## Taylor series and real-analytic functions.

The study of Taylor series of differentiable functions on the real line were investigated by Borel and Denjoy who established several results during the years 1910-1922. Among these we recall the following result. Consider a real-valued $C^\infty$-function $f$ on a bounded open interval $(a,b)$ whose derivatives have have finite maximum norms. For each non-negative integer $k$ we put

$$C_k(f) = \left(|f^{(k)}|\right)^{\frac{1}{k}} \tag{*}$$

where $|f^{(k)}|$ is the maximum of the $k$:th order derivative of $f$ taken over $(a,b)$. The question posed by Borel and Denjoy was to find conditions in order that some a prioro estimates on the sequence $\{C_k(f)\}$ implies that $f$ cannot be flat at a point $x_0 \in (a,b)$ , unless it is identically zero. To say that $f$ is flat at $x_0$ means that

$$f^{(k)}(x_0) = 0 \quad : \quad k = 0, 1, 2, \ldots$$

Denjoy proved that if

$$\sum_{k=0}^{\infty} \frac{1}{C_k(f)} = +\infty \tag{**}$$

then $f$ cannot be flat at a point $x_0 \in (a,b)$ unless it is identically zero. In 1923 Carleman established necessary and sufficient conditions for quasi.analyticity in his lectures at Sorbonnen which goes as follows:

Let $\mathcal{A} = \{\alpha_\nu\}$ be a non-decreasing sequence of positive real numbers. Denote by $\mathcal{C}_\mathcal{A}$ the family of all $f \in C^\infty[0,1]$ for which there exists a constants $M$ and $k$ which may depend on $f$ such that

$$\max_{0 \leq x \leq 1} |f^{(\nu)}(x)| \leq M \cdot k^\nu \cdot \alpha_\nu^\nu \quad : \quad \nu = 0, 1, \ldots \tag{0.4.1}$$

One says that $\mathcal{C}_\mathcal{A}$ is a quasi-analytic class if every $f \in C_\mathcal{A}$ whose Taylor series is identically zero at $x = 0$ vanishes identically on $[0,1]$. The following was proved by Carleman in 1922:

**Theorem.** *The class $C_\mathcal{A}$ is quasi-analytic if and only if*

$$\int_1^\infty \log\Big[\sum_{\nu=1}^{\infty} \frac{r^{2\nu}}{a_\nu^{2\nu}}\Big] \cdot \frac{dr}{r^2} = +\infty \tag{***}$$

The proof is presented in my notes *Mathematics by Carleman.*

### 0.1 Carleman's a priori inequality.

Now we announce a crucial resut while one studies Taylor series. Let $n$ be a positive integer and denote by $\mathcal{F}_n$ the family of $n$ times continuously differentiable functions $f$ on the closed unit interval such that

$$f^{(k)}(0) = f^{(k)}(1) = 0 \quad : 0 \leq k \leq n-1 \tag{0.1.1}$$

and the $L^2$-integral is normalised so that

$$\int_0^1 f(t)^2\, dt = 1 \tag{0.1.2}$$

**0.2.Theorem**. *For each $n \geq 1$ and every $f \in \mathcal{F}_n$ one has the inequality*

$$\sum_{k=1}^{k=n} \frac{1}{C_k(f)} \leq \pi \cdot e \tag{0.2.1}$$

*where $e$ is Neper's constant.*

We prove this result in § 1 and remark that Denjoy's theorem expressed via (**) above is an easy consequence. See § xx for details.

**0.3 Carleman's reconstruction theorem for real-analytic functions.** A real-valued $C^\infty$-function $f$ on the closed unit interval is real analytic if and only if there exist constants $C$ and $M$ such that

$$\max_{0\leq x\leq 1} |f^{(k)}(x)| \leq M\cdot k!\cdot C^k \quad : k=1,2,\ldots \tag{0.3.1}$$

It is wellknown, and easily seen, that the real-analyticity implies that $f$ is determined by its derivatives at the origin. However, the Taylor series

$$\sum_{k\geq 0} f^{(k)}(0)\cdot \frac{x^k}{k!}$$

is in general only convergent for in a small interval $0\leq x<\delta$. In 1921 Borel posed the question how on determines $f(x)$ on the whole interval from the sequence $\{f^{(k)}(0)\}$. An affimative answer was given by Carleman in 1923. He considered solutions to a family of variational problems which goes as follows:

