IV. Nevanlinna-Pick theory

Contents

Introduction.

In the unit disc D there exists the a metric defined by

$$\frac{|dz|}{1-|z|^2}$$

In a joint article from 1916, Lindelöf and Pick discovered that if $f(z) \in \mathcal{O}(D)$ has maximum norm ≤ 1, then the map $z \to f(z)$ does not increase the metric (*). This result turns out to be very useful and is applied in section 2 to give a proof of a theorem due Julia. In Section 3 we prove some results due to Löwner about geometric properties of analytic mappings. Section 0 is devoted to an interpolation theorem due to Nevanlinna and Pick. Here we give a detailed proof since the result has a wide range of applications beyond analytic function theory in various optimization problems.

0. The Nevanlinna-Pick Interpolation Theorem

Let D be the open unit disc. Given an n-tuple of distinct points $z_1, \ldots, z_n$ in D and some n-tuple $w_1, \ldots, w_n$ of complex numbers we put:

(*) $$\rho(z(\cdot), w(\cdot)) = \min_{f \in \mathcal{O}(D)} |f|_D \quad : \quad f(z_\nu) = w_\nu \quad : 1 \leq \nu \leq n$$

Thus we seek to interpolate preassigned values at the points $\{z_k\}$ with an analytic function $f(z)$ whose maximum norm is minimal. If $n = 1$ the constant function $f(z) = w_1$ minimizes (*) so $\rho(z_1, w_1) = |w_1|$ hold for all $z_1 \in D$. If $n \geq 2$ there exists at least some $f \in \mathcal{O}(D)$ which gives a minimum. For let $\{f_\nu\}$ be a sequence of functions which solve the interpolation while their maximum norms tend to $\rho(z(\cdot), w(\cdot))$. This is a normal family and hence we extract a subsequence which converges to a limit function f_* whose maximum norm is equal to $\rho(z(\cdot), w(\cdot))$. Denote by $\mathfrak{B}_{n-1}$ tha familly of functions of the form:

(**) $$f(z) = e^{i\theta} \cdot \prod_{\nu=1}^{\nu=n-1} \frac{z - \alpha_\nu}{1 - \bar{\alpha}_\nu \cdot z}$$

where $0 \leq \theta \leq 2\pi$ and $(\alpha_1, \ldots, \alpha_{n-1})$ is some $(n-1)$-tuple of points in D which are not necessarily distinct.

0.1. Theorem *For each pair of n-tuples $z(\cdot)$ and $w(\cdot)$ there exists a unique $f_* \in \mathfrak{B}_{n-1}$ and a positive real number ρ such that the $\rho \cdot f_*(z)$ minimizes the interpolation (*).*

Remark. With $\rho = \rho(z(\cdot), w(\cdot))$ the uniquenss means that if $g \in \mathcal{O}(D)$ is an arbitrary interpolating function which is $\neq f_*$ then $|g|_D > \rho$.

The proof of Theorem 0.1 requires several steps. First we shall establish a result about Blaschke products.

0.3 Proposition. *Let f be a function in $\mathfrak{B}_{n-1}$. For every $k(z) \in \mathcal{O}(D)$ with maximum norm $|k|_D \leq 1$ such that $f - k$ has at least n zeros counted with multiplicity in D, it follows that $f = k$.*

Proof. We argue by a contradiction. If $k \neq f$ we denote by $N(f-k:r)$ the number of zeros of $f-k$ in $|z|<r$ counted with multiplicities. The hypothesis gives some $r_* < 1$ such that

(ii) $$N(k-f, r_*) \geq n$$

Next, to each $\epsilon > 0$ we consider the function

(iii) $$h_\epsilon(z) = \epsilon \cdot f(z) + \frac{1}{2}(f(z)-k(z)) = (1+\epsilon)\cdot f(z) - \frac{1}{2}(f(z)+k(z))$$

Since f is a Blascke product we have

$$\lim_{r \to 1} \min_\theta |f(re^{i\theta})| = 1$$

Since $|k|_D \leq 1$ it follows that $(1+\epsilon)\cdot|f(z)| > \frac{1}{2}\cdot|f(z)+k(z)|$ when $|z|=r$ and r is close to one. Rouche's theorem implies that

(iv) $$N(h_\epsilon, r) = n-1 \quad : r \text{ close to } 1$$

Next, since $k \neq f$ we find $r_* < r < 1$ where

$$m_*(r) = \min_\theta |f(re^{i\theta} - k(re^{i\theta})| > 0$$

In (iii) we take $\epsilon < \frac{1}{4}m_*(r)$ and then Rouche's theorem gives

(v) $$N(h_\epsilon, r) = N(f-k, r) \geq n$$

But this contradicts (iv) and hence k must be identical to f.

