On spectra of some compact operators.

Introduction. We shall establish some scattered results aobut spectra of linear operators. A crucual tool is Carleman's inequality for norms of resolvents of matrices, expressed by their Hilbert-Schmidt norm. Since this important result is seldom treated in text-books devoted to "abstract linear algebra" we have included details in a separate appendix. In § xx we give an application where one regards compact operators on Hlbert spaces. In § xx we study a specific case where the positviity of a kernel for an integral and compact operator on $L^2[0,1)$ implies that the spectral values are qyite ample. In § 3 we prove the Fredolm's resolvents yiled entire analytic function whose growth is controlled in an optimal fashgion. Here analytic function theory intervenes, i.e. the proof relies upon several basic results, foremost due to Lindelöf, Phragmén and Wiman.

§ 1. On the spectra of compact operators on Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space. Recall that if T is a compact operator on $\mathcal{H}$, then there exists the compact self-adjoint operator $\sqrt{T^*T}$. Denote by $\{\mu_n(T)\}$ its discrete spectrum arranged so that sequence $\{\mu_n(T)\}$ is non-increasing, where eventual multiplicitiesare counted as usual.

1.1 Definition. *For each $p > 0$ we denote by $\mathcal{C}_p$ the class of compact operators on $\mathcal{H}$ such that*

$$\big(\sum_{n=1}^{\infty} \mu_n(T)^p\,\big)^{\frac{1}{p}} < \infty$$

Next, denote by $\mathrm{sp}(T)$ the set of all vectors $x \in \mathcal{H}$ for which there exists some non-zero complex number λ and some integer $n \geq 1$ such that

$$(*) \qquad (\lambda E - T)^n(x) = 0$$

For every positive integer N we have operator T^N and get the image space $T^N(\mathcal{H})$. We shall find sufficient conditions in order that

$$(**) \qquad T^N\mathcal{H} \subset \mathrm{sp}(T)$$

holds for some positive integer N. To achieve this we shall study the resolvent operator

$$R(\lambda) = (\lambda \cdot E - T)^{-1}$$

defined outside the discrete spectrum of T. Let γ be a simple Jordan arc which has the origin as one end-point while $\gamma^* = \gamma \setminus \{0\}$ stays outside the spectrum of T. Now $R(\lambda)$ exists for every $\lambda \in \gamma^*$ and we can compute operator norms which leads to:

1.2 Definition. *A Jordan arc γ as above is called T-escaping of order N if there exists a constant C such that*

$$||R(\lambda)|| \leq C \cdot |\lambda|^{-N} \quad : \quad \lambda \in \gamma_*$$

Next, let $\gamma_1, \ldots, \gamma_s$ be a finite family of Jordan arcs as above whose intersections with a small punctured disc $D^*(\delta) = \{0 < |z| < \delta\}$ gives a disjoint famlily of curves $\{\gamma^*_\nu\}$. Then $D^*(\delta)$ is decomposed into s many pairwise disjoint Jordan domains, each of which is bordered by a pair of γ^*-curves. Let $\rho > 0$ be some positive number. We impose the geometric condition that every Jordan domain above is contained in a sector where $\arg(z)$ stays in an interval of length $< \rho$ as z varies in the Jordan domain. Denote by $\mathcal{J}(\rho)$ the class of all finite families of Jordan curves for which these sector conditions hold.

1.3 Theorem. *Let T be a compact operator of class C_p for some $p > 0$ and suppose there exists a family $\{\gamma_\nu\}$ which belongs to $\mathcal{J}(\pi/p)$ where each γ_ν is T-escaping order N. Then one has the inclusion*

$$T^N(\mathcal{H}) \subset \mathrm{sp}(T)$$

Remark. This theorem is announced and proved in [Dunford-Schwartz: Theorem XI.9.29 on page 1115]. The first step in the proof is straightforward reduces the proof to the case when the compact operator T is of Hilbert-Schmidt type. We remark only that for this reduction one uses the fact that T belongs to C_p for some $p > 0$. The major step is to extend Carleman's inequality for matrices from the appendix to Hilbert-Schmidt operators acting on infinite-dimensional Hilbert spaces. After this one easily finishes the proof is by standard applications of the Phragmén-Lindelöf inequalities.

