions ϕ, ψ in $\mathbf{R}^3$ with compact support:

$$\int L(\phi) \cdot \psi \, dx = \int \phi \cdot L^*(\psi) \, dx \tag{0.2}$$

where the volume integrals are taken over $\mathbf{R}^3$. A locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $L(x, \partial_x)$ if

$$\psi(0) = \int \Phi \cdot L^*(\psi) \, dx \tag{0.3}$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $L - \kappa^2$ and a function $x \mapsto \Phi(x; \kappa)$ is a fundamental solution to $L - \kappa^2$ if

$$\psi(0) = \int \Phi(x : \kappa) \cdot (L^* - \kappa^2)(\psi(x)) \, dx \tag{0.4}$$

hold for compactly supported C^2-functions ψ. Next, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi^*(x, \xi; \kappa)$ with the property that

$$\psi(\xi) = \int \Phi(x, \xi; \kappa) \cdot (L^*(x, \partial_x) - \kappa^2)(\psi(x)) \, dx \tag{*}$$

hold for all $\xi \in \mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x, \xi; \kappa)$ is a function of six variables defined in $\mathbf{R}^3 \times \mathbf{R}^3$. With these notations we announce the main result:

Main Theorem. *There exists a constant κ_* such that for every $\kappa \geq \kappa_*$ one can find a fundaemtal solution $\Phi(x, \xi; \kappa)$ which is locally integrable in the 6-dimensional (x, ξ)-space. Moreover, there exist positive constants C and k and for each $0 < \gamma \leq 2$ a constant C_γ such that*

$$|\Phi(x, \xi; \kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x - \xi|} + \frac{C_\gamma}{(\kappa|x - \xi|)^\gamma}$$

hold for all pairs (x, ξ) in $\mathbf{R}^3$ and every where the constants k and C do not depend upon κ.

1. The construction of $\Phi(x, \xi; \kappa)$.

The subsequent constructions are based upon a classic formula due to Newton together with solutions to integral equations found by a convergent Neumann series. When L has constant coefficients the construction of fundamental solutions given by Newton goes as follows: Consider a positive and symmetric 3×3-matrix $A = \{a_{pq}\}$. Let $\{b_{pq}\}$ be the elements of the inverse matrix which gives the quadratic form

$$B(x) = \sum_{p,q} b_{pq} a_p x_q$$

Put

$$\alpha = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq} a_p a_q - a_0}$$

where κ is chosen so large that the term under the square-root is > 0. Finally, put

$$\Delta = \det(A)$$

With these notations we get a function:

$$H(x;\kappa) = \frac{1}{4\pi \cdot \sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq} a_p \cdot x_q} \tag{1.1}$$

Exercise. Verify by Stokes formula that $H(x;\kappa)$ yields a fundamental solution to the PDE-operator $L(\partial_x) - \kappa^2$.

1.2 The case with variable coefficients. Now L has variable coefficients. For each $\xi \in \mathbf{R}^3$ the elements of the inverse matrix to $\{a_{pq}(\xi)$ are denoted by $\{b_{pq}(\xi)\}$. Choose $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3$$

and for every $\kappa \geq \kappa_0$ we set

$$\alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2} \sum_{p,q} b_{pq}(\xi) a_p(\xi) a_q(\xi) - b(\xi)} \tag{i}$$

Following Newton's construction in (1.1) we put:

$$H(x,\xi;\kappa) = \frac{1}{4\pi} \cdot \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi) a_p(\xi) \cdot x_q} \tag{ii}$$

When ξ is kept the function of $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. We are going to construct a fundamental solution which takes the form

$$\Phi(x,\xi;\kappa) = H(x-\xi,\xi;\kappa) + \int_{\mathbf{R}^3} H(x-y,\xi;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy \tag{iii}$$

where the Ψ-function is the solution to an integral equation which we define in (1.5). But first we need another construction.

1.3 The function $F(x,\xi;\kappa)$. For every fixed ξ we get the differential operator in the x-space:

$$L_*(x, \partial_x, \xi; \kappa) =$$

$$\sum_{p=1}^{p=3} \sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi)) \frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

Apply L_* to the function $x \mapsto H(x-\xi,\xi;\kappa)$ and put

$$F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)) \tag{1.3.1}$$

1.4 Two estimates. The limit conditions in (0.0) give positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \quad : \ [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|^2} \tag{1.4.1}$$

The verification of (1.4.1) is left as an exercise.

1.5 An integral equation. We seek $\Psi(x,\xi;\kappa)$ which satisfies the equation:

$$\Psi(x,\xi;\kappa)=\int_{\mathbf{R}^3}F(x,y;\kappa)\cdot\Psi(y,\xi;\kappa)\,dy+F(x,\xi;\kappa) \tag{1.5.1}$$

To solve (1.5.1) we construct the Neumann series of F. Thus, starting with $F^{(1)}=F$ we set

$$F^{(k)}(x,\xi;\kappa)=\int_{\mathbf{R}^3}F(x,y;\kappa)\cdot F^{(k-1)}(y,\xi;\kappa)\,dy\quad:\quad k\geq 2 \tag{1.5.2}$$

Then (1.4.1) gives the inequality

$$|F^{(2)}(x,\xi;\kappa)|\leq C_1^2\iiint\frac{e^{-k\kappa|\xi-y|}}{|x-y|^2\cdot|\xi-y|^2}\cdot dy \tag{i}$$

