14. Sets of harmonic measure zero

Introduction. The study of harmonic measures and other areas in potential theory goes back to a problem raised by G. Robin in the article [Rob] from 1886 which had physical background in electric engineering. The problem is: *Let E be a compact set in $\mathbf{C}$. Find a probability measure μ on E such that the function*

$$(*) \qquad U_\mu(z) = \int_E \log \frac{1}{[z-\zeta|} \cdot d\mu(\zeta) \quad : z \in \mathbf{C} \setminus E$$

takes constant boundary values on E. For every probability measure μ on E, i.e. a non-negative Riesz measure of unit mass supported by E, the integral (*) is defined for points in E where the value can be finite or infinite. To be precise, if $z_* \in E$ is fixed and n is a positive integer we put $E_n = E \setminus \{|z - z_*| \geq 1/n\}$. Since $\log \frac{1}{|z_* - \zeta|} \geq 0$ when $|\zeta - z_*| \leq 1$ it follows

$$n \mapsto \int_{E_n} \log \frac{1}{|z_* - \zeta|} \cdot d\mu(\zeta)$$

is increasing and by definition the integral (*) taken on E with $z = z_*$ is equal to the limit if (1) which therefore is finite or $+\infty$.

Remark. The limits above which compute $U_\mu(z)$ at points in E imply that the function

$$z \mapsto U_\mu(z)$$

is superharmonic function which is harmonic in the open complement of the support of μ. Recall also from § XX that the Laplacian of U_μ taken in the distribution sense is equal to the negative measure $-\pi \cdot \mu$. One refers to $U_\mu(z)$ as the logarithmic potential of μ. After measure theory had been developed and the notion of subharmonic functions had been consolidated a first major discovery is due to Lindeberg who proved that a compact set E whose Hausdorff measure with respect to the function $h(r) = \log \frac{1}{r}$ ois zero is a set of removable singularities for bounded harmonic functions, i.e. a bounded harmonic function H defined in $U \setminus E$ for some open neighborhood U extends to a harmonic function in U. In § A we expose more precise results due to Myhrberg about the connection between sets which are removable siungularities for bounded harmonic functions and the class of harmonic null sets, i.e. compact sets with a vanishing logarithmic capacity. In § C we expose a result due to H. Cartan (the son of Elie Cartan) which gives a *necessary condition* in order that a compact set E has harmonic measure zero. For several decades it was an open question of Cartan's metric conditions entail the converse. More precisely, let E be a compact set which is thin in Cartan's sense, i.e. its Hausdorff measure is zero for every h-function in Cartan's class. Then one may ask if E has logartihmic capacity zero. In 1952 Carleson constructed a totally disconnected comppact set E on the real line which is thin in Cartan's sense and has positive capacity. This example shows that the search for metric conditions in order that a compact set has harmonic measure zero is more or less hopeless, unless one adds some extra kind of properties. A notable point is that if E is an "ordinary Cantor set" on the real line then it is a harmonic null set if and only if its logarithmic length is zero, i.e. its Hausdorff measure with respect to $\log \frac{1}{r}$ is zero. We prove this result in § XX and remark that Carleson's construction in § XX is far more subtle and general compared to the standard construction of Cantor sets.

About literature. Many text-books treat potential theory with special emphasis on the present study of the logarithmic potential in $\mathbf{C}$. The text-book [Nevanlinna] covers the results in this chapter except for Carleson's constructions in § F. Passing to potential theory in dimension ≥ 3, Frostman's book *xxx* is a veritable master-piece which contains concise and yet very complete proofs leading to existence results of equilibrium measures in a quite general context. The reader may also consult the text-book [Garnett-Marshall] where §§ XX contains material from lectures by Carleson at UCLA during the years 19xx-xx devoted to complex potential theory. In this chapter we foremost follow material from Nevanlinna's text-book [nev].

§ 0. Energy integrals.

We can integrate U_μ with respect to μ and put

$$J(\mu) = \int U_\mu(z) \cdot d\mu(z)$$

$J(\mu)$ is called the energy integral of μ. Notice that the energy is expressed by a double integral:

$$J(\mu) = \iint \log \frac{1}{|z-\zeta|} \cdot d\mu(\zeta) \cdot d\mu(z)$$

In general the energy can be $+\infty$, i.e. it is not always true that U_μ belongs to $L^1(\mu)$. If there exists at least one probability measure μ on E with finite energy one is led to the variational problem

$$\min_\mu J(\mu)$$

with the minimum taken over all probability measures on E. Let $\mathcal{J}_*(E)$ denote the minimum and set

$$(*) \qquad \text{Cap}(E) = e^{-\mathcal{J}_*(\mu)}$$

One refers to (*) as the logarithmic capacity of E. In § E we prove that Cap(E) is equal to the transfinite diameter of E which is obtained via a limit of well-posed variational problems. One is also led to ask if there exists a probability measure μ_* on E such that $J(\mu_*) = \mathcal{J}_*(E)$. This is indeed true and can be seen as follows: Let $\{\mu_n\}$ be a sequence of probability measires on E such that $\{J(\mu_n)\}$ converge to $\mathcal{J}_*(E)$. Passing to a subsequence we can assume that $\{\mu_n\}$ converge weakly to a limit measure μ_*. For each positive integer N we wet

$$J_N(\mu_*) = \iint_{|z-\zeta|\geq N^{-1}} \log \frac{1}{|z_* - \zeta|} \cdot d\mu_*(\zeta) \cdot d\mu_*(z)$$

Then it is clear that $\{J_N(\mu_*)\}$ is increasing and the limit is $J(\mu_*)$. Next, for each fixed N we have

$$J_N(\mu_*) = \lim_{n\to\infty} J_N(\mu_n) \leq \lim_{n\to\infty} J(\mu_n) = \mathcal{J}_*(E)$$

Since this hold for every N we conclude that $J(\mu_*) = \mathcal{J}_*(E)$.

0.1 Equilibrium measures. Let E be a compact set with positive capacity. in § xx we prove that there exists a unique probability measure μ_* on E such that $J(\mu_*) = \mathcal{J}_*(E)$. Moreover, the potential function $U_\mu(z)$ regarded as a function in $L^1(\mu)$ is almost everywhere equal to the constant $\mathcal{J}_*(E)$. This result settles Robin's problem. In the case when E is a closed rectifiable Jordan curve Γ we remark that the equilibrium measure is found via Riemann's conformal mapping theorem as explained in § XX.