For every positive integer $N$ we denote by $\mathcal{H}_N$ the Hilbert space whose elements are $N-1$-times continuous differentiabke functions $g$ on $[0,1]$, and in addition $g^{(N)}$ is square integrable, i.e. it belongs to $L^2[0,1]$. Inside $\mathcal{H}_N$, Carleman introduced the subspace $\mathcal{H}_N(f)$ of functions $g$ such that

$$g^{(k)}(0)=f^{(k)}(0) \quad : k=0,\ldots,N-1 \tag{0.3.1}$$

With these notatons one regards the variational problem

$$\min_{g\in\mathcal{H}_N(f)} \sum_{k=0}^{k=N} (\log(k+2))^{-2k}\cdot (k!)^{-2k}\cdot \int_0^1 g^{(k)}(x)^2\,dx \tag{0.3.2}$$

Elementary Hilbert space methods yield a unique function in $\mathcal{H}_N(f)$ which minimizes the right hand side above. Let us denote it by $f_N$. Hence one has a sequence $\{f_N\}$ where each $f_N)$ has at least $N-1$ continuous derivatives. Now the following hold:

**0.3.3 Theorem.** *When $f$ is real-analytic the sequence $\{f_N\}$ converges uniformly together with all derivatives to $f$, i.e. for every $m\geq 0$ it holds that*

$$\lim_{N\to\infty} |f_N^{(m)}-f^{(m)}|_{0,1}=0$$

**Remark.** Since every individual function $f_N$ is determined by derivative of $f$ up to order $N-1$ at $x=0$, it means that one has obtained a reconstruction of the real-analytic function $f$ on $[0,1]$ via the derivatives a $x=0$. Let us also remark that the proof of Theworem 0.3.3 leads to a description of the family of power series

$$\sum c_n x^n$$

which have some positive radius of convergence, and at the same time

$$c_n=\frac{f^{(n)}(0)}{n\,!}$$

for some real-analytic function $f$ defined on the closed unit intyerval $[0,1]$. More precisely, if we take the sequence $\{c_n\}$ in the variational integrals above, then the proof in § 2 shows that (0.3.1) holds for some $f$ if and only if there exists a constangt $J^*$ such that (0.3.3) is $\leq J^*$ for every $N$, wherfe one now has taken $g$-functions in the Hilbert space, and (0.3.1) is replaced by

$$g^{(k)}(0)=k\,!\cdot c_k \quad : \quad k=0,1,2,\ldots$$

## § 1. Proof of Theorem 0.2

We shall first establish a general inequality for certain analytic functions. Let $0 < b_1 < \ldots < b_n$ be a strictly increasing sequence of positive real numbers where $n \geq 1$ is some integer. Let $\phi(z)$ be an analytic function in the right half-plane $\mathfrak{Re}\, z > 0$ which extends to a continuous function on the imaginary axis. Assume that its maximun norm over the right half-plane is $\leq 1$ and in addition

$$|z|^k \cdot \phi(z) \leq b_k^k \quad : k = 1, \ldots, n \tag{1.1}$$

**1.2 Theorem.** *For each $\phi$ as above and every real $a > 0$ one has the inequality*

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq \log \frac{1}{\phi(a)} \tag{1.2.1}$$

*Proof.* On the imaginary axis we consider the intervals

(i) $$\ell_k = [e \cdot b_k, e \cdot eb_{k+1}] \quad : k = 1, \ldots, n-1 \quad \& \quad \ell_n = [eb_n, +\infty)$$

Now (1.1) gives the following for every $1 \leq k \leq n$:

(ii) $$k \cdot \log |y| + \log |\phi(iy)| \leq k \cdot \log b_k \quad : y \in \ell_k$$

Notice also that

(iii) $$k \cdot \log |y| \geq k \cdot \log (e \cdot b_k) = k + k \cdot \log b_k \quad : y \in \ell_k$$

where we used the equality $\log (e \cdot b_k) = 1 + \log b_k$. Together (ii-iii) give

