Mathematics by Torsten Carleman (1892-1949)

Introduction. Carleman's collected work covers fifty articles of high standard together with several monographs. He became professor at Stockholm University in 1924 when he replaced the chair held by Helge von Koch. From 1927 he also served director at Institute Mittag-Leffler where he delivered frequent seminars during the years 1928-1938. His text-book *Lärobok i differential och integralkalkyl jämte geometeriska och mekaniska tillämpningar* for undergraduate mathematics was published in 1928 and used for several decades in Sweden. Personally I find it outstanding as a beginner's text. A good example is the material in probability theory [ibid: p. 275-296] which contains an elegant proof of the central limit theorem for sums of independent binomial distributions.with sharp remainder terms. His last major publication was *Sur un probléme d'unicité pour les systèmes d'équations aux dérivees partielles à deux variables indépendantes* [Arkiv för matematik 1938] concerned with a uniqueness theorem for elliptic PDE-systems where the variable coefficients of the PDE-operators are non-analytic which extended an earlier result by Erik Holmgren when the coefficients are real-analytic. Let us recall that Holmgren served as supervisor while Carleman prepared his thesis entitled *Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, presented at Uppsala University in 1916 when he was 23 years old. Here appears a studies of double layer potentials which extended earlier work by Poincaré to the case when the boundary in Neumann's boundary value problem no longer is smooth. Many results, foremost precise inequalities in this thesis certainly merit a study up to the present date. Let me remark that already in —ibid] appears naturally defined bounded linear operators whose spectra consist of compact sets with interior points which at that time was a new feature in operator theory.

After World War I, Carleman visited several universities in Europe, such as Zürich, Göttingen, Oxford and Paris. As pointed out in Fritz Carlson's memorial article [Acta Mathematica 1950], Paris was always Carleman's favourite place where he got inspiration from prominent mathematicians such as Borel, Denjoy and Picard. On several occasions Carleman delivered lecture series at Sorbonne. For example, in the spring 1930 he lectured about singular integral operators and spectral theory for unbounded self-adjoint operators on Hilbert spaces, and in 1937 lectures were devoted to Boltzmann's kinetic gaz theory.

Carleman was aware of the interplay between "pure mathematics" and experimental sciences which inspired his own research. In 1920-21 he studied at an engineering school in Paris. An outcome of this is the article entitled *Sur les équations différentielles de la mecanique d'avion* published in [La Technique Aéronautique, vol. 10 1921), inspired by Lanchester's pioneering work *Le vol aérien* which played a significant role while airplanes were designed at an early stage. Carleman's article ends with the following conclusion after an investigation of integral curves to a certain non-linear differential system: *Quelle que soit la vitesse initiale, l'avion, après avoir éxecuté s'il y lieu, un nombre fini des loopings, prend un mouvement qui s'approche indéfinement du régime de descente rectiligne et uniforme.*

Carleman's lecture held 1944 at the Academy of Science in Sweden entitled *Sur l'action réciproque entre les mathématiques et les sciences expérimentales exactes* underlines his concern for applications of mathematics.

During his last years in life Carleman suffered from health problems which caused his decease on January 11 1949 at the age of 56 years. A memorial article about his scientific achievement appears in [Acta. Math. 1950] by Fritz Carlson who was Carleman's collegue at the department of mathematics at Stockholm university for several decades. See also his collected work published by Institute Mittag-Leffler in 1960 and the posthumous work *Problémes mathématiques dans la théorie cinétique des gaz* which was printed in 1957 by Institute Mittag Leffler and contains previously unpublished material about the Boltzmann equation. Let us now describe some contributions by Carleman in mathematical physics.

Results about the Bohr-Schrödinger equation.

Carleman's most valuable work is the monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923]. Further applications of this theory than those treated by Carleman in 1923, were proposed in a lecture by Niels Bohr at the 6:th Scandianvian congress in mathematics held at Copenhagen 1925 devoted to the "new-born" quantum mechanics. Recall that one of the fundamental points is the hypothesis on energy levels which correspond to orbits in Bohr's theory of atoms. For an account about the physical background the reader may consult Bohr's plenary talk when he received the Nobel Prize in physics 1923. Mathematically the Bohr-Schrödinger equation is given by:

$$(*) \qquad \Delta\phi + 2m \cdot \big(E - U\big)\big(\frac{2\pi}{h}\big)^2 \cdot \phi = 0$$

where Δ is the Laplace operator in the 3-dimensional (x, y, z)-space, m the mass of a particle and h Planck's constant while $U(x, y, z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Let us cite an excerpt from Carlemans lectures in Paris at Institut Henri Poincaré held in 1930 where he treated (*) and its associated equation (**) below when a time variable enters.

Dans ces dernières années l'ínteret de la question qui nous occupe a considérablement augmenté. C'est en effet un instrument mathématique indispensable pour development de la mechanique moderne crée par M.M. de Brogile, Heisenberg et Schrödinger. Etude de l'équation integrale:

$$\phi(x) = \lambda \cdot \int_a^b K(x, y)\phi(y)dy + f(x) \quad : \lambda \in \mathbf{C} \setminus \mathbf{R}$$

So a basic equation which emerges from quantum mechanics is to find solutions $u(p, t)$ defined in $\mathbf{R}^3 \times \mathbf{R}^+$ where t is a time variable and $p = (x, y, z)$ which satisfies the PDE-equation

$$(**) \qquad i \cdot \frac{\partial u}{\partial t} = \Delta(u)(p, t) - c(p) \cdot u(p, t) = 0 \quad t > 0$$

and the initial condition

$$u(p, 0) = f(p)$$

Here $f(p)$ belongs to $L^2(\mathbf{R}^3)$ and $c(p)$ is a real-valued and locally square integrable function. Carleman proved that the symmetric and densely defined operator $\Delta + c$ has a self-adjoint extension in $L^2(\mathbf{R}^3)$ if

$$(***) \qquad \limsup_{p\to\infty} c(p) \leq M$$

When (***) holds the spectrum of the densely defined sefl-adjoint operator $\Delta - c$ on the Hilbert space $L^2(\mathbf{R}^3)$ is confined to an interval $[\lambda_1, +\infty)$ on the positive real line, i.e. $\lambda_1 > 0$. Applied to the equation (**) Carleman proved that the solution u is given by an equation

$$u(p, t) = \int_{\mathbf{R}^3} \big[\int_{\lambda_1}^\infty e^{i\lambda t} \cdot d\theta(p, q, \lambda)\big] \cdot f(q)\, dq$$

where $\lambda \mapsto \theta(p, q, \lambda)$ is the non-decreasing spectral function associated to $\Delta + c$. Moreover, he found an asymptotic expansion which recaptures the θ-function.

Later hundreds of text-books and many thousands of articles have treated similar equations as above, Various methods for solutions have been employed, one is to associate the diffusion process to $\Delta + c$ and state results by probabilistic terminology. But personally I think Carleman's original proofs give the best insight, and the condition (***) has not been improved in more recent work, i.e. it appears as an "almost necessary condition" in order that $\Delta + c$ is self-adjoint. Concerning the special equation (*) the proof is much simpler and is for exampe decribed in a few lines during Carleman's plenary tak at the IMU-congrss held at Zürich 1932.

A wave equation.

Let Ω be a bounded cdomain in $\mathbf{R}^3$ with a C^1-boundary $\partial\Omega$. We seek functions $u(x,t)$ where $x=(x_1,y_2,x_3)$ defined in $\mathbf{R}^3\setminus\Omega\times\{t\geq 0\}$ where t is a time variable satisfying the wave equation

$$\frac{\partial^2 u}{\partial t^2}=\Delta(u)$$

when $t>0$ and $x\in\mathbf{R}^3\setminus\overline{\Omega}$. The initial conditions when $t=0$ is that

(i) $$u(x,0)=f_1(x) \quad : \frac{\partial u}{\partial t}(x,0)=f_2(x)$$

where f_1,f_2 are C^2-functions in $\mathbf{R}^3\setminus\Omega$ and $\Delta(f_1)$ and $\Delta(f_2)$ are square integrable, i.e.

$$\iiint_{\mathbf{R}^3\setminus\Omega}|\Delta(f_\nu|^2\,dx<\infty$$

Finally the normal derivatives along $\partial\Omega$ satisfy

$$\frac{\partial f_\nu}{\partial n}=0 \quad : \nu=1,2$$

Given such a pair f_1,f_2 there exists a unique solution $u(x,t)$ which satisfies the wave equation above and the two initial conditions (i) together with the boundary value equation

$$\frac{\partial u}{\partial n}(x,t)=0$$

for every $x\in\partial\Omega$ and each $t\geq 0$. Carleman's cited monograph gives an expression of the solution by an integral formula using the spectral measure of a certain from a densely defined and self-adjoint operator on the Hilbert space $L^2(\mathbf{R}^3\setminus\Omega)$. A crucial result during this construction is that the spectral measure is absolutely continuous which Carleman used to consolidate the physcially evident fact that

$$\lim_{x\to\infty}\nabla(u)(x)=0$$

A final comment.

Personally I find few mathematical texts (if any) which superseed the fundamental approach to state and solve problems in Carleman's work. Several of his articles merit a study up to the present date. My personal favourite is the differential inequality exposed in § 6 from chapter 1 which can be regarded as a truly classic result in real analysis. In these notes we refrain from exposing his work devoted to quasi-analytic functions and Bolzmann's kinetic gaz theory. The second chapter is devoted to Carlemans studies of densely defined self-adjoint opeartors on separable Hilbert spaces with special emphasis on the moment problem of Stieltjes and Hamburger. The chapter devoted to PDE-equations contains material which is a bit more "sophisticated compared to the contents under the headline *Results from the world of analysis*. The reader will recognize that the focus is upon inequalities rather than general theoretical results. To this I would like to add a personal remark. While entering "higher studies" in mathematics my opinion is that it is more valuable to pursue details of proofs rather than digesting "general concepts". Let me finish the introduction by a result in this spirit. The theorem below is elementary and yet the proof requires some rather involved steps.

A result about Cesaro limits.

Consider a sequence of real numbers $a_0,a_1,\ldots$ whose associated power series

(i) $$f(x)=\sum_{n=0}^{\infty}a_nx^n$$

converges when $0<x<1$. Thus, for every $\delta>0$ there exists a constant C_δ such that

$$|a_n|\leq C_\delta(1+\delta)^n$$

Suppose that

$$\text{(ii)} \qquad \lim_{x\to 1} f(x) = 0$$

This condition need not imply that the partial sums

$$s_n = a_0 + a_1 + \ldots + a_n$$

converge to zero as $n \to \infty$. An example is to take $a_n = (-1)^n : n = 1, 2, \ldots$ while $a_0 = 1/2$. A sufficient condition in order that (ii) implies that $s_n \to 0$ was proved by Abel in 1824. Namely, if $n \cdot a_n \to 0$ then (ii) entails that $s_n \to 0$. In a joint article from 1913, Hardy and Littelwood extended Abel's theorem and proved that if (ii) holds and there exists a constant C such that

$$\text{(iii)} \qquad \frac{a_n}{n} \leq C \quad : \ n = 1, 2, \ldots$$

then $s_n \to 0$. Notice that one only requires an upper bound in (iii) while the sequence $\{a_n\}$ in general can contain negative terms for which no constraint has been imposed. The proof of the Hardy-Littlewood theorem is given in Chapter 1 from my notes on analytic function theory. Next, with $f(x)$ given in (i) and $r \geq 1$ one can write

$$f(x) = \frac{(1-x)^{r+1}}{\Gamma(r+1)} \cdot \sum_{n=0}^{\infty} s_n^{(r)} \cdot n^r \cdot x^n$$

The sequence $\{s_n^{(r)}\}$ is determined by a recursion formula from $\{a_n\}$ and one refers to $\{s_n^{(r)}\}$ as the arithemetic means of order r. With $r = 0$ we notice that

$$s_n^{(0)} = a_0 + \ldots + a_n$$

Theorem. *Assume (ii) and that there exists some $r \geq 1$ such that*

$$\lim_{n\to\infty} s_n^{(r)} = 0$$

Then it follows that $s_n \to 0$.

This result appears in Carleman's article *Some theorems concerning the convergence of power-series on the circle of convergence* [Arkiv för matematik, astronomi och fysik. Vol 16]. Theorem 0.1 is closely related to the summation procedure introduced by Cesaro in 1880, and the reader may compare Theorem 0.1 with results from Chapter 2 in Landau's text-book *xxxx* which contains a proof that Cesaro's summability in any order is equivalent to that of Hölder of the same order.

Proof of Theorem 0.1. With $r \geq 1$ kept fixed we set

$$g_m(x) = \frac{(1-x)^{r+1}}{\Gamma(r+1)} \cdot \sum_{n=m}^{\infty} s_n^{(r)} \cdot n^r \cdot x^n$$

for each $m \geq 1$. A classic result which goes back to work by Riemann shows that (ii) entails that $s_n \to 0$ if there to each $\epsilon > 0$ exists a positive integers n_0 such that

$$(*) \qquad \Big| \int_0^1 \frac{\sin 2p\pi x}{1-x} \cdot g_{n_0}(x)\, dx \Big| \leq \epsilon \quad : \ p \geq n_o$$

To analyze when Riemann's condition holds we first notice that the triangle inequality implies that the left hand side in (*) is majorised by

$$\sum_{n=n_0}^{\infty} |s_n^{(r)}| \cdot \frac{n^r}{\Gamma(r+1)} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^n (1-x)^r\, dx \Big|$$

Let $\epsilon > 0$ be given. To find n_0) such that (*) holds we consider the integrals

$$J(p, n) = \int_0^1 \sin(2p\pi x) \cdot x^r (1-x)^n\, dx$$

Since the sine function is periodic one has

$$|J(p,n)| = \Big| \int_0^1 \sin(2p\pi x) \cdot x^n (1-x)^r \, dx \,\Big|$$

Exercise. Verify that

$$|J(p,n)| \leq 2\pi \cdot \Gamma(r+2) \cdot \frac{p}{n^{r+2}} \tag{1}$$

and that there exists a constant $C(r)$ which depends on r only such that

$$|J(p,n)| \leq \frac{C(r)}{p \cdot n^r} \tag{2}$$

Above (1) will be used when $n \geq p$ while (2) is used if $n < p$. Next, if $n_0 \geq 1$ the left hand side in (*) is majorised by

$$\frac{1}{\Gamma(r+1)} \cdot \sum_{n=n_0}^{\infty} |s_n^{(r)}| \cdot n^r \cdot |J(p,n)| \tag{3}$$

Put

$$\delta(n_0) = \max_{n \geq n_0} |s_n^{(r)}|$$

By hypothesis $\delta(n_0) \to 0$ as n_0 increases. At the same time we use (1-2) from the exercise and estimate

$$\sum_{n=n_0}^{\infty} n^r \cdot |J(p,n)|$$

for every $p \geq n_0$. First (2) entails that a summation with $n \leq p$ is majorised by the constant $C(r)$. Next, (1) gives

$$\sum_{n=p+1}^{\infty} n^r \cdot |J(p,n) \leq 2\pi \cdot \Gamma(r+2) \cdot p \cdot \sum_{n=p+1}^{\infty} n^{-2} \leq 2\pi \cdot \Gamma(r+2)$$

We conclude that there exists a constant $C^*(r)$ which only depends on r such that (3) is majorised by $C^*(r) \cdot \delta(n_0)$ and now we choose n_0 so large that

$$\delta(n_0) \leq C^*(r)^{-1} \cdot \epsilon$$

which gives (*).

Results from the world of analysis

In § 0-xx we expose results from Carleman's collected work. Tbe proofs are essentially self-contained but some results in analytic function theory of one complex variable, basic Fourier analysis and measure theory are taken for granted. Relevant background is covered in my notes [Björk]. As said in the introduction the main reward for the reader is to pursue details in the proofs, i.e. the methods which are used tend to have greater value than the actual results. For the reader's convenicence we have inserted an appendix dealing with entire functions of exponential type and some Tauberian theorems due to Beurling and Wiener.

§ 0. Two problems in the calculus of variation.

We shall consider two problems with a geometric content which are classic in the sense that they were alreadt treated by Steiner and Weierstrass at an early stage.

§ 0.1 An isoperimetric inequality

Recall that a planar domain whose boundary curve has prescribed length has a maximal area when it it is a disc. It turns out that discs solve a more extensive class of extremal problems. Consider a function $f(r)$ defined for $r > 0$ which is continuous and increasing with $f(0) = 0$. If p and q are two points in $\mathbf{R}^2$ their euclidian distance is denoted by $|p - q|$. When U is a bounded open domain we set

$$J(U) = \iint_{U \times U} f(|p - q|) \cdot dA_p \cdot dA_q$$

where dA_p and dA_q denote area measures. Given a positive number $\mathcal{A}$ one seeks to maximize the J-functional in the family of domains with prescribed area $\mathcal{A}$. The J-number is unchanged under a translation or a rotation of a domain and the family of discs is stable under these operations. So the following result makes sense:

1.Theorem. *The J-functional takes its maximum on discs D of radius r with $\pi r^2 = \mathcal{A}$. Moreover, for every domain U with area $\mathcal{A}$ which is not a disc one has a strict inequality*

$$J(U) < J(D)$$

When $f(r)$ is a strictly convex function Theorem 1 was established by Blaschke. For a general f-function which need not be convex the theorem was proved by Carleman in [Car] using the symmetrisation process by W. Gros from the article (Monatshefte math.physik 1917). In § xx we explain why Theorem 1 leads to another property of discs.

2.Theorem. *Let Ω be a domain in the family $\mathcal{D}(C^1)$ and denote by ds the arc-length measure on its boundary. Then, if the function*

$$p \mapsto \int_{\partial\Omega} f(|p - q|) \cdot ds(q)$$

is constant as p varies in $\partial\Omega$ it follows that Ω is a disc.

Remark. Theorem 1 can be extended to any dimension $n \geq 3$ using successive symmetrisations of domains taken in different directions converge to the unit ball in $\mathbf{R}^n$. Here discs are replaced by $n - 1$-dimensional spheres.

2. A variational inequality

We first establish some inequalties which will be used in § 3 to finish the proof of Theorem 1. Let $M > 0$ and on the vertical lines $\{x = 0\}$ and $\{x = M\}$ we consider two subsets G_* and G^* which both consist of a finite union of closed intervals. Let $\{[a_\nu, b_\nu]\}$ be the G_*-intervals taken in the y-coordinates and $\{[c_j, d_j]\}$ are the G^*-intervals. Here $a_\nu < b_\nu < a_{\nu+1}$ holds, and similarly the G^*-intervals are ordered with increasing y-coordinates. The number of intervals of the two sets are arbitrary and need not be the same. Given $f(r)$ as in the Theorem 1 we set

$$I(G_*, G^*) = \sum_\nu \sum_j \int_{a_\nu}^{b_\nu} \int_{c_j}^{d_j} f(|y - y'|) \cdot dy dy'$$

Consider the variational problem where we seek to minimize these I-integrals for pairs (G_*, G^*) as above under the constraints:

$$\sum (b_\nu - a_\nu) = \ell_* \quad \text{and} \quad \sum (d_j - c_j) = \ell^*$$

That is, the sum of the lengths of the intervals are prescribed on G_* and G^*.

2.1 Proposition. *For every pair (ℓ_*, ℓ^*) the I-integral is minimized when both G_* and G^* consist of a single interval and the mid-points of the two intervals have equal y-coordinate.*

Proof. First we prove the result when both $G_* = (a, b)$ and $G^* = (c, d)$ both are intervals. We must prove that the I-integral is a minimum when

(i) $$\frac{a+b}{2} = \frac{c+d}{2}$$

Suppose that inequality holds. Since the I-integral is symmetric with respect to the pair of intervals, we may assume that

$$\frac{c+d}{2} = s + \frac{a+b}{2} \quad \text{where} \quad s > 0$$

Now $I(G_*, G^*)$ is unchanged when we translate the two intervals, i.e. if we for some number ξ take $(a+\xi, b+\xi)$ and $(c+\xi, d+\xi)$. By such a translation we can assume that $a = -b$ so the mid-point of G_* becomes $y = 0$ and we have:

$$I = \int_{-b}^{b} \int_{c}^{d} f(\sqrt{M^2 + (y-y')^2}) \cdot dy dy'$$

Using the variable substitutions $u = y' - y$ and $v = y' + y$ we see that

$$-b + c \leq u \leq d + b$$

and obtain

$$I = 2b \int_{-b+c}^{d+b} f(\sqrt{M^2 + v^2}) \cdot dv$$

With

$$s = d - \frac{d+c}{2} = \frac{d-c}{2}$$

we can write

$$I = 2b \cdot \int_{w-s}^{w+s} f(\sqrt{M^2 + u^2}) \cdot dv \quad : w = b + \frac{d+c}{2}$$

The last integral is a function of s, i.e. for every $s \geq 0$ we set

$$\Phi(s) = 2b \cdot \int_{w-s}^{w+s} f(\sqrt{M^2 + u^2}) \cdot dv \quad : w = b + \frac{d+c}{2}$$

The derivative of s becomes

$$\Phi'(s) = f(\sqrt{M^2 + (w+s)^2}) - f(\sqrt{M^2 + (w-s)^2})$$

Since $f(r)$ was increasing the derivative is > 0 when $s > 0$. Hence the minimum is achieved when $s = 0$ which means that G_* and G^* have a common mid-point and Proposition 2.1 is proved for the case of an interval pair.

The general case. If $G_* = \{(a_\nu, b_\nu)\}$ and $G^* = \{(c_k, d_k)\}$ we make an induction over the total number of intervals which appear in the two families. Let

$$\xi^* = \frac{c^* + d^*}{2}$$

be the largest mid-point from the G^*-family which means that k is maximal, In the G_*-family we also get the largest mid-point is

$$\eta^* = \frac{a^* + b^*}{2}$$

If $\xi^* > \eta^*$ the previous case shows that the double sum representing I decreases as long as when the interval (c^*, d^*) is lowered. In this process two cases can occur: First, suppose that the lowered

(c^*,d^*)-interval hits (c_{k-1},d_{k-1}) before the mid-point equality appears. To be precise, this occurs if

$$c^*-d_{k-1}<\xi^*-\eta^*$$

In this case we replace G^* by a union of intervals where the number of intervals therefore has decreased by one and we lower (c^*,d^*) until $\xi^*=\eta^*$. After this we lower the two top-intervals at the same time until one of them hits the second largest G-interval and in this way the total number of intervals is decreased while the double sum for I is not enlarged. This gives the requested induction step and the proof of Proposition 2.1 is finished.

3. Proof of Theorem 1.

Consider a domain U defined by

$$g_1(x)\le y\le g_2(x) \quad : \quad a\le x\le b \tag{1}$$

where $g_1(a)=g_2(a)$ and $g_1(b)=g_2(b)$. To U we associate the symmetric domain U^* defined by

$$-\frac{1}{2}\big[g_2(x)-g_1(x)\big]\le y\le \frac{1}{2}\big[g_2(x)-g_1(x)\big] \quad : \quad a\le x\le b \tag{2}$$

Notice that U and U^* have the same area. Set

$$J=\iint_{U\times U} f\big(\sqrt{(x-x')^2+(y-y')^2}\big)\cdot dxdx'dydy'$$

$$J^*=\iint_{U^*\times U^*} f\big(\sqrt{(x-x')^2+(y-y')^2}\big)\cdot dxdx'dydy'$$

Lemma 3.1. *One has the inequality $J\le J^*$.*

Proof. Set $h(x)=\frac{1}{2}[g_2(x)-g_1(x)]$ and introduce the function

$$H^*(x,x')=\int_{y=-h(x)}^{h(x)}\int_{y'=-h(x')}^{h(x')}\rho\big(\sqrt{(x-x')^2+(y-y')^2}\big)\cdot dydy'$$

We have also the function

$$H(x,x')=\int_{y=g_1(x)}^{g_2(x)}\int_{y'=g_1(x')}^{g_2(x')} f\,\big(\sqrt{(x-x')^2+(y-y')^2}\big)\cdot dydy'$$

It is clear that

$$J=\int_a^b\int_a^b H(x,x')dxdx' \quad \text{and} \quad J^*=\int_a^b\int_a^b H^*(x,x')dxdx'$$

Lemma 3.1 follows if we have proved the inequality

$$H(x,x')\le H^*(x,x') \tag{*}$$

for all pairs x,x' in $[a,b]$. But this follows via Fubini's theorem when Proposition 2.1 applied in the special case where G_* and G^* both consist of a single interval.

3.2 Variation of convex sets. Let $\mathcal{A}$ be the prescribed area in Theorem 1 and consider a convex domain U whose area is $\mathcal{A}$. By elementary geometry we see that after a translation and a rotation the convex domain U can be represented as in(1) above. We construct U_* as above and notice that it is a new convex domain. Moreover, Proposition 2.1 gives $J(U_*)\le J(U)$. In the next step we perform a symmetrisation of U_* along some other line which cuts U_* to get a new domain U_{**} where we now have $J(U_{**})\le J(U_*)\le J(U)$. Finally we use the geometric result due to Steiner for convex domains which asserts that when symmetrisations as above are repeated infinitely often while the angles of the directions to the x-axis change with some irrational multiple of 2π, then the resulting sequence of convex domains converge to a disc. This proves that the J-functional on a disc is $\leq J(U)$ for every convex domain.

3.3 The non-convex case Here we use the symmetrisation process by Gros. Let U be a domain. Its symmetrisation in the x-direction is defined as follows: To every x we get the open set

$$\ell_U(x) = \{y : (x,y) \in U\} \tag{1}$$

Let $\{(a_\nu, b_\nu)\}$ be the disjoint intervals of $\ell_U(x)$ and put

$$d(x) = \frac{1}{2} \cdot \sum (b_\nu - a_\nu)$$

We get the domain U^* which is symmetric with respect to the x-axis where

$$\ell_{U^*}(x) = (-d(x), d(x))$$

Notice that U and U^* have equal area. Proposition 2.1 applies and gives the inequality

$$J(U) \leq J(U^*) \tag{2}$$

Now Theorem 1 follows when we start from a non-convex domain U. Namely, by the result proved in [Gros], it holds that after inifinitely many symmetrizations as above using different directions, the sequence of U-sets converge to a disc.

§ 0.2 On minimal surfaces.

We shall consider an isoperimetric problem with a fixed boundary curve. More involved situations arise when the minimal surfaces are bordered by closed Jordan curves which are free to move on prescribed manifolds. This leads to problems by Plateau and Douglas and for an account about this general case we refer to Courant's article *The existence of minimal surfaces of given toplogical structure under prescribed boundary conditions.* (Acta. Math. Vol 72 1940]) where the reader also finds an extensive references to relevant literature.

