§ 6. The Riemann-Schwarz inequality.

By the uniformisation theorem for Riemann surfaces it suffices to prove the result in §§ when $X = D$ and u is a continuous and subharmonic function in D where $\lambda(z) = e^{u(z)}$ which yields a metric. Consider a pair of points a, b in D and a pair of rectifiable Jordan arcs γ_1, γ_2 in $\mathcal{C}(a, b)$. Suppose for the moment that the intersection of the γ-curves only contains the end-points a and b. Their union gives a closed Jordan curve Γ which borders a Jordan domain Ω. The inequality (*) in Theorem XX follows if there to each point $p \in \gamma_1$ exist a Jordan arc β in Ω which joins p with some $q \in \gamma_2$ and a Jordan arc α in Ω which joins a and b such that

$$(1) \qquad \big(\int_\alpha \lambda(z)\,|dz|\big)^2 + \big(\int_\beta \lambda(z)\,|dz|\big)^2 \leq \frac{1}{2} \cdot \big[\big(\int_{\gamma_1} \lambda(z)\,|dz|\big)^2 + \big(\int_{\gamma_2} \lambda(z)\,|dz|\big)^2\big]$$

To prove (1) we employ a conformal mapping $\psi \colon D \to \Omega$ where D is another unit disc with the complex coordinate w. Recall that the Koebe-Lindelöf theorem entails that ψ extends to a homeomorphism from D onto the closed Jordan domain $\bar{\Omega}$ where the inverse images of γ_1 and γ_2 is a pair of closed intervals on the unit circle $|w| = 1$ which instersect at two points. Now $\log \lambda \circ \psi$ is subharmonic in D and if γ is a rectifiable arc in Ω one has the equation:

$$\int_\gamma \lambda(z) \cdot |dz| = \int_{\psi(\gamma)} \lambda \circ \psi(w) \cdot \frac{|dw|}{|\psi'(w)|}$$

Set

$$\lambda^*(w) = \frac{1}{|\psi'(w)|} \cdot \lambda \circ \psi(w)$$

Then

$$\log \lambda^* = -\log |\psi'(w)| + \log \lambda \circ \psi$$

Here $\log |\psi'(w)|$ is the real part of the analytic function $\log \psi'(w)$ in D and hence harmonic which entails that $\log \lambda^*$ is subharmonic. The proof of (1) is now reduced to the case when Ω is replaced by the unit w-disc where λ is replaced by λ^*. Moreover, the Cauchy-Schwarz inequality gives

$$\big(\int_{\gamma_1} \lambda^*(w)\,|dw|\big)^2 + \big(\int_{\gamma_2} \lambda^*(w)\,|dw|\big)^2 \geq \frac{1}{2} \cdot \big(\int_{|w|=1} \lambda^*(w)\,|dw|\big)^2$$

where we used that $\gamma_1 \cup \gamma_2$ is equal to the whole unit circle $|w| = 1$. There remains only to find a pair of curves α^* and β^* in the w-disc where α^* joins the two points $\{e^{i\theta_\nu}\}$ on the unit circle where the boundary arcs $\psi^{-1}(\gamma_1)$ and $\psi^{-1}(\gamma_2)$ intersect and the β^*-curve joins some point $p \in \psi^{-1}(\gamma_1)$ with a point $q \in \psi^{-1}(\gamma_2)$.

The choice of α^* and β^*. Let α^* be the circular arc with end-points at $e^{i\theta_1}$ and $e^{i\theta_2}$ which intersects $\{|w| = 1\}$ at right angles. Next, given a point $p \in \psi^{-1}(\gamma_1)$ there exists the unique circular arc β^* which intersects both α^* and $\{|w| = 1\}$ at right angles. See figure § XX. With this choice there remains to prove the inequality

$$(2) \qquad \big(\int_{\alpha^*} \lambda^*(w) \cdot |dw|\,\big)^2 + \big(\int_{\beta^*} \lambda^*(w) \cdot |dw|\,\big)^2 \leq \frac{1}{4} \cdot \big(\int_{|w|=1\}} \lambda^*(w)\,|dw|\,\big)^2$$

To prove (2) we shall use a symmetrisation of λ^*. Namely, there exists a Möbius transformation T_1 on the unit disc which is a reflection of α^*, i.e. it restricts to the identity map on α^* and the composed map T_1^2 is the identity in D. Similarly we find the reflection T_2 of β^*. One easily verifies that $T_2 \circ T_1 = T_1 \circ T_2$. Let $S_0(w) = w$ be the identity while $S_1 = T_1$, $S_2 = T_2$ and $S_3 = T_2 \circ T_1$. Set

$$(3) \qquad \lambda^{**}(w) = \frac{1}{4} \cdot \sum_{\nu=0}^{\nu=3} \lambda(S_\nu(w)) \cdot |\frac{dS_\nu(w)}{dw}|$$

Now T_1 maps α^* into itself, and similarly T_2 maps β^* into itself while the composed Möbius transformation $S_3 = T_2 \circ T_1$ interchanges the two curves. Hence the left hand side in (2) is unchanged when λ^* is replaced by λ^{**} and the right hand side is unchanged since the S-transformations map $\{|w| = 1\}$ onto itself. Hence it suffices to prove (2) when λ^* from the start is invariant with respect to the S-transformations. In this case we solve the Dirichlet problem using the boundary value function $\log \lambda^*$ on $\{|w|\}$ so that

$$\log \lambda^* = u + H$$

where H is harmonic in D while u is subharmonic and zero on $\{|w|\}$. The maximum principle entails that $u \leq 0$ in D which gives

$$\lambda^* = e^u \cdot e^H \leq e^H \tag{4}$$

Since $u = 0$ the right hand side in (2) is does not change while the left hand side is majorised when λ^* is replaced by e^H. The S-invariance of λ^* implies that H also is S-invariant and there exists an analytic function $g(w)$ in D such that

$$e^{H(w)} = |g'(w)|$$

where the map $w \mapsto g(w)$ sends α^* to a real interval $[-A, A]$ and β^* to an imaginary interval $[-iB, iB]$. Hence (4) implies that the left hand side in (2) is majorized by

$$(\int_\alpha |g'(w)|\,|dw|)^2 + \big(\int_\beta |g'(w)|\,|dw|\,\big)^2 = 4A^2 + 4B^2 \tag{5}$$

Next, since $\lambda^* = |g'|$ holds on $\{|w|\}$ the right hand side in (2) becomes

$$\frac{1}{4} \cdot \big(\int_{|w|=1} |g'(w)|\,|dw|\,\big)^2 \tag{6}$$

The S-symmetry of g entails thst $\int_{|w|=1} |g'(w)|\,|dw|$ is four times the integral taken along a subarc of T which joins consequtive points where α^* and β^* intersect and every suvh integral is the euclidian length of the image curvr under g which by the above joins the real point A with iB. So its euclidian length is $\geq \sqrt{A^2 + B^2}$, i.e. we have used that the shortest distance between a pair of points is a straight line and then applied Pythagoras' theorem. Hence (6) majorizes

$$\frac{1}{4} \cdot \big(4 \cdot \sqrt{A^2 + B^2}\big)^2 = 4(A^2 + B^2) \tag{7}$$

Then (5) and (7) give the requested inequality (2).

Abel's theorem.

A divisor D on X consists of an assignement of integers $\{\mu_\nu\}$ to a finite set of points $\{p_\nu\}$ in X. One writes

$$D = \sum \mu_\nu \cdot \delta(p_\nu)$$

The μ-integers may be positive or negative. The degree is defined by

$$\deg D = \sum \mu_\nu$$

while the finite set $\{p_\nu\}$ is called the support of D. Next we construct a class of 1-currents. In general, let $\gamma \colon [0,1] \to X$ be an Xvalued function on the closed unit interval We assume that γ is continuous and has a finite total variation. The last condition means that there exists a constant C such that

$$\sum_{\nu=0}^{\nu=N-1} d(\gamma(t_{\nu+1}, \gamma(t_\nu) \leq C$$

for all sequences $0 = t_0 < t-1 < \ldots < t_N = 1$ Here d is some distance function on X regarded as a metric space which in charts is equivsoent to the euclidian distance. Every such γ gives a current of degree one acting on $\mathcal{E}^1(X)$ by

$$(*) \qquad \alpha \mapsto \int_\gamma \alpha$$

Exercise. Explain how the classical Borel-Stieltjes integrals for functions with bounded variation defines the integral in the right hand side. Show also that

$$\int_\gamma dg = g(\gamma(1)) - g(\gamma(0))$$

hold for every $g \in \mathcal{E}()$. One refers to (*) as an integration current of degree one. More generally we can take a finite sum of such currents and get the integration current Γ defined by

$$\Gamma(\alpha) = \sum \int_{\gamma_j} \alpha$$

Let $D = \sum \mu_\nu \cdot \delta(p_\nu)$ be a divisor of degreee zero. An integration current Γ is said to be associated with D if the equality below hold for every $g \in \mathcal{E}(X)$:

$$\Gamma(dg) = \sum \mu_\nu \cdot g(p_\nu)$$

F.1 Principal divisors. Let f be a non-constant meromorphic function. Now we have the finite set of poles $\{p_\nu\}$ and the finite set of zeros $\{q_j\}$. We associate the divisor

$$\text{div}(f) = \sum \mu_k \cdot \delta(p_k) - \sum \mu_j \cdot \delta(q_j)$$

Here μ_k is the order of the pole of f at every p_ν, while μ_j is the order of a zero at q_j. By the result in § xx this divisor has degree zero. A divisor D of degree zero is called principal if it is equal to $\text{div}(f)$ for some $f \in \mathcal{M}(X)$. Notice that if f and g is a pair of non-constant meromorphic functions such that $\text{div}(f) = \text{div}(g)$, then f/g is a holomorphic function on X and hence reduced to a non-zero constant. Introducing the multiplicative group $\mathcal{M}(X)^*$ of non-constant meromorphic functions this means that one has an injective map

$$\frac{\mathcal{M}(X)^*}{\mathbf{C}^*} \to \mathcal{D}_0$$

where $\mathcal{D}_0$ is the additive group of divisors whose degree are zero. The theorem below describes the range of this map.

F.2 Theorem. *A divisor D of degree zero is principal if and only if there exists an integration current Γ associated with D such that*

$$\Gamma(\omega) = 0 \quad : \ \gamma \in \Omega(X)$$

The proof requires several steps. First we shall show the "if part", i.e. if Γ exists so that (*) holds then the divisor D is principal. To achieve this we first perform a local construction in the complex plane.

F.3 A class of currents in C. In the complex z-plane we consider a point z_0 where

$$0 < |z_0| < r < 1$$

holds for some $0 < r < 1$. It is easily seen that there exists a C^∞-function $a(z)$ in $\mathbf{C}$ which never is zero and the function

$$\phi(z) = \frac{z}{z - z_0} \cdot a(z)$$

is identically one when $|z| \geq r$. Now $z \mapsto \frac{1}{z-z_0}$ is locally integrable around z_0 and hence ϕ belongs to $L^1_{\text{loc}}(\mathbf{C})$ and is therefore a distribution.

Cauchy's residue formula can be expressed by saying the $\bar\partial$-image of the $\frac{1}{z}$ is equal to the $(0,1)$-current $2\pi i \cdot \delta(0) d\bar z$ where $\delta(0)$ is the Diraqc distribution at the origin. Using this the reader can check that

$$\text{(i)} \qquad \bar\partial(\phi) = \frac{z}{z - z_0} \cdot \bar\partial(a) + z_0 a(z_0) \cdot \delta(z_0) d\bar z$$

from (i) the reader should also check that

$$\text{(ii)} \qquad \phi^{-1} \cdot \bar\partial(\phi) = a^{-1} \cdot \bar\partial(a)$$

F.4 The current $\phi^{-1} \cdot \partial(\phi)$. To begin we one has

$$\text{(F.4.1)} \qquad \partial(\phi) = \frac{a(z)}{z - z_0} \cdot dz - \frac{za}{(z - z_0)^2)} \cdot dz + \frac{z}{z - z_0} \cdot \partial a$$

Above $(z - z_0)^{-2}$ is the principal value distribution defined as in § xx.

Exercise. Use (F.4.1) to establish the equation

$$\text{(F.4.2)} \qquad \phi^{-1} \cdot \partial(\phi) = z^{-1} \cdot dz - (z - z_0)^{-1} \cdot dz + a^{-1} \cdot \partial(a)$$

Exercise. Use Stokes Theorenm and Cauchy's residue fornula to prove that when the 1-current

$$\phi^{-1} \cdot d\phi = \phi^{-1} \cdot \partial(\phi) + \phi^{-1} \cdot \bar\partial(\phi)$$

is applied to dg for some $g \in C_0^\infty(\mathbf{C}$, then

$$\text{(F.4.3)} \qquad \phi^{-1} \cdot d\phi < dg >= 2\pi i \cdot (g(0) - g(z_0)$$

Let us now consider an integration current Γ given by a finite sum $\sum \gamma_j$ where each γ_j has compact support in a chart (U_j, z_j) in X and the end-points of γ_j are $z_j = 0$ and $z_j = z_j^*$ with $0 < |z_j^*| < r < 1$ and U_j is a chart defined by $\{|z_j| < 1|]$. In X the point $z_j = 0$ is denoted by q_j while z_j^* corresponds to a point p_j. It follows that

$$\text{(F.5)} \qquad \int_\Gamma dg = \sum_{j=1}^{j=N} g(p_j) - g(q_j)$$

for every $g \in \mathcal{E}(X)$. Next, for each j we apply the local contruction the chart and find a function ϕ_j in X which is identically one in $X \setminus \{|z_j| \leq r\}$ such that

$$\phi_j^{-1} \cdot d\phi_j < dg >= 2\pi i \cdot \int_{\gamma_j} dg \quad : g \in \mathcal{E}(X)$$

More generally, since d-closed 1-forms in the chart U_j are d-exact the re4ader should check that (F.5) implies that

$$\phi_j^{-1} \cdot d\phi_j < \alpha >= 2\pi i \cdot \int_{\gamma_j} \alpha \tag{F.6}$$

hold for every d-closed differential 1-form α. Let us now consider the function

$$\Phi = \prod_{j=1}^{j=N} \phi_j$$

F.7 Exercise. From the above the reader should check that the construction of the ϕ-functions imply that Φ is a zero-free C^∞-function in $X \setminus \{q_j, p_j\}$, i.e. in the open complement of the finite set which is the union of these q and p points. Moreover, additivity for logarithmic derivatives and (F.6) above gives

$$\Phi^{-1} \cdot d\Phi < \alpha >= \int_\Gamma \alpha \tag{F.7.1}$$

for every d-closed 1-form α. Finally, show that

$$\Phi^{-1} \cdot \bar{\partial}(\Phi) = \sum a_j^{-1} \cdot \bar{\partial}(a_j) \tag{F.7.2}$$

where the right hand side is a smooth $(0,1)$-form.

F.8 A special case. Suppose that Γ is such that

$$\int_\Gamma \omega = 0 \quad : \quad \omega \in \Omega(X) \tag{F.8.1}$$

The equality in (F.7.1) applied to d-closed holomorphic 1-forms and Theorem § xx entail that the smooth $(0,1)$-form $\Phi^{-1} \cdot \bar{\partial}(\Phi)$ from (F.7.2) is $\bar{\partial}$-exact. Thus, we can find $G \in \mathcal{E}(X)$ such that

$$\bar{\partial}(G) = \Phi^{-1} \cdot \bar{\partial}(\Phi) \tag{F.8.2}$$

Exercise. Put

$$\Psi = e^{-G} \cdot \Phi$$

deduce from the above that $\bar{\partial}(\Psi) = 0$ holds in $X \setminus \{q_j, p_j\}$ and hence Ψ is holomorphic in this open subset of X Show also that the local constructions of the ϕ-functions inside the charts $\{U_j z_j\}$ entail that ψ extends to a mermorphic function in X whose princial divisor is equal to

$$\sum \delta(q_j) - \sum \delta(p_j)$$

Funally the reader should confirm that the constructions above prove the "if part" in Theorem F.2.

F.9 Proof of the "only if part"

Here we are given a non-constant meromorphic funvtion f. We shall condtruct an integration chain Γ as follows: Frst $f \colon X \to \mathbf{P}^1$ is a holomorphic map. With $s = f(x)$ we recall from (xx) that the number critical s-points in the complex s-plane is finite. In $bf[P)^1$ we choose a simple curve γ_* with intial point at $s = 0$ and end-point at $s = |nfty$, while γ_* avoids the critical points $s \neq 0$ in $\mathbf{C}$. of course, while this is done we can choose γ_* so that it is a smooth curve on the C^∞-manifold $\mathbf{P}^1$.

The inverse image $f^{-1}(\gamma_*)$. Let $N = \deg(f)$. Since γ_* avoids f-cirtical points we see that

$$f^{-1}(\gamma_* \setminus \{0, \infty\}) = \gamma_1 \cup \ldots \cup \gamma_N$$

where $\{\gamma_j\}$ are disjoint curves in X, each of which is oriented via γ_* which from the start moves from $s = 0$ to $s = \infty$.

Exercise. Show that the closure $\bar{\gamma}_j$ yields a rectifiable curve whose initial point is zero of f and the end-point a pole. Put

$$\Gamma = \sum \bar{\gamma}_j$$

and conclude from the above that Γ is associated to the principal divisor of f. The proof of the "onmy if part" is therefore finished if we show that

$$(*) \qquad \int_\Gamma \omega = 0 \quad : \quad \omega \in \Omega(X)$$

To prove (*) we shall use a certain trace map. First, a holomorphic 1-form ω on X is regared as a $\bar{\partial}$-closed current with bi-degree $(0,1)$. Since the amp $f \colon X \to \mathbf{P}^1$ is proper there exists the direct image current $f_*(\Omega$ and we recal from general facts that the passage to direct image currents commute with the $\bar{\partial}$-operator. hence $\bar{\partial}(f_*(\Omega)) ==$ and since $\bar{\partial}$ is elliptic it follows that $f_*(\omega)$ is a holomorphic 1-form on $\mathbf{P}^1$. By the result in Exercise xx there does not exist non-zero globally defined holomorphic 1-forms on the projective line. hence

$$(**) \qquad f_*(\omega) = 0$$

In partocuar we have

$$\int_{\gamma_*(\epsilon)} f_*(\omega) = 0$$

where $\gamma_*(\epsilon)$ is the closed curve given by γ intersected with $\{\epsilon \leq |s| \leq \epsilon^{-1}\}$ for a small $\epsilon > 0$.

Exercise. Put $\Gamma(\epsilon) = f^{-1}(\gamma(\epsilon)$ and dediuce from (xx) that

$$\int_{\Gamma(\epsilon)} \omega = 0$$

Finally, pass to the limit as $\epsilon \to 0$ and conclude that (*) holds which finishes the proof of the "only if part".

Holomorphic differential operators

With $\partial = \partial/\partial z$ a holomorphic differential operator Q defined in a connected open subset of $\mathbf{C}$ can be written in a unique way as

$$(*) \qquad Q(z, \partial) = \sum_{\nu=0}^{\nu=k} q_\nu(z) \cdot \partial^\nu$$

where $\{q_\nu(z)\}$ are holomorphic functions in Ω. The largest integer k for which $q_k \neq 0$ is called the order of the differential operator Q, and the set of all holomorphic differential operators in Ω is denoted by $\mathcal{D}(\Omega)$. A basic result in analytic $\mathcal{D}$-module theory is the following:

1. Theorem. *Let Ω be a connected open set in the complex z-plane which contains the origin and let $f_1, \ldots, f_k$ be $\mathbf{C}$-linearly independent in the complex vector space $\mathcal{O}(\Omega)$. Then there exists $Q \in \mathcal{D}(\Omega)$ of order k such that*

$$(1.1) \qquad Q(f_\nu) = 0 \quad : \quad 1 \leq \nu \leq k$$

Moreover, we can always choose Q as in (*) where the holomrphic q-functions have no common zero in Ω, and when Q_1 and Q_2 are two such differential operators which both satisfy (1.1) we have

$$Q_2 = \phi(z) \cdot Q_1$$

where ϕ is a zero-free holomorphic function in Ω.

Remark. When $Q = \sum q_\nu(z) \cdot \partial^\nu$ satisfies (1.1) and the q-functions have no common zeros, the uniquenss in the theorem entails that the zero divisor of $q_k(z)$ is uniquely determined. In the special case when $q_k(z)$ only has a finite number of zeros, we find a *unique* Q satisfying (1.1) of the form

$$Q(z,\partial)=\sum_{\nu=0}^{\nu=k-1} q_\nu(z)\cdot\partial^m\nu+q_m(z)\cdot\partial^m$$

where $q_m(z)$ is a monic polynomial

$$q_m(z)=\rho\,(z-z_\nu)^{\mu_\nu}$$

Here $\{z-\nu\}$ is a finite set of points in Ω snd $\{\mu_\nu\}$ are positive integers, while eventual common zeors of $q_0,\ldots,q_{k-1}\}$ do not contain the zeros of $q_m(z)$. In $\mathcal{D}$-module theory one refers to this unique differential operator as the minimal annihilator of the k-tuple $\{f_\nu\}$ in $\mathcal{O}(\Omega)$. One has also a duality whch means that if $g\in\mathcal{O}(\Omega$ satisfies $Q(g)=0$, then g belongs to the k-dimensionsl vector space generated by $f_1,\ldots,f_k$.

A proof of this result appears in chapter 1 from my book *Analytic $\mathcal{D}$-modules.* Via analyticity it suffices to prove a local version where one regards the local ring $\mathcal{O}=\mathbf{C}\{z\}$ of germs of holomorphic functions at the origin. If $f_1,\ldots,f_k$ sre $\mathbf{C}$-linearly independent, the local version of the theorem above assert that there exists a unique differential operator

$$Q()z,\partial)=z^m\partial^k+\sum$$

where $m\geq 0$ snd $\{q_\nu(z)\}$ is a k-tupe in $\mathcal{O}$ such thst $q_\nu(0)\neq 0$ hold for at least some $0\leq\nu\leq k-1$. An example where $m\geq 1$ already occurs with $k=1$. Namely, let $f_1(z)=z^2$. Now the attached minimal Q-operator is given by

$$z\partial-\frac{1}{2}$$

A special family of local minimal Q-operators appears when the leading q-function $q_k(z)$ satisfies $q_k(0)\neq 0$. Namely,let $k\geq 1$ and consider a differential operator

$$Q(z,\partial)=\partial^k+$$

where $\{q_\nu\}$ is an arbitrary k-tuple in $\mathcal{O}$. Cauch'ys classic result for analytic ODE:s asserts that the Q-kernel on $\mathcal{O}$ is k-dimensional. Moreover, we csn choose soluyions $f-1,\ldots,f_k$ whose derivatives at the origin satisfy

$$\partial^\nu(f_j=\text{Kronecker's delta}\quad:0\leq j,\nu\leq k-1$$

The case when $q_k(0)=$ leads to a more involved analysis which was carried out by Fuchs and Picard around 1880. To begin with, if Q is the minimal operstor to some k-dimensional subspace of $\mathcal{O}$, then it turns out that $m\leq k$ and we can write

$$z^{k-m}\cdot Q(z,\partial)=\nabla^m+\sum p_\nu(z)\cdot\nabla^\nu$$

where $\nabla=z\partial$ is called the Fuchsian operator. One refers to the right hand side as a Fuchsian operstor of order k. The p-functions are not arbitrary if the Q-kernel on $\mathcal{O}$ is k-dimensional To see this we notice thsa if j is a positive integer then

$$Q(z^j)=(j^m+p_{m-1}(0)j^{m-1}+\cdots+p_0(0)\cdot z^j+z^{j+1}\cdot\phi_j(z)$$

where $\phi_j(z)\in\mathcal{O}$. Using this one can prove that the Q-kernel on $\mathcal{O}$ is k-dimensional if and only if the zeros of the polynomial

$$\rho_Q(\lambda)=\lambda^m+\sum p_\nu(0)\cdot\lambda^\nu$$

consists of a k-tuple of distinct non-negstive integers.

An example. Consider the differential operator

$$Q=z\partial^2+\partial\implies z\cdot Q=z^2\partial^2+\nabla=\nabla^2$$

So here $\rho_Q(\lambda)=\lambda^2$ which has a double zero at $\lambda=0$. The constant function $f(z)=1$ is a solution to $Q(f)=$ and generates the 1-dimensional Q-kernel on $\mathcal{O}$. Outside the origin the multi-valued function $\log z$ satisfies

$$(z\partial^2+\partial)(\log z)=z\partial(1/z)+1/z=0$$

In gneral Fuchs studied arbitrary differentisl operators expressed via the right hand side in (xx). When the p-functions are holomorphic in some disc $\{|z|<\delta\}$, we can regard the action by Q on germs of holomorphic functions at points in the punctured disc $\dot{D}=\{0<|z|<\delta\}$. Cauchy's theorem applies and gives a k-dimensional Q-kernel in $\mathcal{O}(z_0)$ for every $z_0\in\dot{D}$. A major result due to Fuchs is that these locally defined germs in the punctured disc are local branches of multi-valued analytic functions of a special form. Let us explain this in more detail.

Analytic function theory.

The Cauchy-Riemann equations in $\mathbf{C}$ are expressed by

(0.1) $$u_x = v_y \quad \& \quad u_y = -v_x$$

for a pair of real-valued functions $u(x,y)$ snd $v(x,y)$, and (0.1) means that the complex-valued function $f(z) = u(x,y) + iv(x,y)$ is holomorphic. More generally one can consider elliptic first order systems. With $n \geq 2$ one seeks an n-tuple of real-valued functions $u_1, \ldots, u_n$ which satisfy the system

(*) $$\frac{\partial u_p}{\partial x} + \sum_{q=1}^{q=n} A_{p,q}(x,y) \cdot \frac{\partial u_q}{\partial q} + \sum_{q=1}^{q=n} B_{p,q}(x,y) \cdot u_q = 0 \quad : \quad 1 \leq p \leq n$$

where $\{A_{p,q}\}$ and $\{B_{p,q}\}$ are real-valued functions of (x,y) which give elements in $n\times$-matrices $\mathcal{A}(x,y)$ and $\mathcal{B}(x,y)$. The system is called elliptic if the determinant polynomial

$$\lambda \mapsto \det(\mathcal{A}(x,y) - \lambda \cdot E_n)$$

has no real roots for every point (x,y). Notice that $n = 2m$ must be an even integer when the system is elliptic. Such general elliptic systems will not be treated here. Let us only remark that a deep result was proved by Carleman in 1937 which yields a uniquness theorem for solutions to the elliptic system above. More precisely, he proved that a u-solution to (*) where $y \mapsto u_k(0,y) = 0$ hold for every k. must vanish identically in a an open neighborhood of the line $\{x = 0\}$. The non-trivial fact is that this uniqueness holds under the sole condition that the A-function are of class C^2 and the B-functions are continuous. Prior to Carleman's work, the uniqueness was proved by Erik Holmgren when the A- and the B-functions are real-analytic. Holmgren's proof from an article published in 1901 by the Royal Scademy of Sweden is very elegant and and offers an instructive lesson for readers who enter studies of PDE-equations, and Carleman's proof of uniqueness in the non-analytic case gives a new perspective upon the Cauchy-Riemann equations and has inspired more recent more recent work devoted to linear elliptic PDE-systems in a wider context which foremost has been put forward by Lars Hörmander.

