Some topics for the second half of the coures in advanced analysis

Introduction. The material in the three separate sections in these notes describe some interplay between analysis and algebra. So this might also be of interest to students who are more concerned with algebra than analysis. However, most of the material which will be treated in the later part of the course deal with "hard analysis" such as Siu's theorem which gives the existence of Lelong numbers to plurisubharmonic functions.

Contents

I. Algebraic sets and currents

Introduction. Let $n \geq 2$ and consider an irreeducble algebraic set V in $\mathbf{C}^n$ of some dimension k where $1 \leq k \leq n-1$. By a classic normalisation theorem due to Max Noether it follows that after a suitable linear transformation the projection

$$\pi \colon V \to \mathbf{C}^k$$

is proper with finite fibers. Here $z' = (z_1, \ldots, z_k)$. Moreover, performing a linear transformation of the remaining $(n-k)$-coordinates and Abel's constructions of primitive elements in field extensions from his fundamental article (Crelle Journal vol 1) which proved that the general algebraic equationof degree 5 cannot be solved by radicals, it follows that there exists an irreducible polynomial

$$Q(z', z_{k+1}) = z_{k+1}^e + q_1(z')z_{k+1}^{e-1} + \ldots q_k(z')$$

where $\deg(q_j) \leq j$ for every j such that $Q = 0$ on V and the map

$$\pi_* \colon V \to Q^{-1}(0) \tag{1}$$

which sends (x', x'') to $(x', x_{k+1}$ is surjective and generically 1-1. Here $S = Q^{-1}(0)$ is a hypersurfce in the $(k+1)$-dimension (x', x_{k+1})-space. Moreover, introduce the discrimant

$$\delta(x') = \prod_{\nu \neq j} (\alpha_\nu(x') - \alpha_j(x'))$$

with the product taken over the k-tuple of zeros of Q for each x'. Then (1) is bijective outside the discriminant locus. At points where $\delta(x') = 0$ it may occur that the inverse fiber contains more than one point, but in any case it is a finite number.

Some integration currents. Assume that $k \leq n-2$. To each $2 \leq j \leq n-k$ there exists the current of bi-degree $(n, n-k)$ defined by

$$\gamma(\phi^{0,k}) = \int_V x_{k+j} \cdot dX' \wedge \phi^{0,k}$$

where $\phi^{0,k}$ are test-forms in $\mathbf{C}^n$ of bi-degree $(0,k)$. A result due to Lelong asserts that this current is $\bar\partial$-closed. Hence we get a $\bar\partial$-closed direct image current $\pi_*(\gamma)$ supported by S. Since the polynomial Q vasnishes on V it follows that

$$Q \cdot \pi_*(\gamma) = 0$$

By the general structure theorem for $\bar\partial$-closed currents on S which are annihilated by Q it follows that $\pi_*(\gamma)$ is a principal value current of the form

$$\phi^{0,k} \mapsto \lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{a(x', x_{k+j})}{\partial_{k+1}(Q)(x', x_{k+1})} \cdot dX' \wedge \phi^{0,k}$$

where a is a polynomial. We notice that this entails that

$$z_{k+j} - \frac{a(x', x_{k+j})}{\partial_{k+1}(Q)(x', x_{k+1})}$$

vanishes on $V \setminus \delta^{-1}(0)$. So if $\mathfrak{p}$ is the prime ideal in the polynomial ring in $\mathbf{C}^n$, then

$$\partial_{k+1}(Q)(x', x_{k+1}) \cdot z_{k+j} - a(x', x_{k+j}) \in \mathfrak{p} \tag{*}$$

Remark.. The inclusion in (*) is classic and can be proved via elimination theory where explicit constructions were made by Kronecker. But it is interesting to see it follows directly from general current theory.

Abelian currents. Consider a polynomial ρ which does not belong to the prime ideal $\mathfrak{p}$. It gives a current supported by V given via a principal value

$$\lim_{\epsilon\to 0}\int_{V_\rho(\epsilon)}\frac{1}{\rho(x)}\cdot dX'\wedge\phi^{0,k}$$

where $V_\rho(\epsilon)=V\cap\{|p|>\epsilon\}$. This current is in general not $\bar\partial$-closed. But we can introduce the ideal of polynomials $p(x)$ such that the current

$$\lim_{\epsilon\to 0}\int_{V_\rho(\epsilon)}\frac{p(x)}{\rho(x)}\cdot dX'\wedge\phi^{0,k}$$

is $\bar\partial$-closed. Notice that it has the additional property that it is annihilated by polynomials in the prime ideal $\mathfrak{p}$. Since we perform an integration over V the current is also annihilated by complex conjugates of polynomials in the prime ideal. Denote by $\mathcal{A}_V$ the family of all $\bar\partial$-closed currents of bi-degree $(n,n-k)$ which are annihilated by polynomials and their complex conjugate functions taken from $\mathfrak{p}$. One refers to $\mathcal{A}_V$ as the space of abelian currents on V. Notice that $\mathcal{A}_V$ is a module over the polynomial ring $\mathbf{C}^n$. By homological algebra one proves that

$$\mathcal{A}_V\simeq \mathrm{Ext}^p_{\mathbf{C}[x]}(\mathbf{C}[x]/\mathfrak{p},\mathbf{C}[x])$$

In fact, this follows from Malgrange's theorem which asserts that the space of tempered distributions in $\mathbf{C}^n$ is an injective module over the polynomial ring. In contrast to the case of hypersurfaces the module $\mathcal{A}_V$ over the polynomial ring is in general not cyclic, and unless one has constructed an explicit projective resolution of the $\mathbf{C}[x]$- module $\mathbf{C}[x]/\mathfrak{p}$ there does not exist an explicit description of generators for $\mathcal{A}_V$. But notice that the passage to direct image currents gives a map

$$\pi_*\colon \mathcal{A}_V\to\mathcal{A}_S$$

where we have seen that $\mathcal{A}_S$ is a cyclic module. It is obvious that the map (*) is injective. Examples show that surjectivity in general fails. The failure of surjectivity can be understood via another consideration. Namely, let γ be a principal value current on V which is $\bar\partial$-closed in $V\setminus\delta^{-1}(0)$ and annihilated by Q and its complex conjugate, i.e. the current is concentrated to V. Examples show that it may ccur that the direct image current

$$\pi_*(\gamma)\in\mathcal{A}_S$$

This leads to several problems. To begin with Noether's normalisation can be carried out via different choices of coordinates. One is therefore led to ask if there exists a finite family of pairs (V,S_ν) where the hypersurfaces S_ν are as above and the following test is valid: If γ is a principal value current on V whose direct images under the projections to each hypersiurface in this finite family is $\bar\partial$-closed, then γ is so. When this holds we say that the family $\mathcal{S}=\{S_\nu\}$ is ample with respect to the $\bar\partial$-operator.

It is not difficult to show that there exist finite ample families. But it appears that the search for minimal famlies is not well understood. In particular one can ask for the least number $w(V)$ for which there exists an ample family with $w(V)$ many hypersurfaces. The case when V is a curve is already non-trivial and even in this situation I do not know about upper bounds for the integer $w(V)$ expressed by other invariants of the curve. Let me only remark that the problem is related to certain polarisations and a local analytic case is treated in my article from Abel Legacy where conclusive results appear when one instead regards ramified maps with finite fibers. So it appears that the problem posed above deserves a further consideration.