0.3 A consequence. Let $z(\cdot)$ and $w(\cdot)$ be some pair of n-tuples and suppose there exists some $f_* \in \mathfrak{B}_{n-1}$ and some $\rho > 0$ such that $\rho \cdot f_*(z_k) = w_k$ hold for each k. Then the function $f = \rho \cdot f_*$ not only interpolates but it has also the minimal maximum norm, i.e. we have the equality $\rho = \rho(z(\cdot), w(\cdot))$. For if $\rho(z(\cdot), w(\cdot)) < \rho$ we find an interpolating function $k(z)$ with maximum norm $|k|_D < \rho$. Now

$$f - k = \rho(f_* - k/\rho)$$

has at least n zeros and the maximum norm of k/ρ is ≤ 1. Proipostion xx entails that $f_* = k/\rho$ which gives a contraidiction, i.e. we must have the equality $\rho = \rho(z(\cdot), w(\cdot))$ and at the same time we have proved the uniqueness part in Theorem XX.

0.4 Proof of existence.

By the above there remains to show that for every given n-tuple $z(\cdot)$ in D and an arbitrary n-tuple $w(\cdot)$ of complex numbers, there exists a pair $f \in \mathfrak{B}_{n-1}$ and $\rho > 0$ such that

(0.4.1) $$\rho \cdot f(z_k) = w_k \quad : \ 1 \leq k \leq n$$

To prove the existence we shall use an induction over n. Let us first notice that if $|a| < 1$ and

$$M_a(z) = \frac{z-a}{1-\bar{a}z}$$

then we have a bijective map on $\mathcal{B}_{n-1}$ given by

$$f \mapsto f \circ M_a$$

With $a = z_1$ and $\{\zeta_k = M_a(z_k)\}$ it suffices to find $g \in \mathcal{B}_{n-1}$ and some ρ such that

(0.4.2) $$\rho \cdot g(\zeta_k) = w_k \quad : \ 1 \leq k \leq n$$

Hence we may assume that $z_1 = 0$ in (0.4.1). When this holds we already knoe that there exists some $f_* \in \mathcal{O}(D)$ with $|f|_D = \rho$ and $\{f(z_k) = w_k\}$. In particular $f(0) = w_1$ and if $|w_1| = \rho$ this entails that f is the constant function, i.e. $f(z) = w_1$ holds in d and hence $w_k = w_1$ for each $k \geq 2$. Thus trivial interpolation is excluded so from now on we can assume that

$$|w_1| < \rho$$

The case $w_1 = 0$. By the induction over n there exists $f_* \in \mathcal{B}_{n-2}$ and some $\rho > 0$ such that

$$\rho \cdot f_*(z_k) = \frac{w_k}{z_k} \quad : 2 \leq k \leq n$$

Then $f = zf_*(z)$ belongs to $\mathcal{B}_{n-1}$ and (0.4.1) holds.

The case $w_1 \neq 0$. As above $z_1 = 0$ and put $\rho = \rho(z(\cdot), w(\cdot))$. We find some $f \in \mathcal{O}(D)$ where $|f|_D = \rho$ and $f(z_k) = w_k$ hold for each k. Consider also the $(n-1)$-tuple

(1) $$\mu_k = \frac{w_k - w_1}{1 - \rho^{-2} \cdot \bar{w}_1 \cdot w_k} \quad : 2 \leq k \leq n$$

With $\mu_1 = 0$ we set

$$\gamma = \rho(z(\cdot), \mu(\cdot))$$

Next, we have the analytic function

$$F(z) = \frac{f(z) - w_1}{1 - \rho^{-2}\bar{w}_1 f(z)}$$

It is clear that the maximum norm

$$|F|_D = \rho$$

Moreover $F(z_k) = \mu_k$ hold for all k and hence (xx) gives

$$\gamma \leq \rho$$

Next, by the case xx above there exists $f_* \in \mathcal{B}_{n-2}$ such that

$$g(z) = \gamma \cdot z \cdot f_*(z) \quad : g(z_k) = \mu_k \quad : 1 \leq k \leq n$$

The equality $\gamma = \rho$. To show this we consider the analytic function

$$\phi(z) = \frac{g_*(z) + w_1}{1 + \rho^{-2} \cdot \bar{w}_1 \cdot g_*(z)}$$

If $\gamma < \rho$ it is clear that the maximum norm $|\phi|_D < \rho$. At the same time (1) entails that

$$\phi(z_k) = w_k \quad : 1 \leq k \leq n$$

This gives a contradiction since $\rho = \rho(z(\cdot), w(\cdot))$ and hence $\gamma = \rho$.