0.2 Harmonic null sets. Let E be compact and totally disconnected which means that if $\Omega = \mathbf{C} \setminus E$ is the open complement. So here the connected component of every boundary point to Ω is reduced to a singleton set and therefore it is not sure that the Dirichlet problem can be solved. But we can consider harmonic functions defined in open complementary sets to E. If $z_* \in E$ there exist arbitrary small

Jordan domains U which where z_* is an interior point while the closed Jordan curve ∂U has empty intersection with E. This leads to the following:

0.3 Definition. *A compact and totally disconnected set E is a removable singularity for bounded harmonic functions if every bounded harmonic function H in $U \setminus E$ for a pair (z_*, U) as above extends to a harmonic function in the whole Jordan domain U.*

The following result is due to Myhrberg.

0.4 Theorem. *A compact and totally disconnected set E is a removable singularity for bounded harmonic functions if and only if there exists a probability measure μ on E such that $U_\mu(z) = +\infty$ for every $z \in E$.*

When the two equivalent conditions hold in Theorem 0.4 we say that E is a harmonic null-set and denote by $\mathcal{N}_{\text{harm}}$ the family of totally disconnected sets harmonic null-sets.

Remark. In the literature one also refers to sets of capacity zero, i.e. a compact and totally disconnected subset E if $\mathbf{C}$ has capacity zero if and only if its is a harmonic null set. One may ask for metric conditions in order that a given compact and totally disconnected set E belongs to $\mathcal{N}_{\text{harm}}$. To analyze this we consider Hausdorff measures. In general, $h(r)$ be a continuous and non-decreasing function defined for $r > 0$ and $h(0) = 0$. If F is a compact set we consider open coverings of F by discs and define its outer h-measure by

$$h^*(F) = \min \sum h(r_\nu)$$

where the minimum is taken over coverings of F by open discs $\{D_\nu\}$ of radius $\{r_\nu\}$. The family of compact sets whose outer h-measure is zero is denoted by $\mathcal{N}(h)$. The case $h(r) = r^2$ means precisely that F has planar Lebesgue measure zero. If $h(r)$ tends more slowly to zero as $r \to 0$ we get a more restrictive class. If

$$h(r) = \frac{1}{\log \frac{1}{r}}$$

one says that a compact set in $\mathcal{N}(h)$ has logarithmic capacity zero . The first major result about harmonic null-sets was proved by Lindeberg in 1918:

0.5 Theorem. *Let E be a compact set whose logarithmic measure zero. Then E has harmonic measure zero.*

Theorem 0.5 was improved in the article *Note on the transfinite diameter* [London.Math. Soc. Vol. 12: 1937) by Erdös and Gillis improved Theorem 0.5 where it was shown that if E has a finite logarithmic measure then it has harmonic measure zero. We prove this result in § XX.

Metric conditions.

A necessary metric condition for a set E to be a harmonic null-set was established by Henri Cartan in [Cartan]. First we give:

0.6 Definition *Let $\mathfrak{H}_*$ denote the class of non-decreasing and continuous function $h(r)$ satisfying*

$$\int_0^1 \frac{h(r)}{r} \cdot dr < \infty$$

0.7 Theorem. *For every $E \in \mathcal{N}_{harm}$ it follows that*

$$E \in \mathcal{N}(h) = 0 \quad : \forall\, h \in \mathfrak{H}_*$$

Remark. Cartan's result is close to Lindeberg's sufficiency result. Namely, if $\eta > 0$ we set

$$h(r) = \frac{1}{[\text{Log}\,\frac{1}{r}]^{1+\eta}}$$

It is clear that $h \in \mathfrak{H}_*$ and hence $h^*(E) = 0$ for every $E \in \mathcal{N}_{\text{harm}}$. With η small this comes close to say that the logarithmic capacity of E is zero. However, Cartan's Theorem does not give sufficient conditions in order that a compact and totally disconnected set E has harmonic measure zero.

0.8 Carleson's example. In the article *On the connection between Hausdorff measures and capacity* [Arkiv för matemtaik: vol. 3 (1956)], Carleson constructed a compact totally disconnected subset E of the unit interval $[0,1]$ which is outside the class $\mathcal{N}_{\text{harm}}$ while the Hausdorff measure $h(E) = 0$ is zero for every $h \in \mathfrak{H}_*$. Carleson's construction shows that the search for both necessary and sufficient metric condtions in order that a given totally disconnected compact set belongs to $\mathcal{N}_{\text{harm}}$ appears to be more or less hopeless. See also Carleson's text-book Exceptional sets for a discussion. Let us also remark that Carelson's set E above is not a standard Cantor set. In fact, if $\mathcal{C}$ is a Cantor set in $[0,1]$ as described in § D below then the vanishing of its Hausdorff measure for every $h \in \mathcal{H}_*$ implies that $\mathcal{C}$ is a harmonic null set. So via cartan's theorem there exists a necessary and sufficient metric condition in order that Cantor sets on the real line belongs to $\mathcal{N}_{\text{harm}}$.

A. Proof of Myhrberg's theorem

A.1 Nested coverings We shall employ a construction which was originally introduced by De Vallé Poussin. Let E be a totally disconnected and compact set and consider some $z_* \in E$. Choose a small Jordan domain U which contains z_* while $\partial U \cap E = \emptyset$. For each positive integer N one has the dyadic grid $\mathcal{D}_N$ of closed squares whose sides are 2^{-N}. We consider only those N such that

$$2^{-N} < \text{dist}(E, \partial U)$$

We get the finite family $\mathcal{D}_N(E \cap U)$ of squares in $\mathcal{D}_N$ which have a non-empty intersection with $E \cap U$ and let Let V_N be the union of these squares. Next, denote by Ω_N^* the connected component of $D \setminus \bar{V}_N$ whose closure contains the Jordan curve ∂U. It follows that Ω_N^* is a doubly connected domain whose boundary is the disjoint union of ∂U and a closed Jordan curve Γ_N formed by line segments from squares in the finite family $\mathcal{D}_N(E \cap U)$. Notice also that $\{\Omega_N^*\}$ is an increasing sequence of open sets where Γ_N appears as a compact subset of Ω_{N+1}^* for each N and finally:

$$\cup\, \Omega_N^* = D \setminus E$$

Next, fix some point $z_0 \in D \setminus E$ and from now on N are so large that $z_0 \in \Omega_N^*$ hold. The Dirichlet problem has a solution in each domain Ω_N^*. This gives a unique pair of non-negative measures μ_N, ρ_N where μ_N is supported by Γ_N and ρ_N by ∂U such that

$$(*) \qquad h(z_0) = \int_{\Gamma_N} h(\zeta) \cdot d\mu_N(\zeta) + \int_{\partial U} h(\zeta) \cdot d\rho_N(\zeta)$$

hold for every h-function which is harmonic in Ω_N^* with continuous boundary values. In particular we let h be the harmonic function which is zero on ∂U and one on Γ_N. This gives the harmonic measure

$$(**) \qquad \mathfrak{m}_{\Gamma_N}(z_0) = \int_{\Gamma_N} d\mu_N(\zeta)|$$

Since $\Gamma_N \subset \Omega^*_{N+1}$ we have $\mathfrak{m}_{N+1} \leq \mathfrak{m}_N$ in Ω^*_N so the masses of the non-negative μ-measures descrease, i.e.

$$||\mu_{N+1}|| \leq ||\mu_N||$$

Hence there exists the limit

$$\alpha = \lim_{N\to\infty} ||\mu_N|| \tag{***}$$

A.2 The case $\alpha = 0$. When this holds the mass of ρ_N tends to one and since (*) in particular hold for h-functions which are harmonic in the whole set U with continuous boundary values on ∂U the reader may verify:

A.3 Proposition. *If $\alpha = 0$ the sequence $\{\rho_N\}$ converges weakly to the representing measure $m(z_0)$ for which*

$$H(z_0) = \int_{\partial U} H(\zeta) \cdot dm(z_0, \zeta)$$

when H is harmonic in U and continuous on $\bar{U}$.