From now on we discuss the restricted problem when a boundary curve is fixed in $\mathbf{R}^3$ with coordinates (x, y, z). Consider a rectifiable closed Jordan curve C and denote by $\mathcal{S}(C)$ the family of surfaces which are bordered by C. A surface M in this family is minimal if it has smallest possible area. To find such a minimal surface corresponds to a problem in the calculus of variation and was studied by Weierstrass in a series of articles starting from *Untersuchungen über die Flächen deren mittlere Krümmung überall gleich null ist* from 1866. A revised version written by Weierstrass himself appears in volume I of his collected work. He proved that if M is a minimal surface in $\mathcal{S}(C)$ then its mean curvature vanishes identically. Moreover, M has no singular points and is simply connected. More precisely, the exists a homeomorphic parametrization of M above an open disc in the complex u-plane which can be achieved via complex analytic functions, or as expressed by Weierstrass in the introduction to [Wei]:

Ich habe mich mit der Theorie die Flächen, deren mittlere Krümmung überall gleich null ist, besonders auf dem grunde eingehender beschäftigt, weil sie, wie ich zeigen werde, auf das Innigste mit der Theorie der analytischen funktionen einer komplexen Argumentz zusammenhhängt. Or shortly phrased: *The theory about minimal surfaces is closely linked to the theory of analytic functions in one complex variable.*

The isoperimetric inequality. Using Weierstrass' description of minimal surfaces the following was proved by Carleman in the article *Zur Theorie der Minimalflächen* in 1920:

Theorem. *For every rectifiable simple closed curve C the area A of the minimal surface in $\mathcal{S}(C)$ satisfies the inequality*

$$A \leq \frac{\ell(C)^2}{4\pi}$$

where $\ell(C)$ is the arc-length of C.

Remark. For historic reasons one may wonder why this result was not already discovered by Weierstrass. The reason might be that certain facts in analytic function theory was not yet enough developed. Carleaman's proof relies upon the Jensen-Blascke factorisation of analytic functions which was not know prior to 1900. Another obstacle was the discovery by Hermann Schwarz that the minimal surface in the family $\mathcal{S}(C)$ is not determined by vanishing mean curvarute alone. See Volume II, page 264 and 151-167 in the collected work of Hermann Schwarz for this "ugly phenomenon" which was one reason why Weierstrass paid much attention to existence problems in the calculus of variation. As remarked by Carleman at the end of his article, an alternative (and simpler) proof was given by Blaschke after the publication of [Carleman]. However, this proof is restricted to a special class of minimal surfaces where the "ugly phenomena" do not occur so here we rely upon Weierstrass' original parametristions which lead to a proof the the theorem above.

The case when C is piecewise linear. Suppose that the boundary curve consists of n many line segements $L_1, \ldots, L_n$. Following Weierstrass it means that one regards the problem: *Es soll ein einfach zusamenhängenden Minimalflächenstück M analytish bestimmt werden, dessen vorgeschiebenen begrenzung C aus n geradlinigen strecken bestecht, welche eine einfache, geschlossene, nicht verknotete Linie bilden.*

In [Weierstrass] appears a far reaching study of this problem. The main result shows that M is determined via a pair of analytic functions $G(u)$ and $H(u)$ defined in the lower half-plane $\mathfrak{Im}\, u < 0$ for which the three functions defined by

$$\phi_1(u) = \det \begin{pmatrix} G(u) & H'(u) \\ G'(u) & H'(u) \end{pmatrix}$$

$$\phi_2(u) = \det \begin{pmatrix} G(u) & H'(u) \\ G''(u) & H''(u) \end{pmatrix}$$

$$\phi_3(u) = \det \begin{pmatrix} G'(u) & H'(u) \\ G''(u) & H''(u) \end{pmatrix}$$

become rational functions of u. Moreover, [ibid] exhibits second order differential equations of the Fuchsian type satisfied by the rational ϕ-functions and the position of their poles are described in terms of the geometric configuration of C. It would lead us too far to enter the material in [Weierstrass] so its rich contents is left to the interested reader for further studies.

The planar case. If C is a simple closed curve in the complex z-plane the isoperimetric inequality follows easily via analytic function theory. Namely, let M be the Jordan domain bordered by C. By Riemann's theorem there exists a conformal mapping $\phi\colon D \to M$ and we have

$$\ell(C) = \int_0^{2\pi} |\phi'(e^{i\theta})|\, d\theta \quad : \quad \text{area}(M) = \iint_D |\phi'(z)|^2\, dxdy$$

Hence the isoperimetric inequality for planar domains boils down to show that

$$\text{(i)} \qquad 4\pi \cdot \iint_D |\phi'(z)|^2\, dxdy \leq \Big(\int_0^{2\pi} |\phi'(e^{i\theta})|\, d\theta \Big)^2$$

To prove (i) we use that the derivative $\phi'(z)$ is zero-free and hence it has a suingle-valued square root $f = \sqrt{\phi}$. We have a series expansion

$$f(z) = a_1 z + a_2 z^2 + \ldots$$

The right hand side in (i) becomes

$$4\pi^2 \cdot \Big(\int_0^{2\pi} |\sum a_\nu e^{i\nu\theta}|^2\, d\theta \Big)^2 = 4\pi^2 \cdot (\sum |a_\nu|^2)^2$$

The left hand side becomes

$$4\pi \cdot \iint_D (\sum |a_\nu z^\nu|)^4\, dxdy$$

Set

$$b_m = \sum_{\nu=1}^{\nu=m} a_\nu \cdot a_{m-\nu} \quad : \ m \geq 2$$

Then (xx) becomes

$$4\pi \cdot \iint_D \sum |b_m z^m|)^2\, dxdy = 4\pi \sum |b_m|^2 \cdot 2\pi \cdot \iint_D r^{2m+1}\, dr = 8\pi^2 \cdot \sum \frac{|b_m|^2}{2m+2}$$

Hence (i) follows if

(ii) $$\sum_{m=2}^{\infty} \frac{|b_m|^2}{m+1} \leq \sum_{\nu=1}^{\infty} |a_\nu|^2)^2$$

At this stage we leave it to the reader to verify the planar isoperimetric inequality in Theorem 1 and that equality holds if and only $\phi(z)$ is such that the complex derivative takes the form

$$\phi'(z) = \frac{a}{(1-qz)^2}$$

for a pair of constants a, b. This means that ϕ is is a Möbius transform and hence C must be a circle, i.e. equality in Theorem 1 for a planar curve holds if and only if C borders a disc,

B. Proof of Theorem 1.

The crucial step in the proof relies upon the following result which is due to Weierstrass:

B.1 Proposition. *Let M be a minimal surface in $\mathcal{S}(C)$. Then there exists an analytic function $F(u)$ in the open unit disc such that points $(x, y, z) \in M$ are given by the equations:*

$$x = \mathfrak{Re} \int (1-u^2)F(u)\, du \quad : \quad y = \mathfrak{Re} \int i(1+u^2)F(u)\, du \quad : \quad z = \mathfrak{Re} \int 2F(u)\, du$$

The proof of this result occupies five pages in [Weierstrass]. We remark that he employed Riemann's mapping theoren for simply connected domains during the proof. Let us indicate some details. To begin with Weierstrass proved that there exists a planar domain Σ with real coordinates (p, q) and a diffeomorphism between M and Σ which is conformal, i.e. M is defined by the equations

(i) $$x = x(p, q) \quad : y = y(p, q) \quad : z = z(p, q)$$

where the vectors $(\frac{\partial x}{\partial p}, (\frac{\partial y}{\partial p}, (\frac{\partial x}{\partial p},)$ and $(\frac{\partial x}{\partial q}, (\frac{\partial y}{\partial q}, (\frac{\partial z}{\partial q})$ are pairwise orthoginal unit vectors. Moreover, when M is minimal the mean curvature of M vanishes which means that the three functions in (1) are harmonic, i.e.

(ii) $$\Delta(x) = \frac{\partial^2 x}{\partial p^2} + \frac{\partial^2 x}{\partial q^2} = 0$$

and similarly for y and z. Next, the harmonic functions above are real parts of analytic functions which yields a triple f, g, h in $\mathcal{O}(\Sigma)$ such that

$$x = \mathfrak{Re}\, f(u)$$

The orthogonality of the vectors $\mathbf{v}$ and $\mathbf{w}$ above entails via the Cauchy Riemann equations that

$$(f'(u))^2 + (g'(u))^2 + (h'(u))^2 = 0$$

Starting from this, Weierstrass used sterographic projections and Riemann's conformal mapping theorem to construct an analytic function $F(u)$ which gives the equations in Proposition B.1. Admitting Weierstrass' result the following hold:

B.2 Propostion. *When the minimal surface M is parametrized as in Proposition B.1 one has the equations*

$$\text{area}(M) = \iint_D (1+|u|^2)^2 \cdot |F(u)|^2\, d\xi d\eta \quad : \quad \ell(C) = 2 \cdot \int_0^{2\pi} |F(e^{i\theta})|\, d\theta$$

Proof. With $u = \alpha + i\beta$ this amounts to show that

(i) $$dx^2 + dy^2 + dz^2 = (1 + |u|^2)|F(u)|^2 \cdot (d\alpha^2 + d\beta^2)$$

To prove (i) we consider some point $u \in D$. Set $F(u) = |F(u)| \cdot e^{i\theta}$ and $u = se^{i\alpha}$. With $du = d\alpha$ real we have

$$dx = \mathfrak{Re}(1 - u^2)F(u)) \cdot d\alpha = |F(u)| \cdot \big(\cos\theta - |u|^2 \cos\theta \cdot \cos 2\alpha - |u|^2 \sin\theta \cdot \sin 2\alpha\big) \cdot d\alpha$$

Trigonometric formulas give

(i) $$(dx)^2 = |F(u)| \cdot \big[\cos^2\theta + |u|^4 \cos^2(2\alpha - \theta) - 2|u|^2 \cos\theta \cdot \cos(2\alpha - \theta)\,\big] \cdot (d\alpha)^2$$

(ii) $$(dy)^2 = |F(u)|^2 \cdot \big[\sin^2\theta + |u|^4 \sin^2(2\alpha - \theta) + 2|u|^2 \sin\theta \cdot \sin(2\alpha - \theta)\big] \cdot d\alpha$$

(iii) $$(dz)^2 = 4|F(u)|^2 \cdot |u|^2 \big(\cos^2(\theta - \alpha)\big] \cdot (d\alpha)^2$$

Adding (i-ii) we get

$$(dx)^2 + (dy)^2 = |F(u)|^2 \cdot \big[1 + |u|^4 - 2 \cdot |u|^2 \cos(2\theta - 2\alpha)\big] \cdot (d\alpha)^2$$

Finally, the trigonometric formula

$$4\cos^2\phi = 2 - 2\cos 2\phi$$

and (iii) entail that

(iv) $$(dx)^2 + (dy)^2 + (dz)^2 = |F(u)|^2 \cdot (1 + |u|^2)^2 \cdot (d\alpha)^2$$

The same infinitesmal equality as in (iv) is proved when $u = id\beta$ for some small real β and then we can read off Proposition B.2.

Final part of the proof

Put $f_1(u) = F(u)u^2$ and $f_2(u) = F(u)$. Proposition B.2 gives

(i) $$\text{area}(M) = \iint_D \big(\,|f_1(u)|^2 + |f_2(u)|^2\,\big)\, d\xi d\eta + 2 \cdot \iint_D |f_1(u)| \cdot |f_2(u)|\, d\xi d\eta$$

Since $|f_1| = |f_2| = |F|$ holds on the unit circle we also get

(ii) $$\ell(C)^2 = \big[\int_0^{2\pi} |f_1(e^{i\theta})|\, d\theta\big]^2 + \big[\int_0^{2\pi} |f_2(e^{i\theta})|\, d\theta\big]^2 + 2 \cdot \int_0^{2\pi} |f_1(e^{i\theta})|\, d\theta \cdot \int_0^{2\pi} |f_2(e^{i\theta})|\, d\theta$$

Using (i-ii) Carleman derived the isoperimetric inequality from the following:

B.3 Lemma. *For each pair of analytic functions g, h in the unit disc one has*

$$\iint_D \big[g(u)| \cdot |h(u)|\, d\xi d\eta \leq \frac{1}{4\pi} \cdot \int_0^{2\pi} |g(e^{i\theta})|\, d\theta \cdot \int_0^{2\pi} |h(e^{i\theta})|\, d\theta$$

Let us first notice that Lemma B.3 applied to the pairs $g = h = f_1$, $g = h = f_2$ and the pair $g = f_1$ and $h = f_2$ together with (i-ii) give Theorem 1. So there remains to prove Lemma B.3. We can write

$$g = B_1 \cdot g^* \quad : \quad h = B_2 \cdot h^*$$

where B_1, B_2 are Blaschke products and the analytic functions g^* and h^* are zero free in the unit disc. Since $|B_1| = |B_2| = 1$ hold on the unit circle it suffices to prove Lemma B.2 for the pair g^*, h^*, i.e. we may assume that both g and h are zero-free. Then they posses square roots so we can find analytic functions G, H in the unit disc where

$$g = G^2 \quad : \quad h = H^2$$

Consider the Taylor series

$$G(z) = \sum a_k u^k \quad : \quad H(z) = \sum b_k u^k$$

Now $GH = \sum c_k u^k$ where

(i) $$c_k = a_0 b_k + \ldots + a_k b_0$$

Using polar coordintes to perform double integrals it follows that

$$\iint_D |G^2(u)| \cdot |H^2(u)| \, d\xi d\eta = \pi \cdot \sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1}$$

At the same time one has

$$\int_0^{2\pi} |G^2(e^{i\theta})| \, d\theta = 2\pi \cdot \sum_{k=0}^{\infty} |a_k|^2$$

with a similar formula for the integral of H^2. Hence Lemma B.2 follows if we have proved the inequality

(ii) $$\sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1} \leq \sum_{k=0}^{\infty} |a_k|^2 \cdot \sum_{k=0}^{\infty} |b_k|^2$$

To get (ii) we use (i) which for every k gives:

$$|c_k|^2 \leq (|a_0||b_k| + \ldots + |a_k||b_0|)^2 \leq (k+1) \cdot (|a_0|^2|b_k|^2 + \ldots + |a_k|^2|b_0|^2)$$

Finally, a summation over k entails (ii) and Lemma B.3 is proved.

§ 1. An inversion formula.

A fundamental result due to Abel gives an inversion formula for the potential function $U(x)$ in a conservative field of forces. More precisely, let $U(x)$ be an even function of x with $U(0) = 0$ and $x \to U(x)$ is stricitly increasing and convex on $x \geq 0$. A particle of unit mass which moves on the real x-line satisfies Newton's equation

$$\ddot{x}(t) = -U'(x(t))$$

where the initial conditions are $x(0) = 0$ and $\dot{x}(0) = v > 0$. It follows that

$$\frac{\dot{x}(t)^2}{2} + U(x(t)) = \frac{v^2}{2} \implies \dot{x}(t) = \sqrt{v^2 - 2U(x(t))} \tag{1}$$

Here (1) holds during a time interval $[0, T]$ where $\dot{x}(t) > 0$ when $0 \leq t < T$ and $\dot{x}(T) = 0$. From (1) we get the equation

$$T = \int_0^{x(T)} \frac{dx}{\sqrt{v^2 - 2U(x)}} \quad : \; 2 \cdot U(x(T)) = v^2$$

With U given this means that $T = T(v)$ is a function of v. In a work from 1824, Abel established an inversion formula which recaptures U when the function $v \mapsto T(v)$ is known. The reader should consult the literature for this famous and very important result.

In the article *Abelsche Intergalgleichung mot konstanten Integralgrenzen* [Mathematische Zeitschrift 1922], Carleman established inversion formulas in Abel's spirit where one example is as follows: For every fixed real $0 \leq x \leq 1$

$$t \mapsto \log |x - t|$$

is integrable on the unit interval $[0, 1]$ and yields a bounded linear operator on the Banach space $C^0[0, 1]$ sending every $g \in C^0[0, 1]$ to

$$T_g(x) = \int_0^1 \log |x - t| \cdot g(t) \, dt$$

It is not difficult to show that T_g is injective. less obvious is the inversion formula below which at the same time gives a description of the range of T.

0.1 Theorem. *With $f = T_g$ one has the inversion formula*

$$\sqrt{x(1-x)} \cdot g(x) = \frac{1}{\pi^2} \cdot \int_0^1 \frac{f'(t) \cdot \sqrt{t(1-t)}}{x - t} \, dt + \frac{1}{\pi} \cdot \int_0^1 g(t) \, dt \tag{*}$$

Remark. In (*) the first order derivative $f'(t)$ appears in an integral where we have taken a principal value. The inversion formula therefore shows that a function f in the range of T must satisfy certain regularity properties.

Proof of Theorem 0.1. For each $0 \leq t \leq 1$. the complex log-function

$$z \mapsto \log (z - t)$$

is defined when $z \in \mathbf{C} \setminus (-\infty, 1]$ where single-valued branches are chosen so that the argument of these log-functions stay in $(-\pi, \pi)$ and $\log x - t$ is real if $x > t$. It follows that

$$\lim_{\epsilon \to 0} \log(x + i\epsilon - t) = \log |x - t| + \pi i \quad : x < t \tag{1}$$

where the limit is taken as $\epsilon > 0$ decrease to zero. Let $g(t)$ be a continuous function on $[0, 1]$ and put

$$G(z) = \int_0^1 \log(z - t) \cdot g(t) \, dt \tag{2}$$

A. Exercise. Show that (1) gives:

(i) $$G(x+i0) = T_g(x) + \pi i \cdot \int_x^1 g(t)\,dt \quad : 0 < x < 1$$

where $G(x+i0)$ is the limit as $z = x + i\epsilon$ and $\epsilon > 0$ decrease to zero. Show in a similar way that

(ii) $$G(x-i0) = T_g(x) - \pi i \cdot \int_x^1 g(t)\,dt \quad : 0 < x < 1$$

Next, outside $[0,1]$ the complex derivative of G becomes

$$G'(z) = \int_0^1 \frac{g(t)}{z-t}\,dt$$

Then (i-ii) give

(iii) $$G'(x+i0) + G'(x-i0) = 2 \cdot \frac{dT_g(x)}{dx} \quad : \quad G'(x+i0) - G'(x-i0) = -2\pi \cdot g(x)$$

B. The Φ-function. In $\mathbf{C} \setminus [0,1]$ we have the analytic function $h(z) = \sqrt{z(z-1)}$ whose branch is chosen so that it is real and positive when $z = x > 1$. It follows that

(b.1) $$h(x+i0) = i \cdot \sqrt{x(1-x)} \quad : \quad h(x-i0) = i \cdot \sqrt{x(1-x)} \quad : 0 < x < 1$$

Consider the analytic function

$$\Phi(z) = \sqrt{z(z-1)} \cdot G'(z)$$

With $f = T_g$ we see that (iii-iv) give the two equations

(b.2) $$\Phi(x+i0) + \Phi(x-i0) = 2\pi \cdot \sqrt{x(1-x)} \cdot g(x)$$

(b.3) $$\Phi(x+i0) - \Phi(x-i0) = 2i \cdot f'(x) \cdot \sqrt{x(1-x)}$$

C. The Ψ-function. Set

(c.1) $$\Psi(z) = \int_0^1 \frac{1}{z-t} \cdot f'(t) \cdot \sqrt{t(1-t)}\,dt$$

The equation (b.3) and the general formula from § XX give

(c.2) $$\Phi(x+i0) - \Phi(x-i0) = \Psi(x+i0) - \Psi(x-i0) \quad : \ 0 < x < 1$$

D. Exercise. Deduce from (c.2) that

$$\Phi(z) = \Psi(z) + \int_0^1 g(t)\,dt$$

where the equality holds when $z \in \mathbf{C} \setminus (-\infty, 1]$ and conclude from the above that

(d.1) $$2\pi \cdot \sqrt{x(1-x)} \cdot g(x) = \Psi(x+i0) + \Psi(x-i0) + 2 \cdot \int_0^1 g(t)\,dt$$

Next, (c.1) and the general formula in § XX give

(d.2) $$\Psi(x+i0) + \Psi(x-i0) = \frac{2}{\pi} \cdot \int_0^1 \frac{f'(t) \cdot \sqrt{t(1-t)}}{x-t}\,dt$$

where the last integral is taken as a principal value. Together (d.1-2) give (*) in Theorem 0.1.

The proof will show that one also has the equation

(**) $$\int_0^1 g(t)\,dt = -\frac{1}{2\pi \cdot \log 2} \cdot \int_0^1 \frac{f(x)}{\sqrt{(1-x)x}}\,dx$$

2. An extension of Weierstrass' approximation theorem.

The result below was proved in the article *Sur un théorème de Weierstrass* [Arkiv för mathematik och fysik. vol 20 (1927]:

Theorem. *Let f be a continuous and complex valued function on the real x-line. To each $\epsilon > 0$ there exists an entire function $\phi(z) = \phi(x + iy)$ such that*

$$\max_{x \in \mathbf{R}} |f(x) - \phi(x)| < \epsilon$$

Carleman gave an elementary proof using Cauchys integral formula. But his constructions can be extended to cover a more general situation which goes as follows. Let K be an unbounded closed null-set in $\mathbf{C}$. If $0 < R < R^*$ we put

$$K[R, R^*] = K \cap \{R \leq |z| \leq R^*\}$$

and if $R > 0$ we put $K_R = K \cap \bar{D}_R$ where $\bar{D}_R = \{|z| \leq R\}$.

2.1 Theorem. *Suppose there exists a strictly increasing sequence $\{R_\nu\}$ where $R_\nu \to +\infty$ such that $\mathbf{C} \setminus K_{R_1}$ and the sets*

$$\Omega_\nu = \mathbf{C} \setminus \bar{D}_{R_\nu} \cup K[R_\nu, R_{\nu+1}]$$

are connected for each $\nu \geq 1$. Then every continuous function on K can be uniformly approximated by entire functions.

To prove this result we first establish the following.

2.2 Lemma. *Consider some $\nu \geq 1$ a continuous function ψ on $S = \bar{D}_{R_n} \cup K[R_\nu, R_{\nu+1}]$ where ψ is analytic in the open disc D_{R_ν}. Then ψ can be uniformly approximated on S by polynomials in z.*

Proof. If we have found a sequence of polynomials $\{p_k\}$ which approximate ψ uniformly on $S_* = \{|z| = R_\nu\} \cup K[R_\nu, R_{\nu+1}]$ then this sequence approximates ψ on S. In fact, this follows since ψ is analytic in the disc D_{R_ν} so by the maximum principle for analytic functions in a disc we have

$$||p_k - \psi||_S = ||p_k - \psi||_{S_*}$$

for each k. Next, if uniform approximation on S_* fails there exists a Riesz-measure μ supported by S_* which is $\perp$ to all analytic polynomials while

$$\int \psi \cdot d\mu \neq 0 \tag{1}$$

To see that this cannot occur we consider the Cauchy transform

$$\mathcal{C}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

Since $\int \zeta^n \cdot d\mu(\zeta) = 0$ for every $n \geq 0$ we see that $\mathcal{C}(z) = 0$ in the exterior disc $|z| > R_{\nu+1}$. The connectivity hypothesis implies that $\mathcal{C}(z) = 0$ in the whole open complement of S. Now K was a null set which means that the L^1_{loc}-function $\mathcal{C}(z)$ is zero in the exterior disc $|z| > R_\nu$ and hence its distribution derivative $\bar{\partial}(\mathcal{C}_\nu)$ also vanishes in this exterior disc. At the same time we have the equality

$$\bar{\partial}(C_\nu) = \mu$$

We conclude that the support of μ is confined to the circle $\{|z| = R_\nu\}$. But then (1) cannot hold since the restriction of ψ to this circle by assumption extends to be analytic in the disc D_{R_ν} and therefore can be uniformly approximated by polynomials on the circle.

Proof of Theorem 2.1. Let $\epsilon > 0$ and $\{\alpha_\nu\}$ is a sequence of positive numbers such that $\sum \alpha_\nu < \epsilon$. Consider some $f \in C^0(K)$. Starting with the set K_{R_1} we use the assumption that its complement is connected and using Cauchy transforms as in Lemma A.3 one shows that the restriction of f to this compact set can be uniformly approximated by polynomials. So we find $P_1(z)$ such that

$$||P_1 - f||_{K_{R_1}} < \alpha_1 \tag{i}$$

From (i) one easily construct a continuous function ψ on $\bar{D}_{R_1} \cup K[R_1, R_2]$ such that $\psi = P_1$ holds in the disc $\bar{D}_{R_1}$ and the maximum norm

$$||\psi - f||_{K[R_1,R_2]} \leq \alpha_1$$

Lemma A.3 gives a polynomial P_2 such that

$$||P_2 - P_1||_{D_{R_1}} < \alpha_2 \quad \text{and} \quad ||P_2 - f||_{K[R_1,R_2]} \leq \alpha_1 + \alpha_2$$

Repeat the construction where Lemma A.3 is used as ν increases. This gives a sequence of polynomials $\{P_\nu\}$ such that

$$||P_\nu - P_{\nu-1}||_{D_{R_\nu}} < \alpha_\nu \quad \text{and} \quad ||P_\nu - f||_{K[R_{\nu-1},R_\nu]} < \alpha_1 + \ldots + \alpha_\nu$$

hold for all ν. From this it is easily seen that we obtain an entire function

$$P^*(z) = P_1(z) + \sum_{\nu=1}^{\infty} P_{\nu+1}(z) - P_\nu(z)$$

Finally the reader can check that the inequalities above imply that the maximum norm

$$||P^* - f||_K \leq \alpha_1 + \sum_{\nu=1}^{\infty} \alpha_\nu$$

Since the last sum is $\leq 2\epsilon$ and $\epsilon > 0$ was arbitrary we have proved Theorem A.3.

2.3 Exercise. Use similar methods as above to show that if $f(z)$ is analytic in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$ and has continuous boundary values on the real line, then f can be uniformly approximated by an entire function, i.e. to every $\epsilon > 0$ there exists an entire function $F(z)$ such that

$$\max_{z \in U^+} |F(z) - f(z)| \leq \epsilon$$

3. An inequality for differentiable functions.

The result below was proved in the article *Sur un théorème de M. Denjoy* [C.R. Acad. Sci. Paris 1922]

3.1 Theorem. *There exists an absolute constant $\mathcal{C}$ such that the inequality below holds for every pair (f,n), where n is a positive integer and f a non-negative real-valued C^∞- function defined on the closed unit interval $[0,1]$ whose derivatives up to order n vanish at the two end points.*

$$(*) \qquad \sum_{\nu=1}^{\nu=n} \frac{1}{[\beta_\nu]^{\frac{1}{\nu}}} \leq \mathcal{C} \cdot \int_0^1 f(x)dx \quad : \quad \beta_\nu = \sqrt{\int_0^1 [f^{(\nu)}(x)]^2 \cdot dx}$$

Remark. The proof below shows that one can take

$$(*) \qquad C \leq 2e\pi \cdot (1 + \frac{1}{4\pi^2 e^2 - 1})$$

The best constant $\mathcal{C}_*$ which would give

$$\text{(i)} \qquad \sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \leq \mathcal{C}^*$$

for all pairs (f,n) is not known. Let us also remark that the inequality (*) is sharp in the sense that there exists a constant C_* such that for every $n \geq 2$ there exists a function $f_n(x)$ as above so that the opposed inequality (*) holds with C_*. Hence (*) demonstrates that the standard cut-off functions which are used in many applications to keep maximum norms of derivatives small up to order n small, are optimal up to a constant. So the theoretical result in (*) plays a role in numerical analysis where one often uses smoothing methods. The proof of (*) employs estimates for harmonic measures applied to the subharmonic Log-function of the absolute value of the Laplace transform of f, i.e. via a "detour into the complex domain" which in 1922 appeared as a "revolutionary method".

Proof of Theorem 3.1.