Now we can write

$$\phi(z) = \rho \cdot \frac{zf_*(z) + \frac{w_1}{\rho}}{1 + \frac{\bar{w}_1}{\rho} \cdot zf_*(z)}$$

By (xx) the absolue value $|\frac{w-1}{\rho}| < 1$ and since $z \cdot f_*(z) \in \mathcal{B}_{n-1}$ it follows from (xx) that ϕ is equal to ρ times a function i$\mathcal{B}_{n-1}$ which finishes the induction over n.

Remark. The induction above ahas shown that the requiested interpolation function for the n-tupe is found from (xx) above via the inductive step, i.e. the solution in the Nevanlinna-Pick theorem is found by an explicit inductive construction.

An example. Consider the case $n = 2$ where $z_1 = 0$ and $z_2 \neq 0$ are given in D and the interpolating values $w_1 \neq w_2$. Theorem 1 gives a unique triple a, θ, ρ where $a \in D$, $0 \leq \theta < 2\pi$ and $\rho > 0$ such that

$$f(z) = \rho \cdot e^{i\theta} \cdot \frac{z - a}{1 - \bar{a} \cdot z}$$

solves the interpolation problem. Here $f(0) = w_1$ and with $\zeta = \rho \cdot e^{i\theta}$ we obtain

$$f(z) = \zeta \cdot \frac{z + \frac{w_1}{\zeta}}{1 + \frac{\bar{w}_1 \cdot z}{\zeta}}$$

The equation $f(z_2 = w_2$ gives

$$z_2 \cdot \zeta + w_1 = w_2 + w_2\bar{w}_1 \cdot z_2 \cdot \frac{1}{\bar{\zeta}}$$

Writing $w_2 = w_1 + \gamma$ this amounts to solve the equation

$$(*) \qquad z_2 \cdot |\zeta|^2 = \gamma \cdot \bar{\zeta} + |w_1|^2 \cdot z_2 + \gamma \cdot \bar{w}_1 \cdot z_2$$

Since $w_2 \neq w_1$ is assumed the minimizing function f is not reduced to a constant which entails that

$$|\zeta| > |w_1$$

Dividing (*) with z_2 and regarding $\lambda = \gamma/z_2$ as a parameter we are led to the equation

$$|\zeta|^2 - \lambda \cdot \bar{\zeta} = |w_1|^2 + \lambda \cdot z_2 \cdot \bar{w}_1$$

A specific example. Let $\gamma = 1$ and $z_2 = \epsilon$ for some small positive ϵ while $w_1 = a$ is real and positive. So the equation becomes

$$|\zeta|^2 - \frac{\bar{\zeta}}{\epsilon} = a^2 + a$$

The solution ζ is therefore real and we are led to the algebraic equation

$$s^2 - \frac{s}{\epsilon} = a^2 + a$$

Notice that we require that $|\zeta| > |w_1| = a$ so we seek the unique root which is $> a$ and it is given by

$$s = \frac{1}{2\epsilon} + \sqrt{a + a^2 + 4^{-1}\epsilon^{-2}}$$

With a kept fixed we obtain $|\zeta| \simeq \frac{1}{\epsilon}$ as $\epsilon \to 0$ which illustrates that the maximum norm of the interpolating function increases when $\epsilon \to 0$.

1. The Lindelöf-Pick principle.

Introduction. The non-euclidian metric on D is defined by

$$(0.1) \qquad \frac{|dz|}{1 - |z|^2} \quad : \; |z| < 1$$

When D is equipped with this metric one gets a model of hyperbolic geometry in the sense of Bolyai and Lobatschevsky which led to an intense geometric study around 1890 by F. Klein and H. Poincaré. We shall not enter a detailed discussion about hyperbolic geometry since our main concern is to apply the metric (0.1) to derive inequalities for analytic functions. In a work from 1916, Lindelöf and Pick discovered that every analytic function $\phi(z)$ in the unit disc with maximum norm one decreases the metric (0.1). This result is called the Lindelöf-Pick principle and is proved in Theorem XX below. In section XX it is used to prove a result by Caratheodory and Julia concerned with the boundary behaviour of analytic functions.

