Some results about elliptic PDE-opertors.

Introduction. We expose material from two of Carleman's articles devoted to asymptotic distributions of eigenvalues to elliptic boundary value problems. The first section deals with the Laplace operator in the plane and in § B studies second order elliptic operators in $\mathbf{R}^3$ with variabe coefficients. The crucial steps in the proofs rely upon analytic functiuon theory and some ratehr explicit contructions where precise estimates for certain Green's functions are needed. in addition to this one derives asymptotic formulas using general facts about series, i.e. Tauberian theorems. Apart from the results in Theorem A.1 and Theorem B.1 the subsequent material is worth studying in detail by readers interested in PDE-theory since it contains constructions of fundasemtnal soilutions which rely upon solutions to integral equations via convergent Neumann series. The merit is that this gives sharp estimates for the fundamental solutions, and after also for Greens' functions while one reagrds eigenfunctions to an elliptic operator in bounded domains. Let us remark that in § B we treat elliptic operators which in general are not symmetric. The subsequent material is not entirely self-contained since we admit the existence of solutions to the Dirichlet problem for positive elliptic operators while Greens' functions are constructed. The existence of solution to the Dirchlet problem is exposed in many text-books so we refrain from further comments about this wellknown result. In addition the asymptotic formulas in Theorem A.1 and Theorem B.0 are derived after the "hard wortk" where various esitamtes are proved, together with a general *Taueberian Theorem* which also is taken for granted, i.e. here we only recall that this is also exposed in many text-.books and here we shall use one version of a Tauberian Theorem due to Norbert Wiener.

§ A. Eigenvalues and eigenfunctions for the Laplace operator in $\mathbf{R}^3$

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The result below were presented by Carleman at the Scandinavian Congress in mathematics held in Stockholm 1934: In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. For every fixed point $p \in \Omega$ one regards the continuous function

$$q \mapsto \log \frac{1}{|p-q|} \quad : q \in \partial\Omega$$

Thus gives a unique harmonic function $x \mapsto H(p,x)$ in Ω such that

$$H(p,q) + \frac{1}{|p-q|} = 0 \quad : q \in \partial\Omega$$

A wellknown fact established by G. Neumann and H. Poincaré during the years 1879-1895 shows that the H-fuyction is symmetric, i.e.

$$H(p,q) = H(q,p)$$

holds, and moreover it extends to a continou function on the product set $\overline{\Omega} \times \overline{\Omega}$. The Greens' function is defiuned by

$$G(p,q) = \log \frac{1}{|p-q|} + H(p,q)$$

For each fixed $p \in \Omega$, the function $q \mapsto G(p,q)$ is super-harmonic and zero when $q \in \partial\Omega$. Hence the minimum principe for superharmonic functions entails that

$$G(p,q) > 0 \quad : p,q \in \Omega$$

Next, it is obvious that

$$\iint_{\Omega\times\Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence the linear operator $\mathcal{G}$ on the Hilbert space $L^2(\Omega)$ defined by the symmetric kernel $G(p,q)$ is of Hilbert-Schmidt type and therefore compact on the Hilbert space $L^2(\Omega)$. Since the kernel positive the eigenvalues are positive, and wellknown facts about such nice integral operators give a sequence of pairwise orthogonal functions $\{\phi_n\}$ whose L^2-norms are one and

$$\mathcal{G}(\phi_n) = \mu_n \cdot \phi_n \tag{1}$$

where $\{\mu_n\}$ is a non-increasing sequence of positive eigenvalues which tend to zero. Moreover, since the kernel $G(p,q)$ is positive it follows - again by general facts - that $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$, i.e. each real-valued L^2-function f has an expansion

$$f = \sum a_n \cdot \phi_n \quad : a_n = \int_\Omega f_n(p) \cdot \phi_n(p)\, dp \tag{2}$$

Exercise. Verify that each ϕ-function extends to a continuous function on $\overline{\Omega}$ whose boundary values are zero.

Next, let Δ be the Laplace operator. Since $\frac{1}{2\pi} \cdot \log |z|$ is a fundamental solution, it follows that

$$\Delta \circ \mathcal{G}_f = -f \quad : f \in L^2(\Omega) \tag{3}$$

Set

$$\lambda_n = \mu_n^{-1}$$

Then (1) and (3) give

$$\Delta(\phi_n) = -\lambda_n \cdot \phi_n \quad : n = 1, 2, \ldots \tag{4}$$

where we now have $0 < \lambda_1 \leq \lambda_2 \leq \ldots \}$.

