§ A. Eigenvalues and eigenfunctions for the Laplace operator in $\mathbf{R}^2$

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Theorem 1 below was presented by Carleman at the Scandinavian Congress in mathematics held in Stockholm 1934: In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact which goes back to original work by Dirichlet gives the following: There exists the Greens' function

$$(*) \qquad G(p,q) = \log \frac{1}{|p-q|} - H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$. Moreover, $H(p,q) = H(q,p)$ is symmetric and when $q \in \Omega$ is fixed, then $p \mapsto H(p,q)$ is harmonic in Ω and

$$H(p,q) = \log \frac{1}{|p-q|} \quad : p \in \partial\Omega$$

This means that $p \mapsto G(p,q)$ vanishes on the boundary. Next, (*) shows that $p \mapsto G(p,q)$ is superharmonic and and the minimum principe for superharmonic functions plus symmetry entail that

$$G(p,q) > 0$$

hold in $\Omega \times \Omega$. Next, it is obvious that

$$\iint_{\Omega\times\Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence the linear operator on the Hilbert space $L^2(\Omega)$ defined by the symmetric kernel $G(p,q)$ is a Hilbert-Schmidt operator and therefore a compact operator. Since the kernel is symmetric and positive the eigenvalues are positive, General Hilbert space theory applied to the symmetric G-kernel gives a a sequence of pairwise orthogonal functions $\{\phi_n\}$ whose L^2-norms are one and

$$(1) \qquad \int_\Omega G(p,q)\phi_n(q)\, dq = 2\pi \cdot \mu_n \cdot \phi_n(p)$$

where $\{\mu_n\}$ is a non-increasing sequence of positive eigenvalues which tend to zero. Next, we apply the Laplace operator on both sides. Recall that

$$\Delta(\log \frac{1}{|z|}) = -2\pi \cdot \delta_0$$

where δ_0 is the Dirac measure. It follows that the Laplacian of the left hand side in (1) becomes $-2\pi \cdot \phi(p)$ and hence (1) gives the equation

$$\Delta(\phi_n)(p) + \frac{1}{\mu_n} \cdot \phi_n(p) = 0$$

We prefer to use $\lambda_n = \frac{1}{\mu_n}$. Then $\{\lambda_n\}$ is a non-decreasing sequence of real numbers which tends to $+\infty$. Since the kernel $G(p,q)$ is positive it follows, again by general Hilbert space theory, that $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$, i.e. each L^2-function f has an expansion

$$(2) \qquad f = \sum a_n \cdot \phi_n \quad : \quad a_n = \int_\Omega f_n(p) \cdot \overline{\phi_n}(p)\, dp$$

Exercise. Verify from the above that each ϕ-function is a continuous function in Ω whose boundary values on $\partial\Omega$ are zero. Show also that

$$(0.1) \qquad G(p,q) = \sum_{n=1}^{\infty} \frac{\phi_n(p) \cdot \phi_n(q)}{\lambda_n}$$

where the right hand side is a convergent series when $p \neq q$.

Notice that $G(p,p) = +\infty$ so the series above with $p = q$ is divergent. However, there exists a limit when we employ larger denominators.

Main Theorem. *For $p \in \Omega$ one has the limit formula*

$$(*)\qquad \lim_{N\to\infty} \lambda_N^{-1}\cdot \sum_{n=1}^{n=N} \phi_n^2(p) = \frac{1}{4\pi}$$

Remark. Carleman was inspired by an earlier result due to H. Weyl which asserts that the set of eigenvalues satisfy the asymptotic formula

$$\lim_{N\to\infty} \frac{\lambda_N}{N} = \frac{\text{Area}(\Omega)}{4\pi}$$

Notice that Weyl's asymptotic formula together with (*) gives

$$\lim_{N\to\infty} N^{-1}\cdot \sum_{n=1}^{n=N} \phi_n^2(p) = xxxx$$

The notable point is that this asymptotic limit is the same for *every* point $p \in \Omega$.

To get the main Theorem we proceed as follows. First, since $\mathcal{G}$ is a Hilbert-Schmidt operator a result due to Schur gives

$$\text{(i)}\qquad \sum \lambda_n^{-2} < \infty$$

Let us also remark that since each ϕ_n is harmonic we have the mean-value equality

$$\phi_n(p) = \frac{1}{\pi r^2}\cdot \int_{D_p(r)} \phi(q)\,dq$$

where $D_p(r)$ is the disc of radius r centered at p and r is chosen so small that the disc stays in Ω. Since the L^2-norms of the ϕ-functions are equal to one, the Cauchy-Schwarz inequality gives a constant C such that

$$\text{(ii)}\qquad |\phi_n(p)| \leq C \quad : \; n = 1, 2, \ldots$$

Now (i-ii) entail that that the Dirichlet series

$$\Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n^2(p)}{\lambda_n^s}$$

is an analytic function of the complex variable s in the half-plane $\mathfrak{Re}\, s > 2$. With these notations we shall prove:

Theorem 2. *For each $p \in \Omega$ there exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Remark. In § xx we explain how Theorem 2 gives Theorem 1 from Ikehara's limit formula. So the main task is to establish Theorem 2. The proof of Theorem 2 employs analytic function theory and is inspired by Riemann's work on the ζ-function. One can establish more general results, where Δ is replaced by a higher order elliptic operator with constant coefficients in $\mathbf{R}^n$ where $n \geq 3$ can hold. and the reader is invited to continue this analysis which does not seem to be covered in the literature. In the cited article such extensions are pointed out by Carleman

Remarquons que la méthode dont nous nous sommes servis est aussi appliquable à une equation elliptique quleconque à un nombre quleconque de dimensions.

