Series

Contents

Introduction.

The study of infinite series is fundamental. The concepts are initially easy to grap but many results require quite involved proofs. In § 1-3 we study general facts about additive series, power series, and finally product series. Then we enter more advanced material. An example is the Hardy-Littlewood theorem in § 6, and § 8,9 and 12 expose resiults due to Carelman whose proofs also are quite technical. The discussion in § 13 goes beyond the present scope where we only describe som facts from Carleman's deep studies about quasi-analytic functions. We shall work with complex series. The reason is that this is useful when one begins to study analytic functions of one complex variable. *Blaschke products* play a central role in analytic function theory and are studied in § 4. Another example where analytic function theory is used occurs in the proof of Thorin's convexity theorem in § xx.

The literature about series is extensive and treated in many text-books where the reader may find examples and applications. Apart from the results due to Carleman which were established after 1920, the material to be presented is inspired by classic texts-books by Landau and de Vallé-Poussin where the Hardy-Littlewood Theorem from 1913 is of the most recent origin.

A remark. Some students prefer to enter abstract, or general theories. here we are confronted with material whose underlying concpets are almost trivial. But I would like to point out that the results are fundamental and used in many areas of mathematics. Personally I think that the best way to advance studies in mathematics is to learn details of proofs which do not require sophistaceted definitions. It does not means that the proofs are simple. To digest details in a section like § 8 is for example quite demanding.

I. Additive series

1. Partial sums and convergent series. To each sequence $\{a_\nu\}$ of complex numbers indexed by non-negative integers one associates the partial sums:

$$S_N = \sum_{\nu=0}^{\nu=N} a_\nu \quad : \quad N = 1, 2, \ldots$$

If $\{S_N\}$ converges converge to a limit S_* one says that $\{a_\nu\}$ yields a convergent series and writes

$$S_* = \sum_{\nu=0}^{\infty} a_\nu \tag{1}$$

2. Absolute convergence. The series is absolutely convergent if

$$\sum_{\nu=0}^{\infty} |a_\nu| < \infty$$

3. A majorant principle. Let $\{b_\nu\}$ be a bounded sequence and $\{a_\nu\}$ a sequence such that $\sum |a_\nu| < \infty$. Then it is obvious that

$$\sum |a_\nu \cdot b_\nu| < \infty$$

Warning. Absolute convergence implies that $\sum a_\nu$ converges. For if $\{S_N\}$ are the partial sums the triangle inequality gives

$$|S_M - S_N| = |\sum_{\nu=N+1}^{M} a_\nu| \leq \sum_{\nu=N+1}^{M} |a_\nu| \quad : \quad M > N \geq 0$$

The absolute convergence therefore implies that the sequence of partial sums is a *Cauchy sequence* of complex numbers and hence has a limit. The converse is false. The series defined by the sequence $\{a_\nu = \frac{(-1)^\nu}{\nu} : \ \nu \geq 1\}$ is not absolutely convergent since

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu} = \infty$$

4. Alternating series. Let $a_0, a_1, \ldots$ be a sequence of positive real numbers which is strictly decreasing, i.e. $a_0 > a_1 > \ldots$. Assume also that $\lim a_n = 0$. Then one gets a convergent series by taking alternating signs, i.e. the series below converges:

$$\sum_{\nu=0}^{\infty} (-1)^\nu \cdot a_\nu \tag{*}$$

Proof. Even partial sums are expressed by a positive series:

$$S_{2N} = (a_0 - a_1) + \ldots + (a_{2N} - a_{2N-1}) > 0 \quad : N \geq 1$$

At the same time

$$S_{2N} = a_0 - [(a_1 - a_2) + \ldots + a_{2N-1} - a_{2N}]$$

where the last term is the negative of a positive series. Hence $\{S_{2N}\}$ is a non-increasing sequence of positive real numbers which implies that there exists a limit:

$$\lim_{N \to \infty} S_{2N} = S_*$$

Finally, since $a_n \to 0$ we get the same limit using odd indices and hence (*) is convergent.

5. The partial sum formula. Consider two sequences $\{a_\nu\}$ and $\{b_\nu\}$. Set

$$S_N = \sum_{\nu=0}^{\nu=N} a_\nu \quad : \quad T_N = \sum_{\nu=0}^{\nu=N} a_\nu \cdot b_\nu$$

Since $a_\nu = S_\nu - S_{\nu-1}$ it follows that

$$(*) \qquad T_N = \sum_{\nu=0}^{\nu=N} (S_\nu - S_{\nu-1}) \cdot b_\nu = a_0 b_0 + \sum_{\nu=1}^{\nu=N-1} S_\nu \cdot (b_\nu - b_{\nu+1}) + S_N \cdot b_N$$

This formula which resembles partial integration of functions is quite useful.

6. Exercise Let $b_1 \geq b_2 \geq \ldots$ be a non-increasing sequence of positive real numbers. Show that for every finite n-tuple $a_1, \ldots, a_n$ of complex numbers one has the inequality

$$\big| b_1 a_1 + \ldots + b_n a_n \big| \leq b_1 \cdot M \quad \text{where} \quad M = \max_{1 \leq k \leq n} |a_1 + \ldots + a_k|$$

7. Theorem by Abel. *Assume that the partial sums $\{S_N\}$ of $\{a_\nu\}$ is a bounded sequence and that the positive series $\sum |b_\nu - b_{\nu+1}| < \infty$ where $b_\nu \to 0$ as $\nu \to \infty$. Then the series below is convergent*

$$\sum_{\nu=0}^{\infty} a_\nu \cdot b_\nu$$

Proof. By (3) the series $\sum_{\nu=0}^{\infty} S_\nu \cdot (b_\nu - b_{\nu+1})$ is absolutely convergent and by the hypothesis we also have $S_N \cdot b_N \to 0$ as $N \to \infty$. Hence (*) in (5) shows that $\{T_N\}$ has a limit T_* expressed by the absolutely convergent series

$$T_* = a_0 b_0 + \sum_{\nu=1}^{\infty} S_\nu \cdot (b_\nu - b_{\nu+1})$$

Next, let $\{b_\nu(p)\}$ be a doubly indexed sequence of non-negative numbers which satisfies

$$\nu \mapsto b_\nu(p) \quad \text{is non-increacing for each} \quad p = 0, 1, \ldots$$

$$\lim_{p \to \infty} b_\nu(p) = 1 \quad \text{for each} \quad \nu = 0, 1, \ldots$$

8. Theorem. *For each convergent series $\sum a_\nu$ it follows that*

$$\sum_{\nu=0}^{\infty} a_\nu \cdot b_\nu(p)$$

converges and one has the limit formula

$$\lim_{p \to \infty} \sum_{\nu=0}^{\infty} a_\nu \cdot b_\nu(p) = \sum_{\nu=0}^{\infty} a_\nu$$

9. Exercise. Prove Theorem 8.. The hint is to employ the partial sum formula.

We finish with another useful result.

10. Theorem *Let $\{a_\nu\}$ be a non-decreasing sequence of real numbers such that*

$$(*) \qquad \sum_{k=1}^{\infty} 2^{-k}(a_{k+1} - a_k) < \infty$$

Then

$$(**) \qquad \lim_{k \to \infty} 2^{-k} a_k \to 0$$

Proof. Let S_N denote partial sums of (*). If $M > N$ the partial summation formula gives:

$$\text{(i)} \qquad S_M - S_N = 2^{-M} a_{M+1} - 2^{-N} a_N + \sum_{\nu=N+1}^{M} 2^{-\nu} a_\nu$$

By the assumption $a_N \leq a_\nu$ when $\nu > N$ which gives:

$$S_M - S_N \geq 2^{-M} a_{M+1} - 2^{-N} a_N + a_N \cdot \sum_{\nu=N+1}^{\nu=M} 2^{-\nu} =$$

$$2^{-M} a_{M+1} - 2^{-N} a_N + a_N \cdot 2^{-N}(1 - 2^{-M+N}) = 2^{-M} a_{M+1} - a_N \cdot^{-M}$$

Now we can argue as follows. Since (*) holds the partial sums is a Cauchy sequence. So if $\epsilon > 0$ there exists N_* such that $S_M < S_{N_*} + \epsilon$ for every $M > N_*$. With $N = N_*$ above we therefore get

(iii) $$2^{-M} a_{M+1} \leq \epsilon + a_{N_*} \cdot 2^{-M} \quad : \quad M > N_*$$

With N_* fixed we find a large M such that (iii) entails

$$2^{-M-1} a_{M+1} \leq 2 \cdot \epsilon$$

Since ϵ is arbitrary we get the limit (**).

B. Counting functions.

A counting function $N(s)$ is an integer-valued function with unit jumps at some strictly increasing sequence $0 < s_1 < s_2 < \ldots$. Suppose that $N(s)$ is defined for $0 < s < 1$ and assume that:

(*) $$\int_0^1 (1-s) \cdot dN(s) < \infty$$

1.B Theorem. When (*) holds it follows that

(**) $$\lim_{s \to 1} (1-s)N(s) = 0$$

Proof. Put $a_k = N(1-2^{-k})$. The interval $[0,1]$ can be divided into the intervals $[1-2^{-k}, 1-2^{-k-1})$. It is easily seen that (*) implies that

$$\sum 2^{-k}(a_{k+1} - a_k) < \infty$$

Hence Theorem 7 gives

(i) $$\lim_{k \to \infty} 2^{-k} N(1-2^{-k}) = 0$$

Next, if $s < 1$ we choose k such that $1 - 2^{-k} \leq s < 1 - 2^{-k-1}$ and then

(ii) $$(1-s)N(s) \leq 2^{-k} \cdot N(1-2^{-k-1}) = 2 \cdot 2^{-k-1} \cdot N(1-2^{-k-1})$$

Hence (i) implies that (**) tends to zero as required.

2.B A study of $\sum (1-a_k)$

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Let $0 < a_k < 1$ and suppose that the series

(1) $$\sum (1-a_k) < \infty$$

Let $N(s)$ be the counting function with unit jumps at $\{a_k\}$. So (1) means that

(2) $$\int_0^1 (1-s) dN(s) < \infty$$

Exercise. With $s = 1-\xi$ close to 1, i.e. when ξ is small the Taylor expansion of the Log-function at $s = 1$ gives

$$\log \frac{1}{s} = \log \frac{1}{1-\xi} = \xi + \text{higher order terms in } \xi$$

Use this to prove that (2) holds if and only if

(3) $$\int_0^1 \log(\frac{1}{s}) \cdot dN(s) < \infty$$

Next, assume that (2) holds and for each $0 < r < 1$ we set

(4) $$S(r) = \int_0^r \log \frac{1}{s} \cdot dN(s) \quad \text{and} \quad T(r) = \int_0^r \log \frac{r}{s} \cdot dN(s)$$

Since $\log(\frac{1}{s}) - \log(\frac{r}{s}) = \log\frac{1}{r}$ it follows that

(5) $$S(r) - T(r) = \log\frac{1}{r}\int_0^r dN(s) = \log\frac{1}{r}\cdot N(r)$$

Now $\log\frac{1}{r} \simeq 1 - r$ as $r \to 1$ and since (2) is assumed, it follows from Theorem 1.B that:

$$\lim_{r\to 1}\log\frac{1}{r}\cdot N(r) = 0$$

Hence (5) gives

(6) $$\lim_{r\to 1} S(r) - T(r) = 0$$

With the notations above we have therefore proved

3.B Theorem. *Assume that (2) holds. Then the following limit formula holds*

$$\lim_{r\to 1}\int_0^r \log\,(\frac{r}{s})\cdot dN(s) = \int_0^1 \log(\frac{1}{s})\cdot dN(s)$$

4.B Remark. By the multiplicative property of the Log-function the last term becomes

(i) $$\sum \log\frac{1}{a_k} = \log\prod\frac{1}{a_k}$$

In the right hand side there appears a *product series* defined by the a-sequence. In section III we study product series in more detail but already here we have seen an example of the interplay between additive series and product series. Notice also that via the equivalence of (1) and (2) above, Theorem 3.B gives the following:

5.B Theorem. *Let $\{a_k\}$ be a sequence with each $0 < a_k < 1$. Then the additive series $\sum(1 - a_k)$ is convergent if and only if the product series*

(i) $$\prod_{k=1}^{\infty}\frac{1}{a_k} < \infty$$

Moreover, when (i) holds one has the limit formula

(ii) $$\lim_{r\to 1}\prod_r\frac{r}{a_k} = \prod_{\nu=1}^{\infty}\frac{1}{a_k} < \infty$$

where $\prod_r$ is extended over those k for which $a_k \leq r$.

6.B Asymptotic formulas.

Here we announce a deeper result without proof. Let $\{\lambda_\nu\}$ be a strictly increasing sequence of positive numbers where $\lambda_\nu = +\infty$ as $\nu \to \infty$. Consider another sequence of positive numbers $\{a_\nu\}$ and assume that the series

$$\sum_{\nu=1}^{\infty}\frac{a_\nu}{\lambda_\nu} < \infty$$

For each $s > 0$ we denote by $\Lambda(s)$ be the largest integer ν such that $\lambda_\nu \leq s$. So the Λ-function is a non-decreasing integer-valued function with jumps at every λ_ν. Next, define the following pair of functions when $0 < x < \infty$:

$$\mathcal{A}(x) = \sum_{\nu<\Lambda(x)} a_\nu \quad\text{and}\quad f(x) = \sum_{\nu=1}^{\infty}\frac{a_\nu}{\lambda_\nu + x}$$

Notice that $\mathcal{A}(x)$ is non-decreasing while $f(x)$ is decreasing. With these notations the following hold:

7.B Theorem. *Assume that there exists some $0 < \alpha < 1$ and a positive constant C such that*

$$\lim_{x\to\infty} x^\alpha\cdot f(x) = C$$

Then there also exists the limit

$$\lim_{x\to\infty} x^{\alpha-1}\cdot\mathcal{A}(x)=\frac{C}{\pi}\cdot\frac{\sin\pi\alpha}{1-\alpha}$$

Remark. Theorem 7.B is due to Carleman and was established in his lectures at Institute Mittag-Leffler in 1935. The proof requires analytic methods based upon Fourier transforms and is given in § XX from Special Topics. In the special case when $\lambda_\nu=\nu$ we see that $\mathcal{A}(n)$ is equal to the partial sum S_n of $\{a_\nu\}$ for positive integers n. So in this special case Theorem 7.B asserts that if the positive series

$$\sum\frac{a_\nu}{\nu}<\infty$$

and if there exists

$$\lim_{x\to\infty} x^{\alpha}\cdot\sum\frac{a_\nu}{\nu+x}=C$$

then

$$\lim_{n\to\infty}\frac{n^{\alpha}\cdot S_n}{n}=\frac{C}{\pi}\cdot\frac{\sin\pi\alpha}{1-\alpha}$$

II. Power series.

Starting with a sequence $\{a_\nu\}$ and a complex number $z \neq 0$ we get the sequence $\{a_\nu \cdot z^\nu\}$. If this sequence yields a convergent additive series the sum is denoted by $S(z)$.

