Operator theory

1. Bounded self-adjoint operators.

Let $\mathcal{H}$ be a complex Hilbert space. A bounded linear operator S on $\mathcal{H}$ is self-adjoint if $S = S^*$, or equivalently

(*) $$\langle x, Sy\rangle = \text{the complex conjugate of } \langle Sx, y\rangle \quad : \quad x, y \in \mathcal{H}$$

If S is self-adjoint we have the equality of operator norms:

(i) $$||S||^2 = ||S^2||$$

To see this we notice that if $x \in \mathcal{H}$ has norm one then

(ii) $$\langle Sx, Sx\rangle = \langle x, S^*Sx\rangle = \langle x, S^2x\rangle$$

By the Cauchy-Schwarz inequality the last term is $\leq ||x| \cdot ||S^2||$. Since (ii) holds for every x of norm one we conclude that

$$||S||^2 \leq ||S^2||$$

Now (i) follows from the multiplicative inequality for operator norms. Next, by induction over n we get the equalities

$$||S||^{2n} = ||S^n||^2 \quad : \; n \geq 1$$

Taking the n:th root and passing to the limit the spectral radius formula gives

(*) $$||S|| = \max_{z \in \sigma(S)} |z|$$

Next, we consider the spectrum of self-adjoint operators.

1.1 Theorem. *The spectrum of a bounded self-adjoint operator is a compact real interval.*

Proof. Let λ be a complex number and for a given x we set $y = \lambda x - Sx$. It follows that

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 + \lambda \cdot \langle x, Sx\rangle + \bar{\lambda} \cdot \langle Sx, x\rangle$$

Since S is self-adjoint we get

$$\lambda \cdot \langle x, Sx\rangle + \bar{\lambda} \cdot \langle Sx, x\rangle = 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle Sx, x\rangle$$

Now $|\langle Sx, x\rangle| \leq ||Sx|| \cdot ||x||$ so the triangle inequality gives

(i) $$||y||^2 \geq |\lambda|^2 \cdot ||x||^2 + ||Sx|^2 - 2|\mathfrak{Re}(\lambda)|| \cdot ||Sx|| \cdot ||x||$$

With $\lambda = a + ib$ the right hand side becomes

$$b^2||x||^2 + a^2||x||^2 + ||Sx||^2 - 2a \cdot ||Sx|| \cdot ||x|| \geq b^2||x||^2$$

Hence we have proved that

(ii) $$||\lambda x - Sx||^2 \geq (\mathfrak{Im}\,\lambda)^2 \cdot ||x||^2$$

This implies that $\lambda E - S$ is invertible for every non-real λ which proves Theorem 1.1. Notioce that the proof also gives

(iii) $$||(\lambda E - S)^{-1}|| \leq \frac{1}{|\mathfrak{Im}\,\lambda|}$$

Theorem 1.1 together with general results about uniform algebras in § XX give the following:

1.2 Theorem. *Denote by* $\mathbf{S}$ *the closed subalgebra of* $L(\mathcal{H}, \mathcal{H})$ *generated by* S *and the identity operator. Then* $\mathbf{S}$ *is a sup-norm algebra which is isomorphic to the sup-norm algebra* $C^0(\sigma(S))$.

Exercise. Let T be an arbitary bounded opertor on $\mathcal{H}$. Show that the operator $A = T^*T$ is self-adjoint and that $\sigma(A)$ is a compact subset of $[0, +\infty)$, i.e. every point in its spectrum os real and non-negative. A hint is to use the biduality formula $T = T^{**}$ and if s is real the reader should verify that

$$||sx + T^*Tx||^2 = s^2||x||^2 + 2s \cdot ||Tx||^2 + ||T^*Tx||^2$$

1.3 Normal operators.

A bounded linear operator A is normal if it commutes with its adjoint A^*. Let A be normal and put $S = A^*A$ which yields a self-adjoint by Exercise 1.2. Hence (*) above Theorem 1.1 gives

(i) $$||S||^2 = ||S^2|| = ||A^2 \cdot A(^*)^2|| \leq ||A^2|| \cdot ||(A^*)^2||$$

where we used the multiplicative inequality for operator norms. Now $(A^*)^2$ is the adjoint of A^2 and we recall from § xx that the norms of an operator and its adjoint are equal. Hence the right hand side in (1) is equal to $||A^2||^2$. At the same time

$$||S|| = ||A^*A|| = ||A||^2$$

and we conclude that (i) gives

(ii) $$||A||^2 \leq ||A^2||$$

Exactly as in the self-adjoint case we can take higher powers and obtain the equality

(1.3.1) $$||A|| = \max_{z \in \sigma(A)} |z|$$

Since every polynomial in A again is a normal operator for which (1.3.1) holds we have proved the following:

1.4 Theorem *Let A be a normal operator. Then the closed subalgebra* $\mathbf{A}$ *generated by A in $L(\mathcal{H}, \mathcal{H})$ is a sup-norm algebra.*

Remark. The spectrum $\sigma(A)$ is some compact subset of $\mathbf{C}$ and in general analytic polynomials restricted to $\sigma(A)$ do not generate a dense subalgebra of $C^0(\sigma(A))$. To get a more extensive algebra we consider the closed subalgebra $\mathcal{B}$ of $L(\mathcal{H}, \mathcal{H})$ which is generated by A and A^*. Since every polynomial in A and A^* again is a normal operator it follows that $\mathcal{B}$ is a sup-norm algebra and here the following holds:

1.5 Theorem. *The sup-norm algebra $\mathcal{B}$ is via the Gelfand transform isomorphic with $C^0(\sigma(A))$.*

Proof. Let $Q \in \mathcal{B}$ be arbitrary. Now $S = Q + Q^*$ is self-adjoint and Theorem 1.1 entails that its Gelfand transform is real-valued, i.e. the function $\widehat{Q}(p) + \hat{Q}^*(p)$ is real. So if with $\widehat{Q}(p) = a + ib$ we must have $\hat{Q}^* = a_1 - ib$ for some real number a_1. Next, QQ^* is also self-adjoint and hence $(a + ib)(a_1 - ib)$ is real. This gives $a = a_1$ and which shows that the Gelfand transform of Q^* is the complex conjugate function of $\widehat{Q}$. Hence the Gelfand transforms of $\mathcal{B}$-elements is a self-adjoint algebra and the Stone-Weierstrass theorem implies that the Glefand transforms of $\mathcal{B}$-elements is equal to the whole algebra $C^0(\mathfrak{M}_{\mathcal{B}}$. Finally, since $\widehat{A}^*$ is the complex conjugate function of $\widehat{A}$ it follows that the Gelfand transform $\widehat{A}$ separates points on $\mathfrak{M}_{\mathcal{B}}$ which means that this maximal ideal space can be identified with $\sigma(A)$.

1.6 Spectral measures.

Let A be a normal operator and $\mathcal{B}$ is the Banach algebra above. Each pair of vectors x, y in $\mathcal{H}$ yields a linear functional on $\mathcal{B}$ defined by

$$T \mapsto \langle Tx, y \rangle$$

Identifying $\mathcal{B}$ with $C^0(\sigma(A))$, the Riesz representation formula gives a unique Riesz measure $\mu_{x,y}$ on $\sigma(A)$ such that

(1.6.1) $$\langle Tx, y \rangle = \int_{\sigma(A)} \widehat{T}(z) \cdot d\mu_{x,y}(z)$$

hold for every $T \in \mathcal{B}$. Since $\widehat{A}(z) = z$ we have

$$\langle Ax, y \rangle = \int z \cdot d\mu_{x,y}(z)$$

Similarly one has

$$\langle A^* x, y\rangle = \int \bar{z} \cdot d\mu_{x,y}(z)$$

1.7 The operators $E(\delta)$. Notice that (1.6.1) implies that the map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures on $\sigma(A)$ is bi-linear. We have for example:

$$\mu_{x_1+x_2,y} = \mu_{x_1,y} + \mu_{x_2,y}$$

Moreover, since $\mathcal{B}$ is the sup-norm algebra $C^0(\sigma(A))$ the total variations of the μ-measures satisfy the equations:

$$||\mu_{x,y}|| \leq \max_{T \in \mathcal{B}_*} |\langle Tx, y\rangle| \tag{1.7.1}$$

where $\mathcal{B}_*$ is the unit ball in $\mathcal{B}$. From this we obtain

$$||\mu_{x,y}|| \leq ||x|| \cdot ||y|| \tag{1.7.2}$$

Next, let δ be a Borel subset of $\sigma(A)$. Keeping y fixed in $\mathcal{H}$ we obtain a linear functional on $\mathcal{H}$ defined by

$$x \mapsto \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta)$$

By (1.7.2) it has norm $\leq ||y||$ and is represented by a vector $E(\delta)x$ in $\mathcal{H}$. More precisely

$$\langle E(\delta)x, y\rangle = \int_\delta d\mu_{x,y}(z) = \mu_{x,y}(\delta) \tag{1.7.3}$$

1.7.4 Exercise. Show that $x \mapsto E(\delta)x$ is linear and that the resulting linear operator $E(\delta)$ commutes with all operators in $\mathcal{B}$. Moreover, show that it is a self-adjoint projection, i.e.

$$E(\delta)^2 = E(\delta) \quad \text{and} \quad E(\delta)^* = E(\delta)$$

Finally, show that

$$E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$$

holds for every pair of Borel subsets and with $\delta = \sigma(A)$ one gets the identity operator.