(iv) $$\log |\phi(iy)| \leq -k \quad : y \in \ell_k$$

Regarding the negative intervals

$$-\ell_k = [-e \cdot b_{k+1}, -e \cdot b_k] \quad \& \quad -\ell_n = (-\infty, -eb_n)$$

one also gets

(v) $$\log |\phi(iy)| \leq -k \quad : y \in -\ell_k \quad \& \quad 1 \leq k \leq n$$

Finally, since the maximum norm of $\phi$ is $\leq 1$ one has

(vi) $$\log |\phi(iy)| \leq 0 \quad : -b_1 \leq y \leq b_1$$

Next, solving the Dirichlet problem we find the harmonic function $u$ in the open right half-plane whose boundary values on $(-b_1, b_1)$ is zero ,while $u = -k$ in the the open intervals $\ell_k$ and $-\ell_k$ for every $k$. The principle of harmonic majorisation applied to the subharmonic function $\log |\phi(z)|$ entails that

(vii) $$\log |\phi(a) \leq u(a)$$

when $a$ is real and positive. Now we evaluate $u(a)$ using Poisson's formula for harmonic functions in the right half-plane. For each $1 \leq k \leq n-1$ we denote by $\theta_a(k)$ the angle between the two vectors which join $a$ to the end-points $ieb_k$ and $ieb_{k+1}$. Computing the area of the triangle with corner points at $a, ieb_k, ieb_{k+1}$ the reader may check that

$$\sqrt{a^2 + e^2 b_k^2} \cdot \sqrt{a^2 + e^2 b_{k+1}^2} \cdot \sin \theta_a(k) = a \cdot e \cdot (b_{k+1} - b_k) \tag{1}$$

Finally, let $\theta_a(n)$ be the angle between the vector which joins $a$ with $ieb_n$ and the vertical line $\{x = a$. The reader may check with the aid of a figure that

$$\sin \theta_a(n) = \frac{a}{\sqrt{a^2 + e^2 b_n^2}} \tag{2}$$

Next, Poisson's formula applied to the harmonic function $u$ gives

$$u(a) = -\frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \theta_a(k) \tag{3}$$

which together with (vii) entails that

$$\frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \theta_a(k) \leq \log \frac{1}{\phi(a)|} \tag{4}$$

The inequality $\sin t \leq t$ for every $t > 0$ implies that

$$\frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \sin(\theta_a(k)) \leq \log \frac{1}{\phi(a)|} \tag{5}$$

Next we use (1-2) to estimate $\{\sin(\theta_a(k))\}$. When $1 \leq k \leq n-1$ we have from (1)

$$e^2 \cdot b_k \cdot b_{k+1} \cdot \sqrt{1 + \frac{a^2}{e^2 b_k^2}} \cdot \sqrt{1 + \frac{a^2}{e^2 b_{k+1}^2}} \cdot \sin\theta_a(k) = a \cdot e \cdot (b_{k+1} - b_k) \implies$$
$$e \cdot (1 + \frac{a^2}{e^2 b_1^2}) \cdot \sin\,\theta_a(k) \leq a \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}})$$

where the last inequality follows since $b_k \geq b_1$ for every $k$. We conclude that the left hand side in (5) majorizes

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n-1} k \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}}) + \frac{2}{\pi} \cdot n \cdot \sin\theta_a(n) \tag{6}$$

Next, (2) and the inequality $b_1 \leq b_n$ give

$$\sin\theta_a(n) = \frac{a}{eb_n} \cdot \frac{1}{\sqrt{1 + \frac{a^2}{e^2 b_n^2}}} \geq \frac{a}{eb_n} \cdot \frac{1}{1 + \sqrt{\frac{a^2}{e^2 b_1^2}}} \tag{7}$$

Together (6-7) imply that the left hand side in (5) majorizes

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot (\sum_{k=1}^{k=n-1} k \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}}) + n \cdot \frac{1}{b_n}) = \frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k}$$

where the last equality follows via Abel's summation formula. Hence

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq \log \frac{1}{\phi(a)} \tag{*}$$

which is the requested inequality (1.2.1).

