1. Distributions and boundary values of analytic functions

Introduction.

Distributions were introduced by Laurent Schwartz in 1945 and the basic theory was presented in his book *Théorie des distributions* from 1951. As pointed out by Lars Gårding in Chapter 12 from [History], Schwartz' broad attack, his radical use of infinitely differentiable functions and his conviction that distributions would be useful almost everywhere made the difference compared to earlier work where special cases had adopted the idea of distributions but never in full generality. In a broad sense distribution theory is a minor extension of the Lebegue-Riesz theory since a distribution in $\mathbf{R}^n$ is locally a finite sum of derivatives of Riesz measures. But the great merit is that calculus based upon distributions clarifies many constructions and is a very useful tool to study differential equations. For a general account about distributions and their role in differential equations we refer to Lars Hörmander's eminent text-book [Hörmander]. The text-books *Generalized functions I-IV* by Gelfand and Shilov offer an account about special classes of distributions in arbitrary dimension, while the subsequent examples are restricted to the distributions on the real line or the two-dimensional (x, y)-space which as usual is identified with $\mathbf{C}$.

Outline of the contents. In § 1-xx we define tempered distributions on the real line and construct their Fourier transforms. Two major results are Fouriers' inversion formula and the existence of boundary value distributions of analytic functions defined in rectangles above or below the real axis. Among more advanced results we mention Theorem XX which leads to the Fourier-Carleman transform which is used in § xx to establish uniqueness results for distributions whose Fourier transforms have certain gaps. § 4 is devoted to the Paley-Wiener theorem and § 5 treats Runge's approximation theorem and results about the inhomogeneous $\bar{\partial}$-equation. § 6 extends Fourier's inversion formula to non-tempered situations which lead to hyperfunctions describing the dual to the space of real-analytic functions. § 7 is devoted to an inequality for differentiable functions concerned with an inequality for L^2-norms of higher order derivatives of differentiable functions which vanish up to a a certain order at the end-points of a bounded interval. The remaining sections treat topics dealing with specific results where distribution theory appears during the analysis. The reader can pass directly to the headline *Origin of distributions.* However, the material in § 0 illustrates aspects of distribution theory related to differential equations and analytic function theory and the discussions reveal several fundamental equations while one computes distribution derivatives. Distributions on the real x-line appear naturally while one regards ordinary differential equations. In § 0 we expose some facts to illustrate the calculus with distibutions. To grasp Theorem 0.0.1 below can be regarded as a veritable aim for the less experienced reader.

0.0. ODE-equations on the real line

To grasp the notion of distributions it is natural to start with a study of distribution solutions to ordinary differential operators which leads to more systematic results as compared to studies before distribution theory was established. An example is the confluent hypergeometric function which arises as a solution to a differential operator of the form

$$P = x\partial^2 + (\gamma - x)\partial - a$$

where γ is a non-zero complex number while a is arbitrary. In the classic literature one solves this equation via the Laplace method which involves a rather cumbersome use of residue calculus. More information about the operator P arises when one determines its kernel on the space of distributions on the real x-line. The result in Theorem 0.0.1 below shows that this P-kernel is a 3-dimensional subspace of $\mathfrak{Db}$. Moreover, there exists a fundamental solution supported by the half-line $\{x \geq 0\}$.

Let us now discuss the general situation where one regards a differential operator with polynomial coefficients

$$(*) \qquad P(x, \partial) = q_m(x) \cdot \partial^m + q_{m-1}(x)\partial^{m-1} + \ldots + q_0(x)$$

where $m \geq 1$ and $q_0(x), \ldots, q_m(x)$ are polynomials which in general have complex coefficients. Let $\mathfrak{Db}$ be the space of distributions on the real x-line. A first question is to determine the P-kernel, i.e. one seeks all distributions μ such that $P(\mu) = 0$. Following material from the thesis by Ismael (xxx - University of xxx) we expose some general facts. The reader may postpone the subsequent discussion until later since we shall use notions such as analytic wave front sets and boundary value distributions whose contructions are given later on. But in anny case it is instructive to pursue the results below and the reader who has learnt the details in the examples 0.0.4-0.0.5 has begun to master distribution theory.

The local Fuchsian condition. We shall restrict the study to operators P which are *locally Fuchsian* at every real zero of the leading polynomial $q_m(x)$. This means the following: Let a be a real zero of $q_m(x)$ with some multiplicity $e \geq 1$ so that $q_m(x) = q(x)(x-a)^e$ where the polynomial q is $\neq 0$ at a. Then we can write

$$P(x, \partial) = q(x) \cdot [(x-a)^e \partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_0(x)]$$

where $\{r_\nu = \frac{p_\nu}{q}\}$ are rational functions with no pole at a and therefore define analytic functions in a neighbrohood of a. Hence

$$P_*(x, \partial) = (x-a)^e \partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_0(x)$$

can be identified with a germ of a differential operator with coefficients in the local ring $\mathcal{O}(a)$ of germs of analytic functions at a. The ring $\mathcal{D}$ of such germs of differential operatoes is studied in § x where we define the subfamily of Fuchsian operators. For example, if $a = 0$ then a Fuchsian operator in $\mathcal{D}$ is of the form

$$\rho(x) \cdot [\nabla^m + g_{m-1}(x)\nabla^{m-1} + \ldots + g_0(x)]$$

where $\rho, g_{m-1}, \ldots, g_0$ belong to $\mathcal{O}$ and $\nabla = x\partial$ is the first order Fuchsiuan operator.

From the above we can announce the following conclusive result:

0.0.1 Theorem *Let $P(x, \partial)$ in (*) above be locally Fuchsian at the real zeros of p_m. Then $Ker_P(\mathfrak{Db})$ is a complex vector space of dimension is equal to $m + e_1 + \ldots + e_k$ where $\{e_\nu\}$ are the multiplicities at the real zeros of p_m. Moreover, for each real zero of p_m there exists a distribution μ supported by $\{x \geq a\}$ such that $P(\mu) = \delta_a$.*

Remark. The crucial part in the proof of Theorem 0.1 relies upon constructions of boundary values of analytic functions. Moreover the following supplement to Theorem 0.0.1 hold. For each real zero a of $p_m(x)$ with some multiplicity e there exists a distinguished e-dimensional subspace V_a of $\mathrm{Ker}_P(\mathfrak{Db})$ which consists of distributions μ supported by the closed half-line $[a, +\infty]$ whose analytic wave front sets satisfy the following: First it contains the whole fiber above a and the remaining part of the analytic wave front set is either empty or a union of half-lines above some of the real zeros of p_m which are $> a$. Moreover, one has a direct sum decomposition

$$(**) \qquad \mathrm{Ker}_P(\mathfrak{Db}) = \mathcal{F}_+ \oplus V_{a_\nu}$$

where the last direct sum is taken over the real zeros of p_m, and $\mathcal{F}_+$ is an m-dimensional subspace of $\mathfrak{Db}$ with a basis given by an m-tuple of boundary value distributions $\{\phi_k(x + i0)\}$. Here $\{\phi_k(z)\}$ are analytic functions in a strip domain

$U = \{-\infty < x < +\infty\} \times \{0 < y < A\}$ with $A > 0$ chosen so that the complex polynomial $p_m(z)$ is zero-free in this domain and each $\phi_k(z)$ satisfies the homogeneous equation $P(z, \partial)(\phi) = 0$ in U.

Example. Consider the first order differential operator

$$P = x\partial + 1$$

Outside $x = 0$ the density x^{-1} is a solution. In §§ we shall learn hos to construct the Euler distribution x_+^{-1} which is suppored by $[0, +\infty)$ and find that

$$P(x_+^{-1}) = \delta_0$$

The 1-dimensional $\mathcal{F}_+$-space in (**) is generated by the boundary value distribution $(x + i0)^{-1}$.

0.0.2 Tempered solutions. The P-kernel in Theorem 0.0.1 need not consust of tempered distributions. The reason is that we have not imposed the condition that P is locally Fuchsian at infinity. So if $\mathcal{S}^*$ denotes the space of tempered distributions, then $\text{Ker}_P(\mathcal{S}^*)$ can have strictly smaller dimension than $m + k$ and the determination of the tempered solution space leads to a more involved analysis. Already the case $P = \partial - 1$ illustrates the situation. Here the P-kernel on $\mathfrak{Db}$ is the 1-dimensional space given by the exponential density e^x which is not tempered so the P-kernel on $\mathcal{S}^*$ is reduced to zero. During the search for tempered fundamental solutions to P supported by half-lines $\{x \geq a\}$ one can use a result due to Poincaré under the extra assumption that $\deg p_k \leq \deg p_m$ hold for every $0 \leq k < m$. For in this case there are series expansions when x is large and positive:

$$\frac{p_k(x)}{p_m(x)} = c_k + \sum_{\nu=1}^{\infty} c_{k\nu} x^{-\nu} \quad : 0 \leq k \leq m - 1$$

The leading coeficients $c_0, \dots, c_{m-1}$ give a monic polynomial

$$\phi(\alpha) = \alpha^m + c_{m-1}\alpha^{m-1} + \dots c_0$$

Let us also chose $A > 0$ so large that the leading polynomial p_m has no real zeros on $[A, +\infty]$. This gives an m-dimensional space of null solutions where a basis consists of real-analytic densities $u_1(x), \dots, u_m(x))$ on this interval.

0.0.3 Poincaré's theorem. *Suppose that ϕ has simple simple zeros $\alpha_1, \dots, \alpha_m$. Then, with A as above one can arrange the u-basis so that*

$$u_k(x) = e^{\alpha_k x} \cdot g_k(x)$$

and there exists a non-negative integer w and a constant C such that

$$|g_k(x) \leq C \cdot (1 + x)^w : 1 \leq k \leq m$$

hold for all $x \geq A$.

So for indices k such that $\mathfrak{Re}\, \alpha_k \leq 0$, it follows that $u_k(x)$ has tempered growth as $x \to +\infty$. In particular Poincaré's result entails that if the real parts are all ≤ 0, then the fundamental solutions from Theorem 0.0.1 are all tempered.

0.0.4 Example. Consider the operator

$$P = x\partial^2 - x\partial - B$$

where $B > 0$. In this case

$$\phi(\alpha) = \alpha^2 - \alpha = \alpha(\alpha - 1)$$

so one of the u-solutions above increse exponentially while the other has tempered growth as $x \to +\infty$. It is easily seen that the there exists an entire solution

$$\text{(i)} \qquad f(x) = x + c_2 x^2 + \dots$$

such that $P(f)=0$, whose coefficients are found by the recursive formulas

$$k(k-1)c_k = (k-1+B(c_{k-1} \quad : k \geq 2$$

Hence $\{c_k\}$ are positive and it is clear that f has exponential growth as $x \to +\infty$. In addition we have a solution on $x>0$ of the form

$$g(x) = f(x) \cdot \log x + a(x)$$

In §§ we explain that $P(g_+) = a \cdot \delta_0$ hold for a non-zero constant while $P(f_+)=0$. Next, let et u_1 be the tempered solution and u_2 the non-tempered solution in Poincaré's theorem on the half-line $x>0$. There are constants c_1, c_2 such that

$$f(x) = c_1 u_1(x) + c_2 u_2(x)$$

Here $c_2 \neq 0$ because f increases exponentially on $(0, +\infty)$. At the same time

$$g(x) = d_1 u_1(x) + d_2 u_2(x)$$

hold for some constants d_1, d_2. Set

$$\gamma = g_+ - \frac{d_2}{c_2} \cdot f_+$$

From the above γ has tempered growth as $x \to +\infty$ and $P(\gamma) = a \cdot \delta_0$ with $a \neq 0$. Hence $\mu = a^{-1} \cdot \gamma$ yields a tempered fundamental solution supported by $\{x \geq 0\}$.

In § xx we give further examples of tempered fundamental solutions.

0.5 Another example. Here we take

$$P = \nabla^2 + q(x) \tag{0.3.1}$$

where $q(x)$ is a polynomial such that $q(0) = -1$ and $q'(0) = 0$. For example, if $q(x) = x^2 - 1$ we encounter a wellknown Bessel operator. It is easily seen that there exists a unique entire solution $f(x)$ which satisfies $P(f)=0$ with a series expansion

$$f(x) = x + c_3 x^3 + \ldots$$

Moreover, one verifies easily that there exists another entire function $g(x)$ with $g(0) = 0$ such that the multi-valued function

$$\phi(z) = f(z) \cdot \log z + g(z) \tag{i}$$

satisfies $P(\phi) = 0$. Theorem 0.0.1 predics that the P-kernel on $\mathfrak{Db}$ is 4-dimensional. To begin with f restricts to a real analytic densitiy on the x-line and gives a null solution. A second solution is obtained by the boundary value distribution

$$\gamma = \phi(x+i0) = f(x) \cdot \log(x+i0) + g(x)$$

Together they give a basis in the 2-dimensional space $\mathcal{F}_+$ from (*) in the remark after Theorem 0.0.1. There remains to find two linearly independent distributions in V_0 since the leading polynomial of P has a double zero at $x=0$. To attain such a pair we first consider the boundary value distribution

$$\gamma_* = f(x) \cdot \log(x-i0) + g(x)$$

which also is a null solution. Here the multi-valuedness of the complex log-function entails that

$$\gamma - \gamma_* = 2\pi i \cdot f(x) \cdot H_-(x)$$

where $H_-(x)$ is the Heaviside distribution supported by the negative half-line. Then

$$\gamma^* = \gamma - \gamma_* - 2\pi i \cdot f(x)$$

is a null solution supportec by the half-line $x \geq 0$ and hence belongs to the 2-dimensional space V_0. A second null solution in V_0 is given by the Dirac measure δ_0. To see that δ_0. is a null solution for P we recall that in the non-commutative ring of differential operators one has the equality $\nabla = \partial x \circ x - 1$. Since $x \cdot \delta_0 = 0$ we get the distribution equation

$$\nabla(\delta_0) = -\delta_0 \implies \nabla^2(\delta_0) = \delta_0$$

Since $q(0) = -1$ is assumed in (0.3.1) it follows that $P(\delta_0) = 0$. Hence we have found four linearly independent null solutions $f_+ . f_-, \gamma^*, \delta_0$ in accordance with Theolrem 0.0.1.

The fundamental solution. A fundamental solution μ supported by $x \geq 0$ is found as follows: From (i) we have the real-analytic density $\phi(x)$ on the open half-line $\{x > 0\}$ which gives the distribution ϕ_+ supported by $\{x \geq 0\}$ defined by

$$\phi_+ = f(x) \cdot (\log x) \cdot H_+ g(x) \cdot H_+$$

In § xx we shall explain that

$$\nabla(\log x \cdot H_+) = \delta_0$$

and from this deduce that

$$P(\phi_+) = -\delta_0$$

Hence $\mu = -\phi_+$ gives the requested fundamental solution.

0.1 The distribution z^{-1}

The complex z-plane is identified with the real (x, y) space when $z = x + iy$. Here $\frac{1}{z}$ is locally integrable and defines a distribution in $\mathbf{R}^2$. Consider the first order differential operator $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Cauchy's residue formula entails that

$$-\iint \frac{\bar{\partial}(g)}{z} \, dxdy = \pi \cdot g(0)$$

hold for every test-function $g(x, y)$. This means that the distribution derivative

$$(*) \qquad \bar{\partial}(\frac{1}{z}) = \pi \cdot \delta_0$$

where δ_0 is the Dirac measure at the origin. Next, let $f(x, y)$ be an integrable function with a compact support in $\mathbf{R}^2$. One says that the distribution derivative $\partial f / \partial x$ has order zero if there exists a Riesz measure μ such that

$$\iint \partial f/\partial x \cdot g \, dxdy = \int g \cdot d\mu$$

hold for every test-function g. A similar condition can be imposed for y, i.e. suppose also that there is a Riesz measure ν such that

$$\iint \partial f/\partial y \cdot g \, dxdy = \int g \cdot d\nu$$

Passing to the operator $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ the two conditions above entail that the distribution derivative $\bar{\partial}(f)$ is a Riesz measure γ. When this holds we introduce the Cauchy transform of the compactly supported measure γ:

$$\text{(i)} \qquad \widehat{\gamma}(z) = \int \frac{d\gamma(\zeta)}{z - \zeta}$$

This yields an analytic function in the open complement if the support of γ. Moreover, since $\widehat{\gamma}$ is the convolution of a measure and a locally integrable function it is also locally intergable in the (x, y)-space and there exists the distribution derivative $\bar{\partial}(\widehat{\gamma})$. Using (*) it is equal to γ which gives the equality

$$\text{(ii)} \qquad \bar{\partial}(f - \widehat{\gamma}) = 0$$

In § xx we shall learn that $\bar{\partial}$ is an elliptic operator and hence (ii) entails that $f - \widehat{\gamma}$ is an analytic function of z. Since z^{-1} is locally integrable it follows in particular that if γ is in L^1, i.e. it is absolutely continuous in the sense of Lebesgue, then general results in measure theory entails that that the convolution which defines $\widehat{\gamma}$ in (i) is a continuous function and hence f is also continuous.

Another example. A merit in distribution theory is that Fourier transforms exist in a wide sense. This will be explained in § §. Consider some L^1-function $\phi(x,y)$ with compact support and construct the convolution of ϕ and z^{-1}. The Fourier transform of this convolution is the product of the Fourier transforms of ϕ and z^{-1}. Since ϕ has compact support its Fourier transform

$$\widehat{\phi}(\xi,\eta) = \iint e^{-i(x\xi+y\eta)} \cdot \phi(x,y)\, dxdy$$

is a a real-analytic function in the (ξ,η)-space. In § xx we shall learn that

$$\widehat{z^{-1}}(\xi,\eta) = xxx \cdot \frac{1}{\xi + i\eta}$$

Hence $\phi * z^{-1}$ is recaptured by Fourier's inversion formula:

$$\phi * z^{-1}(x,y) = \frac{1}{xx} \cdot \iint e^{i(x\xi+y\eta)} \cdot \frac{\widehat{\phi}(\xi,\eta)}{\xi + i\eta}\, d\xi d\eta$$

However, here some caution occurd because it is in general not true that the integrand above is absolutely convergent, i.e. there exist ϕ-functions as above such that

$$\iint \left| \frac{\widehat{\phi}(\xi,\eta)}{\xi + i\eta} \right| d\xi d\eta = +\infty$$

To compensate for this we shall learn that distribution theory enable us to give a meaning of Fourier's inversion formula even when the integral above is divergent.

0.1 The distributions x_+^s

If s is a complex number where $\mathfrak{Re}\, s > -1$ the function defined by x^s for $x > 0$ and zero on the half-line $x \leq 0$ is locally integrable and defines a distribution denoted by x_+^s acting on test-functions g by

$$x_+^s(g) = \int_0^\infty x^s \cdot g(x)\, dx$$

The distribution-valued function $s \mapsto x_+^s$ is analytic in $\mathfrak{Re}\, s > -1$. Indeed if $x < 0$ we have $\frac{d}{ds}(x^s = \log x \cdot x^s$ which entails that the complex derivative of x_+^s is the distribution defined by

$$g \mapsto \int_0^\infty \log x \cdot x^s \cdot g(x)\, dx$$

It turns out that x_+^s extends to a meromorphic distribution-valued function in the whole s-plane. To prove this we perform a partial integration which gives

$$(0.0.1) \qquad x_+^{s+1}(g') = \int_0^\infty x^{s+1} \cdot g(x)\, dx = -(s+1) \cdot \int_0^\infty x^s \cdot g(x)\, dx$$

By the construction of distribution derivatives this means that

$$\frac{d}{dx}(x_+^s + 1) = (s+1) \cdot x_+^s$$

Euler's functional equation. Set $\partial = \frac{d}{dx}$. We can iterate (0.0.1) which for every positive integer m gives

$$(0.0.2) \qquad (s+1) \cdots (s+m) x_+^s = \partial^m (x_+^{s+m})$$

We refer to (0.0.2) as Euler's functional equation. It entails that the distribution-valued function x_+^s extends to a meromorphic function with at most simple poles at negative integers. Let us investigate the situation close to a negative integer. With $s = -m + t$ and t small one has

$$t(t-1) \cdot (t - m + 1) x_+^{-m+t} = \partial^m (x_+^t)$$

When $x > 0$ one has the expansion

$$x^t = 1 + t \log x + \frac{t^2}{2} \cdot (\log x)^2 + \frac{t^3}{3!} \cdot (\log x)^3 + \ldots$$

From this we obtain a series expansion

$$x_+^{-m+t} = t^{-1} \cdot \rho_m + \gamma_0 + t\gamma_1 + \ldots$$

where ρ_m and $\{\gamma_\nu\}$ are distributions. In particular the reader may verify that

$$(-1)^{m-1}(m-1)! \cdot \rho_m = \partial^m (H_+)$$

Let us then consider the constant term γ_0. The linear t-term in the expansion of $t(t-1) \cdot (t - m + 1) x_+^{-m+t}$ becomes

$$(-1)^{m-1}(m-1)! \cdot \gamma_0 + \frac{m(m-1)}{2} \cdot \rho_m$$

If $x > 0$ we notice that

$$\partial^m (\log x) = (-1)^{m-1} \cdot (m-1)! \cdot x^{-m}$$

From the above γ_0 restricts to the density x^{-m} when $x > 0$. At the same time γ_0 is a distribution defined on the whole x–line supported by $\{x \geq 0\}$. We set

$$(*) \qquad x_+^{-m} = \gamma_0$$

and refer to this as Euler's extension of the density x^{-m} which from the start is defined on $\{x > 0\}$. So in (*) we have found distributions for every positive integer m.

0.1.2 Further formulas. With $s = -1 + z$ where z is a small non-zero complex number one has

(i) $$z \cdot x_+^{-1+z} = \partial(x_+^z)$$

Next, if $x > 0$ we have

$$x^z = e^{z \log x} = 1 + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!} \cdot z^k$$

Introducing the Heaviside distribution H_+ which is 1 on $x \geq 0$ and zero on $x < 0$ this means that

(ii) $$\partial(x_+^z) = \partial(H_+) + \sum_{k=1}^{\infty} \partial(\frac{(\log x)^k}{k!} \cdot H_+(x))z^k$$

From this we get a Laurent expansion of x_+^s at $s = -1$. The crucial point is that the distribution derivative

(iii) $$\partial(H_+) = \delta_0$$

where δ_0 is the Dirac distribution at $x = 0$. It follows that

(iv) $$x_+^{-1+z} = z^{-1} \cdot \delta_0 + \sum_{k=1}^{\infty} \partial(\frac{(\log x)^k}{k!} \cdot H_+(x))z^{k-1}$$

In particular the constant term becomes

(v) $$\partial(\log x \cdot H_+(x))$$

To find this distribution we take a test-function g and a partial integration gives

$$-\int_0^{\infty} (\log x \cdot g'(x)\, dx = \int_0^1 \frac{g(x) - g(0)}{x}\, dx + \int_1^{\infty} \frac{g(x)}{x}\, dx$$

From this we conclude that the distribution x_+^{-1} is defined on test-functions by the formula:

$$x_+^{-1}(g) = \frac{(-1)^{m-1}}{(m-1)!} \cdot \int_0^1 \frac{g(x) - g(0)}{x}\, dx + \int_1^{\infty} \frac{g(x)}{x}\, dx$$

0.1.3 Exercise. For each test-function g and integer $m \geq 2$ we set

$$T_{m-1}(g)(x) = g(0) + g'(0)x \ldots \frac{g^{(m-1)}(0)}{(m-1)!} \cdot x^{m-1}$$

Show from the above via partial integrations that

$$x_+^{-m}(g) = \frac{(-1)^{m-1}}{(m-1)!} \cdot \int_0^1 \frac{g(x) - T_{m-1}(g)(x)}{x^m}\, dx + \int_1^{\infty} \frac{g(x)}{x^m}\, dx$$

0.1.4 The distributions $(x + i0)^\lambda$ and $(x - i0)^\lambda$. In the upper half plane there exists the single valued branch of $\log z$ whose argument stays in $(0, \pi)$ and for every complex number λ we have

$$z^\lambda = e^{\lambda \cdot \log z}$$

In § 3 we shall learn how to contruct boundary value distributions of analytic functions defined in strip domains above or below the real x-line. In particular there exists the distribution $(x + i0)^\lambda$ defined on test-functions $g(x)$ by the limit formula

$$\lim_{\epsilon \to 0} \int (x + i\epsilon)^\lambda \cdot g(x)\, dx$$

Notice that this limit exists for all complex λ, i.e even when the real part becomes very negative. In the same way we have the single valued branch of $\log z$ in the lower half-plane whose argument stays in $(-\pi, 0)$ and construct the distribution $(x - i0)^\lambda$ defined by

$$(x - i0)^\lambda(g) = \lim_{\epsilon \to 0} \int (x - i\epsilon)^\lambda \cdot g(x)\, dx$$

Since $\lambda \mapsto e^{\lambda \cdot \log z}$ are entire in λ, we get two entire distribution valued functions by $(x - i0)^\lambda$ and $(x - i0)^\lambda$. Regarding the choice of branches for the log-functions we see that

$$(x - i0)^\lambda = e^{-2\pi i \lambda} \cdot (x + i0)^\lambda \quad : x < 0$$

At the same time

$$(x + i0)^\lambda = (x - i0)^\lambda = x^\lambda \quad : x > 0$$

From this we see that the distribution

$$(x + i0)^\lambda - e^{2\pi i \lambda} \cdot (x - i0)^\lambda$$

is supported by $x \geq 0$ and expressed by the density $(1 - e^{2\pi i \lambda}) \cdot x^\lambda$. The conclusion is that one has the equation

$$\mu_\lambda = \frac{(x + i0)^\lambda - e^{2\pi i \lambda} \cdot (x - i0)^\lambda}{1 - e^{2\pi i \lambda}}$$

Remark. The equation (xx) is more involved compared to the previous description of the meromorphic μ-function found via Euler's functional equation. But (xx) has the merit that the denominator is an entire distribution valued function and when one passes to Fourier transforms it turns out that (xx) is quite useful.

Principal value integrals. If $g(x)$ is a test-function there exists a limit

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{g(x)}{x}\, dx$$

This yields a distribution denoted by $\mathrm{VP}(x^{-1})$. Outside $\{x = 0\}$ it is given bt the density x^{-1} where it agrees with $(x + i0)^{-1}$ and hence the difference

$$\mu = \mathrm{VP}(x^{-1}) - (x + i0)^{\cdot -1}$$

is supported by $\{x = 0\}$.