In § xx we shall present Carleman's asymptotic formula for eigenvalues of a second order ellitpic operator in $\mathbf{R}^3$ which in general has variable coefficiebts and need not be self-adjoint. Of course, to get a resyukt such as Theorem 1 with an asymptotic limit formula which is independent of the point p in the domain where the eigenfunctions appear, usually requires that the elliptic PDE-operator has constant coefficients. It goes without saying that many specific problems desevere to be analyzed in more detail. A broader perpective concerning asymptotic reperesentations arises when one for example regards spectral functions associated to self-adjoint operators defined via elliptic PDE:s. See § xx below whgere we give some comments about Carlea'ns discussion of the Schrödinger equation

$$\Delta(u) - c(x,y,z) \cdot u = i\frac{\partial u}{\partial t}$$

Here $\Delta istheLaplaceoperatorand$c(x,y,z) is a real-valued function which is locally square integrable and there exist constants R and M such that

$$c(x,y,z) \leq M \quad : x^2 + y^2 + z2 \geq R^2$$

When (*) holds it was proved by Carleman in 1931 that the densely defined operator $\Delta - c$ is self-adjoint on $L^2(\mathbf{R}^3)$ whose spectrum is confined to $[\ell, +\infty)$ for some real number ℓ. If Θ is the associated spectral function we get a solution to (xx) wiuth initial condition $u(p,0) = f(p)$ by

$$u(p,t) = \int_\ell^\infty e^{it\lambda} \cdot \big[\int_{\mathbf{R}}^3 \Theta(p,q;\lambda) \cdot f(q)\, dq \,\big]\, d\lambda$$

Proof of Theorem 2

For each λ outside the discrete set $\{\lambda_n\}$ we put

$$G(p,q;\lambda)=G(p,q)+2\pi\lambda\cdot\sum_{n=1}^{\infty}\frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda-\lambda_n)} \tag{1}$$

Notice that (i-ii) above entail that the last sum is converges and gives a meromorohic function of the complex variable λ whose poles are at most simple and confied to the set $\{\lambda_n\}$ Moreover, we get the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

$$\mathcal{G}_\lambda(f)(p)=\frac{1}{2\pi}\cdot\iint_\Omega G(p,q;\lambda)\cdot f(q)\,dq \tag{2}$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta+\lambda)\cdot\mathcal{G}_\lambda=-E$$

B. The function $F(p,\lambda)$. Set

$$F(p,q,\lambda)=G(p,q;\lambda)-G(p,q)$$

Keeping p fixed we see that (1) gives

$$\lim_{q\to p}F(p,q,\lambda)=2\pi\lambda\cdot\sum_{n=1}^{\infty}\frac{\phi_n(p)^2}{\lambda_n(\lambda-\lambda_n)} \tag{B.1}$$

Set

$$F(p,\lambda)=\lim_{q\to p}F(p,q,\lambda) \tag{B.2}$$

From (i-ii) above we see that $F(p,\lambda)$ is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0<a<\lambda_1$. Use residue calculus to show the equality below in the half-space $\mathfrak{Re}\,s>2$:

$$\Phi_p(s)=\frac{1}{4\pi^2\cdot i}\cdot\int_{a-i\infty}^{a+i\infty}F(p,\lambda)\cdot\lambda^{-s}\,d\lambda \tag{C.1}$$

where the line integral is taken on the vertical line $\mathfrak{Re}\,\lambda=a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda)<a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty,a]$ followed by the upper half-circle $\{\lambda=ae^{i\theta}:0\le\theta\le\pi\}$ and the half-line $\{\lambda=a+it:t\ge 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+}F(p,\lambda)\cdot\lambda^{-s}\,d\lambda=0$$

We leave it to the reader to contruct the similar curve $\gamma_-=\bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *When $\mathfrak{Re}\,s$ is sufficientyl large one has the equality*

$$\Phi(s)=\frac{a^{s-1}}{4\pi}\cdot\int_{-\pi}^{\pi}F(ae^{i\theta})\cdot e^{(i(1-s)\theta}\,d\theta+\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty F(p,-x)\cdot x^{-s}\,dx \tag{E.1}$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So Theorem 2 follows if

$$s\mapsto\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty F(p,-x)\cdot x^{-s}\,dx \tag{E.2}$$

is meromorphic with a single pole at $s = 1$ whose residue is $\frac{1}{4\pi}$. To prove this we shall express $F(p, -x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z > 0$ there exists the analytic function

$$K(z) = \int_1^\infty \frac{e^{-zt}}{\sqrt{t^2 - 1}}\, dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z = 0\}$ given by

$$K(z) = -I_0(z) \cdot \log z + I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

$$I_0(z) = \sum_{m=0}^{\infty} \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \tag{i}$$

$$I_1(z) = \sum_{m=0}^{\infty} \rho(m) \cdot \frac{2^{-2m}}{(m!)^2} \cdot z^{2m} \quad : \rho(m) = 1 + \frac{1}{2} + \ldots + \frac{1}{m} - \gamma \tag{ii}$$

where γ is the usual Euler constant.

Next, with p kept fixed and $\kappa > 0$ we solve the Dirichlet problem and find a function $q \mapsto H(p, q; \kappa)$ which satisfies the equation

$$\Delta(H) - \kappa \cdot H = 0 \tag{F.2}$$

in Ω with boundary values

$$H(p, q; \kappa) = K(\sqrt{\kappa}|p - q|) \quad : q \in \partial\Omega$$

G. Exercise. Verify the equation

$$G(p, q; -\kappa) = K(\sqrt{\kappa} \cdot |p - q|) - H(p, q; \kappa) \quad : \kappa > 0 \tag{G.1}$$

Together with the construction of $G(p, q)$ the reader can verify the equation

$$F(p, -\kappa) = \lim_{q \to p} [K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q|] + \lim_{q \to p} [H(p, q) - H(p, q, \kappa)] \tag{G.2}$$

The last term above has the "nice limit" $H(p, p) + H(p, p, \kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q \to p} \big(K(\sqrt{\kappa} \cdot |p - q|) + \log\, |p - q| \big) = -\frac{1}{2} \cdot \log \kappa + \log 2 - \gamma \tag{G.3}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A = +\log 2 - \gamma + H(p, p)$. Then (G.1) and (G.2) give

$$F(p, -\kappa) = -\frac{1}{2} \cdot \log \kappa + A + H(p, p; \kappa)$$

With $x = \kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s \mapsto A \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty x^{-s}\, dx$$

is an entire function of s. Next, consider the function

$$\rho(s) = -\frac{1}{2} \cdot \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty \log x \cdot x^{-s}\, dx$$

Notice that the complex derivative

$$\frac{d}{ds} \int_a^\infty x^{-s}\, dx = -\int_a^\infty \log x \cdot x^{-s}\, dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem 2 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty H(p,p,\kappa) \cdot \kappa^{-s}\, d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \le H(p,q;\kappa) \le \max_{q \in \partial\Omega} K(\kappa|p-q|)$$

With $p \in \Omega$ fixed there is a positive number δ such that $|p-q| \ge \delta : q \in \partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \le e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.

Neumann's resolvent operators and the spectral theorem for self-adjoint operators

Introduction. We shall review basiuc facts from operator theory where the spectral theorem for self-adjoint operators on Hilbert spaces is in the focus. The notable point is that one in general allows unbounded self-adjoint operators.