1.7.5 Resolution of the identity. If $\delta_1, \ldots, \delta_N$ is any finite family of disjoint Borel sets whose union is $\sigma(A)$ then

$$1 = E(\delta_1) + \ldots + E(\delta_N)$$

At the same tine we get a decomposition of the operator A:

$$A = A_1 + \ldots + A_N \quad \text{where} \quad A_k = E(\delta_k) \cdot A$$

For each k the spectrum $\sigma(A_k)$ is equal to the closure of δ_k. So the normal operator is represented by a sum of normal operators where the individual operators have small spectra when the δ-partition is fine.

2. Unbounded operators on Hilbert spaces

Let T be a densely defined linear operator on a complex Hilbert space $\mathcal{H}$. We suppose that T is unbounded so that:

$$\max_{x\in\mathcal{D}_*(T)} ||Tx|| = +\infty \quad \mathcal{D}_*(T) = \text{ the set of unit vectors in } \quad \mathcal{D}(T)$$

2.1 The adjoint T^*. If $y \in \mathcal{H}$ we get a linear functional on $\mathcal{D}(T)$ defined by

(i) $$x \mapsto \langle Tx, y\rangle$$

If there exists a constant $C(y)$ such that the absolute value of (i) is $\leq C(y)\cdot||x||$ for every $x \in \mathcal{D}(T)$, then (i) extends to a continuous linear functional on $\mathcal{H}$. The extension is unique because $\mathcal{D}(T)$ is dense and since $\mathcal{H}$ is self-dual there exists a unique vector T^*y such that

(2.1.1) $$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad : \quad x \in \mathcal{D}(T)$$

This gives a linear operator T^* where $\mathcal{D}(T^*)$ is characterised as above. Now we shall describe the graph of T^*. For this purpose we consider the Hilbert space $\mathcal{H} \times \mathcal{H}$ equipped with the inner product

$$\langle (x,y),(x_1,y_1)\rangle = \langle x,x_1\rangle + \langle y,y_1\rangle$$

On $\mathcal{H} \times \mathcal{H}$ we define the linear operator

$$J(x,y) = (-y,x)$$

2.2 Proposition. *For every densely defined operator T one has the equality*

$$\Gamma(T^*) = J(\Gamma(T))^\perp$$

Proof. Let (y,T^*y) be a vector in $\Gamma(T^*)$. If $x \in \mathcal{D}(T)$ the equality (2.1.1) and the construction of J give

$$\langle (y,-Tx\rangle + \langle T^*y, x\rangle = 0$$

This proves that $\Gamma(T^)* \perp J(\Gamma(T))$. Conversely, if $(y,z) \perp J(\Gamma(T))$ we have

(i) $$\langle y,-Tx\rangle + \langle z,x\rangle = 0 \quad : x \in \mathcal{D}(T)$$

This shows that $y \in \mathcal{D}(T^*)$ and $z = T^*y$ which proves Proposition 2.2.

2.3 Consequences. The orthogonal complement of a subspace in a Hilbert space is always closed. Hence Proposition 2.2 entails that T^* has a closed graph. Passing to the closure of $\Gamma(T)$ the decompostion of a Hilbert space into a direct sum of a closed subpace and its orthogonal complement gives

(2.3.1) $$\mathcal{H} \times \mathcal{H} = \overline{J(\Gamma(T))} \oplus \Gamma(T^*)$$

Notice also that

(2.3.2) $$\Gamma(T^*)^\perp = \overline{J(\Gamma(T))}$$

2.4 Closed extensions of operators. A closed operator S is called a closed extension of T if

$$\Gamma(T) \subset \Gamma(S)$$

2.4.1 Exercise. Show that if S is a closed extension of T then

$$S^* = T^*$$

2.4.2 Theorem. *A densely defined operator T has a closed extension if and only if $\mathcal{D}(T^*)$ is dense. Moreover, if T is closed one has the biduality formula $T = T^{**}$.*

Proof. Suppose first that T has a closed extension. If $\mathcal{D}(T^*)$ is not dense there exists a non-zero vector $0 \neq h \perp \mathcal{D}(T^*)$ and (2.3.2) gives

(ii) $$(h,0) \in \Gamma(T^*)^\perp = J(\Gamma(T))$$

By the construction of J this would give $x \in \mathcal{D}(T)$ such that $(h,0) = (-Tx, x)$ which cannot hold since this equation first gives $x = 0$ and then $h = T(0) = 0$. Hence closedness of T implies that $\mathcal{D}(T^*)$ is dense. Conversely, assume that $\mathcal{D}(T^*)$ is dense. Starting from T^* we construct its adjoint T^{**} and Proposition 2.3.2 applied with T^* gives

(i) $$\Gamma(T^{**}) = J(\Gamma(T^*)^{\perp}$$

At the same time $J(\Gamma(T^*)^{\perp}$ is equal to the closure of $\Gamma(T)$ so (i) gives

(ii) $$\overline{\Gamma(T)} = \Gamma(T^{**})$$

which proves that T^{**} is a closed extension of T.

2.4.3 The biduality formula. Let T be closed. and densely defined operator. from the above T^* also is densely defined and closed. Hence its dual exists. It is denoted by T^{**} and called the bi-dual of T. With these notations one has:

(*) $$T = T^{**}$$

2.4.4 Exercise. Prove the equality (*).

2.5 Inverse operators.

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Denote by $\mathfrak{I}(\mathcal{H})$ the set of closed and densely defined operators T such that T is injective on $\mathcal{D}(T)$ and the range $T(\mathcal{D}(T))$ is dense in $\mathcal{H}$. If $T \in \mathfrak{I}(\mathcal{H})$ there exists the densely defined operator S where $\mathcal{D}(S)$ is the range of T and

$$S(Tx) = x \quad : \quad x \in \mathcal{D}(T)$$

By this construction the range of S is equal to $\mathcal{D}(T)$. Next, on $\mathcal{H} \times \mathcal{H}$ we have the isometry defined by $I(x,y) = (y,x)$, i.e we interchange the pair of vectors. The construction of S gives

(i) $$\Gamma(S) = I(\Gamma(T))$$

Since $\Gamma(T)$ by hypothesis is closed it follows that S has a closed graph and we conclude that $S \in \mathfrak{I}(\mathcal{H})$. Moreover, since I^2 is the identity on $\mathcal{H} \times \mathcal{H}$ we have

(ii) $$\Gamma(T) = I(\Gamma(S))$$

We refer to S is as the inverse of T. It is denoted by T^{-1} and (ii) entails that T is the inverse of T^{-1}, i.e. one has

(*) $$T = (T^{-1})^{-1}$$

2.5.1 Exercise. Let T belong to $\mathfrak{I}(\mathcal{H})$. Use the description of $\Gamma(T^*)$ in Proposition 2.3 to show that T^* belongs to $\mathfrak{I}(\mathcal{H})$ and the equality

(**) $$(T^{-1})^* = (T^*)^{-1}$$

2.6 The operator T^*T

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Each $h \in \mathcal{H}$ gives the vector $(h,0)$ in $\mathcal{H} \times \mathcal{H}$ and (2.3.1) gives a pair $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$. such that

$$(h,0) = (x, Tx) + (-T^*y, y) = (x - T^*y, Tx + y)$$

With $u = -y$ we get $Tx = u$ and obtain

(2.6.1) $$h = x + T^*(Tx)$$

2.6.2 Proposition. *The vector x in (2.6.1) is uniquely determined by h.*

Proof. Uniqueness follows if we show that

$$x + T^*(Tx) \implies x = 0$$

But this is clear since the construction of T^* gives

$$0 = \langle x, x)\rangle + \langle x, T^*(Tx)\rangle = \langle x, x)\rangle + \langle Tx, Tx)\rangle \implies x = 0$$

2.7 The density of $\mathcal{D}(T^*T)$. This is the subspace of $\mathcal{D}(T)$ where the extra condition for a vector $x \in \mathcal{D}(T)$ is that $Tx \in \mathcal{D}(T^*)$. To prove that $\mathcal{D}(T^*T)$ is dense we consider some orthogonal vector h. Proposition 2.6 gives some $x \in \mathcal{D}(T)$ such that $h = x + T^*(Tx)$ and for every $g \in \mathcal{D}(T^*T)$ we have

(i) $$0 = \langle x, g\rangle + \langle T^*Tx, g\rangle = \langle x, g\rangle + \langle Tx, Tg\rangle = \langle x, g\rangle + \langle x, T^*Tg\rangle$$

Here (i) hold for every $g \in \mathcal{D}(T^*T)$ and by another application of Proposition 2.6 we find g so that $x = g + T^*Tg$ and then (i) gives $\langle x, x\rangle = 0$ so that $x = 0$. But then we also have $h = 0$ and the requested density follows.