Example and a question. Let $\mathcal{H} = L^2[0,1]$ and consider the operator T defined by

$$T_u(x) = \int_0^1 \frac{k(x,y) \cdot u(y)}{|x-y|^\alpha} \, dy$$

where $1/2 < \alpha < 1$ and $k(x,y)$ is a real-valued continuous function on the closed unit square. Then T is compact and we leave to the reader to verify the existence of a p-number which depend upon α such that T belongs to the class C_p. It would be interesting to investigate how one can produce integers N to get the inclusion in Theorem 1.3. In such an investigations one can try specified k-functions and in particular regard the symmetric case when $k(x,y) = k(y,x)$.

§ 2. Distribution of eigenvalues for a class of singular operators.

Let $f(x,y)$ be a continuous function on the unit square $0 \leq x,y \leq 1$ which is symmetric, i.e. $f(x,y) = f(y,x)$. When $0 < \alpha < 1$ we set

$$k(x,y) = \frac{f(x,y)}{|x-y|^\alpha}$$

and get the linear operator

$$K_\alpha(u)(x) = \int_0^1 k(x,y)u(y)\,dy$$

Obne easily verifies that K_α is a compact operator on the Hilbert space $L^2[0,1]$, and since k is symmetric the eigenvalues are real and non-zero. Let $\{\lambda_n^+\}$ be the positive eigenvalues arranged in a non-decreasing order. Similarly $\{\lambda_n^-\}$ is the set of negative eigenvalues where the sequence $\{-\lambda_n^-\}$ is non-decreasing. The eigenvalues correspond eigenfunctions $\{\phi_n^+\}$ and $\phi_n^-\}$. In particular

$$K_\alpha(\phi_n^+) = \lambda_n^+ \cdot \phi_n^+$$

2.1 Theorem. *If $f(x,x) > 0$ hold on some open interval $x_0 < x < x_1$ it follows that*

$$\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n^+}\right)^{\frac{1}{1-\alpha}} = +\infty$$

During the proof we use the following notation for real-valued functions u in $L^2[0,1]$:

$$\langle K_\alpha u, u\rangle = \iint f(x,y)u(x)u(y)\,dxdy$$

and for a pair of real-valued L^2-functions u,v we set

$$\langle u,v\rangle = \int_0^1 u(x)v(x)\,dx$$

We shall need the following result whose proof is left as an exercise to the reader:

2.2 Proposition. *For each $u \in L^2[0,1]$ one has the equality*

$$(*) \qquad \langle Ku,u\rangle = \sum \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2 + \sum \frac{1}{\lambda_n^-} \cdot \langle u,\phi_n^-\rangle^2$$

Proof of Theorem 2.1. Let m be a positive integer. Since $\{\lambda_n^-\}$ are negative (*) gives:

$$\langle Ku,u\rangle \leq \sum_{n=1}^{m} \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2 + \sum_{n=m+1}^{\infty} \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2$$

Since $\{\lambda_n^+\}$ is non-decreasing the last sum above is majorized by

$$\frac{1}{\lambda_m^+} \cdot \sum_{n=m+1}^{\infty} \langle u,\phi_n^+\rangle^2 \leq \frac{1}{\lambda_{m+1}^+} \cdot \langle u,u\rangle$$

where the last inequality follows from Bessel's inequality since the eigenfunctions $\{\phi_n^+\}$ form an orthonormal family. Hence the following inequality holds for every positive integer m:

$$\text{(i)} \qquad \langle Ku,u\rangle \leq \sum_{n=1}^{m} \frac{1}{\lambda_n^+} \cdot \langle u,\phi_n^+\rangle^2 + \frac{\langle u,u\rangle}{\lambda_{m+1}^+}$$

Let $\psi_1,\ldots,\psi_m$ be some orthonormal m-tuple in $L^2[0,1]$. We can apply (i) to each ψ-function and a summation over $1 \leq k \leq m$ gives:

$$\text{(ii)} \qquad \sum_{k=1}^{k=m} \langle K\psi_k, \psi_k\rangle \leq \sum_{k=1}^{k=m}\sum_{n=1}^{n=m} \frac{1}{\lambda_n^+}\cdot \langle \psi_k, \phi_n^+\rangle^2 + \frac{m}{\lambda_{m+1}^+}$$