Exercise. Apply Harnakck's inequality inthe domains Ω^*_N to show that when $\alpha = 0$ holds for one chosenpoint $z_0 \in U \setminus e$ then we get a similar vanishing for every other point $z_1 \in U \setminus E$. Thus, the condition that $\alpha) = 0$ is intrinsic.

A.4 Extending bounded harmonic functions. Suppose that $\alpha = 0$ and let H be a bounded harmonic function defined in $U \setminus E$ which in addition extends to a continuous function on ∂U. For each $z_0 \in U \setminus E$ we start with large N so that $z_0 \in \Omega^*_N$ and represent $H(z_0)$ by (1). Since H is bounded the hypothesis $\alpha = 0$ entails that

$$\lim_{N\to\infty} \int_{\Gamma_N} H(\zeta) \cdot d\mu_N(\zeta) = 0$$

It follows that

$$H(z_0) = \lim_{N\to\infty} \int_{\partial U} H(\zeta) \cdot d\rho_N(\zeta)$$

DSince this hold for every $z_0 \in U \setminus E$ it follows from Propostion A.3 that H is equal to the everywhere defined harminic function in u which extends the continuous boundary value function $H|\partial U$. This shows that if $\alpha = 0$ then E is a removable singularity for buounded harmonic functions.

A.6 An infinite potential function. The condition that $\alpha = 0$ is intrinsic and by a conformal mapping we may assume that U is the open unit disc which is convenient because now we can write out more explicit formulas.

A.5 Proposition. *When $\alpha = 0$ there exists a pair of positive numbers $0 < a < A$ such that*

$$a \leq \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) \leq A$$

hold for all $w \in E$ and every N.

Proof. Fix some $z_0 \in D \setminus E$ and consider a point $w \in E$. Since $|e^{i\theta} - w| = |1 - \bar{w} \cdot e^{i\theta}|$ hold for all $e^{i\theta}$ on the unit circle we get

$$\int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta) = \int_T \log \frac{1}{|1 - \bar{w}\zeta|} \cdot d\rho_N(\zeta) \quad N = 1, 2 \ldots \tag{i}$$

Next, in D we have the harmonic function $H(z) = \log \frac{1}{|1-\bar{w}z|}$ in D and Proposition A.3 entails that

$$\log \frac{1}{|1 - \bar{w}z_0|} = \lim_{N\to\infty} \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta) \tag{ii}$$

At the same time (*) in § A.1 applied with $h(z) = \log \frac{1}{|z-w|}$ gives

$$\text{(iii)} \qquad \log \frac{1}{|z_0 - w|} = \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) + \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta)$$

By (ii) the last integral converges to $\log \frac{1}{1-\bar{w}z_0|}$ and hence (iii) gives the limit formula

$$\lim_{N\to\infty} \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) = \log \frac{1 - \bar{w}z_0|}{|z_0 - w|}$$

Since $|z_0| < 1$ and there is some $r < 1$ such that $|w| \leq r$ for every $w \in E$ the last term is between a and A for a pair of positive numbers which proves Proposition A.5.

A.7 The limit measure μ_*. Assume as above that $\alpha = 0$ and set $\alpha_N = ||\mu_N||$. On E we get the probability measures

$$\nu_N = \frac{1}{\alpha_N} \cdot \mu_N$$

We can extract a subsequence which converges weakly to a probability measure μ which by (A.1.0) is supported by E. The left hand side in Proposition A.5 gives the inequality

$$\min_{w\in E} \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\nu_N(\zeta) \geq \frac{a}{\alpha_N}$$

Since $|alpha_N|to0$ while $a > 0$ is a fixed positive constant, it follows that the potential function of the weak limit μ_* is everywhere $+\infty$ on E, i.e.

$$\int_E \log \frac{1}{|\zeta - w|} \cdot d\mu_*(\zeta) = +\infty$$

hold for every $w \in E$.

A.9 Proof of Theorem 0.4

Suppose first that E is a removable singularity for bounded harmonic functions. Working locally as above around some $z_* \in E$ and a Jordan domain U, we obtain for each N the harmonic measure function $\mathfrak{m}_N$ in Ω_N^*. They take values in $(0, 1)$ and passing to a subsequence we obtain a bounded harmonic limit function $\mathfrak{m}_*$ defined in $U \setminus E$. By construction $\mathfrak{m}_* = 0$ on ∂U so if it extends to a harmonic function in U it must be identically zero which entails that

$$\lim_{N\to\infty} \mathfrak{m}_N(z_0) = 0$$

This means precisely that $\alpha = 0$ and from the above we construct a potential function which is everywhere $+\infty$ on E.

The converse. Suppose there exists a probability measure μ on E whose potential $U_\mu(w) = +\infty$ for all $w \in E$. Let $z_* \in E$ and choose some small Jordan domain U around z_* as in § A.1 where we have constructed the nested sequence of Γ-curves. Put

$$C_N = \min_{z\in\Gamma_N} U_\mu(z)$$

Since the distances from Γ_N to E tend to zero it follows that $C_N \to +\infty$. Next, U_μ restricts to a continuous function on ∂U and we find its harmonic extension $H(z)$ to the Jordan domain U. If C_* is the maximum of U_μ on ∂Uwe have $H \leq C_*$ in the whole Jordan domain. With large N we have $C_N > C_*$ and at the same time

$$U_\mu(z_0) - H(z_0) = \int_{\Gamma_N} (U_\mu - H)\ddot{d}\mu_N \geq (C_N - C_*) \cdot ||\mu_N||$$

Since $C_N \to +\infty$ we conclude that $||\mu_N|| \to 0$ which means tha $\alpha = 0$ and then we have proved that bounded harmonic functions outside E can be extended which finishes the proof of Myhrberg's theorem.