To estimate (i) we notice that the triple integral after the substitution $y-\xi\to u$ becomes

$$C_1^2\iiint\frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2\cdot|u|^2}\cdot du \tag{ii}$$

In (ii) the volume integral can be integrated in polar coordinates and becomes

$$C_1^2\cdot\int_0^\infty\int_{S^2}\frac{e^{-k\kappa r^2}}{|x-r\cdot w-\xi|^2}\cdot dwdr \tag{iii}$$

where S^2 is the unit sphere and dw the area measure on S^2. It follows that (iii) becomes

$$2\pi C_1^2\cdot\int_0^\infty\int_0^\pi\frac{e^{-k\kappa r}}{(x-\xi)^2+r^2-2r\cdot|x-\xi|\cdot\sin\theta}\cdot d\theta dr=$$

$$\frac{2\pi C_1^2}{|x-\xi|}\cdot\int_0^\infty e^{-k\kappa|x-\xi|t}\cdot\log|\frac{1+t}{1-t}|\cdot\frac{dt}{t} \tag{iv}$$

where the last equality follows by a straightforward computation.

1.6 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)|\leq\frac{2\pi\cdot C_1^2\cdot C_1^*}{\kappa\cdot|x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

$$|F^{(n)}(x,\xi;\kappa)|\leq\frac{C_1}{|x-\xi|^2}\cdot\Big[\frac{2\pi C_1^2\cdot C_1^*}{\kappa}\Big]^{n-1}\quad\text{for every}\quad n\geq 2 \tag{*}$$

1.6 Conclusion. Choose κ_0^* so large that

$$2\pi C_1^2\cdot C_1^*<\kappa_0^* \tag{1.6.1}$$

Then (*) implies that the Neumann series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $\kappa\geq\kappa_0^*$ and gives the requested solution $\Psi(x,\xi;\kappa)$ in (1.5.1).

1.7 Exercise. We have found Ψ which satisfies the integral equation in § 1.5.1. Next, since the H-function in (ii) from § 1.2 is everywhere positive the integral equation (iii)in § 1.2 has a unique solution $\Phi(x,\xi;\kappa)$. Using Green's formula the reader can check that $\Phi(x,\xi;\kappa)$ yields a fundamental solution of $L(x,\partial_x)-\kappa^2$.

1.8 Some estimates. The constructions above show that the functions

$$x\mapsto\Phi(x,\xi;\kappa)\quad\text{and}\quad x\mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x = \xi$. Consider the difference

$$Q(x, \xi; \kappa) = \Phi(x, \xi; \kappa) - H(x - \xi, \xi; \kappa) \tag{1.8.1}$$

1.8.2 Exercise. Use the previous constructions to show that for every $0 < \gamma \leq 2$ there is a constant C_γ such that

$$\big|Q(x, \xi; \kappa)\big| \leq \frac{C_\gamma}{(\kappa|x - \xi|)^\gamma}$$

hold for every pair (x, ξ) and every $\kappa \geq \kappa_0$. Finally, the reader can apply the inequality for the H-function in (1.4.1) to conclude the results in the Main Theorem.

§ 2. Green's functions.

Let Ω be a bounded domain in $\mathbf{R}^3$, and L an elliptic differential operator as in § 1. Let $\kappa > 0$ and suppose we have found a function $G(x, y; \kappa)$ defined when $(x, y) \in \Omega \times \Omega$ with the property that $G(x, y) = 0$ if $x \in \partial\Omega$ and $y \in \Omega$. Moreover

$$(L(x, \partial_x) - \kappa^2)(G(x, y; \kappa)) = \delta(x - y)$$

where δ denotes the usual Dirac distribution. Taking G as a kernel we get the integral operator

$$\mathcal{G}(f)(x) = \int_\Omega G(x, y; \kappa)\, f(y)\, dy$$

Then $\mathcal{G}(f)(x) = 0$ on $\partial\Omega$ and the composed operator

$$(L(x, \partial_x) - \kappa^2) \circ \mathcal{G} = E$$

To construct G we use the fundamental solution $\Phi(x, y; \kappa)$ from § 1 which satisfies

$$(L(x, \partial_x) - \kappa^2)(\Phi(x, y; \kappa)) = \delta(x - y) \tag{2.0.0}$$

Next, with $y \in \Omega$ kept fixed we have the continuous boundsry functon

$$x \mapsto \Phi(x, y; \kappa)$$

Solving the Dirchlet problem we find $w(x)$ such that $w(x) = \Phi(x, y; \kappa)$ on the boundary while $(L(x, \partial_x) - \kappa^2)(w) = 0$ holds in Ω. Then we can take

$$G(x, y; \kappa) = \Phi(x, y; \kappa) - w(x) \tag{2.0.1}$$

Using the estimates for the Φ-function from § 1 we get estimates for this G-function. We can also choose a sufficiently large κ_0 so that $\Phi(x, \xi; \kappa_0)$ is a positive function of (x, ξ). Then the following hold:

2.1 Theorem. *One has*

$$G(x, \xi; \kappa_0) = \frac{1}{\sqrt{\Delta(x)} \cdot \sqrt{\Phi(x, \xi; \kappa_0)}} + R(x, \xi)$$

where the remainder function satisfies the following for all pairs (x, ξ) in Ω:

$$|R(x, \xi)| \leq C \cdot |x - \xi|^{-\frac{1}{4}}$$

and the constant C only depends on the domain Ω and the PDE-operator L.

