1. Distributions and boundary values of analytic functions

Contents

Introduction.

Distributions and their basic properties were announced by Laurent Schwartz in 1945 and presented in his book *Théorie des distributions* from 1951. As pointed out by Lars Gårding in Chapter 12 from [History], Schwartz' broad attack, his radical use of infinitely differentiable functions and his conviction that distributions would be useful almost everywhere made the difference compared to earlier work where special cases had adopted the idea of distributions but never in full generality. In a broad sense distribution theory is a minor extension of the Lebegue-Riesz theory since a distribution in $\mathbf{R}^n$ is locally a finite sum of derivatives of Riesz measures. However, the notion and the calculus with distributions clarifies many constructions and is essential in the theory of differential operators. Already the case of an ordinary differential operator $P(D)$ with constant coefficients in one variable illustrates the new phenomena which arise when one extends to equation of smooth solutions to distributions and already for differential operators $P(x, D)$ whose coefficients are polynomials in a single variable x the study of solutions in the space of disttibutions becomes quite involved. We give some comments about this in § XX.

The reader may now pass directly to the headline entitled *The origin of distributions* which togehter with § B and the two sections 1-2 provides basic material about distrubtions on the realoine and the plane. Since we are foremost concerened with applications in analytic function theory of one complex variable we have not tried to expose distribution theory in full generality. But we insert some material of a more advanced nature in § xx which includes a discussion about curtrents on manifolds. Let us finish this introduction with some examples which illustrate the flavour of distributions.

Stokes Theorem. Let Ω be a bounded open set in $\mathbf{R}^n$ whose boundary satisfies the Federer condition in § XX. The characteristic function χ_Ω yields a distribution and Stokes Theorem implies that for each $1 \leq k \leq n$ one has the equality of distributions:

$$\frac{\partial \chi_\Omega}{\partial x_k} = \mathbf{n}_k \cdot dS$$

where dS is the area measure on the regular part of $\partial\Omega$ and $\mathbf{n}_k$ is the k:th component of the outer normal. Above the right hand side is a Riesz measure supported by the compact boundary of $\partial\Omega$. It is now tempting to continue and search higher order derivatives. The case when Ω is a convex polygon in $\mathbf{R}^n$ where $n \geq 3$ leads to interesting formuas which one ofgen prefers to express via currents. Currents are by definition linear functionals defined on test-forms rather than test-functions. For example, above one starts with the distribution χ_Ω whose differential is the 1-current which maps a test-form γ^{n-1} of degree $(n-1)$ to

$$\int_{\partial\Omega} \gamma^{n-1}$$

Here one employs the orientation on the $(n-1)$-dimensional pieces of $\partial\Omega$ where the integration takes places. An example where one arrives at a distribution of order one occurs if Ω is as above and H a harmonic function defined in some open neighborhood of $\bar{\Omega}$. Now there exists the distribution $H \cdot \chi_\Omega$. Applying the Laplace operator Greens' formula gives

$$\int_\Omega H \cdot \Delta(f)\, dx = \int_{\partial\Omega} H \cdot \frac{\partial f}{\partial \mathbf{n}} \cdot dS - \int_{\partial\Omega} \frac{\partial H}{\partial \mathbf{n}} \cdot f \cdot dS \quad : \; f \in C_0^\infty(\mathbf{R}^n)$$

The right hand side gives an expression for the distribution $\Delta(H \cdot \chi_\Omega)$. The first integral corresponds to the distribution defined by

(i) $$f \mapsto \int_{\partial\Omega} H \cdot \frac{\partial f}{\partial \mathbf{n}} \cdot dS$$

The construction of normal derivatives means that (i) is the distribution given by

(ii) $$H \cdot \sum_{k=1}^{k=n} \mathbf{n}_k \cdot \frac{\partial dS}{\partial x_k}$$

Thus, in the last sum one has taken partial distribution derivatives of the non-negative Riesz measure dS which from the start has compact support given by the boundary of Ω.

Regularisations. Every distribution μ in $\mathbf{R}^n$ can be approximated in the weak topology by C^∞-densities. To achieve this one takes a C^∞-function $\phi(x)$ with a compact support such that

$$\int_{\mathbf{R^n}} \phi(x)\, dx = 1 \quad \text{and} \quad \phi(0) = 1$$

For every $\delta > 0$ there exists the C^∞-function μ_δ which for each $a \in \mathbf{R}^n$ takes the value

$$\mu_\delta(a) = \mu(\phi_{\delta;a}) \quad \text{where} \quad \phi_{\delta;a}(x) = \delta^{-n} \cdot \phi(\frac{x-a}{\delta})$$

One easily verifies the limit formula

$$\lim_{\delta \to 0} \mu_\delta(f) = \mu(f) \quad : f \in C_0^\infty \mathbf{R}^n)$$

These regularisations reveal why calculus for differentiable functions extends to distributions. We shall foremost study distributions on the real line and occasionally also in 2-dimensional case where $\mathbf{R}^2$ can be identified with $\mathbf{C}$. For a general account about distributions we refer to Hörmander's text-book [Hörmander]. Major results in this chapter are due to Beurling, Carleman, Paley and Wiener. We have also included a proof of Lindeberg's sharp version of the Central Limit Theorem in § 18 where the passagr to the limit is more precise as compared to a "weak limit" in the sense of distributions. In § 1 we construct the Fourier transform of tempered distributions and derive Fourier's inversion formula. The proof employs the Schwartz class $\mathcal{S}$ of rapidly decreasing C^∞-functions which enable us to define the Fourier transform on a wider class than integrable functions.

Example. Let ξ be the variable when we pass to the Fourier transform. If $\epsilon > 0$ there exists the integrable function on the ξ-line defined by $e^{-\epsilon\xi}$ when $\xi \geq 0$ and zero on $(-\infty, 0)$. Its inverse Fourier transform is given by the function

$$J_\epsilon(x) = \frac{1}{2\pi} \cdot \int_0^\infty e^{ix\xi} \cdot e^{-\epsilon\xi}\, d\xi = -\frac{1}{2\pi} \cdot \frac{1}{ix - \epsilon} = -\frac{1}{2\pi i} \cdot \frac{1}{x + i\epsilon}$$

When $g(\xi)$ belongs to the Schwarz class on the ξ-line it is obvious that there exists a limit

$$\lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon\xi} \cdot g(\xi)\, d\xi = \int_0^\infty g(\xi)\, d\xi$$

From this we shall learn that Fourier's inversion formua entails that for every $f(x)$ in the Schwartz class on the real x-line there exists the limit

$$\lim_{\epsilon \to 0} \int \frac{f(x)}{x + i\epsilon}\, dx = -2\pi i \cdot \int_0^\infty \widehat{f}(\xi)\, d\xi$$

Passing to the limit we get a distribution on the real x-line denoted by

(1) $$\mu = \frac{1}{x + i0}$$

whose Fourier transform is becomes

$$(2) \qquad -2\pi i \cdot H_+(\xi)$$

where H_+ is the Heaviside distribution on the ξ-line expressed by the density 1 on $\xi \geq 0$ and zero on $\{\xi < 0\}$ In a similar way we construct the distribution $\frac{1}{x-i0}$ and here one finds that

$$(2) \qquad \widehat{\frac{1}{x-i0}}(\xi) = 2\pi i \cdot H_-(\xi)$$

where $H_-(\xi) = 1$ if $\xi \leq 0$ and zero on $(0, +\infty)$. With

$$\mu = \frac{1}{x+i0} - \frac{1}{x+i0}$$

it follows from the above that $\widehat{\mu} = -2\pi i \cdot 1_\xi$ where 1_ξ is the identity on the ξ-line. Fourier's inversion formula entails that

$$\mu = -2\pi i \cdot \delta_0$$

where δ_0 is the Dirac measure at $x = 0$. As we shall see in § XX this reflects Cauchy's residue formula and illustrates how analytic funtion theory intervenes with distributions.

Boundary values of analytic functions

.

This is a major issue in this chapter. The basic constructions appear in § 2 and are used to obtain boundary values of analytic functions $f(x+iy)$ with a moderate growth as $y \to 0$. Theorem 2.7 gives a uniqueness properties when boundary values of analytic functions are taken from the upper, respectively the lower half plane. § 3 describes general procedure to get Fourier's inversion formula where we follow Carleman's lectures at Institute Mittag Leffler in 1935. Theorem 3.5 leads to the Fourier-Carleman transform which can be used to establish uniqueness results for distributions whose Fourier transforms have certain gaps. § 4 is devoted to the *Paley-Wiener theorem* and § 5 treats Runge's approximation theorem and results about the inhomogeneous $\bar{\partial}$-equation. § 6 extends Fourier's inversion formula to non-tempered situations which lead to hyperfunctions describing the dual to the space of real-analytic functions. § 7 is devoted to a fundamental inequality for differentiable functions concerned with an inequality for L^2-norms of higher order derivatives of differentiable functions which vanish up to a a certain order at the end-points of a bounded interval. The remaining sections treat topics which foremost deal with problems related to analytic functions and Fourier transforms.

The meromorphic family ξ_+^λ

In the upper half plane of the complex z-plane there exists for each complex λ the analytic function

$$z^\lambda = e^{\lambda \cdot \log z}$$

where the single-valued branch of $\log z$ is chosen so that its imaginary part stays in $(0, \pi)$. We shall learn that z^λ has a boundary value distribution which is denoted by $(x+i0)^\lambda$. Moreover, these distributions are tempered and their Fourier transforms are supported by the half-line $\{\xi \geq 0\}$. To find the Fourier transforms one introduces distributions on the ξ-line as follows: If $\mathfrak{Re}\,\lambda > -1$ the function $\xi^{-\lambda}$ is locally integrable on $\{\xi \geq 0\}$ and we get a tempered distribution ξ_+^λ defined by

$$\text{(i)} \qquad \xi_+^\lambda(g) = \int_0^\infty \xi^\lambda \cdot g(\xi)\, d\xi$$

where g are Schwartz functions on the real ξ-line. Here $\lambda \mapsto \xi_+^\lambda$ is a distribution-valued anaytic function in the half-plane $\mathfrak{Re}\,\lambda > -1$ whose complex derivatives are given by

$$\frac{d}{d\lambda}\xi_+^\lambda(g) = \int_0^\infty \xi^\lambda \cdot \log\,\xi \cdot g(\xi)\,d\xi$$

In § xx we shall prove that (i) extends to a meromorphic function defined in the whole complex plane. More precisely one has the equation

$$(*) \qquad \xi_+^\lambda = e^{\lambda+1)\pi i} \cdot \frac{\Gamma(\lambda+1)}{2\pi} \cdot \widehat{(x+i0)}^{-\lambda-1}$$

where the last factor is the Fourier transform of $(x+i0)^{-\lambda-1}$. Since $\lambda \mapsto (x+i0)^\lambda$ is a distribution-valued entire function of λ the same holds when we pass to the Fourier transform. The equation (*) therefore entails that the poles of ξ_+^λ ony occur at poles of the Γ-function which are simple and occur at non-positive integers.

If M is a positive integer then ξ_+^λ has a Laurent series expansion at $\lambda = -M$:

$$\xi_+^{-M+\tau} = \frac{\gamma_M}{\tau} + \mu_M + \sum_{k=1}^\infty \mu_{k,M} \cdot \tau^k$$

If a test-fiunction $g(\xi)$ has compact support on an interval $[a,b]$ where $0 < a < b$, then the integrals in the right hand side of (i) are entire functions of λ. It follows that the polar distributtion γ_M in the Laurent series expansion is supported by the singleton set $\{\xi = 0\}$, i.e. it is a so called Dirac distribution. To find these polar γ-distributions one regards distribution derivatives. When $\mathfrak{Re}(\lambda)$ is a large positive number amd g a test-function, then the construction of distribution derivatives give

$$\text{(i)} \qquad \partial(\xi_+^\lambda(g) = -\int_0^\infty \xi^\lambda \cdot g'(\xi)\,d\xi = -\xi^\lambda \cdot g(\xi)|_0^\infty + \lambda \cdot \int_0^\infty \xi^{\lambda-1} \cdot g(\xi)\,d\xi$$

By analyticity this gives the equation

$$\partial(\xi_+^\lambda) = \lambda \cdot \xi_+^{\lambda-1}$$

which now holds in the space of tempered distributions. With $\lambda = -M + \tau$ the reader may check that repetated use of (x) gives the equation

$$\partial^M(\xi_+^\tau) = \tau(\tau-1)\cdots(\tau-M+1) \cdot \xi_+^{-M+\tau}$$

Passing to the limit as $\tau \to 0$ this entails that

$$(-1)\cdots(-M+1) \cdot \gamma_M = \partial^M(\xi_+^0)$$

Here ξ_+^0 is the Heaviside distribution H_+ given by the density 1 on $\xi \geq 0$. The equality

$$-\int_0^\infty g'(\xi)\,d\xi = g(0)$$

for test-functions means that the first order distribution derivative $\partial(H_+) = \delta_0$. So if $M \geq 1$ we obtain the equation

$$\gamma_M = \frac{(-1)^{M-1}}{(M_1)!} \cdot \partial^{M-1}(\delta_0)$$

To find μ_M in (*) we use the expansion

$$\xi_+^\tau = H_+ + \tau \cdot (\log\xi)_+ + \text{higher order terms}$$

This gives the equation

$$\mu_M = \frac{(-1)^{M-1}}{(M-1)!} \cdot \partial^M(\log\xi)_+)$$

In § xx we shall compute the right hand side and the result is that the action by μ_M is found via Taylor expansions of test-functions at $\xi = 0$. More precisely one has:

$$(*)\quad \mu_M(g) = \int_0^1 [g(\xi) - \xi g'(0) - \ldots - \frac{\xi^{M-1}}{(M-1)!} \cdot g^{(M-1)}(0)] \cdot \frac{d\xi}{\xi^M} + \int_1^\infty \frac{g(\xi)}{\xi^M}\, d\xi$$

Thus, one removes a piece of the Taylor expansion of g at $\xi = 0$ to get a denominator which can be integrated agains ξ^{-M} close to $\xi = 0$.

Example. With $M = 1$ we see that μ_1 is the distribution defined by

$$\mu_1(g) = \int_0^1 g(\xi) \cdot \log(\xi)\, d\xi$$

At the same time (*) gives

$$(*)\qquad \xi_+^{-1+\tau} = e^{\tau\pi i} \cdot \frac{\Gamma(\tau)}{2\pi} \cdot \widehat{(x+i0)}^{-\tau}$$

Since the γ-function has a simpe pole with residue 1 at $\tau = 0$ a computation gives

$$\mu_1 = \frac{1}{2\pi} \cdot (\pi - \log\widehat{(x+i0)})$$

Remark. In § xx we give further details about the computations above. The discussion above has been inserted at this early stage to illustrate a major issue in this chapter which demonstrates how analytic function theory intervenes with distribution theory.

A. The origin of distributions.

Here follows an excerpt from Jean Dieudonne's article *300 years of analyticity* presented at the Symposium on the Occasion of the *Proof of Bieberbach's conjecture.* See [XX].

Since Cauchy and Weierstrass, the central fact in complex analysis has been the one-to-one correspondence

$$\{c_n\}_{n\geq 0} \mapsto \sum_{n=0}^{\infty} c_n z^n$$

between sequences of complex numbers which do not increase too fast and functions holomorphic in a neighborhood of 0. When you turn to Fourier series, you immediately meet the same kind of correspondence

$$\{c_n\}_{n\in Z} \mapsto \sum_{-\infty}^{\infty} c_n e^{in\theta}$$

between families of coefficients and sums of trigonometric series, which has been one of the most unsatisfactory and thickest jungles of classical analysis. A situation as satisfactory as in the analytic case has only been achieved by substituting distributions in place of functions. More precisely, there is a one-to-one correspondence above when, on the left hand side, only families of *polynomial growth* are considered, that is, families such that

$$|c_n| \leq C(1+|n|)^k \quad : \text{for some} \quad k>0 \quad \text{and some constant } C$$

and the right hand side is replaced by *any periodic distribution* T on $\mathbf{R}$. The beauty of this correspondence is that it is *stable* for derivative, primitive, and convolution; so Euler was perfectly justified in taking derivatives of Fourier series and considering them again as Fourier series !

Assuming that the sequence $\{c_n\}$ has polynomial growth the right hand side is a distribution T which can be split into a sum of two periodic distributions $T_1 + T_2$, corresponding to families $\{c_n\}$ having $c_n = 0$ for all $n < 0$ (resp. $n \geq 0$), and which have holomorphic extensions

$$f_1(z) = \sum_{n=0}^{\infty} \quad \text{for} \quad |z|<1 \quad : \quad f_2(z) = \sum_{-\infty}^{-1} \quad \text{for} \quad |z|<1$$

and $f_1(re^{i\theta})$ (resp. $f_2(r^{-1}e^{i\theta})$), defined for $0<r<1$, *tends* indeed to T_1 (resp. T_2) when $r \to 1$ for the *weak topology* of distributions.

Remark. Dieudonne's concise remarks about distributions on the unit circle is treated in many text-books and easy to grasp. Namely, let $C^\infty(T)$ be the linear space of infinitely differentiable functions on the unit circle T, or equivalently 2π-period functions on the real line. Partial integration shows that Fourier coefficients have rapid decay, i.e. with

$$\widehat{f}(n) = \int_0^{2\pi} e^{-in\theta} f(e^{i\theta})dx \quad : f \in C^\infty(T) \quad : \quad n \text{ any integer}$$

it follows that

$$n^k \cdot |\hat{f}(n)| \leq \max_{0\leq\theta\leq 2\pi} |f^{(k)}(e^{i\theta})| \quad : \quad k = 0,1,2,\ldots$$

The topology on $C^\infty(T)$ is defined by the metric

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{|f-g|_k}{1+|f-g|_k}$$

In this way $C^\infty(T)$ is a Frechet space and for every continuous linear form L there exists some integer k and a constant C so that

$$[L(f)| \leq C \cdot |f|_k \quad : \quad f \in C^\infty(T)$$

From this follows the assertion by Dieudonné which identifies distributions on T and sequences $\{c_n\}$ with polynomial growth, i.e. any such sequence yields a distribution L defined by

$$L(f) = \sum_{-\infty}^{+\infty} c_n \cdot \widehat{f}(n)$$

This enable us to define boundary values of analytic functions $g(z)$ in the open unit disc satisfying the growth condition

$$(1) \qquad |g(z)| \leq C \cdot (1 - |z|^{-m}$$

for some $m \geq 2$. Namely, (1) implies that the coefficients $\{c_n\}$ in the series expansion $g(z) = \sum c_n \cdot z^n$ have polynomial growth and define a distribution $\mathfrak{b}_g$ by the formula

$$(*) \qquad \mathfrak{b}_g(f) = \sum_{n=0}^{\infty} c_n \cdot \widehat{f}(n)$$

Notice that the sum above extends over non-negative integers only. Let $\mathcal{O}_{\text{temp}}(D)$ be the space of analytic functions in D satisfying the growth (1) where m can be taken as an arbitrary integer. Then (*) yields an injective map from $\mathcal{O}_{\text{temp}}(D)$ into a subspace of distributions on T. Using conformal mappings between the unit disc and the upper, respectively the lower half-plane we are able to describe tempered distributions on the real line via $\{c_n\}$-sequences as above. So the periodic case essentially covers the study on the real line. We explain this in § XX.

A.1 Poisson's formula. For every function $f(x)$ in the Schwartz class $\mathcal{S}$ on the real x-line there exists a limit of the integrals

$$(1) \qquad \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\sin(x + i\epsilon)}{\cos(x + i\epsilon)} \cdot f(x) \cdot dx$$

taken as $\epsilon > 0$ decrease to zero. With $z = x + iy$ we consider the analytic function in the upper half-plane

$$\frac{\sin(z)}{\cos(z)} = \frac{1}{i} \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} =$$

$$(2) \qquad i \cdot \frac{1 - e^{2iz}}{1 + e^{2iz}} = i + 2i \cdot \sum_{k=1}^{\infty} (-1)^k \cdot e^{2ikz}$$

The last sum converges as long as $z = x + i\epsilon$ with $\epsilon > 0$. Let $\phi(\xi)$ be the Schwartz function on the real ξ-line such that

$$f(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot \phi(\xi) \cdot d\xi$$

So here $\phi = \widehat{f}$ by Fourier's inversion formula. Now (2) implies that the limit in (1) is equal to

$$(3) \qquad i \cdot \phi(0) + 2i \cdot \sum_{k=1}^{\infty} \phi(-2k)$$

This means that the distribution on the ξ-line expressed by the discrete measure

$$\mu = i \cdot \delta_0 + \sum_{k=1}^{\infty} \delta_{-2k}$$

is equal to the Fourier transform of the distribution on the ξ-line obtained as the boundary value function of the analytic function in (2). A special case occurs of $f(x)$ is a test-function with compact support in $[-\pi/2, \pi/2]$. Then the equality between (1) and (3) gives Poisson's classical formula:

$$(*) \qquad \int_{-\pi/2}^{\pi/2} \frac{\sin(x)}{\cos(x)} \cdot f(x) \cdot dx = i \cdot \widehat{f}(0) + 2i \cdot \sum_{k=1}^{\infty} \widehat{f}(-2k)$$

B. A bird's view upon distributions.

Laurent Schwartz defined the class $\mathcal{S}$ of C^∞-functions on the real line which together with all their derivatives have a rapid decay, i.e. when $f \in \mathcal{S}$ there exists for any pair of positive integers N, M a constant $C_{N,M}$ such that

$$|f^{(N)}(x)| \leq C_{N,M} \cdot (1+|x|^{-M} \quad : \quad x \in \mathbf{R}$$

In § 1 we explain how $\mathcal{S}$ is equipped with a sequence of semi-norms which gives it a structure as a Frechet space whose dual is denoted by $\mathcal{S}^*$ and consists of *tempered distributions.* which Riesz representation formula entails that for each $\gamma \in \mathcal{S}^*$ there exists some integer $N \geq 0$ and a Riesz measure μ_N such that

$$(1) \qquad \int_{-\infty}^{\infty} (1+|x|)^{-m} \cdot \big|d\mu_N(x)\big| < \infty$$

hold for some non-negative integer m and

$$(*) \qquad \gamma(f) = \int_{-\infty}^{\infty} f^{(N)}(x) \cdot d\mu_N(x)$$

Let us remark that the given tempered distribution can be represented by different pairs (N, μ_N) as N changes. For example, the Dirac distribution at $x = 0$ is also represented by

$$f \mapsto \int_{-\infty}^{0} f'(x) \cdot dx$$

B.0 The Fourier transform. The usefulness of $\mathcal{S}$ is that the Fourier transform

$$f \mapsto \widehat{f}(\xi) = \int e^{-ix\xi} f(x)dx$$

yields a 1-1 map from $\mathcal{S}$ to the corresponding $\mathcal{S}$-class on the real ξ-line. Moreover one has Fourier's inversion formula:

$$f(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi}\, \widehat{f}(\xi)d\xi \quad : \quad f \in \mathcal{S}$$

In § XX we explain how the Fourier transform extends to tempered distributions and in this way one gets a bijective mapping between the $\mathcal{S}^*$-spaces on the real x-line, respectively the real ξ-line. More precisely, if μ is a tempered distribution on the x-liner then $\widehat{\mu}$ is the distribution on the ξ-line for which

$$\widehat{\mu}(\phi) = \mu(\phi^*) \quad : \phi(\xi) \in \mathcal{S}_\xi$$

where

$$\phi^*(x) = \int e^{-ix\xi} \cdot \phi(\xi)\, d\xi$$

An important example is when μ is the Gaussian density

$$g(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

Then we obtain

$$\widehat{\mu}(\phi) = \iint e^{-x^2/2} \cdot e^{-ix\xi} \cdot \phi(\xi)\, dxd\xi = \int e^{-\xi^2/2} \cdot \phi(\xi)\, d\xi$$

Thus, apart from a constant multiple the Fourier transform of μ is another Gaussian density.

