10. Approximation theorems in complex domains

0. Introduction.

A. Weierstrass approximation theorem

B. Polynomial approximation with bounds

C. Approximation by fractional powers

D. Theorem of Müntz

0. Introduction.

This chapter is devoted to results concerned with approximation by analytic functions due to Carleman, Lindelöf and Müntz.

A. Weierstrass approximation theorem.

A wellknown result due to Weierstrass asserts that if $f(x)$ is a complex valued continuous function on a bounded interval $[a,b]$ then it can be uniformly approximated by polynomials. It turns out that uniform approximations exist on the whole real line where the approximating functions are entire.

A.1 Theorem. *Let $f(x)$ be a continuous function on the real x-line. To every $\epsilon > 0$ there exists an entire function $G(z)$ such that*

$$\max_{x \in \mathbf{R}} \big| G(x) - f(x) \big| \leq \epsilon$$

An elementary proof using Cauchys integral formula only was given in [Carleman]. Here we extend Theorem A.1 to a more general situation. Let K be a closed null-set in $\mathbf{C}$ which in general is unbounded. If $0 < R < R^*$ we put

$$K[R, R^*] = K \cap \{R \leq |z| \leq R^*\}$$

If $R > 0$ we denote by $\bar{D}_R$ the disc of radius R and $K_R = K \cap \bar{D}_R$. With these notations one has:

A.2 Theorem. *Assume there exists a strictly increasing sequence $\{R_\nu\}$ where $R_\nu \to +\infty$ such that the sets*

$$\Omega_\nu = \mathbf{C} \setminus \bar{D}_{R_\nu} \cup K[R_\nu, R_{\nu+1}]$$

are connected for each $\nu \geq 1$ together with the set $\mathbf{C} \setminus K_{R_1}$. Then every continuous function on K can be uniformly approximated by entire functions.

To prove this result we first establish the following.

A. 3 Lemma. *Consider some $\nu \geq 1$ a continuous function ψ on $S = \bar{D}_{R_n} \cup K[R_\nu, R_{\nu+1}]$ where ψ is analytic in the open disc D_{R_ν}. Then ψ can be uniformly approximated on S by polynomials in z.*

Proof. Notice first that if we have found a sequence of polynomials $\{p_k\}$ which approximate ψ uniformly on $S_* = \{|z| = R_\nu\} \cup K[R_\nu, R_{\nu+1}]$ then this sequence approximates ψ on S. In fact, this follows since ψ is analytic in the disc D_{R_ν} so by the maximum principle for analytic functions in a disc we have

$$||p_k - \psi||_S = ||p_k - \psi||_{S_*}$$

for each k. Next, if uniform approximation on S_* fails there exists a Riesz-measure μ supported by S_* which is $\perp$ to all analytic polynomials while

$$\int \psi \cdot d\mu \neq 0 \tag{1}$$

To see that this cannot occur we consider the Cauchy transform

$$\mathcal{C}(z) = \int \frac{d\mu(\zeta)}{z - \zeta}$$

Since $\int \zeta^n \cdot d\mu(\zeta) = 0$ for every $n \geq 0$ we see that $\mathcal{C}(z) = 0$ in the exterior disc $|z| > R_{\nu+1}$. The connectivity hypothesis implies that $\mathcal{C}(z) = 0$ in the whole open complement of S. Now K was a null set which means that the L^1_{loc}-function $\mathcal{C}(z)$ is zero in the exterior disc $|z| > R_\nu$ and hence its distribution derivative $\bar{\partial}(\mathcal{C}_\nu)$ also vanishes in this exterior disc. At the same time we have the equality

$$\bar{\partial}(C_\nu) = \mu$$

We conclude that the support of μ is confined to the circle $\{|z| = R_\nu\}$. But then (1) cannot hold since the restriction of ψ to this circle by assumption extends to be analytic in the disc D_{R_ν} and therefore can be uniformly approximated by polynomials on the circle.