**1.3 A special case.** Assume in addition to (1.1) that

$$\phi(a) \geq e^{-a} \quad : a > 0 \tag{1.3.1}$$

It follows that

$$\log \frac{1}{\phi(a)} \leq a$$

and then a division with $a$ in Theorem 1.2 gives

$$\frac{2}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq 1 \tag{1.3.2}$$

Passing to the limit as $a \to 0$ we get

$$\sum_{k=1}^{k=n} \frac{1}{b_k} \leq \frac{e\pi}{2} \tag{1.3.3}$$

**Proof of Theorem 0.2.**

We are given $f \in \mathcal{F}_n$ and put

$$\phi(z) = \int_0^1 e^{-zt} \cdot f(t)^2 \, dt$$

When $\mathfrak{Re} z \geq 0$ the absolute value $|e^{-zt}| \leq 1$ for all $t$ on the unit interval. Hence the normalisation in (0.1.2) implies that

$$|\phi(z)| \leq 1 \quad : \mathfrak{Re} z \geq 0$$

Next, if $1 \leq k \leq n$ the vanishing in (0.1.1) and partial integration give

$$z^k \cdot \phi(z) = \sum_{\nu=0}^{\nu=k} \binom{k}{\nu} \int_0^1 f^{(\nu)}(t) \cdot f^{(k-\nu)}(t)(t) \, dt \tag{i}$$

The Cauchy-Schwarz inequality estimates the absolute value of the right hand side by

$$\sum_{\nu=0}^{\nu=k} \binom{k}{\nu} \cdot ||f^{(\nu)}||_2 \cdot ||f^{(k-\nu)}||_2 \tag{ii}$$

At this stage we notice that

$$||f^{(\nu)}||_2 \leq ||f^{(k)}||_2 \quad : 0 \leq \nu \leq k \leq n \tag{*}$$

From (*) the reader can check that (ii) is majorised by $2^k \cdot ||f^{(\nu)}||_k^2$. Hence (i) gives

$$|z|^k \cdot |\phi(z)| \leq 2^k \cdot (||f^{(k)}||_2)^2 \quad : \ k = 1, 2, \ldots \tag{iii}$$

Put

$$b_k = 2 \cdot (||f^{(k)}||_2)^{\frac{2}{k}} \implies |z|^k \cdot |\phi(z)| \leq b_k^k \tag{iv}$$

Next, if $a > 0$ we have

$$\phi(a) = \int_0^1 e^{-at} \cdot f(t)^2 \, dt \geq e^{-a} \cdot \int_0^1 f(t)^2 \, dt = e^{-a} \tag{v}$$

where the last equality holds by the normalsation in (0.1.2). Hence (iv-v) and the case from (1.3) give

$$\sum_{k=1}^{k=n} \frac{1}{b_k} \leq \frac{e\pi}{2} \tag{vi}$$

Finally, the trivial inequalities $||f^{(k)}||_2 \leq \max_{0 \leq t \leq 1} |f^{(k)}(t)|$ imply that

$$\frac{1}{b_k} \geq \frac{1}{2 \cdot C_k(f)}$$

and then (vi) gives the requested inequality in Theorem 0.2

**Proof of Theorem 0.3.3**

For each $N$ we denote by $J_*(N)$ the minimum in the variational problem from (0.3.3). Among the competing functions we can choose $f$ and hence $J_*(N)$ is majorised by the right hand side in (0.3.3) when we take $g = f$. Now there exist constants $C$ and $M$ from (0.3.1) which entails that

$$J_*(N) \leq M \cdot \sum_{k=0}^{N} (\log(k+2))^{-k} \cdot C^{2k}$$

Since $\log(k+2)$ tends to $+\infty$, it is clear that the series

$$\sum_{k=0}^{\infty} (\log(k+2))^{-k} \cdot C^{2k} < \infty$$

Hence there exists a constant $J^*$ such that

(i) $$J_*(N) \leq J^* \quad : N = 1, 2, \ldots$$

So if $m$ is some positive integer and $N \geq m$ we have

(ii) $$\sum_{k=0}^{k=m} (\log(k+2))^{-2} \int_0^1 f_N^{(k)}(x)^2 \, dx \leq J_*(N) \leq J^*$$

