Mathematics by Torsten Carleman (1892-1949)

Introduction. Carleman's collected work covers fifty articles of high standard together with several monographs which are cited in the list of references. He became professor at Stockholm University in 1924 when he replaced the chair previously held by Helge von Koch, and in 1927 also director at Institute Mittag-Leffler. He held both positions until his decease 1949. Carleman's last major publication was *Sur un probléme d'unicité pour les systèmes d'équations aux dérivees partielles à deux variables indépendantes* [Arkiv för matematik 1938] concerned with a uniqueness theorem for elliptic PDE-systems where the coefficients of the PDE-operators are non-analytic. His result extended an earlier theorem by Erik Holmgren restricted to the case when the coefficients of the elliptic PDE-system are real-analytic. We shall expose Carleman's article in § xx and remark that the methods he used have become an essential ingredient in more recent literature. The use of "Carleman's weight inequalities" to establish uniqueness theorems have for example been adapted in a more general context by Lars Hörmander. Erik Holmgren served as supervisor while Carleman prepared his thesis entitled *Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, presented at Uppsala University in 1916. Here double layer potentials are studied in dimension 2 and 3, and Carleman's thesis extended earlier work by Gustav Neumann and Henri Poincaré to the case when the boundary no longer is smooth. Many subtle inequalities in this thesis merit a study up to the present date.

The spectral theorem for unbounded self-adjoint operators. The thesis from 1916 led Carleman to analyze the spectrum of unbounded linear operators which in a general set-op was a new feature at that time in operator theory. His monograph *Sur les équations singulières à noyaux réel et symmetrique* [Uppsala University 1923] contains the construction and properties of spectral measure attached to a - in general unbounded - self-adjoint operator on a Hilbert space. Carleman's main concern for developing this new theory was to analyze the moment problem by Stieltjes. Further applications than those treated in [ibid] were proposed in a lecture by Niels Bohr at the 6:th Scandianvian congress in mathematics held at Copenhagen 1925 devoted to the "new-born" quantum mechanics. Recall that one of the fundamental points is the hypothesis on energy levels which correspond to orbits in Bohr's theory of atoms. For an account about the physical background the reader may consult Bohr's plenary talk when he received the Nobel Prize in physics 1922. Mathematically the Bohr-Schrödinger equation is given by:

$$(*)\qquad \Delta\phi + 2m\cdot\big(E-U\big)\big(\frac{2\pi}{h}\big)^2\cdot\phi = 0$$

where Δ is the Laplace operator in the 3-dimensional (x,y,z)-space, m the mass of a particle and h Planck's constant while $U(x,y,z)$ is a potential function. Finally E is a parameter and one seeks values on E such that (*) has a solution ϕ which belongs to $L^2(\mathbf{R}^3)$. Carleman gave a response to Boihr in his own plenary talk at Cpenhagen which applies to the case when the function c below is a finite sum of Newtonian potentials.

Let us cite an excerpt from Carleman's lectures in Paris at Institut Henri Poincaré held in 1930 where he treated (*) and its associated equation (**) below when a time variable enters.

Dans ces dernières années l'ínteret de la question qui nous occupe a considérablement augmenté. C'est en effet un instrument mathématique indispensable pour development de la mechanique moderne crée par M.M. de Brogile, Heisenberg et Schrödinger. Etude de l'équation integrale:

$$\phi(x) = \lambda\cdot\int_a^b K(x,y)\phi(y)dy + f(x) \quad : \lambda\in\mathbf{C}\setminus\mathbf{R}$$

So a basic equation which emerges from quantum mechanics is to find solutions $u(p,t)$ defined in $\mathbf{R}^3\times\mathbf{R}^+$ where t is a time variable and $p=(x,y,z)$ which satisfies the PDE-equation

$$(**)\qquad i\cdot\frac{\partial u}{\partial t} = \Delta(u)(p,t) - c(p)\cdot u(p,t) = 0 \quad t>0$$

and the initial condition

$$u(p,0) = f(p)$$

Here $f(p)$ belongs to $L^2(\mathbf{R}^3)$ and $c(p)$ is a real-valued and locally square integrable function. Carleman proved that the symmetric and densely defined operator $\Delta + c$ has a self-adjoint extension in $L^2(\mathbf{R}^3)$ if