Neumann resolvents.

Let X be a Banach space and $T\colon X \to X$ is an unbounded and densely defined linear operstor. It means that the domain of definition is a dense subspace denoted by $\mathcal{D}(T)$, while thr norms $||Tx||$ can take arbirary large values whiule x varies over vectors of unit norm in $\mathcal{D}(T)$. We say that T has an inverse in Neumann's sense if the range $T(\mathcal{D}(T)) = X$ and there exists a posotiove constant c such that

$$||Tx|| \geq c \cdot ||x|| \quad : \, x \in \mathcal{D}(T)$$

When this hold it is easily seen that nthere exists a bounded inear operator R whose range is equalm tom $\mathcal{D}(T)$ and

$$R \circ T(x) = x \quad : x \in \mathcal{D}(T) \quad \& \quad T \circ R(x) = x \quad : \quad x \in X$$

We put $R = T^{-1}$ and refer to R as Neumann's resolvent of the densely defined operator T. The reader should notice that R is not invertible as a bounded operator, i.e. $\{0\}$ belongs to its compact spectrum $\sigma(R)$.

Exercise. Let E be the identity operator on X. Keeping T fixed we regard the set of non-zero complex numbers λ for which $\lambda \cdot E - T$ has an inverse in Neumann's sense. The reader may verify that $(\lambda \cdot E - T)^{-1}$ exists if and only if the bounded operator $E - \lambda \cdot R$ is invertible and that

$$(*) \qquad (\lambda \cdot E - T)^{-1} = (E - \lambda \cdot R)$$

If $\lambda| \cdot ||R|| < 1$ then the operator $E - \lambda \cdot R$ is invertible and hence $(\lambda \cdot E - T)^1$ exists when λ belongs to the disc of radius $||R||^{-1}$ centered at the origin. The closed complement where $(\lambda \cdot E - T)^{-1}$ does not exist is denoted by $\sigma(T)$. It is in general an unbunded closed subet of the complex λ-plane. More precisely we have

$$\sigma(T) = \{\lambda \neq 0 : \lambda^{-1} \in \sigma(R)\}$$

Remark. Above we regard the spectrum of T in the finite complex plane, i.e. the special point at infinity may be included to add that T itself is an unbounded operator. Conversely, let R be a bounded linear operator whose kernel is the zero space and the range $R(X)$ is dense in X. Then we see that $R = T^{-1}$ where T is the densely defined operstor for ehich $T(Rx) = x$ for all $x \in X$.

Exercise. Find a pair (R, X) where R as above is injective with a dense range and $\sigma(R)$ is reduced to the origin. So in that case th unbounded operator T is such that $(\lambda \cdot E - T)^1$ exists for all $\lambda \in \mathbf{C}$.

The case when X is a Hilbert space. Now we assume that $X = \mathcal{H}$ is a complex Hilbert space. Let T as above be densely defined and unbounded where T^{-1} exists. Wr impose the extra condition that Neumann's resolvent R is a *normal operator*, i.e. R commutes with the adjoint R^* in the algebra of bounded linear opertors on $\mathcal{H}$.

To begin with, let $\mathcal{H}$ be a complex Hilbert space equiped with a hermitian inner priduict, i.e. to each pair of vectors x, y one assigns a complex number $\langle x, y \rangle$ which satisfies

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \& \quad \langle x, x \rangle > 0 : \, x \neq 0$$

Moreover, $\mathcal{H}$ is complete under the norm defined by $|x| = \sqrt{\langle x, x \rangle}$. Denoten by $L(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$ equipped with the norm

$$||T|| = \max_{|x|=1} \, |T(x)|$$

Recall that every bounded inear operator T has a spectrum $\sigma(T)$ which appears as a compact subset of $\mathbf{C}$. A complex number λ is outside $\sigma(T)$ if and only if the operator $\lambda \cdot E_T)$ is invertible. The inverse operator

$$R_T(\lambda) = (\lambda \cdot E - T)^{-1}$$

is called a Neumann resolvent of T. A wellknown fact asserts that the $L(\mathcal{H})$-valued function

$$\lambda \mapsto R_T(\lambda)$$

is analytic in $\mathbf{C} \setminus \sigma(T)$. Moreover, Neumann's resolvent operators commute weith each other, and also with the given operator T. Thus follows from Neuann's equation

$$R(\mu) - R(\lambda) = \frac{R(\mu) \cdot R(\lambda)}{\lambda - \mu}$$

In particular the complex derivatives satsify

$$\frac{d}{d\lambda}(R_T(\lambda) = -R_T(\lambda)^2$$

for every λ outside $\sigma(T)$.

Jext, recall the spectral radius formula which is valid for elements in arbitrary commutative Banach algebras which posses an identty element. For a bounded operator T on $\mathcal{H}$ this gives the equation

$$\lim_{n \to \infty} ||T^n||^{\frac{1}{n}} = \max_{\lambda \in \sigma(T)} |\lambda| \tag{0.1}$$

The common value in (0.1) is denoted by $\rho(T)$. Notice that $\rho(T) = 0$ can occur, i.e this holds when $\sigma(T)$ is reduced to the origin. The multiplicative inequality for operators norms gives

$$\rho(T) \leq ||T|| \tag{0.2}$$

In general this is a strict inequality. Next, let

$$p(T) = a_0 \cdot E + a_1 T \ldots + a_n T^n$$

be a polynomial in T. At the same time we have the polynomial $P(\lambda)$. of the complex variable λ. The following is left to the reader:

Exercise. Show that

$$\sigma(P(T)) = P(\sigma(T)) \tag{0.3}$$

A. Self-adjoint operators.

A bounded linear operator A is self-adjoint if

$$\langle Ax, y \rangle = \langle x, Ay \rangle \tag{A.1}$$

hold for all pairs x, y. it follows that

$$|Ax|^2 = \langle Ax, Ax \rangle = \langle A^2 x, x \rangle \tag{i}$$

hold for every vector x. Form this the reader can check the following equality for operator norms:

$$||A||^2 = ||A^2|| \tag{A.2}$$

A.3 Exercise. Deduce from (A.2) that

$$||A|| = \rho(A) \tag{A3.1}$$

Show also that when A is self-adjoint, then $\sigma(A)$ is a compact set on the real λ-line. A hint is that if λ is complex and x a unit vector then

$$\langle \lambda x - Ax, \lambda x - Ax \rangle = |\lambda|^2 + |Ax|^2 - 2\mathfrak{Re}(\lambda) \cdot \langle Ax, x \rangle \geq |\mathfrak{Im}\lambda|^2 \tag{A.3.2}$$

Finally, show that if $p(A)$ is a polynomial with real coefficients, then $p(A)$ is self-adjoint and

(A.3.3) $$||p(A)|| = \max_{\lambda\in\sigma(A)} |p(\lambda)|$$

B. Normal operators.

A bounded linear operator R is normal if it connutes with its adjoint R^*. Since $R^{**} = R$ and $(ST)^* = T^*S^*$ hold for every pair S, T in $L(\mathcal{H})$, it follows that the operator RR^* is self-adjoint.

