Chapter 4: Multi-valued analytic functions

Introduction.

In § 1 we define winding numbers of curves in $\mathbf{R^2}$ where no complex variables appear. After his we study complex line integrals where Theorem 1.7 which is the first main result in this chapter. The second main result is Theorem 2.1, referred to as the *argument principle*. It gives a gateway to find zeros of an analytic function since the counting function of its zeros is expressed by winding numbers arising from the image under the given function of closed boundary curves to a domain where we seek the number of zeros of f.

§ 3 starts with a construction due to Weierstrass and leads to the total analytic continuation of germs of analytic functions. A picture arises when we consider the *total sheaf space* $\widehat{\mathcal{O}}$ whose stalks are germs of analytic functions at points in $\mathbf{C}$. This topological manifold is locally homemorphic to small discs in $\mathbf{C}$ which express germs of multi-valued functions f. More precisely, if ρ denotes the local homemorphism from $\widehat{\mathcal{O}}$ onto $\mathbf{C}$, then there is a 1-1 correspondence between connected open subsets of $\rho^{-1}(\Omega)$ and the class of multi-valued analytic functions in Ω, where on does not exclude those functions f which may have analytic continuation to larger sets. Using the Weierstrass procedure to perform analytic continuation along a curve by regarding local power series in small discs placed in succesion along the curve, one can prove that various classes of multi-valued analytic functions are *normal* in the sense of Montel. In other words, results for single-valued analytic functions from § X in Chapter III can be extended to the multi-vaued case.

In § 4 we prove the *Monodromy Theorem* and describe multi-valued functions in a punctured disc. After this we insert material from topology in § 5 and announce the uniformisation theorem in § 6. In §7 we construct the $\mathbf{p}^*$-function which will be used to solve the Dirichlet problem in Chapter 5 and § 8 is devoted to a result due to Eisenstein. The proof is instructive since it teaches how to manipulate with multiple-valued algebraic root functions. In § 9 we recall the *Spiegelungsprinzip* by Hermann Schwarz which is applied in § 10 to construct the *modular function* and after can be used to establish the uniformisation theorem for domains in $\mathbf{C}$. A special local study about algebraic functions is made in § 11 where the results are used to construct Riemann surfaces of algebraic function fields. The final Section 12 contains a brief exposittion about contributions by Poincaré, foremost his constructions of Fuchsian groups.

1. Angular variation and Winding numbers

Let (x, y) be the coordinates in $\mathbf{R}^2$ and consider a vector-valued function of the real parameter t:

$$t \mapsto (x(t), y(t)) \quad : \quad 0 \leq t \leq T$$

We assume that the image does not contain the origin, i.e. $x^2(t) + y^2(t) > 0$ and $x(t)$ and $y(t)$ are both continuously differentiable. Set

$$\dot{x}(t) = \frac{dx}{dt} \quad \text{and} \quad \dot{y}(t) = \frac{dy}{dt} \quad \text{and} \quad r(t) = \sqrt{x(t)^2 + y(t)^2}$$

1.1 Proposition. *There exists a unique continuous map $t \mapsto \phi(t)$ such that*

(*) $$x(t) = r(t) \cdot \cos\phi(t) \quad : \quad y(t) = r(t) \cdot \sin\phi(t) \quad : 0 \leq t \leq T$$

where the ϕ-function satisfies the initial condition:

$$x(0) = r(0) \cdot \cos\phi(0) \quad \text{and} \quad y(0) = r(0) \cdot \sin\phi(0$$

Proof. We can solve the first order ODE-equation.

$$\dot{\phi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}$$

with initial condition $\phi(0)$ as above. There remains to show that (*) holds for all t. Let us for example verify the $x(t) = r(t) \cdot \cos\phi(t)$. It suffices to prove that

(1) $$\frac{d}{dt}(r(t) \cdot \cos\phi(t)) = \dot{x}$$

To prove this we notice that the left hand side becomes

(2) $$\dot{r} \cdot \cos(\phi) - r \cdot \sin(\phi)\dot{\phi} = \frac{\dot{r} \cdot x}{r} - y \cdot \dot{\phi}$$

Next, since $r = \sqrt{x^2 + y^2}$ we have

$$r\dot{r} = x\dot{x} + y\dot{y}$$

Hence (2) is equal to

$$\frac{x^2\dot{x} + xy\dot{y}}{r^2} - \frac{y(x\dot{y} - y\dot{x})}{r^2} = \frac{(x^2 + y^2)\dot{x}}{r^2} = \dot{x}$$

This proves Proposition 1.1

1.2 The angular variation. Since the sine and the cosine functions are 2π-periodic, the ϕ-function above is only uniquely determined up to integer multiples of 2π. A specific choice arises from the value $\phi(0)$. However, we get an *intrinsic number* by the difference

(**) $$\phi(T) - \phi(0)$$

This number is called the *angular variation* of the function $t \mapsto (x(t), y(t))$. If we choose another parametrization where $t = t(\tau)$ is non-decreasing and $0 \leq \tau \leq T^*$, then we start with the vector valued function $\tau \mapsto x(t(\tau), y(t(\tau))$ and find $\phi(\tau)$. Calculus shows that (**) is the same. Thus, the angular variation of an oriented parametrized C^1-curve is intrinsically defined.

A notation. The angular variation along a parametrized curve γ is denoted by $\mathfrak{a}(\gamma)$. If γ is a curve we can construct the curve γ^* with the opposite direction:

$$\gamma^*(t) = \gamma(T - t)$$

It is clear that

$$\mathfrak{a}(\gamma^*) = -\mathfrak{a}(\gamma)$$

In other words, up to a sign the angular variation is determined by the orientation of the curve.

1.3 The case of closed curves.

If $x(0) = x(T)$ and $y(0) = y(T)$ the variation is an integer multiple of 2π. So if γ is a closed parametrized curve then $\mathfrak{a}(\gamma)$ is an integer multiple of 2π and we set

$$\mathfrak{w}(\gamma) = \frac{\mathfrak{a}(\gamma)}{2\pi}$$

We refer to $\mathfrak{w}(\gamma)$ as the *winding number* of γ. By the construction one has

(1.1) $$\mathfrak{w}(\gamma) = \frac{1}{2\pi}\int_0^T \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \cdot dt$$

Example. Let m be a positive integer and

$$x(t) = \cos mt \quad : \quad y(t) = \sin mt$$

Notice that $x^2 + y^2 = 1$. It follows that

$$\frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \cdot dt = \cos mt \cdot m \cdot \cos mt - \sin mt \cdot (-m \cdot \sin mt) = m$$

Hence the winding number is equal to m.

1.4 Homotopy invariance. Consider a family of closed curves

$$\{\gamma_s \colon 0 \leq s \leq 1\} \quad : \gamma_s(0) = \gamma_s(T) : \quad 0 \leq s \leq 1$$

Let $t \mapsto (x_s(t), y_s(t))$ be the parametrization of γ_s. For each fixed s the curve $t \mapsto \gamma_s(t)$ has a winding number $\mathfrak{w}(\gamma_s)$. *Assume* that the two C^1-functions depend continuously upon s. Then $s \mapsto \mathfrak{w}(\gamma_s)$ is a continuous function and since it is integer-valued it must be a constant. Hence we have proved

1.5 Theorem *Let $\{\gamma_s\}$ be a homotopic family of closed curves Then they have the same winding number.*

Remark. In topology one refers to this by saying that the winding number is the same in each homotopy class of closed parametrized curves which surround the origin. A parametrized curve γ is defined by a map $t \mapsto \gamma(t)$ which need not be 1-1, i.e. we only assume that $\gamma(0) = \gamma(T)$. One may think of an insect which takes a walk on the horisontal (x, y)-plane starting at point p at time $t = 0$ and returns to p after a certain time interval T. During this walk the insect may cross an earlier path several times and even walk in the same path but in opposed direction for a while. The sole constraint is that the insect never attains the origin, i.e. $x(t)^2 + y(t)^2 > 0$ must hold in order to construct the winding number.

1.6 Rouche's principle.

Let γ_* be a parametrized closed curve. Suppose that γ is another closed curve such that

(i) $$|\gamma_*(t) - \gamma(t)| < |\gamma_*(t)| \quad : \quad 0 \leq t \leq T$$

To each $0 \leq s \leq 1$ we obtain the closed curve $\gamma_s(t) = s \cdot \gamma_* + (1 - s)(\gamma_*(t) - \gamma(t))$ which by (i) also the surrounds the origin. This gives a homotopic family and Theorem 1.5 gives:

(*) $$\mathfrak{w}(\gamma_*) = \mathfrak{w}(\gamma)$$

The case of non-closed curves. Let p and q be two points outside the origin. Consider two curves γ_1 and γ_2 where p is the common initial point and q the common end-point. Now we get the closed curve ρ defined by

$$\rho(s) = \gamma_1(2s) \quad : 0 \leq s \leq T/2\text{and} \quad \rho(s) = \gamma_2(2T - 2s) \quad : T/2 \leq s \leq T$$

Here we find that

$$\mathfrak{w}(\rho) = \mathfrak{a}(\gamma_1) - \mathfrak{a}(\gamma_2)$$

Next, keeping p and q fixed we consider a continuous family of curves $\{\gamma_s\}$ where $\gamma_s(0) = p$ and $\gamma_s(T) = q$ for all $0 \le s \le 1$. To each s we get the two curves γ_0 and γ_s and construct the closed curve ρ as above. Theorem 1.5 implies that the difference

$$\mathfrak{a}(\gamma_0) - \mathfrak{a}(\gamma_s)$$

is a constant function of s. Since the difference obviously is zero when $s = 0$ we conclude that

$$\mathfrak{a}(\gamma_0) = \mathfrak{a}(\gamma_s) \quad : 0 \le s \le 1$$

Thus, the angular variation is constant in a homotopic family of curves which join a pair of points p and q.

1.7 Variation of vector-valued functions Let γ be a parametrized C^1-curve. Here we do not exclude that $\gamma(t) = (0,0)$ for some values of t, i.e. γ is an arbitrary C^1-curve. Consider a pair of C^1-functions $u(x,y)$ and $v(x,y)$ defined in some neighborhood of the compact image set $\Gamma = \gamma([0,T])$. If $u^2 + v^2 \neq 0$ on Γ we get a curve γ^* which surrounds the origin defined by

(i) $$t \mapsto (u(\gamma(t)), v(\gamma(t))$$

Write $\gamma(t) = (x(t), y(t))$ and set

$$\xi(t) = u(x(t), y(t)) \quad : \quad \eta(t) = v(x(t), y(t))$$

Then we have

(ii) $$\mathfrak{a}(\gamma^*) = \int_0^T \frac{\xi\dot{\eta} - \eta\dot{\xi}}{\xi^2 + \eta^2} \cdot dt$$

Now $\dot{\xi} = u_x\ddot{x} + u_y\dot{y}$ and similarly for $\dot{\eta}$. So the last integral becomes

(*) $$\int_0^T \frac{u(v_x\dot{x} + v_y\dot{y}) - v(u_x\dot{x} + u_y\dot{y})}{u^2 + v^2} \cdot dt$$

This yields an integer which we refer to as the variation of the vector valued function (u,v) along the closed curve γ. We denote this integer by a subscript notation and write $\mathfrak{a}_{(u,v)}(\gamma)$. In the case when γ is a closed curve we define the winding number

$$\mathfrak{w}_{(u,v)}(\gamma) = \frac{1}{2\pi} \cdot \mathfrak{a}_{(u,v)}(\gamma)$$

Notice that this integer depends upon the pair (u,v) while γ is kept fixed.

1.8 The case of CR-pairs Let γ be a curve and $f(z) = u + iv$ an analytic in a neighborhood of $\gamma(T)$ where we assume that $f(\gamma(t)) \neq 0$ for all t. Hence $u^2 + v^2 \neq 0$ on γ so we can define $\mathfrak{a}_{(u,v)}(\gamma)$. Now (u,v) satisfy the Cauchy-Riemann equations which enables us to express $\mathfrak{a}_{(u,v)}(\gamma)$ in an elegant way. Namely let $t \mapsto (x(t), y(t))$ be a parametrization of γ and write $z(t) = x(t) + iy(t))$. Then

$$\dot{z} = \dot{x} + i\dot{y}$$

Now we regard the function

(i) $$t \mapsto \mathfrak{Im}\,[\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t)\,]$$

Since the complex derivative $f'(z) = u_x + iv_x$ we obtain

(ii) $$\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t) = \frac{(u_x + iv_x)(u - iv)(\dot{x} + i \cdot \dot{y})}{u^2 + v^2}$$

The imaginary part becomes

(iii) $$\frac{u_x u\dot{y} - u_x v\dot{x} + v_x u\dot{x} + v_x v\dot{y}}{u^2 + v^2} = \frac{u(u_x\dot{y} + v_x\dot{x}) - v(u_x\dot{x} - v_x\dot{y})}{u^2 + v^2}$$

Next, we can apply the Cauchy-Riemann equations and replace u_x with v_y and $-v_x$ by u_y. Then we see that (iii) is equal to the integrand which appears in (*) in 1.7. Hence we have proved the following:.

1.9 Theorem *Let $f(z) = u + iv$ be holomorphic in a neighborhood of γ and set $\mathfrak{a}_f(\gamma) = \mathfrak{a}_{(u,v)}(\gamma)$. Then*

$$(*) \qquad \mathfrak{a}_f(\gamma) = \int_0^T \mathfrak{Im}\,\Big[\frac{f'(z(t))}{f(z(t))} \cdot \dot{z}(t)\,\Big] \cdot dt$$

1.10 Remark By the construction of complex line integrals, (*) can be written as

$$\frac{1}{i} \cdot \int_\gamma \mathfrak{Im}\,\Big[\frac{f'(z)}{f(z)}\,\Big] \cdot dz$$

This complex notation is often used. When γ is a closed curve we get the winding number

$$\mathfrak{w}_f(\gamma) = \frac{1}{2\pi i} \cdot \int_\gamma \mathfrak{Im}\,\Big[\frac{f'(z)}{f(z)}\,\Big] \cdot dz$$

So this complex line integral always is an integer whenever $f(z)$ is analytic and $\neq 0$ in some open neighborhood of the compact set $\gamma([0,T])$.

1.11 Jordan's curve theorem Let γ be a closed C^1-curve and set $\Gamma = \gamma([0,T])$. To each $a \in \mathbf{C} \setminus \Gamma$ the closed curve

$$t \mapsto \frac{1}{\gamma(t) - a}$$

surrounds the origin. Its winding number denoted by $\mathfrak{w}_a(\gamma)$. From (*) in 1.4 we see that this winding number is constant in every connected component if $\mathbf{C} \setminus \Gamma$.