1. Definition *The set of all $z \in \mathbf{C}$ for which the series*

$$\sum_{\nu=0}^{\infty} a_\nu \cdot z^\nu$$

converges is denoted by $\mathfrak{conv}(\{a_\nu\})$ and called the set of convergence for the a-sequence.

Remark. It may occur that $\mathfrak{conv}(\{a_\nu\})$ just contains $z = 0$. An example is when $a_\nu = \nu\,!$. But if the absolute values $|a_\nu|$ do not increase too fast, then $\mathfrak{conv}(\{a_\nu\})$ contains non-zero complex numbers. Since the terms of a convergent sequence is bounded, each $z_0 \in \mathfrak{conv}(\{a_\nu\})$ gives a constant M such that

$$(1) \qquad |a_\nu| \cdot |z_0|^\nu \leq M \quad : \quad \nu = 0, 1, \ldots$$

If $|z| < |z_0|$ it follows that the series defined by $\{a_\nu \cdot z^\nu\}$ is absolutely convergent. Indeed, we have

$$|a_\nu \cdot z^\nu| \leq M \cdot \frac{|z|^\nu}{|z_0|^\nu}$$

Here $r = \frac{|z|}{|z_0|} < 1$ and the geometric series $\sum r^\nu$ is convergent. Hence the *Majorant principle* from I.3 yields the the absolute convergence of $\{a_\nu \cdot z^\nu\}$.

2. The radius of convergence. Above we saw that if $z_0 \in \mathfrak{conv}(\{a_\nu\})$ then the domain of convergence contains the open disc of radius $|z_0|$. Put

$$(*) \qquad \mathfrak{r} = \max |z| \quad : \quad z \in \mathfrak{conv}(\{a_\nu\})$$

Assume that $\mathfrak{conv}(\{a_\nu\})$ is not reduced to $z = 0$. Then $\mathfrak{r}$ is a positive number or $+\infty$. It is called the radius of convergence for $\{a_\nu\}$). The case $\mathfrak{r} = +\infty$ means that the series

$$\sum a_\nu \cdot z^\nu$$

converges for all $z \in \mathbf{C}$.

3. Hadamard's formula for $\mathfrak{r}$. Given a sequence $\{a_\nu\}$ its radius of convergence is found by taking a limes superior. More precisely

$$(*) \qquad \frac{1}{\mathfrak{r}} = \limsup_{\nu \to \infty} \, |a_\nu|^{\frac{1}{n}}$$

3.1 Exercise. Prove Hadamard's formula.

3.2 Remark. A sufficient condition in order that $\mathfrak{r} \geq 1$ for a given sequence $\{a_\nu\}$ can be checked as follows. Suppose that

$$|a_\nu| \leq e^{\rho(\nu)}$$

for some sequence $\{\rho(\nu)\}$. With $r < 1$ we can write $r = e^{-\delta}$ for some $\delta > 0$ and obtain

$$|a_n| \cdot r^n \leq \exp(\rho(n) - \delta \cdot n) \quad : \, n = 1, 2, \ldots$$

From this we conclude that $\mathfrak{r} \geq 1$ holds if

$$(**) \qquad \lim_{n \to \infty} \rho(n) - \delta \cdot n = -\infty \quad \text{for each} \quad \delta > 0$$

4. Application. Let $\sum a_n \cdot z^n$ be a power series whose radius of convergence is one. Let $\{b_n\}$ be some other sequence of complex numbers. We seek for conditions in order that the series $\sum b_n a_n \cdot z^n$ also converges when $|z| < 1$. The result below gives a sufficient condition for this to hold.

5. Theorem. *Let $\{\gamma_n\}$ be a sequence of positive numbers such that*

(i) $$\lim_{n\to\infty} \frac{\gamma_n}{n} \cdot \log(n) = 0$$

Then the $\mathfrak{r}$-number of $\{b_\nu \cdot a_\nu\}$ is ≥ 1 for every b-sequence such that

$$|b_n| \leq n^{\gamma_n} \quad : \quad n = 1, 2, \ldots$$

6. Exercise. Prove this theorem. It applies in particular when $\gamma_n = k$ for some positive integer k and hence the radius of convergence of $\{\nu^k \cdot a_\nu\}$ is at least one. Of course, this can be seen directly from the formula in (3) above since

$$\lim_{n\to\infty} n^{\frac{k}{n}} = 1$$

hold for every positive integer k.

7. Hadamard's Lemma. Let $\{c_n\}$ be a sequence if numbers such that the following two conditions hold:

(i) $$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 1$$

There exists some $0 < \alpha < 1$ such that

(ii) $$\left|c_{n+1}^2 - c_{n+2} \cdot c_n\right| \leq \alpha^n \quad : \quad n = 1, 2, \ldots$$

Show that (i-ii) imply that one has an unrestricted limit:

(*) $$\lim_{n\to\infty} |c_n|^{\frac{1}{n}} = 1$$

8. Exercise. Let $a_0, a_1, \ldots$ be a sequence of positive real numbers. Suppose there exists an integer m and a constant C such that

$$a_k \leq \frac{a_{k-1} + \ldots + a_{k-m}}{k} \quad \text{for all} \quad k \geq m$$

Show that no matter how $a_0, \ldots, a_{m-1}$ are determined initially it follows that the power series

$$\sum a_\nu \cdot z^\nu$$

has an infinite radius of convergence, i.e. for every $R > 0$ the positive series $\sum a_\nu \cdot R^\nu < \infty$.

II.B Convergence at the boundary

Let $\{a_\nu\}$ be a sequence with $\mathfrak{r} = 1$. Given some $0 \leq \theta \leq 2\pi$ we have the complex number $e^{i\theta}$ whose absolute value is one. It is not always true that the series

(1) $$\sum_{\nu=}^{\infty} a_\nu \cdot e^{i\nu\theta}$$

converges. So we have a possibly empty subset of $[0, 2\pi]$ defined by

(2) $$\mathcal{F} = \{0 \leq \theta \leq 2\pi\} \quad : \quad \text{The series (1) converges for } \theta$$

1. Example Let $\{a_\nu = \frac{1}{\nu}\}$. Here $\mathfrak{r} = 1$ and the series $\sum \frac{1}{\nu}$ is divergent. On the other hand

$$\sum \frac{e^{i\nu\theta}}{\nu}$$

converges for each $0 < \theta < 2\pi$. In other words

(i) $$\mathcal{F} = (0, 2\pi)$$

To see this we notice that if $b_\nu = e^{i\nu\theta}$ with $0 < \theta < 2\pi$ then the partial sums are:

$$S_N = \frac{1 - e^{i(N+1)\theta}}{e^{i\theta} - 1}$$

This sequence is bounded and since the positive series $\sum \left(\frac{1}{\nu} - \frac{1}{\nu+1}\right)$ converges, the reader can deduce the inclusion (i) from Abel's theorem in A.7

2. Radial limits Let $\{a_\nu\}$ be a sequence whose radius of convergence is 1. If $0 < r < 1$ and $0 \leq \theta \leq 2\pi$ we get the convergent series

$$S(r, \theta) = \sum_{\nu=0}^{\infty} a_\nu \cdot r^\nu e^{i\nu\theta}$$

Keeping θ fixed we say that one has a radial limit if there exists

(*) $$\lim_{r \to 1} S(r, \theta) = S_*(\theta)$$

Denote by Let $\mathfrak{rad}(\{a_\nu\})$ the set of θ for which the limit above exists. The question arises if $\theta \in \mathfrak{rad}(\{a_\nu\})$ implies that the series $\sum a_\nu e^{i\nu\theta}$ converges. This is not true in general. The simplest example is to take $a_\nu = (-1)^\nu$ and $\theta = 1$. Here $S(r, 0) = \frac{1}{1+r}$ whose limit is $\frac{1}{2}$ while $\sum a_\nu$ diverges since the a-sequence does not tend to zero. But the converse is true, i.e. one has:

3. Theorem *Let $\{a_\nu\}$ give a convergent additive series with sum S_*. Then there exists the limit*

$$\lim_{x \to 1} \sum a_n \cdot x^n$$

Moreover, the radial limit is equal to the series sum S_* of the additive series.

Proof. We can always modify a_0 and assume that $S_* = 0$. Set

(i) $$\rho_N = \max_{\nu \geq N} |S_\nu|$$

So the hypothesis is now that $\rho_N \to 0$ as $N \to \infty$. For each $0 < x < 1$ we set:

$$S_N(x) = \sum_{nu=0}^{\nu=N} a_\nu \cdot x^\nu$$

When $0 < x < 1$ is fixed the infinite power series

(ii) $$S_*(x) = \sum_{\nu=0}^{\infty} a_\nu \cdot x^\nu$$

converges. Next, when $0 < x < 1$ then the sequence $\{b_\nu = x^\nu - x^{\nu+1}\}$ is non-increasing. Hence Exercise 6 from [Additive Series] implies that that for each pair $M > N$ and every $0 < x < 1$ one has

$$|S_M(x) - S_N(x)| \leq \rho_N$$

Since this holds for every $M > N$ and the series (ii) converges we obtain

(iii) $$[S_*(x) - S_N(x)| \leq \rho_N$$

Next, the triangle inequality gives:

$$|S_*(x)| \leq |S_*(x) - S_N(x)| + |S_N(x) - S_N| + |S_N| \leq$$

$$2 \cdot \rho_N + |S_N(x) - S_N|$$

Finally, if $\epsilon > 0$ we first choose N so that $2 \cdot \rho_N < \epsilon/2$ and with N fixed we have

$$\lim_{x \to 1} S_N(x) = S_N$$

This proves the requested limit formula

$$\lim_{x \to 1} S_*(x) = 0$$

4. A theorem of Landau.

One can also study limits on sparse sets which converge to a boundary point. Results of this nature appear in the article *Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande des Konvergenzgebietes* by Landau from 1907. Here we announce and prove one of these results. Consider a sequence of complex numbers $\{z_k\}$ in the open unit disc D which converge to 1. We say that the sequence is of Landau type if there exists a constant $\mathbf{L}$ such that

(i) $$\frac{|1 - z_k|}{1 - |z_k|} \leq \mathbf{L} \quad : \quad \frac{1}{\mathbf{L}} \leq k \cdot |1 - z_k| \leq \mathbf{L} \quad : \; k = 0, 1, 2, \ldots$$

Remark. The first inequality means that z_k come close to the real axis as $|z_k| \to 1$. The second condition means that the sequence of absolute values $1 - |z_k|$ decreases in a regular fashion.

4.1 Theorem. *Let $\{a_\nu\}$ be a sequence such that $\nu \cdot a_\nu \to 0$ as $\nu \to +\infty$ and suppose there exists a sequence $\{z_k\}$ of the Landau type such that there exists a limit*

$$\lim_{k \to \infty} \sum a_\nu \cdot z_k^\nu = A$$

Then the series $\sum a_\nu$ is convergent and the series sum is equal to A.

Proof. Since $\nu \cdot a_\nu \to 0$ it follows that

(i) $$\lim_{k \to \infty} \frac{1}{k} \cdot \sum_{\nu=1}^{k} a_\nu = 0$$

Next, set

(ii) $$f(k) = \sum_{\nu=1}^{\nu=k} a_\nu z_k^\nu \quad \text{and} \quad S_k = \sum_{\nu=1}^{\nu=k} a_\nu$$

The triangle inequality gives

$$|S_k - f(k)| \leq |\sum_{\nu=1}^{\nu=k} a_\nu (1 - z_k^\nu) - \sum_{\nu > k} a_\nu z_k^\nu| \leq$$

(iii) $$\sum_{\nu=1}^{\nu=k} |a_\nu| |1 - z_k| \cdot \nu + \sum_{\nu > k} |a_\nu| \cdot |z_k|^\nu = W(k)_* + W(k)^*$$

Put

(iv) $$\epsilon(k) = \max\{\nu \cdot |a_\nu| : \ \nu \geq k+1\} \implies |a_\nu| \leq \frac{\epsilon(k)}{k} \quad : \nu \geq k+1$$

Since we also have $|z_k|^{k+1} \leq 1$ it follows from (iv) that

(v) $$W^*(k) \leq \frac{\epsilon(k)}{k} \cdot \frac{1}{1-|z_k|} \leq \frac{\mathbf{L} \cdot \epsilon(k)}{k \cdot |1-z_k|} \leq \mathbf{L}^2 \cdot \epsilon(k)$$

At the same time we have

(vi) $$W_*(k) \leq k \cdot |1-z_k| \cdot \frac{\sum_{\nu=1}^{\nu=k} \nu \cdot |a_\nu|}{k} \leq \mathbf{L} \cdot \frac{\sum_{\nu=1}^{\nu=k} \nu \cdot |a_\nu|}{k}$$

Now we are done, i.e. $W_*(k) \to 0$ by the observation in (*) and $W^*(k) \to 0$ since the hypothesis on $\{a_\nu\}$ gives $\epsilon(k) \to 0$.