B.1 Boundary values of analytic functions. Every tempered distribution is recaptured as a sum of boundary values of analytic functions defined in the upper, respectively the lower half-plane. Consider a Riesz measure μ on the real ξ-line such that

$$\int_{-\infty}^{\infty} (1+|\xi|)^{-m} \cdot d|\mu|(\xi) < \infty \tag{1}$$

holds for some integer $m \geq 0$. If $z = x + iy$ belongs to the upper half-plane we set

$$G_+(z) = \int_0^{\infty} e^{iz\cdot\xi} \cdot d\mu(\xi) = \int_0^{\infty} e^{ix\cdot\xi - y\cdot\xi} \cdot d\mu(\xi)$$

This integral is absolutely convergent since $\xi \to e^{-y\xi}$ tends to zero more rapidly than any power of ξ as $\xi \to \infty$ when $y > 0$ is fixed. Hence $G_+(z)$ is an analytic function in $\mathfrak{Im}(z) > 0$. In a similar fashion we get the analytic function $G_-(z)$in the lower half-plane defined by

$$G_-(z) = \int_{-\infty}^{0} e^{iz\cdot\xi} \cdot d\mu(\xi)$$

In § 2 we prove that these two G-functions have boundary values on the real x-line given by tempered distributions. More precisely, there exists $\gamma_+ \in \mathcal{S}^*$ defined by

$$\gamma_+(f) = \lim_{y\to 0} \int_0^{\infty} G_+(x+iy) \cdot f(x) \cdot dx \quad : \quad f(x) \in \mathcal{S}$$

and in a similar fashion the tempered distribution γ_- defined by

$$\gamma_-(f) = \lim_{y\to 0} \int_0^{\infty} G_-(x-iy) \cdot f(x) \cdot dx \quad : \quad f(x) \in \mathcal{S}$$

Excluding the case when μ has an atom at $x = 0$, i.e. its discrete part has no mass at $x = 0$, it follows via Fourier's inversion formula that the tempered distribution defined by μ is equal to the sum $\gamma_+ + \gamma_-$. Conversely, let $g(z)$ be a bounded analytic function in the upper half-plane U_+. Then there exists the tempered distribution $\mathbf{b}_g$ defined by

$$\mathbf{b}_g(f) = \lim_{y\to 0} \int_0^{\infty} g(x+iy) \cdot f(x) \cdot dx \quad : \quad f \in \mathcal{S}$$

Indeed, this follows since the bounded analytic function $g(z)$ has Fatou limits almost everywhere on the real x-line and if $g(x) = \lim_{y\to 0} g(x+iy)$ is the limit function then the distribution $\mathbf{b}_g$ is the L^1_{loc}-density $g(x) \cdot dx$. More generally we shall prove that an analytic function $G(z)$ in the upper half-plane has a boundary value given by a tempered distribution if there exists a pair of integers N, m and a constant C such that

$$|G(x+iy)| \leq C \cdot y^{-m} \cdot (1+|z|)^N \tag{B.1 *}$$

hold for all $z = x + iy$ in the upper half-plane. denote by $\mathcal{O}_{\text{temp}}(U_+)$ the class analytic functions satisfying (B.1.*for some piar N, m). Then we shall learn in § xx that one has an injective map:

$$\mathfrak{bf}_+ : \mathcal{O}_{\text{temp}}(U_+) \to \mathcal{S}^* \tag{1}$$

Moreover, the image under this map consists of temepred distributions μ for which $\widehat{\mu}$ is supported by the closed half-line $\xi \geq 0$. In a similar fashoin we introduce the space $\mathcal{O}_{\text{temp}}(U_-)$ of analytic functions in the lower half-pane satisfying (B.1.*) with y replaced by $|y|$ since we now have $y < 0$. Here one has an injective mapping

$$\mathfrak{bf}_- : \mathcal{O}_{\text{temp}}(U_-) \to \mathcal{S}^* \tag{2}$$

whose image consists of tempered distributions μ where $\widehat{\mu}$ is supported by $\{\xi \leq 0\}$.

Remark. The intersection of the two image spaces above consist of distributions μ where the support of $\widehat{\mu}$ is reduced to $\{\xi = 0\}$ which means that μ is a polynomial density function. A notable fact is that the $\mathfrak{bf}$-maps commute with derivations.

More precisely, if $g \in \mathcal{O}_{\text{temp}}(U_+)$ its complex derivative $\frac{\partial g}{\partial z}$ also belongs to this space and one has

$$\mathfrak{bf}_+(\frac{\partial g}{\partial z}) = \frac{d}{dx}(\mathfrak{bf}_+(g))$$

Using this one arrives at the following:

B.1.2 Theorem. *Every tempered distribution on the real x-line can be expressed by a sum*

$$p(\frac{d}{dx})\mathbf{b}_{g_+} + q(\frac{d}{dx})\mathbf{b}_{g_-} + P(x)$$

where $P(x)$ is a polynomial and the g-functions are bounded analytic functions in U_+ and U_- while p and q are differential operators with constant coefficients.

B.2 Parseval's formula. In § XX we show that Fourier's inversion formula gives the equality:

(1) $$\int |f(x)|^2 \, dx = \frac{1}{2\pi} \int |\widehat{f}(\xi)|^2 \, d\xi \quad : f \in \mathcal{S}$$

Next, on the real x-line we have the Hilbert space $L^2(\mathbf{R})$. Since $\mathcal{S}$ is a dense subspace the construction of the Fourier transform for tempered distributions will show that if $f(x) \in L^2(\mathbf{R})$, then its Fourier transform is an L^2-function on the ξ-line and the equality (1) holds. Suppose now that $f(x) \in L^2$ and that the Fourier transform $\widehat{f}(\xi)$ is supported by $\xi \geq 0$. In this case we can get an inverse transform defined in the upper half-plane by

(2) $$F(z) = \frac{1}{2\pi} \cdot \int_0^\infty e^{iz\xi} \cdot \widehat{f}(\xi) \cdot d\xi \quad : \quad y > 0$$

Fourier' inversion formula and the existence of Fatou limits gives the limit formula:

$$\lim_{y \to 0} F(x + iy) = f(x) \quad \text{for almost every} \quad x$$

At this stage we encounter an example where complex analysis enable us to get a result which goes beyond real variable methods. Namely, the L^2-function $f(x)$ whose Fourier transform by assumption is supported by $\xi \geq 0$ cannot be too small in average. More precisely, Carleman's formula from [XX] gives:

(*) $$\int_{-\infty}^\infty \log^+ \frac{1}{|f(x)|} \cdot \frac{dx}{1 + x^2} < \infty$$

In § 9 we prove a converse result from Carleman's article [Car:xx].

B.2.1 Theorem. *Let $f(x) \in L^2(\mathbf{R})$ be such that (*) holds. Then there exists an L^2-function $g(\xi)$ supported by $\xi \geq 0$ such that*

$$|f(x)| = |\int_0^\infty e^{ix\xi} \cdot g(\xi) \, d\xi \, |$$

B.3 Spectral gaps and Cauchy-Fourier formulas. Let γ be a tempered distribution whose support has gaps, i.e. the complement of $\text{Supp}(\gamma)$ is a union of disjoint open intervals $\{(a_\nu, b_\nu)\}$. From B.1 we find a pair G_+ and G_- such that

$$\gamma = \mathfrak{bf}_+(G_+) - \mathfrak{bf}_-(G_-)$$

For each open interval (a_ν, b_ν) we shall learn that G_+ and G_- extend each other across this interval on the real x-axis. The conclusion is that there exists an analytic function $G^*(z)$ defined in the connected set $\mathbf{C} \setminus \cup [a_\nu, b_\nu]$ such that $G^* = G_+$ in U_+ and $G^* = G_-$ in U_-. Next, on the ξ-line the space of test-functions is dense in the Frechet space $\mathcal{S}_\xi$. Using the Fourier transform it follows that $\mathcal{S}_x$ contains a dense subspace of those funtion $f(x)$ for which $\widehat{f}$ has compact support. Let us denote this space by $\mathcal{S}_x(c)$. Notice that every $f \in \mathcal{S}_x(c)$ is the restriction to the real x-line of the entire function

$$f(z) = \frac{1}{2\pi} \cdot \int e^{iz\xi}\, \widehat{f}(\xi)\, d\xi$$

With γ as above we suppose that R is a positive number such that both R and $-R$ are outside the support of γ and define

$$\gamma_R(f) = \int_{|z|=R} G^*(z) \cdot f(z)\, dz \quad : \, f \in \mathcal{S}_x(c)$$

By Cauchy's Theorem the complex line integral above is equal to

$$\lim_{\epsilon \to 0} \Big[\int_{-R}^{R} G_+(x+i\epsilon) \cdot f(x+i\epsilon)\, dx - \int_{-R}^{R} G_-(x-i\epsilon) \cdot f(x-i\epsilon)\, dx \Big]$$

If we there exists a sequence $\{R_\nu\}$ as above where $R_\nu \to +\infty$ we get the limit formula:

(B.3. *) $$\gamma(f) = \lim_{\nu \to \infty} \gamma_{R_\nu}(f) \quad : \, f \in \mathcal{S}_x(c)$$

This limit formula will be applied in § 16 to establish uniqueness results for tempered distributions whose support have gaps.

B.4 Derivatives of integrable functions.

Denote by $C_0^\infty(0,1)$ the space of test-functions with compact support in the open interval $(0,1)$ and $\psi(x)$ is a real-valued and integrable function on the unit interval. Suppose there exists a constant K such that

(*) $$\Big| \int_0^1 \phi'(x) \cdot \psi(x) \cdot dx \Big| \leq K \cdot \int_0^1 |\phi(x)| \cdot dx$$

hold for every $\phi \in C_0^\infty(0,1)$. Now $C_0^\infty(0,1)$ is a dense subspace of $L^1[0,1]$ whose dual is $L^\infty[0,1]$. Hence (*) gives a unique $q(x) \in L^\infty[0,1]$ such that

(1) $$\int_0^1 \phi'(x) \cdot \psi(x) \cdot dx = \int_0^1 q(x) \cdot \phi(x) \cdot dx \quad \text{for all} \quad \phi \in C_*^\infty(0,1)$$

Let $Q(x) = \int_0^x q(s) \cdot ds$ be the primitive function. A partial integration identifies the right hand side in (1) with

(2) $$-\int_0^1 Q(x) \cdot \phi'(x) \cdot dx$$

First order derivatives of C_0^∞-functions generate a closed hyperplane in $L^1[0,1]$ which consists of all integrable functions on $[0,1]$ with mean-value zero. Hence (1) and (2) imply that the L^1-function $\psi - Q$ is reduced to a constant. So after ψ has been changed on a null-set, it is equal to an absolutely continuous function whose derivative by Lebegue's theorem exists almost everywhere and is given by the bounded and measurable function $q(x)$ in (1).

Up to a constant the inequality (*) describes the family of absolutely continuous ψ-functions whose derivative in Lebesgue's sense is bounded. This illustrates a typcial flavour of distribution theory where one can use an operative definition as in (*) to exhibit regularity conditions on functions.

B.5 Logarithmic potentials.

For each absolutely integrable function g on the unit interval $0 \leq t \leq 1$ we set

$$T_g(x) = \int_0^1 \log|x-t| \cdot g(t)\,dt$$

Here $g \mapsto T_g$ is an injective linear operator from $L^1[0,1]$ into itself. One easily shows that it has a dense range but the T-operator is not surjective so one may ask for a precise description of the range. To achive this analytic function theory will be used. In § 14 we derive an inversion formula which shows that a function f belongs to the range of T if and only if its is absolutely continuous and its L^1-derivative f' is such that

$$\frac{1}{\sqrt{x(1-x)}} \cdot \mathcal{P}_{f'} \in L^1[0,1]$$

where $\mathcal{P}_{f'}$ is a principal value distribution on $(0,1)$ obtained via the limit formula

$$\mathcal{P}_{f'}(x) = \lim_{\epsilon \to 0} \int_0^1 \frac{x-t}{(x-t)^2+\epsilon^2}\, f'(t)\,dt$$

Remark. The example above ilustrates that in more refined questions one must perform a more delicate analysis, ie. the mere notion of distributions is helpful but does not provide immediate answers to more involved problems.

B.6 Principal values

A construction which were carried out prior to the general notion of distributions was introduced is the integral operator which sends an L^2-function f on the real line to

$$(*) \qquad T_f(x) = \frac{1}{\pi} \cdot \text{PV} \int_{-\infty}^{\infty} \frac{f(y)\,dy}{x-y}$$

where one takes a principal value. It turns out that T is an isometry on $L^2(\mathbf{R})$ and the squared operator T^2 is minus the identity. The proof of uses boundary values of analytic functions More precisely, the principal value integral in (*) can be defined by

$$(**) \qquad T_f(x) = \frac{1}{\pi} \cdot \lim_{\epsilon \to 0} \Big[\int_{-\infty}^{\infty} \frac{f(y)\,dy}{x+i\epsilon-y} + \int_{-\infty}^{\infty} \frac{f(y)\,dy}{x-i\epsilon-y} \Big]$$

In § xx we shall also learn that the limit of the left hand side yields an L^2-function whose Fourier transform becomes

$$i \cdot \text{sign}(\xi) \cdot \widehat{f}(\xi)$$

So passing to Fourier transforms the T-operator corresponds to multiplication of functions on the ξ-line by $i \cdot \text{sign}(\xi)$ and Parseval's identity for L^2-norms implies that T is an isometry on $L^2\mathbf{R}$).

B.7 Distributions in $\mathbf{R}^2$.

Consider a continuous function $f(x,y)$ defined on the square $\square = \{0 \leq x, y \leq 1\}$. Prior to distribution theory various conditions were suggested to express that f is an absolutely continuous function of the two real variables. See for example [xxx-page xxx] for conditions introduced by by Caratheodory and Tonelli. Using distributions one can ignore this and simply impose the condition that the distribution derivatives f'_x and f'_y are L^1-densities which means that there exists a constant C such that for every C^∞-function $\phi(x,y)$ in $\mathbf{R}^2$ it holds that

$$\Big| \iint_\square f \cdot \phi'_x \,dxdy \Big| \leq C \cdot \max_{x,y} |\phi(x,y)|$$

and a similar inequality with ϕ'_x replaced by ϕ'_y. This *operative definition* was put forward by Laurent Schwartz when he created distribution theory. Just as in dimension one we can define tempered distributions in $\mathbf{R}^2$ which by defintion are continuous linear forms on the Frechet space $\mathcal{S}(\mathbf{R}^2)$ of Schwartz funtion in two variables. Fourier's inversion formula in dimension two enable us to construct Fourier transforms of tempered distributions just as in the 1-dimensional case. For applications in analytic function theory one often identifies $\mathbf{C}$ with the real (x, y)-space. The complex-valued function z^{-1} is locally integrable in $\mathbf{C}$ and defines a tempered distribution Let us recall from § XX that z^{-1} yields a fundametnal solution to the $\bar{\partial}$-operator which may be expressed by the distribution equation

$$\text{(i)} \qquad \frac{1}{2}(\partial_x + i \cdot \partial_y)(z^{-1}) = \pi \cdot \delta_0$$

where δ_0 is the Dirac measure at the origin in $\mathbf{C}$. Now we emplly interchange rule during the passage to the Fourier transform which will be explained in § XX. It entails that ∂_x corresponds to multiplication with $i\xi$, and similarly ∂_y corresponds to multiplication with $i\eta$. So if we consider the Fourier transform $\mu = \widehat{z^{-1}}$ and use that $\widehat{\delta_0}$ is the identity function in the (ξ, η)-space then (i) gives the equation

$$\frac{1}{2}(i\xi - \eta) \cdot \mu = \pi$$

From this one verifies that μ is the distribution expressed by the L^1_{loc}-function $\frac{2\pi}{i\xi - \eta}$. In the next section we shall give an example where this fomula is applied.

B.8 The Hilbert transform in two variables.

Let E be a compact set in $\mathbf{C}$. The Hilbert transform of the characteristic function χ_E is defined by

$$\text{(1)} \qquad \mathcal{H}_E(z) = \iint_E \frac{dudv}{(z-w)^2}$$

where $w = u + iv$. Outside E this gives an analytic function which decays like $|z|^{-2}$. when $|z| \to +\infty$. There remains to see if the integral (1) makes sense when $z \in E$. For this purpose we introduce the functions

$$\text{(2)} \qquad H_n(z) = \iint_{\{|w-z|>\frac{1}{n}\}\cap E} \frac{dudv}{(z-w)^2}$$

These functions are defined for all z and one may ask if there exists an almost everywhere defined limit function

$$H_*(z) = \lim_{n\to\infty} H_n(z)$$

Outside E it is clear that the limit exists and here $H_* = \mathcal{H}_E$. So if (2) exists almost everywhere then H_* is the candidate for defining $\mathcal{H}_E$ at points in E. Let us analyze the eventual existence of a limit when $z = 0$. Suppose that E is contained in the disc $\{|z| \leq R\}$ and for each r we set $E[r] = E \cap \{|z[= r\}$. In polar coordinates we get

$$H_n(0) = \int_{\frac{1}{n}}^R \Big[\int_{E[r]} e^{-2i\theta}\, d\theta\Big] \frac{dr}{r}$$

The function $\rho_E(r) = \int_{E[r]} e^{-2i\theta}\, d\theta$ is bounded but need not entail that there exists the limit

$$\text{(3)} \qquad \lim_{n\to\infty} \int_{\frac{1}{n}}^R \frac{\rho_E(r)}{r}\, dr$$

Convergence in L^2. To avoid the subtle question about existence of point-wise limits we consider convergence in L^2 where one has:

B.8.1 Theorem. *The sequence $\{H_n\}$ converges in L^2 to a limit function H_* in the L^2-norm, i.e.*

$$\lim_{n\to\infty} \iint |H_n(z) - H_*(z)|^2\, dxdy = 0 \quad \text{and} \quad \iint |H_*(z)|^2\, dxdy = |E|_2$$

where the right hand side is the 2-dimensional Lebesgue measure of E.

About the proof. Theorem B.8.1 is a consequence of Parseval's formula after certain Fourier transforms have been computed. Moreover, one can replace characteristic functions of compact sets by L^2-functions in the (x,y)-space and define the Hilbert transform of two variables by

$$\mathcal{H}_f(z) = \frac{1}{\pi} \cdot \iint \frac{f(w)}{(w-z)^2}\, dudv \tag{4}$$

The conclusive result is that $f \mapsto \mathcal{H}_f$ is an isometry on the Hilbert space $L^2(\mathbf{C})$. To prove this we must learn how to compute the Fourier transform of the distribution z^{-2} which is defined via a principal value, i.e. when $f(x,y)$ is a test-function we set

$$z^{-2}(f) = \lim_{\epsilon\to 0} \iint_{|z|>\epsilon} \frac{f(x,y)}{z^2}\, dxdy$$

Using Stokes formula it follows that one has the distribution equation

$$\frac{1}{2}(\partial_x - \partial_y)(z^{-1}) = -z^{-2}$$

Passing to Fourier transforms where the interchange rules are used, it follows that

$$\widehat{z^{-2}}(\xi,\eta) = \frac{1}{2} \cdot (i\xi + \eta) \cdot \frac{2\pi}{i\xi - \eta} = \pi \cdot \frac{i\xi + \eta}{i\xi - \eta} \tag{5}$$

The last factor has aboslute value one and the isometric property the $\mathcal{H}$-operator in (4) follows from Parseval's identity.

B.9 Weak limits.

In distribution theory limits are mostly taken in a weak sense. For example, consider the family of complex-valued Riesz measures of finite total variation on the real line. Let $\{\mu_n\}$ be a sequence of such measures for which there exists a constant M such that $||\mu_n|| \leq M$ hold for every n. The sequence is said to converge weakly to a Riesz measure μ when a pointwise limit hold for the Fourier transforms, i.e. if

$$\lim_{n\to\infty} \widehat{\mu}_n(\xi) = \widehat{\mu}(\xi) \quad \text{holds pointwise} \tag{1}$$

The uniform bound on $\{||\mu_n||\}$ implies that (1) holds if and only if

$$\lim_{n\to\infty} \int f(x) \cdot d\mu_n(x) = \int f(x) \cdot d\mu(x)$$

for every continuous function $f(x)$ with a compact support. Next, let $\phi(x)$ be a bounded and uniformly continuous function on the real x-line. We do not assume that it has compact support but the integrals below exist for all n:

$$\int \phi(x) \cdot d\mu_n(x) \tag{2}$$

The question arises if the pointwise convergence in (1) entails that

$$\lim_{n\to\infty} \int \phi(x) \cdot d\mu_n(x) = \int \phi(x) \cdot d\mu(x) \tag{*}$$

Beurling proved that (*) holds for every weakly convergent sequence $\{\mu_n\}$ if and only if ϕ can be uniformly approximated on the whole x-line by functions of the form

$$x \mapsto \int e^{i\xi x} \cdot d\gamma(\xi)$$

where γ is a Riesz measure on the ξ-line with a finite total variation. This result is proved in § 17. The proof requires considerable work which relies upon solutions to variational problems so for more subtle problems it is not sufficient to just employ the calculus based upon distributions.

0. Examples related to distribution theory.

Below we give a number of examples which illustrate how distributions appear naturally in many situations. The reader may return to this material after the basic theory has been exposed in § XX-XX.

0.1 The Laplace operator. Let $\Delta=\partial_x^2+\partial_y^2+\partial_z^2$ be the Laplace operator in $\mathbf{R}^3$ where x,y,z are the coordinates. For each $\rho>0$ we define the kernel function:

$$A_\rho(p,q)=\frac{2}{\rho}-\frac{[p-q|}{\rho^2}-\frac{1}{|p-q|}$$

where p and q are points in $\mathbf{R}^3$. The singularity of the last term is not too bad for if p is fixed then $q\mapsto\frac{1}{|p-q|}$ is locally square integrable as a function of q. Let D be a bounded open domain in $\mathbf{R}^3$ and ϕ a square integrable function in D. Suppose that u is a continuous function on the closure $\bar{D}$ such that for each point $p\in D$ and every $\rho<\operatorname{dist}(p,\partial D)$ one has the equality

$$u(p)=\int_{B_p(\rho)}\frac{u(q)}{|p-q|}\cdot dq+\int_{B_p(\rho)}A_\rho(p,q)\cdot\phi(q)\cdot dq \tag{1}$$

Here $dq=dxdydz$ is the Lebesgue measure and $B_p(\rho)$ the open ball of radius ρ centered at p. The last integral in (1) is defined since $A_\rho(p,q)\cdot\phi(q)$ belongs to $L^1(B_p(\rho))$ by the Cauchy-Schwarz inequality. If v is a C^2-function in D then Green's formula gives the equality

$$v(p)=\int_{B_p(\rho)}\frac{v(q)}{|p-q|}\cdot dq+\int_{B_p(\rho)}A_\rho(p,q)\cdot\Delta(v)(q)\cdot dq \tag{2}$$

It follows that that (1) gives the equality below in the sense of distributions:

$$\Delta(u)=\phi \tag{3}$$

Before distributions were introduced, one used the integral formula (1) to express a solution u to the inhomogeneous equation (3) when the ϕ-function is given in $L^2(D)$. In other words, the inhomogeneous equation (3) is solved via the integral equation (1). This device appears in Ivar Fredholm in pioneering work who proved that extensive classes of linear PDE-equations can be solved via integral equations. So the inhomogeneous equation (3) is an example from Fredholm's theory.

0.2 Kernels in analytic function theory. Riemann posed the problem to find an analytic function $f(z)$ in a bounded domain Ω whose real and imaginary parts restrict to preassigned linearly dependent functions on the boundary. Hilbert investigated the more general problem when $\mathcal{C}(z)=\{c_{pq}(z)\}$ is an $n\times n$-matrix of continuous complex valued function defined on $\partial\Omega$ and asked for a pair of n-tuples $\{f_p(z)\}$ and $\{g_q(z)\}$ where the f-functions are meromorphic in Ω with a finite set of poles and the g-functions are meromorphic in the exterior domain with a finite number of poles, such that

$$f_p(z)=\sum_{q=1}^{q=n}c_{pq}(z)\cdot g_q(z)$$

hold on $\partial\Omega$ for every $1\leq p\leq n$. The system above also appears in work by Plemelj devoted to the problem of finding systems of linear differential equations whose solutions have a prescribed monodromy. A more general account was later given by Hasemann and one should also mention work by Uhler who used integral equations to extend the theory about zeta-functions of Fuchsian type. Cases where singular kernels appear lead to equations where regularisations in the spirit of general distribution theory are used and we remark that when smoothness of $\partial\Omega$ is relaxed many open problems remain to be analyzed.

0.3 The Pompieu formula. Let $f(z)$ be a continuous complex-valued function in a domain Ω which belongs to the class $\mathcal{D}(\mathbf{C}^1)$. In Chapter III we established the Pompieu formula under the assumption that f is a C^1-function. This regularity assumption can be relaxed. Namely, assume only that f is continuous and that the *distribution derivative* $\bar\partial(f)$ belongs to $L^1_{\text{loc}}(\Omega)$ which means that there exists an L^1_{loc}-function ϕ in Ω such that the following equality for area integrals hold:

$$\iint \bar\partial(g)\cdot f\cdot dxdy=-\iint g\cdot\phi\cdot dxdy \tag{1}$$

for each $g(x,y)\in C_0^\infty(\Omega)$. When $z\in\Omega$ is given we consider test-functions

$$g_\epsilon(\zeta)=\frac{\bar\zeta-\bar z}{|\zeta-z|^2+\epsilon}\cdot\rho$$

where $\rho\in C_0^\infty(\Omega)$ is identical 1 in the compact set of points in Ω with distance $\geq\epsilon$ to $\partial\Omega$. Passing to the limit as $\epsilon\to 0$ the same reasoning as in Chapter 3: § XX gives the equation:

$$f(z)=\frac{1}{\pi}\iint_\Omega\frac{f(\zeta)\cdot\bar\partial(\rho)}{z-\zeta}\cdot d\xi d\eta+\frac{1}{\pi}\iint_\Omega\frac{\rho(\zeta)\cdot\phi(\zeta)}{z-\zeta}\cdot d\xi d\eta \tag{2}$$

The Pompieu formula can be extended a bit further. Namely, assume only that f from the start is a *bounded* Borel function. It can be identified with an L^1_{loc}-function which therefore has a distribution derivative and if we again assume that $\bar\partial(f)=\phi$ for some L^1_{loc}-function ϕ then (2) still holds. To be precise, we get a pointwise equality at every *Lebesgue point* of f.

0.4 The elliptic property of $\bar\partial$. Let $f(x,y)\in L^1_{\text{Loc}}(\Omega)$ where Ω belongs to $\mathcal{D}(C^1)$. Suppose that

$$\iint f\cdot\bar\partial(g)\cdot dxdy=0 \tag{*}$$

hold for all text-functions $g(x,y)$ with compact support in Ω. Then one says that $\bar\partial(f)=0$ holds in the distribution sense. Here $\phi=0$ holds in (2) above and using ρ-functions in $C_0^\infty(\Omega)$ which are identically equal to 1 on arbitrary large compact subsets of Ω we see that f belongs to $\mathcal{O}(\Omega)$. The remarkable fact is that (*) therefore implies that the L^1_{loc}-function f *automatically is analytic* in Ω. If necessary one has only to change its values on a null-set which is harmless since two L^1_{loc}-functions are considered to be equal if they coincide outside a null set. A similar result is valid for the Laplacian, i.e. let f again be in $L^1_{\text{loc}}(\Omega)$ and suppose that

$$\iiint f\cdot\Delta(g)\cdot dxdy=0 \tag{**}$$

holds for every test-function g in Ω. Then f is a harmonic function in Ω which entails that it is a real-analytic function of the two real variables x and y.