The subsequent material deals foremost with analysis on compact Riemann surfacas and expose clasic facts due to Abel and Riemann. More recent work by Weierstrass, Poincaré. Koebe and Weyl give certain simplifications of the original discoveries by Abel and Riemann. We shall work in the complex z-plane and write $z = x + iy$ which identifies the complex z-plane with $\mathbf{R}^2$. Now

(i) $$z^{-1} = \frac{1}{x+iy}$$

is locally integrable with respect to planar Lebesgue measure. Let us compute some distribution derivatives of (i). Consider the first order differential operator

$$\bar\partial = \frac{1}{2} \cdot (\partial_x + i\partial_y)$$

With these notations one has the equality

(*) $$\bar\partial(z^{-1}) = \pi \cdot \delta_0$$

where δ_0 is the Dirac measure at the origin. To prove (*) we consider for each $\epsilon > 0$ the C^∞-function

$$f_\epsilon(z) = \frac{\bar z}{|z|^2 + \epsilon}$$

The reader can check that

$$\bar\partial(f_\epsilon)(z) = \frac{\epsilon}{(|z|^2+\epsilon)^2}$$

The function in the right hand side tends to zero outside the origin as $\epsilon \to 0$. At the same time

$$\iint f_\epsilon(z)\, dxdy = \epsilon \cdot 2\pi \cdot \int_0^\infty \frac{rdr}{(r^2+\epsilon)^2} = \pi$$

and now the reader can check the equation (*).

Pompieu's formula. Let $f(x,y)$ be a test-function in $\mathbf{R}^2$. There exists the convolution

$$\text{(i)} \qquad \phi(z)=\iint f(\zeta)\cdot\frac{1}{\zeta-z}\,d\xi d\eta$$

where we have put $\zeta=\xi+i\eta$. Notice that the double integral is equal to

$$\text{(ii)} \qquad \iint f(\zeta+z)\cdot\frac{1}{\zeta}\,d\xi d\eta$$

From this we see thst ϕ is a C^∞-function and the reader can check that

$$\text{(ii)} \qquad \bar\partial(\phi)(z)=\iint \bar\partial(f)(\zeta)\cdot\frac{1}{\zeta-z}\,d\xi d\eta$$

At the same time, using (*) and (i) the reader can check that

$$\text{(iii)} \qquad \bar\partial(\phi)(z)=-\pi\cdot f(z)$$

Hence we obtain the formula

$$\text{(iv)} \qquad f(z)=\frac{1}{\pi}\cdot\iint \bar\partial(f)(\zeta)\cdot\frac{1}{z-\zeta}\,d\xi d\eta$$

Next, in the complex ζ-plane one has the differential 1-forms

$$d\zeta=d\xi+id\eta \quad \& \quad d\bar\zeta=d\xi-id\eta$$

From this the reader can conclude that

$$-2i\cdot d\xi d\eta=d\zeta\wedge d\bar\zeta$$

Since $i^2=-1$ it follows that (iv) gives

$$\text{(**)} \qquad f(z)=\frac{1}{2\pi i}\cdot\iint \bar\partial(f)(\zeta)\cdot\frac{1}{z-\zeta}\,d\zeta\wedge d\bar\zeta$$

Notice also that (iv) can be expressed by the equation

$$\text{(***)} \qquad f=\frac{1}{\pi}\cdot\bar\partial(z^{-1}*f)$$

where * denoted the convolution of functions in $\mathbf{R}^2$. This means that the $\bar\partial$-equation

$$\bar\partial(u)=f$$

has a solution u for every test-function f obtained from the formula in (***).

1. A passage to distributions. Let μ be a diistribution with compact support. Then there exists a distribution γ such that

$$\bar\partial(\gamma)=\mu$$

To prove this we shall define $z^{-1}*\mu$. Since μ in general is a sum of distribution derivatives of measures, it is not obvious how to construct this convolution. However, it is schieved via the Fourier transform. Namely, let $\widehat{\mu}$ be the Fourier transofrm of μ, i.e. it is for every (ξ,η) found by evaluating μ on the C^∞-function

$$(x,y)\mapsto e^{-i(x\xi+y\eta)}$$

When μ has compact support we know that $\widehat{\mu}$ is a real-analytic function on the (ξ,η)-space. So in this space there exists the locally integrable function

$$\text{(i)} \qquad \frac{\widehat{\mu}}{\xi+i\eta}$$

Fourier's inversion formula for tempered distributions implies that (i) is the Fourier transform of a tempered distribution ρ in the (x,y)-space. Rules for the Fourier transform give

$$\widehat{\bar\partial(\rho)}=\frac{i}{2}\cdot(\xi+i\eta)\cdot\widehat{\rho}=\frac{i}{2}\cdot\widehat{\mu}$$

So

$$\text{(1.1)} \qquad \gamma=\frac{2}{i}\cdot\rho\implies\bar\partial(\gamma)=\mu$$

1.2 A convolution formula. Consider the distribution. z^{-1}. It is tempered and has therefore a Fourier transform. It turns out that there exists as constant c_0 such that

(i) $$\widehat{z^{-1}} = c_0 \cdot \frac{1}{\xi + i\eta}$$

To prove (i) we use that the product

$$\frac{i}{2} \cdot (\xi + i\eta) \cdot \widehat{z^{-1}}$$

is equal to the Fourier transform of $\bar\partial(z^{-1})$. Next, since the Fouier transform of the Dirac distribution δ_0 is the identity function in the (ξ, η)-space, it follows from (*) above that (i) holds with

$$c_0 = \frac{2\pi}{i}$$

Exercise. Use the results above to extend Pompieu's formula to distributions with a compact support, i.e. one has

$$\mu = \frac{1}{\pi} \cdot z^{-1} * \bar\partial(\mu)$$

The elliptic property of $\bar\partial$. In general, let μ be a distribution with a compact support K and consider the distribution

$$\gamma = z^{-1} * \mu$$

If z is outside K, then

$$\zeta \mapsto \frac{1}{z - \zeta}$$

restricts to a $C^\infty-$function on K and we can evaluate μ on this funtion. The construction of the convolution gives

$$\gamma(z) = \mu < \frac{1}{z - \zeta} >$$

In the special case when μ has order zero, i.e. given by a measure with compact support one has

$$\gamma(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

and it is clear that the right hand side is a holomorphic function of z outside K. In general μ is expressed as a finite sum of derivatives of mesures supported by K Suppose for example that $\mu = \partial_x(\rho)$ where ρ is a measure with compact support confined to K. In this case the construction of distribution derivatives give

$$\gamma(z) = \int \frac{d\rho(\zeta)}{(z - \zeta)^2}$$

From this we see that γ again is holomorphic outside K and in general we conclude that (xx) gives a holomorphic γ-function in the open complement of K. Thus, a distribution μ which satisfies $\bar\partial(\mu) = 0$ in an open set is "truly holomorphic".

Analysis on a compact Riemann surface.

Let X be a compact Riemann surface. Regarding its underlying structure as a real manifold we introduce certain $\mathcal{E}$-spaces. By $\mathcal{E}(X)$ we denote the space of complex-valued C^∞-functions and $\mathcal{E}^1(X)$ the space of differential 1-forms woth C^∞-coefficients, If (U,z) is an analytic chart in X and $w \in \mathcal{E}^1(X)$, its restriction to U is expressed by a sum:

$$w|U = f \cdot dz + g \cdot d\bar{z} \quad : \quad f,g \in C^\infty(U)$$

In this way w is a sum of two globally defined differential 1-forms denoted by $w^{1,0}$ and $w^{0,1}$, and in a chart as above

$$w^{1,0}|U = f \cdot dz \quad \& \quad w^{0,1} = g \cdot d\bar{z}$$

It means that there is a decomposition

$$\mathcal{E}^1(X) = \mathcal{E}^{1,0}(X) \oplus \mathcal{E}^{0,1}(X)$$

We refer to $\mathcal{E}^{1,0}$ as the space of differential 1-forms of bi-degree $(1,0)$, and $\mathcal{E}^{0,1}$ is the space of differential 1-forms of bi-degree $(0,1)$, When $f \in \mathcal{E}(X)$ its usual exterior differential df is dcomposed as a sum

$$df = \partial f + \bar{\partial} f$$

Here ∂f is of bi-degree $(1,0)$ whose restriction to a chart becomes

$$\partial f|U = \partial(f) \cdot dz$$

where $\partial(f)$ is the C^∞-function

$$\partial(f) = \frac{1}{2}(f'_x - i f'_y)$$

The $(0,1)$-form $\bar{\partial} f$ is found in a similar fashion. We have also the space $\mathcal{E}^2(X)$ of differential 2-forms, and the exterior differential map:

$$d \colon \mathcal{E}^1(X) \to \mathcal{E}^2(X)$$

1. Some cohomology spaces. We can cover X by s finite family of charts $\{U_\alpha, z_\alpha)\}$. By a zero-cochain we mean a family of complex numbers $\{c_\alpha\}$ assigned to the open sets $\{U_\alpha\}$. This is a complex vector space denoted by $C^0(\mathfrak{U})$ whose dimension is the number of charts in the covering. A 1-cochain assigns complex numbers $c_{\alpha,\beta}$ to pairs when $U_\alpha \cap U_\beta \neq \emptyset$ which is alternating, i.e.

$$c_{\beta,\alpha} = c_{\alpha,\beta}$$

Each zero-cochain $\{c_\alpha\}$ gives the 1-cochain

$$\delta(\{c_\alpha\}) = \{c_\alpha - c_\beta\}$$

Next, a 1-cochain is δ-closed if

$$c_{\alpha,\beta} + c_{\beta,\gamma} + c_{\gamma,\alpha} = 0$$

for every triple where $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. The set of δ-closed 1-cochains is denoted by $Z^1(\mathfrak{U})$. It is clear thst the δ-image of zero-cochains yield δ-closed 1-cochains. Hence there exists the quotient space

$$H^1(\mathfrak{U}) = \frac{Z^1(\mathfrak{U})}{\delta C^0(\mathfrak{U})} \tag{1.1}$$

Next, we have the quotient space of d-closed modulu d-exact differential 1-forms. It turns out that there is an isomorphism:

$$H^1(\mathfrak{U}) \simeq \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)} \tag{*}$$

To prove (*) we choose a C^∞-partition of the unity, i.e. a family $\{\phi_\alpha \in C_0^\infty(U_\alpha)\}$ such that $\sum \phi_\alpha = 1$ on X. Consider a δ-closed 1-cochain $\{c_{\alpha,\beta}\}$. To each fixed α we put

$$h_\alpha = \sum c_{\alpha,\beta} \cdot \phi_\beta$$

In a non-empty intersection $U_\alpha \cap U_\gamma$ we have

$$(h_\alpha - h_\gamma)|U_\alpha \cap U_\gamma = \sum (c_{\alpha,\beta} - c_{\gamma,\beta})\phi_\beta = c_{\alpha,\gamma} \cdot \sum \phi_\beta = c_{\alpha,\gamma}$$

The right hand side are a constants and hence the differentials dh_α snd dh_γ are equal on the intersection $U_\alpha \cap U_\gamma$. This gives a globslly defined 1-form w such that

$$w|U_\alpha = dh_\alpha$$

hold for every chart in the covering $\mathfrak{U}$. Since $d^2 = 0$ the 1-form w is d-closed. Hence we have constructed a map

(i) $$\rho\colon\ Z^1(\mathfrak{U}) \to \mathcal{E}^1_{cl}(X)$$

It turns out that (i) is surjective. For let w be a d-closed 1-form. In every chart it is d-exst, i.e. we find $h_\alpha \in C^\infty(U_\alpha)$ and

$$w|U_\alpha = dh_\alpha$$

Now

$$\{c_{\alpha,\beta} = h_\alpha - h_\beta | U_\alpha \cap U_\beta\} \in Z^1(\mathfrak{U})$$

is a δ-closed 1-cochiasn and the reader can check that its its ρ-image is equal to w. Next, suppose thst $\{c_\alpha - c_\beta\}$ is a δ-exact 1-cochain. Now

$$h_\alpha = \sum (c_\alpha - c_\beta) \cdot \phi_\beta = c_\alpha - \sum \phi_\beta \cdot c_\beta$$

It follows that

$$dh_\alpha||U_\alpha = d(\sum c_\beta \cdot \phi_\beta)$$

The right hand side is the d-image of the globally defined C^∞-function $\Phi = \sum c_\beta \cdot \phi_\beta$ and we get

$$\rho(\{c_\alpha - c\beta\}) = d(\Phi)$$

Hence (i) induces a map

$$\bar{\rho}\colon\ H^1(\mathfrak{U}) \to \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)}$$

From the surjectivity of ρ it follows that $\bar{\rho}$ is surjective, and we leave as an exercise to check that it also is injective which proves the requested isomorphism in (*). Notice that this implies that $H^1(\mathfrak{U})$ is independent of the chosen covering of X by charts. This intrinsically defined complex vector space is denoted by $H^1(X)$ and called the cohomology space over X in in degree one. In the literature one refers to (*) as the Dolbeault isomorphism.

2. Holomorphic 1-forms. They consist of differential forms of bi-degree (0,1) which are $\bar{\partial}$-closed. This space is denoted by $\Omega(X)$. If ω is a holomorphic 1-form and $(U; z)$ a chart in X we have

$$\omega|U = f(z)dz$$

where $\bar{\partial}(\omega) = 0$ entails that f is holomorphic in U. Notice that every holomorphic 1-form is d-closed, i.e. one has the inclusion

$$\Omega(X) \subset \mathcal{E}^1_{cl}(X)$$

Next, we show that a holomorphic 1-fiorm ω cannot be d-exact so that (i) yields an injective map

(i) $$\Omega(X) \to \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)}$$

To see this we suppose that

$$\omega = dg$$

for some $g \in \mathcal{E}(X)$. Since ω has bi-degree $(1,0)$ this gives the equality

$$\omega = \partial g \implies 0 = \bar{\partial}\omega = \bar{\partial}\partial g$$

As explained in § xx this means that g is a harmonic function in every chart U_α and by the maximum principle for iharmonic functions it follows from the compactness of X that g is a compact and hence $w = 0$. This proves that the map (i) is injective.

We can also consider ∂-closed $(0,1)$-forms. This family is denoted by $\overline{\Omega(X)}$. If γ is such a form its restriction to a chart becomes

$$\gamma = g(z) \cdot d\bar{z} \quad : \quad \partial(g) = 0$$

The last equation means that g is the complex conjugate of a holomorphic function in the chart. Exactly as above one prove that there is sn injective map

$$\overline{\Omega(X)} \to \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)}$$

2.1 Exercise. Shiow that one also has an injective map

$$\Omega(X) \oplus \overline{\Omega(X)} \to \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)}$$

whuch amounts to check that a 1-form

$$w = \omega + \gamma$$

with $\Omega \in \Omega(X)$ and $\gamma \in \overline{\Omega(X)}$ cannot be d-exact, unless $\omega = \gamma = 0$.

3. Riemann's decomposition theorem. It turns out that the injective map from (2.1) is surjective, i,e. one has a direct sum:

$$\mathcal{E}^1_{cl}() = d\mathcal{E}(X) \oplus \Omega(X) \oplus \overline{\Omega(X)} \tag{3.1}$$

To prove (3.1) we proceed as follows. First we have the map

$$w \mapsto w^{0,1}$$

from $\mathcal{E}^1_{cl}(X)$ into $\mathcal{E}^{0,1}(X)$. Passing to a quotient we get a map

$$\rho\colon\ \mathcal{E}^1_{cl}(D) \to \frac{\mathcal{E}^{0,1}}{\bar{\partial}\mathcal{E}(X)}$$

Let us determine the ρ-kernel. So let w be a d-closed 1-form and suppose thst $w^{0,1}$ is $\bar{\partial}$-exact:

$$w^{0,1} = \bar{\partial}g \implies w - dg = w^{1,0} - \partial g$$

Since $d(w - dg) = 0$ we see that the (1,0)-form $w^{1,0} - \partial g$ is $\bar{\partial}$-closed and hence a holomorphic 1-form ω, and from the above

$$w = dg + \omega$$

Conversely the reader can check that $d\mathcal{E}(X) \oplus \Omega(X)$ belongs to the ρ-kernel.

3.2 Exercise. Deduce from the above that there is a left exact sequence

$$0 \to \Omega(X) \to \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)} \to \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}\mathcal{E}(X)}$$

and conclude the inequality

$$\dim \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)} \leq \dim \Omega(X) + \dim \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}\mathcal{E}(X)} \tag{i}$$

At the same time Exercise (2.1) gives the inequality

$$2 \cdot \dim \Omega(X) \leq \dim \frac{\mathcal{E}^1_{cl}(X)}{d\mathcal{E}(X)} \tag{ii}$$

where we used the obvious equslity $\dim \Omega(X) = \dim \overline{\Omega(X)}$.

From the above we see that (3.1) follows if we have proved the equality

$$\dim \Omega(X) = \dim \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}\mathcal{E}(X)} \tag{3.2.1}$$

3.3 A duality formula. To get (3.2.1) we regard the dual space of $\frac{\mathcal{E}^{0,1}(X)}{\bar\partial\mathcal{E}(X)}$. Recall that $\mathcal{E}^{0,1}(X)$ is a Frechet space whose dual $\mathbf{c}^{1,0}(X)$ consist of currents of bi-degree (0,1). Moreover, $\bar\partial\mathcal{E}(X)$ is a closed subspace of $\mathcal{E}^{0,1}(X)$. The construction of $\bar\partial$ on currents gives

$$[\,\frac{\mathcal{E}^{0,1}(X)}{\bar\partial\mathcal{E}(X)}]^* = \text{Ker}_{\bar\partial}(\mathbf{c}^{1,0}(X))$$

At this stage we use that $\bar\partial$ is elliptic and by the result in (xx) applied to charts the reader should confirm that the equality

$$\text{Ker}_{\bar\partial}(\mathbf{c}^{1,0}(X)) = \Omega(X)$$

Hence we have the duality formula:

$$\Omega(X) \simeq [\,\frac{\mathcal{E}^{0,1}(X)}{\bar\partial\mathcal{E}(X)}]^*$$

which gives (3.2.1) and finishes the proof of Riemann's decomposition theorem.

4. The cohomology space $H^1(\mathfrak{U},\mathcal{O})$. By $C^1(\mathfrak{U},\mathcal{O})$ we denote the space of families $\{g_{\alpha,\beta}\}$ where $g_{\alpha,\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$, and this doubly indexed family is alternating. Exactly as before we choose a partition of the uniity by ϕ-functions, and for each α we put

$$h_\alpha = \sum g_{\alpha,\beta} \cdot \phi_\beta$$

Now we construict the (0,1)-forms $\bar\partial(h_\alpha$. Since the holomorphic g-functions are annhilated by $\bar\partial$ we get

$$\bar\partial(h_\alpha) = \sum g_{\alpha,\beta} \cdot \bar\partial(\phi_\beta)$$

In each non-empty intersection the reader can check that these forms are matching and hence there exists a globally defined $(0,1)$-form w with

$$w|U_\alpha = \bar\partial(h_\alpha)$$

So we have constructed a map

$$\rho\colon\ Z^1(\mathfrak{U},\mathcal{O}) \to \mathcal{E}^{0,1}(X) \tag{4.1}$$

It turns out that ρ is surjective. For let w be a globally defined $(1,0)$-form. Pompieu's result applied to charts gives for each U_α some funtion f_α so that

$$w|U_\alpha = \bar\partial(f_\alpha)$$

Then $\{f_\alpha - f_\beta\}$ yields a holomorphic 1-cocycle whose ρ-image is w. Next, consider some δ-exact 1-cochain $\{g_\alpha - g_\beta\}$. Now the h-funtions above become

$$h_\alpha = \sum (g_\alpha - g_\beta) \cdot \phi_\beta = g_\alpha - \sum g_\beta \cdot \phi_\beta$$

Since the g-functions are holomorphic we get

$$\bar\partial(h_\alpha) = \sum g_\beta \cdot \bar\partial(\phi_\beta)$$

whivh holds for every α, while the right hand side is a globally defined $(0,1)$-form. We conclude that the ρ-image of a δ-exact 1-cochain is $\bar\partial$-exact. Hence there exists an induced map

$$\bar\rho\colon\ \frac{Z^1(\mathfrak{U},\mathcal{O})}{\delta(C^0(\mathfrak{U},\mathcal{O})} \to \frac{\mathcal{E}^{0,1}(X)}{\bar\partial\mathcal{E}(X)} \tag{4.2}$$

We leave to the reader to check that this map is bijective. Hence there exists an isomorphism

$$H^1(\mathfrak{U},\mathcal{O}) \simeq \frac{\mathcal{E}^{0,1}(X)}{\bar\partial\mathcal{E}(X)} \tag{4.3}$$

5. Passage to currents. We have the map $\bar\partial\colon \mathbf{c}(X) \to \mathbf{c}^{0,1}(X)$ and shall prove that

$$H^1(\mathfrak{U},\mathcal{O}) \simeq \frac{\mathbf{c}^{0,1}(X)}{\bar{\mathbf{c}}(X)} \tag{5.1}$$

To get (5.1) we consider a (0,1)-current γ. Pompieu's theorem for distributions applied to charts gives for every U_α some $\mu_\alpha \in \mathbf{c}(U_\alpha)$ such that

$$\gamma|U_\alpha = \bar{\partial}(\mu_\alpha)$$

Since $\bar{\partial}$ is elliptic it follows that

$$\mu_\alpha - \mu_\beta|U_\alpha \cap U_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)$$

and now $\{\mu_\alpha - \mu_\beta\}$ is a δ-closed 1-cochain, which has an image in $H^1(\mathfrak{U}, \mathcal{O})$. Exactly as in § 4 the reader can check that we get an injective map

$$\frac{\mathbf{c}^{0,1}}{\bar{\partial}\mathbf{c}(X)} \to H^1(\mathfrak{U}, \mathcal{O}) \tag{5.1}$$

Finally, the inclusion $\mathcal{E}^{0,1}(X) \subset \mathbf{c}^{0,1}(X)$ and (4.3) entail that (5.1) is surjective. Hence

$$H^1(\mathfrak{U}, \mathcal{O}) \simeq \frac{\mathbf{c}^{0,1}(X)}{\bar{\partial}\mathbf{c}(X)} \tag{5.2}$$

6. Exercises. Deduce from (4.3(and (5.1) that

$$\mathbf{c}^{0,1}(X) = \mathcal{E}^{0,1}(X) + \bar{\partial}\mathbf{c}(X)$$

Thus, up to a $\bar{\partial}$-exact (0,1)-image, every (0,1)-current coincides with a smooth (0,1)-current. Moreover, the dulity from § 3 extends and gives

$$\Omega(X) = [\frac{\mathbf{c}^{0,1}}{\bar{\partial}\mathbf{c}(X)}]^*$$

It can be expressed by saying that a $(o,1)$-current γ is $\bar{\partial}$-exact if and only if its evaluation on holomorphic 1-forms is zero.

Meromorphic functions.

Let X be a compact Riemann surface of some genus $g \geq 1$. For each point $p \in X$ we denote by $\mathcal{M}_p(X)$ the space of meromorphic functions on X whose polar set is reduced to the single point p. It turns out that $\mathcal{M}_p(X)$ contains non-constant functions, and we shall prove that there exists $f \in \mathcal{M}_p(X)$ with a pole at p of order $g+1$ at most. To prove this we shall consider $(0,1)$-currents supported by $\{p\}$. Choose a chart (U, z) around p where p corresponds to the origin in the z-disc U. If $\Phi \in \mathcal{E}^{1,0}(X)$ we can consider its restriction to U and get

$$\Phi|U = \phi(z) \cdot dz \quad : \phi \in C^\infty(U)$$

We can evaluate ϕ at $z = 0$ which gives a (1,0)-current on X denoted by ρ_p defined by

$$\rho_p(\Phi) = \phi(0)$$

We can also regard derivstives of ϕ which for each positive integer k gives a current $\rho_p^{(k)}$ defined by

$$\rho_p^{(k)}(\Phi) = \partial^k \phi / \partial \bar{z}^k(0)$$

Now $\{\rho_p^{(k)} :\ 0 \leq k \leq g\}$ are $g+1$ many (0,1)-currents. By Exercise 6 the quotient space

$$\frac{\mathbf{c}^{0,1}(X)}{\bar{\partial}\mathbf{c}(X)}$$

is g-dimensionsl. Hence there exists a $(g+1)$-tuple of complex numbers such that the current

$$\gamma = c_0 \cdot \rho_p^{(0)} + \cdots + c_g \cdot \rho_p^{(g)}$$

is $\bar{\partial}$-exact, i.e we find a distribution μ on X so that

$$\bar{\partial}(\mu) = \gamma$$

Since $\gamma = 0$ in $X \setminus \{p\}$ and the $\bar{\partial}$-operator is ellitpic, it follows that $\mu|X \setminus \{p\} = f$ for some holomorphic function f in the open set $X \setminus \{p\}$. Working in the chart U local residue formulas entail that f extends to a globally defined meromorphic function, i.e. it belongs to $\mathcal{M}_p(X)$, and gives the $(0,1)$-current

$$\bar{\partial} f = \gamma$$

Moreover, lcoal residue calculus inside the chart U together with the expression of γ above, imply that the pole of f at p has order $g+1$ at most.

Weierstrass' points. The question arises if $\mathcal{M}_p(X)$ contains a non-constant meromorphic function f with a pole of order $\leq g$ at p. To decide when such a meromorphic function exists we proceed as follows. From the above, if a non-constant funtion f exists in $\mathcal{M}_p(X)$ with a pole fo some order $m \leq g$ at p, then the $(0,1)$-current

$$\bar{\partial}(f) = c_0 \cdot \rho_p^{(0)} + \cdots + c_m \cdot \rho_p^{(m)}$$

for some constants $\{c_\nu\}$. So if the g-tuple of currents $\{rho_p^{(k)} : 0 \leq k \leq g-1\}$ are $\mathbf{C}$-linearly independent in the g-dimensional quotient space from Exercise 6, then no f as above exists.

A point $p \in X$ is called a Weierstrass point if $\mathcal{M}_p(X)$ contains a non-constant meromorphic function with a pole of order $\leq g$ at p. Let $\mathcal{W}(X)$ denote this set. We shall prove that it is a finite subset of X. To achieve this we choose a basis $\omega_1, \ldots, \omega_p$ in the g-dimensional vector space $\Omega(X)$. Given a point $p \in X$ we choose a chart (U, z) around p and get

$$\omega_\nu = g_\nu(z) \cdot dz$$

where $\{g_\nu(z)\}$ sre holomorphic functions in U. Now we get the $g \times g$-matrix $A(z)$ with elements

$$a_{k\nu}(z) = \partial^{k-1(}g_\nu)(z) \quad : 1 \leq k, \nu \leq g$$

From Exercise 6 the reader can check that p is a Weiertrass point if and only if

$$\det A(z) = 0$$

It turnd out thst the set of Weierstrass points is a non-empty finite set in X as so0n as the genus is ≥ 2.

Hörmander's L^2-estimate in dimension one

Theorem 1 below will not be covered during my lecture devoted to Riemann surfaces. It has been inserted together with a detailed proof to illustrate the flavour of some of the subsequent seminars. Those who appreciate Lsrs Hörmander's theorem below may find the seminars devoted to pluricomplex analysis interesting, while those who prefer to contemplae upon more abstract theories and algebra whree no hard analysis appears might hesitate to attend the them..