Exercise. Notoice that

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{1}{x + i\epsilon}\, dx = \log(1 + i\epsilon) - \log(-1 + i\epsilon) = -\pi i$$

and use this to show that

$$\mu = -\pi i \cdot \delta_0$$

0.1 ODE-equations.

Ordinary differential equations are best treated by distributions. As a first example we take the first order differential operator $\nabla = x \cdot \frac{\partial}{\partial x}$. When $x \neq 0$ the equation $\nabla(f) = 0$ has solutions given by constant functions. To pass beyond $x = 0$ one introduces the Heaviside densities H^+ and H_- where $H^+(x) = 1$ when $x > 0$ and it is zero if $x < 0$. Similarly $H_- = 1 - H^+$. It turns out that these two linearly independent distributions on the x-line generate the vector space of all distribution solutions to the equation $\nabla(\mu) = 0$. See § xx below. On the other hand, if $f(x)$ is a C^1-function, i.e. continuously differentiable and satisfies $\nabla(f) = 0$ it is easily seen that f must be a constant. So the set of distribution solutions is more extensive. Next, let s be a complex number which is not an integer. If $\mathfrak{s} > -1$ then x^s is integrable on intervals $(0, a)$ with $a > 0$ and there exists a distribution denoted by x_+^s acting as a linear functional on test-functions $\phi(x)$ by

$$x_+^s(\phi) = \int_0^\infty x^s \cdot \phi(x)\, dx$$

On the open interval $(0, +\infty)$ we notice that

$$(\nabla - s)(x^s) = 0 \quad : x > 0$$

It we apply $\nabla - s$ to the distribution x_+^s the construction of distribution derivatives means that $(\nabla - s)(x_+^s)$ acts on test-functions ϕ by

$$\text{(i)} \qquad \phi \mapsto \int_0^\infty x^s \cdot (-\partial(x\phi) - s\phi)\, dx$$

When $\mathfrak{Re}\, s > -1$ a partial integration shows that (i) is zero. Hence $(\nabla - s)(x_+^s) = 0$. It turns out that there exist more distributions μ such that $(\nabla - s)(\mu) = 0$. In § XX we construct boundary value distributions of analytic functions and find that the distribution $\mu = (x + i0)^s$ satisfies the equation $(\nabla - s)(\mu) = 0$. Here μ is defined on test-functions $\phi(x)$ by

$$\text{(ii)} \qquad \mu(\phi) = \lim_{\epsilon \to 0} \int (x + i\epsilon)^s \cdot \phi(x)\, dx$$

where the limit is taken as over positive ϵ which decrease to zero and in the upper half-plane one has taken the usual single-valued branch of $\log z$ so that the complex powers

$$z^s = e^{s \log z}$$

are defined for all $s \in \mathbf{C}$. A notable fact that the limit in (ii) exists for all test-functions ϕ and every complex number s, i.e. even when $\mathfrak{Re}\, s < -1$. Hence we have found two linearly independent distribution solutions to the equation $\nabla - s(\mu) = 0$. It turns out that they give a basis, i.e.

$$\dim_{\mathbf{C}} (\mathrm{Ker}_{\nabla - s}(\mathfrak{Db})) = 2$$

In § xx we extend this result to a more extensive class of ODE-equations. More precisely, let $m \geq 1$ and $p_0(x), \ldots, p_m(x)$ are polynomials and set

$$P(x, \partial) = p_m(x)\partial^m + \ldots + p_0(x)$$

Under the assumption that the real zeros of the leading polynomial $p_m(x)$ are simple and consists of some k-tuple $\{a_1 < \ldots a_k\}$ it follows that the P-kernel on $\mathfrak{Db}$ has dimension $k + m$. Moreover, a basis is given by a k-tuple of distributions $\mu_1, \ldots, \mu_k$ where μ_ν is supported by the half-line $(-\infty, a_\nu]$ for every ν. In addition there is an m-tuple of distributions solutions found as boundary values of analytic functions $\{\phi_1(x + i0), \ldots, \phi_m(x + i0)\}$ where $\{\phi_\nu(z)\}$ are analytic in a strip domain $0 < \mathfrak{Im}\, z < A$ and found as holomorphic solutions to the complex differential operator $P(z, \partial_z)$.

The Γ-funtion is used to construct other distributions. For example, there exist distributions x_+^s for all complex numbers s which exclude the set of negative integers. More precisely, if m is a positive integer such that $\mathfrak{Re}\, s > -m - 1$ then the Euler-Riemann distribution x_+^s is defined by

$$\text{(iii)} \qquad \frac{1}{(s+1) \cdots (s+m)} \cdot \partial^m (x_+^{s+m})$$

The conclusion is that there exists an entire distribution valued funtion

$$s \mapsto \frac{1}{\Gamma(s+1)} \cdot x_+^s$$

At the same time the material in § xx wil show that there exists the entire distribution-valued functions $s \mapsto (x+i0)^s$ and $s \mapsto (x-i0)^s$. On the half-line $\{x<0\}$ one has the equations

$$(x+i0)^s = e^{\pi is} \cdot |x|^s \quad : (x-i0)^s = e^{-\pi is} \cdot |x|^s$$

At the same time $(x+i0)^s = (x-i0)^s = x^s$ hold when $x>0$. From this it follows that

$$\text{(iv)} \qquad e^{-\pi is} \cdot (x+i0)^s - e^{\pi is} \cdot (x-i0)^s = -2i \sin \pi s \cdot x_+^s$$

Now the sine-funtion has zeros at $\{-n\pi : \ n=1,2,\ldots\}$ and from this one can again deduce that the distribution-valued function $s \mapsto x_+^s$ extends from $\mathfrak{Re}\, s > -1$ to the complex s-plane where it has simple poles at all negative integers. It is interesting to notice that this meromorphic extension can be achieved in teo ways, i.e. either via partial integration and (iii) or by (iv). We shall learn tha the existence of boundary values of analytic functions is quite powerful and in more involved situations they are used to handle more general situations.

0.1.1 A first order ODE-equation. Let p and q be a pair of polynomials and set

$$Q = q(x) \cdot \partial - p(x)$$

Assume that q has degree $k \geq 2$ whose zeros are real and simple and arranged in strictly increasing order $\{a_1 < a_2 < \ldots < a_k\}$. At the same time $p(a_\nu) \neq 0$ hold for every ν. Here q is real-valued while p in general can be complex-valued. Now one seeks distributions μ on the x-line such that $Q(\mu)=0$. One such solution is found as the boundary value of the analytic function defined in the upper half-plane by

$$f(z) = e^{\int_i^z \frac{p(\zeta)}{q(\zeta)}\, d\zeta}$$

The point is here that when $\mathfrak{Im} z > 0$ it is evident that the complex defivative

$$\frac{\partial f}{\partial z} = \frac{p(z)}{q(z)}$$

In § x we shall learn that the passage to boundary value distributions commute with derivations and therefore (x) entails that $f(x+i0)$ solves the homogeneous ODE-equation. In § xx we shall determine more distriibution solutions. For example, let a_ν be a zero of q and assume that $\frac{p(a_\nu)}{q'(a_\nu)}$ is not a positive integer Then we shall prove that there exists a distribution μ such that $Q(\mu)=0$ and the support of μ is the half-line $(-\infty, a_\nu]$. When the condition above hold for every zero of q it turns out that the solution space to the equation $Q(\mu))0$ has dimension $k+1$. Without using boundary values of analytic functions it is not easy to discover this result.

0.1.2 Higher order equations. In § xx we consider a differential operator $Q(x, \partial)$ of order $m \geq 2$ with polynomial coefficients:

$$Q(x, \partial) = q_m(x)\partial^m + \ldots + q_1(x)\partial + q_0(x)$$

Notice that we get a holomorphic differential operator when we pass to the complex variable $z = x + iy$ and set

$$Q(z, \partial_z) = q_m(z)\partial_z^m + \ldots + q_1(z)\partial_z + q_0(z)$$

Assume that the polynomials $\{q_\nu(x)\}$ have no common zero in the complex plane and that the leading polynomial q_m has real and simple zeros $\{a_1 < \ldots < a_k\}$ for some positive integer k. Since $q_m(z) \neq 0$ when z is outside the real axis it is easily shown that in the upper half-plane $U_+ = \mathfrak{Im}\, z > 0$ there exists an m-dimensional subspace of $\mathcal{O}(U_+)$ whose functions $f(z)$ satisfy $Q(z, \partial_z)(f) = 0$. Let $\{f_1, \ldots, f_m\}$ be the basis for the solutions in $\mathcal{O}(U_+)$. To each zero a_ν we consider small open disc $D = \{|z - a_\nu| < r\}$ with r so small that $|a_j - a_\nu| \geq r$ for all $j \neq \nu$. Here two case can occur: The first is that the restriction of every f_ν to the half-disc D_+

extends to be analytic in D and then a_ν is called a negligable singular point for the ODE-equation. If some f-function fails to extend. Malgrange's index formula from § xx shows that there exists an $m-1$-dimensional subspace of these f-functions whose restrictions extend to be holomorphic in D. Using this local index formula we show in § xx that if k_* is the number of zeros of q_m which are not negligable then the space of distributions μ defined on the whole real line such that $Q(\mu)=0$ is a vector space of dimension $m+k_*$.

0.1.3 The equation $\nabla^2(\mu)=0$. The Fuchsian operator is defined by $\nabla = x\partial$. It turns out that the space of distributions μ satisfying $\nabla^2(\mu)$ is a 4-dimensional vector space. One solution is the Heavisde function H_+ defined by the density 1 if $x>0$ and zero if $x\leq 0$. Here

$$\partial(H_+)(g) = -\int_0^\infty g'(x)\,dx = g(0) \quad : g \in C_0^\infty(\mathbf{R})$$

This means that the distribution derivative $\partial(H_+)=\delta_0$ and since $x\cdot\delta_0=0$ we have $\nabla(H_+)=0$. Similarly the negative Heaviside function H_- satisfies $\nabla(H_-)=0$. Next, on $\{x>0\}$ we see that the density $\log x$ satisfies $\nabla^2(\log x)=0$. It is tempting to extend the locally integrable function $\log x$ on the positive half-line to $\mathbf{R}$ by setting the value zero if $x\leq 0$. Denote the resulting distribution by $\log_+ x$. Now

$$\nabla(\log_+ x)(g) = -\int_0^\infty -\partial(xg)\cdot \log x\,dx = \int_0^\infty xg\cdot\frac{1}{x}\,dx = \int_0^\infty gdx$$

Hence $\nabla(\log_+ x) = H_+$ Since $\nabla(H_+)=0$ we get $\nabla^2(\log_+ x)=0$. In a similar fashion one extends $\log|x|$ on the negative half-line by zero and find that $\log_-|x|$ also is annihilated by ∇^2. Hence we have found four linearly independent solutions to the homogeneous equation given by the pair $(H_+, \log_+ x)$ supported by $x\geq 0$ and the pair $(H_-, \log_-|x|)$ supported by $x\leq 0$.

0.1.4 Higher order Fuchsian equations. Let $m\geq 2$ and consider an operator of the form

$$Q = \nabla^m + q_{m-1}(x)\nabla^{m-1} + \ldots + q_1(x)\nabla + q_0(x)$$

where $\{q_\nu(x)\}$ are polynomials. With $\{c_\nu = q_\nu(0)\}$ we associate the polynomial

$$Q^*(s) = s^m + c_{m-1}s^{m-1} + \ldots + c_1 s + c_0$$

Under the assumption that $Q^*(k)\neq 0$ for all non-negative integers the solution space $\mathcal{S}=\{\mu\colon Q(\mu)=0\}$ has dimension $2m$ at least. Moreover, there exists an m-tuple of linearly independent solutions supported by $\{x\geq 0\}$ and an m-tuple supported by $\{x\leq 0\}$. The strategy of the proof is as follows: In the upper half-plane the Picard-Fuchs theory about holomorphic differential equations entails that there exists an m-tuple of linearly independent analytic functions $\{\phi_\nu(z)\}$ which solve $Q(z,\partial_z)(\phi_\nu)=0$. Similarly one finds an m-tuple $\{\psi_\nu\}$ of solutions in the lower half-plane. The boundary value distributions $\{\phi_\nu(x+i0)\}$ and $\{\psi_\nu(x-i0)\}$ belong to $\mathcal{S}$ and are linearly indepedent. For if $\sum c_\nu\phi_\nu(x+i0)+\sum d_\nu\psi_\nu(x-i0)=0$ where at least some c_ν or d_ν is $\neq 0$ then

$$\text{(i)} \qquad \phi_*(x+i0) = \psi_*(x-i0) = 0$$

where $\phi_* = \sum c_\nu\phi(x+i0)\neq 0$ and $\psi_* = -\sum d_\nu\psi(x-i0)\neq 0$. Now (i) cannot hold. The reason is that the assumption about $Q^*(s)$ first entails that the equation $Q(z,\partial_z)(f)=0$ has no holomorphic solutions at $z=0$ which by Schwarz reflection principle implies that the analytic wave front sets of the distributions ϕ_* and ψ_* both are non-empty. On the other hand we shall learn in § xx that these non-empty wave fronts have opposed directions so the equality (i) cannot hold. This proves that $\mathcal{S}$ is at least $2m$-dimensional and as explained in § xx a one can find m linearly independent solutions supported by $\{x\geq 0\}$ and another by $\{x\leq 0\}$.

In addition to this $2m$-dimensional subspace of $\mathcal{S}$ there may occur Dirac solutions, i.e. distributions supported by $\{x = 0\}$. If δ_0 is the Dirac measure at $\{x = 0\}$ the distribution $\nabla(\delta_0) = -\delta_0$. From this follows that Dirac solutions appear for positive integers k such that $Q^*(-k) \neq 0$. Ths means that there even occur cases when $\mathcal{S}$ has dimension $3m$.

Example. If $Q = \nabla + 1$ then $\mathcal{S}$ is 3-dimensional where a basis is given by δ_0 and the two boundary value distributions $(x + i0)^{-1}$ and $(x - i0)^{-1}$.

0.1.5 Remark. The results above illustrate the interplay between analytic function theory and distributions which becomes even more powerful when we also employ Fourier analysis. An example is the notion of analytic wave front sets of distributions on the real x-line. This is treated in § XX where Hörmander's construction of analytic wave front sets of distributions gives a new perspective on Schwarz's reflection principle.

0.2 PDE-equations with constant coefficients.

The study of PDE-equations with contant coefficients in $\mathbf{R}^n$ for arbitrary $n \geq 2$ is a rich subject. The interested reader may consult Chapter xx in [Hörmander:Vol 2] for an extensive study of PDE-equations with constant coefficients. Here we shall give a construction from Hörmander's article [xxx] which illustrates how analytic function theory can be used with PDE-theory. Fourier's inversion formula for an arbitrary $n \geq 1$ asserts the following: Let $f(x) = f(x_1, \ldots, x_n)$ be a C^∞-function which is rapidly decreasing as $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ tends to $+\infty$. Then

$$(*) \qquad f(x) = \frac{1}{(2\pi)^n} \cdot \int e^{i\langle x,\xi\rangle} \cdot \widehat{f}(\xi)\, d\xi \quad \text{where} \quad \widehat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} \cdot f(x)\, dx$$

The inversion inversion formula (*) entails that the Fourier transform of the partial derivative $\frac{\partial f}{\partial x_j}(x)$ is equal to $i\xi_j \cdot \widehat{f}(\xi)$. In PDE-theory one introduces the first order differential operators

$$D_j = -i \cdot \frac{\partial}{\partial x_j} \quad : 1 \leq j \leq n$$

When $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index we get the higher order differential operator

$$D^\alpha = D_1^{\alpha_1} \cdots D_1^{\alpha_n}$$

We can take polynomials of these and get differential operatators with constant coefficients

$$P(D) = \sum c_\alpha \cdot D^\alpha$$

Fourier's inversion formula gives

$$(**) \qquad P(D)(f)(x) = \frac{1}{2\pi^n} \cdot \int e^{i\langle x,\xi\rangle} \cdot P(\xi) \cdot \widehat{f}(\xi)\, d\xi$$

Thus, applying a differential operator with constant coefficients to f corresponds to the product of its Fourier transform with the polynomial $P(\xi)$ and (**) can be used to construct solutions of the homogeneous equation $P(D)(f) = 0$. Following [Hörmander] we construct distributions μ such that $P(D)(\mu) = 0$ for a suitable class of PDE-operators. Let $\phi_1(s), \ldots, \phi_n(s)$ be some n-tuple of analytic functions of the complex variable s which extend to continuous functions on the boundary of the domain

$$\Omega = \{\mathfrak{Im}(s) < 0\} \cap \{|s| > M\}$$

where M is some positive number. Assume that the ϕ-functions satisfy the growth conditions

$$\text{(i)} \qquad |\phi_k(s)| \leq C|s|^a$$

for a constant C and some $0 < a < 1$. If $a < \rho < 1$ there exists the analytic function in Ω defined by

$$\psi(s) = e^{-(is)^\rho}$$

As explained in § xx one has the estimate

$$|\psi(s)| \leq e^{-\cos\frac{\pi\rho}{2}\cdot|s|^\rho} \quad : \mathfrak{Im}\, s \leq 0$$

The inequality $a < \rho$ and (i) entail that the functions

$$s \mapsto e^{a_1\cdot\phi_1(s)+\cdots+a_n\phi_n(s)} \cdot \psi(s)$$

decrease like $e^{-\cos\frac{\pi\rho}{2}\cdot|s|^\rho}$ in Ω.

Exercise. Verify that the complex line integrals below converge absolutely for every s-polynomial $Q(s)$ and every n-tuple of real numbers $x_1, \ldots, x_n$:

$$\text{(ii)} \qquad \frac{1}{(2\pi)^n} \cdot \int_{\partial\Omega} e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_n s} \cdot Q(s) \cdot \psi(s)\, ds$$

and show that when x varies in $\mathbf{R}^n$ this gives a C^∞-function $f(x)$. If $1 \leq j \leq n-1$ one has for example

$$\frac{\partial f}{\partial x_j} = \frac{1}{(2\pi)^n} \cdot \int_{\partial\Omega} \phi_j(s) \cdot e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_n s} \cdot Q(s) \cdot \psi(s)\, ds$$

Less obvious is that the C^∞-function $f(x)$ is supported by the half-space $\{x_n \geq 0\}$. To prove it one uses the analyticity of the integrand as a function of s which enable us to shift the contour of integration so that (ii) is unchanged while we integrate on a horisontal line $\mathfrak{Im}\, s = -N$ for every $N > M$. With $s = u - iN$ we have

$$|e^{ix_n\cdot s}| = e^{N\cdot x_n}$$

If $x_n < 0$ this term tends to zero as $N \to +\infty$ and from this the resder should confirm that the C^∞-function $f(x)$ is identically zero in $\{x_n < 0\}$.

Suppose now that we are given a PDE-operator $P(D)$ and the ϕ-functions are chosen so that

$$s \mapsto P(\phi_1(s) \ldots \phi_{n-1}(s), \phi_n(s) + s) = 0 \quad : s \in \Omega$$

Then it is clear that

$$P(D)(e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_n s}) = 0$$

hold for all $x \in \mathbf{R}^n$ and $s \in \Omega$. Hence $P(D)(f) = 0$ where f is a C^∞-function supported by the half-space $\{x_n \geq 0\}$. In § xx we will show that the construction of solutions as above is not so special for PDE-operators P such that the hyperplane $\{x_n = 0\}$ is non-characteristic.

0.3 Elliptic boundary value problems

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Hilbert space methods applied to elliptic operators were put forward at an early stage by Courant and Weyl. Later progress, foremost carried carried out between 1945-1955 led to conclusive results about elliptic boundary value problems. We describe the major results below and remark only that the proofs after a "perfect organisation" which foremost is due Gårding leads to relatively easy proofs. More precisely, one emplos all the time Dtokes Tjeorem togehter with suitable C^∞-partitions of the unity and requested L^2-estiamtes while higher order partial derivatives of functions are studied are handled by Parseval's formula applied to Fourier series expansions. So the rwesults below may be left as exercises to the reader. If necessary, consult Chapter xx in the second volume of Dunford-Schwartz for detailed profs.

The set-up is as follows: With $n \geq 2$ one considers a bounded open and connected set Ω in $\mathbf{R}^n$ with a smooth boundary. It means that there exists a real-valued C^∞-function ρ in $\mathbf{R}^n$ such that $\Omega = \{\rho < 0\}$ while $\partial\Omega = \{\rho = 0\}$ and the gradient vector $\nabla(\rho) \neq 0$ on $\partial\Omega$.

0.3.1 Some Hilbert spaces. If k is a positive integer one defines the L^2-norm of order k on functions $f \in C^\infty(\Omega)$ by:

(i) $$||f||_k^2 = \sum_{|\alpha|\leq k} \int_\Omega |\partial^\alpha(f)|^2 \, dx$$

The completion yields the Hilbert space $H^{(k)}(\Omega))$ of functions $f \in L^2(\Omega)$ whose distribution derivatives $\partial^\alpha(f)$ belong to $L^2(\Omega)$ for every multi-index $|\alpha| \leq k$. Inside $H^{(k)}(\Omega)$ we have the subspace $C_0^\infty(\Omega)$ of test-functions with compact support in Ω. Its closure in $H^{(k)}(\Omega)$ yields a a Hilbert space denoted by $H_0^{(k)}(\Omega)$. Let us remark that already the case $n = 1$ where Ω is the unit interval $[0, 1]$ shows that the inclusion $H_0^{(k)}(\Omega) \subset H^{(k)}(\Omega)$ is strict.

0.3.2 Elliptic operators. Let p be a positive integer and consider a PDE-operator of order $2p$:

$$P(x, \partial) = \sum_{|\alpha|\leq 2p} a_\alpha(x) \cdot \partial^\alpha$$

The a-functions are in general complex-valued C^∞-functions but assumed to be defined in a neighborhood of the closure of Ω. One says that P is elliptic if

(*) $$(-1)^p \cdot \mathfrak{Re} \sum_{|\alpha|=2p} a_\alpha(x) \cdot \xi^\alpha > 0$$

hold for all non-zero real n-vectors ξ and every $x \in \bar{\Omega}$. Let us denote this class of PDE-operators by $\mathcal{E}(2p; \Omega)$.

Closed extensions. Restrictions of complex-valued C^∞-functions in $\mathbf{R}^n$ to $\bar{\Omega}$ yields a space $C^\infty(\bar{\Omega})$. Every such function has normal derivatives along $\partial\Omega$ and we set

$$\mathcal{C}_* = \{f \in C^\infty(\bar{\Omega}) : \frac{\partial^\nu f}{\partial \mathbf{n}^\nu} |\partial\Omega = 0 \quad : 0 \leq \nu \leq p-1\}$$

Each $P \in \mathcal{E}(2p; \Omega)$ sends functions $f \in \mathcal{C}_*$ to a square integrable functions in Ω. Hence we have a linear mapping

(i) $$P \colon \mathcal{C}_* \to L^2(\Omega)$$

Now $\mathcal{C}_*$ is a dense subspace of $L^2(\Omega)$ and taking the closure of the graph of P in $L^2(\Omega) \times L^2(\Omega)$ gives a densely defined and closed linear operator on $L^2(\Omega)$ denoted by T_P whose domain of definition is denoted by $\mathcal{D}(T_P)$.

Adjoint operators. Given P as above we introduce the PDE-operator

$$P^*(x, \partial) = \sum (-1)^{|\alpha|} \partial^\alpha \circ \bar{a}_\alpha(x)$$

Stokes theorem gives the equality below for every pair f, g in $\mathcal{C}_*$:

$$\int P(f) \cdot \bar{g} = \int f \cdot \overline{P^*(g)} \, dx$$

The closure of the graph of P^* in $L^2(\Omega)$ gives the densely defined and closed operator T_{P^*}. By the general constructions from § xx T_P has an adjoint operator T_P^* where an L^2-function g belongs to $\mathcal{D}(T_P^*)$ if and only if there exists a constant C such that

$$\Big|\int_\Omega P(f) \cdot g \, dx\Big| \leq C \cdot \sqrt{\int_\Omega |f|^2 \, dx} \quad : \; f \in C^\infty(\Omega)$$

Main results. With the notations above the following hold:

0.3.4.1 Theorem *For every elliptic operator P of order $2p$ one has the equality*

$$\mathcal{D}(T_P) = H^{(2p)}(\Omega) \cap H_0^{(p)}(\Omega)$$

0.3.4.2 Theorem. *Let f be a function in $H_0^{(p)}(\Omega)$ such that $P(f)$ belongs to $H^{(m)}(\Omega)$ for some non-negative integer m. Then $f \in H^{(m+p)}(\Omega)$.*

0.3.4.3 Theorem. *The adjoint operator T_P^* is equal to T_{P^*}. Moreover, the spectrum $\sigma(T_P)$ is a discrete subset of $\mathbf{C}$ with no finite cluster point and for each λ outside $\sigma(T_P)$ the resolvent $(\lambda \cdot E - T_P)^{-1}$ is a compact operator on $L^2(\Omega)$.*

0.3.5 Gårding's inequality. The crucial step to prove the results above relies upon the following:

0.3.5.1 Theorem. *For each $P \in \mathcal{E}(2p; \Omega)$ there exist positive constants C and c such that the following hold for every $f \in C_0^\infty(\Omega)$:*

$$\mathfrak{Re} \int_\Omega P(f) \cdot f \, dx + C \int_\Omega f^2 \, dx \geq c \cdot ||f||_p^2 \tag{0.3.1}$$

where $||f||_p$ is the norm taken in the Hilbert space $H_0^p(\Omega)$.