Remark. In a broader context the two results above hold because the differential operators $\bar\partial$ and Δ are *elliptic*.

Exercise. Show that

$$-\iint \bar{\partial}(g) \cdot \frac{1}{z} \cdot dxdxy = \pi \cdot g(0)$$

hold for every test function $g(x,y)$ which by the construction of distribution derivatives means that

$$(*) \qquad \bar{\partial}(\frac{1}{z}) = \pi \cdot \delta_0$$

It follows that the L^1-density function $\frac{1}{\pi \cdot z}$ is a *fundamental solution* to the $\bar{\partial}$-operator. For the Laplace operator we find a fundamental solution via the formula

$$(**) \qquad \Delta(g)(z) = \frac{1}{\pi} \cdot \iint \operatorname{Log}|z-\zeta| \cdot g(z+\zeta) \cdot d\xi d\eta \quad : g(x,y) \in C_0^\infty(\mathbf{C})$$

0.5 Riesz measures and positive harmonic functions. Let $u(x,y)$ be a non-negative harmonic function defined in the open unit disc. The mean-value property gives

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \cdot d\theta \quad \text{for all} \quad 0 < r < 1$$

In general the u-function is unbounded and examples show that the radial limits $\lim_{r \to 1} u(re^{i\theta})$ can be quite irregular. It may also occur that these radial limits exist and are zero for all θ-angles outside a null set on the periodic interval $[0, 2\pi]$. However, a consistent and unique limit exists in the distribution sense. Namely, there exists a unique non-negative Riesz measure μ on the unit circle whose total mass is $u(0)$ and

$$(*) \qquad \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \cdot u(re^{i\theta}) \cdot d\theta = \int_0^{2\pi} g(\theta) \cdot d\mu(\theta)$$

hold for every 2π-periodic and continuous g-function. Thus, "boundary values" of u expressed by the equation (*). Moreover, Herglotz' formula implies that the map which sends a positive harmonic function u to the boundary measure μ is bijective. We can express as follows:

0.5.1 Theorem. *There exists a one-to-one correspondence between the family of positive harmonic functions in D and positive Riesz measures on T.*

0.6 Cauchy problems.

A basic boundary value problem for a second order PDE-equation is to find a function $u(x,y)$ which for $x > 0$ satisfies some PDE-equation, and at $x = 0$ the two boundary conditions:

$$(*) \qquad u(0,y) = \phi(y) \quad \text{and} \quad \frac{\partial u}{\partial x}(0,y) = \psi(y)$$

Here ϕ and ψ are given functions of the real y-variable and when $x > 0$ the u-function solves a PDE-equation of the *Riemann type*:

$$(**) \qquad \frac{\partial^2 u}{\partial x^2} = F(\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u, x, y)$$

Above F is a real valued polynomial of seven variables or more generally some analytic series. If ϕ and ψ both are analytic functions of the y-variable the Cauchy-Kovalevsky theorem gives existence and uniqueness of a solution $u(x,y)$ defined in some open interval $0 \leq x < \delta$ when the analytic Cauchy data is given on some y-interval. The study of this boundary value problem when ϕ and ψ no longer are analytic leads to a more involved situation. This was demonstrated by counter-examples in the article [xx] by Sophie Kovalevsky from 1874. See also the [HS] by Harold Shapiro for an account about the history of the Cauchy-Kovalevsky theorem.

Conditions on non-analytic pairs ϕ, ψ in order that the Cauchy problem has a solution were investigated by Gevrey, Hadamard and Holmgren. In his article [XX] from 1902 Hadamard considered the case when F is linear and elliptic. For example, let F be the Laplace operator. So here we seek a harmonic function $u(x, y)$ defined in some open rectangle $\{0 < x < \delta\} \cap \{a < y < b\}$ which satisfies (*). Using Schwarz reflection principle and the fundamental solution of Δ expressed by log $|z|$, Hadamard proved that the boundary value problem has a solution u if and only if the function defined for $a < y < b$ by

$$y \mapsto \phi(y) + \frac{1}{\pi} \int_a^b \log \frac{1}{|y-s|} \cdot \psi(s) \cdot ds$$

is real-analytic on the interval (a, b). Next, consider the heat equation. So here u satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial y} \quad : x > 0$$

In the article *Sur l'extension de la méthode d'integration de Riemann* from 1904, Holmgren showed that the boundary value problem for the heat operator has solutions if and only if ϕ and ψ are related to each other by an equation

$$\psi(y) = -\frac{1}{\sqrt{\pi}} \cdot \int_0^y \frac{\phi'(u)}{\sqrt{y-u}} \cdot du + g((y)$$

where the g-function must be C^∞ and higher order derivatives satisfy:

$$\left|g^{(n)}(y)\right| \leq M \cdot K^n \cdot (2n+1)! \quad : n = 1, 2, \ldots$$

for some pair of constants M and K. These estimates on the higher order derivatives are sufficiently relaxed in order that there exists g-functions with arbitrary small compact supports, i.e. the class is not quasi-analytic.

Remark. The discussion above shows that one should not always restrict the attention to the analytic case. It would lead us too far to discuss results concerned with well-posedness of boundary value problems in general. Volume II in Lars Hörmander's text-book series [Hörmander I-IV] contains a wealth of results together with an extensive list of references.

0.6.1 Uniqueness problems. Consider a C^∞-function $u(x, y)$ which satisfies the heat equation in a strip domain

$$B = \{-\infty < x < \infty\} \times \{0 < y < A\}$$

The question arises if the vanishing of u on some line $\{y = a\} : 0 < a < A$ implies that u is identically zero in B. It turns out that this is related to growth conditions on u. A sufficient and "almost necessary" condition for this uniquness to be valid was proved by Erik Holmgren in the article [1924:Arkiv]:

0.6.2 Theorem. *Suppose there exists a constant k such that*

$$|u(x, y]| \leq e^{kx^2 \cdot \log(e+|x|)} \quad : (x, y) \in B$$

Then, if $u(x, a) = 0$ holds identically in x for some $0 < a < A$, it follows that u is identically zero.

Remark. This result cannot be proved by a mere abstract reasoning but relies on analytic function theory. A complete investigation of the uniqeness problem for the heat equation was carried out by Täcklund in [Täcklund].

0.6.3 The analytic case. This occurs when u satisfies an inequality

(i) $$|u(x,y]| \leq e^{kx^2}$$

A result due to Hadamard shows that (i) entials that $u(x,y)$ is an analytic function with respect to y. Moreover, for each $0 < a < A$ one recovers u from its values on $\{y = a\}$ via the integral equation

(ii) $$u(x,y) = \frac{1}{2\sqrt{\pi(y-a)}} \int_{-\infty}^{\infty} u(\xi, a) \cdot U(x-\xi, y-a)\, d\xi$$

where $U(x,y)$ is the function which is zero when $y \leq 0$ and

$$U(x,y) = e^{-x^2/y} \,:\, y > 0$$

Exercise. Show that (i) implies that that the integral in the right hand side of (ii) exists and that the resulting function satisfies the heat equation. Moreover, it coincides with u on the line $\{y = a\}$.

0.6.4 Non-analytic integral formulas. Suppose that u satisfies the heat equation in B and that there exists a function $\epsilon(r)$ which tends to zero as $r \to +\infty$ such that

(i) $$u(x,y)| \leq e^{\epsilon(x) \cdot x^2 \cdot \log(e+|x|)}$$

This is a stronger estimate than in Theorem 0.6.2. Hence uniquenss holds and the question arises if u is recaputred by an integral formua. This time one cannot employ the U-kernel because (i) does not ensure that the integral in the right hand side from (0.6.3) converges. However, Beurling has found such an integral formula using another kernel function than U. See § xx for the detailed construction.

0.7 Cauchy transforms and the logarithmic potential

Let μ be a Riesz measure on the unit interval $\{0 \leq t \leq 1\}$. With $z = x + iy$ we get the Cauchy transform

(*) $$\mathcal{C}_\mu(z) = \int_0^1 \frac{d\mu(t)}{z-t}$$

We shall learn that there exist the two boundary value distributions

(1) $$\mathcal{C}_\mu(x+i0) = \lim_{\epsilon \to 0} \mathcal{C}_\mu(x+i\epsilon) \quad \text{and} \quad \mathcal{C}_\mu(x-i0) = \lim_{\epsilon \to 0} \mathcal{C}_\mu(x-i\epsilon)$$

For example, when $f(x)$ is a differentiable function on the real x-line with compact support on some interval $[-a,a]$ then

(2) $$\mathcal{C}_\mu(x+i0)(f) = \lim_{\epsilon \to 0} \int_0^1 \Big[\int_{-a}^{a} \frac{f(x) \cdot dx}{x - t + i\epsilon}\Big] \cdot d\mu(t)$$

The fact that the limit above exists will be demonstrated in § 2. Next, we have the equation

(3) $$\mathcal{C}_\mu(x+i\epsilon) - \mathcal{C}_\mu(x-i\epsilon) = -2i \cdot \int_0^1 \frac{\epsilon}{(x-t)^2+\epsilon^2} \cdot d\mu(t)$$

When $\epsilon > 0$ the right hand side is a function of x which we denote by $\rho_\epsilon(x)$. If $g(x)$ is a test-function on the real x-line we get:

(4) $$\int \rho_\epsilon(x) \cdot g(x) \cdot dx = 2i \cdot \int_0^1 \Big[\int \frac{\epsilon}{(x-t)^2+\epsilon^2} \cdot g(x)dx\Big] \cdot d\mu(t)$$

The limit of the inner integral is found for each t since g is a test-function. More precisely the reader can verify that

(5) $$\lim_{\epsilon \to 0} \int \frac{\epsilon}{(x-t)^2+\epsilon^2} \cdot g(x)dx = \pi \cdot g(t)$$

where the convergence even holds uniformly with respect to t. Hence

$$\lim_{\epsilon\to 0}\int \rho_\epsilon(x)\cdot g(x)\cdot dx = 2\pi i\cdot\int g(t)\cdot d\mu(t) \tag{6}$$

In terms of distributions this gives the equality

$$\mathcal{C}_\mu(x+i0)-\mathcal{C}_\mu(x-i0)=2\pi i\cdot\mu \tag{**}$$

Hence μ is expressed as a difference of two distributions which arise via boundary values of analytic functions. Next we consider the sum

$$\mathcal{C}_\mu(x+i\epsilon)+\mathcal{C}_\mu(x-i\epsilon)=\int_0^1\frac{2(x-t)}{(x-t)^2+\epsilon^2}\cdot d\mu(t) \tag{7}$$

To get a limit formula we introduce the function

$$F(z)=\int_0^1\log(z-t)\cdot d\mu(t) \tag{8}$$

Here F is defined outside the real interval $[0,1]$ as a multi-valued analytic function. But its complex derivative is single-valued and is given by

$$F'(z)=\mathcal{C}_\mu(z) \tag{9}$$

At the same time we can choose single-valued branches of $\log(z-t)$ in the the half-planes $\mathfrak{Im}(z)>0$ and $\mathfrak{Im}(z)<0$. If $0<t<1$ is fixed we get the limit formula

$$\lim_{\epsilon\to 0}\big(\log(x+i\epsilon-t)+\log(x-i\epsilon-t)\big)=2\cdot\log[x-t|+\pi\cdot i$$

This yields the limit formula:

$$\lim_{\epsilon\to 0}F(x+i\epsilon)+F(x-i\epsilon)=2\cdot\int_0^1\log|x-t|\cdot d\mu(t)\cdot dt+i\pi\cdot\int_0^1 d\mu(t) \tag{10}$$

In XX we shall learn that the passage to boundary value distributions commute with derivations. So (9) and (10) give the following equality of distributions on the real x-line:

$$\frac{1}{2}\big[\mathcal{C}_\mu(x+i0)+\mathcal{C}_\mu(x-i0)\big]=\partial_x(f) \tag{***}$$

where the right hand side is the distribution derivative of function $f(x)$ defined by:

$$f(x)=\int_0^1\log|x-t|\cdot d\mu(t) \tag{11}$$

0.8 The Fourier transform.

The fact that one can define the Fourier transform of a distribution is very useful. An example occurs in § 9 where we consider the integral equation

$$f*\phi(x)=\int_0^\infty f(x-y)\phi(y)dy=0$$

Here f is a function in $L^1(\mathbf{R})$ and we seek solutions ϕ in the space $L^\infty(\mathbf{R})$, i.e. bounded and Lebesgue measurable functions on the real line. When the zeros of the Fourier transform $\widehat{f}(\xi)$ is a discrete set on the real ξ-line we shall find all ϕ-solutions. More precisely, every such solution is the limit of functions given by finite sums of the exponential functions $\{e_\alpha(y)=e^{i\alpha\cdot x}\}$ where $\{\alpha\}$ are the zeros of $\widehat{f}$. This result is expected but the systematic use of distribution fascilates the proof. One can also reverse the consideration and start from some $\phi\in L^\infty(\mathbf{R})$ and seek the set of all $f\in L^1(\mathbf{R})$ such that $f*\phi=0$. This leads to the problem of *spectral synthesis* which is treated in § 10, and again the notion of distributions is helpful to analyze this problem.

0.8.1 Plancherel's theorem. The construction of Fourier transforms of tempered distributions gives in particular the existence of Fourier transforms for functions in $L^2(\mathbf{R})$. This was actually achieved by Plancherel before the general notion of distributions had appeared. More precisely, for every $A > 0$ one defines the operator $\mathcal{P}_A$ which sends a square integrable function $f(x)$ to

$$(*) \qquad \mathcal{P}_A(f)(\xi) = \int_{-A}^{A} e^{-ix\xi} \cdot f(x) \cdot fx$$

Plancherel proved that there exists an L^2-function $g(\xi)$ on the real ξ-line such that

$$(**) \qquad \lim_{A\to\infty} ||g - \mathcal{P}_A(f)||_2 = 0$$

In the context of distribution theory, g is the Fourier transform of the tempered distribution on the x-line expressed by the L^2-density $f(x)$ and we set $g = \widehat{f}$.

0.8.2 Parseval's formula. By $f \mapsto \widehat{f}$ one gets a linear isomorphism beween the L^2-spaces on the real x- respectively the real ξ-line. More precisely, introducing the inner product on these complex Hilbert spaces one has the equality

$$(***) \qquad \langle\, f, g\,\rangle = \frac{1}{2\pi} \cdot \langle\, \widehat{f}, \widehat{g}\,\rangle \quad \text{for all pairs} \quad f, g \in L^2(\mathbf{R})$$

Remark. In the literature one sometimes normalises the Fourier transform of L^2-functions to attain an isometry, i.e. one employs $\frac{1}{\sqrt{2\pi}} \cdot \widehat{f}$ rather than $\widehat{f}$. However, to fit everything with the general construction of Fourier transforms of tempered distributions we prefer to define $\widehat{f}$ without this normalizing factor which entails that the factor $\frac{1}{2\pi}$ appears in Parseval's formula.

0.8.3 Comment to Plancherel's theorem. The Fourier transform of the characteristic function for the interval $[-A, A]$ becomes

$$(1) \qquad \rho_A(\xi) = 2 \cdot \frac{\sin(A\xi)}{\xi}$$

Parseval's formula via a convolution gives

$$(2) \qquad \mathcal{P}_A(f)(\xi) = \frac{1}{\pi} \cdot \int \rho_A(\eta) \cdot \widehat{f}(\xi + \eta) \cdot d\eta$$

The limit in (**) with $g = \widehat{f}$ can be established using the oscillatory behaviour of the ρ_A-function as $A \to +\infty$ and the continuity under translations of L^2-functions, i.e. one uses that:

$$(3) \qquad \lim_{\delta\to 0} \int \left|\widehat{f}(\xi + \delta) - \widehat{f}(\xi)\right|^2 \cdot d\xi = 0$$

Remark. In harmonic analysis one investigates the rate of convergence in (**) as $A \to +\infty$ via the behaviour of (3) as $\delta \to 0$. Such results are not covered by the mere passage to weak limits of distributions. An example is the *Central Limit Theorem* in probability theory. Here one can easily establish the existence of a limit expressed in a distribution theoretic context. But further analysis is needed to attain a more precise information about rate of convergence. This occurs in Lindeberg's Central Limit Theorem where on allows "fat tails" during the passage to the limit of sums of independent random variables. So the reader should be aware of the fact that when one refers to limits in spaces of distributions they are often taken in a weak sense, while more precise limit formulas require additional work. The next example illustrates this.

0.8.4 A pointwise limit formula. Above the Fourier transform was considered on L^2-functions and inversion formulas expressed in terms of L^2-norms. When $f(x)$

from the start is a continuous function which vanishes outside an interval one gets pointwise limit formulas provided that the Dini condition holds below. Let us recall this classical result. For simplicity we assume that $f(x)$ is an even and continuous funtion which is zero outside $[-1,1]$ and impose:

Dini's condition. *It holds at $x=0$ when*

$$\int_0^1 \frac{|f(x)|}{x}\cdot dx<\infty \tag{*}$$

From now on (*) is assumed. Since f is even we have:

$$\widehat{f}(\xi)=2\cdot\int_0^1 \cos(x\xi)\cdot f(x)\cdot dx$$

With $A>0$ we set

$$\gamma(A)=\frac{1}{2\pi}\int_{-A}^{A}\widehat{f}(\xi)\cdot d\xi \tag{1}$$

Our aim is to show that Dini's condition implies that

$$\lim_{A\to\infty}\gamma(A)=f(0) \tag{**}$$

To prove (**) we first evaluate (1) which gives

$$\gamma(A)=\frac{2}{\pi}\int_0^1 \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx \tag{2}$$

Next, we have the limit formula

$$\lim_{A\to\infty}\frac{2}{\pi}\int_0^1\frac{\sin(Ax)}{x}\cdot dx=\frac{2}{\pi}\int_0^A\frac{\sin(t)}{t}\cdot dt=1 \tag{3}$$

So in order to get $\gamma(A)\to f(0)$ we can replace f by $f(x)-f(0)$, i.e. it suffices to show that $\gamma(A)\to 0$ when $f(0)=0$ is assumed. To obtain this we fix some $0<\delta<1$ and put

$$\gamma_\delta(A)=\frac{2}{\pi}\int_0^\delta \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx \tag{4}$$

Since $|\sin(Ax)|\leq 1$ the triangle inequality gives

$$\gamma_\delta(A)\leq\int_0^\delta\frac{|f(x)|}{x}\cdot dx<\infty \tag{5}$$

for all A and every $\delta>0$. Dini's condition implies that the right hand side tends to zero as $\delta\to 0$. Next, we set

$$\gamma^\delta(A)=\frac{2}{\pi}\int_\delta^1 \frac{\sin(Ax)}{x}\cdot f(x)\cdot dx \tag{6}$$

Here $\frac{f(x)}{x}$ is continuous on $[\delta,1]$ and has therefore a finite modulus of continuity, i.e. we get the function

$$\omega_\delta(r)=\max_{\delta\leq x_1,x_2\leq 1}\Big|\frac{f(x_1)}{x_1}-\frac{f(x_2)}{x_2}\Big|\quad\text{where}\quad |x_1-x_2|\leq r \tag{7}$$

With these notations one has the inequality:

$$\gamma^\delta(A)\leq\frac{8\pi}{\pi}\cdot\omega_\delta(\frac{2\pi}{A}) \tag{***}$$

The verification is left to the reader as an exercise. We remark only that the extra factor 4 replacing 2 by 8 comes form $4=\int_0^{2\pi}|\sin(t)|\cdot dt$. Hence we have

$$\gamma(A)\leq\gamma_\delta(A)+\frac{8\pi}{\pi}\cdot\omega_\delta(\frac{2\pi}{A}) \tag{8}$$

This holds for all pairs δ and A and now we conclude that Dini's condition indeed gives the limit formula in (**).

Remark. Above $x = 0$. More generally we can impose Dini's condition for f at an arbitrary point a, i.e. for every a we set

$$D_f(a) = \int \frac{|f(x) - f(a)|}{|x - a|} \cdot dx$$

The results above show that whenever $D(a) < \infty$ one has a pointwise limit

$$(1) \qquad f(a) = \lim_{A \to \infty} \frac{1}{2\pi} \cdot \int_{-A}^{A} e^{ia\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

An example when this occurs is when $f(x)$ is Hölder continuous of some order > 0.

0.8.5 Carleson's Theorem. With no other assumption than continuity on f the question about the pointwise limits in (*) was unclear for more than a century. In 1965 Carleson proved that pointwise limit in (*) holds almost everywhere for an arbitrary continuous function $f(x)$ with a compact support. This famous result from [Carleson] goes far beyond the scope of this book. In the sequel we define the Fourier transform of distributions where the resulting inversion formula is taken in the distribution theoretic sense which means that one can *ignore* precise pointwise limits and so on. In (1) from the Remark above we are content to assert that the right hand side as functions of a converge to f in the L^2-norm on the real a-line.

0.8.6 A theorem by Carleman and Hardy. A result about the existence of pointwise convergence goes as follows: We are given some L^1-function $u(x)$ which is even and zero outside $[-1, 1]$ and of class C^2 when $x \neq 0$. Moreover, assume that there exists a constant C such that

$$(*) \qquad |u''(x)| \leq \frac{C}{x^2} \quad : \quad x \neq 0$$

Since u is an L^1-function we can construct the Fourier series

$$F_u(x) = \sum_{n=0}^{\infty} A_n \cdot \cos nx \quad \text{where} \quad A_n = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\xi) \cdot u(\xi) \cdot d\xi$$

Since u is a C^2-function when $x \neq 0$ the series converges uniformly to u on every interval $[\delta, 1]$ when $0 < \delta < 1$. There remains to analyze the situation at $x = 0$. The following result is due to Carleman and Hardy and will be proved in § 13:

0.8.7 Theorem. *The series $\sum A_n$ converges if and only if there exists the limit*

$$\lim_{x \to 0} u(x) = S_*$$

and here $S_* = \sum A_n$.

0.8.8 The central limit theorem.

Consider a sequence of independent random variables $\chi_1, \ldots, \chi_N$, where each individual variable has mean-value zero and a finite variance σ_ν. The sum variable

$$(*) \qquad S_N = \frac{\chi_1 + \ldots + \chi_N}{\sqrt{\sigma_1^2 + \ldots + \sigma_N^2}}$$

has been normalised so that its variance is one. In 1812 Simon Laplace stated that $\{S_N\}$ converges to the normal distribution whose frequency function is

$$\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

under the condition that each individual variable yields a *relatively small contribution.* This is for example valid in the sense of Laplace if there exists a constant M such that

$$\sigma_\nu \leq M \tag{1}$$

hold for all ν. However, one must assume a bit more than (1) to get ensure convergence of $\{S_N\}$ to the normal distribution. The conclusive result was established in 1920 by Lindeberg who proved the convergence with under an extra hypothesis that tails of second order moments admit a certain limit as $n \to \infty$. Lindeberg's theorem is proved in §18.

A crucial step in the proof of Lindeberg's theorem is the inequality (**) below. To motivate its relevance we consider two probability measures μ and ν on the real line expressing distributions of two random variables. For a pair of real numbers $a < b$ and a small $\delta > 0$ there exists a non-negative C^2-function $g_\delta(x)$ which is identically one on $[a, b]$ and vanishes outside $[a - \delta, b + \delta]$. Elementary calculus shows that g_δ can be constructed so that the maximum norm of its second derivative is bounded by $\frac{2}{\delta^2}$. At the same time we have the Fourier transforms $\widehat{\mu}$ and $\widehat{\nu}$. With these notations, Parseval's formula gives equality

$$2\pi \cdot \int g_\delta(x) \cdot [d\mu(x) - d\nu(x)] = \int \widehat{g}_\delta(\xi) \cdot (\widehat{\mu}(\xi) - \widehat{\nu}(\xi)) \cdot d\xi \tag{2}$$

Next, if $|\xi| \geq 1$ we have the inequality

$$|\widehat{g}_\delta(\xi)| \leq \frac{1}{\xi^2} \cdot \int |g''_\delta(x)| \cdot dx \leq \frac{4}{\delta \cdot \xi^2}$$

where the last inequality follow since the support of g''_δ is contained in a union of two δ-intervals while is maximum norm is $\leq \frac{2}{\delta^2}$. At the same time the maximum norm of $\widehat{\mu} - \widehat{\nu}$ is bounded by 2. Moreover, the L^2-norm of $\widehat{g}_\delta$ is $\sqrt{2\pi}$ times the L^2-norm of g_δ and the latter is bounded by $\sqrt{b - a + 2\delta}$. So if $A \geq 1$ the triangle inequality together with the Cauchy-Schwarz inequality show that the absolute value of the right hand side in (2) is majorised by

$$\sqrt{(2\pi(b - a + 2\delta)} \cdot [\int_{-A}^{A} |\widehat{\mu}(\xi) - \widehat{\nu}(\xi)|^2 \cdot d\xi]^{\frac{1}{2}} + 2 \cdot \frac{4}{\delta} \cdot 2 \cdot \int_A^\infty \frac{d\xi}{\xi^2} \tag{3}$$

Notice that the last term is $\frac{16}{A \cdot \delta}$ which for a given $\delta > 0$ can be made arbitrary small when $A >> 1$. From this one can derive limit formulas when one has an L^2-convergence of Fourier transforms of probability measures over arbitrary large ξ-intervals. A variant of (3) is to get rid of the factor $\sqrt{(2\pi(b - a + 2\delta)}$. Namely, since the maximum norm of the g_δ-function is one one has the majorisation

$$\int_{-A}^{A} |\widehat{\mu}(\xi) - \widehat{\nu}(\xi)| \cdot d\xi + \frac{16}{\delta \cdot A} \tag{4}$$

Example. Let $\{\mu_N\}$ be the sequence of probability measures expressing distributions of the sum variables $\{S_N\}$ in (*) above. Then (4) yields the Central Limit Formula if if

$$\lim_{N \to \infty} \int_{-A}^{A} |\widehat{\mu}_N(\xi) - e^{-\xi^2/2}| \cdot d\xi = 0 \tag{i}$$

holds for every $A \geq 1$. In 1733 de Moivre in 1733 proved the central limit theorem when S_N is the sum of N many Bernoulli variables which with probability 1/2 take the values + or $-\frac{1}{\sqrt{N}}$. Here

$$\widehat{\mu}_N(\xi) = \left[\cos(\frac{\xi}{\sqrt{N}})\right]^N \tag{ii}$$

Regarding the Taylor series of the cosine function at $\xi = 0$ we get (i) from Neper's limit formula

$$\lim_{N\to\infty} (1 - \frac{\xi^2}{2N})^N = e^{-\xi^2/2} \tag{3}$$

Remark. De Moivre proved actually the central limit theorem when S_N is the sum of N equally distributed random variables with some finite probability distribution. The original proof was based upon Stirling's formula and Wallis' formula for products of sine-functions and has the extra merit that the rate of convergence is estimated in a quite sharp manner. The interested reader may consult the literature for de Moivre's proof. See for example the elegant account in Carleman's text-book [Car: page 278-282] where the proof gives a precise upper bound for the rate of convergence of $\{S_N\}$ to the normal distribution.

