III. A. Hardy-Littlewood's maximal function

Contents

Introduction. The results below are foremost due to Hardy, Littlewood and Fatou. Before we describe results about harmonic and analytic functions we expose some general facts about maximal functions from measure theory.

1. The weak type estimate

Let $f(x)$ be a non-negative function and integrable function on the real x-line with support in a finite interval $[0, A]$. The forward maximal function is defined for ever $x \geq 0$ by

$$f^*(x) = \max_{h>0} \frac{1}{h} \int_x^{x+h} f(t) \cdot dt$$

It is clear that f^* is non-negative and supported by $[0, A]$. To each $\lambda > 0$ we get the set $\{f^* > \lambda\}$.

1.1 Theorem *For each $\lambda > 0$ one has the inequality*

$$\mathbf{m}(\{f^* > \lambda\}) \leq \frac{1}{\lambda} \cdot \int_{\{f^*>\lambda\}} f(x) \cdot dx$$

Proof. Introduce the primitive function

$$F(x) = \int_0^x f(t) \cdot dt$$

With $\lambda > 0$ we have the continuous function $F(x) - \lambda$ and define the forward Riesz set by:

$$\mathcal{E}_\lambda = \{0 \leq x < A \colon \exists\, y > x \quad \text{and} \quad F(y) - \lambda y > F(x) - \lambda x\}$$

1.2 Exercise. Show the equality

$$\mathcal{E}_\lambda = \{f^* > \lambda\}$$

Now $\mathcal{E}_\lambda$ is an open set and hence a disjoint union of intervals $\{(a_k, b_k)\}$. With these notations one has

1.3 Exercise. Show the following for each interval (a_k, b_k):

$$F(b_k) - \lambda \cdot b_k = \max_{a_k \leq x \leq b_k} F(x) - \lambda$$

In particular one has

$$\lambda(b_k - a_k) \leq F(b_k) - F(a_k)$$

This holds for each k and after a summation over the forward Riesz intervals the requested inequality in Theorem 1.1 follows.

Using Theorem 1 can prove the following L^2-inequality.

1.4 Theorem. *One has*

$$||f^*||_2 \leq |f||_2$$

Proof. By the general formulas for distribution functions from XX we have:

$$\int_0^A f^*(x)^2 \cdot dx = \int_0^\infty \lambda \cdot \mathbf{m}(\{f^* > \lambda\}) \cdot d\lambda$$

By Theorem 1.1 the last integral is majorised by

$$\int_0^\infty [\int_{\mathbf{m}(\{f^*>\lambda\}} f(x) \cdot dx\,] \cdot d\lambda) = \iint_{\{f^*(x)>\lambda\}} f(x) \cdot dx d\lambda =$$
$$\int_0^A [\int_0^{f^*(x)} d\lambda] \cdot f(x) \cdot dx = \int_0^A f^*(x) \cdot f(x) \cdot dx$$

By the Cauchy-Schwarts in equality the last integral is majorised by the product of L^2-norms

$$||f^*||_2 \cdot |f||_2$$

Hence

$$||f^*||_2^2 = \int_0^A f^*(x)^2 \cdot dx \leq ||f^*||_2 \cdot |f||_2$$

and Theorem 1.4 follows after division with $||f^*||_2$.

1.5 Remark. In a similar way we get an L^2-inequality using the backward maximal function

$$f_*(x) = \max_{h>0} \frac{1}{h} \int_{x-h}^x f(t) \cdot dt$$

and also the full maximal function

$$(**) \qquad f^{**}(x) = \max_{a,b} \frac{1}{a+b} \int_{x-a}^{x+b} |f(t)| \cdot dt$$

with the maximum taken over pairs $a, b > 0$. Then we get the L^2-inequality

$$(1.6) \qquad ||f^{**}||_2 \leq 2 \cdot |f||_2$$

2. A study of harmonic functions.

Let $f(t)$ be complex-valued function on the real t-line such that

$$\int_{-\infty}^\infty \frac{|f(t)|}{1+t^2} \cdot dt < \infty$$

We also assume that

$$\max \frac{1}{b+a} \cdot \int_{-a}^b |f(t)| \cdot dt < \infty$$

where the maximum is taken over all pairs $a, b > 0$. Define the function $V(z) = V(x+iy)$ in the upper half-plane $y > 0$ by

$$V(z) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{y}{(x-t)^2 + y^2} \cdot f(t) \cdot dt$$