B.1 Exercise. Use the equality $||T|| = ||T^*||$ for every bounded operator, together with (A.3.1) applied to $A = RR^*$ to conclude that

(B.1.1) $$\rho(R) = ||R||$$

Next, let

$$p(R) = c_0 + c_1 R + \ldots + c_m R^m$$

be a polynomial where $\{c_\nu\}$ are complex numbers. The adjoint becomes

$$p(R)^* = \bar{c}_0 + \bar{c}_1 R^* + \ldots + \bar{c}_m (R^*)^m$$

Conclude that when R is normal, it follows that $p(R)$ also is normal, and that (0.3) gives

(B.1.2) $$||p(R)|| = \max_{\lambda\in\sigma(R)} |p(\lambda)|$$

More generally, consider a polynomial in λ and its complex conjugarte:

$$Q = \sum\sum c_{\nu,k} \cdot \lambda^\nu \cdot \bar{\lambda}^k$$

To Q we associate the operator

$$\widehat{Q} = \sum\sum c_{\nu,k} \cdot R^\nu \cdot (R^*)^k$$

Te reader shouod check that this gives a normal operator and also verify the equality

(B.1.3) $$||Q|| = \max_{\lambda\in\sigma(R)} |Q(\lambda)|$$

B.2 The algebra $\mathcal{B}^\infty(R)$. Keeping R fixed we have the compact set $K = \sigma(R)$ in the complex λ-plane. Let $C^0(K)$ be the Banach algebra of continuous and complex-valued functions on K equipped with the maxium norm. The Stone-Weierstrass theorem asserts that every $\phi \in C^0(K)$ can be uniformly approximated by polynomials in λ and its complex conjugate $\bar{\lambda}$. So given $\epsilon > 0$ we can find

$$Q = \sum\sum c_{\nu,k} \cdot \lambda^\nu \cdot \bar{\lambda}^k$$

such that the maximum norm $|\phi - Q|_K < \epsilon$. Let us choose a sequence of such polynomials $\{Q_n\}$ where $|\Phi - Q_n|_K \to 0$ Now (B.1.3) entials that

$$\lim ||Q_n - Q_m|| = 0$$

when n and m both tend to infty. Since $L(\mathcal{H})$ is compete under the operator norm there exists a bounded linear operator T such that

$$\lim ||Q_n - T|| = 0$$

The reader should check that T does not depend upon the sequence $\{Q_n\}$ which approximates the given Φ-function. We set $T = \widehat{\Phi}$ and in this way we have constructed a bounded linear opertor starting from an arbitrary continuous function on K. Moreover, we have the equality

(B.2.1) $$|\Phi|_K = ||\widehat{\Phi}||$$

B.3 Exercise. Use (B.2.1) to conclude that $\Phi \mapsto \widehat{\Phi}$ is a norm-preseribnfg algebra isomorphism from $C^0(K)$ onto a closed subalgebra of $L(\mathcal{H})$ which we denote by $\mathcal{B}(R)$. Moreover, every operator in this algebra is normal and commutes with R. In this algebra R corresponds to the Φ-function defined by λ, while $R^* = \widehat{\Psi}$ with $\Psi(\lambda) = \bar{\lambda}$.

B.4 The algebra $\mathcal{B}^\infty(R)$. Let x, y be a pair of vectors in $\mathcal{H}$.. To each $\Phi \in C^0(K)$ we get the complex number

(B.4.1) $$\langle \widehat{\Phi}(x), y \rangle$$

From (B.3) we see that the absolute value of this inner product is

(B.4.2) $$\leq |x| \cdot |y| \cdot |\Phi|_K$$

Tjs emans that (B.4.1) in defines a linear functional on the complex vector space $C^0(K)$ whose norm is majorised by $|x| \cdot |y|$. The famous represetnationm theoremn by F. Riesz gives a compex Borel measure μ of totalvariation $\leq |x| \cdot |y|$ such that

$$\langle \widehat{\Phi}(x), y \rangle = \int_K \Phi(\lambda)\, d\mu(\lambda)$$

The measure depends on the pair x, y and is denoted by $\mu_{\{x,y\}}$. In this way we have condtructed a map

(B.4.3) $$\mathcal{H} \times \mathcal{H} \to M(K)$$

where $M(K) = nC^0(K)^*$ is the space of Borel measure of finite total variation on K. let us now consider a bounded complex-valued Borel function Ψ on K. In other words, for every open set U in the complex λ-plane the inverse image $\Psi^m{-1}(U)$ belongs to the Boolean σ-algebra of subsets of K generated by its compact subsets. Now we apply classic measure theory from original work by Emile Borel In particular there exist integrals

(B.4.4) $$\int_K \Psi(\lambda) \cdot d\mu_{\{x,y\}}(\lambda)$$

for every pair x, y in $\mathcal{H}$.

The operator $\widehat{\Psi}$. Keeping Ψ and y fixed we notice that the integral in (B.4.4) depends linearly upon x. Recall that a Hibert space is self-dual. This gives a unique vector $\xi \in \mathcal{H}$ such that

$$\int_K \Psi(\lambda) \cdot d\mu_{\{x,y\}}(\lambda) = \langle x, \xi \rangle \quad : \forall\, x \in \mathcal{H}$$

Next, ξ depends upon the pair Ψ and y and is dentoed by $\xi(\Psi; y)$. Keeping Ψ fixed while y varies the reader should verify that

$$y \mapsto \xi(\Psi; y)$$

is linear with resepct to y. The conclusion is that there exists a linear operator $\widehat{\Psi}$ such that

$$\int_K \Psi(\lambda) \cdot d\mu_{\{x,y\}}(\lambda) = \langle x, \widehat{\Psi}(y) \rangle$$

hold for every pair x, y in $\mathcal{H}$. The reader should alsomcheck that thisoinear operator is bounded and the operator norm satisfies

$$||\widehat{\Psi}|| \leq |\Psi|_K$$

Self-adjoint operators.