Let $n \geq 1$ and keeping f fixed we put $\beta_p = \beta_p(f)$ to simplify notations. Using partial integrations and the Cauchy-Schwarz inequality one shows that the β-numbers are non-decreasing. i.e.

$$(*) \qquad 1 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_{n+1}$$

Define the complex Laplace transform

$$\Phi(z) = \int_0^1 e^{-zt} f(t)dt$$

Since f by assumption is n-flat at the end-ponts, integration by parts p times gives:

$$\Phi(z) = z^{-p} \int_0^1 e^{-zt} \cdot \partial^p(f^2)(t)dt \quad : \quad 1 \leq p \leq n+1$$

where $\partial^p(f^2)$ is the derivative of order p of f^2. We have

$$(1) \qquad \partial^p(f^2) = \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot f^{(\nu)} \cdot f^{(p-\nu)} \quad : \ 1 \leq p \leq n+1$$

Now we study the absolute value of Φ on the vertical line $\mathfrak{Re}(z) = -1$. Since $|e^{t-iyt}| = e^t$ for all y, the triangle inequality gives

$$(2) \qquad |-1+iy|^p \cdot |\Phi(-1+iy)| \leq \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \int_0^1 e^t \cdot |f^{(\nu)}(t)| \cdot |f^{(p-\nu)}(t)| \cdot dt$$

Since $e^t \leq e$ on $[0,1]$, the Cauchy-Schwarz inequality and the definition of the β-numbers give:

$$(3) \qquad |-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu}$$

From (*) it follows that $\beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu} \leq \beta_p^p$ for each ν and since $\sum_{\nu=0}^{\nu=p} \binom{p}{\nu} = 2^p$ we obtain

(4) $$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot 2^p \cdot \beta_p^p$$

Passing to the logarithm we get

(5) $$\log|\Phi(-1+iy)| \leq 1 + p \cdot \log \frac{2\beta_p}{|-1+iy|}$$

Here (5) holds when $1 \leq p \leq n+1$ and the assumption that $\beta_0 = 1$ also gives

(6) $$\log|\Phi(-1+iy)| \leq 1$$

The ω-function. To each $1 \leq p \leq n+1$ we find a positve number y_p such that

$$|-1+iy_p| = 2e\beta_p$$

Now we define a function $\omega(y)$ where $\omega(y) = 0$ when $y < y_1$ and

$$\omega(y) = p \quad : \quad y_p \leq y < y_{p+1}$$

and finally $\omega(y) = n+1$ when $y \geq y_{n+1}$. Then (5-6) give the inequality

(7) $$\log|\phi(-1+iy)| \leq 1 - \omega(y) \quad : \quad -\infty < y < +\infty$$

A harmonic majorisation. With $1-\omega(y)$ as boundary function in the half-plane $\mathfrak{Re}(z) > -1$ we construct the harmonic extension $H(z)$ which by Poisson's formula is given by:

$$H(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\omega(y)}{1+y^2} \cdot dy$$

Now $\log|\Phi(z)|$ is subharmonic in this half-plane and hence (7) gives:

$$0 = \log|\Phi(0)| \leq H(0)$$

We conclude that

(8) $$\int_{-\infty}^{\infty} \frac{\omega(y)}{1+y^2} \cdot dy \leq \pi$$

Since $\omega(y) = 0$ when $y \leq y_1$ we see that (8) gives the inequality

(9) $$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy \leq \frac{y_1^2}{1+y_1^2} \cdot \pi$$

The construction of the ω-function gives the equation

(10) $$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy = \frac{1}{y_1} + \ldots + \frac{1}{y_{n+1}}$$

Next, the construction of the y_p-numbers entail that $y_p \leq 2e\beta_p$ so (9-10) give

(11) $$\frac{1}{\beta_1} + \ldots + \frac{1}{\beta_{n+1}} \leq 2e\pi \cdot \frac{1}{1+\frac{1}{y_1^2}}$$

Finally, we have $1 + y_1^2 = 4e^2\beta_1^2$ and recall that Writinger's inequality implies that $\beta_1 \geq \pi$. Hence

(12) $$\frac{1}{1+\frac{1}{y_1^2}} \leq 1 + \frac{1}{4\pi^2 e^2 - 1}$$

and then (11-12) give the requested inequality in Theorem 2.1.

4. An inequality for inverse Fourier transforms in $L^2(\mathbf{R}^+)$.

By Parsevel's theorem the Fourier transform sends L^2-functions on the ξ-line to L^2-functions on the x-line. Consider the class of non-negative L^2-functions $\phi(x)$ such that there exists an L^2-function $F(\xi)$ supported by the half-line $\xi \geq 0$ and

$$(*) \qquad \phi(x) = \Big| \int_0^\infty e^{ix\xi} \cdot F(\xi) \cdot d\xi \Big|$$

The theorem below was proved in [Carleman:xx] which apart from applications to quasi-analytic functions has several other consequences which are put forward by Paley and Wiener in their text-book [Pa-Wi]. The proof employs a detour in the complex domain where Jensen's formula plays a crucial role.

4.1 Theorem. *An L^2-function $\phi(x)$ satisfies (*) if and only if*

$$\text{(i)} \qquad \int_{-\infty}^\infty \log^+ \Big[\frac{1}{\phi(x)} \Big] \cdot \frac{dx}{1+x^2} < \infty$$

Moreover, when (*) holds and $F(\xi)$ satisfies the weighted mean-value equality

$$\text{(ii)} \qquad \int_0^\infty F(\xi) \cdot e^{-\xi}\, d\xi = 1$$

then

$$\text{(iii)} \qquad \int_{-\infty}^\infty \log^+ \Big[\frac{1}{\phi(x)} \Big] \cdot \frac{dx}{1+x^2} \leq \int_{-\infty}^\infty \frac{\phi(x)^2}{1+x^2} \cdot dx$$

Proof. First we prove the suffiency. Let $\phi(x)$ be a non-negative L^2-function where the integral (i) is finite. The harmonic extension of $\log\, \phi(x)$ to the upper half-plane is given by:

$$(1) \qquad \lambda(x+iy) = \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\log\ \phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0$$

Let $\mu(z)$ be the conjugate harmonic function of λ and set

$$(2) \qquad h(z) = e^{\lambda(z) + i\mu(z)}$$

Fatou's theorem gives for almost every x a limit

$$(3) \qquad \lim_{y \to 0} \lambda(x+iy) = \log\ \phi(x)$$

Or, equivalently

$$(4) \qquad \lim_{y \to 0} |h(x+iy)| = \phi(x)$$

From (1) and the fact that the geometric mean value of positive numbers cannot exceed their arithmetic mean value, one has

$$(5) \qquad \big|h(x+iy)\big| = e^{\lambda(x+iy)} \leq \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0$$

Then (5) the Schwarz inequality give:

$$(6) \qquad \int_{-\infty}^\infty \big|h(x+iy)\big|^2\, dx \leq \int_{-\infty}^\infty \big|\phi(x)\big|^2\, dx \quad : y > 0$$

Here $h(z)$ is analytic in the upper half-plane so that (6) and Cauchy's formula entail that if $\xi < 0$, then the integrals

$$(7) \qquad J(y) = \int_{-\infty}^\infty h(x+iy) \cdot e^{-ix\xi + y\xi} \cdot dx \quad : y > 0$$

are independent of y. Passing to the limit as $y \to \infty$ and using the uniform upper bounds on the L^2-norms of the functions $h_y(x) \mapsto h(x+iy)$, it follows that $J(y)$ vanishes identically. So the

Fourier transforms of $h_y(x)$ are supported by $\xi \geq 0$ for all $y > 0$. Passing to the limit as $y \to 0$ the same holds for the Fourier transform of $h(x)$. Finally (4) gives

$$\phi(x) = |h(x)| \tag{8}$$

By Parseval's theorem $\widehat{h}(\xi)$ is an L^2-function and hence $\phi(x)$ has the requested form (*).

Necessity. Since F is in L^2 there exists the Plancherel limit

$$\psi(x) = \lim_{N\to\infty} \frac{1}{2\pi} \cdot \int_0^N e^{ix\xi} \cdot F(\xi)\, d\xi \tag{9}$$

and in the upper half plane we get the analytic function

$$\psi(x+iy) = \frac{1}{2\pi} \cdot \int_0^\infty e^{ix\xi - y\xi} \cdot F(\xi)\, d\xi \tag{10}$$

Suppose that $F(\xi)$ satisfies (ii) in the Theorem which gives

$$\psi(i) = 1$$

Consider the conformal map from the upper half-plane into the unit disc where

$$w = \frac{z-i}{z+i}$$

Here $\psi(x)$ corresponds to a function $\Phi(e^{is})$ on the unit circe $|w| = 1$ and:

$$\int_{-\pi}^{\pi} |\Phi(e^{is})|^2\, ds = 2 \cdot \int_{-\infty}^{\infty} \frac{|\phi(x)|^2}{1+x^2}\, dx \tag{11}$$

Similarly let $\Psi(w)$ be the analytic function in $|w| < 1$ which corresponds to $\psi(z)$. From (10-11) it follows that $\Psi(w)$ is the Poisson extension of Φ, i.e.

$$\Psi(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-|w|^2}{|e^{is}-w|^2} \cdot \Phi(e^{is}) \cdot ds \tag{12}$$

If $0 < r < 1$ it follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}| \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\Psi(re^{is}\right|^2 \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{is}|^2 \cdot ds \tag{13}$$

Now (12) gives:

$$\lim_{r\to 1} \Psi(re^{is}) = \Phi(e^{is}) \quad : \text{almost everywhere} \quad 0 \leq s \leq 2\pi \tag{14}$$

Next, since $\psi(i) = 1$ we have $\Psi(0) = 1$ which gives the inequality

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{|\Psi(re^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}| \cdot ds \quad : 0 < r < 1 \tag{15}$$

By (13-15) a passage to the limit as $r \to 1$ gives

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{|\Phi(e^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \left| \Phi(e^{is}\right|^2 \cdot ds \tag{16}$$

Returning to the real x-line the inequality (iii) follows which at the same time finishes the proof of the theorem.

5. The Bergman kernel.

Let Ω be a bounded and simply connected domain in $\mathbf{C}$. If $a \in \Omega$ a result due to Stefan Bergman gives the conformal mapping function $f_a \colon \Omega \to D$ via the kernel function of the Hilbert space $H^2(\Omega)$.

5.1 Bergman's Theorem. *The conformal map f_a is given by*

$$f_a(z) = \sqrt{\frac{\pi}{K(a,a)}} \cdot \int_a^z K(z,a)\,dz$$

Remark. Recall that Bergman's kernel function $K(z,\zeta)$ satisfies

$$f(z) = \int_\Omega K(z,\zeta) \cdot f(\zeta)\,d\xi d\eta$$

for every square integrable analytic function f in Ω. This is a Hilbert space denoted by $H^2(\Omega)$.

The Gram-Schmidt construction gives an orthonormal basis $\{P_n(z)\}$ in $H^2(\Omega)$ where $P_n(z)$ is a polynomial of degree n and

$$\iint_\Omega P_k \cdot \bar{P}_m \cdot dxdy = \text{Kronecker's delta function}$$

It follows that

$$K(z,\zeta) = \sum P_n(z) \cdot \overline{P_n(\zeta)}$$

Bergman's result indicates that the polynomials above are related to a conformal mapping. We shall consider the case when Ω is a Jordan domain whose boundary curve Γ is *real-analytic*. Let ϕ be the conformal map from the *exterior domain* $\Omega^* = \Sigma \setminus \bar{\Omega}$ onto the exterior disc $|z| > 1$. Here ϕ is normalised so that it maps the point at infinity into itself. The inverse conformal mapping function ψ is defined in $|z| > 1$ and has a series expansion

$$(*) \qquad \psi(z) = \tau \cdot z + \tau_0 + \sum_{\nu=1}^\infty \tau_\nu \cdot \frac{1}{z^\nu}$$

where τ is a positive real number. The assumption that Γ is real-analytic gives some $\rho_1 < 1$ such that ψ extends to a conformal map from the exterior disc $|z| > \rho_1$ onto a domain whose compact complement is contained in Ω.

Inspired by Faber's article *Über Tschebyscheffsche Polynome* [Crelle. J. 1920], Carleman proved an asymptotic result in the article *Über die approximation analytischer funktionen durch linearen aggregaten von vorgegebenen potenzen* [Arkiv för matematik och fysik. 1920].

5.2 Theorem. *There exists a constant C which depends upon Ω only such that to every $n \geq 1$ there is an analytic function $\omega_n(z)$ defined in Ω^* and*

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \cdot \phi'(z) \cdot \phi(z)^n \cdot \big(1 + \omega_n(z)\big) \quad : z \in \partial\Omega$$

where

$$\max_{z \in \partial\Omega} |\omega_n(z)| \leq C \cdot \sqrt{n} \cdot \rho_1^n \quad : \quad n = 1, 2, \ldots$$

Proof.

For each $n \geq 2$ we denote by $\mathcal{M}_n$ the space of monic polynomials of degree n:

$$Q(z) = z^n + b_{n-1} z^{n-1} + \ldots + b_0$$

Put

$$I(Q) = \iint_\Omega |Q(z)|^2\,dxdy$$

and with n kept fixed we set

$$I_*(n) = \min_{Q \in \mathcal{M}_n} I(Q)$$

To each Q we introduce the primitive polynomial

$$\widehat{Q}(z) = \frac{z^{n+1}}{n+1} + \frac{b_{n-1}}{n} z^n + \ldots + b_0 z$$

1. Exercise. Use Green's formula to show that

$$I(Q) = \frac{1}{4} \int_{\partial\Omega} |\partial_n(\widehat{Q})|^2 \, ds$$

where ds is the arc-length measure on $\partial\Omega$ and we have taken the outer normal derivative of $\widehat{Q}$. Next, take the inverse conformal map $\psi(\zeta)$ in (*) and set

$$F(\zeta) = \widehat{Q}(\psi(\zeta))$$

Then F is analytic in the exterior disc $|\zeta| > 1$ and by (*) above, F has a series expansion

$$F(\zeta) = \tau^{n+1} \Big[\frac{\zeta^{n+1}}{n+1} + A_n \zeta^n + \ldots + A_1 \zeta + A_0 + \sum_{\nu=1}^{\infty} \alpha_\nu \cdot \zeta^{-\nu} \Big] \tag{1.1}$$

2. Exercise. Use a variable substitution via ψ to show that

$$I(Q) = \int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2) \, d\theta$$

Show also that the series expansion (1.1) identifies the right hand side with

$$\pi \cdot \tau^{2n+2} \cdot \Big[\frac{1}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 - \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \Big] \tag{2.1}$$

3. An upper bound for $I_*(n)$. In (2.1) the coefficients $A_1, \ldots, A_n$ are determined via Q and the reader may verify that there exists $Q \in \mathcal{M}_n$ such that $A_1 = \ldots = A_n = 0$. It follows that

$$I_*(n) \leq \pi \cdot \tau^{2n+2} \cdot \Big[\frac{1}{n+1} - \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \Big] \leq \pi \cdot \tau^{2n+2} \cdot \frac{1}{n+1} \tag{3.1}$$

4. A lower bound for I_*. The upper bound (3.1) did not use that $\partial\Omega$ is real- analytic, i.e. (3.1) is valid for every Jordan domain whose boundary curve is of class C^1. To get a lower bound we use the constant $\rho - 1 < 1$ from the above and choose $\rho_1 < \rho < 1$. Now ψ maps the exterior disc $|\zeta| > \rho$ conformally to an exterior domain $U^* = \Sigma \setminus \bar{U}$ where U is a relatively compact Jordan domain inside Ω. Choose $Q_n \in \mathcal{M}_n$ so that

$$I(Q_n) = I_*(n)$$

Since $\Omega \setminus \bar{U} \subset \Omega$ we have

$$I_* > \iint_{\Omega \setminus \bar{U}} |Q_n(z)|^2 \, dx dx \tag{4.1}$$

5. Exercise. Show that (4.1) is equal to

$$\int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot d\theta - \int_{|\zeta|=\rho} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot \rho \cdot d\theta =$$

$$\pi \cdot \tau^{2n+2} \cdot \Big[\frac{1 - \rho^{2n+2}}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1 - \rho^{2\nu}) + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1) \Big]$$

and conclude that one has the lower bound

$$I_*(n) \geq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot (1 - \rho^{2n+2}) \tag{5.1}$$

Together (4.1) and (5.1) give the inequality

$$\sum_{k=1}^{k=n} k\cdot|A_k|^2\cdot(1-\rho^{2\nu})+\sum_{\nu=1}^{\infty}\nu\cdot|\alpha_\nu|^2\cdot(\frac{1}{\rho^{2\nu}}-1)\ \leq\ \frac{\pi}{n+1}\cdot\rho^{2n+2} \tag{5.2}$$

Since $1-\rho^2\leq 1-\rho^{2\nu}$ for every $\nu\geq 1$ it follows that

$$\sum_{k=1}^{k=n} k\cdot|A_k|^2+\sum_{\nu=1}^{\infty}\nu\cdot|\alpha_\nu|^2\leq\frac{\pi}{(1-\rho^2)\cdot n+1}\cdot\rho^{2n+2} \tag{5.3}$$

6. Conclusion. Recall that $F(\zeta)=\widehat{Q_n}(\psi(\zeta)$. So after a derivation we get

$$F'(\zeta)=\psi'(\zeta)\cdot Q_n(\psi(\zeta))$$

Hence the series expansion of $F(\zeta)$ gives

$$Q_n(\psi(\zeta))=\frac{\tau^{n+1}}{\psi'(\zeta)}\cdot\big[\zeta^n+\sum_{k=1}^{k=n}k\cdot A_k\zeta^{k-1}+\sum_{\nu=1}^{\infty}\nu\cdot\alpha_\nu\cdot\zeta^{-\nu-1}\,\big] \tag{6.1}$$

where the equality holds for $|\zeta|>\rho$. Put

$$\omega^*(\zeta)=\sum_{k=1}^{k=n}k\cdot A_k\zeta^{k-1}+\sum_{\nu=1}^{\infty}\nu\cdot\alpha_\nu\cdot\zeta^{-\nu-1}$$

When $|\zeta|=1$ the triangle inequality gives

$$|\omega^*(\zeta)|\leq\sum_{k=1}^{k=n}k\cdot|A_k|+\sum_{\nu=1}^{\infty}\nu\cdot|\alpha_\nu| \tag{6.2}$$

7. Exercise. Notice that (5.3) holds for every $\rho>\rho_1$ and use this together with suitable Cauchy-Schwarz inequalities to show that (6.1) gives a constant C which is independent of n such that

$$|\omega^*(\zeta)|\leq C\cdot\sqrt{n}\cdot\rho_1^{n+1} \tag{7.1}$$

Final part of the proof. Since ψ is the inverse of ϕ we have

$$\psi'(\phi(z))\cdot Q_n(\psi(\phi(z)=\frac{Q_n(z)}{\phi'(z)}$$

Define the function on $\partial\Omega$ by

$$\omega_n(z)=\frac{\omega^*(\phi(z))}{\phi'(z)} \tag{i}$$

Then (6.2) gives

$$Q_n(z)=\tau^{n+1}\cdot\phi'(z)\cdot[\phi(z)^n+\omega_n(z)\,] \tag{ii}$$

where Exercise 7 shows that $|\omega_n(z)|$ satisfies the estimate in Theorem 2. Finally, the polynomial Q_n minimized the L^2-norm under the constraint that the leading term is z^n and for this variational problem the upper and the lower bounds in (4.1-5.1) imply that

$$|I_*(n)-\frac{\pi}{n+1}\cdot\tau^{2n+2}|\leq\frac{\pi}{n+1}\cdot\tau^{2n+2}\cdot\rho^{2n+2}$$

If we normalise Q_n so that its L^2-norm is one gets a polynomial $P_n(z)$ where the factor τ^{n+1} is replaced by $\frac{\sqrt{n+1}}{\sqrt{\pi}}$ which finishes the proof of Theorem B.

6. Partial sums of Fourier series.

Let $f(x)$ be a continuous function on $0 \leq x \leq 2\pi$ with $f(0) = f(2\pi)$ and consider the Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

Gibbs gave examples which show that the partial sums

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{k=n} a_k \cdot \cos kx + \sum b_k \cdot \sin kx$$

can fail to converge to $f(x)$ for certain x-values. To analyze the situation we introduce the maximum norms

$$\rho_f(n) = \max_{0 \leq x \leq 2\pi} |s_n(x) - f(x)| \quad : n = 1, 2, \ldots$$

In general they behave in an irregular fashion as $n \to +\infty$ and Hardy constructed examples of continuous functions f such that

$$\limsup_{n \to \infty} \rho_f(n) = +\infty$$

This led Carleman to study the behaviour in the mean. For each $n \geq 1$ we set

$$\mathcal{C}_n(f) = \sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} \rho_f(\nu)^2}$$

Next, put

$$\omega_f(\delta) = \max_{x,y} |f(x) - f(y)| : |x - y| \leq \delta$$

Since f is uniformly continuous $\omega_f(\delta) \to 0$ as $\delta \to 0$. The following result is proved in the article *A theorem concerninig Fourier series* [London math.soc. 1923]:

6.1 Theorem. *There exists an absolute constant K such that the following hold for every continuous function f with maximum norm ≤ 1:*

$$\mathcal{D}_n(f) \leq K \cdot [\frac{1}{\sqrt{n}} + \omega_f(\frac{1}{n})] \quad : \ n = 1, 2, \ldots$$

Remark. This result shows that Gibbs' phenomenon for a given continuous function f only arises for a sparse sequence of n-values. In a joint article *Fourier's series and analytic functions* [Proc. of the Royal Soc. 1923] with Hardy, pointwise convergence was studied where the limit is unrestricted as $n \to \infty$. However, these results were incomplete and it was not until 1965 when Lennart Carleson proved that for every continuous function $f(x)$ there exists a null set $\mathcal{N}$ in $[0, 2\pi]$ such that

(*) $$\lim_{n \to \infty} s_n(x) = f(x) \quad : \ x \in [0, 2\pi] \setminus \mathcal{N}$$

Carleson's result constitutes one of the greatest achievements ever in mathematical analysis and we shall not try to enter details of the proof. Recall that Carleson also proved the almost everywhere convergence when $f \in L^2[0, 2\pi]$. When pointwise convergence holds in (*) is obvious that

(**) $$\lim_{n \to \infty} \sqrt{\frac{1}{n+1} \cdot \sum_{\nu=0}^{\nu=n} |s_n(x) - f(x)|^2} = 0$$

Here (**) is much weaker than (*). On the other hand, the null-set which appears in (*) for a given L^2-function f is in general strictly larger than the set of Lebesgue points for f, i.e. those x for which

$$f(x) = \lim_{\delta \to 0} \frac{1}{2\delta} \cdot \int_{x-\delta}^{x+\delta} |f(s) - f(x)| \, ds = 0$$

In the cited article from 1922, Carleman proved that (**) holds for every Lebesgue point of the L^2-function f which for "ugly Lebegue points" which belong to $\mathcal{N}$ gives an averaged substitute to Carleson's result.

6.2 A convergence result for L^2-functions. Let $f(x)$ be a real-valued and square integrable function on $(-\pi,\pi)$, i.e.

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx < \infty$$

We say that f has a determined value $A = f(0)$ at $x = 0$ if the following two conditions hold:

$$\text{(i)} \qquad \lim_{\delta\to 0} \frac{1}{\delta}\cdot \int_0^{\delta} |f(x)+f(-x)-2A|\,dx = 0$$

$$\text{(ii)} \qquad \int_0^{\delta} |f(x)+f(-x)-2A|^2\,dx \leq C\cdot\delta \quad \text{holds for some constant} \quad C$$

Remark. In the same way we can impose this condition at every point $-\pi < x_0 < \pi$. To simplify the subsequent notations we take $x = 0$. If $x = 0$ is a Lebesgue point for f and A the Lebesgue value we have (i). Hence Lebesgue's Theorem entails that (i) holds almost everywhere when $x = 0$ is replaced by other points x_0. We leave it to the reader to show that the second condition also is valid almost everywhere when f is square integrable. Next, expand f in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k\cdot\cos kx + \sum b_k\cdot \sin kx$$

and with $x = 0$ we consider the partial sums

$$s_n(0) = \frac{a_0}{2} + a_1 + \ldots + a_n + b_1 + \ldots + b_n$$

The result below is proved in Carleman's cited article.

6.3 Theorem. *Assume that f has a determined value A at $x = 0$. Then the following hold for every positive integer k*

$$(*) \qquad \lim_{n\to\infty} \frac{1}{n+1}\cdot\sum_{\nu=0}^{\nu=n} |s_\nu - A|^k = 0$$

Proof. Set $A = f(0)$ and $s_n = s_n(0)$. Introduce the function: $\phi(x) = f(x)+f(-x)-2A$. It follows that

$$(0) \qquad s_n - A = \int_0^{\pi} \frac{\sin(n+1/2)x}{\sin x/2}\cdot\phi(x)\cdot dx$$

where Dini's kernel was used. Trigonometric formulas express (0) as a sum of three terms for each $0 < \delta < \pi$:

$$(1) \qquad \alpha_n = \frac{1}{\pi}\cdot\int_0^{\delta} \sin nx\cdot\cot x/2\cdot\phi(x)\cdot dx$$

$$(2) \qquad \beta_n = \frac{1}{\pi}\cdot\int_{\delta}^{\pi} \sin nx\cdot\cot x/2\cdot\phi(x)\cdot dx$$

$$(3) \qquad \gamma_n = \frac{1}{\pi}\cdot\int_0^{\pi} \cos nx\cdot\phi(x)\cdot dx$$

By Hölder's inequality it suffices to show Theorem 6.3 if $k = 2p$ is an even integer and Minkowski's inequality gives

$$(4) \qquad \Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq \Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} + \Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p}$$

Denote by $o(\delta)$ small ordo and $O(\delta)$ is big ordo. When $\delta \to 0$ we shall establish the following:

(i) $$\Big[\sum_{\nu=0}^{\nu=n} |\alpha_\nu|^{2p}\Big]^{1/2p} = n^{1+1/2p)} \cdot o(\delta)$$

(ii) $$\Big[\sum_{\nu=0}^{\nu=n} |\beta_\nu|^{2p}\Big]^{1/2p} \leq K \cdot p \cdot \delta^{-1/2p}$$

(iii) $$\Big[\sum_{\nu=0}^{\nu=n} |\gamma_\nu|^{2p}\Big]^{1/2p} \leq K$$

In (ii-iii) K is an absolute constant which is independent of p, n and δ. Let us first see why (i-iii) give Theorem 6.3. Write $o(\delta) = \epsilon(\delta) \cdot \delta$ where $\epsilon(\delta) \to 0$. With these notations (4) gives:

(5) $$\Big[\sum_{\nu=0}^{\nu=n} |s_\nu - A|^{2p}\Big]^{1/2p} \leq n^{1+1/2p} \cdot \delta \cdot \epsilon(\delta) + Kp \cdot \delta^{-1/2p} + K$$

Next, let $\rho > 0$ and choose b so large that

$$pKb^{-1/2p} < \rho/3$$

Take $\delta = b/n$ and with n large it follows that $\epsilon(\delta)$ is so small that

$$b \cdot \epsilon(b/n) < \rho/3$$

Then right hand side in (5) is majorized by

$$\frac{2\rho}{3} \cdot n^{1/2p} + K$$

When n is large we also have

$$K \leq \frac{\rho}{3} \cdot n^{1/2p}$$

Hence the left hand side in (*) is majorized by $\rho \cdot n^{1/2p}$ for all sufficiently large n. Since $\rho > 0$ was arbitrary Theorem 6.3 follows when the power is raised by $2p$.