Hörmander's proof below teaches an instructive lesson. Here is the situation: Let Ω be an open set in $\mathbf{C}$. A real-valued continuous and non-negative function ϕ on Ω gives the Hilbert space $\mathcal{H}_\phi$ whose elements are complex-valued Lebesgue measurable functions f in Ω such that

$$(1) \qquad \int_\Omega |f|^2 \cdot e^{-\phi}\, dxdy < \infty$$

The square root yields norm denoted by $||f||_\phi$. Let ψ be another continuous and non-negative function which gives the Hilbert space $\mathcal{H}_\psi$ where the norm of an element g is denoted by $||g||_\psi$. We consider the $\bar\partial$-operator which sends a function $f \in \mathcal{H}_\phi$ to $\bar\partial(f) = df/d\bar z$ and study the equation

$$(2) \qquad \bar\partial(f) = g \quad : g \in H_\psi$$

One seeks conditions for the pair (ϕ, ψ) in order that there exists a constant C such that (2) has a solution f for every g where

$$(3) \qquad ||f||_\phi \leq C \cdot ||g||_\psi$$

Notice that (2) does not have a unique solution since f can be replaced by $f + a(z)$ where a is a holomorphic function which belongs to $\mathcal{H}_\phi$. For example, non-uniquneness fails when Ω is a bounded open set and the function $e^{-\phi}$ is bounded in Ω. For then f can be replaced by $f + p$ for an arbitrary polynomial $p(z)$. We shall find a sufficient condition in order that (2-3) above hold.

4. Hörmander's condition. The pair ϕ, ψ satisfies the Hörmander condition if ψ is a C^2-function and ϕ is at least a C^1-function, and there exists a positive constant c_0 such that the following pointwise inequality holds in Ω:

$$(4.0) \qquad \Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y \geq 2 \cdot c_0^2 \cdot e^{\psi(z)-\phi(z)}$$

where we have put $|\nabla(\psi)|^2 = \psi_x^2 + \psi_y^2$.

4.1 Theorem. *If the pair (ϕ, ψ) satisfies (4.0) the equation $\bar\partial(f) = g$ has a solution for every $g \in \mathcal{H}_\psi$ where*

$$||f||_\phi \leq \frac{1}{c_0} \cdot ||q||_\psi$$

Before the proof starts we recall some facts sbout linear operators between Hilbert spaces. In general, let $\mathcal{H}_0$ and $\mathcal{H}_1$ be a pair of complex Hilbert spaces and $T \colon \mathcal{H}_0 \to \mathcal{H}_1$ is a densely defined linear operator. Following Torsten Carlemsn's famous monograph about unbounded operators on Hilbert spaces published by Uppsala university in 1923, we recall the construction of an adjoint. Namely, a vector $y \in \mathcal{H}_1$ belongs to the domain of definition for the adjoint operator T^* if and only if there exists a constant C such that

$$(i) \qquad \big|\langle Tx, y\rangle_1\big| \leq C \cdot |x|_o \quad : x \in \mathcal{D}(T)$$

where $|x_|$ is the norm of the vector x taken in $\mathcal{H}_0$, and in the left hand side we considered the hermitiain inner product on $\mathcal{H}_1$. Since $\mathcal{D}(T)$ is dense and Hilbert spaces are self-dual, each y for which (i) holds yields a unique vector $T^*(y) \in \mathcal{H}_0$ such that

$$(ii) \qquad \langle Tx, y\rangle_1 = \langle x, T^*y\rangle_0$$

In general $\mathcal{D}(T^*)$ is not a dense subspace of $\mathcal{H}_1$. But let us add this as an hypothesis on T. Moreover, assume that the two densely defined operators T and T^* both are closed, i.e. their graphs are closed in the product of the two Hilbert spaces.

Exercise. Suppose as above that both T and T^* are closed with dense domains of definition. Assume in addition that there exists a positive constant c such that

$$|T^*y|_0 \geq |y|_1 \quad : \quad y \in \mathcal{D}(T^*)$$

Show that this implies that the range $T^*(\mathcal{D}(T^*))$ is a closed subspace of $\mathcal{H}_0$ which is equal to the orthogonal complement of the nullspace $\text{Ker}(T)$ Moreover, show that for each $y \in \mathcal{H}_1$ we can find $x \in \mathcal{D}(T)$ such that

$$Tx = y \quad \& \quad |x|_0 \leq c^{-1} \cdot |y|_1$$

Proof of Theorem 4.2

Since $C_0^\infty(\Omega)$ is a dense subspace of $\mathcal{H}_\phi$ the linear operator $T\colon f \mapsto \bar\partial(f)$ from $\mathcal{H}_\phi$ to $\mathcal{H}_\psi$ is densely defined and we leave as an exercise to the reader to check that T also is closed, i.e. this relies on a genrral fact about closedness of operators defined by differential operators. The reader may also check that Stokes Teorem entails that test-functions in Ω belong to $\mathcal{D}(T^*)$ and since $C_0^\infty(\Omega)$ is dense in the Hilbert space $\mathcal{H}_\psi$ the adjoint is also densely defined. Let us then consider some $g \in \mathcal{D}(T^*)$. For each $f \in C_0^\infty(\Omega)$ Stokes theorem gives

(i) $$\langle T(f), g\rangle = \int \bar\partial(f) \cdot \bar g \cdot e^{-\psi}\, dxdy = -\int f \cdot \big[\bar\partial(\bar g) - \bar g \cdot \bar\partial(\psi)\big] \cdot e^{-\psi}\, dxdy$$

Since ψ is real-valued, $\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)$ is equal to the complex conjugate of $\partial(w) - w \cdot \partial(\psi)$. We conclude that (i) gives

(ii) $$T^*(g) = -\big[\partial(g) - g \cdot \partial(\psi)\big] \cdot e^{\phi-\psi}$$

In particular T^* is defined via a differential operator and has therefore a closed graph. Taking the squared L^2-norm in $\mathcal{H}_\phi$ we obtain

$$||T^*(g)||_\phi^2 = \int |\partial(g) - g \cdot \partial(\psi)|^2 \cdot e^{\phi-2\psi} =$$

(iii) $$\int \big(|\partial(g)|^2 + |g|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi-2\psi} - 2 \cdot \mathfrak{Re}\Big(\int \partial(g) \cdot \bar g \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}\Big)$$

By partial integration the last integral in (iii) is equal to

(iv) $$2 \cdot \mathfrak{Re}\Big(\int g \cdot [\partial(\bar w) \cdot \bar\partial(\psi) + \bar g \cdot \partial\bar\partial(\psi) - 2\bar w \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar g \cdot \bar\partial(\psi) \cdot \partial(\phi)] \cdot e^{\phi-2\psi}\Big)$$

Next, the Cauchy-Schwarz inequality gives

(v) $$\Big|2 \cdot \int g \cdot \partial(\bar g) \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}\Big| \leq \int \big(|\partial(g)|^2 + |g|^2 \cdot |\partial(\psi)|^2\big) \cdot e^{\phi-2\psi}$$

Together (iii-v) give

$$||T^*(g)||_\phi^2 \geq 2 \cdot \mathfrak{Re} \int |g|^2 \cdot \big[\,\partial\bar\partial(\psi) - 2 \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar\partial(\psi) \cdot \partial(\phi)\,\big] \cdot e^{\phi-2\psi} =$$

(vi) $$2 \cdot \mathfrak{Re} \int |g|^2 \cdot \frac{1}{4}\big[\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y\,\big] \cdot e^{\phi-2\psi}$$

where the last equality follows since ϕ and ψ are real-valued. Funally, since (4.1) is assumed it follows that

(vi) $$||T^*(g)||_\phi^2 \geq c_0^2 \cdot \int |g|^2 \cdot e^{\psi-\phi} \cdot e^{\phi-2\psi} = c_0^2 \cdot ||g||_\psi^2$$

Now we aåply the Exercise above and get Theorem 4.2.

Remark. Let Ω be an open subset of a disc $\{|z| < r\}$ for somne $r < 1$ which is centered at the origin. We find a pair with $\phi = \psi$ satisfying (4.1) by:

$$\phi(z) = \log(|z|^2 + 1) = \log(x^2 + y^2 + 1)$$

For then we see that

$$\Delta(\psi) = \frac{4}{(x^2+y^2+1)^2}$$

At the same time

$$\psi_x^2 + \psi_y^2 = \frac{4x^2+4y^2}{(x^2+y^2+1)^2}$$

The reader can check that one gets a constant c_0 so that Hörmander's condition (4.1) holds.

Final remsrk. The full strength of L^2-estimates appears in dimension $n \geq 2$ where one works with *plurisubharmonic functions* and impose the condition that Ω is a strictly pesudo-convex set in $\mathbf{C}^n$ and solve inhomogeneous $\bar{\partial}$-equations for differential forms of bi-degree (p, q). We refer to Chapter 4 in Hörmander's cited text-book for details.

Pluricomplex analysis

The seminars continue throughout the academic yesr 2011-12. Focus is upon complex analysis of one and several variables. They will take place every second week, except for the first two lectures below. Here are titles of these two lectures with J-E Björk as speaker.

1. November 21: Riemann surfaces.

2. November 28: Volumes and limits of analytic sets in $\mathbf{C}^n$.

Time and place. Each lecture takes place on a Tuesday $15^{30} - 17^{30}$ at room 304 in House 6 at the department of mathematics at SU.

Remarks about the lectures. Distributions, and more generally currents play a central role during the whole analysis. From a historic point of view the study of Riemann surfaces, i.e. 1-dimensional complex manifolds has paved the way towards many areas in mathematics, such as cohomology of sheaves and the determination of cohomology on manifolds via the de Rham complex of smooth differential forms. For example, duality in the sense of Poincaré is best expressed via the de Rham complex of currents. The study of Riemann surfaces, i.e. 1-dimensional complex manifolds offer instructive lessons about currents on C^∞-manifolds of arbitrary dimension, and the aim of Lecure 1 is to expose how the sydtemstic use of currents on a Riemann surface - compact or non-compact - enabe us to establish many remarkable results. Among these we mention the Behnke-Sdtrin theorrmn whivassertd thst thr inhomogrnouus $\bar{\partial}$-equstion is sovsbe on onevety non-compsvt Riesnn surface, the fsmoud reut due to Abelwhigves as necesssry nd sufficient conditio for the podtion of es snd xerod of s meromprphic funvtion on a compsvt riemsnn surface.

Basic facts from calculus, foremost in dimension two will be used without hesitation. Let me recall some results expressed via currents We work in the complex z-plane with $z = x + iy$. Consider a closed and dimle curve γ with vontonous tangent i.e. a closed Jordan curve of class C^1. It borders a bounded domain U and we recall that γ is oriented in a natural way vis the rule of the thumb. If $w = A \cdot dx + B \cdot dy$ is a differential 1-form, where $A(x, y)$ and $B(x, y)$ in general are complex-vslued C^∞.funvtions, we obtain a current of degree 1 via the linear functional

$$w \mapsto \int_\gamma w$$

One refers to this as the integration current attached to the oriented curve γ. Stokes theorem gives the equality

$$\iint_U dw = \int_\gamma w$$

It means that the distribution defined by the characteristic function χ_U has a differentisl whivh is equal to the integration current, i.e. in the space $\mathbf{c}^1$ of current of degree one, we have the equslity

$$(*) \qquad d\chi_U = \int_\gamma$$

This is a convenient way to express Stokes theorem and we shall learn that similar formulas extend to manifolds.

Next, in analytic function theory the fsmous residue formula due to Cauchy can be expressed and later on extended to a formula related to currents. Consider the function z^{-1} which is meromorphic with a simple pole at the origin If $f(x, y)$ is a complex-valued C^∞-function a Taylor expansion of f at the origin gives

$$f(0) = \frac{1}{2\pi i} \cdot \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{f(z)}{z}\, dz$$

This local residue formula leads to an equality expressed via currents, Namely, first z^{-1} is a locally integrable function in the (x, y).space and gived therefore a distirbution. By definition $\mathfrak{Db}(\mathbf{C})$ is the dual of test-form of maximal degree 2. Thus space to be denoted by $\mathcal{E}^2(\mathbf{C})$ consists of differential 2-forms

$$\psi(x, y) \cdot dx \wedge dy$$

where ψ is a complex-valued C^∞-function with compact support. Now z^{-1} is the distribution degined by

$$\psi \cdot dx \wedge dy \mapsto \iint \frac{\psi(x, y)}{x + iy}\, dxdy$$

Recall that we can construct derivatives of distributions. In particular we regard the first order differential operator

$$\bar{\partial} = \frac{1}{2} \cdot (\partial_x + i \cdot \partial_y)$$

Now there exists the 1-current $\bar{\partial}(z^{-1})$ which by defintion is a linear functional on test-forms of degreee one, defined by

$$\bar{\partial}(z^{-1}) < w >= -\iint \frac{\bar{\partial}(w)}{z}$$

In the right hand side we have constructed the differrential 2-form $\bar{\partial}(w)$. Te 1-form $w = A \cdot dx + B \cdot dy$ and we shall express it using the 1-forms dz and $d\bar{z}$. By definition

$$dz = dx + idy \quad \& \quad d\bar{z} = dx - idy$$

It follows thst

$$A \cdot dx + B \cdot dy = \frac{1}{2}[A \cdot (dz + d\bar{z}) - i \cdot B \cdot (dz - d\bar{z})] = \frac{1}{2}[(A - iB) \cdot dz + (A + iB) \cdot d\bar{z}]$$

So in this way every differential 1-form can be expressed via the 1-forms dz and $d\bar{z}$, i.e. an arbitrary 1-form w csn be expressed as

$$w = A \cdot dz + B \cdot d\bar{z}$$

where A, B is a pair of C^∞-functions. Now the $\bar{\partial}$-operator from 1-forms into 2-forms becomes

$$\bar{\partial}(w) = \bar{\partial}(A) \cdot d\bar{z} \wedge dz$$

where $dz \wedge d\bar{z}$ is the 2-form given by

$$dz \wedge d\bar{z} = -2i \cdot dx \wedge dy$$

Remark. Denote by $\mathcal{E}^1$ the space of differential 1-forms in the complex z-plane. From the aboce one has a direct dum decompositon

$$\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1} \quad : \quad \mathcal{E}^{1,0} = \mathcal{E} \cdot dz \,\&\, \mathcal{E}^{0,1} = \mathcal{E} \cdot d\bar{z}$$

where $\mathcal{E}$ denotes complex valued C^∞-functions. With these notyions the exterior differntisl of a 1-fotm w can be found as follows: First we decompose w as

$$w = w^{1,0} + w^{0.1}$$

and then

$$dw = \bar{\partial}(w^{1.0}) + \partial(w^{0,1}$$

Above we therefore find the exterior differentials

$$\bar{\partial} \colon \mathcal{E}^{1,0} \to \mathcal{E}^2$$

In both lectures attention is given to the notion of currents on a complex analytic manifold where § 1 treats the 1-dimensional case. So several classic results, such as Abel's theorrn which gives a necessary and sufficient condition for the positions of poles and zeros of a non-constant meromorphic function on a compact Riemann surface. Baisc fvts from anakytic funvtion theory in dimension one eill be used eithout hesitstion,and in addition the resder dhould be fsmlisir eith basic didtributyion throty whivh for exsmple is covered in chsoter 1-3 in Lars Hörmsnderäs tect-book on linesr psrtisl differentisl equstions. Cincering the importstn interplsy between anslytic funvtyion theory in one complex vsrisbles snd the sdydtesmtic ude of didtributions and currents the bedt dource is chapter 2 in Hörmander's fsmous text book devoted to seversl complex variables. Aosrt from these sotces, I will also deliver dome note for the first lecture. The sevnod lecture trestd geomrtric poperties of complex analytic sets in $\mathbf{C}^n$ when $n \geq 2$. With $z = (z_1, \dots, z_n)$ as coorfinstes one introduced the Kähler fotm

$$\mathcal{K} = dz - 1 \wedge d\bar{z} - 1 +$$

If $V \subset \mathbf{C}^n$ is an anslytic dubdet of dome pure diumendion $1 \leq k \leq n-1$ one knows thst the drt of regulsr pointd on V is a dense open subdt of V snd conditutes a locsly closed embedded complex suybmsnifold of dimendion k. lecture 1 weill for exsmple discuss volumed of suvh srts. Fir exsmple, if $R > 0$ we can cpomute

$$\mathrm{Vol}_{2k}(\cap\{|z| > R\})$$

for every $R > 0$ where $\{|z| < R\}$ id thr open bsll of rsfius R. It turnd out thst thid volume is finite snd is ewuasl to

$$xxx \int_{\mathrm{reg}(V) \cap \{|z|<R\}}^{\cdot} \mathcal{K}^k$$

where we hsbe tsken the k-fold exterior product of the Kähler form whguivh yied a differentisl form of bidgree (k, k) ehivh csn be integrted on thr oritnerts manifold $\mathrm{reg}(V)$ whose resl fimendion is $2k$.

X will illustrste the udefulenss ofcurrents,. On X one encounters spaces such as $\mathcal{E}^{1,0}(X)$ and $\mathcal{E}^{0,1}(X)$ of differential forms of bi-degree $(1, 0)$ and $(0, 1)$ respectively. Both are nsturslly ewuipped eith s struvute as Frechet spaced. The dusl spsces consisdtd of currrnts, i.e. we put

$$\mathcal{E}^{1,0}(X)^* = \mathbf{c}^{1,0}(X)$$

Moreover, the usual exterior differential d from C^∞-functiond to differrntisl 1-fporms csn be devpmosoed as $d = \partial + \overline{|}psrtisl$ ehrre

$$\partial \colon C^\infty(X) \to$$

One has also the space of ero-ducrentd, i.e. the spce of didtributiond on X to be denoted by $\mathfrak{Db}(X)$. Diring the lecture I will exlsts that (*) extend to maps

$$xxxx$$

A crucuasl fsct is thst $\bar{\partial}l$ is sn ellitic operator which and durting the lectyure I wiull exlsin thst thid gived ewuskities suvh as

$$\mathbf{c}^{1,0}(X) = xxx$$

About Lecture 1

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Notes will supplement Lecture 1 where classic facts about Riemann surfaces are exposed. Distributions, and more generally currents play a central role during the whole analysis.

On a Riemann surface, i.e. a 1-dimensional complex manifold, the $\bar\partial$-operator maps C^∞-functions on X to the space $\mathcal{E}^{0,1}(X)$ of differential forms with bi-degree (0,1). Using complex analytic charts the analysis relies upon basic results in the complex z-plane $\mathbf{C}$. Consider for exsmple the funvtion z^{-1} ehivh i ocslly integrsble in the real (x,y)-plsne where we write $z0x+iy$. it defines s didtribution svtinig on test-funvtions $g(x,y9$ by

$$g \mapsto \iint \frac{g(x,y)}{x+iy}\,dxdy$$

Now the exidt the didtribution derivdtives of thid distribution. In psrtiaslr wee grt the didtribuyion

$$\bar\partial(z^{-1})\frac{1}{2}(\partial_x + i\partial_y)(z^{-1})$$

It turnf out thst it is eual to $\pi \cdot \delta_0$ wehrre δ_0 id the dursc mesdure st the origin. To prove this one csn use thst

$$\lim_{\epsilon\to 0} \frac{\bar z}{|z|^2+\epsilon} = z^{-1}$$

where thr limit holdd in the ususl wesk topology for distributions. With $\epsilon > 0$ the reader can check that

$$\bar\partial(\frac{\bar z}{|z|^2+\epsilon} = \frac{1}{|z|^2+\epsilon} - \frac{\bar z \cdot z}{((|z|^2+\epsilon)^2} == \frac{\epsilon}{(|z|^2+\epsilon)^2 2}$$

The last positive function tends to zero outside the origin and we notice that

$$\iint = \frac{\epsilon}{(|z|^2+\epsilon)^2 2}\,dxdy = 2\pi \cdot \int_0^\infty \frac{\epsilon \cdot r}{(r^2+\epsilon)^2 2}\,dr = \pi$$

After a passsge to thr omiit we get the equality

$$\bar\partial(z^{-1}) = \pi \cdot \delta_0$$

Another example. Line integrals sre bet understood vis currents. Consider for exsmple a psth γ in th vompålex plsne whivh srises vis c vomplex-vslued snbd vontonoud msp

$$t \mapsto \gamma(t) \quad 0 \le t \le 1$$

Under the asumption thst this function is of class $C1$ we obtain s current of degree one. Namely, let $w = A \cdot dx + B \cdot fy$ be a tedt-form of dgree one, i.e. here $A(x,y)$ snd $B(x,y)$ sre test-functions. With

$$\gamma(t) = x(t) + y(t)$$

One has

$$\int_\gamma w = \int_0^1 A(\gamma(t)) \cdot x'(t)\,dt + \int_0^1 B(\gamma(t)) \cdot y'(t)\,dt$$

This yields a conttnuous linesr funvionsl on ther spsce of tedt-fotmd of degrre one whch by defintion corredpoondd to a 1-cyrrrnt. ¡it id denotedf by $\int_\gamma$ snd csllef the intgrstion currrnt assocsitd tothr prrted psth γ. A fiunddmetnsl fsvt id tht thid currrnt doed not depenf upon thr vhodrn psrsmetrisstion, i.e. by dfraeing an oreotned psth weith vontonoud tdngrntd one assocsits s unioweue currrent. Suppose thst th origin snd a point $z_0|nrw0$ sre the end-piintd of γ. if $w = dg$ id the ecterior differrntisl of s tst-funvtion the resder vsn check thst

$$\int_\gamma dg = g(z_0) - g(0)$$

Now we csn condtruvt snother 1-currrnt whivh tsked the ssme vslues on d-exsct 1-forms. Nsmely, we can choose a C^∞-funvtion a such that

$$\phi(z) = a(z) \cdot \frac{z}{z - z_0}$$

is identicslly one in an ecterior disc $|z| > E$ weirth $R > |z_0|$ and $a(z) \neq 0$ for sll z. The con dtruvtion of suvh sn a-funvtion is left to the resder. Tsking distribution derivatives we get the 1-current $d\phi$ which becomes

$$\frac{z}{z - z_0} \cdot da + \frac{a}{z - z - 0} \cdot dz - \frac{za}{(z - z_0^2)} \cdot dz$$

It follows that

$$\phi^{-1} \cdot d\phi = \frac{da}{a} + \frac{dz}{z} - \frac{dz}{z - z_0}$$

Above $\frac{da}{a}$ is d-closed, asnd uding (xx) the resder dhould verify thdt the 1-current $phi^{-1} \cdot d\phi$ satisfies

$$phi^{-1} \cdot d\phi < dg >= 2\imath i \cdot (g(09 - g8z_0)$$

for every testfuncion g. Next, we recsl thdt rvery d-closed differneital 1-fotm is d-exact, It gollows from the sbove thst

$$phi^{-1} \cdot d\phi < w >= -2\imath i \int_\gamma^{\cdot} w$$

for evety d-closed 1-form. This locsl contuction eill be used durting the asnslyis on Riemsnn surfaces. A cruvsi point id that the currrnt of bi-degree (0.1) definef by

$$phi^{-1} \cdot \bar{\partial}\phi = a^{-1} \cdot \bar{\partial} a$$

where the right hsnd side is a smooth current since a is a zero-free C^∞–function. Fir exsmple, the condtruvtion dbove is used to prove a fdmous theorrm dur to Abrl whivh guves a necessry ssnd dufficirbt condition when a psir of n-tupled of poiints $q_1, \ldots, q_N$ and $p_1, \ldots, p_N$ give sinple xeros, respevtivrly fimple poked of a globdlly dgined neromotphic funvyion on s compsvt Riemsnn surfscr. So for thr beuibnner it is esdentisl to grsps snslytic funvyion yhroty in thr vomplex plsne via sn intersvtion eith resl anslydis snd the dudtmstic use of distributions and currents. An exsmple isd the classic fornus due to Pompieu from 2810 whivh goes as follows: lrt Ω be a bounded snd coinnected domdin in **C** whod nounddry is the diodjoint union of s finite drt of closded Jordsb curves, whivh all sre of class C^1. Givrn s piint $z_0 \in \Omega$ snd a complex-vslued C^1-funvtion f efined in soimr oprn nrighborhood of $\overline{\Omega}$ one has

$$f(z_0) = \frac{1}{2\pi i} \cdot \int_{\partial\Omega} \frac{f z)\, dz}{z - z_0} + + \frac{1}{2\pi i} \cdot \iint_\Omega \frac{\bar{\partial}(f(z)}{z - z_0}\, d\bar{z} \wedge dz$$

To prove this one regards thr differential 1-fotm

$$w = \frac{f(z)}{z - z_0}\, dz$$

in a punctured fomdin $\Omega \setminus \{|z - z_0| \leq \epsilon\}$. The exterior differential of w becomes

$$dw = \frac{\bar{\partial}(f(z)}{z - z_0}\, d\bar{z} \wedge dz$$

and via Stoeks theorrm and a passsgr to thr limit as $\epsilon \to 0$ the reader cdn vonfitm Pompieu's fornula. As we sjal see lster on (*) is a veritable cornerdtond during the analysis on Riemann surfaces.