§ A. Eigenvalues and eigenfunctions for the Laplace operator in $\mathbf{R}^2$

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An instructive exampe of a densely defined but unbounded self-adjoint operator appears in PDE-thery where we consider propagation of sound. More precisely, let U be a bounded open domain in $\mathbf{R}^3$ whose boundsry $S = \partial U$ is a union of a finite number of surfaces, each of class C^2. Let $\Omega = \mathbf{R}\setminus\overline{U}$ be the unbounded complement. With t as a time variable we seek solutions to the wave equation

$$\partial^2 u/\partial t^2 = Delta(u)$$

where Δ is the Laplace operator in $\mathbf{R}3$. Above $u = u(x,t)$ is defined for $t \geq 0$ and $x \in \Omega$ and satisfies the following boundary conditions: To begin with $x \mapsto u(x,t)$ belongs to $L^2(\Omega 9$ for every $t \geq 0$. Moreover, applying the Lapace operator we also request that $x \mapsto \Delta u(x,t)$ belongs to $L^2(\Omega)$ for every $t \geq 0$. Finally

$$\frac{\partial u}{\partial n}(p,t) = 0$$

for every $p \in S$ and $t \geq 0$ where we have taken outer nortmal derivatives along S. Denote by $\mathfrak{U}$ the family of all functions $u(x,t)$ satisfying the wave equation and the boundary condition (*) together with the impposed L^2-conditions. With these notations one has the following classic result:

Theorem. *For each pair of functions f_0, f_1 in the family $\mathcal{F}(\Omega)$ there exists a unique $u \in \mathfrak{U}$ such that*

$$u(p,0) = f_0(p) \quad : \frac{\partial u}{\partial t}(p,0) = f_1(p)$$

In [ibid] Carleman gave a proof of this theorem using his theory about unbounded self-adjoint operators.It has the merit that it yields an expression of the unique solution u and at the same time clarifies the physically expected fact that for every u-solution one has

$$\lim_{t \to +infty} \nabla u(p,t) = 0$$

where the convergence holds uniformly while p satays in a compact subset of Ω. Here $\nabla(u) = (u_x, u_y, u_z)$ is the gradient vector and (*) can be expressed by saying that as $t \to +infty$ then every compact portion of Ω comes to rest. The novelty in Carelan's demonstration of (*) was to analyze the spectral function assocviated toa certain symmetric Green's function. This method has later been adotpoted in PDE-theory dealing with various boundary value problems. Following (ibid; page 1xx-1xx] we expose Carleman's proof of Theorem x and of (*) which in my opinion offers a very instructiuve lesson about constructions of unbounded self-adjoint operators. First we recall some facts which were established by G. Neumann and H.Poincaré around 1880.

The Green's function $G(p,q)$.

Given Ω as above with its C^2-boundary S there exists a unique function $G(p,q)$ defined in $\Omega \times \Omega$ which is symmetric and has Newton's singulairty along the diagonal, i.e.

$$G(p,q) = \frac{1}{[p-q|} + H(p,q)$$

where the H-function is symmetric and continuous on $\overline{\Omega} \times \overline{\Omega}$ and

$$\lim_{|p| \to \infty} |p-q| \cdot H(p,q) = 0$$

hold for every fixed q. Moreover, for every fixed $p \in \Omega$ one has

$$\frac{\partial G(p,q)}{\partial n_q} = 0 \quad : q \in S$$

let us also recall that during the construction of the syymeric G-ffun ction, Poincaré established the following estimate: For every $p \in \Omega$ there exists a constant $C(p)$ such that

$$\max_{q\in\overline{\Omega}} |p-q|\cdot|G(p,q)| \leq C(p)$$

Keeping p fixed one has

$$G^2(p,q) \simeq |p-q|^{-2}$$

when $|q|$ tends to infinity. So the integral

$$\int_\Omega G^2(p,q)\,dq$$

is divergent. However, if p and p' is a pair of points in Ω then Poincaré proved that

$$\int_\Omega (G(p,q)-G(p',q))^2\,dq < \infty$$

The finiteness of these L^2-integrals will be sued below to analyze the unbounded operator defined via the symmetric G-function. More precisely, we notice that there exists a densely defined operator on $L^2(\Omega$ defined by

$$\mathcal{G}(f)(p) = \int_\Omega G(p,q)\cdot f(q)\,dq$$

So far we have collected classic results due to G. Neumann and H. Poincaré. The new feature in Carleman's work goes as follows:

Theorem. *The densely defined linear operator $\mathcal{G}$ is self-adjoint and its spectrum is confined to the non-negative real line. Moreover, $\mathcal{G}$ is complete in the sense that*

$$f = \int_0^\infty \frac{d\Theta}{d\lambda}(f)$$

hold for every $f\in\mathcal{F}(\Omega)$ where $\lambda\to\Theta(\lambda)$ is the spectral function attached to $\mathcal{G}$.

Remark. See § xx for the meaning of the operator-valued Θ-function attched to the self-adjoint operator $\mathcal{G}$. The theorem above implies that the unique solution u in Theorem xx is given by

$$u = \int_0^\infty \cos\sqrt{\lambda}\,t\cdot\frac{d\Theta}{d\lambda}(f_0) + \int_0^\infty \sin\sqrt{\lambda}\,t\cdot\frac{d\Theta}{d\lambda}(f_1)$$

Finally, (*) follows via Riemann's Lemma for Fourier series if the spectral function Θ is absolutely continuous with respect to λ.