B. Equilibrium distributions and Robin's constant.

Let E be a compact set in $\mathbf{C}$. To each probability measure μ supported by E we get the potential function

$$U_\mu(z) = \int \log \frac{1}{|z-\zeta|} \cdot d\mu(\zeta)$$

We are going to construct a special μ for which U_μ either is identically $+\infty$ or else takes a constant value almost everywhere on E with respect to μ. First we carry out the construction in the special case when E is a finite union of pairwise disjoint and closed Jordan domains $U_1, \ldots, U_m$ for some $m \geq 1$. We also assume that each Jordan curve ∂U_k is of class C^1. When this holds we get the connected exterior domain

$$\Omega^* = \mathbf{C} \cup \{\infty\} \setminus \cup \bar{U}_k$$

Here we can solve Dirichlet's problem. in particular we obtain the unique probability measure μ on $\partial\omega^*$ such that

$$H(\infty) = \int H \cdot d\mu$$

for every harmonic function H in Ω^* with continuous boundary values. If z_1 and z_2 are two points in $/cup\, U_k$ which may or may not belong to the same Jordan domain then we notice that the function

$$H(z) = \log|z - z_1| - \log|z - z_2|$$

is harmonic i Ω^*. Moreover, as $|z| \to \infty$ we notice that

$$H(z) = \log|1 - \frac{z_1}{z}| - \log|1 - \frac{z_2}{z}|$$

and in the limit we have $H(\infty) = 0$. Since $\log r = -\log \frac{1}{r}$ for each $r > 0$ it follows that

$$U_\mu(z_1) = U_\mu(z_2)$$

Hence the function $z \mapsto U_\mu(z)$ is constant in the interior of E Since the boundary curves $\{\partial U_k\}$ are C^1 it follows that U_μ extends to a continuous function with constant value on the whole set E. Of course, U_μ is also continuous outside E where it is harmonic. In fact, we conclude that U_μ is a globally defined and continuous super-harmonic function in $\mathbf{C}$. The measure μ is called the equilibrium distribution of E.

Remark. If E is contained in the unit disc it is clear that the constant value of U_μ is positive. On the other hand, let $R > 1$ and E is the disc $|z \leq R$. here μ is the measure $\frac{1}{\pi} \cdot d\theta$ on the circle of radius R and we find that the constant value is $-\log R$.

Notation. If a is the constant value of U_μ we set

$$\text{cap}(E) = e^{-a}$$

and refer to this as the capacity of E. For example, if E is the disc $|z| \leq r$ where r is small we see that the capacity becomes r.

The general case. Now E is an arbitrary compact set. To construct a special probability measure μ_E we use a similar construction as in section A. Thus, for

$N \geq 1$ we get the family of cubes in $\mathcal{D}_N$ which have a non-empty intersection with E and then we construct the outer boundary curves of thus set which borders a connected exterior domain Ω_N^* whose boundary now will be a union of closed and piecewise linear Jordan curves where two of these may interest at corner points. We solve the Dirichlet problem and exactly as above we find the equilibrium measure μ_N supported by $\partial\Omega_N^*$.

C. Cartan's theorem

We shall actually establish an inequality in Theorem C.1 below which has independent interest since it applies to compact sets E which are not necessarily harmonic null sets. Consider a pair (h, μ) where μ is a probability measure with compact support in a compact set E of $\mathbf{C}$ with planar Lebesgue measure zero while $h \in \mathfrak{H}_*$. To each point $a \in E$ and every $r > 0$ we have the open disc $D_r(a)$ centered at a and can regard its μ-mass. This gives anon-decreasing function

$$r \mapsto \mu(D_r(a)) \quad : r > 0$$

Put

(1) $$\mathcal{U}^* = \{a \in E \quad : \exists\, r > 0 \quad : \mu(D_r(a)) > h(r)\}$$

We assume that the pair (h, μ) is such that this set is non-empty. Since μ is a Riesz measure one has the limit formula

$$\lim_{\rho \to r} \mu(D_\rho(a) = \mu(D_r(a)$$

for each $r > 0$ where the limit is taken as ρ increases to r. From this it is obvious that $\mathcal{U}^*$ is a relatively open subset of E and in the closed complement we have

(2) $$a \in E \setminus \mathcal{U}^* \implies \mu(D_r(a)) \leq h(r) \quad : \forall\, r > 0$$

Now the size of $\mathcal{U}^*$ is controlled as follows:

C.1 Cartan's Covering Lemma. *There exists a sequence $\{a_\nu\}$ in E and a sequence of positive numbers $\{r_\nu\}$ such that the following hold:*

$$\mathcal{U}^* \subset \cup \bar{D}_{r_\nu}(a_\nu) \quad \text{and} \quad \sum h(r_\nu) \leq 6$$

Moreover, for each $z \in \mathbf{C}$ at most five discs from the family $\{D_{r_\nu}(a_\nu)\}$ contains z.

Proof We may assume that $\mathcal{U}^* \neq \emptyset$. Set

(1) $$\lambda_1^*(r) = \max_{a \in E} \mu(D_r(a))$$

Since the functions $r \mapsto \mu(D_r(a))$ are lower semi-continuous for each a, it follows that the maximum function $\lambda_1^*(r)$ also is lower semi-continuous. Hence the set $\{r\colon\, \lambda_1^*(r) > h(r)\}$ is open and we find its least upper bound r_1^*. Thus,

(2) $$\lambda_1^*(r_1^*) = h(r_1^*) \quad : \lambda_1^*(r) < h(r) \text{ for all } r > r_1^*$$

Pick $a_1 \in E$ so that

(3) $$\lambda_1^*(r_1^*) < \mu(D_{r_1^*}(a_1)) + 1/2$$

Next, set $E_1 = E \setminus D_{r_1^*}(a_1)$ and define

$$\lambda_2^*(r) = \max_{a \in E_1} \mu(D_r(a))$$

If $\lambda_2^*(r) \leq h(r)$ for every r we stop the process. Otherwise we find the unique largest r_2^* such that

$$\lambda_2^*(r_2^*) = h(r_2^*)$$

Notice that $r_2^* \leq r_1^*$ holds since h is non-decreasing while it is obvious that $\lambda_2^* \leq \lambda_1^*$. This time we pick $a_2 \in E$ so that

$$\lambda_2^*(r_2^*) < \mu(D_{r_2^*}(a_2)) + 2^{-2}$$

Put $E_2 = E_1 \setminus D_{r_2^*}$ and continue as above, i.e. inductively we get E_n and set

$$\lambda_{n+1}(r) = \max_{a \in E_n} \mu(D_r(a))$$

The process continues if we have found r_{n+1}^* so that $\lambda_{n+1}(r_{n+1}^*) = h(r_{n+1}^*)$, then we pick $a_{n+1} \in E_n$ where

$$\lambda_{n+1}(r_{n+1}^*) \leq \mu(D_{r_{n+1}^*}(a_{n+1})) + 2^{-n-1} \tag{4}$$

In this way we get the sequence $r_1^* \geq r_2^* \geq \ldots$ and a family of discs $\{D_{r_\nu^*}(a_\nu)\}$. To simplify notations we set

$$D_\nu^* = D_{r_\nu^*}(a_\nu)$$

Sublemma Every point $a \in E$ belongs to at most five many D^*-discs.