Coleff-Herrera currents. Given the irreducible algebraic set V there exists the space of currents which are $\bar{\partial}$-closed outside the discrimant locus and annihilated by complex conjugates of polynomials in the associated prime ideal, and obtained via principal value currents of the form

$$\gamma(\phi) = \lim_{\epsilon \to 0} \int_{V_\rho(\epsilon)} \frac{1}{\delta(x')^N} \cdot dX' \wedge Q(x, \partial)(\phi^{0,k})$$

Here δ as above is the discriminant polynomial, N some positive integer and $Q(x, \partial)$ belongs to the Weyl algebra A_n of differential operators with polynomial coefficients. For suitable pairs N and Q this yields a $\bar{\partial}$-closed current which is said to be of the Coleff-Herrera type. Let $\mathbf{CH}_V$ denote the family of all Coleff-Herrera currents. A wellknown result in residue theory asserts that one has a canonical isomorphism

$$\mathbf{CH}_V \simeq H_V^p(\mathbf{C}[x])$$

where the right hand side is the usual cohomology module given via the inductive limit of Ext-groups as in (§ xx) when one takes arbitrary powers of $\mathfrak{p}$. Suppose that the γ-current in (*) above is of the Coleff–Herrera type. Its annihialting ideal in $\mathbf{C}[x]$ becomes

$$(0 : \gamma) = \{p(x) : \; Q(x, \partial) \cdot p(x) \in \mathfrak{p} \cdot A_n$$

Let us recall that basic residue calculus entails that the ideai $(0 : \gamma)$ is primary. In general one can start with a differential operator $Q(x, \partial')$ and for a given positive integer N one gets the ideal in $\mathbf{C}[x]$ of polynomials $p(x)$ for which the current

$$\gamma_p(\phi) = \lim_{\epsilon \to 0} \int_{V_\rho(\epsilon)} \frac{p(x)}{\delta(x')^N} \cdot dX' \wedge Q(x, \partial'')(\phi^{0,k})$$

is of the Coleff-Herrera type. This yields a primary ideal which depends on the pair N and Q. It goes without saying that the determination of this primary ideal in general situations is outside the scope of ordinary calculus and it appears that contemporary computer programs are not developed enough to cover this. Let us finally remark that the passage to direct image currents of Coleff-Herrera currents via proper maps π_* as above is best understood via the systematic use of direct images of differential systems, i.e. one should employ the constructions which in the general context were given by Kashiwara around 1970. Moreover, the study of the polynomial ideal $(0 : \gamma)$ is is special. Namely , if R is a differential operator in A_n we get the current defined by

$$\phi \mapsto \gamma(R(\phi^{0,k}))$$

where we used the obvious left A_n-module structure on the space of test-forms of bi-degree $(0, k)$. This gives a right annhilating ideal of γ in the Weyl algebra denoted by $J(\gamma)$. It contains more information about the current. A deep result due to the late E. Andronikof asserts that the analytic wave of γ is equal to the common zeros in the complex cotangent bundle over $\mathbf{C}^n$ given by common zeros of the principal symbols of the differential operators in $J(\gamma)$. Moreover, the cyclic A_n-module $A_n/J(\gamma)$ is regular holonomic. A special class of Coleff-Herrera modules are those for which the A_n-module $A_n/J(\gamma)$ is simple. A wellknown resultmin $\mathcal{D}$-module theory assets that these Coleff-Herrera currents all belong to the right modue generted by Lelong's integration current $\square_V$ whch arises when we take $N = 0$ and $Q = 1$ in (xx). Let us remark that an open problem of considerable relevance is the description of the analytic wave front set of $\square_V$, i.e. this reamis as one challenge in general $\mathcal{D}$-module theory. Via the Riemann-Hilbert correspondence and "modern sheaf-theory" based upon systematic use of micro-localisations, an equivalent problem is to determine the micro-support in the sense of Kashiwara and Schapira of the intersection complex of V.

II. Null solutions of PDE:s with constant coefficients.

Introduction. We expose a result from the article *Null solutions to partial differential operators* [Arkiv för matematik. 1959] by Hörmander. Apart from the conclusive result in Theorem 0 below, the proof is instructive. The crucial step is the construction of null solutions to a differential operator with constant coefficients using suitable complex line integrals taken over contours adapted to the complex zeros of the polynomial of n independent complex variables which corresponds to $P(D)$. To attain this Hörmander employed Puiseux series constructed via embedded curves in the zeros of $P(\zeta)$. The remaining part of the proof is based upon the Paley-Wiener theorem and straighforward duality arguments from general distribution theory. However, one crucial technical point appears. Namely, thanks to wellknown constructions by Gevrey, there exists test-functions whose higher order derivatives have a "good control" which entails that their Fourier transforms enjoy certain decay conditions. Using this, Hörmander's subsequent constructions, which require certain factors with exponential decay to define complex line integrals, can be used after a passage to a certain limit and then the proof of Theorem 0 below is finished. Let us remark that we also will need a density result which goes back to work by Pusieeux which goes as follows. For every irreducible algebraic hypersurface $S = \{P(\zeta) = 0\}$ in the n-dimensional complex ζ-space there exists an ample family of curves of two independent complex contained in S with the property that if $g(\zeta)$ is an entire function which vanishes on all these curves. then is is identically zero on S. This entails that

$$g(\zeta) = P(\zeta)h(\zeta)$$

for another entire function h. In the case when g is the Fourier-Laplace transform of a distribution μ with compact support, the Paley-Wiener theorem entails that $h = \widehat{\gamma}$ for another distribution whose compact support is contained in the convex hull of μ which is used during the final step in the proof.

Before we announce Hörmander's result we need some notations. Let $n \geq 2$ and in $\mathbf{R}^n$ we consider the hyperplane $H = \{x_n = 0\}$. Let $P(D)$ be a differential operator with constant coefficients. Here $D_k = -i \cdot \partial/\partial x_k$ and by Fourier's inversion formula

$$P(D)f(x) = (2\pi)^{-n} \cdot \int e^{i\langle x,\xi\rangle}\, \widehat{f}(\xi)\, d\xi$$

for test-functions $f(x)$. Let m be the order of $P(D)$ which means that

$$P(D) = \sum c_\alpha \cdot D^\alpha$$

where the sum is taken over multi-indices α for which $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq m$. The leading form is defined by

$$P_m(D) = \sum_{|\alpha|=m} c_\alpha \cdot D^\alpha$$

The hyperplane H is characteristic if $P_m(N) = 0$ where $N = (0, \ldots, 1)$, i.e. the term D_n^m does not appear in $P_m(D)$ with a non-zero coefficient. Put $H_+ = \{x_n > 0\}$ and

$$\mathcal{N}_+ = \{g \in C^\infty(H_+) : \; P(D)(g) = 0$$

Thus, we consider C^∞-functions in the open half-plane H_+ which are null solutions to $P(D)$. A smaller space is given by

$$\mathcal{N}_* = \{g \in C^\infty(\mathbf{R}^n) : \ P(D)(g) = 0 \text{ and } \mathrm{Supp}(g) \subset \overline{H_+}\}$$

Next, denote by $\mathcal{N}_*^\perp$ the set of all distributions μ with compact support in H_+ which are zero on $\mathcal{N}_*$.