III. Product series

Consider a sequence of positive real numbers $\{q_\nu\}$. To each $N \geq 1$ we define the partial product

$$\Pi_N = \prod_{\nu=1}^{\nu=N} q_\nu$$

If $\lim_{N\to\infty} \Pi_N$ exists we say that the infinite product converges and put

$$\Pi_* = \prod_{\nu=1}^{\infty} q_\nu$$

It is clear that if the product converges then $\lim_{\nu\to\infty} q_\nu = 1$. A very useful result goes as follows:

1. Theorem. *Let $\{q_\nu\}$ be a sequence where $0 < q_\nu < 1$ hold for all ν. Then the following three conditions are equivalent:*

$$\sum (1 - q_\nu) < \infty \quad : \quad \sum \operatorname{Log} \frac{1}{q_\nu} < \infty \quad : \quad \prod_{\nu=1}^{\infty} q_\nu > 0$$

Exercise Prove this theorem. A hint is that the function $\log r$ has the Taylor expansion close to $r = 1$ given by

$$\log r = (r-1) + (r-1)^2/2 + \ldots$$

2. Proposition *One has the inequality*

$$|\operatorname{Log}(1+z) - z| \leq |z|^2 \quad : \quad |z| \leq 1/2$$

Exercise. Prove this inequality. A hint is to use the series expansion of rhe complex log-function:

$$\log(1+z) = z - z^2/2 + z^3/3 + \ldots$$

Next, consider a complex sequence $a(\cdot)$ where $|a_\nu| \leq \frac{1}{2}$ hold for all ν and put:

$$\Pi_N = \prod_{\nu=1}^{\nu=N} (1 - a_\nu) \implies \log(\Pi_N) = \sum_{\nu=0}^{\nu=N} \log(1 - a_\nu)$$

Proposition 2 gives the inequality

(*) $$|\log(1-a_\nu) + a_\nu| \leq |a_\nu|^2$$

This enable us to investigate the convergence of the product series with the aid of the additive series for $\{a_\nu\}$. We get for example

(**) $$|\log(\Pi_N) + \sum_{\nu=1}^{\nu=N} a_\nu| \leq \sum_{\nu=1}^{\nu=N} |a_\nu|^2$$

From (**) we can conclude:

3. Theorem. *Let $\{a_\nu\})$ be a sequence where each $|a_\nu| \leq \frac{1}{2}$ and $\sum |a_\nu|^2 < \infty$. Then $\sum a_\nu$ converges if and only if the product series $\Pi(1 - a_\nu)$ converges. Moreover, when convergence holds one has the equality*

$$\log(\Pi_*) = \sum_{\nu=1}^{\infty} \log(1 - a_\nu)$$

IV. Blaschke products.

Let $\{a_\nu\}$ be a sequene in the open unit disc D which are enumerated so that their absolute values are non-decreasing. But repetitions may occur, i.e. several a-numbers can be equal. We always assume that $|a_\nu| \to 1$ as $\nu \to +\infty$. Hence $\{a_\nu\}$ is a discrete subset of D. To each ν we set

(1) $$\beta_\nu(\theta) = \frac{e^{i\theta} - a_\nu}{1 - e^{i\theta} \cdot \bar{a}_\nu} \cdot \frac{\bar{a}_\nu}{|a_\nu|} \quad : \quad 0 \leq \theta \leq 2\pi$$

The *Blaschke product of order* N is the partial product

(2) $$B_N(\theta) = \prod_{\nu=1}^{\nu=N} \beta_\nu(\theta)$$

The question arises when the product series converges and gives a limit

(3) $$B_*(\theta) = \prod_{\nu=1}^{\infty} \beta_\nu(\theta)$$

To analyze this we use polar coordinates and put

$$a_\nu = r_\nu e^{i\theta\nu}$$

Each β-number has absolute value one and if $\theta \neq \theta_\nu$ for every ν we have

(4) $$\beta_\nu(\theta) = e^{i \cdot \gamma(r_\nu, \theta - \theta_\nu)} \quad : \quad 0 < \gamma(r_\nu, \theta - \theta_\nu) < 2\pi$$

Exercise. Show that when $-\pi/2 < \theta - \theta_\nu < \pi/2$ then the construction of the arctan-function gives

(4) $$\gamma(r, \theta - \theta_\nu) = \text{arctg}\,[\frac{(1 - r^2) \cdot \sin(\theta_\nu - \theta)}{1 + r^2 - 2r\cos(\theta_\nu - \theta)}\,]$$

4.2. Blashke's condition We impose the condition that the positive series

(*) $$\sum (1 - r_\nu) < \infty$$

4.3 Exercise. If x is a real number we set

$$\{x\} = \min_{k \in \mathbf{Z}} [x - 2\pi k|$$

Assume that (*) holds. Show that the Blaschke product has a radial limit at $\theta = 0$ if and only if there exists the limit

(4.3.1) $$\lim_{N \to \infty} \{\sum_{\nu=1}^{\nu=N} \frac{(1 - r_\nu) \cdot \theta_\nu}{(1 - r_\nu)^2 + \theta_\nu^2}\}$$

In other words, (4.3.1) holds if and only if the infinite product $B_*(0)$ exists and

$$\lim_{r \to 1} B(r) = B_*(0)$$

4.4.Remark. Notice that θ_ν may be < 0 or > 0 and it is not necessary that all of them become close to 0. To determine all sequence of pairs (r_ν, θ_ν) where 4.3.1 holds and $\theta_\nu \to 0$ appears to be a very difficult problem.

In complex analysis one considers the analytic function defined in the open unit disc $|z| < 1$ by

(i) $$B(z) = \prod_{\nu=0}^{\infty} \frac{z - a_\nu}{1 - \bar{a}_\nu \cdot z} \cdot e^{-i\arg(a_\nu)}$$

which exists under the sole condition that Blascke's condition (*) is valid. A major resultdue tio Blaschke asserts that the

$$\lim_{r \to 1} B(re^{i\theta}) = B_*(\theta)$$

exists for almost every θ, and the absolute value of the limit valkue $B_*(\theta)$ is equal to one almost everywhere, taken in the sens of Lebesgue. But the determination of the set of all $0 \leq \theta \leq 2\pi$

for which the radial limit exists is not clear when no special assumptions are imposed on the $\{\theta_\nu\}$-sequence. For example, divergence may appear when many θ_ν:s are close to θ even if $\{r_\nu\}$ tend rapidly to 1.

V. Estimates using the counting function.

Let $\{\alpha_\nu\}$ be a complex sequence where $0 < |\alpha_1| \leq |\alpha_2| \leq \ldots\}$, amnd assume that the absolute values tend to $+\infty$. We get the counting function $N(R)$ which for every $R > 0$ is the number of α_ν with absolute value $\leq R$. Consider the situation when there exists a constant C such that

$$(*) \qquad N(R) \leq C \cdot R \quad \text{for all} \quad R > 0$$

5.1. The first estimate. To each $R > 0$ we set

$$(2) \qquad S(R) = \prod (1 + \frac{R}{|\alpha_\nu|}) \quad : \text{product taken over all } |\alpha_\nu| \leq 2R$$

Then we have

$$(*) \qquad S(R) \leq e^{KR} \quad \text{where} \quad K = 2C(1 + \text{Log}\,\frac{3}{2})$$

To prove this we consider log $S(R)$. A partial integration gives:

$$\log\, S(R) = \int_0^{2R} \log\,(1 + \frac{R}{t}) \cdot dN(t) = \log\,(1 + \frac{1}{2}) \cdot N(2R) + \int_0^{2R} \frac{R \cdot N(t)}{t(t+R)} \cdot dt$$

Since $\frac{R}{t+R} \leq 1$ for all t, the last integral is estimated by $2R \cdot C$ and (*) follows.

5.2. The second estimate. For each $R > 0$ we consider infinite tail products:

$$(i) \qquad S^*(R) = \prod (1 + \frac{R}{\alpha_\nu}) \cdot e^{-\frac{R}{\alpha_\nu}} \quad : \text{product taken over all } |\alpha_\nu| \geq 2R$$

To estimate (i) we notice that the analytic function $(1+\zeta)e^{-\zeta} - 1$ has a double zero at the origin. This gives a constant A such that

$$(ii) \qquad |(1+\zeta)e^{-\zeta} - 1| \leq A \cdot |\zeta|^2 \quad : \ |\zeta| \leq \frac{1}{2}$$

Since $|\alpha_\nu| \geq 2R$ for every ν we obtain:

$$(iii) \qquad \log^+ |(1 + \frac{R}{\alpha_\nu}) \cdot e^{-\frac{R}{\alpha_\nu}}| \leq \log\,[1 + A\frac{R^2}{|\alpha_\nu|^2}) \leq A \cdot \frac{R^2}{|\alpha_\nu|^2}$$

From (6) we get

$$(iv) \qquad \log^+(S^*(R)) \leq AR^2 \int_{2R}^\infty \frac{dN(t)}{t^2} = A \cdot N(2R) + 2AR^2 \cdot \int_{2R}^\infty \frac{N(t)}{t^3}$$

The last term is estimated by

$$(8) \qquad 2AR^2 \cdot C \cdot \int_{2R}^\infty \frac{dt}{t^2} = AC \cdot R$$

Adding up the result we get

5.3 Theorem. *One has the inequality*

$$S^*(R) \leq \frac{5A}{4} \cdot C \cdot R$$

VI. Theorems by Abel, Tauber, Hardy and Littlewood

Introduction. Consider a power series $f(z) = \sum a_n z^n$ whose radius of convergence is one. If $r < 1$ and $0 \leq \theta \leq 2\pi$ the series

$$f(re^{i\theta}) = \sum a_n r^n e^{in\theta}$$

Passing to $r = 1$ it is in general not true that the series $\sum a_n e^{in\theta}$ is convergent. An example arises if we consider the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

So here $a_n = 1$ for all n and hence the absolute values $|a_n re^{in\theta}| = 1$ for sll n so the series $\sum a_n e^{in\theta}$ diverges. At the same time we notice that when $0 < \theta < 2\pi$ there exists the limit

$$\lim_{r \to 1} \sum r^n e^{in\theta} = \frac{1}{1 - re^{i\theta}}$$

This leads to the following problem where we without loss of generality can take $\theta = 0$. Consider as above a convergent power series and assume that there exists the limit

$$(*) \qquad \lim_{r \to 1} \sum a_n r^n$$

When can we conclude that the series $\sum a_n$ also is convergent and that one has the equality

$$(**) \qquad \sum a_n = \lim_{r \to 1} \sum a_n r^n$$

The first result in this direction was established by Abel in a work from 1823:

A. Theorem *Let $\{a_n\}$ be a sequence such that $\frac{a_n}{n} \to 0$ as $n \to \infty$ and there exists*

$$A = \lim_{r \to 1} \sum a_n r^n$$

Then $\sum a_n$ is convergent and the sum is A.

An extension of Abel's result was established by Tauber in 1897.

B. Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that*

$$\lim_{n \to \infty} \frac{a_1 + 2a_2 + \ldots + na_n}{n} = 0$$

Then, if there exits the radial limit

$$A = \lim_{r \to 1} \sum a_n r^n$$

it follows that the series $\sum a_n$ is convergent and the sum is A.

C. Results by Hardy and Littlewood.

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The following extension of Abel's result was proved by Hardy and Littlewood in 1913:

C Theorem. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a constant C so that $\frac{a_n}{n} \leq C$ for all $n \geq 1$. Assume also that the power series $\sum a_n z^n$ converges when $|z| < 1$. Then the same conclusion as in Abel's theorem holds.*

Remark. The proof of Theorem C requires several steps where an essential ingredient is a result about positive series from the cited article which has independent interest.

D. Theorem. *Assume that each $a_n \geq 0$ and that there exists the limit:*

(*) $$A = \lim_{r \to 1} (1-r) \cdot \sum a_n r^n$$

Then there exists the limit

(**) $$A = \lim_{N \to \infty} \frac{a_1 + \ldots + a_N}{N}$$

Notice that we do not impose any growth condition on $\{a_n\}$ above, i.e. the sole assumption is the existence of the limit (*).

Remark. The proofs of Abel's and Tauber's results are rather easy while C and D require more effort.

1. Proof of Abel's theorem.

Without loss of generality we can assume that $a_0 = 0$ and set $S_N = a_1 + \ldots + a_N$. Given $0 < r < 1$ we let $f(r) = \sum a_n r^n$. For every positive integer N the triangle inequality gives:

$$\big|S_N - f(r)\big| \leq \sum_{n=1}^{n=N} |a_n|(1-r^n) + \sum_{n \geq N+1} |a_n| r^n$$

Set $\delta(N) = \max_{n \geq N} \frac{|a_n|}{n}$. Since $1 - r^n) = (1-r)(1 + \ldots + r^{n-1} \leq (1-r)n$ the last sum is majorised by

$$(1-r) \cdot \sum_{n=1}^{n=N} n \cdot |a_n| + \delta(N+1) \cdot \sum_{n \geq N+1} \frac{r^n}{n}$$

Next, the obvious inequality $\sum_{n \geq N+1} \frac{r^n}{n} \leq \frac{1}{N+1} \cdot \frac{1}{1-r}$ gives the new majorisation

(1) $$(1-r) \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \frac{\delta(N+1)}{N+1} \cdot \frac{1}{1-r}$$

This hold for all pairs N and r. To each $N \geq 2$ we take $r = 1 - \frac{1}{N}$ and hence the right hand side in (1) is majorised by

$$\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n} + \delta(N+1) \cdot \frac{N}{N+1}$$

Here both terms tend to zero as $N \to \infty$. Indeed, Abel's condition $\frac{a_n}{n} \to 0$ implies that $\frac{1}{N} \cdot \sum_{n=1}^{n=N} \frac{|a_n|}{n}$ tends to zero as $N \to \infty$. Hence we have proved the limit formula:

(*) $$\lim_{N \to \infty} \big|s_N - f(1 - \frac{1}{N})\big| = 0$$

Finally it is clear that (*) gives Abel's result.