Remark. Above the negative power of $|x - \xi|$ is a fourth-root which means that the remainder term R is more regular compared to the first term which behaves like $|x - \xi|^{-1}$ on the diagonal $x = \xi$.

2.2 Exercise. Prove Theorem 2.1 If necessary, consult [Carleman: page xx-xx] for details.

2.3. Almost reality of eigenvalues.

Consider the set of eigenvalues λ for which there exists a function u in Ω which is zero on $\partial\Omega$ while

$$L(u) + \lambda \cdot u = 0$$

holds in Ω.

2.3.1 Proposition. *There exist positive constants C_* and c_* such that every eigenvalue λ above satisfies*

$$|\mathfrak{Im}\, \lambda|^2 \leq C_*(\mathfrak{Re}\, \lambda) + c_*)$$

Proof. Let u be an eigenfunction where $L(u) + \lambda \cdot u = 0$. Stokes theorem and the vanishing of $u|\partial\Omega$ give:

$$0 = \int_\Omega \bar{u} \cdot (L + \lambda)(u)\, dx = -\int_\Omega \sum_{p,q} a_{pq}(x) \cdot \frac{\partial u}{\partial x_p} \frac{\partial \bar{u}}{\partial x_q}\, dx + \int_\Omega \bar{u} \cdot (\sum a_p(x) \frac{\partial u}{\partial x_p}\,)\, dx +$$

$$\int_\Omega |u(x)|^2 \cdot b(x)\, dx + \lambda \cdot \int |u(x)|^2\, dx$$

Write $\lambda = \xi + i\eta$. Separating real and imaginary parts we find the two equations:

$$\text{(i)} \qquad \xi \int |u|^2\, dx = \int \sum_{p,q} a_{p,q}(x)\, \frac{\partial u}{\partial x_p} \cdot \frac{\partial \bar{u}}{\partial x_q}\, dx + \int \big(\frac{1}{2} \cdot \sum \frac{\partial a_p}{\partial x_p} - b\big) \cdot |u|^2\, dx$$

$$\text{(ii)} \qquad \eta \int |u|^2\, dx = \frac{1}{2i} \int \sum a_p\big(u \frac{\partial \bar{u}}{\partial x_p} - \bar{u} \frac{\partial u}{\partial x_p}\,\big)\, dx$$

Set

$$A = \int |u|^2\, dx \quad : \quad B = \int |\nabla(u)|^2\, dx$$

Since L is elliptic there exists a positive constant k such that

$$\sum_{p,q} a_{p,q}(x)\, \frac{\partial u}{\partial x_p} > k \cdot |\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants c_1, c_2, c_3 such that

$$\text{(iii)} \qquad A\xi > c_1 B - c_2 B \quad : \quad A|\eta| < c_3 \cdot \sqrt{AB}$$

Here (iii) implies that $\xi > -c_2$ and the reader can also confirm that

$$\text{(iv)} \qquad B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A \cdot c_2 \cdot \sqrt{\frac{\xi + c_2}{c_1}} \quad : \quad |\eta| < c_3 \cdot \sqrt{\frac{\xi + c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 2.3.1.

2.4. Asymptotic formula for eigenvalues

Consider a function f which satisfies

$$\mathcal{G}(f) = -\frac{1}{\lambda} \cdot f$$

for some non-zero complex number λ. With $u = \mathcal{G}(f)$ it follows from the previous material that

$$(L - \kappa^2)(u) = f = -\lambda \cdot u$$

Hence

$$L(u) + (\lambda - \kappa^2)u = 0$$

From the above the asymptotic formula in the Main Theorem from the introduction can be derived from asymptotic properties of eigenvalues to the integral operator $\mathcal{G}$. More precisely, using Theorem 2.1 and the estimates for the fundamental solution Φ in § 1, one can proceed as in the the next section and employ a Tauberian theorem. The reader may try to supply details or consult [Carleman: page xx-xx] for details after reading the proof of another asymptotic formula in the next section.

§ 3. A study of $\Delta(\phi) + \lambda \cdot \phi$.

Introduction. We expose material from Carleman's article *xxx* presented at the Scandinavian Congress in Stockholm 1934. In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact established by G. Neumann and H. Poincaré during the years 1879-1895 gives the following: First there exists the Greens' function

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$ with the property that the operator $\mathcal{G}$ defined on $L^2(\Omega)$ by

$$f \mapsto \mathcal{G}_f(p) = \frac{1}{2\pi} \dot{\iint} G(p,q) f(q)\, dq$$

satisfies

$$\Delta \circ \mathcal{G}_f = -f \quad : f \in L^2(\Omega)$$

Moreover, $\mathcal{G}$ is a compact operator on the Hilbert space $L^2(\Omega)$ and there exists a sequence $\{f_n\}$ in $L^2(\Omega$ such that $\{\phi_n = \mathcal{G}_{f_n}\}$ is an orthonormal basis in $L^2(\Omega$ and

$$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : n = 1, 2, \ldots$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots \}$. When eigenspaces have dimension ≥ 2, the eigenvalues are repeated by their multiplicity.