0.3.6 Sobolev inequalities. The Hilbert spaces in (0.3.1) are related to differentiable functions. Taylor expansions show that if m is a non-negative integer and

$$k > m + \frac{n}{2} \tag{i}$$

then each $f \in H^k(\Omega)$ is automatically m times continuously differentiable in $\bar{\Omega}$, i.e. (i) gives the inclusion

$$H^k(\Omega) \subset C^m(\bar{\Omega}) \tag{ii}$$

Moreover one has

0.3.6.1 Theorem. *Let k be a posotive integer and f a function in the intersection $C^k(\bar{\Omega}) \cap H_0^k(\bar{\Omega})$. Then the normal derivatives of f along $\partial\Omega$ up to order $k-1$ vanish identically on $\partial\Omega$.*

0.3.7 Brouwder's completeness theorem. Let P be given in $\mathcal{E}(2p; \Omega)$. Denote by $\mathcal{C}_p$ the space of functions $f \in C^{2p}(\bar{\Omega})$ whose normal derivatives vanish on $\partial\Omega$ up to order $p-1$. Now P is defined on $\mathcal{C}_p$ and taking the closure of its graph in $L^2(\Omega)| \times L^2(\Omega)$ we get a densely defined and closed linear operator on $L^2(\Omega)$ denoted by $\mathcal{P}_p$. By this construction the domain of definition of $\mathcal{P}_p$ contains that of T_P.

0.3.7.1 Theorem. *The spectrum $\sigma(\mathcal{P}_p)$ is a discrete set and the resolvent operators are compact. Moreover, for each $\mu \in \sigma(\mathcal{P}_p)$ there exists a finite dimensional subspace H_μ of $L^2(\Omega)$ where $f \in H_\mu$ entails that*

$$(P(f) - \mu \cdot f)^k = 0$$

for some positive integer k and the inclusion below holds for each μ:

$$H_\mu \subset \mathcal{C}_*$$

Finally, the linear space generated by $\{H_\mu \colon \mu \in \sigma(\mathcal{P})\}$ is dense in $L^2(\Omega)$.

0.3.8 Example With $n = 2$ we consider elliptic operators of the form

$$P = -\Delta + a(x,y)\partial_x + b(x,y)\partial_y + c(x,y)$$

Here Δ is the Laplace operator and we suppose that a, b, c are real-valued C^∞-functions. If Ω is a bounded domain in $\mathbf{C}$ with a smooth boundary we seek f where

$P(f) = 0$ in Ω and $f|\partial\Omega$ is equal to a given function g. If c is positive function a maximum principle holds. Namely, if g is real-valued and f a solution to the boundary value problem then its maximum norm satisfies

$$|f|_{\bar{\Omega}} = |g|_{\partial\Omega}$$

The easy proof relies upon Taylor's formula in calculus. In particular the solution f for a given boundary function g is unique if it exists. To find conditions for existence one considers the adjoint operator

$$P^* = P = -\Delta - a(x,y)\partial_x - b(x,y)\partial_y + c(x,y) - \partial_x(a) - \partial_y(b)$$

If the function $c - \partial_x(a) - \partial_y(b)$ is positive in Ω one verifies that the adjoint has no kernel and the Dirichlet problem has a unique solution. See § xx for a further discussion of this special case which is instructive since it illuminates the general results above.

0.4 Preliminaries about distributions on the real line and C.

The space $\mathfrak{Db}(\mathbf{R})$ of distributions on the real line is by definition the topological dual of test-functions. A continuous complex valued function f on the real x-line defines a distribution μ_f by

$$\phi \mapsto \int f(x)\phi(x)\,dx \quad : \phi \in C_0^\infty(\mathbf{R})$$

The distribution derivative is the linear functional defined by

$$\phi \mapsto -\int f(x)\phi'(x)\,dx \tag{1}$$

If f is of class C^1 we can perform a partial integration and (1) is equal to

$$\int f'(x)\phi'(x)\,dx$$

So the distribution derivative of μ_f is expressed by the continuous density $f'(x)$. In general there exist continuous functions f which are nowhere differentiable and then one gives an operative meaning to the distribution derivative $\frac{d\mu_f}{dx}$. The reader should be aware of the fact that the notion of distributions is quite abstract but the merit is that distributions posses derivatives via the operative definition above. Concerning limits of distributions one often employs the weak topology which means that a sequence $\{\mu_n\}$ in $\mathfrak{Db}(\mathbf{R})$ converges weakly to a limit distribution μ if

$$\lim_{n\to\infty} \mu_n(\phi) = \mu(\phi)$$

hold for all test-functions ϕ For example, if k is a positive integer and μ_n is the density $n^k \cdot \sin nx$ then

$$\mu_n(\phi) = n^k \cdot \int \sin nx \cdot \phi(x)\ dx \to 0$$

hold for every test-function ϕ. So $\{\mu_n\}$ converges weakly to zero while the total variations over compact intervals tend to $+\infty$. One has for example

$$n^k \cdot \int_0^{2\pi} |\sin(nx)|\,dx = n^k \cdot \pi \quad : n \geq 1$$

0.2.1 The Fourier transform. If μ is a Riesz measure on a real ξ-line with a finite total variation its inverse Fourier transform is defined by

$$\mathcal{F}_\mu(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot d\mu(\xi)$$

The triangle inequality entails that the maximum norm of $\mathcal{F}_\mu$ is $\leq \frac{1}{2\pi} \cdot ||\mu||$. It is also clear that $\mathcal{F}_\mu$ is uniformly continuous, i,e. to every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x_1 - x_2| < \delta \implies |\mathcal{F}_\mu(x_1) - \mathcal{F}D_\mu(x_2)| < \epsilon$$

A result about weak limits. Let $\mathcal{C}_*$ be the space of continuous functions $\psi(x)$ on the real x-line which are bounded and uniformly continuous. If $\mathcal{M}$ is the space of Riesz measures on the ξ-line with finite total variation then the operator $\mathcal{F}$ sends $\mathcal{M}$ into a subspace of $\mathcal{C}_*$. The inclusion $\mathcal{F}(\mathcal{M}) \subset \mathcal{C}_*$ is strict. See § xx for examples. A necessary and sufficient condition in order that a function $\psi \in \mathcal{C}_*$ belongs to $\mathcal{F}(\mathcal{M})$ goes as follows: A sequence $\{\gamma_n\}$ of Riesz measures on the x-line is said to converge weakly to zero if there exists a constant C such that $||\gamma_n|| \leq C$ every n and their Fourier transforms converge pointwise to zero:

$$\lim_{n\to\infty} \int e^{-i\xi x}\, d\gamma_n(x) = 0 \quad \text{hold for each } \xi$$

0.2.2 Theorem. *A function $\psi(x) \in C_*$ belongs to $\mathcal{F}(\mathcal{M})$ if and only if*

$$\lim_{n\to\infty} \int \psi(x) \cdot d\gamma_n(x) = 0$$

hold for every weakly convergent sequence $\{\gamma_n\}$.

This theorem is due to Beurling and the proof is given in § XX. The proof relies upon solutions to a variational problem so the result above is not a straightforward consequence of general topological arguments. We shall encounter other situations where distributions appear but the passage to limits and rate of convergence require extra work. An example is Lindeberg's sharp version of the Central Limit Theorem. Even though it can be established in a "weak sense" via distributions the rate of convergence is important. So Lindeberg's result goes beyond a mere appeal to distribution theory which serves as a helpful tool but is not "the end of the story".

0.2.3 Parseval's formula. It expresses an L^2-equality between functions and their Fourier transforms:

$$\widehat{f}(\xi) = \int e^{-ix\xi} \cdot f(x)\, dx$$

More precisely one has

$$(*) \qquad \int |f(x)|^2\, dx = \frac{1}{2\pi} \cdot \int |\widehat{f}(\xi)|^2\, d\xi$$

This is proved in § X.X.

0.2.4 Planck's equation. An example where Parseval's equation is used appears in Planck's equation whose physical relevance will be recalled in § XX. After a change of scale factors Planck's equation boils down to find a function $\Phi(u)$ on the non-negative real u-line such that the function

$$(*) \qquad s \mapsto \int_0^\infty \frac{us}{e^{us}-1} \cdot \Phi(u)\, du$$

is equal to a given function $\Psi(s)$ defined on $\{s \geq 0\}$ satisfying

$$(1) \qquad \int_0^\infty |\Psi(s)|^2 \cdot \frac{ds}{s} < \infty$$

It turns out that (1) implies that (*) has a unique solution $\Phi(u)$ satsfying

$$(2) \qquad \int_0^\infty |\Phi(u)|^2 \cdot \frac{du}{u} < \infty$$

The proof employs a multiplicative version of Parseval's formula which shows that (1) is equal to

$$(3) \qquad \int_{-\infty}^\infty |\widehat{\Psi}(\xi)|^2\, d\xi \quad : \; \widehat{\Psi}(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty s^{-i\xi-1/2} \cdot \Psi(s)\, ds$$

The requested solution $\Phi(u)$ is found by an inverse transform. Namely, suppose that (*) is equal to $\Psi(s)$ and apply the transform (3) which gives a function

$$(4) \qquad \xi \mapsto \int_0^\infty \Big[\int_0^\infty \frac{us}{e^{us}-1} \cdot s^{-i\xi-1/2}\, ds\Big] \cdot \Phi(u)\, du$$

After the substitution $us = t$ the inner integral becomes

$$u^{i\xi-1/2}\cdot\int_0^\infty \frac{t}{e^t-1}\cdot t^{-i\xi-1/2}\,dt \tag{5}$$

The last factor is computed using the expansion

$$\frac{1}{e^t-1}=\sum_{k=1}^\infty e^{-kt}\quad : t>0$$

As explained in § xx this entails that the last factor in (5) is equal to

$$\Gamma(\frac{3}{2}-i\xi)\cdot\zeta(\frac{3}{2}-i\xi) \tag{6}$$

This leads to the equation

$$\widehat{\Psi}(\xi)=\Gamma(\frac{3}{2}-i\xi)\cdot\zeta(\frac{3}{2}-i\xi)\cdot\int_0^\infty u^{i\xi-1/2}\cdot\Phi(u)\,du \tag{7}$$

From this one finds Φ via the multiplicative version of Fourier's inversion formula and Parseval's equation gives the integrability condition in (2) because the function $\xi\mapsto\Gamma(\frac{3}{2}-i\xi)\cdot\zeta(\frac{3}{2}-i\xi)$ is bounded below on the real ξ-line. See § XX for details.

0.2.5 The Pompieu formula. With $\epsilon>0$ we set

$$g_\epsilon(z)=\frac{1}{\pi}\cdot\frac{\bar z}{|z|^2+\epsilon}$$

Apply the Cauchy-Riemann operator $\bar\partial=\frac{1}{2}(\partial_x+i\partial_y)$ which gives:

$$\bar\partial(g_\epsilon)z)=\frac{\epsilon}{\pi}\cdot\frac{1}{(|z|^2+\epsilon)^2}$$

The area integral $\iint g_\epsilon\,dxdy=1$ and the distribution of mass becomes concentrated to the origin which means that

$$\lim_{\epsilon\to 0}\iint\bar\partial(g_\epsilon)\cdot\phi\,dxdy=\phi(0)$$

hold for every test-function ϕ. So the C^∞-densities $\{\bar\partial(g_\epsilon)\}$ converge weakly to the Dirac distribution δ_0 which assigns a unit point mass at the origin. Next, consider a bounded open set Ω in $\mathbf{C}$ and let f be a locally integrable function in Ω. With a small $\delta>0$ we find a C^∞-function ρ which vanishes outside the compact subset of Ω with distance $\geq\delta$ to the boundary, while $\rho=1$ on the compact subset of Ω whose points have distance $\geq 2\delta$ to $\partial\Omega$. Let $\zeta\in\Omega$ where $\text{dist}(\zeta,\partial\Omega)>2\delta$ and put

$$g_\epsilon(z)=\frac{1}{\pi}\cdot\frac{\bar z-\bar\zeta}{|z-\zeta|^2+\epsilon}$$

Now $\{\rho\cdot g_\epsilon\}$ are test-functions in $\mathbf{C}$ and introducing the distribution derivative $\bar\partial(f)$ one has the equation

$$\bar\partial(f)(\rho\cdot g_\epsilon)=-\iint_\Omega f\cdot\bar\partial(\rho\cdot g_\epsilon)\,dxdy$$

Now $\bar\partial(\rho\cdot g_\epsilon)=\bar\partial(\rho)\cdot g_\epsilon+\rho\cdot\bar\partial(g_\epsilon)$ so if ζ is a Lebesgue point of the L^1-function f it is clear that

$$\lim_{\epsilon\to 0}\iint f\cdot\rho\cdot\bar\partial(g_\epsilon)\,dxdy=f(\zeta)$$

This entails that the limit in the right hand side becomes:

$$f(\zeta)+\frac{1}{\pi}\iint\frac{f(z)\cdot\bar\partial(\rho)(z)}{z-\zeta}\,dxdy$$

So if $\bar{\partial}(f) = 0$ holds in the sense of distributions we get the Pompieu formula

$$(*) \qquad f(\zeta) = \frac{1}{\pi} \iint \frac{f(z) \cdot \bar{\partial}(\rho)(z)}{\zeta - z}\, dxdy$$

Here (*) hold for all Lebesgue points ζ in the open subset Ω_* of points in Ω whose distance to $\partial\Omega$ is $> 2\delta$ and at the same time the right hand side in (*) is an analytic function of ζ in Ω_*. Hence the vanishing of$\bar{\partial}(f)$ in the distribution sense entails that f restricted to Ω_* is almost everywhere equal to the analytic function in the right hand side above. it means that the L^1_{loc}-function f can be identfied with this analytic functuon in Ω_*. Moreover, since δ can be arbitrary small we conclude that f belongs to $\mathcal{O}(\Omega)$. This is expressed by saying that the $\bar{\partial}$-operator is elliptic.

Remark. Above we started from a locally integrable function f. More generally, let μ be a distribution in Ω and suppose that $\bar{\partial}(\mu) = 0$. Then μ is a holomorphic density in Ω. To prove this one empoys regularisations of μ, i.e. as in § xx we get C^∞-densities $\psi_s * \mu$ where we choose $0 < s < \delta$ so that the distributions $\{\psi_s * \mu\}$ can be applied to test-functions in Ω supported by $\Omega[\delta]$. Then

$$(*) \qquad \psi_s * \mu(\zeta) = \frac{1}{\pi} \iint_\Omega \frac{\psi_s * \mu(z) \cdot \bar{\partial}(\rho)(z)}{\zeta - z}\, dxdy \quad : \zeta \in \Omega_*$$

With $\zeta \in \Omega$ kept fixed in the right hand side one has applied $\psi_s * \mu$ to the test function

$$z \mapsto \frac{\bar{\partial}(\rho)(z)}{\zeta - z}$$

Since $\psi_s * \mu$ converge weakly to μ we obtain limit as $s \to 0$ in the right hand side which yields an analytic function of ζ:

$$(**) \qquad \zeta \mapsto \mu\Big(\frac{\bar{\partial}(\rho)(z)}{\zeta - z}\Big)$$

At the same time $\psi_s * \mu$ converges weakly to μ. The conclusion is that the restriction of μ to Ω_* is expressed by the holomorphic density (**) which proves the elliptic property of the $\bar{\partial}$-operator.

0.5 Boundary values of analytic functions

.

Basic constructions appear in § 2 which give boundary values of analytic functions $f(z) = f(x + iy)$ defined in open rectangle $\{0 < x < A\} \times \{0 < y < a\}$ where a is positive and there exists some integer $N \geq 0$ anda constant C such that

$$[f(x + iy)|\, leqC \cdot y^{-N}$$

for all $x + iy$ in the rectangle. When this holds we show that f has a boundary value given by a distribution defined on the x-interval $(0, A)$. More precisely, for every $g(x)(\in C_0^\infty(0, A)$ there exists the limit

$$\int_0^A g(x) \cdot f(x + i\epsilon)\, dx$$

and the resulting linear functional on $C_0^\infty(0, A)$ is the boundary value distribution of f denoted by $f(x + i0)$ to indicate that the boundary value is taken as y decreases to zero. After we have introduced tempered distributions on the real x-line the construction of boundary values from analytic functions in the upper, respectively the lower half-plane has many implications. For example, in $U_+ = \{\mathfrak{Im}\, z > 0\}$ there exists the analytic function

$$z^\lambda = e^{\lambda \cdot \log z}$$

Here $\log z = \log |z| + i \arg z$ is analytic in U_+ where the argument of z stays in $(0, \pi)$. We shall learn that this analytic function has a boundsry value denoted by

$(x+i0)^\lambda$ which yields a tempered distribution on the real x-line. After this we pass to its Fourier transform $\widehat{(x+i0)}^\lambda$ which yields a tempered distribtuion on the real ξ-line. In § x we prove the Euler-Riemann formula

$$(*) \qquad \widehat{(x+i0)}^\lambda = \frac{2\pi \cdot i^\lambda}{\Gamma(-\lambda)} \cdot \xi_+^{-\lambda-1}$$

Here (*) hold for all complex numbers λ and in the right hand side one encounters Euler's distributions which for complex s where $\mathfrak{Re}\, s > -1$ are defined on the ξ-line by

$$\xi_+^s(\phi) = \int_0^\infty \xi^s \cdot \phi(\xi)\, d\xi$$

where ϕ are Schwartz functions on the ξ-line.

Remark. Boundary values and various Fourier transforms were already considered and computed by Riemann in many situations. Of course, he did not use the vocabulary of distributions but one can easily interpretate manyof Riemann's results in the distribution sense. Notable examples occur during the study of the ζ-function.

0.6 Stokes Theorem.

Let Ω be a bounded open set in $\mathbf{R}^n$ whose boundary satisfies the Federer condition in § XX. The characteristic function χ_Ω yields a distribution and Stokes Theorem means that for each $1 \leq k \leq n$ one has the equality of distributions:

$$\frac{\partial \chi_\Omega}{\partial x_k} = \mathbf{n}_k \cdot dS$$

where dS is the area measure on the regular part of $\partial\Omega$ and $\mathbf{n}_k$ is the k:th component of the outer normal. It is tempting to continue and search higher order derivatives. When Ω is a convex polygon in $\mathbf{R}^n$ where $n \geq 3$ leads to interesting formuas expressed by currents. A current is is a linear functional defined on test-forms rather than test-functions. For example, above one starts with the distribution χ_Ω and its differential is the 1-current which maps a test-form γ^{n-1} of degree $(n-1)$ to

$$\int_{\partial\Omega} \gamma^{n-1}$$

Here one employs the orientation on the $(n-1)$-dimensional pieces of $\partial\Omega$ where the integration takes places. It would bring us too far to eneter a detailed dsicusison about currents but they will be used in lower dimensions, especially in the chapter devoted to Riemann surfaces. An example where one arrives at a distribution of order one occurs if Ω is as above and H is a harmonic function defined in some open neighborhood of $\bar{\Omega}$ which gives the distribution $H \cdot \chi_\Omega$. Apply the Laplace operator so that Greens' formula gives

$$\int_\Omega H \cdot \Delta(f)\, dx = \int_{\partial\Omega} H \cdot \frac{\partial f}{\partial \mathbf{n}} \cdot dS - \int_{\partial\Omega} \frac{\partial H}{\partial \mathbf{n}} \cdot f \cdot dS \quad : \; f \in C_0^\infty(\mathbf{R}^n)$$

The right hand side gives an expression for the distribution $\Delta(H \cdot \chi_\Omega)$. The first integral corresponds to the distribution defined by

$$\text{(i)} \qquad f \mapsto \int_{\partial\Omega} H \cdot \frac{\partial f}{\partial \mathbf{n}} \cdot dS$$

The construction of normal derivatives means that (i) is the distribution given by

$$H \cdot \sum_{k=1}^{k=n} \mathbf{n}_k \cdot \frac{\partial dS}{\partial x_k}$$

Thus, in the last sum one has taken partial distribution derivatives of the non-negative Riesz measure dS.

0.7 Regularisations.

Every distribution μ in $\mathbf{R}^n$ ican be approximated in the weak topology by C^∞-densities. To achieve this one takes a C^∞-function $\phi(x)$ with a compact support such that

$$\int_{\mathbf{R^n}} \phi(x)\,dx = 1 \quad \text{and} \quad \phi(0) = 1$$

For every $\delta > 0$ there exists the C^∞-function μ_δ which for each $a \in \mathbf{R}^n$ takes the value

$$\mu_\delta(a) = \mu(\phi_{\delta;a}) \quad \text{where} \quad \phi_{\delta;a}(x) = \delta^{-n} \cdot \phi(\frac{x-a}{\delta})$$

Then one easily verifies that

$$\lim_{\delta \to 0} \mu_\delta(f) = \mu(f) \quad : \ f \in C_0^\infty \mathbf{R}^n)$$

These regularisations reveal why calculus for differentiable functions extends to distributions.

Example. With $n = 1$ we have the Heaviside distribution H_+ on the real x-line given by the constant density 1 on $\{x \geq 0\}$ and zero on $\{x < 0\}$. If $\epsilon > 0$ there exists the C^∞-function $g_\epsilon(x)$ defined to be zero if $x \leq 0$ while

$$g_\epsilon(x) = e^{-\frac{\epsilon}{x}} \quad : \ x > 0$$

Here

$$\lim_{\epsilon \to 0} g_\epsilon(x) = 1$$

holds when $x > 0$ and the reader xan verify that the C^∞-densities $\{g_\epsilon\}$ give a regularisation of H_+ in the sense that

$$\lim_{\epsilon \to 0} \int g_\epsilon(x) \cdot f(x)\,dx = \int_0^\infty f(x)\,dx = H_+(f)$$

hold for every test-function $f(x)$ This is an example of a one-sided regularisation. Another example arises when we start with the principal value distribution defined by

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{f(x)}{x}\,dx$$

Here we get regularisations using the C^∞-densities

$$\phi_\epsilon(x) = \frac{x}{x^2 + \epsilon^2}$$

These examples show that thery exist several ways to perform regularisations of distributions.

0.8 On L^2-derivatives.

Consider the open unit interval $(0,1)$ and the space $C_0^\infty(0,1)$ of test-functions with compact support in $(0,1)$. If $f(x)$ is such a test-function its derivative $f'(x)$ is denoted by $\partial(f)$. Now $\partial(f)$ has an L^2-norm and we put

$$||f||_{(2,1)} = \sqrt{\int_0^1 \partial(f)^2\,dx}$$

Since f has compact support we have

$$f(x) = \int_0^x f'(t)\,dt \implies |f(x)| \leq x \cdot ||f||_{(2,1)} \quad : 0 < x < 1$$

where the Cauchy-Schwarz inequality was used. In particular

(i) $$\int_0^1 f^2\,dx \leq \int_0^1 x^2\,dx \cdot ||f||^2_{(2,1)}$$

Suppose now that μ is a linear functional on $C_0^\infty(0,1)$ such that

(*) $$|\mu(f)| \leq C \cdot \sqrt{\int_0^1 (f^2 + \partial(f)^2)\,dx}$$

hold for some constant C. By (i) this is equivalent to the condition that there exists a constant C such that

(ii) $$|\mu(f)| \leq C \cdot ||f||_{2,1}$$

Now $\{\partial(f)\colon\ f \in C_0^\infty(0,1)\}$ is a subspace of $L^2(0,1)$ so by the Hahn-Banach theorem and the self-duality of the L^2-space there exists $\phi \in L^2(0,1)$ such that

(iii) $$\mu(f) = -\int_0^1 \phi \cdot \partial(f)\,dx$$

0.8.1 Exercise. If $f \in C_0^\infty(0,1)$ the mean-value $\int_0^1 \partial(f)\ dx = 0$. In $L^2(0,1)$ one has the hyperplane of L^2-functions with mean-value zero. Show that the closure of $\{\partial(f)\colon\ f \in C_0^\infty(0,1)\}$ is dense in this subspace and conclude that the ϕ-function which represents μ in (iii) is unique if it is chosen with mean-value is zero.

The unique L^2-function ϕ representing μ in (iii) is the first order derivative of the L^2-density ϕ taken in the sense of distributions and one writes

$$\mu = \frac{d\phi}{dx}$$

In this way the space $L^2_*(0,1)$ of L^2-functions with mean-value zero is identified with the space of distributions on $(0,1)$ for which (*) holds for some constant C.

0.8.2 Higher order derivatives. If $k \geq 2$ we set

$$||f||_{(2,k)} = \sqrt{\int_0^1 \partial^k(f)^2\,dx}$$

We leave it as an exercise to verify that there exists a constant C such that

$$\sqrt{\int_0^1 \partial^\nu(f)^2\,dx} \leq C \cdot ||f||_{(2,k)} \quad : 0 \leq \nu \leq k-1$$

Next, let μ be a linear functional on $C_0^\infty(0,1)$ for which there exists a constant C such that

(**) $$|\mu(f)| \leq C \cdot ||f||_{(2,k)}$$

Exactly as above we find some $\phi \in L^2(0,1)$ such that

$$\mu(f) =) - 1)^k \cdot \int_0^1 \phi \cdot \partial^k(f)\,dx$$

The reader can verify that ϕ is uniquely determined when it satisfies

(0.7.2) $$\int_0^1 x^\nu \dot{\phi}\ dx = 0 \quad : \ 0 \leq \nu \leq k-1$$

Hence ∂^k gives a linear map from $L^2(0,1)$ to $\mathfrak{Db}(0,1)$ whose range consists of those distributions μ for which (**) holds for some constant C.

0.8.3 The space $H^k(0,1)$. Above we considered test-functions with compact support in $(0,1)$. We have also the space $C^\infty[0,1]$ which by definition consists of restrictions to the closed unit interval of C^∞-functions on the real line. To each $f \in C^\infty[0,1]$ we take L^2-norms of its derivatives up to order k and set

$$||f||_{2,k} = \sqrt{\int_0^1 (f^2 + \partial(f)^2 + \ldots + \partial^k(f)^2)\,dx}$$

This yields a norm on $C^\infty[0,1]$ whose completetion is a Hilbert space denoted by $H^{(k)}$. It consists of distributions μ supported by $[0,1]$ such that μ and all its distribution derivatives up to order k are expressed by functions in $L^2[0,1]$. Let us also consider C^∞-functions$f(x)$ which are identically zero on some interval $[0,\delta)$ with $\delta > 0$ and let $H_0^{(k)}$ be the closure taken in $H^{(k)}$.

0.8.4 Exercise. Show that $H_0^{(k)}$ is a proper subspae of $H^{(k)}$ whose codimension is k and one has a direct sum decomposition

$$H^{(k)} = H_0^{(k+1)} \oplus \mathcal{P}_k \tag{0.7.4}$$

where $\mathcal{P}_k$ is the (k)-dimensional vector space of polynomials in x of degree $\leq k-1$.

0.8.5 Exercise. Show that there exists a constant C such that

$$\max_{[} 0 \leq x \leq 1]|\partial^\nu(f)(x)| \leq C \cdot ||f||_{2,k} \quad : 0 \leq \nu \leq k-1$$

hold for all C^∞-functions f. Conclude that each $\mu \in H^{(k)}$ is expressed by a function in $C^{k-1}[0,1]$.