After these prelminary remarks from classical theory we announce Carleman's theorem.

A.1. Theorem. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N\to\infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n(p)^2 = \frac{1}{4\pi} \tag{*}$$

To prove this we consider some point $p \in \Omega$. Since every $\phi_n$is harmonic and has L^2-norm one, the resder can check that eith a fixed p there exists a constant $C(p)$ such that

$$\phi_n(p)^2 \leq C(p) \quad : p = 1, 2, \ldots$$

Hence there exists the Dirichlet series

$$\Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n(p)^2}{\lambda_n^s}$$

which is analytic in the half-space $\mathfrak{Re}\, s > 1$. Less trivial is the following:

A. 2 Theorem. *There exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Let us first remark that Theorem A.2 gives Theorem A.1 by a general result due to Norbert Wiener in the article *Tauberian theorem* [Annals of Math.1932]. His theorem asserts that if $\{\lambda_n\}$ is a non-decreasing sequence of positive numbers which tends to infinity and $\{a_n\}$ are non-negative real numbers such that there exists the limit

$$\lim_{s\to 1} (s-1) \cdot \sum \frac{a_n}{\lambda_n^s} = A$$

then it follows that

$$\lim_{n\to\infty} \lambda_n^{-1} \cdot \sum_{k=1}^{k=n} a_k = A$$

Proof of Theorem A. 2

Since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

(i) $$\sum \lambda_n^{-2} < \infty$$

This convergence entails that various constructions below are defined. For each λ outside the discrete set $\{\lambda_n\}$ we put

(ii) $$G(p,q;\lambda) = G(p,q) + 2\pi\lambda \cdot \sum_{n=1}^{\infty} \frac{\phi_n(p)\cdot\phi_n(q)}{\lambda_n(\lambda-\lambda_n)}$$

This gives the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

(iii) $$\mathcal{G}_\lambda(f)(p) = \frac{1}{2\pi}\cdot\iint_\Omega G(p,q;\lambda)\cdot f(q)\,dq$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta+\lambda)\cdot\mathcal{G}_\lambda = -E$$

B. The function $F(p,\lambda)$. Set

$$F(p,q,\lambda) = G(p,q;\lambda) - G(p,q)$$

Keeping p fixed we see that (ii) gives

(B.1) $$\lim_{q\to p} F(p,q,\lambda) = 2\pi\lambda\cdot\sum_{n=1}^{\infty}\frac{\phi_n(p)^2}{\lambda_n(\lambda-\lambda_n)}$$

Set

(B.2) $$F(p,\lambda) = \lim_{q\to p} F(p,q,\lambda)$$

From (i) and (B.1) it follows that (B.2) is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0 < a < \lambda_1$. Use residue calculus to show the equality below in the half-space $\mathfrak{Re}\, s > 2$:

(C.1) $$\Phi(s) = \frac{1}{4\pi^2\cdot i}\cdot\int_{a-i\infty}^{a+i\infty} F(p,\lambda)\cdot\lambda^{-s}\,d\lambda$$

where the line integral is taken on the vertical line $\mathfrak{Re}\,\lambda = a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda) < a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty, a]$ followed by the upper half-circle $\{\lambda = ae^{i\theta} : 0 \leq \theta \leq \pi\}$ and the half-line $\{\lambda = a + it : t \geq 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+} F(p,\lambda)\cdot\lambda^{-s}\,d\lambda = 0$$

We leave it to the reader to contruct the similar curve $\gamma_- = \bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *One has the equality*

(E.1) $$\Phi(s) = \frac{a^{s-1}}{4\pi}\cdot\int_{-\pi}^{\pi} F(ae^{i\theta})\cdot e^{(i(1-s)\theta}\,d\theta + \frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty F(p,-x)\cdot x^{-s}\,dx$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So there remains to prove that

(E.2) $$s \mapsto \frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty F(p,-x)\cdot x^{-s}\,dx$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To attain this we express $F(p, -x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^\infty \frac{e^{-zt}}{\sqrt{t^2-1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$I_0(z) = \sum_{m=0}^\infty \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \tag{i}$$

$$I_1(z) = \sum_{m=0}^\infty \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma \tag{ii}$$

where γ is the usual Euler constant.