1.12 The case when γ is 1-1 Assume that $\gamma(t)$ is 1-1 except for the common end-values. This means that the image curve $t \mapsto \gamma(t)$ is a *closed Jordan curve.* For each $a \in \mathbf{C} \setminus \Gamma$ we notice that $t \mapsto \gamma(t) - a$ is 1-1. In the equation from XX which determines the ϕ-function for a given a where we may take $\phi(0) = 0$ as initial value shows that $t \mapsto \gamma(t)$ is 1-1 on the open interval $(0,T)$. Hence $\phi(t)$ cannot be an integer multiple of 2π when $0 < t < T$. Starting with $\phi(0) = 0$ it follows that

$$-2\pi < \phi(t) < 2\pi \quad : \quad 0 < t < 2\pi$$

Hence $\phi(T)$ can only attain one of the values $-2\pi, 0, 2\pi$. The *Jordan curve theorem* tells us that the value zero is never attained. Moreover, the set of points a for which the winding number equals 1 is a conneced open set, called the Jordan domain bounded by Γ. The complementary set is also connected and here $\mathfrak{w}_a(\gamma) = 0$. This can be expressed by saying that the closed Jordan curve Γ divides $\mathbf{C}$ into two component. This topological result was proved by Camille Jordan in 1850 and it is is actually valid under the relaxed assumption that the γ-function is only continuous. In that case the proof of Jordan's Curve Theorem is more demanding. For a detailed proof of the continuous version of Jordan's Curve Theorem we refer [Newmann] where methods of algebraic topology are used. We remark that Jordan's theorem in the plane is subtle in view of a quite remarkable discovery in dimension 3 due to X. Alexander who constructed a *homeomorphic copy* of the unit sphere in R^3 where the analogue of Jordan's theorem is not valid. This goes beyond the scope of these notes. A recommended text-book in algebraic topology is Alexander's classic text-book [Al] which gives an excellent introduction to the subtle parts of the theory.

1.13 The case of a simple polygon. Let $p_1, \ldots, p_N$ be distinct points in $\mathbf{C}$ where $N \geq 3$. To each $1 \leq \nu \leq N-1$ we get a line segment $\ell_\nu = [p_\nu, p_{\nu+1}]$ and we also get the line segment $\ell_N = [p_N, p_1]$. Assume that they do not intersect.Then they give sides of a simple closed curve Γ whose corner points are $p_1, \ldots, p_N$. The circle Γ is oriented where one travels in the positive direction from p_ν to $p_{\nu+1}$ when $1 \leq \nu \leq N-1$ and makes the final positive travel from p_N to p_1. We can imagine a narrow channel $\mathcal{C}_+$ which surrounds Γ and from this one can "escape" to the point at infinity. For example at a corner point p_ν where $|p_\nu|$ is maximal the channel contains points of absolute value > 1. From this picture it is clear the the *outer component* Ω_∞ of Γ is connected - and even simply connected if one adds the point at infinity. Rouche's principle shows that the winding number is zero for all points in the exterior component. If we instead construct a narrow channel $\mathcal{C}_*$ which moves "just inside" Γ then the channel itself is obviously connected. But their remains to see why the whole interior is connected and that the common winding number is equal to one. This, if Ω_* is the open complement of $\Gamma \cup \Omega_\infty$ we must first prove that Ω_* is connected. Since the narrow channel $\mathcal{C}_*$ is connected it suffices to show that when $p \in \Omega_*$ then there exists some curve γ from p which reaches $\mathcal{C}_*$. To obtain γ we consider a point $p^* \in \Gamma$ such that $|p - p^*|$ is the distance of p to Γ, i.e. we pick a point nearest to p. Now we draw the straight line L through p and p^* and by a picture the reader discovers that if we travel along L from p towards p^* then we reach $\mathcal{C}_*$ prior to the arrival at p^*. This proves that Ω_* is connected. The proof that the common winding number for points in Ω_* is equal to one is left as an *exercise* to the reader.

2. The argument principle

Let $\Omega \in \mathcal{D}(C^1)$ and $f(z)$ is an analytic function in Ω which extends to a C^1-function on its closure. Denote by $\mathcal{N}_\Omega(f)$ the number of zeros of f in Ω. We also assume that $f \neq 0$ on $\partial\Omega$.

2.1 Theorem. *Let $\Omega \in \mathcal{D}(C^1)$ and let $\Gamma_1, \ldots, \Gamma_k$ be its simple and closed boundary curves. Then*

$$N_\Omega(f) = \sum_{\nu=1}^{\nu=k} \mathfrak{w}_f(\Gamma_\nu)$$

Proof. By Theorem III.XX we have

$$N_\Omega(f) = \sum \frac{1}{2\pi i} \cdot \int_{\Gamma_\nu} \frac{f'(z)dz}{f(z)}$$

Since $N_\Omega(f)$ is an integer and hence a real nmber this gives

$$N_\Omega(f) = \sum \frac{1}{2\pi} \cdot \int_{\Gamma_\nu} \mathfrak{Im}\,[\frac{f'(z)dz}{f(z)}\,]$$

Theorem 1.9 expressed with complex notations shows each term of the sum above is equal to $\mathfrak{w}_f(\Gamma_\nu)$ and Theorem 2.1 follows.

2.2 Rouche's Theorem. *Let Ω and f be as above and let g be another holomorphic function in Ω which extends to be C^1 on the closure. Then, if $|g| < |f|$ holds on $\partial\Omega$, it follows that*

$$\mathcal{N}_{f+g}(\Omega) = \mathcal{N}_f(\Omega)$$

Proof. Apply the result in 1.6.

2.3 An application to trigonometric series. Let $1 \leq m < n$ be a pair of positive integers. Consider a trigonometric polynomial

$$P(\theta) = \sum_{\nu=m}^{\nu=n} a_\nu \cos(\nu\theta) + b_\nu \sin(\nu\theta) \quad : \quad a_\nu, b_\nu \in \mathbf{R}$$

We assume that at least one of the coefficients a_m or b_m is $\neq 0$, and similarly at least one of the numbers a_n or b_n is $\neq 0$. Then one has

2.4 Theorem *P has at least $2m$ zeros on $[0, 2\pi]$ and at most $2n$ zeros.*

Proof Consider the polynomial

$$Q(z) = (a_m - ib - m)z^m + \ldots + (a_n - ib_n)z^n$$

Notice that $\mathfrak{Re}(Q(e^{i\theta}) = P(\theta)$. The polynomial Q has a zero of multiplicity m at the origin. Let us now consider some $r < 1$ with $r \simeq 1$ and chosen so that $Q \neq 0$ on the circle $T_r = \{|z| = r\}$. Since $Q(z)$ has at least m zeros counted with multiplicity in the disc D_r, it follows from Theorem 2.2that

$$\mathfrak{w}_{T_r}(Q) \geq m$$

Regarding a picture the reader discovers that the curve $\theta \mapsto Q(re^{i\theta})$ must intersect the real line at least $2m$ times. This proves the lower bound. The upper bound $2n$ is easily proved by elementary Calculus and left to the reader.

2.5 A special estimate. Theorem 2.1 can be used to give upper bounds for the counting function $\mathcal{N}_\Omega(f)$. Suppose that Ω is a rectangle

$$\{z = x + iy \colon\ a < x < b \colon\ 0 < y < T\}$$

Here $\partial\Omega$ contains the vertical line $\ell = \{x = b :\ 0 < y < T\}$. The line integral along ℓ contributes to the evaluation of $\mathcal{N}_\Omega(f)$ by

$$\frac{1}{2\pi} \cdot \int_\ell \mathfrak{Im}\,(\frac{f'(z)dz}{f(z)}\,)$$

Now $dz = idy$ along ℓ and therefore the integral above is equal to

$$\frac{1}{2\pi} \cdot \int_0^T \mathfrak{Re}[\frac{f'(b+iy)}{f(b+iy)}]\, dy$$

Assume that $\mathfrak{Re}\, f(b+iy) \geq c_0 > 0$ for all $0 \leq y \leq T$ which gives a single valued branch of the Log-function, i.e.

$$\log f(b+iy) = \log |f(b+iy)| + i \cdot \arg(f(b+iy)) \ : \ -\pi/2 < \arg(f(b+iy)) < \pi/2$$

Since $f'(z) = \frac{1}{i} \cdot \partial_y(f)$ it follows that

$$\frac{f'(b+iy)}{fb+iy)} = \frac{1}{i} \cdot [\partial_y(\log |f(b+iy)|) + i \cdot \partial_y(\arg(f(b+iy)))]$$

Hence we obtain

2.6 Proposition. *One has the equality*

$$\mathfrak{Re}\, \frac{f'(b+iy)}{f(b+iy)} = \partial_y(\arg(f(b+iy)))$$

2.7 Remark. Proposition 2.6 gives therefore

$$(*) \qquad \frac{1}{2\pi} \cdot \int_\ell \mathfrak{Im}\, (\frac{f'(z)dz}{f(z)}) = \frac{1}{2\pi} \cdot \arg(f(b+iT))) - \arg(f(b))$$

The right hand side is a real number in $(-1/4, 1/4)$ and hence we get a small contribution from the line integral in the left hand side when we regard whole line integral over $\partial\Omega$ which evaluates $\mathcal{N}_f(\Omega)$. This will be used to study the zeros of Riemann's ζ-function.

2.8 An application. Let $m \geq 2$ and $g_2(z), \ldots, g_m(z)$ are analytic functions defined in an open disc D centered at $z = 0$ and where $g_\nu(0) = 0$ for every ν. Let also $\phi(z)$ be another analytic function in D with $\phi(0) = 0$. Consider the algebraic equation

$$(*) \qquad y + g_2(z)y^2 + \ldots + g_m(z)y^m = \phi(z)$$

Thus, we seek $y(z)$ so that (*) holds. It turns out that there exists a unique analytic function $y(z)$ defined in some open disc D_* centered at $z = 0$ where $y(0) = 0$ and (*) hold for every $z \in D_*$. To attain this we set

$$P(y,z) = y + g_2(z)y^2 + \ldots + g_m(z)y^m$$

Now we can find $\delta > 0$ such that if $z \in D(\delta)$ then

$$\text{(i)} \qquad |\phi(z)| < |P(e^{i\theta}, z)| \quad \text{for all} \quad 0 \leq \theta \leq 2\pi$$

Next, let us put

$$P'_y(y,z) = 1 + 2g_2(z)y + \ldots + mg_m(z)y^{m-1}$$

From (i) there exists the integral

$$(1) \qquad \frac{1}{2\pi i} \cdot \int_{|y|=1} \frac{P'_y(y,z)}{P(y,z) - \phi(z)} \cdot dy \quad : z \in D(\delta)$$

Rouche's Theorem shows that this integer-valued function is constant as z varies in $D(\delta)$. When $z = 0$ we see tha the integrand is $\frac{1}{y}$ and hence the constant integer is one. But this means precisely that when $z \in D(\delta)$ is fixed, then the analytic function

$$y \mapsto P(y,z) - \phi(z)$$

has exactly one simple zero in $|y| < 1$. Denote this zero by $y(z)$. By the residue formula we get

$$(2) \qquad y(z) = \frac{1}{2\pi i} \cdot \int_{|y|=1} \frac{y \cdot P'_y(y,z)}{P(y,z) - \phi(z)} \cdot dy$$

It is clear that $y(z)$ is analytic in $D(\delta)$ and by the construction $P(y(z), z) = 0$. Thus, $y(z)$ is the required solution. The proof of its uniqueness is left as an exercise.

3. Multi-valued functions

Let Ω be an open connected subset of $\mathbf{C}$ and $D \subset \Omega$ is an open disc of some radius r centered at a point z_0. The material about power series in Chapter XX shows that $\mathcal{O}(D)$ is identified with convergent power series

$$\sum c_\nu (z-z_0)^\nu \quad : \quad \text{radius of convergenece } \geq r$$

So if $f \in \mathcal{O}(\Omega)$ its restriction to any disc $D \subset \Omega$ determines a convergent power series. These power series must be matching when two discs have a non-empty intersection. This observation is the starting point for a general construction due to Weierstrass.

3.1 Analytic continuation along paths Let $s \mapsto \gamma(s)$ be a continuous and complex valued function with values in Ω. We do not require that $\gamma(0) = \gamma(1)$ or that γ is 1-1. The points $p = \gamma(0)$ and $q = \gamma(1)$ are called the terminal points of γ. Let $f_0 \in \mathcal{O}(D_r(p))$ for some $r > 0$, i.e. f is analytic in a small disc centered at p. Consider a strictly increasing sequence $0 = s_0 < s_1 < \ldots < s_N = 1$ and to each $p_\nu = \gamma(s_\nu)$ we choose a small disc $D_{p_\nu}(r_\nu)$ such that:

$$D_{p_\nu}(r_\nu) \cap D_{p_{\nu+1}}(r_{\nu+1}) \neq \emptyset \qquad 0 \leq \nu \leq N-1$$

Assume that for each $1 \leq \nu \leq N$ exists $f_\nu \in \mathcal{O}(D_{r_\nu}(p_\nu))$ such that

$$f_\nu = f_{\nu+1} \text{ holds in } D_{p_\nu}(r_\nu) \cap D_{p_{\nu+1}}(r_{\nu+1})$$

After N many *direct analytic continuations over pairs of intersecting discs* we arrive at f_N which is analytic in an open disc centered at $\gamma(1)$. From the uniqeness of each direct analytic continuation, it follows that the locally defined analytic function f_N at $\gamma(1)$ is the same if we instead have chosen a *refined* partition of $[0,1]$. Since two coverings of γ via finite families of discs have a common refinement, we conclude that locally defined analytic function at the end-point is unique. Thus, the construction yields a map T_γ map which sends an analytic function f defined in a disc around $\gamma(0)$ to an analytic function $T_\gamma(f)$ defined is some disc centered at $\gamma(1)$. Of course, here T_γ is only defined on those f at $\gamma(0)$ which have an analytic continuation along γ in the sense of Weierstrass.

3.2 Example Let γ be the closed unit circle centered at the origin where $\gamma(0) = \gamma(1)$. At $\gamma(0)$ we start with a chosen branch of the complex Log-function with the power series representation

$$f_0(1+z) = z - z^2/2 + z^3/3 + \ldots$$

This power series converges when $|z| = 1$ So let the first disc D_1 be centered at $z = 1$ and of radius 1. Let D_2 be the disc centered at i of radius i. In this disc we have the analytic function $f_1(z)$ given by the power series

$$f_1(i+z) = i\frac{\pi}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot i^{-n} \cdot z^n/n$$

Drawing a figure the two discs intersect and in particular $D_1 \cap D_2$ contains an open arc of the unit circle around $e^{i\pi/4}$. With θ close to $\pi/4$ we get the two series expansion

$$f_0(e^{i\theta}) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot (e^{i\theta} - 1)^n \quad : \quad f_1(e^{i\theta}) = i\frac{\pi}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot i^{-n} \cdot (e^{i\theta} - i)^n/n$$

Exercise. Verify that the two series expansions give the same θ-function, i.e. that one has the equality

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot (e^{i\theta} - 1)^n = i\frac{\pi}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot i^{-n} \cdot (e^{i\theta} - i)^n/n \quad : \quad |\theta - \frac{\pi}{4}| < \delta$$

where $\delta > 0$ is chosen so that $|e^{i\pi/4} - 1| < 1$. This series expansion starts the analytic continuation of the complex Log-function along the closed unit circle which after one turn around the origin gives the new local branch $f_0(z) + 2\pi i$.