Proof of (i-iii)

To obtain (i) we use the triangle inequality which gives the following for every integer $\nu \geq 1$:

$$|a_\nu| \leq \frac{2}{\pi} \cdot \max_{0 \leq x \leq \delta} |\sin \nu x \cdot \cot x/2| \cdot \int_0^\delta |\phi(x)|\, dx = \nu \cdot o(\delta)$$

where the small ordo δ-term comes from the hypothesis expressed by (*) in the introduction. Hence the left hand side above is majorized by

$$\Big[\sum_{\nu=1}^{\nu=n} \nu^{2p}\Big]^{\frac{1}{2p}} \cdot o(\delta) = n^{1+1/2p} \cdot o(\delta)$$

which was requested to get (i). To prove (iii) we notice that

$$\gamma_0^2 + 2 \cdot \sum_{\nu=1}^{\infty} \gamma_\nu^2 = \frac{1}{\pi} \int_0^\pi |\phi(x)|^2\, dx$$

Next, we have

$$\sum_{\nu=1}^{\infty} |\gamma_\nu|^{2p} \leq \Big[\sum_{\nu=1}^{\infty} |\gamma_\nu|^2\Big]^{1/2p} \leq K$$

where K exists since ϕ is square-intergable on $[0, \pi]$ and (iii) follows.

Proof of (ii). Here several steps are required. For each $0 < s < \pi$ we define the function $\phi_s(x)$ by

$$\phi_s(x) = \phi(x) \quad : \quad 0 < x < s$$

and extend it to an odd function, i.e. $\phi_s(-x) = -\phi_s(x)$ while $\phi_s(x) = 0$ when $|x| > s$. This odd function has a sine series

$$\text{(A)} \qquad \phi_s(x) = \sum_{\nu=1}^{\infty} c_\nu(s) \cdot \sin x$$

Let us also introduce the functions

$$\text{(B)} \qquad \rho(s) = \int_0^s |\phi(x)| \cdot dx \quad \text{and} \quad \Theta(s) = \int_0^s |\phi(x)|^2 \cdot dx$$

The crucial step towards the proof of (ii) is the following:

Sublemma. *One has the inequality*

$$\sum_{|nu=1}^{\infty} |c_\nu(s)|^{2p} \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot \rho(s)^{2p-2}$$

Proof. We employ convolutions and define inductively a sequence of functions $\{\phi_{n,s}(x)\}$ where $\phi_{1,s}(x) = \phi_s(x)$ and

$$\phi_{n+1,s}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_{n,s}(y)\phi_s(x+y) \cdot dy$$

Since convolution yield products of the Fourier coefficients and $2p$ is an even integer we have the standard formula:

$$\text{(1)} \qquad \sum_{\nu=1}^{\infty} c_n(s)^{2p} = \phi_{2p,s}(0)$$

Next, using the Cauchy-Schwarz inequality the reader may verify that

$$|\phi_{2,s}(x)| \leq \frac{2}{\pi} \cdot \Theta(x)$$

This entails that

$$\phi_{3,s}(x) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |\phi_{2,s}(y)| \cdot |\phi_s(x+y)| \cdot dy \leq \frac{2}{\pi^2} \cdot \Theta(s) \cdot \int_{-\pi}^{\pi} |\phi_s(x+y)| \cdot dy = (\frac{2}{\pi})^2 \cdot \Theta(s) \cdot \rho(s)$$

Proceeding in this way it follows by an induction that

$$\phi_{2p,s}(x) \leq (\frac{2}{\pi})^{2p-1} \cdot \Theta(s) \cdot (\rho(s))^{2p-2}$$

This holds in particular when $x = 0$ and then (1) above gives the sublemma.

Proof continued. We have by definition

$$\beta_\nu = \frac{2}{\pi} \int_\delta^\pi \sin \nu x \cdot \frac{1}{2} \cot(\frac{x}{2}) \cdot \phi(x) \cdot dx$$

An integration by parts and the construction of the Fourier coefficients $\{c_\nu(s)\}$ which applies with $s = \delta$ give:

$$\text{(C)} \qquad \beta_\nu = -\frac{1}{2} \cdot \cot \delta/2 \cdot c_\nu(\delta) + +\frac{1}{4} \int_\delta^\pi c_\nu(x) \cdot \operatorname{cosec}^2(\frac{x}{2}) \cdot dx$$

Now we profit upon Minkowski's inequality. Let q be the conjugate of $2p$, i.e $\frac{1}{q} + \frac{1}{2p} = 1$ and choose $\{\xi_\nu\}$ to be the sequence in ℓ^q of unit norm such that

$$|\sum \xi_\nu \cdot \beta_\nu| = ||\beta_\bullet||_{2p}$$

where the last term is the left hand side in (ii). At the same time (*) above and the triangle inequality give

$$||\beta_\bullet||_{2p} \leq -\frac{1}{2} \cdot \cot(\delta/2) \cdot \sum |c_\nu(\delta)| \cdot |\xi_\nu| + \frac{1}{4} \int_\delta^\pi \operatorname{cosec}^2(\frac{x}{2}) \cdot \sum |c_\nu(x) \cdot \xi_\nu| \cdot dx \leq$$

$$\text{(D)} \qquad \frac{1}{2}\cdot\cot(\delta/2)\cdot||c_\bullet(\delta)||_{2p}+\frac{1}{4}\int_\delta^\pi \operatorname{cosec}^2(\frac{x}{2})\cdot||c_\bullet(x)||_{2p}\cdot dx$$

At this stage we apply the sublemma and the assumption which give a constant K such that

$$\text{(E)} \qquad \Theta(s)\leq K \quad\text{and}\quad \rho(s)\leq K\cdot s$$

The last estimate actually is weaker than the hypothesis but it will be sufficient to get the requested estimate of the ℓ^{2p}-norm in (ii). Namely, the sublemma gives a constant K_1 such that

$$||c_\bullet(\delta)||_{2p}\leq K_1\cdot\delta^{1-1/p}$$

At the same time we have a constant K_2 such that

$$\cot(\delta/2)\leq\frac{K_2}{\delta}$$

The product in the first term from (D) is therefore majorized by $K_1K_2\cdot\delta^{-1/2p}$ as requested in (ii). For the second term we use again the sublemma which first gives

$$||c_\bullet(x)||_p\leq K\cdot x^{-1/2p}$$

At this stage we leave it to the reader to verify that we get a constant K so that

$$\int_\delta^\pi x^{-1/2p}\cdot\operatorname{cosec}^2(\frac{x}{2})\cdot dx\leq K\cdot\delta^{-1/2p}$$

which finishes the proof of (ii).

The case when f is continuous. Under the normalisation that the L^2-integral of f is ≤ 1 the inequalities (ii-iii) hold for an absolute constant K. In (i) we notice that the construction of ϕ and the definition of ω_f give the estimates

$$|a_\nu|\leq\nu\cdot\delta\cdot\omega_f(\delta)$$

With $p=2$ the proof of Theorem 5.3 entails that (i) is majorised by

$$n^{1+1/2}\cdot\delta\cdot\omega_f(\delta)$$

This holds for every $0\leq x\leq 2\pi$ and from the previous proof we conclude that the following hold for each $n\geq 2$ and every $0<\delta<\pi$:

$$\text{(i)} \qquad \mathcal{D}_n(f)\leq\frac{1}{\sqrt{n+1}}\cdot[n^{1+1/2}\cdot\delta\cdot\omega_f(\delta)+2K\delta^{-1/2}+K]$$

With $n\geq 2$ we take $\delta=n^{-1}$ and see that (i) gives a requested constnt in Theorem 5.1

7. The Denjoy conjecture

Introduction. Let ρ be a positive integer and $f(z)$ is an entire function such that there exists some $0<\epsilon<1/2$ and a constant A_ϵ such that

$$|f(z)| \leq A_\epsilon \cdot e^{|z|^{\rho+\epsilon}} \tag{0.1}$$

hold for every z. Then we say that f has integral order $\leq \rho$. Next, the entire function f has an asymptotic value a if there exists a Jordan curve Γ parametrized by $t \mapsto \gamma(t)$ for $t \geq 0$ such that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and

$$\lim_{t\to+\infty} f(\gamma(t)) = a \tag{0.2}$$

In 1920 Denjoy raised the conjecture that (0.1) implies that the entire function f has at most 2ρ many different asymptotic values. Examples show that this upper bound is sharp. The Denjoy conjecture was proved in 1930 by Ahlfors in [Ahl]. A few years later T. Carleman found an alternative proof based upon a certain differential inequality. Theorem A.3 below has applications beyond the proof of the Denjoy conjecture for estimates of harmonic measures. See [Ga-Marsh].

A. The differential inequality.

Let Ω be a connected open set in $\mathbf{C}$ whose intersection S_x between a vertical line $\{\mathfrak{Re}\, z = x\}$ is a bounded set on the real y-line for every x. When $S_x \neq \emptyset$ it is the disjoint union of open intervals $\{(a_\nu, b_\nu)\}$ and we set

$$\ell(x) = \max_\nu (b_\nu - a_\nu) \tag{*}$$

Next, let $u(x,y)$ be a positive harmonic function in Ω which extends to a continuous function on the closure $\bar\Omega$ with the boundary values identical to zero. Define the function ϕ by:

$$\phi(x) = \int_{S_x} u^2(x,y)\cdot dy \tag{1}$$

The Federer-Stokes theorem gives the following formula for the derivatives of ϕ:

$$\phi'(x) = 2\int_{S_x} u_x \cdot u(x,y)dy \tag{2}$$

$$\phi''(x) = 2\int_{S_x} u_{xx}\cdot u(x,y)dy + 2\int_{S_x} u_x^2 \cdot dy \tag{3}$$

Since $\Delta(u) = 0$ when $u > 0$ we have

$$2\int_{S_x} u_{xx}\cdot u(x,y)dy = -2\int_{S_x} u_{yy}\cdot u(x,y)dy = 2\int u_y^2 dy \tag{4}$$

The Cauchy-Schwarz inequality applied in (2) gives

$$\phi'(x)^2 \leq 4\cdot \int_{S_x} u_x^2 \cdot \int_{S_x} u^2(x,y)dy = 4\cdot\phi(x)\cdot\int_{S_x} u_x^2 dy \tag{5}$$

Hence (4) and (5) give:

$$\phi''(x) \geq 2\int_{S_x} u_y^2(x,y)\cdot dy + \frac{1}{2}\cdot\frac{\phi'^2(x)}{\phi(x)} \tag{6}$$

Next, since $u(x,y) = 0$ at the end-points of all intervals of S_x, *Wirtinger's inequality* and the definition of $\ell(x)$ give:

$$\int_{S_x} u_y^2(x,y)\cdot dy \geq \frac{\pi^2}{\ell(x)^2}\cdot\phi(x) \tag{7}$$

Inserting (7) in (6) we have proved

A.1 Proposition *The ϕ-function satisfies the differential inequality*

$$\phi''(x) \geq \frac{2\pi^2}{\ell(x)^2} \cdot \phi(x) + \frac{\phi'^2(x)}{2\phi(x)}$$

Proof continued. The maximum principle for harmonic functions implies that the $\phi(x) > 0$ when $x > 0$ and hence there exists a ψ-function where $\phi(x) = e^{\psi(x)}$. It follows that

$$\phi' = \psi' e^{\psi} \quad \text{and} \quad \phi'' = \psi'' e^{\psi} + \psi'^2 e^{\psi}$$

Now Proposition A.1 gives

$$(*) \qquad \psi'' + \frac{\psi'^2}{2} \geq \frac{2\pi^2}{\ell(x)^2}$$

A.2 An integral inequality. From (*) we obtain

$$\frac{2\pi}{\ell(x)} \leq \sqrt{\psi'(x)^2 + 2\psi''(x)} \leq \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

Taking the integral we get

$$(**) \qquad 2\pi \cdot \int_0^x \frac{dt}{\ell(t)} \leq \psi(x) + \log \psi'(x) + O(1) \leq \psi(x) +, \psi'(x) + O(1)$$

where $O(1)$ is a remainder term which is bounded independent of x. Taking the integral once more we obtain:

A.3 Theorem. *The following inequality holds:*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + O(x)$$

where the remainder term $O(x)$ is bounded by Cx for a fixed constant.

B. Solution to the Denjoy conjecture

B.1 Theorem. *Let $f(z)$ be entire of some integral order $\rho \geq 1$. Then f has at most 2ρ many different asymptotic values.*

Proof. Suppose f has n different asymptotic values $a_1, \ldots, a_n$. To each a_ν there exists a Jordan arc Γ_ν as described in the introduction. Since the a-values are different the n-tuple of Γ-arcs are separated from each other when $|z|$ is large. So we can find some R such that the arcs are disjoint in the exterior disc $|z| > R$. We may also consider the tail of each arc, i.e. starting from the last point on Γ_ν which intersects the circle $|z| = R$. So now we have an n-tuple of disjoint Jordan curves in $|z| \geq R$ where each curve intersects $|z| = R$ at some point p_ν and after the curves moves to the point at infinity. See figure. Next, we take one of these curves, say Γ_1. Let D_R^* be the exterior disc $|\zeta| > R$. In the domain $\Omega = \mathbf{C} \setminus \Gamma_1 \cup D_R^*$ we can choose a single-valued branch of $\log \zeta$ and with $z = \log \zeta$ the image of Ω is a simply connected domain Ω^* where S_x for each x has length strictly less than 2π The images of the Γ-curves separate Ω^* into n many disjoint connected domains denoted by $D_1, \ldots, D_n$ where each D_ν is bordered by a pair of images of Γ-curves and a portion of the vertical line $x = \log R$.

Let $\zeta = \xi + i\eta$ be the complex coordinate in Ω^*. Here we get the analytic function $F(\zeta)$ where

$$F(\log(z)) = f(z)$$

We notice that F may have more growth than f. Indeed, we get

$$(1) \qquad |F(\xi + i\eta)| \leq \exp\big(e^{(\rho+\epsilon)\xi}\big)$$

With $u = \text{Log}^+ |F|$ it follows that

(2) $$u(\xi, \eta) \leq e^{(\rho+\epsilon)\xi}$$

Hence the ϕ-function constructed during the proof of Theorem A.3 satisfies

$$\phi(\xi) \leq e^{2(\rho+\epsilon)\xi}$$

It follows that the ψ-function satisfies

(3) $$\psi(\xi) = 2 \cdot (\rho + \epsilon)\xi + O(1)$$

Now we apply Theorem A.3 in each region D_ν where we have a function $\ell_\nu(\xi)$ constructed by (0) in section A. This gives the inequality

(4) $$2\pi \cdot \int_R^\xi \frac{\xi - s}{\ell_\nu(s)} \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1) \quad : \quad 1 \leq \nu \leq n$$

Next, recall the elementary inequality which asserts that if $a_1, \ldots, a_n$ is an arbitrary n-tuple of positive numbers then

(5) $$\sum a_\nu \cdot \sum \frac{1}{a_\nu} \geq n^2$$

For each s we apply this to the n-tuple $\{\ell_\nu(s)\}$ where we also have

$$\sum \ell_\nu(s) \leq 2\pi$$

So a summation in (4) over $1 \leq \nu \leq n$ gives

(6) $$n \cdot \int_R^\xi (\xi - s) \cdot ds \leq \int_R^\xi (\rho + \epsilon)s \cdot ds + (\rho + \epsilon)\xi + O(1)$$

Another integration gives:

(7) $$n \cdot \frac{\xi^2}{2} \leq (\rho + \epsilon) \cdot \xi^2 + O(\xi)$$

This inequality can only hold for large ξ if $n \leq 2(\rho + \epsilon)$ and since $\epsilon < 1/2$ is assumed it follows that $n \leq 2\rho$ which finishes the proof of the Denjoy conjecture.

8. Lindelöf functions.

Introduction. For each real number $0 < a \leq 1$ there exists the entire function

$$Ea_(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+na)}$$

Growth properties of the E-functions were investigated in a series of articles by Mittag-Leffler between 1900-1904 using integral formulas for the entire function $\frac{1}{\Gamma}$. This inspired Phragmén to study entire functions $f(z)$ such that there are constants C and $0 < a < 1$ with:

$$\log|f(re^{i\theta})| \leq C \cdot (1 + |r|)^a) \quad : -\alpha < \theta < \alpha$$

for some $0 < \alpha < \pi/2$ while $f(z)| \leq C$ for all $z \in \mathbf{C} \setminus S$. When this holds we get the entire function

$$g(z) = \int_0^{\infty} f(sz) \cdot e^{-s} \cdot ds$$

If z is outside the sector S it is clear that $|g(z)|$ is bounded by $C \cdot \int_0^{\infty} e^{-s} ds = C$. When $z = re^{i\theta}$ is in the sector we still get a bound from (1) since $0 < a < 1$ and conclude that the entire function g is bounded and hence a constant. Since the Taylor coefficients of f are recaptured from g it follows that f must be constant. More general results of this kind were obtained in the joint article [PL] by Phragmén and Lindelöf from 1908 and led to the Phragmén-Linedlöf principle. A continuation of [PL] appears in Lindelöf's article *Remarques sur la croissance de la fonction* $\zeta(s)$ (Bull. des scienes mathématiques 1908] devoted to the growth of Riemann's ζ-function along vertical lines in the strip $0 < \mathfrak{Re}(z) < 1$. This leads to the study of various indicator functions attached to analytic functions and we shall expose material from Care,an's article *Sur la fonction* $\mu(\sigma)$ *de M. Lindelöf* which was published in 1930 and attributed toi A. Wiman,. Here is the set-up: Consider a strip domain in the complex s-plane:

$$\Omega = \{s = \sigma + it \quad : t > 0 \quad \text{and} \quad 0 \leq a < \sigma < b\}$$

An analytic function $f(z)$ in Ω is of *finite type* if there exists some integer k, a constant C and some $t_0 > 0$ such that

$$|f(\sigma + it)| \leq C \cdot t^k \quad \text{hold for} \quad t \geq t_0$$

To every such f we define the Lindelöf function

(*) $$\mu_f(\sigma) = \limsup_{t \to \infty} \frac{\operatorname{Log}|f(\sigma + it)|}{\operatorname{Log} t}$$

Lindelöf and Phragmén proved that μ_f is a continuous and convex function on (a, b). No further restrictions occur on the μ-function because one has:

1. Theorem. *For every convex and continuous function $\mu(\sigma)$ defined in $[a, b]$ there exists an analytic function $f(z)$ without zeros in Ω such that $\mu_f = \mu$.*

2. Exercise. Prove this result using the Γ-function. First, to a pair of real numbers (ρ, α) we set

(i) $$f(s) = e^{-\frac{\pi i \cdot \rho s}{2}} \cdot \Gamma(\rho(s-a) + \frac{1}{2})$$

Use properties of the Γ-function to show that f has finite type in Ω and its indicator function becomes a linear function:

$$\mu_f(\sigma) = \rho \cdot (\sigma - a)$$

More generally one gets a function f where μ_f is piecewise linear by:

(ii) $$f = \sum_{k=1}^{k=m} c_k e^{-\frac{\pi i \cdot \rho_\nu s}{2}} \Gamma(\rho_k(s - a_k) + \frac{1}{2})$$

where $\{c_k\}$, $\{\rho_k\}$ and $\{a_k\}$ are m-tuples of real numbers. Finally, starting from an arbitrary convex curve we can choose some dense and enumerable set of enveloping tangents to this curve. Then an infinite series of the form above gives an analytic function $f(s)$ such that

$$\sigma \mapsto \mu_f(\sigma)$$

yields an arbitrarily given convex μ-function on (a,b).

1. A construction of harmonic functions.

Let $U(x,y)$ be a bounded harmonic function in the strip domain Ω and V its harmonic conjugate. Set

$$(*) \qquad f(s) = \exp\left[(\log(s) - \frac{\pi i}{2})(U(s) + iV(s))\right]$$

It is easily see that $f(z)$ has finite type in Ω. With $s = \sigma + it$ we have

$$|f(\sigma + it)| =$$

$$\exp(\frac{1}{2}\log(\sigma^2 + t^2) \cdot U(\sigma + it) \cdot \exp(-(\frac{\pi}{2} - \arg(\sigma + it)) \cdot V(\sigma + it))$$

It follows that

$$\frac{\log\,|f(\sigma + it)|}{t} = \frac{\log\,\sqrt{\sigma^2 + t^2} \cdot U(\sigma + it)]}{\log t} + \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t}$$

1.1 Exercise. With σ kept fixed one has

$$\arg(\sigma + it) = \tan\,\frac{t}{s}$$

which tends to $\pi/2$ as $t \to +\infty$. Next, $V(\sigma + it)$ is for large $t > 0$ up to a constant the primitive of

$$\int_1^t \frac{\partial V}{\partial u}(\sigma + iu) \cdot du$$

Here the partial derivative of V is equal to the partial derivative $\partial U/\partial\sigma(\sigma, u)$ taken along $\mathfrak{Re}\, s = \sigma$. Since U is bounded in the strip domain it follows from Harnack's inequalities that this partial derivative stays bounded when $1 \le u \le t$ by a constant which is independent of t. Putting this together the reader can verify that

$$(1.2) \qquad \lim_{t \to +\infty} \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t} = 0$$

From (1.2) we obtain the equality

$$(*) \qquad \mu_f(\sigma) = \limsup_{t \to \infty}\, U(\sigma + it)$$

This suggests a further study of growth properties of bounded harmonic functions in strip domains.

2. The M and the m-functions.

To a bounded harmonic function U in Ω we associate the maximum and the minimum functions:

$$M(\sigma) = \limsup_{t \to \infty}\, U(\sigma + it) \quad \text{and} \quad \liminf_{t \to \infty}\, U(\sigma + it)$$

2.1 Proposition. *$M(\sigma)$ is a convex function while $m(\sigma)$ is concave.*

We prove the convexity of $M(\sigma)$. The concavity of m follows when we replace U by $-U$. Consider a pair α, β with $a < \alpha < \beta < b$. Replacing U by $U + A + \cdot Bx$ for suitable constants A and B we may assume that $M(\alpha) = M(\beta) = 0$ and the requested convexity follows if we can show that

$$M(\sigma) \le 0 \quad : \quad \alpha < \sigma < \beta$$

To see this we consider rectangles

$$\mathcal{R}[T_*, T^*] = \{\sigma + it \quad \alpha \le \sigma \le \beta \quad \text{and} \quad T_* \le t \le T^*\}$$

Let $\epsilon > 0$ and start with a large T_* so that

(i) $$t \geq T_* \implies U(\alpha + it) \leq \epsilon$$

and similarly with α replaced by β. Next, we have a constant M such that $|U|_\Omega \leq M$. If $z = \sigma + it$ is an interior point of the rectangle above it follows by harmonic majorisation that

$$U(\sigma + it) \leq \epsilon + M \cdot \mathfrak{m}_z(J_* \cup J^*)$$

where the last term is the harmonic measure at z which evaluates the harmonic function in the rectangle at z with boundary values zero on the two verical lines of the rectangle which it is equal to 1 on the horizontal intervals $J^* = (\alpha, \beta) + iT^*$ and $J_* = (\alpha, \beta) + iT_*$

Exercise. Show (via the aid of figure that with $T^* = 2T_*$ one has

$$\lim_{T_* \to +\infty} \mathfrak{m}_{\sigma + 3iT_*/2}(J_* \cup J^*) = 0$$

where this limit is uniform when $\alpha \leq \sigma \leq \beta$. Since $\epsilon > 0$ is arbitrary in (i) the reader can conclude that $M(\sigma) \leq 0$ for every $\sigma \in (\alpha, \beta)$.

A special case. Suppose that we have the equalities

(1) $$m(\alpha) = M(\alpha) \quad \text{and} \quad m(\beta) = M(\beta)$$

Using rectangles as above and harmonic majorization the reader can verify that this implies that

$$m(\sigma) = M(\sigma) \quad : \quad \alpha < \sigma < \beta$$

Let us remark that this result was originally proved by Hardy and Littlewood in [H-L].

The case when $M(\sigma) - m(\sigma)$ has a tangential zero. Put $\phi(\sigma) = M(\sigma) - m(\sigma)$ and suppose that this non-negative function in (a, b) has a zero at some $a < \sigma_0 < b$ whose graph has a tangent at σ_0. This means that if:

$$h(r) = \max_{-r \leq |\sigma - \sigma_0| \leq r} \phi(\sigma)$$

then

(*) $$\lim_{r \to 0} \frac{h(r)}{r} = 0$$

Under this hypothesis the following result is proved in [Carleman].

2.2 Theorem. *When (*) holds we have*

$$m(\sigma) = M(\sigma) \quad : a < \sigma < b$$

The subsequent proof from [Carleman] was given at a lecture by Carleman in Copenhagen 1931 which has the merit that a similar reasoning can be applied in dimension ≥ 3. Adding some linear function to U we may assume that $M(\sigma_0) = m(\sigma_0) = 0$ which means that

(1) $$\limsup_{t \to \infty} U(\sigma_0, t) = 0$$

Next, consider the function

(1) $$\phi \colon t \mapsto \partial U / \partial \sigma(\sigma_0, t)$$

The assumption (*) and the result in XXX gives:

(2) $$\lim_{t \to \infty} \partial U / \partial \sigma(\sigma_0, t) = 0$$

Next, consider some $a < \sigma < b$ and let $\epsilon > 0$. By the result from XX there exist finite tuples of constants $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$ and some N-tuple $\{\tau_\nu\}$ which stays in a $[0, 1]$ such that

(5) $$\left| U(\sigma, t) - \sum a_\nu \cdot U(\sigma_0, t_\nu + t) - \sum b_\nu \cdot \partial U / \partial \sigma(\sigma_0, t_\nu + t) \right| < \epsilon \quad \text{hold for all} \quad t \geq 1$$

Since ϵ is arbitrary it follows from (1-2) that

(5) $$\lim_{t \to \infty} U(\sigma, t) = 0$$

for every $a < \sigma < b$ which obviously gives the requested equality in Theorem 2.2.

2.3. Integral indicator funtions.

Let $f(s)$ be an analytic function of finite order in the strip domain Ω and fix some $t_0 > 0$ which does not affect the subsequent constructions. For a pair (σ, p) where $a < \sigma < b$ and $p > 0$ we associate the set of of positive numbers χ such that the integral

$$(*) \qquad \int_{t_0}^{\infty} \frac{|f(\sigma + it)|^p}{t^{\chi}} \cdot dt < \infty$$

We get a critical smallest non-negative number $\chi_*(\sigma, p)$ such that (*) converges when $\chi > \chi_*(\sigma, p)$. In the case $p = 1$ a result due to Landau asserts that $\chi(\sigma, 1)$ determines the half-plane of the complex z-plane where the function

$$\gamma(z) = \int_{t_0}^{\infty} \frac{f(\sigma + it)}{t^z} \cdot dt$$

is analytic and $\sigma \mapsto \chi(\sigma, 1)$ is a convex function on (a, b). A more general convexity result holds when p also varies.

2.4 Theorem. *Define the ω-function by*:

$$\omega(\sigma, \eta) = \eta \cdot \chi(\sigma, \frac{1}{\eta}) \quad : a < \sigma < b \quad : \eta > 0$$

Then ω is a continuous and convex function of the two variables (σ, η) in the product set $(a, b) \times \mathbf{R}^+$.

2.5 Remark. Theorem 2.4 is proved using Hölder inequalities and factorisations of analytic functions which reduces the proof to the case when f has no zeros. The reader is invited to supply details of the proof or consult [Carleman].