3. Proof of Tauber's theorem.

We may assume that $a_0 = 0$. Notice that

$$a_n = \frac{\omega_n - \omega_{n-1}}{n} \quad : n \geq 1$$

It follows that

$$f(r) = \sum \frac{\omega_n - \omega_{n-1}}{n} \cdot r^n = \sum \omega_n \big(\frac{r^n}{n} - \frac{r^{n+1}}{n+1}\big)$$

Using the equality $\frac{1}{n} = \frac{1}{n+1} = \frac{1}{n(n+1)}$ we can rewrite the right hand side as follows:

$$\sum \omega_n \big(\frac{r^n - r^{n+1}}{n+1} + \frac{r^n}{n(n+1)}\big)$$

Set

$$g_1(r) = \sum \omega_n \cdot \frac{r^n - r^{n+1}}{n+1} = (1-r) \cdot \sum \frac{\omega_n}{n+1} \cdot r^n$$

By the hypothesis $\lim_{n\to\infty} \frac{\omega_n}{n+1} = 0$ and then it is clear that we get

$$\lim_{r\to 1} g_1(r) = 0$$

Since we also have $f(r) \to 0$ as $r \to 1$ we conclude that

(1)
$$\lim_{r\to 1} \sum \frac{\omega_n}{n(n+1)} \cdot r^n = 0$$

Next, with $b_n = \frac{\omega_n}{n(n+1)}$ we have $nb_n = \frac{\omega_n}{n+1} \to 0$. Hence Abel's theorem applies so (1) gives convergent series

(2)
$$\sum \frac{\omega_n}{n(n+1)} = 0$$

If $N \geq 1$ we have the partial sum

$$S_N = \sum_{n=1}^{n=N} \frac{\omega_n}{n(n+1)} = \sum_{n=1}^{n=N}, \omega_n \cdot (\frac{1}{n} - \frac{1}{n+1})$$

The last term becomes

$$\sum_{n=1}^{n=N} \frac{1}{n}(\omega_n - \omega_{n-1}) - \frac{\omega_N}{N+1} = \sum_{n=1}^{n=N} a_n - \frac{\omega_N}{N+1}$$

Again, since $\frac{\omega_N}{N+1} \to 0$ as $N \to \infty$ we conclude that the convergent series from (2) implies that the series $\sum a_n$ also is converges and has sum equal to zero. This finishes the proof of Tauber's result.

we need some results from calculus in one variable. So before we enter the proofs of the theorems above insert some preliminaries.

3. Results from calculus

To prove the theorems by Hardy sand Littlewood we need some results from calculus in one variable. So before we enter the proofs of Theorem C and D we insert some preliminaries. Below $g(x)$ is a real-valued function defined on $(0,1)$ and of class C^2 at least.

3.1 Lemma *Assume that there exists a constant $C > 0$ such that*

$$g''(x) \leq C(1-x)^{-2} \quad : 0 < x < 1 \quad \text{and} \quad \lim_{x\to 1} g(x) = 0$$

Then one has the limit formula:

$$\lim_{x\to 1} (1-x) \cdot g'(x) = 0$$

3.2 Lemma *Assume that the second order derivative $g''(x) > 0$. Then the following implication holds for each $\alpha > 0$:*

$$\lim_{x\to 1} (1-x)^\alpha \cdot g(x) = 1 \implies \lim_{x\to 1} (1-x)^{\alpha+1} \cdot g'(x) = \alpha$$

Remark. If $g(x)$ has higher order derivatives which all are > 0 on $(0,1)$ we can iterate the conclusion in Lemma 1.2 where we take α to be positive integers. More precisely, by an induction over ν the reader may verify that if

$$\lim_{x\to 1} (1-x) \cdot g(x) = 1$$

exists and if $\{g^{(\nu)}(x) > 0\}$ for all every $\nu \geq 2$ then

(*)
$$\lim_{x\to 1} (1-x)^{\nu+1} \cdot g^{(\nu)}(x) = \nu\,! \quad : \nu \geq 2$$

Next, to each integer $\nu \geq 1$ we denote by $[\nu - \nu^{2/3}]$ the largest integer $\leq (\nu - \nu^{2/3})$. Set

$$J_*(\nu) = \sum_{n \leq [\nu - \nu^{2/3}]} n^\nu e^{-\nu} \quad : \quad J^*(\nu) = \sum_{n \geq [\nu + \nu^{2/3}]} n^\nu e^{-\nu}$$

3.3 Lemma *There exists a constant C such that*

$$\frac{J^*(\nu) + J_*(\nu)}{\nu\,!} \leq \delta(\nu) \quad : \quad \delta(\nu) = C \cdot \exp\big(-\frac{1}{2} \cdot \nu^{\frac{1}{3}}\big) \quad : \nu = 1, 2, \ldots$$

Proofs

We prove only Lemma 1.1 which is a bit tricky while the proofs of Lemma 1.2 and 1.3 are left as exercises to the reader. Fix $0 < \theta < 1$. Let $0 < x < 1$ and set

$$x_1 = x + (1-x)\theta$$

The mean-value theorem in calculus gives

(i) $$g(x_1) - g(x) = \theta(1-x)g'(x) + \frac{\theta^2}{2}(1-x)^2 \cdot g''(\xi) \quad \text{for some} \quad x < \xi < x_1$$

By the hypothesis

$$g''(\xi) \leq C(1-\xi)^{-2} \leq C)1 - x_1)^{-2}$$

Hence (i) gives

$$(1-x)g'(x) \geq \frac{1}{\theta}(g(x_1) - g(x)) - C \cdot \frac{\theta}{2}\frac{(1-x)^2}{1-x_1)^2} =$$

$$\frac{1}{\theta}(g(x_1) - g(x)) - \frac{C \cdot \theta}{2(1-\theta)^2}$$

Keeping θ fixed we have by assumption

$$\lim_{x \to 1} g(x) = 0$$

Notice also that $x \to 1 \implies x_1 \to 1$. It follows that

$$\liminf_{x \to 1} (1-x)g'(x) \geq -\frac{C \cdot \theta}{2(1-\theta)^2}$$

Above $0 < \theta < 1$ is arbitrary, i.e. we can choose small $\theta > 0$ and hence we have proved that

(*) $$\liminf_{x \to 1} (1-x)g'(x) \geq 0$$

Next we prove the opposed inequality

(**) $$\limsup_{x \to 1} (1-x)g'(x) \leq 0$$

To get (**) we apply the mean value theorem in the form

(ii) $$g(x_1) - g(x) = \theta(1-x)g'(x_1) - \frac{\theta^2}{2}(1-x)^2 \cdot g''(\eta) \quad : x < \eta < x_1$$

Since $(1 - x_1) = \theta(1-x)(1-\theta)$ we get

(iii) $$(1-x_1)g'(x_1) = \frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + \frac{(1-\theta)\theta}{2} \cdot (1-x)^2 g''(\eta)$$

Now $g''(\eta) \leq C(1-\eta)^{-2} \leq C(1-x_1)^{-2}$ so the right hand side in (iii) is majorized by

$$\frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{(1-\theta)\theta}{2} \cdot (1-x)^2(1-x_1)^2 =$$

(iv) $$\frac{1-\theta}{\theta} \cdot (g(x_1) - g(x)) + C \cdot \frac{\theta}{2(1-\theta}$$

Keeping θ fixed while $x \to 1$ we obtain:

$$\liminf_{x \to 1} (1-x)g'(x) \leq C \cdot \frac{\theta}{2(1-\theta}$$

Again we can choose arbitrary small θ and hence (**) holds which finishes the proof of Lemma 3.1.

4. Proof of Theorem D.

Set $g(x) = \sum s_n x^n$ which is defined when $0 < x < 1$. Notice that

$$(1-x)g(x) = \sum a_n x^n$$

Since $s_n \geq 0$ for all n all the higher order derivatives

$$g^{(p)}(x) = \sum_{n=p}^{\infty} n(n-1)\cdots(n-p+1)s_n x^{n-p} > 0$$

when $0 < x < 1$. The hypothesis (*) and the inductive result in the remark after Lemma 1.2 give:

(1) $$\lim_{x \to 1} (1-x)^{\nu+2} \cdot \sum s_n \cdot n^{\nu} x^n = (\nu+1)! \quad : \nu \geq 1$$

We shall use the substitution $e^{-t} = x$ where $t > 0$. Since $t \simeq 1 - x$ when $x \to 1$ we see that (1) gives

(2) $$\lim_{t \to 0} t^{\nu+2} \cdot \sum s_n \cdot n^{\nu} e^{-nt} = (\nu+1)! \quad : \nu \geq 1$$

Put

$$J_*(\nu, t) = \frac{t^{\nu+2}}{(\nu+1)!} \cdot \sum_{n=1}^{\infty} s_n \cdot n^{\nu} e^{-nt}$$

So (2) gives for each fixed ν

(3) $$\lim_{t \to 0} J_*(\nu, t) = 1$$

Next, for each pair $\nu \geq 2$ and $0 < t < 1$ we define the integer

(*) $$N(\nu, t) = \left[\frac{\nu - \nu^{2/3}}{t}\right]$$

Since the sequence $\{s_n\}$ is non-decreasing we get

(i) $$s_{N(\nu,t)} \cdot \sum_{n \geq N(\nu,t)} n^{\nu} e^{-nt} \leq \sum_{n \geq N(\nu,t)} s_n \cdot n^{\nu} e^{-nt} \leq \frac{(\nu+1)! \cdot J_*(\nu, t)}{t^{\nu+2}}$$

Next, the construction of $N(\nu, t)$ and Lemma 1.3 give:

(ii) $$\sum_{n \geq N(\nu,t)} n^{\nu} e^{-nt} \geq \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta(\nu))$$

where the δ function is independent of t and tends to zero as $\nu \to \infty$. Hence (i-ii) give

(iii) $$s_{N(\nu,t)} \leq \frac{(\nu+1)}{t} \cdot \frac{1}{1 - \delta(\nu)} \cdot J_*(\nu, t)$$

Next, by the construction of N one has

$$N(\nu, t) + 1 \geq \frac{\nu - \nu^{2/3}}{t} = \frac{\nu}{t} \cdot (1 - \nu^{-1/3})$$

It follows that (iii) gives

(iv) $$\frac{s_{N(\nu,t)}}{N(\nu,t)+1} \leq \frac{\nu+1}{\nu} \cdot \frac{1}{1-\nu^{-1/3}} \cdot \frac{1}{1-\delta(\nu)} \cdot J_*(\nu, t)$$

Since $\delta(\nu) \to 0$ it follows that for any $\epsilon > 0$ there exists some ν_* such that

(v) $$\frac{\nu_* + 1}{\nu_*} \cdot \frac{1}{1 - \nu_*^{-1/3}} \cdot \frac{1}{1 - \delta(\nu_*)} < 1 + \epsilon$$

Increasing ν_* if necesssty we also notice that the construction of $N(\nu_*, t)$ gives

$$\big|\, N(\nu_*, t) - \frac{\nu_* + 1}{t} \big| < \epsilon$$

CVhpoisng a fixed ν_* as above,we find for each large postive integer N some t_N such that $N = N(\nu_*, t_N)$, and notice that

(vi) $$N \to +\infty \implies t_N \to 0$$

Next, (iv) and (v) yield:

(vii) $$\frac{s_N}{N+1} < (1+\epsilon) \cdot J_*(\nu_*, t_N)$$

Now (vi) and the limit in (3) which applies with ν_* is kept fixed while while $t_N \to 0$ entail that

(viii) $$\lim_{N \to \infty} J(\nu_*, t_N) = 1$$

At the same time $\frac{N}{N+1} \to 1$ and since $\epsilon > 0$ was arbitrary we conclude from (vii) that:

(4) $$\limsup_{N \to \infty} \frac{s_N}{N} \leq 1$$

So Theorem 2 follows if we also prove that

(5) $$\liminf_{N \to \infty} \frac{s_N}{N} \geq 1$$

The proof of (II) is similar where we now define the integers

$$N(\nu, t) = \big[\frac{\nu + \nu^{2/3}}{t}\big]$$

Then

$$S_{N(\nu,t)} \cdot \sum_{n \leq N(\nu,t)} n^\nu e^{-nt} \geq \frac{(\nu+1)! \cdot J_*(\nu, t)}{t^{\nu+2}} - \sum_{n > N(\nu,t)} s_n \cdot n^\nu e^{-nt}$$

Here the last term can be estimated since the Lim.sup inequality (4) gives a constant C such that $s_n \leq Cn$ for all n and then

$$\sum_{n > N(\nu,t)} s_n \cdot n^\nu e^{-nt} \leq C \cdot \sum_{n > N(\nu,t)} n^{\nu+1} e^{-nt} \leq C \cdot \delta(\nu) \cdot \frac{(\nu+1)!}{t^{\nu+2}}$$

where Lemma 1.3 entails that $\delta(\nu) \to 0$ as ν increases. At the same time Lemma 1.3 also gives

$$\sum_{n \leq N(\nu,t)} n^\nu \cdot e^{-nt} = \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta_*(\nu)$$

where $\delta_*(\nu) \to 0$ as $\nu \to +\infty$. Given $\epsilon > 0$ we choose ν_* large so that $C \cdot \delta(\nu_*) < \epsilon$ and $\delta_*(\nu_*) < \epsilon$. Increasing ν_* if necesssty we also notice that the construction of $N(\nu_*, t)$ gives

$$\big|\, N(\nu_*, t) - \frac{\nu_* + 1}{t} \big| < \epsilon$$

Choosing a fixed ν_* as above,we find for each large postive integer N some t_N such that $N = N(\nu_*, t_N)$, and notice that Keeping ν_* fixed we conclude that

$$S_{(N(\nu_*,t)} \cdot \frac{\nu_*!}{t^{\nu_*+1}} \cdot (1-\epsilon) \geq \frac{(\nu_*+1)!}{t^{\nu_*+2}} \cdot [J_*(\nu_*, t) - \epsilon] \implies$$

$$S_{(N(\nu_*,t)} \cdot (1-\epsilon) \geq \frac{(\nu_*+1)}{t} \cdot [J_*(\nu_*, t) - \epsilon]$$

For large integers N we find t_N so that $N(\nu_*, t_N) = N$ and we notice that the construction of $N(\nu_*, t)$