About the proof of Theorem xx

As expected Green'sformuka wull be applied in several situations. To begin with we shal need the following:

A. Lemma*For each $f\in\mathcal{F}(\Omega)$ it follows that that the Dirichlet integral*

$$\int_\Omega |\nabla(f)|^2\,dq < \infty$$

Proof. Consider large R so that $\mathbf{R}^3\setminus\Omega$ is a compact subset of the ball $B(R)$. Greens formula gives

$$\int_{\Omega\cap B(R)} f\cdot\Delta(f)\,dq + \int_{\Omega\cap B(R)} |\nabla(f)|^2\,dq = \int_{S(R)} u\cdot\frac{\partial u}{\partial r}\,d\sigma - \int_{\partial\Omega} u\cdot\frac{\partial u}{\partial n}\,d\sigma$$

where $S=\partial\Omega$ and $S(R)$ is the shere of radius R. Since both u and $\Delta(u)$ belong to $L^2(\Omega)$ the Cauchy-Schwarz inequaslity entials that the first term above is bounded as a funvtion of r and the last area integral over $\partial\Omega$ is also bounded by (xx). Now we see that lemma A follows if

(A.1) $$\liminf_{R\to\infty} \int_{S(R)} u\cdot\frac{\partial r}{\partial n}\,d\sigma = 0$$

To orve (A.1) we introduce the function

$$\psi(R) = \int_{\Omega \cap B(R)} u^2 \, dq$$

It follows that

$$\psi'(R) = \int -S(R)\, u^2 \, d\sigma$$

Passing to the second derivative the reader can check that

$$\psi''(R) = \frac{2}{R} \cdot \psi'(R) + 2 \cdot \int_{S(R)} u \cdot \frac{\partial u}{\partial r} \, d\sigma$$

So (A.1) follows if

$$\liminf_{R \to \infty} \frac{\psi''(R)}{2} - \frac{1}{R} \cdot \psi'(R) = 0$$

To prove this we frist notice that $\psi(R)$ is a positive and increasing function. Moreover, since $u \in L^2(\Omega)$ the ψ-function has a finite limit A as $R \to \infty$. Now two csses can occur:

Case 1. Here we assume that $R \mapsto \psi'(R)$ is non-increasing and tends to zero. With $R_2 > R_1$ we apply Rolle's theorem and obtain

$$\psi'(R_2) - \psi'(R_1) = (R_2 - R_1) \cdot \psi''(r) \quad : R_1 < r < R_2$$

We can apply this for pair $R_2 >> R_1 >> 1$ and from this the reader can check (xx).

Case 2. Here the non-negative function $\psi'(R)$ takes infinitely many local minima. Say that they occur at points $q_1 < q_2 < \ldots$ where $q_\nu \to \infty$. At these points we have $\psi''(q_\nu)) = 0$.

exercise. deducde from the above that if f and g is a pair in $\mathcal{F}(\Omega)$ then

$$\int_\Omega g \cdot \delta(f) \ dq = \int_\Omega f \cdot \delta(q) \ dq$$

An integral equation. Keeping $p_* \in \Omega$ fixed we consider a complex number λ and seek complex-valued functions u in $\mathcal{F}(\Omega)$ which satisfy

$$u(p) - u(p_*) = \frac{\lambda}{4\pi} \cdot \int_\Omega (G(p,q) - G(p_*, q)) u(q) \, dq$$

From the results in § xx it is clear that every such u-function satisfies the equation

$$\Delta(u) + \lambda \cdot u = 0$$

and the outer normal derivative $\frac{\partial u}{\partial n}$ is zero on $\partial\Omega$. Moreover, using Lemma xx the reader can check that

$$\lambda \cdot \int_\Omega |u|^2 \, dq = \int_\Omega \big[\, |u_x|^2 + |u_y|^2 + (u_z|^2 \big] \, dq$$

This entails that λ must be a real and positive number when u is not identically zero.

Conclusion. The fact that (xx) above has no non-trivial solution when $\mathfrak{Im}\, \lambda \neq 0$ entails via Carleman's general theory in [ibid] that the densely defined operator $\mathcal{G}$ is self-adjoint. Moroever, we leave it to the reader to check that every $g \in L^2(\Omega)$ for which the function

$$p \mapsto \int_\Omega \big(G(p,q) - G(p_*, q) \big) \cdot g(q) \, dq = 0$$

must be a zero function in $L^2(\Omega)$. From this it follows that the self-adjoint operator $\mathcal{G}$ is complete, i.e. one has the integral represetnation formua (xx) for every $f \in \mathcal{F}(\Omega)$.

§ A. Eigenvalues and eigenfunctions for the Laplace operator in $\mathbf{R}^2$

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Theorem 1 below was presented by Carleman at the Scandinavian Congress in mathematics held in Stockholm 1934: In $\mathbf{R}^2$ we consider a bounded Dirichlet regular domain Ω, i.e. every $f \in C^0(\partial\Omega)$ has a harmonic extension to Ω. A wellknown fact which goes back to original work by Dirichlet gives the following: There exists the Greens' function

$$(*) \qquad G(p,q) = \log \frac{1}{|p-q|} - H(p,q)$$

where $H(p,q) = H(q,p)$ is continuous in the product set $\overline{\Omega} \times \overline{\Omega}$. Moreover, $H(p,q) = H(q,p)$ is symmetric and when $q \in \Omega$ is fixed, then $p \mapsto H(p,q)$ is harmonic in Ω and

$$H(p,q) = \log \frac{1}{|p-q|} \quad : \, p \in \partial\Omega$$

This means that $p \mapsto G(p,q)$ vanishes on the boundary. next, (*) means that $p \mapsto G(p,q)$ is superharmonic and and the minimum principe for superharmonic functions plus symmetry entail that

$$G(p,q) > 0$$

hold in $\Omega \times \Omega$. Next, it is obvious that

$$\iint_{\Omega\times\Omega} |G(p,q)|^2 \, dpdq < \infty$$

Hence the linear operator on the Hilbert space $L^2(\Omega)$ defined by the symmetric kernel $G(p,q)$ is a Hilbert-Schmidt operator on the Hilbert space $L^2(\Omega)$ and therefore a compact operator. Since the kernel symmetric and positive the eigenvalues are positive, and general Hilbert space theory applied to the symmetric G-kernel gives a a sequence of pairwise orthogonal functions $\{\phi_n\}$ whose L^2-norms are one and

$$(1) \qquad \int_\Omega G(p,q)\phi_n(q)\, dq = 2\pi \cdot \mu_n \cdot \phi_n(p)$$

where $\{\mu_n\}$ is a non-increasing sequence of positive eigenvalues which tend to zero. Next, we apply the Laplace operator on both sides. Recall that

$$\Delta(\log \frac{1}{|z|}) = -2\pi \cdot \delta_0$$

where δ_0 is the Dirac measure. It follows that the Laplacian of the left hand side in (1) becomes $-2\pi \cdot \phi(p)$ and hence (1) gives the equation

$$\Delta(\phi_n)(p) + \frac{1}{\mu_n} \cdot \phi_n(p) = 0$$

We prefer to use $\lambda_n = \frac{1}{\mu_n}$. Then $\{\lambda_n\}$ is a non-decreasing sequence of real numbers which tends to $+\infty$. Since the kernel $G(p,q)$ is positive it follows by general Hilbert space theory - that $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$, i.e. each L^2-function f has an expansion

$$(2) \qquad f = \sum a_n \cdot \phi_n \quad : \, a_n = \int_\Omega f_n(p) \cdot \overline{\phi_n}(p)\, dp$$

0. Exercise. Verify from the above that each ϕ-function is a continous function in Ω whose boundary values on $\partial\Omega$ are zero. Show also that

$$(0.1) \qquad G(p,q) = \sum_{n=1}^{\infty} \frac{\phi_n(p) \cdot \phi_n(q)}{\lambda_n}$$

where the right hand side is a convergent series when $p \neq q$.