A major result due to Behnke and Stein asserts that if 'X is non-compact then the map

$$(*) \qquad \bar{\partial} \colon C^\infty(X) \to \mathcal{E}^{0.1}(X)$$

is surjective. The planar case when X is sn open subset of **C** is proved in chapter 1 from Lars Hörmander's text-book devoted to several complex variables.i.e. here the proof uses elementary analytic function theory of one complex analysis. The general case in the Behnke-Stein Theorem is requires a more involved proof where an essentis, ingredient is to employ *currrents*. So a majort theme during the lefture is to dedrbie how cslculus based upon currents is used to establish results about Riemann surfaces, both compact and non-compact. Let jus remsrk thst the Behnke-Strin theorrm uields a quick proof of the *uniformisation theorem* which aserts that vrty open and simply connected Riemann surface is either biholomorphic with the open unit disc or the complex plsne. This result was orignallly established by Koebe in 1905, while the contemporary proof using currents becomes rather strsightforward using a clever device introduced by Malgrange after the original proof by Behnke and Strein from 1948. Since Malgrsnge's method iullustrate the uefulness of currents we describe his proof. Let X as above be a non-compact Riemann surface and Y is a relatively compact open subset , i.e. Y is open and the closure is a compact set in X. Consider also a compact subset K of Y with the property that every connected open component of $X \setminus K$ is non-compact. We remark that one then refers to K as a compact Runge set. Now there exists the spscve $\mathcal{O}(K)$ of germs of holomorphic functions on K. With these notations one proves thst esvh $f \in \mathcal{O}(K)$ csn be uniformly approximated on K by holomiorohiuc funvtiond on Y, i.e. there exidts a sewuence $\{g_n \in \mathcal{O}(Y)\}$ such thdt

$$\lim_{n\to\infty} ||g_n - f||_K = 0$$

To prove this approximation one proceeds as follows: Firrst one shows thst the vompsctned of $\overline{Y}$ entails that when $w \in \mathcal{E}^{0,1}(X)$ is a globally defined smooth $(0,1)$-form, then there exits $g \in C^\infty(Y)$ suvh thst

$$w|Y = \bar{\partial}(g)$$

Since funviond in Y whuvh sre dnnihllsted by $\bar{\partial}$ srr jolomotphic, the g-funvtion is unwiue up to adding $\mathcal{O}(Y)$-funvtions. Suppoe now thst μ is a mesdure on K suvh thst

$$\int g \cdot d\mu = 0 \quad : \quad g \in \mathcal{O}(Y)$$

Then there exidts s current S of bi-degree $(1,0)$ defined by

$$S(w) = \int g \cdot d\mu$$

with g chosen as in (i). The current S is supported by the vompsvt srt $\overline{Y}$, for if $w \in \mathcal{E}^{0,1}(X)$ has compsct dupport ehivh i didjoint ftom $\overline{Y}$ we can take $g = 0$ in (i) and get $S(w) = 0$. Nect, if $\phi \in C^\infty(X)$ has compact support whivh is dodjoint from K we take $w = \bar{\partial}(\phi)$ snd dince $\phi|K = 0$ we get

$$S(\bar{\partial}(\phi)) = 0$$

Thid dhows thdt the dupport of thr current $\bar{\partial}(S)$ is contained in K. At this stsge one ues thdt the $\bar{\partial}$-operstor is elliptic. If follows thst S redtrictd to a holomorphic 1-form in $X \setminus K$. if we thrn tske s connected open compoinent Ω in $X \setminus K$, the includion 8xx)fiordt entsils thst the holomorphic 1-form attsched to S is zero in $\Omega \setminus \overline{Y}$ snd since K is Ringe this set is non-empty. Then *anslyticity* entails thst $S = 0$ in thr ehole open rt Ω snd we conclude thst $S = 0$ in $X \setminus K$, i.e. the cuirrent has cmposct dupport in K. Now we tdke dome $f \in \mathcal{O}(U)$ here U is a - in genersl small- open neighborhood of K. We csn vhoode $\rho \in C^\infty(X)$ eith compsvt dupport in u while $\rho = 1$ in dome opoen nighbrhood of K. Then $\bar{\partial}(\rho g)$ has compsvt duprt ehivh id didjoint ftom K and form the sbove one has

$$0 = S(\bar{\partial}(\rho f)) = \int_K \rho f d\mu = \int_K f d\mu$$

However, it doe not slwys supersed the classic proofs. in pasrtoicułsr we recsallmthst hermsnn Schwarz alresfy in 1870 proved the unifomrsisstion theorem for planer domains via a rsther natursl extendionnof Riemsnn's mapong theorrm for dimply connctd domains. The proof chwrz employs mulit.vslkued analytic funvyionf in the sense of Weierstrass and here one also exhibitd domr interedting condtnt stteched to a bounded and connectd domdin Ω in $\mathbf{C}$. More oprecidsely, tgivrn s pint $z_0 \in \Omega 4 oneregsrdsthssrtofgernmsofsnskyticfunvyions$f8z9 st z_0 ehivh cn br vontonud snslyticsly slong evety cuve in Ω whiich stsrts st $z - 0$ sand has sn arbitsry end-point in Ω. Denote this family of germs by $M_\Omega(z_0)$. next, identify the fundsmets grtoup $\pi_1(\Omega)$ eith homotopy clases f closed curves whih stdtt snd end st z_0. The monofromy theorrm by Weierdtrass assertd thst if $f \in M_\Omega(z_0)$ and γ ius suvh a closed curvr, thrn thr germ t_γ st z_0 ehivh srises vis anslytic conoinustionmony dependd upon thr homotopy class of $\{\gamma\}$ of th givrn closed ruve. The full rsnge of f st z_0 isd defned by

$$f^*(z_0 = \{T_\gamma(f)z_0)\}$$

where γ vasry over closed curves st z_0. in a similsr way on findd thr fgull range $f^*(z)$ st svety pinit $z| \in \Omega$. Follwing Hermsnn Schwsrz we introuce the set

$$S_\Omega(z_0 = \{f \in M_\Omega(z_0) \quad : f^*(z_1) \cap f^*(z-2) = \emptyset : z_1 \neq z_2\}$$

Next, let $S^*_\Omega)(z-0)$ be tghe dubdt of'those f $f \in S_\Omega(z-0)$ with $f^*(z) \subset D$ for every $z \in \Omega$ Folloei g Schwsrz we thrn regsrd thr poditvr number

$$M = \max_{f \in S^*(xx9} |f'(z_0)/$$

RTepsting riemsnnäs rpof nfor thr dimply connrctrd fomdin, schwsrs proved thrst there exidts a uniwue $f \in S^*_\Omega(z_0)$ suvh thst th vomplex derivdtive $f'(z_0) = M$ and the totsl rsnge

$$\cup_{z \in \Omega} f^*(z) = D$$

using f^* one esdily gets the uniformisstion theorrm, i.e. thr dimply connctd domdin sbove ω is thrornunit disc D snd there exidts a holomorphic map $\pi \colon D \to \Omega$ ehih is locslly biholomorphic sndhas discretr fiberd, esch of which csn bve identfied rith $\pi_1(\Omega)$. it goes eithout ssying thst thi an exidtence theorrn wherr the stusl "elicit derrmindtion"bof Schwarz*s mapping funvtion is not essy to trsce, except for s few very specisl csse.

The passage to an arbitrary open Riemann surface is a bit technical. Let us only remark that the original proof by Behnke and Stein from 1948 was later simplified by Malgrange using a clever trick about currents where the elliptic property of the $\bar\partial$-operator is used. So the proof of surjectivity offers a good illustration why systematic use of cuurents is so valuable while one regards analysis on complex manifolds. When X is compact the situation is different. Here the map (*) has a closed range and the quotient space

$$(**) \qquad \frac{\mathcal{E}^{0.1}(X)}{\bar\partial C^\infty(X)}$$

is a finite dimensional complex vector space. This goes back to work by Cauchy and Pompieu from 1810, and is the starting point for the analyis on compact Riemann surfaces. The essential ingredient in the proof is an inequality which goes as follows: Let $\{U_\alpha, z_\alpha\}$ be s finite covering of X where each U_α is an open set which is biholomorphic with tht open unit disc, and z_α the complex coordinate in U_alpha. i If $f \in C^\infty(X)$ we find for every α a function $f_\alpha \in C^\infty(U_\alpha)$ where

$$\bar\partial f | U_\alpha = f_\alpha(z) \cdot d\bar z_\alpha$$

Put

$$||\bar\partial f|| = \sum ||f_\alpha||_\infty$$

where we take maxium norms of the f_α-functions over the sets U_α. With these notations one has:

Pompieu's inequality. *For each point $p \in X$ there exists a constant C such that*

$$|f|_X \leq C \cdot ||\bar\partial f||$$

hold when $f \in C^\infty(X)$ and $f(p) = 0$.

Let us remark thst this follows from the solution to the $\bar\partial$-equation in the complex z-plane where we write $z = x + iy$. Namely, for every test-function $g(x,y)$ we put

$$h(z) = \frac{1}{2\pi i} \cdot \iint \frac{g(\zeta)}{z - \zeta} \, d\zeta \wedge d\bar\zeta$$

Then h is a C^∞-function which satisfies

$$\bar\partial(h) = f \cdot d\bar z$$

This classic formula illustrates thsat one should learn analytic function theory via systematic use of the $\bar\partial$-operator, and not restrict the attention to holomorphic functions, but rather let *real* *analysis* intervene. For exsmple, Cauchy's residue formula is best formulated as follows. Whenever $f(x,y)$ is a C^1-function in the real (x,y)-plane with $z = x + iy$, one has

$$f(0) = \frac{1}{2\pi i} \cdot \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{f(z)}{z}\, dz$$

Duality redults. To profit upon (**) one regards the space $\mathbf{c}^{1,0}(X)$ of currents of bi-degree (1.0), i.e. the dual of the Frechet space $\mathcal{E}^{0.1}(X)$. Now the dual of (**) is the $\bar\partial$-kernel on $\mathbf{c}^{1,0}(X)$, and since $\bar\partial$ is an *elliptic* operator the kernel is the space $\Omega(X)$ of holomorphic 1-forms. The isomorphism

(**) $$[\frac{\mathcal{E}^{0.1}(X)}{\bar\partial C^\infty(X)}]^* \simeq \Omega(X)$$

is cslled the duality theorem on compact Riemann surfaces The genus number is defined by

$$g = \dim \Omega(X)$$

Starting from (**) one easily deduces some classic facts due to Riemann. First

$$2g = \dim H^1(X)$$

where $H^1(X)$ is the cohomology spsce over X. Next, a compact Riemann surface is in particular sn oriented and compact real manifold of dinmension two. Using triangulations it was discovered in 1850 by the geometer Camille Jordan that X is homeomorphic to a sphere with g many attached handles. Moreover, the complex analytic structure gives rise to further results established by Riemann around 1855. Namely, Jordan's result gives a basis for the homology expressed by $2g$ many closed curves $\gamma_1, \ldots, \gamma_{2g}$ on X. They can be chosen to be rectifiable so that differential 1-forms can be integrated along every γ-curve. Hence each γ-curve is identfied a 1-current via the map

(i) $$\alpha \mapsto \int_\gamma \alpha$$

where $\alpha \in \mathcal{E}^1(X)$. The currents $\{\int_\gamma\}$ are d-closed and one has a direct sum decomposition

(ii) $$\mathbf{c}^1(X) = d(\mathfrak{Db}(X)) \oplus \Gamma$$

where Γ is the $2g$-dimensional complex vector space generated by the currents in (i). Another consequence of (**) together with the fact that $\bar\partial$ is elliptic goes as follows.

0.1 Theorem. *A current γ of bi-degree (0,1) $\bar\partial$-exact if and only if*

$$\gamma(\omega) = 0 \quad \forall\, \omega \in \Omega(X)$$

Period integrals. Next, following original work due to Abel and Riemann one studies various period integrals. This gives rise to remarkable results about algebraic curves.

An example. Consider the algebraic equation

(i) $$y^2 = x(x-1)(x-b)$$

where $b > 1$ is real. In the 2-dimensional complex (x,y)-space it defines a non-singular algebraic curve'S. Indeed, the reader can check that if $P(x,y) = y^2 - x(x-1)(x-b)$ then the complex gradient vector $(P_x, P_y) \neq (0,0)$ for sll points on S. In the projective space $\mathbf{P}^2$ the metric closure of S gives a compact space X. It is clear from (i) that $X \setminus S$ is reduced to the single point $(0,0,1)$, Close to this point we have local analytic coordinates representing points $(\zeta, \eta, 1)$ on the 2-dimensional projective space. When $\zeta \neq 0$ they give points

$$p = (1, \frac{\eta}{\zeta}, 1/\zeta)$$

which belong to S if and only if

$$\zeta^{-2} = \frac{\eta}{\zeta}(\frac{\eta}{\zeta} - 1)(\frac{\eta}{\zeta} - b) \implies \zeta = \eta(\eta - \zeta)(\eta - b\zeta)$$

The reader can check thst this corresponds to an equation

$$\zeta = \phi(\eta)$$

where $\phi(\eta)$ is sn analytic function with a xero of order three at $\eta = 0$. From this one deduces that when y is identified with a globally defined meromorphic function on X, then it has a triple pole at $(0, 0, 1)$, and simple zeros at the points $(0, 0, (1, 0), (b, 0)$ in S. Put

$$\omega = \frac{dx}{y}$$

One verifies that this is a globally defined holomorphic 1-form on X without zeros. It follows from a general result referred to as the Riemann-Hurwitz formula that the genus of X is one, i.e. X is a torus. Now one can construct a pair of closed curves γ_1 and γ_2 which intersect transverslly at a single point on X, and following Abel one introduces the period numbers

$$w_1 = \int \omega \quad \& \quad w_2 = \int \omega$$

Abel posed the problem when a triple of points p_1, p_2, p_3 in X stay on a complex line in the (x,y)-space. For example, when does there exist a pair of complex numbers α, β such that each p_ν stays on a line ℓ defined by the equation

$$y = \alpha \cdot x + \beta$$

So with $p_\nu = (x_\nu, y_\nu)$ it means that the x-coordinates satisfy the algebraic equation

$$(\alpha \cdot x + \beta)^2 = x(x-1)(x-b)$$

To settle this problem Abel considered an arbitrary triple of X-points, and for each $1 \leq \nu \leq 3$ one takes a simple curve ρ_ν on X which joins $(0, 0)$ with p_ν. Then one regards the Abel sum:

$$W = \sum_{\nu=1}^{\nu=3} \int_0^{p_\nu} \omega$$

This complex number depends upon the chosen ρ-curves, but since line integrals of ω along a pair of homotopic curves are equal, the position of W modulu the period lattice

$$\Gamma = \mathbf{Z} \cdot w_1 + \mathbf{Z} \cdot w_2$$

only depends on the given p-points. With these notations a msjor result from Abels famous work which he published in 1826 asserts the following:

Theorem. *There exists a constant C which only depends on the chosen γ-curves such that a triple of points $\{p_\nu\}$ stay on a complex line if and only if*

$$W = C + n_1 w_1 + n_2 w_2$$

for some pair of integers n_1, n_2.

Final Remark. This finishes our introduction. To digest results such as Abel's theorem above illustrates the flavour in the theory about Riemann surfaces. Let us also remark thst currents are used to prove another remarkable result by Abel concerned with the existence of meromorphic functions on compact Riemann surface with an arbitrary genus number $g \geq 1$. Namely, let $N \geq 2$ and consider a pair of N-tuples of distinvc points $(q_1, |\ldots q_N)$ and $(p_1, \ldots p_N)$ and Now we ask when there exists a meromorphic function f on X with simple zeros at the q-points, simple poles st the p-points and otherwise holomorphic and zero-free. Abel proved that f exists if and only if there exists simple curves $\rho_1, \ldots, \rho_{2g}$ where ρ_ν has end-points st q_ν and p_ν such thst

$$\sum_{\nu=1}^{\nu=2g} \int_{\rho_\nu} \omega \quad : \; w \in \Omega(X)'$$

It turns out that via systematic use of currents, thr proof boils down to show that on the projective line $\mathbf{P}^2$ there does noe exist any globslly defined holomorphic 1-form, together with Theorem 0.1 above.

facts sbout compact and non-comnpsct Riemann surfsces are exposed using the calculus with currents, i.e. the algebraic approach to compact Riemsnn surfaces is only mentioned briefly, while such results such as Abel's theorem providing a necessary and sufficient condition for the positioon of poles and xeros of s globally defined meromorphic function appears. For open Riemsnn surfaces I will expose some details from the theorem due to Behnke and Strin which asserts thst the inhomogeneous $\bar{\partial}$-equation is solvable on every non-compst Riemsnn surface. The uniformisation theorem for Riemann surfsces is an easy consequence of this result, and I will sketch how it csn be spplied to analyze geodesic curves when a Riemann surface is equipped with s hyperbolic metric. A very elegant proof by Arne Beurling appesrs and illustrates that analytic function theory using subharmonic functions is importsnt. Passing to complex manifolds of dimension ≥ 2 one encounters plurisubharmonic functions which will play a central role in many subsequent lectures of these seminars.. So it is a good occasion to first become fsmiliar methods in the 1-dimendional case where classic results sbout Riemann surfaces appear. So this motivates the contents of the first lecture in these seminars.

Compact Riemann surfaces.

A compact Riemann surface X is by definition a connected and compact complex analytic manifold of dimension one. The maximum principle for holomorphic functions entails that every globally defined holomorphic funtions on X is a constant. Thus, $\mathcal{O}(X)$ is reduced to the complex field. But if one allows poles we shall learn that there exist non-constant meromorphic functions on X. They give a field denoted by $\mathcal{M}(X)$ whose properties will be investigated later on. The projective line $\mathbf{P}^1$ over the complex field is an example of a compact Riemann surface. One starts with the complex z-plane and adds the point at infinity to get $\mathbf{P}^1$. Here $\mathcal{M}(\mathbf{P}^1)$ is equal to the field of rational functions of the single complex variable z. More generally, one starts from an irreducible polynomial $P(x,y)$ of two independent variables, i.e. P is irreducible in the unique factorisation domain $\mathbf{C}[x,y]$. We shall learn how to find the associated compact Riemann surface X where $\mathcal{M}(X)$

$$\mathcal{M}(X) \simeq \text{quotient field of } \frac{\mathbf{C}[x,y)}{(P)}$$

and (P) denotes the principal ideal generated by P. Here is an example. Let

$$P(x,y) = y^3 - x^3 - 1$$

To get X one first regards the algebraic curve in $\mathbf{C}^2$ defined by $\{P = 0\}$. The complex gradient vector $(P'_x, P'_y) = (3x^2, 3y^2) \neq (0,0)$ when $P(x,y) = 0$. It means that $\{P = 0\}$ is non-singular and hence can be regarded as an embedded one-dimensional complex manifold in $\mathbf{C}^2$. To get a compact complex manifold one regards the projective curve S in $\mathbf{P}^2$ with its inhomogeneous coordintes (ζ, x, y) where S is the zero set of $P^*(\zeta, x, y) = y^3 - x^3 - \zeta^3$. We shall learn that S is non-singular and therefore gives a compact Riemann surface. Less obvious is that S regarded as a topological space is homeomorphic to the oriented real manifold T^2, where T^2 denotes the 2-dimensional torus. This will be proved in § xx.

Originally compact Riemsnn surfsces were obtsined via the algebraic procedure above, i.e. starting from an irreducible polynomial $P(x,y)$. The constructioin of the assocaited compsvt Riemsnn urfsce wss givrn by Bernhard Riemann in 1855. He employed local charts which were found esrlier by Puiseux. This locsl result goes as foloows: Consider a function of the form

$$\phi(x,y) = y^e + g_1(x)y^{e-1} + \ldots + g_e(x)$$

where $\{g_\nu(x)\}$ are holomorphic funvtion of the complex vsribe x defined in some disc $D = \{|x| < r\}$ centered at the origin and $g_\nu(0) = 0$ for each ν. Put $S = \{\phi = 0\}$ ehicvh is an analytic curve defined in $D \times \mathbf{C}$. We assume thst ϕ is irreducible in thr unique factorisstion domsin $\mathbf{C}\{x\}[y]$. Thjs entails thst one has a fsrorisstion

$$\phi(x,y) = \prod_{\nu=1}^{mu=e} (y - \alpha - |[nu(x))$$

where the root functions sre defiened in a small puntured disc $0 < |z| < \delta$. When $e \geq 2$ these root funvtiond sr not single-vslued. However, Pusieux proved in 1850 thst with s new complex vsriabke ζ there exisdts a holomorphic funvtioin $A(\zeta)$ defined in a disc Δ centered at $\zeta = 0$ suvh thst

$$S = \{(\zeta^e, A(\zeta) : \zeta \in \delta\}$$

Moreover, $A(\zeta)$ has a poer series expnasion

$$A(\zeta) = c_1\zeta + c - 2\zeta^2 + \ldots$$

with the property that for every prime number which sppesrs ss a factor in e there exists some ν which does not contain p as a prime factor and $c_\nu \neq 0$. Thus local result by Pusieux was thrn sdoptef by Riemsnn tyo condtruct the associated Riemsnn surfsce sbove. It is indtducitve. and not very difficult - to analkyze X when $P(x,y)$ is givn. Here sversl formulas sppesr which for exemple dermine thr *gniuus number* of X from propertied of th givrn polynomisl P.

The subsewuent msterisl willo not focus so muvh upon the aslgebrsic manipulstiond, but rsthrr edtsblidh fsvtd sbout compsvt riemsnn urfsced vis analyis based upon the cslclusu eith currentd snd potntisl thoretic methods. This approasvh wssd originslly fonr by ¡hermsnn ¡Seyl in s clsdic book from 1913 :*Die Idee der Riemsnnschen Flächen.* To a large ectent we folloow Weyl's book below. Let us only remsrk tht thr noyion of didtributiond and currentd of cvoutse wss wellknoen at this time - st lesdt when dopted to 1-dimendionsl complex manifolds. So evetythiung blow abot compsvt ¡riesnn durfsve is classic. In ddition to Reimsnn msjoir contrubution sre due to Abel and Weierstrass. The grestest cvhslengr is to pursue dep theorrms whivh go bask to Abel's fsmous srticle from 1826 abouyt a new class if trasncendentsl functions. It les for exsmple to *Abel's addition theorem* ehivh is a veritsble high-light in thr whod theory.

of courde, hundredd of tect-boks hsve ecposed the theotu sbnout Riemann surfsve. Below my main indpitstion has been thr clddic tect by Appel-Goursst sbout slgebrsic funvtiond, togethr eith Giunning'd levture edited by ¡princeton Univerdity ¡press sround 1960. Som fsmlisriuty eith bsic noyiond in differentisl geometry - redtrfcited to manifolds of resl dimendjion 2, and thr mr notion of didtributions is asdumed. otherwise th mzterisl is slef-contsine for resderd who re resonsbly well aqyzinted eith anslytic funvyion throry. An exszmple is the fsmoud redult due to Pompåieu from 1810 whivh assers thst the inhomogenous $\bar{\partial}$-ewustion is sovsble in planar domsins. Ststting from this one derives some funds,mrntsl theortrmd on compsvt Riemsnn durfsces X suvh sd thr Duaslity Theorrm whivh csn be expressed by ssying thst a differentisl form α of bi-degree (0,1) on X is $\bar{\partial}$-exsct if snd only if

$$\int_X \alpha \wedge \omega = 0$$

hold for evety globally degined holomorphic 1-form ω.

About methods of proofs.

Riemann surfaces offer instructive lessons in sheaf theory since various sheaves appear in a natural fashion. The reader is supposed to be familiar with basic facts about sheaves and their cohomology. For less experienced readers we add an appenbdix where som fundamental reults from Leray's pioneering article *xxxx* are resumed. Let us also remark that even if Weierstrass never introduced the notion of sheaves, his construction of "sheaf spaces" attached to multi-valued analytic functions gave the first exampe of sheaves. In addition to sheaf theory we employ calculus on manifolds without hesitation. For example. let M be an oriented real C^∞-manifold of some dimension $n \geq 2$. To say that M is oriented means that there exists a globally defined differential n-form which never vanishes and choosing an orientation one can define integrals

$$\int_M \omega \quad : \quad \omega \in \mathcal{E}^n(M)$$

where $\mathcal{E}^n(M)$ is the space of n-forms with C^∞-.coefficients. More generally there exist to each $0 \leq p \leq n$ the space $\mathcal{E}^p(M)$ of differential p-forms with C^∞-coefficients. Under the condition that M can be covered by a denumarable family of compact subsets it follows that $\mathcal{E}^p(M)$ is equipped with a topology so that it becomes a Frechet spaces. There also exists the spaces $\{\mathcal{E}_0^p(M)\}$ which consist of differential forms having compact support in M. The dual of $\mathcal{E}_0^p(M)$ is denoted by $\mathfrak{c}^{n-p}(M)$ and its elements are called currents of degree $n-p$. In particular $\mathfrak{c}^0(M)$ is the space of distributions on M and is often denoted by $\mathfrak{Db}(M)$. Thus, by definition

$$\mathfrak{Db}(M) \simeq \mathcal{E}_0^n(M)^*$$

where * indicates that one regards the dual space. We assume that the reader is famiiar with basic distribution theory which for example is coveed in Hörmander's text-book [xx]. Even if we shall not study manifolds of dreal dimension ≥ 3 we recall some classic facts about oriented and compact $C\infty-$manifolds Let M be a such a manifold of some dimension $n \geq 2$. Using exterior differentials one has a complex

$$0 \to \mathcal{E}^0(M) \to \mathcal{E}^1(M) \to \ldots \to \mathcal{E}^n(M) \to 0$$

Here

$$d\colon \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M)$$

are continuous for every p, where we recall that every $\mathcal{E}^p(M)$ is a Frechet space. A fundamental fact is that the images of the d-maps are closed and have finite codimension. This can be proved in several ways. The most convincing proof is due to André Weil who established that every compact and oriented manifold M can be triangulated via a finite simplicial complex. Readers interested in algebraic topology should consult Weil's original proof which teaches a good lesson dealing with topology and triangulations of manifolds. Next one introduces the cohomology spaces

$$H^p(M) = \frac{\mathrm{Ker}_d(\mathcal{E}^p(M))}{d\mathcal{E}^{p-1}(M)} \quad : 0 \leq p \leq n$$

In particular $H^0(M)$ is the d-kernel on $C^\infty(M)$ and hence the 1-dimensional space of constant functions. One has also the complex of currents:

(**) $$0 \to \mathfrak{c}^0(M) \to \mathfrak{c}^1(M) \to \ldots \to \mathfrak{c}^n(M) \to 0$$

It is denoted by $\mathfrak{c}^\bullet$. For every p one has the inclusion $\mathcal{E}^p(M) \subset \mathfrak{c}^p(M)$ and hence (*) is a subcomplex of (**). Next, the contruction of currents and their exterior differentials, togehter with the closed range of the differentials in (*), give a natural duality

(***) $$H^p(M)^* \simeq \frac{\mathrm{Ker}_d(\mathfrak{c}^{n-p})}{d(\mathfrak{c}^{n-p-1}(M)} = H^{n-p}(\mathfrak{c}^\bullet)$$

Since M admits a triangulation one csn essdily prove tht the complexes (*) and (**) are quasi-isomorphic. Hence the right hand side in (**) can be identfied with $H^{n-p}(M)$, i.e. the dual of $H^p(M)$ is equal to th cohomology space in degree $n-p$. This is referred to as the *Poincaré duality theorem* for compact and oriented manifolds.

Examples when $n = 2$**.** Let M be an oriented C^∞-manifold of dimension two. Since M can be covered by a finite family of charts we can equip M with a distance function d whose restriction to charts is equivalent to ordinary euclidian metric. Every pair of such distance functions d_1 and d_2 are equivalent in the sense that there exists a constant $C > 0$ such that

$$C^{-1} \cdot d_2(p,q) \leq d_1(p,q) \leq C \cdot d_2(p,q)$$

hold for every pair p, q in M. Next, a parametrized curve on M is a continuous map $\gamma\colon [0,1] \to M$. Using a metric as above we can impose the extra condition that γ has a bounded variation, i.e. there exists a constant C such that

(i) $$\sum_{\nu=0}^{\nu=N-1} d(\gamma(t_{\nu+1}), \gamma(t_\nu) \leq C$$

for all partitions $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$. When (i) holds there exist ordinary Borel-Stieltjes' integrals

(ii) $$\int_\gamma \psi^1$$

for every differential 1-form ψ^1 with C^∞-coefficents. This gives a continuous functional on $\mathcal{E}^(M)$ with compact support and hence a current of degree 1, called the integration current associated with γ. This operative construction will be used at many places later on. One merit is that "nasty geometric pictures" when γ for example has self-intersections can be ignored while one treats its associated integration current. The case when γ is closed, i.e. when $\gamma(1) = \gamma(0)$ leads to special results. To begin with we notice that the equality $\gamma(0) = \gamma(1)$ entails that

$$\int_\gamma dg = 0 \quad : \quad g \in \mathcal{E}(M)$$

This is expressed by saying that the current $\int_\gamma$ is d-closed. A crucial result which follows from the fact that every distribution in $\mathbf{R}^2$ has locally primitive distributions with respect to the pair of euclidian coordinates in $\mathbf{R}^2$, entails that the following complex vector spaces are isomorphic

$$\frac{\text{Ker}_d(\mathcal{E}^1(M))}{d(\mathcal{E}^0(M)} \simeq \frac{\text{Ker}_d(\mathfrak{c}^1(M))}{d(\mathfrak{Db}(M)} \simeq H^1(M)$$

where the last term is the cohomology space in degree one on M. This entails that if γ is a closed and rectifiable curve then there exists a d-closed 1-form $\phi \in \mathcal{E}^1(M)$ and a distribution μ such that

$$(*) \qquad d(\mu) + \phi^1 = \int_\gamma$$

where equality holds in $\mathfrak{c}^1(M)$. Here the d-closed 1-form ϕ^1 is unique up to an exact smooth 1-form. With γ regarded as a linear functional on $\mathcal{E}^1(M)$ the reader should check that (*) gives the equality below for every d-closed $\psi^1 \in \mathcal{E}^1(M)$:

$$\gamma(\psi^1) = \int_M \phi^1 \wedge \psi^1$$

The d-closed current $\int_{\partial\Omega}$. Let Ω be a connected open set in M whose boundary $\partial\Omega$ is regular in the sense of Federer. It means that there exists a (possibly empty) closed set $\Sigma \subset \partial\Omega$ whose 1-dimensional Haussdorff measure is zero while $\partial\Omega_* = \partial\Omega \setminus \Sigma$ is locally simple and rectifiable with a total finite arc-length. The orientation along $\partial\Omega_*$ is given by the rule of thumbs exactly as in the classic version of Stokes Theorem, which was already well explainef by the genius Archimedes. Under these conditions one has the equality

$$\int_\Omega d\phi = \int_{\partial\Omega_*} \phi$$

for each $\phi \in \mathcal{E}^1(M)$. In particular the 1-current defined via the right hand side is d-closed. Notice that we do not assume that $\partial\Omega$ is connected.