0.9 Dirac's δ-function and plane waves

A basic object in distribution theory is Dirac's δ-function which for $n \geq 1$ is the unit point mass at the origin of $\mathbf{R}^n$. We restrict the discussion to the case $n = 2$ and mention only that similar results hold more or less verbatim in dimension ≥ 3. By definition δ_0 evaluates test functions $\phi(x, y)$ in $\mathbf{R}^2$ at the origin:

$$\delta_0(\phi) = \phi(0)$$

A far more important fact is that δ_0 is embedded in an analytic family which leads to *plane wave decompositions.* With $z = x + iy$ we consider a parametrised family of functions:

$$G_\lambda(z) = \frac{2 \cdot |z|^\lambda}{2\pi \cdot \Gamma(\lambda/2 + 1)} \quad : \quad z \in \mathbf{C} \quad : \lambda \in \mathbf{C}$$

where Γ is the usual Gamma-function.

0.9.1 The analytic continuation of G_λ. If $\mathfrak{Re}\,\lambda > 0$ it is clear that $G_\lambda(z)$ is a continuous function in $\mathbf{C}$ which yields a distribution on the underlying real (x, y)-space. Moreover, this distribution valued function is analytic as λ varies in the right half plane where the complex derivative becomes:

$$\frac{d}{d\lambda}\big(G_\lambda(z)\big) = \text{Log}\,|z| \cdot G_\lambda(z) - \frac{|z|^\lambda \cdot \Gamma'(\lambda/2 + 1)}{2\pi \cdot \Gamma(\lambda/2 + 1)^2}$$

It turns out that this distribution-valued function extends to the whole complex λ-plane.

0.9.2 Theorem *The distribution valued function G_λ extends to an entire function and at $\lambda = -2$ one has the equality $G_{-2} = \delta_0$.*

We prove this in XXX. The fact that G_λ is an entire function becomes useful when we write out the integral formula:

$$G_\lambda(x + iy) = \frac{2}{2\pi \cdot \Gamma(\lambda/2 + 1)} \cdot \int_0^{2\pi} \big|\cos(\theta) \cdot x + \sin(\theta) \cdot y\big|^\lambda \cdot d\theta \tag{*}$$

Together with Theorem 0.9.2 this constitute the plane wave decomposition of δ_0 in dimension two.

0.9.3 Fundamental solutions. Using (*) and Theorem 0.9.2 we shall construct fundamental solutions to elliptic PDE-operators of even order with constant coefficients:

$$L(\partial) = \sum_{j+k\leq 2m} c_{j,k} \cdot \partial_x^j \cdot \partial_y^k$$

Here m is a positive integers and the c-coefficients are complex numbers and the elliptic property means that the leading polynomial

$$L^*(\theta) = \sum_{j+k=2m} c_{j,k} \cdot \cos^j(\theta) \cdot \sin^k(\theta) \neq 0 \quad \text{for all} \quad 0 \leq \theta \leq 2\pi$$

To find a distribution μ_L such that $L(\mu_L) = \delta_0$ we first construct a certain distribution-valued entire function

(1) $$\lambda \mapsto \mu_\lambda$$

where μ_L will be the constant term μ_{-2} at $\lambda = -2$. To get (1) one proceeds as follows: Consider for each $0 \leq \theta \leq 2\pi$ the ordinary differential operator of the single real variable s:

$$L_\theta(\frac{d}{ds}) = \sum_{j+k \leq 2m} c_{j,k} \cdot \cos^j(\theta) \cdot \sin^k(\theta) \cdot (\frac{d}{ds})^{j+k}$$

It has order $2m$ and the elliptic property of $L(\partial)$ means that

$$L_\theta(\frac{d}{ds}) = c_{2m}(\theta) \cdot (\frac{d}{ds})^{2m} + \ldots + c_0(\theta) \quad \text{where} \quad c_{2m}(\theta) \neq 0 \quad \text{for all} \quad \theta$$

Elementary ODE-theory gives for each fixed $0 \leq \theta \leq 2\pi$ and every $\lambda \in \mathbf{C}$ a unique C^∞-function $v_{\lambda,\theta}(s)$ on the real s-line such that $(\frac{d}{ds})^{2m}(v_{\lambda,\theta})(0) = 1$ and

$$L_\theta(\frac{d}{ds})(v_{\lambda,\theta}) = 0 \quad \text{where} \quad (\frac{d}{ds})^j(v_{\lambda,\theta})(0) = 0 \ : \ 0 \leq j \leq 2m-1$$

Moreover, classical formulas for solutions to inhomogeneous ODE-equations imply that the map:

$$\lambda \mapsto v_{\lambda,\theta}$$

is an entire function of λ with values in the space of C^∞-functions on the s-line, and at the same time $\theta \mapsto v_{\lambda,\theta}(s)$ is a real-analytic function of θ for every pair (λ, s). Define the functions

$$V_{\lambda,\theta}(z) = v_{\lambda,\theta}\big(\cos(\theta) \cdot x + \sin(\theta) \cdot y\big)$$

In § XX we show that

(1) $$L(\partial)\big(V_{\lambda,\theta}\big)(z) = \frac{1}{\Gamma(\lambda/2+1)} \cdot \big|\cos(\theta) \cdot x + \sin(\theta) \cdot y\big|^\lambda$$

Next, let us put

$$u_\lambda(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} V_{\lambda,\theta}(z) \cdot d\theta$$

Now $\lambda \to u_\lambda$ is an entire distribution-valued function. Using (1) and the plane wave decomposition (*) the reader can verify that when $\lambda = -2$ then the constant term u_{-2} gives a fundamental solution to $L(\partial)$.

1. Tempered distributions on the real line.

The Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions on the real line is equipped with a topology defined by the sequence of norms $\{\rho_k\}_0^\infty$ where

$$\rho_k(f) = \max_{x\in\mathbf{R}} \ (1+|x|)^k \cdot \sum_{\nu=0}^{\nu=k} |f^{(\nu)}(x)|$$

The distance function on $\mathcal{S}$ defined by

$$d(f,g) = \sum_{k=1}^\infty 2^{-k} \cdot \frac{\rho_k(f-g)}{1+\rho_k(f-g)}$$

One verifies that this metric above is complete, i.e. $\mathcal{S}$ is a Frechet space.

1.1 Exercise. Prove that there exists a constant C such that the following hold for each pair of integers $0 \leq \nu < k$ and every $f \in \mathcal{S}$.

$$\max_x \,[1+|x|)^\nu \cdot |f^{(\nu)}(x)| \leq C^{k-\nu} \cdot \max_x \,[1+|x|)^k \cdot |f^{(k)}(x)|$$

1.2 An isomorphism. We have the bicontinuous map from the unit circle with the point 1 removed onto the real x-axis defined by

$$e^{i\theta} \mapsto i \cdot \frac{e^{i\theta}+1}{e^{i\theta}-1} \quad : 0 < \theta < 2\pi$$

If $f(x) \in \mathcal{S}$ we define the function

$$f_*(\theta) = f(i \cdot \frac{e^{i\theta}+1}{e^{i\theta}-1})$$

The rapid decay of f as $|x| \to +\infty$ implies that f_* extends to a C^∞-function on the whole unit circle which is flat at 1, i.e. the derivatives $f^\nu(1) = 0$ for all ν, or equivalently

$$\lim_{\theta\to 0} \frac{f(e^{i\theta})}{(e^{i\theta}-1)^n} = 0$$

for every positive integer n. Denote by $C_*^\infty(T)$ the class of C^∞-functions on T which are flat at $\theta = 0$. The constructions above gives a bijection

$$\mathcal{S} \simeq C_*^\infty(T)$$

which in addition is an isomorphism of Frechet spaces. More precisely, here $C_*^\infty(T)$ is regarded as a closed subspace of the Frechet space $C^\infty(T)$.

1.3 The dual space $\mathcal{S}^*$. Exercise 1.1 shows that to every continuous linear functional γ on $\mathcal{S}$ there exists some pair of integers $N, M \geq 0$ and a Riesz measure μ such that

$$(*) \qquad \gamma(f) = \int f^{(N)}(x) \cdot d\mu_N(x) \quad \text{where} \quad \int [1+|x|)^{-M} \cdot |d\mu_N(x)| < \infty$$

Remark. The integer N and the associated Riesz measure μ are not uniquely determined by γ. For example, let δ_0 be the Dirac distribution at $x = 0$. It can also be defined by

$$\delta_0(f) = \int_0^\infty f'(x)\,dx$$

Another caseis the distribution γ defined by a principal value:

$$\gamma(f) = \lim_{\epsilon\to 0} \text{PV} \int_{-\infty}^\infty \frac{f(x)\cdot dx}{x}$$

Here the reader may verify that

$$\gamma(f) = \int_{-\infty}^{\infty} \log|x| \cdot f'(x)\,dx$$

Thus, with $N = 1$ we get the locally integrable function $\log|x|$ and notice that we can take $M = 2$ above, i.e. the integral

$$\int \big|\log|x|\,\big| \cdot (1+|x|)^{-2}dx < \infty$$

1.4 The Fourier transform on $\mathcal{S}$.

If $f \in \mathcal{S}$ its Fourier transform is defined by

$$\widehat{f}(\xi) = \int e^{-ix\xi} \cdot f(x) \cdot dx$$

Since $f(x)$ is rapidly decreasing we can differentiate with respect to ξ and obtain

$$\partial_\xi^k(\hat{f}) = \int (-ix)^k \cdot e^{-ix\xi} \cdot f(x) \cdot dx \quad : \quad k \geq 1$$

Next, partial integration with respect to x gives

$$i \cdot \xi \cdot \hat{f}(\xi) = \int e^{-ix\xi} \cdot f'(x) \cdot dx$$

Denote by $\mathcal{F}$ the Fourier operator from $\mathcal{S}$ on the x-line to the corresponding $\mathcal{S}$-space on the ξ-line. Then the formulas above can be expressed as follows:

1.4.1 Proposition. *The following two interchange formulas hold:*

(ii) $$-i\partial_\xi \circ \mathcal{F} = \mathcal{F} \circ x \quad : \quad i\xi \circ \mathcal{F} = \mathcal{F} \circ \partial_x$$

1.4.2 Fourier's inversion formula. *Let $f(x) \in \mathcal{S}$ and set*

$$F(x) = \frac{1}{2\pi} \cdot \int e^{ix\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

Then one has the equality

(*) $$f(x) = F(x)$$

Proof. First we establish the equality when $x = 0$. Notice that $f \mapsto F(0)$ is a linear functional on $\mathcal{S}$. Next, a function $f \in \mathcal{S}$ such that $f(0) = 0$ can be divided by x, i.e. $f = x \cdot \phi(x)$ with $\phi \in \mathcal{S}$. When this holds we have

(i) $$\widehat{f} = -\partial_\xi(\widehat{\phi})$$

The Fundamental Theorem of Calculus gives

$$\int_{-\infty}^{\infty} \partial_\xi(g) \cdot d\xi = 0$$

for all $g(\xi) \in \mathcal{S}$. Applied to $\widehat{\phi}$ and using (i) we conclude that

$$f(0) = 0 \implies F(0) = 0$$

But then the linear functional on the vector space $\mathcal{S}$ defined by $f \mapsto f(0)$ must be a constant times the functional $f \mapsto f(0)$. Hence there exists a constant c such that

$$f(0) = c \cdot \int \widehat{f}(\xi) \cdot d\xi$$

There remains to determine c. For this purpose we choose the special function

$$f(x) = e^{-x^2/2}$$

A verification which is left to the reader yields

$$\widehat{f}(\xi) = 2\pi \cdot e^{-\xi^2/2}$$

From this we deduce that $c = \frac{1}{2\pi}$.

The general case. With a fixed real number a and $f \in \mathcal{S}$ we set

$$f_a(x) = f(x+a)$$

It follows that

$$f(a) = f_a(0) = \frac{1}{2\pi} \int \widehat{f}_a(\xi) \cdot d\xi$$

Next, notice that a variable substitution gives:

$$\hat{f}_a(\xi) = \int f(x+a) \cdot e^{-ix\xi} \cdot dx = e^{ia\xi} \int f(x) \cdot e^{-ix\xi} \cdot dx = e^{ia\xi} \cdot \hat{f}(\xi)$$

From this we get the equality

$$f(a) = \frac{1}{2\pi} \int e^{ia\xi} \widehat{f}_a(\xi) \cdot d\xi$$

Since a is an arbitrary real number we have proved Fourier's inversion formula.

Exercise. If $n \geq 2$ we define the Schwarz class of rapidly decreasing C^∞-functions in $\mathbf{R}^n$. The Fourier transform is defined by

$$\widehat{f}(\xi) = \int e^{-i\langle \xi, x\rangle} \cdot f(x)dx$$

where the integration now is over $\mathbf{R}^n$. Fourier's inversion formula in dimension $n \geq 2$ amounts to show that

$$f(0) = \frac{1}{(2\pi)^n} \cdot \int \widehat{f}(\xi) \cdot d\xi$$

This formula can be proved via the fundamental theorem of calculus by an induction over n. Let us give the details when $n = 2$ where (x, y) are the coordinates in $\mathbf{R}^2$. Let $f(x, y)$ be given in $\mathcal{S}(\mathbf{R}^2)$. Define the partial Fourier transform

$$f^*(\xi, y) = \int e^{-ix\xi} f(x, y)\, dx$$

With ξ kept fixed we notice that the Fourier transform of the function $y \mapsto f^*(\xi, y)$ is equal to $\hat{f}(\xi, \eta)$. The 1-dimensional case applied to the y-variable gives for every ξ:

(i) $$f^*(\xi, 0) = \frac{1}{2\pi} \int \widehat{f}(\xi, \eta)\, d\eta$$

Next, the 1-variable case is also applied to the x-variable which gives

(ii) $$f(0, 0) = \frac{1}{2\pi} \int f^*(\xi, 0) d\xi$$

Now (i-ii) give the required formula

$$f(0, 0) = \frac{1}{(2\pi)^2} \iint \widehat{f}(\xi, \eta)\, d\xi d\eta$$

1.5 The Fourier transform of temperate distributions.

Let $\gamma \in \mathcal{S}^*$ be given. Since the Fourier transform on $\mathcal{S}$ is bijective and bi-continuous with respect to the Frechet metric there exists a unique tempered distribution $\widehat{\gamma}$ on the real ξ-line defined on functions $g(\xi) \in \mathcal{S}$ by:

$$(*) \qquad \widehat{\gamma}(g) = \gamma(g_*) \quad : \quad g_*(x) = \int e^{-ix\xi} g(\xi) d\xi \quad :$$

Remark. Let $f(x) \in \mathcal{S}$ and denote by γ_f the distribution defined by the density $f(x)dx$. Then (*) gives

$$\widehat{\gamma}_f(g) = \iint f(x) \cdot e^{-ix\xi} g(\xi)\, dx d\xi = \int \hat{f}(\xi) \cdot g(\xi)\, d\xi$$

Thus, under the inclusion $\mathcal{S} \cdot dx \subset \mathcal{S}^*$, the construction of the Fourier transform of functions in $\mathcal{S}$ extend to temperate distributions via (*).

1.5.1 The Fourier transform of H_+. On the real x-line we have the Heaviside distribution H_+ defined by the density 1 when $x \geq 0$ and zero if $x < 0$. To find its Fourier transform we shall perform certain limits. To begin with, for every large real number N we have the distribution on the x-line defined by

$$\mu_N(f) = \int_0^N f(x) \cdot dx$$

Its Fourier transform becomes

$$\widehat{\mu}_N(\xi) = \int_0^N e^{-ix\xi} \cdot dx = \frac{1 - e^{-iN\xi}}{i\xi}$$

It is clear that

$$\lim_{N \to \infty} \mu_N(f) = H_+(f)$$

hold for every $f \in \mathcal{S}$. It follows that $\{\widehat{\mu}_N\}$ converges weakly to $\widehat{H}_+$, i.e. for each $g(\xi) \in \mathcal{S}$ one has

$$(ii) \qquad \widehat{H}_+(g) = \lim_{N \to \infty} \int_{-\infty}^{\infty} \frac{1 - e^{-iN\xi}}{i\xi} \cdot g(\xi) \cdot d\xi$$

We can also find $\widehat{H}_+$ by other limit formulas. Namely, if $\epsilon > 0$ we define the distribution γ_ϵ on the x-line by

$$\gamma_\epsilon(f) = \int_0^\infty e^{-\epsilon x} f(x) dx$$

Here we find that

$$\widehat{\gamma}_\epsilon(\xi) = \int_0^\infty e^{-\epsilon x - i\xi x} dx = \frac{1}{\epsilon + i\xi}$$

This gives the limit formula

$$(ii) \qquad \widehat{H}_+(g) = \lim_{\epsilon \to 0} \int \frac{g(\xi) \cdot d\xi}{i\xi + \epsilon}$$

Hence $\widehat{H}_+$ can be found by different limit formulas. Both (i) and (ii) are useful.

1.5.2 A passage to the unit circle. in § X we identfied $\mathcal{S}$ with $C_*^\infty(T)$. The Hahn-bansch theorem implies that every $\mu \in \mathcal{S}^*$ corresponds to a distribution γ on the unit circle whose restriction to $C_*^\infty(T)$ represents μ via the topological isomorphism in (xx). Here γ is determined up to those distributions on T which vanish identically on $C_*^\infty(T)$. It is clear that evety such distribution γ on T is supported by the singleton set $\{1\}$ and is therefore a finite sum of derivatives of the

Dirac measure at this point which means that their periodic Fourier coefficients are finite $\mathbf{C}$-linear sums

$$\widehat{\gamma}(n) = \sum_{\nu=0}^{\nu=k} a_\nu \cdot i^\nu \cdot n^\nu$$

where $\{a_\nu\}$ are complex numbers. Thus, if $\text{Temp}(\mathbf{Z})$ denotes the linear space of all complex sequences $\{c_n\}$ with moderate growth then one has an isomorphism of vector spaces:

$$\mathcal{S}^* \simeq \frac{\text{Temp}(\mathbf{Z})}{\mathcal{P}}$$

where $\mathcal{P}$ is the subspace of polynomial sequences as described above. It leads to some rather interesting constructions. Consider for example the subspace $c_+(\mathbf{Z})$ of sequences $\{c_n\}$ where $c_n = 0$ for every $n < 0$. It has empty intersection with $\mathcal{P}$ and can therefore be identified with a subspace of $\mathcal{S}^*$. Every sequence $\{c_n\}$ in $c_+(\mathbf{Z})$ gives the analytic function in the unit disc $\{|w| < 1\}$:

$$g(w) = \sum c_n w^n$$

One has the conformal mapping where

$$w = \frac{z - i}{z + i}$$

from the upper half-plane $\mathfrak{Im}\, z > 0$ onto the unit w-disc. We find the analytic function $G_+(z)$ in the upper half-plane where

$$G_+(z) = g(\frac{z - i}{z + i})$$

Now one verifies that G_+ satisfies the growth condition from (XX) which yields the tempered distribution $\mathbf{b}G_+$ on the real x-line and corresponds to the distributionon T defined by the sequence $\{c_n\}$. So in this way it is possible to transform the study of tempered distributions on the real line to situations on the unit circle where the Nevannlina-Jensen class of analytic functions in the unit disc and the corrspoindfing class in the exterior disc together exhibit all termpered distributions on the real line.

1.6 Tempered distributions in $\mathbf{R}^2$

One has the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions $g(x,y)$ of the two real variables x and y. The dual $\mathcal{S}^*$ consists of tempered distributions and every such distribution γ is represented as a finite sum

$$\gamma(f) = \iint_{\mathbf{R}^2} g^{(\alpha)}(x,y) \cdot d\mu_\alpha(x,y)$$

where $\{\mu_\alpha\}$ is a finite family of Riesz measures and α are multi-indices which yield higher order derivatives of g. Finally, there exists an integer N such that

$$\iint_{\mathbf{R}^2} (1+x^2+y^2)^N \cdot |d\mu_\alpha(x,y)| < \infty$$

hold for every α. Fourier's inversion formula was established in § 1.4.1 and leads to the construction of Fourier transform of tempered distributions. Let us give some examples.

Identifying $\mathbf{R}^2$ with $\mathbf{C}$ we have the first order differential operators $\partial = \frac{1}{2}[\partial_x - i\partial_y)$ and $\bar\partial = \frac{1}{2}[\partial_x + i\partial_y)$. The locally integrable function $g = \frac{1}{z}$ yields a tempered distribution. The interchange rules under Fourier transforms give

(i) $$\partial_\xi(\hat g) = -i \cdot \widehat{xg}(\xi,\eta) \quad \text{and} \quad \partial_\eta(\hat g) = -i \cdot \widehat{yg}(\xi,\eta)$$

In the (ξ,η)-space we have the complex variable $\zeta = \xi + i\eta$ and $\bar\partial_\zeta = \frac{1}{2}(\partial_\xi + i\partial_\eta)$. We can express (i) by the equation:

(ii) $$i \cdot \bar\partial_\zeta(\hat g) = \text{The Fourier transform of} \quad (x+iy)g = 1$$

Next, the Fourier transform of the identity function 1 in the (x,y)-space becomes

$$\widehat{1} = \delta_0$$

where δ_0 is the Dirac measure at the origin in the (ξ,η)-space. Hence (ii) means that

$$\bar\partial_\zeta(\widehat{g}) = \frac{1}{i} \cdot \delta_0$$

in particular $\widehat{g}$ is satisfies the Cacuhy-Riemann equation outside the origin so we expect that it is given by a holomorphic density in the punctured complex ζ-plane. This is indeed true and the precise formula is:

1.6.1 Theorem. *The Fourier transform of* $\frac{1}{x+iy}$ *is equal to* $\frac{2\pi i}{\zeta}$.

1.6.2 Exercise. Prove the formula above.

2. Boundary values of analytic functions

Introduction. We shall construct boundary values of analytic functions $f(z)$ defined in an open rectangle $\{-a < x < a \quad : 0 < y < b\}$. One says that f has moderate growth when the real axis is approached if there exists some integer $N \geq 0$ and a constant C such that

$$(*) \qquad |f(x+iy)| \leq C \cdot y^{-N}$$

When (*) holds we prove that f has a boundary value given by a distribution $\mathbf{b}(f)$ defined on the interval (a,b) of the real x-axis. Moreover, the map $f \mapsto \mathbf{b}(f)$ commutes with derivations, i.e. if $\partial_z = d/dz$ while ∂_x is the derivation on the real x-axis then

$$\mathbf{b}(\partial_z(f)) = \partial_x(\mathbf{b}(f))$$

where the right hand side is a distribution derivative.

2.1 The construction. Let $f(z)$ be analytic in a rectangle

$$\square = \{(x+iy\colon \quad -A < x < A \quad : 0 < y < B\}$$

where A, B are positive constants and

$$|f(x+iy)| \leq C \cdot y^{-N} \quad : \quad x+iy \in \square$$

where C is a constant and N some non-negative integer. Under this condition f has a boundary value as $y \to 0$ expressed by a distribution acting on test-functions $g(x)$ with compact support in $(0,A)$. To achieve this we extend a test-function $g(x)$ in such a way that $\bar\partial$-derivatives are small when $y \to 0$.