Main Theorem. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N \to \infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

The strategy in the proof is to consider the function of a complex variable s defined by

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

and show that it is a meromorphic function in the whole complex s-plane with a simple pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. More precisely we shall prove:

3.1 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem 3.1 gives ther Main Theorem by a result due to Wiener in the article *Tauberian theorem* [Annals of Math.1932]. Wiener's theorem asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s \to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n \to \infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Exercise. Derive the main theorem from Wiener's result and Theorem 3.1.

About Wiener's result. It is a version of a famous theorem proved by Hardy and Littlewood in 1913 which goes as follows:

0.2 The Hardy-Littlewood theorem. *Let $\{a_n\}$ be a sequence of non-negative real numbers such that*

$$(*) \qquad A = \lim_{r \to 1} (1-r) \cdot \sum a_n r^n$$

exists. Then there also exists the limit

$$(**) \qquad A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Notice that no growth condition is imposed on the sequence $\{a_n\}$, i.e. the sole assumption is the existing limit (*). The proof is quite demanding and does not follow by "abstract nonsense" from functional analysis. For the reader's convenience we include details of the proof in a separate appendix since courses devoted to such a basic and very delicate topic as the study of series rarely appear in contemporary education where too much attention often is given to more "soft snalysis".

§ 1. Proof of Theorem 3.1.

Let Ω be a bounded and Dirichlet regular domain. For ech fixed point $p \in \Omega$ we get the continuous function on $\partial\Omega$ defined by

$$q \mapsto \log \frac{1}{|p-q|}$$

We find the harmonic function $u_p(q)$ in Ω such that $u_p(q) = \log \frac{1}{|p-q|} \, : \, q \in \partial\Omega$. Green's function is defined for pairs $p \neq q$ in $\Omega \times \Omega$ by

$$(1) \qquad G(p,q) = \log \frac{1}{|p-q|} - u_p(q)$$

Keeping if $p \in \Omega$ fixed, the function $q \mapsto G(p,q)$ extends to the closure of Ω where it vanishes if $q \in \partial\Omega$. If $f \in L^2(\Omega$ we set

$$(2) \qquad \mathcal{G}_f(p) = \frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot f(q)\, dq$$

where $q = (x,y)$ so that $dq = dxdy$ when the double integral is evaluated. From (1) we see that

$$\iint_{\Omega \times \Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence $\mathcal{G}$ is of the Hilbert-Schmidt type and therefore a compact operator on $L^2(\Omega)$. Next, recall that $\frac{1}{2\pi} \cdot \log \sqrt{x^2+y^2}$ is a fundamental solution to the Laplace operator. From this the reader can deduce the following:

1.1 Theorem. *For each $f \in L^2(\Omega)$ the Lapacian of $\mathcal{G}_f$ taken in the distribution sense belongs to $L^2(\Omega)$ and one has the equality*

$$(*) \qquad \Delta(\mathcal{G}_f) = -f$$

The equation (*) means that the composed operator $\Delta \circ \mathcal{G}$ is minus the identity on $L^2(\Omega)$. We are led to introduce the linear operator S on $L^2(\Omega)$ defined by Δ, where $\mathcal{D}(S)$ is the range of $\mathcal{G}$. If $g \in C_0^2(\Omega)$, i.e. twice differentiable and with compact support, it follows via Greens' formula that

$$\frac{1}{2\pi} \cdot \int_\Omega G(p,q) \cdot \Delta(g)(q)\, dq = -g(p)$$

In particular $C_0^2(\Omega) \subset \mathcal{D}(S)$ which implies that S is densely defined and we leave it to the reader to verify that

$$\mathcal{G}(\Delta(f)) = -f \quad : f \in \mathcal{D}(S)$$

Remark. By Carl Neumann's classic construction of resolvent operators from 1879, the result above means that $-\mathcal{G}$ is Neumann's inverse of S. Since $-\mathcal{G}$ is compact it follows by Neumann's formula for spectra that S has a discrete spectrum,and we recall the following wellknown fact which goes back to work by Poincaré:

1.2 Proposition. *There exists an orthonormal basis $\{\phi_n\}$ in $L^2(\Omega)$ where each $\phi_n \in \mathcal{D}(S)$ is an eigenfunction, and a non-decreasing sequence of positive real numbers $\{\lambda_n\}$ such that*

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0 \quad : n = 1, 2, \ldots \tag{1.2.1}$$

Remark. Above (1.2.1) means that

$$\mathcal{G}(\phi_n) = \frac{1}{\lambda_n} \cdot \phi_n$$

This, $\{\lambda_n^{-1}\}$ are eigenvalues of the compact operator $\mathcal{G}$ whose sole cluster point is $\lambda = 0$. As usual eigenvalues whose eigenspaces have dimension $e > 1$ are repeated e times.