3.3 The class $M\mathcal{O}(\Omega)$.

Let Ω be an open subset of $\mathbf{C}$. At each point $z \in \Omega$ we denote by $\mathcal{O}(z_0)$ the germs of analytic functions at z_0 and recall that this set is identified with power series $\sum c_\nu (z-z_0)^\nu$ which have some positive radius of convergence. In $\mathcal{O}(z_0)$ we can consider those germs which have analytic continuation along *every* curve in Ω whose initial point is z_0 while the end-point is arbitrary. This leads to:

3.4 Definition *A germ $f \in \mathcal{O}(z_0)$ generates a multi-valued analytic function in Ω if it can be extended in the sense of Weierstrass along every curve $\gamma \subset \Omega$ which has z_0 as initial point. The set of all these germs is denoted by $M\mathcal{O}(\Omega)(z_0)$.*

Remark. Notice that $M\mathcal{O}(\Omega)(z_0)$ contains those germs at z_0 which are induced by *single-valued* analytic functions in Ω. If $f \in M\mathcal{O}(\Omega)(z_0)$ and γ is a curve in Ω with z_0 as initial point and z_1 as end-point, then the germ $T_\gamma(f)$ at z_1 belongs to $M\mathcal{O}(\Omega)(z_1)$. This is obviuos since if γ_1 is a curve starting at z_1 with end-point at z_2, then f extends along the composed curve $\gamma_1 \circ \gamma_1$ and one has the composition formula:

$$(*) \qquad T_{\gamma_1}(T_\gamma(f)) = T_{\gamma\circ\gamma}(f)$$

3.5 The total sheaf space $\widehat{\mathcal{O}}$. Let us introduce a big topological space $\widehat{\mathcal{O}}$ defined as follows: One has a map ρ from $\widehat{\mathcal{O}}$ onto $\mathbf{C}$. The inverse fiber $\rho^{-1}(z) = \mathcal{O}(z)$ for each $z \in \mathbf{C}$. An open neighborhood of a "point" $f \in \rho^{-1}(z_0)$ consists of a pair (f, D) where D is a small disc centered at z_0 such that the germ f extends to an analytic function in D. Then its induced germ at a point $z \in D$ belongs to $\rho^{-1}(z)$. The set of points in $\widehat{\mathcal{O}}$ obtained in this way yields the subset (f, D) and as D shrinks to z_0 they give by definition a fundamental system of open neighborhoods of the point f in $\widehat{\mathcal{O}}$. With this topology on $\widehat{\mathcal{O}}$ the map ρ is a *local homeomorphism* and each inverse fiber $\rho^{-1}(z_0)$ appears as a *discrete* subset of $\widehat{\mathcal{O}}$.

Remark Above $\widehat{\mathcal{O}}$ is the first example of a sheaf which has led to the general construction of sheaves which is presented in elementary text on topology. The construction of the sheaf topology on $\widehat{\mathcal{O}}$ yields the following elegant description of multi-valued functions.

3.6 Proposition. *Let Ω be an open and connected subset of $\mathbf{C}$. Let $z_0 \in \Omega$ and $f \in M(\Omega)(z_0)$. Then f appears in the inverse fiber $\rho^{-1}(z_0)$ of an open and connected set $\mathcal{W}(f)$ of $\rho^{-1}(\Omega)$ called Weierstrass Analytische Gebilde of the germ of this multi-valued funtion. For each $z \in \Omega$ the set $\mathcal{W}(f) \cap \rho^{-1}(z)$ consists of all germs at z obtained by analytic continuation of f along some curve with end-point at z.*

Some notations. Let f be as above. If $z \in \Omega$ we denote by $W(f:z)$ the set of germs at z which arise via all analytic continuations of f. Thus, $W(f:z)$ is equal to $\mathcal{W}(f) \cap \rho^{-1}(z)$. In addition to this we can consider the set of values at z which are attained by these germs. So we have also the set

$$R_f(z) = \{T_\gamma(f)(z) \quad : \quad T_\gamma(f) \in W(f:z)\}$$

Example Let $\Omega = \mathbf{C}$ minus the origin, i.e. the punctured complex plane. Then we have the multi-valued Log-function. At each point $z \in \Omega$ it has an infinite set of local branches which differ by integer multiples of $2\pi i$. The resulting connected set $\mathcal{W}(\text{Log}(z))$ can be regarded as a 2-dimensional connected manifold. In topology one learns that this is the *universal covering space* of Ω, so that $\mathcal{W}(\text{Log}(z))$ is a *simply connected* manifold. On the other hand, if $N \geq 2$ is an integer we have the multi-valued function $z^{\frac{1}{N}}$ Here $\mathcal{W}(z^{\frac{1}{N}})$ is an N-fold unramified covering map of Ω, i.e. the ρ-map is a local homeomorphism whose inverse fibers contain N points.

3.7 Normal families.

Let Ω be some connected open set in $\mathbf{C}$. Let $x_0 \in \Omega$ and consider some germ $f \in M\mathcal{O}(\Omega)(x_0)$. We say that f yields a bounded multi-valued function if there exists a constant K such that

$$(*) \qquad \big|T_\gamma(f)(x)\big| \leq K$$

holds for all pairs x, γ where $x \in \Omega$ and γ is any curve from x_0 to x. Suppose that $\{f_\nu\}$ is a sequence of germs in $M\mathcal{O}(\Omega)(x_0)$ which are uniformly bounded, i.e. (*) above holds for some constant K and every ν. If we to begin with consider a small open disc D centered at x_0 we get the unique single-valued branches of each f_ν in $\mathcal{O}(D)$. This family in $\mathcal{O}(D)$ is normal by the results in XXX. Passing to a subsequence we may assume that there exists a limit function $g \in \mathcal{O}(D)$, i.e. shrinking D if necessary we may assume that

$$\text{(i)} \qquad \lim_{\nu\to\infty} ||f_\nu - g||_D \to 0$$

Next, if γ is a curve in which starts at x_0 and has some end-point x we cover γ with a finite number of open discs and each f_ν has its analytic continuation along γ by the Weierstrass procedure. From the material in XXX it is clear that during these analytic continuations the local series expansions of the sequence $\{f_\nu\}$ converge uniformly and as a result we find that g has an analytic extension along γ. Hence the germ of g at x_0 belongs to $M\mathcal{O}(\Omega)(x_0)$. Moreover, the uniform convergence "propagates". For example, if γ is a closed curve at x_0 we get the sequence $\{T_\gamma(f_\nu)\}$ after the analytic continuation along γ, and similarly $T_\gamma(g)$. Then

$$\text{(ii)} \qquad \lim_{\nu\to\infty} ||T_\gamma(f_\nu) - T_\gamma(g)||_D \to 0$$

holds for a small disc D centered at the end point of γ.

3.8 Algebraic root functions

Consider a polynomial $P(z,w)$ in two variables:

$$P(z,w) = p_m(z)w^m + p_{m-1}(z)w^{m-1} + \ldots + p_1(z)w + p_0(w)$$

Here $m \geq 2$ is assumed. The zeros of leading polynomial $p_m(z)$ is a finite subset of $\mathbf{C}$ which we denote by $\mathcal{Z}_P$. If z is outside this zero set we have an algebraic equation $P(z_0, w)$ which by the Fundamental Theorem of algebra has m roots denoted by $\alpha_1(z), \ldots, \alpha_m(z)$. By the Newton formula from XXX in Chapter I the symmetric product

$$D(z) = \prod_{i \neq \nu} (\alpha_i(z) - \alpha_\nu(z))$$

is a rational function of z, unless it happens to be identically zero. We shall exclude this case and remark that elementary algebra shows that the D-function is not identically zero if and only if the polynomial $P(z,w)$ has no multiple factors in the unique factorisation domain $C[z,w]$. One refers to $D(z)$ as the *discrimant* of P. Put

$$\Sigma_P = \mathcal{Z}_P \cup D^{-1}(0)$$

This is a finite set in $\mathbf{C}$. If Ω is its open complement we get the multi-valued root functions. Assuming that $P(z,w)$ from the start is an irreducible polynomial in $C[z,w]$ these root functions are local branches of a multi-valued function f in Ω. The fibers of $W(f)$ above a point $z_0 \in \Omega$ consists of the k-tuple of germs induced by the m-many distinct α-roots at z_0. The *connectivity* of $W(f)$ means precisely that the polynomial $P(z,w)$ is irreducible.

Remark The realisation of roots to $P(z,w)$ as multi-valued analytic functions is classic. Historically it led to the construction of the *closed Riemann surface* attached to the polynomial $P(z,w)$. We return to this in the chapter about compact Riemann surfaces.

4. The Monodromy Theorem

Let $f \in M\mathcal{O}(\Omega)$. If $z_0 \in \Omega$ and γ is a curve starting at z_0 we obtain the germ $T_\gamma(f)$ at the end-point z_1 of γ. The analytic continuation is obtained by the Weierstrass procedure and since γ is a compact subset of Ω it can be covered by a finite set of discs $D_0, D_1, \ldots, D_N$ where $D_\nu \cap D_{\nu+1}$ are non-empty and the analytic continuation of f is achieved by succesive direct continuations of analytic functions $\{g_\nu \in \mathcal{O}(D_\nu)\}$ where $g_\nu = g_{\nu+1}$ holds in $D_\nu \cap D_{\nu+1}$. The discs are chosen so small that they are relatively compact in Ω. If γ_1 is another curve from z_0 to z_1 which stays so close to γ that the discs $D_0, \ldots, D_N$ again can be used to perform the analytic continuation of f along γ_1, then it is clear that $T_\gamma(f) = T_{\gamma_1}$. This observation gives:

4.1 Theorem *Let (z_0, z_1) be a pair in Ω and $\Gamma(s,t)$ a continuous map from the unit square in the (s,t)-space into Ω where*

$$\Gamma(s,0) = z_0 \quad : , \gamma(s,1) = z_1 \quad : \quad 0 \leq s \leq 1$$

Then, if $\{\gamma_s\}$ is the family of curves defined by $t \mapsto \Gamma(s,t)$, it follows that

$$T_{\gamma_s}(f) = T_{\gamma_0}(f) \quad : \quad 0 \leq s \leq 1$$

Remark This is called the monodromy theorem and can be expressed by saying that analytic continuation along a curve which joins a given pair of points only depends on the *homotopy* class of the curve, taken in the family of all curves which joint the two given points. Of course, when we deal with some multi-valued function in an open set Ω we are obliged to use curves inside Ω only.

4.2 The case of finite determination. Let $f \in M(\Omega)$. If $z_0 \in \Omega$ we get the set of germs $W(f : z_0)$ at z_0. This is a subset of $\mathcal{O}(z_0)$ and we can regard the complex vector space it generates. It is denoted by $\mathcal{H}_f(z_0)$. Suppose that this complex vector space has a finite dimension k. Then we can choose a k-tuple of germs $g_1, \ldots, g_k$ in $W(f : z_0)$ which give a basis of $\mathcal{H}_f(z_0)$. Thus, one has to begin with

$$\mathcal{H}_f(z_0) = Cg_1 + \ldots + Cg_k$$

Let z_1 be another point in Ω and fix some curve γ which joins z_0 and z_1. At z_1 we get the germs $T_\gamma(g_1), \ldots, T_\gamma(g_k)$. By the remark in XXX T_γ is a *bijective map* from $W(f : z_0)$ to $W(f : z_1)$. Moreover, if $\phi = c_1 g_1 + \ldots + c_k g_k$ belongs to $\mathcal{H}_f(z_0)$ we have

$$T_\gamma(\phi) = c_1 T_\gamma(g_1) + \ldots + c_k T_\gamma(g_k)$$

Hence the k-tuple $\{T_\gamma(g_\nu)\}$ generates the vector space $\mathcal{H}_f(z_1)$. Since we also can use the inverse map $T_{\gamma^{-1}}$ it follows that the k-tuple $\{T_\gamma(g_\nu)\}$ yields a basis of $\mathcal{H}_f(z_1)$. In particular the vector space $\mathcal{H}_f(z)$ have common dimension k as z varies in the connected open set Ω. *Summing up*, we can conclude the following:

4.3 Proposition *If $f \in M(\Omega)$ has finite determination the complex vector spaces $\mathcal{H}_f(z)$ have common dimension. Moreover, one gets a basis of these by starting at any point z_0 and choose some k-tuple of C-linearly germs $g_1, \ldots, g_k$ in $W(f : z_0)$. Then we obtain a basis in $\mathcal{H}_f(z)$ for any point $z \in \Omega$ by a k-tuple $\{T_\gamma(g_\nu)\}$ where γ is any curve which joins z_0 and z.*

4.4 The case of a punctured disc Let $\dot{D} = \{0 < |z| < R\}$ be a punctured disc centered at the origin. Consider some $f \in \mathcal{MO}(\dot{D})$ of finite determination and let k be its rank. In a punctured open disc every closed curve is homotopic to a closed circle parametrized by $\theta \mapsto re^{i\theta}$. Another way to express this is that the fundamental group $\pi_1(\dot{D})$ is isomorphic to the abelian group of integers. Thus, the multi-valuedness is determined by a sole T-operator which arises when we let γ be a circle surrounding the origin in the positive sense. Given $z_0 \in \dot{D}$ we consider the **C**-linear operator

$$T_\gamma \colon\ \mathcal{H}_f(z_0) \mapsto \mathcal{H}_f(z_0)$$

By Jordan's decomposition theorem we can choose a basis in $\mathcal{H}_f(z_0)$ such that the matrix representing T_γ is of Jordan's normal form. This means that we have a direct sum

$$\mathcal{H}_f(z_0) = \oplus \mathcal{K}_\nu(z_0)$$

where $\{\mathcal{K}_\nu(z_0)\}$ are T_γ-invariant subspaces and the restriction of T_γ $\mathcal{K}_\nu(z_0)$ is represented by an elementary Jordan matrix $J(m, \lambda)$ for some complex number λ and $m \geq 1$. Given the pair m, λ we consider a local branch of the function

$$f(z) = z^\alpha \cdot \ [\operatorname{Log} z\,]^{m-1} \quad : \, e^{2\pi i \alpha} = \lambda$$

which for example is defined close to $z = 1$ where $f(1) = 0$. After one turn around the origin we get a new local branch of the form

$$f_1(z) = \lambda \cdot z^\alpha \cdot [\operatorname{Log} z \, + 2\pi i]^{m-1}$$

Continuing in this way m times we see that the local branches of f generate an m-dimensional complex vector space whose monodromy is determined by the matrix $J(m, \lambda)$. Using this fact it follows that if $f(z)$ is any local branch of a multi-valued function of finite determination, then it can be expressed as:

$$(*) \qquad f(z) = \sum_{\nu=1}^{k} \sum_{j} g_{\alpha_\nu, j}(z) \cdot z^{\alpha_\nu} \cdot \ [\operatorname{Log} z\,]^j$$

Here $0 \leq \mathfrak{Re}(\alpha_1) < \ldots < \mathfrak{Re}(\alpha_k) < 1$ and $\{j\}$ is a finite set of non-negative integers and the g-functions are *single-valued* in the punctured disc $\dot{D}$. Moreover these g-functions are uniquely determined provided a specific local branch of the Log-function is chosen. For example, when f is a local branch at some real point $0 < a < R$ where $\operatorname{Log} a$ is chosen to be real.