Now we recall the classic resut due to Arzela-Ascoli which implies that bounded sets in the Hilbert space $H_m$ are relatively compact subsets of $C^{m-1}[0,1]$. Since (ii) hold for each $m$, it follows by a standard diagonal procedure that there exist a subsequence $\{g_\nu = f_{N_\nu}\}$ such that the sequence of derivatives $\{g_\nu^{(m)}\}$ converge uniformly for every $m$, i.e. we find $g_* \in C^\infty[0,1)$ such that

$$g_\nu \to g_*$$

holds in the Frechet space $C^\infty[0,1]$. Next, by (0.3.2) we have for each fixed integer $k \geq 0$:

$$f^{(k)}(0) = f_N^{(k)}(0) \quad : N \geq k+1$$

It follows that

(iii) $$f^{(k)}(0) = g_*^{(k)}(0) \quad : k = 0, 1, 2 \ldots$$

Hence the $C^\infty$-function

$$\phi = f - g_*$$

is flat at $x = 0$. Next, for a fixed integer $k$ the uniform bound in (ii) and the triangle inequality for squared $L^2$-norms give

(iv) $$\int_0^1 \phi^{(k)}(x)^2 \, dx \leq 2 \cdot J_* \cdot (\log(k+2))^{2k} \cdot (k!)^2$$

Moreover, for each $0 < x \leq 1$ the Cauchy-Schwartz inequality gives

$$\phi^{(k)}(x) = \int_0^x \phi^{(k+1)}(t) \, dt \leq \sqrt{\int_0^1 \phi^{(k)}(x)^2 \, dx}$$

and since (iv) hold for every $k$ it follows that

$$\max_x |\phi^{(k)}(x)| \leq 2 \cdot J_* \cdot (\log(k+2))^k \cdot k!$$

Since $k\,! \leq k^k$ this entails that

$$\mathcal{C}_k(\phi) \leq J_*^{\frac{1}{k}} \cdot k \cdot (\log(k+2))$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k \log k}$ is divergent we conclude that

(v) $$\sum_{k=1}^{\infty} \frac{1}{\mathcal{C}_k(\phi)} = +\infty$$

Hence Denjoy's result in (**) form the introduction entails that $\phi$ is identically zero which means that

$$\lim_{k\to\infty} f_{N_k} = f \tag{vi}$$

where the convergence holds in the Frechet space $C^\infty[0,1]$. Finally, by the Arzela-Aascoli's compactness result above, the fact that (vi)= hold for every convergent subsequence entails that the whole sequence $\{f_N\}$ converges to $f$ which finishes the proof of Theorem 0.3.4.

## § 0.5. Quasi-analytic boundary values.

We consider boundary values of analytic functions on analytic functions in the unit disc $\{z| < 1\}$ whose set of non-zero Taylor-coefficients at the origin is sparse. In general, consider a power series $\sum a_n z^n$ whose radius of convergence equal to one and set

$$f = \sum a_n \cdot z^n$$

Assume that there exists an interval $\ell$ on the unit circle such that the analytic function $f(z)$ extends to a continuous function in the closed sector where $\arg(z) \in \ell$. So on $\ell$ we get a continuous boundary value function $f^*(\theta)$. Suppose in addition that gaps occur in the $a$-sequence, i.e. non-zero terms are $\{a_{n_1}, a_{n_2} \ldots\}$ where $k \mapsto n_k$ is a strictly increasing sequence. With these notations the following result is due to Hadamard:

**0.5.1 Theorem.** *Let $f(z)$ be as above where the boundary function $f^*(\theta)$ is real-analytic on $\ell$. Then there exists an integer $M$ such that*

$$n_{k+1} - n_k \leq M \quad : k = 1, 2, \ldots$$

Hadamard's result was extended to the quasi-analytic case in [Carleman] . In particular we may consider the case when $f^*$ belongs to a Denjoy class $\mathcal{D}_{\mathcal{A}}$ for a sequence $\mathcal{A} = \{\alpha_\nu\}$ where the series

$$\sum \frac{1}{\alpha_\nu} = +\infty$$

In [ibid] it is proved that when this hold and $f(z)$ is not identically zero, then the gaps of its Taylor coefficients cannot be too sparse. However, in contrast to Hadamard's theorem the result is more involved and and in Carlean's extension of Hadamard's theorem there also occur a condition on the rate of increase of the absolute values $\{|\alpha_\nu|\}$. Up to the present date it appears that no precise descriptions of the growth of $k \mapsto n_k$ which would ensure unicity is known while one regards an arbitrary Denjoy classes as above. So there remains many basic questions concerned with quasi-analyticity, and readers who would like to pursue this should first consult the subtle analysis which appears in Carleman's original work