1. Preliminary results. The non-euclidian distance between two point z_1 and z_2 in D will be denoted by

$$(1.1) \qquad \mathfrak{h}(z_1, z_2)$$

To grasp this distance function we first notice the equality:

$$(*) \qquad \mathfrak{h}(0, z) = \frac{1}{2} \cdot \text{Log}\, \frac{1 + |z|}{1 - |z|}$$

Indeed, (*) follows since it is obvious from (0.1) that the geodesic curve from the origin to a point $z \in D$ is the ray from 0 to z. So with $|z| = r$ one computes

$$\int_0^r \frac{ds}{1 - s^2}$$

which after integration gives (*). Next, consider a Möbius transformation:

$$w = \frac{z - a}{1 - \bar{a} \cdot z} \quad : \quad a \in D \implies \frac{dw}{dz} = \frac{1 - |a|^2}{(1 - \bar{a} \cdot z)^2}$$

At the same time we notice that

$$1 - |w|^2 = \frac{|1 - \bar{a}z|^2 - |z - a|^2}{|1 - \bar{a} \cdot z|^2} = (1 - |a|^2) \cdot \frac{1 - |z|^2}{|1 - \bar{a} \cdot z|^2}$$

From this the reader may deduce that the Möbius transform preserves the $\mathfrak{h}$-metric.

1.1 Example. Take $z_1 = 1/2$ and $z_2 = e^{i\theta}/2$ with some $0 < \theta < \pi$. Now

$$z \mapsto \frac{z - 1/2}{1 - z/4}$$

sends z_1 to the origin. It follows that

$$\mathfrak{h}(1/2, e^{i\theta}/2) = \frac{1}{2} \cdot \mathrm{Log}\, \frac{1 + r}{1 - r} \quad : \ r = \frac{2 \cdot |e^{i\theta} - 1|}{|2 - e^{i\theta}|}$$

The following consequence of Schwarz Lemma was discovered by G. Pick in 1915.

1.2 Theorem. *Let $\phi \colon D \to \Omega$ be a conformal map from the unit disc onto a simply connected domain contained in $|w| < 1$. Then the non-euclidian metric decreases.*

Proof. Let $z_0 \in D$ and set $w_0 = \phi(z_0)$. The quotient

$$G(z) = \frac{\phi(z) - w_0}{1 - \bar{w}_0 \phi(z)} : \frac{z - z_0}{1 - \bar{z}_0 z}$$

Since

$$\lim \frac{|z - z_0|}{|1 - \bar{z}_0 z|} = 1 \quad \text{as} \quad |z| \to 1$$

we see that $|G(z)| \leq 1$ holds for all $z \in D$. With $z = z_0$ we have

$$G(z_0) = \phi'(z_0) \cdot \frac{1 - |z_0|^2}{1 - |\phi(z_0)^2|}$$

Since $z_0 \in D$ was arbitrary we get the differential inequality

$$\frac{|d\phi(z)|}{|1 - \phi(z)|^2} \leq \frac{|dz|}{1 - |z|^2}$$

and this is precisely the assertion in Pick's theorem.

The Lindelöf-Pick principle. Above ϕ was a conformal mapping. Since the $\mathfrak{h}$-metric is defined locally the inequality in Pick's theorem extends to analytic functions in D of absolute value < 1 and leads to the following general result:

1.3 Theorem *Let $\phi(z) \in \mathcal{O}(D)$ have maximum norm ≤ 1. Then ϕ decreases the $\mathfrak{h}$-metric.*

Remark. Thus, if we set $w = \phi(z)$ and z_1, z_2 is a pair in the unit disc D_z one has

(*) $$\mathfrak{h}(\phi(z_1), \phi(z_2)) \leq \mathfrak{h}(z_1), z_2)$$

1.4 The $\mathfrak{h}$-metric in half-spaces. Passing to the right half-plane U_+ where $\mathfrak{Re}(w) > 0$, the non-euclidian metric is obtained via the conformal map

$$z \mapsto w = \frac{1 + z}{1 - z}$$

From this it follows that

$$\frac{|dz|}{1 - |z|^2} \mapsto 2 \cdot \frac{|w + 1|^4 \cdot |dw|}{|w + 1|^2 - |w - 1|^2}$$

So with $w = \xi + i\eta$ the non-euclidian metric in the right half-plane becomes

(*) $$\frac{|w + 1|^4 \cdot |dw|}{2\xi}$$

Next, the Lindelöf-Pick principle applies after a conformal mapping from D onto any other simply connected domain Ω where one then regards analytic functions $g \in \mathcal{O}(\Omega)$ such that $g(\Omega) \subset \Omega$.

1.5 Example. Let $\Phi(z) = u(x, y) + iv(x, y) \in \mathcal{O}(U^+)$ be such that its real part u is positive in U_+. The Lindelöf-Pick principle applies to Φ and using (*) in (1.4) one has the following result:

1.6 Proposition. *To every $k > 0$ there exists another constant k^* such that the following inequality holds for every pair of points $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$ in U_+:*

$$|\Phi(x_1 + iy_1)| \leq |v(x_0 + iy_0)| + k^* \cdot \frac{x_1 \cdot u(x_0, y_0)}{x_0} \quad : \ |y_1| < k \cdot x_1$$

1.7 Exercise. Try to prove this result. If necessary, consult the text-book [Nevanlinna: page 59-61] for a proof where it is also shown that for each $k > 0$ one can take

(*) $$k^* = 3 + 2(k+1)^2 \quad : \text{ provided that } x_1 > x_0 \quad \text{and} \quad x_1 > |y_0|$$

2. A result by Julia.

Let $\phi \in \mathcal{O}(D)$ be such that $|\phi(z)| < 1$ when $z \in D$ and consider the boundary point $z = 1$.