Notice that $G(p,p) = +\infty$ so the series above with $p = q$ is divergent. However, there exists a limit when we employ larger denominators.

Theorem 1. *For every Dirichlet regular domain Ω and each $p \in \Omega$ one has the limit formula*

$$\lim_{N\to\infty} \lambda_N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n^2(p) = \frac{1}{4\pi} \tag{*}$$

Remark. Carleman was iinspired by an earlier result due to H. Weyl which asserts that the set of eigenvalues satisfy the asymptotic formula

$$\lim_{N\to\infty} \frac{\lambda_N}{N} = \frac{\text{Area}(\Omega)}{4\pi}$$

Notice that Weyl's asymptotic formula together with (*) gives

$$\lim_{N\to\infty} N^{-1} \cdot \sum_{n=1}^{n=N} \phi_n^2(p) = xxxx$$

The notable point is that this asymptotic limit is the same for *every* point $p \in \Omega$. The proof of Theorem 1 requires several steps. First, since $\mathcal{G}$ is a Hilbert-Schmidt operator a wellknown result due to Schur gives

$$\sum \lambda_n^{-2} < \infty \tag{i}$$

Let us also remark that since each ϕ_n is harmonic we have the mean-value equality

$$\phi_n(p) = \frac{1}{\pi r^2} \cdot \int_{D_p(r)} \phi(q)\, dq$$

where $D_p(r)$ is the disc of radius r centered at p and r is chosen so small that the disc stays in Ω. Since the L^2-norms of the ϕ-functions are equal to one, the Cauchy-Schwarz inequality gives a constant C such that

$$|\phi_n(p)| \leq C \quad : \; n = 1, 2, \ldots \tag{ii}$$

Now (i-ii) entail that that the Dirichlet series

$$\Phi_p(s) = \sum_{n=1}^{\infty} \frac{\phi_n^2(p)}{\lambda_n^s}$$

is an analytic function of the complex variable s in the half-plane $\mathfrak{Re}\, s > 2$. Less obvious is the following:

Theorem 2. *For each $p \in \Omega$ there exists an entire function $\Psi_p(s)$ such that*

$$\Phi_p(s) = \Psi_p(s) + \frac{1}{4\pi(s-1)}$$

Remark. In § xx we explain how Theorem 2 gives Theorem 1 from Ikehara's limit formula. So the main task is to establish Theorem 2.

Remark. The proof of Theorem 2 employs analytic function theory and is inspiredby Riemann's work on the ζ-function. The interested reader is invited to establish more general results, where Δ is replaced by a higher order elliptic operator in $\mathbf{R}^n$ where $n \geq 3$ can hold. In the cited article such extensions are poibnted out by Carleman on p. xx after the proof of Theorem 1.

Remarquons que la méthode dont nous nous sommes servis est aussi appliquable à une equation elliptique quleconque à un nombre quleconque de dimensions.

In § xx we shall present Carleman's asymptotic formula for eigenvalues of a second order ellitpic operator in $\mathbf{R}^3$ which in general has variable coefficiebts and need not be self-adjoint. Of course, to get a resyukt such as Theorem 1 with an asymptotic limit formula which is independent of the point p in the domain where the eigenfunctions appear, usually requires that the elliptic PDE-operator has constant coefficients. It goes without saying that many specific problems desevere to be analyzed in more detail. A broader perpective concerning asymptotic reperesentations arises when one for example regards spectral functions associated to self-adjoint operators defined via elliptic PDE:s. See § xx below whgere we give some comments about Carlea'ns discussion of the Schrödinger equation

$$\Delta(u) - c(x,y,z) \cdot u = i\frac{\partial u}{\partial t}$$

Here $\Delta is the Laplace operator and$c(x,y,z) is a real-valued function which is locally square integrable and there exist constants R and M such that

$$c(x,y,z) \leq M \quad : x^2 + y^2 + z2 \geq R^2$$

When (*) holds it was proved by Carleman in 1931 that the densely defined operator $\Delta - c$ is self-adjoint on $L^2(\mathbf{R}^3)$ whose spectrum is confined to $[\ell, +\infty)$ for some real number ℓ. If Θ is the associated spectral function we get a solution to (xx) wiuth initial condition $u(p,0) = f(p)$ by

$$u(p,t) = \int_\ell^\infty e^{it\lambda} \cdot \big[\int_{\mathbf{R}}^3 \Theta(p,q;\lambda) \cdot f(q)\, dq \,\big]\, d\lambda$$

Proof of Theorem 2

For each λ outside the discrete set $\{\lambda_n\}$ we put

(1) $$G(p,q;\lambda)=G(p,q)+2\pi\lambda\cdot\sum_{n=1}^{\infty}\frac{\phi_n(p)\phi_n(q)}{\lambda_n(\lambda-\lambda_n)}$$

Notice that (i-ii) above entail that the last sum is converges and gives a meromorohic function of the complex variable λ whose poles are at most simple and confied to the set $\{\lambda_n\}$ Moreover, we get the integral operator $\mathcal{G}_\lambda$ defined on $L^2(\Omega)$ by

(2) $$\mathcal{G}_\lambda(f)(p)=\frac{1}{2\pi}\cdot\iint_\Omega G(p,q;\lambda)\cdot f(q)\,dq$$

A. Exercise. Use that the eigenfunctions $\{\phi_n\}$ is an orthonormal basis in $L^2(\Omega)$ to show that

$$(\Delta+\lambda)\cdot\mathcal{G}_\lambda=-E$$

B. The function $F(p,\lambda)$. Set

$$F(p,q,\lambda)=G(p,q;\lambda)-G(p,q)$$

Keeping p fixed we see that (1) gives

(B.1) $$\lim_{q\to p}F(p,q,\lambda)=2\pi\lambda\cdot\sum_{n=1}^{\infty}\frac{\phi_n(p)^2}{\lambda_n(\lambda-\lambda_n)}$$

Set

(B.2) $$F(p,\lambda)=\lim_{q\to p}F(p,q,\lambda)$$

From (i-ii) above we see that $F(p,\lambda)$ is a meromorphic function in the complex λ-plane with at most simple poles at $\{\lambda_n\}$.