Proof. If some a belongs to six discs then elementary geometry gives a pair a_k, a_ν such that the angle between the lines $[a, a_k]$ and $[a.a_\nu]$ is $< \pi/3$. Suppose that for example that $|a - a_k| \geq [a - a_\nu|$. Euclidian geometry gives

$$|a_k - a_\nu| < |a - a_k|$$

But this is impossible. For say that $k < \nu$. Now the disc D_k^* was removed and a_ν is picked from the subset E_ν of E_k while $E_k \cap \Delta_k = \emptyset$.

Proof continued. The Sublemma implies that

$$\sum \mu(D_\nu^*) \leq 5 \cdot \mu(E) = 5 \tag{5}$$

The convergence of (5) and (4) imply that $\lim_{\nu \to \infty} r_\nu^* = 0$. From this it follows that

$$\mathcal{U}^* \subset \cup \bar{D}_{r_\nu}(a_\nu) \tag{6}$$

Finally we have

$$\sum h(r_\nu^*) = \sum \lambda_\nu^*(r_\nu^*) \leq \sum [\mu(D_\nu^*) + 2^{-\nu}] \leq 5 \cdot \mu(E) + \sum 2^{-\nu} = 6 \tag{7}$$

This completes the proof of Cartan' s Covering Lemma.

The family $\mathcal{G}_h$. Let $g(r)$ be a positive function defined on $(0, +\infty)$ which satisfies:

$$\lim_{r \to 0} g(r) = +\infty$$

In this family we get those g-functions for which

$$\int_0^1 g(r) \cdot dh(r) < \infty \tag{*}$$

This family is denoted by $\mathcal{G}_h$. With this notation we have:

C.2 Lemma *For each $g \in \mathcal{G}_h$ and every point $a \in E \setminus \mathcal{U}^*$ one has*

$$\int_E g(|z - a|) d\mu(z) \leq \int_0^\rho g(r) dh(r) \quad \text{where } h(\rho) = 1$$

Proof. Since a is outside $\mathcal{U}^*$ we have

$$\mu(D_r(a)) \leq h(r)$$

for every $r > 0$. Moreover, we recall that μ has total mass one and now the reader can verify the inequality in Lemma C.2 b using a partial integration.

C.3 A special choice of g. Let us take

$$g(r) = \text{Log}\,\frac{1}{r} \quad : \ 0 < r < 1 \quad : \quad g(r) = 0 \quad : \ r \geq 1$$

This g-function belongs to $\mathcal{G}_h$ by the condition on h-functions in Cartan's theorem. Next, for every $\lambda > 1$ we get the function $h_\lambda = \lambda \cdot h$ in $\mathfrak{H}_*$ and set:

$$E \setminus \mathcal{U}^*(\lambda) = \{a \in E \quad : \mu(D_r(a)) \leq \lambda \cdot h(r) \, : \, \forall \, r > 0\}$$

Proposition XX(measure general) applied with h_λ gives:

$$(1) \qquad \int_E g(|z-a|) d\mu(z) \leq \lambda \cdot \int_0^{\rho/\lambda} g(r) dh(r) \quad : \, a \in E \setminus \mathcal{U}^*(\lambda)$$

A partial integration shows that the right hand side in (1) becomes

$$g(\rho) + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r}$$

Hence we have the inequality

$$(2) \qquad \int g(|z-a|) \cdot d\mu(z) \leq g(\rho) + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r} \quad : \quad a \in E \setminus \mathcal{U}^*(\lambda)$$

In addition to this, the Covering Lemma gives an inclusion

$$(3) \qquad \mathcal{U}^*(\lambda) \subset \cup \, D_{r_\nu}(a_\nu) \quad \text{where} \sum \; h_\lambda(r_\nu) < 6$$

Since $h_\lambda = \lambda \cdot h$ this means that the outer h-measure

$$(4) \qquad h^*(\mathcal{U}^*(\lambda)) \leq \frac{6}{\lambda}$$

Hence we have proved the following where we recall that $g(r) = \text{Log}\, \frac{1}{r}$:

C.4 Theorem. *For every triple (E, μ, h), where μ is a probability measure supported by E and $h \in \mathfrak{H}_*$, and any $\lambda > 1$ there exists a relatively open subset $\mathcal{U}^*(\lambda) \subset E$ such that the following two inequalities hold:*

$$\text{(i)} \qquad \int_E \log \frac{1}{|z-a|} \cdot d\mu(z) \leq \log \frac{1}{\rho} + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r} \quad : \quad a \in E \setminus \mathcal{U}_*(\lambda)$$

$$\text{(ii)} \qquad h^*(U_*(\lambda)) < \frac{6}{\lambda}$$

C.11 Proof of Theorem 0.5. Let $E \in \mathcal{N}_{\text{harm}}$ which by Theorem 0.2 gives a probability measure μ supported by E such that that the left hand side in (i) is $+\infty$ for every $a \in E$. It follows that the set $E \setminus \mathcal{U}^*(\lambda)$ is empty for every $\lambda > 1$. With a fixed λ we apply Cartan's covering Lemma and since $E = \mathcal{U}^*(\lambda)$ it follows that

$$h^*(E) \leq \frac{6}{\lambda}$$

Here $\lambda > 1$ is arbitrary which gives $h^*(E) = 0$ as required and of Cartan's theorem follows.

D. Cantor sets.

We construct a family of closed subsets of $[0,1]$ as follows. Let $1 < p_1 < p_2 < \ldots$ be some strictly increasing sequence of real numbers such that the products $\{p_1 \cdots p_n\}$ tend to $+\infty$ as n increases. Then we can construct a decreasing sequence of closed sets $E_1, E_2, \ldots$ where each E_n is the union of 2^n-many closed intervals with equal length

$$\ell_n = 2^{-n} \cdot \frac{1}{p_1 \cdots p_n}$$

D.1 The construction. First E_1 is any closed interval $[a_1, b_1]$ with

$$b_1 - a_1 = \frac{1}{2p_1}$$

Inside this closed interval we pick two pairwise disjoint closed interval both of length ℓ_2 and let E_2 be their union. In the next step we pick a pair of closed intervals both of length ℓ_3 from each of the two intervals in E_2. Their union gives the set E_3 and we continue in the same way for every n and arrive at a closed set

$$\mathcal{E} = \cap E_n$$

We refer to $\mathcal{E}$ as a Cantor set. The construction is flexible since we do not impose any condition on specific positions while we at stage n pick pairs of intervals of length ℓ_{n+1} from each of the 2^n many intervals of E_n. Thus, for a given p-sequence we obtain a whole family of Cantor sets denoted by $\text{Cantor}(p_\bullet)$. The next result gives a condition for such Cantor sets to have harmonic measure zero.