$$\limsup_{p\to\infty} c(p) \leq M \tag{***}$$

When (***) holds the spectrum of the densely defined sefl-adjoint operator $\Delta - c$ on the Hilbert space $L^2(\mathbf{R}^3)$ is confined to an interval $[\lambda_1, +\infty)$ on the positive real line, i.e. $\lambda_1 > 0$. Applied to the equation (**) Carleman proved that the solution u is given by an equation

$$u(p,t) = \int_{\mathbf{R}^3} [\int_{\lambda_1}^{\infty} e^{i\lambda t} \cdot d\theta(p,q,\lambda)\,] \cdot f(q)\,dq$$

where $\lambda \mapsto \theta(p,q,\lambda)$ is the non-decreasing spectral function associated to $\Delta + c$. Moreover, he found an asymptotic expansion which recaptures the θ-function.

Asymptotic behavior of solutions to the exterior wave-equation. Let us also give an example from [ibid] where Carleman applied his general theory to a wave equation.

Let Ω be a bounded domain in $\mathbf{R}^3$ with a C^1-boundary $\partial\Omega$. With $x = (x_1, y_2, x_3)$ we seek functions $u(x,t)$ defined in $\mathbf{R}^3 \setminus \Omega \times \{t \geq 0\}$ where t is a time variable satisfying the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta(u)$$

when $t > 0$ and $x \in \mathbf{R}^3 \setminus \overline{\Omega}$. The initial conditions when $t = 0$ is that

$$u(x,0) = f_1(x) \quad : \frac{\partial u}{\partial t}(x,0) = f_2(x) \tag{i}$$

where f_1, f_2 are C^2-functions in $\mathbf{R}^3 \setminus \Omega$ and $\Delta(f_1)$ and $\Delta(f_2)$ are square integrable, i.e.

$$\iiint_{\mathbf{R}^3\setminus\Omega} |\Delta(f_\nu|^2\,dx < \infty$$

Finally the normal derivatives along $\partial\Omega$ satisfy

$$\frac{\partial f_\nu}{\partial n} = 0 \quad : \nu = 1,2$$

Given such a pair f_1, f_2 there exists a unique solution $u(x,t)$ which satisfies the wave equation above and the two initial conditions (i) together with the boundary value equation

$$\frac{\partial u}{\partial n}(x,t) = 0$$

for every $x \in \partial\Omega$ and each $t \geq 0$. Carleman's cited monograph gives an expression of the solution by an integral formula using the spectral measure of a densely defined and self-adjoint operator on the Hilbert space $L^2(\mathbf{R}^3 \setminus \Omega)$. A crucial fact is that the spectral measure is absolutely continuous. Carleman's proof of this property of the spectral measure consolidated the physcially evident fact that

$$\lim_{x\to\infty} \nabla(u)(x) = 0$$

It goes without saying that this way to obtain asymptotic behaviour has become fashion in hundreds of text-book and many thousands articles devoted to PDE:s like the wave equation.

Some biographical data. After the end of World War I, Carleman visited several universities in Europe, such as Zürich, Göttingen, Oxford and Paris. As pointed out in Fritz Carlson's memorial article [Acta Mathematica 1950], Paris was always Carleman's favourite place where he met and received inspiration from mathematicians such as Borel, Denjoy, Hadamard and Picard. On several occasions he delivered lecture series at Sorbonne. For example, in the spring 1930 about singular integral operators and spectral theory for unbounded self-adjoint operators on Hilbert spaces, and in 1937 his lectures were devoted to Boltzmann's kinetic gaz theory. Carleman was always aware of the interplay between pure mathematics and experimental sciences. While still an unemployed doctor in mathematics he attended an engineering school in Paris during two semesters 1920-21. An outcome of these studies is the article entitled *Sur les équations différentielles de la mecanique d'avion* published in [La Technique Aéronautique, vol. 10 1921), inspired by Lanchester's pioneering work *Le vol aérien* which played a significant role while airplanes were designed at this early stage. His article ends with the following conclusion after an investigation of integral curves to a certain non-linear differential system: *Quelle que soit la vitesse initiale, l'avion, après avoir éxecuté s'il y lieu, un nombre fini des loopings, prend un mouvement qui s'approche indéfinement du régime de descente rectiligne et uniforme.* Carleman's lecture held 1944 at the Academy of Science in Sweden entitled *Sur l'action réciproque entre les mathématiques et les sciences expérimentales exactes* underlines his concern for applications of mathematics.