3. Lindelöf estimates in the unit disc.

Let $f(z)$ be analytic in the open unit disc given by a power series

$$f(z) = \sum a_n \cdot z^n$$

We assume that the sequence $\{a_n\}$ has temperate growth, i.e. there exists some integer $N \geq 0$ and a constant K such that

$$|a_n| \leq K \cdot n^N \quad : \quad n = 1, 2, \ldots$$

In addition we assume that the sequence $\{a_n\}$ is not too small in the sense that

$$(*) \qquad \sum_{n=1}^{\infty} |a_n|^2 \cdot n^s = +\infty \quad : \quad \forall\, s > 0$$

Now there exists the smallest number $s_* \geq 0$ such that the Dirichlet series

$$\sum_{n=1}^{\infty} |a_n|^2 \cdot \frac{1}{n^s} < \infty, \quad \text{for all } s > s_*$$

To each $0 \leq \theta \leq 2\pi$ we set

$$(1) \qquad \chi(\theta) = \min_s \int_0^1 \left|f(re^{i\theta})\right| \cdot (1-r)^{s-1} \cdot dr < \infty$$

$$(2) \qquad \mu(\theta) = \text{Lim.sup}_{r \to 1} \frac{\text{Log}\,|f(re^{i\theta})|}{\text{Log}\,\frac{1}{1-r}}$$

We shall study the two functions χ and μ. The first result is left as an exercise.

3.1. Theorem. *The inequality*

$$\chi(\theta) \leq \frac{s^*}{2}$$

holds almost everywhere, i.e. for all $0 \leq \theta \leq 2\pi$ *outside a null set on* $[0, 2\pi]$.

Hint. Use the formula

$$\frac{1}{2\pi} \cdot \int_{=}^{2\pi} |f(re^{i\theta})|^2 \cdot d\theta = \sum |a_n|^2$$

For the μ-function a corresponding result holds:

3.2. Theorem. *The inequality below holds almost everywhere.*

$$\mu(\theta) \leq \frac{s^*}{2}$$

Proof. Let $\epsilon > 0$ and introduce the function

$$\Phi(z) = \sum a_n \cdot \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{s^*}{2}+\epsilon)} \cdot z^n = \sum c_n \cdot z^n$$

It is clear that the construction of s^* entails

$$\sum |c_n|^2 < \infty$$

Next, set $\Phi_0 = \Phi$ and define inductively the sequence $\Phi_0, \Phi_1, \ldots$ by

$$\Phi_\nu(z) = z^{\nu-1} \cdot \frac{d}{dz}\big[z^\nu \cdot \Phi_{\nu-1}(z)\big) \quad : \quad \nu = 1, 2, \ldots$$

3.3 Exercise. Show that for almost every $0 \leq \theta \leq 2\pi$ there exists a constant $K = K(\theta)$ such that

$$|\Phi_\nu(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{1-r)^\nu} \quad : 0 < r < 1$$

Next, with s^* and ϵ given we define the integers ν and ρ:

$$\nu = \big[\,\frac{s^*}{2} + \epsilon\big] + 1 \quad : \quad \rho = \frac{s^*}{2} + \epsilon - \big[\,\frac{s^*}{2} + \epsilon\big]$$

where the bracket term is the usual notation for the smallest integer $\geq \frac{s^*}{2} + 1$.

Exercise Show that with ν and ρ chosen as above one has

$$\Phi_\nu(z) = \sum a_n \cdot \frac{\Gamma(n+1+\nu)}{\Gamma(n+1+\rho-1)} \cdot z^n$$

and use this to show the inversion formula

(*) $$f(z) = \frac{1}{z^\nu \cdot \Gamma(1-\rho)} \cdot \int_0^z (z-\zeta)^{-\rho}\dot{\zeta}^{\nu+\rho-1} \cdot \Phi_\nu(\zeta) \cdot d\zeta$$

3.4 Exercise. Deduce from the above that for almost every θ there exists a constant $K(\theta)$ such that

(**) $$|f(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{(1-r)^{\nu+\rho-1}}$$

Conclusion. From (**) and the construction of ν and ρ the reader can confirm Theorem 3. 2.

3.5 Example. Consider the function

$$f(z) = \sum_{n=1}^{\infty} z^{n^2}$$

Show that $s^* = \frac{1}{2}$ holds in this case. Hence Theorem B.2 shows that for each $\epsilon > 0$ one has

$$\text{(E)} \qquad \max_r \, (1-r)^{\frac{1}{4}+\epsilon} \cdot |f(re^{i\theta}| < \infty$$

for almost every θ.

3.6 Exercise. Use the inequality above to show the following: For a complex number $x+iy$ with $y > 0$ we set

$$q = e^{\pi ix - \pi y}$$

Define the function

$$\Theta(x+iy) = 1 + q + q^2 + \dots$$

Show that when $\epsilon > 0$ then there exists a constant $K = K(\epsilon, x)$ for almost all x such that

$$y^{\frac{1}{4}+\epsilon} \cdot |\theta(x+iy)| \leq K \quad : \quad y > 0$$

9. Approximation theorems in complex domains

0. Introduction.

This chapter is devoted to results concerned with approximation by analytic functions due to Carleman, Lindelöf and Müntz.

B. Polynomial approximation with bounds

Introduction. We begin with a result due to Lindelöf. Let U be a Jordan domain and set $\Gamma = \partial U$. For each $f(z) \in \mathcal{O}(U)$, Runge's theorem gives a sequence of polynomials $\{Q_\nu(z)\}$ which approximates f uniformly over each compact subset of U. If we impose some bound on f one may ask if an approximation exists where the Q-polynomials satisfy a similar bound as f. Lindelöf proved that bounds exist for many different norms on the given function f in the article *Sur un principe géneral de l'analyse et ses applications à la theorie de la représentation conforme* from 1915. Let us announce two results from [Lindelöf] of this nature. Let $p > 0$ and consider the H^p-space of analytic functions in U for which

$$||g||_p = \iint_U |g(z)|^p \cdot dxdy < \infty \tag{1}$$

B.1 Theorem *Let $p > 0$ and suppose that $f(z)$ has a finite H^p-norm. Then there exists a sequence of polynomials $\{Q_n(z)\}$ which converge uniformly to f in compact subsets of U while*

$$||Q_n||_p \leq ||f||_p \quad : \quad n = 1, 2, \ldots$$

A similar approximation holds when the H^p-norm is replaced by the maximum norm. Thus, if f is a bounded analytic function in U there exists a sequence of polynomials $\{Q_n\}$ which converge to f in every relatively compact subset of U and at the same time the maximum norms satisfy:

$$||Q_n||_U \leq ||f||_U \quad : \quad n = 1, 2, \ldots$$

To prove Theorem B.1 one constructs for each $n \geq 1$ a Jordan curve Γ_n which surrounds $\bar{U}$, i.e. its interior Jordan domain U_n contains $\bar{U}$ and for every point $p \in \Gamma_n$ the distance of p to Γ is $< 1/n$. It is trivial to see that such a family of Jordan curves exist where the domains $U_1, U_2, \ldots$ decrease. Next, fix a point $z_0 \in U$. There exists the unique conformal map ψ_n from U_n onto U such that

$$\psi_n(z_0) = z_0 \quad : \quad \psi_n'(z_0) \text{ is real and positive}$$

With these notations Lindelöf used the following lemma whose proof is left as an exercise:

B.2. Lemma *For each compact subset K of U the maximum norms $|\psi_n(z) - z|_K$ tend to zero as $n \to \infty$. Moreover, the complex derivatives $\psi_n'(z_0) \to 1$.*

Proof of Theorem B.1. To each n we set

$$F_n(z) = f(\psi_n(z)) \cdot \psi_n'(z)^{\frac{2}{p}}$$

By Lemma B.2 the sequence $\{F_n\}$ converges uniformly to f on compact subsets of U. Moreover, each $F_n \in \mathcal{O}(U_n)$ and it is clear that

$$\iint_U |f(z)|^p \cdot dxdy = \iint_{U_n} |F_n(z)|^p \cdot dxdy$$

hold for every n. To get the required polynomials $\{Q_n\}$ in Theorem B.1 for H^p-spaces it suffices to apply Runge's theorem for each single F_n. This detail of the proof is left to the reader. For maximum norms we use the functions

$$F_n(z) = f(\psi_n(z))$$

and after apply Runge's theorem in the domains $\{U_n\}$.

B.3 Remark. More delicate approximations by polynomials where other norms such as the modulus of continuity, were established later by Lindelöf and De Vallé-Poussin. We shall not pursue this any further. The reader can consult articles by De Vallé-Poussin which contain many interesting results concerned with approximation theorems.

C. Approximation by fractional powers

Here is the set-up in the article *Über die approximation analytischer funktionen* by Carleman from 1922. Let $0 < \lambda_1 < \lambda_2 < \dots$ be a sequence of positive real numbers and Ω is a simply connected domain contained in the right half-space $\mathfrak{Re}(z) > 0$. The functions $\{q_\nu(z) = z^{\lambda_\nu}\}$ are analytic in the half-plane, i.e. with $z = re^{i\theta}$ and $-\pi/2 < \theta < \pi/2$ we have:

$$q_\nu(z) = r^{\lambda_\nu} \cdot e^{i\lambda_\nu \cdot \theta}$$

C.1 Definition. *The sequence $\Lambda = \{\lambda_\nu\}$ is said to be dense for approximation if there for each $f \in \mathcal{O}(\Omega)$ exists a sequence of functions of the form*

$$Q_N(z) = \sum_{\nu=1}^{N} c_\nu(N) \cdot q_\nu(z) \quad : \quad N = 1, 2, \dots$$

which converges uniformly to f on compact subsets of Ω.

C.2 Theorem. *A sequence Λ is dense if*

$$(*) \qquad \limsup_{R \to \infty} \frac{\sum_R \frac{1}{\lambda_\nu}}{\operatorname{Log} R} > 0$$

where $\sum_R$ means that we take the sum over all $\lambda_\nu < R$.

Remark. Above condition (*) is the same for every simply connected domain Ω. Theorem C.2 gives a *sufficient* condition for an approximation. To get necessary condition one must specify the domain Ω and we shall not try to discuss this more involved problem. The proof of Theorem C.2 requires several steps, the crucial is the uniqueness theorem in C.4 while the proof of Theorem C.2 is postponed until C.5.

C.3 A uniqueness theorem.

Consider a closed Jordan curve Γ of class C^1 contained in $\mathfrak{Re}\, z > 0$. When $z = re^{i\theta}$ stays in the right half-plane we get an entire function of the complex variable λ defined by:

$$\lambda \mapsto z^\lambda = r^\lambda \cdot e^{i\theta \cdot \lambda}$$

We conclude that a real-valued and continuous function g on Γ gives an entire function of λ defined by:

$$G(\lambda) = \int_\Gamma g(z) \cdot z^\lambda \cdot \big|dz\big|_\Gamma$$

where $|dz|_\Gamma$ is the arc-length on Γ. With these notations one has

C.4 Theorem. *Assume that Λ satisfies the condition in Theorem C.2. Then, if $G(\lambda_\nu) = 0$ for every ν it follows that the g-function is identically zero.*

Proof. We leave as an exercise to the reader to verify that if the G-function is identically zero then $g = 0$. There remains to show that if $G(\lambda_\nu) = 0$ for every ν then $G = 0$. To oprove this we assume the contrary, i.e. that G is not identically zero. We leave it to the reader to verify that there exist constants A, K and $0 < a < \frac{\pi}{2}$ such that:

(i) $$|G(\lambda)| \leq K \cdot e^{|\lambda|} \quad \text{and} \quad |G(is)| \leq K \cdot e^{|s| \cdot a} \quad : \lambda \in \mathbf{C} : s \in \mathbf{R}$$

Set

(ii) $$U(r, \phi) = \log \left| G(re^{i\phi}) \right|$$

and let $\{r_\nu e^{i\phi_\nu}\}$ be the zeros of G in $\mathfrak{Re}(z) > 0$. Carleman's integral formula from § xx formula gives the following for each $R > 1$:

$$\sum_{1 < r_\nu < R} \left[\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\right] \cdot \cos\theta_\nu = \frac{1}{\pi R} \cdot \int_{-\pi/2}^{\pi/2} U(R, \phi) \cdot \cos\phi \cdot d\phi +$$

(iii) $$\frac{1}{2\pi R} \cdot \int_1^R \left(\frac{1}{r^2} - \frac{1}{R^2}\right) \cdot \left[U(r, \pi/2) + U(r, -\pi/2) \right] \cdot dr + c_*(R)$$

where the remainder term $c_*(R)$ satisfies

$$|c_*(R)| \leq K$$

for a constant which is independent of R. It follows easily from (i) that there exists a consntat K^* such that the right hand side is $\leq K^*$ for all $R > 1$. Next, since $\{\lambda_\nu\}$ occur in the zeros of G, the left hand side in (iii) majorizes

(iv) $$\sum_{1 < \lambda_\nu < R} \left[\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right]$$

Now we get a contradiction because the reader can easily verify that (*) entails that (iv) cannot be $\leq K^*$ for a constant as $R \to \|infty$.

Proof of Theorem C.2

Denote by $\mathcal{O}^*(\Lambda)$ the linear space of analytic functions in the right half-plane given by finite $\mathbf{C}$-linear combinations of the fractional powers $\{z^{\lambda_\nu}\}$. To obtain uniform approximations over relatively compact subsets when Ω is a simply connected domain in $\mathfrak{Re}(z) > 0$, it suffices to regard a closed Jordan arc Γ which borders a Jordan domain U where U is a relatively compact subset of Ω. In particular Γ has a positive distance to the imaginary axis and there remains to show that when (*) holds in Theorem C.2, then an arbitrary analytic function $f(z)$ defined in some open neighborhood of $\bar{U}$ can be uniformly approximated by $\mathcal{O}^*(\Lambda)$-functions over a relatively compact subset U_* of U. To achieve this we shall use a trick which reduces the proof of uniform approximation to a problem concerned with L^2-approximation on Γ. To begin with we have

C.5 Lemma. *The uniqueness in Theorem C.4 implies that if V is a real-valued function on Γ then there exists a sequence $\{Q_n\}$ from the family $\mathcal{O}(\Lambda)$ such that*

$$\lim_{n \to \infty} \int_\Gamma \left| Q_n - V \right|^2 \cdot |dz| = 0$$

The proof is left as an exercise.

C.6 A tricky construction. Let $f(z)$ be analytic in a neighborhood of the closed Jordan domain $\bar{U}$ bordered by Γ. Define a new analytic function

(1) $$F(z) = \int_{z_*}^{z} \frac{f(\zeta)}{\zeta} \cdot d\zeta$$

where z_* is some point in $\bar{U}$ whose specific choice does not affect the subsequent discussion. We can write $F = V + iW$ where $V = \mathfrak{Re}(F)$. Lemma C.5 gives a sequence $\{Q_n\}$ which approximates V in the L^2-norm on Γ. Using this L^2-approximation we get

Lemma C.7 *Let U_0 be relatively compact in U. Then there exists a sequence of real numbers $\{\gamma_n\}$ such that*

$$\lim_{n\to\infty} \big||Q_n(z) - i\cdot\gamma_n - F(z)\big|_{U_0} = 0$$

Again we leave out the proof as an exercise. Next, taking complex derivatives Lemma C.7 implies that if U_* is even smaller, i.e. taken to be a relatively compact in U_0, then there exists a uniform approximation of derivatives:

$$Q_n'(z) \to F'(z) = \frac{f(z)}{z}$$

This implies that

$$z\cdot Q_n' \to f(z)$$

holds uniformly in U_*. Next, notice that

$$z\cdot \frac{d}{dz}(z^{\lambda_\nu}) = \lambda_\nu \cdot z^{\lambda_\nu}$$

hold for each ν. Hence $\{z\cdot Q_n'(z)\}$ again belong to the $\mathcal{O}(\Lambda)$-family. So we achieve the required uniform approximation of the given f function on U_* which completes the proof of Theorem C.2.

D. Theorem of Müntz

Theorem D.1 below is due to Müntz in the article *Über den Approximationssatz von Weierstrass* from 1914. The simplified version of the original proof below is given in [Car]. Here is the set up: Let $0<\lambda_1<\lambda_2<\ldots$. To each ν we get the function x^{λ_ν} defined on the real unit interval $0\leq x\leq 1$. We say that the sequence $\Lambda=\{\lambda_\nu\}$ is L^2-dense if the family $\{x^{\lambda_\nu}\}$ generate a dense linear subspace of the Hilbert space of square integrable functions on $[0,1]$.

D.1 Theorem. *The necessary and sufficient condition for Λ to be L^2-dense is that $\sum \frac{1}{\lambda_\nu}$ is convergent.*

D.2 Proof of necessity. If Λ is not L^2-dense there exists some $h(x)\in L^2[0,1]$ which is not identically zero while

(1) $$\int_0^1 h(x)\cdot x^{\lambda_\nu}\cdot dx = 0 \quad : \quad \nu=1,2,\ldots$$

Now consider the function

(2) $$\Phi(\lambda) = \int_0^1 h(x)\cdot x^{-i\lambda}\cdot dx$$

It is clear that Φ is analytic in the right half plane $\mathfrak{Im}\,\lambda>0$. If $\lambda=s+it$ with $t>0$ we have

$$|x^\lambda| = x^t \leq 1$$

for all $0\leq x\leq 1$. From this and the Cauchy-Schwarz inequality we see that

(3) $$|\Phi(\lambda)| \leq ||h||_2 \quad : \lambda \in U_+$$

Hence Φ is a bounded analytic function in the upper half-plane. At the same time (1) means that the zero set of Φ contains the sequence $\{\lambda_\nu\cdot i\}$. By the integral formula formula we have seen in XX that this entails that

(*) $$\sum \frac{1}{\lambda_\nu} < \infty$$

which proves the necessity.

Proof of sufficiency. There remains to show that ifwe have the convergence in (*) above then there exists a non-zero h-function i $L^2[0,1]$ such that (1) above holds. To find h we first construct an analytic function Φ by

$$\text{(i)} \qquad \Phi(z) = \frac{\prod_{\nu=1}^{\infty}\left(1-\frac{z}{\lambda_\nu}\right)}{\prod_{\nu=1}^{\infty}\left(1+\frac{z}{\lambda_\nu}\right)} \cdot \frac{1}{(1+z)^2} \quad : \quad \mathfrak{Re}\, z > 0$$

Notice that $\Phi(z)$ is defined in the right half-plane since the series (*) is convergent. When $\mathfrak{Re}(z) \geq 0$ we notice that each quotient

$$\frac{1-\frac{z}{\lambda_\nu}}{1+\frac{z}{\lambda_\nu}}$$

has absolute value ≤ 1. It follows that

$$\text{(ii)} \qquad |\Phi(x+iy)| \leq \frac{1}{1+x+iy|^2} = \frac{1}{(1+x)^2+y^2}$$

In particular the function $y \mapsto \Phi(iy)$ belongs to L^2 on the real y-line. Now we set

$$\text{(ii)} \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ity} \cdot \Phi(iy) \cdot dy$$

using the inequality (ii) If $t < 0$ we can move the line integral of $e^{tz} \cdot \Phi(z)$ from the imaginary axis to a line $\mathfrak{Re}(z) = a$ for every $a > 0$ and it is clear that

$$\lim_{a \to +\infty} \int_{-\infty}^{\infty} e^{-at+ity} \cdot \Phi(a+iy) \cdot dy = 0$$

We conclude that $f(t) = 0$ when $t < 0$. Next, since $y \mapsto \Phi(iy)$ is an L^2-function it follows by Parseval's equality that

$$\int_0^{\infty} |f(t)|^2 \cdot dt < \infty$$

Moreover, for a fixed λ_ν we have

$$\int_0^{\infty} f(t)e^{-\lambda_\nu t} \cdot dt = \frac{1}{2\pi} \cdot \int_0^{\infty} [\int_{-\infty}^{\infty} e^{ity} \cdot \Phi(iy) \cdot dy\,] \cdot e^{-\lambda\nu t} \cdot dt =$$

$$\int_{-\infty}^{\infty} \frac{1}{iy-\lambda_\nu} \cdot \Phi(iy) \cdot dy$$

where the last equality follows when the repeated integral is reversed. By construction $\Phi(z)$ has a zero at λ_ν and therefore (xx) above remains true with Φ replaced by $\frac{\Phi(z)}{z-\lambda_\nu}$ which entails that

$$\int_0^{\infty} f(t) \cdot e^{-\lambda_\nu t} \cdot dt = 0$$

At this stage we obtain the requested h-function. Namely, since $t \mapsto e^{-t}$ identifies $(0,+\infty)$ with $(0,1)$ we get a function $h(x)$ on $(0,1)$ such that

$$h(e^{-t}) = e^{t/2} \cdot f(t)$$

The reader may verify that

$$\int_0^1 |h(x)|^2 \cdot dx = \int_0^{\infty} |f(t)|^2 \cdot dt$$

and hence h belongs to $L^2(0,1)$. Moreover, one verifies that the vanishing in (xx) above entails that

$$\int_0^1 h(x) \cdot x^{\lambda_\nu} \cdot dx = 0$$

Since this holds for every ν we have proved the sufficiency which therefore finishes the proof of Theorem D.1.

D.2 Another density result

Density results using exponential functions appear in many applications. For example, in the the *Sampling Theorem* by Shanning used in telecommunication engineering. Expressed in analytic function theory it can be formulated as follows:

D.3 Theorem. *Let $T > 0$ and $g(t)$ is an L^2-function on the interval $[0,T]$ which is not identically zero and suppose that $a > 0$ is such that*

$$\int_0^T e^{inat} \cdot g(t) \cdot dt = 0 \quad : \quad n \in \mathbf{Z}$$

Then we must have

$$a \geq \frac{2\pi}{T}$$

Remark This result is due to Fritz Carlson in his article [xx] from 1914. Carlson's result was later improved by Titchmarsh and goes as follows:

D.4 Theorem. *Let $0 < m_1 < m_2 < \ldots$ be an increasing sequence of positive real numbers such that*

(*) $$\limsup_{n \to \infty} \frac{n}{m_n} > 1$$

Then if $f(x) \in L^2(0,1)$ and

(**) $$\int_0^1 e^{im_n x} \cdot f(x) \cdot dx = \int_0^1 e^{-im_n x} \cdot f(x) \cdot dx = 0$$

hold for each n, it follows that $f = 0$.

Proof. Notice that (**) implies that we also have vanishing integrals using $f(-x)$. Replacing f by $f(x) + f(-x)$ or by $f(x) - f) - x)$, it suffices to prove the result when f is even or odd. Let us show that there cannot exist an even L^2-function f such that the integrals (**) vanish while

(i) $$\int_0^1 f(x) \cdot dx = 1$$

We leave it to the reader to verify that this gives Theorem D.4. Put

(ii) $$\phi(z) = \int_{-\pi}^{\pi} e^{izt} \cdot f(t) \cdot dt$$

Now the entire function ϕ is even and (i) means that $\phi(0) = 1$. Moreover, ϕ is an entire function of exponential type and Cauchy-Schwarz inequality gives

(iii) $$|\phi(x + iy)| \leq ||f||_2 \cdot e^{\pi|y|}$$

for all $z = x + iy$. Moreover, Parseval's equality shows that the restriction of ϕ to the real x-line belongs to L^2 which by the result in § X implies that ϕ belong to the Carleman class $\mathcal{N}$. So Theorem § XX gives the existence of a limit

(iv) $$\lim_{R \to \infty} \frac{N_\phi(R)}{R} = A$$

where $N_\phi(R)$ is the counting function of zeros of ϕ. The inequality (iiii) and the result in XXX entails that

(v) $$A \leq 2$$

Next, since the zeros of ϕ contains the even sequence $\{-m_\nu\} \cup \{m_\nu\}$ we have the inequality

(vi) $$N_\phi(m_n) \geq 2n$$

At the same time the limi formula (iv) gives:

(vii) $$A = \lim_{n \to \infty} \frac{N_\phi(m_n)}{m_n}$$

Finally, (vi) and the hypothesis (*) in the Theorem give

(viii) $$\limsup_{n \to \infty} \frac{N_\phi(m_n)}{m_n} \geq 2 \cdot \limsup_{n \to \infty} \frac{n}{m_n} > 2$$

This contradicts (v) and we conclude that the non-zero function ϕ cannot exist. Finally from (ii) and Fourier's inversion formula the vanishing of ϕ entails that $f = 0$ which finishes the proof.

10. Convergence under substitution.

Introduction. Let $\{a_k\}$ be a sequence of complex numbers where the additive series $\sum a_k$ is convergent. This gives an analytic function $f(z)$ defined in the open disc by

$$f(z) = \sum a_n \cdot z^n \tag{1}$$

If $0 < b < 1$ we can expand f around b and obtain another series

$$f(b+z) = \sum c_n \cdot z^n \tag{2}$$

From the convergence of $\sum a_k$ one expects that the series

$$\sum c_n \cdot (1-b)^n \tag{3}$$

also is convergent. This is indeed true and was proved by Hardy and Littlewood in (H-L]. A more general result was established in [Carleman] and we are going to expose results from Carleman's article. In general, consider a power series

$$\phi(z) = \sum b_\nu \cdot z^\nu \tag{1}$$

which represents an analytic function D where $|\phi(z)| < 1$ hold when $|z| < 1$. There exists the composed analytic function

$$f(\phi(z)) = \sum_{k=0}^{\infty} c_k \cdot z^k \tag{*}$$

We seek conditions on ϕ in order that the convergence of $\{a_k\}$ entails that the series

$$\sum c_k \quad \text{also converges} \tag{**}$$

First we consider the special case when the b-coefficients are real and non-negative.

8.1. Theorem. *Assume that $\{b_\nu \geq 0\}$ and that $\sum b_\nu = 1$. Then (**) converges and the sum is equal to $\sum a_k$.*

Proof. Since $\{b_\nu\}$ are real and non-negative the Taylor series for ϕ^k also has non-negative real coefficients for every $k \geq 2$. Put

$$\phi^k(z) = \sum B_{k\nu} \cdot z^\nu$$

and for each pair of integers k, p we set

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=p} B_{k\nu}$$

If $k \geq 2$ we notice that

$$B_{kj} = \sum_{\nu=0}^{j} b_{j-\nu} \cdot B_{k-1,\nu}$$

Since $\{b_\nu\}$ are non.-begstive with sum equal to one the reader can easiy verify that the following hold:

$$\lim_{N\to\infty} \Omega_{N,p} = 0 \quad \text{for every} \quad p \tag{i}$$

$$k \mapsto \Omega_{k,p} \quad \text{decreases for every} \quad p \tag{ii}$$

$$\sum_{\nu=0}^{\infty} B_{k\nu} = 1 \quad \text{hold for every} \quad k \tag{iii}$$

Next, the Taylor series of $f(\phi(z))$ is given by

$$\sum a_k \cdot \phi^k(z) = \sum_{\nu=0}^{\infty} \big[\sum_{k=0}^{\infty} a_k \cdot B_{k\nu}\big] \cdot z^{\nu}$$

For each positive integer n^* we set

$$\sigma_p[n^*] = \sum_{\nu=0}^{\nu=p} \big[\sum_{k=0}^{k=n^*} a_k \cdot B_{k,\nu}\big] \tag{1}$$

$$\sigma_p(n^*) = \sum_{\nu=0}^{\nu=p} \big[\sum_{k=n^*+1}^{\infty} a_k \cdot B_{k,\nu}\big] = \sum_{k=n^*+1}^{\infty} a_k \cdot \Omega_{k,p} \tag{2}$$

Notice that

$$\sigma_p[n^*] + \sigma_p(n^*) = \sum_{k=0}^{k=p} c_k \quad \text{hold for each} \quad p$$

Our aim is to show that the last partial sums converge to 1 as $p \to \infty$. To obtain this we study the σ-terms separately. Introduce the partial sums

$$s_n = \sum_{k=0}^{k=n} a_k$$

By assumption there exists a limit $s_n \to S$ where $S = 1$. This entails that the sequence $\{s_k\}$ is bounded and so is the sequence $\{a_k = s_k - s_{k-1}\}$. By (i) above it follows that the last term in (2) tends to zero when n^* increases. So if $\epsilon > 0$ we find $n*$ such that

$$n \geq n^* \implies |\sigma_p(n)| \leq \epsilon \tag{3}$$

A study of $\sigma_p[n^*]$. Keeping n^* and ϵ fixed we apply (iii) for each $0 \leq k \leq n^*$ and find an integer p^* such that

$$1 - \sum_{\nu=0}^{\nu=p} B_{k,\nu} \leq \frac{\epsilon}{n^*+1} \quad \text{for all pairs} \quad p \geq p^* : 0 \leq k \leq n^*$$

The triangle inequality gives

$$|\sigma_p(n^*) - s_{n^*}| \leq \frac{\epsilon}{n^*+1} \cdot \sum_{k=0}^{k=n^*} |a_k| \quad \text{for all} \quad p \geq p^* \tag{4}$$

Since $\sum a_k$ converges the sequence $\{a_k\}$ is bounded, i.e. we have a constant M such that $|a_k| \leq M$ for all k. Hence (4) gives

$$|\sigma_p(n^*) - S| \leq \epsilon + \epsilon \cdot M \quad : \quad p \geq p^* \tag{5}$$

Together with (3) this entails that

$$n \geq n^* \implies |\sum_{k=0}^{n^*} |c_k - s| \leq 2\epsilon + M \cdot \epsilon$$

Since we can chose ϵ arbitrary small we conclude that $\sum c_k$ converges and the limit is equal to S which finishes the proof of Theorem 8.1.