0. Theorem. *Every distribution μ in $\mathcal{N}_*^\perp$ is zero on $\mathcal{N}_+$*

Remark. This means that $\mathcal{N}_*$ appears as a dense subspace of $\mathcal{N}_+$. The proof requires several steps. The crucial step is to construct functions in $\mathcal{N}_*$ and after prove that they give a dense subspace of $\mathcal{N}_+$. So we begin with:

1. A construction of Puiseux expansions.

Let ξ_0 be a real n-vector such that $P_m(\xi_0) \neq 0$ and ζ_0 some complex n-vector. Let s and t be independent complex variables and set

$$p(s,t) = P(s \cdot N + t\xi_0 + \zeta_0)$$

This gives a polynomial of two variables where the term s^m appears since $P_m(\xi) \neq 0$. At the same time t^m does not appear because $P_m(N) =$ is assumed. A classic result due to Pusieux from 1852 shows that there exists a positive integer p and a series

$$t(s) = s^{k/p} \cdot \sum_{j=0}^{\infty} c_j \cdot s^{-j/p} \tag{1.1}$$

where $0 \leq k < p$ which converges when $|s|$ is large, i.e. there exists some $M > 0$ such that

$$\sum_{j=0}^{\infty} |c_j| \cdot M^{-j/p} < \infty$$

Moreover,

$$P(s \cdot N + t(s)\xi_0 + \zeta_0) = 0 \quad : \ |s| \geq M \tag{1.2}$$

In the lower half-plane $\mathfrak{Im}(s) < 0$ we choose a single valued branch of $s^{1/p}$ which takes positive real values when s is real and positive. Next, choose a number

$$1 - 1/p < \rho < 1$$

Now $(is)^\rho$ has a single valued branch in the lower half-plane such that if $a > 0$ and $s = -ia$ is on the negative imaginary axis, then $(-i^2 a)^\rho = a^\rho$ is real and positive. Notice that this chosen branch gives

$$\mathfrak{Re}((is)^\rho) = \cos \frac{\rho\pi}{2} \cdot |s|^\rho \tag{1.3}$$

when s is on the real line. So if $\epsilon > 0$ we have

$$|e^{-\epsilon(is)^\rho}| = e^{-\epsilon \cdot \mathfrak{Re}(is)^\rho} = e^{-\epsilon \cos \frac{\rho\pi}{2} \cdot |s|^\rho} \tag{1.4}$$

for all real s. Let M be as above and denote by C_* the circle in the lower half-pane which consists of the two real intervals $(-\infty, -M)$ and $(M, +\infty)$ and the lower half-circle where $|s| = R$. For each $x \in \mathbf{R}^n$ and every non-negative integer ν we get the complex line integral

$$\int_{C_*} e^{i\langle x, sN + t(s)\xi_0 + \zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho} \, ds \tag{*}$$

This integral is absolutely convergent. Namely, during the integration on the real interval $(-\infty, -M)$ the absolute value of the integrand is equal to

$$\text{(i)} \qquad |s|^{\nu/p} \cdot |e^{i\langle x,\zeta_0\rangle}| \cdot |e^{it(s)\langle x,\xi_0\rangle}| \cdot e^{-\epsilon\cos\frac{\rho\pi}{2}\cdot|s|^\rho}$$

Above the second factor does not depend on s and the third factor is bounded by

$$e^{|\langle x,\zeta_0\rangle\cdot|t(s)|}$$

Moreover, the Puiseux expansion entails that

$$|t(s) \leq A|s|^{1-1/p}$$

for some constant A. Since $\rho > 1 - 1/p$ the product of the last three terms in (i) decrease exponentially and we conclude that the line integral convrges absolutely for all positive integers ν. A similar convergence holds during the integration on $(M, +\infty)$.

Exercise. Show by Cauchy's theorem in analytic function theory that the line integral (*) does not depend on M as soon as it has been chosen so that the Puiseux series definiing $t(s)$ exists. The resulting value of (*) is therefore a function of x and ϵ and gives a function $u_\epsilon(x)$ defined for all x in $\mathbf{R}^n$. Moreover, the reader should check that when $\epsilon > 0$ kept fixed this yields a C^∞-function of x. In particular

$$(**) \quad P(D)(u_\epsilon)(x) = \int_{C_*} P(sN + t(s)\xi_0 + \zeta_0) \cdot e^{i\langle x,sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

Since $P(sN + t(s)\xi_0 + \zeta_0) = 0$ when $|s| \geq M$ we conclude that $P(D)(u_\epsilon) = 0$, i.e. u_ϵ is a null solution.

The inclusion Supp$(u) \subset \overline{H}_+$. In (*) we perform a line integral whose integrand is an analytic function in the lower half-plane. Using Cauchy's theorem the reader can check that for any $M^* > M$ we have

$$u_\epsilon(x) = \int_{\mathfrak{Im}(s)=-M^*} e^{i\langle x,sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

With $s = t - iM^*$ we have

$$|e^{i\langle x,sN\rangle}| = e^{M^*\langle x,N\rangle}$$

If $\langle x, N\rangle < 0$ this decreases exponentially to zero as $M^* \to +\infty$ and then the reader can check that the limit of (**) as $M^* \to +\infty$ is zero. This proves that the null solution u_ϵ is supported by the half-plane $\overline{H}_+$ and hence belongs to $\mathcal{N}_*$.

§ 2. A study of $\mathcal{N}_*^\perp$.

Consider a test-function ϕ with a compact support in H_+ such that $\phi(\mathcal{N}_*) = 0$. In particular $\mu(u_\epsilon) =$ for every null solution as above. We have also the entire function in the n-dimensional complex ζ-space:

$$\Phi(\zeta) = \int e^{i\langle x,\zeta\rangle}\,\phi(x)\,dx$$

Using the convergence of the line integrals in (*) the reader should verify that Fubini's theorem gives the equation

$$(2.1) \qquad \int u_\epsilon(x)\phi(x)\,dx = \int_{C_*} \Phi(sN + t(s)\xi_0 + \zeta_0) \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

Since $\phi(\mathcal{N}_*) = 0$ is assumed it follows that the last integral is zero for all non-negative integers ν snd each $\epsilon > 0$.

2.2 Another vanishing integral. In the upper half-plane $\mathfrak{Im}(s) > 0$ we can also choose single-valued branches of $s^{1/p}$ and $(-is)^\rho$, where the last branch is chosen so that the value is $a^\rho > 0$ when $s = ai$ for $a > 0$. Then we construct the contour C^* given by the real intervals $(\infty, -M)$ and $(M, +\infty)$ together with the upper half circle of radius M, which for each non-negative integer ν gives the function

$$(*) \qquad v_\epsilon(x) = \int_{C^*} e^{i\langle x, sN + t(s)\xi_0 + \zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(-is)^\rho}\, ds$$

Exactly as in § 1 one verifies that this gives a C^∞-functionof x supported by the right half space $\{x_n \leq 0\}$. Since ϕ has compact support in H_+ it follows that

$$(2.2.1) \qquad 0 = \int v_\epsilon(x)\phi(x)\, dx = \int_{C^*} \Phi(sN + t(s)\xi_0 + \zeta_0) \cdot s^{\nu/p} \cdot e^{-\epsilon(-is)^\rho}\, ds$$

2.3 The limit as $\epsilon \to 0$. In (2.2.1) we have vanshing integrals for each $\epsilon > 0$. If the test-function $\phi(x)$ belongs to a suitable Gevrey class with more regularity than an arbitrary test-function, then the entire function $\Phi(\zeta)$ enjoys a decay condition which enable us to pass to the limit as $\epsilon \to 0$ in (2.2.1). To find a sufficient decay condition we set $\zeta = \xi + i\eta$, and with M kept fixed we study the function

$$s \mapsto \Phi(sN + t(s)\xi_0 + \zeta_0)$$

We already know that there is a constant C such that $|t(s)| \leq C|s|^{1-1/p}$ when $|s| \geq M$. Since ξ_0 and ζ_0 are fixed this gives a constant C_1 such that

$$(2.3.1) \qquad |\mathfrak{Im}(sN + t(s)\xi_0 + \zeta - 0)| \leq C_1(1 + |s|)^{1-1/p}$$