0.8.6 The higher dimensional case

Let $n \geq 1$ and consider the n-dimensional torus T^n, i.e. the n-fold product of the periodic interval $[0,2\pi]$. Let t be another real variable and consider C^∞-functions $f(t,x_1,\ldots,x_n)$ which are 2π-periodic in the x-variables. We can construct partial derivatives of f in these ($n+1$ many variables and if $k \geq 1$ we set

$$||f||_{2,k} = \sqrt{\sum_{|\alpha|\leq k} \int_\Box |\partial^\alpha(f)|^2\,dxdt}$$

where $\Box = \{0 \leq t \leq 1\} \times T^n$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ are multi-indices producing partial derivatives of f. Exactly as in the 1-dimensional case we take the closure under the norm above and get a Hilbert space $H^{(k)}(\Box)$.

Here $H^{(k)}(\Box)$ is a subspace of $L^2(\Box)$ where functions $\phi \in H^{(k)}(\Box)$ posses the property that the distribution derivatives $\{\partial^\alpha(\phi)\}$ are L^2-densitites for every $|\alpha| \leq k$. An equivalent condition that an L^2-function ϕ belongs to $H^{(k)}(\Box)$ is that there exists a constant C such that the following hold for every multi-index $|\alpha| \leq k$ and every $g \in C_0^\infty(\Box)$:

$$|\int_\Box \phi \cdot \partial^\alpha(g)\,dtdx| \leq C \cdot \sqrt{\int_\Box |g|^2\,dtdx}$$

0.8.7 The space $H_0^{(k)}(\Box)$. If we start from functions f such that $f(t,x) = 0$ when $0 \leq t < \delta$ for some $\delta > 0$ the closure with respect to the norm in (0.8.6) yields a a subspace of $H^{(k)}(\Box)$ denoted by $H_0^{(k)}(\Box)$.

0.8.8 Exercise. Show that there exists a direct sum decomposition

$$H^{(k)}(\Box) = H_0^{(k)}(\Box) \oplus \mathcal{P}_k$$

where $\mathcal{P}_k$ are functions of the form

$$\phi_0(x) + \phi_1(x)t + \ldots + \phi_{k-1}(x)t^{k-1} \quad : \phi_\nu(x) \in H^{(k)}(T^n)$$

0.8.9 The space $H^{(-k)}(\Box)$. If $g \in L^2(\Box)$ there exists the distribution derivatives $\partial^\alpha((g)$. If $|\alpha| \leq k$ and $\phi \in H^{(k)}(\Box)$ the definition of distribution derivatives means that
$$\partial^\alpha(g)(\phi) = (-1)^{|\alpha|} \cdot \int_\Box g \cdot \partial^\alpha(\phi)\, dtdx$$
The Cauchy-Schwarz inequality entails that the right hand side is majorized by the L^2-norm of g times $||\phi||_{(2,k)}$. So derivatives up to order k of L^2-functions on $\Box$ yield continuous linear functionals on $H^{(k)}(\Box)$. In this way the dual space of $H^{(k)}(\Box)$ can be identified with distributions given by finite sums
$$\mu = \sum_{|\alpha| \leq k} \partial^\alpha(g_\alpha) \quad : g_\alpha \in L^2(\Box)$$
The resulting subspace of $\mathfrak{Db}(\Box)$ is denoted by $H^{(-k)}(\Box)$.

0.8.10 Sobolev's inequality. Let $n \geq 1$ and consider 2π-periodic functions $f(x)$ on the torus T^n. To each n-tuple of integers $k = (k_1, \ldots, k_n)$ we get the Fourier coefficient
$$\widehat{f}(k) = (2\pi)^{-n} \cdot \int_{T^n} e^{-i\langle x,k\rangle}\, f(x)\, dx$$
Fouire's inversion formula in the peridic case entails that
$$f(x) = \sum_{k \in \mathbf{Z}^n} \widehat{f}(k) \cdot e^{i\langle x,k\rangle}$$
If f has an absolutely convergent Fouier series, i.e. when
$$\sum_{k \in \mathbf{Z}^n} |\widehat{f}(k)| < \infty$$
then it is clear that f is a continuous function on T^n. Next, if α is a multi-undex we notice that
$$\widehat{D^\alpha(f)}(k) = k^\alpha \cdot \widehat{f}(k)$$
If m is a positive integer the squared norm of f taken in $H^m(T^n)$ is given by
$$||f||^2_{2,m} = \sum_{|\alpha| \leq m} \sum |k|^{2\alpha}\, |\widehat{f}(k)|^2$$

0.8.11 Exercise. Show that (xx) gives a constant C_m which depends on m only such that
$$\sum ||k||^{2m}\, |\widehat{f}(k)|^2 \leq C_m^2 \cdot ||f||^2_{2,m}$$
where $||k|| = \sqrt{k_1^2 + \ldots + k_n^2}$. Next, the Cauchy-Schwarz inequality entails that if we exlude $k = 0$ then
$$\sum_{k \neq 0} |\widehat{f}(k)| \leq C_m \cdot ||f||_{(m)} \cdot \sqrt{\sum_{k \neq 0} ||k||^{-2m}}$$
The last term convergens if and ony if the n-dimensional integral
$$\int_{\mathbf{R}^n} \frac{dx}{1 + |x||)^{2m}} < \infty$$
Taking polar coordinates an equivalent condition is that
$$\int_0^\infty \frac{r^{n-1} dr}{(1+r)^{2m}} < \infty$$
We conlcude that (xx) is finite if $2m > n-1$. So if we take the integral part of $\frac{n-1}{2}$ and set
$$n_* = [\frac{n-1}{2}]$$

then convergence in xx hold for every $m \geq n_* + 1$. In particular it follows that if $m = n_* + 1$ then every function f in $H^m(T^n)$ has an absolutely convergent Fourier series. If m increases we also get highet order dertivtives. In particular has the inclusions

$$C^k(T) \subset H^{k+n_*+1}(T^n) \quad : k \geq 1$$

A. The origin of distributions.

Here follows an excerpt from Jean Dieudonne's article *300 years of analyticity* presented at the Symposium *On the Occasion of the Proof of Bieberbach's Conjecture.*

Since Cauchy and Weierstrass, the central fact in complex analysis has been the one-to-one correspondence

$$\{c_n\}_{n\geq 0} \mapsto \sum_{n=0}^{\infty} c_n z^n$$

between sequences of complex numbers which do not increase too fast and functions holomorphic in a neighborhood of 0. When you turn to Fourier series, you immediately meet the same kind of correspondence

$$\{c_n\}_{n\in Z} \mapsto \sum_{-\infty}^{\infty} c_n e^{in\theta}$$

between families of coefficients and sums of trigonometric series, which has been one of the most unsatisfactory and thickest jungles of classical analysis. A situation as satisfactory as in the analytic case has been achieved by substituting distributions in place of functions. More precisely, there is a one-to-one correspondence above when, on the left hand side, only families of *polynomial growth* are considered, that is, families such that

$$|c_n| \leq C(1+|n|)^k \quad : \text{for some} \quad k>0 \quad \text{and some constant } C$$

and the right hand side is replaced by *any periodic distribution* T on $\mathbf{R}$. The beauty of this correspondence is that it is stable for derivative, primitive, and convolution; so Euler was perfectly justified in taking derivatives of Fourier series and considering them again as Fourier series !

Assuming that the sequence $\{c_n\}$ has polynomial growth the right hand side is a distribution T which can be split into a sum of two periodic distributions T_1+T_2, corresponding to families $\{c_n\}$ having $c_n=0$ for all $n<0$ (resp. $n\geq 0$), and which have holomorphic extensions

$$f_1(z)=\sum_{n=0}^{\infty} \quad \text{for} \quad |z|<1 \quad : \quad f_2(z)=\sum_{-\infty}^{-1} \quad \text{for} \quad |z|<1$$

and $f_1(re^{i\theta})$ (resp. $f_2(r^{-1}e^{i\theta})$), defined for $0<r<1$, tends indeed to T_1 (resp. T_2) when $r\to 1$ for the weak topology of distributions.

Remark. To this historic account one should add that Riemann performed an accurate calculus while he derived inversion formulas related to the Fourier transform during studies of the ζ-functions. For this reason we refer to the Euler-Riemann equation in § xx where Fourier transforms of boundary value distributions $(x+i0)^\lambda$ appear for arbitrary complex λ. Dieudonne's remarks about distributions on the unit circle is easy to grasp. Namely, let $C^\infty(T)$ be the linear space of infinitely differentiable functions on the unit circle T, or equivalently 2π-period functions on the real line. Partial integration shows that Fourier coefficients have rapid decay, i.e. with

$$\widehat{f}(n)=\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})dx \quad : f\in C^\infty(T) \quad : \quad n \text{ any integer}$$

It follows that

$$n^k \cdot |\hat{f}(n)| \leq \max_{0\leq\theta\leq 2\pi} |f^{(k)}(e^{i\theta})| \quad : \quad k=0,1,2,\ldots$$

To each $k \geq 0$ we introduce the C^k-norm defined by

$$||f||_k = \sum_{\nu=0}^{\nu=k} \max_\theta |f^{(\nu)}(e^{i\theta})|$$

The topology on $C^\infty(T)$ is defined by the metric

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{||f-g||_k}{1+||f-g||_k}$$

In this way $C^\infty(T)$ becomes a Frechet space and for every continuous linear form L there exists some integer k and a constant C so that

$$[L(f)| \leq C \cdot ||f||_k \quad : \quad f \in C^\infty(T)$$

From this follows Dieudonné's assertion above which identifies distributions on T and sequences $\{c_n\}$ with polynomial growth, i.e. every such sequence yields a distribution L defined by

$$L(f) = \sum_{-\infty}^{+\infty} c_n \cdot \widehat{f}(n)$$

A.1 Example. Let $g(z)$ be an analytic function in the unit disc satisfying the growth condition

$$(1) \qquad |g(z)| \leq C \cdot (1-|z|)^{-m}$$

for some $m \geq 2$. Consider the series expansion $g(z) = \sum c_n \cdot z^n$. Cauchy's integral formula and the triangle inequality entail that

$$|c_n| \leq (\frac{n+m}{m})^m \cdot (1 - \frac{m}{m+n})^{-n}$$

for every n. Hence (1) gives a constant C_m such that

$$|c_n| \leq C_m \cdot (n+1)^m \quad : n = 0, 1, \ldots$$

We conclude that $\{c_n\}$ defines a distribution on T defined by

$$(2) \qquad \mathfrak{b}_g(f) = \sum c_n \cdot \widehat{f}(-n) \quad : f \in C^\infty(T)$$

One refers to $\mathfrak{b}_g$ as the boundary value distribution of g. The point is that the rapid decay of $\{\widehat{f}(n)\}$ ensures that (2) is a convergent series, i.e. it is even absolutely convergent and the reader may verify the limit formula

$$\mathfrak{b}_g(f) = \lim_{r \to 1} \int_0^{2\pi} g(re^{i\theta}) \cdot f(e^{i\theta})\, d\theta$$

Denote by $\mathcal{O}_{\text{temp}}(D)$ the space of analytic functions g in D satisfying (1) for some integer m. By the above we have found an injective map from $\mathcal{O}_{\text{temp}}(D)$ into a subspace of distributions on T. The subspace is special since (2) shows that $\mathfrak{b}_g(f) = 0$ when $f \in C^\infty(T)$ is such that the Fourier coefficients vanish when $n \leq 0$ which means that f itself is the boundary value of an analytic function in the unit disc which is zero at $z = 0$, i.e. f is a C^∞-function in the disc algebra whose constant term is zero. Another class of distributions on T arises via boundary values of analytic functions $h(z)$ defined in the exterior disc $|z| > 1$. Using conformal mappings between the unit disc and the upper, respectively the lower half-plane we shall later on construct boundary value distributuins of analytic functions from the upper- respectively the lower halfplane which have moderate growth when one approaches the real axis.

A.2 Example. Let Γ be a closed Jordan curve which borders a bounded Jordan domain Ω. If μ is a Riesz measure in $\mathbf{C}$ supported by Γ there exists an analytic function defined in $\mathbf{C} \setminus \Gamma$ by

$$\text{(i)} \qquad \mathcal{C}_\mu(z) = \int_\Gamma \frac{d\mu(\zeta)}{z-\zeta}$$

One refers to (i) as the Cauchy transform of μ. We can restrict this analytic function to the bounded open set Ω and the triangle inequality gives

$$|\mathcal{C}_\mu(z)| \leq \int_\Gamma \frac{[d\mu(\zeta)|}{|z-\zeta|}$$

Notice that the area integrals

$$\iint_\Omega \frac{dxdy}{|z-w|} \leq C \quad : w \in \Gamma$$

where C is independent of w. It follows that

$$\iint_\Omega |\mathcal{C}_\mu(z)|\, dxdy \leq C \cdot ||\mu||$$

where the last factor is the total variation of μ. So if if χ_Ω is the characteristic function of Ω then $\chi_\Omega \cdot \mathcal{C}_\mu$ is an integrable function with compact support and yields a distribution defined by

$$f \mapsto \iint_\Omega f(z) \cdot \mathcal{C}_\mu(z)\, dxdy \quad : \ f \in C_0^\infty(\mathbf{C})$$

Let us denote this distribution by γ. Now there exist distribution derivatives. In particular we find the distribution

$$\mu_+ = \bar{\partial}(\gamma)$$

Since $\mathcal{C}_\mu(z)$ is analytic in Ω the support of μ_+ is contained in Γ. The construction of distribution derivatives gives the existence. of μ_+ but it is not clear how to describe it. We refer to § XX for a further discussion about properties of μ_+ and how it can be attained via a limit process.

1. The Schwartz space on the real line.

Denote by $\mathcal{S}$ the space of C^∞-functions on the real x-line which together with all derivatives are rapidly decreasing. For each non-negative integer k we set

$$\rho_k(f) = \max_{x\in\mathbf{R}} \, (1+|x|)^k \cdot \sum_{\nu=0}^{\nu=k} |f^{(\nu)}(x)| \quad : \; f \in \mathcal{S}$$

The distance function on $\mathcal{S}$ defined by

$$d(f,g) = \sum_{k=1}^{\infty} 2^{-k} \cdot \frac{\rho_k(f-g)}{1+\rho_k(f-g)}$$

One verifies that this metric above is complete, i.e. $\mathcal{S}$ is a Frechet space.

1.1 Exercise. Prove that there exists a constant C such that the following hold for each pair of integers $0 \leq \nu < k$ and every $f \in \mathcal{S}$.

$$\max_x \, [1+|x|)^\nu \cdot |f^{(\nu)}(x)| \leq C^{k-\nu} \cdot \max_x \, [1+|x|)^k \cdot |f^{(k)}(x)|$$

1.2 An isomorphism. Let w be a new complex variable and set

$$w = \frac{z-i}{z+i}$$

This is a conformal mapping from the upper half-plane onto the unit disc $\{|w|<1\}$. When $z = x$ is real one has $w = e^{i\theta}$ and now

$$\text{(i)} \qquad ie^{i\theta} \cdot \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{x-i}{x+i}\right) = \frac{2i}{(x+i)^2} \implies \frac{d\theta}{dx} = \frac{2}{1+x^2}$$

In particular we get a map from the unit circle with the point 1 removed onto the real x-axis defined by

$$\text{(ii)} \qquad e^{i\theta} \mapsto i \cdot \frac{e^{i\theta}+1}{1-e^{i\theta}} = -\frac{2\sin\theta}{2-2\cos\theta} \quad : 0 < \theta < 2\pi$$

Above $i = e^{\pi i/2}$ is mapped to $x = -1$ while $\theta = \pi$ is mapped to $x = 0$. If $f(x) \in \mathcal{S}$ we put

$$\text{(1.2.1)} \qquad f_*(\theta) = f\left(i \cdot \frac{e^{i\theta}+1}{1-e^{i\theta}}\right)$$

The rapid decay of $f(x)$ as $|x| \to +\infty$ shows that f_* extends to a C^∞-function on the unit circle which is flat at 1, i.e. the derivatives $f^{(\nu)}(1) = 0$ for all ν. Denote by $C^\infty_*(T)$ the class of C^∞-functions on T which are flat at $\theta = 0$.

Exercise. Show that the constructions above gives an isomorphism of Frechet spaces:

$$\mathcal{S} \simeq C^\infty_*(T)$$

where $C^\infty_*(T)$ is regarded as a closed subspace of the Frechet space $C^\infty(T)$.

1.3 Tempered distributions.

The topological dual of $\mathcal{S}$ is denoted by $\mathcal{S}^*$ and called the space of tempered distributions on the x-line. Exercise 1.1 shows that to every $\gamma \in \mathcal{S}^*$ there exists some pair of integers $N, M \geq 0$ and a Riesz measure μ such that

$$(*) \qquad \gamma(f) = \int f^{(N)}(x) \cdot d\mu_N(x) \quad \text{where} \quad \int [1+|x|)^{-M} \cdot |d\mu_N(x)| < \infty$$

Remark. The integer N and the associated Riesz measure μ are not uniquely determined by γ. For example, let δ_0 be the Dirac distribution at $x = 0$. It can also be defined by

$$\delta_0(f) = \int_0^\infty f'(x)\, dx$$

Another example is the distribution γ defined by a principal value:

$$\gamma(f) = \lim_{\epsilon \to 0} \text{PV} \int_{-\infty}^{\infty} \frac{f(x) \cdot dx}{x}$$

Here the reader may verify that

$$\gamma(f) = \int_{-\infty}^{\infty} \log|x| \cdot f'(x)\, dx$$

Thus, with $N = 1$ we get the locally integrable function $\log|x|$ and notice that we can take $M = 2$ above, i.e. the integral

$$\int \big|\log|x|\,\big| \cdot (1 + |x|)^{-2} dx < \infty$$

1.3.1 A passage to the unit circle. In § 1.2 we identified $\mathcal{S}$ with $C_*^\infty(T)$. The Hahn-Banach theorem implies that every $\mu \in \mathcal{S}^*$ corresponds to a distribution γ on the unit circle whose restriction to $C_*^\infty(T)$ represents μ via the isomorphism in (1.2). Here γ is determined up to those distributions on T which vanish identically on $C_*^\infty(T)$. These distributions on T are supported by the singleton set $\{1\}$ and therefore finite sums of derivatives of the Dirac measure at this point. Hence their periodic Fourier coefficients are finite $\mathbf{C}$-linear sums

$$\widehat{\gamma}(n) = \sum_{\nu=0}^{\nu=k} a_\nu \cdot i^\nu \cdot n^\nu$$

where $\{a_\nu\}$ are complex numbers. Thus, if $\text{Temp}(\mathbf{Z})$ denotes the linear space of all complex sequences $\{c_n\}$ with moderate growth then one has an isomorphism of vector spaces:

$$\mathcal{S}^* \simeq \frac{\text{Temp}(\mathbf{Z})}{\mathcal{P}}$$

where $\mathcal{P}$ is the subspace of polynomial sequences as described above. It leads to some rather interesting constructions. Consider the subspace $c_+(\mathbf{Z})$ of sequences $\{c_n\}$ where $c_n = 0$ for every $n \leq 0$. It has empty intersection with $\mathcal{P}$ and can therefore be identified with a subspace of $\mathcal{S}^*$. Every sequence $\{c_n\}$ in $c_+(\mathbf{Z})$ gives the analytic function in the unit disc $\{|w| < 1\}$:

$$g(w) = \sum c_n w^n$$

In this way $\mathcal{O}_{\text{temp}}(D)$ is identified with a subspace of $\mathcal{S}^*$.

1.3.2 Example. Let $n \neq 0$ be an integer. The map $f \to f_*$ in (1.2.1) gives a tempered distribution γ_n defined by

$$\gamma_n(f) = \frac{1}{2\pi} \cdot \int_0^{2\pi} e^{in\theta} \cdot f_*(\theta)\, d\theta$$

From (i) in § 1.2 we conclude that

$$\gamma_n(f) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} f(x) \cdot (\frac{x-i}{x+i})^n \cdot \frac{dx}{1+x^2}$$

1.4 The Fourier transform on $\mathcal{S}$.

If $f \in \mathcal{S}$ its Fourier transform is defined by

$$\widehat{f}(\xi) = \int e^{-ix\xi} \cdot f(x) \cdot dx$$

Since $f(x)$ is rapidly decreasing we can differentiate with respect to ξ and obtain

$$\partial_\xi(\hat{f}) = \int (-ix) \cdot e^{-ix\xi} \cdot f(x) \cdot dx$$

Next, partial integration with respect to x gives

$$i \cdot \xi \cdot \hat{f}(\xi) = \int e^{-ix\xi} \cdot f'(x) \cdot dx$$

Denote by $\mathcal{F}$ the Fourier operator from $\mathcal{S}$ on the x-line to the $\mathcal{S}$-space on the ξ-line. The equations above can be expressed as follows:

1.4.1 Proposition. *The following two interchange formulas hold:*

(ii) $$i\partial_\xi \circ \mathcal{F} = \mathcal{F} \circ x \quad : \quad i\xi \circ \mathcal{F} = \mathcal{F} \circ \partial_x$$

1.4.2 Fourier's inversion formula. *Let* $f(x) \in \mathcal{S}$ *and set*

$$F(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

Then one has the equality

(*) $$f(x) = F(x)$$

Proof. First we establish the equality when $x = 0$. Notice that $f \mapsto F(0)$ is a linear functional on $\mathcal{S}$. Next, a function $f \in \mathcal{S}$ such that $f(0) = 0$ can be divided by x, i.e. $f = x \cdot \phi(x)$ with $\phi \in \mathcal{S}$. When this holds we have

(i) $$\widehat{f} = -\partial_\xi(\widehat{\phi})$$

The Fundamental Theorem of Calculus gives

$$\int_{-\infty}^{\infty} \partial_\xi(g) \cdot d\xi = 0$$

for all $g(\xi) \in \mathcal{S}$. Applied to $\widehat{\phi}$ and using (i) we conclude that

$$f(0) = 0 \implies F(0) = 0$$

But then the linear functional on the vector space $\mathcal{S}$ defined by $f \mapsto f(0)$ must be a constant times the functional $f \mapsto f(0)$. Hence there exists a constant c such that

$$f(0) = c \cdot \int \widehat{f}(\xi) \cdot d\xi$$

There remains to determine c. For this purpose we choose the special function

$$f(x) = e^{-x^2/2}$$

A verification which is left to the reader yields

$$\widehat{f}(\xi) = 2\pi \cdot e^{-\xi^2/2}$$

From this we deduce that $c = \frac{1}{2\pi}$.

The general case. With a fixed real number a and $f \in \mathcal{S}$ we set

$$f_a(x) = f(x + a)$$

It follows that

$$f(a) = f_a(0) = \frac{1}{2\pi} \int \widehat{f_a}(\xi) \cdot d\xi$$

Next, notice that a variable substitution gives:

$$\hat{f}_a(\xi) = \int f(x+a) \cdot e^{-ix\xi} \cdot dx = e^{ia\xi} \int f(x) \cdot e^{-ix\xi} \cdot dx = e^{ia\xi} \cdot \hat{f}(\xi)$$

From this we get the equality

$$f(a) = \frac{1}{2\pi} \int e^{ia\xi} \widehat{f_a}(\xi) \cdot d\xi$$

Since a is an arbitrary real number we have proved Fourier's inversion formula.

1.4.3 Exercise. If $n \geq 2$ we define the Schwarz class of rapidly decreasing C^∞-functions in $\mathbf{R}^n$. The Fourier transform is defined by

$$\widehat{f}(\xi) = \int e^{-i\langle \xi, x\rangle} \cdot f(x)dx$$

where the integration now is over $\mathbf{R}^n$. Fourier's inversion formula in dimension $n \geq 2$ amounts to show that

$$f(0) = \frac{1}{(2\pi)^n} \cdot \int \widehat{f}(\xi) \cdot d\xi$$

This formula can be proved via the fundamental theorem of calculus by an induction over n. Let us give the details when $n = 2$ where (x, y) are the coordinates in $\mathbf{R}^2$. Let $f(x, y)$ be given in $\mathcal{S}(\mathbf{R}^2)$. Define the partial Fourier transform

$$f^*(\xi, y) = \int e^{-ix\xi} f(x, y)\, dx$$

With ξ kept fixed we notice that the Fourier transform of the function $y \mapsto f^*(\xi, y)$ is equal to $\hat{f}(\xi, \eta)$. The 1-dimensional case applied to the y-variable gives for every ξ:

(i) $$f^*(\xi, 0) = \frac{1}{2\pi} \int \widehat{f}(\xi, \eta)\, d\eta$$

Next, the 1-variable case is also applied to the x-variable which gives

(ii) $$f(0, 0) = \frac{1}{2\pi} \int f^*(\xi, 0) d\xi$$

Now (i-ii) give the required formula

$$f(0, 0) = \frac{1}{(2\pi)^2} \iint \widehat{f}(\xi, \eta)\, d\xi d\eta$$

1.5 The Fourier transform of temperate distributions.

Let $\gamma \in \mathcal{S}^*$ be given. Since the Fourier transform is a linear isomorphism there exists a unique tempered distribution $\widehat{\gamma}$ on the real ξ-line defined on functions $g(\xi) \in \mathcal{S}$ by:

(*) $$\widehat{\gamma}(g) = \gamma(g_*) \quad : \quad g_*(x) = \int e^{-ix\xi} g(\xi) d\xi \quad :$$

Remark. Let $f(x) \in \mathcal{S}$ and denote by γ_f the distribution defined by the density $f(x)dx$. Then (*) gives

$$\widehat{\gamma}_f(g) = \iint f(x) \cdot e^{-ix\xi} g(\xi)\, dx d\xi = \int \hat{f}(\xi) \cdot g(\xi)\, d\xi$$

So the construction of the Fourier transform of functions in $\mathcal{S}$ extend to temperate distributions via (*).