Proof of Theorem A.2. Let $\epsilon > 0$ and $\{\alpha_\nu\}$ is a sequence of positive numbers such that $\sum \alpha_\nu < \epsilon$. Consider some $f \in C^0(K)$. Starting with the set K_{R_1} we use the assumption that its complement is connected and using Cauchy transforms as in Lemma A.3 one shows that the restriction of f to this compact set can be uniformly approximated by polynomials. So we find $P_1(z)$ such that

(i) $$||P_1 - f||_{K_{R_1}} < \alpha_1$$

From (i) one easily construct a continuous function ψ on $\bar{D}_{R_1} \cup K[R_1, R_2]$ such that $\psi = P_1$ holds in the disc $\bar{D}_{R_1}$ and the maximum norm

$$||\psi - f||_{K[R_1,R_2]} \leq \alpha_1$$

Lemma A.3 gives a polynomial P_2 such that

$$||P_2 - P_1||_{D_{R_1}} < \alpha_2 \quad \text{and} \quad ||P_2 - f||_{K[R_1,R_2]} \leq \alpha_1 + \alpha_2$$

Repeat the construction where Lemma A.3 is used as ν increases. This gives a sequence of polynomials $\{P_\nu\}$ such that

$$||P_\nu - P_{\nu-1}||_{D_{R_\nu}} < \alpha_\nu \quad \text{and} \quad ||P_\nu - f||_{K[R_{\nu-1},R_\nu]} < \alpha_1 + \ldots + \alpha_\nu$$

hold for all ν. From this it is easily seen that we obtain an entire function

$$P^*(z) = P_1(z) + \sum_{\nu=1}^{\infty} P_{\nu+1}(z) - P_\nu(z)$$

Finally the reader can check that the inequalities above imply that the maximum norm

$$||P^* - f||_K \leq \alpha_1 + \sum_{\nu=1}^{\infty} \alpha_\nu$$

Since the last sum is $\leq 2\epsilon$ and $\epsilon > 0$ was arbitrary we have proved Theorem A.3.

Exercise. Use similar methods as above to show that if $f(z)$ is analytic in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$ and has continuous boundary values on the real line, then f can be uniformly approximated by an entire function, i.e. to every $\epsilon > 0$ there exists an entire function $F(z)$ such that

$$\max_{z \in U^+} |F(z) - f(z)| \leq \epsilon$$

B. Polynomial approximation with bounds

Introduction. We begin with a result due to Lindelöf. Let U be a Jordan domain and set $\Gamma = \partial U$. For each $f(z) \in \mathcal{O}(U)$, Runge's theorem gives a sequence of polynomials $\{Q_\nu(z)\}$ which approximates f uniformly over each compact subset of U. If we impose some bound on f one may ask if an approximation exists where the Q-polynomials satisfy a similar bound as f. Lindelöf proved that bounds exist for many different norms on the given function f in the article *Sur un principe géneral de l'analyse et ses applications à la theorie de la représentation conforme* from

1915. Let us announce two results from [Lindelöf] of this nature. Let $p > 0$ and consider the H^p-space of analytic functions in U for which

$$||g||_p = \iint_U |g(z)|^p \cdot dxdy < \infty \tag{1}$$

B.1 Theorem *Let $p > 0$ and suppose that $f(z)$ has a finite H^p-norm. Then there exists a sequence of polynomials $\{Q_n(z)\}$ which converge uniformly to f in compact subsets of U while*

$$||Q_n||_p \leq ||f||_p \quad : \quad n = 1, 2, \ldots$$

A similar approximation holds when the H^p-norm is replaced by the maximum norm. Thus, if f is a bounded analytic function in U there exists a sequence of polynomials $\{Q_n\}$ which converge to f in every relatively compact subset of U and at the same time the maximum norms satisfy:

$$||Q_n||_U \leq ||f||_U \quad : \quad n = 1, 2, \ldots$$

To prove Theorem B.1 one constructs for each $n \geq 1$ a Jordan curve Γ_n which surrounds $\bar{U}$, i.e. its interior Jordan domain U_n contains $\bar{U}$ and for every point $p \in \Gamma_n$ the distance of p to Γ is $< 1/n$. It is trivial to see that such a family of Jordan curves exist where the domains $U_1, U_2, \ldots$ decrease. Next, fix a point $z_0 \in U$. There exists the unique conformal map ψ_n from U_n onto U such that

$$\psi_n(z_0) = z_0 \quad : \quad \psi_n'(z_0) \text{ is real and positive}$$

With these notations Lindelöf used the following lemma whose proof is left as an exercise:

B.2. Lemma *For each compact subset K of U the maximum norms $|\psi_n(z) - z|_K$ tend to zero as $n \to \infty$. Moreover, the complex derivatives $\psi_n'(z_0) \to 1$.*