4. Proof of Theorem D.

Set $g(x) = \sum s_n x^n$ which is defined when $0 < x < 1$. Notice that

$$(1-x)g(x) = \sum a_n x^n$$

Since $s_n \geq 0$ for all n all the higher order derivatives

$$g^{(p)}(x) = \sum_{n=p}^{\infty} n(n-1)\cdots(n-p+1)s_n x^{n-p} > 0$$

when $0 < x < 1$. The hypothesis (*) and the inductive result in the remark after Lemma 1.2 give:

(1) $$\lim_{x\to 1}(1-x)^{\nu+2}\cdot\sum s_n\cdot n^\nu x^n = (\nu+1)! \quad : \nu \geq 1$$

We shall use the substitution $e^{-t} = x$ where $t > 0$. Since $t \simeq 1-x$ when $x \to 1$ we see that (1) gives

(2) $$\lim_{t\to 0} t^{\nu+2}\cdot\sum s_n\cdot n^\nu e^{-nt} = (\nu+1)! \quad : \nu \geq 1$$

Put

$$J_*(\nu,t) = \frac{t^{\nu+2}}{(\nu+1)!}\cdot\sum_{n=1}^{\infty} s_n\cdot n^\nu e^{-nt}$$

So (2) gives for each fixed ν

(3) $$\lim_{t\to 0} J_*(\nu,t) = 1$$

Next, for each pair $\nu \geq 2$ and $0 < t < 1$ we define the integer

(*) $$N(\nu,t) = [\frac{\nu-\nu^{2/3}}{t}]$$

Since the sequence $\{s_n\}$ is non-decreasing we get

(i) $$s_{N(\nu,t)}\cdot\sum_{n\geq N(\nu,t)} n^\nu e^{-nt} \leq \sum_{n\geq N(\nu,t)} s_n\cdot n^\nu e^{-nt} \leq \frac{(\nu+1)!\cdot J_*(\nu,t)}{t^{\nu+2}}$$

Next, the construction of $N(\nu,t)$ and Lemma 1.3 give:

(ii) $$\sum_{n\geq N(\nu,t)} n^\nu e^{-nt} \geq \frac{\nu!}{t^{\nu+1}}\cdot(1-\delta(\nu))$$

where the δ function is independent of t and tends to zero as $\nu \to \infty$. Hence (i-ii) give

(iii) $$s_{N(\nu,t)} \leq \frac{(\nu+1)}{t}\cdot\frac{1}{1-\delta(\nu)}\cdot J_*(\nu,t)$$

Next, by the construction of N one has

$$N(\nu,t)+1 \geq \frac{\nu-\nu^{2/3}}{t} = \frac{\nu}{t}\cdot(1-\nu^{-1/3})$$

It follows that (iii) gives

(iv) $$\frac{s_{N(\nu,t)}}{N(\nu,t)+1} \leq \frac{\nu+1}{\nu}\cdot\frac{1}{1-\nu^{-1/3}}\cdot\frac{1}{1-\delta(\nu)}\cdot J_*(\nu,t)$$

Since $\delta(\nu) \to 0$ it follows that for any $\epsilon > 0$ there exists some ν_* such that

(v) $$\frac{\nu_*+1}{\nu_*}\cdot\frac{1}{1-\nu_*^{-1/3}}\cdot\frac{1}{1-\delta(\nu_*)} < 1+\epsilon$$

Increasing ν_* if necesssty we also notice that the construction of $N(\nu_*,t)$ gives

$$\Big|\, N(\nu_*,t) - \frac{\nu_*+1}{t}\Big| < \epsilon$$

CVhpoisng a fixed ν_* as above,we find for each large postive integer N some t_N such that $N = N(\nu_*, t_N)$, and notice that

(vi) $$N \to +\infty \implies t_N \to 0$$

Next, (iv) and (v) yield:

(vii) $$\frac{s_N}{N+1} < (1+\epsilon) \cdot J_*(\nu_*, t_N)$$

Now (vi) and the limit in (3) which applies with ν_* is kept fixed while while $t_N \to 0$ entail that

(viii) $$\lim_{N\to\infty} J(\nu_*, t_N) = 1$$

At the same time $\frac{N}{N+1} \to 1$ and since $\epsilon > 0$ was arbitrary we conclude from (vii) that:

(4) $$\limsup_{N\to\infty} \frac{s_N}{N} \leq 1$$

So Theorem 2 follows if we also prove that

(5) $$\liminf_{N\to\infty} \frac{s_N}{N} \geq 1$$

The proof of (II) is similar where we now define the integers

$$N(\nu, t) = [\frac{\nu + \nu^{2/3}}{t}]$$

Then

$$S_{N(\nu,t)} \cdot \sum_{n \leq N(\nu,t)} n^\nu e^{-nt} \geq \frac{(\nu+1)! \cdot J_*(\nu, t)}{t^{\nu+2}} - \sum_{n > N(\nu,t)} s_n \cdot n^\nu e^{-nt}$$

Here the last term can be estimated since the Lim.sup inequality (4) gives a constant C such that $s_n \leq Cn$ for all n and then

$$\sum_{n>N(\nu,t)} s_n \cdot n^\nu e^{-nt} \leq C \cdot \sum_{n>N(\nu,t)} n^{\nu+1} e^{-nt} \leq C \cdot \delta(\nu) \cdot \frac{(\nu+1)!}{t^{\nu+2}}$$

where Lemma 1.3 entails that $\delta(\nu) \to 0$ as ν increases. At the same time Lemma 1.3 also gives

$$\sum_{n \leq N(\nu,t)} n^\nu \cdot e^{-nt} = \frac{\nu!}{t^{\nu+1}} \cdot (1 - \delta_*(\nu)$$

where $\delta_*(\nu) \to 0$ as $\nu \to +\infty$. Given $\epsilon > 0$ we choose ν_* large so that $C \cdot \delta(\nu_*) < \epsilon$ and $\delta_*(\nu_*) < \epsilon$. Increasing ν_* if necesssty we also notice that the construction of $N(\nu_*, t)$ gives

$$\big| \, N(\nu_*, t) - \frac{\nu_* + 1}{t} \big| < \epsilon$$

Choosing a fixed ν_* as above,we find for each large postive integer N some t_N such that $N = N(\nu_*, t_N)$, and notice that Keeping ν_* fixed we conclude that

$$S_{(N(\nu_*,t)} \cdot \frac{\nu_*!}{t^{\nu_*+1}} \cdot (1-\epsilon) \geq \frac{(\nu_*+1)!}{t^{\nu_*+2}} \cdot [J_*(\nu_*, t) - \epsilon] \implies$$

$$S_{(N(\nu_*,t)} \cdot (1-\epsilon) \geq \frac{(\nu_*+1)}{t} \cdot [J_*(\nu_*, t) - \epsilon]$$

For large integers N we find t_N so that $N(\nu_*, t_N) = N$ and we notice that the construction of $N(\nu_*, t)$

5. Proof of Theorem C

Set $f(x) = \sum a_n x^n$. Notice that it suffices to prove Theorem C when the limit value

$$\lim_{x\to 1} \sum a_n x^n = 0$$

Next, the assumption that $a_n \leq \frac{c}{n}$ for a constant c gives

$$f''(x) = \sum n(n-1)a_n x^{n-2} \leq c \sum (n-1)x^{n-2} = \frac{c}{1-x)^2}$$

The hypothesis $\lim_{x\to 1} f(x) = 0$ and Lemma xx therefore gives

(i) $$\lim_{x\to 1} (1-x)f'(x) = 0$$

Next, notice the equality

(ii) $$\sum_{n=1}^{\infty} \frac{na_n}{c} x^n = \frac{x}{c} \cdot f'(x)$$

At the same time $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ and hence (i-ii) together give:

$$\lim_{x\to 1} (1-x) \cdot \sum (1 - \frac{na_n}{c}) \cdot x^n = 1$$

Here $1 - \frac{na-n}{c} \geq 0$ so Theorem 2 gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n=N} (1 - \frac{na_n}{c}) = 1$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{n=N} na_n = 0$$

This means precisely that the condition in Tauber's Theorem holds and hence $\sum a_n$ converges and has series sum equal to 0 which finishes the proof of Theorem C.

VII. An example by Hardy

Consider the series expansion of

$$(1-z)^\alpha = \sum b_n z^n$$

where α in general is a complex number. Newton's binomial formula gives:

$$(*) \qquad b_n = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!}$$

Apply this with $\alpha = i$. If $n \geq 1$ which entails that

$$|n \cdot b_n| = \frac{|(i+1)|\ldots|i+n-1|}{n!} = \sqrt{(1+\frac{1}{1^2})\cdot(1+\frac{1}{2^2})\cdots(1+\frac{1}{(n-1)^2}}$$

It follows that

$$\lim_{n\to\infty} |n \cdot b_n| = \sqrt{\prod_{\nu=1}^{\infty}(1+\frac{1}{\nu^2})}$$

In particular $|b_n| \simeq \frac{1}{n}$ when n is large and therefore the series

$$\text{(i)} \qquad \sum_{n=2}^{\infty} \frac{|b_n|}{\operatorname{Log} n} = +\infty$$

In spite of the divergence above one has:

Theorem 7.1. *The series*

$$\sum_{n=2}^{\infty} \frac{b_n}{\operatorname{Log} n} \cdot e^{in\phi}$$

converges uniformly when $0 \leq \phi \leq 2\pi$.

Before we give the proof of Theorem 7.1 we need a result if independent interest.

7.2. Theorem. *Let* $\{a_n\}$ *be a sequence of complex numbers such that* $|a_n| \leq \frac{C}{n}$ *hold for all* n *and some constant* C *and the analytic function* $f(z) = \sum a_n z^n$ *is bounded in* D*, i.e.*

$$|f(z)| \leq M \quad : z \in D$$

Then, if $B_n(z) = a_0 + a_1 + \ldots + a_n z^n$ *are the partial sums one has the inequality*

$$\max_\theta |B_m(e^{i\theta})| \leq M + 2C \quad \text{for all} \quad m = 1, 2, \ldots$$

Proof. When $0 < r < 1$ and θ are given we have

$$\left|B_m(\theta) - f(re^{i\theta})\right| = \sum_{n=0}^{n=m} a_n e^{in\theta}(1-r^n) - \sum_{n=m+1}^{\infty} a_n e^{in\theta} \cdot r^n \leq$$

$$(1-r)\sum_{n=0}^{n=m} n \cdot |a_n| + \sum_{n=m+1}^{\infty} |a_n| \cdot r^n$$

With m given we apply this when $r = 1 - 1/m$. Then the last sum above is estimated above by

$$(*) \qquad \frac{1}{m} \cdot \sum_{n=1}^{n=m} n \cdot |a_n| + \frac{c}{m} \cdot \sum_{n=m+1}^{\infty} r^n \leq C + \frac{c}{m} \cdot \sum_{n=0}^{\infty} r^n = 2C$$

Finally, since the maximum norm of f is $\leq M$ the triangle inequality gives

$$|B_m(e^{i\theta})| \leq 2C + M$$

Here m and θ are arbitrary so Theorem 7.2 follows.

Proof of Theorem 7.1 To each $m \geq 2$ we consider the partial sum series

$$S_m(\phi) = \sum_{n=2}^{n=m} \frac{b_n}{\operatorname{Log} n} \cdot e^{in\phi}$$

Theorem 7.1 follows if there to every $\epsilon > 0$ exists an integer M such that

(1) $$\max_{0 \leq \phi \leq 2\pi} |S_m(\phi) - S_M(\phi)| < \epsilon \quad : \quad \forall\ m > M$$

To prove (1) we employ the partial sums

(2) $$B_n(\phi) = \sum_{\nu=1}^{\nu=n} b_\nu \cdot e^{i\nu\phi}$$

For each pair $m > M \geq 2$, the partial summation formula in gives

$$S_m(\phi) - S_M(\phi) =$$

(3) $$\sum_{n=M}^{n=m} B_n(\phi) \cdot \big[\frac{1}{\operatorname{Log} n} - \frac{1}{\operatorname{Log}(n+1)}\big] - \frac{B_{M-1}(\phi)}{\operatorname{Log} M} + \frac{B_m(\phi)}{\operatorname{Log}(m+1)}$$

Now we can apply Theorem 7.2 and find a constant K such that

(4) $$|B_n(\phi)| \leq K \quad >: \quad n \geq 2 \quad \text{and} \quad 0 \leq \phi \leq 2\pi$$

Notice that if (4) holds then (3) gives the inequality

$$\big|S_m(\phi) - S_M(\phi)\big| \leq K \cdot \sum_{n=M}^{n=m} \big[\frac{1}{\operatorname{Log} n} - \frac{1}{\operatorname{Log}(n+1)}\big] + \frac{1}{\operatorname{Log} M} + \frac{1}{\operatorname{Log}(m+1)} = \frac{2K}{\operatorname{Log} M}$$

Hence Theorem 7.1 is proved if we establish the inequality (4). To prove this we study the analytic function defined in the open unit disc D by the convergent power series:

(*) $$(1-z)^i = \sum c_n \cdot z^n$$

Since $\mathfrak{Re}(1-z) > 0$ when $|z| < 1$ there exists a single valued branch of $\log(1-z)$ and the function above can be written as

$$g(z) = e^{i \cdot \log(1-z)}$$

Now the argument of $\log(1-z)$ stays in $(-\pi/2, \pi/2$ and we conclude that

$$|g(z)| \leq e^{\pi/2} \quad : z \in D$$

Hence the g-function is bounded in D. Now

$$g(z) = \sum b_n z^n$$

and Newton's binomial formula gives:

$$|b_n| \leq \frac{C}{n} \quad : n \geq 1$$

Then it is clear that Theorem 7.2 applied to g gives (4) above.