D.2 Theorem. *The following are equivalent for an arbitrary sequence $p_\bullet$ as above*:

$$\text{Cantor}(p_\bullet) \subset \mathcal{N}_{\text{harm}} \text{ holds if and only if } \sum_{\nu=1}^{\infty} \frac{\text{Log}\, p_\nu}{2^\nu} = +\infty$$

The proof uses the explicit formulas for Robin constants of intervals on the real line. For the detailed proof we refer to page xx-xx in [Nevanlinna].

E. Transfinite diameters and the logarithmic capacity.

Let E be a compact set where we do not assume that E is totally disconnected. To each n-tuple of distinct points $z_1, \ldots, z_n$ we put:

$$L_n(z_\bullet) = \frac{1}{n(n-1)} \cdot \sum_{k \neq j} \log \frac{1}{|z_j - z_k|} \quad : \quad \mathcal{L}_n(E) = \min L_n(z_\bullet)$$

where the minimum in the right hand side is taken over all n-tuples in E. Since $\log \frac{1}{r}$ is large when $r \simeq 0$ this means intuitively that we use separated n-tuples to minimize the L_n-function. For example, if $n = 2$ the minium is achieved for a pair of points in E whose distance is maximal, i.e. $\mathcal{L}_2$ is the diameter of E.

E.1 Proposition. *The sequence $\{\mathcal{L}_n\}$ is non-decreasing.*

Proof. Let $z_1, \ldots, z_{n+1}$ minimize the L_{n+1}-function which gives

$$\mathcal{L}_{n+1}(E) = \frac{1}{n(n+1)} \cdot \sum_{k \neq j}^{(1)} \log \frac{1}{|z_j - z_k|} + \frac{2}{n(n+1)} \cdot \sum_{k=2}^{k=n+1} \log \frac{1}{z_1 - z_k|}$$

where (1) above the sum above means that we only consider pairs k, j which both are ≥ 2. Since $z_2, \ldots, z_{n+1}$ is an n-tuple we get the inequality

$$\mathcal{L}_{n+1}(E) \geq \frac{1}{n(n+1)} \cdot n(n-1) \cdot \mathcal{L}_n(E) + \frac{2}{n(n+1)} \cdot \sum_{k=2}^{k=n+1} \log \frac{1}{z_1 - z_k|}$$

The same inequaliiy holds when when some $z_j : 2 \leq j \leq n+1$ is deleted. Taking the sum of the resulting inequalities we obtain

$$(n+1)\mathcal{L}_{n+1}(E) \geq \frac{1}{n} \cdot n(n-1) \cdot \mathcal{L}_n(E) + \frac{2}{n(n+1)} \cdot \sum_{k \neq j} \log \frac{1}{|z_j - z_k|}$$

The last term is $2 \cdot \mathcal{L}_{n+1}$ which gives:

$$(n-1) \cdot \mathcal{L}_{n+1}(E) \geq \frac{1}{n} \cdot n(n-1) \cdot \mathcal{L}_n(E) = (n-1)\mathcal{L}_n(E)$$

A division by $n-1$ gives the requested inequality.

E.2 Definition. *Put*

$$\mathfrak{D}(E) = \lim_{n \to \infty} e^{-\mathcal{L}_n(E)}$$

This non-negative number is called the transfinite diameter of E.

Remark. The definition means that $\mathfrak{D}(E) = 0$ if and only if $\mathcal{L}_n(E)$ tends to $+\infty$ as n increases. Intuitively this means that we are not able to choose large tuples in E separated enough to keep the sum of the log-terms bounded. Another number is associated to E is defined by:

$$\mathcal{J}_*(E) = \min_{\mu} J(\mu)$$

where the minimum is taken over all probability measures in E and $J(\mu)$ are the energy integrals from § 0.2.

E.3 Definition. *The logarithmic capacity of E is defined by:*

$$\text{Cap}(E) = e^{-J_*(E)}$$

E.4 Theorem. *For each compact set E one has the equality*

$$\text{Cap}(E) = \mathfrak{D}(E)$$

Proof. Let $n \geq 2$ and $z_1^*, \ldots, z_n^*$ is some n-tuple where $L_n(z_\bullet) = \mathcal{L}_n(E)$. Now we have the probability measure

$$\mu = \frac{1}{n} \cdot \sum_{k=1}^{k=n} \delta_{z_k^*}$$

It is clear that the energy

$$J(\mu) = \frac{n(n-1)}{n^2} \cdot L_n(z_\bullet)$$

Hence we have the inequality

$$\mathcal{J}_*(E) \leq \frac{n(n-1)}{n^2} \cdot \mathcal{L}_n(E)$$

Since $\frac{n(n-1)}{n^2}$ tends to one as $n \to \infty$ a passage to the limit gives:

$$\mathcal{J}_*(E) \leq \lim_{n \to \infty} \mathcal{L}_n(E)$$

Taking exponentials and recalling the negative signs in E.2 and E.3 we conclude that

$$\text{(i)} \qquad \mathfrak{D}(E) \leq \text{Cap}(E)$$

Exercise. Prove the opposite inequality. The hint is that probability measures on E can be approximated by discrete measures.

F. Thin sets with positive capacity on the real line.

Introduction. We expose results from the article [Carleson]. In contrast to "ordinary Cantor sets" the inductive process while open intervals are removed is far more flexible in Carleson's construction below. This leads to an extensive class of totally disconnected compact subsets of the interval $[0, 1]$ which are null sets for all h in the family from Definition 0.4. Foloowing [ibid] we show in § X how to construct a compact totally disconnected set E with a positive capacity while $h(E) = 0$ for all h in Cartan's family. Let us remark that the construction of E gave a negative answer to previously open question whether a converse to Cartan's theorem for sets on a real line was valid or not.