After these preliminaries we embark upon the proof of Theorem 0.1. First, since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\sum \lambda_n^{-2} < \infty \tag{i}$$

This convergence entails that various constructions below are defined. For each complex number λ outside $\{\lambda_n\}$ we set

$$G(p, q; \lambda) = G(p, q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda - \lambda_n)} \tag{ii}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi} \cdot \iint_\Omega G(p, q; \lambda) \cdot f(q)\, dq \tag{iii}$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta + \lambda) \cdot \mathcal{G}_\lambda = -E$$

B. The function $F(p, \lambda)$. Set

$$F(p, q, \lambda) = G(p, q; \lambda) - G(p, q)$$

Keeping p fixed we see that (ii) gives

$$\lim_{q \to p} F(p, q, \lambda) = 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n(\lambda - \lambda_n)} \tag{B.1}$$

Set

$$F(p, \lambda) = \lim_{q \to p} F(p, q, \lambda)$$

From (i) and (B.1) it follows that it is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Show via residue calculus that one has the equality below in a half-space $\mathfrak{Re}\, s > 2$:

$$\Phi(s) = \frac{1}{4\pi^2 \cdot i} \cdot \int_{a - i\infty}^{a + i\infty} F(p, \lambda) \cdot \lambda^{-s}\, d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\, \lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p, \lambda) \cdot \lambda^{-s}\, d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

$$\Phi(s) = \frac{a^{s-1}}{4\pi} \cdot \int_{-\pi}^{\pi} F(ae^{i\theta}) \cdot e^{(i(1-s)\theta}\, d\theta + \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx \tag{E.1}$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty F(p,-x) \cdot x^{-s}\, dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p,-x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^\infty \frac{e^{-zt}}{\sqrt{t^2-1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$\text{(i)} \qquad I_0(z) = \sum_{m=0}^\infty \frac{2^{-2m}}{(m!)^2} \cdot z^{2m}$$

$$\text{(ii)} \qquad I_1(z) = \sum_{m=0}^\infty \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p,q;\kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p,q;\kappa) = K(\sqrt{\kappa}|p-q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p,q;-\kappa) = K(\sqrt{\kappa} \cdot |p-q|) - H(q;\kappa) \quad : \kappa > 0$$

Next, the construction of $G(p,q)$ gives

$$F(p,-\kappa) = \lim_{q \to p} [K(\sqrt{\kappa} \cdot |p-q|) + \log |p-q|] + \lim_{q \to p} [u_p(q) + H(p,q,\kappa)] \tag{G.1}$$

The last term above has the "nice limit" $u_p(p) + H(p,p,\kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q \to p} [K(\sqrt{\kappa} \cdot |p-q|) + \log |p-q|] = -\frac{1}{2} \cdot \log \kappa + \log 2 - \gamma \tag{G.2}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p,-\kappa) = -\frac{1}{2} \cdot \log \kappa + A + H(p,p;-\kappa)$$

With $x=\kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty x^{-s}\, dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2} \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty \log x \cdot x^{-s}\, dx$$

Notice that the complex derivative

$$\frac{d}{ds} \int_a^\infty x^{-s}\, dx = -\int_a^\infty \log x \cdot x^{-s}\, dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem 0.1 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty H(p,p,\kappa) \cdot \kappa^{-s}\, d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \leq H(p,q;\kappa) \leq \max_{q \in \partial\Omega} K(\kappa|p-q|)$$

With $p \in \Omega$ fixed there is a positive number δ such that $|p-q| \geq \delta : q \in \partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.

Appendix. Theorems by Abel, Tauber, Hardy and Littlewood

Introduction. Consider a power series $f(z) = \sum a_n z^n$ whose radius of convergence is one. If $r < 1$ and $0 \leq \theta \leq 2\pi$ we are sure that the series

$$f(re^{i\theta}) = \sum a_n r^n e^{in\theta}$$

is convergent. In fact, it is even absolutely convergent since the assumption implies that

$$\sum |a_n| \cdot r^n < \infty \quad \text{for all} \quad r < 1$$

Passing to $r = 1$ it is in general not true that the series $\sum a_n e^{in\theta}$ is convergent. An example arises if we consider the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

This leads to the following problem where we without loss of generality can take $\theta = 0$. Consider as above a convergent power series and assume that there exists the limit

$$(*) \qquad \lim_{r \to 1} \sum a_n r^n$$

When can we conclude that the series $\sum a_n$ also is convergent and that one has the equality

$$(**) \qquad \sum a_n = \lim_{r \to 1} \sum a_n r^n$$

The first result in this direction was established by Abel in a work from 1823:

A. Theorem *Let $\{a_n\}$ be a sequence such that $\frac{a_n}{n} \to 0$ as $n \to \infty$ and there exists*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Then $\sum a_n$ is convergent and the sum is A.

An extension of Abel's result was established by Tauber in 1897.

B. Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists the limit*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Set

$$\omega_n = a_1 + 2a_2 + \ldots + na_n \quad : \; n \geq 1$$

If $\lim_{n \to \infty} \omega_n = 0$ it follows that the series $\sum a_n$ is convergent and the sum is A.

C. Results by Hardy and Littlewood. In their joint article *xxx* from 1913 the following extension of Abel's result was proved by Hardy and Littlewood:

C. Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a constant C so that $\frac{a_n}{n} \leq C$ for all $n \geq 1$. Assume also that the power series $\sum a_n z^n$ converges when $|z| < 1$. Then the same conclusion as in Abel's theorem holds.*

Remark. In addition to this they proved a result about positive series from the cited article which has independent interest.

D. Theorem. *Assume that each $a_n \geq 0$ and that there exists the limit:*

$$(*) \qquad A = \lim_{r \to 1} (1 - r) \cdot \sum a_n r^n$$

Then there exists the limit

$$(**) \qquad A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Remark. The proofs of Abel's and Tauber's results are easy while C and D require more effort and rely upon results from calculus in one variable. So before we enter the proofs of the theorems above insert some preliminaries.