5. Homotopy and Covering spaces

First we recall some facts in topology. Let X be a metric space, i.e. the topology is defined by some distance function. By a curve in X we mean a continuous map γ from the closed unit interval $[0, 1]$ into X. In general γ need not be 1-1. The initial point is $\gamma(0)$ and the end point is $\gamma(1)$. If $\gamma(0) = \gamma(1)$ we say that γ is a *closed* curve. We say that X is *arcwise connected* if there to each pair of points x_0, x_1 exists some curve γ with $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$.

A notation. Given a point $x_0 \in X$ we denote by $\mathcal{C}(x_0)$ the family of all closed curves γ where $\gamma(0) = \gamma(1) = x_0$.

5.1 Definition. *A pair of closed curves γ_0 and γ_1 in $\mathcal{C}(x_0)$ are homotopic if there exists a continuos map Γ from the unit square $\square = \{(t, s) \quad : 0 \leq t, s \leq 1\}$ into X such that*

$$\Gamma(t, 0) = \gamma_0(t) \text{ and } \Gamma(t, 1) = \gamma_1(t) \quad \Gamma(0, s) = \Gamma(1, s) = x_0 \ : \ 0 \leq s \leq 1$$

It is clear that homotopy yields an equivalence relation on $\mathcal{C}(x_0)$. If $\gamma \in \mathcal{C}(x_0)$ then $\{\gamma\}$ denotes its homotopy class. Next, if γ_0 and γ_1 are two closed curves at x_0 we get a new closed curve γ_2 defined by

$$\gamma_2(t) = \gamma_1(2t) : \ 0 \leq t \leq \frac{1}{2} \text{ and } \gamma_2(t) = \gamma_2(2t - 1) : \ \frac{1}{2} \leq t \leq 1$$

We refer to γ_2 as the composed curve and it is denoted by $\gamma_1 \circ \gamma_0$. One verifies easily that the homotopy class of γ_2 depends upon $\{\gamma_1\}$ and $\{\gamma_1\}$ only. In this way we obtain a composition law on the set of homotopy classes of closed curves at x_0 defined by

$$\{\gamma_1\} \cdot \{\gamma_0\} = \{\gamma_1 \circ \gamma_0\}$$

One verifies easily that this composition satisfies the associative law. A neutral element is the closed curve γ_* for which $\gamma_*(t) = x_0$ for every t. Finally, if $\gamma(t)$ is any closed curve at x_0 we get a new closed curve by reversing the direction, i.e. set

$$\gamma^{-1}(t) = \gamma(1 - t)$$

Exercise. Show that the composed curve $\gamma^{-1} \circ \gamma$ is homotopic to γ_*.

5.2 The fundamental group. The construction of composed closed curves and the exercise above show that homotopy classes of closed curves at x_0 give elements of a group to be denoted by $\pi_1(X : x_0)$.

Remark. The group $\pi_1(X : x_0)$ is intrinsic in the sense that it does not depend upon the chosen point x_0. Namely, let x_1 be another point in X and fix a curve λ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$. Then we obtain a map from $\mathcal{C}(x_1)$ to $\mathcal{C}(x_0)$ defined by

(i) $$\gamma \mapsto \lambda^{-1} \circ \gamma \circ \lambda$$

One verifies that (i) sends homotopic curves to homotopic curves and by considering homotopy classes we obtain an isomorphism between $\pi_1(X : x_0)$ and $\pi_1(X : x_1)$. Hence there exists an intrinsically defined group denoted by $\pi_1(X)$. It is called the fundamental group of the metric space X. If $\pi_1(X)$ is reduced to a single element, i.e. when all closed curves in $\mathcal{C}(x_0)$ are homotopic we say that X is *simply connected.*

Exercise. Let x_0 and x_1 be two distinct points in X. Denote by $\mathcal{C}(x_0, x_1)$ the family of curves γ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Two such curves γ_0 and γ_1 are homotopic if there exists a continuos map Γ from the square $\square$ such that

$$\Gamma(t, 0) = \gamma_0(t) \text{ and } \Gamma(t, 1) = \gamma_1(t) \quad \Gamma(0, s) = x_0 \text{ and } \Gamma(1, s) = x_1 \;:\; 0 \le s \le 1$$

Show that a pair γ_0 and γ_1 are homotopic in $\mathcal{C}(x_0, x_1)$ if and only if the closed curve $\gamma_1^{-1} \circ \gamma_0$ is homotopic to γ^* in $\mathcal{C}(x_0)$ where

$$\gamma_1^{-1} = \gamma_1(1 - t)$$

In particular each pair of curves in $\mathcal{C}(x_0, x_1)$ are homotopic if X is simply connected.

5.3 Covering spaces. Let X and Y be two arcwise connected metric spaces. A continuous map ϕ from X onto Y is called a *covering map* if the following hold: For each $y_0 \in Y$ there exists an open neighborhood U such that the inverse image $\phi^{-1}(U)$ is a union of pairwise disjoint open sets $\{U^*_\alpha\}$ and the restriction of ϕ to each U^*_α is a homeomorphism from this set onto U.

5.4 Lifting of curves. Suppose that $\phi\colon X \to Y$ is a covering map. Let γ be a curve in Y defined by a continuous map $t \to \gamma(t)$ from the closed unit interval $[0, 1]$ into Y with some initial point $y_0 = \gamma(0)$ and some end-point $y_1 = \gamma(1)$. The case $y_0 = y_1$ is not excluded, i.e. γ may be a closed curve. Next, in X we chose a point x_0 such that $\phi(x_0) = y_0$. By assumption there exists an open neighborhood U of y_0 in X a unique open neighborhood U^* of x_0 such that $\phi\colon U^* \to U$ is a homeomorphism. Since $t \to \gamma(t)$is continuous there exists some $\delta > 0$ such that

(i) $$\gamma(t) \in U\,, \quad 0 \le t \le \delta$$

Then we get a *unique* curve γ^* in X defined for $0 \le t \le \delta$ such that

(ii) $$\phi(\gamma^*(t)) = \gamma(t)\,, \quad 0 \le t \le \delta \quad \text{and } \gamma^*(0) = x_0\,.$$

If this lifting process can continued for all $0 \le t \le 1$ we say that γ has a lifted curve γ^*. This means that there exists a curve $t \mapsto \gamma^*(t)$ from $[0, 1]$ into X such that

(*) $$\phi(\gamma^*(t)) = \gamma(t)\,, \quad 0 \le t \le 1 \quad \text{and } \gamma^*(0) = x_0\,.$$

Exercise. Show that the curve γ^* is unique if it exists. The hint is to use that ϕ is a local homeomorphism.

The whole discussion above leads to

5.5 Definition. *A covering map $\phi\colon X \to Y$ is of class $\mathcal{L}$ if the following hold: For each pair of points $y_0 \in Y$ and $x_0 \in \phi^{-1}(0)$, every curve γ in Y with initial point y_0 can be lifted to a curve in X with initial point x_0.*

5.6 The case when X is simply connected. Assume this and let $\phi\colon X \to Y$ be a covering map of class $\mathcal{L}$. Let $y_0 \in Y$ and choose some point $x_0 \in \phi^{-1}(y_0)$. Next, let γ be a closed curve in Y with $\gamma(0) = \gamma(1) = y_0$. By assumption there exists a unique lifted curve γ^* in X with $\gamma^*(0) = x_0$. Suppose that $\gamma^*(1) = x_0$, i.e. the lifted curve is closed. Since X is simply connected it is homotopic to the trivial curve which stays at x_0, i.e. there exists a continuos map Γ^* from $\square$ into X such that

(i) $$\Gamma^*(t,0) = \gamma^*(t) \,\text{and}\, \Gamma^*(t,1) = x_0 \quad \Gamma^*(0,s) = \Gamma(1,s) = x_0 \,:\, 0 \leq s \leq 1$$

Now $\Gamma(t,s) = \phi(\Gamma^*(t,s)$ is a continuous map from $\square$ into Y and from (i) we see that Γ yields a homotopy between γ_0 and γ_1. Using this observation we arrive at:

5.7 Proposition. *Let γ_0 and γ_1 be two closed curves at y_0. Then they are homotopic if and only if $\gamma^*(1) = \gamma^*(1)$.*

Proof. We have already seen that if $\gamma^*(1) = \gamma^*(1)$ then the two curves are homotopic. Conversely, if they are homotopic we get a continus map $\Gamma(s,t)$ from $\square$ into Y and for each $0 \leq s \leq 1$ we have the closed curve $\gamma_s(t) = \Gamma(t,s)$ at y_0. Since the inverse fiber $\phi^1{}_1(y_0)$ by assumption is a discrete set in X, it follows by continuity and the unique path lifting that $s \mapsto \gamma_s^*(1)$ is constant and hence $\gamma_0^*(1) = \gamma_1^*(1)$.

5.8 Conclusion. Proposition 5.7 shows that homotopy classes of closed curves γ at y_0 are in a 1-1 correspondence with their end-points in X. Notice also that if x belongs to $\phi^{-1}(y_0)$ then the arc-wise connectivity of X gives a curve ρ where $\rho(0) = x_0$ and $\rho(1) = x$. Now $\gamma(t) = \phi(\rho(t))$ is a closed curve at y_0 and here $\gamma^*(t) = \rho(t)$ and hence x appears as an end-point for at least one closed curve at y_0. Identifying $\pi_1(Y)$ with homotopy classes of closed curves at y_0 we have therefore proved the following:

5.9 Theorem. *The map $\gamma \to \gamma^*(1)$ yields a bijective correspondence between the fundamental group $\pi_1(Y)$ and the inverse fiber $\phi^{-1}(y_0)$.*

Exercise. Let X and Z be two simply connected metric spaces. Suppose that $\phi\colon X \to Y$ and $\psi\colon Z \to Y$ are two covering maps which both belong to the class $\mathcal{L}$. Fix some $y_0 \in Y$. Choose $x_0 \in \phi^{-1}(y_0)$ and $z_0 \in \psi^{-1}(y_0)$. Next, let $y \in Y$ and consider some curve γ in Y with $\gamma(0) = y_0$ and $\gamma(1) = y$. Its unique lifted curve to X is denoted by γ^* and we get the end-point

$$\gamma^*(1) \in \phi^{-1}(y)$$

Similarly. we get a unique lifted curve γ^{**} in Z and the end-point

$$\gamma^{**}(1) \in \psi^{-1}(y)$$

From the above these two end-points only depend on the homotopy class of γ. Use this to conclude that we obtain a *unique and bijective* map from the discrete fiber $\phi^{-1}(y)$ to $\psi^{-1}(y)$. Moreover, as y varies in Y this gives a unique homeomorphism G from X onto Z with $G(x_0) = z_0$.

5.10 The universal covering space.

The exercise above shows that up to a homeomorphism the metric space Y with a non-trivial fundamental group has a unique simply connected covering space X where the map $\phi\colon X \to Y$ is of class $\mathcal{L}$. In topology one refers to X as the *universal covering space* of Y. There remains the question if there exists such a universal covering space. We shall not try to investigate this existence problem in full generality but consider the case when Y is an open and connected subset of $\mathbf{C}$. In this case we can show that Y has a universal covering space by the following construction. First, if $y_0 \in Y$ we find an open disc $D(y_0)$ centered at y_0 whose radius is equal to $\text{dist}(y_0, \partial Y)$. Next, let G denote the fundamental group of Y. To each $y_0 \in Y$ we consider the product set

$$D(y_0) \times G$$

It is considered as a disjoint union of copies of the discs given by

$$D(y_0) \times \{g\} \quad g \in G$$

Let us now consider two circles $D(y_0)$ and $D(y_1)$ with a non-empty intersection. Let $y \in D(y_1) \cap D(y_1)$. If $g \in G$ we have the disc $D(y) \times g$. Let $g = \{\gamma\}$ for some closed curve at y. In the disc $D(y_0$ we have the line ℓ_0 from y_0 to y. Now we get a closed curve at y_0 defined by

$$\gamma_0 = \ell_0^{-1} \circ \gamma \circ \ell_0$$

Identifying G with $\pi_1(Y : y_0)$ we get the G-element $g_0 = \{\gamma_0\}$. We say that g_0 is *a neighbor* of g. In the same way we use the straight line ℓ_1 from y_1 to y and construct a neighbor g_1 to g when G is identified with $\pi_1(Y : y_1)$. Set

$$W_0(g) = \{(y, g_0) \in D(y_0) \times \{g_0\} \quad : y \in D(y_0) \cap D(y-1)\}$$

Similarly we set

$$W_1(g) = \{(y, g_0) \in D(y_1) \times \{g_1\} \quad : y \in D(y_0) \cap D(y-1)\}$$

Now we agree to identify $W_0(G)$ and $W_1(g)$. This identification takes place for each $g \in G$. At this stage it is clear how one constructs the universal covering space over Y where the details are left to the reader.