With p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p, q; \kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p, q; \kappa) = K(\sqrt{\kappa}|p - q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p, q; -\kappa) = K(\sqrt{\kappa} \cdot |p - q|) - H(q; \kappa) \quad : \kappa > 0$$

Next, the construction of $G(p, q)$ gives

$$F(p, -\kappa) = \lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q|] + \lim_{q \to p} [u_p(q) + H(p, q, \kappa)] \tag{G.1}$$

The last term above has the "nice limit" $u_p(p) + H(p, p, \kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q|] = -\frac{1}{2} \cdot \log \kappa + \log 2 - \gamma \tag{G.2}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + u_p(p)$. Then (G.1) and (G.2) give

$$F(p, -\kappa) = -\frac{1}{2} \cdot \log \kappa + A + H(p, p; -\kappa)$$

With $x = \kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty x^{-s}\, dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2} \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty \log x \cdot x^{-s}\, dx$$

Notice that the complex derivative

$$\frac{d}{ds} \int_a^\infty x^{-s}\, dx = -\int_a^\infty \log x \cdot x^{-s}\, dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem D.2 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty H(p,p,\kappa) \cdot \kappa^{-s}\, d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \leq H(p,q;\kappa) \leq \max_{q \in \partial\Omega} K(\kappa|p-q|)$$

With $p \in \Omega$ fixed there is a positive number δ such that $|p-q| \geq \delta : q \in \partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.

A final remark. As pointed out by Carleman in his cited article *La méthode dont nous sommes servis est aussi applicable à une équation elliptique à un nombre quelconque de dimensions.* The reader is for example invited to find the companion to Theorem A.1 for Dirichlet regular domains in $\mathbf{R}^n$ when $n \geq 3$.

§ B. Elliptic operators in $\mathbf{R}^3$

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Consider an elliptic operator

$$(*) \qquad L = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x)\cdot\frac{\partial^2}{\partial x_p\partial x_q} + \sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p} + a_0(x)$$

where the a-functions are real-valued and defined in a neighborhood of the closure of a bounded domain Ω with a C^1-boundary and satisfy the symmetry

$$a_{pq} = a_{qp}$$

Moreover, we assume that $\{a_{pq}\}$ are of class C^2, $\{a_p\}$ of class C^1 and a_0 is continuous. The elliptic property of L means that for every $x \in \overline{\Omega}$ the eigenvalues of the symmetric matrix $A(x)$ with elements $\{a_{pq}(x)\}$ are positive. Under these conditions, a result which goes back to work by Neumann and Poincaré, gives a positive constant κ_0 such that if $\kappa \geq \kappa_0$ then the inhomogeneous equation

$$(0,1) \qquad L(u) - \kappa^2\cdot u = f \quad : f \in L^2(\Omega)$$

has a unique solution u which is a C^2.-function in Ω and extends to the closure where it is zero on $\partial\Omega$. Moreover, there exists some κ_0 and for each $\kappa \geq \kappa_0$ a Green's function $G(x, y; \kappa)$ such that

$$(0.2) \qquad (L-\kappa^2)(\frac{1}{4\pi}\cdot\int_\Omega G(x,y;\kappa)\, f(y)\, dy) = -f(x) \quad : f \in L^2(\Omega)$$

This means that the bounded linear operator on $L^2(\Omega$ defined by

$$(0.3) \qquad f \mapsto -\frac{1}{4\pi}\cdot\int_\Omega G(x,y;\kappa)\, f(y)\, dy$$

is Neumann's resolvent to the densely defined operator $L - \kappa^2$ on the Hilbert space $L^2(\Omega)$. The discrete sequence of eigenvalues $\{\lambda_n\}$ are in general and arranged with non-decreasing the absolute values. Notice that the operator L in general is not symmetric since first order variabla coefficients appear in (*). However, the eigenvalues are not too far from real numbers. More precisely, in § xx we show that that there exist positive constants C and c such that

$$(0.4) \qquad |\mathfrak{Im}(\lambda_n)| \leq C\cdot(\mathfrak{Re}(\lambda_n) + c)$$

hold for every n. Next, the elliptic hypothesis means that the determinant function

$$(0.5) \qquad D(x) = \det(a_{p,q}(x))$$

is positive in Ω. With these notations one has

B.0 Theorem. *The following limit formula holds:*

$$(*) \qquad \lim_{n\to\infty}\frac{\mathfrak{Re}(\lambda_n)}{n^{\frac{2}{3}}} = \frac{1}{6\pi^2}\cdot\int_\Omega \frac{1}{\sqrt{D(x)}}\, dx$$