1. Results from calculus

Below $g(x)$ is a real-valued function defined on $(0,1)$ and of class C^2 at least.

1.1 Lemma *Assume that there exists a constant $C>0$ such that*

$$g''(x)\leq C(1-x)^{-2} \quad : 0<x<1 \quad \text{and} \quad \lim_{x\to 1} g(x)=0$$

Then one has the limit formula:

$$\lim_{x\to 1}\,(1-x)\cdot g'(x)=0$$

1.2 Lemma *Assume that the second order derivative $g''(x)>0$. Then the following implication holds for each $\alpha>0$:*

$$\lim_{x\to 1}\,(1-x)^{\alpha}\cdot g(x)=1 \implies \lim_{x\to 1}\,(1-x)^{\alpha+1}\cdot g'(x)=\alpha$$

Remark. If $g(x)$ has higher order derivatives which all are >0 on $(0,1)$ we can iterate the conclusion in Lemma 1.2 where we take α to be positive integers. More precisely, by an induction over ν the reader may verify that if

$$\lim_{x\to 1}\,(1-x)\cdot g(x)=1$$

exists and if $\{g^{(\nu)}(x)>0\}$ for all every $\nu\geq 2$ then

$$\lim_{x\to 1}\,(1-x)^{\nu+1}\cdot g^{(\nu)}(x)=\nu\,! \quad : \; \nu\geq 2 \tag{*}$$

Next, to each integer $\nu\geq 1$ we denote by $[\nu-\nu^{2/3}]$ the largest integer $\leq (\nu-\nu^{2/3})$. Set

$$J_*(\nu)=\sum_{n\leq[\nu-\nu^{2/3}]} n^{\nu}e^{-\nu} \quad : \quad J^*(\nu)=\sum_{n\geq[\nu+\nu^{2/3}]} n^{\nu}e^{-\nu}$$

1.3 Lemma *There exists a constant C such that*

$$\frac{J^*(\nu)+J_*(\nu)}{\nu\,!}\leq\delta(\nu) \quad : \quad \delta(\nu)=C\cdot\exp\big(-\frac{1}{2}\cdot\nu^{\frac{1}{3}}\big) \quad : \; \nu=1,2,\ldots$$

Proofs

We prove only Lemma 1.1 which is a bit tricky while the proofs of Lemma 1.2 and 1.3 are left as exercises to the reader. Fix $0<\theta<1$. Let $0<x<1$ and set

$$x_1=x+(1-x)\theta$$

The mean-value theorem in calculus gives

$$g(x_1)-g(x)=\theta(1-x)g'(x)+\frac{\theta^2}{2}(1-x)^2\cdot g''(\xi) \quad \text{for some} \quad x<\xi<x_1 \tag{i}$$

By the hypothesis

$$g''(\xi)\leq C(1-\xi)^{-2}\leq C)1-x_1)^{-2}$$

Hence (i) gives

$$(1-x)g'(x)\geq\frac{1}{\theta}(g(x_1)-g(x))-C\cdot\frac{\theta}{2}\frac{(1-x)^2}{1-x_1)^2}=$$

$$\frac{1}{\theta}(g(x_1)-g(x))-\frac{C\cdot\theta}{2(1-\theta)^2}$$

Keeping θ fixed we have by assumption

$$\lim_{x\to 1}\,g(x)=0$$

Notice also that $x\to 1 \implies x_1\to 1$. It follows that

$$\liminf_{x\to 1}\,(1-x)g'(x)\geq-\frac{C\cdot\theta}{2(1-\theta)^2}$$

Above $0 < \theta < 1$ is arbitrary, i.e. we can choose small $\theta > 0$ and hence we have proved that

(*) $$\liminf_{x \to 1} (1-x)g'(x) \geq 0$$

Next we prove the opposed inequality

(**) $$\limsup_{x \to 1} (1-x)g'(x) \leq 0$$

To get (**) we apply the mean value theorem in the form

(ii) $$g(x_1) - g(x) = \theta(1-x)g'(x_1) - \frac{\theta^2}{2}(1-x)^2 \cdot g''(\eta) \quad : \; x < \eta < x_1$$

Since $(1 - x_1) = \theta(1-x)(1-\theta)$ we get

(iii) $$(1-x_1)g'(x_1) = \frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + \frac{(1-\theta)\theta}{2} \cdot (1-x)^2 g''(\eta)$$

Now $g''(\eta) \leq C(1-\eta)^{-2} \leq C(1-x_1)^{-2}$ so the right hand side in (iii) is majorized by

$$\frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{(1-\theta)\theta}{2} \cdot (1-x)^2(1-x_1)^2 =$$

(iv) $$\frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{\theta}{2(1-\theta}$$

Keeping θ fixed while $x \to 1$ we obtain:

$$\liminf_{x \to 1} (1-x)g'(x) \leq C \cdot \frac{\theta}{2(1-\theta}$$

Again we can choose arbitrary small θ and hence (**) holds which finishes the proof of Lemma 1.1.