2.1 Exercise. Prove the inequality

$$(1) \qquad |V(x+iy)| \leq \big(\frac{|x|}{y} + 2\big) \cdot f^{**}(0)$$

where

$$f^{**}(0) = \max_{a,b>0} \frac{1}{b-a} | \cdot \int_{-a}^b |f(t) \cdot dt$$

Next, define Fatou's maximal function on the real x-line by

$$(3) \qquad V^*(x) = \max_{y \leq |s|} |V(x+iy)|$$

We also introduce the function $f^{**}(x)$ defined for each real x by:

$$f^{**}(x) = \max \frac{1}{b-a}| \cdot \int_{x-a}^{x+b} |f(t) \cdot dt$$

2.2 Exercise. Show the inequality

$$V^*(x) \leq 3 \cdot f^*(x)$$

Next, apply the inequality (1.6) which together with Exercise 2.2 give

$$\text{(i)} \qquad \int_{-\infty}^{\infty} V^*(x)^2 \cdot dx \leq 18 \cdot \int_{-\infty}^{\infty} f^2(x) \cdot dx$$

Since $V(x) = f(x)$ holds on the real line we conclude the following:

2.3 Theorem. *One has the inequality*

$$(*) \qquad ||V^*||_2 \leq 3\sqrt{2} \cdot \sqrt{\int_{-\infty}^{\infty} V(x)^2 \cdot dx}$$

3. Application to analytic functions.

Let $f(z)$ be analytic in $\mathfrak{Im}(z) > 0$ and assume that there is a constant C such that

$$\int_{-\infty}^{\infty} \frac{|f(x+iy)|}{1+x^2} \cdot dx \leq C \quad \text{for all} \quad y > 0$$

It means that f belongs to the Hardy space H^1 in the upper half-plane U_+. We can divide out the zeros via a Blaschke product and write

$$f = B_f \cdot g$$

where g again belongs to H^1 and has no zeros in U_+. Then $\sqrt{g}$ is defined which gives a complex-valued harmonic function

$$V(z) = \sqrt{g(z)}$$

3.1 Exercise. Apply Theorem 2.3 to the V-function and use that $|f(z)| \leq |g(z)| \leq |V^2(z)|$ to show that

$$\text{(1)} \qquad \int_{-\infty}^{\infty} |f^*(x)| \cdot dx \leq 3\sqrt{2} \cdot \int_{-\infty}^{\infty} |f(x)| \cdot dx$$

where $f^*(x)$ is Fatou's maximal function for f defined for each real x by

$$f^*(x) = \max_{y \leq |s|} |f(x+iy)|/tag2$$

3.2 Exercise. Use the conformal map from U_+ to the unit disc D defined by

$$w = \frac{z-i}{z+i}$$

Explain how the previous result is translated when we start from an analytic function f in D for which the boundary value function $f(e^{i\theta})$ is in $L^1(T)$.

4. Hardy spaces and conformal maps

Let $g(z) = \sum a_n z^n$ be analytic in D and assume that there exists a constant C such that

$$\int_0^{2\pi} |g(re^{i\theta})| \cdot d\theta \leq C$$

for every $r < 1$. Thus, by the Brothers Riesz Theorem g belongs to the Hardy space $H^1(T)$. In D there exists a single-valued brach of $\log(1-z)$ whose imaginary part stays in $(-\pi/2, \pi/2)$ and with $z = re^{i\theta}$ we have

$$\mathfrak{Im}\ \log(1-z) = -\frac{1}{2i} \cdot \sum_{n=1}^{\infty} \frac{r^n}{n}(e^{in\theta} - e^{-in\theta})$$

4.1 Exercise. Deduce from the above that

$$(*) \qquad \int_0^{2\pi} \mathfrak{Im}\ \log(1-re^{i\theta}) \cdot g(re^{i\theta}) \cdot d\theta = -\pi i \cdot \sum_{n=1}^{\infty} \frac{a_n}{n} \cdot r^{2n}$$

The case when $\{b_n\}$ are real and ≥ 0. If this holds then (*) and the triangle inequality yield:

$$\pi \cdot \sum_{n=1}^{\infty} \frac{a_n}{n} \cdot r^{2n} \leq \frac{\pi}{2} \cdot \int_0^{2\pi} |g(re^{i\theta})| \cdot d\theta$$