Remark. This asymptotic formula above was established by Courant and Weyl when L is symmetric and was extended to non-symmetric operators by Carleman during lectures at Institute Mittag-Leffler in 1935. Weyl and Courant used calculus of variation while Carleman employed different methods which have the merit that the passage to the non-symmetric case does not cause any trouble. As pointed out by Carleman the methods in the proof give similar asymptotic formulas in other boundary value problems such as those considered by Neumann where one imposes boundary value conditions on outer normals, and so on. A crucial step during the proof of the theorem above relies upon *constructions of a fundamental solutions* and we start with this while the proof of Theorem B.0 is postponed until § XX. Let us remark that the subsewu8ent material includes a proof of the existence of Green's function and the cited results above from original work by Carl Neumann and Henri Poincaré.

§ B.1. Fundamental solutions to second order Elliptic operators.

Consider as an elliptic operator

$$
\text{(B.1.1)} \qquad L=\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq}(x)\cdot \frac{\partial^2}{\partial x_p\partial x_q}+\sum_{p=1}^{p=3} a_p(x)\frac{\partial}{\partial x_p}+a_0(x)
$$

where a-functions are real-valued and one has the symmetry $a_{pq}=a_{qp}$. To ensure existence of a *globally defined* fundamental solutions we suppose the a-functions are defined in the whole space $\mathbf{R}^3$ and the following limit formulas hold as $|x|\to\infty$ in $\mathbf{R}^3$.

$$
\lim a_\nu(x)=0\colon 0\leq p\leq 3 \quad : \quad \lim a_{pq}(x)=\text{Kronecker's delta function}
$$

Thus, L approaches the Laplace operator as $|x\to+\infty$. The elliptic property means that the eigenvalues of the symmetric matrix with elements $\{a_{pq}(x)\}$ are positive for every x. Next, recall the notion of fundamental solutions. First the adjoint operator is defined by:

$$
\text{(0.1)} \qquad L^*(x,\partial_x)=L-2\cdot(\sum_{p=1}^{p=3}(\sum_{q=1}^{q=3}\frac{\partial a_{pq}}{\partial x_q})\cdot\frac{\partial}{\partial x_p}-\sum_{p=1}^{p=3}\frac{\partial a_p}{\partial x_p}+2\cdot\sum\sum\frac{\partial^2 a_{pq}}{\partial x_p\partial x_q}
$$

Partial integration gives the equation below for every pair of C^2-functions ϕ,ψ in $\mathbf{R}^3$ with compact support:

$$
\text{(0.2)} \qquad \int L(\phi)\cdot\psi\,dx=\int\phi\cdot L^*(\psi)\,dx
$$

with volume integrals taken over $\mathbf{R}^3$. By definition a locally integrable function $\Phi(x)$ in $\mathbf{R}^3$ is a fundamental solution to $L(x,\partial_x)$ if

$$
\text{(0.3)} \qquad \psi(0)=\int\Phi\cdot L^*(\psi)\,dx
$$

hold for every C^2-function ψ with compact support. Next, to each positive number κ we get the PDE-operator $L-\kappa^2$ and a function $x\mapsto\Phi(x;\kappa)$ is a fundamental solution to $L-\kappa^2$ if

$$
\text{(0.4)} \qquad \psi(0)=\int\Phi(x:\kappa)\cdot(L^*-\kappa^2)(\psi(x))\,dx
$$

hold for compactly supported C^2-functions ψ. Finally, the origin can replaced by a variable point ξ in $\mathbf{R}^3$ and then one seeks a function $\Phi(x,\xi;\kappa)$ with the property that

$$
\text{(0.5)} \qquad \psi(\xi)=\int\Phi(x,\xi;\kappa)\cdot(L^*(x,\partial_x)-\kappa^2)(\psi(x))\,dx
$$

hold for all $\xi\in\mathbf{R}^3$ and every C^2-function ψ with compact support. Keeping κ fixed this means that $\Phi(x,\xi;\kappa)$ is a function of 6 variables defined in $\mathbf{R}^3\times\mathbf{R}^3$. We are going to construct $\Phi(x,\xi;\kappa)$ in a canonical way, starting from Isaac Newton's formulas for elliptic operators with constant coefficients.