F.1. A set-theoretic construction Let $J = [a, b]$ be a subinterval of $[0, 1]$ and (m, q) is a pair of positive integers such that

$$\text{(i)} \qquad \sum_{\nu=0}^{\nu=m} e^{-m-\nu} + m \cdot e^{-q} = b - a$$

Then $[a, b]$ is covered by $2m + 1$ intervals taken in increasing order with lengths

$$\text{(ii)} \qquad e^{-m}, e^{-q}, e^{-m-1} \ldots, e^{-q}, e^{-2m}$$

We refer to this as a decomposition of type (m, q) of the interval $[a, b]$. When this has been done we remove the q-intervals and are left with $m + 1$ many intervals $J_0, J_1, \ldots, J_m$ where J_ν has length $e^{-m-\nu}$. Consider a pair (m_0^*, q_0^*) adapated to the length of J_0, i.e.

$$\sum_{\nu=0}^{\nu=m_0^*} e^{-m_0^*-\nu} + m_0 \cdot e^{-q_0^*} = e^{-m}$$

Exactly as above we get a partion of J_0 and remove the m_0^* many open q-intervals from J_0 which gives a closed set J_0^* formed by $m_0^* + 1$ many closed intervals of length $\{e^{-m_0^*-\nu}\}$ Next, consider some pair (m_1^*, q_1^*) adapted to J_1 and proceed as above. Removing m_1^* many q-intervals from J_1 there remains a set J_1^*. In the next stage we chooise a pair (m_2^*, q_2^*) apated to J_2 and remove q-intervals. At the final stage we have a pair m_m^*, q_m^* where q-intervals are removed from J_m. In this way a chosen $(m + 1)$-tuple $\{(m_k^*, q_k^*)\}$ leads to the removal of q-intervals on the family $J_0, J_1, \ldots, J_m$ and we get a smaller closed set which consists of $M = m_0 + m_1 + \ldots + m_m^*$ many disjoint and closed intervals denoted by $J_1^*, \ldots, J_M^*$. We repeat the construction starting from this M-tuple and begin with a pair $m_0^{**}, q_0^{**})$ adapated to J_0^* and so on. After one has removed q-intervals from $J_0^*, J_1^*, \ldots, J_M^*$ there remains a family J^{**} which consists of $M_0^* + M_1^* + \cdots + M_M^*$ many disjoint intervals. This process can be repeated an infinite number of times and yields a decreasing sequence of closed sets $\{E_n\}$ where E_n is the union of J-intervals after n many constructions. At last we obtain the intersection $E_* = \cap E_n$.

F.2 How to make E_* thin. At every single step above where one has some J-interval in the n-th partition of some length ℓ we choose an adapted pair (m, q) to this intervall where $m > 2\dot{\ell}$. Under this condition one verifies that if we for an arbitrary large n pick a finite set of intervals which appear in E_n with lengths $e^{-s_1}, \ldots, e^{-s_p}$ then the integers $s_1, \ldots, s_p$ are all distinct ! As pointed out by in [Carleson: page 405] this is *a crucial of the construction.*. Let us now consider some Cartan-function h. To estimate the h-measure of E_* we choose n large and cover E_* by a finite set of intervals with lenghts $\{e^{-s_\nu}\}$ where $s_1 < \ldots < s_p$ are different integers and s_1 can be made arbitrary large. This entails that

$$\sum_{j=1}^{j=p} h(e^{-s_j}) \leq \sum_{\nu=s_1}^{\infty} h(e^{-\nu}) < \int_0^{e^{-s_1}} \frac{h(r)}{r}\,dr$$

So if h belongs to the family from Definition 0.4 we conclude that $\mathfrak{h}(E_*) = 0$.

The construction of a μ-measure. By the above one has a flexible way to obtain sets E_* with vanishing h-measures for $h \in \mathfrak{H}$. There remains to make one such construction along with an inductively defined sequence of positive measures $\{\mu_n\}$ supported by $\{E_n\}$ and then pass to a limit measure μ_* supported by E_* whose energy integral

$$\iint \log\left|\frac{1}{x-y}\right| d\mu(x)d\mu(y) < \infty$$

TO FINISH

E. Further result and someexamples.

A. The Robin constant

.

First we consider a finite union of pairwise disjoint Jordan domains $U_1, \ldots U_p$ and get the compact boundary $\Gamma = \cup \partial U_\nu$. The closed Jordan curves $\{\partial U_\nu\}$ are in general disjoint but we may also allow that a pair intersect at some finite set of points. We also impose the condition that the open complement

$$\Omega = \mathbf{C} \setminus \cup \bar{U}_\nu$$

is connected. We add the point at infinity to Ω where $w = 1/z$ is a new coordinate. A harmonic function $H(z)$ defined in Ω becomes harmonic at infinity if $w \mapsto H(1/w)$ extends to a harmonic function in a whole disc centered at $w = 0$.

A.1 A basic construction In the exterior domain $\Omega^* = \Omega \cup \{\infty\}$ the Dirichlet problem is well posed. Hence there exists a unique probability measure μ on $\partial\Omega^*$ such that

$$(*) \qquad H(\infty) = \int H(\zeta) \cdot d\mu(\zeta)$$

holds for every harmonic function H in Ω^*. Notice that $\partial\Omega^*$ is equal to the union of the closed Jordan curves $\{\partial U_\nu\}$. Let us then consider two points a and b in in $\cup U_\nu$. It may occur that they belong to a common Jordan domain or in two different U-domains. In any case we get a harmonic function in Ω^* defined by:

$$H(z) = \operatorname{Log}|z - a| - \operatorname{Log}|z - b|$$

Here H is harmonic at the point at infinity. For if $w = 1/z$ we have

$$H(w) = \operatorname{Log}|1 - aw| - \operatorname{Log}|1 - bw|$$

which is harmonic in a disc $|w| < \delta$ and zero if $w = 0$, i.e. $H(\infty) = 0$. Since the pair a, b above was arbitrary we conclude that the function

$$(**) \qquad a \mapsto \int \operatorname{Log} \frac{1}{|\zeta - a|} \cdot d\mu(\zeta)$$

is constant as a varies in $\cup U_\nu$. Here μ is supported by the union of the Jordan curves $\{\partial U_\nu\}$ Since every point on a single Jordan curve can be approximated from the inside by a-points in U_ν, it follows by continuity that the potential function (**) is constant on $\partial\Omega^*$. So when E is the union of the closed Jordan curves $\{\partial U_\nu\}$ we conclude that μ is a probability measure which settles Robin's problem, i.e. $U_\mu(z)$ is constant on the compact set $E = \cup\, \partial U_\nu$.