At the same time we have the unit vector N and get a positive constant C_2 such that

$$(2.3.2) \qquad |\mathfrak{Re}(sN + t(s)\xi_0 + \zeta - 0)| \geq C_1(1 + |s|)$$

when $|s|$ is large. Suppose now that the test-function ϕ has been chosen so that

$$(2.3.3) \qquad |\Phi(\xi + i\eta) \leq C \cdot e^{A|\eta|) - B|\xi|^a}$$

hold for some constants C, A, B, a where $a < 1$. From (2.3.1-2.3.2) this gives with other positive constants

$$(2.3.4) \qquad |\Phi(sN + t(s)\xi_0 + \zeta_0) \leq C_1 e^{A_1|s|^{1-1/p} - B_1|s|^a}$$

With ρ chosen as in § 1 where the equality (1.3) is used, it follows that as sson as

$$a > \rho$$

then we get absoutely convergent integrals

$$\int_{|s| \geq M} |\Phi(sN + t(s)\xi_0 + \zeta_0) \cdot |s|^w|\, ds < \infty$$

for every positive integer w. This enable us to pass to the limit in (2.2) and conclude that

$$(2.3.5) \qquad \int_{C^*} \Phi(sN + t(s)\xi_0 + \zeta_0) \cdot s^{\nu/p}\, ds = 0$$

for every non-negative integer ν. The vanishing of these integrals for all $\nu \geq 0$ entails via an application of Cauchy integrals that the entire Φ-function vanishes on the Puisieux curves constructed in § 1 and by the Puiseux improvement of the usual Nullstellen Satz, it follows that Φ vanishes identically on the hypersurface $P^{-1}(0)$. Then a classic division theorem with bounds due to Lindelöf, together with the Paley-Wiener theorem imply that the entire quotient

$$\frac{\Phi}{P} = \Psi \tag{i}$$

where Ψ is given as in (2.0) for some test-function ψ supported by the convex hull of the support of ϕ. Moreover, (i) entails that

$$P(-D)(\psi) = \phi$$

and then it is obvious that ϕ annihilates $\mathcal{N}_+$. Hence we have provec the implication in Theorem 0 for distributions which are defined by test-functions ϕ whose associated entire Φ-function satisfies (2.3.3) with some $a > 1 - 1/p$. But this gives the implication in the theorem for every distribution μ in $\mathcal{N}_*^\perp$. Indeed, to see this we fix some a as above and put

$$\delta = 1/a$$

Now $\delta > 1$ which by a classic construction due to Gevrey enable us to construct an ample family of test-functions ϕ for which (2.3.3) hold and at the same time this family is weak-star dense in the space of distributions with compact support in H_+ and the proof of Theorem 0 is finished. For details about this the reader can consult Hörmander's article if necessary.

III. Lech's theorem on zeros of a polynomial ideal.

Let $n \geq 3$ and $\mathfrak{p}$ is a prime ideal in $\mathbf{C}[z_1, \ldots, z_n]$ whose locus $\mathfrak{p}^{-1}(0) = V$ is an irreducible algebraic set. We assume that $\dim(V) = d$ where $1 \leq d \leq n-2$. Set $p = n - d$ and let $f_1, \ldots, f_m$ be a finite family in $\mathfrak{p}$ whose set of common zeros is V. To each polynomial $p(z) = \sum c_\alpha z^\alpha$ we can take complex conjugates of the coefficients and get the polynomial $\widehat{p}(z) = \sum \bar{c}_\alpha \cdot z^\alpha$. In particular we get

$$f_*(z) = \sum \widehat{f}_j \cdot f_j$$

If $x = (x_1, \ldots, x_n)$ is a real point in $\mathbf{C}^n$ we have $f_*(x) = \sum |f_j(x)|^2$ and hence the intersections

$$\mathbf{R}^n \cap V = \mathbf{R}^n \cap f_*^{-1}(0)$$

Less obvious is the following result due to Christer Lech which gave an affirmative answer to a question raised by by Lars Hörmander in 1956. We remark that Lech's theorem below has applications to over-determined systems of hyperbolic PDE-equations.

Main Theorem *There exists a finite set $f_1, \ldots, f_m$ in $\mathfrak{p}$ and a constant C such that the following inequality hold for each $x \in \mathbf{R}^n$:*

$$\operatorname{dist}(x, V) \leq C \cdot \operatorname{dist}(x, f_*^{-1}(0))$$

The analytic case. It is an open question if a similar result is valid in the local analytic case. Here one starts with a prime ideal $\mathfrak{p}$ in the local ring $\mathcal{O}_n$ of germs of holomorphic functions at the origin in $\mathbf{C}^n$. In a small polydisc D^n we get the analytic set $V = \mathfrak{p}^{-1}(0)$ and one asks if there exists a pair $f \in \mathcal{O}(D^n)$ which vanishes on V and a constant C as in Theorem 1. The difficulty in the local analytic case is to find a sufficently generic family in the given prime ideal and yet consolidate that $\mathcal{M}$ is a finite dimensional complex vector space. The open problem is of special relevance in questions concerned with micro-hyperbolicity and has for example been put forward in work by Kashiwara and Kawai.

The intuitive proof. Let $\mathfrak{p}^*$ be the ideal in $\mathbf{C}[z_1, \ldots, z_n]$ generated by the leading forms of polynomials in $\mathfrak{p}$. The Zariski cone V^* is the set of common zeros of these polynomials. It is wellknown that V^* is an algebraic set whose dimension is equal to that of V. Let $m = n - d$ be the codimension and consider the Grassmanian $\mathcal{G}$ of $(m-1)$-dimensional subspaces of $\mathbf{C}^n$. A subspace $\Pi \in \mathcal{G}$ is transversal to V if the intersection $V^* \cap \Pi$ is reduced to the origin in $\mathbf{C}^n$. A classical result due to M. Noether asserts that for every transversal Π, the set-theoretic sum

$$V + \Pi = \{z + w : \; z \in V : w \in \Pi|]$$

is an algebraic hypersurface given as the zero set of a polynomial $P_\Pi(z)$ whose degree is majorized by the multiplicity $e(V)$ of the given algebraic set. It follows that the family $\{P_\Pi\}$ generates a finite dimensional complex vector space. Let $f_1, \ldots, f_m$ be a basis. Now Lech's polynomial is given by

$$f_*(z) = \sum \widehat{f}_\nu(z) \cdot f_\nu(z)$$

There remains to prove that f_* satisfies (*) in the main theorem. To achieve this several steps are needed where one ingredient is Lech's Inequality in § 0 below. We shall also use the linear systems attached to the prime ideal $\mathfrak{p}$ which .gives the finite dimensional vector space $\mathcal{M}_{\mathfrak{p}}$ generated by the Noether's polynomials $\{P_\pi\}$ above.

Example. The first non-trivial case arises when $n = 3$ and $\mathfrak{p}^{-1}(0) = S$ is an algebraic curve. An example is the non-singular curve S parametrised by a complex variable t so that

$$S = \{(t, t^2, i \cdot t^3\}$$

It is defined by $z_2 = z_1^2$ and $z_3 = iz_1^3$. So the prime ideal $\mathfrak{p}$ is easily described. But distances between real points and S behave in a rather irregular fashion. For example, let R be a large positive real number and take the point $a = (0, 0, R)$. Now we seek a complex number t so that

$$\text{dist}(a : S)^2 = \min_t |t|^2 + |t|^4 + |R + it^3|^2$$

where the minimum is taken over complex t. It is clear that one should choose t so that it^3 is real and negative which does not affect the first two terms, i.e. (1) becomes

$$\min_x x^2 + x^4 + (R - x^3)^2$$

where x are real and positive. The reader may check that the minimum behaves like $R^{\frac{4}{3}}$. One can continue to analyze different directions as a are real and get large euclidian distances from the origin in $\mathbf{R}^3$. With this kept in mind it is not obvious how to prove that there exists a polynomial f_* in the ideal generated by $z_1 - z_2^2$ and $z_3 - iz_1^3$. The reader is invited to carry out further computations and eventually find f_* which works in the main theorem.