8.2 A general case. Now we relax the condition that $\{b_\nu\}$ are real and non-negative but impose extra conditions on ϕ. First we assume that $\phi(z)$ extends to a continuous function on the closed disc, i.e. ϕ belongs to the disc-algebra. Moreover, we assume that $\phi(1) = 1$ while $|\phi(z)| < 1$ for all $z \in \bar{D} \setminus \{1\}$ which means that $z = 1$ is a peak point for ϕ. Consider also the function $\theta \mapsto \phi(e^{i\theta})$ where θ is close to zero. The final condition on ϕ is that there exists some positive real number β and a constant C such that

$$|\phi(e^{i\theta}) - 1 - i\beta| \leq C \cdot \theta^2 \tag{8.2.1}$$

holds in some interval $-\ell \leq \theta \geq \ell$. This implies that for every integer $n \geq 2$ we get another constant C_n so that

(8.2.2) $$|\phi^n(e^{i\theta}) - 1 - in\beta| \leq C_n \cdot \theta^2$$

Hence the following integrals exist for all pairs of integers $p \geq 0$ and $n \geq 1$:

(8.2.3) $$J(n,p) = \int_{-\ell}^{\ell} \frac{\phi(e^{i\theta})^n \cdot (1 - \phi(e^{i\theta}))}{e^{ip\theta} \cdot (1 - e^{i\theta})} \cdot d\theta$$

With these notations one has

8.2 Theorem. *Let ϕ satisfy the conditions above. Then, if there exists a constant C such that*

(*) $$\sum_{k=0}^{\infty} |J(k,p)| \leq C \quad \text{for all} \quad p \geq 0$$

it follows that the series (**) from the introduction converges and the sum is equal to $\sum a_k$.

Proof With similar notations as in the previous proof we introduce the Ω-numbers by:

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=k} B_{k\nu}$$

Repeating the proof of Theorem 8.1 the reader may verify that the series $\sum c_k$ converges and has the limit S if the following two conditions hold:

(i) $$\lim_{N\to\infty} \Omega_{N,p} = 0 \quad \text{holds for every} \quad p$$

(ii) $$\sum_{k=0}^{\infty} \big|\Omega_{k+1,p} - \Omega_{k,p}\big| \leq C \quad \text{for a constant} \quad C$$

where C is is independent of p. Here (i) is clear since $\{g_N(z) = \phi^N(z)\}$ converge uniformly to zero in compact subsets of the unit disc and therefore their Taylor coefficients tend to zero with N. To get (ii) we use residue calculus which gives:

(iii) $$\Omega_{k+1,p} - \Omega_{k,p} = \frac{1}{2\pi i} \int_{|z|=1} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz$$

Let ℓ be a small positive number and T_ℓ denotes the portion of the unit circle where $\ell \leq \theta \leq 2\pi - \ell$. Since 1 is a peak -point for ϕ there exists some $\mu < 1$ such that

$$\max_{z \in T_\ell} |\phi(z)| \leq \mu$$

This gives

(iv) $$\frac{1}{2\pi} \cdot \Big| \int_{z \in T_\ell} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz \Big| \leq \mu^k \cdot \frac{2}{|e^{i\ell} - 1} \Big|$$

Since the geometric series $\sum \mu^k$ converges it follows from (iii) and the construction of the J-integrals in (3) that (ii) above holds precisely when

$$\sum_{k=0}^{\infty} |J_\ell(k,p]| \leq C$$

for a constant C which is independent of p. This finishes the the proof of Theorem 8.2.

8.3. Oscillatory integrals. Condition (*) in Theorem 8.2.1 is implicit. We shall therefore find a sufficient condition in order that the J-integrals satisfy (*) which is expressed by local conditions on the ϕ-function close to $z = 1$. To begin with (8.2.1) enable us to write

(i) $$\phi(e^{i\theta}) = e^{i\beta\theta + \rho(\theta)}$$

when $\theta \simeq 0$ and

(ii) $$\rho(\theta) = O(\theta^2)$$

The next result gives the requested convergence of the composed series expressed by an additional condition on the ρ-function in (i) above.

8.4. Theorem. *Assume that $\rho(\theta)$ is a C^2-function on some interval $-\ell < \theta < \ell$ and that the second derivative $\rho''(0)$ is real and negative. Then (*) in Theorem 8.2 holds.*

Remark. The proof is left as a (hard) exercise to the reader. If necessary, consult Carleman's article [Car] for a detailed proof.

11. The series $\sum \big[a_1 \cdots a_\nu\big]^{\frac{1}{\nu}}$

Introduction. We shall prove a result from [Carleman:xx. Note V page 112-115]. Let $\{a_\nu\}$ be a sequence of positive real numbers such that $\sum a_\nu < \infty$ and e denotes Neper's constant.

9.1 Theorem. *Assume that the series $\sum a_\nu$ is convergent and let S be the sum. Then one has the strict inequality*

(*) $$\sum_{\nu=1}^{\infty} \big[a_1 \cdots a_\nu\big]^{\frac{1}{\nu}} < e \cdot S$$

Remark. The result is sharp in the sense that e cannot be replaced by a smaller constant. To see this we consider a large positive integer N and take the finite series $\{a_\nu = \frac{1}{\nu} \quad : 1 \leq \nu \leq N\}$. Stirling's limit formula gives:

$$\big[a_1 \cdots a_\nu\big]^{\frac{1}{\nu}} \simeq \frac{e}{\nu} \quad : \nu >> 1$$

Since the harmonic series $\sum \frac{1}{\nu}$ is divergent we conclude that for every $\epsilon > 0$ there exists some large integer N such that $\{a_\nu = \frac{1}{\nu}\}$ gives

$$\sum_{\nu=1}^{\nu=N} \big[a_1 \cdots a_\nu\big]^{\frac{1}{\nu}} > (e - \epsilon) \cdot \sum_{\nu=1}^{\nu=N} \frac{1}{\nu}$$

There remains to prove the strict upper bound (*) when $\sum a_\nu$ is a convergent positive series. To attain this we first establish inequalities for finite series. Given a positive integer m we consider the variational problem

(1) $$\max_{a_1,\ldots,a_m} \sum_{\nu=1}^{\nu=m} \big[a_1 \cdots a_\nu\big]^{\frac{1}{\nu}} \quad \text{when} \quad a_1 + \ldots + a_m = 1$$

Let $a_1^*, \ldots, a_m^*$ give a maximum and set $a_\nu^* = e^{-x_\nu}$. The Lagrange multiplier theorem gives a number $\lambda^*(m)$ such that if

$$y_\nu = \frac{x_\nu + \ldots + x_m}{\nu}$$

then

(2) $$\lambda^*(m) \cdot e^{-x_\nu} = \frac{1}{\nu} \cdot e^{-y_\nu} + \ldots + \frac{1}{m} \cdot e^{-y_m} \quad : \quad 1 \leq \nu \leq m$$

A summation over all ν gives

$$\lambda^*(m) = e^{-y_1} + \ldots + e^{-y_m} = \sum_{\nu=1}^{\nu=m} \big[a_1^* \cdots a_\nu^*\big]^{\frac{1}{\nu}}$$

Hence $\lambda^*(m)$ gives the maximum for the variational problem which is no surprise since $\lambda^*(m)$ is Lagrange's multiplier. Now we shall prove the strict inequality

(3) $$\lambda^*(m) < e$$

We prove (3) by contradiction. To begin with, subtracting the successive equalities in (2) we get the following equations:

$$\lambda^*(m)\cdot[e^{-x_\nu}-e^{-x_{\nu+1}}]=\frac{1}{\nu}\cdot e^{-y_\nu}\quad:\quad 1\leq\nu\leq m-1 \tag{4}$$

$$m\cdot\lambda^*(m)=e^{x_m-y_m} \tag{5}$$

Next, set

$$\omega_\nu=\nu(1-\frac{a_{\nu+1}}{a_\nu}):\quad 1\leq\nu\leq m-1 \tag{6}$$

With these notations it is clear that (4) gives

$$\lambda^*(m)\cdot\omega_\nu=e^{x_\nu-y_\nu}\quad:\quad 1\leq\nu\leq m-1 \tag{7}$$

It is clear that (7) gives:

$$\big(\lambda^*(m)\cdot\omega_\nu\big)^\nu=e^{\nu(x_\nu-y_\nu)}=\frac{a_1\cdots a_{\nu-1}}{a_\nu^{\nu-1}} \tag{8}$$

By an induction over ν which is left to the reader it follows the ω-sequence satisfies the recurrence equations:

$$\omega_\nu^\nu=\frac{1}{\lambda^*(m)}\cdot\big(\frac{\omega_{\nu-1}}{1-\frac{\omega_{\nu-1}}{\nu-1}}\big)^{\nu-1}\quad:\quad 1\leq\nu\leq m-1 \tag{9}$$

Notice that we also have

$$\omega_1=\frac{1}{\lambda^*(m)} \tag{10}$$

A special construction. With λ as a parameter we define a sequence $\{\mu_\nu(\lambda)\}$ by the recursion formula:

$$\mu_1(\lambda)=\frac{1}{\lambda}\quad\text{and}\quad[\mu_\nu(\lambda)]^\nu=\frac{1}{\lambda}\cdot\big[\frac{\mu_{\nu-1}(\lambda)}{1-\frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1}\quad:\quad\nu\geq 2 \tag{**}$$

From (5) and (9) it is clear that $\lambda=\lambda^*(m)$ gives the equality

$$\mu_m(\lambda^*(m))=m \tag{***}$$

Now we come to the key point during the whole proof.

Lemma *If $\lambda\geq e$ then the $\mu(\lambda)$ -sequence satisfies*

$$\mu_\nu(\lambda)<\frac{\nu}{\nu+1}\quad:\quad\nu=1,2,\dots$$

Proof. We use an induction over ν. With $\lambda\geq e$ we have $\frac{1}{\lambda}<\frac{1}{2}$ so the case $\nu=1$ is okay. If $\nu\geq 1$ and the lemma holds for $\nu-1$, then the recursion formula (**) and the hypothesis $\lambda\geq e$ give:

$$[\mu_\nu(\lambda)]^\nu=\frac{1}{\lambda}\cdot\big[\frac{\mu_{\nu-1}(\lambda)}{1-\frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1}<\frac{1}{e}\cdot\big[\frac{\frac{\nu-1}{\nu}}{1-\frac{\nu-1}{\nu(\nu-1)}}\big]^{\nu-1}$$

Notice that the last factor is 1 and hence:

$$[\mu_\nu(\lambda)]^\nu<e<\big(1+\frac{1}{\nu}\big)^{-\nu}$$

where the last inequality follows from the wellknown limit of Neper's constant. Taking the ν:th root we get $\mu_\nu(\lambda)<\frac{\nu}{\nu+1}$ which finishes the induction.

Conclusion. If $\lambda^*(m)\geq e$ then the lemma above and the equality (***) would entail that

$$m=\mu(\lambda^*(m))<\frac{m}{m+1}$$

This is impossible when m is a positive integer and hence we must have proved the strict inequality $\lambda^*(m)<e$.

The strict inequality for an infinite series. It remains to prove that the strict inequality holds for a convergent series with an infinite number of terms. So assume that we have an equality

$$\text{(i)} \qquad \sum_{\nu=1}^{\infty} \big[a_1 \cdots a_\nu\big]^{\frac{1}{\nu}} = e \cdot \sum_{\nu=1}^{\infty} a_\nu$$

Put as as above

$$\text{(ii)} \qquad \omega_n = n(1 - \frac{a_{n+1}}{a_n})$$

Since we already know that the left hand side is at least equal to the right hand side one can apply Lagrange multipliers and we leave it to the reader to verify that the equality (i) gives the recursion formulas

$$\text{(iii)} \qquad \omega_n^n = \frac{1}{e} \cdot \big[\frac{\omega_{n-1}}{1 - \frac{\omega_{n-1}}{n-1}}\big]^{n-1}$$

Repeating the proof of the Lemma above it follows that

$$\text{(iv)} \qquad \omega_n < \frac{n}{n+1} \implies \frac{a_{n+1}}{a_n} > \frac{n}{n+1}$$

where (ii) gives the implication. So with $N \geq 2$ one has:

$$\frac{a_{N+1}}{a_1} > \frac{1 \cdots N}{1 \cdots N(N+1)} = \frac{1}{N+1}$$

Now $a_1 > 0$ and since the harmonic series $\sum \frac{1}{N}$ is divergent it would follows that $\sum a_n$ is divergent. This contradiction shows that a strict inequality must hold in Theorem 9.1.

Appendix

.

A. Entire functions of exponential type

The class $\mathcal{E}$ of entire functions of exponential type is defined as follows:

A.0 Definition. *An entire function f belongs to $\mathcal{E}$ if and only if there exists constants A and C such that*

$$(*) \qquad |f(z)| \leq C \cdot e^{A|z|} \quad : \ z \in \mathbf{C}$$

The results in Sections A-B are due to Hadamard and Lindelöf. The class $\mathcal{N}$ which appears in Section 3 was introduced by Carleman who used it to prove certain approximation theorems related to moment problems. The main results deal with Tauberian theorems which is treated in section D and based upon Chapter V in [Paley-Wiener]. Let us present some of the results to be proved in § D while we refer to § A-B for more elementary material about the class $\mathcal{E}$. Consider a non-decreasing sequence $\{\lambda_\nu\}$ of positive real numbers such that the series

$$(1) \qquad \sum \lambda_\nu^{-2} < \infty$$

When this holds there exists the entire function given by a product series:

$$H(z) = \prod (1 - \frac{z^2}{\lambda\nu^2})$$

Notice that $H(z)$ is positive on the imaginary axis. We get the function defined for real $y > 0$:

$$y \mapsto \frac{\log H(iy)}{y} = \frac{1}{y} \sum \log (1 + \frac{y^2}{\lambda\nu^2})$$

At the same time we consider the intergals

$$J(R) = \int_{-R}^{R} \frac{\log |H(x)| \cdot dx}{x^2}$$

0.1 Theorem. *The existence of a constant A such that*

$$\text{(i)} \qquad \lim_{y \to \infty} \frac{\log H(iy)}{y} = \pi A$$

and

$$\text{(ii)} \qquad \lim_{R \to \infty} J(R) = -\pi^2 A$$

are completely equivalent

Remark. Of special interest is the case when the limit in (ii) is automatic via an integrability condition, i.e. when

$$(*) \qquad \int_{-\infty}^{\infty} \frac{\big|\log |H(x)|\,\big| \cdot dx}{x^2} < \infty$$

In this case the J-integrals converge and as a consequence there is a limit in (i) for some $A \geq 0$. It turns out that further conclusions can be made. namely, the convergence of (*) implies that the sequence $\{\lambda_\nu\}$ also has a regular growth in the sense that if $N(r)$ is the counting function which for every $r > 0$ counts the number of $\lambda\nu \leq r$, then there exists the limit

$$\lim_{R \to \infty} \frac{N(R)}{R} = A$$

with A determined via Theorem 0.1. We shall prove this in Section D and remark that the integrability condition (*) is related to the study of the Carleman class in Section C.

Growth of entire functions.

Each entire function $f(z)$ can be written in the form

$$f(z) = az^m \cdot f_*(z)$$

where f_* is entire and $f_*(0) = 1$. The case when $f(0) = 1$ is therefore not so special and several formulas below take a simpler form when this holds.

A.1 The functions $T_f(R)$ and $m_f(R)$. They are defined for every $R > 0$ by

(i)
$$T_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \log^+ \big|f(R(e^{i\theta})\big| \, d\theta$$

(ii)
$$m_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \log^+ \big[\frac{1}{\big|f(R(e^{i\theta})\big|}\big] \, d\theta$$

A.2 The maximum modulus function. It is defined by

$$M_f(R) = \max_{0 \leq \theta \leq 2\pi} |f(Re^{i\theta})|$$

A.3 The counting function $N_f(R)$. To each $R > 0$ we count the number of zeros of f in the punctured disc $0 < |z| < R$. This integer is denoted by $N_f(R)$, where multiple zeros are counted according to their multiplicities. Jensen's formula shows that if $f(0) = 1$ then

(A.3.1)
$$\int_0^R \frac{N_f(s)}{s} \cdot ds = \frac{1}{2\pi} \cdot \int_0^{2\pi} \log \big|f(R(e^{i\theta})\big| \cdot d\theta = T_f(R) - m_f(R)$$

Since the left hand side always is ≥ 0, the inequality below holds under the hypothesis that $f(0) = 1$:

(A.3.2)
$$m_f(R) \leq T_f(R)$$

Next, since $N_f(R)$ is increasing we get

$$\log 2 \cdot N_f(R) \leq \int_R^{2R} \frac{N_f(s)}{s} \cdot ds \leq T_f(2R) \implies$$

(A.3.3)
$$N_f(R) \leq \frac{T_f(2R)}{\log 2}$$

A.4 Harnack's inequality. The function $\log^+ |f|$ is subharmonic which implies that whenever $0 < r < R$ then one has

$$\log^+ |f(re^{i\alpha})| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{R+r}{R-r} \cdot \log^+ \big|f(R(e^{i\theta})\big| \cdot d\theta$$

It follows that

$$M_f(r) \leq \frac{R+r}{R-r} \cdot T_f(R)$$

In particular we can take $R = 2r$ and conclude that

$$M_f(r) \leq 3 \cdot T_f(2r) \quad \text{hold for every} \quad r > 0$$

The last inequality gives:

A.5 Theorem. *An entire function f belongs to $\mathcal{E}$ if and only if there exists a constant A such that*

$$T_f(R) \leq A \cdot R$$

holds for every R.

A.6 A division theorem. Let f and g be in $\mathcal{E}$ and assume that $h = \frac{f}{g}$ is entire. Now

(i) $$\log^+ |h| \leq \log^+ |f| + \log^+ |g|$$

In the case when $g(0) = 1$ we apply (**) in A.3 and conclude that

$$T_h(R) \leq T_f(R) + T_g(R)$$

Hence Theorem A.5 implies that h belongs to $\mathcal{E}$. We leave it to the reader to verify that this conclusion holds in general, i.e. without any assumption on $g(0)$.

A.7 Hadamard products. Let $\{\alpha_\nu\}$ be a sequence of complex numbers arranged so that the absolute values are non-decreasing. The counting function of the sequence is denoted by $N_{\alpha(\bullet)}(R)$. Suppose that the counting function satisfies:

(*) $$N_{\alpha(\bullet)}(R) \leq A \cdot R \quad \text{for all} \quad R \geq 1$$

A.8 Theorem *When (*) holds the infinite product*

$$\prod (1 - \frac{z}{\alpha_\nu}) \cdot e^{\frac{z}{\alpha_\nu}}$$

converges for every z and gives an entire function to be denoted by $H_{\alpha(\bullet)}$ and called the Hadamard product of the α-sequence.

A.9 Exercise. Prove this theorem and show also that there exists a constant C which is independent of A such that the Hadamard product satisfies the growth condition:

$$\big|H_{\alpha(\bullet)}(z)\big| \leq C \cdot e^{A\cdot|z|\cdot\log\,|z|} \quad \text{for all} \quad |z| \geq e$$

A.10 Lindelöf's condition. For a sequence $\{\alpha_\nu\}$ we define the Lindelöf function

$$L(R) = \sum_{|\alpha_\nu|<R} \frac{1}{\alpha_\nu}$$

We say that $\{\alpha_\nu\}$ is of Lindelöf type if there there exists a constant L^* such that

(A.10.1) $$\big|L(R)\big| \leq L^* \quad \text{hold for all} \quad R.$$

A.11 Theorem. *If the α-sequence satisfies (*) above A.8 and is of the Lindelöf type, then there exists a constant C such that the maximum modulos function of $H_{\alpha(\bullet)}$ satisfies*

$$M_{H_{\alpha(\bullet)}}(R) \leq C \cdot e^{AR}$$

and hence the entire function $H_{\alpha(\bullet)}(z)$ belongs to $\mathcal{E}$.

A.12 Exercise. Prove this result. A hint is to study the products

$$\prod_{|\alpha_\nu|<2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}} \quad \text{and} \quad \prod_{|\alpha_\nu|\geq 2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}}$$

separately for every $R \geq 1$. Try also to find an upper bound for C expressed by A and L^*.

A converse result. Let f belong to $\mathcal{E}$. Then the set of zeros $\{\alpha_\nu\}$ satisfies the Lindelöf condition, i.e. (**) in A.10 holds for a constant L^*. To prove this we shall use:

A.13 An integral formula. With $R > 0$ we put $g(z) = \frac{1}{z} - \frac{\bar{z}}{R^2}$. This is a is harmonic function in $\{0 < |z| > R\}$ and $g = 0$ on $|z| = R$. Let $f(z)$ be an entire function with $f(0) = 1$ and consider a pair $0 < \epsilon < R$ where f has not zeros in $|z| \leq \epsilon$. Applying Green's formula to g and $\log |f|$ on an annulus $\{\epsilon < |z| < R\}$ we get:

(A.13.1) $$\sum_{|\alpha_\nu|<R} [\frac{1}{\alpha_\nu} - \frac{\bar{\alpha}_\nu}{R^2}] = \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \log |f(Re^{i\theta})| \cdot e^{-i\theta} \cdot d\theta - f'(0)$$

where the sum is taken over zeros of f repeated with multiplicities in the disc $\{|z| < R\}$.

A.14 The case $f \in \mathcal{E}$. From (A.3.3) it follows that the counting function $N_f(R)$ is bounded by $C \cdot R$ for some constant C. With $f(0) = 1$ and $R > 0$ one has

$$|\sum_{|\alpha_\nu|<R} \frac{\bar{\alpha}_\nu}{R^2}| \leq \int_0^R s \cdot dN(s) \leq R \cdot N(R)$$

Hence

$$R \mapsto R^{-2} \cdot \sum_{|\alpha_\nu|<R} \frac{\bar{\alpha}_\nu}{R^2}$$

is a bounded function. By (A.13.1) we conclude that the Lindelöf functionof its zeros satisfies (A.10.1) if

$$R \mapsto \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \log |f(Re^{i\theta})| \cdot e^{-i\theta} \cdot d\theta$$

is bounded and this follows from Theorem A.5 and the inequality (A.3.2).

B. The factorisation theorem for $\mathcal{E}$

Consider some $f \in \mathcal{E}$. If f has a zero at the origin we can write

$$f(z) = az^m \cdot f_*(z) \quad \text{where} \quad f_*(0) = 1$$

It is clear that f_* again belongs to $\mathcal{E}$ and in this way we essentially reduce the study of $\mathcal{E}$-functions f to the case when $f(0) = 1$. Above we proved that the set of zeros satisfies Lindelöf's condition and therefore the Hadamard product

$$H_f(z) = \prod (1 - \frac{z}{\alpha_\nu}) \cdot e^{\frac{z}{\alpha_\nu}}$$

taken over all zeros of f outside the origin belongs to $\mathcal{E}$. Now the quotient f/H_f is entire and we shall prove:

B.1 Theorem *Let $f \in \mathcal{E}$ where $f(0) = 1$. Then there exists a complex number b such that*

$$f(z) = e^{bz} \cdot H_f(z)$$

Proof. The division in A.6 shows that the function

$$G = \frac{f}{H_f}$$

is entire and belongs to $\mathcal{E}$. By construction G is zero-free which gives the entire function $g = \log G$ for we have the inequality

$$|g(z)| \leq 1 + \log^+ |G(z)| \leq 1 + C|z|$$

Since $G \in \mathcal{E}$ we see that $|g|$ increases at most like a linear function so by Liouvile's theorem it is a polynomial of degree 1. Since $f(0) = 1$ we have $g(0) = 0$ and hence $g(z) = bz$ for a complex number b and the formula in Theorem B.1 follows.

C. The Carleman class $\mathcal{N}$

Let $f \in \mathcal{E}$. On the real x-axis we have the non-negative function $\log^+ |f(x)|$. If the integral

$$(*) \qquad \int_{-\infty}^{\infty} \frac{\log^+ |f(x)| \cdot dx}{1 + x^2} < \infty$$

we say that f belongs to the Carleman class denoted by $\mathcal{N}$. To study $\mathcal{N}$ the following integral formula plays an important role.

C.1 Integral formula in a half-plane. Let $g(z)$ be analytic in the half plane $\mathfrak{Im}(z) > 0$ which extends continuously to the boundary $y = 0$, i.e. to the real x-axis and that $g(0) = 1$. For each pair $0 < \ell < R$ we consider the domain

$$\Omega_{\ell,R} = \{\ell^2 < x^2 + y^2 < R^2\} \cap \{y > 0\}$$

With $z = re^{i\theta}$ we have the harmonic function

$$v(r,\theta) = (\frac{1}{r} - \frac{r}{R^2})\sin\theta = \frac{y}{x^2+y^2} - \frac{y}{R^2}$$

Here $v = 0$ on the upper half circle where $|z| = R$ and $y > 0$ and the outer normal derivative along the x-axis becomes

$$\partial_n(v) = -\partial_y(v) = -\frac{1}{x^2} + \frac{1}{R^2} \quad : \quad x \neq 0$$

Let $\{\alpha_\nu\}$ be the zeros of g counted with multiplicites in the upper half-plane.

C.2 Proposition. *One has the equation*

$$2\pi \cdot \sum \frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha_\nu|^2} - \frac{\mathfrak{Im}\,\alpha_\nu}{R^2} =$$

$$\text{(C.2.1)} \qquad \int_\ell^R (\frac{1}{R^2} - \frac{1}{x^2}) \cdot \log|g(x) \cdot g(-x)|\,dx - \frac{2}{R}\int_0^\pi \sin(\theta) \cdot \log|g(Re^{i\theta})|\,d\theta + \chi(\ell)$$

where $\chi(\ell)$ is a contribution from line integrals along the half circle $|z| = \ell$ with $y > 0$.