2.2 Small $\bar\partial$-extensions Given a positive integer N and some $g(x) \in C_0^\infty(-A,A)$ we get a function $G_N(x+iy)$:

$$G_N(x+iy) = g(x) + \sum_{\nu=1}^{\nu=N} i^\nu \cdot \frac{g^{(\nu)}(x) \cdot y^\nu}{\nu\,!}$$

Since $\bar\partial = \frac{1}{2}[\partial_x + i\partial_y]$ one has the equality

$$2 \cdot \bar\partial(G_N)(x+iy) =$$

$$(*) \qquad \sum_{\nu=0}^{N} i^\nu \cdot \frac{g^{(\nu+1}(x) \cdot y^\nu}{\nu\,!} \sum_{\nu=1}^{N} i^{\nu+1} \cdot \frac{g^{(\nu}(x) \cdot y^{\nu-1}}{(\nu-1)\,!} = i^N \cdot \frac{g^{(N+1)}(x) \cdot y^N}{N\,!}$$

2.3 The distribution $\mathfrak{b}(f)$. Let $f(z)$ satisfy the growth condition above. Given $g \in C_0^\infty(-A,A)$ we construct G_N. With $0 < \epsilon < b < B$ we apply Stokes formula and obtain:

$$\int_{-A}^{A} G_N(x+i\epsilon) f(x+i\epsilon) dx =$$

$$\int_{-A}^{A} G_N(x+ib) f(x+ib) dx + 2i \cdot \int_0^A \int_\epsilon^b \bar\partial(G_N)(x+iy) f(x+iy) dx dy$$

The growth condition on f and (*) imply that the absolute value of the double integral is majorized by

$$(**) \qquad \frac{Cb}{N\,!} \cdot \int_0^A |g^{(N+1)}(x)| \cdot dx$$

Since this holds for any $\epsilon > 0$ we can pass to the limit $\epsilon \to 0$ while the absolutely integrable double integral is computed. Hence we have proved

2.4 Proposition *There exists the limit*

$$\lim_{\epsilon\to 0}\int_{-A}^{A} G_N(x+i\epsilon)f(x+i\epsilon)dx$$

Moreover, the limit is equal to

$$\int_{-A}^{A} G_N(x+ib)f(x+ib)dx+2i\cdot\int_{-A}^{A}\int_0^b \bar\partial(G_N)(x+iy)f(x+iy)dxdy \quad : \quad 0<b<B$$

where the absolute value of the double integral is majorized by (**) above.

2.5 Definition. *The limit integrals above yield a distribution on the open interval* $(-A,A)$. *It is denoted by* $\mathfrak{b}(f)$ *and called the boundary value distribution of* f.

2.6 Use of primitive functions. Starting with $f\in\mathcal{O}(\square)$ we construct primitive functions which behave better as we approach the real x-axis. For example, fix a point $p=ia$ with $a>0$ and set

$$F(z)=\int_{ia}^{z} f(\zeta)d\zeta$$

If $|f(x+iy)|\leq C\cdot y^{-N}$ for some $N\geq 2$ we get $|F(i+iy)\leq C_1\cdot y^{-N+1}$ for another constant C_1. In the case $N=1$ we get $[F(x+iy)|\leq C_1\cdot \text{Log}\,\frac{1}{|y|}$. So by choosing N sufficiently large and taking the N:th order primitive F_N of f it has even continuous boundary values and $\mathfrak{b}(F_N)$ is just the density function $F_N(x)$. Then one can take distribution derivatives on the real x-line and get

$$\mathfrak{b}(f)=\frac{d^N}{dx^N}(\mathfrak{b}(F_N(x))$$

So this is an alternative procedure to define $\mathfrak{b}(f)$ without small $\bar\partial$-extensions. Both methods have their advantage depending on the situation at hand.

2.7 Example. Let $f(z)=\log z$ where the single valued branch is chosen in $\mathfrak{Im}(z)>0$ so that the argument is between 0 and π. Then $\mathfrak{b}(f)$ is the distribution defined by the density $\log x$ when $x>0$ and if $x<0$ by

$$\log|x|+\pi\cdot i$$

The complex derivative of f is $\frac{1}{z}$. Here one finds that $\mathfrak{b}(\frac{1}{z})$ is the distribution defined by

$$(1)\qquad g\mapsto \lim_{\epsilon|\to 0}\int\frac{g(x)dx}{x+i\epsilon}=\int\frac{(g(x)-g(0))\cdot dx}{x}+\pi i\cdot g(0)$$

It is an instructive exercise to take the distribution derivative of $\mathfrak{b}(\log z)$ and verify that it is equal to the distribution (1).

The reflection principle.

Let

$$\square_-=\{(x+iy\colon\quad 0<x<A\quad:\ -B<y<0\}$$

be an opposed rectangle in the lower half-plane and let $h\in\mathcal{O}(\square_-)$ satisfy the moderate growth condition. We construct $\mathfrak{b}(h)$ in the same way as above.

2.8 Theorem. *Let* $f\in\mathcal{O}(\square)$ *and* $h\in\mathcal{O}(\square_-)$ *be a pair such that* $\mathfrak{b}(f)=\mathfrak{b}(h)$ *holds as distributions. Then they are analytic continuations of each other, i.e. there exists an analytic function* Φ *defined in* $\{-B<y<B\colon\ -A<x<A\}$ *such that* $\Phi=f$ *in* $\square$ *and* $\Phi=h$ *in* $\square_-$.

Proof. We choose a large N so that th N:th order primitive functions F_N and H_N both extend continuously to the real x-axis. The equality $\mathfrak{b}(f) = \mathfrak{b}(h)$ entails that

$$\frac{d^N}{dx^N}(\mathfrak{b}(F_N) - \mathfrak{b}(H_N) = 0$$

Now a distribution on the real x-line whose N:th order derivative is zero is a polynomial $p(x)$ of degree $\leq N - 1$. So the pair of continuous functions $F(x)$ and $H(x)$ satisfy

$$H(x) = F(x) + p(x)$$

Hence the analytic functions $F(z) + p(z)$ and $H(z)$ have a common continuous boundary value function so by the Schwarz reflection principle they are analytic continuations of each other. Let $G(z)$ be the resulting analytic function defined in the open domain where $-B < y < B$ now holds. Its N:th order complex derivative is also analytic in this domain and equal to f in $\square_+$ and to h in $\square_-$. This proves Theorem 2.8 with $\Phi = G^{(N)}$.

3. Fourier-Carleman transforms

Introduction. The Fourier transform can be obtained from a pair of analytic functions defined in the upper - resp. the lower half plane. The idea is that a Fourier transform

$$\widehat{g}(\xi) = \int e^{-ix\xi} \cdot g(x)\, dx$$

becomes a sum when we integrate over $(-\infty, 0)$ respectively $(0, +\infty)$. For each of these we get analytic functions $G_+(\zeta$ and $G_-(\zeta)$ in the upper, resp. the lower half plane of the complex ζ-plane where $\zeta = \xi + i\eta$. After one can take their boundary values. This construction has special interest when the support of the Fourier transform has gaps, i.e. when its complement consists of many open intervals and leads to certain uniqueness results which are presented at the end of this section based upon Beurling's lectures at Stanford University in 1961.

3.1 The functions G_+ and G_-. Consider a continuous complex-valued function $g(x)$ defined on the real x-line which is absolutely integrable:

$$\int_{-\infty}^{\infty} |g(x)|\, dx < \infty$$

We obtain analytic functions $G_+(\zeta)$ and $G_-(\zeta)$ defined in the upper, respectively the lower half-plane of the complex ζ-plane where $\zeta = \xi + i\eta$.

$$(*) \qquad G_+(\zeta) = \int_{-\infty}^{0} g(x) e^{-i\zeta x}\, dx \quad : \quad G_-(\zeta) = \int_{0}^{\infty} g(x) e^{-i\zeta x}\, dx$$

With $\zeta = \xi + i\eta$ we have $|e^{-i\zeta x}| = e^{\eta x}$. This number is < 1 when $x < 0$ and $\eta > 0$, and vice versa. We conclude that $G_+(\zeta)$ is analytic in $\mathfrak{Im}(\zeta) > 0$ while $G_-(\zeta)$ is analytic in $\mathfrak{Im}(\zeta) < 0$. Since $|g|$ is integrable we see that G_+ extends continuously to the closed upper half plane where

$$G_+(\xi) = \int_{-\infty}^{0} g(x) e^{-i\xi x} dx$$

Similarly G_- extends to $\mathfrak{Im}(\zeta) \leq 0$ and we have:

$$(**) \qquad G_+(\xi) + G_-(\xi) = \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx = \widehat{g}(\xi)$$

3.2 The case when $\widehat{g}$ has compact support. Then there are two intervals $(-\infty, a)$ and $(b, +\infty)$ and and family of bounded interval $\{(\alpha_\nu, \beta_\nu)\}$ whose union is the open complement of $\text{Supp}(\widehat{g})$. On each such interval $G_+(\xi)+G_-(\xi)$ is identically zero. Hence we get

3.3 Theorem *Put $\Omega = \mathbf{C} \setminus Supp(\widehat{g})$. Then there exists a function $\mathcal{G} \in \mathcal{O}(\Omega)$ such that $\mathcal{G} = G_+$ in the upper half plane and $\mathcal{G} = -G_-$ in the lower half plane.*

Consider some $R > 0$ which is chosen so large that the open disc D_R centered at the origin in the ζ-space contains the compact set $\text{Supp}(\widehat{g})$. For each real x the function $e^{ix\zeta}\mathcal{G}(\zeta)$ is again analytic in Ω. Put

$$\text{(i)} \qquad J_R(x) = \int_{|\zeta|=R} e^{ix\zeta} \cdot \mathcal{G}(\zeta)\, d\zeta$$

Exercise. Show that if $[-R,R]$ contains the support of g then

$$J_R(x)=\int_{-R}^{R}e^{ix\xi}G_+(\xi)\,d\xi+\int_{-R}^{R}e^{ix\xi}G_-(\xi)\,d\xi=\int_{-R}^{R}e^{ix\xi}\cdot\widehat{g}(\xi)\,d\xi$$

The last integral appears in Fourier's inversion formula which gives:

3.4 Theorem *Let $g(x)\in L^1(\mathbf{R})$ be such that $\widehat{g}$ has compact support in the interval $[-R_*,R_*]$. Then*

$$g(x)=\frac{1}{2\pi}\cdot\int_{|\zeta|=R}e^{ix\zeta}\cdot\mathcal{G}(\zeta)\,d\zeta\quad:\quad R>R_*$$

3.5 A more general case. The condition that $\widehat{g}$ has compact support is restrictive since it implies that $g(x)$ extends to an entire function in the complex z-plane. A relaxed condition is that the complement of $\mathrm{Supp}(\widehat{g})$ contains some open intervals both on positive and the negative real ξ-axis. With $\Omega=C\setminus\mathrm{Supp}(\widehat{g})$ we have $\mathcal{G}\in\mathcal{O}(\Omega)$. So if $R>0$ is a positive number such that R and $-R$ both are outside the support of $\widehat{g}$ we have the equality

$$(*)\qquad \frac{1}{2\pi}\cdot\int_{|\zeta|=R}e^{ix\zeta}\mathcal{G}(\zeta)\,d\zeta=\frac{1}{2\pi}\cdot\int_{-R}^{R}e^{ix\xi}\cdot\widehat{g}(\xi)\,d\xi$$

If $\widehat{g}$ is absolutely integrable Fourier's inversion formula gives

$$g(x)=\lim_{R\to\infty}\frac{1}{2\pi}\cdot\int_{-R}^{R}e^{ix\xi}\,\widehat{g}(\xi)\,d\xi$$

So when $\widehat{g}\in L^1(\mathbf{R})$ and there exists some sequence $\{R_\nu\}$ where R_ν and $-R_\nu$ both are outside $\mathrm{Supp}(\widehat{g})$, then

$$(**)\qquad g(x)=\lim_{\nu\to\infty}\frac{1}{2\pi}\cdot\int_{|\zeta|=R_\nu}e^{ix\zeta}\cdot\mathcal{G}(\zeta)\,d\zeta$$

3.6 Further extensions

Above we assumed that $g(x)$ was absolutely integrable which implies that $\widehat{g}$ is a bounded and continuous function. Suppose now that $g(x)$ is a continuous function such that

$$\int_{-\infty}^{\infty}\frac{|g(x)|}{1+|x|^N}\cdot x<\infty$$

holds for some positive integer N. We can still define the two analytic functions G_+ and G_-. Consider the behaviour of G_+ as ζ approaches the real ξ-line. We have by definition

$$G_+(\xi+i\eta)=\int_{-\infty}^{0}g(x)^{-i\xi x}e^{\eta x}dx$$

Taking absolute values we get for $\eta>0$:

$$|G_+(\xi+i\eta)|\leq\int_{-\infty}^{0}\frac{|g(x)|}{(1+|x|)^N}\cdot(1+|x|)^N\cdot e^{\eta x}dx$$

Notice that when $\alpha>0$, then the function

$$t\mapsto(1+t)^Ne^{-\alpha t}:t\geq 0$$

takes its maximum when $1+t=\frac{N}{\alpha}$ so the maximum value over $[0,+\infty)$ is $\leq\frac{N^N}{\alpha^N}$. Apply this with $\eta>0$ and $x<0$ above which gives a constant C such that

$$(*)\qquad |G_+(\xi+i\eta)|\leq\frac{C}{\eta^N}\cdot\int_{-\infty}^{\infty}\frac{|g(x)|\cdot dx}{1+|x|^N}$$

Hence G_+ has temperate growth as $\eta \to 0$ so its boundary value distribution $\mathfrak{b}(G_+)$ exists. Similarly we find the boundary value distribution $\mathfrak{b}(G_-)$. The Fourier transform of $g(x)$ regarded as a tempered distribution is equal to $\mathfrak{b}(G_+) + \mathfrak{b}(G_-)$. Again, if $\text{Supp}(\widehat{g})$ has gaps we can proceed as in 3.5 and construct the complex line integral

$$J_R(x) = \frac{1}{2\pi} \cdot \int_{|\zeta|=R} e^{ix\zeta} \mathcal{G}(\zeta)\, d\zeta$$

for those values of R such that $-R$ and R are outside the support of $\hat{g}$.

Exercise. Let g be as above and assume that there exists a sequence $\{R_\nu\}$ where $-R_\nu$ and R_ν are outside the support of g. Show that the following hold for every test-function $f(x)$

$$\int g(x) \cdot f(-x)\, dx = \lim_{\nu \to \infty} \frac{1}{2\pi} \cdot \int_{|\zeta|=R_\nu} \mathcal{G}(\zeta) \cdot \widehat{f}(\zeta)\, d\zeta$$

where $\widehat{f}(\zeta)$ is the entire Fourier-Laplace transform of f.

3.7 Use of Fourier's inversion formula.

Consider the following situation: Let $f(x)$ be a function in the Schwartz class and assume that $\widehat{f}(\xi)$ vanishes on some open interval $a < \xi < b$. Set $c = \frac{a+b}{2}$ and $g(x) = e^{ixc} \cdot f(x)$. Then we get

$$\widehat{g}(\xi) = \widehat{f}(\xi + c)$$

Here $\widehat{g}$ is zero on an interval centered at $\xi = 0$ and we may therefore assume from the start that $\widehat{f}$ is zero on some interval $(-A, A)$. Set

$$F_+(x+iy) = \frac{1}{2\pi} \cdot \int_A^\infty e^{(x+iy)i\xi}\, \widehat{f}(\xi)\, d\xi \quad : \quad F_-(x+iy) = -\frac{1}{2\pi} \cdot \int_{-\infty}^{-A} e^{(x+iy)i\xi}\, \widehat{f}(\xi)\, d\xi$$

When $y = 0$ we see that

$$(*) \qquad F_+(x) - F_-(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{ix\xi}\, \widehat{f}(\xi)\, d\xi$$

By Fourier's inversion formula the last integral is equal to $f(x)$ since $\widehat{f} = 0$ on $(-A, A)$. Hence $f(x)$ is represented as a difference of two analytic functions defined in the upper and the lower half-plane respectively where one has the estimates:

$$(1) \qquad \big|F_+(x+iy)\big| \leq \int_A^\infty e^{-y\xi} \cdot |\widehat{f}(\xi)|\, d\xi \leq e^{-Ay} \cdot \int_A^\infty |\widehat{f}(\xi)|\, d\xi$$

$$(2) \qquad |F_-(x+iy)| \leq e^{-A|y|} \cdot \int_{-\infty}^{-A} |\widehat{f}(\xi)|\, d\xi$$

Suppose now that $f(x)$ also is zero on some interval, say $a < x < b$. This means that the two analytic functions $F_+(z)$ and $F_-(z)$ agree on this interval and by the Schwarz reflection principle they are analytic continuations of each other. Hence, we get the following:

3.8 Proposition. *Assume that Supp(f) is a proper subset of* $\mathbf{R}$ *and consider the open complement*

$$U = \cup\, (a_\nu, b_\nu)$$

where $\{(a_\nu, b_\nu)\}$ is a family of disjoint open intervals. Then there exists an analytic function $\mathcal{F}(z)$ defined in the connected set $\mathbf{C} \setminus$ Supp(f) where

$$\mathcal{F}(z) = F_+(z) \quad : z \in U_+ \quad : \quad \mathcal{F}(z) = F_-(z) \quad : z \in U_*$$

We refer to $\mathcal{F}$ as the inverse Fourier-Carleman transform of $\widehat{f}(\xi)$.

3.9 A local estimate. Consider an open interval (a_ν, b_ν) in U. Set

$$r = \frac{b_\nu - a_\nu}{2} \quad : \quad c = \frac{a_\nu + b_\nu}{2}$$

Hence the open disc $D_r(c)$ stays in the open set $\Omega = \mathbf{C} \setminus \mathrm{Supp}(f)$. Next, when $0 < \phi < \pi$ we have

$$\mathcal{F}(c + re^{i\phi}) = \frac{1}{2\pi} \cdot \int_A^\infty e^{(c+r\cos\phi)i\xi - r\sin\phi\cdot\xi} \cdot \widehat{f}(\xi) \cdot d\xi$$

Since $\left|e^{(c+r\cos\phi)i\xi}\right| = 1$ the triangle inequality gives

$$|\mathcal{F}(c + re^{i\phi})| \leq \frac{1}{2\pi} \cdot e^{-rA\cdot\sin\phi} \cdot \int_A^\infty \left|\widehat{f}(\xi)\right| \cdot d\xi$$

When $-\pi \leq \phi \leq 0$ we get a similar estimate where we now use that $\mathcal{F} = F_-$. Introducing the L^1-norm of $\widehat{f}$ we conclude

3.10 Proposition. *One has the inequality*

$$\mathcal{F}(c + re^{i\phi}\Big| \leq \frac{||\widehat{f}||_1}{2\pi} \cdot e^{-Ar\cdot|\sin\phi|} \quad : \quad 0 \leq \phi \leq 2\pi$$

3.11 The subharmonic function $U = \mathbf{Log}\,|\mathcal{F}|$. Replacing f by $c \cdot f$ with some constant c we assume that $\frac{||\widehat{f}||_1}{2\pi} \leq 1$. Then Proposition 3.10 gives the inequality

$$U(c + re^{i\phi}) \leq -Ar \cdot |\sin\phi| \quad : \quad -\pi \leq \phi \leq \pi$$

Since U is subharmonic we can apply Harnack's inequality from XX and conclude

3.12 Proposition. *One has the inequality*

$$U(x) \leq -\frac{Ar}{2\pi} \cdot \quad : \quad c - \frac{r}{2} \leq x \leq c + \frac{r}{2}$$

3.13 A vanishing theorem. In addition to the inequality in Proposition 3.12 which is valid for every open interval of the x-axis outside the support of f, we also have the estimate from Proposition 3.10. This gives

$$(*) \qquad U(x + iy) \leq -A|y|$$

when $||\widehat{f}||_1 \leq 2\pi$. Now we can apply the general result from XX. Namely, if suppose that there is a sequence of disjoint intervals $\{(a_\nu, b_\nu)$ are outside the support of f. Then

$$(**) \qquad \sum (b_\nu - a_\nu) \cdot \int_{a_\nu}^{b_\nu} \frac{dx}{1 + x^2} < \infty$$

must hold unless f is identically zero. This gives a uniqueness theorem which can be phrased as follows. Let $0 < c_1 < c_2 \ldots$ where each c_ν is the mid-point of an interval (a_ν, b_ν) and these intervals are disjoint. We say that this interval family is thick if

$$\sum \frac{(b_\nu - a_\nu)^2}{c_\nu^2} = +\infty$$

3.14 Theorem. *Let $f(x)$ be a continuous function on the x-line be such its support is disjoint from a thick union of intervals and $\widehat{f}(\xi)$ is integrable. Then $\widehat{f}$ cannot be identically zero on any open subinterval of the ξ-line unless f is identically zero.*

Remark. The proofs above are taken from i [Benedicks] and we refer to [loc.cit] for further gap-theorems which are derived using the Fourier-Carleman transform.

4. The Paley-Wiener theorem

Let μ be a distribution on the real-x-line whose support is contained in an intervall $[-B, B)$. We consider the complex ζ-space where $\zeta = \xi + i\eta$. For each fixed ζ we have the C^∞-function $x \mapsto e^{-ix\zeta}$. If μ is a Riesz measure with compact support we get the entire function

$$\widehat{\mu}(\zeta) = \int e^{-i\zeta x}\, d\mu(x)$$

Moreover one has the estimate

$$|\widehat{\mu}(\xi + i\eta)| \leq e^{B\cdot|\eta|} \cdot ||\mu||$$

More generally, if the distribution μ is a sum of derivatives of Riesz measures up to some order m the entire function $\widehat{\mu}$ satsfies

$$|\widehat{\mu}(\xi + i\eta)| \leq C_m \cdot (1 + |\zeta|^m) \cdot e^{B\cdot|\eta|}$$

for some constant C_m. The Paley-Wiener theorem gives a converse to this result.

4.1 Theorem. *Let $H(\zeta)$ be an entire function for which there exist B and some integer m such that*

(4.1.1) $$|H(\zeta)| \leq (1 + |\zeta|)^m \cdot e^{B|\eta|} \quad :$$

Then $H(\zeta) = \widehat{\mu}(\zeta)$ for a distribution supported by $[-B, B]$.

The proof has two ingredients. The entire H-function is of exponential type and has also a polynomial growth on the real ξ-line. By the reeult from §§ there exists for a given integer m a constant C_m such that (4.1.1) above gives

(2) $$|H(\zeta)| \leq C_m \cdot (1 + |\zeta|)^m \cdot e^{B|\eta|} \quad :$$

Using (2) we shall prove that $H = \widehat{\mu}$ for a distribution μ supported by $[-B, B]$. To begin with $H(\xi)$ has a polynomial growth on the ξ-line whose inverse Fourier transform is a tempered distribution μ. If $g(x)$ is a test-function this means that

(i) $$\mu(g) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \widehat{g}(-\xi) \cdot H(\xi) \cdot d\xi$$

There remains to prove that μ is supported by $[-B, B]$. To show this we use that $\widehat{g}$ extends to an entire function and from (xx) we find a constant C such that

(ii) $$|\widehat{g}(\zeta)| \leq C(1 + |\zeta|)^{-m-2} \cdot e^{|\mathfrak{Im}\zeta|}$$

Next, consider rectangles in the ζ-space defined by

$$\square_{R:\rho} = \ -R < \xi < R\} \times \{0 < \eta < \rho\}$$

where (R, ρ) is a pair of positive numbers. The decay in (ii) and (2) entail that

$$\lim_{R\to\infty} \int_0^\rho \widehat{g}(-R - is) \cdot H(R + is) \cdot ds = 0$$

and a similar vanishing of the limit holds for line integrals along $\mathfrak{Re}(\zeta) = -R$. So for each $\rho > 0$ Cauchy's integral theorem gives

(iii) $$\mu(g) = \frac{1}{2\pi} \cdot \lim_{R\to\infty} \int_{-R}^{R} \widehat{g}(-\xi - i\rho) \cdot H(\xi + i\rho) \cdot d\xi$$

Next, suppose that the support of g is contained in $[B^*, +\infty)$ for some $B^* > B$. By (xx) we have a constant C such that

(iv) $$|\widehat{g}(\xi + i\rho)| \leq C \cdot (1 + |\xi| + \rho)^{-m-2} \cdot e^{-B^*\rho}$$

hold for all pairs ξ and $\rho > 0$. Hence (iii) and the triangle inequality give

$$\text{(v)} \qquad |\mu(g)| \leq C \cdot \frac{1}{2\pi} \cdot e^{(B-B^*)\rho} \cdot \int (1+|\xi|+\rho)^{-2} \cdot d\xi$$

The last integral factor is bounded above by the convergent integral $\int (1+|\xi|)^{-2} \cdot d\xi$. Since we can choose ρ arbitrary large in (5) it follows that $\mu(g) = 0$. Here $B^* > B$ was arbitrary so $\text{Supp}(\mu)$ is contained in $[B, +\infty)$. In the same way one proves that the support is contained $(-\infty, -B]$ and hence μ is supported by $[-B, B]$ which finishes the proof.

4.2 A division problem Let μ and γ be a pair of distributions with compact support. Assume that the quotient

$$\frac{\widehat{\gamma}}{\widehat{\mu}} \in \mathcal{O}(\mathbf{C})$$

Since $\widehat{\mu}$ and $\widehat{\gamma}$ both are entire functions of exponential type the division theorem by Lindelöf in § XX implies that the quotient is of exponential type, i.e. there exists some B and a constant C so that

$$\frac{|\widehat{\gamma}(\zeta)|}{|\widehat{\mu}(\zeta)|} \leq Ce^{B|\zeta|}$$

However, the polynomial growth which is required in (*) from 4.3 on the real ξ-line fails in general. To compensate for this failure, A. Beurling and P. Malliavin made an intensive study of functions in the Carleman class in [B-M] to analyze inverse formulas for entuire quotients as above. In these notes we refrain from a further discussion of these more advanced reults. In addition to [ibid] the interested reader can also consult the article [Malliavin:xx].

5. Runge's theorem and the inhomogeneous $\bar{\partial}$-equation

The main result in this section goes as follows:

5.1 Theorem. *For every open set* Ω *in* $\mathbf{C}$ *and each* $g \in C^\infty(\Omega)$ *there exists* $\phi \in C^\infty(\Omega)$ *such that*

$$\bar{\partial}(\phi) = g \tag{1}$$

The proof relies upon an approximation result of independent interest. To each compact subset K of $\mathbf{C}$ we consider the algebra $\mathcal{O}(K)$ of germs of analytic functions on K. We can restrict every $f \in \mathcal{O}(K)$ to K. These restrictions yield a subalgebra of $C^0(K)$ whose uniform closure is denoted by $\mathcal{H}(K)$. Let us also consider some open set Ω which contains K as a compact subset. Each $g \in \mathcal{O}(\Omega)$ can be restricted to K and these restrictions give a subalgebra of $C^0(K)$ whose uniform closure is denoted by $\mathcal{H}_\Omega(K)$. We have the obvious inclusion

$$\mathcal{H}_\Omega(K) \subset \mathcal{H}(K) \tag{0.1}$$

The question arises when equality holds. The affirmative answer is as follows:

5.2 Theorem. *The equality* $\mathcal{H}_\Omega(K) = \mathcal{H}(K)$ *holds if and only if* $\bar{U} \cap \partial\Omega \neq \emptyset$ *for every connected component* U *of* $\mathbf{C} \setminus K$.