Another application of Bessel's inequality gives for each $1 \leq n \leq m$:

$$\sum_{k=1}^{k=m} \langle \psi_k, \phi_n^+\rangle^2 \leq \langle \phi_n^+, \phi_n^+\rangle = 1$$

Hence (ii) entails that

$$\text{(iii)} \qquad \sum_{k=1}^{k=m} \langle K\psi_k, \psi_k\rangle \leq \sum_{n=1}^{n=m} \frac{1}{\lambda_n^+} + \frac{m}{\lambda_{m+1}^+}$$

A choice of ψ-functions. By assumption we find an interval $[x_0, x_1]$ where $f(x,x) > 0$. Set $d = x_1 - x_0$ and when m is a positive integer we define $\psi_1, \ldots, \psi_m$ where

$$\psi_k(x) = \sqrt{\frac{m}{d}} \quad \text{when} \quad x_0 + (k-1)\frac{d}{m} < x < x_0 + k\frac{d}{m}$$

while $\psi_k = 0$ outside the intervals above. The continuity of f gives some large integer m_* and a positive constant δ such that if $m \geq m_*$ then $f(x,y) \geq \delta$ on each small square where $\psi_k(y)\cdot\psi_k(x) \neq 0$. Denote this small square by $\square_k$ which gives the inequality below for each $1 \leq k \leq m$:

$$\langle K\psi_k, \psi_k\rangle \geq \delta\cdot\frac{m}{d}\cdot \iint_{\square_k} \frac{dxdy}{|x-y|^\alpha}$$

An easy calculation shows that the double integral over $\square_k$ becomes

$$\frac{2}{(1-\alpha)(2-\alpha)}\cdot\big(\frac{d}{m}\big)^{2-\alpha}$$

So with $A = \delta\cdot\frac{2}{(1-\alpha)(2-\alpha)}$ one has the inequality

$$\text{(iv)} \qquad \sum_{k=1}^{k=m} \langle K\psi_k, \psi_k\rangle \geq A\cdot\big(\frac{d}{m}\big)^{1-\alpha}\cdot m = Ad^{1-\alpha}\cdot m^\alpha$$

By construction $\psi_1, \ldots, \psi_m$ is an orthonormal family and hence (iii) holds which together with (iv) gives the inequality below for every $m \geq m_*$:

$$\text{(v)} \qquad Ad^{1-\alpha}\cdot m^\alpha \leq \sum_{n=1}^{n=m} \frac{1}{\lambda_n^+} + \frac{m}{\lambda_{m+1}^+}$$

At this stage we shall argue by a contradiction, i.e. we prove that (v) prevents that the positive series in Theorem 1 converges. Namely, suppose that

$$\sum \big(\frac{1}{\lambda_n^+}\big)^{\frac{1}{1-\alpha}} < \infty$$

Since the terms in this positive series decrease with n it follows that

$$\lim_{m\to\infty} m\cdot\big(\frac{1}{\lambda_m^+}\big)^{\frac{1}{1-\alpha}} = 0$$

So if $\epsilon > 0$ we can find $m^* \geq m_*$ such that

$$m\cdot\big(\frac{1}{\lambda_m^+}\big)^{\frac{1}{1-\alpha}} < \epsilon \implies \frac{1}{\lambda_m^+} < \big(\frac{\epsilon}{m}\big)^{1-\alpha}$$

Hence (v) gives the following when $m > m^*$:

$$Ad^{1-\alpha} \cdot m^{\alpha} \leq \sum_{n=1}^{n=m^*} \frac{1}{\lambda_n^+} + \sum_{\nu=m^*+1}^{n=m} (\frac{\epsilon}{\nu})^{1-\alpha} + m \cdot (\frac{\epsilon}{m+1})^{1-\alpha}$$

The middle sum above is majorized by

$$\epsilon^{1-\alpha} \cdot \int_{m^*}^{m} \frac{dx}{x^{1-\alpha}} = \frac{\epsilon^{1-\alpha}}{\alpha} \cdot m^{\alpha}$$

At the same time we notice that the last term is $\leq \epsilon^{1-\alpha} \cdot m^{\alpha}$ and after a division with m^{α} we obtain

$$Ad^{1-\alpha} \leq m^{-\alpha} \cdot \sum_{n=1}^{n=m^*} \frac{1}{\lambda_n^+} + \epsilon^{1-\alpha}(\frac{1}{\alpha} + 1)$$

Above A and d are fixed positive constants while we can choose arbitary large m and arbitrary small ϵ. This gives a contradiction and Theorem 2.1 is proved.