C.3 Exercise Prove this equation using Green's theorem. Above the term $\chi(\ell)$ is independent of R so (*) can be used to study the asymptotic behaviour as $R \to +\infty$.

Next, the family of analytic functions $g(z)$ in the upper half-plane is identified with $\mathcal{O}(D)$ using a conformal map, i.e. with a given g we get $g_* \in \mathcal{O}(D)$ where

$$g_*(\frac{z-i}{z+i}) = g(z)$$

holds when $\mathfrak{Im}(z) > 0$. When g extends to a continuous function on the real x-axis the reader can verify the equality

$$\text{(i)} \qquad \int_0^{2\pi} \log^+|g_*(e^{i\theta})|\,d\theta = 2 \cdot \int_{-\infty}^{\infty} \frac{\log^+|g(x)|}{1+x^2}\,dx$$

This means that the last integral is finite if and only if g_* belongs to the Jensen-Nevanlinna class and in XX we proved that this entails that

$$\text{(ii)} \qquad \int_0^{2\pi} \log^+ \frac{1}{|g_*(e^{i\theta})|} \cdot d\theta < \infty$$

In particular we conclude that if an entire function f satisfies (*) above then it follows that

$$\text{(iii)} \qquad \int_{-\infty}^{\infty} \log^+ \frac{1}{|f(x)|} \cdot \frac{dx}{1+x^2}$$

Since the absolute value of $\log|f(x)|$ is equal to the sum

$$\log^+ \frac{1}{|f(x)|} + \log^+|f(x)|$$

we conclude from (iii) above that (*) entails that the absolute value of $\log|f(x)|$ is integrable with respect to the density $\frac{1}{1+x^2}$. Using this we can prove:

C.4 Theorem *Let $f \in \mathcal{N}$. Then*

$$\sum^* \mathfrak{Im}\frac{1}{\alpha_\nu} < \infty$$

where the sum is taken over all zeros of f which belong to the upper half-plane.

Proof. Since $f \in \mathcal{E}$ there exists a constant C such that $N_f(R) \leq C \cdot R$. If $R \geq 1$ it follows that

$$\left|R^{-2} \sum \bar{\alpha}_\nu\right| \leq R^{-2} \cdot R \cdot N_f(R) \leq C$$

where the sum is taken over zeros in $\Omega_{\ell,R}$. Next, since $\mathfrak{Im}\,\alpha_\nu > 0$ in this open set it follows that

$$\frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha_\nu|^2} > 0$$

for every zero in the upper half-plane. In particular this holds for the zeros in $\Omega_{\ell,R}$ and passing to the limit as $R \to \infty$ it suffices to establish an upper bound in the right hand side of Proposition C.2 with $g = f$. The integral taken over the half-circle where $|z| = R$ is uniformly bounded with respect to R since $f \in \mathcal{E}$ and we have the inequality XX from A.XX. For the integral on the x-axis we therefore only need an upper bound. Since $R^{-2} - x^{-2} \leq 0$ during the integration it suffices to find a constant C such that

$$\int_\ell^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \cdot \log^+ \frac{1}{|f(x) \cdot f(-x)|} \cdot dx \leq C \quad \text{hold for all} \quad R \geq 1$$

The reader may verify that such a constant C since (iii) above Theorem C.4 holds.

D. A Tauberian Theorem

Let Λ be a non-decreasing and discrete sequence of positive real numbers $\{t_\nu\}$ whose counting function satisfies $\mathcal{N}_\Lambda(R) \leq C \cdot R$ for some constant. We get the entire function

$$f(z) = \prod \left(1 - \frac{z^2}{t_\nu^2}\right)$$

which by the results in § A belongs to $\mathcal{E}$. If $R > 0$ we set:

(*) $$J_1(R) = \frac{\log f(iR)}{R} \quad \text{and} \quad J_2(R) = \int_{-R}^R \frac{\operatorname{Log}|f(x)|}{x^2} \cdot dx$$

D.1 Theorem. *There exists a limit*

$$\lim_{R \to \infty} \frac{N_f(R)}{R} = 2A$$

if and only if at least one of the J-functions has a limit as $R \to \infty$. Moreover, when this holds one has the equalities:

$$\lim_{R \to \infty} J_1(R) = \frac{\pi \cdot A}{2} \quad \text{and} \quad \lim_{R \to \infty} J_2(R) = -\frac{\pi^2 \cdot A}{2}$$

The proof reauires seveeral steps. First we introduce the following:

D.2 The W-functions. On the positive real t-line we define the following functions:

(1) $$W_0(t) = \frac{1}{t} \quad : \; t \geq 1 \quad \text{and} \quad W_0(t) = 0 \quad \text{when} \; t < 1$$

(2) $$W_1(t) = \frac{\log(1 + t^2)}{t}$$

(3) $$W_2(t) = \int_0^t \frac{\log|1 - x^2|}{x^2} \cdot dx$$

Next, the real sequence $\Lambda = \{t_\nu\}$ gives a discrete measure on the positive real axis where one assigns a unit point mass at every t_ν. If repetitions occur, i.e. if some t-numbers are equal we add these unit point-masses. Let ρ denote the resulting discrete measure. The constructions of the J-functions obviously give:

(4) $$J_k(R) = \int_0^\infty W_k\left(\frac{R}{t}\right) \cdot \frac{d\rho(t)}{t} \quad : \; k = 1, 2$$

Moreover, the reader can verify that

$$\frac{\mathcal{N}_\Lambda(R)}{R} = 2 \cdot \int_0^\infty W_0(R/t) \cdot \frac{d\rho(t)}{t} \tag{5}$$

D.3 Exercise. Show that under the assumption that the function $\frac{\mathcal{N}_\Lambda(R)}{R}$ is bounded, it follows the three $\mathcal{W}$-functions belong to the $\mathcal{BW}$-algebra defined by the measure ρ as explained in § XXX.

D.4 Fourier transforms. Recall that on $\{t > 0\}$ we have the Haar measure $\frac{dt}{t}$. We leave it to the reader to verify that all the W-functions above belong to $L^1(\mathbf{R}^+)$, i.e.

$$\int_0^\infty \big|W_k(t)\big| \cdot \frac{dt}{t} < \infty \quad : k = 0, 1, 2 \tag{i}$$

The Fourier transforms are defined by

$$\widehat{W}_k(s) = \int_0^\infty W_k(t) \cdot t^{-(is+1)} \cdot dt \tag{ii}$$

We shall prefer to use the functions with reversed sign on s, i.e. set

$$\mathcal{F}\,W_k(s) = \int_0^\infty W_k(t) \cdot t^{is-1} \cdot dt \tag{iii}$$

D.5 Proposition *One has the formulas*

$$\mathcal{F}\,W_0(s) = \frac{1}{1 - is} \tag{i}$$

$$\mathcal{F}\,W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1 - is) \cdot (1 + e^{-\pi s})}$$

$$\mathcal{F}\,W_2(s) = \frac{1}{is} \cdot \Big[\frac{i\pi}{1 - is} + \frac{2\pi}{(i + s) \cdot (e^{\pi s/2} + e^{-\pi s/2})}\Big] \tag{iii}$$

Proof. Equation (i) is easily verified and left to the reader. To prove (ii) we use a partial integration which gives

$$\mathcal{F}\,W_1(s) = \frac{1}{is - 1} \cdot \int_0^\infty \frac{2 \cdot t^{is} \cdot dt}{1 + t^2}$$

To compute this integral we employ residue calculus where we shall use the function

$$\phi(z) = \frac{z^{is}}{1 + z^2}$$

We perform line integrals over large half-circles where $z = Re^{i\theta}$ and $0 \leq \theta \leq \pi$. A residue occurs at $z = i$. Notice also that if $t > 0$ then

$$(-t)^{is} = t^{is} \cdot e^{-\pi s}$$

This gives

$$\mathcal{F}W_1(s) = \frac{1}{1 - is) \cdot (1 + e^{-\pi s})} \cdot \lim_{R \to \infty} \int_{-R}^R \phi(t) \cdot dt$$

Here ϕ has a simple pole at $z = 1$ so by residue calculus the last integral becomes

$$-2\pi i \cdot (i)^{is} \cdot \frac{1}{2i} = -\pi \cdot e^{-\pi s/2}$$

Taking the minus sign into the account we conclude that

$$\mathcal{F}W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1 - is) \cdot (1 + e^{-\pi s})}$$

For (iii) a partial integration gives

$$\mathcal{F}W_2(s) = -\frac{1}{is} \cdot \int_0^\infty \log|1-t^2| \cdot t^{is-2} \cdot dt$$

Here we computed the right hand side in [Residue Calculus] which gives (iii).

D.6 Proof of Theorem D.1

The formulas for the Fourier transforms in Proposition D.5 show that each of them is $\neq 0$ on the real s-line. Hence we can apply the general result in XX to the discrete measure ρ since the $\mathcal{W}$-functions belong to the $\mathcal{BW}$-algebra from § XXX. This implies that if one of the three limits in Theorem D.1 exists, so do the other. To get the relation between the limit values we only have to evaluate the Fourier transform at $s=0$. From Proposition D.5 we see that

$$\mathcal{F}W_0(0) = 1 \quad : \quad F\mathcal{W}_1(0) = \frac{\pi}{2}$$

Finally, (iii) in (D.4) and a computation which is left to the reader gives

$$(**) \qquad F\mathcal{W}_2(0) = -\frac{\pi^2}{2}$$

This gives the formulas in Theorem D.3 by the general result for $\mathcal{BW}$-algebras in XXX.

D.7 An application. Using Theorem D.1 we can prove the following:

D.8 Theorem *For each $f \in \mathcal{N}$ there exists the limit:*

$$\lim_{R\to\infty} \frac{N_f(R)}{R}$$

Proof. The product $f(z)\cdot f(-z)$ also belongs to $\mathcal{N}$ and for this even function the counting function is twice that of f. So it suffices to prove Theorem D.8 when f is even. We may also assume that $f(0)=1$ and since $f \in \mathcal{E}$ it is given by a Hadamard product

$$(1) \qquad f(z) = \prod^* (1 - \frac{z^2}{\alpha_\nu^2})$$

where $\prod^*$ indicates the we take the product of zeros whose real part is >0 and if they are purely imaginary they are of the form $b \cdot i$ with $b > 0$. We can replace the zeros by their absolute values and construct

$$(2) \qquad f_*(z) = \prod^* (1 - \frac{z^2}{|\alpha_\nu|^2})$$

If x is real we see that

$$(3) \qquad |f_*[x)| \leq |f(x)|$$

We conclude that if f belongs to $\mathcal{N}$ so does f_*. At the same time their counting functions of zeros are equal. This reduces the proof to the special case when f is even and the zeros are real and at this stage it is clear that Theorem D.1 gives existence of the limit in Theorem D.8.

E. Application to measures with compact support.

Le μ be a Riesz measure on the real t-line with compact support in an interval $[-a,a]$ where we assume that both end-points belong to the support. The measure is in general complex-valued. Now we get the entire function

$$f(z) = \int_{-a}^{a} e^{-izt} \cdot d\mu(t)$$

Here f restricts to a bounded function on the real x-axis with maximum norm $\leq ||\mu||$. Hence f belongs to $\mathcal{N}$ which means that Theorem D.8 holds and the reader may now verify the following:

E.1 Theorem. *One has the equality*

$$\lim_{R\to\infty} \frac{N_f(R)}{R} = \frac{a}{\pi}$$

F. Tauberian theorems with a remainder term

An extension of Theorem D.1 which contains remainder terms were established by Beurling in 1936. An example of Beurling's results goes as follows: Let

$$f(z) = \prod (1 - \frac{z^2}{t_\nu^2})$$

be an even and entire function of exponential type with real zeros as in section D.

F.1 Theorem. *Let $A > 0$ and $0 < a < 1$ and assume that there exists a constant C_0 such that*

$$\Big|-\frac{1}{\pi^2} \cdot \int_0^R \frac{\log|f(x)|}{x^2} \cdot dx - A\,\Big| \leq C_0 \cdot R^{-a}$$

hold for all $R \geq 1$. Then there is another constant C such that

$$\big|\,N_f(R) - R\big| \leq C_1 \cdot R^{1-a/2}$$

Remark. Beurling's original manuscript which contains a proof of Theorem F.1 as well as other results dealing with remainder terms has remained unpublished. It was resumed with details of proofs in a Master's Thesis at Stockholm University by F. Gülkan in 1994. The interested reader should also consult articles by Beurlings former Ph.d student S. Lyttkens which prove various Tauberian theorems with remainder terms. See also work by T. Ganelius for closely related material.

XVI.. Beurling-Wiener algebras

Contents

Introduction.

The cornerstone in this section is Wiener's general Tauberian Theorem. It will be applied to the class of Beurling-Wiener algebras where the ordinary convolution algebra $L^1(\mathbf{R})$ is replaced by weight algebras introduced by Beurling in the article [Beurling: 1938]. Here follows the set-up in this section. Consider the Banach space $L^1(\mathbf{R})$ of Lebesgue measurable and absolutely integrable functions whose product is defined by convolutions:

$$f * g(x) = \int f(x-y)g(y)\,dy$$

A.1 The space $\mathcal{F}_0^\infty$. On the ξ-line we have the space C_0^∞ of infintely differentiable functions with compact support. Each $g(\xi) \in C_0^\infty$ yields an L^1-function on the real x-line defined by

$$(*) \qquad \mathcal{F}(g)(x) = \frac{1}{2\pi}\int e^{ix\xi}\, g(\xi)\, d\xi$$

The resulting subspace of L^1 is denoted by $\mathcal{F}_0^\infty$.

A.2 Beurling-Wiener algebras. A subalgebra B of L^1 is called a Beurling-Wiener algebra - for short a $\mathcal{BW}$-algebra - if the following two conditions hold:

Condition 1. B is equipped with a complete norm denoted by $||\cdot||_B$ such that

$$||f*g||_B \leq ||f||_B \cdot ||g||_B \quad : \quad f,g \in B \quad \text{and} \quad ||f||_1 \leq ||f||_B$$

Condition 2. $\mathcal{F}_0^\infty$ is a dense subalgebra of B.

A.3 Theorem *Let B be a $\mathcal{BW}$-algebra. For each multiplicative and continuous functional λ on B which is not identically zero there exists a unique $\xi \in \mathbf{R}$ such that*

$$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

Proof. First we prove uniqueness. For suppose there exists some ξ such that

$$\text{(i)} \qquad \lambda(f) = 0 \implies \widehat{f}(\xi) = 0$$

This means that the linear form $f \mapsto \widehat{f}(\xi)$ has the same kernel as λ and hence there exists some constant c such that

$$\text{(ii)} \qquad \lambda(f) = c \cdot \hat{f}(\xi) \quad \text{for all } f \in B\,.$$

Since λ is multiplicative it follows that $c = c^n$ for every positive integer n which gives $c = 1$. Next, since B contains $\mathcal{F}_0^\infty$ and test-functions on the ξ-line separate points, it is clear that ξ is uniquely determined. There remains to prove the existence of some ξ for which (i) holds. To prove (i) we use the density of $\mathcal{F}_0^\infty$ in B which by the continuity of λ gives some $g \in \mathcal{F}_0^\infty$ such that $\lambda(g) \neq 0$. Let K be the compact support of the test-function $\widehat{g}(\xi)$ and suppose that (i) fails for each point $\xi \in K$. The density of $\mathcal{F}_0^\infty$ gives for each $\xi \in K$ some $f_\xi \in \mathcal{F}_0^\infty$ such that

$$\text{(iii)} \qquad \widehat{f_\xi}(\xi) \neq 0 \quad \text{and } \lambda(f) = 0$$

Heine-Borel's Lemma yields a finite set of points $\xi_1, \ldots, \xi_N$ in K such that $\{\widehat{f}_{\xi_k}\}$ have no common zero on K. To simplify notations we set $f_k = f_{\xi_k}$. The complex conjugates of $\{\widehat{f}_k\}$ are again test-functions. So for each k we find $h_k \in B$ such that $\widehat{h}_k$ is the is complex conjugate of $\widehat{f}_k$. Set

$$\phi(\xi) = \sum_{k=1}^{k=N} \widehat{h}_k(\xi) \cdot \widehat{f}_k(\xi)$$

This test-function is > 0 on the support of $\widehat{g}$ and hence there exists the test-function

$$\text{(iv)} \qquad Q(\xi) = \frac{\widehat{g}}{\phi}$$

By Condition 2, Q is the Fourier transform of some B-element q. Since $L^1(\mathbf{R})$-functions are uniquely determined by their Fourier transforms, it follows from (iv) that

$$\text{(v)} \qquad \sum_{k=1}^{k=N} q * h_k * f_k = g$$

Now we get a contradiction since $\lambda(f_k) = 0$ for each k while $\lambda(g) \neq 0$.

A.4 The algebra B_a.

Let $a > 0$ be a positive real number. Given a Beurling-Wiener algebra B we set

$$J_a = \{f \in B \quad : \; \widehat{f}(\xi) = 0 \text{ for all } -a \leq \xi \leq a\}$$

Condition 1 and the continuity of the Fourier transform on L^1-functions imply that J_a is a closed ideal in B. Hence we get the Banach algebra $\frac{B}{J_a}$ which we denote by B_a. Let $g \in \mathcal{F}_0^\infty$ be such that $\widehat{g}(\xi) = 1$ on $[-a, a]$. For every $f \in B$ it follows that $g * f - f$ belongs to J_a which means that the image of f in B_a is equal to the image of $g * f$. We conclude that the g-image yields an identity in the algebra B_a and hence B_a is a Banach algebra with a unit element.

A.5 Theorem. *The Gelfand space of B_a is equal to the compact interval $[-a, a]$.*

A.6 Exercise. Prove this using Theorem A.3

A.7. Examples of $\mathcal{B}W$-algebras

Let B be the space of all continuous functions $f(x)$ on the real x-line such that the positive series:

$$(*) \qquad \sum_{-\infty}^{\infty} ||f||_{[\nu,\nu+1]} < \infty$$

where $||f||_{[\nu,\nu+1]}$ is the maximum norm of f on the closed interval $[\nu.\nu + 1]$ and the sum extends over all integers. The norm on B-elements is defined by the sum of the series above. It is obvious that this norm dominates the L^1-norm. Moreover, one easily verifies that

$$\text{(i)} \qquad ||f * g||_B \leq ||f||| \cdot ||g||_B$$

for pairs in B. Hence B satisfies Condition 1 from B.

Exercise. Show that the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on the real x-line is a dense subalgebra of B.

Next, since $\mathcal{F}_0^\infty \subset \mathcal{S}$ we have the inclusion

$$\text{(ii)} \qquad \mathcal{F}_0^\infty \subset B$$

There remains to see why $\mathcal{F}_0^\infty$ is dense in B. To prove this we construct some special functions on the x-line whose Fourier transforms have compact support. If $b > 0$ we set

$$f_b(x) = \frac{1}{2\pi} \int_{-b}^{b} e^{ix\xi} \cdot (1 - \frac{|\xi|}{b}) \cdot d\xi$$

By Fourier's inversion formula this means that

$$\widehat{f}_b(\xi) = 1 - \frac{|\xi|}{b} \qquad -b \leq \xi \leq b \quad \text{and zero if } |\xi| > b$$

A computation which is left to the reader gives

$$f_b(x) = \frac{1}{\pi} \cdot \frac{1 - \cos bx}{bx^2}$$

From this expression it is clear that $f_b(x)$ belongs to B and we leave it to the reader to verify that

(iii) $$\lim_{b \to +\infty} ||f_b * g - g||_B = 0 \quad \text{for all } g \in B$$

Next, the functions $\widehat{f}_b(\xi)$ have compact support but they are not smooth, i.e. they do not belong to $\mathcal{F}_0^\infty$. However, we can perform a smoothing of these functions as follows: Let $\phi(\xi)$ be an even and non-negative C_0^∞-function with support in $-1 \leq \xi \leq 1$ such that the integral

$$\int \phi(\xi) \cdot d\xi = 1$$

With $\delta > 0$ we set $\phi_\delta(\xi) = \frac{1}{\delta} \cdot \phi(\xi/\delta)$ and for each pair δ, b we get the test-function on the ξ-line defined by

$$\psi_{\delta,b}(\xi) = \int_{-b}^{b} \phi_\delta(\xi - \eta) \cdot (1 - \frac{|\eta|}{b}) \cdot d\eta$$

The inverse Fourier transforms

$$f_{\delta,b}(x) = \frac{1}{2\pi} \int e^{ix\xi} \cdot \psi_{\delta,b}(\xi) \cdot d\xi$$

yield functions in $\mathcal{F}_0^\infty$ for all pairs δ, b. Next, if $g \in B$ then the Fourier transform of the B-element $f_{\delta,b} * g$ is equal to the *convolution* of $\phi_\delta(\xi)$ and the Fourier transform of $f_b * g$. This implies that

$$f_{\delta,b} * g \in \mathcal{F}_0^\infty \,.$$

At this stage we leave it to the reader to verify that

$$\lim_{(\delta,b) \to (0,0)} f_{\delta,b} * g = g$$

holds for every $g \in B$. Hence the required density of $\mathcal{F}_0^\infty$ is proved and B is a Beurling-Winer algebra.

A.8 Adding discrete measures

Let $M_d(\mathbf{R})$ be the Banach algebra of discrete measures of finite total variation, i.e. measures of the form

$$\mu = \sum c_\nu \cdot \delta_{x_\nu} \quad : \; ||\mu|| = \sum |c_\nu| < \infty$$

As explained in XX the Gelfand space is the compact Bohr group $\mathfrak{B}$, where the real ξ-line via the Fourier transform appears as a dense subset. Now we adjoin some $\mathcal{BW}$-algebra B and obtain a Banach algebra B_d which consists of measures of the form

$$f + \mu \quad : \quad f \in B \text{ and } \mu \in M_d(\mathbf{R})$$

where the norm of $f + \mu$ is the sum of the B-norm of f and the total variation of μ. Since B is a subspace of L^1 one easuly checks that this yields a complete norm. next, by condition (2) in A.2 it follows that if $f \in b$ and $\mu \in M_d(\mathbf{R})$ then the convolution $f * \mu$ belongs to B. This means that B appears as a closed ideal in B_d.

A.9 The Gelfand space $\mathcal{M}_{B_d}$. Let λ is a multiplicative functional on B_d which is not identically zero on B. Theorem A.3 gives a unique ξ such that

(i) $$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

If a is a real number then $\delta_a * f$ has the Fourier transform becomes $e^{ia\xi} \cdot \widehat{f}(\xi)$. It follows that

(ii) $$\lambda(\delta_a) \cdot \widehat{f}(\xi) = \lambda(\delta_a * f) = e^{-ia\xi} \cdot \widehat{f}(\xi)$$

We conclude that $\lambda(\delta_a) = e^{-ia\xi}$ and hence the restriction of λ is the evaluation of the Fourier transform at ξ on the whole algebra B_d. In this way the real ξ-line is embedded in $\mathcal{M}_B$ where a point $\lambda \in \mathcal{M}_B$ belongs to this subset if and only if $\lambda(f) \neq 0$ for some $f \in B$. The construction of the Gelfand topology shows that this copy of the real ξ-line appears as an *open* subset of $\mathcal{M}_{B_d}$ denoted by $\mathbf{R}_\xi$.

A.10 The set $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$. If λ belongs to this closed subset it is identically zero on the ideal B and its restriction to $M_d(\mathbf{R})$ corresponds to a point γ in the Bohr group $\mathfrak{B}$. Conversely, every point in $\mathfrak{B}$ yields a $\lambda \in \mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ since the quotient algebra

$$\frac{B_d}{B} \simeq M_d(\mathbf{R})$$

Hence we have the set-theoretic equality

(*) $$\mathcal{M}_{B_d} = \mathbf{R}_\xi \cup \mathfrak{B}$$

A.11 Proposition. *The open subset $\mathbf{R}_\xi$ is dense in $\mathcal{M}_B$.*

Proof. Let λ be a point in $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ which therefore corresponds to a point $\gamma \in \mathfrak{B}$. By the result in XX we know that for every finite set $\mu_1, \ldots, \mu_N$ of discrete measures, there exists a sequence $\{\xi_\nu\}$ such that

$$\lim_{\nu \to \infty} \widehat{\mu}_i(\xi_\nu) = \gamma(\mu_i) \quad \text{and } |\xi_\nu| \to \infty$$

At the same time the Riemann-Lebesgue Lemma entails that

$$\lim_{\nu \to \infty} \widehat{f}(\xi_\nu) = 0$$

for every $f \in B$. Hence the construction of the Gelfand topology on $\mathcal{M}_{B_d}$ gives the requested density in Proposition A.11

A.12 An inversion formula. Let $f \in B$ and μ is some discrete measure. Suppose that there exists $\delta > 0$ such that the Fourier transform of $f + \mu$ has absolute value $\geq \delta$ for all ξ. Proposition A.11 implies that its Gelfand transform has no zeros and hence this B_d-element is invertible, i.e. there exist $g \in B$ and a discrete measure γ such that

(i) $$\delta_0 = (f + \mu) * (g + \gamma)$$

Notice that the right hand side becomes

$$f * g + f * \gamma + g * \mu + \mu * \gamma$$

Here $f * g + f * \gamma + g * \mu$ belongs to B while $\mu * \gamma$ is a discrete measure. So (i) implies that γ must be the inverse of μ in $M_d(\mathbf{R})$ and hence (i) also gives the equality:

(ii) $$f * g + f * \mu^{-1} + g * \mu = 0$$

B. A Tauberian Theorem.

Consider the Banach algebra B in (A.7). The dual space B^* consists of Riesz measures μ on the real line for which there exists a constant A such that

$$\int_\nu^{\nu+1} |d\mu(x)| \leq A \quad \text{for all integers } \nu\,.$$

The smallest A above is the norm of μ in B^* and duality is expressed by:

$$\mu(f) = \int f(x) \cdot d\mu(x) \quad : \quad f \in B \text{ and } \mu \in B^*$$

Let $f \in B$ be such that $\widehat{f}(\xi) \neq 0$ for all ξ. For each $a > 0$ it follows from Theorem A.5 that the f-image in B_a generates the whole algebra. Since this hold for every $a > 0$ it follows that each $\phi \in \mathcal{F}_0^\infty$ belongs to the principal ideal generated by f in B, i.e. there exists some $g \in B$ such that

$$(*) \qquad \phi = g * f$$

Since $\mathcal{F}_0^\infty$ is dense in B we conclude that $B \cdot f$ is dense in B. Using this density we have:

B.1 Theorem *Let $\mu \in B^*$ be such that*

$$\lim_{y \to +\infty} \int f(y-x) \cdot d\mu(x) = A \text{ exists.}$$

Then, for each $g \in B$ it follows that

$$\lim_{y \to +\infty} \int g(y-x) \cdot d\mu(x) = B \quad \text{where} \quad B = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

Proof. Let $g \in B$. If $\epsilon > 0$ we find $h_\epsilon \in B$ such that $||g - f * h_\epsilon||_B < \epsilon$. When $y > 0$ we get:

$$\text{(i)} \qquad \int (f * h_\epsilon)(y-x) \cdot d\mu(x) =$$

$$\int \big[f(y-s-x) h_\epsilon(s) \cdot ds \,\big] \cdot d\mu(x) = \int h_\epsilon(s) \cdot \big[\int f(y-s-x)\mu(x) \,\big] \cdot ds$$

By the hypothesis the inner integral converges to A when $y \to +\infty$ every fixed s. Since h belongs to B it follows easily that the limit of (i) when $y \to +\infty$ is equal to

$$\text{(ii)} \qquad A \cdot \int h_\epsilon(s) \cdot ds = A \cdot \widehat{h}_\epsilon(0)$$

Next, since the B-norm is stronger than the L^1-norm it follows that

$$\text{(iii)} \qquad |\widehat{g}(0) - \hat{h}_\epsilon(0) \cdot \widehat{f}(0)| < \epsilon$$

Moreover, since the B-norm is invariant under translations we have

$$\text{(iv)} \qquad \Big| \int g(y-x) d\mu(x) - \int (f * h_\epsilon)(y-x) \cdot d\mu(x) \Big| \leq \epsilon \cdot ||\mu|| \quad \text{for all } y$$

where $||\mu||$ is the norm of μ in the dual space B^*. Notice also that (iii) gives:

$$\lim_{\epsilon \to} \hat{h}_\epsilon(0) = \frac{\hat{g}(0)}{\hat{f}(0)}$$

Finally, since $\epsilon > 0$ is arbitrary we see that the limit formula for (i) when $y \to +\infty$ expressed by (ii) and (iv) above together imply that

$$\lim_{y \to +\infty} \int g(y-x) d\mu(x) = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

This finishes the proof of Theorem A.9

B.2 The multiplicative version

Let $\mathbf{R}^+$ be the multiplicative group of positive real numbers. To each function $f(t)$ on $\mathbf{R}^+$ we get the function $E_f(x) = f(e^x)$ on the real x-line. Since $dt = e^x dx$ under the exponential map we have

$$\int_0^\infty f(t) \frac{dt}{t} = \int_{-\infty}^\infty E_f(x) dx$$

provided that f is integrable. On $\mathbf{R}^+$ we get the convolution algebra $L^1(\mathbf{R}^+)$ where

$$f * g(t) = \int_0^\infty f(\frac{t}{s}) \cdot g(s) \cdot \frac{ds}{s}$$

This convolution commutes with the E map from $L^1(\mathbf{R}^+$ into $L^1(\mathbf{R}^1)$, i.e.

$$E_{f*g} = E_f * E_g$$

Next, recall that the Fourier transform on $L^1(\mathbf{R}^+)$ is defined by

$$\widehat{f}(\xi) = \int_0^\infty t^{-i\xi} \cdot f(t) \cdot \frac{dt}{t}$$

B.3 The Banach algebra B_m. The companion to B on $\mathbf{R}^+$ consists of continuous functions $f(t)$ for which

$$\sum ||f||_{[2^\nu, 2^{\nu+1}]} < \infty$$

where the is taken over all integers. Notice that with $\nu < 0$ one takes small intervals approaching $t = 0$. Just as in Theorem A.9 we obtain a Tauberian Theorem for functions $f \in B_m$ whose Fourier transform is everywhere $\neq 0$. Here we the dual space B_m^* consists of Riesz measures μ on $\mathbf{R}^+$ for which there exists a constant C such that

$$\int_{2^m}^{2^{m+1}} |d\mu(t)| \leq C$$

for all integers m.