§ 0. Lech's inequality.

Let $\phi(z_1, \ldots, z_n)$ be a polynomial of n variables of some degree $m \geq 1$. To each point $\alpha \in \mathbf{C}^n \setminus \phi^{-1}(0)$ we set consider the euclidian distance

$$\text{dist}(\alpha, \phi^{-1}(0)) = \min_{z \in \phi^{-1}(0)} |z - \alpha|$$

where $z - \alpha|$ is the euclidian distance. Next, let $\Theta = (\theta_1, \ldots, \theta_n)$ be a complex vector of unit length, i.e. $|\theta_1|^2 + \ldots + \theta_n|^2 = 1$. This gives a polynomial $t \mapsto \phi(\alpha + t \cdot \Theta)$ and a Taylor expansion yields

$$\phi(\alpha + t \cdot \Theta) = \phi(\alpha) + \sum_{k=1}^{k=m} D_k(\alpha; \Theta) \cdot t^k$$

where $\{D_k(\alpha; \Theta)\}$ is an m-tuple of complex numbers. For each $1 \leq k \leq m$ we set

$$\mathcal{D}_k^*(\phi; \alpha) = \max_\Theta |D_k(\alpha; \Theta)|$$

Definition. *The Lech number of ϕ at the point α is defined by*

$$\mathcal{L}(\phi; \alpha) = \max_{1 \leq k \leq m} \Big[\frac{|\phi(\alpha)|}{\mathcal{D}_k^*(\phi; \alpha)}\Big]^{\frac{1}{k}}$$

0.1 Theorem. *For every positive integer m and each polynomial ϕ of degree m the following inequality holds when α is outside the zero-set of ϕ:*

$$(*) \qquad \frac{1}{2} \leq \frac{\text{dist}(\alpha, \phi^{-1}(0))}{\mathcal{L}(\phi; \alpha)} \leq m$$

Proof Replacing ϕ by $c \cdot \phi$ for some constant we may assume that $\mathcal{L}(\phi; \alpha) = 1$ which means that

$$(1) \qquad |\phi(\alpha)| = \max_{k, \Theta} \; D_k(\alpha; \Theta)$$

with the maximum taken over $1 \leq k \leq m$ and every Θ of unit length. Each complex number ζ with absolute value $\leq 1/2$ and every unit vector Θ give the ζ-polynomial

$$(2) \qquad \phi(\alpha + \Theta \cdot \zeta) = \phi(\alpha) + \sum D_k(\alpha; \Theta) \cdot \zeta^k$$

Now (1) and the triangle inequality give:

$$|\phi(\alpha + \Theta \cdot \zeta)| \geq |\phi(\alpha)| \cdot (1 - 2^{-1} + 2^{-2} + \cdots + 2^{-m}) > 0$$

Hence $\text{dist}(\alpha, \phi^{-1}(0)) \geq 1/2$ which shows the lower bound in (*). To get the upper bound we choose a pair k_*, Θ_* such that

$$|\phi(\alpha)| = D_{k_*}(\alpha; \Theta_*)$$

and consider the polynomial

$$(3) \qquad g(\zeta) = \zeta^m \cdot \phi(\alpha + \Theta_* \cdot \zeta^{-1})$$

Write

$$g(\zeta) = c_m \zeta^m + \ldots + c_0$$

Now (1) implies that the absolute values of c_m and c_{m-k} are equal. Let $\beta_1, \ldots, \beta - m$ be the zeros of g where eventual multiple zeros are repeated. the symmetric polynomial of order $m-k$ of this m-tuple is a sum of c monomials in the roots of degree k and equal to

$$(-1)^{m-k} \cdot \frac{c_{m-k}}{c_m}$$

whose absolute value is 1. if all the zeros have absolute value $\leq 1/m$ the absolute value of the symmetric sum above is majorised by

$$m^{-k} \binom{m}{k}$$

Since this term is < 1 we conclude that g must have a zero of absolute value $> 1/m$ which by (3) implies that the right hand side in (*) is $< m$.

An application. For each finite family of polynomials $\{g_1, \ldots, g_k\}$ and every real point a we set

$$\text{dist}(a, g_\bullet^{-1}(0)) = \min_\nu \; \text{dist}(a, g_\nu^{-1}(0))$$

0.2 Lech's Lemma *Let M be a finite dimensional subspace of $\mathbf{C}[z_1, \ldots, z_n]$. and $f_1, \ldots, f_m$ some basis of M. $f_* = \sum \bar{f}_\nu \cdot f_\nu$. Then, if $\{g_1, \ldots, g_m\}$ is another basis in M there exists a constant $c > 0$ such that the inequality below holds for every real point a:*

$$\text{dist}(a, f_*^{-1}(0)) \geq c \cdot \text{dist}(a, g_\bullet^{-1}(0))$$

where $f_* = \sum \bar{f}_\nu \cdot f_\nu$.

Proof. Applying Lech's inequality to the the g-functions and f_* we can reformulate Lech's Lemma as follows: There exists a constant C which is independent of the real point a such that the following hold: If

$$|g_\nu(\alpha) \geq A^k \cdot D_k(g_\nu)(\alpha) \tag{1}$$

hold for all $k \geq 1$ and all ν and some constant A, then

$$|f_*(\alpha) \geq (CA)^k \cdot D_k(f_*)(\alpha) \tag{2}$$

Since $f_1, \ldots, f_m$ is a k-basis in $\mathcal{M}$ there exists a constant C_0 which is independent of α and

$$|g_\nu(\alpha) \leq C_0 \cdot \sum_{k=1}^{k=m} |f_k(\alpha)| \quad : \, 1 \leq \nu \leq m$$

Conversly, since the g-polynomials also is a basis of $\mathcal{M}$ it is clear that (1) gives a constant $C_1 > 0$ which again is independent of α such that

$$C_1 \cdot \max_\nu |f_\nu(\alpha)| \geq A^k \cdot \sum_{\nu=1}^{\nu=m} D_k(f_\nu; \alpha) \tag{3}$$

Next, since α is real we have

$$f_*(\alpha) = \sum |f_\nu(\alpha)|^2 \implies C_1 \cdot \sqrt{f_*(\alpha)} \geq A^k \cdot \sum_{\nu=1}^{\nu=m} D_k(f_\nu; \alpha) \tag{4}$$

Put

$$D_k^*(\alpha) = \sum_{\nu=1}^{\nu=N} D_k(f_\nu; \alpha) \tag{5}$$

Notice that when f_ν is replaced by $\bar{f}_\nu$ one has

$$D_k(f_\nu; \alpha) = D_k(\bar{f}_\nu; \alpha) \quad : \, k = 1, 2, \ldots$$

These Taylor expansions give the inequality

$$D_k(f_*; \alpha) \leq N(k+1) \max_{i+j=k} D_i^*(\alpha) \cdot D_j^*(\alpha) \tag{6}$$

Finally, with $k = i + j$ it is clear that (4) gives

$$A^k \cdot D_i^*(\alpha) \cdot D_j^*(\alpha) \leq C_1^2 |f_*(\alpha)| \tag{7}$$

Then (6-7) give

$$A^k \cdot D_k(f_*; \alpha) \leq N(k+1) \cdot C_1^2 \cdot f_*(\alpha)| \tag{8}$$

Since this hold for each k we get the requested positive constant C in (2) above.