C. Exercise. Let $0<a<\lambda_1$. Use residue calculus to show the equality below in the half-space $\mathfrak{Re}\,s>2$:

(C.1) $$\Phi_p(s)=\frac{1}{4\pi^2\cdot i}\cdot\int_{a-i\infty}^{a+i\infty}F(p,\lambda)\cdot\lambda^{-s}\,d\lambda$$

where the line integral is taken on the vertical line $\mathfrak{Re}\,\lambda=a$.

D. Change of contour integrals. At this stage we employ a device which goes to Riemann and move the integration into the half-space $\mathfrak{Re}(\lambda)<a$. Consider the curve γ_+ defined as the union of the negative real interval $(-\infty,a]$ followed by the upper half-circle $\{\lambda=ae^{i\theta}:0\le\theta\le\pi\}$ and the half-line $\{\lambda=a+it:t\ge 0\}$. Cauchy's theorem entails that

$$\int_{\gamma_+}F(p,\lambda)\cdot\lambda^{-s}\,d\lambda=0$$

We leave it to the reader to contruct the similar curve $\gamma_-=\bar{\gamma}_+$. Using the vanishing of these line integrals and taking the branches of the multi-valued function λ^s into the account the reader should verify the following:

E. Lemma. *When $\mathfrak{Re}\,s$ is sufficientyl large one has the equality*

(E.1) $$\Phi(s)=\frac{a^{s-1}}{4\pi}\cdot\int_{-\pi}^{\pi}F(ae^{i\theta})\cdot e^{(i(1-s)\theta}\,d\theta+\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty F(p,-x)\cdot x^{-s}\,dx$$

The first term in the sum of the right hand side of (E.1) is obviously an entire function of s. So Theorem 2 follows if

(E.2) $$s\mapsto\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty F(p,-x)\cdot x^{-s}\,dx$$

is meromorphic with a single pole at $s=1$ whose residue is $\frac{1}{4\pi}$. To prove this we shall express $F(p,-x)$ when x are real and positive in another way.

F. The K-function. In the half-space $\mathfrak{Re}\, z>0$ there exists the analytic function

$$K(z)=\int_1^\infty \frac{e^{-zt}}{\sqrt{t^2-1}}\,dt$$

Exercise. Show that K extends to a multi-valued analytic function outside $\{z=0\}$ given by

$$K(z)=-I_0(z)\cdot\log z+I_1(z) \tag{F.1}$$

where I_0 and I_1 are entire functions with series expansions

(i)
$$I_0(z)=\sum_{m=0}^\infty \frac{2^{-2m}}{(m!)^2}\cdot z^{2m}$$

(ii)
$$I_1(z)=\sum_{m=0}^\infty \rho(m)\cdot\frac{2^{-2m}}{(m!)^2}\cdot z^{2m}\quad : \rho(m)=1+\frac{1}{2}+\ldots+\frac{1}{m}-\gamma$$

where γ is the usual Euler constant.

Next, with p kept fixed and $\kappa>0$ we solve the Dirichlet problem and find a function $q\mapsto H(p,q;\kappa)$ which satisfies the equation

$$\Delta(H)-\kappa\cdot H=0 \tag{F.2}$$

in Ω with boundary values

$$H(p,q;\kappa)=K(\sqrt{\kappa}|p-q|)\quad : q\in\partial\Omega$$

G. Exercise. Verify the equation

$$G(p,q;-\kappa)=K(\sqrt{\kappa}\cdot|p-q|)-H(p,q;\kappa)\quad : \kappa>0 \tag{G.1}$$

Together with the construction of $G(p,q)$ the reader can verify the equation

$$F(p,-\kappa)=\lim_{q\to p}\,[K(\sqrt{\kappa}\cdot|p-q|)+\log\,|p-q|]+\lim_{q\to p}\,[H(p,q)-H(p,q,\kappa)] \tag{G.2}$$

The last term above has the "nice limit" $H(p,p)+H(p,p,\kappa)$ and from (F.1) the reader can verify the limit formula:

$$\lim_{q\to p}\big(K(\sqrt{\kappa}\cdot|p-q|)+\log\,|p-q|\big)=-\frac{1}{2}\cdot\log\kappa+\log 2-\gamma \tag{G.3}$$

where γ is Euler's constant.

H. Final part of the proof.. Set $A=+\log 2-\gamma+H(p,p)$. Then (G.1) and (G.2) give

$$F(p,-\kappa)=-\frac{1}{2}\cdot\log\kappa+A+H(p,p;\kappa)$$

With $x=\kappa$ in (E.2) we proceed as follows. To begin with it is clear that

$$s\mapsto A\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty x^{-s}\,dx$$

is an entire function of s. Next, consider the function

$$\rho(s)=-\frac{1}{2}\cdot\frac{\sin\pi s}{2\pi^2}\cdot\int_a^\infty \log x\cdot x^{-s}\,dx$$

Notice that the complex derivative

$$\frac{d}{ds}\int_a^\infty x^{-s}\,dx=-\int_a^\infty \log x\cdot x^{-s}\,dx$$

H.1 Exercise. Use the above to show that

$$\rho(s) - \frac{1}{4\pi(s-1)}$$

is an entire function.

From the above we see that Theorem 2 follows if we have proved

H.2 Lemma. *The following function is entire*:

$$s \mapsto \frac{\sin \pi s}{2\pi^2} \cdot \int_a^\infty H(p,p,\kappa) \cdot \kappa^{-s}\, d\kappa$$

Proof. When $\kappa > 0$ the equation (F.1) shows that $q \mapsto H(p,q;\kappa)$ is subharmonic in Ω and the maximum principle gives

(i) $$0 \leq H(p,q;\kappa) \leq \max_{q \in \partial\Omega} K(\kappa|p-q|)$$

With $p \in \Omega$ fixed there is a positive number δ such that $|p-q| \geq \delta : q \in \partial\Omega$ which gives positive constants B and α such that

(ii) $$H(p,p;\kappa) \leq e^{-\alpha\kappa} \quad : \kappa > 0$$

The reader may now check that this exponential decay gives Lemma H.2.