8. Convergence under substitution.

Introduction. Let $\{a_k\}$ be a sequence of complex numbers where $\sum a_k$ is convergent. This gives an analytic function $f(z)$ defined in the open disc by

$$f(z) = \sum a_n \cdot z^n \tag{1}$$

If $0 < s < 1$ we can expand f around s and obtain another series

$$f(s+z) = \sum c_n \cdot z^n \tag{2}$$

From the convergence of $\sum a_k$ one expects that the series

$$\sum c_n \cdot (1-s)^n \tag{3}$$

also is convergent. This is indeed true and was proved by Hardy and Littlewood in (H-L]. A more general result was established in [Carleman] and we are going to expose results from Carleman's article. Let $f(z)$ be give as in (1) and consider another power series

$$\phi(z) = \sum b_\nu \cdot z^\nu \tag{1}$$

which represents an analytic function D where $|\phi(z)| < 1$ hold when $|z| < 1$. Then there exists the composed analytic function

$$f(\phi(z)) = \sum_{k=0}^{\infty} c_k \cdot z^k \tag{*}$$

We seek conditions on ϕ in order that the convergence of $\{a_k\}$ entails that the series

$$\sum c_k \quad \text{also converges} \tag{**}$$

First we consider the special case when the b-coefficients are real and non-negative.

8.1. Theorem. *Assume that $\{b_\nu \geq 0\}$ and that $\sum b_\nu = 1$. Then (**) converges and the sum is equal to $\sum a_k$.*

Proof. Since $\{b_\nu\}$ are real and non-negative the Taylor series for ϕ^k also has non-negative real coefficients for every $k \geq 2$. Put

$$\phi^k(z) = \sum B_{k\nu} \cdot z^\nu$$

and for each pair of integers k, p we set

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=p} B_{k\nu}$$

By assumption on $\{b_\nu\}$ we have $\phi 81) = 1$ which entials that $\phi^k(1) = 01$ for every $k \geq 2$ and hence

$$\sum_{\nu=0}^{\infty} B_{k\nu} = 1 \quad : \quad k \geq 1 \tag{i}$$

Sublemma. *The following hold for every fixed $p \geq 0$*

$$\lim_{N \to \infty} \Omega_{N,p} = 0 \quad \text{and} \quad k \mapsto \Omega_{k,p} \quad \text{decreases} \tag{ii}$$

The proofs of (ii-iii) are left as excercises to the reader.

Next, the Taylor series of the composed analytic function $f(\phi(z))$ becomes

$$\sum a_k \cdot \phi^k(z) = \sum_{\nu=0}^{\infty} [\sum_{k=0}^{\infty} a_k \cdot B_{k\nu}] \cdot z^\nu$$

For each positive integer n^* we set

(1) $$\sigma_p[n^*] = \sum_{\nu=0}^{\nu=p} \Big[\sum_{k=0}^{k=n^*} a_k \cdot B_{k,\nu} \Big]$$

(2) $$\sigma_p(n^*) = \sum_{\nu=0}^{\nu=p} \Big[\sum_{k=n^*+1}^{\infty} a_k \cdot B_{k,\nu} \Big] = \sum_{k=n^*+1}^{\infty} a_k \cdot \Omega_{k,p}$$

Notice that

$$\sigma_p[n^*] + \sigma_p(n^*) = \sum_{k=0}^{k=p} c_k \quad \text{hold for each} \quad p$$

Our aim is to show that the last partial sums have a limit. To obtain this we study the σ-terms separately. Introduce the partial sums

$$s_n = \sum_{k=0}^{k=n} a_k$$

By assumption there exists a limit $s_n \to S$ which entails that the sequence $\{s_k\}$ is bounded and so is the sequence $\{a_k = s_k - s_{k-1}\}$. The first limit formula in the Sublemma entails that the last term in (2) tends to zero when n^* increases. So if $\epsilon > 0$ we find n^* such that

(3) $$n \geq n^* \implies |\sigma_p(n)| \leq \epsilon$$

A study of $\sigma_p[n^*]$. Keeping n^* and ϵ fixed we apply (i) for each $0 \leq k \leq n^*$ and find an integer p^* such that

(4) $$1 - \sum_{\nu=0}^{\nu=p} B_{k,\nu} \leq \frac{\epsilon}{n^*+1} \quad \text{for all pairs} \quad p \geq p^* : \ 0 \leq k \leq n^*$$

The triangle inequality gives

(5) $$|\sigma_p(n^*) - s_{n^*}| \leq \frac{\epsilon}{n^*+1} \cdot \sum_{k=0}^{k=n^*} |a_k| \quad \text{for all} \quad p \geq p^*$$

Since $\sum a_k$ converges the sequence $\{a_k\}$ is bounded, i.e. we have a constant M such that $|a_k| \leq M$ for all k. Hence (4) gives

(6) $$|\sigma_p[n^*] - S| \leq |s_{n^*} - S| + \epsilon \cdot M \quad : \quad p \geq p^*$$

Together with (5) this entails that

$$n \geq n^* \implies |\sum_{k=0}^{n^*} |c_k - S| \leq \epsilon + |s_{n^*} - S| + M \cdot \epsilon$$

Above ϵ is arbitrary small and since n^* csn be chosen large while $s_{n^*} \to S$, we conclude that $\sum c_k$ converges and the limit is equal to S. This finishes the proof of Theorem 8.1.

Another result.

Now we relax the condition that $\{b_\nu\}$ are real and nonnegative but impose extra conditions on ϕ. First we assume that $\phi(z)$ extends to a continuous function on the closed disc, i.e. ϕ belongs to the disc-algebra. Moreover, $\phi(1) = 1$ while $|\phi(z)| < 1$ for all $z \in \bar{D} \setminus \{1\}$ which means that $z = 1$ is a peak point for ϕ. Consider also the function $\theta \mapsto \phi(e^{i\theta})$ where θ is close to zero. The final condition on ϕ is that there exists some positive real number β and a constant C such that

(1) $$|\phi(e^{i\theta}) - 1 - i\beta| \leq C \cdot \theta^2$$

holds in some interval $-\ell \leq \theta \geq \ell$. This gives for every integer $n \geq 2$ another constant C_n so that

(2) $$|\phi^n(e^{i\theta}) - 1 - in\beta| \leq C_n \cdot \theta^2$$

Hence the following integrals exist for all pairs of integers $p \geq 0$ and $n \geq 1$:

(3)
$$J(n.p) = \int_{-\ell}^{\ell} \frac{\phi(e^{i\theta})^n \cdot (1 - \phi(e^{i\theta})}{e^{ip\theta} \cdot (1 - e^{i\theta})} \cdot d\theta$$

With these notations one has

8.2. Theorem. *Let ϕ satisfy the conditions above. Then, if there exists a constant C such that*

(*)
$$\sum_{k=0}^{\infty} |J(k,p)| \leq C \quad \text{for all} \quad p \geq 0$$

it follows that the series (**) from the introduction converges and the sum is equal to $\sum a_k$.

Proof With similar notations as in the proof of Theorem 8.1 we introduce the Ω-numbers by:

$$\Omega_{k,p} = \sum_{\nu=0}^{\nu=k} B_{k\nu}$$

Repeating the proof of Theorem 8.1 the reader may verify that the series $\sum c_k$ converges and has the limit S if the following two conditions hold:

(i)
$$\lim_{N \to \infty} \Omega_{N,p} = 0 \quad \text{holds for every} \quad p$$

(ii)
$$\sum_{k=0}^{\infty} \left| \Omega_{k+1,p} - \Omega_{k,p} \right| \leq C \quad \text{for a constant} \quad C$$

where C is is independent of p. Here (i) is clear since $\{g_N(z) = \phi^N(z)\}$ converge uniformly to zero in compact subsets of the unit disc and therefore their Taylor coefficients tend to zero with N.

Proof of (ii). Residue calculus gives:

(iii)
$$\Omega_{k+1,p} - \Omega_{k,p} = \frac{1}{2\pi i} \int_{|z|=1} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz$$

Let ℓ be a small positive number and T_ℓ denotes the portion of the unit circle where $\ell \leq \theta \leq 2\pi - \ell$. Since 1 is a peak -point for ϕ there exists some $\mu < 1$ such that

$$\max_{z \in T_\ell} |\phi(z)| \leq \mu$$

This gives

(iv)
$$\frac{1}{2\pi} \cdot \Big| \int_{z \in T_\ell} \frac{\phi^k(z)}{z^{p+1}} \cdot \frac{1 - \phi(z)}{1 - z} \cdot dz \Big| \leq \mu^k \cdot \frac{2}{|e^{i\ell} - 1|} \Big|$$

Since the geometric series $\sum \mu^k$ converges it follows from (iii) and the construction of the J_ℓ-functions in Theorem 8.2 that (ii) is valid precisely when

$$\sum_{k=0}^{\infty} |J_\ell(k,p]| \leq C$$

hold for a constant which is independent of p. But this holds by the hypothesis (*) and finishes the proof of Theorem 8.2.

8.3. An oscillatory integrals. Condition (*) Theorem 8.2 is implicit. A sufficient condition that the J-integrals satisfy (*) can be expressed by conditions on the ϕ-function close to $z = 1$. To begin with the condition (1) above Theorem 8.2 entails that

$$\text{(i)} \qquad \phi(e^{i\theta}) = e^{i\beta\theta + \rho(\theta)}$$

holds in a neighborhood of $\theta = 0$ where the ρ-function behaves like big ordo of θ^2 when $\theta \to 0$. The next result gives the requested convergence of the composed series expressed by an additional condition on the ρ-function in (i) above.

8.4. Theorem. *Assume that $\rho(\theta)$ is a C^2-function on some interval $-\ell < \theta < \ell$ and that the second derivative $\rho''(0)$ is real and negative. Then (*) in Theorem 8.2 holds.*

The proof is left as an exercise to the reader. If necessary, consult Carleman's article [Car] which contains a detailed proof.

IX. The series $\sum \big[a_1\cdots a_\nu\big]^{\frac{1}{\nu}}$

We shall prove a result from [Carleman:xx. Note V page 112-115]. Let $\{a_\nu\}$ be a sequence of positive real numbers such that $\sum a_\nu < \infty$ and e denotes Neper's constant.

9.1 Theorem. *Assume that the series $\sum a_\nu$ is convergent and let S be the sum. Then one has the strict inequality*

$$(*) \qquad \sum_{\nu=1}^{\infty} \big[a_1\cdots a_\nu\big]^{\frac{1}{\nu}} < e\cdot S$$

Remark. The result is sharp in the sense that e cannot be replaced by a smaller constant. To see this we consider a large positive integer N and take the finite series $\{a_\nu = \frac{1}{\nu} \quad : 1\leq \nu \leq N\}$. Stirling's limit formula gives:

$$\big[a_1\cdots a_\nu\big]^{\frac{1}{\nu}} \simeq \frac{e}{\nu} \quad : \nu >> 1$$

Since the harmonic series $\sum \frac{1}{\nu}$ is divergent we conclude that for every $\epsilon>0$ there exists some large integer N such that $\{a_\nu = \frac{1}{\nu}\}$ gives

$$\sum_{\nu=1}^{\nu=N} \big[a_1\cdots a_\nu\big]^{\frac{1}{\nu}} > (e-\epsilon)\cdot \sum_{\nu=1}^{\nu=N} \frac{1}{\nu}$$

There remains to prove the strict upper bound (*) when $\sum a_\nu$ is an arbitrary convergent positive series. To attain this we first establish inequalities for finite series. Given a positive integer m we consider the variational problem

$$(1) \qquad \max_{a_1,\ldots,a_m} \sum_{\nu=1}^{\nu=m} \big[a_1\cdots a_\nu\big]^{\frac{1}{\nu}} \quad \text{when} \quad a_1+\ldots+a_m = 1$$

Let $a_1^*,\ldots,a_m^*$ give a maximum and set $a_\nu^* = e^{-x_\nu}$. The Lagrange multiplier theorem gives a number $\lambda^*(m)$ such that if

$$y_\nu = \frac{x_\nu+\ldots+x_m}{\nu}$$

then

$$(2) \qquad \lambda^*(m)\cdot e^{-x_\nu} = \frac{1}{\nu}\cdot e^{-y_\nu}+\ldots+\frac{1}{m}\cdot e^{-y_m} \quad : \quad 1\leq \nu\leq m$$

A summation over all ν gives

$$\lambda^*(m) = e^{-y_1}+\ldots+e^{-y_m} = \sum_{\nu=1}^{\nu=m} \big[a_1^*\cdots a_\nu^*\big]^{\frac{1}{\nu}}$$

Hence $\lambda^*(m)$ gives the maximum for the variational problem which is no surprise since $\lambda^*(m)$ is Lagrange's multiplier. Now we shall prove the strict inequality

$$(3) \qquad \lambda^*(m) < e$$

We prove (3) by contradiction. To begin with, subtracting the successive equalities in (2) we get the following equations:

$$(4) \qquad \lambda^*(m)\cdot\big[e^{-x_\nu} - e^{-x_{\nu+1}}\big] = \frac{1}{\nu}\cdot e^{-y_\nu} \quad : \quad 1\leq\nu\leq m-1$$

$$(5) \qquad m\cdot\lambda^*(m) = e^{x_m - y_m}$$

Next, set

$$(6) \qquad \omega_\nu = \nu\Big(1-\frac{a_{\nu+1}}{a_\nu}\Big): \quad 1\leq\nu\leq m-1$$

With these notations it is clear that (4) gives

$$(7) \qquad \lambda^*(m)\cdot\omega_\nu = e^{x_\nu - y_\nu} \quad : \quad 1\leq\nu\leq m-1$$

It is clear that (7) gives:

$$\big(\lambda^*(m)\cdot\omega_\nu\big)^\nu = e^{\nu(x_\nu-y_\nu)} = \frac{a_1\cdots a_{\nu-1}}{a_\nu^{\nu-1}} \tag{8}$$

By an induction over ν which is left to the reader it follows the ω-sequence satisfies the recurrence equations:

$$\omega_\nu^\nu = \frac{1}{\lambda^*(m)}\cdot\big(\frac{\omega_{\nu-1}}{1-\frac{\omega_{\nu-1}}{\nu-1}}\big)^{\nu-1} \quad : \quad 1\leq\nu\leq m-1 \tag{9}$$

Notice that we also have

$$\omega_1 = \frac{1}{\lambda^*(m)} \tag{10}$$

A special construction. With λ as a parameter we define a sequence $\{\mu_\nu(\lambda)\}$ by the recursion formula:

$$\mu_1(\lambda) = \frac{1}{\lambda} \quad \text{and} \quad [\mu_\nu(\lambda)]^\nu = \frac{1}{\lambda}\cdot\big[\frac{\mu_{\nu-1}(\lambda)}{1-\frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1} \quad : \quad \nu\geq 2 \tag{**}$$

From (5) and (9) it is clear that $\lambda = \lambda^*(m)$ gives the equality

$$\mu_m(\lambda^*(m)) = m \tag{***}$$

Now we come to the key point during the whole proof.