2.8 Conclusion. Set $A = T^*T$. From the above it is densely defined and (2.6.1) entails that the densely defined operator $E + A$ is injective. Moreover, its range is equal to $\mathcal{H}$. Notice that

$$\langle x + Ax, x + Ax\rangle = c + \langle x, Ax\rangle + \langle Ax, x\rangle$$

Here

$$\langle x, Ax\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = ||Tx||^2$$

and from this the reader can conclude that

$$||x + Ax||^2 = ||x||^2 + ||Ax||^2 + 2 \cdot ||Tx||^2 \quad : x \in \mathcal{D}(A)$$

The right hand side is $\geq ||x||^2$ which implies that $E + A$ is invertible in Neumann's sense.

2.9 The equality $A^* = A$. Recall the biduality formula $T = T^{**}$ and apply Proposition 2.6.2 starting with T^*. It follows that $\mathcal{D}(TT^*)$ also is dense and exactly as in (2.6.1) every $h \in \mathcal{H}$ has a unique representation

$$h = y + T(T^*y)$$

2.10. Exercise. Verify from the above that A is self-adjoint, i.e one has the equality $A = A^*$.

§ 2.B Unbounded self-adjoint operators.

A densely defined operator A on the Hilbert space $\mathcal{H}$ for which $A = A^*$ is called self-adjoint.

2.B.1 Proposition *The spectrum of a self-adjoint operator A is contained in the real line, and if λ is non-real the resolvent satisfies the norm inequality*

$$||R_A(\lambda)|| \leq \frac{1}{|\mathfrak{Im}\,\lambda|}$$

Proof. Set $\lambda = a + ib$ where $b \neq 0$. If $x \in \mathcal{D}(A)$ and $y = \lambda x - Ax$ we have

$$||y||^2 = |\lambda|^2 \cdot ||x||^2 + ||Ax||^2 - 2 \cdot \mathfrak{Re}(\lambda) \cdot \langle x, Ax\rangle$$

The Cauchy-Schwarz inequality gives

(i) $$||y||^2 \geq b^2||x||^2 + a^2||x||^2 + ||Ax||^2 - 2|a| \cdot ||Ax|| \cdot ||x|| \geq b^2||x||^2$$

This proves that $x \to \lambda x - Ax$ is injective and since A is closed the range of $\lambda \cdot E - A$ is closed. Next, if y is $\perp$ to this range we have

$$0 = \lambda\langle x, y\rangle - \langle Ax, y\rangle \quad : x \in \mathcal{D}(A)$$

From this we see that y belongs to $\mathcal{D}(A^*)$ and since A is self-adjoint we get

$$0 = \lambda\langle x, y\rangle - \langle x, Ay\rangle$$

This holds for all x in the dense subspace $\mathcal{D}(A)$ which gives $\lambda \cdot y = Ay$ Since λ is non-real we have already seen that this entails that $y = 0$. Hence the range of $\lambda \cdot E - A$ is equal to $\mathcal{H}$ and the inequality (i) entails $R_A(\lambda)$ has norm $\leq \frac{1}{|\mathfrak{Im}\lambda|}$.

2.B.2 A conjugation formula. Let A be self-adjoint. For each complex number λ the hermitiain inner product on $\mathcal{H}$ gives the equation

$$\bar{\lambda} - A = (\lambda \cdot E - A)^*$$

So when we take the complex conjugate of λ it follows that § 2.5 that

$$R_A(\lambda)^* = R_A(\bar{\lambda}) \tag{2.5.1}$$

2.B.3 Properties of resolvents. Let A be self-adjoint. By Neumann's resolvent calculus the family $\{(R_A(\lambda)\}$ consists of pairwise commuting bounded operators outside the spectrum of A. Since $\sigma(A)$ is real there exist operator-valued analytic functions $\lambda \mapsto R_A(\lambda)$ in the upper- respectively the lower half-plane. Moreover, since Neumann's resolvents commute, it follows from (2.5.1) that $R_A(\lambda)$ commutes with its adjoint. Hence every resolvent is a bounded normal operator.

2.B.4 A special resolvent operator. Take $\lambda = i$ and set $R = R_A(i)$. So here

$$R(iE - A)(x) = x \quad : \quad x \in \mathcal{D}(A)$$

2.B.5 Theorem. *The spectrum $\sigma(R)$ is contained in the circle*

$$C_* = \{|\lambda + i/2| = 1/2\}$$

Proof. Since $\sigma(A)$ is confined to the real line, it follows from § 0.0. XX that points in $\sigma(R)$ have the form

$$\lambda = \frac{1}{i - a} \quad : a \in \mathbf{R}$$

This gives

$$\lambda + i/2 = \frac{1}{i - a} + i/2 = \frac{1}{2(i - a)}(2 + i^2 - ia) = \frac{1 - ia}{2i(1 + ia)}$$

and the last term has absolute value $1/2$ for every real a.

2.C. The spectral theorem for unbounded self-adjoint operators.

The operational calculus in § 1.3-1.6 applies to the bounded normal operator R in § 2.14. If N is a positive integer we set

$$C_*(N) = \{\lambda \in C_* : \mathfrak{Im}(\lambda) \leq -\frac{1}{N}\} \quad \text{and} \quad \Gamma_N = C_*(N) \cap \sigma(R)$$

Let χ_{Γ_N} be the characterstic function of Γ_N. Now

$$g_N(\lambda) = \frac{1 - i\lambda}{\lambda} \cdot \chi_{\Gamma_N}$$

is Borel functionon $\sigma(R)$ which by operational calculus in § 1.xx gives a bounded and normal linear operator denoted by G_N. On Γ_N we have $\lambda = -i/2 + \zeta$ where $|\zeta| = 1/2$. This gives

(1)
$$\frac{1 - i\lambda}{\lambda} = \frac{1/2 - i\zeta}{-i/2 + \zeta} = \frac{(1/2 - i\zeta)(i/2 + \bar{\zeta})}{|\zeta - i/2|^2} = \frac{\mathfrak{Re}\,\zeta}{|\zeta - i/2|^2}$$

By § 1.x the spectrum of G_N is the range of the g-function on Γ_N and (1) entails that $\sigma(G_N)$ is real. Since G_N also is normal it follows that it is self-adjoint. Next, notice that

(2)
$$\lambda \cdot (\frac{1 - i\lambda}{\lambda} + i) = 1$$

holds on Γ_N. Hence operational calculus gives the equation

(3)
$$R(G_N + i) = E(\Gamma_N)$$

where $E(\Gamma_N)$ is a self-adjoint projection. Notice also that

(4)
$$R \cdot G_N = (E - iR) \cdot E(\Gamma_N)$$

Hence (3-4) entail that

(5)
$$E(\Gamma_N) - iRE(\Gamma_N) = (E - iR) \cdot E(\Gamma_N)$$

Next, the equation $RA = E - iR$ gives

(*)
$$RAE(\Gamma_N) = (E - iR)E(\Gamma_N) = R \cdot G_N$$

2.C.1 Exercise. Conclude from the above that

(*)
$$AE(\Gamma_N) = G_N$$

Show also that:

(**)
$$\lim_{N\to\infty} AE(\Gamma_N)(x) = A(x) \quad \text{for each} \quad x \in \mathcal{D}(A)$$

2.C.2 A general construction. For each bounded Borel set e on the real line we get a Borel set $e_* \subset \sigma(R)$ given by

$$e_* = \sigma(R) \cap \{\frac{1}{i - a}\, a \in e\}$$

The operational calculus gives the self-adjoint operator G_e constructed via $g \cdot \chi_{e_*}$. We have also the operstor $E(e)$ given by χ_{e_*} and exactly as above we get

$$AE(e) = G_e$$

The bounded self-adjoint operators $E(e)$ and G_e commute with A and $\sigma(G_e)$ is contained in the closure of the bounded Borel set e. Moreover each $E(e)$ is a self-adjoint projection and for each pair of bounded Borel sets we have

$$E(e_1)E(e_2) = E(e_1 \cap E(e_1)$$

In particular the composed operators

$$E(e_1) \circ E(e_2) = 0$$

when the Borel sets are disjoint.