6. The uniformisation theorem.

Introduction. Let Ω be a connected open subset of $\mathbf{C}$. If the closed complement contains at least two points then the universal covering space can be taken as the unit disc D. Moreover there exists a a covering map $f\colon\ D \to \Omega$ of $\mathcal{L}$-type given by an analytic function. This will be proved in Chapter 6. Here we take this existence for granted and analyze some consequences. In particular we discuss some properties of such an analytic uniformisation. More precisely, in Chapter VI we prove Riemann's mapping theorem for connected domains which goes as follows:

6.1 Theorem. *For every $z_0 \in \Omega$ there exists a unique analytic covering map f of $\mathcal{L}$-type where $f(0) = z_0$ and $f'(0)$ is real and positive.*

6.2 The multi-valued inverse to f. We take the theorem above for granted and discuss some consequences. Let f be a an analytic covering map as above. To distinguish the z-coordinate in Ω from D we let w be the complex coordinate in D. To begin with f yields a biholomorphic map from a small open disc D_* centered at the origin in D to a small open neighborhood U_0 of z_0. It gives the inverse analytic function $F(z)$ defined in U_0 such that

$$F(f(w)) = w \quad w \in D\,.$$

Next, let γ be a curve in Ω with $\gamma(0) = z_0$. Since f is of $\mathcal{L}$-type there exists a unique lifted curve γ^* in D with $\gamma^*(0) = 0$. Now the germ of F at z_0 can be continued analytically along γ where

(i) $$T_{\gamma(t)}(F(\gamma(t)) = \gamma^*(t) \quad : \ 0 \leq t \leq 1$$

Hence we get a multi-valued analytic function F in Ω. It gives an inverse to f in the following sense: Let $w \in D$ and consider the curve $t \mapsto t \cdot w$ in D. Now $t \mapsto f(t \cdot w)$ is a curve γ in Ω and by the construction (i) we have

(ii) $$T_{\gamma(t)}(F(f(t \cdot w)) = tw \quad : \ 0 \leq t \leq 1$$

We may express this by saying that the composed function $F \circ f$ is the identity on D.

6.3 Example. To begin with the unit disc D is conformally equivalent to the upper half-plane so in Theorem we can just as well consider an analytic covering map f from U_+ to Ω. Suppose that Ω is the punctured unit disc $\dot{D} = D \setminus \{0\}$. in U_+ we have the analytic function $f(z) = e^{iz}$ and it is clear that it gives an analytic covering map from U_+ onto $\dot{D}$ where we have

$$f(i) = e^{-1} = z_0$$

In $\dot{D}$ we have the multi-valued Log-function

$$F(z) = -i \cdot \text{Log}(z)$$

So here

$$\mathfrak{Im}(F) = - \cdot \text{Log}|z|$$

When $0 < |z| < 1$ it means that the imaginary part is > 0 and it is clear that the F-image is U_+. We also get

$$-i \cdot \text{Log}(e^{iw}) = w \quad : \ w \in U_+$$

6.4 Constructing single-valued functions. Return to the situation in Theorem 6.2, i.e. f is a covering map from D into Ω. Let $g(w)$ be some analytic function in D whose range $g(D) \subset \Omega$ and $g(0) = 0$. We use F to construct another single-valued analytic function $F \circ g$ in D. Namely, let $w \in D$ which gives the curve γ_w $t \mapsto g(t \cdot w)$ in Ω where $\gamma_w(0) = z_0$. We can continue F along this curve and when $t = 1$ we get the value

$$T_{\gamma_w(1)}(F(g(w))$$

It is clear that this gives an analytic function in D defined by

$$F \circ g(w) = T_{\gamma_w(1)}(F(g(w))$$

This construction can be performed for every $g \in \mathcal{O}(D)$ such that $g(0) = z_0$ and $g(D) \subset \Omega$. Hence we have proved the following:

6.5 Proposition. *Let $\mathcal{O}_*(D:\Omega)$ denote the family of analytic functions g in D where $g(0) = z_0$ and $g(D) \subset \Omega$. Then there exists a map from $\mathcal{O}_*(D:\Omega)$ into $\mathcal{O}(D)$ given by:*

$$g \mapsto F \circ g\,.$$

Here $F \circ g(0) = 0$ and the range $(F \circ g)(D) \subset D$.

6.6 Möbius transforms. Let f be a covering map as in Theorem 6.1. Identify the fundamental group $\pi_1(\Omega)$ with homotopy classes of closed curves at z_0. Theorem xx gives a bijective map between elements in the group $\pi_1(\Omega)$ and the discrete subset $f^{-1}(z_0)$ of D. Consider a point a in this inverse fiber, i.e. here $f(a) = z_0$. For each $0 \leq \theta < 2\pi$ we get a new covering map g defined by

$$g(w) = f(e^{i\theta} \cdot \frac{w+a}{1+\bar{a}\cdot w})$$

Here $g(0) = f(a) = z_0$ and the complex derivative at $w = 0$ becomes

$$g'(0) = f'(a) \cdot e^{i\theta} \cdot (1 - |a|^2)$$

We can choose θ so that $f'(a) \cdot e^{i\theta}$ is real and positive. With this choice of θ it follows from the uniqueness in Theorem 6.1 that $g = f$. Hence the function f satisfies

$$(*) \qquad f(w) = f(e^{i\theta} \cdot \frac{w+a}{1+\bar{a}\cdot w})$$

6.7 Inverse multi-valued functions.

Let ϕ be an analytic function defined in some open and connected subset U of $\mathbf{C}$. We assume that the derivative is $\neq 0$ at every point and get the open image domain $\Omega = \phi(U)$. Since ϕ is locally conformal it gives a covering map from Ω onto U. Consider some $\zeta_0 \in \Omega$ and put $x_0 = \phi(\zeta_0)$. We get a germ $f(x)$ of an analytic function at x_0 using the local inverse of ϕ, i.e. since $\phi'(\zeta_0) \neq 0$ there exists a small open disc $D_\delta(\zeta_0)$ such that

$$f(\phi(\zeta) = \zeta \quad : \quad |\zeta - \zeta_0| < \delta$$

In fact, we simply find the convergent power series

$$f(x) = \sum c_\nu (x - x_0)^\nu$$

where $c_0, c_1, \ldots$ are determined so that

$$\sum c_\nu \big(\phi(\zeta) - \phi(\zeta_0)\big)^\nu = \zeta$$

Less obviousis the following

6.8 Proposition. *The germ f at x_0 extends to a multi-valued analytic function in U.*

Proof. Let γ be a curve in U having x_0 as initial point. The lifting lemma gives a unique curve γ^* in Ω. The required analytic continuation of f along γ now follows when we apply the Heine-Borel Lemma cover the compact set γ with a finite set of discs which are homemorphic images of discs in Ω whose consequtive union covers γ^*. Then we use that ϕ is everywhere analytic. The result is that the germ $T_\gamma(f)$ at the end-point $\zeta_1 = \gamma(1)$ satisfies

$$T_\gamma(f)(\phi(x)) = x$$

where x is close to the point $\phi(\gamma^*(1))$ in Ω. So in particular

$$T_\gamma(f)(\gamma(1)) = \gamma^*(1)$$

which clarifies how to determine values of the multi-valued analytic function.

6.9 Remark. It is instructive to consider some specific cases. Consider the entire function $\phi(\zeta) = e^\zeta$. With $\Omega = \mathbf{C}$ the image domain U is the punctured complex plane. If we take $x_0 = 1$ and $\zeta_0 = 0$ we find that f is the multi-valued Log-function where we start with the local branch at $x_0 = 1$ for which $\log 1 = 0$. Next, let us regard the polynomial $\phi(\zeta) = \zeta^2$. in order to get a covering we must exclude the origin to ensure that $\phi'(\zeta) \neq 0$. So if $\Omega = \mathbf{C} \setminus \{0\}$ we get a covering whose image set U also becomes the punctured complex plane. In this case the inverse fiber consists of two points and the function $f(z)$ is the multi-valued square-root of z. More involved examples occur when $\phi(\zeta)$ is a polynomial of degree ≥ 3. Consider the case

$$\phi(\zeta) = \zeta^3 - 3\zeta - 2$$

Here $\phi'(\zeta) = 3\zeta^2 - 3$ and to avoid zeros we take $\Omega = \mathbf{C} \setminus \{-1, 1\}$. In this case the multi-valued inverse function is defined in the open set given as the image under ϕ. Thus, we seek the domain:

$$U = \{x\colon \quad x = \zeta^3 - 3\zeta - 2 \quad : \zeta \neq -1, +1\}$$

The reader is invited to find U. Next, notice that ϕ has a simple zero when $\zeta = 2$. Now we can start with the germ and f_0 at $x = 0$ which satisfies

$$f_0(\zeta^3 - 3\zeta + 1) = \zeta \quad : \quad f_0(0) = 0$$

for a small disc centered at the $\zeta = 2$ in the complex ζ-plane. After analytic continuation the total number of local branches of f at $x = 0$ is 3, i.e. this follows since the inverse fibers under the covering map ϕ contain three points. However, the determination of these local branches is not so easy. The reason is that the fundamental group of U no longer is generated by a single loop around a point, i.e. above we have removed two points. So one must study the analytic continuation of f along several closed curves. To begin with the two simple closed curves in U which surround +1 and -1 respectively. It is clear from this "intuitive discussion" that one needs a more systematic theory to analyze analytic continuations. The most efficient procedure is to use $\mathcal{D}$-module theory where one starts with the polynomial map

$$\phi\colon \zeta \mapsto x = \zeta^3 - 3\zeta + 1$$

and regards not only the special multi-valued inverses function f but more objects which have direct images and to achieve this the basic role is played by the direct image of $\mathcal{D}$-modules under ϕ which after is used to determine the set of differential operators in the Weyl algebra $\mathbf{C}\langle x, \partial_x\rangle$ which annihilate f. Let us remark that $\mathcal{D}$-module theory has a rather recent origin. The foundations of the theory appeared for the first time in the thesis [Kash] by Masaki Kashiwara from 1970. We refer to Chapter for further comments on $\mathcal{D}$-module theory and describe how we use it to investigate the local branches of f in the specific example above.

6.10 Constructing single-valued functions.

Let Ω be a connected open set and consider some multi-valued analytic function F in Ω. Let U be some open and *simply connected* set. Consider some $h \in \mathcal{O}(U)$ whose image set $h(U)$ is contained in Ω. No further conditions on h are imposed, i.e. the inclusion $h(U) \subset \Omega$ may be strict and the derivative of h may have zeros. Using F we produce single valued analytic functions in U by the following procedure. Let us fix a point $\zeta_0 \in U$ and put $x_0 = h(\zeta_0)$. At x_0 we haver the family of local branches of F. Let f_* be one such local branch. Next, let γ be a curve in U where x_0 is the initial point and $x = \gamma(1)$ denotes the end-point. In Ω we get the image curve

(i) $$t \mapsto h(\gamma(t))$$

Now f_* has an analytic continuation along the curve in (i). When $t = 1$ we arrive at the endpoint $\gamma(1)$ which we denote by x. At x we can evaluate the local branch $T_\gamma(f_*)$. Next, let $\gamma_1(t)$ be another curve in U with the same end-point x as γ. By assumption $U$is simply connected which means that the tewo curves γ and γ_1 are homotopic. It is clear that the homotopy in U implies that the two image curves obtained via (i) are homotopic in the curve family in Ω which joint x_0 and x. It follows that the image curves constructed via (i) are homotopic The monodromy theorem applied to F implies that

(ii) $$T_\gamma(f_*)(x) = T_{\gamma_1}(f_*)(x)$$

Next, given an open and simply connected set U in $\mathbf{C}$ we denote by $\mathcal{O}(U)_\Omega$ the family of analytic functions in U whose image is contained in Ω. With these notations the discussion above gives:

6.11 Proposition. For each point $\zeta_0 \in U$ there exists a map

$$\rho\colon\ \mathcal{O}(U)_\Omega \times M\mathcal{O}(\Omega)(x_0) \to \mathcal{O}(U)$$

where $x_0 = h(\zeta_0)$ and for a pair $h \in \mathcal{O}(U)_\Omega$ and $f_* \in M\mathcal{O}(\Omega)(x_0)$ the analytic function $\rho(h, f_*)$ satisfies

$$\rho(h, f_*)(\zeta) = T_\gamma(f_*)(h(\zeta)) \quad : \quad \zeta \in U$$

where γ is the h-image of any curve in U which joins ζ_0 with ζ.

Remark. Keeping h fixed we notice that the map $f_* \to \rho(h, f_*)$ is a $\mathbf{C}$-algebra homomorphism from the complex $\mathbf{C}$-algebra $M\mathcal{O}(\Omega)(x_0)$ into $\mathcal{O}(U)$.

Example. Let $h(z)$ be analytic in the open unit disc D and assume that $h(D) \subset U = \mathfrak{Im}\, z > 0$. Consider the multi-valued function $\operatorname{Log} z$. I has a single valued branch in U and we therefore get an analytic function in D defined by

$$g(z) = \operatorname{Log} h(z)$$

which by the construction satisfies

$$\mathfrak{Im}\, g(z) > 0 \quad : \quad z \in D$$

7. The p^*-function.

We construct a special harmonic function which will be used to get solutions to the Dirichlet problem in XXX. Let Ω be an open and connected set in $\mathbf{C}$. Its closed complement has connected components. Let E be such a connected component. To each $a \in E$ we get the winding number $\mathfrak{w}_a(\gamma)$. If b is another point in E which is sufficiently close to a it is clear that

$$|\frac{1}{\gamma(t)-a} - \frac{1}{\gamma(t)-b}| < |\frac{1}{\gamma(t)-a}|$$

Rouche's theorem from 1.4 implies that $\mathfrak{w}_a(\gamma) = \mathfrak{w}_b(\gamma)$, i.e. for every closed curve γ in Ω, the winding number stays constant in each connected component of $\mathbf{C} \setminus \Omega$. This enable us to construct single valued Log-functions in Ω. Namely, let $a \in E$ where E is a connected componen in the complement of Ω. Consider $f = \text{Log}\,(z-a)$ and choose a single valued branch f_* at some point $z_0 \in \Omega$. If $\gamma \subset \Omega$ is a closed curve with initial point at z_0 the analytic continuation along γ of the Log-function gives:

(1) $$T_\gamma(f_*) = f_* + 2\pi i \cdot \mathfrak{w}_a(\gamma)$$

Next, if b is another point in E we consider $g_* = \text{Log}(z-b)$ and obtain

(2) $$T_\gamma(g_*) = g_* + 2\pi i \cdot \mathfrak{w}_b(\gamma)$$

Since $\mathfrak{w}_b(\gamma) = \mathfrak{w}_b(\gamma)$ it follows that

(3) $$T_\gamma(f_*) - T_\gamma(g_*) = f_* - g_*$$

Hence the difference $\text{Log}(z-a) - \text{Log}(z-b)$ is a *single valued* analytic function in Ω. Taking is exponential we find $\Psi(z) \in \mathcal{O}(\Omega)$ such that

(4) $$e^{\Psi(z)} = \frac{z-a}{z-b}$$

Since $a \neq b$ we see that $\Psi(z) \neq 0$ for all $z \in \Omega$. Next, we get the harmonic function defined in Ω by

(*) $$p(z) = \mathfrak{Re}(\frac{1}{\Psi(z))}) = \frac{\mathfrak{Re}(\Psi(z)}{|\Psi(z)|^2}$$

Notice that $\mathfrak{Re}(\Psi(z)) = \text{Log}\,|z-a| - \text{Log}|z-b|$ and since $\text{Log}\,|z-a| \to -\infty$ as $z \to a$ we see from (*) that

(**) $$\lim_{z \to a} p(z) = 0$$

Notice also that $\Psi(z)$ exends to a continuous function on $\bar{\Omega} \setminus (a,b)$ and we can choose a small $\delta > 0$ such that

(ii) $$\text{Log}|z-a| - \text{Log}\,|z-b| < -1 \quad : \quad |z-a| \leq \delta$$

Then (i) and (ii) give

7.1 Theorem. *Let $a \in \partial\Omega$ be such that the connected component of $\mathbf{C} \setminus \Omega$ which contains a is not reduced to the single point a. Then there exists a harmonic function $p^*(z)$ in Ω for which*

$$\lim_{z \to a} p^*(z) = 0$$

and there exists $\delta > 0$ such that

$$\max_{\{|z-a|=r\}\,\cap\Omega} p^*(z) < 0 \quad : \quad z \in D_a(r) \cap \Omega$$

8. Eisenstein's theorem.

Introduction. The theorem below was announced by Eisenstein in 1852 and goes as follows: Let

$$w(z) = c_1 z + c_2 z^2 \ldots \tag{0.1}$$

be a convergent power series in a disc centered at the origin where the coefficients $\{c_\nu\}$ are rational numbers. Assume also that w satisfies an algebraic equation:

$$q_m(z) w^m + \ldots + q_1(z) w + q_0(z) = 0 \quad : q_0, \ldots, q_m \text{ are polynomials} \tag{0.2}$$

Eisenstein's Theorem. *Under the assumption above there exists a positive integer k such that*

$$k^\nu \cdot c_\nu \in \mathbf{Z} \tag{*}$$

Or equivalently, we can find k so that $w(kz)$ has a power series expansion with integer coefficients.