§ 1. Linear systems.

We shall perform some constructions using specialisations whose general constructions are due to Zariski and Weil. It has the merit that geometric descriptions are carried over to a calculus where formal proofs replace arguments which are not easily found in an "intuitive fashion". We shall employ the Zariski-Weil theory in characteristic zero and follow material from Weil's book *Foundations of algebraic geometry.* Personally I think the material in this outstanding text should be introduced at an early stage to students interested in systems of algebraic equations, and I have never understood the point in putting emphasis upon "soft" sheaf theory" and the trivial yoga about schemes which tend to hide relevant calculations. An exception is of course computations of cohomology, but again all crucial results boil down to algebraic eliminations. See Weil's comments about these matters in the reprinted version of his text-book from 1962.

We begin to expose material from Weil's text-book which is taken for granted during the subsequent proof of Lech's theorem. Let k be a subfield of $\mathbf{C}$ and $\mathfrak{p}$ a prime ideal in the polynomial ring $k[t_1, \ldots, t_n]$ where $n \geq 2$. Generic specialisations of $\mathfrak{p}$ consist of points $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbf{C}^n$ with the property that a polynomial $f(t)$ in $k[t]$ belongs to $\mathfrak{p}$ if and only if $f(\xi) = 0$. This means that one simply evaluates f at the point $\xi \in \mathbf{C}^n$. The existence of generic specialisations is not difficult to prove and is explained in Chapter 1 in [ibid] and the wellknown text-book by Samuel and Zariski. See also the appendix to the present note. An invariant of $\mathfrak{p}$ is the degree of trancendency of ξ over k. More precisely, one takes the subfield of $\mathbf{C}$ generated by k and the n-tuple $\xi_1, \ldots, \xi_n$. Now the invariant denoted by $\deg(\xi)$ is the maximal number of ξ-coordinates which are algebraiccally independent over k.

Consider prime ideal $\mathfrak{p}$ in a polynomial ring $k[t_1, \ldots, t_n]$ where $n \geq 2$ and k is a subfield of $\mathbf{C}$. Let ξ be a generic specialisation. The case $\deg(\xi) = n$ will be excluded for then $\mathfrak{p}$ is reduced to the zero ideal. If $k = \deg(\xi)$ we set $p = n - k$ and refer to p as the codimension of the prime ideal. The case $p = 1$ occurs when $\mathfrak{p}$ is a principal ideal generated by an irreducible polynomial in $k[t]$. Ignoring this special case we suppose that $p \geq 2$, and following Weil one introduces the following: First one takes $(p-1)$ many n-vectors

$$\tau^j = (\tau_1^j, \ldots, \tau_n^j) \quad : 1 \leq j \leq p-1$$

where $(\tau_1^1, \ldots, \tau_n^1, \ldots, \tau_1^{p-1}, \ldots, \tau_n^{p-1})$ are algebraically independent over k. Next, let $\lambda^1, \ldots, \lambda^{p-1}$ be some new $(p-1)$-tuple so that (τ, λ) are algebraically independent. So we have introduced $n(p-1)+p-1 = (n+1)(p-1)$ many variables. Given a generic specialisation ξ we set

$$\zeta_i = \xi_i + \sum_{j=1}^{j=p-1} \lambda^j \cdot \tau_i^j \quad : 1 \leq i \leq n$$

Let $K = k(\zeta, \tau)$ be the field extension of k generated by $\zeta_1, \ldots, \zeta_n$ and the τ-variables.

1.0 Theorem. *The field K has degree of trancendency equal to $np - 1$. Moreover, $\xi_1, \ldots, \xi_n$ as well as $\lambda^1, \ldots, \lambda^{p-1}$ are algebraic over K.*

1.1 The polynomial F. Since $\deg K = np - 1$ there exists an irreducible polynomial $F(\zeta, \tau)$ in $k[\zeta, \tau]$ such that

$$(*) \qquad F(\zeta, \tau) = 0$$

We expand F as a finite sum

$$(**) \qquad F(\zeta, \tau) = \sum f_\nu(\zeta) \cdot \phi_\nu(\tau)$$

where $\{\phi_\nu(\tau)\}$ are k-linearly independent in $k[\tau]$ and and $\{f_\nu(\zeta)\}$ are k-linearly independent in $k[\zeta]$.

1.2 Proposition. *The polynomials $\{f_\nu\}$ belong to $\mathfrak{p}$ and the vector space over k generated by these polynomials is independent of the polar representation of F above.*

1.3 A special family of polynomials. We can fix the $(p-1)$ many τ-vectors which gives a family $\{\bar{\tau}^j : j = 1, \ldots, p-1\}$ of n-vectors in $\mathbf{C}^n$ and the polynomial:

$$g(z) = F(z, \bar{\tau})$$

in $k[z_1, \ldots, z_n]$. The Zariski-Weil theory shows that for every $(p-1)$-tuple $\bar{\lambda}$ there exists a specialisaton $(\zeta, \tau) \mapsto (\bar{\zeta}, \bar{\tau})$ where one has the equations

$$\bar{\zeta}_i = \xi_i + \sum_{j=1}^{j=p-1} \bar{\lambda}^j \cdot \bar{\tau}_i^j \quad : 1 \leq i \leq n$$

It follows that if $\mathcal{L}$ is the subspace of $\mathbf{C}^n$ generated by the $(p-1)$-many n vectors $\{\bar{\tau}^j = (\bar{\tau}_1^j, \ldots, \bar{\tau}_n^j)\}$ then g-polynomial vanishes at each point in the set

$$\mathfrak{p}^{-1}(0) + \mathcal{L}$$

In other word, g vanishes on the subset of $\mathbf{C}^n$ whose points are of the form $\alpha + \beta$ wihere $\alpha \in \mathfrak{p}^{-1}(0)$ and $\beta \in \mathcal{L}$. In particular every g-polynomial above is zero on $\mathfrak{p}^{-1}(0)$ and hence belongs to $\mathfrak{p}$ by the Nullstellen Satz for prime ideals.

1.4 The vector space $\mathcal{M}_{\mathfrak{p}}$. It is the vector space over k generated by the f-polynomials above. Let m be its dimension. A basis is found as follows: Consider an m-tuple of specialized τ-vectors, i.e. a family $\{\bar{\tau}^j(k) : 1 \leq k \leq m\}$. Evaluating the ϕ-polynomials from (**) give the $m \times m$-matrix with elements $\{\phi_\nu(\bar{\tau}(k)\}$. If the determinant of this matrix is $\neq 0$ the Zariski-Weil theory shows that the m-tuple

$$g_k(z) = F(z, \bar{\tau}(k)) \quad : 1 \leq k \leq m$$

is a basis of the k-space $\mathcal{M}_{\mathfrak{p}}$.