1. The case with constant coefficients.

Following contructions from Newton's famous text-books from 1666. Suppose thst the a-funvtions in (B.1.1) are constants. In particular we have the positive and symmetric 3×3-matrix $A=\{a_{pq}\}$. Let $B=\{b_{pq}\}$ be the inverse matrix and define the quadratic form

$$
B(x)=\sum_{p,q} b_{pq}x_px_q
$$

and noticr that it is positive definite. next, put

$$
\alpha=\sqrt{\kappa^2+\frac{1}{2}\sum_{p,q} b_{pq}a_pa_q-a_0}
$$

where κ is chosen so large that the term under the square-root is > 0. With κ chosen as above this gives the locally integrable function

$$H(x;\kappa) = \frac{1}{4\pi \cdot \sqrt{\Delta \cdot B(x)}} \cdot e^{-\alpha\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}a_p \cdot x_q} \tag{1.1}$$

Exercise. Verify via Stokes formula that $H(x;\kappa)$ yields a fundamental solution to the PDE-operator

$$L(\partial_x) - \kappa^2 = \sum_{p=1}^{p=3}\sum_{q=1}^{q=3} a_{pq} \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} a_p \frac{\partial}{\partial x_p} + a_0 - \kappa^2$$

1.2 The case with variable coefficients.

Now L is given as in (B.1.1) with varaible coeffcients. For each $\xi \in \mathbf{R}^3$ the elements of the inverse matrix to $\{a_{pq}(\xi)$ are denoted by $\{b_{pq}(\xi)\}$. Thee assumption in (B.1.2) entails that there exists some $\kappa_0 > 0$ such that

$$\kappa_0^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi) - b(\xi) > 0 \quad \text{hold for all} \quad \xi \in \mathbf{R}^3 \tag{1.2.1}$$

Next, for every $\kappa \geq \kappa_0$ we set

$$\alpha_\kappa(\xi) = \sqrt{\kappa^2 + \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)a_q(\xi) - b(\xi)} \tag{i}$$

Following Newton's construction in (1.1) we put:

$$H(x,\xi;\kappa) = \frac{1}{4\pi} \cdot \frac{\sqrt{\Delta(\xi)}^{-\frac{1}{2}}}{\sqrt{\sum_{p,q} b_{pq}(\xi) \cdot x_p x_q}} \cdot e^{-\alpha_\kappa(\xi)\sqrt{B(x)} - \frac{1}{2}\sum_{p,q} b_{pq}(\xi)a_p(\xi)\cdot x_q} \tag{ii}$$

When ξ is kept fixed this function of x is real analytic outside the origin and we notice that $x \to H(x,\xi;\kappa)$ is locally integrable as a function of x in a neighborhood of the origin. We are going to find a fundamental solution satisfying (0.5) which takes the form

$$\Phi(x,\xi;\kappa) = H(x-\xi,\xi;\kappa) + \int_{\mathbf{R}^3} H(x-y,\xi;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy \tag{1.2.2}$$

where the Ψ-function is the solution to an integral equation to be given (1.5) below. But first we need some further constructions.

1.3 The function $F(x,\xi;\kappa)$. For every fixed ξ we consider the differential operator in the x-space:

$$L_*(x,\partial_x,\xi;\kappa) =$$

$$\sum_{p=1}^{p=3}\sum_{q=1}^{q=3} (a_{pq}(x) - (a_{pq}(\xi)) \cdot \frac{\partial^2}{\partial x_p \partial x_q} + \sum_{p=1}^{p=3} (a_p(x) - a_p(\xi))\frac{\partial}{\partial x_p} + (b(x) - b(\xi))$$

With ξ fixed we apply L_* to the function $x \mapsto H(x-\xi,\xi;\kappa)$ and put

$$F(x,\xi;\kappa) = \frac{1}{4\pi} \cdot L_*(x,\partial_x,\xi;\kappa)(H(x-\xi,\xi,\kappa)) \tag{1.3.1}$$

1.4 Two estimates. The limit conditions in (0.0) give positive constants C, C_1 and k such that the following hold when $\kappa \geq \kappa_0$:

$$|H(x-\xi,\xi;\kappa)| \leq C \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} \quad : \quad [F(x,\xi;\kappa)| \leq C_1 \cdot \frac{e^{-k\kappa|x-\xi|}}{|x-\xi|^2} \tag{1.4.1}$$

The verification of (1.4.1) is left as an exercise.