§ 3. An entire spectral function.

Introduction. Theorem 1 was proved by Carleman in the article *Sur le genre du dénominateur $D(\lambda)$ de Fredholm*. The proof uses some basic results about entire functions due to Poincaré, Lindelöf, Phragmén and Wiman and offers an instructive lesson in analytic function theory. Let $k(x,y)$ be a continuous function on the unit square $\{0 \leq x, y \leq 1\}$. We do not assume that k is symmetric, i.e. $k(x,y) \neq k(y,x)$ can hold. To each n-tuple of points $\{s_\nu\}$ on $[0,1]$ we assign the determinant function

$$K(s_1,\ldots,s_n) = \det \begin{pmatrix} k(s_1,s_1) & \cdots & k(s_1,s_n) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ k(s_n,s_1) & \cdots & k(s_n,s_n) \end{pmatrix}$$

Set

$$c_n = \int_{\square_n} K(s_1,\ldots,s_n) \cdot ds_1 \cdots ds_n$$

where the integral is taken over the n-dimensional unit cube.

3.1. Theorem. *Put*

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cdot c_n \cdot \lambda^n$$

Then D is an entire function of the form

$$D(\lambda) = e^{a\lambda} \prod (1 - \frac{\lambda}{\lambda_\nu}) \cdot e^{\frac{\lambda_\nu}{\lambda}}$$

where a is some complex constant and

$$\sum \frac{1}{|\lambda_\nu|^2} < \infty$$

Remark. Prior to Carleman's result above, Schur proved that $D(\lambda)$ is an entire function of the form

$$D(\lambda) = e^{a\lambda + b\lambda^2} \prod (1 - \frac{\lambda}{\lambda_\nu}) \cdot e^{\frac{\lambda_\nu}{\lambda}}$$

for some second constant b. The novelty in [Carleman] is that $b = 0$ always holds. Above we assumed that k is a continuous kernel. This condition was later relaxed by Carelan in the article [§ xx. 1919] which gives Theorem 1 when $k(x,y)$ is a kernel of the Hilbert-Schmidt type, i.e. it suffices to assume that

$$\iint |k(x,y)|^2 \, dxdy < \infty$$

Proof of Theorem 3.1

First we approximate k by polynomials. If $\epsilon > 0$ we find a polynomial $P(x,y)$ such that the maximum norm of $k - P$ over the unit square is $< \epsilon$. Write

$$k(x,y) = P(x,y) + B(x,y)$$

So now $|B(x,y)| < \epsilon$ for all $0 \le x, y \le 1$. To each pair $0 \le p \le n$ we set

$$B_p(s_1,\ldots,s_n) = \det \begin{pmatrix} P(s_1,s_1) & \cdots & k(s_1,s_n) \\ \cdots & \cdots & \cdots \\ P(s_p,s_1) & \cdot & P(s_p,s_n) \\ B(s_{p+1},s_1) & \cdots & B(s_{p+1},s_n) \\ \cdots & \cdots & \cdots \\ B(s_n,s_1) & \cdots & B(s_n,s_n) \end{pmatrix}$$

It is easily seen that

(i) $$c_n = \sum_{p=0}^{p=n} \binom{n}{p} \cdot \int_{\square_n} B_p(s_1,\ldots,s_n) \cdot ds_1 \cdots ds_n$$

Next, let N be the degree of the polynomial $P(x,y)$. The reader can verify that the first p row vectors in the matrix which defines $B_p(s_1,\ldots,s_n)$ are linearly independent as soon as $p > N$ which therefore gives $B_p = 0$. So for every $n \ge N$ one has the equality

(ii) $$c_n = \sum_{p=0}^{p=N} \binom{n}{p} \cdot \int_{\square_n} B_p(s_1,\ldots,s_n) \cdot ds_1 \cdots ds_n$$

Next, if M is the maximum norm of $k(x,y)$, and $\epsilon < M$ the maximum norm of P is $\le 2M$. Hadamard's determinant inequality in § xx gives