Proof of Theorem B.1. To each n we set

$$F_n(z) = f(\psi_n(z)) \cdot \psi_n'(z)^{\frac{2}{p}}$$

By Lemma B.2 the sequence $\{F_n\}$ converges uniformly to f on compact subsets of U. Moreover, each $F_n \in \mathcal{O}(U_n)$ and it is clear that

$$\iint_U |f(z)|^p \cdot dxdy = \iint_{U_n} |F_n(z)|^p \cdot dxdy$$

hold for every n. To get the required polynomials $\{Q_n\}$ in Theorem B.1 for H^p-spaces it suffices to apply Runge's theorem for each single F_n. This detail of the proof is left to the reader. For maximum norms we use the functions

$$F_n(z) = f(\psi_n(z))$$

and after apply Runge's theorem in the domains $\{U_n\}$.

B.3 Remark. More delicate approximations by polynomials where other norms such as the modulus of continuity, were established later by Lindelöf and De Vallé-Poussin. We shall not pursue this any further. The reader can consult articles by De Vallé-Poussin which contain many interesting results concerned with approximation theorems.

C. Approximation by fractional powers

Here is the set-up in the article *Über die approximation analytischer funktionen* by Carleman from 1922. Let $0 < \lambda_1 < \lambda_2 < \ldots$ be a sequence of positive real numbers and Ω is a simply connected domain contained in the right half-space $\mathfrak{Re}(z) > 0$. Notice that the functions $q_\nu(z) = z^{\lambda_\nu}$ are analytic in the half-plane, i.e. with $z = re^{i\theta}$ and $-\pi/2 < \theta < \pi/2$ we have:

$$q_\nu(z) = r^{\lambda_\nu} \cdot e^{i\lambda_\nu \cdot \theta}$$

C.1 Definition. *We say that the sequence $\Lambda = \{\lambda_\nu\}$ is dense for approximation if there for each $f \in \mathcal{O}(\Omega)$ exists a sequence of functions of the form*

$$Q_N(z) = \sum_{\nu=1}^{N} c_\nu(N) \cdot q_\nu(z) \quad : \quad N = 1, 2, \ldots$$

which converges uniformly to f on compact subsets of Ω.

C.2 Theorem. *A sequence Λ is dense if*

$$(*) \qquad \limsup_{R \to \infty} \frac{\sum_R \frac{1}{\lambda_\nu}}{\operatorname{Log} R} > 0$$

where $\sum_R$ means that we take the sum over all $\lambda_\nu < R$.

Remark. Above condition (*) is the same for every simply connected domain Ω. Theorem C.2 gives a *sufficient* condition for an approximation. To get necessary condition one must specify the domain Ω and we shall not try to discuss this more involved problem. The proof of Theorem C.2 requires several steps, the crucial is the uniqueness theorem in C.4 while the proof of Theorem C.2 is postponed until C.5.

C.3 A uniqueness theorem.

Consider a closed Jordan curve Γ of class C^1 which is contained in $\mathfrak{Re}\, z > 0$. When $z = re^{i\theta}$ stays in the right half-plane we get an entire function of the complex variable λ defined by:

$$\lambda \mapsto z^\lambda = r^\lambda \cdot e^{i\theta \cdot \lambda}$$

We conclude that a real-valued and continuous function g on Γ gives an entire function of λ defined by:

$$G(\lambda) = \int_\Gamma g(z) \cdot z^\lambda \cdot \left|dz\right|_\Gamma$$

where $|dz|_\Gamma$ is the arc-length on Γ. With these notations one has

C.4 Theorem. *Assume that Λ satisfies the condition in Theorem C.2. Then, if $G(\lambda_\nu) = 0$ for every ν it follows that the g-function is identically zero.*

Proof. If we have shown that the G-function is identically zero then the reader may verify that $g = 0$. There remains to show that if $G(\lambda_\nu) = 0$ for every ν then $G = 0$. To attain this one first shows that there exist constants A, K and $0 < a < \frac{\pi}{2}$ such that:

$$\text{(i)} \qquad |G(\lambda)| \leq K \cdot e^{|\lambda|} \quad \text{and} \quad |G(is)| \leq K \cdot e^{|s| \cdot a} \quad : \lambda \in \mathbf{C} : s \in \mathbf{R}$$