2.1 Theorem. *For every $e^{i\theta}$ there exists the limit*

$$(1) \qquad c(\theta) = \lim_{z \to 1} \frac{|e^{i\theta} - \phi(z)|}{|1 - z|} \quad : 0 \leq c(\theta) \leq +\infty$$

where the limit $z \to 1$ is taken in any Fatou sector at 1. Moreover, if θ is such that the limit $0 < c(\theta) < \infty$ then there exist the Fatou limits:

$$(2) \qquad \phi'(z) \to c(\theta) \cdot e^{i\theta} \quad : \arg \frac{e^{i\theta} - \phi(z)}{1 - z} \to \theta$$

and the following inequality holds

$$(3) \qquad \frac{1 - |\phi(z)|^2}{|e^{i\theta} - \phi(z)|^2} \geq \frac{1}{c} \cdot \frac{1 + |z|}{1 - |z|} \quad : z \in D$$

Remark. Of course, only the case when $c(\theta) < \infty$ is of interest. Notice that this finiteness only can occur for at most one θ-value. The theorem above was the starting point for an extensive study of boundary values of analytic functions in Julia's work [Ju] and has later led to a far-reaching study about Julia sets in complex dynamics. See [Carleson-Garnett] for this more recent and advanced theory in function theory. The reader may also consult Chapter IV in [Caratheodory] for an account of Julia's original theorem where some geometric interpretations appear.

Proof of Theorem 2.1

Applying the two conformal mappings

$$z \mapsto \frac{1 + z}{1 - z} \quad : w \mapsto \frac{e^{i\theta} + w}{e^{i\theta} - w}$$

we can work in the right half plane where $z = 1$ has been mapped into the point at infinity and ϕ has become an analytic function

$$\Phi(x + iy) = u(x + iy) + iv(x + iy) \quad : u(x, y) > 0 \text{ for all } (x, y) \in U_+$$

The crucial step in the proof is to show the result below:

Let $\Phi = u + iv$ be an arbitrary analytic map from U_+ to U_+ and assume that

$$(*) \qquad \min_{x+iy \in U_+} \frac{u(x + iy)}{x} = 0$$

Then it follows that

$$\lim_{x \to +\infty} \frac{u(x + iy)}{x} = 0 \quad : \text{holds uniformly inside any Fatou sector } |y| < kx \quad : k > 0$$

To prove this we take some $k > 0$ and for each $\epsilon > 0$ the hypothesis (*) gives a point $z_0 = x_0 + iy_0$ in U_+ such that

$$(1) \qquad \frac{u(x_0, y_0)}{x_0} < \epsilon$$

Next, if $z = x + iy$ stays in the Fatou sector $|y| < k|x|$ and x_1 is large then Proposition 1.6 gives:

$$|\Phi(x + iy)| \leq |v(x_0 + iy_0)| + k^* \cdot \frac{x \cdot u(x_0 + iy_0)}{x_0} < |v(x_0 + iy_0)| + \epsilon \cdot k^* \cdot x$$

In particular we have

$$\frac{u(x + iy)}{x} < \frac{|v(x_0 + iy_0)|}{x} + \epsilon \cdot k^*$$

Since $\epsilon > 0$ can be chosen arbitrary small the conclusion after (*) follows.

Proof continued. Next, suppose that

$$c = \min_{x+iy\in U_+} \frac{u(x+iy)}{x} > 0 \tag{1}$$

is positive. The result above applies to $\Phi(z) - cz$ and hence $\frac{\Phi(z)}{z} \to c$ holds uniformly as $|z| \to \infty$ inside any Fatou sector $|y| < k|x|$. Moreover, this gives:

$$\liminf_{x\to\infty} \frac{u(x,y)}{x} = c \tag{2}$$

Let us no consider the complex derivative of Φ assuming that (1) above holds for some $c > 0$.

Sublemma One has

$$\lim_{z\to\infty} \Phi'(z) = c$$

where this limit holds uniformly while z stays in any given Fatou sector.