Proof. Suppose first the inclusion (0.1) is strict. Riesz representation formula and the Hahn-Banach theorem give a Riesz measure μ supported by K such that

$$\mu \perp \mathcal{H}_\Omega(K) \quad : \quad \text{and} \quad \exists\, g \in \mathcal{H}(K) : \int_K g \cdot d\mu \neq 0 \tag{1}$$

By density we find $f \in \mathcal{O}(K)$ so that $f|K$ approximates g so close that $\int f \cdot d\mu \neq 0$. By the result in XXX this gives some $z_* \in \mathbf{C} \setminus K$ such that

$$\int_K \frac{d\mu(\zeta)}{\zeta - z_*} \neq 0 \tag{2}$$

Here z_* belongs to a connected component of $\mathbf{C} \setminus K$ which we denote by U_*. Now one has

$$U_* \subset \Omega \tag{i}$$

For if (i) fails we pick a point

$$a \in U_* \setminus \Omega \tag{ii}$$

The functions $\{\frac{1}{(z-a)^m}\}$ belong to $\mathcal{O}(\Omega)$ for all $m \geq 1$ and since $\mu \perp \mathcal{H}_\Omega(K)$ we have:

$$\int_K \frac{d\mu(\zeta)}{(\zeta - a)^m} = 0 \quad : m = 1, 2, \ldots \tag{iii}$$

In U_* there exists the analytic function

$$z \mapsto \int_K \frac{d\mu(\zeta)}{(\zeta - z)} = 0 \tag{iv}$$

Here (iii) means that the series expansion at $z = a$ is identically zero and hence (iv) is identically zero in U_* which contradicts (2) and hence (i) must hold. Next, since Since U_* is a connected component of $\mathbf{C} \setminus K$ we have the inclusion

$$\partial U_* \subset K \tag{v}$$

At the same time $U_* \subset \Omega$ holds by (i) and since K is compact in Ω it follows that the $\partial U_* \cap \partial\Omega \neq \emptyset$. This proves the if part of Theorem 5.2 and there remains to prove the implication

$$\mathcal{H}_\Omega(K) = \mathcal{H}(K) \implies \bar{U} \cap \partial\Omega \neq \emptyset \tag{*}$$

for every connected component U of $\mathbf{C}\setminus K$. To show this we argue by a contradiction and suppose that

(**) $$\bar{U} \cap \partial\Omega = \emptyset$$

Now $\partial U \subset K$ and since K is compact in Ω and U is connected we see that (**) gives the inclusion

(vi) $$\bar{U} \subset \Omega$$

Now we pick $z_0 \in U$ which gives the function

$$f(z) = \frac{1}{z - z_0} \in \mathcal{O}(K)$$

The equality $\mathcal{H}_\Omega(K) = \mathcal{H}(K)$ gives a sequence $\{g_n\}$ in $\mathcal{O}(\Omega$ which converges uniformly to f on K so that

(vii) $$\lim_{n\to\infty} (z - z_0)(g(z) = 1 \quad \text{holds uniformly on} \quad K$$

But this gives a contradiction. For by (vi) and the inclusion $\bar{U} \subset \Omega$ the maximum principle for analytic functions in U gives

$$|g_n(z_0)| \leq |g_n|_K \quad : n = 1, 2, \ldots$$

Since $g_n \to f$ holds uniformly on K it follows that the sequence $\{g_n(z_0)\}$ is bounded. But then it is clear that (vii) cannot hold which gives the requested contradiction and the proof of Theorem 5.2 is finished.

5.3 Proof of Theorem 5.1

Proof. Plane topology gives an increasing sequence of compact subsets $\{K_\nu\}$ such that for every ν one has:

(1) $$\partial U \cap \partial\Omega \neq \emptyset \quad : \forall \text{ connected components of } U \subset \mathbf{C} \setminus K_\nu$$

Next, Theorem §§ XX gives for each ν a C^∞-function ϕ_ν and some small open neighborhood U_ν of K such that

(2) $$\bar{\partial}(\phi_\nu) = g \quad \text{holds in } U_\nu$$

It follows that $\phi_{\nu+1} - \phi_\nu \in \mathcal{O}(K_\nu)$ and Theorem 5.2 gives $h_\nu \in \mathcal{O}(\Omega)$ such that the maximum norm

(3) $$|\phi_{\nu+1} - \phi_\nu - h_\nu|_{K_\nu} \leq 2^{-\nu}$$

For every $p \geq 2$ we therefore get a continuous function defined in U_p by:

(4) $$\psi_p = \phi_p + \sum_{\nu=p}^{\infty} (\phi_{\nu+1} - \phi_\nu - h_\nu) - (h_1 + \ldots + h_{p-1})$$

A trivial calculation shows that $\psi_q = \psi_p$ for all pairs $q > p$ so we get a function ψ_* defined in the whole open set Ω where $\psi_*|U_p = \psi_p$ for each p. Moreover, for each fixed p we have

(5) $$\phi_{\nu+1} - \phi_\nu - h_\nu|U_p \in \mathcal{O}(K_p) \quad : \nu \geq p$$

Next, since $\{h_\nu\}$ are analytic in Ω, we conclude that

(6) $$\psi_* - \phi_p \in \mathcal{O}(K_p) \quad : p = 1, 2, \ldots$$

Finally, since the increasing sequence $\{K_p\}$ exhaust Ω it follows from (6) that $\psi_* \in C^\infty(\Omega)$ and (2) gives $\bar{\partial}(\psi_*) = g$ in Ω. Hence ψ gives the requested solution in Theorem 5.1.

6. The generalised Fourier transform.

Introduction. The book *L'Integrale de Fourier et questions qui s'y rattachent* published in 1944 by Institute Mittag-Leffler is based upon Carleman's lectures at the institute in 1935. In the introduction he writes: *C'est avant tous les travaux fondamentaux de M. Wiener et Paley qui ont attiré mon attention.* The book *Fourier transforms on the complex domain* by Raymond Paley and Norbert Wiener was published the year before. We expose material from Chapter II in [Car] which leads a generalised Fourier transform and goes beyond the ordinary Fourier transform for tempered distributions on the real line. This generalised Fourier transform is used when analytic function theory is applied to study singular integral equations with non-temperate solutions. An example is the Wiener-Hopf equation where one seeks eigenfunctions $f(x)$ to the integral equation

$$(*) \qquad f(x) = \int_0^\infty K(x-y)f(y)dy$$

In many physical applications the kernel K has exponential decay and one seeks eigenfunctions f which are allowed to increase exponentially. The major result about solutions (*) appear in Theorem XVI on page 56 in [Pa-Wi] based upon the article [Ho-Wi]. This inspired Carleman to the constructions in § 1 below. Let us remark that the generalised inversion formula in Theorem 6.3 leads to the calculus of hyperfunctions. For comments about the relation between Carleman's original constructions and later studies of hyperfunctions we refer to the article [Kis] by Christer Kiselman.

6.0 A special construction.. Let $f(z)$ be a bounded analytic in the upper half plane $\mathfrak{Im}\, z > 0$ and suppose it extends to a continuous function on the closed half-plane and that the boundary function $f(x)$ is integrable, i.e.

$$\int_{-\infty}^\infty |f(x)|\, dx < \infty$$

To each $0 \leq \theta \leq \pi$ we set

$$(6.0.1) \qquad G_\theta(\zeta) = \int_0^\infty e^{i\zeta re^{i\theta}} \cdot f(re^{i\theta})e^{i\theta}\, dr$$

With $\zeta = se^{i\phi}$ we have

$$|e^{i\zeta re^{i\theta}}| = e^{-sr\sin(\phi+\theta)}$$

Hence (6.0.1) converges if $\sin(\phi+\theta) > 0$, i.e when

$$\text{(ii)} \qquad -\theta < \phi < \pi - \theta$$

So $G_\theta(z)$ is analytic in a half-space. In particular G_0 is analytic in $\mathfrak{Im}\,\zeta > 0$ while G_π is analytic in the lower half-plane. Moreover these G-functions are glued as θ-varies. To see this we notice that (0.1) is the complex line integral

$$\int_{\ell_+(\theta)} e^{i\zeta z} \cdot f(z)\, dz$$

where $\ell_+(\theta)$ is the half-line $\{re^{i\theta} : r \geq 0\}$. Hence there exists an analytic function $G^*(z)$ in $\mathbf{C} \setminus [0, +\infty)$ which is equal to $G_\theta(z)$ in every half-space defined via (ii). Next, with $\zeta = \xi + i\eta$ there exists a limit

$$\lim_{\epsilon \to 0} G_0(\xi + i\epsilon) = \int_0^\infty e^{i\xi x} f(x)\, dx$$

Similarly the reader may verify that

$$\text{(iii)} \qquad \lim_{\epsilon \to 0} G_\pi(\xi - i\epsilon) = -\int_0^\infty e^{-i\xi r} f(-r)\, dr = -\int_{-\infty}^0 e^{i\xi r} f(r)\, dr$$

Passing to the usual Fourier transform of $f(x)$ we therefore get the equation

(iv) $$\widehat{f}(-\xi) = G_0(\xi + i0) - G_\pi(\xi - i0)$$

where we have taken boundary values of the analytic functions G_0 and G_π. Now

(v) $$G^*(\xi) = G_0(\xi + i0) = G_\pi(\xi - i0) \quad : \xi < 0$$

Hence (iv) entials that $\widehat{f}(-\xi) = 0$ when $\xi < 0$ and reversing signs we conclude that the support of $\widehat{f}$ is contained in the half-line $\{\xi \leq 0\}$. This inclusion has been seen before since the L^1-function $f(x) = f(x + i0)$ is the boundary value of an analytic function in the upper half-plane. Moreover (iv) means that $\widehat{f}$ on $\{\xi < 0\}$ is expressed as the difference of the boundary values of G_0 and G_π taken on the positive real ξ-line. Notoce that this difference is expressing obstructions for the G^*-function to extend across intervals on the positive ξ-line. So in this sense G^* alone determines $\widehat{f}$.

The observations above which were used in work by Plaey and Wiener led Carelan to perform similar cointructions where regularity and growth properties are relaxed.

6.1 Carleman's constructions

Let U^* be the upper half-plane. To each pair of real numbers a, b we denote by $\mathcal{O}_{a,b}(U^*)$ the family of functions $f \in \mathcal{O}(U^*)$ such that for every $0 < \theta_0 < \pi/2$ there exists a constant $A(\theta_0)$ and

(*) $$|f(re^{i\theta})| \leq A(\theta_0) \cdot \big(r^a + \frac{1}{r^b}\big) \quad : \quad \theta_0 < \theta < \pi - \theta_0$$

Remark. No condition is imposed on the A-function as $\theta_0 \to 0$. In particular $f(z)$ need not have tempered growth as one approaches the real line. In the same way we define the family $\mathcal{O}_{a,b}(U_*)$ of analytic functions defined in the lower half-plane U_* satisfying similar estimates as above.

Example. Let $f(z)$ be the ordinary Fourier-Laplace transform of a tempered distribution μ on the real t-line supported by the half-line $t \leq 0$. Recall that this gives an integer N and a constant C such that

$$|f(x + iy)| \leq C \cdot y^{-N} \quad : \quad y > 0$$

Here we can take $a = b = N$ and $A(\theta) = \frac{C}{\sin(\theta)}$ to get $f(z)$ in $\mathcal{O}_{a,b}(U^*)$.

Let us return to the general case. Consider some $f \in \mathcal{O}_{a,b}(U^*)$. If $b \geq 1$ we choose a positive integer m so that $b < 1 + m$ and when $0 < \theta < \pi$ we consider the half space

$$U^*_\theta = \{z = re^{i\phi} \quad : -\pi - \theta < \phi < -\theta\}$$

The choice of m and (*) give an analytic function in $U^*_\theta(z)$ defined by:

(i) $$F_\theta(z) = \frac{i}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-izre^{i\theta}} \cdot r^m \cdot e^{im\theta} \cdot f(re^{i\theta})e^{i\theta} \cdot dr$$

Cauchy's theorem applied to the analytic function $f(z)$ shows that these F_θ-functions are glued together as we rotate the angle θ in the open interval $(0, \pi)$. Notice that

(ii) $$\cup_{0<\theta<\pi} U^*(\theta) = \mathbf{C} \setminus [0, +\infty)$$

Hence there exists an analytic function $F^*(z)$ in $\mathbf{C} \setminus [0, +\infty)$ such that

(iii) $$F^* \big| U^*_\theta = G_\theta \quad : \quad 0 < \theta < \pi$$

Next, let U_* be the lower half-plane where one defines the family $\mathcal{O}_{a,b}(U_*)$. If $g(z)$ belongs to $\mathcal{O}_{a,b}(U_*)$ we obtain exactly as above analytic functions

$$G_\theta(z) = \frac{i}{\sqrt{2\pi}} \cdot \int_0^\infty e^{izre^{i\theta}} \cdot r^m \cdot e^{im\theta} \cdot g(re^{-i\theta})e^{-i\theta} \cdot dr \quad : \quad 0 < \theta < \pi$$

defined in the half-planes

$$U_*(\theta) = \{z = re^{i\phi} \quad : -\pi + \theta < \phi < \theta\}$$

These G_θ-functions are again glued together and give an analytic function $G_*(z)$ in $\mathbf{C} \setminus (-\infty, 0]$ where which satisfies:

(iii) $$G_* \big| U_*(\theta) = G_\theta \quad : \quad 0 < \theta < \pi$$

The $\mathcal{S}$-transform. Consider a pair $f \in \mathcal{O}_{a,b}(U^*)$ and $g \in \mathcal{O}_{a,b}(U_*)$. We get the functions F^* and G_* and here $G_* - F^*$ is analytic outside the real axis and can be restricted to both the upper and the lower half-plane. This enable us to give the following:

6.2 Definition. *To every pair f, g as above we set*

$$\mathcal{S}^*(z) = G_*(z) - F^*(z) \quad : \quad \mathfrak{Im}(z) > 0$$
$$\mathcal{S}_*(z) = G_*(z) - F^*(z) \quad : \quad \mathfrak{Im}(z) < 0$$

Remark. The constructions of F^* and G_* entail that $G_* - F^*$ restricts to a function in $\mathcal{O}_{a,b}(U)$ when U is the upper or the lower half-plane. Hence $\mathcal{S}$ is a map from $O_{a,b}(U^*) \times \mathcal{O}_{a,b}(U_*)$ into itself.

6.3 The reflection operator. If $\phi \in \mathcal{O}(U^*)$ we get the analytic function in the lower half-plane defined by

$$T(\phi)(z) = \bar{\phi}(\bar{z})$$

In the same way T sends an analytic function defined in U_* to an analytic function defined in U^*. The composed operator $T \circ \mathcal{S}$ gives a pair of analytic functions defined by

$$(T \circ \mathcal{S})^*(z) = \bar{\mathcal{S}}_*(\bar{z}) \quad : \quad \mathfrak{Im}(z) > 0$$
$$(T \circ \mathcal{S})_*(z) = \bar{\mathcal{S}}^*(\bar{z}) \quad : \quad \mathfrak{Im}(z) < 0$$

With the notations above the result below extends Fourier's inversion formula for tempered distributions. Below $\simeq$ means that two functions differ by a polynomial in z.

6.3 Inversion Theorem. *For each pair (f, g) in $\mathcal{O}_{a,b}(U^*) \times \mathcal{O}_{a,b}(U^*)$ one has*

$$T \circ \mathcal{S} \circ T \circ \mathcal{S}(f) \simeq f \quad : \quad T \circ \mathcal{S} \circ T \circ \mathcal{S}(g) \simeq g$$

where $\simeq$ means that the differences are polynomials in z.

Remark. Theorem 6.3 is the assertion from p. 49 in [Car]. For details of the proof we refer to [loc.cit. p. 50-52]. The proof relies upon results of analytic extensions across a real interval. Since these results have independent interest we proceed to discuss material from from [ibid] and once this has been done we leave it to the reader to discover the proof of Theorem 6.3 or consult Carleman's proof.

6.4 Some analytic extensions.

Let D be the unit disc centered at the origin and set

$$D^* = D \cap \mathfrak{Im}(z) > 0 \quad \text{and} \quad D_* = D \cap \mathfrak{Im}(z) < 0$$

6.5 Theorem. *Let $f^* \in \mathcal{O}(D^*)$ and $f_* \in \mathcal{O}(D_*)$ be such that*

(*) $$\lim_{y \to 0} f^*(x+iy) - f_*(x-iy) = 0$$

holds uniformly with respect to x. Then there exists $F \in \mathcal{O}(D)$ with $F|D^* = f^*$ and $F|D_* = f_*$.

Remark. No special assumptions are imposed on the two functions except for (*). For example, it is from the start not assumed that they have moderate growth as one approaches the real x-line in (*).

Proof. In D^* we get the analytic function

(i) $$G(z) = f^*(z) - \bar{f}_*(\bar{z})$$

Write $G = U + iV$ and notice that (*) gives

(ii) $$\lim_{y \to 0} U(x,y) = 0$$

Hence the harmonic function U in D^* converges uniformly to zero on the part of ∂D^* defined by $y = 0$. If $\delta > 0$ is small we restrict U to the upper half-disc $D^*(\delta)$ of radius $1 - \delta$. Now (ii) implies that when G is expressed by the Poisson kernel of $D^*(\delta)$ then the boundary integral is only taken over the upper half-circle. It follows by the analyticity of the kernel function for $D^*(\delta)$ that $G(x,y)$ extends to a real analytic function across the the real interval $-1+\delta < x < 1-\delta$. The same holds for the derivatives $\partial G/\partial x$ and $\partial G/\partial y$. The Cauchy Riemann equations show that the complex derivative of $F(z)$ extends analytically across the real interval and the reflection principle by Schwarz finishes the proof.

6.6 Another continuation. We expose another result from [ibid]. See [Car: p. 40: Théorème 3] whose the essential ingredient is a subharmonic property for the radius of convergence of analytic functions. Put

$$\square = \{(x,y) : \ -1 < x < 1 \quad \text{and} \quad 0 < y < 1\}$$

Consider some $F(z) \in \mathcal{O}(\square)$. With a small $\ell > 0$ we put

$$D_+(\ell) = \{|\zeta| < \ell \cap \mathfrak{Im}(\zeta) > 0\}$$

With $z_0 = x_0 + iy_0$ where $-1/2 < x_0 < 1/2$ and $0 < y_0 < 1 - \ell$. we get an analytic function

(i) $$G_\zeta(z) = F(z+\zeta) - F(z)$$

which is defined in some neighborhood of z_0. It has a series expansion:

$$G_\zeta(z) = F(z+\zeta) - F(z) = \sum P_\nu(\zeta)(z-z_0)^\nu \quad \text{where :}$$

(*) $$P_\nu(\zeta) = \frac{1}{\nu!} \cdot [F^{(\nu)}(z_0+\zeta) - F^{(\nu)}(z_0)]$$

Let $\rho(\zeta)$ be the radius of convergence for the series (*). Hadamard's formula gives:

(**) $$\log \frac{1}{\rho(\zeta)} = \limsup_{\nu \to \infty} \frac{\log |P_\nu(\zeta)|}{\nu}$$

Suppose we knew that

(***) $$\rho(\zeta) \geq y_0 \quad : \quad \zeta \in D_+(\ell)$$

Then we can pick $\zeta = \frac{iy_0}{2}$ and conclude that the function

$$\text{(ii)} \qquad z \mapsto F(z + \frac{iy_0}{2}) - F(z)$$

is analytic in the disc $|z - z_0| < y_0$. At the same time the function $z \mapsto F(z + \frac{iy_0}{2})$ is analytic when $\mathfrak{Im}(z) > -\frac{y_0}{2}$ and hence $F(z)$ extends as an analytic function across a small interval on the real x-axis centered at x_0. So if (***) holds for every $-1/2 < x_0 < 1/2$. it follows that $F(z)$ extends analytically across the real interval $-1(2 < x < 1/2$. There remains to find a condition on F in order that (***) holds. Notice that it suffices to get (***) for sufficiently small y_0 if we seek some analytic extension of F across the real x-line. To obtain (***) Carleman imposed the following:

6.7 Hypothesis on F. *There exists a pair $\ell > 0$ and $\delta > 0$ such that if ξ is real with if $|\xi| < \ell$ then $z \mapsto G_\xi(z)$ extends to an analytic function in the domain where $|z| < 1/2$ and $\mathfrak{Im}(y) > -\delta$.*

It is clear that this hypothesis implies that if y_0 is sufficiently small then there exists a constant k such that

$$\text{(1)} \qquad |P_\nu(\zeta) \leq k^\nu \quad : \quad \zeta \in D_+(\ell) \quad : \nu = 1, 2, \ldots$$

Moreover, we see from a figure that the hypothesis also implies that

$$\text{(2)} \qquad \rho(\zeta) \geq y_0 \quad : \quad |\zeta| = \ell \quad : \mathfrak{Im}(\zeta) \geq 0$$

It is also trivial that

$$\text{(3)} \qquad \rho(\zeta) \geq y_0 \quad : |\zeta| = \ell \quad : \mathfrak{Im}(\zeta) = 0$$

Proof that (***) holds

The functions $\zeta \mapsto \log |P_\nu(\zeta)|$ are subharmonic in $D_+(\ell)$ for every ν. So if G is Green's function for $D_+(\ell)$ we have the inequality

$$\text{(i)} \qquad \log |P_\nu(\zeta)| \leq \frac{1}{2\pi} \int_{\partial D_*(\ell)} \frac{\partial G(\zeta, w)}{\partial n_w} \cdot \frac{\log |P_\nu(w)|}{\nu} \cdot |dw|$$

Now (1) above entails that

$$\text{(ii)} \qquad \frac{\log |P_\nu(w)|}{\nu} \leq k \quad : w \in \partial D_+(\ell)$$

At the same time (2-3) and Hadamard's formula give

$$\text{(iii)} \qquad \limsup_{\nu \to \infty} \frac{\log |P_\nu(w)|}{\nu} \leq \log \frac{1}{y_0} \quad : w \in \partial D_+(\ell)$$

Thanks to (ii) we can apply Lebesgue's dominated convergence theorem when we pass to the limes superior in (i) and hence (iii) gives

$$\text{(iv)} \qquad \limsup_{\nu \to \infty} \frac{\log |P_\nu(\zeta)|}{\nu} \leq \log \frac{1}{y_0} \quad : \zeta \in D_+(\ell)$$

Now we apply Hadamard's formula for points in $D_+(\ell)$ and (***) follows.

Remark. The continuation found above can be applied to relax the assumption in Theorem 6.5. For example, there exists an analytic extension for a pair f^*, f_* under the less restrictive condition that

$$\lim_{y \to 0} \int_a^b \big[f^*(x + iy) - f_*(x - iy)\big] \cdot dx = 0$$

This follows when 6.6 is applied to the primitive functions of the pair.

7. Carleman's inequality

Introduction. In the article [Ca:1] from 1923, Carleman proved a result about differentiable functions on the real line which confirms the general philosophy that in order for a polynomial $P(x)$ *of any degree* $\geq n$ to have multiple roots of some order n at two points, say 0 and 1, while it does not vanish identically, the maximum norms of its derivatives up to order n *cannot be too small.* Theorem 1 below gives a conclusive answer to this problem. The crucial key step in the proof is to use *subharmonic majorisations.* In 1923 this was a pioneering idea which after has become a standard tool in analysis. See § 2 in Chapter 4 in R. Nevanlinna's book [Nev[for a further discussion and examples which illustrate *Carleman's Prinzip der Gebietserweiterung.*

Now we begin to announce Carleman's theorem. Let $[0,1]$ be the closed unit interval on the real t-line. To each integer $n \geq 1$ we denote by S_n the class of n-times differentiable functions and non-negative functions on $[0,1]$ satisfying

$$f^{(\nu)}(0) = f^{(\nu)}(1) = 0 \quad : \quad 0 \leq \nu \leq n \quad : \quad \int_0^1 f^2(t)dt = 1 \quad : f(t) \geq 0$$

Thus, we regard non-negative functions which are "flat up to order n" at the end points. Notice that S_n contains all polynomials of the form

$$t^n(1-t)^n \cdot Q(t) \;:\; Q(t) \text{ any polynomial } \geq 0 \quad \text{where} \quad \int_0^1 t^n(1-t)^n Q(t)dt = 1$$

Since the degree of $Q(t)$ can be arbitrarily large the set of such polynomials is dense in S_n. Next, to each $f \in S_n$ we introduce the p:th roots of the L^2-norms of its derivatives of order $1 \leq p \leq n$:

$$\beta_p(f) = [\int_0^1 [f^{(p)}(t)]^2 dt\,]^{\frac{1}{2p}} \quad : \quad 1 \leq p \leq n$$

With these notations Carleman's inequality asserts:

Theorem 1. *For every n and each $f \in S_n$ one has*

$$(*) \qquad \sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \leq 2e\pi \cdot (1 + \frac{1}{4\pi^2 e^2 - 1})$$

Remark. The absolute constant in the right hand side appears as a consequence of the subsequent proof where several majorisations appear. The best constant $\mathcal{C}_*$ which would give

$$(i) \qquad \sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \leq \mathcal{C}^*$$

for all n and every $f \in \mathcal{S}_n$ is not known. Let us remark that (*) is sharp in the sense that there exists a constant C^* such that for every n one can find $f \in \mathcal{S}_n$ for which

$$(ii) \qquad \sum_{p=1}^{p=n} \frac{1}{\beta_p(f)} \geq C_*$$

See§ xx and also Chapter 1 in [Hö:xx] for the construction of such functions. Before we enter the proof if Theorem 1 we cite an excerpt from Emile Borel's comments to [Ca]:

La demonstration donnée par M. Carleman de l'ènoncé que j'avais induit du théorème de Denjoy est remarkquable par sa profondeur et par sa simplicité. Il serait toutefois désirable d'arriver à donner une demonstration sinon algébrique, du moins ne faisant appel qu'aux variables réelles du théorème auquel M. Carlemnan vient d'áttacher son nom. Ce théorème de Carleman me parait en effet devoir être considéré, avec le théorème de Denjoy, comme l'un des théorèmes fondamentaux de la théorie des functions indéfiniment dérivables de variables réelles. Il serait encore plus intèressant de complêter les théorèmes de Denjoy et de Carleman pour une étude asymptotique aussi précise que possible des séries de toute terme géneral quand $n \to \infty$.