The complete proof of (*) was given by Heine from 1854. Since $\mathbf{R}$ is a vector space over the field of rational numbers it is easily seen that (0.1) implies that we can choose the q-polynomials in $Q[z]$, i.e. so that they all have rational coefficients. Multiplying these q-polynomials with a positive integer we remove denominators and may assume from the start that they have integer coefficients.

0.1 A special case. Suppose that the q-polynomials satisfy:

$$q_m(0 = \ldots = q_2(0) = 0 \quad \text{where} \quad q_1(0) = k \neq 0 \tag{**}$$

When (**) holds an easy induction over ν shows that $k^\nu \cdot c_\nu$ are integers for all k. The non-trivial part of the proof is the reduction to the case (**). Heine achieved this by the following construction: For every positive integer p we break the series (0.1) and write

$$w(z) = c_1 z + \ldots + c_{p-1} z^{p-1} + \beta_p(z) \cdot z^p \tag{0.5}$$

Heine showed that if p is sufficiently large then $\beta_p(z)$ satisfies an algebraic equation for which (*) holds and after this Eisenstein's theorem can be derived. Heine's reduction is established in section 2. But first we expose background about algebraic functions which is needed for Heine's result and in section 3 we give the proof of Eisenstein's theorem.

1. On algebraic functions.

Let $\mathbf{C}[z, w]$ be the polynomial ring in two independent variables Introducing the field $K = \mathbf{C}(z)$ of rational functions in z we get the polynomial ring $K[w]$ of one variable. This ring has important properties. Namely, if $Q(w) = \sum k_\nu(z) \cdot w^\nu$ is a w-polynomial with of some degree $m \geq 1$, then every other polynomial $S \in K[w]$ can be written in a unique way as

$$S(w) = A(w) \cdot Q(w) + R(w) \quad : \text{ degree of } R(w) \leq m - 1 \tag{1}$$

This division shows that $K[w]$ is an *euclidian ring* where every ideal is principal and every polynomial can be written in a unique way as a product of irreducible polynomials. Of course, in such a factorisation multiple factors can occur. Suppose now that $Q(w)$ is an irreducible polynomial on $K[w]$. Bringing out the denominators in the coefficients from $\mathbf{C}(z)$ it follows that $Q(w)$ corresponds to a polynomial

$$q(w, z) = \rho(z) \cdot [g_m(z) w^m + \ldots + g_1(z) w + g_0(z)] \tag{2}$$

where $g_0(x), \ldots, g_m(z)$ are polynomials in $\mathbf{C}[x]$ without a common zero. Moreover, this factorisation is unique when we require that the leading polynomial $g_m(x)$ is monic, i.e. if k is its degree then $g_m(x) = x^k +$ lower order monomials. Now (2) is used to construct root functions of $q(z, w)$. Namely, let σ be the finite set of zeros of $g_m(z)$. For each fixed $z_* \in \mathbf{C} \setminus \sigma$ we get a w-polynomial of degree m:

$$g_m(z_*)w^m + \ldots + g_1(z_*)w + g_0(z_*)$$

By the fundamental theorem of algebra it has m roots where eventual multiple roots are repeated. Denote this unordered m-tuple of roots by $\alpha_1(z_*), \ldots, \alpha_m(z_*)$. Then we have

$$q(w, z_*) = \prod_{\nu=1}^{\nu=m} (w - \alpha_\nu(z_*)) \tag{3}$$

To find out if multiple roots occur we set

$$\Delta(z_*) = \prod_{j \neq \nu} (\alpha_\nu(z_*) - \alpha_j(z_*)) \tag{4}$$

This is a product of $m(m-1)/2$ many factors and constructed in a symmetric fashion, i.e. the product does not change when the m-tuple of roots is permuted. Since $q(w, z)$ comes from the irreducible polynomial $Q(w)$ the Δ-function is not identically zero. To see this we regard the w-derivative of $Q(w)$ which has degree $m-1$ in $K[w]$. Since $Q(w)$ is irreducible there exist polynomials $A(w)$ and $B(w)$ in $K[w]$ such that

$$A(w)Q'(w) + B(w)Q(w) = 1 \tag{5}$$

Bringing out common denominators we get an equality in $\mathbf{C}[z, w]$:

$$a(w, z) \cdot \partial(Q(w, z)/\partial w + b(w, z) \cdot Q(w, z) = h(z) \tag{6}$$

where a, b, h are some polynomials and $h(z)$ is not identically zero. Then it is clear that if $z_* \in \mathbf{C} \setminus \sigma$ and $h(z_*) \neq 0$, then $\Delta(z_*) \neq 0$.

1.1 The resolvent polynomial for Δ. We have seen that $\Delta(z_*)$ is a symmetric product of the roots. By a wellknown result in algebra this implies that it can be expressed as a sum of elementary symmetric α-polynomials. Applied to the present situation this gives rational functions $c_0(z) \ldots, c_{m(m-1)/2}(z)$ such that

$$\Delta(z_*) = \sum c_j(z_*) \cdot [\, \alpha_1^j(z_*) + \ldots, \alpha_m^j(z_*)] \quad : \ z_* \in \mathbf{C} \setminus \sigma. \tag{7}$$

Now residue calculus gives:

$$\alpha_1^j(z_*) + \ldots + \alpha_m^j(z_*) = \frac{1}{2\pi i} \cdot \int_{|w|=R} \frac{w^j \cdot \partial q/\partial w(w, z_*) \cdot dw}{q(w, z_*)} \tag{8}$$

where we for each fixed z_* choose R so large that the zeros of $q(w, z_*)$ have absolute value $< R$. From this we conclude that $\Delta(z_*)$ is a rational function of the z-variable and since it stays bounded when the leading polynomial $g_m(z)$ of $q(w, z)$ is non-zero, this rational function has poles contained in $g_m^{-1}(0)$. Finally, using (7-8) and the euclidian division in $K[w]$ we arrive at

1.2 Proposition. *There exists a unique w-polynomial*

$$\mathcal{R}(w, z) = r_{m-1}(z)w^{m-1} + \ldots + r_0(z) \quad : \ r_\nu(z) \in \mathbf{C}|z]$$

such that

$$\Delta(z_*) = \frac{1}{2\pi i} \cdot \int_{|w|=R} \frac{\mathcal{R}(w, z_*) \cdot \partial q/\partial w(w, z_*) \cdot dw}{q(w, z_*)} \quad : \quad z_* \in \mathbf{C} \setminus \sigma$$

Moreover, the poles of the rational r-functions are contained in $g_m^{-1}(0)$.

Proof. The existence of $\mathcal{R}$ has already been settled. To see that uniqueness holds we recall from linear algebra that the *van der Monde determinant* constructed from the $m \times m$-matrix whose rows are $1, \alpha_\nu(z_*), \ldots, \alpha_\nu^{m-1}(z_*)$ is $\neq 0$ when $z_* \in \mathbf{C} \setminus \sigma$.

1.3 The multi-valued root functions. Put $\sigma^* = \sigma \cup h^{-1}(0)$. Outside this set the m-tuple of roots are distinct and at the same time we avoid zeros of the g_m-polynomial. This implies that the root functions yield single valued analytic functions in every disc D contained in $\Omega = \mathbf{C} \setminus \sigma^*$. Moreover, each single root is a germ of a multi-valued analytic function defined in the whole connected set Ω. These multi-valued functions are special since we always stay within roots during an analytic continuation. So if z_* is fixed in Ω and $\alpha_1(z)$ is a local branch of one root at z_*, then $T_\gamma(\alpha_1)$ is again a root of $q(w, z_*)$ whenever γ is a closed curve a z_*. Hence the family $\{T_\gamma(\alpha_1)\}$ is a finite subset of $\mathcal{O}(z_*)$. Moreover, this family consists of *all the root functions* at z_*. For assume the contrary. Then all the analytic continuations of α_1 produce k many root functions at z_*, say $\alpha_1, \ldots, \alpha_k$. This k-tuple is permuted under all analytic continuations along closed curves at z_*. As explained in XX the same holds at other points in $\mathbf{C} \setminus \sigma^*$. Hence their symmetric products become single valued and we get a function

$$B(w,z) = \prod_{\nu=1}^{\nu=k} [w - \alpha_\nu(z))$$

which is a polynomial in w with rational coefficients in $\mathbf{C}$. In the same way get a polynomial $B_1(w,z)$ using the symmetric product over the remaining $m-k$ roots functions. But then the irreducible polynomial $Q(w)$ in $K[w]$ has a factorisation $B \cdot B_1$ which is a contradiction. Hence we have the proved:

1.4 Proposition. *For each single root-function α at a point $z_* \in \mathbf{C}) \setminus \sigma^*$ it follows that $\{T_\gamma(\alpha)\}$ consists of the whole m-tuple of root functions at z^*.*

Remark. This means that in the total sheaf space $\widehat{\mathcal{O}}$ there exists a connected set W contained in $\rho^{-1}(\mathbf{C} \setminus \sigma^*)$ which corresponds to the root functions. In particular the projection $\rho\colon W \to \mathbf{C} \setminus \sigma^*$ is an m-sheeted covering map.

1.5 Associated algebraic functions. We have the multi valued root functions $\{\alpha_\nu(z)\}$ in $\Omega = \mathbf{C} \setminus \sigma_*$. If $z_0 \in \Omega$ each root function is analytic in a disc D centered at z_0 and has some series expansions:

$$\alpha_\nu(z) = \sum_{j=0}^{\infty} c_{\nu,j}(z - z_0)^j \quad : 1 \le \nu \le m \tag{1}$$

Take $\nu = 1$ and start the series expansion of $\alpha_1(z)$. If $p \ge 1$ we can write

$$\alpha_1(z) = c_{1,0} + \ldots + c_{1,p-1} \cdot (z - z_0)^{p-1} + (z - z_0)^p \cdot \beta_p(z) \tag{2}$$

Here $\beta_p(z)$ is a new germ of analytic function at z_0 and we can write

$$\beta_p(z) = \frac{\alpha_1(z) - c_{1,0} + \ldots + c_{1,p-1} \cdot (z - z_0)^{p-1}}{(z - z_0)^p} \tag{3}$$

If γ is an curve in $\mathbf{C} \setminus \sigma^*$ we obtain an analytic continuation of β where

$$T_\gamma(\beta_p) = \frac{T_\gamma(\alpha_1) - c_{1,0} + \ldots + c_{1,p-1} \cdot (z - z_0)^{p-1}}{(z - z_0)^p} \tag{3}$$

The resulting his multi-valued extension produces the same number of different local branches a z_0^* as the germ of α_1, i.e. the number is equal to m. From this we conclude that β_p is a root function associated to an irreducible polynomial $S[w]$ of degree m in $K[w]$.

2. Heine's reduction.

We establish a result due to Heine which will be used to prove Eisenstein's theorem. With the notations from § 1.5 we have $\beta_p(z)$ which is analytic at z_0. When we perform analytic continuations, α_1 changes and it may occur that we arrive to a root α_j where the numerator in (3) from 1.5 does not vanish up to order p at z_0, i.e. we may encounter a pole at z_0 when we regard

$$\frac{\alpha_j(z) - c_{1,0} + \ldots + c_{1,p-1} \cdot (z - z_0)^{p-1}}{(z - z_0)^p} \tag{4}$$

In fact, this will occur for *every* $2 \leq j \leq k$ if p is large enough. To see this we notice that the m-tuple of germs $\alpha_1, \ldots, \alpha_m$ at z_0 are *linearly independent* in the complex vector space $\mathcal{O}(z_0)$. This implies that there exists some p^* such that the power series of $\alpha_1, \ldots, \alpha_m$ at z_0 cannot be identical up to order p^*. So when $p \geq p^*$ it follows that we get a pole at z_0 for each $2 \leq j \leq m$ in (4).