2. Proof of Abel's theorem.

Without loss of generality we can assume that $a_0 = 0$ and set $S_N = a_1 + \ldots + a_N$. Given $0 < r < 1$ we let $f(r) = \sum a_n r^n$. For every positive integer N the triangle inequality gives:

$$\big|S_N - f(r)\big| \leq \sum_{n=1}^{n=N} |a_n|(1-r^n) + \sum_{n \geq N+1} |a_n| r^n$$

Set $\delta(N) = \max_{n \geq N} \frac{|a_n|}{n}$. Since $1 - r^n) = (1-r)(1 + \ldots + r^{n-1} \leq (1-r)n$ the last sum is majorised by

$$(1-r) \cdot \sum_{n=1}^{n=N} n \cdot |a_n| + \delta(N+1) \cdot \sum_{n \geq N+1} \frac{r^n}{n}$$

Next, the obvious inequality $\sum_{n \geq N+1} \frac{r^n}{n} \leq \frac{1}{N+1} \cdot \frac{1}{1-r}$ gives the new majorisation

(1) $$(1-r) \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \frac{\delta(N+1)}{N+1} \cdot \frac{1}{1-r}$$

This hold for all pairs N and r. To each $N \geq 2$ we take $r = 1 - \frac{1}{N}$ and hence the right hand side in (1) is majorised by

$$\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \delta(N+1) \cdot \frac{N}{N+1}$$

Here both terms tend to zero as $N \to \infty$. Indeed, Abel's condition $\frac{a_n}{n} \to 0$ implies that $\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n}$ tends to zero as $N \to \infty$. Hence we have proved the limit formula:

$$(*) \qquad \lim_{N\to\infty} \left| s_N - f(1-\frac{1}{N}) \right| = 0$$

Finally it is clear that (*) gives Abel's result.

3. Proof of Tauber's theorem.

We may assume that $a_0 = 0$. Notice that

$$a_n = \frac{\omega_n - \omega_{n-1}}{n} \quad : \, n \geq 1$$

It follows that

$$f(r) = \sum \frac{\omega_n - \omega_{n-1}}{n} \cdot r^n = \sum \omega_n \big(\frac{r^n}{n} - \frac{r^{n+1}}{n+1} \big)$$

Using the equality $\frac{1}{n} = \frac{1}{n+1} = \frac{1}{n(n+1)}$ we can rewrite the right hand side as follows:

$$\sum \omega_n \big(\frac{r^n - r^{n+1}}{n+1} + \frac{r^n}{n(n+1)} \big)$$

Set

$$g_1(r) = \sum \omega_n \cdot \frac{r^n - r^{n+1}}{n+1} = (1-r) \cdot \sum \frac{\omega_n}{n+1} \cdot r^n$$

By the hypothesis $\lim_{n\to\infty} \frac{\omega_n}{n+1} = 0$ and then it is clear that we get

$$\lim_{r\to 1} g_1(r) = 0$$

Since we also have $f(r) \to 0$ as $r \to 1$ we conclude that

$$(1) \qquad \lim_{r\to 1} \sum \frac{\omega_n}{n(n+1)} \cdot r^n = 0$$

Next, with $b_n = \frac{\omega_n}{n(n+1)}$ we have $nb_n = \frac{\omega_n}{n+1} \to 0$. Hence Abel's theorem applies so (1) gives convergent series

$$(2) \qquad \sum \frac{\omega_n}{n(n+1)} = 0$$

If $N \geq 1$ we have the partial sum

$$S_N = \sum_{n=1}^{n=N} \frac{\omega_n}{n(n+1)} = \sum_{n=1}^{n=N}, \omega_n \cdot \big(\frac{1}{n} - \frac{1}{n+1} \big)$$

The last term becomes

$$\sum_{n=1}^{n=N} \frac{1}{n}(\omega_n - \omega_{n-1}) - \frac{\omega_N}{N+1} = \sum_{n=1}^{n=N} a_n - \frac{\omega_N}{N+1}$$

Again, since $\frac{\omega_N}{N+1} \to 0$ as $N \to \infty$ we conclude that the convergent series from (2) implies that the series $\sum a_n$ also is converges and has sum equal to zero. This finishes the proof of Tauber's result.

4. Proof of Theorem D.

Set $f(x) = \sum a_n x^n$ which is defined when $0 < x < 1$. Notice that

$$(1-x)f(x) = \sum s_n x^n \quad \text{where} \quad s_n = a_1 + \ldots + a_n$$

Set $g(x) = \sum s_n x^n$ which is defined when $0 < x < 1$. Since $s_n \geq 0$ for all n all the higher order derivatives

$$g^{(p)}(x) = \sum_{n=p}^{\infty} n(n-1)\cdots(n-p+1)a_n x^{n-p} > 0$$

when $0 < x < 1$. The hypothesis that $\lim_{x\to 1} g(x) = A$ and Lemma 1.1 and the inductive result in the remark after Lemma 1.2 give:

$$(1) \qquad \lim_{x\to 1}(1-x)^{\nu+2}\cdot \sum s_n\cdot n^\nu x^n = (\nu+1)! \quad : \nu \geq 1$$

We shall use the substitution $e^{-t} = x$ where $t > 0$. Since $t \simeq 1-x$ when $x \to 1$ we see that (1) gives

$$(2) \qquad \lim_{t\to 0} t^{\nu+2}\cdot \sum s_n\cdot n^\nu e^{-nt} = (\nu+1)! \quad : \nu \geq 1$$

Let us put

$$J_*(\nu,t) = \frac{t^{\nu+2}}{(\nu+1)!}\cdot \sum_{n=1}^\infty s_n\cdot n^\nu e^{-nt}$$