1.5.1 Example. On the real x-line the Heaviside distribution H_+ is defined by the density 1 when $x \geq 0$ and zero if $x < 0$. To find its Fourier transform we perform certain limits. To begin with, for every large real number N we have the distribution on the x-line defined by

$$\mu_N(f) = \int_0^N f(x) \cdot dx$$

Its Fourier transform becomes

$$\widehat{\mu}_N(\xi) = \int_0^N e^{-ix\xi} \cdot dx = \frac{1 - e^{-iN\xi}}{i\xi}$$

It is clear that

$$\lim_{N \to \infty} \mu_N(f) = H_+(f)$$

hold for every $f \in \mathcal{S}$. It follows that $\{\widehat{\mu}_N\}$ converges weakly to $\widehat{H}_+$, i.e. for each $g(\xi) \in \mathcal{S}$ one has

(i) $$\widehat{H}_+(g) = \lim_{N \to \infty} \int_{-\infty}^{\infty} \frac{1 - e^{-iN\xi}}{i\xi} \cdot g(\xi) \cdot d\xi$$

We can also find $\widehat{H}_+$ by other limit formulas. Namely, if $\epsilon > 0$ we define the distribution γ_ϵ on the x-line by

$$\gamma_\epsilon(f) = \int_0^\infty e^{-\epsilon x} f(x) dx$$

Here we find that

$$\widehat{\gamma}_\epsilon(\xi) = \int_0^\infty e^{-\epsilon x - i\xi x} dx = \frac{1}{i\xi + \epsilon}$$

This gives the limit formula

(ii) $$\widehat{H}_+(g) = \lim_{\epsilon \to 0} \int \frac{g(\xi) \cdot d\xi}{i\xi + \epsilon}$$

Hence $\widehat{H}_+$ can be found by different limit formulas. Both (i) and (ii) are useful.

1.6 Plancherel's formula.

Let g be given in $\mathcal{S}$. Applying Fourier's inversion formula to g and takings its complex conjugate we get

$$\bar{g}(x) = \frac{1}{2\pi} \int e^{-ix\xi} \cdot \overline{\widehat{g}(\xi)}\, d\xi$$

If $f(x)$ is another function in $\mathcal{S}$ it follows that

$$\int f(x) \cdot \bar{g}(x)\, dx = \frac{1}{2\pi} \cdot \iint e^{-ix\xi} \cdot f(x)\overline{\widehat{g}(\xi)}\, dx d\xi = \frac{1}{2\pi} \cdot \int \widehat{f}(\xi) \cdot \overline{\widehat{g}(\xi)}\, d\xi$$

Introducting the hermitian inner product on the L^2-spaces we get the equality

(*) $$\langle f, g \rangle = 2\pi \cdot \langle \widehat{f}, \widehat{g} \rangle$$

With $f = g$ it follows that

(1.6.1) $$||f||_2^2 = 2\pi \cdot ||\widehat{f}||_2^2$$

So up to a constant the Fourier transform yields an isomorphism of L^2-spaces. In general, if $f(x)$ is an L^2-function on the real x-line it can be approximated in the L^2-norm by a sequence $\{f_n\}$ in $\mathcal{S}$ and by the above $\{\widehat{f_n}\}$ is a Cauchy sequence in the L^2-norm on the ξ-line. This gives a unique $\phi(\xi)$ in L^2 such that $||\widehat{f_n} - \phi||_2 = 0$. Here $\phi = \widehat{f}$ which means that the Fourier transform of the tempered distribution defined by the L^2-density $f(x)$ is an L^2-density on the ξ-line and the equality (1.6.1) holds.

1.7 Tempered distributions in $\mathbf{R}^2$

One has the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions $g(x,y)$ of the two real variables x and y. The dual $\mathcal{S}^*$ consists of tempered distributions and every such distribution γ is represented as a finite sum

$$\gamma(f) = \iint_{\mathbf{R}^2} g^{(\alpha)}(x,y) \cdot d\mu_\alpha(x,y)$$

where $\{\mu_\alpha\}$ is a finite family of Riesz measures and α are multi-indices which yield higher order derivatives of g. Finally, there exists an integer N such that

$$\iint_{\mathbf{R}^2} (1+x^2+y^2)^N \cdot |d\mu_\alpha(x,y)| < \infty$$

hold for every α. Fourier's inversion formula in $\mathbf{R}^2$ was established in § 1.4.1 and exactly as in the case $n=1$ this leads to the construction of Fourier transform of tempered distributions in the (x,y)-space.

1.7.1 The tempered distribution $\frac{1}{z}$. Identifying $\mathbf{R}^2$ with $\mathbf{C}$ the locally integrable function $\frac{1}{z}$ yields a tempered distribution. In $\mathcal{S}^*$ one has:

$$\lim_{\epsilon\to 0} \frac{\bar z}{|z|^2+\epsilon} = \frac{1}{z}$$

Consider the differential operator $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$. A computation gives

$$\bar\partial(\frac{\bar z}{|z|^2+\epsilon}) = \frac{\epsilon}{|z|^2+\epsilon)^2}$$

The right hand side is a positve density and when $\epsilon \to 0$ its mass becomes concentrated around the origin. At the same time

$$\iint \frac{\epsilon}{|z|^2+\epsilon)^2}\,dxdy = \epsilon\cdot 2\pi \int_0^\infty \frac{rdr}{(r^2+\epsilon)^2} = \pi$$

A passage to the limit gives the equality

$$(*) \qquad \bar\partial(\frac{1}{z}) = \pi\cdot\delta_0$$

Let us now consider the distribution derivative $\partial(\frac{1}{z})$. Outside the origin it is the density $-z^{-2}$. It turns out that the distribution is given by a princial value. More precisely the following hold for every test-.function $f(x,y)$:

$$(**) \qquad \partial(\frac{1}{z})(f) = \lim_{\epsilon\to 0} \iint_{|z|>\epsilon} \frac{f(x,y)}{z^2}\,dxdy$$

To prove (**) we use Stokes formula. The definition of distribution derivatives entails that

$$\partial(\frac{1}{z})(f) = -\frac{1}{z}(\partial(f)) = \lim_{\epsilon\to 0}\iint_{|z|>\epsilon} \frac{\partial(f)}{z}\,dxdy$$

where the last limit formula holds since $\frac{1}{z}$ is locally integrable. Now we employ complex differentials and and consider the differential 1-form

$$\omega = \frac{f}{z}\dot{d}\bar z$$

Since f has compact support Stokes Theorem gives for every $\epsilon > 0$:

$$\iint_{|z|>\epsilon} d\omega = -\int_{|z|=\epsilon} \frac{f}{z}\,d\bar z$$

Here we notice that

$$d\omega = \frac{\partial(f)}{z}\cdot dz\wedge d\bar z - \frac{f}{z^2}\cdot dz\wedge d\bar x$$

Hence (**) follows if we have establoshed the limit fornua:

$$\lim_{\epsilon\to 0}\int_{|z|=\epsilon}\frac{f}{z}\,d\bar{z}=0$$

But this is clear for in polar coordinates it amounts to show that

$$\lim_{\epsilon|\to 0}\int_0^{2\pi} f(\epsilon\cdot e^{i\theta})\cdot -i\cdot e^{-2i\theta}\,d\theta=0$$

The easy verification is left to the reader.

1.7.2 Exercise. Show that for every $n\geq 2$ the distribution derivative $\partial^n(\frac{1}{z})$ is given by a prinicial value:

$$\partial^n(\frac{1}{z})(f)=(-1)^{n-1}(n-1)!\cdot\lim_{\epsilon|\to 0}\iint_{|z|>\epsilon}\frac{f}{z^n}\,dxdy$$

1.7.3 Some Fourier transforms. Let g denote the Fourier transform of $\frac{1}{z}$. Interchange rules for derivations under the Fourier transform give

$$\frac{1}{2}(i\xi-\eta)\cdot\widehat{g}=\widehat{\bar{\partial}(\frac{1}{z})}$$

From (*) and the fact that the Fourier transform of δ_0 is the identity the right hand side is π which implies that

$$\widehat{g}=\frac{2\pi}{i\xi-\eta}$$

holds outside the origin in the (ξ,η)-space. Since $\frac{2\pi}{i\xi-\eta}$ is locally integrable it defines a distribution in the (ξ,η)-space now we show that $\widehat{g}$ is equal to this $L^1_{textloc}$-density. To prove this we argue as follows: The equality outside the orihin entails that

$$g=\frac{2\pi}{i\xi-\eta}+\rho$$

where ρ is a distriution supported by the origin and hence a finite sum of derivatives of the Dirac measure in the (ξ,η)-space. Taking the inverse Fourier transform of $\frac{2\pi}{i\xi-\eta}=-\frac{2\pi i}{\xi+i\eta}$ the same arghument as above shows that it is $\frac{1}{z}$ plus an eventual Dirac distribution in the (x,y)-space. If γ denotes this inverse Fourier transform we get

$$\frac{1}{z}=\gamma+\quad\text{the inverse transform of }\ \rho$$

The last inverse transform is a polynomial in x and y and therefore an unbounded function and from this we conclude that $\rho=0$. Hence the Fourier transform of $\frac{1}{z}$ is given by the locally integrable density above.

1.7.4 The Fourier transform of z^{-2}. Since this is minus the ∂-derivative of z^{-1} interchange formulas gives

$$\widehat{z^{-2}}=-\frac{1}{2}(i\xi+\eta)\cdot\frac{2\pi}{i\xi-\eta}=-\pi\cdot\frac{\xi-i\eta}{\xi+i\eta}$$

1.7.5 Exercise. Use the results above and Fourier's inversion formula to show the equality below for each f in $\mathcal{S}(\mathbf{R}^2)$:

$$(*)\qquad\iint\frac{f(x,y)}{x+iy}\,dxdy=\iint\frac{\widehat{f}(-\xi,-\eta)}{\xi+i\eta}\,d\xi d\eta$$

1.8 The wave decomposition of δ_0

Let δ_0 be the Dirac distribution i.e.

$$\delta_0(f) = f(0,0) \quad : f \in C_0^\infty(\mathbf{R}^2)$$

With $z = x + iy$ we shall find δ_0 via a meromorphic family of distribution-valued functions. If λ is a complex number with $\mathfrak{Re}\lambda > -2$ we notice that the function $|z|^\lambda$ is locally integrable around the origin. Indeed, this follows when we employ polar coordinates which for example gives

$$\iint_D |z|^\lambda dxdy = 2\pi \cdot \int_0^1 r^{\lambda+1}\, dr = \frac{2\pi}{\lambda+2}$$

In general, let m, k be a pair of non-negative integers with $m + k \geq 1$ and put

$$c_{m,k} = \int_0^{2\pi} \cos^m \theta \cdot \sin^k \theta \, d\theta$$

The reader may verify that $c_{m,k} = 0$ when $m + k$ is an odd integer. Next, we have Then

$$\iint_D |z|^\lambda \cdot x^m y^k \cdot dxdy = c_{m,k} \cdot \int_0^1 r^{\lambda+1+m+k}\, dr = \frac{c_{m,k}}{\lambda + 2 + m + k}$$

Exercise. Let $f(x, y)$ be a test.function. Use its Taylor expansion at the origin and the above to conclude thathe function

$$\lambda \mapsto \iint |z|^\lambda \cdot f(x,y)\, dxdy$$

is a meromorphic function in the comlex λ-plane whose poles are at most simle and confined to the set $\{-2, -4, -6, \ldots\}$. It means that the density $|z|^\lambda$ extends to a meromorohic distribution-valued function in the comlex λ-plane whose poles are at most simple and confined to the set of even and negative integers. From this we get an entire distribution-valued function defined by

$$\mathcal{G}_\lambda = \frac{|z|^\lambda}{2\pi \cdot \Gamma(\lambda/2 + 1)}$$

where we use that the entire function $\Gamma^{-1}(\lambda/2{+}1)$ has simple zeros at $\{-2, -4, -6, \ldots\}$.

1.8.1 Theorem. *The distribution $\mathcal{G}_{-2}$ is equal to δ_0.*

Proof. If $f(x, y)$ is a test function we can choose a C^∞-function ρ which only depends upon $r = \sqrt{x^2 + y^2}$ and is identically one for $0 \leq r \leq 1$ and vanishes if $r > 2$ while

$$f = f(0) \cdot \rho + \phi$$

where ϕ is a test-function which is zero at the origin. From the above the reader can check that

$$\lambda \mapsto \iint |z|^\lambda \cdot \phi(x,y)\, dxdy$$

is holomorphic at $\lambda = -2$. In fact, it is even holomorphic in $\mathfrak{Re}\, \lambda > -4$. At the same time

$$\iint |z|^\lambda \cdot \rho(x,y)\, dxdy = 2\pi \cdot \int_0^1 r^{\lambda+1}\, dr + 2\pi \int_{r \geq 1} r^{\lambda+1} \cdot \rho(r)\, dr$$

The first integral in the right hand side is $\frac{2\pi}{\lambda+2}$ and from this the reader can conclude that $\mathcal{G}_{-2}(f) = f(0)$ which proves Theorem 1.8.1.

1.8.2 Remark. Theorem 1.8 gives the limit formula

$$\lim_{\lambda \to -2} (\lambda - 2) \cdot \iint |z|^\lambda \cdot f(x,y)\, dxdy = 2\pi \cdot f(0)$$

The reader may also notice that the integrals

$$\iint |z|^{-2+i\delta} \cdot f(x,y)\, dxdy$$

exists for all real $\delta \neq 0$ and by the above

$$\lim_{\delta| \to 0} \delta \cdot \iint |z|^{-2+i\delta} \cdot f(x,y)\, dxdy = -2\pi i \cdot f(0)$$

2. Boundary values of analytic functions

Introduction. We consider analytic functions $f(z)$ defined in open rectangles $\square = \{-A < x < A \quad : 0 < y < B\}$. One says that f has moderate growth when the real axis is approached if there exists some integer $N \geq 0$ and a constant C such that

$$(*) \qquad |f(x+iy)| \leq C \cdot y^{-N}$$

When (*) holds we shall prove that f has a boundary value given by a distribution $\mathbf{b}(f)$ defined on the interval (a,b) of the real x-axis. Moreover, the map $f \mapsto \mathbf{b}(f)$ commutes with derivations, i.e. if $\partial_z = d/dz$ while ∂_x is the derivation on the real x-axis then

$$\mathbf{b}(\partial_z(f)) = \partial_x(\mathbf{b}(f))$$

where the right hand side is a distribution derivative. To achieve this we extend test-funtions on the x-line in a special way.

2.1 Small $\bar\partial$-extensions Given a positive integer N and some $g(x) \in C_0^\infty(-A,A)$ we set

$$G_N(x+iy) = g(x) + \sum_{\nu=1}^{\nu=N} i^\nu \cdot \frac{g^{(\nu)}(x) \cdot y^\nu}{\nu\,!}$$

Since $\bar\partial = \frac{1}{2}[\partial_x + i\partial_y]$ one has the equality

$$2 \cdot \bar\partial(G_N)(x+iy) =$$

$$(**) \qquad \sum_{\nu=0}^{N} i^\nu \cdot \frac{g^{(\nu+1}(x) \cdot y^\nu}{\nu\,!} \sum_{\nu=1}^{N} i^{\nu+1} \cdot \frac{g^{(\nu}(x) \cdot y^{\nu-1}}{(\nu-1)\,!} = i^N \cdot \frac{g^{(N+1)}(x) \cdot y^N}{N\,!}$$

2.2 The distribution $\mathfrak{b}(f)$. Let $f(z)$ satisfy (*) above. To each $g \in C_0^\infty(-A,A)$ we construct G_N and Stokes formula gives for every $0 < \epsilon < b < B$

$$\int_{-A}^{A} G_N(x+i\epsilon) f(x+i\epsilon)dx =$$

$$\int_{-A}^{A} G_N(x+ib) f(x+ib)dx + 2i \cdot \int_0^A \int_\epsilon^b \bar\partial(G_N)(x+iy) f(x+iy)dxdy$$

The growth condition on f and (**) imply that the absolute value of the double integral is majorized by

$$(**) \qquad \frac{Cb}{N\,!} \cdot \int_0^A |g^{(N+1)}(x)| \cdot dx$$

Passing to the limit $\epsilon \to 0$ while the absolutely integrable double integral is computed we get:

2.3 Proposition *To each $0 < b < B$ one has the equality*

$$\lim_{\epsilon \to 0} \int_{-A}^{A} G_N(x+i\epsilon) f(x+i\epsilon)dx =$$

$$\int_{-A}^{A} G_N(x+ib) f(x+ib)dx + 2i \cdot \int_{-A}^{A} \int_0^b \bar\partial(G_N)(x+iy) f(x+iy)dxdy \quad : \quad 0 < b < B$$

where the absolute value of the double integral is majorized by (**) above.

2.4 Definition. *The limit integrals above yield a distribution on the open interval $(-A,A)$. It is denoted by $\mathfrak{b}(f)$ and called the boundary value distribution of f.*

2.5 Primitive functions. Starting with $f \in \mathcal{O}(\square)$ we construct primitive functions which behave better as we approach the real x-axis. For example, fix a point $p = ia$ with $0 < a < A$ and set

$$F(z) = \int_{ia}^{z} f(\zeta)d\zeta$$

If $|f(x+iy)| \leq C \cdot y^{-N}$ for some $N \geq 2$ we get $|F(i+iy) \leq C_1 \cdot y^{-N+1}$ for another constant C_1. In the case $N = 1$ we get $[F(x+iy)| \leq C_1 \cdot \text{Log}\,\frac{1}{|y|}$. So by choosing N sufficiently large and taking the N:th order primitive F_N of f it has even continuous boundary values and $\mathfrak{b}(F_N)$ is just the density function $F_N(x)$. Then one can take distribution derivatives on the real x-line and get

$$\mathfrak{b}(f) = \frac{d^N}{dx^N}(\mathfrak{b}(F_N(x))$$

So this is an alternative procedure to define $\mathfrak{b}(f)$ without small $\bar{\partial}$-extensions. Both methods have their advantage depending on the situation at hand.

2.6 Examples. Let $f(z) = \log z$ where the single valued branch is chosen in $\mathfrak{Im}(z) > 0$ so that the argument is between 0 and π. Then $\mathfrak{b}(f)$ is the distribution defined by the density $\log x$ when $x > 0$ and if $x < 0$ by

$$\log|x| + \pi \cdot i$$

The complex derivative of f is $\frac{1}{z}$. Hence the boundary value distribution $\mathfrak{b}(\frac{1}{z})$ is given by

$$g \mapsto -\int \log|x| \cdot g'(x)\,dx - \pi i \int_{-\infty}^{0} g'(x)\,dx \tag{2.6.1}$$

Performing a partial integration the reader sohlud verify that (2.6.1) is equal to

$$\int \frac{(g(x) - g(0)) \cdot dx}{x} + \pi i \cdot g(0) \tag{2.6.2}$$

The reader may also verify the equation

$$\lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{g(x)}{x}\,dx = \int \frac{(g(x) - g(0)) \cdot dx}{x} \tag{2.6.2}$$

where the left hand side is referred to as a principal value integral. Next, starting with the analytic function z^{-1} in the upper half-plane we denote its boundary value distribution by $(x+i0)^{-1}$ and with $N = 1$ in (2.2) we have

$$(x+i0)^{-1}(g) = \lim_{\epsilon \to 0}\Big[\int \frac{g(x)}{x+i\epsilon}\,dx - \int \frac{\epsilon \cdot g'(x)}{x+i\epsilon}\,dx\Big] \tag{2.6.3}$$

At the same time the previous calculations show that

$$(x+i0)^{-1}(g) = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{g(x)}{x}\,dx + \pi i \cdot g(0) \tag{2.6.4}$$

Exercise. Show that

$$\lim_{\epsilon \to 0} -\int \frac{\epsilon \cdot g'(x)}{x+i\epsilon}\,dx = 0$$

and conclude that one has the equation

$$\lim_{\epsilon \to 0} \int \frac{g(x)}{x+i\epsilon}\,dx = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{g(x)}{x}\,dx + \pi i \cdot g(0) \tag{2.6.5}$$

It is instructive to check (*) when the g-function is identically one in a neighborhood of $x = 0$. In this case (*) corresponds to a residue formula expressed by the equation

$$\lim_{\epsilon\to 0}\int_{-a}^{a}\frac{dx}{x+i\epsilon} = \pi i \quad : a > 0$$

which the reader should verify for every $a > 0$ using branches of the complex log-function.

Exercise. Use that the passage to boundsry value distribtuins commute with derivaations and conclude from the above that if $N \geq 2$ and we start with the analytic function z^{-N} in the upper half-plane then

$$(x+i0)^{-N}(g) = \int \frac{g(x)}{(x+i\epsilon)^N}\,dx \tag{2.6.6}$$

A notable point is that the limit in the right hand side exists for all test-functions g. With $N = 2$ the construction of distribution derivatives and (2.6.4) give:

$$(x+i0)^{-2}(g) = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{g'(x)}{x}\,dx + \pi i\cdot g'(0)$$

2.7 The reflection principle.

Let

$$\square_- = \{(x+iy\colon \quad 0 < x < A \quad : \; - B < y < 0\}$$

be the opposed rectangle in the lower half-plane and let $h \in \mathcal{O}(\square_-)$ satisfy the moderate growth condition. We construct $\mathfrak{b}(h)$ in the same way as above.

2.8 Theorem. *Let $f \in \mathcal{O}(\square)$ and $h \in \mathcal{O}(\square_-)$ be a pair such that $\mathfrak{b}(f) = \mathfrak{b}(h)$ holds as distributions. Then they are analytic continuations of each other, i.e. there exists an analytic function Φ defined in $\{-B < y < B\colon \; - A < x < A\}$ such that $\Phi = f$ in $\square$ and $\Phi = h$ in $\square_-$.*

Proof. Choose a large N so that th N:th order primitive functions F_N and H_N both extend continuously to the real x-axis. The equality $\mathfrak{b}(f) = \mathfrak{b}(h)$ entails that

$$\frac{d^N}{dx^N}(\mathfrak{b}(F_N) - \mathfrak{b}(H_N) = 0$$

Now a distribution on the real x-line whose N:th order derivative is zero is a polynomial $p(x)$ of degree $\leq N-1$. So the pair of continuous functions $F(x)$ and $H(x)$ satisfy

$$H(x) = F(x) + p(x)$$

Hence the analytic functions $F(z) + p(z)$ and $H(z)$ have a common continuous boundary value function so by the ordinary reflection principle they are analytic continuations of each other. Let $G(z)$ be the resulting analytic function defined in the open domain where $-B < y < B$ now holds. Its N:th order complex derivative is also analytic in this domain and equal to f in $\square_+$ and to h in $\square_-$. This proves Theorem 2.8 with $\Phi = G^{(N)}$.

2.9 Principal value integrals.

Let μ be a Riesz measure supported by the unit interval $[0, 1]$. In $\mathbf{C} \setminus [0, 1]$ there exists the analytic function

$$F(z) = \int_0^1 \frac{d\mu(t)}{z-t}$$

It is clear that F has moderate growth as we approach the real x-line from above or below. Hence there exists two boundary value distributions denoted by $F(x+i0)$ and $F(x-i0)$. Their sum is given as a limit of the density functions

$$x \mapsto \lim_{\epsilon \to 0} \int_0^1 [\frac{1}{x-t+i\epsilon} - \frac{1}{x-t-i\epsilon}] \cdot d\mu(t) = \lim_{\epsilon \to 0} \int_0^1 \frac{x-t}{(x-t)^2+\epsilon^2} \cdot d\mu(t)$$

If $g(x)$ is a C^∞-function with compact support there exists the limit

$$(*) \qquad \lim_{\epsilon \to 0} \iint \frac{x-t}{(x-t)^2+\epsilon^2} \cdot g(x) \cdot d\mu(t)$$

This yields a distribution which we refer to as the principal value distribution associated to the given measure μ.

2.9.1 Exercise. For each $\epsilon > 0$ we put

$$L_\epsilon(x) = -\int_0^1 \log \sqrt{(x-t)^2+\epsilon^2} \cdot d\mu(t)$$

Show that a partial integration identifies (*) with

$$-\lim_{\epsilon \to 0} \int_0^1 L_\epsilon(x) \cdot g'(x)\, dx$$

Moreover, $\{L_\epsilon(x)\}$ converge in the measure theoretic sense to the L^1-function

$$(2.9.1) \qquad x \mapsto \int_0^1 \log |x-t| \cdot d\mu(t)$$

arising via a convolution. Hence the principal value distribution in (*) is the derivative of function from (2.9.1) which by construction is the convolution of μ nd the locally integrable log-function on the real line. Notice also that (2.9.1) is a continuous function of x.

2.9.2 Another boundary value. Consider the analytic function $\log z$ in the upper half plane whose branch is defined so that the argument of $\log z$ stays between 0 and π. Now

$$G(z) = \int_0^1 \log(z-t) \cdot d\mu(t)$$

is an analytic function of z in the upper half-plane which gives the distribution $G(x+i0)$. The reader should verify that it is expressed by the L^1-density on the x-line defined by:

$$x \mapsto \int_0^1 \log |x-t| \cdot d\mu(t) + \pi i \cdot \int_0^x d\mu(t)$$

In a similar fashion we use the branch of $\log z$ in the lower half-plane where the argument of $\log z$ stays in $(-\pi, 0)$. Now we get the boundary value distribution

$$G(x-i0) = \int_0^1 \log |x-t| \cdot d\mu(t) - \pi i \cdot \int_0^x d\mu(t)$$

It follows that the difference

$$(2.9.3) \qquad G(x+i0) - G(x-i0) = 2\pi i \cdot \int_0^x d\mu(t)$$

2.9.4 The distribution $F(x+i0)-F(x-i0) =$. Taking the distribution derivative and using (xx) above one has

$$F(x+i0) - F(x-i0) = 2\pi i \cdot d\mu(x)$$

Thius, the difference of $F(x+i0)$ and $F(x-i0)$ it he distribution supported by $[0,1]$ given by $2\pi i$ times the Riesz measure μ.

2.9.5 Higher order derivatives. Let $n \geq 2$ and put

$$F_n(z) = \int_0^1 \frac{d\mu(t)}{(z-t)^n}$$

In the upper and the lower half-planes we notice that $F_n(z)$ is a complex derivative of F, i.e.

$$F_n(z) = (-1)^{n-1} \cdot (n-1)! \cdot \partial_z^{n-1}(F)$$

Passing to boundsry value distributions it follows that

$$F_n(x+i0) - F_n(x-i0) = (-1)^{n-1} \cdot (n-1)! \cdot 2\pi i \cdot \mu^{(n-1)}$$

where the last term is the distribution derivative of order $n-1$ of μ.