A.2 An inequality. Consider a pair of open sets Ω and Ω_1 which both consist of a finite union of Jordan domains as above and Ω is a relatively compact subset of Ω_1, i.e. the closure of each Jordan domain in Ω is a compact subset of Ω_1. Pick some point $a \in \Omega$. In the exterior domain Ω^* we find the harmonic function $H_a(z)$ whose boundary values on $\partial\Omega^*$ is equal to the continuous function $z \mapsto \log|z - a|$. Set

$$(1) \qquad g_a(z) = \log|z - a| - H_a(z)$$

This gives a super-harmonic function in Ω^* with a singularity at ∞ while $g_a = 0$ on the boundary. It follows from the minimum principle for super-harmonic functions that

$$(2) \qquad g_a(z) > 0$$

for all $z \in \Omega^*$. Next, replacing Ω by Ω_1 we find the harmonic function $H_a^1(z)$ which equals $\log|z - a|$ on $\partial\Omega_1^*$ and the super-harmonic function

$$(3) \qquad g_a^1(z) = \log|z - a| - H_a^1(z)$$

Now $\partial\Omega_1$ is contained in Ω^* so g_a restricts to a non-negative function on $\partial\Omega_1$. At ∞ the two g-functions have the same logarithmic singularity and hence $G_a = g_a - g_a^1$ is harmonic in Ω_1^*. Now $\partial\Omega_1^*$ is contained in Ω^* and (2) entails that $G_a > 0$ on $\partial\Omega_1^*$. We conclude that $G_a(\infty) \geq 0$. At the same time we notice that

$$G_a(\infty) = H_a^1(\infty) - H_a(\infty)$$

It follows that

$$H_a^1(\infty) \geq H_a(\infty) \tag{4}$$

The monotonic property expressed by (4) above will be used to study the case when E is a totally disconnected and compact subset of $\mathbf{C}$. Let Ω^* be the exterior domain of E, i.e. the union of ∞ and $\mathbf{C} \setminus E$. Since E is totally disconnected the boundary points $a \in \partial\Omega^*$ fail to satisfy the Perron condition for Dirichlet's problem. However, we shall define Robin's constant for E by a limit process and find a probability measure μ on E such that the potential

$$z \mapsto \int_E \operatorname{Log}|1 - \frac{\zeta}{z}| \cdot \mu(\zeta) \tag{*}$$

is constant as z varies in E. But in contrast to the case case when E is a union of Jordan arcs, it may occur that the "constant value" is $+\infty$. To attain this we use a construction which goes back to De Vallé-Poussin.

A.2 Nested coverings Let E be a totally disconnected and compact set. Without essential loss of generality we may assume that E is a compact subset of the open unit disc D. We construct a sequence of open sets $\{V_N\}$ as follows. For each positive integer N one has the dyadic grid $\mathcal{D}_N$ of open squares whose sides are 2^{-N}. We get the finite family $\mathcal{D}_N(E)$ of dyadic squares in $\mathcal{D}_N$ which have a non-empty intersection with E. The union of this finite family of open squares gives an open neighborhood V_N of E. If necessary one starts with a large N so that 2^{-N} is strictly larger than the distance of E to the unit circle T which entails that the closure of V_N is a compact subset of D. Let Ω_N^* be the exterior connected component of $D \setminus \bar{V}_N$ whose closure contains the unit circle. So here Ω_N^* contains an annulus $r < |z| < 1$ for some $0 < r < 1$ and $\partial\Omega_N$ is the union of T and a closed subset Γ_N of ∂V_N which is a finite union of line segments with a finite set of corner points. During this construction it is clear that the open sets $\{\Omega_N^*\}$ increase and since $D \setminus E$ is connected we have

$$\cup\, \Omega_N^* = D \setminus E$$

Next, fix some point $z_0 \in D \setminus E$ which is chosen relatively close to T. For all sufficiently large N we have $z_0 \in \Omega_N^*$ and get the unique probability measure $\mathfrak{m}_N$ which is supported by $\partial\Omega_N$ and satisfies:

$$g(z_0) = \int_T g(e^{i\theta}) \cdot d\mathfrak{m}_N(\theta) + \int_{\Gamma_N} g(\zeta) \cdot d\mathfrak{m}_N(\zeta)$$

for every g-function which is harmonic in Ω_N^* with continuous boundary values. Since $\{\Omega_N^*\}$ is a nested sequence it follows that the masses

$$\rho_N = \int_{\Gamma_N} d\mathfrak{m}_N(\zeta)$$

is a decreasing sequence of positive real numbers. Hence there exists the limit

$$\rho_* = \lim_{N\to\infty} \rho_N \tag{*}$$

Above two cases may occur. Either $\rho_* > 0$ or it is zero. The condition that $\rho_* = 0$ is intrinsic since Harnack's inequality shows that the choice of z_0 in $D \setminus E$ is irrelevant.

A.3 The case $\rho_* = 0$. If $a \in E$ is kept fixed then $g(z) = \mathrm{Log}\frac{|1-\bar{a}z|}{|z-a|}$ is a harmonic function in the domain Ω_N^* for every integer N which is identically zero on the unit circle. Hence

$$g(z_0) = \int_{\Gamma_N} g(\zeta) \cdot d\mathfrak{m}_N(\zeta) \tag{1}$$

Consider the probability measure $\mu_N = \frac{1}{\rho_N} \cdot \mathfrak{m}_N$ which gives:

$$\frac{1}{\rho_N} \cdot g(z_0) = \int_{\Gamma_N} \log \frac{1}{|\zeta - a|} \cdot d\mu_N(\zeta) + \int_{\Gamma_N} \log |1 - \bar{a}\zeta| \cdot d\mu_N(\zeta) \tag{2}$$

Next, from $\{\mu_N\}$ we can extract a subsequence which converges weakly to a Riesz measure μ_*. It is clear that μ_* is supported by E and the weak convergence gives:

$$\lim_{N\to\infty} \int_{\Gamma_N} \log |1 - \bar{a}\zeta| \cdot d\mu_N(\zeta) = \int \log |1 - \bar{a}\zeta| \cdot d\mu_*(\zeta) \tag{3}$$

Indeed, this holds since the integrand $\mathrm{Log}\,|1 - \bar{a}\zeta z|$ is continuous on E. Next, when $\rho_N \to 0$ is assumed it follows from (2) that

$$\lim_{N\to\infty} \int_{\Gamma_N} \log \frac{1}{|\zeta - a|} \cdot d\mu_N(\zeta) = +\infty \tag{4}$$

A.4 Exercise. Show that (4) together with the weak convergence of $\{\mu_N\}$ to μ_* imply that

$$\int_E \log \frac{1}{|\zeta - a|} \cdot d\mu_*(\zeta) = +\infty$$

This holds for an arbitrary a in E and hence we have proved:

A.5 Theorem. *Let E be a totally disconnected and compact set where $\rho_* = 0$. Then there exists at least one probability measure μ supported by E such that*

$$\int_E \log \frac{1}{|a - \zeta|} \cdot d\mu(\zeta) = +\infty \quad \text{for all} \quad a \in E$$

B. Removable singularities

.

Let E be a totally disconnected and compact subset of some open domain Ω. We say that E is a removable singularity if every bounded harmonic function in $\Omega \setminus E$ extends to be harmonic in the whole domain Ω. Now we shall use Theorem A.5 to prove Myrberg's result.

Proof. XXXX TO BE GIVEN