2.C.3 The spectral measure. Exactly as for bounded self-adjoint operators the results above give rise to a map from $\mathcal{H} \times \mathcal{H}$ into the space of Riesz measures:

$$(x, y) \mapsto \mu_{x,y}$$

For each real-valued and bounded Borel function $\phi(t)$ on the real line with compact support there exists a bounded self-adjoint operator ϕ such that

$$\langle \Phi(x), y \rangle = \int g(t) \cdot d\mu_{x,y}(t)$$

All these Φ operators commute with A. If $x \in \mathcal{D}(A)$ and y is a vector in $\mathcal{H}$ one has

$$\langle A(x), y \rangle = \lim_{M \to \infty} \int_{-M}^{M} t \cdot d\mu_{x,y}(t)$$

§ 3. Symmetric operators

A densely defined and closed operator T on a Hilbert space $\mathcal{H}$ is symmetric if

$$(*) \qquad \langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{hold for all pairs} \quad x, y \in \mathcal{D}(T)$$

The symmetry means that the adjoint T^* extends T, i.e.

$$\Gamma(T) \subset \Gamma(T^*)$$

Recall that adjoints always are closed operators. Hence $\Gamma(T^*)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$ and becomes a Hilbert space equipped with the inner product

$$\{x, y\} = \langle x, y\rangle + \langle T^*x, T^*y\rangle$$

Moreover, since T is closed. it follows that $\Gamma(T)$ appears as a closed subspace of this Hilbert space. Consider the eigenspaces:

$$\mathcal{D}_+ = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = ix\} \quad \text{and} \quad \mathcal{D}_- = \{x \in \mathcal{D}(T^*) \quad : T^*(x) = -ix\}$$

3.1 Proposition. *The following orthogonal decomposition exists in the Hilbert space $\Gamma(T^*)$:*

$$(*) \qquad \Gamma(T^*) = \Gamma(T) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$$

Proof. The verification that the three subspaces are pairwise orthogonal is left to the reader. To show that the direct sum above is equal to $\Gamma(T^*)$ we use duality and there remains only to prove that

$$(1) \qquad \Gamma(T)^\perp = \mathcal{D}_+ \oplus \mathcal{D}_-$$

To show (1) we pick a vector $y \in \Gamma(T)^\perp$. Here $(y, T^*y) \in \Gamma(T^*)$ and the definition of orthogonal complements gives:

$$\langle x, y\rangle + \langle Tx, T^*y\rangle = 0 \quad : \quad x \in \mathcal{D}(T)$$

From this we see that $T^*y \in \mathcal{D}(T^*)$ and obtain

$$\langle x, y\rangle + \langle x, T^*T^*y\rangle = 0$$

The density of $\mathcal{D}(T)$ entails that

$$0 = y + T^*T^*y = (T^* + iE)(T^* - iE)(y) \implies$$

$$\xi = T^*y - iy \in \mathcal{D}_- \quad \text{and} \quad \eta = T^*y + iy \in \mathcal{D}_+ \implies$$

$$y = \frac{1}{2i}(\eta - \xi) \in \mathcal{D}_- \oplus \mathcal{D}_+$$

which proves (1).

3.2 The case $\dim(\mathcal{D}_+) = \dim(\mathcal{D}_-)$. Suppose that $\mathcal{D}_+$ and $\mathcal{D}_-$ are finite dimensional with equal dimension $n \geq 1$. Then self-adjoint extensions of T are found as follows: Let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathcal{D}_+$ and $f_1, \ldots, f_n$ a similar basis in $\mathcal{D}_-$. For each n-tuple $e^{i\theta_1}, \ldots, e^{i\theta_n}$ of complex numbers with absolute value one we have the subspace of $\mathcal{H}$ generated by $\mathcal{D}(T)$ and the vectors

$$\xi_k = e_k + e^{i\theta_k} \cdot f_k \quad : \quad 1 \leq k \leq n$$

On this subspace we define a linear operator A_θ where $A_\theta = T$ on $\mathcal{D}(T)$ while

$$A_\theta(\xi_k) = ie_k - ie^{i\theta_k} \cdot f_k$$

3.3 Exercise. Verify that A_θ is self-adjoint and prove the converse, i.e. if A is an arbitrary self-adjoint operator such that

$$\Gamma(T) \subset \Gamma(A) \subset \Gamma(T^*)$$

then there exists a unique n-tuple $\{e^{i\theta_\nu}\}$ such that

$$A = A_\theta$$

3.4 Example. Let $\mathcal{H}$ be the Hilbert space $L^2[0,1]$ of square-integrable functions on the unit interval $[0,1]$ with the coordinate t. A dense subspace $\mathcal{H}_*$ consists of functions $f(t) \in C^1[0,1]$ such that $f(0) = f(1) = 0$. On $\mathcal{H}_*$ we define the operator T by

$$T(f) = if'(t)$$

A partial integration gives

$$\langle T(f), g \rangle = i \int_0^1 f'(t) \cdot \bar{g}(t) \cdot dt = \int_0^1 \bar{g}'(t) \cdot f(t) dt = \langle f, T(g) \rangle$$

Hence T is symmetric. Next, an L^2-function h belongs to $\mathcal{D}(T^*)$ if and only if there exists a constant $C(h)$ such that

$$\Big| \int_0^1 if'(t) \cdot \bar{h}(t) dt \Big| \leq C(h) \cdot ||f||_2 \quad : f \in \mathcal{H}_*$$

This means that $\mathcal{D}(T^*)$ consists of all L^2-functions h such that the distribution derivative $\frac{dh}{dt}$ again belongs to L^2.

Exercise. Show that

$$\mathfrak{D}_+ = \{h \in L^2 \quad : \frac{dh}{dt} = ih\}$$

is a 1-dimensional vector space generated by the L^2-function e^{ix}. Similarly, $\mathcal{D}_-$ is 1-dimensional and generated by e^{-ix}.

Self-adjoint extensions of T. For each complex number $e^{i\theta}$ we get the linear space $\mathcal{D}_\theta$ of functions $f(t) \in \mathcal{D}(T^*)$ such that

$$f(1) = e^{i\theta} f(0)$$

Exercise. Verify that one gets a self-adjoint operator T_θ which extends T where is $\mathcal{D}(T_\theta) = \mathcal{D}_\theta$. Conversely, show every self-adjoint extension of T is equal to T_θ for some θ. Hence the family $\{T_\theta\}$ give all self-adjoint extensions of T with their graphs contained in $\Gamma(T^*)$.

3.5 Semi-bounded symmetric operators.

Let T be closed, densely defined and symmetric. It is said to be bounded below if there exists some positive constant k such that

(*) $$\langle Tx, x \rangle \geq k \cdot ||x||^2 \quad : \quad x \in \mathcal{D}(T)$$

On $\mathcal{D}(T)$ we have the Hermitian bilinear form:

(1) $$\{x, y\} = \langle Tx, y \rangle \quad \text{where (*) entails that} \quad \{x, x\} \geq k \cdot ||x||^2$$

In particular a Cauchy sequence with respect to this inner product is a Caichy sequence in the given Hilbert space $\mathcal{H}$. So if $\mathcal{D}_*$ is the completion of $\mathcal{D}(T)$ with respect to the inner product above, then it appears as a subspace of $\mathcal{H}$. Put

$$\mathcal{D}_0 = \mathcal{D}(T^*) \cap \mathcal{D}_*$$

3.5.1 Proposition. *One has the equality*

(*) $$T^*(\mathcal{D}_0) = \mathcal{H}$$

Proof. A vector $x \in \mathcal{H}$ gives a linear functional on $\mathcal{D}_*$ defined by

$$y \mapsto \langle y, x \rangle$$

We have

(i) $$|\langle y, x \rangle| \leq ||x| \cdot ||y|| \leq ||x||| \cdot \frac{1}{\sqrt{k}} \cdot \sqrt{\{y, y\}}$$

where we used (1) above. The Hilbert space $\mathcal{D}_*$ is self-dual. This gives a vector $z \in \mathcal{D}_*$ such that

(iii) $$\langle y, x \rangle = \{y, z\} = \langle Ty, z \rangle$$

Since $\mathcal{D}(T) \subset \mathcal{D}_*$ we have (iii) for every vector $y \in \mathcal{D}(T)$, and the construction of T^* entails that $z \in \mathcal{D}(T^*)$ so that (iii) gives

(iv) $$\langle y, x \rangle = \langle y, T^*(z) \rangle$$

The density of $\mathcal{D}_*$ in $\mathcal{H}$ implies that $x = T^*(z)$ and since $x \in \mathcal{H}$ was arbitrary we get (*) in the proposition.