2.9.6 Example. Let μ be the absolutely continuous density $\log x$ on $0 \leq x \leq 1$. The distribution derivative becomes

$$g \mapsto \int_0^1 \frac{g(x) - g(0)}{x} \, dx$$

which by the above corresponds to the boundary value distribution

$$-\frac{1}{2\pi i} \cdot [F_2(x+i0) - F_2(x-i0)]$$

3. Tempered boundary values

Let $f(z)$ be analytic in the upper half-plane. Suppose there exist non-negative integers N, M and a constant C such that

$$(*) \qquad |f(x+iy)| \leq C \cdot (1+|x|+|y|)^N \cdot y^{-M}$$

for all $z = x+iy$ in the upper half-plane. While x stays in a bounded interval we have a moderate growth as $y \to 0$ and from § 2.1 we obtain the boundary value distribution $\mathbf{b}(f) = f(x+i0)$. Using (*) and the estimate (**) from § 2.1 we get another constant C^* such that

$$|\mathfrak{b}(f)(g)| \leq C^* \int (1+|x|)^N \cdot |g^{M+1)}(x)|\, dx$$

for all test-functions g. It follows that $\mathfrak{b}(f)$ is a tempered distribution and hence its Fourier transform exists. It turns out that

$$(3.1) \qquad \mathrm{Supp}(\widehat{\mathfrak{b}(f)}) \subset [0, +\infty)$$

To prove (3.1) we consider a test-function $g(\xi)$ with a compact support contained in $[-b, -a]$) with $0 < a < b$. The inverse Fourier transform

$$G(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot g(\xi)\, d\xi$$

extends to an entire function in the complex z-plane and partial integration give for every positive integer m:

$$(iz)^M \cdot G(z) = (-1)^M \cdot \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot g^{(M)}(\xi)\, d\xi$$

From this it is clear that for every positive integer m there exists a constant C_m such that

$$(3) \qquad |G(x+iy)| \leq C_m \dot{(}1+|z|)^{-m} \cdot e^{-ay}$$

Now (*) holds for $f(z)$ and choosing $m \geq 2+M$ it follows via Cacuhy integrals applied to the pair of analytic functions f and G in the upper half-plane that

$$(4) \qquad \int_{-\infty}^{\infty} f(x+i\epsilon) \cdot G(x+i\epsilon)\ dx = \int_{-\infty}^{\infty} f(x+iB) G(x+iB)\ dx$$

hold for all pairs $0 < \epsilon < B$. The exponential decay with the factor e^{-aB} from (3) entails that the last integral becomes arbitrary small when B is large and hence (4) vanishes. Taking the limit when $\epsilon \to 0$ the left hand side in (4) evaluates $\mathfrak{b}(f)(G)$ which therefore is zero and Fourier's inversion formula entails that $\widehat{\mathfrak{b}(f)}(g) = 0$. Since this vanishing hold for every the test-function g as above we get (3.1).

3.2 A converse. Let μ be a Riesz measure on the ξ-line supported by $\{\xi \geq 0\}$ and

$$(\mathrm{i}) \qquad \int_0^{\infty} (1+\xi)^{-M} \cdot d|\mu|(\xi) < \infty$$

holds for some non-negative integer. This gives an analytic function in $\mathfrak{Im}\, z > 0$:

$$U(z) = \frac{1}{2\pi} \cdot \int_0^{\infty} e^{iz\xi}\, d\mu(\xi)$$

With $z = x+iy$ where $y > 0$ we have

$$|U(x+iy) \leq \int_0^{\infty} e^{-y\xi} \cdot d|\mu|(\xi)$$

With M as in (i) we put

$$\rho_M(y) = \max_{\xi \geq 0} e^{-y\xi} \cdot (1+\xi)^M$$

The maximum is attained when $(1+\xi)y = M$ which gives

$$\rho_M(y) \leq \frac{M^M}{y^M}$$

It follows that

$$U(x+iy)| \leq C \cdot M^M \cdot y^{-M}$$

Hence $U(z)$ satisfies the estimate (*). More generally, let γ be a tempered distribution on the ξ-line supported by $\{\xi \geq 0\}$. It is represented by a derivative of some order N of a Riesz measure μ satisfying (i) above and from this the reader may verify that we get an analytic function $U(z)$ in the upper halfplane defined by

$$z \mapsto \frac{1}{2\pi} \cdot \gamma(e^{iz\xi})$$

where there exists a pair of non-negative integers N and M and a constant C such that

(ii) $$|U(z)| \leq C \cdot (1+|z|)^N \cdot y^{-M}$$

Together with the previous facts we get the following conclusive result.

3.3 Theorem. *There is a 1-1 correspondence between tempered distributions on the x-line whose Fourier transforms are supported by $\{\xi \geq 0\}$ and the family of analytic functions $U(z)$ in the upper half-plane satisfying (*).*

3.4 The distribution $(x+i0)^\lambda$

.

In the upper half-plane $\mathfrak{Im}\, z > 0$ there exists the single-vaued branch of $\log z$ whose argument stays in the interval $(0,\pi)$. If $x<0$ is real we have for example

$$\lim_{\epsilon \to 0} \log(x+i\epsilon) = \log|x| + \pi i$$

If λ is a complex number there exists the analytic function in the upper half-plane defined by

$$z^\lambda = e^{\lambda \cdot \log z}$$

Now we can take its boundary value distribution

$$(x+i0)^\lambda = \lim_{\epsilon \to 0} (x+i\epsilon)^\lambda$$

This is a tempered distribution on the real x-line. Following Euler and Riemann we shall find this Fourier transform for every complex number λ. To attain this we consider the ξ-line where ξ^s is lovally integrable on $(0,+\infty)$ when $\mathfrak{Re}\, s > -1$. This yields a tempered distribution supported on $\{\xi \geq 0\}$ and defined by

$$\xi_+^s(\phi) = \int_0^\infty \xi^s \cdot \phi(\xi)\, d\xi$$

where $\phi(\xi)$ are Schwartz functions on the ξ-line Keeping s in $\mathfrak{Re}\, s > -1$ a partial integration gives

$$(s+1) \cdot \xi_+^s(\phi) = -\int_0^\infty \xi^{s+1} \cdot \phi'(\xi)\, d\xi$$

The definition of distribution derivatives entails that

$$(s+1) \cdot \xi_+^s = \partial_\xi(\xi_+^{s+1})$$

We can continue and for every positive integer m one has

(3.4.*) $$(s+1)\cdots(s+m) \cdot \xi_+^s = \partial_\xi^m(\xi_+^{s+m})$$

We refer to (3.4.*) as Euler's functional equation for the distribution-valued function $s \mapsto \xi_+^s$. Next, recall that $\Gamma^{-1}(s)$ is an entire function with of s with simple zeros at non-posistive integers. Euler's equation therefore gives an entire function of s with values in the space of tempered distributions on the ξ-line defined by

$$s \mapsto \frac{1}{\Gamma(s+1)} \cdot \xi_+^s$$

At the same time

$$\lambda \mapsto (x+i0)^\lambda$$

is an entire function of λ with values in the space of tempered distributions on the x-line whose complex derivative becomes

$$\frac{d}{d\lambda}((x+i0)^\lambda) = \lambda \cdot (x+i0)^{\lambda-1}$$

Passing to the Fourier transform we get the entire function

$$\lambda \mapsto \widehat{(x+i0)^\lambda}$$

with values in the space of tempered distributions on the ξ-line. With these notations one has the Euler-Rimeann equation:

3.4.1 Theorem. *One has the equality*

$$\widehat{(x+i0)^\lambda} = \frac{2\pi \cdot i^\lambda}{\Gamma(-\lambda)} \cdot \xi_+^{-\lambda-1}$$

Proof. When $\mathfrak{Re}\, s > -1$ the inverse Fourier transform of ξ_+^s becomes

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} \cdot \int_0^\infty e^{ix\xi - \epsilon\xi} \cdot \xi^s \, d\xi$$

The variable substitution $(\epsilon - ix)\xi = \eta$ identifies the integral with

$$\frac{1}{2\pi} \cdot (\epsilon - ix)^{-s-1} \cdot \int_0^\infty e^{-\eta} \cdot \eta^s \, d\eta = \frac{1}{2\pi} \cdot (\epsilon - ix)^{-s-1} \cdot \Gamma(s+1)$$

Notice that

$$i^{s+1} \cdot (\epsilon - ix)^{s+1} = (x + i\epsilon)^{s+1}$$

After a passage to the limit when $\epsilon \to 0$ and the substitution $\lambda \to -s-1$ we get the Euler-Riemann equation.

Example. Let us consider the limit when $\lambda \to 0$. Here

$$\lim_{\lambda \to 0} (x+i0)^\lambda = 1$$

Now $\widehat{1} = 2\pi \cdot \delta_0$ where δ_0 is the Dirac distribution on the ξ-line. Next, in the right hand side of the Euler-Riemann equation we use that

$$\xi_+^{-\lambda-1} = -\frac{\partial_\xi(\xi^{-\lambda})}{\lambda}$$

So with λ close to zero the right hand side in (3.4.1) becomes

(i) $$2\pi \cdot i^\lambda \cdot \frac{1}{-\lambda \cdot \Gamma(-\lambda)} \cdot \partial_\xi(\xi^{-\lambda})$$

Now we notice that

$$\lim_{\lambda \to 0} \xi_+^\lambda = H^+(\xi)$$

where $H^* + (\xi)$ is the Heaviside distribution on the ξ-line supported by $\xi \geq 0$. At the time $-i^\lambda \cdot \lambda \cdot \Gamma(-\lambda)$ tends to 1 as $\lambda \to 0$. Hence the limit in (i) becomes

$$2\pi\partial_\xi(H^+(\xi)) = 2\pi \cdot \delta_0$$

which confirms the equality $\widehat{1} = 2\pi \cdot \delta_0$.

3.4.2 The Fourier transform of $x+i0)^{-1}$**.** When $\lambda=-1$ in the right hand side of the Euler-Riemann equation we get the Fourier transform

$$2\pi\cdot i\cdot H^+(\xi)$$

Hence we have the equation

$$\frac{1}{2\pi i}\cdot\widehat{\frac{1}{x+i0)}}=H^+(\xi)$$

3.4.3 Further examples. Let $\lambda=-1-z$ with z close to zero. The right hand side in Theorem 3.4.1 becomes

$$\frac{2\pi i\cdot i^z}{\Gamma(1+z)}\xi_+^z$$

We can regard the series expansion of this funtion of z. Since $\xi^z=1+z\log\xi+O(z2)$ when $\xi>0$ the distribution valued function ξ_+^z has a series expansion

$$\xi_+^z=H^+(\xi)+z\cdot(\log\xi)_++O(z^2)$$

We have also an expansion

$$\frac{2\pi i\cdot i^z}{\Gamma(1+z)}=2\pi i+c_1z+O(z^2)$$

Next we have

$$(x+i0)^{-1-z}=(x+i0)^{-1}(1-z\log(x+i0)+O(z^2)$$

Identifying the linear z-term the Fourier transform of $-(x+i0)^{-1}\cdot\log(x+i0)$ is equal to

$$2\pi i\cdot(\log\xi)_++c_1\cdot H^+(\xi)$$

Exercise. Determine the constant c_1 and consider also the integral

(i) $$\frac{1}{2\pi}\cdot\int_0^\infty e^{iz\xi}\cdot\log\xi\cdot d\xi$$

when z varies in the upper half-plane. From the Euler-Riemann equation we get the answer via the above and Fourier's inversion formula. A direct evaluation of (i) is less obvious and the reader may try to perform it by sutiable partial integrations. The point is that one does not need to invoke such tricks since everything already has been identified via Taylor series expansions.

3.4.4 The Fourier transform of $\log(x+i0)$**.** In the upper half-plane the complex derivative of $\log z$ is z^{-1}. It follows that

$$\partial_x(\log(x+i0))=(x+i0)^{-1}$$

Passing to the Fourier transform and using (3.4.2) we have

$$i\xi\cdot\log\widehat{(x+i0})^{-1}=2\pi i\cdot H^+(\xi)+c\cdot\delta_0$$

for some constant c where we used that $\widehat{1}=\delta_0$. So $\log\widehat{(x+i0)}$ is supported by $\{\xi\geq 0\}$ and if $\xi>0$ it is expressed by the density $\frac{2\pi}{\xi}$. There remains to give the complete description. On $\{\xi\geq 0\}$ we have the locally integrable function $\log\xi$ whose distributuion derivative on the open half-line $\{|xi>0\}$ is ξ^{-1}. It follows from the above that

(ii) $$\log\widehat{(x+i0)}=2\pi\cdot\partial_\xi(\log\xi)+\gamma$$

where γ is supported by $\{\xi=0\}$. To determine γ we take the inverse Fourier transform of $2\pi\cdot\partial_\xi(\log\xi)$ which becomes

(iii) $$\mu = -ix \cdot \int_0^\infty e^{ix\xi} \cdot \log \xi \cdot d\xi$$

From the above μ differs from $\log(x + i0)$ by a constant. To find this constant we can simply perform an evaluation at $z = i$. In (iii) we therefore get

(iv) $$c_1 = \int_0^\infty e^{-\xi} \cdot \log \xi \cdot d\xi$$

At the same time $\log i = \pi i/2$. So if $\gamma = c \cdot \delta_0$ the constant c satisfies

$$c = \pi i/2 - c_1$$

Exercise. Determine c_1 and write out the Fourier transform of $\log(x + i0)$.

A tricky convolution. On the ξ-line we have the Heaviside distributions $H_+(\xi)$ and $H_-(|xi)$. Their convolution cannot be defined in a direct manner. If it exists it means that one can define integrals

$$\iint g(\xi_1 + \xi_2)\, d\xi_1 d\xi_2$$

taken over the quadrant $\{\xi_1 \geq 0\} \times \{|xi_2 \leq 0\}$. Even if g has a compact support this integral may diverge. On there other hand there exist welldefined convolutions

$$\xi_+^a * \xi_-^b$$

for pairs of complex numbers whose real parts are < -1. See Exercise xx below. Using Theorem 3.4.1 we can give a meaning to convolutions as above for all pairs of complex numbers. Namely, on the x-line an analogue problem is to construct products of distributions

(i) $$(x + i0)^\alpha \cdot (x - i0)^\beta$$

where α and β in general are complex numbers. If both have real part > 0 we simply take the product of the corresponding continuous densities. To procced one uses the equations

$$(x + i0)^\alpha = e^{\pi i\alpha} \cdot |x|^\alpha \quad : x < 0 \quad : \quad (x - i0)^\alpha = e^{-\pi i\alpha} \cdot |x|^\alpha \quad : x < 0$$

It follows that

$$(x + i0)^\alpha - e^{-2\pi i\alpha} \cdot (x - i0)^\alpha = (1 - e^{-2\pi i\alpha}) \cdot |x|_+^\alpha$$

Hence (i) can be written as

$$e^{-2\pi i\alpha} \cdot (x - i0)^\alpha \cdot (x - i0)^\beta + (1 - e^{-2\pi i\alpha}) \cdot |x|_+^\alpha \cdot (x - i0)^\beta$$

Since $z^{|alpha+\beta}$ is analytic in the ower half-plane the furst product above is defined for all pairs α, β and gives an entire distribution-valued function in the 2-dimensional complex (α, β)-space. The last erm is equal to

$$(1 - e^{-2\pi i\alpha}) \cdot |x|_+^{\alpha+\beta}$$

By § xx this is a meromorphic distribution valued function and after multilication with $\Gamma^{-1}(\alpha+\beta)$ it becomes an entire function. hence there exists the entire function

(*) $$(\alpha, \beta) \mapsto \frac{1}{\Gamma(\alpha + \beta)} \cdot (x + i0)^\alpha \cdot (x - i0)^\beta$$

Via Fourier transforms we can use this entire function to give a meaning to convolutions on the ξ-line whixh in general will be defined via analytic continuations.

3.4 The distributions $\mathfrak{b}(z^\lambda)$

For every complex number λ there exists a single valued branch of z^λ in the upper half-plane:

$$z^\lambda = e^{\lambda\cdot\log z}$$

where the single-valued branch of $\log z$ is chosen so that

$$\log z = \log|z| + i\arg z$$

where the argument is in $(0,\pi)$. This analytic function satisfies (ii) above and weget the tempered distribution $\mathfrak{b}_+(z^\lambda)$ whose Fourier transform is supported by $\{\xi \geq 0$. We prefer to use the notation

$$(x+i0)^\lambda = \mathfrak{b}_+(z^\lambda)$$

Put

$$\mu(\lambda) = \widehat{(x+i0)}^\lambda$$

Now $\lambda \mapsto \mu(\lambda)$ is an entire function of λ with values in the space of tempered distributions supported by $\{\xi \geq 0\}$. Next, for a fixed λ the complex derivative

$$\frac{d}{dz}(z^\lambda) = \lambda\cdot z^{\lambda-1}$$

Passing to the boundary value distributions it folllows that

$$x\partial_x(x+i0)^\lambda) = \lambda\cdot(x+i0)^\lambda$$

Interchange rules under the Fourier transform give

$$(i\partial_\xi \circ i\xi(\mu(\lambda)) = \lambda\cdot\mu$$

Now $i\partial_\xi \circ i\xi = -\xi\partial_\xi - 1$ and hence $\mu(\lambda$ satisfies the differential equation

(2) $$\xi\partial_\xi(\mu(\lambda)) = -(\lambda+1)\cdot\mu(\lambda)$$

This differential equation entails that there exists a constant $c(\lambda)$ such that

(3) $$\mu(\lambda) = c(\lambda)\cdot\xi^{-\lambda-1} \quad : \ \xi > 0$$

To fiond the constants $c(\lambda)$ we shall study a reversed situation. When s is a complex number and $\mathfrak{Re}\, s > -1$ there exists the locally intergable densityξ^{-s} on $\xi \geq 0$. It is extended by zero on $\{\xi < 0\}$ and the resulting tempered distribution is denoted by ξ_+^s. Its inverse Fourier transform is the boundary value distribution of the analytic function in the upper half-plane defined by

$$V_s(z) = \frac{1}{2\pi}\cdot\int_0^\infty e^{iz\xi}\cdot\xi^s\,d\xi$$

3.4.1 Exercise. Show that a variable substitution gives

$$V_s(z) = \frac{1}{2\pi}\cdot(-iz)^{-s-1}\Gamma(s+1)$$

So with $\lambda = -s-1$ we get the equality

$$2\pi\cdot(-i)^{-\lambda}\cdot\Gamma(-\lambda)\cdot\widehat{(x+i0)}^\lambda = \xi_+^{-1-\lambda} \quad : \ \mathfrak{Re}\lambda < 0$$

Hence we have

(4) $$c(\lambda) = \frac{(-i)^\lambda}{2\pi}\cdot\frac{1}{\Gamma(-\lambda)} \quad : \mathfrak{Re}\lambda < 0$$

3.5 A meromorphic extension. To extend the situation to the case when $\mathfrak{Re}\,\lambda \geq 0$ we first study the distribtuins $\{\xi_+^s\}$ defined as above when $\mathfrak{Re}\, s > -1$. If $g(\xi)$ is a test-function on the ξ-line and $\mathfrak{Re}(s) > -1$ a partial integration gives:

$$(s+1)\cdot\xi_+^s(g) = -\int_0^\infty g'(\xi)\cdot\xi^{s+1}\,d\xi$$

We can continue and for every positive integer m one has

$$(s+m)\ldots(s+1)\xi_+^s(g)=(-1)^m\cdot\int_0^\infty g^{(m)}(\xi)\cdot\xi^{s+m}\,d\xi \tag{3.5.1}$$

This is called Euler's functional equation and entails that

$$s\mapsto\xi_+^s$$

extends to a meromorphic distribution valued function with at most simple poles at negative integers. Next, we recall that $\Gamma^{-1}(s)$ is an entire function with simple zeros at the negative integers. From the above it follows that there exists an entire distribution-valued function defined by

$$s\mapsto\frac{1}{\Gamma(s)}\cdot\xi_+^s \tag{3.5.2}$$

Together with (3.4.1) and analyticity we arrive at the following:

3.6 Theorem. *For every complex λ one has the equality*

$$\mu(\lambda)=2\pi\cdot(-i)^\lambda)\cdot\frac{1}{\Gamma(-\lambda-1)}\cdot\xi_+^{-1-\lambda}$$

Example. Consider the case $\lambda=-1$. Since $\Gamma(0)=1$ the right hand side becomes

$$-2\pi i\cdot\xi_+^0=-2\pi i\cdot H_+(\xi)$$

where $H_+(\xi)$ is the Heaviside distribution. Hence we have the equation

$$\widehat{(x+i0)}^{-1}=-2\pi i\cdot H_+(\xi)$$

The last equality can be verified directly via the Inverse Fourier transform using the equation

$$\int_0^\infty e^{-\epsilon\xi+ix\xi}\,d\xi=-\frac{1}{\epsilon-ix}=\frac{i}{x+i\epsilon}$$

Next, let us take $\lambda=0$ which means that we start with $z^0=1$ in the upper half-plane which gives the identity as a boundsry value distribution. Now $\widehat{1}$ is the Dirac distribution δ_0 on the ξ-line. At the same time the function Γ^{-1} has a simpe zero at $\lambda=-1$ which compensates the pole of the meromorphic extension of ξ_+^s at $s=-1$. To catch the right hand side in (xx) we use Euler's equation from (xx) which with a small λ gives

$$\lambda\cdot\xi_+^{-1-\lambda}=\partial_\xi(\xi_+^\lambda)$$

When $\lambda\to 0$ then ξ_+^λ converges to $H_+(\xi)$ and we recall that $\partial_\xi(H_+)$ yields the Dirrac measure on the ξ-line. From this the reader can confirm the formula in (xx).

TO BE revised

Laurent expansions. Consider a positive integer m and with $\zeta \simeq 0$ one has a Laurent expansion

$$\xi_+^{-m+\zeta} = \frac{\rho_m}{\zeta} + \gamma_{m,0} + \gamma_{m,1} \cdot \zeta + \dots$$

where ρ_m and $\{\gamma_{m,\nu}\}$ are tempered distributions. We seek $\gamma_{m,0}$. For a test-function g we have

$$\zeta \cdot (-1+\zeta) \cdots (-m-1+\zeta)\xi_+^{-m+\zeta} = (-1)^m \cdot \int_0^\infty g^{(m)}(\xi) \cdot \xi^\zeta \, d\xi$$

The expansion $\xi^\zeta = 1 + \zeta \cdot \log \xi +$ higher order terms gives

$$\rho_m(g) = \frac{(-1)^m}{(m-1)!} \cdot \int_0^\infty g^{(m)}(\xi)\, d\xi \quad : \gamma_0(g) = \frac{(-1)^m}{(m-1)!} \cdot \int_0^\infty g^{(m)}(\xi) \cdot \log \xi \, d\xi$$

The case $m = 1$. Here partial integrations give

$$\int_1^\infty g'(\xi){\cdot}\log \xi \, d\xi = -\int_1^\infty \frac{g(\xi)}{\xi}\, d\xi \quad : \quad \int_0^1 g'(\xi){\cdot}\log \xi \, d\xi = -\int_0^1 \frac{g(\xi) - g(0)}{\xi}\, d\xi \implies$$

$$\gamma_{,1,0}(g) = \int_0^1 \frac{g(\xi) - g(0)}{\xi}\, d\xi + \int_1^\infty \frac{g(\xi)}{\xi}\, d\xi$$

At the same time we notice that

$$\rho_1(g) = -\int_0^\infty g'(\xi)\, d\xi = g(0)$$

Hence ρ_1 is the Dirac distribution δ_0.

Exercise. If $m \geq 2$ and g is a test function we get the Taylor polynomial

$$T_{m-1}(g;\xi) = g(0) + g'(0)\xi + \ldots + \frac{g^{(m-1)}(0)}{(m-1)!} \cdot \xi^{m-1}$$

Show that the distribution $\gamma_m(0)$ in (xx) is given by

$$\gamma_{m,0}(g) = \int_0^1 \frac{g(\xi) - T_{m-1}(g;\xi)}{\xi^m}\, d\xi + \int_1^\infty \frac{g(\xi)}{\xi^m}\, d\xi$$

The distribution $\log(x + i0)$. In the upper half-plane we choose the single-valued branch of $\log z$ whose boundary value yields a tempered distribution. When $x > 0$ we get the continuous density $\log x$. If $x < 0$ the branch of $\log z$ entails that one has the density $\log |x| + \pi i$. Thus

$$\log(x + i0) = \log |x| + \pi i \cdot H_-(x)$$

where $H_-(x)$ is one for $x < 0$ and zero if $x \geq 0$ while $\log |x|$ is an even and locally integrable function. In the upper half-plane

$$\frac{d \log z}{dz} = \frac{1}{z}$$

Since the passage to boundary value distributions commute with derivatives we have

$$\frac{d}{dx}(\log(x + i0) = \frac{1}{x + i0}$$

The Fourier transform of $\log(x + i0)$**.** The interchange rule and (xx) give

$$i\xi \cdot (\widehat{\log(x + i0)} = -i \cdot H_+(\xi)$$

Hence the Fourier transform of $\log(x+i0)$ is the density $-\frac{1}{\xi}$ when $\xi > 0$. There remains to describe its extension to $\xi = 0$. To attain it we use the expansion below when $\lambda \simeq 0$:

$$z^\lambda = 1 + \lambda \cdot \log z + \text{higher order terms}$$

With $\gamma = \log\widehat{(x+i0)}$ we get

$$\mu(\lambda) = 1 + \lambda \cdot \gamma + \text{higher order terms}$$

Now Theorem xx identifies (xx) with

$$(-i)^\lambda \cdot \frac{1}{\Gamma(\lambda)} \cdot \xi_+^{-1-\lambda}$$

From xx one has

$$\xi_+^{-1-\lambda} = \frac{\delta_0}{\lambda} + \gamma_{1,0} + \text{higher order terms}$$

At the same time one has an expansion

$$(-i)^\lambda \cdot \frac{1}{\Gamma(\lambda)} = \lambda + c_2\lambda^2 + \text{higher order terms}$$

Hence (xx) becomes

$$\delta_0 + \lambda \cdot (\gamma_{1,0} + c_2 \cdot \delta_0) + \text{higher order terms}$$

We conclude that

$$\gamma = \gamma_{1,0} + c_2 \cdot \delta_0$$

The constant c_2 can be found when we first regard the inverse Fourier transform of $\gamma_{1,0}$ whose value at $x = 1$ becomes

$$A = \frac{1}{2\pi} \cdot \Big[\int_0^1 \frac{e^{i\xi} - 1}{\xi}\, d\xi + \int_0^1 \frac{e^{i\xi}}{\xi}\, d\xi\,\Big]$$

At the same time $\log 1 = 0$ and since the inverse Fourier transform of δ_0 is the identity times $\frac{1}{2\pi}$ it follows that

$$c_2 = -A$$

TWO options to find c_2 from the above.