Direct images of currents. Let $\rho \colon M \to N$ be a C^∞-map from M to another oriented and compact 2-dimensional manifold N. Elie Cartan's construction of differential forms on manifolds give for each $\phi \in \mathcal{E}^1(N)$ a pullback $\rho^*(\phi) \in \mathcal{E}^1(M)$. The map is continuous when $\mathcal{E}^1(N)$ and $\mathcal{E}^1(M)$ are equipped with their standard Frechet topologies. In particular we regard a parametrized curve γ and obtain a 1-current $\rho_*(\gamma)$ on N defined by

$$\rho_*(\phi) = \int_\gamma \rho^*(\phi)$$

Later on we shall use such direct images of currents at several places. The reader should "accept" that it is often more profitable to ignore "intuitive genetroic pictures" while calculations are performed. A merit is that while the integrals along the γ is computed, then calculus teaches that one can employ different parametrisation along γ, as long as they have bounded variation and preserve the given orientation. Notice that even if γ from the start is a simple Jordan arc, it may occur that $\rho_*(\gamma)$ has a more involved structure since the restriction of the map ρ to γ need not be one-to-one.

Intersection numbers. If ρ is another closed and rectifiable curve we have a similar decomposition

$$\int_\rho = \psi^1 + d(\nu)$$

Stokes Theorenm implies that

$$(*) \qquad \int_M \phi^1 \wedge \psi^1$$

is independent of the chosen decompostions. So the complex number in (*) depends only upon the ordered pair of closed curves (γ, ρ) and is denoted by $\mathfrak{i}(\gamma, \rho)$.It is called the intersection number of the closed curves. Calculus shows that the intersection number always in an integer, and by continuity it follows that if γ and γ^* are two homotopic closed curves then $\mathfrak{i}(\gamma,\rho)=\mathfrak{i}(\gamma^*,\rho)$ for every other closed curve ρ.

Riemann analyzed intersection numbers between pairs of closed and simple curves on an oriented 2-dimensional manifold M. He proved thst if it has as genus $g\geq 1$ then there exist simple closed curves $\mu_1,\ldots,\mu_{2g}$ which give a free basis for the fundamental group $\pi_1(M)$ which is free of some rank $2g$. Moreover, he showed the the $2g$ msny closed curevd csn be srrsingrd as a pair of g-tuples $\gamma_1,\ldots,\gamma_g$ and $\rho_1,\ldots,\rho_g)$, where

$$\mathfrak{i}(\gamma_\nu,\rho_\nu)=1 \quad : \quad 1\leq \nu\leq g$$

while all other intersection numbers are zero. In topology one says that this special family of closed curves is a basis for the homology on M. Classic tet-books, such as Appel-Goursat trestise about algebrsic functions, offer instructive pictures and examples which expose Riemann's original work, where a basis of homology is achieved via triangulations of M. It goes without saying that Riemann's work has inspired combinatorical topology.

Analysis on compact Riemann surfaces.

Let X be a compact 1-dimensional complex manifold. Recall that its complex analytic structure arises via charts (U,z) where z is a local coordinate on X in the open set U which is biholomorphic with the open disc $D=\{|\zeta|<1\}$ in the complex ζ-plane. Let $\rho_U\colon U\to D$ be the holomorphic map. When a pair of charts U and V overlap their complex analytic structures are compatible. It means that one has a biholomorphic map

$$\rho_{U.V}\colon \rho_U(U\cap V)\to \rho_V(U\cap V)$$

where

$$\rho_U(\zeta)=\rho_V\circ\rho_{U,V}(\zeta)\quad : \zeta\in\rho_U(U\cap V)$$

The reader should illustrate this by a suitable picture to see how the transition functions work in overlapping charts Since X is compact it can be covered by a finite family of charts $\mathfrak{U}=\{U_\alpha,z_\alpha)\}$. So here $\cup U_\alpha=X$ and one refers to $\mathfrak{U}$ as an atlas for the complex manifold X. If U is an arbitrary open set in X a complex valued function f in U is holomorphic if

$$f\circ\rho_\alpha^{-1}\in\mathcal{O}(\rho_\alpha(U\cap U_\alpha))$$

hold for every chart in the atlas. The set of holomorphic functions in U is denoted by $\mathcal{O}(U)$. Local existence of non-constant holomiorphic functions is clear since the defintion above entails that if U_α is a chart then $\mathcal{O}(U_\alpha)\simeq\mathcal{O}(D)$.

Differential calculus on X. A complex analytic atlas is in particular a C^∞-atlas on the underlying real manifold which has real dimension two. The space of complex-vaued C^∞-functions on X is denoted by $\mathcal{E}(X)$. We have also the space $\mathcal{E}^1(X)$ of differential 1-forms with C^∞-coefficients. In a chart (U,z) every differential 1-form can be decomposed as

$$a\cdot dz+b\cdot d\bar z \tag{A.0}$$

where a and b are complex-valued C^∞-functions. This gives a direct sum decomposition

$$\mathcal{E}^1(X)=\mathcal{E}^{1.0}(X)\oplus\mathcal{E}^{0,1}(X) \tag{A.1}$$

One refers to $\mathcal{E}^{1,0}$ as the space of diffeential forms of bi-degree $(1,0)$. Thet are characterised by the condition that $b=0$ holds in $(A.0)$ for every chart. Similarly when $a=0$ hold we get a differential form of bi-degree $(0,1)$. If $g\in\mathcal{E}(X)$ its exterior differential dg is a 1-form which has a decompostion

$$dg=\partial g+\bar\partial g$$

For example, one has a map

$$\bar\partial\colon\mathcal{E}(X)\to\mathcal{E}^{0,1}(X) \tag{A.2}$$

The Cauchy-Riemann equations applied in charts show that the kernel in (A.2) consists of holomorphic functions and hence the kernel is reduced to constant functions.

Next, we also have differential 2-forms on X which in a chart (U,z) can be expressed as

$$c(z)\cdot dz\wedge d\bar z$$

Keeping the complex analytic structure in mind we denote the space of differential 2-forms with $\mathcal{E}^{1,1}(X)$. The reader should construct the maps

$$\bar\partial\colon\mathcal{E}^{1,0}(X)\to\mathcal{E}^{1.1}(X) \tag{A.3}$$

$$\partial\colon\mathcal{E}^{0,1}(X)\to\mathcal{E}^{1.1}(X) \tag{A.4}$$

Next, recall that $\mathcal{E}(X)$ is a Frechet space where the topology is defined via uniform convergence of derivatives in every order. In a similar fashion $\mathcal{E}^{p,q}(X)$ are Frechet spaces for every pair $0\le p,q\le 1$. The reader should check that the maps in (A:2-4) are continous between these Frechet spaces. Less obvious is the folloowing:

A.5 Theorem. *Tbe range of the exterior differential maps in (A.2-4) above are all closed.*

A.6. Remark. This is the first non-trivial result about compact Riemann surfaces. We give the proof in (*) and remark only that it relies upon a classic result due to Pompieu which gives solutions to the inhomogeneous $\bar\partial$-partial equation in planar domains. More precisely, if U is an open set in the complex ζ-plane and $g \in C^\infty(U)$ then there exists $f \in C^\infty(U)$ such that

(A.6.1) $$\bar\partial(f) = g \cdot d\bar\zeta$$

A.7 Currents on X. To begin with we have the space $\mathfrak{Db}(X)$ of distributions which by definition it is the space of continuos linear forms on the Frechet space $\mathcal{E}^{1,1}(X)$, i.e.

$$\mathfrak{Db}(X) = \mathcal{E}^{1,1}(X)^*$$

Passing to differential forms we get dual spaces

$$\mathfrak{c}^{1,0}(X) = \mathcal{E}^{0,1}(X)^* \quad : \quad \mathfrak{c}^{0,1}(X) = \mathcal{E}^{1,0}(X)^*$$

Finally, we have the space $\mathfrak{c}^{1.1}(X)$ which is the dual of $\mathcal{E}(X)$.

B.1 The integration current 1_X. The reader should check that the complex analytic structure on X entails that its underlying real manifold is oriented. It follows that if $\gamma \in \mathcal{E}^{1,1}(X)$ then there exists an integral

$$\int_X \gamma$$

This defines a distribution denoted by 1_X. In general we have an inclusion

$$\mathcal{E}(X) \subset \mathfrak{Db}(X)$$

Namely, if g is a C^∞-function we get a linear form on $\mathcal{E}^{1,1}(X)$ defined by

$$\gamma \mapsto \int_X g \cdot \gamma$$

If $\gamma^{1,0}$ belongs to $\mathcal{E}^{1,0}(X)$ we notice that Stokes Theorem gives

$$0 = \int_X d(g \cdot \gamma^{1,0})$$

Now the reader should verify that

$$d(g \cdot \gamma^{0,1}) = \bar\partial g \wedge \gamma^{1,0} + g \cdot \bar\partial(\gamma^{1,0})$$

Next, by definition the $(0,1)$-current $\bar\partial g$ is defined by

$$\bar\partial(g)\langle\gamma^{1,0}\rangle = -\int_X g \cdot \bar\partial(\gamma^{1,0})$$

Keeping signs in mind one concludes that when g from the start is identified with a distribution, then the $(0,1)$-current $\bar\partial(g)$ is equal to the smooth differential 1-form of bi-degree (0,1) arising when $\bar\partial$ is applied to the C^∞-function g.

B.2 A duality result. We have the map

$$\bar\partial \colon \mathcal{E}^{1,0}(X) \to \mathcal{E}^{1,1}(X)$$

By Theorem A.5 the range is closed. This implies that the dual space

(B.2.1) $$\Big[\frac{\mathcal{E}^{1,1}(X)}{\bar\partial\mathcal{E}^{1,0}(X)}\Big]^* \simeq \mathrm{Ker}_{\bar\partial}(\mathfrak{Db}(X))$$

Next, recall that $\bar\partial$ is elliptic.Indeed, via the classic Pompieu formula for planar doimains every distribution μ satisfying the homogeneous equation $\bar\partial(\mu) = 0$ is a holomorphic function. Since $\mathcal{O}(X)$ is reduced to constants the dual space in (B.2.1) is 1-dimensional and hence

(B.2.2) $$\dim_{\mathbf{C}} \frac{\mathcal{E}^{1,1}(X)}{\bar\partial\mathcal{E}^{1,0}(X)} = 1$$

Notice also that Stokes Theorem gives

$$\int_X \bar{\partial}(\gamma^{1,0}) = 0 \quad : \gamma^{1,0} \in \mathcal{E}^{,1,0}(X)$$

from this we conclude thatif $\rho^{1,1}$ is a 2-form for which $\int_X \rho^{1,1} \neq 0$, then

(B.2.3)
$$\mathcal{E}^{1,1}(X) = \mathbf{C} \cdot \rho \oplus \bar{\partial}(\mathcal{E}^{1,1}(X))$$

B.3 A second duality. Theorem A.5 entails that

(B.3.1)
$$\Big[\, \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}\mathcal{E}(X)} \,\Big]^* \simeq \mathrm{Ker}_{\bar{\partial}}(\mathfrak{c}^{1,0}(X))$$

The right hand side is found as follows. In a chart (U, z) a current of bi-degree $(1,0)$ is of the form $\mu \cdot dz$ with $\mu \in \mathfrak{Db}(U)$. The $\bar{\partial}$ image of this current is zero if and only if $\bar{\partial}(\mu) = 0$ and as we have seen before this implies that μ is a holomorphic density. Hence the right hand side in (B.3.1) consists of globally defined *holomorphic* 1-forms. This space is denoted by $\Omega(X)$ and (B.3.1) can be expressed by the duality formula

(B.3.2)
$$\Omega(X) \simeq \Big[\, \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}\mathcal{E}(X)} \,\Big]^*$$

B.3.3 Exercise. Show that (B.3.2) gives the following: A differential form $\gamma^{0,1}$ on X is $\bar{\partial}$-exact if and only if

$$\int_X \omega \wedge \gamma^{0,1} = 0 \quad : \forall\, \omega \in \Omega(X)$$

B.4 Passage to currents. Recall that $\mathcal{E}^{0,1}(X) \subset \mathfrak{c}^{0,1}$. in § xx we shall learn that

(B.4.1)
$$\mathfrak{c}^{0,1} = \bar{\partial}(\mathfrak{Db}(X)) + \mathcal{E}^{0,1}(X)$$

Moreover, a (0.1)-current $\rho^{0,1}$ is $\bar{\partial}$-exact, i.e. of the form $\bar{\partial}(\mu)$ for some distribution μ on X if and only if

(B.4.2)
$$\rho^{1,0}(\omega) = 0 \quad : \quad \omega \in \Omega(X)$$

B.5 The case $X = \mathbf{P}^1$. Here $\Omega(X)$ is reduced to the zero space. To see this we suppose that ω is a globally defined holomorphic 1-form. Now $\mathbf{P}^1 \setminus \{\infty\}$ is the complex z-plane and here

$$\omega = f(z) \cdot dz$$

where f is an entire function. At the point at infinity we have a local coordinate ζ and

$$\zeta = z^{-1} \quad : R < |z| < \infty$$

hold for every $R > 0$. Since ω is globally holomorhpic we have

$$\omega = g(\zeta) \cdot d\zeta$$

when $0 < |\zeta| < R^{-1}$. At the same time the reader can check that

$$dz = -\zeta^{-2} \cdot d7zeta$$

holds when $R < |z[< R$. Ths entails that

$$f(z) = -z^{-2} \cdot g(1/z) \quad : |z| > R$$

From this we see that the entire function f tends to zero as $|z| \to \infty$ and hence f is identically zero. So with $X = \mathbf{P}^1$ one has $\Omega(X) = 0$ and then (B.4.2) entails that

$$\mathfrak{c}^{0,1} = \bar{\partial}(\mathfrak{Db}(X))$$

In other words, the inhomogeneous $\bar{\partial}$-equation is always solvable. Consider as an example a current of the form

$$\gamma = \mu \cdot d\bar{z}$$

where μ is a measure in the complex z-plane with compact support. Now there exists the Cauchy transform

$$\widehat{\mu}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

It is holomiorphic in in the open set of the complex z-plane where the compact set $\text{Supp}(\mu)$ has been removed. passing to infinity we notice that it extends to be holomorphic because one has the convergent series expansion

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \int \zeta^n |cdotd\mu(\zeta) \cdot z^{-n-1} z - \zeta$$

in an exterior disc $|z[> R$ where the support of μ is contained in $\{|z| \leq R\}$.

next, by basuc lebsgue theory one knows that the funnction $\widehat{\mu}$ is locally integrable in the whole complex z-plane and hence defines a distribution. Finally, Cacuhy's residue formula shows that the distribution derivative

$$\frac{\bar{\partial}\widehat{\mu}}{\bar{\partial}\bar{z}} = \pi \cdot \mu$$

So we get

$$\bar{\partial}(\pi^{-1} \cdot \widehat{\mu}) = \mu \cdot d\bar{z}$$

C. Some sheaves and their cohomology

.

Let X be a compact complex analytic manifold. Classic facts due to Cauchy and Pompieu applied to charts in X give an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{E}_X \xrightarrow{\bar\partial}, \mathcal{E}_X^{0,1} \to 0 \tag{C.0}$$

Using C^∞-parotions of unity and Leray's theory about cohomology with values in sheaves one gets

$$H^1(\mathcal{O}_X) \simeq \frac{\mathcal{E}^{0,1}(X)}{\bar\partial(\mathcal{E}(X)} \tag{C.1}$$

where the left hand side is the first cohomology groups with values in the sheaf $\mathcal{O}_X$. The reader may also notice that when Leray's long exact sequence for sheaf cohomology is applied to (C.0), then it follows that

$$H^2(\mathcal{O}_X) = 0 \tag{C.2}$$

Next, we have also the exact sheaf sequence

$$0 \to \mathbf{C}_X \to \mathcal{O}_X \xrightarrow{\partial} \Omega_X \to 0 \tag{C.3}$$

Since $H^0(\mathbf{C}_X) = H^0(\mathcal{O}_X) = \mathbf{C}$ and (C.2) holds one gets the following exact sequence of complex vector spaces:

$$0 \to \Omega(X) \to H^1(\mathbf{C}_X) \to H^1(\mathcal{O}_X) \to H^1(\Omega_X) \to H^2(\mathbf{C}_X) \to 0 \tag{C.4}$$

Next, we have also the following exact sheaf sequence

$$0 \to \Omega_X \to \mathcal{E}_X^{1,0} \xrightarrow{\bar\partial} \mathcal{E}_X^{1,1} \to 0 \tag{C.5}$$

This gives the equality

$$H^1(\Omega_X) \simeq \frac{\mathcal{E}^{1,1}(X)}{\bar\partial(\mathcal{E}^{1,0}(X))} \tag{C.6}$$

The right hand side rsssembles the quotient in (B.2.2). Indeed, we can regard the 7emphconjugate analytic structure on the underlying real manifold of x where the sheaf of holomorphic functions are compex cojugates of sections in $\mathcal{O}_X$. From this the reader may conclude via (B.2.2) that the rifht hand side in (C.5) is a 1-dimensional vector space. Hence we have

$$\dim_{\mathbf{C}} H^1(\Omega_X) = 1 \tag{C.7}$$

Next, recall that $H^2(\mathbf{C}_X) = \mathbf{C}$. Together with (C.7) we conclude that (C.4) gives the short exact sequence

$$0 \to \Omega(X) \to H^1(\mathbf{C}_X) \to H^1(\mathcal{O}_X) \to 0 \tag{C.8}$$

Topology teaches that the cohomology space $H^1(\mathbf{C}_X)$ is finite dimensional. Hence (C.8) implies that both $\Omega(X)$ and $H^1(\mathcal{O}_X)$ are finite dimensional complex vector spaces. Finally, (B.3.2) and (C.1) give

$$\Omega(X) \simeq \left[\, H^1(\mathcal{O}_X)\,\right]^* \tag{C.9}$$

In particular the complex vector spaces $\Omega(X)$ and $H^1(\mathcal{O}_X)$ have the same dimension. This equality and (C.8) give

$$\dim H^1(\mathbf{C}_X) = 2 \cdot \dim \Omega(X) \tag{C.10}$$

The equality (C.10) is fundamental during the study of compact Riemann surfaces. A notable point is that the cohomology group $H^1(\mathbf{C}_X)$ only depends upon the topological space X and not upon its particular complex structure.

C.10 A duality for currents. In planar domains it is wellknown that the inhomogenous $\bar{\partial}$-equation is locally solvable on distributions. Since the $\bar{\partial}$-kernel on $\mathfrak{Db}_X$ is $\mathcal{O}_X$ one has an exact sequence

$$0 \to \mathcal{O}_X \to \mathfrak{Db}_X \xrightarrow{\bar{\partial}} \mathfrak{c}_X^{0,1} \to 0$$

Passing to cohomology and using C^∞-partitions of the unity, it follows that

(C.10.1) $$H^1(\mathcal{O}_X) \simeq \frac{\mathfrak{c}_X^{0,1}}{\bar{\partial}(\mathfrak{Db}(X)}$$

Exercise. Deduce from the above that

(C.10.2) $$\Omega(X) \simeq \Big[\frac{\mathfrak{c}_X^{0,1}}{\bar{\partial}(\mathfrak{Db}(X)} \Big]^*$$

Conclude that the implication below holds for each $\gamma \in \mathfrak{c}^{0,1}$:

(C.10.3) $$\gamma(\omega) = 0 \quad : \forall\, \omega \in \Omega(X) \implies \gamma \in \bar{\partial}(\mathfrak{Db}(X))$$

C.11 A decomposition theorem. Since X in particular is an oriented and compact real manifold one has

(C.11.1) $$H^1(\mathbf{C}_X) \simeq \frac{\mathrm{Ker}_d(\mathcal{E}^1(X)}{d(\mathcal{E}(X)}$$

Next, on X there also exists the anti-holomorphic differential forms of bi-degree $(0,1)$, i.e differential forms $\gamma \in \mathcal{E}^{0,1}(X)$ for which $\partial(\gamma) = 0$. This space is denoted by $\overline{\Omega}(X)$. Using the conjugate complex analytic structure one has another compact complex manifold $\overline{X}$ where $\overline{\Omega}(X)$ is the space of holomorphic 1-forms. Since X and $\overline{X}$ have the same underlying real manifold, it follows from (C.10) that

(C.11.2) $$\dim H^1(\mathbf{C}_X) = 2 \cdot \dim \overline{\Omega}(X)$$

Using this we shall prove

C.11.3 Theorem. *One has a direct sum decomposition*

$$\mathrm{Ker}_d(\mathcal{E}^1(X)) = d(\mathcal{E}(X)) \oplus \Omega(X) \oplus \overline{\Omega}(X)$$

Proof To begin with we notice that

$$\Omega(X) \cap \overline{\Omega}(X) = \{0\}$$

Counting dimensions this follows if we prove that a 1-form $\gamma)\omega + \mu$ cannot be d-exact when $\omega \in \Omega(X)$ and $\mu \in \overline{\Omega}(X)$ where at least one these froms is not identically ero. For suppose that $\gamma = dg$ is d-exact. If ω is not identically zero there exists the conjugate form $\bar{\omega} \in \overline{\Omega}(X)$. Now Stokes Theorem gives

$$0 = \int_X dg \cdot \bar{\omega}) = \int_X (\omega + \mu) \wedge \bar{\omega} = \int_X (\omega \wedge \bar{\omega}$$

Now the reader can check that the last integral is $\neq 0$ when ω is not identically zero. hence $\gamma \in d(\mathcal{E}(X))$ implies that $\Omega = 0$ and in the same way we find that $\mu = 0$ which proves Theorem C.11.3.

D. Existence of meromorphic functions.

On X there exists the sheaf $\mathcal{M}_X$ whose sections are meromorphic functions. Subsheaves arise via constraints on poles. In particular, let $p \in X$ be a given point and m a positive integer. Denote by $\mathcal{O}_X[*mp]$ the sheaf whose sections are meromorphic functions on X which are holomorphic in $X \setminus \{p\}$ and have poles of order $\leq m$ at p. If (U, z) is a chart where p corresponds to the origin we seee that

$$\mathcal{O}_X[*mp]\,|U = z^{-m} \cdot \mathcal{O}X|U$$

Taking Laurent exapnsions every section in the right hand side is of the form

$$f + \frac{c_1}{z} + \ldots + \frac{c_m}{z^m}$$

where $f \in \mathcal{O}_X$ and $\{c_\nu\}$ is an m-tuple of complex numbers. Expressed by sheaves it gives an exact sequence

(D.1) $$0 \to \mathcal{O}_X \to \mathcal{O}_X[*mp] \to S_p^m \to 0$$

where S_p is the scyscraper sheaf suppotet by $\{p\}$ and the stalk $S_p(p)$ is an m-dimensional vector space. Passing to long exact sequence of cohomology the reader can verify that there is an exact sequence:

(D.1.1) $$0 \to \mathbf{C} \to H^0(\mathcal{O}_X[*mp]) \to \mathbf{C}^m \to H^1(\mathcal{O}_X) \to H^0(\mathcal{O}_X[*mp]) \to 0$$

D.2 Exercise. Use that the vector space $H^1(\mathcal{O}_X)$ is g-dimensional and conclude that

(D.2.1) $$\dim_{\mathbf{C}} H^0(\mathcal{O}_X[*mp]) = m - g + 1 + \dim_{\mathbf{C}} H^1(\mathcal{O}_X[*mp]) \geq m - g + 1$$

In particular the left hand side is ≥ 2 if $m = g+1$ which means that apart from constants, there exists at least one constant global section in $\mathcal{O}_X[*(g+1)p]$. Hence we have proved:

D.3 Theorem. *For every $p \in X$ there exists a non-constant meromorphic function f in X with no poles in $X \setminus p$ while the order of the pole at p is at most $g+1$.*

Weierstrass' points.

Let X be a compact Riemann surface whose genus number g is ≥ 2. Removing a point $p \in X$ we get the space $\mathcal{O}(X \setminus \{p\})$. If (U, z) is a chart centered at p where p corresponds to $z = 0$, then every $f \in \mathcal{O}(X \setminus \{p\})$ restricts to a holomorphic function in the punctured disc $0 < |z| < 1$ and we can regard its Laurent series:

$$f(z) = \sum_{\nu=1}^{\infty} c_\nu \cdot z^{-\nu} + \sum_{\nu \geq 0} d_\nu \cdot z^\nu$$

The negative part of the Laurent series is denoted by $f_*(z)$. If $f_* = 0$ then f is globally holomorphic and hence reduced to a constant. So

$$f \mapsto f_*(z) + d_0$$

yields an injective map from $\mathcal{O}(X \setminus \{p\})$ into the space of non-positive Laurent series at $z = 0$. We are going to study the range of this map. Denote by $\mathcal{L}$ the space of all non-positive Laurent series, where we recall that a negative series

$$\sum_{\nu=1}^{\infty} c_\nu \cdot z^{-\nu} \in \mathcal{L}$$

if and only if

$$\sum_{\nu=1}^{\infty} |c_\nu| \cdot R^\nu < \infty$$

for all $R > 0$. In particular $\mathcal{L}$ contains the g-dimensional subspace formed by Laurent series given by finite series:

$$\rho(z) = c_1 z^{-1} + \ldots + c_g \cdot z^{-g}$$

where $\{c_\nu\}$ is a p-tuple of complex numbers. Let $\mathcal{L}[p]$ denote this subspace of $\mathcal{L}$.