(iii) $$|B_p(s_1,\ldots,s_n)| \le (2M)^p \epsilon^{n-p} \cdot n^{\frac{n}{2}}$$

Next, when $n \ge p$ and $0 \le p \le n$ we set

$$c_n(p) = \binom{n}{p} \cdot \int_{\square_n} B_p(s_1,\ldots,s_n) \cdot ds_1 \cdots ds_n$$

Then (iii) gives:

(iv) $$|c_n(p)| \le \binom{n}{p} \cdot (2M)^p \epsilon^{n-p} \cdot n^{\frac{n}{2}}$$

Next, recall that $\binom{n}{p} \le \frac{n^p}{p!}$ and hence (iv) gives

(v) $$|c_n(p)| \le \frac{(2M)^p}{p!} \cdot \epsilon^{n-p} \cdot n^{\frac{n}{2}+p}$$

At this stage we return to the D-function. To each $0 \le p \le N$ we set

$$D_p(\lambda) = \sum_{n=p}^{\infty} \frac{-1)^n}{n!} \cdot c_n(p)\lambda^n$$

Sublemma. *For each p we have*

(*) $$\lim_{|\lambda| \to +\infty} e^{-4\epsilon|\lambda|^2} \cdot D_p(\lambda) = 0$$

Exercise. Prove (*). The hint is to use (v) above and Lindelöf's wellknown asymptotic formula for entire functions. See my notes in analytic function theory if necessary.

Next, we notice that (ii) gives an equation

$$\text{(vi)} \qquad D(\lambda) = q(\lambda) + \sum_{p=0}^{p=N} D_p(\lambda)$$

where $q(\lambda)$ is a polynomial of degre $N-1$ at most. This entails that the entire function $D(\lambda)$ also satisfies (*) in the Sublemma. Let $\{\lambda_\nu\}$ be the zeros of $D(\lambda)$. Then (*) and a classic result due to Poincaré gives the entire function

$$\text{(**)} \qquad F(\lambda) = \prod (1 - \frac{\lambda}{\lambda_\nu}) \cdot e^{\frac{\lambda}{\lambda_\nu}} \quad \& \quad \lim_{|\lambda| \to +\infty} e^{-\delta|\lambda|^2} \cdot F(\lambda) = 0 \quad \text{for all} \quad \delta > 0$$

To profit upon (**) we use a device introduced by Lindelöf. Let $\omega = e^{2\pi i/5}$ which gives the entire function

$$\text{(vii)} \qquad G(\zeta) = F(\zeta^5) \cdot F(\omega\zeta^5) \cdots F(\omega^4\zeta^5)$$

The right hand side in (**) entails that the entire function G has order $< 1/2$. Then a result due to Wiman gives the existence of an increasing sequence $\{R_k\}$ which tends to $+\infty$ such that

$$\text{(viii)} \qquad \min_\theta |G(R_k e^{i\theta})| \geq 1 \quad : \ k = 1, 2, \ldots$$

¡Choose λ-circles with $r_k^5 = R_k$ and Poincaré's limit from (**). Then it is clear that (vii) and (viii) entail that for every $\delta > 0$ there exist some k_* such that

$$\text{(ix)} \qquad k \geq k_* \implies \max_\theta \frac{1}{|F(r_k e^{i\theta})}| \leq e^{\delta r_k^2}$$

Finally, we have the zero-free entire function

$$H(\lambda) = \frac{D(\lambda)}{F(\lambda)}$$

In (ix) we can take $\delta = \epsilon$ which gives

$$\limsup_{k \to +\infty} e^{-5\epsilon \cdot r_k^2} \cdot \max_\theta |H(r_k e^{i\theta})| = 0$$

Liouville's theorem entails that the entire function $\log H(z)$ must be a linear polynomial and since $D(0) = 1$ we conclude that

$$D(\lambda) = e^{a\lambda} \cdot F(\lambda)$$

for a constant a which finishes the proof of Theorem 1.

Remark. Above we took several facts from analytic function theory for granted. At the end we appealed to the classic Liouiville Theorem. A much stronger version was proved by Carleman and for readers who are not so familiar with analytic function theory we include this result together with some details of the proof.

The general Liouville theorem.