Proof. Replacing Φ by $\Psi(z) = \Phi(z) - cz$ it suffices to show that

(i) $$\lim_{z} \Psi'(z) = c \quad : \text{uniformly when the limit is in a Fatou sector}$$

To show (i) we proceed as follows. Consider some $0 < p < 1$ and choose also some q so that $p < q < 1$. For every $r > 0$ we consider the disc

$$\Delta_r = \{|z-r| < q \cdot r\}$$

Since $q < 1$ this disc stays in a fixed Fatou sector for all large r and Cauchy's inequality gives

(ii) $$|\Psi'(z)| \leq \frac{qr}{2\pi} \int_0^{2\pi} \frac{|\Psi(r+qre^{i\theta})|}{|r+qre^{i\theta} - z|^2} \cdot d\theta \quad : \quad z \in \Delta_r$$

Next, if $\epsilon > 0$ Propostion 1.6 gives some large r^* such that

(iii) $$|\frac{\Psi(\zeta)}{\zeta}| < \epsilon \quad : |\zeta - r| = qr \quad : \ r \geq r^*$$

Hence, if $|z-r| \leq pr$, the Cauchy inequality from (ii) and a computation which is left to the reader gives:

(iii) $$|\Psi'(z)| \leq \epsilon \cdot \frac{q(1+q)}{(q-p)^2}$$

This proves that $\Psi'(z) \to 0$ holds uniformly when z stays in the sector

$$|\arg z| < \text{arc.sin}(p)$$

Above $p < 1$ is arbitrary which therefore gives the Caratheodory-Julia theorem after we have returned to the unit disc via a conformal map between D and U_+.

3. Some geometric results

3.1 A study of convex domains. Let Ω be a bounded convex domain and $p \in \Omega$ an interior point. The convexity implies that if we start at some boundary point $q_0 \in \partial\Omega$ where $q_0 - p$ is real and positive, then we obtain a function

(*) $$\phi \mapsto q(\phi) \quad : \ \arg\,[q(\phi) - p\,] = \phi \quad : q(\phi) \in \partial\Omega$$

where $q(2\pi) = q_0$ holds after one turn. The q-function is continuous and 1-1, i.e. a homeomorphism between the unit circle and $\partial\Omega$. Let $g(\phi)$ be a non-negative continuous function on T, i.e here $g(2\pi) = g(0)$. We get $g^* \in C^0(\partial\Omega)$ satisfying

$$g^*(q(\phi)) = g(\phi)$$

Starting from g^* we solve the Dirichlet problem and find the harmonic function G^* in Ω which extends g^*. With these notations we have

Theorem 3.2 *One has the inequality*

$$G^*(p) \leq \frac{1}{\pi} \int_0^{2\pi} g(\phi) d\phi$$

Remark. The inequality is of special interest when p approaches the boundary. Before Theorem 3.2 is proved we consider a general situation. Let W be any bounded Jordan domain and $p \in W$ an interior point. Let a, b be two points on ∂W. Denote by γ the Jordan subarc of ∂W which joins a and b. Let L be the line passing through these two points. Suppose that the two infinite half lines from a and b are outside W, i.e. $W \cap L$ is contained in the line segment (a, b). Now L cuts W into two halfs. Let W^* be one of these. Given a point $p \in W^*$ we shall find an upper bound for the harmonic measure $\mathfrak{m}_W(p; \gamma)$. After a rotation and a translation we may assume that $a = m$ and $b = -m$ for some $m > 0$, i.e. $[a, b]$ is an interval on the real axis and that W^* is contained in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$. Now $W \subset U$ and Carleman's principle from XX gives:

$$\mathfrak{m}_W(p; \gamma) \leq \mathfrak{m}_{W^*}(p : [a.b]) \leq \mathfrak{m}_U^+(p : [a.b]) \tag{1}$$

By the result in XXX the last term is equal to $\frac{1}{\pi} \cdot \alpha$ where α is the angle formed by $a - p$ and $b - p$.

Proof of Theorem 3.2. Consider a small arc $\gamma \subset \partial\Omega$ which by the parametrisation (*) above is defined by some ϕ-interval $\phi_* \leq \phi \leq \phi^*$. Let $\mathfrak{m}_\Omega(p : \gamma)$ be the harmonic measure at p with respect to this boundary arc. We can apply the inequality (1) and conclude that

$$\mathfrak{m}_\Omega(p : \gamma) \leq \phi^* - \phi_*$$

Now the Theorem 3.2 follows after an integration over $0 \leq \phi \leq 2\pi$ where we use that $G^*(p)$ is evaluated by the integral of g^* over $\partial\Omega$ with respect to the positive measure on $\partial\Omega$ defined by the harmonic measure at p.