Proof of Theorem 1.

Let $n \geq 1$ and $f \in S_n$. Keeping f fixed we put $\beta_p = \beta_p(f)$ to simplify notations. The result in § 7.A shows that the β-numbers are non-decreasing. i.e. we have

$$1 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_{n+1} \tag{*}$$

Define the complex Laplace transform

$$\Phi(z) = \int_0^1 e^{-zt} f(t) dt$$

Since f is n-flat at the end-ponts, integration by parts p times gives:

$$\Phi(z) = z^{-p} \int_0^1 e^{-zt} \cdot \partial^p(f^2)(t) dt \quad : \quad 1 \leq p \leq n+1$$

where $\partial^p(f^2)$ is the derivative of order p of f^2. Now we study the absolute value of Φ on the vertical line $\mathfrak{Re}(z) = -1$. To each $1 \leq p \leq n+1$ we have

$$\partial^p(f^2) = \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot f^{(\nu)} \cdot f^{(p-\nu)} \tag{1}$$

Since $|e^{t-iyt}| = e^t$ for all y, the triangle inequality gives

$$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \int_0^1 e^t \cdot |f^{(\nu)}(t)| \cdot |f^{(p-\nu)}(t)| \cdot dt \tag{2}$$

Now we use that $e^t \leq e$ on $[0,1]$ and apply the Cauchy-Schwarz inequality which gives by the definition of the β-numbers give:

$$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot \sum_{\nu=0}^{\nu=p} \binom{p}{\nu} \cdot \beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu} \tag{3}$$

From (*) it follows that $\beta_\nu^\nu \cdot \beta_{p-\nu}^{p-\nu} \leq \beta_p^p$ for each ν and since $\sum_{\nu=0}^{\nu=p} \binom{p}{\nu} = 2^p$ we obtain

$$|-1+iy|^p \cdot |\Phi(-1+iy)| \leq e \cdot 2^p \cdot \beta_p^p \tag{4}$$

Passing to the logarithm we get

$$\log |\Phi(-1+iy)| \leq 1 + p \cdot \log \frac{2\beta_p}{|-1+iy|} \tag{5}$$

Here (5) holds when $1 \leq p \leq n+1$ and the assumption that $\beta_0 = 1$ also gives

$$\log |\Phi(-1+iy)| \leq 1 \tag{6}$$

The ω-function. To each $1 \leq p \leq n+1$ we find a positve number y_p such that

$$|-1+iy_p| = 2e\beta_p$$

Now we define a function $\omega(y)$ where $\omega(y) = 0$ when $y < y_1$ and

$$\omega(y) = p \quad : \quad y_p \leq y < y_{p+1}$$

and finally $\omega(y) = n + 1$ when $y \geq y_{n+1}$. From this (5-6) give the inequality

$$\log |\phi(-1 + iy)| \leq 1 - \omega(y) \tag{7}$$

for all $-\infty < y < +\infty$.

A harmonic majorisation. With $1 - \omega(y)$ as boundary function in the half-plane $\mathfrak{Re}(z) > -1$ we construct the harmonic extension $H(z)$ where Poisson's formula in a half-plane gives:

$$H(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \omega(y)}{1 + y^2} \cdot dy$$

Since the function $\log |\Phi(z)|$ is subharmonic in this half-plane it follows from (7) that

$$0 = \log |\Phi(0)| \leq H(0)$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{\omega(y)}{1 + y^2} \cdot dy \leq \pi \tag{8}$$

Now $\omega(y) = 0$ when $y \leq y_1$ and hence (8) gives the inequality

$$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy \leq \frac{y_1^2}{1 + y_1^2} \cdot \pi \tag{9}$$

Next, by the construction of the ω-function we have the equality

$$\int_{y_1}^{\infty} \frac{\omega(y)}{y^2} \cdot dy = \frac{1}{y_1} + \ldots + \frac{1}{y_{n+1}} \tag{10}$$

Next, the construction of the y_p-numbers entail that $y_p \leq 2e\beta_p$ so (9-10) give

$$\frac{1}{\beta_1} + \ldots + \frac{1}{\beta_{n+1}} \leq 2e\pi \cdot \frac{1}{1 + \frac{1}{y_1^2}} \tag{11}$$

Finally, we have $1 + y_1^2 = 4e^2\beta_1^2$ and by 7:A below we have $\beta_1 \geq \pi$ which gives

$$\frac{1}{1 + \frac{1}{y_1^2}} \leq 1 + \frac{1}{4\pi^2 e^2 - 1} \tag{12}$$

Now (11-12) give the requested inequality in Theorem 1.

7.A. The proof above used an elementary result which asserts that the β-sequence increases when $f \in \mathcal{S}_n$. To see this, let $1 \leq p \leq n - 1$ and a partial integration gives:

$$\beta_p^2 = \int_0^1 f^{(p-1)}(t) f^{(p+1)}(t)\, dt$$

The Cauchy-Schwarz inequality gives

$$\beta_p^2 \leq \beta_{p-1}\beta_{p+1}$$

By an induction over p it follows that $\beta_1 \leq \ldots \leq \beta_n$ provided that we prove the inequality

$$\beta_0^2 = \int_0^1 f(t)^2\, dt \leq \beta_1^2 \tag{*}$$

Here (*) follows easily by regarding Fourier's development of f into a sine series $\sum a_\nu \cdot \sin(\nu\pi t)$ and shows that equality in (*) only occurs when $f(t) = \sin(\nu\pi t)$.

8. Carleman's inequality for inverse Fourier transforms in $L^2(\mathbf{R}^+)$.

Introduction. By Parsevel's theorem the Fourier transform sends L^2-functions on the ξ-line to L^2-functions on the x-line. We shall determine the class of non-negative L^2-functions $\phi(x)$ such that there exists an L^2-function $F(\xi)$ supported by the half-line $\xi \geq 0$ and

$$\phi(x) = \Big|\int_0^\infty e^{ix\xi} \cdot F(\xi) \cdot d\xi\Big| \tag{*}$$

The theorem below was proved in [Carleman]. Apart from applications to quasi-analytic functions this result has several other consequences which are put forward by Paley and Wiener in [Pe-Wi].

Theorem 8.1. *An L^2-function $\phi(x)$ satisfies (*) if and only if*

$$\int_{-\infty}^\infty \log^+\Big[\frac{1}{\phi(x)}\Big] \cdot \frac{dx}{1+x^2} < \infty$$

8.2 Remark. Theorem 8.1 means that $\phi(x)$ in the average cannot be too small when (*) holds. In addition one has the inequality (ii) below. Namely, suppose that $F(\xi)$ satisfies the weighted mean-value equality

$$\int_0^\infty F(\xi) \cdot e^{-\xi}\, d\xi = 1 \tag{i}$$

The proof of Theorem 8.1 will show that when $\phi(x)$ is defined by (*) then

$$\int_{-\infty}^\infty \log^+\Big[\frac{1}{\phi(x)}\Big] \cdot \frac{dx}{1+x^2} \leq \int_{-\infty}^\infty \frac{\phi(x)^2}{1+x^2} \cdot dx \tag{ii}$$

Proof of Theorem 8.1

First we prove the suffiency. Let $\phi(x)$ be a non-negative L^2-function where the integral in Theorem 8.1 is finite. Then there exists the harmonic extension of $\log\ \phi(x)$ to the upper half-plane;

$$\lambda(x+iy) = \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\log\ \phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0 \tag{1}$$

Let $\mu(z)$ be the conjugate harmonic function of λ and set

$$h(z) = e^{\lambda(z)+i\mu(z)} \tag{2}$$

Fatou's theorem gives for almost every x a limit

$$\lim_{y\to 0} \lambda(x+iy) = \log\ \phi(x) \tag{3}$$

Or, equivalently

$$\lim_{y\to 0} |h(x+iy)| = \phi(x) \tag{4}$$

From (1) and the fact that the geometric mean value of positive numbers cannot exceed their arithmetic mean value, one has

$$\big|h(x+iy)\big| = e^{\lambda(x+iy)} \leq \frac{y}{\pi} \cdot \int_{-\infty}^\infty \frac{\phi(t)}{(x-t)^2+y^2} \cdot dt \quad : y > 0 \tag{5}$$

Then (5) the Schwarz inequality give:

$$\int_{-\infty}^\infty \big|h(x+iy)\big|^2\, dx \leq \int_{-\infty}^\infty \big|\phi(x)\big|^2\, dx \quad : y > 0 \tag{6}$$

Since $h(z)$ is analytic in the upper half-plane it follows from (6) and Cauchy's formula that if $\xi < 0$, then the integrals

$$J(y) = \int_{-\infty}^{\infty} h(x+iy) \cdot e^{-ix\xi + y\xi} \cdot dx \quad : \quad y > 0 \tag{7}$$

are independent of y. Passing to the limit as $y \to \infty$ and using the uniform upper bounds on the L^2-norms of the functions $h_y(x) \mapsto h(x+iy)$, it follows that $J(y)$ vanishes identically. So the Fourier transforms of $h_y(x)$ are supported by $\xi \geq 0$ for all $y > 0$. Passing to the limit as $y \to 0$ the same holds for the Fourier transform of $h(x)$. Finally (4) gives

$$\phi(x) = |h(x)| \tag{8}$$

By Parseval's theorem $\widehat{h}(\xi)$ is an L^2-function and we conclude that $\phi(x)$ has the requested form (*).

Necessity. Since F is in L^2 there exists the Plancherel limit

$$\psi(x) = \lim_{N \to \infty} \frac{1}{2\pi} \cdot \int_0^N e^{ix\xi} \cdot F(\xi)\, d\xi \tag{9}$$

and in the upper half plane we get the analytic function

$$\psi(x+iy) = \frac{1}{2\pi} \cdot \int_0^{\infty} e^{ix\xi - y\xi} \cdot F(\xi)\, d\xi \tag{10}$$

When $F(\xi)$ satisfies (i) in the remark after Theorem 8.1 we have:

$$\psi(i) = 1$$

Consider the conformal map from the upper half-plane into the unit disc where

$$w = \frac{z-i}{z+i}$$

Here $\psi(x)$ corresponds to a function $\Phi(e^{is})$ on the unit circe $|w| = 1$ and:

$$\int_{-\pi}^{\pi} |\Phi(e^{is})|^2\, ds = 2 \cdot \int_{-\infty}^{\infty} \frac{|\phi(x)|^2}{1+x^2}\, dx \tag{11}$$

Similarly let $\Psi(w)$ be the analytic function in $|w| < 1$ which corresponds to $\psi(z)$. From (10-11) it follows that $\Psi(w)$ is the Poisson extension of Φ, i.e.

$$\Psi(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |w|^2}{|e^{is} - w|^2} \cdot \Phi(e^{is}) \cdot ds \tag{12}$$

If $0 < r < 1$ it follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}| \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\Psi(re^{is}\right|^2 \cdot ds \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{is}|^2 \cdot ds \tag{13}$$

Now (12) gives:

$$\lim_{r \to 1} \Psi(re^{is}) = \Phi(e^{is}) \quad : \text{almost everywhere} \quad 0 \leq s \leq 2\pi \tag{14}$$

Next, since $\psi(i) = 1$ we have $\Psi(0) = 1$ which gives the inequality

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{|\Psi(re^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \log^+ |\Psi(re^{is}| \cdot ds \quad : 0 < r < 1 \tag{15}$$

By (13-15) a passage to the limit as $r \to 1$ gives

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{|\Phi(e^{is}|} \cdot ds \leq \int_{-\pi}^{\pi} \left| \Phi(e^{is}\right|^2 \cdot ds \tag{16}$$

Returning to the real x-line we get the inequality (ii) in Remark 8.2 which at the same time finishes the proof of Theorem 8.1.

9. An integral equation.

Let $f(x)$ be an L^1-function on the real x-line where the zeros of $\widehat{f}$ is a discrete subset $\{\alpha_\nu\}$ of $\mathbf{R}$ and enumerated so that their absolute values are non-decreasing. Consider the set of all L^∞-functions $\phi(x)$ which satisfy the convolution equation

$$(*) \qquad f * \phi(x) = \int f(x-y)\phi(y)dy = 0 \quad : \ -\infty < x < +\infty$$

The exponential functions $\{e^{-i\alpha_\nu x}\}$ give solutions to (*) for every zero of $\widehat{f}$. More generally an exponential polynomial

$$P(x) = \sum c_\nu \cdot e^{-i\alpha_\nu x}$$

solves (*). It turns out that these exponential polynomials is a dense subset of all L^∞-solutions to (*).

9.1 Theorem. *To every L^∞-solution ϕ there exists a sequence of exponential solutions $\{P_N(x)\}$ such that $\lim_{N\to\infty} P_N(x) = \phi(x)$ holds almost everywhere where $\{P_N\}$ can be chosen so that their maximum norms over the whole x-line satisfy*

$$||P_N||_\infty \leq ||\phi||_\infty$$

The crucial point in the subsequent proof is to use the local invertibility for L^1-functions whose Fourier transforms are $\neq 0$ over open intervals on the ξ-line. To begin with the L^∞-function ϕ which solves (*) is a tempered distribution on the real x-axis. Since ϕ is not integrable we cannot find its Fourier transform in an explicit way. But the following holds:

9.2 Proposition. *The support of $\widehat{\phi}$ is contained in the discrete set $\{\alpha_\nu\}$.*

Proof. Let $J = (a, b)$ be an open interval on the ξ-axis where $\widehat{f}(\xi)$ has no zeros on the closed interval $[a, b]$. The result in XX about the Banach algebra $L^1(\mathbf{R}$ gives an L^1-function g such that

$$\widehat{g}(\xi) \cdot \widehat{f}(\xi) = 1 \quad : \ a \leq \xi \leq b$$

Next, consider some $\psi \in \mathcal{S}$ whose Fourier transform $\widehat{\psi}$ has support contained in $[a, b]$. Now we take the convolution $\psi * g * f$ in L^1 whose Fourier transform becomes

$$\widehat{\psi} \cdot \widehat{g} \cdot \widehat{f} = \widehat{\psi}$$

Since $f * \phi = 0$ it follows that $\psi * g * f * \phi = 0$ and hence also $\psi * \phi = 0$. Now ϕ belongs to $\mathcal{S}$ and the space of tempered distributions is a module over $\mathcal{S}$. So in the space of tempered distributions on the ξ-line there exists the product $\widehat{\psi} \cdot \widehat{\phi}$ and the vanishing of the convolution $\psi * \phi$ on the x-line entails that $\widehat{\psi} \cdot \widehat{\phi} = 0$. Above $\widehat{\psi}(\xi)$ can be taken as an arbitrary test-function in $C_0^\infty(a, b)$. It follows that the support of the distribution $\widehat{\phi}$ does not intersect the open interval (a, b). Since (a, b) can be an arbitrary interval in the complement of the discrete set of zeros Proposition 9.2 follows.

Proposition 9.2 shows that $\widehat{\phi}$ is a sum of Dirac distributions at points from the discrete set $\{\alpha_\nu\}$. Let us show that every such Dirac distribution has order zero, i.e. of the form $c \cdot \delta_{\alpha_\nu}$ for some constant c. To prove this we consider some α_ν and choose $\psi(x) \in \mathcal{S}$ such that its Fourier transform a $\hat{\psi}(\xi)$ is a test-function which is identically one in a small neighborhood of α_ν while its support does not contain any other zero. Now the distribution

$$\text{(i)} \qquad \hat{\psi} \cdot \hat{\phi} = \rho_\nu$$

where ρ_ν is the Dirac distribution determined by $\hat{\phi}$ at α_ν. At the same time (i) is the Fourier transform of $\psi * \phi$ which is a bounded function on the x-line, Namely, by the the L^1-norm of ψ times the sup-norm of ϕ. Hence the inverse Fourier transform of the Dirac distribution ρ_ν is a bounded function on the x-line. But this can only occur if ρ_ν has order zero.

Summing up we have proved the following:

9.3 Proposition. *The Fourier transform $\widehat{\phi}$ is given by:*

$$\widehat{\phi} = \sum a_\nu \cdot \delta_{\alpha_\nu}$$

where $\{a_\nu\}$ is some sequence of complex numbers.

Remark. By Fouriers inversion the a-numbers are determined via the equations

$$\sum a_\nu \cdot \widehat{g}(\alpha_\nu) = \frac{1}{2\pi} \int \phi(x) \cdot g(-x) dx \quad : g \in \mathcal{S}$$

The right hand side is defined for every L^1-function. So if the maximum norm $||\phi||_\infty = 1$ then the series

$$\sum a_\nu \cdot \widehat{g}(\alpha_\nu)$$

converges for every L^1-function g and the absolute value of the sum is $\leq ||g||_1$.

9.4 Approximation by exponential solutions. At this stage we can finish the proof of Theorem 9.1. First, fix some non-negative function $g \in \mathcal{S}$ such that $\widehat{g}(\xi)$ is a test-function and $\hat{g}(0) = 1$. If $N \geq 2$ we set $g_N(x) = Ng(Nx)$ which gives

$$\widehat{g}_N(\xi) = \widehat{g}(\xi/N)$$

Next, the Fourier transform of the convolution $g_N * \phi$ is equal to

$$\widehat{g}_N(\xi) \cdot \widehat{\phi}(\xi)$$

Since $\hat{g}_N$ has compact support the left hand side is a finite sum

$$\sum \hat{g}_N(\alpha_\nu) \cdot a_\nu \cdot \delta_{\alpha_\nu}$$

This gives the exponential polynomial

$$P_N(x) = (g_N * \phi)(x) = \frac{1}{2\pi} \cdot \sum \widehat{g}_N(\alpha_\nu) \cdot a_\nu \cdot e^{i\alpha_\nu x}$$

Finally, by Lebesgue's theorem shows that

$$\lim_{N \to \infty} P_N(x) = \phi(x)$$

holds almost everywhere. For the maximum norms we get

$$||P_N||_\infty \leq |g||_1 \cdot ||\phi||_\infty$$

and Theorem 9.1 follows since since $||g||_1 = \widehat{g}(0) = 1$.

10. Spectral synthesis.

The subsequent material comes from Beurling's article [Beur] presented in 1938 at the Scandinavian Congress in Helsinki. For each pair $g \in L^1(\mathbf{R})$ and $\phi \in L^\infty(\mathbf{R})$ the convolution

$$g * \phi(x) = \int g(x-y) \cdot \phi(y) \cdot dy$$

yields a bounded and continuous function whose maximum norm is majorised by $||g||_1 \cdot ||\psi||_\infty$. So each $\phi(x)$ gives an ideal in the convolution algebra $L^1(\mathbf{R})$:

$$J_\phi = \{f \in L^1(\mathbf{R} \quad : \ f * \phi = 0\}$$

10.1 Definition. *The set of common zeros of Fourier transforms of functions in J_ϕ is denoted by $\sigma(\phi)$ and called the spectrum of ϕ.*

The spectrum is non-empty unless $\phi = 0$. For if $\sigma(\phi) = \emptyset$ then the result from XXX shows that the closed ideal $J_\phi = L^1(\mathbf{R})$ and since $L^\infty(\mathbf{R})$ is the dual space of L^1 we get $\phi = 0$.

Exercise. Show that $\sigma(\phi)$ is equal to the support of the temperate distribution $\widehat{\phi}$. The hint is to use the fact that the inverse Fourier transform of every test-funtion on the ξ-line is an L^1-function on the x-line.

10.2 The spectral synthesis problem. *Let $f \in L^1(\mathbf{R})$ be such that $\widehat{f}(\xi) = 0$ on $\sigma(\phi)$. Does this imply that $f \in J_\phi$.*

In general the answer is negative, i.e. there exist ϕ-functions for which the spectral synthesis fails. See XXX (??). But if $\sigma(\phi)$ satisfies a certain topological condition the spectral synthesis holds. A closed set K without interior points on the real ξ-line is called perfect if the closure of $K \setminus \{p\}$ is equal to K for every $p \in K$. So if F is a closed set with empty interior which does not contain any perfect subset, then F must contain at least one isolated point. This property was used in [Beu] to establish the following:

10.3 Theorem. *Let $\phi \in L^\infty(\mathbf{R})$ be such that the boundary of $\sigma(\phi)$ does not contain any perfect subset. Then spectral synthesis holds for the ideal J_ϕ.*

Before the proof of Theorem 10.3 we establish a result of independent interest.

10.4 Theorem. *Consider a pair $\phi \in L^\infty(\mathbf{R})$ and $h \in L^1(\mathbf{R})$. Let a be a real number such that the Fourier transform $\widehat{h}(a) = 0$. Then the Fourier transform of $h * \phi$ cannot be a constant times the Dirac measure δ_a unless $h * \phi$ is identically zero.*

Proof. Replacing h by $e^{-iax} \cdot h(x)$ we may take $a = 0$. So now $\widehat{h}(0) = 0$ and we argue by a contradiction. Thus, suppose that the Fourier transform of $h*\phi$ is $c_0 \cdot \delta_0$ for some constant $c_0 \neq 0$. We shall prove that this is impossible. The vanishing $\widehat{h}(0) = 0$ and the result in § XX gives a sequence $\{g_N\}$ in $\mathcal{S}$ such that $\widehat{g}_N(0) = 0$ for all N and at the same time:

(i) $$\lim_{N \to \infty} ||g_N * h - h||_1 = 0$$

Next, since every $g_N \in \mathcal{S}$ we get

$$\widehat{g_N * h * \phi} = \widehat{g}_N \cdot \widehat{h * \phi} = \widehat{g}_N \cdot c_0 \cdot \delta_0 = 0$$

where the last equality holds since $\widehat{g}_N(0) = 0$. Hence the convolutions $g_N * h * \phi = 0$ for all N. Finally, since $\phi \in L^\infty$ (i) would give $h * \phi = 0$ which contradicts the hypothesis that the Fourier transform is a non-zero constant times δ_0.

Proof of Theorem 10.3. Let f be an L^1-function such that $\widehat{f} = 0$ on $\sigma(\phi)$. We must prove that $f * \phi = 0$. Assume the contrary and put $\psi = f * \phi$. One has the inclusion

(1) $$\sigma(\psi) \subset \sigma(\phi)$$

Indeed, this follows since the commutative law for convolutions imply that $h \in J_\phi$ gives $h * (f * \phi) = f * (h * \phi) = 0$. Hence $J_\phi \subset J_\psi$ and (1) follows. Next we shall improve (1) and establish the inclusion

(2) $$\sigma(\psi) \subset \partial(\sigma(\phi))$$

To prove (2) may assume that the interior of $\sigma(\phi) \neq \emptyset$ and let $(a, b) \subset \sigma(\phi)$ be an open interval. To every $\rho(\xi) \in C_0^\infty(a, b)$ the inverse Fourier transform is an L^1-function which we denote by ρ_*. Since $\hat{f} = 0$ on $\sigma(\phi)$ we get $\rho \cdot \widehat{f} = 0$ which entails $\rho_* * f = 0$ and then

$$0 = (\rho_* * f) * \phi = \rho_* * (f * \phi) = \rho_* * \psi \implies \rho \cdot \widehat{\psi} = 0$$

where the last product is defined since $\hat{\psi}$ is a tempered distribution on the ξ-line while ρ is a test-function so we can use that $\mathcal{S}^*$ is a module over the space of test-functions. Since $\rho \in C_0^\infty(a, b)$ was arbitrary we conclude that the support of $\hat{\psi}$ does not intersect (a, b), and since (a, b) was an arbitrary interval of the interior of $\sigma(\phi)$ (2) follows from Exercise 1.

Next, (2) and the hypothesis on the topology of $\partial\sigma(\phi)$ imply that $\sigma(\psi)$ contains at least one isolated point a. Choose a test-function $\rho(\xi)$ which is identically one in a small neighborhood of a while the support of ρ has empty intersection with the rest of $\sigma(\psi)$. Then the support of the tempered distribution $\rho \cdot \hat{\psi}$ is reduced to $\{a\}$. Let g be the inverse Fourier transform of ρ so that $\rho \cdot \hat{\psi}$ is the Fourier transform of $g * \psi$. Here $g * \psi$ is a bounded continuous function and exactly as in the proof of Theorem 10.4 we conclude that the Dirac distribution defined by $\rho \cdot \hat{\psi}$ can only be a constant times δ_a. This means that

$$g * \psi(x) = c_0 \cdot e^{aix}$$

Let us then consider the L^1-function $h = g * f$. Since $\hat{f} = 0$ holds on $\sigma(\phi)$ we have $\hat{h}(a) = 0$. At the same time:

(*) $$h * \phi(x) = c_0 \cdot e^{aix} \quad \text{where} \quad c_0 \neq 0$$

But this contradicts the result in Theorem 10.4 and Theorem 10.3 is proved.