The easy verification of (i) is left to the reader. Next, the first inequality in (i) means that G is an entire function of exponential type one. By assumption $G(\lambda_\nu) = 0$ for every ν. Now we can use Carleman's formula for analytic functions in a half-space from XXX to conclude that $G = 0$. Namely, set

$$\text{(ii)} \qquad U(r, \phi) = \log \left|G(re^{i\phi})\right|$$

Let $\{r_\nu e^{i\phi_\nu}\}$ be the zeros of G in $\mathfrak{Re}(z) > 0$ which by the hypothesis contains the set Λ. By Carleman's formula the following hold for each $R > 1$:

$$\sum_{1 < r_\nu < R} \left[\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\right] \cdot \cos\theta_\nu = \frac{1}{\pi R} \cdot \int_{-\pi/2}^{\pi/2} U(R, \phi) \cdot \cos\phi \cdot d\phi +$$

$$\frac{1}{2\pi R} \cdot \int_1^R \left(\frac{1}{r^2} - \frac{1}{R^2}\right) \cdot \left[\, U(r, \pi/2) + U(r, -\pi/2) \,\right] \cdot dr + c_*(R)$$

where $c_*(R) \leq K$ holds for some constant which is independent of R. Finally, the set Λ satisfies (*) in Theorem C.2 and the sum over zeros in Carleman's formula above majorizes the sum extended over the real λ-numbers from Λ satisfying $1 < \lambda_\nu < R$. At this stage we leave it to the reader to verify that the second inequality in (i) above implies that G must be identically zero.

Proof of Theorem C.2

Denote by $\mathcal{O}^*(\Lambda)$ the linear space of analytic functions in the right half-plane given by finite $\mathbf{C}$-linear combinations of the fractional powers $\{z^{\lambda_\nu}\}$. To obtain uniform approximations over relatively compact subsets when Ω is a simply connected domain in $\mathfrak{Re}(z) > 0$, it suffices to regard a closed Jordan arc Γ which borders a Jordan domain U where U is a relatively compact subset of Ω. In particular Γ has a positive distance to the imaginary axis and there remains to show that when (*) holds in Theorem C.2, then an arbitrary analytic function $f(z)$ defined in some open neighborhood of $\bar{U}$ can be uniformly approximated by $\mathcal{O}^*(\Lambda)$-functions over a relatively compact subset U_* of U. To achieve this we shall use a trick which reduces the proof of uniform approximation to a problem concerned with L^2-approximation on Γ. To begin with we have

C.5 Lemma. *The uniqueness in Theorem C.4 implies that if V is a real-valued function on Γ then there exists a sequence $\{Q_n\}$ from the family $\mathcal{O}(\Lambda)$ such that*

$$\lim_{n\to\infty} \int_\Gamma |Q_n - V|^2 \cdot |dz| = 0$$

The proof of this result is left as an exercise.

C.6 A tricky construction. Let $f(z)$ be analytic in a neighborhood of the closed Jordan domain $\bar{U}$ bordered by Γ. Define a new analytic function

$$F(z) = \int_{z_*}^{z} \frac{f(\zeta)}{\zeta} \cdot d\zeta \tag{1}$$

where z_* is some point in $\bar{U}$ whose specific choice does not affect the subsequent discussion. We can write $F = V + iW$ where $V = \mathfrak{Re}(F)$. Lemma 6.6 gives a sequence $\{Q_n\}$ which approximates V in the L^2-norm on Γ. Using this L^2-approximation we get

Lemma C.8 *Let U_0 be relatively compact in U. Then there exists a sequence of real numbers $\{\gamma_n\}$ such that*

$$\lim_{n\to\infty} \big||Q_n(z) - i\cdot\gamma_n - F(z)\big|_{U_0} = 0$$

Again we leave out the proof as an exercise. Next, taking complex derivatives Lemma C.8 implies that if U_* is even smaller, i.e. taken to be a relatively compact in U_0, then we a get uniform approximation of derivatives:

$$Q_n'(z) \to F'(z) = \frac{f(z)}{z}$$

Well, this means that

$$z \cdot Q_n' \to f(z)$$

holds uniformly in U_*. Next, notice that

$$z \cdot \frac{d}{dz}(z^{\lambda_\nu}) = \lambda_\nu \cdot z^{\lambda_\nu}$$

hold for each ν. Hence $\{z \cdot Q_n'(z)\}$ again belong to the $\mathcal{O}(\Lambda)$-family. So we achieve the required uniform approximation of the given f function on U_*. This completes the proof of Theorem C.2.