During his last years in life Carleman suffered from health problems which caused his decease on January 11 1949 at the age of 56 years. A memorial article appears in [Acta. Math. 1950] by Fritz Carlson who was Carleman's collegue at the department of mathematics at Stockholm university for several decades. See also his Collected Work published by Institute Mittag-Leffler in 1960 and the posthumous work *Problémes mathématiques dans la théorie cinétique des gaz* which was printed in 1957 by Institute Mittag Leffler and contains previously unpublished material about the Boltzmann equation.

Author's personal comment.

Personally I find few mathematical texts which superseed the approach to state and solve fundamental problems which the reader meets in Carleman's work. Several of his articles merit a study up to the present date. The reader will recognize that focus in Carleman's work is upon inequalities ,which I personally consider as the core of mathematics. To this I would like to add a remark to less experienced readers. While entering higher studies in mathematics my opinion is that it is more valuable to pursue details of individual proofs rather than digesting general concepts. As a Phd-student during the years 1964-68 I was often excited by elegant proofs and remarkable conclusions from Carlemans work. Here are two examples:

In [Ca:xx] one finds an extension of Weierstrass' approximation theorem which states that every complex-valued and continuous function on the real line can be uniformly approximated on the whole real line by entire functions. Even today I meet professional mathematicians who are not aware of this result which ought to put forward in text-books and learnt by every phd-student. My first advisor in mathematics was Otto Frostman who replaced Carleman's chair at Stockholm University. Inspired by Frostman's lectures in potential theory I proved as a graduate student the following extension of the Carleman-Weierstrass approximation:

Theorem. *Let K be a closed subset of $\mathbf{C}$ whose planar Lebesgue measure is zero, and assume there exists a strictly increasing sequence $0 < R_1 < R_2 < \ldots$ where $R_n \to +\infty$ such that the open complements of the sets*

$$D(R_k) \cup (D(R_{k+1}) \cap K)$$

are connected for every $k \geq 2$,where $D(R_k)$ denote the closed discs of radius R_k, and $k = 1$ means that $\mathbf{C} \setminus (D(R_1) \cap K)$ is connected. Then every $f \in C^0(K)$ can be uniformly approximated by entire functions.

Around 1965 both Frostman and myself considered a result of this kind to be standard and it was therefore never published, but just included as a summary from seminars in analysis held at Stockholm University. Today, almost half an century later I am astonished that an approximation theorem as above still remains absent in text-books. So the reader is invited to prove the theorem above as an exercise.

A result about Fourier's partial sums. Let $f(x)$ be a 2π-periodic and continuous function on $[0, 2\pi]$. For each $n \geq 0$ we get the partial sum function $S_n(x)$. It is wellknown that the sequence

$\{S_n(x)\}$ in general fails to convergence point-wise to f on the whole interval $[0, 2\pi]$. However, a limit formulas hold for certain average sums. Namely, to each $N \geq 1$ we put

$$E_N = \frac{1}{N} \cdot \sum_{n=0}^{n=N} ||S_n - f||_\infty$$

where we have taken maximum norms of the differences $S_n - f$ over $[0, 2\pi]$. By a careful study of the Dini kernel, Carleman proved that $E_N \to 0$ as $N \to +\infty$ hold for every f as above. Moreover, the rate of this zero limit is controlled via the modulus of continuity of f. See § xx for details. Let us remark that this limit theorem for averaged sums is far more ready to prove than Lennart Carleson's deep theorem about the almost everywhere convergence of Fourier series. More precisely in 1965 Carleson proved that there exists a constant S such that for every f as above one has the inequality

$$||S^*||_2 \leq C \cdot ||f||_2$$

where we have taken L^2-norms and S^* denoted the maximal function defined by

$$S^*(x) = \max_n |S_n(x)|$$

Quasi-analytic functions.

Around 1920 the French mathematicians Emile Borel and xx Denjoy raised several problems about Taylor expansions of C^∞-functions on the real line. One issue was when the Taylor series at a point determines a C^∞- function on a whole interval centered at this point. Carlean obtained conclusive results which are exposed in his monograph XXX and based upon his lectures at Sorbonne from this time. The pioneering device by Carleman was to employ the notion of *harmonic measure* and then use the principle of subharmonic majorisation. The central result fromCarlean's work about quasi.-abnalyticty is the following remarkable inequality:

Theorem. *There exists a constant $\mathcal{C}$ such that the following hold for each positive integer n and every n times continuously differentiable function f whose derivatives up to order n vanish at $x = 0$ and $x = 1$, while $\int_0^1 f(x)^2\,dx = 1$:*

$$\sum_{\nu=1}^{\nu=n} \frac{1}{(||f^{(\nu)}||_2)^{1/\nu}} \leq \mathcal{C}$$

where $||f^{(\nu)}||_2$ denote the L^2-norms of the derivatives of order ν.