So if we introduce the $H^1(T)$-norm

$$||g||_1 = \int_0^{2\pi} |g(e^{i\theta})| \cdot d\theta$$

it follows after a passage to the limit when $r \to 1$ that

$$(**) \qquad \sum_{n=1}^{\infty} \frac{b_n}{n} \leq \pi \cdot |g||_1$$

4.2 A study of conformal mappings. Let $\phi\colon D \to \Omega$ be a conformal mappng and assume that the complex derivative $\phi'(z)$ belongs to the Hardy space H^1 as above. Since $\phi' \neq 0$ in D there exists a single-valued analytic square-root:

$$\psi(z) = \sqrt{\phi'(z)}$$

Then $\psi \in H^2(T)$ so if

$$\psi(z) = \sum b_n z^n \implies \sum |b_n|^2 < \infty$$

Let us then consider the H^2-function

$$\Psi(z) = \sum |b_n| z^n$$

We get

$$\Psi^2(z) = \sum A_n z^n \quad \text{where} \quad A_n = \sum_{k=0}^{k=n} |b_k| \cdot |b_{n-k}|$$

and (**) in Exercise 4.1 gives:

$$(1) \qquad \sum_{n=1}^{\infty} \frac{A_n}{n} \leq \pi \cdot \int_0^{2\pi} |\Psi(e^{i\theta})|^2 \cdot d\theta$$

Next, consider the Taylor series

$$\phi'(z) = \sum a_n z^n \implies a_n = \sum_{k=0}^{k=n} b_k \cdot b_{n-k}$$

The triangle inequality gives $|a_n| \le A_n$ for each n so (1) entails that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty \tag{2}$$

Finally, consider the Taylor expansion of $\phi(z)$:

$$\phi(z) = \sum c_n z^n$$

Here

$$nc_n = a_{n-1} \quad : \quad n \ge 1$$

Then it is clear that (2) implies that the series $\sum |c_n| < \infty$. Hence we have proved the following result which is due to Hardy:

4.3 Theorem. *Let $\phi(z)$ be a conformal map such that ϕ' belongs to H^1. Then the Taylor series of ϕ is absolutely convergent.*

4.4 Exercise. Let Ω be a Jordan domain whose boundary curve $\Gamma = \partial\Omega$ has a finite arc-length. Let $\phi\colon D \to \Omega$ be the con formal mapping which by results from (xx) extends to a homeomorphism from the closed disc $\bar{D}$ onto $\bar{\Omega}$.' Let $\ell(\Gamma)$ be the arc-length of Γ. Show that the derivative $\phi'(z)$ belongs to the Hardy space and

$$\int_0^{2\pi} |\phi'(e^{i\theta})| \cdot d\theta \le \ell(\Gamma)$$

From this it follows that the Taylor series of $\phi(z)$ is absolutely convergent.

A hint for the exercise. To each $n \ge 1$ we set $\epsilon = e^{2\pi i/n}$, i.e. the n:th root of the unity. Now ϕ yields a homeomorphism from T onto Γ. The definition of $\ell(\Gamma)$ gives the inequality below where we set $\epsilon^0 = 1$.

$$\sum_{k=1}^{n} |\phi(\epsilon^k \cdot e^{i\theta}) - \phi(\epsilon^{k-1} \cdot e^{i\theta})| \le \ell(\Gamma) \quad \text{for every} \quad 0 \le \theta \le 2\pi \tag{1}$$

Keeping n fixed we notice that the function

$$s_n(z) = \sum_{k=1}^{n} |\phi(\epsilon^k \cdot z) - \phi(\epsilon^{k-1} \cdot z)|$$

is subharmonic in D. So the maximum principle for subharmonic functions and (1) give

$$\max_{\theta} s_n(re^{i\theta}) \le \ell(\Gamma) \tag{2}$$

for each $r < 1$. Next, with $r < 1$ fixed the reader may verify the limit formula:

$$\lim_{n\to\infty} s_n(r) = \int_0^{2\pi} |\phi'(re^{i\theta})| \cdot d\theta \tag{3}$$

Hence (2-3) give

$$\int_0^{2\pi} |\phi'(re^{i\theta})| \cdot d\theta \le \ell(\Gamma)$$

Now the Brothers Riesz theorem implies that $\phi'(z)$ belongs to $H^1(T)$, i.e. the boundary value function $\phi'(e^{\theta})$ exists and belongs to $L^1(T)$.