D. Theorem of Müntz

Introduction. Theorem D.1 below is due to Müntz. See his article *Über den Approximationssatz von Weierstrass* from 1914. The simplified version of the original proof below is given in [Car]. Here is the set up: Let $0 < \lambda_1 < \lambda_2 < \ldots$. To each ν we get the function x^{λ_ν} defined on the real unit interval $0 \leq x \leq 1$. We say that the sequence $\Lambda = \{\lambda_\nu\}$ is L^2-dense if the family $\{x^{\lambda_\nu}\}$ generate a dense linear subspace of the Hilbert space of square integrable functions on $[0,1]$.

D.1 Theorem. *The necessary and sufficient condition for Λ to be L^2-dense is that $\sum \frac{1}{\lambda_\nu}$ is convergent.*

D.2 Proof of necessity. If Λ is not L^2-dense there exists some $h(x) \in L^2[0,1]$ which is not identically zero while

$$\int_0^1 h(x) \cdot x^{\lambda_\nu} \cdot dx = 0 \quad : \quad \nu = 1, 2, \ldots \tag{1}$$

Now consider the function

$$\Phi(\lambda) = \int_0^1 h(x) \cdot x^{-i\lambda} \cdot dx \tag{2}$$

It is clear that Φ is analytic in the right half plane $\mathfrak{Im}\,\lambda > 0$. If $\lambda = s + it$ with $t > 0$ we have

$$|x^\lambda| = x^t \leq 1$$

for all $0 \leq x \leq 1$. From this and the Cauchy-Schwarz inequality we see that

$$|\Phi(\lambda)| \leq ||h||_2 \quad : \lambda \in U_+ \tag{3}$$

Hence Φ is a bounded analytic function in the upper half-plane. At the same time (1) means that the zero set of Φ contains the sequence $\{\lambda_\nu \cdot i\}$. By the integral formula formula we have seen in XX that this entails that

$$\sum \frac{1}{\lambda_\nu} < \infty \tag{*}$$

which proves the necessity.

Proof of sufficiency. There remains to show that ifwe have the convergence in (*) above then there exists a non-zero h-function i $L^2[0,1]$ such that (1) above holds. To find h we first construct an analytic function Φ by

$$\Phi(z) = \frac{\prod_{\nu=1}^\infty \left(1 - \frac{z}{\lambda_\nu}\right)}{\prod_{\nu=1}^\infty \left(1 + \frac{z}{\lambda_\nu}\right)} \cdot \frac{1}{(1+z)^2} \quad : \quad \mathfrak{Re}\,z > 0 \tag{i}$$

Notice that $\Phi(z)$ is defined in the right half-plane since the series (*) is convergent. When $\mathfrak{Re}(z) \geq 0$ we notice that each quotient

$$\frac{1 - \frac{z}{\lambda_\nu}}{1 + \frac{z}{\lambda_\nu}}$$

has absolute value ≤ 1. It follows that

$$|\Phi(x + iy)| \leq \frac{1}{1 + x + iy|^2} = \frac{1}{(1+x)^2 + y^2} \tag{ii}$$

In particular the function $y \mapsto \Phi(iy)$ belongs to L^2 on the real y-line. Now we set

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ity} \cdot \Phi(iy) \cdot dy \tag{ii}$$

using the inequality (ii) If $t < 0$ we can move the line integral of $e^{tz} \cdot \Phi(z)$ from the imaginary axis to a line $\mathfrak{Re}(z) = a$ for every $a > 0$ and it is clear that

$$\lim_{a \to +\infty} \int_{-\infty}^\infty e^{-at+ity} \cdot \Phi(a + iy) \cdot dy = 0$$

We conclude that $f(t) = 0$ when $t < 0$. Next, since $y \mapsto \Phi(iy)$ is an L^2-function it follows by Parseval's equality that

$$\int_0^\infty |f(t)|^2 \cdot dt < \infty$$

Moreover, for a fixed λ_ν we have

$$\int_0^\infty f(t)e^{-\lambda_\nu t}\cdot dt = \frac{1}{2\pi}\cdot\int_0^\infty [\int_{-\infty}^\infty e^{ity}\cdot\Phi(iy)\cdot dy\,]\cdot e^{-\lambda \nu t}\cdot dt =$$