3.5.2 A self-adjoint extension. Let T_1 be the restriction of T^* to $\mathcal{D}_0$. We leqve it to the reader to check that T_1 is symmetric and has a closed graph. Moroever, since $\mathcal{D}(T) \subset \mathcal{D}_0$ and T^* is an extension of T we have

$$\Gamma(T) \subset \Gamma(T_1)$$

Next, Proposition 4.2.1 gives

$$T_1(\mathcal{D}(T_1) = \mathcal{H}$$

i.e. the T_1 is surjective. But then T_1 is self-adjoint by the general result below.

3.5.3 Theorem . *Let S be a densely defined, closed and symmetric operator such that*

(*) $$S(\mathcal{D}(S)) = \mathcal{H}$$

Then S is self-adjoint.

Proof. Let S^* be the adjoint of S. When $y \in \mathcal{D}(S^*)$ we have by definition

$$\langle Sx, y \rangle = \langle x, S^*y \rangle \quad : \quad x \in \mathcal{D}(S)$$

If $S^*y = 0$ this entails that $\langle Sx, y \rangle = 0$ for all $x \in \mathcal{D}(S)$ so the assumption that $S(\mathcal{D}(S)) = \mathcal{H}$ gives $y = 0$ and hence S^* is injective. Finally, if $x \in \mathcal{D}(S^*)$ the hypothesis (*) gives $\xi \in \mathcal{D}(S)$ such that

(i) $$S(\xi) = S^*(x)$$

Since S is symmetric, S^* extends S so that (i) gives $S^*(x - \xi) = 0$. Since we already proved that S^* is injective we have $x = \xi$. This proves that $\mathcal{D}(S) = \mathcal{D}(S^*)$ which means that S is self-adjoint.

§ 4. Contractions and the Nagy-Szegö theorem

A linear operator A on the Hilbert space $\mathcal{H}$ is a contraction if its operator norm is ≤ 1, i.e.

(1) $$||Ax|| \leq ||x|| \quad : \quad x \in \mathcal{H}$$

Let E be the identity operator on $\mathcal{H}$. Now $E - A^*A$ is a bounded self-adjoint operator and (1) gives:

$$\langle x - A^*Ax, x\rangle = ||x||^2 - ||Ax||^2 \geq 0$$

From the result in § 1.xx it follows that this non-negative self-adjoint operator has a square root:

$$B_1 = \sqrt{E - A^*A}$$

Next, the operator norms of A and A^* are equal so A^* is also a contraction and the equation $A^{**} = A$ gives the self-adjoint operator

$$B_2 = \sqrt{E - AA^*}$$

Since $AA^* = A^*A$ is not assumed the self-adjoint operators B_1, B_2 need not be equal. However, the following hold:

4.3.1 Propostion. *One has the equations*

$$AB_1 = B_2A \quad \text{and} \quad A^*B_2 = B_1A^*$$

Proof. If n is a positive integer we notice that

(i) $$A(A^*A)^n = (AA^*)^nA$$

Now A^*A is a self-adjoint operator whose compact spectrum is confined to the closed unit interval $[0, 1]$. If $f \in C^0[0, 1]$ is a real-valued continuous function it can be approximated uniformly by a sequence of polynomials $\{p_n\}$ and the operational calculus from § XX yields an operator $f(A^*A)$ where

$$\lim_{n\to\infty} ||p_n(A^*A) - f(A^*A)|| = 0$$

Since the spectrum of AA^* also is confined to $[0, 1]$, the same polynomial sequence $\{p_n\}$ gives an operator $f(AA^*)$ where

$$\lim ||p_n(AA^*) - f(AA^*)|| = 0$$

Now (i) and the two limit formulas above give:

(ii) $$A \circ f(A^*A) = f(AA^*) \circ A$$

In particular we can take $f(t) = \sqrt{1-t}$ and Proposition 4.3.1 follows.

4.2 The unitary operator U_A. On the Hilbert space $\mathcal{H} \times \mathcal{H}$ we define a linear operator U_A represented by the block matrix

(*) $$U_A = \begin{pmatrix} A & B_2 \\ B_1 & -A^* \end{pmatrix}$$

4.3 Proposition. *U_A is a unitary operator on $\mathcal{H} \times \mathcal{H}$.*

Proof. For a pair of vectors x, y in $\mathcal{H}$ we must prove the equality

(i) $$||U_A(x \oplus y)||^2 = ||x||^2 + ||y||^2$$

To get (i) we notice that for every vector $h \in \mathcal{H}$ the self-adjointness of B_1 gives

(ii) $$||B_1h||^2 = \langle B_1h, B_1h\rangle = \langle B_1^2h, h\rangle = \langle h - A^*Ah, h\rangle = ||h||^2 - ||Ah||^2$$

where the last equality holds since we have $\langle A^*Ah, h\rangle = \langle Ah, A^{**}h\rangle = ||Ah||^2$ and the biduality formula $A = A^{**}$. In the same way one has:

(iii) $$||B_2h||^2 = ||h||^2 - ||A^*h||^2$$

Next, by the construction of U_A the left hand side in (i) becomes

(iv) $$||Ax + B_2y||^2 + ||B_1x - A^*y||^2$$

Using (iii) we have

$$||Ax + B_2y||^2 = ||Ax||^2 + ||y||^2 - ||A^*y||^2 + \langle Ax, B_2y\rangle + \langle B_2y, Ax\rangle$$

Similarly, (ii) gives

$$||B_1x - A^*y||^2 = ||x||^2 - ||Ax||^2 + ||A^*y||^2 - \langle B_1x, A^*y\rangle - \langle A^*y, B_x\rangle$$

Adding these two equations we conclude that (i) follows from the equality

(v) $$\langle Ax, B_2y\rangle + \langle B_2y, Ax\rangle = \langle B_1x, A^*y\rangle + \langle A^*y, B_x\rangle$$

To get (v) we use Proposition 4.5.1 which gives

$$\langle Ax, B_2y\rangle = \langle x, A^*B_2y\rangle = \langle x, B_1A^*y\rangle = \langle B_1x, A^*y\rangle$$

where the last equality used that B_1 is self-adjoint. In the same way one verifies that

$$\langle B_2y, Ax\rangle = \langle A^*y, B_x\rangle$$

and (v) follows.

4.4 The Nagy-Szegö theorem.

The constructions above were applied by Nagy and Szegö to give:

4.4.1 Theorem *For every bounded linear operator A on a Hilbert space $\mathcal{H}$ there exists a Hilbert space $\mathcal{H}^*$ which contains $\mathcal{H}$ and a unitary operator U_A on $\mathcal{H}^*$ such that*

$$A^n = \mathcal{P}\cdot U_A^n \quad : \quad n = 1, 2, \ldots$$

where $\mathcal{P}\colon \mathcal{H}^* \to \mathcal{H}$ is the orthogonal projection.

Proof. On the product $\mathcal{H}_1 = \mathcal{H}\times\mathcal{H}$ we have the unitary operator U_A from (*) in 4.3.2. Let $\mathcal{P}(x, y) = x$ be the projection onto the first factor. Then (*) in (4.3.2) gives $A = \mathcal{P}U_A$ and the constructions from the proof of Propostion 4.3.4 imply that $A^n = \mathcal{P}\cdot U^n$ hold for every $n \geq 1$ which finishes the proof.

The Nagy-Szegö result has an interesting consequence. Let A be a contraction. If $p(z) = c_0 + c_1 < + \ldots + c_nz^n$ is an arbitrary polynomial with complex coefficients we get the operator $p(A) = \sum c_\nu A^\nu$ and with these notations one has:

4.4.2 Theorem *For every pair $A, p(z)$ as above one has*

$$||p(A)|| \leq \max_{z\in D} |p(z)|$$

where the the maximum in the right hand side is taken on the unit disc.

Proof. Theorem 4.4.1 gives $p(A) = \mathcal{P}\cdot p(U_A)$. Since the orthogonal $\mathcal{P}$-projection is norm decreasing we get

$$||p(A)(\xi)||^2 \leq ||p(U_A)(\xi, 0)||^2$$

Let ξ be a unit vector such that $||p(A)(\xi)|| = ||p(A)||$. The operational calculus in § 7 XX applied to the unitary operator U_A yields a probablity measure μ_ξ on the unit circle such that

$$||p(U_A)(\xi, 0)||^2 = \int_0^{2\pi} |p(e^{i\theta})|^2 \cdot d\mu_\xi(\theta)$$

The right hand side is majorized by $|p|_D^2$ and Theorem 4.4.2 follows.