1.5 An integral equation.

With F defined in (1.3.1) we shall prove the following:

1.5.0 Theorem. *There exists a positive number κ_0^* such that the integral equation below has a solution for each $\kappa \geq \kappa_0^*$*

$$\Psi(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot \Psi(y,\xi;\kappa)\, dy + F(x,\xi;\kappa) \tag{1.5.1}$$

Proof. We construct the Neumann series of F. Thus, starting with $F^{(1)} = F$ we set

$$F^{(\nu)}(x,\xi;\kappa) = \int_{\mathbf{R}^3} F(x,y;\kappa) \cdot F^{(\nu-1)}(y,\xi;\kappa)\, dy \quad : \quad \nu \geq 2 \tag{1.5.2}$$

The last inequality in (1.4.1) gives

$$|F^{(2)}(x,\xi;\kappa)| \leq C_1^2 \iiint \frac{e^{-k\kappa|\xi-y|}}{|x-y|^2 \cdot |\xi-y|^2} \cdot dy \tag{i}$$

To estimate (i) we notice that the triple integral after the substitution $y - \xi \to u$ becomes

$$C_1^2 \iiint \frac{e^{-k\kappa|u|^2}}{|x-u-\xi|^2 \cdot |u|^2} \cdot du \tag{ii}$$

In (ii) the volume integral can be integrated in polar coordinates and becomes

$$C_1^2 \cdot \int_0^\infty \int_{S^2} \frac{e^{-k\kappa r^2}}{|x - r\cdot w - \xi|^2} \cdot dwdr \tag{iii}$$

where S^2 is the unit sphere and dw the area measure on S^2 and we see that (iii) becomes

$$2\pi C_1^2 \cdot \int_0^\infty \int_0^\pi \frac{e^{-k\kappa r}}{(x-\xi)^2 + r^2 - 2r \cdot |x-\xi| \cdot \sin\theta} \cdot d\theta dr =$$

$$\frac{2\pi C_1^2}{|x-\xi|} \cdot \int_0^\infty e^{-k\kappa|x-\xi|t} \cdot \log|\frac{1+t}{1-t}| \cdot \frac{dt}{t} \tag{iv}$$

where the last equality follows by a straightforward computation.

1.6 Exercise. Show that (iv) gives the estimate

$$|F^{(2)}(x,\xi;\kappa)| \leq \frac{2\pi \cdot C_1^2 \cdot C_1^*}{\kappa \cdot |x-\xi|^2}$$

where C_1^* is a fixed positive constant which is independent of x and ξ and show by an induction over n that one has:

$$|F^{(n)}(x,\xi;\kappa)| \leq \frac{C_1}{|x-\xi|^2} \cdot [\frac{2\pi C_1^2 \cdot C_1^*}{\kappa}]^{n-1} \quad \text{hold for every} \quad n \geq 2 \tag{1.6.1}$$

The choice of κ_0^*. It is taken so that κ_0^* so large that

$$2\pi C_1^2 \cdot C_1^* < \kappa_0^*$$

Then (1.6.1) entails that the Neumann series

$$\sum_{n=1}^\infty F^{(n)}(x,\xi;\kappa)$$

converges when $\kappa \geq \kappa_0^*$ and gives the requested solution $\Psi(x,\xi;\kappa)$ in (1.5.1).

1.7 Conclusion. Above we have found Ψ which satisfies the integral equation in § 1.5.1 Using Green's formula the reader may verify:

1.8 Proposition. *With Ψ as above it follows that the function $\Phi(x,\xi;\kappa)$ defined in (1.2.2) is a fundamental solution of $L(x,\partial_x)-\kappa^2$.*

1.9 Some estimates.

The constructions above show that the functions

$$x \mapsto \Phi(x,\xi;\kappa) \quad \text{and} \quad x \mapsto H(x-\xi,\xi;\kappa)$$

have the same singularities at $x=\xi$. Consider the difference

$$G(x,\xi;\kappa) = \Phi(x,\xi;\kappa) - H(x-\xi,\xi;\kappa) \tag{1.9.1}$$

1.9.2 Exercise. Use the previous constructions to show that for every $0<\gamma\leq 2$ there is a constant C_γ such that

$$\big|G(x,\xi;\kappa)\,\big| \leq \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for every pair (x,ξ) and every $\kappa\geq\kappa_0$. Together with the the inequality for the H-function in (1.4.1) we arrive at the following estimate for the fundamental solution Φ.