Lemma *If $\lambda\geq e$ then the $\mu(\lambda)$ -sequence satisfies*

$$\mu_\nu(\lambda) < \frac{\nu}{\nu+1} \quad : \quad \nu = 1,2,\ldots$$

Proof. We use an induction over ν. With $\lambda\geq e$ we have $\frac{1}{\lambda}<\frac{1}{2}$ so the case $\nu=1$ is okay. If $\nu\geq 1$ and the lemma holds for $\nu-1$, then the recursion formula (**) and the hypothesis $\lambda\geq e$ give:

$$[\mu_\nu(\lambda)]^\nu = \frac{1}{\lambda}\cdot\big[\frac{\mu_{\nu-1}(\lambda)}{1-\frac{\mu_{\nu-1}(\lambda)}{\nu-1}}\big]^{\nu-1} < \frac{1}{e}\cdot\big[\frac{\frac{\nu-1}{\nu}}{1-\frac{\nu-1}{\nu(\nu-1)}}\big]^{\nu-1}$$

Notice that the last factor is 1 and hence:

$$[\mu_\nu(\lambda)]^\nu < e < \big(1+\frac{1}{\nu}\big)^{-\nu}$$

where the last inequality follows from the wellknown limit of Neper's constant. Taking the ν:th root we get $\mu_\nu(\lambda) < \frac{\nu}{\nu+1}$ which finishes the induction.

Conclusion. If $\lambda^*(m)\geq e$ then the lemma above and the equality (***) would entail that

$$m = \mu(\lambda^*(m)) < \frac{m}{m+1}$$

This is impossible when m is a positive integer and hence we must have proved the strict inequality $\lambda^*(m) < e$.

The strict inequality for an infinite series. It remains to prove that the strict inequality holds for a convergent series with an infinite number of terms. So assume that we have an equality

$$\sum_{\nu=1}^\infty \big[a_1\cdots a_\nu\big]^{\frac{1}{\nu}} = e\cdot\sum_{\nu=1}^\infty a_\nu \tag{i}$$

Put as as above

$$\omega_n = n(1-\frac{a_{n+1}}{a_n}) \tag{ii}$$

Since we already know that the left hand side is at least equal to the right hand side one can apply Lagrange multipliers and we leave it to the reader to verify that the equality (i) gives the recursion formulas

$$\omega_n^n = \frac{1}{e}\cdot\big[\frac{\omega_{n-1}}{1-\frac{\omega_{n-1}}{n-1}}\big]^{n-1} \tag{iii}$$

Repeating the proof of the Lemma above it follows that

(iv)
$$\omega_n < \frac{n}{n+1} \implies \frac{a_{n+1}}{a_n} > \frac{n}{n+1}$$

where (ii) gives the implication. So with $N \geq 2$ one has:

$$\frac{a_{N+1}}{a_1} > \frac{1 \cdots N}{1 \cdots N(N+1)} = \frac{1}{N+1}$$

Now $a_1 > 0$ and since the harmonic series $\sum \frac{1}{N}$ is divergent it would follows that $\sum a_n$ is divergent. This contradiction shows that a strict inequality must hold in Theorem 9.1.

10. Thorin's convexity theorem.

Introduction. In the article [Thorin] a convexity theorem was established which goes as follows: Let $N \geq 2$ be a positive integer and $\mathcal{A} = \{A_{\nu k}\}$ a complex $N \times N$-matrix. To each pair of real numbers a, b in the square $\square = \{0 < a, b < 1\}$ we set

$$M(a,b) = \max_{x,y} |\sum\sum A_{\nu k} \cdot x_k \cdot y_\nu| \quad : \quad \sum |x_\nu|^{1/a} = \sum |y_k|^{1/b} = 1$$

10.1 Theorem *The function* $(a, b) \mapsto \log M(a, b)$ *is convex in* $\square$.

The proof relies upon Hadamard's inequality for maximum norms of bounded analytic functions in strip domains. More precisely, let $f(w)$ be an entire function which is bounded in the infinite strip domain

$$\Omega = \{\sigma + is \quad : 0 \leq \sigma \leq 1 : \; -\infty < s < \infty\}$$

Set

$$M_f(\sigma) = \max_s f(\sigma + is)| \quad : \quad 0 \leq \sigma \leq 1$$

Then the following is proved in § XX:

(*) $$M_f(\sigma) \leq M_f(0)^{1-\sigma} \cdot M_f(1)^\sigma$$

Proof of Theorem 10.1. With $0 < a, b < 1$ fixed we consider N-tuples $x_\bullet$ and $y_\bullet$ in $\mathbf{C}^N$ and write

$$x_\nu = c_\nu^a \cdot e^{i\theta_\nu} \quad \text{and} \quad y_k = d_k e^{i\phi_k}$$

where the c-and the d-numbers are real and positive whenever they are $\neq 0$. It is is clear that

(1) $$M(a,b) = \max_{c,d,\theta,\phi} |\sum\sum A_{\nu k} \cdot c_\nu^a \cdot d_k^b \cdot e^{i\theta_\nu} e^{i\phi_k}|$$

where the maximum is taken over N-tuples $\{c_\bullet\}$ and $\{d_\bullet\}$ of non-negative real numbers such that

(2) $$\sum c_\nu = \sum d_k = 1$$

and $\{\theta_\nu\}$ and $\{\phi_k\}$ are arbitrary N-tuples from the periodic interval $[0, 2\pi]$. Consider a pair (a_1, b_1) and (a_2, b_2) in $\square$ and let $(\bar{a}, \bar{b})$ be the middle point. Then we find $c^*, d^*, \theta^*, \phi^*$ so that

(3) $$M(\bar{a}, \bar{b}) = |\sum\sum A_{\nu k} \cdot (c_\nu^*)^a \cdot (d_k^*)^b \cdot e^{i\theta_\nu^* + i\phi_k^*}|$$

Let $w = \sigma + is$ be a complex variable and define the analytic function f by

(4) $$f(w) = \sum\sum A_{\nu k} \cdot (c_\nu^*)^{a_1 + w(a_2 - a_1)} \cdot (d_k^*)^{b_1 + w(b_2 - b_1)} \cdot e^{i\theta_\nu^* + i\phi_k^*}$$

It is clear that $f(w)$ is an entire analytic function and $|f(1/2)| = M(\bar{a}, \bar{b})$. Next, with $w = is$ purely imaginary we have

(5) $$f(is) = \sum\sum A_{\nu k} \cdot (c_\nu^*)^{a_1} \cdot (d_k^*)^{b_1} \cdot e^{is(a_2 - a_1)\log c_\nu^* + is(b_2 - b_1)\log d_k^*} \cdot e^{i\theta_\nu^* + i\phi_k^*}$$

For each pair ν, k the exponential product

$$e^{is(a_2 - a_1)\log c_\nu^* + is(b_2 - b_1)\log d_k^*} \cdot e^{i\theta_\nu^* + i\phi_k^*} = e^{i(\theta_\nu(s) + \phi_k(s))}$$

for some pair $\theta_\nu(s), \phi_k(s)$. From (1) we see that

(6) $$\max_s |f(is)| \leq M(a_1, b_1)$$

In the same way the reader can verify that

(7) $$\max_s |f(1 + is)| \leq M(a_2, b_2)$$

Now Hadamard's inequality (*) entails that

$$\log M(\bar{a}, \bar{b}) \leq \frac{1}{2} \cdot [\log M(a_1, b_1) + \log M(a_2, b_2)$$

This proves the required convexity.

11. Cesaro and Hölder limits

Introduction. In 1880 Cesaro introduced a certain summation procedure which which is a substitute for divergent series and leads to the notion of Cesaro summability to be defined below. Another summability was introduced by Hölder and later Knopp and Schnee proved that the conditions for Cesaro-respectively Hölder are equivalent. In Theorem 11.8 we present the elegant proof due to Schur taken from [Landau; Chapter 2]. For a given sequence of complex numbers $a_0, a_1, a_2, \ldots$ we put:

$$S_n = a_0 + \ldots + a_n$$

If $k \geq 0$ we define inductively

$$S_n^{(k+1)} = S_0^{(k)} + \ldots + S_n^{(k)} \quad \text{where} \quad S_n^{0)} = S_n$$

11.1 Definition. *For a given integer $k \geq 0$ we say that the sequence $\{a_n\}$ is Cesaro summable of order k if there exists a limit*

(*) $$s_*(k) = \lim_{n \to \infty} \frac{k!}{n^k} \cdot S_n^{(k)}$$

11.2 Exercise. Assume that $\{a_n\}$ is Cesaro summable of some order k. Show that

$$a_n = O(n^k)$$

11.3 The power series $f(x) = \sum a_n x^n$. Assume that $\{a_n\}$ is Cesaro summable of some order k. Exercise 11.2 shows that the power series $f(x)$ has a radius of convergence which is at least one and for every integer $k \geq 0$ the reader can verify the equality

(*) $$f(x) = \sum a_n x^n = (1-x)^{k+1} \cdot \sum S_n^{(k)} \cdot x^n$$

11.4 Exercise. Deduce from the above that if $\{a_n\}$ is Cesaro summable of some order k_* with limit value $s_*(k_*)$ then one has the limit formula:

$$s_*(k_*) = \lim_{x \to 1} f(x)$$

Now we prove that Cesaro summability of some order implies the summability for every higher order.

11.5 Proposition. *If (*) holds for some k_* then the Cesaro limit exists for every $k \geq k_*$ and one has the equality $s_*(k) = s_*(k_*)$.*

Proof. Cesaro summability of some order k with a limit $s_*(k)$ means that

(i) $$S_n^{(k)} = \frac{n^k}{k!} \cdot s_*(k) + o(n^k)$$

where the last term is small ordo. If (i) holds we get

$$S_n^{(k+1)} = \frac{s_*(k)}{k!} \sum_{\nu=0}^{n} n^\nu + o\big(\sum_{\nu=0}^{n} n^\nu\big) = \frac{s_*(k)}{k!} \cdot [\frac{n^{k+1}}{k+1} - 1] + o(n^{k+1})$$

From this the reader discovers the requested induction step and Proposition 11.5 follows.

11.6 Hölder's summation. To each sequence of complex numbers $a_0, a_1, a_2, \ldots$ we put

$$H_n^{(0)} = a_0 + \ldots + a_n$$

and if $k \geq 0$ we define inductively

$$H_n^{(k+1)} = \frac{H_0^{(k)} + \ldots + H_n^{(k)}}{n+1}$$

11.7 Definition. *The sequence $\{a_n\}$ is Hölder summable of order k if there exists a limit*

(**) $$\lim_{n \to \infty} H_n^{(k)}$$

11.8 Theorem *A sequence $\{a_n\}$ is Cesaro summable of of some order k if and only if it is Hölder summable of the same order and there respectively limits are the same.*

The proof of Theorem 11.8 requires several steps. First we introduce arithmetic mean value sequences attached to every sequence $\{x_0, x_1, \ldots\}$:

$$M(\{x_\nu\})[n] = \frac{x_0 + \ldots + x_n}{n+1}$$

Next, to each $k \geq 1$ we construct the sequence $T_k(\{x_\nu\})$ by

$$T_k(\{x_\nu\})[n] = \frac{k-1}{k} \cdot M(\{x_\nu\})[n] + \frac{x_n}{k}$$

So above M and $\{T_k\}$ are linear operators which send a complex sequence to another complex sequence. The reader may verify that these operators commute, i.e.

$$T_k \circ M = M \circ T_k$$

hold for every k and similarly the T-operators commute. For a given k we can also regard the passage to the Cesaro sequence $\{S_n^{[k)}\}$ as a linear operator which we denote by $\mathcal{C}^{(k)}$. Similarly we get the Hölder operators $\mathcal{H}^{(k)}$ for every $k \geq 1$.

11.9 Proposition. *The following identities hold*

(i) $$T_k \circ \mathcal{C}^{(k-1)} = M \circ \mathcal{C}^{(k)} \quad : \quad k \geq 1$$

(ii) $$\mathcal{H}^{(k)} = T_2 \circ \ldots \circ T_k \circ \mathcal{C}^{(k)} \quad : \quad k \geq 2$$

11.10 Exercise. Prove (i) and (ii) above.

As a last preparation towards the proof of Theorem 11.8 we need certain limit formulas which show that the T-operators have robust properties. First we have:

11.11 Lemma *Let $\{x_1, x_2, \ldots\}$ be a sequence of complex numbers and q a positive integer such that*

$$\lim_{n\to\infty} q \cdot \frac{x_1 + \ldots + x_n}{n} + x_n = 0$$

Then it follows that

$$\lim_{n\to\infty} x_n = 0$$

Proof. Set $y_n = q(x_1 + \ldots + x_n) + nx_n$. By an induction over n one verifies that

(1) $$\sum_{\nu=1}^{\nu=n} (\nu+1)\cdots(\nu+q-1) \cdot y_\nu = (n+1)\cdots(n+q) \cdot \sum_{\nu=1}^{\nu=n} x_\nu$$

hold for every $n \geq 1$. By the hypothesis $y_n = o(n)$ where $o(n)$ is small ordo of n. It follows that the left hand side in (1) is $o(n^{q+1})$ and since the product $(n+1)\cdots(n+q) \simeq n^q$ we conclude that

(2) $$\sum_{\nu=1}^{\nu=n} x_\nu = o(n)$$

Finally, we have

$$nx_n = y_n - q \cdot \sum_{\nu=1}^{\nu=n} x_\nu$$

and by (2) and the hypothesis the right hand side is $o(n)$ which after division with n gives $x_n = o(1)$ as required.