4.4.3 An application. Let $A(D)$ be the disc algebra. Since each $f \in A(D)$ can be uniformly approximated by analytic polynomials, Theorem 4.4.2 entails that if a linear operator A on the Hilbert space $\mathcal{H}$ is a contraction then each $f \in A(D)$ gives a bounded linear operator $f(A)$, i.e. we have norm-preserving map from the supnorm algebra $A(D)$ into the space of bounded linear operators on $\mathcal{H}$.

§ 5 Miscellanous results

Before Theorem 5.x is announced we recall that the product formula for matrices in § X asserts the following. Let $N \geq 2$ and T is some $N \times N$-matrix whose elements are complex numbers which as usual is regarded as a linear operator on the Hermitian space $\mathbf{C}^N$. Then there exists the self-adjoint matrix $\sqrt{T^*T}$ whose eigenvalues are non-negative. Notice that for every vector x one has

(i) $$||T^*T(x)||^||Tx||^2 \implies ||\sqrt{T^*T}(x)|| = ||Tx||$$

and since $\sqrt{T^*T}$ is self-adjoint we have an orthogonal decomposition

(ii) $$\sqrt{T^*T}(\mathbf{C}^N) \oplus \mathrm{Ker}(\sqrt{T^*T}) = \mathbf{C}^N$$

where the self-adjointness gives the equality

(iii) $$\mathrm{Ker}(\sqrt{T^*T}) = \sqrt{T^*T}(\mathbf{C}^N)^\perp$$

The partial isometry operator. Show that there exists a unique linear operator P such that

(*) $$T = P \cdot \sqrt{T^*T}$$

where the P-kernel is the orthogonal complement of the range of $\sqrt{T^*T}$. Moreover, from (i) it follows that

$$||P(y)|| = ||y||$$

for each vector in the range of $\sqrt{T^*T}$. One refers to P as a partial isometry attached to T.

Extension to operators on Hilbert spaces.. Let T be a bounded opertor on the Hilbert space $\mathcal{H}$. The spectral theorem for bounded and self-adjoint operators gives a similar equation as in (*) above using the non-negative and self-adjoint operator $\sqrt{T^*T}$. More generally, let T be densely defined and closed. From § XX there exists the densely defined self-adjoint operator T^*T and we can also take its square root.

5.1 Theorem. *There exists a bounded partial isometry P such that*

$$T = P \cdot \sqrt{T^*T}$$

Proof. Since T has closed graph we have the Hilbert space $\Gamma(T)$. For each $x \in \mathcal{D}(T)$ we get the vector $x_* = (x, Tx)$ in $\Gamma(T)$. Now

$$(x_*.y_*) \mapsto \langle x, y \rangle$$

is a bounded Hermitiain bi-linear form on the Hilbert space $\Gamma(T)$. The self-duality of Hilbert spaces gives bounded and self-adjoint operator A on $\Gamma(T)$ such that

$$\langle x, y \rangle = \{Ax_*, y_*\}$$

where the right hand side is the inner product between vectors in $\Gamma(T)$. Let

$$j : (x, Tx) \mapsto x$$

be the projection from $\Gamma(T)$ onto $\mathcal{D}(T)$ and for each $x \in \mathcal{D}(T)$ we put

$$Bx = j(Ax_*)$$

Then B is a linear operator from $\mathcal{D}(T)$ into itself where

(i) $$\langle Bx, y \rangle = \{Ax_*, y_*\} = \{x_*, Ay_*\} = \langle x, By \rangle \quad : \quad x, y \in \mathcal{D}(T)$$

We have also

$$\langle Bx, x \rangle = \{A^2x_*, x_*\} = \{Ax_*, Ax_*\} = \langle Bx, Bx \rangle + \langle TBx, TBx \rangle \implies$$

$$||Bx||^2 = \langle Bx, Bx \rangle \leq \langle Bx, x \rangle \leq ||Bx|| \cdot ||x||$$

where the Cauchy-Schwarz inequality was used in the last step. Hence

$$||Bx|| \leq ||x|| \quad : \quad x \in \mathcal{D}(T)$$

This entails that that the densey defined operator B extends uniquely to $\mathcal{H}$ as a bounded operator of norm ≤ 1. Moreover, since (i) hold for pairs x, y in the dense subspace $\mathcal{D}(T)$, it follows that B is self-adjoint. Next, consider a pair x, y in $\mathcal{D}(T)$ which gives

$$\langle x, y\rangle = \{Ax_*, y_*\} = \{x_*, Ay_*\} = \langle x, By\rangle + \langle Tx, TBy\rangle$$

Keeping y fixed the linear functiional

$$x \mapsto \langle Tx, TBy\rangle = \langle x, y\rangle - \langle x, By\rangle$$

is bounded on $\mathcal{D}(T)$. By the construction of T^* it follows that $TBy \in \mathcal{D}(T^*)$ and we also get the equality

(ii) $$\langle x, y\rangle = \langle x, By\rangle + \langle x, T^*TBy\rangle$$

Since (ii) holds for all x in the dense subspace $\mathcal{D}(T)$ we conclude that

(iii) $$y = By + T^*TBy = (E + T^*T)(By) \quad : \quad y \in \mathcal{D}(T)$$

Conclusion. From the above we have the inclusion

$$TB(\mathcal{D}(T)) \subset \mathcal{D}(T^*)$$

Hence $\mathcal{D}(T^*T)$ contains $B(\mathcal{D}(T))$ and (iii) means that B is a right inverse of $E + T^*T$ provided that the y-vectors are restricted to $\mathcal{D}(T)$.

FINISH ..

5.2 Positive operators on $C^0(S)$

Let S be a compact Hausdorff space and X the Banach space of continuous and complex-valued functions on S. A linear operator T on X is positive if it sends every non-negative and real-valued function f to another real-valued and non-negative function. Denote by $\mathcal{F}^+$ the family of positive operators T which satisfy the following: First

(1) $$\lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = 0$$

hold for all pairs $x \in X$ and $x^* \in X^*$. The second condition is that $\sigma(T)$ is the union of a compact set in a disc $\{|\lambda| \leq r$ for some $r < 1$, and a finite set of points on the unit circle. The final condition is that $R_T(\lambda)$ is meromorphic in the exterior disc $\{|\lambda > r\}$, i.e. it has poles at the spectral points on the unit circle.

5.2.1. Theorem. *If $T \in \mathcal{F}^+$ then each spectral value $e^{i\theta} \in \sigma(T)$ is a root of unity.*

Proof. First we prove that $R_T(\lambda)$ has a simple pole at each $e^{i\theta} \in \sigma(T)$. Replacing T by $e^{-i\theta} \cdot T$ it suffices to prove this when $e^{i\theta} = 1$. If $R_T(\lambda)$ has a pole of order ≥ 2 at $\lambda = 1$ we know from § XX that there exists $x \in X$ such that

(i) $$Tx \neq x \quad \text{and} \quad (E - T)^2\, x = 0$$

This gives $T^2 + x = 2Tx$ and by an induction

(ii) $$\frac{1}{n} \cdot T^n x = \frac{1}{n} \cdot x + (E - T)x \quad : n = 1, 2, \ldots$$

Condition (1) and (ii) give for each $x^* \in X^*$:

$$0 = \lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = \lim_{n\to\infty} x^*(\frac{1}{n} \cdot x + (E - T)x)$$

It follows that $x^*(E - T)(x) = 0$ and since x^* is arbitrary we get $Tx = x$ which contradicts (i). Hence the pole must be simple.