§ 2. Proof of the Main Theorem

From § 1 we have the vector space $\mathcal{M}_{\mathfrak{p}}$ and the polynomial f_*. Let $\{a_\mu\}$ be a real sequence outside V. For each μ we set

(i) $$c_\mu = \frac{1}{\sum |f_\nu(\alpha_\mu)|}$$

Take a subsequence $1 \leq \mu_1 < \mu_2 < \ldots$ such that there exist limits

(ii) $$d_\nu = \lim_{j\to\infty} c_{\mu_j} \cdot f_\nu(\alpha_{\mu_j}) \quad : 1 \leq \nu \leq m$$

Notice that (i) entails that $|d_1| + \ldots + |d_m| = 1$. To simplify notations we set

$$\beta_j = \alpha_{\mu_j}$$

From § 1.1 we have the polynomial $F(z,t)$ with its polar decomposition. Put

(iii) $$\phi_*(t) = \sum d_\nu \cdot \phi_\nu(t)$$

Here $\phi_*(t)$ is not identically zero since $\{\phi_\nu(t)\}$ are linearly independent and the d-vector is non-zero. Next, we can choose an m-tuple of specialised τ-vectors $\{\bar{\tau}(k)\}$ such that the determinant of the matrix $\{\, \phi_\nu(\bar{\tau}(k)\}$ from § 1.4 is non-zero and in addition

(iv) $$\phi_*(\bar{\tau}(k)) \neq 0 \quad : 1 \leq k \leq m$$

To the specialised m-tuple of $\bar{\tau}$-vectors we assign polynomials

(v) $$g_k(z) = F(z, \bar{\tau}(k))$$

By the result in § 1.4 the non-vanishing determinant above entails that $\{g_k\}$ is a basis in $\mathcal{M}_{\mathfrak{p}}$. Notice that

(vi) $$c_{\mu_j} \cdot g_k(\beta_j) = \sum c_{\mu_j} \cdot f_\nu(\beta_j) \cdot \phi_\nu(\bar{\tau}(k))$$

The limit in (ii) entails that (vi) close to $\sum d_\nu \cdot \phi_\nu(\bar{\tau}(k)) = \phi_*(\bar{\tau}(k))$. This approximative equality will be used to prove the following crucial resut.

3.1 Lemma. *For every k it holds that*

$$\limsup_{j\to\infty} \frac{\operatorname{dist}(\alpha_{\mu_j}, g_k^{-1}(0)}{\operatorname{dist}(\alpha_{\mu_j}, V)} > 0$$

Proof. Consider some k, say $k = 1$. and denote by $\mathcal{L}$ the linear subspace of $\mathbf{C}^n$ spanned by the specialized vector $\bar{\tau}(1)$ and $\mathcal{L}$ is the $(p-1)$-dimensional subspace in $\mathbf{C}^n$ generated by $\bar{\tau}(k)$ as explained in § 1.4 and we also recall that

$$g_1^{-1}(0) = V + \mathcal{L}$$

Now we have

(i) $$\operatorname{dist}(\beta_j, g_1^{-1}(0)) = \min_{z,\zeta} ||\beta_j - z - \zeta||$$

where the minimum is taken over pairs $z \in V$ and $\zeta \in \mathcal{L}$. Notice that the quotient in Lemma 3.1 for each j is majorizes

(ii) $$\min_{z,\zeta} \frac{||\beta_j - z - \zeta||}{||\beta_j - z||}$$

Since $\mathcal{L}$ is a subspace the minimum is the same as

$$\text{(iii)} \qquad \min_{z,\zeta} \left|\left|\frac{\beta_j - z}{||\beta_j - z||} - \zeta\right|\right|$$

Above we measure distances of unit vectors $\frac{\beta_j - z}{||\beta_j - z||}$ to the subspace $\mathcal{L}$ and Lemma 3.1 amounts to prove that this distance is $\geq c_*$ for a positive conatant c_* and all sufficently large j. For if no such c_* exists we can find specialisations $\bar{\tau}$ where the corresponding vector space $\mathcal{L}(\bar{\tau}))$ is close to $\mathcal{L}$ in the corresponding Grassmannian and there exist large j and points $z_j \in V$ such that

$$\text{(iv)} \qquad \frac{\beta_j - z_j}{||\beta_j - z_j||} \in \mathcal{L}(\bar{\tau})$$

This entails that

$$\beta_j \in V + \mathcal{L}(\bar{\tau})$$

As explained in § 1.4 it follows that

$$0 = F(\beta_j, \bar{\tau}) = \sum f_\nu(\beta_j) \cdot \phi_\nu(\bar{\tau}) \implies$$

$$\text{(v)} \qquad 0 = \sum c_{\mu_j} \cdot f_\nu(\beta_j) \cdot \phi_\nu(\bar{\tau})$$

Next, recall that

$$\lim_{j \to \infty} c_{\mu_j} \cdot f_\nu(\beta_j) = d_\nu$$

hold for every ν and hence (v) would entail that

$$\phi_*(\bar{\tau}) = \sum d_\nu \cdot \phi_\nu(\bar{\tau}) \simeq 0$$

This gives a contradiction since continuity implies that $\phi_*(\bar{\tau})$ stays away from zero when $\bar{\tau}$ is close to $\bar{\tau}(k)$ and Lemma 3.1 is proved.

3.2 Final part of the proof.

If the Main Theorem fails there exists a sequence of real points $\{a_\mu\}$ for which

$$\lim_{\mu} \frac{\text{dist}(a_\mu, f_*^{-1}(0))}{\text{dist}(a_\mu, V)} = 0$$

But this is impossible in view of Lech's Lemma in § 0 and Lemma 3.1 which finishes the proof of the Main Theorem.

Appendix. Prime ideals and specialisations.

Let us give a brief account of material from the cited text-books by Weil and Samuel-Zariski. Let n be some positive integer. Denote by $\mathcal{K}$ the family of all subfields k of $\mathbf{C}$. Notice that every such field contains the rational number field Q. To each $k \in \mathcal{K}$ one has the k-algebra $k[t_1, \ldots, t_n]$ in n variables. Prime ideals in this commutative algebra can be constructed as follows: Let $\xi = (\xi_1, \ldots, \xi_n)$ be a point in $\mathbf{C}^n$. Now we can evaluate polynomials $p(t) \in k[t]$ at ξ and the map

$$\text{(i)} \qquad p(t) \to p(\xi)$$

is called a specialisation. Its kernel is obviously a prime ideal in $k[t]$ which we denote $\mathfrak{p}_\xi$. If $\mathfrak{p}_\xi = \mathfrak{p}_\eta$. for a pair of distinct complex n-vectors ξ and η one says that ξ and η are k-equivalent. It may also occur that one has an inclusion

$$\mathfrak{p}_\xi \subset \mathfrak{p}_\eta$$

A warning. A polynomial $p(t)$ in $k[t_1, \ldots, t_n]$ can be irreducible and yet p may be factorised in polynomial rings over larger fields than k. A first example occurs when $n = 1$ and $k = Q$ where the polynomial $p(t) = t^2 + 1$ is irreducible in $Q[t]$ but if we take the field $k = Q(i)$ where the imaginary unit is adjoined, then $p(t) = (t - i)(t + i)$ has a factorisation in $k[t]$. On the other hand, if $n = 2$ then the polynomial $p(t_1, t_2) = t_1 \cdot t_2 + 1$ is *absolutely irreducible polynomial* which means that it stays irreducible in $k[t_1, t_2]$ for every subfield k of $\mathbf{C}$, i.e. p is even irreducible in $\mathbf{C}[t_1, t_2]$. Recall also the classic result which asserts that if k is a field and $n \geq 1$ then the polynomial ring $k[t_1, \ldots, t_n]$ is a unique factorisation domain. The proof is carried out by an induction and is exposed in text-books. One refers to this as *Gauss's theorem* since he was the first to recognize and use the result in a general set-up.

The existence of specialisations.