2.1 A consequence. Let p be so large that we get poles at z_0 in (4) for every $2 \leq j \leq m$. Regard analytic continuations along closed curves γ^* which stay in Ω^*. Then (2) implies that

$$T_{\gamma^*}(\alpha_1) = c_{1,0} + \ldots + c_{1,p-1} \cdot (z - z_0)^{p-1} + (z - z_0)^p \cdot T_{\gamma^*}(\beta_p) \tag{5}$$

Choose γ^* so that $T_{\gamma^*}(\beta_p)$ is another root of the algebraic equation satisfied by β_p. Let us call it ρ for the moment. Since $\rho \neq \beta_p$ in $\mathcal{O}(z_0^*)$ it follows that $T_{\gamma^*}(\alpha_1) \neq \alpha_1$ and hence $T_{\gamma^*}(\alpha_1) = \alpha_j$ for some $j \geq 2$. So at z_0^* we have

$$\alpha_j(z) = c_{1,0} + \ldots + c_{1,p-1} \cdot (z - z_0)^{p-1} + \rho(z)(z - z_0)^p \tag{6}$$

But now (4) gives a pole at z_0 and therefore the germ ρ *cannot* extend to be analytic a z_0. This holds for every root except β_p. Hence we have proved

2.2 Proposition. *If $S[w)$ is the irreducible polynomial in $K[w]$ which produces the root functions generated by β_p, then β_p is the only root which can be extended analytically at z_0. Moreover, if Ω is a simply connected sector in D and $\rho_2, \ldots, \rho_m$ the remaining roots then their single-valued restrictions to Ω are unbounded when z_0 i approached.*

2.3 Structure of $S[w]$ To $S[w]$ we get exactly as in (0.2) a polynomial

$$s(w, z) = \rho^*(z) \cdot s_m(z) w^m + \ldots + s_1(z) w + s_0(z)] \tag{i}$$

where $\beta_p(z)$ appears as one root function. By analyticity any germ $\phi \in \mathcal{O}(z_0)$ which satisfies

$$s_m(z) \cdot \phi^m(z) + \ldots + s_1(z) \cdot \phi(z) + s_0(z) = 0$$

in a neighborhood of z_0 yields a root function of $S[w]$. Now Proposition 2.2 and the result in XX shows that we must have

$$s_m(0) = \ldots = s_2(0) = 0 \tag{**}$$

This finishes the proof of Heine's reduction

3. Proof of Eisenstein's theorem.

Let $w(z)$ satisfy an algebraic equation $P(z, w) = 0$ and assume that it has a rational series expansion (0.1) at some point. Now $P(z, w) = \sum c_j k z^j w^k$ with complex coefficients. But **C** is a vector space over the field Q of rational numbers. So regarding the finite set of complex coefficients they can be expanded in a basis, i.e. we can write

$$c_{jk} = \sum q_{j,k;\alpha} \cdot \xi_\alpha \quad : \; q_{i,j;\alpha} \in Q \quad , : \; m\{\xi_\alpha\} \text{ are linearly independent over } Q$$

If $P_\alpha(z, w) = \sum q_{j,k;\alpha}(z, w)$ it follows that $\xi_\alpha \cdot \sum P_\alpha(z, w) = 0$. Using the rational series for w and the Q-linear independence of the ξ-numbers, we see that $P_\alpha(z, w) = 0$ for eah α. Now we simply use that at least one of these polynomials is not identically zero. hence w satisfies t an algebraic equation defined by a polynomial in $Q[z, w]$. So now P has rational coefficients. Next, in (0.5) the expansion of $\beta_p(z)$ contains all coefficients in the expansion in (0.1) except for $c_0, \ldots, c_{p-1}$. if $k \geq 1$ is found so that $\beta_p(kz)$ has an integer expansion the same holds for $w(M \cdot z)$ when M is an integer such that $M \cdot c_\nu \in \mathbf{Z}$ for each $0 \leq \nu \leq p-1$. There remains only to show that k exists for the algebraic function $\beta_p(z)$ which satisfies an equation $Q(z, \beta_p) = 0$. Here P has rational coefficients and multiplying these by some integer we can assume that the polynomial has integer coefficients. To simplyify notations we set $w = \beta_p$ and we are in the favourable case described by (*) in (0.3). The rational expansion of $w(z)$ is given as in (0.1).

(i) $$\sum_{\nu=2}^{\nu=m} q_\nu(z) \cdot w(z)^\nu + q_1(z)w(z) + q_0(z) = 0$$

Let $2 \leq \nu \leq m$ be given. Since $q_j(0) = 0$ we have

(ii) $$q_\nu(z) \cdot w(z)^\nu = \sum_{j \geq 1} q_{j,\nu} z^j \cdot (c_0 + c_1 z + \ldots)^\nu = \sum \rho_{\nu,N} \cdot z^N$$

By assumption $q_{j,\nu} \in \mathbf{Z}$ and since $j \geq 1$ always occurs it is clear that if $N \geq 1$ then the ρ-coefficient is expressed by

$$\rho_{\nu,N} = \sum_{j \geq 1} q_{j,\nu} z^j \cdot B_j[c_0, \ldots, c_{N-j}]$$

where each B_j is a homogeneous polynomial of degree $N - j$. This implies that if k is a positive integer such that $k^j \cdot c_j \in \mathbf{Z}$ hold for $0 \leq j \leq N-1$, then $k^{N-1} \cdot \rho_{\nu,N} \in \mathbf{Z}$. This conclusion holds for every $\nu \geq 2$. Together with (i) we get the following

Sublemma. Let $N \geq 2$ and assume let k be an integer k is an integer such that $k^j c_j \in \mathbf{Z}$ hold for each $0 \leq j \leq N-1$. Then

$$k^{N-1} \cdot \rho_{0,N} \in \mathbf{Z}$$

The Sublemma enable us to cary out an induction. For first we have the zero coefficient for z^N in (i) which gives:

(iii) $$q_{0,1} \cdot c_N + \sum_{j \geq 1} q_{j,1} c_{N-j} + \sum_{\nu \geq 2} \rho_{\nu,N} + q_{N,0} = 0$$

Here all the doubly-indexed q-numbers are integers. If $k = q_{0,1}$ and the induction holds up to N_1 we therefore get $k^N \cdot c_N$ and hence Eisenstein's Theorem will follow. Well, of course we must settle the start, i.e. we must also show that

$$q_{0,1} \cdot c_1 \in \mathbf{Z}$$

But this is clear for now we only have to identify the z-coefficient in (0.1) and get

$$\sum_{\nu \geq 2} q_{1,\nu} \cdot c_0 + q_{0,1} \cdot c_1 + q_{1,0} = 0$$

Since we also may assume that c_0 is an integer we get (x) as required. So this finishes the proof of Eisenstein's Theorem.

9. Extensions by reflection

Introduction. *Das Spiegelungsprinzip* is due to H. Schwartz. It is frequently used to obtain analytic continuations. First we describe the standard case. Let $f(z)$ be an analytic function in the upper half plane $U_+ = \{\mathfrak{Im}\, z > 0\}$. Let $J(a,b) = \{a < x < b\}$ be an interval, on the real axis. Suppose that f extends to a continuous function to this open interval and takes real values. In the lower half-plane U_- we get the analytic function

(i) $$f_*(z) = \bar{f}(\bar{z})$$

By the result in XX the two functions are analytic continuations of each other over (a,b). So this means that f itself has an analytic extension to the open set $\Omega = \mathbf{C} \setminus J$, where $J_* = (-\infty, a] \cup [b, +\infty)$ is the closed complement of $J(a,b)$ on the x-axis. Next, suppose that $e^{i\theta} f(z)$ extends to a real-valued function on (a,b) for some θ. After multiplication with $e^{-i\theta}$ we get an extension of f. That is, one has only to require that the argument of f is constant to obtain an analytic continuation. Suppose now that the argument of f is constant over a family of pairwise disjoint intervals $\{J(a_\nu, b_\nu)\}$. Then we get analytic continuations across each interval. In particular one has:

9.1 Theorem. *Let $a_1 < \ldots < a_N$ be a finite set of real numbers and assume that f extends to a continuous function on each of the intervals*

$$J_0 = (-\infty, a_1) \quad : \; J_\nu = (a_\nu, a_{\nu+1} : 2 \leq \nu \leq N_1 \quad : \; J_N = (a_N, +\infty)$$

and on every such interval the argument of f is some constant. By reflection over one such interval we obtain an analytic of f_*^ν to the set $\mathbf{C} \setminus (a_1, \ldots, a_N)$.

Example. In the upper half-plane U_+ we consider the analytic function

$$f(z) = \sqrt{z} \cdot \sqrt{1-z}$$

The single-valued branches of the root functions are chosen so that

$$\sqrt{z} = \sqrt{r} \cdot e^{i\theta/2} \quad : \quad \sqrt{z-1}) = \sqrt{1 + r^2 - 2r \cdot \cos\theta} \cdot e^{i\phi} \quad : z = re^{i\theta}$$

where $0 < \theta < \pi$ and ϕ is the outer angle of the triangle in figure xx. So here $0 < \phi < \pi$ holds. As we approach a point $0 < x < 1$ we get the boundary value

$$f(x) = \sqrt{x} \cdot i \cdot \sqrt{1-x}$$

Now get the analytic continuation f_*^1 across the interval $J_1 = (0,1)$ which becomes an analytic function defined in the lower half-plane U_- by

$$f_*^1(z) = -\bar{f}(\bar{z})$$

Notice that the minus-sign appears in order that $f(x) = f_*^1(x)$ holds for $0 < x < 1$. Suppose now that $x > 1$. Then we get

$$\lim_{y \to 0} f_*^1(x - iy) = \lim_{y \to 0} -\bar{f}(1(x+iy) = -\sqrt{x} \cdot \sqrt{x-1}$$

So f and f_*^1 do not agree on the real interval $(1, +\infty)$. At the same time f_*^1 can be continued analytically across $(1, +\infty$ and gives an analytic function f_+** defined in U_+ where we obtain

$$f_+^{**}(z) = -f(z) \quad : \quad z \in U_+$$

Another analytic continuation of f_*^1 takes place across $(-\infty, 0)$. When $x < 0$ we have

$$\lim_{y \to 0} f_*^1(x - iy) = \lim_{y \to 0} -f(1(x+iy) =$$

After f has been extended to the lower half-plane where we get an analytic function denoted by f_* we notice that f_* by the construction also has boundary values with a constant argument as we approach points on the real axis form below. So f_* also extends to the upper half-plane where we encounter a new analytic function $f^*(z)$. Next, we can continue f^* to the lower half-plane and so on. The result is that f from the start extends to a multi-valued function in $\mathbf{C} \setminus (a_1, \ldots, a_N)$.

9.2 The use of conformal maps. For local extensions there exists a general result. Let D be an open disc and γ a Jordan arc which joins two points on ∂D and separates $D \setminus \gamma$ into a pair of disjoint Jordan domains. Let f is analytic in one of the Jordan domains, say D^*. Assume also that f extends to a continuous and real-valued function on γ. Now there exists a conformal map from D^* to the upper half-plane and using this it follows that f extends analytically across γ. of course, the extension f_* will in general only exist in a small domain close to γ, i.e. this is governed via the conformal mapping. But here exists at least a locally defined analytic continuation across γ.

9.3 Boundary values on circles Let $f(z)$ be as above and suppose it extends continuously to γ where the absolute value is constant, say 1. Using a conformal map from D_* to the unit disc we may assume that γ is an interval of the unit disc D and f is analytic in a small region $U \subset D$ where γ appears as a relatively open subset of ∂U. By hypothesis $f(\gamma)$ is a subset of another unit circle and using a conformal map from the disc bordered by this unit circle we get the situation in 7.2. and conclude that f continues analytically across γ. More generally, using a locally defined conformal map there exists an analytic extension of f across γ if we only assume that the continuous boundary values of f on γ are contained in some locally defined *real-analytic curve.* Finally, by a two-fold application of conformal mappings we get the following quite general result:

9.4 Theorem. *Let $f(z)$ be analytic in a Jordan domain Ω and suppose that $\gamma$is an open arc of $\partial\Omega$ such that f extends continuosly from Ω to $\Omega \cup \gamma$ and the restriction $f|\gamma$has a range $f(\gamma)$ contained in a simple real-analytic curve γ^*. Then f extends analytically across γ, i.e. there exists an open and connected neighborhood U of γ such that the original f-function extends to the connected domain $\Omega \cup U$.*

Remark. Theorem 9.4 follows from the fact that if Ω_2 and Ω_2 are two Jordan domains whose boundaries both are *real analytic* closed Jordan curves, then a conformal map from Ω_1 to Ω_2 extends to a conformal map from an open neighborhood of $\bar{\Omega}_1$ to an open neighborhood of $\bar{\Omega}_2$.

10. The elliptic modular function

Introduction. We construct an analytic function $\lambda(z)$ in the upper half-plane $U_+ = \mathfrak{Im}\, z > 0$ whose complex derivative of λ is $\neq 0$ and the image set $\lambda(U_+)$ is equal to the connected open set $\Omega = \mathbf{C} \setminus \{0, 1\}$, i.e. the two points 0 and 1 are removed from the complex plane.

8.1 An initial construction. Consider the simply connected open set

$$V_0 = U_+ \cap \ |z - 1/2| > 1/2 \cap \{0 < \mathfrak{Re}\, z < 1\}$$

Here ∂V_0 consists of three pieces. One is the vertical line ℓ_0 on which $x = 0$ and $y > 0$. The second is the line ℓ_1 on which $x = 1$ and $y > 0$. Finally we have the half-circle

$$T_0 = \{1/2 + 1/2 \cdot e^{i\theta}\} \quad : \quad 0 < \theta < \pi$$

We admit Riemann's mapping theorem for simply connected domains and find a unique conformal mapping λ_0 from V_0 onto U_+ when we require that λ_0 yields a 1-1 map from ℓ_0 to $(-\infty, 0)$ and after it maps T_0 onto $(0, 1)$ and finally ℓ_1 is mapped onto $(1, +\infty)$. See Figure XXX.

8.2 Reflection on vertical lines. The boundary values along ℓ_0 of the mapping function λ_0 stay on the real interval $(-\infty, 0)$. We can therefore apply the reflection principle to continue λ_0 across ℓ_0. To see this we consider the domain

$$V_{-1} = V_0 - \{1\}$$

That is, points in V_{-1} are of the form $z - 1$ with $z \in V_0$. Notice that we get a 1-1 from V_{-1} onto V_0 by:

$$z \mapsto -\bar{z} \quad : \quad z \in V_{-1}$$

By the Schwarz reflection principle we get an analytic function in V_{-1} defined by

$$g(z) = \bar{\lambda}_0(-\bar{z})$$

Since $-\bar{z} = z$ hold when $z = iy$ we conclude that λ_0 extends to an analytic function in the domain formed by the union $V_0 \cup V_{-1}$ and the real line where $x = 0$ and $y > 0$. Since conjugate values appear in (ii) this extended function has an image given by $\mathbf{C} \setminus [0, +\infty)$ and by the construction this map is even conformal.