Next, with $e^{i\theta} \in \sigma(T)$ we have seen that R_T has a simple pole. By the general result in § xx there exists some $f \in C^0(S)$ which is not identically zero and

$$T(f) = e^{i\theta} \cdot f$$

Multiplying f with a complex scalar we may assume that its maximum norm on S is one and there exists a point $s_0 \in S$ such that

$$f(s_0) = 1$$

For each $n \geq 1$ we have a linear functional on X defined by $g \mapsto T^n(g)(s_0)$ which gives a Riesz measure μ_n such that

$$\int_S g \cdot d\mu_n = T^n g(s_0) \quad : g \in C^0(S)$$

Since T^n by the hypothesis is positive, the integrals in the left hand side are ≥ 0 when g are real-valued and non-negative. This entails that the measures $\{\mu_n\}$ are real-valued and non-negative. For each $n \geq 1$ we put

$$A_n = \{x : \, e^{-in\theta} \cdot f(x) \neq 1\}$$

Since the sup-norm of f is one we notice that

(iii) $$A_n = \{x : \, \mathfrak{Re}(e^{-in\theta} f(x)) < 1\}$$

Now

(iv) $$0 = f(s_0) - e^{-in\theta} \cdot T^n f(s_0) = \int_S [1 - e^{-in\theta} f(s)] \cdot d\mu_n(s)$$

Taking real parts we get

(v) $$0 = \int_S [1 - \mathfrak{Re}(e^{-in\theta} f(s))] \cdot d\mu_n(s)$$

By (iii) the integrand in (v) is non-negative and since the whole integral is zero it follows that

(vi) $$\mu_n(A_n) = \mu_n(\{\mathfrak{Re}(e^{-in\theta} < 1\} = 0$$

Suppose now that there exists a pair $n \neq m$ such that

(vii) $$(S \setminus A_n) \cap (S_m \setminus A_m) \neq \emptyset$$

A point s_* in this non-empty intersection gives

$$1 = e^{in\theta} f(s_*) = e^{im\theta} \cdot f(s_*) \implies e^{in\theta} = e^{im\theta}$$

and hence $e^{i\theta}$ is a root of unity. $m - n \neq 0$. So the proof of Theorem 5.2.1 is finished if we have established the following

Sublemma. The sets $\{S \setminus A_n\}$ cannot be pairwise disjoint.

Proof. First, f has maximum norm and by the above:

$$\int_S f \cdot d\mu_n = e^{in\theta}$$

Hence the total mass $\mu_n(S)$ is at least one. Next, for each $n \geq 2$ we set

$$\pi_n = \frac{1}{n} \cdot (\mu_1 + \ldots + \mu_n)$$

Since $\mu_n(S) \geq 1$ for each n we get $\pi_n(S) \geq 1$. Put

$$\mathcal{A} = \bigcap A_n$$

Above we proved that $\mu_n(A_n) = 0$ hold for every n which gives

(*) $$\pi_n(\mathcal{A}) = 0 \quad : \, n = 1, 2, \ldots$$

Next, when the sets $\{S \setminus A_k\}$ are pairwise disjoint one has the inclusions

$$S \setminus A_k \subset A_\nu \quad \forall \, \nu \neq k$$

Keeping k fixed it follows that $\pi_\nu(S \setminus A_k) = 0$ for every $\nu \geq 0$. So when n is large while k is kept fixed we obtain

(**) $$\pi_n(S \setminus A_k)) = \frac{1}{n} \cdot \mu_k(S \setminus A_k)) \implies \lim_{n\to\infty} \pi_n(S \setminus A_k)) = 0 \quad : k = 1, 2, \ldots$$

next, recall that we already proved that $R_T(\lambda)$ has at most a simple pole at $\lambda = 1$. With $\epsilon > 0$ the Neumann series expansion gives

$$E + \sum_{k=1}^{\infty} \frac{T^k}{(1+\epsilon)^k} = R_T(1+\epsilon)) = \frac{1}{\epsilon} \cdot Q + W(1+\epsilon))$$

where $W(\lambda)$ is an operator-valued analytic function in an open disc centered at $\lambda = 1$ while Q is a bounded linear operator on $C^0(S)$. Keeping $\epsilon > 0$ fixed we apply both sides to the identity function 1_S on S and the construction of the measures $\{\mu_n\}$ gives

$$1 + \sum_{k=1}^{\infty} \frac{\mu_k(S)}{(1+\epsilon)^k} = \frac{1}{\epsilon} \cdot Q(1_S)(s_0) + W(1+\epsilon))(1_S)(s_0)$$

If $n \geq 2$ is an integer and $\epsilon = \frac{1}{n}$ one gets the inequality

$$\sum_{k=1}^{k=n} \frac{\mu_k(S)}{(1+\frac{1}{n})^k} \leq n \cdot |Q(1_S)(s_0)| + |W(1+1/n))(1_S)(s_0)| \leq n||Q|| + ||W(1+1/n)|| \implies$$

$$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq (1+\frac{1}{n})^n \cdot (||Q|| + \frac{||W(1+1/n)||}{n}$$

Since Neper's constant $e \geq (1+\frac{1}{n})^n$ for every n we find a constant C which is independent of n such that

$$\frac{1}{n} \cdot \sum_{k=1}^{k=n} \mu_k(S) \leq C$$

Hence the sequence $\{\pi_n(S)\}$ is bounded and we can pass to a subsequence which converges weakly to a limit measure μ_*. For this σ-additive measure the limit formula in (**) above entails that

(i) $$\mu_*(S \setminus A_k) = 0 \quad : k = 1, 2, \ldots$$

Moreover, by (*) we also have

(ii) $$\pi_*(\mathcal{A}) = 0$$

Now $S = \mathcal{A} \cup A_k$ so (i-ii) give:

$$\mu_*(S) = 0$$

But this is impossible for at the same time we have already seen that $\pi_n(S) \geq 1$ for each n and hence also $\mu_*(S) \geq 1$. This finishes the proof of Theorem 5.2.1.

5.2.2 The family $\mathcal{F}(X)$. if X is a banach space this family consists of bounded liner operators T on X such that

(*) $$\lim_{n\to\infty} \frac{1}{n} \cdot x^*(T^n x) = 0$$

hold for all pairs $x \in X$ and $x^* \in X^*$. The Banach-Steinhaus theorem implies that if $T \in \mathcal{F}(X)$, then there exists a constant M such that the operator norms satisfy

$$||T^n|| \leq M \cdot n \quad : n = 1, 2, \ldots$$

Since the n:th root of $M \cdot n$ tends to one as $n \to +\infty$, the spectral radius formula entails that the spectrum $\sigma(T)$ is contained in the closed unit disc.

5.2.3 The class $\mathcal{F}_*$. It consists of those T in $\mathcal{F}(X)$ for which there exists some $\alpha < 1$ such that $R_T(\lambda)$ extends to a meromorphic function in the exterior disc $\{|\lambda| > \alpha\}$. Since $\sigma(T) \subset \{|\lambda| \leq 1\}$

it follows that when $T \in \mathcal{F}_*$ then the set of points in $\sigma(T)$ which belongs to the unit circle in the complex λ-plane is empty or finite and after we can always choose $\alpha < 1$ such that

$$\sigma(T) \cap \{\alpha < |\lambda| < 1\} = \emptyset$$

Exactly as in the beginning of the proof of Theorem 5.2.1 one has

5.2.4 Proposition. *If $T \in \mathcal{F}_*$ and $e^{i\theta} \in \sigma(T)$ for some θ, then Neumann's resolvent $R_T(\lambda)$ has a simple pole at $e^{i\theta}$.*

5.2.5.Theorem. *Let $T \in \mathcal{F}(X)$ be such that there exists a compact operator K where $||T+K|| < 1$. Then $T \in \mathcal{F}_*$ and for every $e^{i\theta} \in \sigma(T)$ the eigenspace $E_T(\theta) = \{x \in X : Tx = e^{i\theta}x\}$ is finite dimensional.*

Proof. Set $S = T + K$ and for a complex number λ we write $\lambda \cdot E - T = \lambda \cdot E - T - K + K$. Outside $\sigma(S)$ we get

(i) $$R_S(\lambda)(\lambda \cdot E - T) = E + R_S(\lambda) \cdot K$$

The Neumann series for large absolute values $|\lambda|$ applied to $R_S(\lambda)$ gives some $\rho > 0$ and

(ii) $$(E + R_S(\lambda) \cdot K)^{-1} = E + R_S(\lambda) \cdot K\dot{(}E + R_S(\lambda) \cdot K)^{-1} \quad : \ |\lambda| > \rho$$

Next, when $|\lambda|$ is large we notice that (i) gives

(iii) $$R_T(\lambda) = (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Together with (ii) we obtain

(iv) $$R_T(\lambda) = R_S(\lambda) + R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Set $\alpha = ||S||$ which by assumption is < 1. Now $R_S(\lambda)$ is analytic in the exterior disc $\{\lambda| > \alpha\}$ so in this exterior disc $R_\lambda(T)$ differs from the analytic function $R_\lambda(S)$ by

(v) $$\lambda \mapsto R_S(\lambda) \cdot (E + R_S(\lambda) \cdot K)^{-1} \cdot R_S(\lambda)$$

Here K is a compact operator so the result in § XX entails that this function extends to be meromorphic in $\{|\lambda| > \alpha\}$. There remains to prove that eigenspaces at spectral points on the unit circle are finite dimensional. To prove this we use (iv). Let $e^{i\theta} \in \sigma(T)$. By Proposition 5.2.3 it is a simple pole so we have a Laurent series expansion

$$R_T(e^{i\theta} + z) = \frac{A_{-1}}{z} + A_0 + A_1 z + \ldots$$

By the general results from §§ there remains to show that A_{-1} has finite dimensional range. To see this we apply (iv) which gives the equation

$$R_S(e^{i\theta} + z) + R_S(e^{i\theta} + z) \cdot (E + R_S(e^{i\theta} + z) \cdot K)^{-1} \cdot R_S(e^{i\theta} + z)$$