If k is a field and $n \geq 1$ we denote by $\mathrm{Spec}(k; n)$ the family of all prime ideals in $k[t_1, \ldots, t_n]$. Each prime ideal $\mathfrak{p}$ in this spectral set is equal to $\mathfrak{p}_\xi$ for some $\xi \in \mathbf{C}^n$. The proof relies upon Gauss' theorem and since it is such an essential result we expose details and start with:

The case of principal ideals. Let $n \geq 2$ and $p(t)$ is an irreducible polynomial in $k[t_1, \ldots, t_n]$. Gauss' theorem entails that the principal ideal generated by p is a prime ideal. If p has some degree m we perform a k-linear transformation on the vector space k^n which yields an obvious k-algebra isomorphism on $k[t]$ and assume that $p(t)$ is of the form

$$\text{(1)} \qquad p(t) = t_n^m + q_1(t')t_n^{m1} + \ldots + q_m(t')$$

where the q-polynomials depend on $t' = (t_1, \ldots, t_{n-1})$ and $\deg q_j \leq j$ hold for each j. To p we associate the polynomial

$$\frac{\partial p}{\partial t_n} = m \cdot t_n^{m-1} + \ldots + q_{m-1}(t')$$

Let $K = k(t_1, \ldots, t_{n-1})$ be the field of rational functions in the t'-variables. Gauss' theorem entails that p regarded as an element in thr polynomial ring $K[t_n]$ is irreducible and therefore the ideal in $K[t_n]$ generated by p and its partial derivative above is equal to $K([t_n]$. This gives an equation

$$S(t_n) \cdot p(t', t_n) + R(t_n) \cdot \frac{\partial p}{\partial t_n} = 1$$

where $S(t_n)$ is a polynomial in $K[t_n]$ of degree $\leq m-2$ and R has degree $\leq m-1$. Taking the product of denominators in the $k[t']$-coefficients of S and R we find a non-zero polynomial $\delta(t') \in k[t_1, \ldots, t_{n-1}]$ such that

$$\text{(i)} \qquad \delta(t') = A(t_n) \cdot p(t', t_n) + B(t_n) \cdot \frac{\partial p}{\partial t_n}$$

where A and B are polynomials in the the n many t-variables. Let us then consider a point $\xi^* = (\xi_1, \ldots, \xi_{n-1})$ in $\mathbf{C}^{n-1}$ where $\xi_1, \ldots, \xi_{n-1}$ are *algebraically independent* over k. This implies that $\delta(\xi^*) \neq 0$. With ξ^* chosen we consider the polynomial $p(\xi^*, t_n)$ which has degree m in the variable t_n with complex coefficients. The fundamental theorem of algebra gives a factorisation

$$p(\xi^*, t_n) = \prod_{\nu=1}^{\nu=m} (t_n - \alpha_j)$$

The equation (i) gives for each α-root

$$\text{(ii)} \qquad \delta(\xi^*) = B(\xi^*, \alpha_j) \cdot \frac{\partial p}{\partial t_n}(\xi^*, \alpha_j)$$

Hence the t_n-derivative of $p(\xi^*, t)$ is $\neq 0$ at all these roots which means that they are simple, i.e. the complex numbers $\alpha_1, \ldots, \alpha_m$ are distinct. Let us choose one of these simple roots, say α_1 and put

$$\xi = (\xi^*, \alpha_1)$$

Exercise. Show that the prime ideal $\mathfrak{p}_\xi$ in $k[t_1, \ldots, t_n]$ is equal to the principal ideal generated by p.

Specialisations of arbitrary prime ideals. Let $\mathfrak{p}$ be non-zero in $\text{Spec}(k; n)$ where $n \geq 2$. Let m_* be the smallest integer such that $\mathfrak{p}$ contains a polynomial of degree m_*. The case $m_* = 0$ is excluded for then $\mathfrak{p}$ is the whole polynomial ring. After a k-linear transformations we get an irreducible polynomial $p(t) \in \mathfrak{p}$ as above. Now we consider the intersection

$$k[t_1, \ldots, t_{n-1}] \cap \mathfrak{p}$$

This is obviously zero or a prime ideal in the polynomial ring with $n-1$ many variables. Let K be a field which contains k. To each $\mathfrak{p} \in \text{Spec}(K : n)$ we put

$$\mathfrak{p} \cap k[t_1, \ldots, t_n]$$

is prime ideal in the polynomial ring over k. Hence there exists a map

$$\text{Spec}(K : n) \to \text{Spec}(k : n)$$

Specialisations. Following [ibid: page 27] a pair of points ξ and η in $\mathbf{C}^n$ are generic specialisations of each other with respect to a field $k \in \mathcal{K}$ when their prime ideals are equal. Next, if $\mathfrak{p}_\xi \subset \mathfrak{p}_\eta$ one says that η is a specialisation of ξ and write

$$\eta \leq_k \xi$$

where the subscript indicates that the given field k is taken into the account.

1.1 The linear system. Let $\mathfrak{p}$ be a prime ideal in $k[t]$ and choose ξ so that $\mathfrak{p} = \mathfrak{p}_\xi$. Next, let $\lambda_1, \ldots, \lambda_{p-1}$ and $\{\bar{\tau} = (\bar{\tau}_1 \ldots, \bar{\tau}_{p-1} : 1 \leq j \leq n\}$ be variables over k, i.e. the field generated by these points has degree of trancendency $(n+1)(p-1)$ over k. Above $\bar{\tau}_k = (\tau_k^1, \ldots, \tau_k^n)$ are n-vectors for each $1 \leq k \leq p-1$. To each $1 \leq i \leq n$ we set

$$\zeta_i = \xi_i + \sum_{k=1}^{k=p-1} \lambda_k \cdot \tau_k^i \tag{1}$$

Witb $N = n + n(p-1) = np$ we obtain variable points $(\zeta, \bar{\tau}) \in \mathbf{C}^N$ where we denote the coordinates by $(\zeta, \tau) = (\zeta_1, \ldots, \zeta_n, t_1, \ldots, t_{n(p-1)})$ as coordinates.

1.2 Theorem. *The prime ideal $\mathfrak{p}_{(\zeta,\tau)}$ in the polynomial ring $k[\zeta, \tau]$ is principal. Moreover, $\{\xi_j\}$and $\{\lambda_k\}$ belong to the algebraic closure of the field generated by k and (ζ, τ).* In the sequely we prefer to use $z_\nu = \zeta_\nu$ and the t-letter for the τ-variables.

Remark. A detailed proof appears in Weil's text-book. In the sequel we prefer to use $z_\nu = \zeta_\nu$ and the t-letter for the τ-variables. Theorem 1.2 gives the irreducible polynomial $F(z, t)$ of N-variables in the polynomial ring $k[z, t]$ such that $F(\zeta, \bar{\tau}) = 0$ We can express F in as a sum

$$F(z, t) = \sum_{\nu=1}^{\nu=m} f_\nu(z) \cdot \phi_\nu(t) \tag{i}$$

where $\{f_\nu(z)\}$ are k-linearly independentin $k[z_1, \ldots, z_n]$ and $\{\phi_\nu(t)\}$ are k-linearly inependent in the polynomial ring of the $n(p-1)$ many t-variables. The polar decomposition is not unique. However, the vector space over k generated by the f-polynomials in $k[z]$ does not depend upon the chosen polar representation of F. It is denoted by $\mathcal{M}$ and called the Weil-space attached to the prime ideal $\mathfrak{p}$. Since ξ is algebraic of the field generated by (ζ, τ) it follows that

$$F(\xi, t) = \sum f_\nu(\xi) \cdot \phi_\nu(t) = 0 \quad : \forall t \in \mathbf{C}^{N-n} \tag{*}$$

Since $\{\phi_\nu(t)\}$ are k-linearly independent it follows that $f_\nu(\xi) = 0$ for each ν, i.e. these polynomials belong to $\mathfrak{p}$.