C. Ikehara's theorem.

Let ν be a non-negative Riesz measure supported on $[1, +\infty)$ which satisfies

$$\int_1^\infty x^{-1-\delta} \cdot d\nu(x) < \infty \quad \text{for all } \delta > 0$$

This gives an analytic function $f(s)$ defined in the half plane $\mathfrak{Re}(s) > 1$ by

$$f(s) = \int_1^\infty x^{-s} \cdot d\nu(x)$$

D.1 Theorem. *Assume that there exists a constant A and a continuous function $G(u)$ defined on the real u-line such that*

$$(*) \qquad \lim_{\epsilon \to 0} \big[f(1+\epsilon+iu) - \frac{A}{1+\epsilon+iu} \big] = G(u)$$

where the limit holds uniformly on every bounded interval $-b \leq u \leq b$. Then

$$(**) \qquad \lim_{x \to +\infty} \frac{1}{x} \int_1^x d\nu(t) = A$$

We shall prove a sharper version of Ikehara's result where the assumption on $G(u)$ is relaxed. Namely, replace (*) by the weaker assumption that there exists a locally integrable function $G(u)$ such that

$$(***) \qquad \lim_{\epsilon \to 0} \int_{-b}^b \big| f(1+\epsilon+iu) - \frac{A}{1+\epsilon+iu} - G(u) \big| \cdot du = 0 \quad \text{holds for each } b > 0$$

Proof that (***) gives (**). The variable substitution $x \mapsto e^\xi$ gives

$$f(s) = \int_0^\infty e^{-\xi s} \cdot d\nu(e^\xi)$$

Define the measure μ on the non-negative real ξ-line by

$$(1) \qquad d\mu(\xi) = e^{-\xi} \cdot d\nu(e^\xi) - A \cdot d\xi \quad : \ \xi \geq 0$$

Then we see that

$$f(s) - \frac{A}{s-1} = \int_0^\infty e^{(1-s)\xi} d\mu(\xi) \tag{2}$$

It is clear that (**) holds in Theorem D.1 if and only if

$$\lim_{\eta \to \infty} \int_0^\eta e^{-\eta+\xi} \cdot d\mu(\xi) = 0 \tag{3}$$

A reformulation of Ikehara's theorem. From the equations above we can restate the sharp version of Ikehara's theorem as follows. Let ν^* be a non-negative measure on $0 \leq \xi < +\infty$ such that

$$\int_0^\infty e^{-\delta \cdot \xi} \cdot d\nu^*(\xi) < \infty \quad \text{for all } \delta > 0 \tag{4}$$

Next, let $A > 0$ be some positive constant and put $d\mu(\xi) = d\nu^*(\xi) - A \cdot d\xi$. Then (1) gives the analytic function $g(s)$ defined in $\mathfrak{Re}(s) > 0$ by

$$g(s) = \int_0^\infty e^{-s \cdot \xi} \cdot d\mu(\xi) \tag{5}$$

D.2. Definition. *We say that the measure μ is of the Ikehara type if there exists a locally integrable function $G(u)$ defined on the real u-line such that*

$$\lim_{\epsilon \to 0} \int_{-b}^{b} \big| g(\epsilon + iu) - G(u) \big| \cdot du = 0 \quad \text{holds for each } b > 0$$

D.3. The space $\mathcal{W}$. Let $\mathcal{W}$ be the space of continuous functions $\rho(\xi)$ defined on $\xi \geq 0$ which satisfy:

$$\sum_{n \geq 0} ||\rho||_n < \infty \quad \text{where } ||\rho||_n = \max_{n \leq u \leq n+1} |\rho(u)|$$

The dual space $\mathcal{W}^*$ consists of Riesz measures γ on $[0, +\infty)$ such that

$$\max_{n \geq 0} \int_n^{n+1} |d\gamma(\xi)| < \infty$$

With these notations we have

D.4. Theorem. *Let ν^* be a non-negative meausure on $[0, +\infty)$ and $A \geq 0$ some constant such that the measure $\mu = \nu^* - A \cdot d\xi$ is of Ikehara type. Then $\mu \in \mathcal{W}^*$ and for every function $\rho \in \mathcal{W}$ one has*

$$\lim_{\eta \to +\infty} \int_0^\eta \rho(\eta - \xi) \cdot d\mu(\xi) = 0$$

Exercise. Use the previous observations to show that Theorem D. 4 gives the sharp version of Ikehara's theorem. The hint is to apply Theorem D.4 to the $\mathcal{W}$-function $\rho(s) = e^{-s}$

Proof of Theorem D.4.

Let $b > 0$ and define the function $\omega(u)$ by

$$\omega(u) = 1 - \frac{|u|}{b}\,, \quad -b \leq u \leq b \quad \text{and } \omega(u) = 0 \text{ outside this interval} \tag{i}$$

Set

$$J_b(\epsilon, \eta) = \int_{-b}^{b} e^{i\eta u} \cdot g(\epsilon + iu) \cdot \omega(u) \cdot du \tag{ii}$$

From Definition D.2 we have the L^1_{loc}-function $G(u)$ and since $\omega(u)$ is a continuous function on the compact interval $[-b,b]$ it follows that

(iii)
$$\lim_{\epsilon\to 0} J_b(\epsilon,\eta)=J_b(0,\eta)=\int_{-b}^b e^{i\eta u}\cdot G(u)\cdot\omega(u)\cdot du$$

With b kept fixed the right hand side is a Fourier transform of an L^1-function and the Riemann-Lebesgue theorem gives:

(iv)
$$\lim_{\eta\to+\infty} J_b(0,\eta)=0$$

Moreover, the triangle inequality gives the inequality:

(v)
$$|J_b(0,\eta)|\leq\int_{-b}^b \big|G(u)\big|\cdot du$$

Some integral formulas. From the above it is clear that

(1)
$$J_b(\epsilon,\eta)=\int_0^\infty \big[\int_{-b}^b e^{i\eta u-i\xi u}\cdot\omega(u)\cdot du\big]\cdot e^{-\epsilon\cdot\xi}\cdot d\mu(\xi)$$

Next, notice that

(2)
$$\int_{-b}^b e^{i\eta u-i\xi u}\cdot\omega(u)\cdot du=2\cdot\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}$$

Hence we obtain

(3)
$$J_b(\epsilon,\eta)=2\cdot\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\mu(\xi)$$

From (iii) above it follows that (3) has a limit as $\epsilon\to 0$ which is equal to the integral in the right hand side in (iii) which is denoted by $J_b(0,\eta)$. Next, it is easily seen that there exists the limit

(4)
$$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot Ad\xi=2\pi\cdot A$$

Hence (3-4) imply that there exists the limit

(5)
$$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\nu^*(\xi)=J_b(0,\eta)+2\pi\cdot A$$

Next, the measure $\nu^*\geq 0$ and the function $\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\geq 0$ for all ξ. So the existence of a finite limit in (5) entails that there exists the convergent integral

(6)
$$\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot d\nu^*(\xi)=J_b(0,\eta)+2\pi\cdot A$$

Proof that $\mu\in\mathcal{W}^*$. Let $\mathcal{W}^*$ be the dual of the Banach space $\mathcal{W}$ from D.3. Since $A\cdot d\xi$ obviously belongs to $\mathcal{W}^*$ it suffices to show that $\nu^*\in\mathcal{W}^*$. To prove this we consider some integer $n\geq 0$ and with $b=1$ it is clear that (6) gives

$$\big|\int_n^{n+1}\frac{1-\cos(\eta-\xi)}{(\eta-\xi)^2}\cdot d\nu^*(\xi)\big|\leq|J_1(0,\eta)|+2\pi=\int_{-1}^1|G(u)|\cdot du+2\pi\cdot A$$

Apply this with $\eta=n+1+\pi/2$ and notice that

$$\frac{1-\cos(n+1+\pi/2-\xi)}{(n+1+\pi/2-\xi)^2}\geq a\quad\text{for all } n\leq\xi\leq n+1$$

This gives a constant K such that

$$\int_n^{n+1} d\nu^*(\xi) \leq K \quad n = 0, 1, \ldots$$

Final part of the proof. We have proved that $\mu \in \mathcal{W}^*$. Moreover, (iv) and the integral formula (6) give

$$(*) \qquad \lim_{\eta \to +\infty} \int_0^\infty \frac{1 - \cos b(\eta - \xi)}{b(\eta - \xi)^2} \cdot d\mu(\xi) = 0 \quad \text{for all } b > 0$$

Next, for each fixed $b > 0$ we notice that the function

$$\rho_b(\xi) = 2 \cdot \frac{1 - \cos(b\xi)}{b \cdot \xi^2}$$

belongs to $\mathcal{W}$ and its Fourier is $\omega_b(u)$. Here $\omega_b(u) \neq 0$ when $-b < u < b$. So the family of these ω-functions have no common zero on the real u-line. By the Remark in XX this entails that the linear subspace of $\mathcal{W}$ generated by the translates of all ρ_b -functions with arbitrary large b is dense in $\mathcal{W}$. Hence (*) above implies that we get a zero limit as $\eta \to +\infty$ for every function $\rho \in \mathcal{W}$. But this is precisely the assertion in Theorem D.4.

E. The algebra $L^1(\mathbf{R}^+)$

Consider the family of L^1-functions on the real x-line supported by the half-line $x \geq 0$. This yields a closed subalgebra of $L^1(\mathbf{R})$ denoted by $L^1(\mathbf{R}^+)$. Adding the unit point mass δ_0 we obtain the commutative Banach algebra

$$B = \mathbf{C} \cdot \delta_0 + L^1(\mathbf{R}^+)$$

E. 1. The Gelfand space $\mathfrak{M}_B$. For each $f(x) \in L^1(\mathbf{R}^+)$ we set:

$$\widehat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot f(x) \cdot dx, \quad \text{where} \quad \mathfrak{Im}(\zeta) \geq 0$$

With $\zeta = \xi + i\eta$ we get

$$|\widehat{f}(\xi + i\eta)| \leq \int_0^\infty |e^{i\xi x - \eta x}| \cdot |f(x)| \cdot dx = \int_0^\infty |e^{-\eta x} \cdot |f(x)| \cdot dx \leq ||f||_1$$

We conclude that for every point $\zeta = \xi + i\eta$ in the closed upper half-plane corresponds to a point in $\zeta^* \in \mathfrak{M}_B$ defined by

$$\zeta^*(f) = \widehat{f}(\zeta) \quad \text{and} \quad \zeta^*(\delta_0) = 1$$

In addition to this $L^1(\mathbf{R}^+)$ is a maximal ideal in B whioch gives a point $\zeta^\infty \in \mathfrak{M}_B$ such that

$$\zeta^\infty(f) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

E.2. Theorem. *The Gelfand space $\mathfrak{M}_B$ can be identified with the union of ζ^∞ and the closed upper half-plane.*

Remark. The theorem asserts that every multiplicative functional on B is either ζ^∞ or determined by a point $\zeta = \xi + i\eta$ where $\eta \geq 0$. Concerning the topological identification ζ^∞ corresponds to the one point compactification of the closed upper half-plane. Thus, whenever $\{\zeta_\nu\}$ is a sequence in $\mathfrak{Im}(\zeta) \geq 0$ such that $|\zeta_\nu| \to \infty$ then $\{\zeta_\nu^*\}$ converges to ζ^* in $\mathfrak{M}_B$. In fact, this follows via the Riemann-Lebegue Lemma which gives

$$\lim_{|\zeta| \to \infty} \widehat{f}(\zeta) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

The general result in XX gives Theorem 2 if we have proved if the following:

E.3. Proposition. *Let $\{g_\nu = \alpha_\nu \cdot \delta_0 + f_\nu\}_1^k$ be a finite family in B such that the k-tuple $\{\hat{g}_\nu\}$ has no common zero in $\bar{U}_+ \cup \{\infty\}$. Then the ideal in B generated by this k-tuple is equal to B.*

The proof requires some preliminary constructions. We use the conformal map from the upper half-plane onto the unit disc defined by

$$w = \frac{\zeta - i}{\zeta + i}$$

So here w is the complex coordinate in D. Next, consider the disc algebra $A(D)$. Via the conformal map each transform $\widehat{f}(\zeta)$ of a function $f \in L^1(\mathbf{R}^+)$ yields an element of $A(D)$ defined by:

$$F(w) = \widehat{f}(\frac{i + iw}{1 - w})$$

It is clear that $F(w) \in A(D)$. Moreover, we notice that

$$w \to 1 \implies |\frac{i + iw)}{1 - w}| \to \infty$$

It follows that the $A(D)$-function $F(w)$ is zero at $w = 1$ and we can conlcude:

E.4. Lemma. *By $f \mapsto F$ we have an algebra homomorphism from $L^1(\mathbf{R}^+)$ to functions in $A(D)$ which are zero at $w = 1$.*

Next, let $\mathcal{H}$ denote the algebra homomorphism in Lemma 4 and consider the function $1 - w$ in $A(D)$. We claim this it belongs to the image under $\mathcal{H}$. To see this we consider the function

$$f(x) = e^{-x} \quad x \geq 0 \quad \text{and } f(x) = 0 \quad \text{when } x < 0$$

Then we see that

$$\hat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot e^{-x} \cdot dx = \frac{1}{1 - i\zeta}$$

It follows that

$$F(w) = \frac{1}{1 - i(\frac{iw+i}{1-w})} = \frac{1 - w}{1 - w + w + 1} = \frac{1 - w}{2}$$

Using $2f$ we conclude that $1 - w$ belongs to the $\mathcal{H}$-image. Next, the identity element δ_0 is mapped to the constant function on D. So via $\mathcal{H}$ we have an algebra homomorphism from B into a subalgebra of $A(D)$ which contains $1 - w$ and the identity function and hence all w-polynomials. Returning to the special B-element f we notice that the convolution

$$f * f(x) = x \cdot e^{-x}$$

We can continue and conclude that the subalgebra of B generated by f and δ_0 contains L^1-functions of the form $p(x) \cdot e^{-x}$ where $p(x)$ are polynomials.

E.5. Exercise. Prove that the linear space $\mathbf{C}[x] \cdot e^{-x}$ is a dense subspace of $L^1(\mathbf{R}^+)$.

From the result in the exercise it follows that the polynomial algebra $\mathbf{C}[w]$ appears as a dense subalgebra of $\mathcal{H}(B)$ when it is equipped with the B-norm. At this stage we are prepared to give:

Proof of Proposition E.3. In $A(D)$ we have the functions $\{G_\nu = \mathcal{H}(g_\nu)\}$. By assumption $\{G_\nu\}$ have no common zero in the closed disc D. Since D is the maximal ideal space of the disc algebra and $\mathbf{C}[w]$ a dense subalgebra, it follows that for every $\epsilon > 0$ there exist polynomials $\{p_\nu(w)\}$ such that the maximum norm

$$|p_1 \cdot G_1 + \ldots + p_k \cdot G_k - 1|_D < \epsilon| \tag{1}$$

where 1 is the identity function. Now $p_\nu = \mathcal{H}(\phi_\nu)$ for B-elements $\{\phi_\nu\}$. So in B we get the element

$$\psi = \phi_1 g_1 + \ldots + \phi_k \cdot g_k \tag{2}$$

Moreover we have $|\mathcal{H}(\psi) - 1|_D < \epsilon$ and here we can choose $\epsilon < 1/4$ and by the previous identifications it follows that

$$|\widehat{\psi}(\xi)| \geq 1/4 \quad \text{for all} \quad -\infty < \xi < \infty \tag{3}$$

The proof of Proposition E.3 is finished if we can show that (3) entails that the B-element ψ is invertible. Multiplying ψ with a non-zero scalar we may assume that

$$\psi = \delta_0 - g \quad : \quad g \in L^1(\mathbf{R}^+)$$

and the Fourier transform $\widehat{\psi}(\xi)$ satisfies

$$|\widehat{\psi}(\xi) - 1| \leq 1/2$$

for all ξ. It means that $|\widehat{g}(\xi)| \leq 1/2$. The spectral radius formula applied to L^1-functions shows that if N is a sufficiently large integer then

(4) $$||g^{(N)}||_1 \leq (3/4)^N$$

where $g^{(N)}$ is the N-fold convolution of g. Now we have

(5) $$(1 + g + \ldots + g^{N_1}) \cdot \psi = 1 - g^{(N)}$$

By (4) the norm of the B-element $g^{(N)}$ is strictly less than one and hence the right hand side is invertible where the inverse is given by a Neumann series, i.e. with $g_* = g^{(N)}$ the inverse is

$$\delta_0 + \sum_{\nu=1}^{\infty} g_*^{\nu}$$

Since convolutions of $L^1(\mathbf{R}^+)$-functions still are supported by $x \geq 0$, it follows from the above that ψ is invertible in B and Proposition E.3 is proved.

0.2 Results about a Schrödinger equation.

Carleman's most valuable work is the monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923]. Here the main results give conditions in order that densely defined symmetric operators on Hilbert spaces have self-adjoint extensions, together with the construction of spectral resolutions for unbounded and self-adjoint operators. A basic equation which emerges from quantum mechanics is to find solutions $u(p,t)$ defined in $\mathbf{R}^3 \times \mathbf{R}^+$ where t is a time variable and $p = (x, y, z)$ which satisfies the PDE-equation

$$i \cdot \frac{\partial u}{\partial t} = \Delta(u)(p,t) - c(p) \cdot u(p,t) = 0 \quad t > 0$$

and the initial condition

$$u(p,0) = f(p)$$

Here $f(p)$ belongs to $L^2(\mathbf{R}^3)$ and $c(p)$ is a real-valued and locally square integrable function. Carleman proved that the symmetric and densely defined operator $\Delta + c$ has a self-adjoint extension in $L^2(\mathbf{R}^3)$ if

$$(*) \qquad \limsup_{p \to \infty} \, c(p) \leq M$$

holds for some constant M As far as I know the sufficiency condition (*) for the existence of a self-adjoint extension remains as the "best possible result". Returning to Schrödinger's equation we assume (*) and that the spectrum of the densely defined sefl-adjoint operator $\Delta - c$ on the Hilbert space $L^2(\mathbf{R}^3)$ is confined to an interval $[\lambda_1, +\infty)$ on the positive real line, i.e. $\lambda_1 > 0$. When this holds, the solution u is given by an equation

$$u(p,t) = \int_{\mathbf{R}^3} \Big[\int_{\lambda_1}^{\infty} e^{i\lambda t} \cdot d\theta(p,q,\lambda) \Big] \cdot f(q)\, dq$$

where $\lambda \mapsto \theta(p,q,\lambda)$ is a non-decreasing function on $[\lambda_1, +\infty)$ for each fixed pair $p.q$. Moreover, Carleman found an asymptotic expansion which recaptures the θ-function. For details we refer to the article [Carleman XX].

0.3 A wave equation.

Let Ω be a bounded cdomain in $\mathbf{R}^3$ with a C^1-boundary $\partial\Omega$. We seek functions $u(x,t)$ where $x = (x_1, y_2, x_3)$ defined in $\mathbf{R}^3 \setminus \Omega \times \{t \geq 0\}$ where t is a time variable satisfying the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta(u)$$

when $t > 0$ and $x \in \mathbf{R}^3 \setminus \overline{\Omega}$. The initial conditions when $t = 0$ is that

$$\text{(i)} \qquad u(x,0) = f_1(x) \quad : \frac{\partial u}{\partial t}(x,0) = f_2(x)$$

where f_1, f_2 are C^2-functions in $\mathbf{R}^3 \setminus \Omega$ and $\Delta(f_1)$ and $\Delta(f_2)$ are square integrable, i.e.

$$\iiint_{\mathbf{R}^3 \setminus \Omega} |\Delta(f_\nu|^2 \, dx < \infty$$

Finally the normal derivatives along $\partial\Omega$ satisfy

$$\frac{\partial f_\nu}{\partial n} = 0 \quad : \nu = 1, 2$$

Given such a pair f_1, f_2 there exists a unique solution $u(x,t)$ which satisfies the wave equation above and the two initial conditions (i) together with the boundary value equation

$$\frac{\partial u}{\partial n}(x,t) = 0$$

for every $x \in \partial\Omega$ and each $t \geq 0$. Carleman's cited monograph gives an expression of the solution by an integral formula using the spectral measure of a certain from a densely defined and self-adjoint operator on the Hilbert space $L^2(\mathbf{R}^3 \setminus \Omega)$. This is used to prove that

$$\lim_{x \to \infty} \nabla(u)(x) = 0$$

In other words, the first order derivatives of u tend to zero as $x \to \infty$. Let us remark that similar spectral functions were employed by Friedrich's in an article from 1944 which proved the existence and uniqueness for solutions to symmetric hyperbolic systems, where Maxwell's equations for electro-magnetic fields is an example.

0.4 Fredholm's resolvent.

In the article *Sur le genre du denominateur $D(\lambda)$ de Fredholm* [Arkiv för matematik 1917], Carleman considered a continuous function $K(x,y)$ defined on the square $\{0 \leq x, y \leq 1\}$ which yields the integral operator

$$f \mapsto \mathcal{K}_f(x) = \int_0^1 K(x,y)f(y)\,dy$$

There exists the discrete sequence of spectral values $\{\lambda_\nu\}$ for which $E - \lambda_\nu \cdot \mathcal{K}_f$ are not invertible where E is the identity operator on the Hilbert space $L^2[0,1]$. Here $\{0 < |\lambda_1| \leq |\lambda_2| \leq \ldots\}$ are arranged with non-decreasing absolute values and eventual eigenspaces of dimension $e \geq 1$ means that the corresponding λ-value is repeated e times in the sequence. Carleman proved that the infinte product

$$(*) \qquad \mathcal{D}(\lambda) = e^{a\lambda} \cdot \prod_{\nu=1}^{\infty} \bigl(1 - \frac{\lambda}{\lambda_\nu}\bigr)$$

yields an entire of function of exponential type. Here $\mathcal{D}(\lambda)$ is the Fredholm resolvent defined for each complex λ by the series

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\,!} \cdot \int_{[0,1]^n} \det(K_n(s_1,\ldots,s_n))\,ds_1 \cdots ds_n$$

where $K_n(s_1,\ldots,s_n)$ is the $n \times n$-matrix with elements $\{K(s_i,s_k) : 1 \leq i,k \leq n\}$. Notice that one does not assume that the kernel function $K(x,y)$ is symmetric, Carleman's proof is presented in § xx and offers an excellent lesson about spectral properties of linear operators. Let us remark that the main ingredients in the proof rely upon analytic function theory where a result due to Arne Wiman plays a crucial role. Recall that Wiman also was professor at Uppsala during this time. So Carleman's career started at Uppsala University in an "optimal enviroment" guided by Holmgren and Wiman. Let us also remark that in a later work Carleman extended the result above to kernels $K(x,y)$ of the Hibert-Schmidt type, i.e. here

$$\iint_0^1 |K(x,y)|^2\,dxdy < \infty$$

Carleman's inequality. The cited article above also contains an inequaliy for norms of resolvents to matrices which goes as follows: Recall first that the Hilbert-Schmidt norm of an $n \times n$-matrix $A = \{a_{ik}\}$ is defined by:

$$||A|| = \sqrt{\sum\sum |a_{ik}|^2}$$

where the double sum extends over all pairs $1 \leq i,k \leq n$. Next, for a linear operator S on $\mathbf{C}^n$ its *operator norm* is defined by

$$||S|| = \max_x ||S(x)|| \quad \text{with the maximum taken over unit vectors.}$$

Theorem. *Let $\lambda_1,\ldots,\lambda_n$ be the roots of $P_A(\lambda)$ and $\lambda \neq 0$ is outside $\sigma(A)$. Then one has the inequality:*

$$|\prod_{i=1}^{i=n} [1 - \frac{\lambda_i}{\lambda}]e^{\lambda_i/\lambda}| \cdot ||(\lambda \cdot E_n - A)^{-1}|| \leq |\lambda| \cdot \exp(\frac{1}{2} + \frac{||A||^2}{2 \cdot |\lambda|^2})$$

Te proof is given in chapoter 1 from my notes on analytic function theory. It gives an instructive lesson for the calculus with matrices and we remark that the inequality in the theorem above can be applied to Hilbert-Schmidt operators in the infinite dimensional case and therefore plays a crucial role for expansions via eigen-functions of such operators.