1.9.3 Theorem. *With κ_0^* chosen from the proof of Theorem 1.5.0 there exist positive constants C and k and for each $0<\gamma\leq 2$ a constant C_γ such that*

$$|\Phi(x,\xi;\kappa)| \leq C\cdot\frac{e^{-k\kappa|x-\xi|}}{|x-\xi|} + \frac{C_\gamma}{(\kappa|x-\xi|)^\gamma}$$

hold for all pairs (x,ξ) in $\mathbf{R}^3$ and every $\kappa\geq\kappa_0^*$.

Remark. Above C and k are independent of κ as soon as κ_0^* has been chosen as above. The size of these constant depend on the C^2-norms of the functions $\{a_{pq}(x)\}$ and as well as the C^1-norms of $\{a_1,a_2,a_3\}$ and the maximum norm of a_0. The merit in Theorem 1.9.3 is that one gets a control both in a finite region as well as the behaviour of Φ when $|x-\xi|$ gets large where one has the damping exponential factor which is useful during constructions of Green's functions for exterior boundary value problems. Notice that the whole construction is canonical. Let us finally remark that similar constructions as above can be carried out for elliptic operators of even degree $2m$ when $m\geq 2$. Here Newton's solution for constant coefficients is replaced by those of Fritz John which arise via the wave deompostion of the Dirac measure. It would be interesting to analyze the resulting version of Theorem 1.9.3, i.e. to exhibit estimates of a similar nature when $m\geq 2$. Of course, one can also extend everything to elliptic operators of n variables were $n\geq 4$ in which case the denominator $|x-\xi|^{-1}$ is replaced by $|x-\xi|^{-n+2}$, starting from Newton's fundamental solution for the case of second order elliptic operators with constant constant coefficients in $\mathbf{R}^n$.

§ B.2. Green's functions.

Let Ω be a bounded domain in $\mathbf{R}^3$. A Green's function $G(x,y;\kappa)$ attached to this domain and the elliptic PDE-operator $L(x,\partial_x)$ is a function which for fixed κ is a function in $\Omega \times \Omega$ with the following properties:

$$(*) \qquad G(x,y;\kappa) = 0 \quad \text{when} \quad x \in \partial\Omega \quad \text{and} \quad y \in \Omega$$

$$(**) \qquad \psi(y) = \int_\Omega (L^*(x,\partial_x) - \kappa^2)(\psi(x)) \cdot G(x,y;\kappa)\, dx \quad : \quad y \in \Omega$$

hold for all C^2-functions ψ with compact support in Ω. To find G one solves a Dirchlet problem. With $\xi \in \Omega$ kept fixed we have a continuous function on $\partial\Omega$: $x \mapsto \Phi(x,\xi;\kappa)$. Solving Dirchlet's problem gives a unique C^2-function $w(x)$ which satisfies:

$$L(x,\partial_x)(w) + \kappa^2 \cdot w = 0 \quad \text{holds in} \quad \Omega \quad \text{and} \quad w(x) = \Phi(x,\xi;\kappa) \quad : x \in \partial\Omega = 0$$

From the above we get the requested Greens'-function, i.e. the reader can check the result below.

2.1 Proposition. *The function* $G(x,\xi;\kappa) = \Phi(x,\xi;\kappa) - w(x)$ satisfies $(* - **)$

Next, using the estimates for the Φ-function in Theorem 1.9.3 we shall establish estimates for the G-function above where we start with a sufficiently large κ_0 so that $\Phi^*(x,\xi;\kappa_0)$ is a positive function of (x,ξ).

2.2 Theorem. *One has*

$$G(x,\xi;\kappa_0) = \frac{1}{\sqrt{\Delta(x)} \cdot \sqrt{\Phi(x,\xi;\kappa_0)}} + R(x,\xi)$$

where the remainder function satisfies the following for all pairs (x,ξ) in Ω:

$$|R(x,\xi)| \leq C \cdot |x-\xi|^{-\frac{1}{4}}$$

with a constant C which depends on the domain Ω and the PDE-operator P.

2.3 Exericse. Above the negative power of $|x-\xi|$ is a fourth-root which means that the remainder term R is more regular compared to the first term which behaves like $|x-\xi|^{-1}$ on the diagonal $x=\xi$. The proof is left to the reader. If necessary, consult [Carleman: page 125-127].