3.3. On the range of analytic functions

Consider a domain $\Omega \in \mathcal{D}(C^1)$. Let $\phi \in \mathcal{O}(\Omega)$ and assume it extends to $C^0(\bar{\Omega})$. The ϕ-function is not supposed to be 1-1. We get the domain

$$W = \phi(\Omega)$$

Now the following may occur: There exists a subset Γ of $\partial\Omega$ given as a finite union of arcs $\{\gamma_\alpha\}$ such that the image set $\phi(\Gamma)$ gives the boundary ∂A of a domain $A \subset W$, i.e. here A is a relatively compact subset of the connected open set W. Put

$$\Omega_* = \{z \in \Omega \quad : \phi(z) \in W \setminus A\}$$

Here $A \subset \partial(W \setminus A)$ and we construct a harmonic measures as follows: If $z \in \Omega_*$ we have $\phi(z) \in W \setminus A$ and get the function

$$z \mapsto \mathfrak{m}_{W \setminus A}(\phi(z); \partial A) \quad : z \in \Omega_*$$

Since $w \mapsto \mathfrak{m}_{W \setminus A}(w; \partial A)$ is a harmonic function in $W \setminus A$ it follows that the function above is harmonic in Ω_*. Let us analyze its boundary values on $\partial\Omega_*$. If $z \in \Omega_*$ approaches Γ, then $\phi(z) \to A$ and hence

$$\lim_{z \to \Gamma} \mathfrak{m}_{W \setminus A}(\phi(z); \partial A) = 1$$

Let us now regard the harmonic measure function

$$z \mapsto \mathfrak{m}_{\Omega_*}(z : \Gamma)$$

By definition it has boundary value 1 along Γ and otherwise it is zero. Hence the maximum principle for harmonic functions gives:

3.4 Theorem. *In the situation above one has the inequality:*

$$\mathfrak{m}_{\Omega_*}(z:\Gamma) \leq \lim_{z\to\Gamma} \mathfrak{m}_{W\setminus A}(\phi(z);\partial A) \quad : \quad z \in \Omega_*$$

Application. Using Theorem 3.4 we prove a result due to Löwner. Let $w(z) \in \mathcal{O}(D)$ where $w(0)=0$ and $|w(z)|<1$. Suppose there exists an arc γ on the unit circle such that $w(z)$ extends continuously up to γ and that

$$|\gamma(e^{i\theta})| = 1 \quad : \quad e^{i\theta} \in \gamma$$

Consider the image $w(\gamma)$ which is an another arc on the unit cicle. With these notations Theorem 3.4 gives

3.5 Löwner's inequality. *The length of $w(\gamma)$ is $\geq$ the length of γ and equality can only hold if $w(z)$ from the start is $e^{i\alpha}z$ for some α.*

3.6 Remark. Actually Löwner proved a more precise result. Before it is announced we insert a preliminary remark. Given $w(z)$ and an arc $\gamma \subset T$ where $|w(z)|=1$ one should expect that $|w(z)|$ must tend to 1 rather quick as $z \in D$ approaches γ. To put this in a precise form, Löwner proceeds as follows: Up to a rotation we may take

$$\gamma = \{e^{i\theta} \quad : \ -a < \theta < a\} \quad : 0 < a < \pi/2$$

Now we consider the family of circles K_λ passing the two end-points e^{ia} and e^{-ia} where $\lambda > 0$ expresses the angle of intersection beteen K_λ and the unit circle T.

The reader should draw a picture to see the situation where the constraint that the λ-numbers are chosen so that obtain a simple connected domain $\Omega_\lambda \subset D$ bordered by γ and a portion of K_λ. Next, regard the image set $w(\Omega_\lambda)$. On its boundary we find the arc $w(\gamma)$ which by the hypothesis that $|w|=1$ on γ, is a sub-arc of T. At the same time we can start with the arc $w(\gamma)$ and take the circle K^*_λ which passes the end-points of $w(\gamma)$. This gives a domain Ω^*_λ bordered by $w(\gamma)$ and a subarc of the circle K^*_λ. With these notations the precise result by Löwner goes as follows:

3.7 Theorem. *For each λ as above one has the inclusion*

$$w(\Omega_\lambda) \subset \Omega^*_\lambda$$

3.8 Exercise. Deduce Theorem 3.7 from Theorem 3.5. The strategy is that if $w(z)$ is *outside* the set Ω_λ while $z \in \Omega^*_\lambda$, then the inequality for harmonic measures is violated. We leave it to the reader to discover this contradiction which gives Löwner's theorem. See also his article [Lö:1]: *Untersuchungen über schlichte konforme Abildungen* for details and further results.