In exactly the same way we can extend λ_0 across ℓ_1,. This gives an analytic function defined in the union of $V_1 = V_0 + \{1\}$ and V_0 which yields a conformal map from this open set onto $\mathbf{C} \setminus (-\infty, 0]$. At this stage it is clear how one can proceed. For example, we can regard the extended λ-function in V_1 and extend it across the line ℓ_2 where $x = 2$ and $y > 0$. Simiarly we get extensions to the left. As explained by Figure xx the result after all such reflections across vertical lines ℓ_ν we arrive at the following. Set

$$V_* = \cup \{V_0 + \nu\}$$

where the union is taken over all integers. Then one has:

8.3 Proposition. *After continuation we obtain a single valued analytic function λ_* defined in the domain V_* whose image set is $\mathbf{C} \setminus [0.1]$.*

Remark. As explained by Figure x we have for each integer k the simply connected domain $V_k = V_0 + \{k\}$. The restriction of λ_* to V_k gives a conformal map onto the upper half-plane if k is an even integer and the lower half-plane if k is odd. Notice also that λ_* is periodic:

$$\lambda_*(z) = \lambda_*(z + 2k) \quad \text{for each integer} \quad k.$$

8.4 Reflections over circles. Let us return to the function λ_0. The semi-circle T_0 borders V_0 and is mapped by λ_0 to the real interval $(0, 1)$. So again we can apply the reflection principle. Namely, introduce a new complex variable w and represent points on T_0 by

$$1/2 + 1/2 \cdot e^{i\theta}$$

Here $\lambda_0(1/2+w/2)$ is defined in a portion of the exterior disc $|w|>1$, i.e. when $1/2+w/2 \in V_0$. Set

$$g(w) = \bar{\lambda}(1+\frac{1}{2\bar{w}})$$

Then $g(w) = \lambda_0(1/2+w/2)$ when $|w|=1$ and g extends to an analytic function in $|w|<1$ as long as

(*) $$1+\frac{1}{2\bar{w}} \in V_0$$

By drawing a figure and using euclidian geometry one sees that (*) holds if and only if the corresponding points $z=1/2+w/2$ stay outside the union of the two half-discs $|z-1/4|<1/4$ and $|z-3/4|<1/4$. The conclusion is that λ_0 extends from V_0 to the larger domain

$$V_0(1) = \{z : 0 < \mathfrak{Re}(z) < 1\} \setminus \{|z-1/4| \leq 1/4\} \cup \{|z-3/4| \leq 1/4\}$$

The extended function gives a conformal map from to $V_0(1)$ onto the simply connected subset of $\mathbf{C}$ where $(-\infty,0]$ and $(1,+\infty)$ are removed. In the same way we apply the reflection principle in each domain V_k and obtain an analytic extension to the set $V_k(1) = \{V_0(1)+k\}$. Taking the whole union we obtain an extension of the λ_*-function to the domain $V_*(1)$ which is a simply connected subset of the upper half-plane bordered by a sequence of half-circle of radius 1/4. See Figure XX.

Let us denote the extended function by λ_{**}. The construction goes on where we now use the refection principle for the extension of λ_{**} across each of the half-circles which borders $V_*(1)$ and as a result we get an extension to a domain $V_*(2)$ which now is bordered by a sequence of half-circles of radius 1/8 centered at integer multiples of 1/8. We can continue and passing to the limit we obtain an analytic function λ defined in the whole upper half-plane with values in $\mathbf{C} \setminus \{0,1\}$. This proves the existence of the required λ-function.

8.5 Invariance under a Möbius group By the construction the λ-function is locally conformal but not 1-1. It turns out that this function is *invariant* under a group of Möbius transaformations U_+.

8.6 Definition. *Denote by G the group of Möbius transformations of the form*

$$z \mapsto \frac{az+b}{cz+d} \quad : \quad ad-bc=1 \quad : a,b,c,d \in \mathbf{Z} \quad : \ a,d \text{ are even} : b,c \text{ are odd}$$

One verifies easily that G is a group and that every G-element yields a conformal map of U_+ onto itself. By scrutinizing the construction of the λ-function one verifies that it is invariant under G. More precisely,

$$\lambda(z) = \lambda(g(z)) \quad : \quad g \in G \quad : z \in U_+$$

This means that λ is an automorphic function with respect to G. Moreover, for each point $\zeta \in \mathbf{C} \setminus \{0,1\}$ the inverse the fiber $\lambda^{-1}(\zeta)$ is an orbit for G. More precisely, we get the inverse fiber by the following procedure: Pick any z_0 with $\lambda(z_0)=\zeta_0$. Then the map

$$g \mapsto g(z_0) \quad : \quad g \in G$$

yields a *bijective* correspondence between the elements of the group G and the inverse fiber $\lambda^{-1}(\zeta_0)$.

Remark. For a further account of the assertions above the reader may consult [Bieberbach : page 40-45].

The multi-valued inverse. Since $\lambda(U_+) = \mathbf{C}\setminus\{0,1\}$ and λ is locally conformal we can construct a multi-valued inverse function to be denoted by $\mathfrak{m}$. Namely, set $\Omega = \mathbf{C} \setminus \{0,1\}$ and consider the point $\zeta_0 = i$. We first find the unique point $z_0 \in V_0$ such that $\lambda(z_0) = i$. At ζ_0 we get a unique germ $\mathfrak{m}_0(\zeta) \in \mathcal{O}(\zeta_0)$ such that

$$\mathfrak{m}_0(\lambda(z)) = z$$

hold for z close to i. Next, let γ be a curve in Ω which starts at i and has some end-point ζ_1. Since λ is locally conformal there exists a unique curve γ^* in U_+ such that

$$\lambda(\gamma^*(t)) = \lambda(t) \quad : \ 0 \leq t \leq 1$$

Since λ is locally conformal it is clear that we can construct an analytic extension of $\mathfrak{m}_0$ along γ which locally produces inverses of the λ-function. For the resulting multi-valued $\mathfrak{m}$-function we get the sets of values $W(\mathfrak{m}, \zeta)$ for every $\zeta \in \Omega$. This set is in a 1-1 correspondence with the inverse fiber $\lambda^{-1}(\zeta)$ and hence also a bijective with the unimodular group G above.

11. Poincare's theory of Fuchsian groups

The theory of Fuchsian groups was created by Poincaré. His two articles *Théorie des groupes fuchsiens* and *Memoire sur les fonctions fuchsiennes* were published 1882 in the first first volume of Acta Mathematica and the article *Memoire sur les groupes kleinéens* appeared in volume III. The last article is more advanced and we shall not discuss Kleinan groups here. Nor do we discuss the article *Memoire sur les fonctions zétafuchsiennes.* The connection to arithmetic was presented in a later article *Les fonctions fuchsiennes et l'Arithmétique* from 1887. One should also mention the article *Les fonctions fuchsiennes et l'équation* $\Delta(u) = e^u$ where Poincaré proved that this second order differential equation has a subharmonic solution with prescribed singularities on every closed Riemann surface attached to an algebraic equation. The last work started potential theoretic analysis on complex manifolds. Here we only discuss material from the first two cited articles.

Remark. Poincaré was inspired by earlier work, foremost by Bernhard Riemann, Hermann Schwarz and Karl Weierstrass. For example, he used the construction of multi-valued analytic extensions by Weierstrass which leads to the *Analytische Gebilde* of a multi-valued function f defined in some connected open subset Ω of $\mathbf{C}$. This *Analytische Gebilde* is a connected complex manifold X on which f becomes a single valued analytic function f^*. More precisely, there exists a locally biholomorphic map

$$\pi\colon\ X \mapsto \Omega$$

When $U \subset \Omega$ is simply connected the inverse image $\pi^{-1}(U)$ is a union of pairwise disjoint open sets U_γ^* where the single-calued analytic function f^* is determined by a branch T_γ of f, i.e. one has

$$T_\gamma(f)(\pi(x) = f^*(x) \quad : x \in U_\gamma^*$$

Major contributions are also due to Schwarz. In 1869 he used the reflection principle and calculus of variation to settle the Dirichlet problem and used this to prove the uniformisation theorem for connected domains bordered by p many real analytic and closed Jordan curves where p in general is ≥ 2. Of special interest is the multi-valued $\mathfrak{m}$-function defined in $\mathbf{C} \setminus \{0,1\}$ which is related to the elliptic integral of the first kind and hence to Jacobi's $\mathfrak{sn}$-function which appears in the equation of motion when a rigid body rotates around a fixed point.

A comment. The theory of Fuchsian functions was not restricted to analytic function theory. The main concern for Poincaré was to develop the theory of differential systems, both linear and non-linear. His research was also directed towards to the general theory about abelian functions and their integrals, inspired by Abel's pioneering work. Hundreds of text-books have appeared after Poincaré. Personally I find that his own and often quite personal presentation superseeds most text-books which individually only treat some fraction from his great visions. See in particular the book *Analyse de ses travaux scientifiques* where Poincaré describes his research areas in the period between 1880 until 1907. It contents has the merit that it not only contains a summary of results but also explanations of the the main ideas and methods which led to the theories.

Of course there exists more recent advancement in function theory. Here one should foremost mention work by Lars Ahlfors. So in addition to the cited reference above I recommend text-books by Ahlfors, especially his book *Conformal Invariants* which contains material about the theory of *extremal length* created by Arne Beurling. From a complex analytic point of view the discoveries by Ahlfors and Beurling have a wider scope and has led to many still unsolved problems in complex analysis. including the study of quasi-conformal mappings. In addition one should also mention the book [A-S] by Ahlfors and Sario about Riemann surfaces.

12. Remarks about Fuchsian groups.

They are constructed via Möbius transforms which give conformal mappings of the unit disc D onto itself. The set up is as follows: To each $a \in D$ we have the Möbius transform

(i) $$M_a(z) = \frac{z+a}{1+\bar{a}\cdot z}$$

If a, b is a pair of points in D, a computation shows that the composed map

(ii) $$M_b \circ M_a = M_c \quad : \quad c = \frac{a+b}{1+\bar{a}\cdot b}$$

In particular M_{-a} is the inverse to M_a and by (ii) the points in the unit disc correspond to elements in a group denoted by $\mathcal{M}$. Let $\mathcal{F}$ be a subgroup of $\mathcal{M}$. To each $z \in D$ we get the orbit:

(*) $$\mathcal{F}_z = \{M_a(z) \quad : M_a \in \mathcal{F}\}$$

Following Poincaré we say that $\mathcal{F}$ is a discrete Fuchsian group if every orbit is a discrete subset of D. It means that if $r < 1$ then $\mathcal{F}_z$ only contains a finite set of points in the disc D_r of radius r. Notice that if $z = 0$ is the origin then the orbit $\mathcal{F}_0$ corresponds to the set of of points $a \in D$ for which the Möbius transform M_a belongs to $\mathcal{F}$.

Fundamental domains. Let $\mathcal{F}$ be a discrete Fuchsian group. We seek open subsets of the unit disc where every pair of distinct points are *non-equivalent.* To find such domains we use the hyperbolic distance function on the unit disc introduced by Hermann Schwarz.

12.1 Definition. *The δ-distance in the unit disc D is defined by*

$$\delta(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - z_1\bar{z}_2|} \quad : \quad z_1, z_2 \in D$$

Let us see how a Möbius transformation affects the δ-function. If $a \in D$ a computation gives:

$$\delta(M_a(z_1), M_a(z_1)) = \frac{1}{1-|a|^2}\cdot \delta(z_1, z_2)$$

So when $|a| \to 1$ then the Möbius transform M_a tends to increase the δ-distance. Given a Fuchsian group $\mathcal{F}$ as above we set

(**) $$\mathfrak{D} = \{z \in D \quad : \quad \delta(z, 0) < \delta(z, a) \quad \forall\, a \in \mathcal{F}_0 \setminus \{0\}\}$$

where $\mathcal{F}_0$ is the orbit which contains the origin.

12.2 Proposition. *Every $\mathcal{F}$-orbit intersects $\mathfrak{D}$ in at most one point.*

Proof. Assume the contrary, i.e. there exists some $b \in \mathfrak{D}$ and $0 \neq a \in \mathcal{F}_0$ such that both b and $M_a(b)$ belong to $\mathfrak{D}$. Since $\mathcal{F}$ is a group we also have $-a \in \mathcal{F}_0$. Now

$$\delta(b, 0) < \delta(b, -a)$$

From (iii) we get

$$\delta(M_a(b), M_a(0)) < \delta(M_a(b), M_a(-a)) = \delta(M_a(b), 0)$$

This gives a contradiction since $M_a(0) = a$ and the inclusion $M_a(b) \in \mathfrak{D}$ means that we have the opposite inequality

$$\delta(M_a(b), 0) < \delta(M_a(b), M_a(0))$$

12.3 The boundary of $\mathfrak{D}$. If z is a boundary point of $\mathfrak{D}$ it follows by continuity that there exists at least some $0 \neq a \in \mathcal{F}_0$ such that

(*) $$\delta(z, 0) = \delta(z, a)$$

The converse also holds, i.e. the reader should verify

12.4 Proposition. *The set $\partial\mathfrak{D} \cap D$ is equal to the set of points $z \in D$ for which there exists some $0 \neq a \in \mathcal{F}_0$ for which (*) above holds.*

12.5 The sets $K(a,b)$. Proposition 12.4 suggests that we consider sets of the form:

(iv) $$K(a,b) = \{z \in D \quad : \quad \delta(z,a) = \delta(z,b)\} \quad : \quad a,b \in D$$

A.5 Proposition. *The set $K(a,b)$ is an arc of a circle which intersects the unit circle at a right angle. Moreover, a and 0 "liegen spiegelbildlich zueinander".*

Proof. The assertion is invariant under a Möbius transform. So it suffices to consider the case when $b = -a$ with $0 < a < 1$, i.e. the pair are real and symmetrically placed with respect to the origin. Then

$$K(a,b) = \{z : \frac{|z-a|}{1-az|} = \frac{z+a|}{1+az|}$$

With $z = x + iy$ an easy computation shows that (i) holds if and only if

$$4ax(1 - x^2 - y^2) = 0$$

It follows that $K(A,-A) \cap D$ is the lie segment $(-i,i)$ on the imaginary axis. It is regarded as a circle which has a $\perp$-intersection with the unit circle and the tow real points a and $-a$ are mutually reflected with each other along the imaginary axis.

A.6 Remark. The reader should illustrate the results above by suitable pictures. For example, describe the sets $K(a,0)$ as a varies where one seeks all $z \in D$ such that

$$|z| = \frac{|z-a|}{1-\bar{a}z|}$$

Up to rotation it is enough to treat the case when $0 < a < 1$ is real and positive.

A.7 The favourable case. In most applications $\partial\mathfrak{D}$ consists of a finite union of circular arcs which belong to $K(0,a)$ for a finite set of points $0 \neq a \in \mathcal{F}_0$. Moreover, every such circular arc has end-points on the unit circle. So on T there exists a finite set of corner points which appear as common end-points of two circular arcs in the boundary of $\mathfrak{D}$. The simplest case is the Fuchsian group which corresponds to the fundamental group of $\mathbf{C} \setminus \{0,1\}$ which was already described by Schwarz after his explicit construction of the modular function with the aid of the reflection principle. For further examples we refer to the cited articles by Poincaré. Of course, the reader may also consult text-books of more recent origin for further examples.

A.8. Automorphic functions. Let $\mathcal{F}$ be a discrete Fuchsian group. An analytic function $F(z)$ in D is called $\mathcal{F}$-automorphic if

$$F(z) = F(M_a(z)) \quad : \quad a \in \mathcal{F}$$

The existence of such automorphic functions can be established when $\mathcal{F}$ is defined via a uniformisation of a connected open domain in $\mathbf{C}$. The construction of automorphic functions for an arbitrary Fuchsian group and further studies about these is a topic which goes beyond the scope of these notes, especially since we do not prove the uniformisation theorem in its full generality. In addition to the cited literature above the reader may consult other text-books. Personally I would recommend [Ford].