3.X Poisson's formula.

In the upper half-plane we have the analytic function

$$U(z) = \frac{\sin(z)}{\cos(z)} = \frac{1}{i} \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} =$$

$$i \cdot \frac{1 - e^{2iz}}{1 + e^{2iz}} = i + 2i \cdot \sum_{k=1}^{\infty} (-1)^k \cdot e^{2ikz} \tag{1}$$

The reader may verify that there exists a constant C such that

$$|U(x+iy)| \le \frac{C}{y}$$

Hence there exists the boundary value distribution $U(x+i0)$ and put

$$\mu = U\widehat{(x+i0)}$$

3.6.1 Theorem. *One has the equation*

$$\mu = i \cdot \delta_0 + \sum_{k=1}^{\infty} \delta_{-2k}$$

The proof follows easily from (1) above. Let us apply the forula in Theorem XX starting from a function $g(\xi)$which belongs to $\mathcal{S}$ on the ξ-line. Put

$$f(x) = \frac{1}{2\pi} \int e^{ix\xi} \cdot g(\xi)\, d\xi$$

Fourier's inversion formula gives the equality

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\sin(x+i\epsilon)}{\cos(x+i\epsilon)} \cdot f(x) \cdot dx = i \cdot g(0) + 2i \cdot \sum_{k=1}^{\infty} g(-2k) \tag{3}$$

A special case occurs of $f(x)$ is a test-function with compact support in an interval $[-a,a]$ where $0 < a < \pi/2$. Then (3) corresponds to the Fourier-Poisson formula:

$$\int_{-\pi/2}^{\pi/2} \frac{\sin(x)}{\cos(x)} \cdot f(x) \cdot dx = i \cdot \widehat{f}(0) + 2i \cdot \sum_{k=1}^{\infty} \widehat{f}(-2k) \tag{*}$$

A special case. Let $f(x) = \cos x \cdot \sin x \cdot e^{-x^2}$. This function belongs to $\mathcal{S}$ and the left hand side in (3) becomes

$$2\int_0^{\infty} \sin^2 x \cdot e^{-x^2}\, dx$$

At the same time the reader can verify the equality

$$\widehat{f(\xi)} = -i \cdot \int_0^{\infty} \sin 2x \cdot \sin(x\xi) \cdot e^{-x^2}\, dx$$

So the right hand side in (3) becomes

$$-2\sum_{k=1}^{\infty} \int_0^{\infty} \sin 2x \cdot \sin\, 2kx \cdot e^{-x2}\, dx$$

Wave front sets of distributions.

Let μ be a distribution on the real line. It can be multipleid with test-functions and to analyze local properties of μ around a point x_0 we use test-functions g which are identically one in some interval centered at x_0 and vanish outside a slightly larger interval. Now $g\mu$ has compact support and its Fourier transform yields a real-analytic function of ξ which extends to an entire function of the complex variable $\zeta = \xi + i\eta$. Let N be a positive integer and suppose there exists a constant C such that

$$|\widehat{g\mu}(\xi)| \leq C \cdot [1 + |\xi|)^{-N-2} \tag{1}$$

Fourier's inversion formula implies that the distribution $g\mu$ is a densitiy expressed by a function which is at least N times differentiable. In fact, this follows from Fourier's inversion formula which gives

$$\frac{d^N g\mu}{dx^N}(x) = \frac{i^N}{2\pi} \cdot \int e^{ix\xi} \cdot \xi^N \cdot \widehat{g\mu}(\xi)\, d\xi \tag{2}$$

where (1) entails that the integral in (2) converges. When g as above is identically one in a neighborhood of x_0 this entils that μ restricts to an N times differentiable function in an interval around x_0. So the decay in (i1) yields a local regularity of μ.

1. A class of cut-off functions. Consider a closed interval $[a, b]$ and let $\delta > 0$. Fix a test-function $\rho(x)$ with the property that $\rho \geq 0$ and has support in $[-1, 1]$. Moreover, $\rho(x)$ is an even function and

$$\int_{-1}^{1} \rho(x)\, dx = 1 \tag{i}$$

Let N be a postive integer and $\delta > 0$ is so small that $\delta < b - a$. We can construct the N-fold convolution of the function $\rho(Nx/\delta)$ and convolve this with the characteristic function of (a, b). Set

$$g_N(x) = \chi_{(a,b)} * \rho(Nx/\delta) * \ldots * \rho(Nx/\delta)$$

Now $\rho(Nx/\delta) = 0$ when $|x| \geq \delta/N$. It follows that $g_N(x)$ is supported by the interval $[a - \delta, b + \delta]$. Moreover (i) entails that $g = 1$ on the interval $[a + \delta, b - \delta]$.

2. Exercise. Let C be the maximum norm of the first order derivative of ρ over $[-1, 1]$. Show that the derivatives of g_N up to order N satisfy

$$|g_N^{(\nu)}(x)| \leq C^\nu \quad : 0 \leq \nu \leq N \tag{2.1}$$

3. A criterion for real-analyticity. Let μ be a distribution whose restriction to an interval containing $[a - \delta, b + \delta]$ is a real-analytic density $f(x)$. This gives a constant $C(f)$ such that the higher order derivatives satisfy the inequalites below when $a - \delta \leq x \leq b + \delta$.

$$|f^{(\nu)}(x)| \leq C(f)^\nu \cdot \nu! \quad : \nu = 0, 1, 2$$

If $N \geq 1$ we consider $g_N f$ and Leibniz's rule gives

$$\partial^N (g_N f)(x) = \sum \binom{N}{\nu} \cdot g_N^{(\nu)}(x) \cdot f^{(N-\nu)}(x)$$

By (2.1) and (ii) the absolute value of the right hand side is majorized by

$$N! \cdot \sum_{\nu=0}^{\nu=N} \frac{(CN/\delta)^\nu}{\nu!} \cdot C(f)^{N-\nu} \tag{3.1}$$

Put

$$C(f)_* = \max\{C(f), C\}$$

Then (3.1) is majorized by

$$C(f)_*^N \cdot N\,! \cdot \sum_{\nu=0}^{\nu=N} \frac{(N/\delta)^\nu}{\nu!} \leq C(f)_*^N \cdot N\,! \cdot e^{N/\delta}$$

So if we set

$$C_* = C(f)_* \cdot e^{1/\delta}$$

we get

$$|\partial^N (g_N f)(x)| \leq C_*^N \cdot N\,! \tag{3.2}$$

This inequality hold for all x in the support of $g_N f$. Passing to the Fourier transform we take derivatives up to order N and conclude that

$$|\xi|^N \cdot |\widehat{g_N f}(\xi)| \leq C_*^N \cdot N\,! \tag{3.2.*}$$

4. A converse result. Let μ be a distribution defined in an interval containing $[a, b]$. With δ sufficiently small the products $\{g_N \mu\}$ are distributions with compact support. Let us assume that there exists a constant C and some non-negative integer k such that

$$|\widehat{g_N \mu}(\xi)| \leq C^N (1 + |\xi|)^{-N+k} \cdot N! \quad : N = 1, 2, \ldots \tag{4.1}$$

If $N \geq k + 2$ Fourier's inversion formula gives

$$\partial^{N-k-2}(g_N \mu)(x) = \frac{i^{N-k-2}}{2\pi} \int \xi^{N-k-2} \cdot \widehat{g_N \mu}(\xi)\, d\xi \tag{i}$$

By (4.1) the absolute value of the right hand side is majorized by

$$C^N \cdot N! \cdot \frac{1}{2\pi} \int |\xi|^{N-k-2} \cdot (1 + |\xi|)^{-N+k}\, d\xi$$

The last integral is bounded by $\int (1 + |\xi|)^{-2}\, d\xi$ so with an absolute constant C_0 we have

$$|\partial^{N-k-2}(g_N \mu)(x)| \leq C_0 \cdot C^N \cdot N! \tag{ii}$$

With $m = N - k - 2$ it entails that

$$|g \mu_N)^{(m)}(x)| \leq C_0 \cdot C^{m+k+2} \cdot (m + k + 2)! \tag{iii}$$

Here (iii) hold when $a + \delta < x < b - \delta$ and since $g_N = 1$ on this interval we obtain estimates for the higher order derivatives of $\mu^{[m)}$ over this interval and we leave it to the reader to check that (iii) entails that μ restricts to a real-analytic density on $(a + \delta, b - \delta)$.

5. One-sided decay conditions. In (4.1) we allow that ξ tends both to $+\infty$ and $-\infty$. A weaker condition is to impose (3.2.*) when $\xi \geq 0$ which means that there exists a constant C_* and some non-negative integer k such that

$$|\widehat{g_N \mu}(\xi)| \leq C_*^N \cdot N! \cdot (1 + \xi)^{-N+k} \quad : \xi \geq 0 \tag{5.1}$$

This time we cannot conclude that μ restricts to a real-analytic density. But we shall prove that μ is expressed as a boundary value distribution from a single half-plane. Let us first assume that

$$\mu = \phi(x + i0) \tag{i}$$

where $\phi(z)$ is an analytic function in a rectangle $\{a-\delta^* < x < b+\delta^*\} \times \{0 < y < b\}$ for some $\delta^* > \delta$ and there exist an integer M and a constant C such that

$$|\phi(x+iy)| \leq C \cdot y^{-M}$$

holds for all $x+iy$ in the rectangle.

5.2 Proposition. *Set $\gamma_N = g_N \cdot \phi(x+i0)$. Then (xx) entails that*

$$(**) \qquad |\widehat{\gamma_N}(\xi)| \leq C_*^N \cdot N! \cdot (1+\xi)^{-N+M} \quad : \xi \geq 0$$

CHECK constants...

6. Analytic wave front sets. Given a distribution μ defined on the interval $(a-\delta^*, b+\delta^*)$ we construct the compactly supported distributions $\{g_N\mu\}$. Now we get the two analytic functions in the upper and the lower half-plane

$$\text{(i)} \qquad \phi(z) = \frac{1}{2\pi}\int_0^\infty e^{ix\xi} \cdot \widehat{g_N\mu}(\xi)\, d\xi \quad : \quad \psi(z) = \frac{1}{2\pi}\int_{-\infty}^0 e^{ix\xi} \cdot \widehat{g_N\mu}(\xi)\, d\xi$$

On the interval $(a+\delta, b-\delta)$ we have the equation

$$\text{(ii)} \qquad g_N\mu = \phi(x+i0) + \psi(x-i0)$$

Let us then assume that the one-sided conditions (5.1) hold for $\{g_N\mu\}$. By Proposition 5.2 this one-sided condition also holds for $\{g_N\phi(x+i0)\}$. Hence (ii) entials that it also hold for $\{g_N \cdot \psi(x-i0)\}$. Now Propostion 5.2 applies to the boundsry value distribution $\psi(x-i0)$ with reversed sign of ξ and hence $\{g_N \cdot \psi(x-i0)\}$ satisfies the *two-sided* condition in (4.1)and then the result in § 3 entails that the boundary value distribution $\psi(x-i0)$ restricts to a real-analytic density on $(a+\delta, b-\delta)$. So on this interval μ is equal to the sum of a real-analytic density and $\phi(x+i0)$. The real-analytic density can be regarded as the boundary value distribution of an analytic function defined in a rectangle placed in the upper half-plane.

6.1 Conclusion. The one-sided condition (5.1) holds if and only if μ is a boundary value distribution of the form $\phi(x+i0)$ on the interval $(a+\delta, b-\delta)$.

6.2 Remark. The conclusion above leads to the notion of analytic wave front sets. More precisely, let μ be a distribution on some interval (a,b). To each $a < x < b$ one takes cut-off functions $\{g_N\}$ as above supported by a small interval centered at x and checks if one-sided decay conditions hold. If none of them hold one says that the whole the analytic wave front fiber of μ above x contains both dx and $-dx$. If the two-sided decay condition hold the analytic wave front fiber above x is empty. In between it may occur that μ is locally a boundary value function $\phi(x+i0)$ or $\psi(x-i0)$. The "bad set of points" x for which dx and $-dx$ both belong to the analytic wave front fiber is a closed subset of (a,b) denoted by $\sigma(\mu)$. The set $\omega(\mu)$ where the analytic wave front fiber is empty is an open set which consists of intervals on which μ restricts to a real-analytic density. The set ω_- where $-dx$ is outside the analytic wave front fiber is open. Put

$$\sigma^+(\mu) = \omega_- \setminus \omega$$

This is the set of points x such that the analytic wave front fiber above x is dx. In the same way $\sigma^-(\mu)$ is the set of points whose analytic wave front fiber is $-dx$. Now we have a disjoint union

$$(*) \qquad \sigma(\mu) \cup \sigma^+(\mu) \cup, \sigma^+(\mu) \cup \omega(\mu)$$

We refer to (*) as an analytic spectral decomposition of μ. Taking boundary values of analytic functions one can exhibit many examples of such decompositions. The construction of analytic wave front fibers for distributions in $\mathbf{R}^n$ with $n \geq 2$ can be carried out in a similar fashion. Here boundary values of analytic functions defined in so called truncated tube domains are used and leads to the analytic singular spectrum to be denoted by $\mathrm{SS}_A(\mu)$ for a distribution defined in some open set in $\mathbf{R}^n$ where the analytic wave front fiber above a point $x_0 \in \Omega$ is a (possibly empty) closed subset of the unit sphere in the n-dimensional ξ-space. The precise construction of the analytic singular spectrum is diue to M, Sato. using the Fourier transform and cut-off functions another analytic wave front set denoted by $\mathrm{WF}_A(\mu)$ was constructed by Hörmander in [Hörmander]. The equality $\mathrm{WF}_A(\mu) = \mathrm{SS}_A(\mu)$ in dimension $n \geq 2$ is not as easy to prove as in the case $n = 1$. It was proved by Bros-Iagonlitser in [xx]. A third construction of analytic wave front sets was introduced by Sjöstrand in [Sjöstrand] which for many applications in PDE-theory serves as the optimal notion of analytic wave front sets. The reader may consult Chapter 8 in [Hörmander] for an account about analytic wave front sets of distributions in arbitrary dimension.

STRUL

wave front sets: General $\mu = \phi(x + i0)$. If $g(x)$ test function compact support $[-a, a]$. Then $g \cdot \mu$ has a Fourier transofrm. Decay conditions can be analyzed using integrals in § xx. Estimate with a positive integer N and

$$|\widehat{g \cdot \mu)}(\xi)| = \int e^{-iz\xi} \cdot xxx$$

We have a constant $b > 0$ where $\phi(z)$ was analytic and get a decay estimate via norm on g. Point is a factor while $z = x + ib$:

$$|e^{-iz\xi}| = e^{b\xi}$$

hence good decay formula. In his way by good cut-off functions a rapid decay found. The converse: Just μ given and rapid decay in $\xi < 0$. Then the inverse Fourier transform of μ is good. To start μ has compact support done before cutting... Point is that half-inverse transform

$$\mu_*(x) = \int e^{ix\xi}\,\widehat{mu}(\xi)\,d\xi$$

decreases so well that $\mu_*(x)$ is real-analyic and μ is a real-analyitc function plus its inverse of non-negative Fourier transform which extends analytically and is a boundary value distribution

From this principle of linear independence property.

and assume that $\phi(z)$ extends analytically while $x \neq 0$ on both sides. But at $x = 0$ we encounter a singularity ,i.e. ϕ does not extend to be holomophic in the whole disc. Now this is recaputered by properties. Choose cut-off function $g(x)$ and $g \cdot \mu$ has a compact support. The Fourier transform exists

$$\gamma(\xi) = \mu(eix\xi \cdot g(x))$$

We know that is is found via a limit value. and one finds a rapid decay in one direction but not in the other. For if $\widehat{mu}$ decreases fast as $\xi \to +\infty$ a good picture avaliable

$$\phi(z) = \int_0^\infty e^{iz\xi}\,\widehat{\mu}(\xi)\,d\xi$$

Now $z = x + iy$ and $y > 0$ gives nice convergence. Opposed directions... Want is really to bound derivatives of the inverse Fourier transform. So at stake is

$$\int |\xi|^N \cdot |\widehat{\mu})\xi)|\,d\xi$$

So a bound lie

$$|\widehat{\mu})(\xi)| \leq C_N(1 + |\xi|)^{-N-2}$$

Then derivative bound is okay. Good analytic case occurs if

$$C_N \leq \frac{c^N}{N!}$$

can be achieved. Point is that we can achieve it via good cut-off functions when $\mu = \phi(x + i0)$ and it becomes intrinsic. This settles everyhing. So in this way the notion of wave front sets is good.

Wee consider an ODE-operator

$$Q(x,\partial)=q_m(x)\partial^m+\ldots+q_1(x)\partial+q_0(x)$$

Assume that the zeros of q_m are real and simple and arranged by increasing order $a_1<\ldots<a_k$. For each $1\leq\nu\leq k$ we have $q_m(a_\nu+z)=(z-a_\nu)\cdot\rho(z)$ where $\rho(0)\neq 0$. Passing to the oocal ring $\mathcal{O}$ of germs of analytic functions at a_ν we can invert ρ and write

$$Q=\rho^{-1}(z)[z\partial^m+g_{m-1}(z)\partial^{m-1}+\ldots+g-0(z)]$$

Here Q is identfied with an element in the ring $\mathcal{D}$ of germs of holomorphic differential operators at $z=0$. Malgransge's index formula from § xx shows that the Q-kerenl on $\mathcal{O}$ is either $m-1$ or m. The case when the Q-kernel is m-dimensional is rather exceptional. See § xx for examples. Let us now consider the case when the kernel has dimension $m-1$. Choose a small disc D centered at $\{z=0\}$ where the g-functions are analytic. We have the open half-disc D^+ and the lower open half-disc D_-. As explained in § xx the Q-kernel on $\mathcal{O}(D^+)$ is m-dimensional and similasrly the Q-kernel on $\mathcal{O}(D^+)$ is m-dimensional. In addition we have $(m-19$ many $\mathbf{C}$-linearly independent functions $f-1,\ldots,f_{m-1}$ in $\mathcal{O}(D)$ such that $Q(f_\nu)=0$. It follows that we can pick $\phi\in\mathcal{O}(D^+)$ which together with the f-functions is a basi for the m-dimensional Q-kernel on $\mathcal{O}(D^+)$. Similarly we pick $\psi\in\mathcal{O}(D_-)$ which together with the f-functions give a basis of the Q-kernel on $\mathcal{O}(D_-)$. If $0<x<r$ we pass to boundary values and find unique constants $a,c_1,\ldots,c_{m-1}$ such that

$$\phi(x)=a\psi(x)+\sum c_\nu\cdot f_\nu(x)\quad:0<x<r$$

On the interval $(-r,0)$ we find unique constants $b,d_1,\ldots,d-m-1$ where find constants $a,c_1,\ldots,c_{m-1}$ such that

$$\phi(x)=b\psi(x)+\sum d_\nu\cdot f_\nu(x)\quad:\;-r<x<0$$

Let us consider the boundary value distribution

$$\mu=\phi(x+i0)-a\cdot\psi(x-i0)-\sum c_\nu\cdot f_\nu(x)$$

From (i) it vanishes on $(0,r)$. On the other hand it cannot vanish identically on $(-r,r)$. The reason is that the wave front set of the boundsry value distribution $\phi(x+i0)$ is non-empty because ϕ does not extend to be analytic acriss $z=0$ and as explained in § xx this wave front set is a half-line $(0,\mathbf{R}^+dx)$. Similarly the wave front set of $\psi(x-i0)$ is the opposed half-line $(0,\mathbf{R}_-dx)$. They cannot cancel each other. So the wave front set of μ contains $(0,\mathbf{R}^+dx)$ and it also contains $(0,\mathbf{R}_-dx)$ since $a\neq 0$ by § xx above. The conclusion is that μ is a distribution supported by $[-r,0]$ and $\{x=0\}$ belongs to its support. At the same time $Q(\mu)=0$ since both $\phi(x+i0)$ and $\psi(x-i0)$ belong to the Q-kernel.

Conclusion. We have found the distribution μ supported by the half-interval $[-r,0]$ which satisfies $Q(\mu)=0$. In addition $f_1,\ldots,f_{m-1}$ and $\phi(x+i0)$ are distributions which belong to the Q-kernel. The whole discussion above gives:

Proposition. *The Q-kernel on $\mathfrak{Db}(-r,r)$ is $m+1$-dimensional whose basis is $\{\mu,\phi(x+i0),f_1,\ldots,f_{m-1}\}$.*

Passing to the whole real x-line we notice that $\phi(x+i0)$ and $\psi(x-i0)$ are globally defined distributions and the same hold for $\{f_\nu(x+i0)\}$. In particular μ is a distribution defined on the whole real x-line with support in $(-\infty,a_1]$ and $Q(\mu)=$ holds. Now we can proceed to the next zero a_2 and repeat the argument under the assumption that the Q-kernel on the space of germs of analytic functions at a_2 also is $(m-1)$-dimensional. At a_2 we find locally a new distribution supported by a half-interval to the left of a_2 and extends to the whole real line where it is supported by $(-\infty, a_2]$ and annihilated by Q. We can continue to later roots of q_m and arrive at the followewing conclusive result:

Theorem. *If the Q-kernels are $(m-1)$-dimensional at $\mathbf{C}\{z-a_\nu\}$ for every zero of q_m it follows that the Q-kernel on $\mathfrak{Db}(\mathbf{R})$ is $m+k$-dimensional. A basis consists of a k-tuple of distributions μ_ν where Supp$(\mu_\nu) \subset (-\infty, a_\nu]$. In addition the Q-kernel consists of an m-dimensional space given by boundary value distributions $\{\phi_j(x+i0)\}$ where $\{\phi_j(z)\}$ are analytic in the upper half-plane where they solve $Q(z,\partial_z)(\phi_j)=0$.*

The case when q_m has multiple zeros. Suppose that q_m has real zeros but this time we allow multiple zeros. If a is a real zero of some multilplicity $e \geq 2$ with $k \leq m$ then Malgrange's index formula appkies to the germ of Q in the ring of differential operators with holomorphic coefficients in the local ring $\mathbf{C}\{z-a\}$. It follows that the Q-kernel has dimension between $m-e$ and m. Suppose the Q-kernel on $\mathbf{C}\{z-a\}$ is $m-e$-dimensional. In this case one find a e-tuple of linearly independent distribution solutions $\mu_1, \ldots, \mu_k\}$ supported by $(a-r, a]$ where $Q(\mu_j) = 0$ Passing to the global case one finds exactly as in Theorem xx that the Q-kernel on $\mathfrak{Db}(\mathbf{R})$ has dimension $m+k$. Here each zero a_ν of q_m with multiplicity $e_\nu \geq 2$ produces e_ν many linearly independent distribution solutions supported by the half-line $(-\infty, a_\nu]$.

The general case. When q_m has a zero at a real point a of multiplicity $e \geq 2$ one encounters new phenomena when the Q-kernel on $\mathbf{C}\{z-a\}$ differs from $m-e$. Then one simply finds more μ-distributions.... Arrive at a formua for the Q-kernel on the space of distributions provided that Q is Fuchsian at a. The case when Q no longer is Fuchsian leads to a more involved situation. The reason is that not all holomorphic solutions in the discs D^+ and D_- have moderate growth as we apporach the real x-line and therefore may fail to give boundsry value distributions. A study of this general case occurs work by Leif Svensson and Malgrange.

Example. Consider the operator $Q = x^2\partial + 1$. Outside $x = 0$ we see that the solution is given by $e^{1/x}$. It fails to have moderate groeth when $x \to 0$ on the positive half-line whie it is a C^∞-fuction on $x < 0$ which extends to $x) = 0$ where all the derivatives are zero. So one finds a C^∞-solution given by a distribution supported by $x \leq 0$ while no other solutions in $\mathfrak{Db}(\mathbf{R})$ occur. So here the Q-kernel is 1-dimensional.

A local analytic situation

With the notations from § xx we consider the ring $\mathcal{D}$ whose elements are germs of holomorphic differential operators at $z = 0$. Consider such a differential operator given in the form:

$$Q = z\partial^m + q_{m-1}(z)\partial^{m-1} + \ldots + q_0(z)$$

So here $\{q_\nu(z)\}$ belong to the local ring $\mathcal{O}$. Malgrange's index formula shows that the Q-kernel on $\mathcal{O}$ has dimesnion $m-1$ or m and is $(m-1)$-dimensional if and only if Q is surjective on $\mathcal{O}$. Let us analyzie when surjectivity holds. Suppose that $m \geq 2$. Now $Q(\mathcal{O}) = \mathcal{O}$ if and only if the differential operator $S = z^{m-1} \cdot Q$ maps the ideal $z^{m-1}\mathcal{O}$ onto itself. Here

$$\text{(i)} \qquad S = z^m\partial^m + +q_{m-1}(z) \cdot z^{m-1}\partial^{m-1} + \sum_{\nu=0}^{\nu=m-2} q_\nu(z) \cdot z^{m-1} \cdot \partial^\nu$$

Let $\nabla = z\partial$ be the Fuchsian operator. If $q_{m-1}(0)$ is the constant term of $q_{m-1}(z)$ we see that (i) gives

$$\text{(ii)} \qquad S = z^m\partial^m + +q_{m-1}(0) \cdot z^{m-1}\partial^{m-1} + \sum_{\nu=0}^{\nu=m-1} \rho_\nu(z) \cdot \nabla^\nu$$

where the ρ-functions have constant term equal to zero. If $k \geq m-1$ we have

$$\text{(iii)} \qquad (z^m\partial^m + +q_{m-1}(0) \cdot z^{m-1}\partial^{m-1})(z^k) = c(k;m) \cdot z^k$$

A computation shows that $c(m-1;m) = (m-1)! \cdot q_{m-1}(0)$ and

$$\text{(iv)} \qquad c(k;m) = k(k-1)\cdots(k-m+2)(k-m+1+q_{m-1}(0)) \quad : k \geq m$$

Since the differential operator $\sum_{\nu=0}^{\nu=m-1} \rho_\nu(z) \cdot \nabla^\nu$ sends z^k into the ideal generated by z^{k+1} it follows from the general result in § xx that the equality $S(z^k\mathcal{O} = z^k\mathcal{O}$ holds if and ony if $c(k;m) \neq 0$ for every $k \geq m-1$. From (iv) this holds if and only if

$$(*) \qquad q_{m-1}(0) \in \mathbf{C} \setminus \{0, -1, -2, \ldots\}$$

So (*) gives a necessary and sufficient condition in order that the Q-kernel has dimension $(m-1)$.