$$\int_{-\infty}^\infty \frac{1}{iy-\lambda_\nu}\cdot\Phi(iy)\cdot dy$$

where the last equality follows when the repeated integral is reversed. By construction $\Phi(z)$ has a zero at λ_ν and therefore (xx) above remains true with Φ replaced by $\frac{\Phi(z)}{z-\lambda_\nu}$ which entails that

$$\int_0^\infty f(t)\cdot e^{-\lambda_\nu t}\cdot dt = 0$$

At this stage we obtain the requested h-function. Namely, since $t\mapsto e^{-t}$ identifies $(0,+\infty)$ with $(0,1)$ we get a function $h(x)$ on $(0,1)$ such that

$$h(e^{-t}) = e^{t/2}\cdot f(t)$$

The reader may verify that

$$\int_0^1 |h(x)|^2\cdot dx = \int_0^\infty |f(t)|^2\cdot dt$$

and hence h belongs to $L^2(0,1)$. Moreover, one verifies that the vanishing in (xx) above entails that

$$\int_0^1 h(x)\cdot x^{\lambda_\nu}\cdot dx = 0$$

Since this holds for every ν we have proved the sufficiency which therefore finishes the proof of Theorem XX.

D.2 Another density result

Density results using exponential functions are used in many applications. For example, equidistant sequences appear in the the *Sampling Theorem* by Shanning used in telecommunication engineering. When this result is expressed in analytic function theory via characteristic functions it can be formulated as follows:

D.3 Theorem. *Let $T>0$ and $g(t)$ is an L^2-function on the interval $[0,T]$ which is not identically zero and suppose that $a>0$ is such that*

$$\int_0^T e^{inat}\cdot g(t)\cdot dt = 0 \quad : \quad n\in\mathbf{Z}$$

Then we must have

$$a \geq \frac{2\pi}{T}$$

Remark This result is due to Fritz Carlson in his article [xx] from 1914. let us recall that Carlson and Carleman later became collegues at Stockholm University for more than two decades. Carlson's result was later improved by Titchmarsh and goves as follows:

D.4 Theorem. *Let $0<m_1<m_2<\ldots$ be an increasing sequence of positive real numbers such that*

(*) $$\limsup_{n\to\infty}\frac{n}{m_n} > 1$$

Then if $f(x)\in L^2(0,1)$ and

(**) $$\int_0^1 e^{+-im_nx}\cdot f(x)\cdot dx = 0$$

hold for each n, it follows that $f=0$.

We give a proof below taken from the text-book [Paley-Wiener: page 84-85]. To begin with we notice that (**) implies that we also have vanishing integrals using $f(-x)$. Replacing f by $f(x)+f(-x)$ or $f)x)-f)-x)$ it suffices to prove the result when f is even or odd. Let us show that there cannot exist an even L^2-function f such that the integrals (**) vanish while

(i) $$\int_0^1 f(x)\cdot dx = 1$$

We leave it to the reader to verify that it suffices to prove this version of Theorem D.3. With f as above we set

(ii) $$\phi(z) = \int_{-\pi}^{\pi} e^{izt}\cdot f(t)\cdot dt$$

We see that the entire function ϕ is even and (i) means that $\phi(0)=1$. Moreover, ϕ is an entire function of exponential type and Cauchy-Schwarz inequality gives

$$|\phi(x+iy)| \leq ||f||_2 \cdot e^{\pi|y|}/tagiii$$

for all $z=x+iy$. Next, Parseval's equality shows that the restriction of ϕ to the real x-line belongs to L^2 which by the Remark in (xx) implies that ϕ belong to the Carleman class $\mathcal{N}$. So Theorem xx gives the existence of a limit

(iv) $$\lim_{R\to\infty} \frac{N_\phi(R)}{R} = A$$

The inequality (iiii) and the result in XXX entails that

(v) $$A \leq 2$$

Next, since the zeros of ϕ contains the even sequence $\{+-m_\nu\}$ we have the inequality

(vi) $$N_\phi(m_n) \geq 2n$$

At the same time the limi formula (iv) gives:

(vii) $$A = \lim_{n\to\infty} \frac{N_\phi(m_n)}{m_n}$$

Finally, (vi) and the hypothesis (*) in the Theorem give

(viii) $$\limsup_{n\to\infty} \frac{N_\phi(m_n)}{m_n} \geq 2\cdot \limsup_{n\to\infty} \frac{n}{m_n} > 2$$

This contradicts (v) and we conclude that a non-zero function ϕ and hence also $f\neq 0$ cannot exist.