E.1 Theorem. *Outside a finite set* $\mathbf{W}$ *in* X *one has*

$$\mathcal{L} = \mathcal{L}[p] \oplus \mathfrak{Im}(\mathcal{O}(X \setminus \{p\})$$

E.2 Remark. In §xx we show that the exceptional set $\mathbf{W}$ is non-empty when $g \geq 2$, If (*) holds above it follows that when $m > g$ then there exists a unique g-tuple $\{c_\nu\}$ such that

$$z^{-m} = c_1 z^{-1} + \cdots + c_g z^{-g} + \mathfrak{Im}(\phi)$$

where ϕ is holomorphic in $X \setminus \{p\}$. Herte the negative Laurent series of ϕ is finite at p which implies that ϕ is a globally defined meromorphic function with a pole of order m at p whose negative Laurent expnasion at p is of the form

$$z^{-m} + c_1 z^{-1} + \cdots + c_g z^{-g}$$

where the g-tuple $\{c_\nu\}$ is uniquely determined and depends on the chosen integer m. At the same time the direct dum in (*) entails that there do not exist non-constant meromorphic functions which are holomorphic in $X \setminus \{p\}$ while the order of the pole at p is $\leq g$.

The proof of Theorem E.1 requires several steps. We are given a point $p \in X$ and let (U, z) be a chart around p where p corresponds to $z = 0$. Residue calculus gives a g-tuple of currents defined by

$$\rho_\nu = \bar{\partial}(z^{-\nu}) \quad : \quad \nu = 1, \ldots, g$$

Each current is supported by the singleton set $\{p\}$ and extends to a current on X supported by p. If $\Psi^{1,0} \in \mathcal{E}^{1,0}(X)$ we can regard its restriction to U and write

$$\Psi^{1,0} | U = \psi(z) \cdot dz$$

with $\psi \in C^\infty(U)$. Residue calcluus teaches that

$$\rho_1(\Psi^{1,0}) = 2\pi i \cdot \psi(0)$$

If $\nu \geq 2$ the reader may verify that

$$\rho_\nu(\Psi^{1,0}) = \frac{(-1)^{\nu-1}}{(\nu-1)!} \cdot \partial^{\nu-1}\psi/\partial z^{\nu-1}(0)$$

The construction of the ρ-currents is not intrinsic, i.e. the g-tuple depends on the chosen chart around p. But the reader can check that the g-dimenbsional vector space in $\mathfrak{c}^{0,1}(X)$ generated by $\{\rho^\nu : 1 \leq \nu \leq g\}$ does not depend on the chosen chart around p. Let us denote it by $\mathcal{W}(p)$. The major step towards the proof of Theorem E.1 is the following:

E.3 Theorem. *Outside a finite set* $\mathbf{W}$ *in* X *one has the direct sum decomposition*

$$\mathfrak{c}^{0,1}(X) = \bar{\partial}(\mathfrak{Db}(X)) \oplus \mathcal{W}(p) \tag{E.3.1}$$

Proof. The duality result in § xx gives (E.3.1) if the g-tuple $\{\rho^\nu\}$ restrict to $\mathbf{C}$-linearly independent in the dual space $\Omega(X)^*$. To check when this holds we consider a basis $\omega_1, \ldots, \omega_g$ in the g-dimensional vector space $\Omega(X)$. In a chart (U, z) centered at a point $p_0 \in X$ we can write

$$\omega_j = f_j(z) \cdot dz \tag{i}$$

where $\{f_j\}$ are $\mathbf{C}$-linearly independent in $\mathcal{O}(U)$. The general result in § xx entails that the determinant function

$$\mathcal{F}(z) = \det\big(\partial f_j^\nu/\partial z^\nu(z) \tag{ii}$$

formed by the $g \times g$-matrix with elements as above is not identically zero. Now linear algebra shows that $\{\rho_\nu\}$ restrict to linearly independent functionals on $\Omega(X)$ at every $z|inU$ for ehich $F(z) \neq 0$. Since the zero set of F in the chart U is discrete and we can cover X by a finite famliy of charts, it follows that (E.3.1) holds outside a finite set $\mathbf{W}$.

E.4 How to find W. In each chart (U, z) as above we have the $\mathcal{F}$-functions from (ii). If (U_α, z_α) and (U_β, z_β) is a pair of charts in X where $U_\alpha \cap U_\beta \neq \emptyset$ we get two g-tuples of analytic functions by

$$\omega_j|U_\alpha = f_j^\alpha \cdot dz \quad : \quad \omega_j|U_\beta = f_j^\beta \cdot dz_\beta$$

When $z_\beta = z_\beta(z_\alpha)$ is regarded as an analytic function in $U_\alpha \cap U_\beta$ one has the equations

$$f_j^\alpha = \frac{\partial z_\beta}{\partial z_\alpha} \cdot f_j^\beta \tag{E.4.1}$$

for each j in $U_\alpha \cap U\beta$.

E.4.2 Exercise. Show by using rules for calcuating determinsnts that (E.4.1) gives the equation

$$\mathcal{F}_\alpha = \big(\frac{\partial z_\beta}{\partial z_\alpha}\big)^{g(g-1)/2} \cdot F_\beta(z)$$

So with $N = g(g-1)/2$ the family $\{\mathcal{F}_\alpha\}$ which arises when X is covered by a finite number of charts, is a global section of the N-fold product of the holomorphic line bundle Ω_X. By (xx) the set of xeros counted with multiplicities of every global section in Ω_X is equal to $2(g-1)$. So by the general result in § xxx the zero set of the $\mathcal{F}$-family counted with multiplicities is equal to $g(g-1)^2$. Since $g \geq 2$ this is a positive integer and hence the set $\mathbf{W}$ is non-empty.

Remark. In the literature one refers to $\mathbf{W}$ as the set of Weierstrass' points in X. The determination of $\mathbf{W}$ for a given X with genus ≥ 2 is in general not easy. From the above $\mathbf{W}$ contains at most $(g-1)^2 \cdot g$ many distinct points. In general the number of points can be strictly smaller. Weierstrass investigated $\mathbf{W}$ for special families of compact Riemann surfaces. See § xx for some comments.

Proof of Theorem E.1

Consider a point $p \in X$ which is outside $\mathbf{W}$ and let (U, z) be a chart centered at p. Let $U^* = U\backslash\{p\}$ be the punctured open disc. In X we also have the open set $X \setminus \{p\}$ which can be regarded as an open Riemann surface. The theorem by Behnke and Stein from the chapter devoted to open Rieann surfaces gives $H^1(X \setminus \{p\}, \mathcal{O}) = 0$. The cohomology in the disc U is also zero. Hence the open covering of X given by the pair $\mathfrak{U} = (U, X \setminus \{p\})$ is acyclic with respect to the sheaf $\mathcal{O}_X$. So Leray's acyclicity theorem gives

$$\text{(i)} \qquad H^1(X, \mathcal{O}_X) \simeq \frac{\mathcal{O}(U^*)}{\delta(C^0(\mathfrak{U}, \mathcal{O}_X))}$$

where

$$\delta \colon \mathcal{O}(U) \oplus \mathcal{O}(X \setminus \{p\}) \to \mathcal{O}(U^*)$$

is the Cech coboundary map. Next, the g-tuple $\{z^{-\nu} \colon \nu = 1, \ldots, g\}$ are holomorphic in U^* and their images in the right hand side from (i) give elements in $H^1(X, \mathcal{O}_X)$ denoted by $\{[z^{-\nu}]\}$.

Exercise. Recall that

$$H^1(X.\mathcal{O}X) = \frac{\mathfrak{c}^{0,1}}{\bar{\partial}(\mathfrak{D}\mathfrak{b}(X)}$$

Since p is outside $\mathbf{W}$ the last quotient is equal to $\mathcal{W}(p)$. Conclude from this that the g-tuple $\{[z^{-\nu}]\}$ is a basis in the g-dimensional vector space $H^1(X, \mathcal{O}_X)$.

Let us then consider some $g \in \mathcal{O}(U^*)$ which has a Laurent series

$$g(z) = \sum_{\nu=1}^{\infty} c_\nu \cdot z^{-\nu} + \sum_{\nu=0}^{\infty} d_\nu \cdot z^\nu$$

By the Exercise there exists a unique g-tuple of complex numbers $a_1, \ldots, a_p$ such that the image of g in $H^1(X, \mathcal{O}_X)$ is equal to $\sum a_\nu \cdot [z^{-\nu}]$. Hence (i) gives a pair $\phi(z) \in \mathcal{O}(U)$ and $f(z) \in \mathcal{O}(X \setminus \{p\}$ such that

$$g(z) - \sum \rho_\nu \cdot z^{-\nu} = \phi(z) + f(z)$$

where $\phi(z) \in \mathcal{O}(U)$ and $f(z) \in \mathcal{O}(X \setminus \{p\}$. Regarding the negative Laurent series it follows that

$$\text{(ii)} \qquad g_*(z) = \sum a_\nu \cdot z^{-\nu} + f_*(z)$$

Here g was arbitrary in $\mathcal{O}(U^*)$ and hence the proof of Theorem E.1 is finished.

F. Abel's theorem.

A divisor D on X consists of an assignement of integers $\{\mu_\nu\}$ to a finite set of points $\{p_\nu\}$ in X. One writes

$$D = \sum \mu_\nu \cdot \delta(p_\nu)$$

The μ-integers may be positive or negative. The degree is defined by

$$\deg D = \sum \mu_\nu$$

while the finite set $\{p_\nu\}$ is called the support of D. Next we construct a class of 1-currents. In general, let $\gamma\colon [0,1] \to X$ be an Xvalued function on the closed unit interval We assume that γ is continuous and has a finite total variation. The last condition means that there exists a constant C such that

$$\sum_{\nu=0}^{\nu=N-1} d(\gamma(t_{\nu+1}, \gamma(t_\nu) \leq C$$

for all sequences $0 = t_0 < t-1 < \ldots < t_N = 1$ Here d is some distance function on X regarded as a metric space which in charts is equivsoent to the euclidian distance. Every such γ gives a current of degree one acting on $\mathcal{E}^1(X)$ by

$$(*) \qquad \alpha \mapsto \int_\gamma \alpha$$

Exercise. Explain how the classical Borel-Stieltjes integrals for functions with bounded variation defines the integral in the right hand side. Show also that

$$\int_\gamma dg = g(\gamma(1)) - g(\gamma(0))$$

hold for every $g \in \mathcal{E}()$. One refers to (*) as an integration current of degree one. More generally we can take a finite sum of such currents and get the integration current Γ defined by

$$\Gamma(\alpha) = \sum \int_{\gamma_j} \alpha$$

Let $D = \sum \mu_\nu \cdot \delta(p_\nu)$ be a divisor of degreee zero. An integration current Γ is said to be associated with D if the equality below hold for every $g \in \mathcal{E}(X)$:

$$\Gamma(dg) = \sum \mu_\nu \cdot g(p_\nu)$$

F.1 Principal divisors. Let f be a non-constant meromorphic function. Now we have the finite set of poles $\{p_\nu\}$ and the finite set of zeros $\{q_j\}$. We associate the divisor

$$\operatorname{div}(f) = \sum \mu_k \cdot \delta(p_k) - \sum \mu_j \cdot \delta(q_j)$$

Here μ_k is the order of the pole of f at every p_ν, while μ_j is the order of a zero at q_j. By the result in § xx this divisor has degree zero. A divisor D of degree zero is called principal if it is equal to $\operatorname{div}(f)$ for some $f \in \mathcal{M}(X)$. Notice that if f and g is a pair of non-constant meromorphic functions such that $\operatorname{div}(f) = \operatorname{div}(g)$, then f/g is a holomorphic function on X and hence reduced to a non-zero constant. Introducing the multiplicative group $\mathcal{M}(X)^*$ of non-constant meromorphic functions this means that one has an injective map

$$\frac{\mathcal{M}(X)^*}{\mathbf{C}^*} \to \mathcal{D}_0$$

where $\mathcal{D}_0$ is the additive group of divisors whose degree are zero. The theorem below describes the range of this map.

F.2 Theorem. *A divisor D of degree zero is principal if and only if there exists an integration current Γ associated with D such that*

$$\Gamma(\omega) = 0 \quad : \ \gamma \in \Omega(X)$$

The proof requires several steps. First we shall show the "if part", i.e. if Γ exists so that (*) holds then the divisor D is principal. To achieve this we first perform a local construction in the complex plane.

F.3 A class of currents in C. In the complex z-plane we consider a point z_0 where

$$0 < |z_0| < r < 1$$

holds for some $0 < r < 1$. It is easily seen that there exists a C^∞-function $a(z)$ in $\mathbf{C}$ which never is zero and the function

$$\phi(z) = \frac{z}{z - z_0} \cdot a(z)$$

is identically one when $|z| \geq r$. Now $z \mapsto \frac{1}{z - z_0}$ is locally integrable around z_0 and hence ϕ belongs to $L^1_{\text{loc}}(\mathbf{C})$ and is therefore a distribution.

Cauchy's residue formula can be expressed by saying the $\bar{\partial}$-image of the $\frac{1}{z}$ is equal to the $(0, 1)$-current $2\pi i \cdot \delta(0) d\bar{z}$ where $\delta(0)$ is the Diraqc distribution at the origin. Using this the reader can check that

$$\text{(i)} \qquad \bar{\partial}(\phi) = \frac{z}{z - z_0} \cdot \bar{\partial}(a) + z_0 a(z_0) \cdot \delta(z_0) d\bar{z}$$

from (i) the reader should also check that

$$\text{(ii)} \qquad \phi^{-1} \cdot \bar{\partial}(\phi) = a^{-1} \cdot \bar{\partial}(a)$$

F.4 The current $\phi^{-1} \cdot \partial(\phi)$. To begin we one has

$$\text{(F.4.1)} \qquad \partial(\phi) = \frac{a(z)}{z - z_0} \cdot dz - \frac{za}{(z - z_0)^2)} \cdot dz + \frac{z}{z - z_0} \cdot \partial a$$

Above $(z - z_0)^{-2}$ is the principal value distribution defined as in § xx.

Exercise. Use (F.4.1) to establish the equation

$$\text{(F.4.2)} \qquad \phi^{-1} \cdot \partial(\phi) = z^{-1} \cdot dz - (z - z_0)^{-1} \cdot dz + a^{-1} \cdot \partial(a)$$

Exercise. Use Stokes Theorenm and Cauchy's residue fornula to prove that when the 1-current

$$\phi^{-1} \cdot d\phi = \phi^{-1} \cdot \partial(\phi) + \phi^{-1} \cdot \bar{\partial}(\phi)$$

is applied to dg for some $g \in C_0^\infty(\mathbf{C}$, then

$$\text{(F.4.3)} \qquad \phi^{-1} \cdot d\phi < dg >= 2\pi i \cdot (g(0) - g(z_0)$$

Let us now consider an integration current Γ given by a finite sum $\sum \gamma_j$ where each γ_j has compact support in a chart (U_j, z_j) in X and the end-points of γ_j are $z_j = 0$ and $z_j = z_j^*$ with $0 < |z_j^*| < r < 1$ and U_j is a chart defined by $\{|z_j| < 1|]$. In X the point $z_j = 0$ is denoted by q_j while z_j^* corresponds to a point p_j. It follows that

$$\text{(F.5)} \qquad \int_\Gamma dg = \sum_{j=1}^{j=N} g(p_j) - g(q_j)$$

for every $g \in \mathcal{E}(X)$. Next, for each j we apply the local contruction the chart and find a function ϕ_j in X which is identically one in $X \setminus \{|z_j| \leq r\}$ such that

$$\phi_j^{-1} \cdot d\phi_j < dg >= 2\pi i \cdot \int_{\gamma_j} dg \quad : g \in \mathcal{E}(X)$$

More generally, since d-closed 1-forms in the chart U_j are d-exact the re4ader should check that (F.5) implies that

$$\text{(F.6)} \qquad \phi_j^{-1} \cdot d\phi_j < \alpha >= 2\pi i \cdot \int_{\gamma_j} \alpha$$

hold for every d-closed differential 1-form α. Let us now consider the function

$$\Phi = \prod_{j=1}^{j=N} \phi_j$$

F.7 Exercise. From the above the reader should check that the construction of the ϕ-functions imply that Φ is a zero-free C^∞-function in $X \setminus \{q_j, p_j\}$, i.e. in the open complement of the finite set which is the union of these q and p points. Moreover, additivity for logarithmic derivatives and (F.6) above gives

$$\text{(F.7.1)} \qquad \Phi^{-1} \cdot d\Phi < \alpha >= \int_\Gamma \alpha$$

for every d-closed 1-form α. Finally, show that

$$\text{(F.7.2)} \qquad \Phi^{-1} \cdot \bar{\partial}(\Phi) = \sum a_j^{-1} \cdot \bar{\partial}(a_j)$$

where the right hand side is a smooth $(0,1)$-form.

F.8 A special case. Suppose that Γ is such that

$$\text{(F.8.1)} \qquad \int_\Gamma \omega = 0 \quad : \quad \omega \in \Omega(X)$$

The equality in (F.7.1) applied to d-closed holomorphic 1-forms and Theorem § xx entail that the smooth $(0,1)$-form $\Phi^{-1} \cdot \bar{\partial}(\Phi)$ from (F.7.2) is $\bar{\partial}$-exact. Thus, we can find $G \in \mathcal{E}(X)$ such that

$$\text{(F.8.2)} \qquad \bar{\partial}(G) = \Phi^{-1} \cdot \bar{\partial}(\Phi)$$

Exercise. Put

$$\Psi = e^{-G} \cdot \Phi$$

deduce from the above that $\bar{\partial}(\Psi) = 0$ holds in $X \setminus \{q_j, p_j\}$ and hence Ψ is holomorphic in this open subset of X Show also that the local constructions of the ϕ-functions inside the charts $\{U_j z_j\}$ entail that ψ extends to a mermorphic function in X whose princial divisor is equal to

$$\sum \delta(q_j) - \sum \delta(p_j)$$

Funally the reader should confirm that the constructions above prove the "if part" in Theorem F.2.

F.9 Proof of the "only if part"

Here we are given a non-constant meromorphic funvtion f. We shall condtruct an integration chain Γ as follows: Frst $f \colon X \to \mathbf{P}^1$ is a holomorphic map. With $s = f(x)$ we recall from (xx) that the number critical s-points in the complex s-plane is finite. In $bf[P)^1$ we choose a simple curve γ_* with intial point at $s = 0$ and end-point at $s = |nfty$, while γ_* avoids the critical points $s \neq 0$ in $\mathbf{C}$. of course, while this is done we can choose γ_* so that it is a smooth curve on the C^∞-manifold $\mathbf{P}^1$.

The inverse image $f^{-1}(\gamma_*)$. Let $N = \deg(f)$. Since γ_* avoids f-cirtical points we see that

$$f^{-1}(\gamma_* \setminus \{0, \infty\}) = \gamma_1 \cup \ldots \cup \gamma_N$$

where $\{\gamma_j\}$ are disjoint curves in X, each of which is oriented via γ_* which from the start moves from $s = 0$ to $s = \infty$.

Exercise. Show that the closure $\bar{\gamma}_j$ yields a rectifiable curve whose initial point is zero of f and the end-point a pole. Put

$$\Gamma = \sum \bar{\gamma}_j$$

and conclude from the above that Γ is associated to the principal divisor of f. The proof of the "onmy if part" is therefore finished if we show that

$$(*) \qquad \int_\Gamma \omega = 0 \quad : \quad \omega \in \Omega(X)$$

To prove (*) we shall use a certain trace map. First, a holomorphic 1-form ω on X is regared as a $\bar{\partial}$-closed current with bi-degree $(0, 1)$. Since the amp $f \colon X \to \mathbf{P}^1$ is proper there exists the direct image current $f_*(\Omega$ and we recal from general facts that the passage to direct image currents commute with the $\bar{\partial}$-operator. hence $\bar{\partial}(f_*(\Omega)) ==$ and since $\bar{\partial}$ is elliptic it follows that $f_*(\omega)$ is a holomorphic 1-form on $\mathbf{P}^1$. By the result in Exercise xx there does not exist non-zero globally defined holomorphic 1-forms on the projective line. hence

$$(**) \qquad f_*(\omega) = 0$$

In partocuar we have

$$\int_{\gamma_*(\epsilon)} f_*(\omega) = 0$$

where $\gamma_*(\epsilon)$ is the closed curve given by γ intersected with $\{\epsilon \leq |s| \leq \epsilon^{-1}\}$ for a small $\epsilon > 0$.

Exercise. Put $\Gamma(\epsilon) = f^{-1}(\gamma(\epsilon)$ and dediuce from (xx) that

$$\int_{\Gamma(\epsilon)} \omega = 0$$

Finally, pass to the limit as $\epsilon \to 0$ and conclude that (*) holds which finishes the proof of the "only if part".

G. Holomorphic line bundles.

There exists the multiplicative sheaf $\mathcal{O}_X^*$ whose sections are zero-free holomorphic functions. every such function is locally e^g with $g \in \mathcal{O}_X$ where g is determined up to an integer multiple of $2\pi i$. This gives an exact sheaf sequence

$$0 \to \mathbf{Z}_X \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 0$$

where exp sends g to $e^{2\pi i g}$ and $\mathbf{Z}_X$ is the sheaf of locally constant integer-valued functions on X. Notice that global sections in $\mathcal{O}_X^*$ are reduced to non-zero complex numbers and we have the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{C} \overset{\exp}{\to} \mathbf{C}^* \to 0$$

Conclude that one has an exact sequence

$$0 \to H^1(\mathbf{Z}_X) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \to H^2(\mathbf{Z}_X) \to 0 \tag{G.1}$$

As explained in § xx the cohomology group $H^1(\mathbf{Z}_X)$ is a free abelian group of rank $2g$ while $H^2(\mathbf{Z}_X) = \mathbf{Z}$.

Compact Riemann surfaces.

A compact Riemann surface X is by definition a connected and compact complex analytic manifold of dimension one. The maximum principle for holomorphic functions entails that every globally defined holomorphic funtions on X is a constant. Thus, $\mathcal{O}(X)$ is reduced to the complex field. But if one allows poles we shall learn that there exist non-constant meromorphic functions on X. They give a field denoted by $\mathcal{M}(X)$ whose properties will be investigated later on. The projective line $\mathbf{P}^1$ over the complex field is an example of a compact Riemann surface. Topologically it is homeomorphic to a where and as a module one takes the As a model one starts with the complex z-plane and adds the point at infinity. Now $\mathcal{M}(\P^1)$ becomes the field of rational functions of the single complex variable z. Other compact Riemann urfaces arise when one starts from an irreducible polynomial $P(x, y)$ of two independent variables, i.e. P is irreducible in the unique factorisation domain $\mathbf{C}[x, y]$. We shall learn how to find the associated compact Riemann surface X where (X) is isomorphic to the quotient field of the integral domain $\frac{\mathbf{C}[x,y]}{(P)}$ where (P) denotes the princial ideal generated by P. As an example we take

$$P(x, y) = y^3 - x^3 - 1$$

To get X one first regards the algebraic curve in $\mathbf{C}^2$ defined by $\{P = 0\}$. Since the complex gradient vector $(P'_x, P'_y) = (3x^2, 3y^2) \neq (0, 0)$ it follows that $\{P0 =\}$ is non-singular and therefore brcomes an embedded one-dimensional complex manifold in $\mathbf{C}^2$. To get a copmpact complex manifold one regards the prjective curve in $\mathbf{P}^2$ with its inhomogeneous coordintes (ζ, x, y) where the zero set of $P^*(\zeta, x, y) = y^3 - x^3 - \zeta^3$ yields a projective curve S. We shall learn that S is non-singular and therefore gives a compact Riemann surface. Less obvious is that S regarded as a topological space is homeomorphic to the oriented real manifold T^2, where T^2 denotes the 2-dimensional torus. The reason why $S \simeq T^2$ as a topoligcal space is that $\mathcal{M}(S)$ contains a meromorohiuc function u which has one double pole and two simple zeros on S and is otherwise holomorphic without zeros. An example of such a meromorphic function is

$$u = \frac{1}{y - x}$$

whose properties as a meromorphic function on S will be described in § xx. In § xxx we expose the pioneering work by Hermann Weyl devoted to "abstract" Riemann surfaces which in particular prove the existence of non-constant meromorphic functions. From this Weyl deduced that $\mathcal{M}(X)$ is a so called algebraiuc functon field, i.e. as a filed extension over $\mathbf{C}$ the degree of trancendency is one and in addition it is finitely generated. Using the existence of primitive elements in finite algebraic extensions of fields in characterstic zero it follows that $\mathcal{M}(X)$ is isomorphic to the quotoent field of an integral domain

$$\frac{\mathbf{C}[x, y]}{P(x, y)}$$

where $P(x, y)$ is an irreducible polynomial in the unique factorisation domain $\mathbf{C}[x, y]$. So every compact Riemann surface arises from an irreducible polynomial $P(x, y)$. A remarkable fact is that a compact Riemann surface X is determiuned by the field $\mathcal{M}(X)$. More precisely, let Y be another compact Riemann surface and assume that there exists a $\mathbf{C}$-linear isimorohism

$$\rho \colon \mathcal{M}(X) \to \mathcal{M}(Y)$$

Thus, the two algebraic function fields are isomorphic. In § xx we show that X and Y are isomorphic as complex manifolds, i.e. ρ yields a *biholomorphic map* from X to Y.

Non-compact Riemann surfaces. In addition to compact Riemann surfaces there exist open -or non-compact- Riemann surfaces X. It means that X is a connected but non-compact complex manifold of dimension one. In this case it turns out that $\mathcal{O}(X)$ contains a quite extensive family of non-constant functions. In § xx we expose results due to Behnke and Stein about non-compact Riemann surfaces which for example show that one can prescribe zeros and poles in an arbitrary fashion and which correspond to a globally defined meromorophic function on X, i.e $\mathcal{M}(X)$ is so to speak "optimally ample". In § xx we also expose the *Uniformisation Theorem* where our proof follows that of Behnke and Stein. It implies for example that every simply connected open Riemann surface is either biholomorphic to the open unit disc or the complex plane. The uniformisation theorem for connected open subsets Ω of the complex z-plane deserve special attention. More precisely, if $\Omega \subset \mathbf{C}$ is open and connected and the complement contains at least two points, then there exists a holomorphic map

$$\phi \colon D \to \Omega$$

where D is the open unit disc, and ϕ has the following two properties: First it is surejctive, i.e. $\phi(D) = \Omega$ and moreover it is locally conformal, i.e. the complex derivative $\phi'(z)$ has no zeros. In § xx we shall learn that this entails that thr universal covering space of Ω is homeomorphic to the open unit disc which is achieved via the locally biholomorphic map ϕ. The uniformisation theorem for planar domains is due to Hermann Schwarz who adopted Riemann's methods for simple connected domains and is be exposed in § xx in connection with Riemann's ordinary conformal mapping theorem for simply connected planar domains. Even though Schwarz's Uniformisation Theorem can be deduced via the general case treated by Behnke and Stein, it is instructive to pursue his original proof which only relies upon analytic function theory. As expected it is in general very hard to obtain explicit formulas. An example is when Ω is the open complement of $\mathbf{C}$ where one has removed three disjoint closed intervals $\{[a_\nu, b_\nu]\}$ where $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$. See § xx for a discussion related to this case.

About methods of proofs.