The remarkable fact is of course that the constant $\mathcal{C}$ is independent of n. Upper bounds for $\mathcal{C}$ will be described in § xx, while a more precise bound for Carleman's constant remains open.

Neumann's resolvent operators. In these notes we shall frequently encounter densely defined unbounded linear operators. The pioneering article by Gustav Neuann in [Neu:xx] has paved the way towards contemporary operator theory where one constructs bounded inverse operators. An example is the densely defined Laplacian. Consider for example a bounded Dirichlet domain Ω in $\mathbf{R}^2$, i.e. a bounded open set where every $f \in C^0(\partial\Omega)$ extends to a harmonic function in Ω. When this holds one gets the Green's function

$$(*) \qquad G(p, q) = \log \frac{1}{|p - q|} + H(p, q)$$

defined in the product of $\overline{\Omega}$. Here $G(p, q) = G(q, p)$ and

$$G(p, q) = 0 \quad : \, p \in \Omega \,\&\, q \in \partial\Omega$$

Finally, $H(p, q)$ is continuous in $\overline{\Omega} \times \overline{\Omega}$ and $q \mapsto H(p, q)$ is harmonic for each fixed p. The symmetric G-kernel defines a linear operator on the Hilbert space $L^2(\Omega)$ by

$$\mathcal{G}(\phi)(p) = \int_\Omega G(p, q) \cdot \phi(q)\, dm(q)$$

As ϕ varies in $L^2(\Omega$ the range f $\mathcal{G}$ consists of functions which are continous in $\overline{\Omega}$ and zero on the boundary. Moreover, in the open domain Ω their Laplacians are square integrable. One therefore takes the range of $\mathcal{G}$ as the domain of definition for Δ and (*) gives the equation

$$\Delta \circ \mathcal{G}(f\phi) = -4\pi \cdot \phi$$

Following Neumann and Poinicarë this can be expressed by saying that the densely defined Δ-operator on $L^2(\Omega)$ is invertible, where

$$\Delta^{-1} = .\frac{1}{4\pi} \cdot \mathcal{G}$$

Let us remark that prior to Carleman's cited monograph [xx], David Hilbert proved the spectral theorem for bounded self-adjoint operators and soon after his general theory in the gret txt-nook *Integralgleichungen* from 1904, the spectrum of Δ in a Dirichlet domain was analyzed by several authors, such as Hermann Weyl and Richard Courant. In § xx we expose a result which Carleman presented at the Scandinavian Congress held at Stockholm in 1934, where he proved the following. Let Ω be a Dirichlet domain in $\mathbf{R}^2$ which gives a non-decraseing sequece if eigenvalues $\{0 < \lambda - 1 \leq \lambda_2 \leq \ldots$ nd eigenfunctions $\{\phi_n\}$ which satisfy

$$\Delta(\phi_n) + \lambda_n \cdot \phi_n = 0$$

So as explained above each ϕ_n extends to be zero on the boundary. With these notations Carleman proved that

$$\lim_{N \to \infty} \frac{1}{\lambda_N} \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi}$$

hold for each point $p \in \Omega$. The proof is given in § xx and is instructive since it teaches the power of the Neumann-Poincaré construction of resolvents, and at the same time Carleman used analytic function theory via certain contour integrals inspired by Bernhard Riemann's pioneering work

Final Comments.

Above we discussed some of the central results from Carleman's work. Various special sections contain more individual results from analytic function theory, as well as PDE:s. Today's student might find some of these sections a bit tedious, since they contain rather technical constructions. For example, one can learn about the mere existence of fundamental solutions to a second order elliptic PDE vía more abstract functional analysis. But in order to attain precise results, for example when elliptic boundary value problems in specific domain appear, it is often essential to exhibit good estimates for the fundamental solutions which after leads to estimates for various Greens' functions. So personaly I find Carleman's constructions in § xx are instructive. Here one gets the fundamental solution to a - in general non-symmetric - second order PDE in a straightforward manner which starts from Isaac Newton's construction in the case of constant coefficients, and after one performs some manipulations to handle the case of variable coefficients where Gusav Neumann's series expansions to solve integral equations lead an "explicit formula" for the requested fundamental solution.