So for each fixed ν one has

$$(3) \qquad \lim_{t\to 0} J_*(\nu,t) = 1$$

Next, for each pair $\nu \geq 2$ and $0 < t < 1$ we define the integer

$$(*) \qquad N = [\frac{\nu - \nu^{2/3}}{t}]$$

Since the sequence $\{s_n\}$ is non-decreasing we get

$$(i) \qquad s_N\cdot \sum_{n\geq N} n^\nu e^{-nt} \leq \sum_{n\geq N} s_n\cdot n^\nu e^{-nt} \leq \frac{(\nu+1)!\cdot J_*(\nu,t)}{t^{\nu+2}}$$

Next, the construction of N and Lemma 1.3 give:

$$(ii) \qquad \sum_{n\geq N} n^\nu e^{-nt} \geq \frac{\nu!}{t^{\nu+1}}\cdot (1-\delta(\nu))$$

where the δ function is independent of ν and tends to zero as $\nu \to \infty$. Hence (i-ii) give

$$(iii) \qquad s_N \leq \frac{(\nu+1)}{t}\cdot \frac{1}{1-\delta(\nu)}\cdot J_*(\nu,t)$$

Next, by the construction of N one has

$$N+1 \geq \frac{\nu-\nu^{2/3}}{t} = \frac{\nu}{t}\cdot (1-\nu^{-1/3})$$

It follows that (iii) gives

$$(iv) \qquad \frac{s_N}{N+1} \leq \frac{\nu+1}{\nu}\cdot \frac{1}{1-\nu^{-1/3}}\cdot \frac{1}{1-\delta(\nu)}\cdot J_*(\nu,t)$$

Since $\delta(\nu) \to 0$ it follows that for any $\epsilon > 0$ there exists some ν_* such that

$$(v) \qquad \frac{\nu_*+1}{\nu_*}\cdot \frac{1}{1-\nu_*^{-1/3}}\cdot \frac{1}{1-\delta(\nu_*)} < 1+\epsilon$$

Keeping ν_* fixed we now consider pairs t_n, N such that (*) above hold with $\nu = \nu_*$. Notice that

$$(vi) \qquad N \to +\infty \implies t_N \to 0$$

It follows from (iv) and (v) that we have:

$$(vii) \qquad \frac{s_N}{N+1} < (1+\epsilon)\cdot J_*(\nu_*, t_N) \quad : N \geq 2$$

Now (vi) and the limit in (3) which applies with ν_* while $t_N \to 0$ entail that

$$\lim_{N\to\infty} J(\nu_*, t_N) = 1$$

We have also that $\frac{N}{N+1} \to 1$ and since $\epsilon > 0$ was arbitrary we see that (vii) proves the inequality

$$\limsup_{N\to\infty} \frac{s_N}{N} \leq 1 \tag{1}$$

So Theorem 2 follows if we also prove that

$$\liminf_{N\to\infty} \frac{s_N}{N} \geq 1 \tag{2}$$

The proof of (II) is similar where we now define the integers N by:

$$N = [\frac{\nu + \nu^{2/3}}{t}]$$

Then we have

$$S_N \cdot \sum_{n\leq N} n^\nu e^{-nt} \geq \frac{(\nu+1)! \cdot J_*(\nu,t)}{t^{\nu+2}} - \sum_{n>N} s_n \cdot n^\nu e^{-nt}$$

Here the last term can be estimated above since the Lim.sup inequality (I) gives a constant C such that $s_n \leq Cn$ for all n and then

$$\sum_{n>N} s_n \cdot n^\nu e^{-nt} \leq C \cdot \sum_{n>N} n^{\nu+1} e^{-nt} \leq C \cdot \delta^*(\nu) \cdot \frac{(\nu+1)!}{t^{\nu+2}}$$

where Lemma 1.3 entails that $\delta^*(\nu) \to 0$ as ν increases. At the same time Lemma 1.3 also gives

$$\sum_{n\leq N} n^\nu \cdot e^{-nt} = \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta_*(\nu)$$

where $\delta(\nu_*) \to 0$. At this stage the reader can verify that (2) by similar methods as in the proof of (I).

5. Proof of Theorem C

Set $f(x) = \sum a_n x^n$. Notice that it suffices to prove Theorem C when the limit value

$$\lim_{x\to 1} \sum a_n x^n = 0$$

Next, the assumption that $a_n \leq \frac{c}{n}$ for a constant c gives

$$f''(x) = \sum n(n-1)a_n x^{n-2} \leq c \sum (n-1)x^{n-2} = \frac{c}{1-x)^2}$$

The hypothesis $\lim_{x\to 1} f(x) = 0$ and Lemma xx therefore gives

$$\lim_{x\to 1} (1-x)f'(x) = 0 \tag{i}$$

Next, notice the equality

$$\sum_{n=1}^{\infty} \frac{na_n}{c} x^n = \frac{x}{c} \cdot f'(x) \tag{ii}$$

At the same time $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ and hence (i-ii) together give:

$$\lim_{x\to 1} (1-x) \cdot \sum (1 - \frac{na_n}{c}) \cdot x^n = 1$$

Here $1 - \frac{na-n}{c} \geq 0$ so Theorem 2 gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n=N} (1 - \frac{na_n}{c}) = 1$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{n=N} na_n = 0$$

This means precisely that the condition in Tauber's Theorem holds and hence $\sum a_n$ converges and has series sum equal to 0 which finishes the proof of Theorem C.