The study of Riemann surfaces offer instructive lessons in sheaf theory since various sheaves appear in a natrual fashion. The reader is supposed to be familiar with basic facts about sheaves and their cohomology. But for less experienced readers we add an appenbdix where som fundamental reults from Leray's pioneering article *xxxx* are resumed. Let us also remark that even if Weierstrass never introduced the notion of sheaves, his construction of "sheaf spaces" attached to in general multiu.valued abnalytic functions gave the first exampe of sheaves. In addition to sheaf theory we employ calculus on manifolds without hesitation. For example. let M be an oriented real C^∞-manifold of some dimension $n \geq 2$. To say that M is oriennted means that there exists a globally defined differential n-form which never vanishes and choosing the orientation one can define integrals

$$\int_M \omega \quad : \quad \omega \in \mathcal{E}^n(M)$$

where $\mathcal{E}^n(M)$ is the space of n-forms with C^∞-.coefficients. More generally there exist to each $0 \leq p \leq n$ the space $\mathcal{E}^p(M)$ of differential p-forms with C^∞-coefficients. Under the condition that M can be covered by a denumarable family of compact subsets it follows that $\mathcal{E}^p(M)$ is equipped with a topology so that it becomes a Frechet spaces. There also exists the spaces $\{\mathcal{E}_0^p(M)\}$ which consist of differential forms having compact support in M. The dual of $\mathcal{E}_0^p(M)$ is denoted by $\mathfrak{c}^{n-p}(M)$ and its elements are called currents of degree $n-p$. in particular $\mathfrak{c}^0(M)$ is the space of distributions on M and is often denoted by $\mathfrak{Db}(M)$. Thus, by definition

$$\mathfrak{Db}(M) \simeq \mathcal{E}_0^n(M)^*$$

where (*) indicates that one regards the dual space. We assume that the reader is famiiar with basic distribution theory which for example is coveed in Hörmander's text-book [xx], where the passage to manifolds and currents become trivial using charts on C^∞-manifolds and C^∞-partitions of the unity. Even if we shall not study manifolds of dreal dimension ≥ 3 we recall some classic facts about oriented and compact $C\infty-$manifolds Let M be a such a manifold of siome dimension $n \geq 2$. using exterior differentials one has a complex

$$0 \to \mathcal{E}^0(M) \to \mathcal{E}^1(M) \to \ldots \to \mathcal{E}^n(M) \to 0$$

Here

$$d \colon \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M)$$

are continuous for every p, where we recall that every $\mathcal{E}^p(M)$ is a Frechet space. A fundamental fact is that the images of the d-maps above are closed and have finite codimension. Ths can be proved in several ways. The most convincing proof is due to André Weil who also established that every compact and oriented manifold M can be tringulated via a finite simplicial complex. So readers interested in algebraic topology should consult Weil's original proof which teaches a good lesson dealing with topology and triangulations of manifolds. Now one introduces the cohomology spaces

$$H^p(M) = \frac{\text{Ker}_d(\mathcal{E}^p(M)}{d\mathcal{E}^{p-1}(M)} \quad : 0 \leq p \leq n$$

In particular $H^0(M)$ is the d-kernel on $C^\infty(M)$ and hence the 1-dimensional space of constant functuons. One has also the complex of currents:

$$(**) \qquad 0 \to \mathfrak{c}^0(M) \to \mathfrak{c}^1(M) \to \ldots \to \mathfrak{c}^n(M) \to 0$$

It is denoted by $\mathfrak{c}^\bullet$. For every p one has the inclusion $\mathcal{E}^p(M) \subset \mathfrak{c}^p(M)$ and hence Hence (*) above is a subcomplex of (**). Next, the contruction of currents and their exterior differentials, togehter eith the closed range of the differentials in (*), give a natural duality

$$H^p(M)^* \simeq \frac{\text{Ker}_d(\mathfrak{c}^{n-p})}{d(\mathfrak{c}^{n-p-1}(M)} = H^{n-p}(\mathfrak{c}^\bullet)$$

In addition to this the fact that M admits a triangulation implies that the complexes (*) and (**) are quasi-isomorphic. Hence the right hsnd side in (**) can be identfied with $H^{n-p}(M)$, i.e. the dual of $H^p(M)$ is equal to th cohomology space in degree $n-p$. This is referred to as Poincaré duality theorem for compact and oriented manifolds.

Some examples when $n = 2$. Let M be an oriented C^∞-manifold of dimension two. Since M can be covered by a finite family of charts we can equip M with a distance function d whose restriction to charts is equivsrnt to ordinsty euclidian metroc. Every pair of such distance functions d_1 and d_2 are equivalent in the sense that there exists a constant $C > 0$ such that

$$C^{-1}|cdotd_2(p,q) \leq d_1(p,q) \leq C \cdot d_2(p,q)$$

hold for every pair p, q in M. Next, a parametrized curve on M is a map $\gamma \colon [0,1] \to M$ which is continuous. Using a metric as above we can impose the extra condition that γ has a bounded variation, i.e. there exists a constant C such that

$$\text{(i)} \qquad \sum_{\nu=0}^{\nu=N-1} d(\gamma(t_{\nu+1}), \gamma(t_\nu) \leq C$$

for all partitions $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$. When (i) holds there exist ordinary Borel-Stieltjes' integrals

$$(*) \qquad \int_\gamma \psi^1$$

for every differential 1-form ψ^1 with C^∞-coefficents. This gives a continuous functional on $\mathcal{E}^(M)$ with compact support and hence a current of degree 1, called the integration current associated with γ. This "operative construction" will be used at many places later on. One merit is that "nasty geometric pictures" when γ for example has self-intersections can be ignored while one treats its associated integration current. The case when γ is closed, i.e. when $\gamma(1) = \gamma(0)$ leads to special results. To begin with we notice that the equality $\gamma(0) = \gamma(1)$ entails that

$$\int_\gamma dg = 0 \quad : \quad g \in \mathcal{E}(M)$$

This is expressed by saying that the current $\int_\gamma$ is d-closed. A crucial result which follows from the fact that every distribution in $\mathbf{R}^2$ has locally primitive distributuions with respect to the pair of euclidian coordinates in $\mathbf{R}^2$, entails that the following complex vector spaces are isomorphic

$$\frac{\mathrm{Ker}_d(\mathcal{E}^1(M)}{d(\mathcal{E}^0(M)} \simeq \frac{\mathrm{Ker}_d(\mathfrak{c}^1(M)}{d(\mathfrak{Db}(M)} \simeq H^1(M)$$

where the last term is the cohomology space in degree one on M. This entials that if γ is a closed and rectifiable curve then there exists a d-closed 1-form $\phi \in \mathcal{E}^1(M)$ and a distribution μ such that

(*) $$\int_\gamma = d(\mu) + \phi^1$$

where this equality takes place in $\mathfrak{c}^1(M)$. Moreover, the d-closed 1-form ϕ^1 is unique up to an exact smooth 1-form. With γ regarded as a linear functional on $\mathcal{E}^1(M)$ the reader should check that (*) gives the equality below for every d-closed $\psi^1 \in \mathcal{E}^1(M)$:

$$\gamma(\psi^1) = \int_M \phi^1 \wedge \psi^1$$

The d-closed current $\int_{\partial\Omega}$. Let Ω be a connected open set in M whose boundary $\partial\Omega$ is regular in the sense of Federer. It means that there exists a (possibly empty) closed set $\Sigma \subset \partial\Omega$ whose 1-dimensional Haussdorff measure is zero while $\partial\Omega_* = \partial\Omega \setminus \Sigma$ is locally simple and rectifiable. That is. for each $p \in \partial\Omega_*$ there exists a bijective map γ from the closed unit interval $[0,1]$ to a connected subset of $\partial\Omega_*$ where $p = \gamma(1/2)$ while the distinct end-points $\gamma(0)$ and $\gamma(1)$ stay in $\partial\Omega_*$. The orientation along$\partial\Omega_*$ is given by the rule of thumbs which appears in the classic version of Stokes Theorem as found already by Archimedes. The extra the condition is that the total variation, or equivalently the arc-length measure of $\partial\Omega_*$ is finite while it is evaluated with respect to a local euclidian distance function, is finite. Under these conditions one has

$$\int_\Omega d\phi = \int_{\partial\Omega_*} \phi$$

for each $\phi \in \mathcal{E}^1(M)$. In particular the 1-current defined via the right hand side is d-closed. Notice that we do not assume that $\partial\Omega$ is connected.

Direct images of currents. Let $\rho\colon M \to N$ be a C^∞-map from M to another oriented and compact 2-dimensional manifold N. Elie Cartan's construction of differential forms on manifolds give for each $\phi \in \mathcal{E}^1(N)$ a pullback $\rho^*(\phi) \in \mathcal{E}^1(M)$. The map is continuous when $\mathcal{E}^1(N)$ and $\mathcal{E}^1(M)$ are equipped with their standard Frechet topologies. In particular we regard a parametrized curve γ and obtain a 1-current $\rho_*(\gamma)$ on N defined by

$$\rho_*(\phi) = \int_\gamma \rho^*(\phi)$$

Later on we shall use such direct images of currents at several places. The reader should "accept" that it is often more profitable to ignore "intuitive genetroic pictures" while calculations are performed. An extra merit is that while the integrals along the γ is computed, then calcluus teaches that one can employ different parametrisatiin along γ. as long as they have bounded variation and preserve the given orientation. Notice that even if γ from the start is a simple Jordan arc, it may occur that $\rho_*(\gamma)$ has a more involved structure since the restriction of the map ρ to γ need not be one-to-one.

Intersection numbers.

If ρ is another closed and rectifiabsle curve we have a similar decomposition

$$\int_\rho = \psi^1 + d(\nu)$$

Stokes Theorenm entials that

(*) $$\int_M \phi^1 \wedge \psi^1$$

is independent of the chosen decompostions. So the complex number in (*) depends only upon the ordered pair of closed curves (γ,ρ) and is denoted by $\mathfrak{i}(\gamma,\rho)$ and called the intersection number of the closed curves. Elementaty calculus shows that the intersection number always in an integer, and by continuity it follows that if γ and γ^* are two homotopic closed curves then $\mathfrak{i}(\gamma,\rho)=\mathfrak{i}(\gamma^*,\rho)$ for every other closed curve ρ.

Following original work by Riemann one regards specific simple closed curves $\gamma_1,\ldots,\gamma_{2g}$ which give a free basis for the fundamental group $\pi_1(M)$ which is free of some rank $2g$ where g is a positive number. Here we exlude the case when $M\simeq S^2$, i.e M is not homoemorphic to the sphere. Instead M is homeomorphic to a sphere where g handles have been attached and with this picture in mind one can draw $2g$-many closed curves on M whose intersection numbers are special. More precisely, there exists a pair of g-tuples $\gamma_1,\ldots,\gamma_g$ and $\rho_1,\ldots,\rho_g)$, where all these $2g$ many closed curves are simple and

$$\mathfrak{i}(\gamma_\nu,\rho_\nu)=1 \quad : \quad 1\leq\nu\leq g$$

while all other intersection numbers are zero.

Analysis on compact Riemann surfaces.

Let X be a compact 1-dimensional complex manifold. Recall that its complex analytic structure arises via charts (U, z) where z is a local coordinate on X in the open set U which is biholomorphic with the open disc $D = \{|\zeta| < 1\}$ in the complex ζ-plane. Let $\rho_U : U \to D$ be the holomorphic map. When a pair of charts U and V overlap their complex analytic structures are compatible. It means that one has a biholomorphic map

$$\rho_{U.V} : \rho_U(U \cap V) \to \rho_V(U \cap V)$$

where

$$\rho_U(\zeta) = \rho_V \circ \rho_{U,V}(\zeta) \quad : \zeta \in \rho_U(U \cap V)$$

The reader should illustrate this by a suitable picture to see how the transition functions work in overlapping charts Since X is compact it can be covered by a finite family of charts $\mathfrak{U} = \{U_\alpha, z_\alpha)\}$. So here $\cup U_\alpha = X$ and one refers to $\mathfrak{U}$ as an atlas for the complex manifold X. If U is an arbitrary open set in X a complex valued function f in U is holomorphic if

$$f \circ \rho_\alpha^{-1} \in \mathcal{O}(\rho_\alpha(U \cap U_\alpha))$$

hold for every chart in the atlas. The set of holomorphic functions in U is denoted by $\mathcal{O}(U)$. Local existence of non-constant holomiorphic functions is clear since the defintion above entails that if U_α is a chart then $\mathcal{O}(U_\alpha) \simeq \mathcal{O}(D)$.

Differential calculus on X. A complex analytic atlas is in particular a C^∞-atlas on the underlying real manifold which has real dimension two. The space of complex-vaued C^∞-functions on X is denoted by $\mathcal{E}(X)$. We have also the space $\mathcal{E}^1(X)$ of differential 1-forms with C^∞-coefficients. In a chart (U, z) every differential 1-form can be decomposed as

$$a \cdot dz + b \cdot d\bar{z} \tag{A.0}$$

where a and b are complex-valued C^∞-functions. This gives a direct sum decomposition

$$\mathcal{E}^1(X) = \mathcal{E}^{1.0}(X) \oplus \mathcal{E}^{0,1}(X) \tag{A.1}$$

One refers to $\mathcal{E}^{1,0}$ as the space of diffeential forms of bi-degree $(1, 0)$. Thet are characterised by the condition that $b = 0$ holds in $(A.0)$ for every chart. Similarly when $a = 0$ hold we get a differential form of bi-degree $(0, 1)$. If $g \in \mathcal{E}(X)$ its exterior differential dg is a 1-form which has a decompostion

$$dg = \partial g + \bar{\partial} g$$

For example, one has a map

$$\bar{\partial} : \mathcal{E}(X) \to \mathcal{E}^{0,1}(X) \tag{A.2}$$

The Cauchy-Riemann equations applied in charts show that the kernel in (A.2) consists of holomorphic functions and hence the kernel is reduced to constant functions.

Next, we also have differential 2-forms on X which in a chart (U, z) can be expressed as

$$c(z) \cdot dz \wedge d\bar{z}$$

Keeping the complex analytic structure in mind we denote the space of differential 2-forms with $\mathcal{E}^{1,1}(X)$. The reader should construct the maps

$$\bar{\partial} : \mathcal{E}^{1,0}(X) \to \mathcal{E}^{1.1}(X) \tag{A.3}$$

$$\partial : \mathcal{E}^{0,1}(X) \to \mathcal{E}^{1.1}(X) \tag{A.4}$$

Next, recall that $\mathcal{E}(X)$ is a Frechet space where the topology is defined via uniform convergence of derivatives in every order. In a similar fashion $\mathcal{E}^{p,q}(X)$ are Frechet spaces for every pair $0 \leq p, q \leq 1$. The reader should check that the maps in (A:2-4) are continous between these Frechet spaces. Less obvious is the folloowing:

A.5 Theorem. *Tbe range of the exterior differential maps in (A.2-4) above are all closed.*

A.6. Remark. This is the first non-trivial result about compact Riemann surfaces. We give the proof in (*) and remark only that it relies upon a classic result due to Pompieu which gives solutions to the inhomogeneous $\bar{\partial}$-partial equation in planar domains. More precisely, if U is an open set in the complex ζ-plane and $g \in C^\infty(U)$ then there exists $f \in C^\infty(U)$ such that

$$\bar{\partial}(f) = g \cdot d\bar{\zeta} \tag{A.6.1}$$

A.7 Currents on X. To begin with we have the space $\mathfrak{Db}(X)$ of distributions which by definition it is the space of continuos linear forms on the Frechet space $\mathcal{E}^{1,1}(X)$, i.e.

$$\mathfrak{Db}(X) = \mathcal{E}^{1,1}(X)^*$$

Passing to differential forms we get dual spaces

$$\mathfrak{c}^{1,0}(X) = \mathcal{E}^{0,1}(X)^* \quad : \quad \mathfrak{c}^{0,1}(X) = \mathcal{E}^{1,0}(X)^*$$

Finally, we have the space $\mathfrak{c}^{1.1}(X)$ which is the dual of $\mathcal{E}(X)$.

B.1 The integration current 1_X. The reader should check that the complex analytic structure on X entails that its underlying real manifold is oriented. It follows that if $\gamma \in \mathcal{E}^{1,1}(X)$ then there exists an integral

$$\int_X \gamma$$

This defines a distribution denoted by 1_X. In general we have an inclusion

$$\mathcal{E}(X) \subset \mathfrak{Db}(X)$$

Namely, if g is a C^∞-function we get a linear form on $\mathcal{E}^{1,1}(X)$ defined by

$$\gamma \mapsto \int_X g \cdot \gamma$$

If $\gamma^{1,0}$ belongs to $\mathcal{E}^{1,0}(X)$ we notice that Stokes Theorem gives

$$0 = \int_X d(g \cdot \gamma^{1,0})$$

Now the reader should verify that

$$d(g \cdot \gamma^{0,1}) = \bar{\partial}g \wedge \gamma^{1,0} + g \cdot \bar{\partial}(\gamma^{1,0})$$

Next, by definition the $(0,1)$-current $\bar{\partial}g$ is defined by

$$\bar{\partial}(g)\langle\gamma^{1,0}\rangle = -\int_X g \cdot \bar{\partial}(\gamma^{1,0})$$

Keeping signs in mind one concludes that when g from the start is identified with a distribution, then the $(0,1)$-current $\bar{\partial}(g)$ is equal to the smooth differential 1-form of bi-degree (0,1) arising when $\bar{\partial}$ is applied to the C^∞-function g.

B.2 A duality result. We have the map

$$\bar{\partial} \colon \mathcal{E}^{1,0}(X) \to \mathcal{E}^{1,1}(X)$$

By Theorem A.5 the range is closed. This implies that the dual space

$$\Big[\, \frac{\mathcal{E}^{1,1}(X)}{\bar{\partial}\mathcal{E}^{1,0}(X)} \,\Big]^* \simeq \mathrm{Ker}_{\bar{\partial}}(\mathfrak{Db}(X)) \tag{B.2.1}$$

Next, recall that $\bar{\partial}$ is elliptic.Indeed, via the classic Pompieu formula for planar doimains every distribution μ satisfying the homogeneous equation $\bar{\partial}(\mu) = 0$ is a holomorphic function. Since $\mathcal{O}(X)$ is reduced to constants the dual space in (B.2.1) is 1-dimensional and hence

$$\dim_{\mathbf{C}} \frac{\mathcal{E}^{1,1}(X)}{\bar{\partial}\mathcal{E}^{1,0}(X)} = 1 \tag{B.2.2}$$

Notice also that Stokes Theorem gives

$$\int_X \bar{\partial}(\gamma^{1,0}) = 0 \quad : \gamma^{1,0} \in \mathcal{E}^{,1,0}(X)$$

from this we conclude thatif $\rho^{1,1}$ is a 2-form for which $\int_X \rho^{1,1} \neq 0$, then

(B.2.3) $$\mathcal{E}^{1,1}(X) = \mathbf{C} \cdot \rho \oplus \bar{\partial}(\mathcal{E}^{1,1}(X))$$

B.3 A second duality. Theorem A.5 entails that

(B.3.1) $$\Big[\, \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}\mathcal{E}(X)} \,\Big]^* \simeq \mathrm{Ker}_{\bar{\partial}}(\mathfrak{c}^{1,0}(X))$$

The right hand side is found as follows. In a chart (U,z) a current of bi-degree $(1,0)$ is of the form $\mu \cdot dz$ with $\mu \in \mathfrak{Db}(U)$. The $\bar{\partial}$ image of this current is zero if and only if $\bar{\partial}(\mu) = 0$ and as we have seen before this implies that μ is a holomorphic density. Hence the right hand side in (B.3.1) consists of globally defined *holomorphic* 1-forms. This space is denoted by $\Omega(X)$ and (B.3.1) can be expressed by the duality formula

(B.3.2) $$\Omega(X) \simeq \Big[\, \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}\mathcal{E}(X)} \,\Big]^*$$

B.3.3 Exercise. Show that (B.3.2) gives the following: A differential form $\gamma^{0,1}$ on X is $\bar{\partial}$-exact if and only if

$$\int_X \omega \wedge \gamma^{0,1} = 0 \quad : \forall\, \omega \in \Omega(X)$$

B.4 Passage to currents. Recall that $\mathcal{E}^{0,1}(X) \subset \mathfrak{c}^{0,1}$. in § xx we shall learn that

(B.4.1) $$\mathfrak{c}^{0,1} = \bar{\partial}(\mathfrak{Db}(X)) + \mathcal{E}^{0,1}(X)$$

Moreover, a (0.1)-current $\rho^{0,1}$ is $\bar{\partial}$-exact, i.e. of the form $\bar{\partial}(\mu)$ for some distribution μ on X if and only if

(B.4.2) $$\rho^{1,0}(\omega) = 0 \quad : \quad \omega \in \Omega(X)$$

B.5 The case $X = \mathbf{P}^1$. Here $\Omega(X)$ is reduced to the zero space. To see this we suppose that ω is a globally defined holomorphic 1-form. Now $\mathbf{P}^1 \setminus \{\infty\}$ is the complex z-plane and here

$$\omega = f(z) \cdot dz$$

where f is an entire function. At the point at infinity we have a local coordinate ζ and

$$\zeta = z^{-1} \quad : R < |z| < \infty$$

hold for every $R > 0$. Since ω is globally holomorhpic we have

$$\omega = g(\zeta) \cdot d\zeta$$

when $0 < |\zeta| < R^{-1}$. At the same time the reader can check that

$$dz = -\zeta^{-2} \cdot d7zeta$$

holds when $R < |z[< R$. Ths entails that

$$f(z) = -z^{-2} \cdot g(1/z) \quad : |z| > R$$

From this we see that the entire function f tends to zero as $|z| \to \infty$ and hence f is identically zero. So with $X = \mathbf{P}^1$ one has $\Omega(X) = 0$ and then (B.4.2) entails that

$$\mathfrak{c}^{0,1} = \bar{\partial}(\mathfrak{Db}(X))$$

In other words, the inhomogeneous $\bar{\partial}$-equation is always solvable. Consider as an example a current of the form

$$\gamma = \mu \cdot d\bar{z}$$

where μ is a measure in the complex z-plane with compact support. Now there exists the Cauchy transform

$$\widehat{\mu}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

It is holomiorphic in in the open set of the complex z-plane where the compact set $\text{Supp}(\mu)$ has been removed. passing to infinity we notice that it extends to be holomorphic because one has the convergent series expansion

$$\widehat{\mu}(z) = \sum_{n=0}^{\infty} \int \zeta^n |cdotd\mu(\zeta) \cdot z^{-n-1} z - \zeta$$

in an exterior disc $|z[> R$ where the support of μ is contained in $\{|z| \leq R\}$.

next, by basuc lebsgue theory one knows that the funnction $\widehat{\mu}$ is locally integrable in the whole complex z-plane and hence defines a distribution. Finally, Cacuhy's residue formula shows that the distribution derivative

$$\frac{\bar{\partial}\widehat{\mu}}{\bar{\partial}\bar{z}} = \pi \cdot \mu$$

So we get

$$\bar{\partial}(\pi^{-1} \cdot \widehat{\mu}) = \mu \cdot d\bar{z}$$

C. Some sheaves and their cohomology

.

Let X be a compact complex analytic manifold. Classic facts due to Cauchy and Pompieu applied to charts in X give an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{E}_X \xrightarrow{\bar{\partial}}, \mathcal{E}_X^{0,1} \to 0 \tag{C.0}$$

Using C^∞-parotions of unity and Leray's theory about cohomology with values in sheaves one gets

$$H^1(\mathcal{O}_X) \simeq \frac{\mathcal{E}^{0,1}(X)}{\bar{\partial}(\mathcal{E}(X)} \tag{C.1}$$

where the left hand side is the first cohomology groups with values in the sheaf $\mathcal{O}_X$. The reader may also notice that when Leray's long exact sequence for sheaf cohomology is applied to (C.0), then it follows that

$$H^2(\mathcal{O}_X) = 0 \tag{C.2}$$

Next, we have also the exact sheaf sequence

$$0 \to \mathbf{C}_X \to \mathcal{O}_X \xrightarrow{\partial} \Omega_X \to 0 \tag{C.3}$$

Since $H^0(\mathbf{C}_X) = H^0(\mathcal{O}_X) = \mathbf{C}$ and (C.2) holds one gets the following exact sequence of complex vector spaces:

$$0 \to \Omega(X) \to H^1(\mathbf{C}_X) \to H^1(\mathcal{O}_X) \to H^1(\Omega_X) \to H^2(\mathbf{C}_X) \to 0 \tag{C.4}$$

Next, we have also the following exact sheaf sequence

$$0 \to \Omega_X \to \mathcal{E}_X^{1,0} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{1,1} \to 0 \tag{C.5}$$

This gives the equality

$$H^1(\Omega_X) \simeq \frac{\mathcal{E}^{1,1}(X)}{\bar{\partial}(\mathcal{E}^{1,0}(X))} \tag{C.6}$$

The right hand side rsssembles the quotient in (B.2.2). Indeed, we can regard the 7emphconjugate analytic structure on the underlying real manifold of x where the sheaf of holomorphic functions are compex cojugates of sections in $\mathcal{O}_X$. From this the reader may conclude via (B.2.2) that the rifht hand side in (C.5) is a 1-dimensional vector space. Hence we have

$$\dim_{\mathbf{C}} H^1(\Omega_X) = 1 \tag{C.7}$$

Next, recall that $H^2(\mathbf{C}_X) = \mathbf{C}$. Together with (C.7) we conclude that (C.4) gives the short exact sequence

$$0 \to \Omega(X) \to H^1(\mathbf{C}_X) \to H^1(\mathcal{O}_X) \to 0 \tag{C.8}$$

Topology teaches that the cohomology space $H^1(\mathbf{C}_X)$ is finite dimensional. Hence (C.8) implies that both $\Omega(X)$ and $H^1(\mathcal{O}_X)$ are finite dimensional complex vector spaces. Finally, (B.3.2) and (C.1) give

$$\Omega(X) \simeq \left[\, H^1(\mathcal{O}_X)\,\right]^* \tag{C.9}$$

In particular the complex vector spaces $\Omega(X)$ and $H^1(\mathcal{O}_X)$ have the same dimension. This equality and (C.8) give

$$\dim H^1(\mathbf{C}_X) = 2 \cdot \dim \Omega(X) \tag{C.10}$$

The equality (C.10) is fundamental during the study of compact Riemann surfaces. A notable point is that the cohomology group $H^1(\mathbf{C}_X)$ only depends upon the topological space X and not upon its particular complex structure.