0.9 Interpolation constants.

Let $E = (z_1, \ldots, z_n)$ be a given n-tuple of distinct points in D. Since every $f \in \mathfrak{B}_{n-1}$ has $n-1$ many roots counted with multiplicities in D it cannot vanish identically on E, i.e the maximum norm

$$|f|_E = \max_k |f(z_k)| > 0$$

This leads us to define the number

$$\tau(E) = \min_{f \in \mathfrak{B}_{n-1}} |f|_E$$

Let us also introduce the interpolation number:

$$\mathsf{int}(E) = \max_{w(\cdot)} \rho(z(\cdot), w(\cdot))$$

with the maximum taken over all w-sequences with $|w_k| \leq 1$ for every k. With these notations one has the following result which is due to Beurling:

0.9 Theorem. *For every finite set E one has the equality*

$$\tau(E) = \frac{1}{\mathsf{int}(E)} \tag{*}$$

Moreover, a function $f \in \mathcal{B}_{n-1}$ which gives $|f|_E = \tau(E)$ is unique up to a constant and for such an extremal f one has $|f(\alpha_k)| = \tau(E)$ for every $1 \leq k \leq n$.

Proof. With n kept fixed the family of $\mathcal{B}_{n-1}$ enjoys normal properties in the sense of Montel so it follows that there exists at least some extremal $f \in \mathcal{B}_{n-1}$ such that $|f|_E = \tau(E)$. Now we prove that $|f(z_k)| = \tau(E)$ for each k. For suppose strict inequality holds at some point in E which we can take to be z_1. Consider the Blaschke product

$$B(z) = \prod_{k=2}^{k=n} \frac{z - z_k}{1 - \bar{z}_k \cdot z}$$

Rouche's theorem gives some $\delta > 0$ such that if $|\zeta| < \delta$ then the analytic function $f(z) + \zeta \cdot B(z)$ has $n-1$ zeros in D and we can therefore write

$$f(z) + \zeta \cdot B(z) = \rho(\zeta) \cdot \psi_\zeta(z) \tag{1}$$

where the ζ-indexed ψ-functions belong to $\mathcal{B}_{n-1}$ and $\rho(\zeta)$ are complex numbers. Notice that

$$f(\alpha_k) = \rho(\zeta) \cdot \psi_\zeta(\alpha_k) \tag{2}$$

hold when $2 \leq k \leq n$. Moreover, since $|f(\alpha_1)| < \tau(E)$ it is clear by continuity that if δ is sufficiently small then $|\psi_\zeta(\alpha_1)| < \tau(E)$ when $|\zeta| < \delta$. Since f is extremal we conclude from (2) that there exists $\delta > 0$ such that

$$|\zeta| < \delta \implies |\rho(\zeta)| \geq 1 \tag{3}$$

This gives a contradiction since the absolute value of the ρ-function cannot have a relative minimum at $\zeta = 0$ by the local complex expansion of this ρ-function in Chapter III:XX.

Uniqueness. Let f and g be two extremal functions so that $|f|_E = |g|_E = \tau(E)$ and suppose they are not identical. For each ζ where $|\zeta| < \delta$ for a sufficiently small δ we can write

$$1 - \zeta) \cdot f + \zeta \cdot g = \rho(\zeta) \cdot \psi_\zeta(z)$$

with $\psi_\zeta \in \mathcal{B}_{n-1}$. The triangle inequality gives

$$|1 - \zeta) \cdot f(\alpha_k) + \zeta \cdot g(\alpha_k)| \leq \tau(E)$$

for every k and since $|\psi_\zeta| \geq \tau(E)$ we get as above that $|\rho(\zeta(\geq 1$ whenever ζ is sufficiently close to zero. This contradicts again the complex expansion of this ρ-function from Chapter III.

The equality $\mathrm{int}(E) = \frac{1}{\tau(E)}$**.** To begin with, let f be the unique extremal above which gives an n-tuple of points on the unit circle so that

$$f(\alpha_k) = \tau(E) \cdot e^{i\theta_k}$$

The Nevanlinna-Pick theorem shows that $\frac{f((z)}{\tau(E)}$ has smallest maximum norm over D when the n-tuple $\{w_k = e^{i\theta_k}\}$. This implies that

$$\mathrm{int}(E) \geq \frac{1}{\tau(E)}$$

To prove the opposite inequality we consider some n-tuple $\{w_\bullet\}$ for which the interpolating function $g(z)$ has the maximum norm $|g|_D = \mathrm{int}(E)$. Theorem 0.1 gives

$$g = \mathrm{int}(E) \cdot f \quad \text{where} \quad f \in \mathcal{B}_{n-1}$$

This entails that

$$\tau(E) \leq |f|_E \leq \frac{1}{\mathrm{int}(E)}$$

and the requested equality (*) in Theorem 0.9 follows.

0.10 Remark. The equality in Theorem 0.9 is the starting point for a study in [Beurling] which leads to a certain mini-max theorem. We return to this in Section XX which gives a new perspective on both on Theorem 0.1. and Theorem 0.9.