11.12 Proposition. *Let $\{x_\nu\}$ be a sequence and $k \geq 1$ an integer such that there exists*

$$\lim_{n\to\infty} T_k(\{x_\nu\}][n] = s$$

Then it follows that $\{x_n\}$ converges to s.

11.13 Exercise. Deduce Proposition 11.12 from Lemma 11.11.

11.14 Proof of Theorem 11.8.

The easy case $k = 1$ is left to the reader and we proceed to prove the theorem when $k \geq 2$. Assume first that $\{a_n\}$ is Cesaro summable of some order $k \geq 2$ with a limit s. Exercise 11.12 implies that $T_k \circ \mathcal{C}^{(k)}$ sends $\{a_n\}$ to a convergent sequence with limit s. If $k \geq 3$ we apply the exercise to T_{k-1} and continue until the composed operator

$$T_2 \circ \cdots \circ T_k \circ \mathcal{C}^{(k)}$$

sends the a-sequence to a convergent sequence with limit s. By (ii) in Proposition 11.8 this entails that $\{a_k\}$ is Hölder summable of order k with limit k. Conversely, assume that $\{a_n\}$ is Hölder summable of some order $k \geq 2$. The equality (ii) from Proposition 11.9 gives

$$\mathcal{H}^{(2)} = T_2 \circ \mathcal{C}^{(2)}$$

Hence Proposition 11.13 applied to T_2 shows that Hölder summability of order 2 entails Cesaro summability of the same order. Next, if $k \geq 3$ we again use (ii) in 11.9 and conclude that the sequence

$$T_3 \circ \ldots T_k \circ \mathcal{C}^{(k)}(\{a_n\})$$

is convergent. By repeated application of (ii) in 11.18 applied to $T_3, \ldots, T_k$ we conclude that the a-sequence is Cesaro summable of order k and has the same limit as the Hölder sum.

12. Power series and arithmetic means.

Consider a power series

$$f(x) = \sum a_n \cdot x^n$$

which converges when $|x| < 1$ and assume also that

(*) $$\lim_{x \to 1} \sum a_n \cdot x^n = 0$$

For each $k \geq 1$ we get the sequence $\{S_n^{(k)}\}$ from the previous section and we prove the following:

12.1 Theorem. *Assume (*) and that there exists some integer $k \geq 1$ such that*

$$\lim_{n \to \infty} S_n^{(k)} = 0$$

Then the series $\sum a_n$ converges.

Example. Consider the case $r = 1$ where

$$S_n^{(1)} = \frac{na_0 + (n-1)a_1 + \ldots + a_n}{n}$$

The sole assumption that $S_n^{(1)} \to 0$ does not imply $\sum a_n$ converges. But in addition (*) is assumed in Theorem 12.1 which will give the convergence. The proof of Theorem 12.1 is based upon the following convergence criterion where (*) above is tacitly assumed.

12.2 Proposition. *The series $\sum a_n$ converges if there to every $\epsilon > 0$ exists a pair (p_0, n_0) such that*

$$p \geq p_0 \implies J(n_0, p) = \Big| \int_0^1 \frac{\sin 2p\pi(x-1)}{x-1} \cdot \sum_{n=n_0}^{\infty} a_n x^n \cdot dx \Big| < \epsilon$$

Exercise. Prove this classic result which already was wellknown to Abel.

Proof of theorem 12.1. To profit upon Proposition 12.2 we need the two inequalities below which are valid for all pairs of positive integers p and n:

(i) $$\Big| \int_0^1 \sin(2p\pi x) \cdot x^k (1-x)^n \cdot dx \Big| \leq 2\pi(k+2)\,! \cdot \frac{p}{n^{k+2}}$$

(ii) $$\Big| \int_0^1 \sin 2p\pi x \cdot x^k (1-x)^n \cdot dx \Big| \leq \frac{C(k)}{p \cdot n^k}$$

where the constant $C(k)$ in (ii) as indicated only depends upon k. The verification of (i-ii) is left to the reader. Next, recall from (*) in § 11.4 that:

(iii) $$f(x) = \frac{(1-x)^{k+1}}{(k+1)!} \cdot \sum S_n^{(k)} n^k \cdot x^n$$

Let $\epsilon > 0$ and choose n_0 such that

(iv) $$n \geq n_0 \implies |S_n^{(k)}| < \epsilon$$

which is possible from the assumption in Theorem 12.1 Notice that (iii) gives the equality

(iii) $$\sum_{n=n_0}^{\infty} a_n x^n = \frac{(1-x)^{k+1}}{(k+1)!} \cdot \sum_{n=n_0}^{\infty} S_n^{(k)} n^k \cdot x^n$$

Hence (iv) and the triangle inequality shows that with n_0 kept fixed, the absolute value of the integral in Proposition 12.2 is majorized as follows for every p:

$$J(n_0, p) \leq \epsilon \cdot \sum_{n=n_0}^{\infty} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big|$$

In (iv) we have chosen n_0 and for an arbitrary $p \geq p_0 = n_0 + 1$ we decompose the sum from n_0 up to p and after we take a sum with $n \geq p+1$ which means that $J(n_0, p)$ is majorized by ϵ times the sum of the following two expressions:

$$\sum_{n=n_0}^{n=p} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big| \tag{1}$$

$$\sum_{n=p+1}^{\infty} \frac{n^k}{(k+1)!} \cdot \Big| \int_0^1 \sin(2p\pi x) \cdot x^k (x-1)^n \cdot dx \Big| \tag{2}$$

Using (i) above it follows that (1) is estimated by

$$2\pi \cdot (k+2)! \cdot \frac{C(k)}{p} \cdot (p - n_0) \leq 2\pi \cdot (k+2)! \cdot C(k) = K_1$$

Next, using (ii) it follows that (2) is estimated by

$$\pi \cdot \frac{k+2}{k+1} \cdot p \cdot \sum_{n=p+1}^{\infty} n^{-2} \leq \pi \cdot \frac{k+2}{k+1} = K_2$$

So with $K = K_1 + K_2$ we have

$$J(n_0, p) \leq 2K \cdot \epsilon$$

for every $p \geq n_0 + 1$ and since $\epsilon > 0$ was arbitrary the proof of Theorem 12.2 is finished.

13. Taylor series and quasi-analytic functions.

Introduction. Let $f(x)$ an infinitely differentiable function defined on the interval $[0,1]$. At $x=0$ we can take the derivatives and set

$$C_\nu = f^{(\nu)}(0)$$

In general the sequence $\{C_\nu\}$ does not determine $f(x)$. The standard example is the C^∞-function defined for $x>0$ by $e^{-1/x}$ and zero on $x \leq 0$. Here $\{C_\nu\}$ is the null sequence and yet the function is no identically zero. But if we impose sufficiently strong growth conditions on the derivatives of f over the whole interval $(-1,1)$ then $\{C_\nu\}$ determines f. In general, let $\mathcal{A} = \{\alpha_\nu\}$ be an increasing sequence of positive real numbers and denote by $\mathcal{C}_\mathcal{A}$ the class of C^∞-functions on $[0,1]$ where the maximum norms of the derivatives satisfy

$$(*) \qquad \max_x |f^{(\nu)}(x)| \leq k^\nu \cdot \alpha_\nu^\nu \quad : \quad \nu = 0,1,\ldots$$

for some $k>0$ which may depend upon f. One says that $\mathcal{C}_\mathcal{A}$ is a quasi-analytic class if every $f \in C_\mathcal{A}$ whose Taylor series is identically zero at $x=0$ vanishes identically on $[0,1)$. In the article [Denjoy 1921), Denjoy proved that $C_\mathcal{A}$ is quasi-analytic if the series

$$(**) \qquad \sum \frac{1}{\alpha_\nu} = +\infty$$

The conclusive result which gives a necessary and sufficient condition on the sequence $\{\alpha_\nu\}$ in order that $C_\mathcal{A}$ is quasi-analytic is proved in Carleman's book [1923]. The criterion is as follows:

Theorem. *Set $A_\nu = \alpha_\nu^\nu$ for each $\nu \geq 1$. Then $C_\mathcal{A}$ is quasi-analytic if and only if*

$$\int_1^\infty \log\,[\sum_{\nu=1}^\infty \frac{r^{2\nu}}{A_\nu^2}\,] \cdot \frac{dr}{r^2} = +\infty$$

For the proof of this result we refer to § XX in Special Topics.

The reconstruction theorem. Since quasi-analytic functions by definition are determined by their Tayor series at a single point there remains the question how to determine $f(x)$ in a given quasi-analytic class $C_\mathcal{A}$ when the sequence of its Taylor coefficients at $x=0$ are given. Such a reconstruction was announced by Carleman in [CR-1923] and the detailed proof appears in [Carleman-book]. Carleman considered a class of variational problems to attain the reconstruction. Let $n \geq 1$ and for a given sequence of real numbers $\{C_0,\ldots,C_{n-1}\}$ we consider the class of n-times differentiable functions f on $[0,1]$ for which

$$\text{(i)} \qquad f^{(\nu)}(0) = C_\nu \quad : \quad \nu = 0,\ldots,n-1$$

Next, let $\{\gamma_0,\gamma_1,\ldots,\gamma_n\}$ be some $n+1$-tuple of positive numbers and consider the variational problem

$$\text{(ii)} \qquad \min_f J_n(f) = \sum_{\nu=0}^{\nu=n} \gamma_\nu^{-2\nu} \cdot \int_0^1 [f^{(\nu)}(x)\,]^2 \cdot dx$$

where the competing family consist of n-times differentiable functions on $[0,1]$ satisfying (i) above. The strict convexity of L^2-norms entail that the variational problem has a unique minimizing function f_n which depends linearly upon $C_0,\ldots,C_n$. In other words, there exists a unique doubly indexed sequence of functions $\{\phi_{p,n}\}$ defined for pairs $0 \leq p \leq n$ such that

$$f_n(x) = \sum_{\nu=0}^{\nu=n-1} C_p \cdot \phi_{p,n-1}(x)$$

where the functions $\{\phi_{0,n-1},\ldots\phi_{n-1,n-1}\}$ only depend upon $\gamma_0,\ldots,\gamma_n$.

A specific choice of the γ-sequence. Let $\mathcal{A} = \{\alpha_\nu\}$ be a Denjoy sequence, i.e. (**) above diverges. Set $\gamma_0 = 1$ and

$$\gamma_\nu = \frac{1}{\alpha_\nu} \cdot \sum_{p=1}^{p=\nu} \alpha_p \quad : \quad \nu \geq 1$$

Given some $F(x) \in C^\infty[0,1]$ we get the sequence $\{C_\nu = F^{(\nu)}(0)\}$ and to each $n \geq 1$ we consider the variational problem above using the n-tuple $\gamma_0, \ldots, \gamma_{n-1}$ which yields the extremal function $f_n(x)$. With these notations Carelan proved the following:

13.1 Theorem. *If $F(x)$ belongs to the class $C_{\mathcal{A}}$ it follows that*

$$\lim_{n \to \infty} f_n(x) = F(x)$$

where the convergence holds uniformly on interval $[0, a]$ for every $a < 1$.

13.2 A series expansion. Using Theorem 13.1 Carleman also proved that when the series (**) diverges, then there exists a doubly indexed sequence $\{a_{\nu,n}\}$ defined for pairs $0 \leq \nu \leq n$ which only depends on the sequence $\{\alpha_\nu\}$ such that if $F(x)$ belongs to $\mathcal{C}_{\mathcal{A}}$ then it is given by a limit of series:

$$F(x) = \lim_{n \to \infty} \sum_{\nu=0}^{\nu=n} a_{\nu,n} \cdot \frac{F^{(\nu)}(0)}{\nu\,!} \cdot x^\nu \quad : \quad 0 \leq x < 1$$

Remark. Above we exposed the reconstruction for quasi-analytic classes of the Denjoy type. For a general quasi-analytic class a similar result is proved in [Carleman]. Here the proof and the result is of a more technical nature so we refrain to present the details. Concerning the doubly indexed a-sequence it is found in a rather implicit manner via solutions to the variational problems and an extra complication is that these a-numbers depend upon the given α-sequence. it appears that several open problems remain concerning effective formulas and the reader may also consult [Carleman: page xxx] for some open questions related to the reconstruction above.

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13.4 Quasi-analytic boundary values. Another problem is concerned with boundary values of analytic functions where the set of non-zero Taylor-coefficients is sparse. In general, consider a power series $\sum a_n z^n$ whose radius of convergence equal to one. Assume that there exists an interval ℓ on the unit circle such that the analytic function $f(z)$ defined by the series extends to a continuous function in the closed sector where $\arg(z) \in \ell$. So on ℓ we get a continuous boundary value function $f^*(\theta)$ and suppose that f^* belongs to some quasi-analytic class on this interval. Let f be given by the series

$$f = \sum a_n \cdot z^n$$

Suppose that gaps occur and write the sequence of non-zero coefficients as $\{a_{n_1}, a_{n_2} \ldots\}$ where $k \mapsto n_k$ is a strictly increasing sequence. With these notations the following result is due to Hadamard:

13.5 Theorem. *Let $f(z)$ be analytic in the open unit disc and assume it has a continuous extension to some open interval on the unit circle where the boundary function $f^*(\theta)$ is real-analytic. Then there exists an integer M such that*

$$n_{k+1} - n_k \leq M$$

for all k. In other words, the sequence of non-zero coefficients cannot be too sparse.

Hadamard's result was extended to the quasi-analytic case in [Carleman] where it is proved that if f^* belongs to some quasi-analytic class determined by a sequence $\{\alpha_\nu\}$ then the gaps cannot be too sparse, i.e. after a rather involved analysis one finds that f must be identically zero if the integer function $k \mapsto n_k$ increases too fast. The rate of increase depends upon $\{\alpha_\nu\}$ and it appears that no precise descriptions of the growth of $k \mapsto n_k$ which would ensure unicity is known for a general quasi-analytic class, i.e. even in the situation considered by Denjoy. So there remains many interesting open questions concerned with quasi-analyticity.