C.10 A duality for currents. In planar domains it is wellknown that the inhomogenous $\bar{\partial}$-equation is locally solvable on distributions. Since the $\bar{\partial}$-kernel on $\mathfrak{Db}_X$ is $\mathcal{O}_X$ one has an exact sequence

$$0 \to \mathcal{O}_X \to \mathfrak{Db}_X \xrightarrow{\bar{\partial}} \mathfrak{c}_X^{0,1} \to 0$$

Passing to cohomology and using C^∞-partitions of the unity, it follows that

(C.10.1) $$H^1(\mathcal{O}_X) \simeq \frac{\mathfrak{c}_X^{0,1}}{\bar{\partial}(\mathfrak{Db}(X)}$$

Exercise. Deduce from the above that

(C.10.2) $$\Omega(X) \simeq \Big[\, \frac{\mathfrak{c}_X^{0,1}}{\bar{\partial}(\mathfrak{Db}(X)} \,\Big]^*$$

Conclude that the implication below holds for each $\gamma \in \mathfrak{c}^{0,1}$:

(C.10.3) $$\gamma(\omega) = 0 \quad : \forall\, \omega \in \Omega(X) \implies \gamma \in \bar{\partial}(\mathfrak{Db}(X))$$

C.11 A decomposition theorem. Since X in particular is an oriented and compact real manifold one has

(C.11.1) $$H^1(\mathbf{C}_X) \simeq \frac{\mathrm{Ker}_d(\mathcal{E}^1(X)}{d(\mathcal{E}(X)}$$

Next, on X there also exists the anti-holomorphic differential forms of bi-degree $(0,1)$, i.e differential forms $\gamma \in \mathcal{E}^{0,1}(X)$ for which $\partial(\gamma) = 0$. This space is denoted by $\overline{\Omega}(X)$. Using the conjugate complex analytic structure one has another compact complex manifold $\overline{X}$ where $\overline{\Omega}(X)$ is the space of holomorphic 1-forms. Since X and $\overline{X}$ have the same underlying real manifold, it follows from (C.10) that

(C.11.2) $$\dim H^1(\mathbf{C}_X) = 2 \cdot \dim \overline{\Omega}(X)$$

Using this we shall prove

C.11.3 Theorem. *One has a direct sum decomposition*

$$\mathrm{Ker}_d(\mathcal{E}^1(X)) = d(\mathcal{E}(X)) \oplus \Omega(X) \oplus \overline{\Omega}(X)$$

Proof To begin with we notice that

$$\Omega(X) \cap \overline{\Omega}(X) = \{0\}$$

Counting dimensions this follows if we prove that a 1-form $\gamma)\omega + \mu$ cannot be d-exact when $\omega \in \Omega(X)$ and $\mu \in \overline{\Omega}(X)$ where at least one these froms is not identically ero. For suppose that $\gamma = dg$ is d-exact. If ω is not identically zero there exists the conjugate form $\bar{\omega} \in \overline{\Omega}(X)$. Now Stokes Theorem gives

$$0 = \int_X dg \cdot \bar{\omega}) = \int_X (\omega + \mu) \wedge \bar{\omega} = \int_X (\omega \wedge \bar{\omega}$$

Now the reader can check that the last integral is $\neq 0$ when ω is not identically zero. hence $\gamma \in d(\mathcal{E}(X))$ implies that $\Omega = 0$ and in the same way we find that $\mu = 0$ which proves Theorem C.11.3.

D. Existence of meromorphic functions.

On X there exists the sheaf $\mathcal{M}_X$ whose sections are meromorphic functions. Subsheaves arise via constraints on poles. In particular, let $p \in X$ be a given point and m a positive integer. Denote by $\mathcal{O}_X[*mp]$ the sheaf whose sections are meromorphic functions on X which are holomorphic in $X \setminus \{p\}$ and have poles of order $\leq m$ at p. If (U, z) is a chart where p corresponds to the origin we seee that

$$\mathcal{O}_X[*mp]\,|U = z^{-m} \cdot \mathcal{O}X|U$$

Taking Laurent exapnsions every section in the right hand side is of the form

$$f + \frac{c_1}{z} + \ldots + \frac{c_m}{z^m}$$

where $f \in \mathcal{O}_X$ and $\{c_\nu\}$ is an m-tuple of complex numbers. Expressed by sheaves it gives an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X[*mp] \to S_p^m \to 0 \tag{D.1}$$

where S_p is the scyscraper sheaf suppotet by $\{p\}$ and the stalk $S_p(p)$ is an m-dimensional vector space. Passing to long exact sequence of cohomology the reader can verify that there is an exact sequence:

$$0 \to \mathbf{C} \to H^0(\mathcal{O}_X[*mp]) \to \mathbf{C}^m \to H^1(\mathcal{O}_X) \to H^0(\mathcal{O}_X[*mp]) \to 0 \tag{D.1.1}$$

D.2 Exercise. Use that the vector space $H^1(\mathcal{O}_X)$ is g-dimensional and conclude that

$$\dim_{\mathbf{C}} H^0(\mathcal{O}_X[*mp]) = m - g + 1 + \dim_{\mathbf{C}} H^1(\mathcal{O}_X[*mp]) \geq m - g + 1 \tag{D.2.1}$$

In particular the left hand side is ≥ 2 if $m = g + 1$ which means that apart from constants, there exists at least one constant global section in $\mathcal{O}_X[*(g+1)p]$. Hence we have proved:

D.3 Theorem. *For every $p \in X$ there exists a non-constant meromorphic function f in X with no poles in $X \setminus p$ while the order of the pole at p is at most $g + 1$.*

Weierstrass' points.

Let X be a compact Riemann surface whose genus number g is ≥ 2. Removing a point $p \in X$ we get the space $\mathcal{O}(X \setminus \{p\})$. If (U, z) is a chart centered at p where p corresponds to $z = 0$, then every $f \in \mathcal{O}(X \setminus \{p\})$ restricts to a holomorphic function in the punctured disc $0 < |z| < 1$ and we can regard its Laurent series:

$$f(z) = \sum_{\nu=1}^{\infty} c_\nu \cdot z^{-\nu} + \sum_{\nu \geq 0} d_\nu \cdot z^\nu$$

The negative part of the Laurent series is denoted by $f_*(z)$. If $f_* = 0$ then f is globally holomorphic and hence reduced to a constant. So

$$f \mapsto f_*(z) + d_0$$

yields an injective map from $\mathcal{O}(X \setminus \{p\})$ into the space of non-positive Laurent series at $z = 0$. We are going to study the range of this map. Denote by $\mathcal{L}$ the space of all non-positive Laurent series, where we recall that a negative series

$$\sum_{\nu=1}^{\infty} c_\nu \cdot z^{-\nu} \in \mathcal{L}$$

if and only if

$$\sum_{\nu=1}^{\infty} |c_\nu| \cdot R^\nu < \infty$$

for all $R > 0$. In particular $\mathcal{L}$ contains the g-dimensional subspace formed by Laurent series given by finite series:

$$\rho(z) = c_1 z^{-1} + \ldots + c_g \cdot z^{-g}$$

where $\{c_\nu\}$ is a p-tuple of complex numbers. Let $\mathcal{L}[p]$ denote this subspace of $\mathcal{L}$.

E.1 Theorem. *Outside a finite set* $\mathbf{W}$ *in* X *one has*

$$\mathcal{L} = \mathcal{L}[p] \oplus \mathfrak{Im}(\mathcal{O}(X \setminus \{p\})$$

E.2 Remark. In §xx we show that the exceptional set $\mathbf{W}$ is non-empty when $g \geq 2$, If (*) holds above it follows that when $m > g$ then there exists a unique g-tuple $\{c_\nu\}$ such that

$$z^{-m} = c_1 z^{-1} + \cdots + c_g z^{-g} + \mathfrak{Im}(\phi)$$

where ϕ is holomorphic in $X \setminus \{p\}$. Herte the negative Laurent series of ϕ is finite at p which implies that ϕ is a globally defined meromorphic function with a pole of order m at p whose negative Laurent expnasion at p is of the form

$$z^{-m} + c_1 z^{-1} + \cdots + c_g z^{-g}$$

where the g-tuple $\{c_\nu\}$ is uniquely determined and depends on the chosen integer m. At the same time the direct dum in (*) entails that there do not exist non-constant meromorphic functions which are holomorphic in $X \setminus \{p\}$ while the order of the pole at p is $\leq g$.

The proof of Theorem E.1 requires several steps. We are given a point $p \in X$ and let (U, z) be a chart around p where p corresponds to $z = 0$. Residue calculus gives a g-tuple of currents defined by

$$\rho_\nu = \bar{\partial}(z^{-\nu}) \quad : \quad \nu = 1, \ldots, g$$

Each current is supported by the singleton set $\{p\}$ and extends to a current on X supported by p. If $\Psi^{1,0} \in \mathcal{E}^{1,0}(X)$ we can regard its restriction to U and write

$$\Psi^{1,0}|U = \psi(z) \cdot dz$$

with $\psi \in C^\infty(U)$. Residue calcluus teaches that

$$\rho_1(\Psi^{1,0}) = 2\pi i \cdot \psi(0)$$

If $\nu \geq 2$ the reader may verify that

$$\rho_\nu(\Psi^{1,0}) = \frac{(-1)^{\nu-1}}{(\nu-1)!} \cdot \partial^{\nu-1}\psi/\partial z^{\nu-1}(0)$$

The construction of the ρ-currents is not intrinsic, i.e. the g-tuple depends on the chosen chart around p. But the reader can check that the g-dimenbsional vector space in $\mathfrak{c}^{0,1}(X)$ generated by $\{\rho^\nu : 1 \leq \nu \leq g\}$ does not depend on the chosen chart around p. Let us denote it by $\mathcal{W}(p)$. The major step towards the proof of Theorem E.1 is the following:

E.3 Theorem. *Outside a finite set* **W** *in* X *one has the direct sum decomposition*

$$\mathfrak{c}^{0,1}(X) = \bar{\partial}(\mathfrak{Db}(X)) \oplus \mathcal{W}(p) \tag{E.3.1}$$

Proof. The duality result in § xx gives (E.3.1) if the g-tuple $\{\rho^\nu\}$ restrict to **C**-linearly independent in the dual space $\Omega(X)^*$. To check when this holds we consider a basis $\omega_1, \ldots, \omega_g$ in the g-dimensional vector space $\Omega(X)$. In a chart (U,z) centered at a point $p_0 \in X$ we can write

$$\omega_j = f_j(z) \cdot dz \tag{i}$$

where $\{f_j\}$ are **C**-linearly independent in $\mathcal{O}(U)$. The general result in § xx entails that the determinant function

$$\mathcal{F}(z) = \det\big(\partial f_j^\nu/\partial z^\nu(z) \tag{ii}$$

formed by the $g \times g$-matrix with elements as above is not identically zero. Now linear algebra shows that $\{\rho_\nu\}$ restrict to linearly independent functionals on $\Omega(X)$ at every $z|inU$ for ehich $F(z) \neq 0$. Since the zero set of F in the chart U is discrete and we can cover X by a finite famliy of charts, it follows that (E.3.1) holds outside a finite set **W**.

E.4 How to find W. In each chart (U,z) as above we have the $\mathcal{F}$-functions from (ii). If (U_α, z_α) and (U_β, z_β) is a pair of charts in X where $U_\alpha \cap U_\beta \neq \emptyset$ we get two g-tuples of analytic functions by

$$\omega_j | U_\alpha = f_j^\alpha \cdot dz \quad : \quad \omega_j | U_\beta = f_j^\beta \cdot dz_\beta$$

When $z_\beta = z_\beta(z_\alpha)$ is regarded as an analytic function in $U_\alpha \cap U_\beta$ one has the equations

$$f_j^\alpha = \frac{\partial z_\beta}{\partial z_\alpha} \cdot f_j^\beta \tag{E.4.1}$$

for each j in $U_\alpha \cap U\beta$.

E.4.2 Exercise. Show by using rules for calcuating determinsnts that (E.4.1) gives the equation

$$\mathcal{F}_\alpha = \big(\frac{\partial z_\beta}{\partial z_\alpha}\big)^{g(g-1)/2} \cdot F_\beta(z)$$

So with $N = g(g-1)/2$ the family $\{\mathcal{F}_\alpha\}$ which arises when X is covered by a finite number of charts, is a global section of the N-fold product of the holomorphic line bundle Ω_X. By (xx) the set of xeros counted with multiplicities of every global section in Ω_X is equal to $2(g-1)$. So by the general result in § xxx the zero set of the $\mathcal{F}$-family counted with multiplicities is equal to $g(g-1)^2$. Since $g \geq 2$ this is a positive integer and hence the set **W** is non-empty.

Remark. In the literature one refers to **W** as the set of Weierstrass' points in X. The determination of **W** for a given X with genus ≥ 2 is in general not easy. From the above **W** contains at most $(g-1)^2 \cdot g$ many distinct points. In general the number of points can be strictly smaller. Weierstrass investigated **W** for special families of compact Riemann surfaces. See § xx for some comments.

Proof of Theorem E.1

Consider a point $p \in X$ which is outside **W** and let (U,z) be a chart centered at p. Let $U^* = U \setminus \{p\}$ be the punctured open disc. In X we also have the open set $X \setminus \{p\}$ which can be regarded as an open Riemann surface. The theorem by Behnke and Stein from the chapter devoted to open Rieann surfaces gives $H^1(X \setminus \{p\}, \mathcal{O}) = 0$. The cohomology in the disc U is also zero. Hence the open covering of X given by the pair $\mathfrak{U} = (U, X \setminus \{p\})$ is acyclic with respect to the sheaf $\mathcal{O}_X$. So Leray's acyclicity theorem gives

$$\text{(i)} \qquad H^1(X, \mathcal{O}_X) \simeq \frac{\mathcal{O}(U^*)}{\delta(C^0(\mathfrak{U}, \mathcal{O}_X))}$$

where

$$\delta \colon \mathcal{O}(U) \oplus \mathcal{O}(X \setminus \{p\}) \to \mathcal{O}(U^*)$$

is the Cech coboundary map. Next, the g-tuple $\{z^{-\nu} \colon \nu = 1, \dots, g\}$ are holomorphic in U^* and their images in the right hand side from (i) give elements in $H^1(X, \mathcal{O}_X)$ denoted by $\{[z^{-\nu}]\}$.

Exercise. Recall that

$$H^1(X.\mathcal{O}X) = \frac{\mathfrak{c}^{0,1}}{\bar{\partial}(\mathfrak{D}\mathfrak{b}(X)}$$

Since p is outside $\mathbf{W}$ the last quotient is equal to $\mathcal{W}(p)$. Conclude from this that the g-tuple $\{[z^{-\nu}]\}$ is a basis in the g-dimensional vector space $H^1(X, \mathcal{O}_X)$.

Let us then consider some $g \in \mathcal{O}(U^*)$ which has a Laurent series

$$g(z) = \sum_{\nu=1}^{\infty} c_\nu \cdot z^{-\nu} + \sum_{\nu=0}^{\infty} d_\nu \cdot z^{\nu}$$

By the Exercise there exists a unique g-tuple of complex numbers $a_1, \dots, a_p$ such that the image of g in $H^1(X, \mathcal{O}_X)$ is equal to $\sum a_\nu \cdot [z^{-\nu}]$. Hence (i) gives a pair $\phi(z) \in \mathcal{O}(U)$ and $f(z) \in \mathcal{O}(X \setminus \{p\}$ such that

$$g(z) - \sum \rho_\nu \cdot z^{-\nu} = \phi(z) + f(z)$$

where $\phi(z) \in \mathcal{O}(U)$ and $f(z) \in \mathcal{O}(X \setminus \{p\}$. Regarding the negative Laurent series it follows that

$$\text{(ii)} \qquad g_*(z) = \sum a_\nu \cdot z^{-\nu} + f_*(z)$$

Here g was arbitrary in $\mathcal{O}(U^*)$ and hence the proof of Theorem E.1 is finished.

F. Abel's theorem.

A divisor D on X consists of an assignement of integers $\{\mu_\nu\}$ to a finite set of points $\{p_\nu\}$ in X. One writes

$$D = \sum \mu_\nu \cdot \delta(p_\nu)$$

The μ-integers may be positive or negative. The degree is defined by

$$\deg D = \sum \mu_\nu$$

while the finite set $\{p_\nu\}$ is called the support of D. Next we construct a class of 1-currents. In general, let $\gamma\colon [0,1] \to X$ be an Xvalued function on the closed unit interval We assume that γ is continuous and has a finite total variation. The last condition means that there exists a constant C such that

$$\sum_{\nu=0}^{\nu=N-1} d(\gamma(t_{\nu+1}, \gamma(t_\nu) \leq C$$

for all sequences $0 = t_0 < t-1 < \ldots < t_N = 1$ Here d is some distance function on X regarded as a metric space which in charts is equivsoent to the euclidian distance. Every such γ gives a current of degree one acting on $\mathcal{E}^1(X)$ by

$$(*) \qquad \alpha \mapsto \int_\gamma \alpha$$

Exercise. Explain how the classical Borel-Stieltjes integrals for functions with bounded variation defines the integral in the right hand side. Show also that

$$\int_\gamma dg = g(\gamma(1)) - g(\gamma(0))$$

hold for every $g \in \mathcal{E}()$. One refers to (*) as an integration current of degree one. More generally we can take a finite sum of such currents and get the integration current Γ defined by

$$\Gamma(\alpha) = \sum \int_{\gamma_j} \alpha$$

Let $D = \sum \mu_\nu \cdot \delta(p_\nu)$ be a divisor of degreee zero. An integration current Γ is said to be associated with D if the equality below hold for every $g \in \mathcal{E}(X)$:

$$\Gamma(dg) = \sum \mu_\nu \cdot g(p_\nu)$$

F.1 Principal divisors. Let f be a non-constant meromorphic function. Now we have the finite set of poles $\{p_\nu\}$ and the finite set of zeros $\{q_j\}$. We associate the divisor

$$\mathrm{div}(f) = \sum \mu_k \cdot \delta(p_k) - \sum \mu_j \cdot \delta(q_j)$$

Here μ_k is the order of the pole of f at every p_ν, while μ_j is the order of a zero at q_j. By the result in § xx this divisor has degree zero. A divisor D of degree zero is called principal if it is equal to $\mathrm{div}(f)$ for some $f \in \mathcal{M}(X)$. Notice that if f and g is a pair of non-constant meromorphic functions such that $\mathrm{div}(f) = \mathrm{div}(g)$, then f/g is a holomorphic function on X and hence reduced to a non-zero constant. Introducing the multiplicative group $\mathcal{M}(X)^*$ of non-constant meromorphic functions this means that one has an injective map

$$\frac{\mathcal{M}(X)^*}{\mathbf{C}^*} \to \mathcal{D}_0$$

where $\mathcal{D}_0$ is the additive group of divisors whose degree are zero. The theorem below describes the range of this map.

F.2 Theorem. *A divisor D of degree zero is principal if and only if there exists an integration current Γ associated with D such that*

$$\Gamma(\omega) = 0 \quad : \ \gamma \in \Omega(X)$$

The proof requires several steps. First we shall show the "if part", i.e. if Γ exists so that (*) holds then the divisor D is principal. To achieve this we first perform a local construction in the complex plane.

F.3 A class of currents in C. In the complex z-plane we consider a point z_0 where

$$0 < |z_0| < r < 1$$

holds for some $0 < r < 1$. It is easily seen that there exists a C^∞-function $a(z)$ in $\mathbf{C}$ which never is zero and the function

$$\phi(z) = \frac{z}{z - z_0} \cdot a(z)$$

is identically one when $|z| \geq r$. Now $z \mapsto \frac{1}{z-z_0}$ is locally integrable around z_0 and hence ϕ belongs to $L^1_{\text{loc}}(\mathbf{C})$ and is therefore a distribution.

Cauchy's residue formula can be expressed by saying the $\bar\partial$-image of the $\frac{1}{z}$ is equal to the $(0,1)$-current $2\pi i \cdot \delta(0) d\bar z$ where $\delta(0)$ is the Diraqc distribution at the origin. Using this the reader can check that

$$\text{(i)} \qquad \bar\partial(\phi) = \frac{z}{z - z_0} \cdot \bar\partial(a) + z_0 a(z_0) \cdot \delta(z_0) d\bar z$$

from (i) the reader should also check that

$$\text{(ii)} \qquad \phi^{-1} \cdot \bar\partial(\phi) = a^{-1} \cdot \bar\partial(a)$$

F.4 The current $\phi^{-1} \cdot \partial(\phi)$. To begin we one has

$$\text{(F.4.1)} \qquad \partial(\phi) = \frac{a(z)}{z - z_0} \cdot dz - \frac{za}{(z - z_0)^2)} \cdot dz + \frac{z}{z - z_0} \cdot \partial a$$

Above $(z - z_0)^{-2}$ is the principal value distribution defined as in § xx.

Exercise. Use (F.4.1) to establish the equation

$$\text{(F.4.2)} \qquad \phi^{-1} \cdot \partial(\phi) = z^{-1} \cdot dz - (z - z_0)^{-1} \cdot dz + a^{-1} \cdot \partial(a)$$

Exercise. Use Stokes Theorenm and Cauchy's residue fornula to prove that when the 1-current

$$\phi^{-1} \cdot d\phi = \phi^{-1} \cdot \partial(\phi) + \phi^{-1} \cdot \bar\partial(\phi)$$

is applied to dg for some $g \in C_0^\infty(\mathbf{C}$, then

$$\text{(F.4.3)} \qquad \phi^{-1} \cdot d\phi < dg >= 2\pi i \cdot (g(0) - g(z_0)$$

Let us now consider an integration current Γ given by a finite sum $\sum \gamma_j$ where each γ_j has compact support in a chart (U_j, z_j) in X and the end-points of γ_j are $z_j = 0$ and $z_j = z_j^*$ with $0 < |z_j^*| < r < 1$ and U_j is a chart defined by $\{|z_j| < 1|\]$. In X the point $z_j = 0$ is denoted by q_j while z_j^* corresponds to a point p_j. It follows that

$$\text{(F.5)} \qquad \int_\Gamma dg = \sum_{j=1}^{j=N} g(p_j) - g(q_j)$$

for every $g \in \mathcal{E}(X)$. Next, for each j we apply the local contruction the chart and find a function ϕ_j in X which is identically one in $X \setminus \{|z_j| \leq r\}$ such that

$$\phi_j^{-1} \cdot d\phi_j < dg >= 2\pi i \cdot \int_{\gamma_j} dg \quad : g \in \mathcal{E}(X)$$

More generally, since d-closed 1-forms in the chart U_j are d-exact the re4ader should check that (F.5) implies that

$$\text{(F.6)} \qquad \phi_j^{-1} \cdot d\phi_j < \alpha >= 2\pi i \cdot \int_{\gamma_j} \alpha$$

hold for every d-closed differential 1-form α. Let us now consider the function

$$\Phi = \prod_{j=1}^{j=N} \phi_j$$

F.7 Exercise. From the above the reader should check that the construction of the ϕ-functions imply that Φ is a zero-free C^∞-function in $X \setminus \{q_j, p_j\}$, i.e. in the open complement of the finite set which is the union of these q and p points. Moreover, additivity for logarithmic derivatives and (F.6) above gives

$$\text{(F.7.1)} \qquad \Phi^{-1} \cdot d\Phi < \alpha >= \int_\Gamma \alpha$$

for every d-closed 1-form α. Finally, show that

$$\text{(F.7.2)} \qquad \Phi^{-1} \cdot \bar{\partial}(\Phi) = \sum a_j^{-1} \cdot \bar{\partial}(a_j)$$

where the right hand side is a smooth $(0,1)$-form.

F.8 A special case. Suppose that Γ is such that

$$\text{(F.8.1)} \qquad \int_\Gamma \omega = 0 \quad : \quad \omega \in \Omega(X)$$

The equality in (F.7.1) applied to d-closed holomorphic 1-forms and Theorem § xx entail that the smooth $(0,1)$-form $\Phi^{-1} \cdot \bar{\partial}(\Phi)$ from (F.7.2) is $\bar{\partial}$-exact. Thus, we can find $G \in \mathcal{E}(X)$ such that

$$\text{(F.8.2)} \qquad \bar{\partial}(G) = \Phi^{-1} \cdot \bar{\partial}(\Phi)$$

Exercise. Put

$$\Psi = e^{-G} \cdot \Phi$$

deduce from the above that $\bar{\partial}(\Psi) = 0$ holds in $X \setminus \{q_j, p_j\}$ and hence Ψ is holomorphic in this open subset of X Show also that the local constructions of the ϕ-functions inside the charts $\{U_j z_j\}$ entail that ψ extends to a mermorphic function in X whose princial divisor is equal to

$$\sum \delta(q_j) - \sum \delta(p_j)$$

Funally the reader should confirm that the constructions above prove the "if part" in Theorem F.2.

F.9 Proof of the "only if part"

Here we are given a non-constant meromorphic funvtion f. We shall condtruct an integration chain Γ as follows: Frst $f \colon X \to \mathbf{P}^1$ is a holomorphic map. With $s = f(x)$ we recall from (xx) that the number critical s-points in the complex s-plane is finite. In $bf[P)^1$ we choose a simple curve γ_* with intial point at $s = 0$ and end-point at $s = |nfty$, while γ_* avoids the critical points $s \neq 0$ in $\mathbf{C}$. of course, while this is done we can choose γ_* so that it is a smooth curve on the C^∞-manifold $\mathbf{P}^1$.

The inverse image $f^{-1}(\gamma_*)$. Let $N = \deg(f)$. Since γ_* avoids f-cirtical points we see that

$$f^{-1}(\gamma_* \setminus \{0, \infty\}) = \gamma_1 \cup \ldots \cup \gamma_N$$

where $\{\gamma_j\}$ are disjoint curves in X, each of which is oriented via γ_* which from the start moves from $s = 0$ to $s = \infty$.

Exercise. Show that the closure $\bar{\gamma}_j$ yields a rectifiable curve whose initial point is zero of f and the end-point a pole. Put

$$\Gamma = \sum \bar{\gamma}_j$$

and conclude from the above that Γ is associated to the principal divisor of f. The proof of the "onmy if part" is therefore finished if we show that

$$(*) \qquad \int_\Gamma \omega = 0 \quad : \quad \omega \in \Omega(X)$$

To prove (*) we shall use a certain trace map. First, a holomorphic 1-form ω on X is regared as a $\bar{\partial}$-closed current with bi-degree $(0,1)$. Since the amp $f \colon X \to \mathbf{P}^1$ is proper there exists the direct image current $f_*(\Omega$ and we recal from general facts that the passage to direct image currents commute with the $\bar{\partial}$-operator. hence $\bar{\partial}(f_*(\Omega)) ==$ and since $\bar{\partial}$ is elliptic it follows that $f_*(\omega)$ is a holomorphic 1-form on $\mathbf{P}^1$. By the result in Exercise xx there does not exist non-zero globally defined holomorphic 1-forms on the projective line. hence

$$(**) \qquad f_*(\omega) = 0$$

In partocuar we have

$$\int_{\gamma_*(\epsilon)} f_*(\omega) = 0$$

where $\gamma_*(\epsilon)$ is the closed curve given by γ intersected with $\{\epsilon \leq |s| \leq \epsilon^{-1}\}$ for a small $\epsilon > 0$.

Exercise. Put $\Gamma(\epsilon) = f^{-1}(\gamma(\epsilon)$ and dediuce from (xx) that

$$\int_{\Gamma(\epsilon)} \omega = 0$$

Finally, pass to the limit as $\epsilon \to 0$ and conclude that (*) holds which finishes the proof of the "only if part".

G. Holomorphic line bundles.

There exists the multiplicative sheaf $\mathcal{O}_X^*$ whose sections are zero-free holomorphic functions. every such function is locally e^g with $g \in \mathcal{O}_X$ where g is determined up to an integer multiple of $2\pi i$. This gives an exact sheaf sequence

$$0 \to \mathbf{Z}_X \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^* \to 0$$

where exp sends g to $e^{2\pi i g}$ and $\mathbf{Z}_X$ is the sheaf of locally constant integer-valued functions on X. Notice that global sections in $\mathcal{O}_X^*$ are reduced to non-zero complex numbers and we have the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{C} \stackrel{\exp}{\to} \mathbf{C}^* \to 0$$

Conclude that one has an exact sequence

$$0 \to H^1(\mathbf{Z}_X) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \to H^2(\mathbf{Z}_X) \to 0 \tag{G.1}$$

As explained in § xx the cohomology group $H^1(\mathbf{Z}_X)$ is a free abelian group of rank $2g$ while $H^2(\mathbf{Z}_X) = \mathbf{Z}$.