To simplify notations we set $B(z) = R_S(e^{i\theta}+z)$ which by assumption is analytic in a neighborhood of $z = 0$. Moreover, the operator $B(0)$ is invertible. So now one has

$$\frac{A_{-1}}{z} + A_0 + A_1 z + \ldots = B(z) + B(z)(E + B(z) \cdot K)^{-1} B(z)$$

Since $B(0)$ is invertible we have a Laurent series expansion

$$(E + B(z) \cdot K)^{-1} = \frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots$$

and identying the coefficient of z^{-1} gives

$$A_{-1} = B(0) A^*_{-1} B(0)$$

Next, from (xx) one has

$$E = (E + B(z) \cdot K)(\frac{A^*_{-1}}{z} + A^*_0 + A^*_1 z + \ldots) \implies (E + B(0) \cdot K) A^*_{-1} = 0$$

Here $B(0) \cdot K$ is a compact operator and hence Fredholm theory implies that A^*_{-1} has a finite dimensional range. Since $B(0)$ is invertible the same is true for A_{-1} which finishes the proof of Theorem 5.2.4.

We finish with noyther result which is used to establish Kakutani's theorem in § xx.

5.2.5 Proposition. *If $T \in \mathcal{F}$ is such that $T^N \in \mathcal{F}_*$ for some integer $N \geq 2$. Then $T \in \mathcal{F}_*$.*

Proof. We have the algebraic equation

$$\lambda^N \cdot E - T^N = (\lambda \cdot E - T)(\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1})$$

It follows that

$$R_T(\lambda) = (\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1}) \cdot R_{T^N}(\lambda^N)$$

Since $T^N B \in \mathcal{F}_*$ there exists $\alpha < 1$ such that

$$\lambda \mapsto R_{T^N}(\lambda^N)$$

extends to be meromorphic in $\{|\lambda| > \alpha\}$. At the same time $(\lambda^{N-1} \cdot E + \lambda^{N-2} \cdot T + \ldots + T^{N_1})$ is a polynomial and hence $R_T(\lambda)$ also extends to be meromorphic in this exterior disc so that $T \in \mathcal{F}_*$.

5.3 Factorizations of non-symmetric kernels.

Recall that the Neumann-Poincaré kernel $K(p,q)$ of a plane C^1-curve $\mathcal{C}$ is given by

$$K(p,q) = \frac{\langle p-q, \mathbf{n}_i(p)\rangle}{|p-q|}$$

This kernel function gives the integral operator $\mathcal{K}$ defined on $C^0(\mathcal{C})$ by

$$\mathcal{K}_g(p) = \int_C K(p,q)\cdot g(q)\, ds(q)$$

where ds is the arc-length measure on C. Let M be a positive number which exceeds the diameter of $\mathcal{C}$ so that $|p-q| < M :\ p,q \in \mathcal{C}$. Set

$$N(p,q) = \int_{\mathcal{C}} K(p,\xi)\cdot \log\, \frac{M}{|q-\xi|}\cdot ds(\xi)$$

Exercise. Verify that N is symmetric, i.e. $N(p,q) = N(q,p)$ hold for all pairs p,q in $\mathcal{C}$. Moreover,

$$S(p,q) = \log\, \frac{M}{|p-q|}$$

is a symmetric and positive kernel function and since $\mathcal{C}$ is of class C^1 the reader should verify that it gives a Hilbert-Schmidt kernel, i.e.

$$\iint_{\mathcal{C}\times\mathcal{C}} S(p,q)^2\, ds(p)ds(q) < \infty$$

Hence the Neuman-Poincaré operator $\mathcal{K}$ appears in an equation

(*) $$\mathcal{N} = \mathcal{K}\circ\mathcal{S}$$

where $\mathcal{S}$ is defined via a positive symmetric Hilbert-Schmidt kernel and $\mathcal{N}$ is symmetric. Following [Carleman: § 4] we give a procedure to determine the spectrum of $\mathcal{K}$.

5.3.1 Spectral properties of non-symmetric kernels.

In general,let $K(x,y)$ be a continuous real-valued function on the closed unit square $\square = \{0 \leq x,y \leq 1\}$. We do not assume that K is symmetric but there exists a positive definite Hilbert-Schmidt kernel $S(x,y)$ such that

$$N(x,y) = \int_0^1 S(x,t)K(t,y)\, dy$$

yields a symmetric kernel function. The Hilbert-Schmidt theory gives an orthonormal basis $\{\phi_n\}$ in $L^2[0,1]$ formed by eigenfunctions to $\mathcal{S}$ where

(1) $$\mathcal{S}\phi_n = \kappa_n\phi_n$$

where the positive κ-numbers tend to zero. Moreover, each $u \in L^2[0,1]$ has a Fourier-Hilbert expansion

(2) $$u = \sum \alpha_n\cdot\phi_n$$

We seek eigenfunctions of the integral operator $\mathcal{K}$. Let u be a function in $L^2[0,1]$ such that:

(3) $$u = \lambda\cdot\mathcal{K}u$$

where λ in general is a complex number. It follows that

(4) $$\lambda\cdot\int N(x,y)u(y)\, dy = \lambda\iint SA(x,t)K(t,y)u(y)\, dtdy = \int S(x,t)u(t)\, dt$$

Multiplying with $\phi_p(x)$ an integration gives

(5) $$\lambda\cdot\int \phi_p(x)N(x,y)u(y)\, dxdy = \iint \phi_p(x)S(x,t)u(t)\, dxdt = \kappa_p\int \phi_p(t)u(t)\, dt$$

Next, using the expansion of u from (2) we get the equations:

$$(6) \qquad \sum_{q=1}^{\infty} \alpha_q \cdot \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy = \kappa_p\alpha_p \quad : \ p = 1, 2, \ldots$$

Set

$$c_{qp} = \iint \phi_q(x)\phi_p(x)N(x,y)\,dxdy$$

It follows that $\{\alpha_p\}$ satisfies the system

$$(7) \qquad \kappa_p\alpha_p = \lambda \cdot \sum_{q=1}^{\infty} c_{qp}\alpha_q$$

Since $N(x,y) = N(y,x)$ the doubly indexed c-sequence is symmetric. Set

$$(1) \qquad \beta_p = \sqrt{\kappa_p} \cdot \alpha_p \implies \beta_p = \lambda \cdot \sum_{q=1}^{\infty} \frac{c_{pq}}{\sqrt{\kappa_p} \cdot \sqrt{\kappa_q}} \cdot \beta_q$$

Next, put

$$(2) \qquad k_{p,q} = \iint K(x,y)\phi_p(x)\phi_q(y)\,dxdy$$

From the above the following hold for each pair p, q:

$$c_{pq} = \iiint \phi_q(x)\phi_p(y)S(x,t)K(t,y)\,dxdydt = \kappa_q k_{p,q} = \kappa_p k_{q,p} \implies$$

$$(3) \qquad \frac{c_{p,q}^2}{\kappa_p \cdot \kappa_q} \leq |k_{p,q} \cdot k_{q,p}| \leq \frac{1}{2}(k_{p,q}^2 + k_{q,p}^2)$$

Here $\{k_{p,q}\}$ are the Fourier-Hilbert coefficients of $K(x,y)$ which entails that

$$\sum\sum k_{p,q}^2 \leq \iint K(x,y)^2\,dxdy$$

Hence the symmetric and doubly indexed sequence

$$(4) \qquad \frac{c_{p,q}}{\sqrt{\kappa_p \cdot \kappa_q}}$$

is of Hilbert-Schmidt type.

5.3.2 Conclusion. The eigenfunctions u in $L^2[0,1]$ associated to the $\mathcal{K}$-kernel have Fourier-Hilbert expansions via the $\{\phi_n\}$-basis which are determined by α-sequences satisfying the system (7)

5.3.3 Remark. When a plane curve $\mathcal{C}$ has corner points the Neumann-Poincaré kernel is unbounded. Here the reduction to the symmetric case is more involved and leads to quite intricate results which appear in Part II from [Carleman]. The interplay between singularities on boundaries in the Neumann-Poincaré equation and the corresponding unbounded kernel functions illustrates the general theory densely defined self-adjoint operators. Much analysis remains to be done and open problems about the Neumann-Poincaré equation remains to be settled in dimension three. So far it appears that only the 2-dimensional case is properly understood via results in [Car:516]. See also § xx for a studiy of Neumann's boundary value problem both in the plane and $\mathbf{R}^3$.