Euler-Riemann distributions.

Introduction. We shall construct some meromorphic families of distributions which arise via boundary values of analytic functions. We shall refer to these as Euler-Roemann distributions since their constructions rely upon methods by these authors. In Riemann's work one encounters calculations of Fourier transforms which go beyond those below and they were in turn inspired by original discoveries due to Euler. With λ as a complex parameter we have the distribution-valued function $\lambda \mapsto x_+^\lambda$ and its Fourier transform satisfies the Euler-Riemann equation

$$(*) \qquad \widehat{x_+^\lambda} = \Gamma(\lambda+1) \cdot (i(\xi - i0))^{-\lambda-1}$$

Similarly we have

$$(**) \qquad \widehat{x_-^\lambda} = \Gamma(\lambda+1) \cdot (i(\xi + i0))^{-\lambda-1}$$

Next, when λ and μ are independent complex parameters we seek convolutions

$$x_+^\lambda * x_-^\mu$$

Passing to the Fourier transforms this amounts to construct products

$$\text{(i)} \qquad (\xi - i0)^{-\lambda-1} \cdot (\xi + i0)^{-\mu-1}$$

In general the product of two distributions is not defined. But if the real parts of λ and μ both are < -1 then (i) is a product of two continuous density functions and Fourier's inversion formula gives a well defined distribution

$$\frac{1}{\Gamma(\lambda+1)\cdot\Gamma(\mu+1)} \cdot x_+^\lambda * x_-^\mu \quad : (\lambda,\mu) \in \{\mathfrak{Re}\,\lambda < -1\} \times \{\mathfrak{Re}\,\mu < -1\}$$

It turns out that we have an extension to the whole complex space $\mathbf{C}^2$. In the theorem below we use the negative Heaviside function $H_-(\xi)$ on the real ξ-line.

Theorem. *There exists an entire distribution valued function of two independent complex parameters defined by*

$$(\lambda,\mu) \mapsto \frac{1}{\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(-\lambda-\mu-1)} \cdot x_+^\lambda * x_-^\mu$$

Proof. If α and β are two complex numbers one has the equation

$$\text{(i)} \qquad (\xi - i0)^\alpha = (\xi + i0)^\alpha - 2i \sin \pi\alpha \cdot H_-(\xi) \cdot |\xi|^\alpha$$

Hence we can write

$$(\xi - i0)^\alpha \cdot (\xi + i0)^\beta =$$
$$(\xi + i0)^{\alpha+\beta} - 2i \sin \pi\alpha \cdot H_-(\xi) \cdot |\xi|^\alpha \cdot (\xi + i0)^\beta$$

The last product is for example defined when $\mathfrak{Re}\,\alpha > -1$ and $\mathfrak{Re}\,\beta > 0$ for then we take a product of a continuous density and a locally integrable function and here

$$H_-(\xi) \cdot |\xi|^\alpha \cdot (\xi + i0)^\beta = e^{\pi i\beta} \cdot |\xi|^{\alpha+\beta} \cdot H_-(\xi)$$

By the result in § xx there exists the entire distribution valued function

$$(\alpha,\beta) \mapsto \frac{1}{\Gamma(\alpha+\beta+1)} \cdot e^{\pi i\beta} \cdot |\xi|^{\alpha+\beta} \cdot H_-(\xi)$$

At the same time $(\alpha,\beta) \mapsto (\xi - i0)^{\alpha+\beta}$ is entire. From this and the Euler-Riemann equation (*) we can read off theorem xx.

The Fourier transform. From the above the Euler-Riemann equation entails that the Fourier transform of the distributions in Theorem xx become

$$\frac{i^{-\lambda-\mu-2}}{\Gamma(-\lambda-\mu-1)} \cdot \big[\frac{1}{i(\xi - i0)}\big]^{\lambda+\mu+2} + \frac{2i \cdot \sin(-\pi\lambda - \pi)}{\Gamma(-\lambda-\mu-1)} \cdot e^{-\pi i(\mu+1)} \cdot |\xi|^{-\lambda-\mu-2} \cdot H_-(\xi)\big]$$

Example. Consider the case $\lambda + \mu + 2 = 0$. Since $\frac{1}{\Gamma(1)} = 1$ the Fourier transform above is a sum of two Heaviside functions. With $\lambda = -1 + is$ and $\mu = -1 - is$ we encounter the distribution

$$\frac{1}{\Gamma(is) \cdot \Gamma(-is)} \cdot x_+^{-1+is} * x_-^{-1-is}$$

Keeping $s|neq0$ fixed the convolution $x_+^{-1+is} * x_-^{-1-is}$ is the distribution defined on test-functions g by the limit formula

$$\lim_{\epsilon \to 0} \iint g(x+y) \cdot x^{-1+\epsilon+is} \cdot y^{-1+\epsilon-is} \, dxdy$$

where the double integral is taken over the quadrant $\{x \geq 0\} \times \{y \leq 0\}$. The reader is invited to investiage thiese double integrals for specific choice of g-functions and also analyze the limit as $s \to 0$ where

$$\frac{1}{\Gamma(is) \cdot \Gamma(-is)} \simeq s^2$$

A more general case. Let $q(\xi)$ and $p(\xi)$ be a pair of polynomials with simple real zeros. Cases where a common real zero occurs is not excluded. On the ξ-line we seek distributions expressed by products

$$q(\xi + i0)^\lambda \cdot p(\xi - i0)^\mu$$

To define such products we denote by $a_1 < a_2 < \ldots < a_m$ the simple real zeros of $q(\xi)$ and suppose that $q(\xi) < 0$ when $\xi < a_1$ while $q(\xi) > 0$ if $\xi > a_m$. On intervals where q is negative we have

$$q(\xi + i0)^\lambda = q(\xi - i0)^\lambda \cdot 2i \sin \pi\lambda \implies$$

$$q(\xi + i0)^\lambda = q(\xi - i0)^\lambda + |q(\xi)|^\lambda \cdot H_{(-\infty, a_1]}(\xi) + \sum 2i \sin \pi\lambda \cdot |q(\xi)|^\lambda \cdot \chi_k(\xi)$$

where $\{\chi_k(\xi)\}$ are the characteristic functions of the intervals $\{[a_{2k}, a_{2k+1}]\}$ on which $q < 0$ Now $q(\xi - i0)^\lambda \cdot p(\xi - i0)^\mu$ is an entire distribution valued function in $\mathbf{C}^2$ and we from the above we are led to study distribution valued functions

$$(*) \qquad \phi(\lambda, \mu) = \chi_{a,b}(\xi) \cdot |q(\xi)|^\lambda \cdot p(\xi - i0)^\mu$$

where $q(a) = q(b) = 0$ and $q \neq 0$ on the open interval (a, b). Here it may occur that p has zeros at a or b and also at points in the open interval (a, b). Here (*) is defined when $\mathfrak{Re}(\lambda)$ and $\mathfrak{Re}(\mu)$ both are > 0, i.e, then we just have a product of two continuous density functions. A meromorphic extension is found by Euler's equations. More precisely

$$\frac{\partial \phi}{\partial \xi}(\lambda + 1, \mu + 1) = (\lambda + 1) \cdot q'(\xi) \cdot \phi(\lambda, \mu + 1) + (\mu + 1) p'(\xi) \cdot \phi(\lambda + 1, \mu)$$

From this it follows that

$$\frac{1}{\Gamma(\lambda + 1)\Gamma(\mu + 1)} \cdot \phi(\lambda, \mu)$$

yields an entire distribution valued function.

Abel integrals.

Now we study distributions of a more involved nature. Define distributions on the real x-line by

$$\mu_\alpha(g) = \int_0^1 \frac{g(x)}{\sqrt{\alpha - x}} \, dx \quad : g \in C_0^\infty(\mathbf{R})$$

When the complex parameter α stays outside $[0,1]$ this yields a compactly supported distribution. Of course, one must also choose branches of $\sqrt{\alpha - x}$. In the exterior disc $|\alpha| > 1$ one has Newton's binomial series when $0 \leq x \leq 1$:

$$\frac{1}{\sqrt{1 - x/\alpha}} = 1 + \sum_{n=1}^{\infty} c_n \cdot \frac{x^n}{\alpha^n}$$

which yields the expansion

$$\mu_\alpha = \frac{1}{\sqrt{\alpha}} \cdot \left[1 + \sum_{n=1}^{\infty} \alpha^{-n} \cdot \gamma_n\right]$$

where $\{\gamma_n\}$ are the distributions defined by the densities $\{x^n\}$ restricted to $[0,1]$. The complex derivative with respect to α becomes

$$\frac{\partial \mu_\alpha}{\partial \alpha}(g) = -\frac{1}{2} \cdot \int_0^1 \frac{g(x)}{(\alpha - x)^{3/2}}\, dx$$

At the same time we can consider distribution derivatives. Here

$$\frac{d\mu_\alpha}{dx}(g) = -\int_0^1 \frac{g'(x)}{\sqrt{\alpha - x}}\, dx = -\frac{g(x)}{\sqrt{\alpha - x}}\Big|_0^1 + \frac{\partial \mu_\alpha}{\partial \alpha}(g)$$

Hence we have the distribution equation

$$\frac{d\mu_\alpha}{dx} = \frac{1}{\sqrt{\alpha}} \cdot \delta_0 - \frac{1}{\sqrt{\alpha - 1}} \cdot \delta_1 + \frac{\partial \mu_\alpha}{\partial \alpha}$$

Exercise. Use the above to show that $\alpha \mapsto \mu_\alpha$ extends to a distribution valued function in $\mathbf{C} \setminus \{0,1\}$ which is multi-valued.

Remark. In § xx we consider the construction from another point of view. Restricting α to real numbers in $[0,1]$ we have the integral operator

$$K_g(\alpha) = \int_0^1 \frac{g(x)}{\sqrt{\alpha - x}}\, dx$$

So here $K_g(\alpha)$ is a function of α and the operator is defined to begin with defined on all continuous g-functions on $[0,1]$. With the notstions above it means that the distribution μ_α expressed by the integrable density function $x \mapsto \frac{1}{\sqrt{\alpha - x}}$ supported by $0 \leq x \leq 1$ gives the equation

$$K_g(\alpha) = \mu_\alpha(g)$$

This illustrates that various integral operators arise from distribution valued functions which depend on the parameter which serves as a coordinate variable for the integrated function. In this context the inversion formulas in § xx give new light upon the analytic extensions of the μ-function above.

Local solutions.

Let $\mathcal{D}$ be the ring of germs of holomorphic differential operators at the origin. Malgrange's index formula asserts that if $m \geq 1$ and

$$Q = q_m(z)\partial^m + q_{m-1}(z)\partial^{m-1} + \ldots + q_0(z)$$

is an operator in $\mathcal{D}$ where the germ q_m has some order k then the analytic index

$$\mathrm{Ker}(Q) - Q(\mathcal{O}) = k - m$$

Consider the special case when $q_m(z) = z$. Then the index is $m-1$ and hence the kernel of Q is either m-dimensional or $(m-1)$-dimensional. The case when $\mathrm{Ker}(Q)$ is $(m-1)$-dimensional is rather special. To begin with, if $q_m(z) = z$ and $q_\nu(0) = 0$ for all $0 \leq \nu \leq m-1$ we can write

$$Q = z \cdot (\partial^m + r_{m-1}(z)\partial^{m-1} + \ldots + r_0(z))$$

where $\{r_j\}$ belong to $\mathcal{O}$ and then it is clear that the Q-kernel is m-dimensional. From now one we assume that $q_j(0) \neq 0$ for at least one $0 \leq j \leq m-1$. If this occurs for a specific j we notice that

$$Q(z^j) - j! \cdot q_j(0) \in \mathfrak{m}$$

where $\mathfrak{m}$ is the maximal ideal in the local ring $\mathcal{O}$. In other words, the image space $Q(\mathcal{O})$ contains a germ with a non-zero constant term. Before we announce Theorem 1 we set:

$$Q_* = (q_{m-1}(z) - q_{m-1}(0))\partial^{m-1} + \ldots + q_0(z)$$

If k is a non-negative integer we apply Q_* to monomials $\{z^j : 0 \leq j \leq m+k-1\}$ and write

$$Q_*(z^j) = \sum_{\nu=0}^{\infty} c_\nu(j) \cdot z^\nu$$

With these notations one has

Theorem. *The equality* $\dim(Ker(Q)) = m-1$ *holds if and only if there exists a non-negative integer* k *such that*

$$k + 1 + q_{m-1} = 0 \quad \text{and} \quad c_{k+1}(j) = 0 : 0 \leq j \leq m+k-1$$

Proof. Let $\mathfrak{m}$ be the maximal ideal in the local ring $\mathcal{O}$. To each $k \geq 0$ we have a constant $c(m:k)$ such that

$$\text{(i)} \qquad Q(z^{m+k}) - c(m;k)z^{k+1} \in \mathfrak{m}^{k+2}$$

where Leibniz's rule gives

$$c(m;k)z^{k+1} = z\partial^m(z^{m+k}) + q_{m-1}(0)\partial^{m-1}(z^{k+1})$$

It follows that $c(m;k) = 0$ if and only if $k + 1 + q_{m-1}(0) = 0$. If $c(m;k) \neq 0$ for all $k \geq 0$ then (i) entails that the Q-image contains $\mathfrak{m}$ On the other hand, if $k + 1 + q_{m-1}(0) = 0$ holds for some $k \geq 0$ it may occur that the $Q(\mathcal{O})$ has codimension one because the coefficient of z^{k+1} is zero for every holomorphic germ in the Q-image. This occurs of and only if the vanishing in right hand side of (*) holds. Hence (*) gives a necessary and sufficient condition in order that $Q(\mathcal{O}) = \mathcal{O}$ and Theorem xx follows from Malgrange's index formula.

Local distribution solutions. Let $\mathfrak{Db}$ be the space of germs of distributions at $x = 0$ on the real x-line. With $z = x + iy$ we consider $Q(x, \partial_x)$ as a differential operator on the real x-line where it acts on $\mathfrak{Db}$ and we seek the dimension of the solution space

$$\mathcal{S} = \{\mu \in \mathfrak{Db} : \; Q(\mu) = 0\}$$

Let $Q = z\partial^m + q_{m-1}(z)\partial^{m-1} + \ldots + q_0(z)$ be as above.

Proposition. *One has the equality*

$$\dim(\mathcal{S}) = 2m - \dim(\mathrm{Ker}(Q))$$

So if $\mathrm{Ker}(Q)$ is m-dimensional so is $\mathcal{S}$ and in this case the local distribution solutions are the real analytic functions on the x-line given by restrictions of the holomorphic solutions to Q. Then no-trivial case occurs if $\dim(\mathrm{Ker}(Q)) = m - 1$ where the Proposition asserts that $\mathcal{S}$ has dimension $m + 1$.

Proof of Proposition xx. We consider a small disc D centered at $z = 0$ where the q-functions in Q are holomorphic. In the upper half-disc $D_+ = D \cap$ $mathfrakImz > 0\}$ we find the m-dimensional subspace of $\mathcal{O}(D_+)$ of holomorphic solutions to $Q(\phi) = 0$. let $\phi_1, \ldots, \phi_m$ be a basis of this solution space. In the same way we consider the lower half-disc D_- and find the m-dimensional solution space in $\mathcal{O}(D_-)$ where $\psi_1, \ldots, \psi_m$ is a basis. We get boundary value distributions from both of these m-tuples. Suppose that the Q-kernel on $\mathcal{O}$ is $m-1$ where $f_1, \ldots, f_{m-1}$ is a basis. Above we can choose $|[\phi_n u\}$ to be $(f_1, \ldots, f_{m-1}, \phi_*)$ where $\phi_* \in \mathcal{O}(D_+)$ cannot be extended to an analytic function in D. It means that the boundary value distribution $\phi_*(x + i0)$ is not expressed by a real-analytic density. In the same way a basis in the solution space in $\mathcal{O}(D_-)$ is of the form $(f_1, \ldots, f_{m-1}, \psi_*)$ where ψ_* does not extend to be holomirphic in D which means that $\psi_*(x - i0)$ is not real-analytic. Now $\mathcal{S}$ contain the distributions $\{f_\nu(x)\}$ and the boundary value distributions $\phi_*(x + i0)$ and $\psi_*(x - i0)$. They are linearly independent so $\mathcal{S}$ has at least dimension $m + 1$ and the proof of Propostion xx is finished if we prove that this $(m + 1)$-tuple is a basis for $\mathcal{S}$. To attain this we consider first consider the restrictions of the distrinutions to the half-line $\{x > 0\}$. Here we find a unique m-tuple of complex constants such that

$$\psi_*(x - i0) = c_1 f_1(x) + \ldots + c_{m-1} f_{m-1}(x) + c_m \phi_*(x + i0) \quad : x > 0$$

Set

$$\mu_* = \psi_*(x - i0) - (c_1 f_1(x) + \ldots + c_{m-1} f_{m-1}(x) + c_m \phi_*(x + i0))$$

Then $\mu_* \in \mathcal{S}$ and it is supported by the half-line $\{x \leq 0\}$. Let us then consider some $\mu \in \mathcal{S}$. Again we find a unique m-tuple of constants such that

$$\mu = d_1 f_1(x) + \ldots + d_{m-1} f_{m-1}(x) + d_m \phi_*(x + i0) \quad : x > 0$$

Then $\gamma = \mu - (d_1 f_1(x) + \ldots + d_{m-1} f_{m-1}(x) + d_m \phi_*(x + i0))$ is a solution and supported by $\{x \leq 0\}$. Moreover we find constants $\{e_\nu\}$ such that

$$\gamma = e_1 f_1(x) + \ldots + e_{m-1} f_{m-1}(x) + e_m \mu_* \quad : \; x < 0$$

here $\gamma - e_m \mu_*$ is supported by $\{x \leq 0\}$ and since $e_1 f_1(x) + \ldots + e_{m-1} f_{m-1}(x)$ is a real-analytic density it must vanish identically which entails that $\gamma = e_m \mu_*$. This proves that a basis for $\mathcal{S}$ is given by the distributions $\{f_\nu\}$ and the pair $\phi(x+i0), \mu_*$.

Distribution solutions.

Introduction. We establish a dimensionforula for distribution solutions to a homogeneous equation $P(\mu) = 0$ where P is a Fuchsian operator. Theorem xx below is of course wellknoen and is a very special case of index formulas for regular holonomic systems in higher dimension. We shall give a proof using sheaf theory, i.e. basic facts from Leray's original work and we assume that the reader is familiar with basic notions such as the contruction of local cohomlogy with support and the spectral sequence associated to a composed derived functor. We remark that this "general nonsense" is easy to grasp and exposed in many text-books. The merit is that one does not need to carry out any comuations. The non-trivial part in the proof of Theorem xx below relies upon Malgrange's index forula for homolomorphic solutions and Sato's sheaf theoretic formula for distributions which expreses the fact that every distribution on the real line is represented in a unique was as the sum of two boundary value distributions.

Let m be a positive integers and consider a Fuchsian operator

$$P = \nabla^m + p_{m-1}(x)\nabla^{m-1} + \ldots + p_0(x) \tag{1}$$

where $\{p_\nu\}$ are holomorphic functions in some disc D centered at the origin. Let $\mathfrak{D}$ be the germs of distributions on the real x-line at $x = 0$ which gives a map

$$P \colon \mathfrak{D} \to \mathfrak{D}$$

The Fuchsian condition entails that P is surjective. In fact, this follows from Sato's formula which identifies the sheaf $\mathfrak{Db}$ on the real x-line with the temperate cohomology sheaf

$$\mathcal{H}^1_{[M]}(\mathcal{O}) \tag{*}$$

where $\mathcal{O}$ is the sheaf of holomorphic functions in the complex z-plane and M the real line defined by $\mathfrak{Im}\, z = 0$. Taking boundary values of analytic functions with moderate growth along the real axis this corresponds to the fact that every distribution is expressed by a sum

$$g(x + i0) + h(x - i0)$$

where $g(z)$ is holomorphic in an upper half-disc D_+ and $h(z)$ in a lower half-disc. The surjectivity in (1) then follows from a classical result due to Fuchs which asserts that when P is Fuchsian then it gives a surjective map on $\mathcal{O}_{\text{temp}}(D_+)$ and similarly on $\mathcal{O}_{\text{temp}}(D_-)$. For the P-kernel on $\mathfrak{D}$ one has

1. Theorem. *The complex vector space $Ker_P(\mathfrak{D})$ is $2m$-dimensional.*

To prove this we use the sheaf $\mathcal{D}$ of holomorphic differential operators in the complex z-plane and consider the solution complex

$$Sol(\mathcal{D}/\mathcal{D}P, \mathcal{O})$$

which by defintion is the derived sheaf complex

$$\mathbf{R}Hom_{\mathcal{D}}((\mathcal{D}/\mathcal{D}P, \mathcal{O}))$$

Since P is Fuchsian and we already know that P is surjective on $\mathfrak{Db}$ one has the interchange formula:

$$\text{Ker}_P(\mathfrak{D})[1] \simeq \mathbf{R}\Gamma_M(Sol(\mathcal{D}/\mathcal{D}P)(0) \tag{i}$$

where the right hand side is the stalk complex at the origin while a shift in degree one is used in the left hand side. Set

$$\mathcal{E}^\nu = Ext^\nu_{\mathcal{D}}((\mathcal{D}/\mathcal{D}P, \mathcal{O})$$

By Cauchy's classical result $P\colon \mathcal{O} \to \mathcal{O}$ is surjective outside the origin which entails that $\mathcal{E}^1$ is a scyscraper sheaf supported by the origin whose dimension is the codimension of $P(\mathcal{O})$. Malgrange's index formula for holomorphic solutions gives:

$$\mathcal{E}^1 = \mathbf{C}^\mu_{\{0\}} \quad : \mu = \dim(\mathcal{E}^0(0))$$

Next, to (i) one associates Laeray's spectral sequence whose second table consist of the sstalks

(ii) $$E_2^{p,q} = \mathcal{H}^p_M(\mathcal{E}^q)(0)$$

Since $\mathcal{E}^1$ is a scyscraper sheaf above we have

$$E_2^{0,1} \simeq \mathbf{C}^\mu_{\{0\}} \quad : \ E_2^{p,1} = 0 : \ p \geq 1$$

Next, the sheaf $\mathcal{E}^0$ is locally free of rank m outside the origin and obviously it does not contain sections supported by the real line M. It follows that

$$E_2^{p,0} = 0 : \ p \neq 1$$

Thus, only two non-zero terms appear in Leray's spectral sequence which both are placed on the diagonal of degree one and Sato's formula gives the equation:

(iii) $$\dim(\mathrm{Ker}_P(\mathfrak{D}) = dim(\mathcal{H}^1_M(\mathcal{E}^0)(0)) + \mu$$

Next, since μ is the dimension of the $\mathcal{O}$-kernel of P one has an exact sequence of sheaves

(iv) $$0 \to \mathbf{C}^\mu_X \to \mathcal{E}^0 \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is locally free of rank $m - \mu$ in a small punctured disc whose stalk at $\{z = 0\}$ vanishes. Now the stalk at $\{z = 0\}$ of $\mathcal{H}^1_M(\mathbf{C}^\mu_X)$ is μ-dimensional while the vanishing of the stalk $\mathcal{L}(0)$ entails that the stalk $\mathcal{H}^1_M(\mathcal{L})(0)$ has dimension $2m - 2\mu$ and Theorem 1 follows from (iii-iv).

A more general formula. Let m and k be a pair of positive integers and consider a differential operatator of the form

(1) $$P = x^k \cdot \partial^m + p_{m-1}(x)\partial^{m-1} + \ldots + p_0(x)$$

where $\{p_\nu(x)\}$ are germs of holomorphic functions at the origin and $p_\nu(=) \neq 0$ hold for at least one ν. We also impose the condition that P is Fuchsian in the sense of § xx. It follows as above that P is surjective on $\mathfrak{D}$ and there remain to determine the P-kernel on $\mathfrak{D}$. With the same notations as in the previous proof we apply Malgrange's index formula which gives

(i) $$\mathcal{E}^1 = \mathbf{C}^\mu_{\{0\}} \quad : \ \mu = \dim(\mathcal{E}^0(0)) + k - m$$

Next, we have exactly as above

(ii) $$\dim(\mathrm{Ker}_P(\mathfrak{D})) = \mu + \dim(\mathcal{H}^1_M(\mathcal{E})(0)$$

Arguing as in (iv) above we find that

(iii) $$\dim(\mathcal{H}^1_M(\mathcal{E})(0) = 2m - \dim(\mathcal{E}^0(0))$$

From this we obtain

$$\dim(\mathrm{Ker}_P(\mathfrak{D})) = m + k$$

Example. Consider the operator

$$P = x\partial^2 - 1$$

It has the holomorphic solution $x^2 + 2x$ and the reader can check that $P\colon \mathcal{O} \to \mathcal{O}$ is surjective so Malgrange's index formula implies that this polynomial generates the holomorphic P-kernel. In the upper half-plane we have the analytic function $\phi(z)$ which solves $P(\phi) = 0$ given by

$$\phi(z) = \int_i^z \log \zeta \, d\zeta$$

where a single valued branch of the log-function has been used. In the same way we find an analytic function ψ in the lower half-plane where we again employ a single-valued branch of the complex log-function. The 3-dimensional P-kernel on $\mathfrak{D}$ has a basis given by the analytic density $x^2 - 2x$ and the boundary value distributions $\phi(x + i0)$ and $\psi(x - i0)$.

Example. Let

$$P = x\partial^2 + q_1(x)\partial + q_0(x)$$

be a second order Fuchsian operator and this time we suppose that the holomorphic P-kernel is 2-dimensional.

The operator $x^2\partial^2 + q(x)x\partial + p(x)$