2.4 The integral operator $\mathcal{J}$. Theorem 2.2 enable us to study the integral operator which sends a function u in Ω to

$$\mathcal{J}_u(x) = \int_\Omega G(x,\xi;\kappa_0) \cdot u(\xi)\, d\xi$$

The construction of the Green's function gives:

$$(2.4.1) \qquad (L(x,\partial_x) - \kappa_0^2)(\mathcal{J}_u)(x) = u(x) \quad : \quad x \in \Omega$$

Thus, if E denotes the identity we have the operator equality

$$(2.4.2) \qquad L(x,\partial_x) \circ \mathcal{J}_u = \kappa_0^2 \cdot \mathcal{J} + E$$

Consider a real number γ and some u-function which satisfies:

$$(2.4.3) \qquad u(x) + \gamma \cdot \mathcal{J}_u(x) = 0 \quad : \quad x \in \Omega$$

The vanishing from (*) in § B.2 for the G-function implies that $J_u(x) = 0$ on $\partial\Omega$. Hence every u-function which satisfies in (2.4.3) for some constant γ vanishes on $\partial\Omega$. Next, when L is applied to (2.4.3) the operator equation (2.4.2) gives

$$0 = P(u) + \gamma \cdot \kappa_0^2 \cdot \mathcal{J}_u + \gamma \cdot u \implies P(u) + (\gamma - \kappa_0^2)u = 0$$

2.4.4 Conclusion. The boundary value problem (*) from 0.B is equivalent to find eigenfunctions of $\mathcal{J}$ via (2.4.3) above.

3. Almost reality of eigenvalues.

Consider the set of eigenvalues λ to (*) in (0.B). Then we have:

3.1 Proposition. *There exist positive constants C_* and c_* such that every eigenvalue λ to (*) in (0.B) satisfies*

$$|\mathfrak{Im}\,\lambda|^2 \leq C_*(\mathfrak{Re}\,\lambda) + c_*)$$

Proof. Let u be an eigenfunction where $P(u) + \lambda \cdot u = 0$. Stokes theorem and the vanishing of $u|\partial\Omega$ give:

$$0 = \int_\Omega \bar{u} \cdot (P+\lambda)(u)\,dx = -\int_\Omega \sum_{p,q} a_{pq}(x) \cdot \frac{\partial u}{\partial x_p}\frac{\partial \bar{u}}{\partial x_q}\,dx + \int_\Omega \bar{u} \cdot (\sum a_p(x)\frac{\partial u}{\partial x_p}\,)\,dx +$$

$$\int_\Omega |u(x)|^2 \cdot b(x)\,dx + \lambda \cdot \int |u(x)|^2\,dx$$

Write $\lambda = \xi + i\eta$. Separating real and imaginary parts we find the two equations:

(i) $$\xi \int |u|^2\,dx = \int \sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} \cdot \frac{\partial \bar{u}}{\partial x_q}\,dx + \int \big(\frac{1}{2} \cdot \sum \frac{\partial a_p}{\partial x_p} - b\big) \cdot |u|^2\,dx$$

(ii) $$\eta \int |u|^2\,dx = \frac{1}{2i} \int \sum a_p \big(u \frac{\partial \bar{u}}{\partial x_p} - \bar{u}\frac{\partial u}{\partial x_p}\,\big)\,dx$$

Set

$$A = \int |u|^2\,dx \quad : \quad B = \int |\nabla(u)|^2\,dx$$

Since P is elliptic there exists a positive constant k such that

$$\sum_{p,q} a_{p,q}(x)\,\frac{\partial u}{\partial x_p} > k \cdot |\nabla(u)|^2$$

From this we see that (i-ii) gives positive constants c_1, c_2, c_3 such that

(iii) $$A\xi > c_1 B - c_2 B \quad : \quad A|\eta| < c_3 \cdot \sqrt{AB}$$

Here (iii) implies that $\xi > -c_2$ and the reader can also confirm that

(iv) $$B < \frac{A}{c-1}(\xi + c - 2) \quad : \quad A|\eta| < A \cdot c_2 \cdot \sqrt{\frac{\xi + c_2}{c_1}} \quad : \quad |\eta| < c_3 \cdot \sqrt{\frac{\xi + c_2}{c_1}}$$

Finally it is obvious that (iv) above gives the requested inequality in Proposition 3.1.

4. Proof of the asymptotic formula.

Using the results above where we have found a good control of the integral operator $\mathcal{J}$ and the identification of eigenvalues to | and those from (*) in (0.B), one can proceed and apply Tauberian theorems to derive the asymptotic formula in Theorem B.0 using similar methods as in § A where we treated the Laplace operator. The details are left as an exercise to the reader. if necessary, consult [Carleman: page xx-xx].