11. On inhomogeneous $\bar\partial$-equations.

Recall that $L^1(\mathbf{R}^2)$ is a convolution algebra, i.e. for a pair of L^1-functions $f(x,y)$ and $g(x,y)$ one defines

$$f*g(x,y)=\iint f(x-s,y-t)g(s,t)dsdt \tag{1}$$

The convolution is commutative and satisfies the associative law. Next, given some L^1-function $f(x,y)$ there exists the *distribution derivatives* $\partial_x(f)$ and $\partial_y(f)$. We shall impose the condition that these distribution derivatives are bounded measurable functions, i.e. both belong to $L^\infty(\mathbf{R}^2)$.

Exercise. Recall that $L^\infty(\mathbf{R}^2)$ is the dual space of $L^1(\mathbf{R}^2)$. Use this and the definition of distribution derivatives to conclude that $\partial_x(f)\in L^\infty(\mathbf{R}^2)$ if and only if there to every compact set K in $\mathbf{R}^2$ exists a constant C_K such that

$$|\iint f(x,y)\cdot\partial_x(g(x,y)\cdot dxdy\leq C_K\cdot||g||_1$$

for every test-function $g(x,y)$ whose support is contained in K.

Next, let f be a complex valued function L^1-function with compact support. With $\bar\partial=\frac{1}{2}(\partial_x+i\partial_y)$ we get the distribution

$$\bar\partial(f)=\frac{1}{2}\big(\partial_x(f)+i\partial_y(f)\big)$$

11.1 Theorem. *The inclusion $\bar\partial(f)\in L^\infty(\mathbf{R}^2)$ implies that f is a continuous function whose modules of continuity is bounded by $C\cdot\delta\cdot\log\frac{1}{\delta}$ for a constant C which only depends on the size of the support of f.*

Proof Recall that

$$\bar\partial(\frac{1}{z})=2\pi i\cdot\delta_0 \tag{i}$$

Hence the hypothesis that $\bar\partial(f)\in L^\infty(\mathbf{R}^2)$ gives:

Sublemma. One has the equality:

$$2\pi i f=\frac{1}{z}*\bar\partial(f)$$

where the right hand side is the convolution of $\frac{1}{z}$ and the L^∞-function $\bar\partial(f)$ and the equality holds in L^1.

Proof continued. Set $g=\bar\partial(f)$. Given a pair of points z_1,z_2 in the complex plane we get

$$|f(z_2)-f(z_1)|\leq\frac{1}{2\pi}\cdot|\iint[\frac{1}{z-z_1}-\frac{1}{z-z_2}]\cdot g(z)dxdy\,|\leq$$
$$\frac{||g||_\infty}{2\pi}\cdot\iint_K|\frac{1}{z-z_1}-\frac{1}{z-z_2}|\cdot dxdy$$

where K is the compact support of f. There remains to estimate the double integral. Suppose for example that K is contained in the disc D_R of radius R centered at the origin. Given a pair of distinct points z_1,z_2 in this disc we notice that

$$\frac{1}{z-z_1}-\frac{1}{z-z_2}=\frac{z_2-z_1}{(z-z_1)(z-z_2)}$$

With $\delta=|z_1-z_2|$ Theorem 11.1 follows if we find a constant C such that

$$\iint_{D_R}\frac{dxdy}{|z-z_1|\cdot|(z-z_2|}\leq C\cdot\operatorname{Log}\frac{1}{\delta} \tag{*}$$

The verification of (*) is left as an exercise to the reader.

12. Some integral equations.

A. Planck's equation. Following [Paley-Wiener. page 40] we recall the physical background for the integral equation in (*) below. By Planck's law the radiation per unit volume in a black cavity at temperature T in a state of steady equilibrium and of frequency between ν and $\nu + d\nu$ is given by

$$\frac{8\pi \cdot h\nu^3}{c^3(e^{\frac{h\nu}{kT}} - 1} \cdot d\nu \tag{1}$$

Here h is Planck's constant, c the velocity of light and k the gas constant reckoned for one molecule. This suggests that the radiation from a source in approximative local equilibrium but consisting of a mixture of black bodies of different temperatures, will have a distribution as a function of ν given by

$$\nu^3 \cdot \int_0^\infty \frac{\phi(T) \cdot dT}{e^{\frac{h\nu}{kT}} - 1}. \tag{2}$$

where $\phi(T)$ represents the amount of radiation coming from black bodies at temperature T. If then, we have an observable radiation with frequency distribution $\psi(\nu)$, the problem of resolving this into its constituent black-body radiations is equivalent to the solution of the equation:

$$\psi(\nu) = \nu^3 \cdot \int_0^\infty \frac{\phi(T) \cdot dT}{e^{\frac{h\nu}{kT}} - 1}. \tag{*}$$

Following [Paley-Wiener] we show how to find ϕ for a given ψ-function. Set

$$\mu = \frac{h}{kT} \quad : \quad \phi(T) \cdot dT = \Phi(\mu) \cdot d\mu \quad : \quad \frac{\psi(\nu)}{\nu^2} = \Psi(\nu)$$

Then (*) assumes the form:

$$\Psi(\nu) = \int_0^\infty \Phi(\mu) \cdot \frac{\mu\nu}{e^{\mu\nu} - 1} \cdot d\mu \tag{**}$$

Above Ψ and Φ are defined on $\mathbf{R}^+$ and recall that on the multiplicative group of positive real numbers with coordinate μ the invariant Haar measure is $\frac{d\mu}{\mu}$. Impose the L^2-condition:

$$\int_0^\infty |\Psi(\nu)|^2 \cdot \frac{d\nu}{\nu} < \infty \tag{iii}$$

When (iii) holds we seek an L^2-function $\Phi(\mu)$ on the multiplicative μ-line such that (ii) holds.

Solution. The idea is to express (**) in terms of Fourier transforms of Ψ and Φ. First Ψ yields the L^2-function defined on the real x-line by

$$\widehat{\Psi}(x) = \frac{1}{\sqrt{2\pi}} \cdot \lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} \Psi(\nu) \cdot \nu^{(ix - 1/2)} \cdot d\nu \tag{2}$$

Inserting this in (**) we are led to evaluate the integral below when $\mu > 0$ and x a real number:

$$\lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} \frac{\mu\nu}{e^{\mu\nu} - 1} \cdot \nu^{(ix - 1/2)} \cdot d\nu \tag{3}$$

Set

$$J(x) = \lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} \frac{s}{e^s - 1} \cdot s^{(ix - 1/2)}\, ds$$

The substitution $\mu \cdot \nu = s$ shows that (3) becomes

$$(4) \qquad \lim_{\epsilon \to 0} \int_{\epsilon}^{1/\epsilon} \frac{s}{e^s - 1} \cdot (s/\mu)^{(ix-1/2)} \cdot \frac{ds}{\mu} = \mu^{-ix-1/2} \cdot J(x)$$

If $\Phi \in L^2(\mathbf{R}^+)$ its normalised Fourier transform on the real x-line is given by:

$$(5) \qquad \widehat{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \cdot \lim_{\epsilon \to 0} \int_{\epsilon}^{1/\epsilon} \Phi(\mu) \cdot \mu^{(ix-1/2)} \cdot d\mu$$

With these notations it is clear that the equation (**) holds if

$$\widehat{\Phi}(-x) = \frac{\widehat{\Psi}(x)}{J(x)}$$

The requested L^2-solution Φ therefore exists if

$$|J(x)| \geq c$$

hold for some positive constant c and all real x. This turns out to be true. More precisely, the crucial formula which by the above solves Planck's equation is as follows:

12.A.2 Theorem. *One has the equation*

$$J(x) = \Gamma(\frac{3}{2} - ix) \cdot \zeta(\frac{3}{2} - ix)$$

where the last factor is Riemann's ζ-function.

Exercise. Show the formula and conclude that there exists a positive constant c as above.

B. The Laplace equation

Let $f(x)$ be a function such that

$$(1) \qquad \int_0^\infty |f(x)|^2 \cdot dx < \infty$$

This gives the analytic function $G(u + iv)$ in the right half-plane defined by

$$(*) \qquad G(u + iv) = \int_0^\infty e^{-ux - ivx} \cdot f(x) \cdot dx$$

Here $G(iv) = \widehat{f}(v)$ so by Parseval's formula (1) entails that G extends to an L^2-function on the imaginary axis. Conversely, suppose we are given an analytic function G in the right half-plane where $G(iv)$ is in L^2. Then we shal find an equatio of its inverse Fourier transform $f(x)$ expressed by the restriction of G to the $\{u > 0\}$. This corresponds to the integral equation

$$(*) \qquad g(u) = \int_0^\infty e^{-ux} \cdot f(x) \cdot dx$$

where $g(u)$ is defined when $u > 0$ and one seeks f with

$$(1) \qquad \int_0^\infty |f(x)|^2 \cdot dx < \infty$$

The solution relies upon the following inversion formula due to Laplace. Introduce the Laplace transform

$$F(u + iv) = \int_0^\infty e^{-ux - ivx} \, f(x) \, dx$$

B.1 Theorem. *For each L^2-function f supported by $\{x \geq 0\}$ one has the inversion formula*

$$(*) \quad f(x) = \lim_{A\to\infty} \frac{1}{2\pi x} \int_0^\infty \rho_A(t) \cdot F(\frac{t}{x}) \cdot \frac{dt}{\sqrt{t}} \quad \text{where} \quad \rho_A(t) = \int_{-A}^{A} \frac{t^{i\xi} \cdot d\xi}{\Gamma(i\xi + \frac{1}{2})}$$

Exercise. Prove Laplace's inversion formula. .

B.2 Widder's solution. In the article *The inversion of the Laplace integral and the related moment problem*, D.V Widder found an exceedingly simple method to solve the Laplace equation (*). Consider first a bounded and continuous function $f(x)$ defined on $x > 0$. Then it is clear that the g-function in (*) is infinitely differentiable on $u > 0$ and for every $n \geq 1$ we have

$$(-1)^n \cdot g^{(n)}(u) = \int_0^\infty x^n \cdot e^{-ux} \cdot f(x)\, dx$$

With $x > 0$ fixed we therefore get the following equality for every $n \geq 1$:

$$(1) \qquad \frac{(-1)^n}{n!} \cdot g^{(n)}(\frac{n}{x}) \cdot (\frac{n}{x})^{n+1} = \frac{\int_0^\infty \xi^n \cdot e^{-n\xi/x} \cdot f(\xi)\, d\xi}{\int_0^\infty \xi^n \cdot e^{-n\xi/x}\, d\xi}$$

B.3 Exercise. Use the assumption that $f(x)$ is a bounded and continuous function on $x > 0$ to show that (1) gives the limit formula:

$$f(x) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \cdot g^{(n)}(\frac{n}{x}) \cdot (\frac{n}{x})^{n+1}$$

The hint is to use the functions $\psi_n(x) = \frac{1}{n!} \cdot x^n \cdot e^{-x}$ which are all ≥ 0 and the integral over $(0, +\infty)$ is one. Here $\psi_n(x)$ takes its maximum when $x = n$. Now we consider the functions

$$\rho_n(x) = \frac{\psi_n(\frac{x}{n})}{n}$$

Then the reader can verify that if $p(x)$ is an arbitrary bounded and continuous function on $x > 0$, then

$$p(x) = \lim_{n\to\infty} \frac{1}{n} \cdot \int_0^\infty \psi_n(\frac{x}{n}) \cdot p(x) \cdot dx \quad \text{hold for every} \quad x > 0$$

13. The Carleman-Hardy theorem.

The proof of Theorem 0.3.7 from the introduction relies upon inequalities about differentiable functions.

13.1 Lemma. *Let $\psi(x)$ be a C^1-function defined for $x>0$ such that*

$$\lim_{x\to 0}\psi(x)=0 \quad\text{and}\quad |\psi'(x)|\leq \frac{C}{x} \quad : x>0$$

holds for some constant C. Then it follows that

$$(*)\qquad \lim_{x\to 0} x\cdot\psi'(x)=0$$

The proof is left as an exercise to the reader. See also the article *Contributions to the Arithmetic Theory of Series* by Hardy and Littlewood for further limit formulas of higher order derivatives. Next we establish a result due to Landau from the article *Einige ungleichungen für zweimal differentierbare Funktionen.*

13.2 Proposition. *Let $\psi(x)$ be a C^3-function defined on $x>0$ such that*

$$\text{(i)}\qquad \lim_{x\to 0}\frac{\psi(x)}{x^2}=0 \quad\text{and}\quad |\psi'''(x)|\leq \frac{C}{x}$$

hold for some constant C. Then it follows that

$$\text{(ii)}\qquad \lim_{x\to 0}\psi''(x)=0$$

Proof. Let $x>0$ and set $\xi=\zeta\cdot x$ where $0<\zeta<1/2$. Keeping these numbers fixed, Taylor's formula gives

$$\psi(x+\xi)+\psi(x-\xi)-2\psi(x)=\xi^2\psi''(x)+\frac{\xi^3}{6}\cdot[\psi'''(x+\theta_1\xi)-\psi'''(x-\theta_2\xi)] \quad : 0<\theta_1,\theta_2<1$$

The triangle inequality gives

$$|\psi''(x)|\leq$$

$$(2)\quad \frac{1}{\xi^2}\cdot[|\psi(x+\xi)|+|\psi(x-\xi)|+2|\psi(x)|]+\frac{\xi^3}{6}\cdot[\big(|\psi'''(x+\theta_1\xi)|+|\psi'''(x-\theta_2\xi)|\big)$$

By the second condition in (i) the last term above is majorised by

$$(3)\qquad \frac{C}{6}\cdot\xi\cdot(\frac{1}{x+\theta_1\xi}+\frac{1}{x-\theta-2\xi}]\leq\frac{C}{6}\cdot\frac{2\zeta}{1-\zeta}$$

Given $\epsilon>0$ we can choose ζ so small that (3) is $<\epsilon/2$. Next, keeping ζ fixed the first term in (2) above is majorised by

$$(4)\qquad \frac{1}{\zeta^2x^2}\cdot\big[(1+\zeta)^2\cdot o(x^2)+(1-\zeta)^2\cdot o(x^2)+2\cdot o(x^2)\big]$$

where the small ordo terms follows from the first condition in (i). Now (ii) in Proposition 13.2 follows from the inequality (2) above.

Proof of Theorem 0.3.7 Assume first that $\sum A_n$ converges and define the following two functions when $x>0$:

$$\text{(i)}\qquad U(x)=\frac{1}{2}A_0\cdot x^2+\sum_{n=1}^{\infty}\frac{A_n}{n^2}\cdot(1-\cos(nx))$$

$$\text{(ii)}\qquad V(x)=\int_0^x\big[\int_0^y u(s)\cdot ds\big]\cdot dy$$

In (i) $U(x)$ is the associated Riemann function of u. Since u is of class C^2 when $x>0$ it is clear that $U''(x)=V''(x)$ when $x>0$ and hence $U(x)-V(x)$ is a linear function $C+Dx$ on $(0,+\infty)$. Since u by assumption is an L^1-function we see that $V(x)=o(x)$ and we also have $U(x)=o(x^2)$ by a classical result known as Riemann's Lemma. It follows that $C = D = 0$, i.e. the functions U and V are identical which gives

$$V(x) = o(x^2)$$

Next, notice that

$$\text{(ii)} \qquad x > 0 \implies V'''(x) = u'(x)$$

Now $|u''(x)| \leq \frac{C}{x^2}$ was assumed in 0.3.6 which gives another constant C^* such that $|u'(x)| \leq \frac{C^*}{x}$. Hence (ii) implies that V satisfies the Landau conditions in Proposition 13.2 which gives:

$$\text{(iii)} \qquad \lim_{x \to 0} V''(x) = 0$$

Finally, since $V''(x) = u(x)$ holds when $x > 0$ we conclude that $\lim_{x \to 0} u(x) = 0$. This proves one half of Theorem 0.3.7.

The case $\lim_{x \to 0} u(x) = 0$. When this is assumed there remains to show that $\sum A_n$ converges. By assumption $|u''(x)| \leq \frac{C}{x^2}$ which gives a constant C^* such that $|u'(x)| \leq \frac{C^*}{x}$ and Lemma 13.1 gives

$$\text{(i)} \qquad u'(x) = 0(x)$$

Next, we use a result by Lebesgue from his book *Lecons des series trigonmétriquees* which asserts that the series $\sum A_n$ converges if

$$\text{(ii)} \qquad \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{|u(x+\epsilon) - u(x)|}{x} \cdot dx = 0$$

To get (ii) we use Rolle's theorem and write

$$\text{(iii)} \qquad u(x+\epsilon) - u(x) = \epsilon \cdot u'(x + \theta \cdot \epsilon)$$

Now it is clear that (i) and (iii) give (ii) which finishes the proof that $\sum A_n$ is convergent.

14. The log-potential and the Abel-Carleman inversion formula

Let us first recall that Abel established a remarkable inversion formula to express potential functions in $U(x)$ in a conservative force field on the real line in his article [Abel] from 1824. Adopting Abel's formula, Carleman's article *Abelsche Integralgleichungen mit konstanten Integrationsgrenzen* give inversion fromulas of a similar kind where the integals are taken over a fixed interval.

First we consider the logarithmic potential. For each $g(t) \in L^1[0,1]$ we

$$T_g(x) = \int_0^1 \log\, |x-t| \cdot g(t)\, dt \tag{1}$$

The function $T_g(x)$ is restricted to $0 < x < 1$. A notable fact due to Legendre is the following inequality where $||g||_1 = \int_0^1\, |g(t)|\, dt$ is the L^1-norm.

14.0.Legendre's inequality. *There exists a constant C such that*

$$\int_0^1 \frac{|T_g(x)|}{\sqrt{x(1-x)}}\, dx \leq C \cdot ||g||_1 \tag{*}$$

Proof. A straightforward calculation shows that the function

$$x \mapsto \int_0^1 \log|x-t| \cdot \frac{1}{\sqrt{t(1-t)}} \cdot dt$$

is a constant function whose value is given by $-2\pi \cdot \log 2$. From this the reader can deduce (*) and notice that one can take $C = 2\pi \cdot \log 2$.

We shall establish an inversion formula which entails that special properties hold for functions in the range of T.

14.1 Theorem. *The necessary and sufficient condition that an L^1-function f is equal to T_g for some $g \in L^1[0,1]$ is that the integral in (*) is absolutely convergent. Moreover, f is absolutely continuous and the principal value convolution of the derivative f' satisfies*

$$\frac{1}{\sqrt{x(1-x)}} \cdot \mathcal{P}_{f'} \in L^1[0,1]$$

Moreover, with $K = (2\pi^2 \cdot \log 2)^{-1}$ one has the inversion formula:

$$g(x) = \frac{1}{\pi^2} \cdot \frac{1}{\sqrt{x(1-x)}} \cdot \mathcal{P}_{f'}(x) + \frac{K}{\sqrt{x(1-x)}} \cdot \int_0^1 \frac{f(t)}{\sqrt{t(1-t)}}\, dt$$

Fractional operators. Let $0 < \alpha < 1$ and this time we set

$$T_g(x) = \int_0^1 \frac{g(t)}{|x-t|^\alpha}\, dt$$

Again we restrict $T_g(x)$ to $[0,1]$ and one seeks an inversion formula. It will be given under the additional hypothesis that

$$\int_0^1 |g(t)| \cdot (t(1-t)^{\frac{1-\alpha}{2}}\, dt < \infty$$

This enable us to define the Cauchy transform

$$H(z) = \int_0^1 \frac{g(t) \cdot (t(1-t)^{\frac{1-\alpha}{2}}}{z-t}\, dt$$

Here $H(z)$ is analytic outside $[0,1]$ and for a fixed $0 < x < 1$ one can perform complex line integrals along rectifiable Jordan curves Γ_x which start and end at x while the remaining part of the curve stays outside $[0,1]$. In particular we can take such Jordan curves whose winding number around $z = 0$ is one. See figure §xx. With this in mind one constructs the function

$$K(x) = \int_{\Gamma_x} \frac{(z(1-z))^{\frac{1-\alpha}{2}}}{(z-x)^{1-\alpha}} \cdot H(z)\, dz \quad : 0 < x < 1$$

Theorem. *The function $K(x)$ is absolutely continuous and one has the inversion formula*

$$g(x) = xxx \cdot \frac{dK}{dx}$$

Remark. In particular the operator in (xx) is injective.

Proof. In the upper half plane $\mathfrak{Im}\, z > 0$ we choose a single valued branch of the complex log-function and get the analytic function

$$G(z) = \int_0^1 \log(z-t) \cdot g(t)\, dt$$

Passing to the boundary value we see that

(i) $$G(x+i0) = T_g(x) + \pi i \cdot \int_x^1 g(t)\, dt \quad : 0 < x < 1$$

In the lower half-plane we have the branch of the complex log-function whose argument stays in $(-\pi, 0)$ and get the boundary value equation

(ii) $$G(x-i0) = T_g(x) - \pi i \cdot \int_x^1 g(t)\, dt \quad : 0 < x < 1$$

Set $f(x) = T_g(x)$. Taking derivatives in (i-ii) we get the equations

(iii) $G'(x-i0) + G'(x-i0) = 2f'(x) \quad : G'(x-i0)' - G'(x-i0)' = -2\pi i \cdot g(x)$

Next, the Cauchy transform gives the single valued analytic function in $\mathbf{C} \setminus [0,1]$:

$$C_g(z) = \int_0^1 \frac{g(t)}{z-t}\, dt$$

Since $\sqrt{z(1-z}$ has a single valued branch in $\mathbf{C} \setminus [0,1]$ we also get the single-valued function

$$\Phi(z) = \sqrt{z(z-1} \cdot C_g(z)$$

Here we notice that

(iv) $\Phi(x+i0) = i \cdot \sqrt{x(1-x} \cdot C_g(x+i0) \quad : \Phi(x-i0) = -i \cdot \sqrt{x(1-x} \cdot C_g(x+i0)$

Together (iii-iv) give the equations below when $0 < x < 1$

(v) $$2i\sqrt{x(1-x)} \cdot f'(x) = \Phi(x+i0) - \Phi(x-i0)$$

(vi) $$2\pi \cdot \sqrt{x(1-x)} \cdot g(x) = \Phi(x+i0) + \Phi(x-i0)$$

Next, when $|z|$ is large we have

$$\Phi(z) = z \cdot \sqrt{1-z^{-1}} \cdot z^{-1} \cdot \int_0^1 \frac{g(t)}{1-t/z}\, dt$$

The right hand side tends to $\int_0^1 g(t)\, dt$ and then (v) entails that

(vii) $$\Phi(z) = \frac{1}{\pi} \cdot \int_0^1 \frac{\sqrt{t(1-t)} \cdot f'(t)}{z-t}\, dt + \int_0^1 g(t)\, dt$$

Passing to boundary values we get

(viii) $$\Phi(x+i0)+\Phi(x-i0)=2\cdot\int_0^1 g(t)\,dt+\frac{2}{\pi}\cdot\int_0^1 \frac{\sqrt{t(1-t)}\cdot f'(t}{x-t}\,dt$$

where the last term is a principal value integral. Finally, (vi) gives

(ix) $$\sqrt{(1-x)x}\cdot g(x)=\frac{1}{\pi^2}\cdot\mathcal{P}_{f'}(x)+2\cdot\int_0^1 g(t)\,dt$$

Division with $\sqrt{x(1-x}$ gives the requested inversion formula via Legendre's constant in the proof of 14.0.

14.2 A Fourier transform. The T-operator is a convolution and let us analyze the Fourier transform

$$\widehat{T}_g(\xi)=\iint_\square e^{-ix\xi}\cdot\log|x-t|\cdot g(t)\,dt$$

under the condition that the mean value $\int_0^1 g(t)\,dt=0$ which entails that $\widehat{g}(\xi)=0$.
14.3 Exercise. Verify the formula

(i) $$\int_{-1}^1 e^{-iu\xi}\cdot\log|u|\cdot du=2\cdot\int_{-1}^1 \cos u\xi\cdot\log u\cdot du=-\frac{2}{\xi}\int_0^\xi \frac{\sin u}{u}\cdot du$$

Use this to conclcude that

(0.3.1) $$\widehat{T}_g(\xi)=-2\cdot\frac{\widehat{g}(\xi)}{\xi}+\rho(\xi)\cdot\widehat{g}(\xi)$$

where the ρ-function decays at infinity like ξ^{-2}, i.e. there is an absolute constant C for which

$$|\rho(\xi)|\leq C\cdot\xi^{-2}\quad:\quad|\xi|\geq 1$$

The equation $f(x)=\int_0^1 \frac{\phi(t)}{|x-t|^\alpha}\,dt$

Let $0<\alpha<1$ and consider the the operator which sends $\phi(t)\in L^1[0,1]$ to

$$T_g(x)=\int_0^1 \frac{\phi(t)}{|x-t|^\alpha}\,dt$$

Since $0<\alpha<1$ this convolution implies that $T_g(x)$ is an almost everywhere defined locally integrable function. In particular it restricts to an L^1-function on $\{0\leq x\leq 1\}$. We shall prove that this operator is injective and exhibit an inversion formula. To achieve this we introduce the function

$$G(z)=\int_0^1 \frac{\phi(t)}{(z-t)^\alpha}\,dt$$

It is analytic in the upper half-plane where branches of $(z-t)^\alpha$ have argument in $(\alpha\cdot\pi)$. Here we get the boundary value function

$$G(x+i0)=\lim_{\epsilon\to 0}\int_0^1 \frac{\phi(t)}{(x-t+i\epsilon)^\alpha}\,dt$$

In the lower half-plane the argument of $(z-t)^\alpha$ stays in $(-\alpha\pi,0)$ we have

$$G(x-i0)=\lim_{\epsilon\to 0}\int_0^1 \frac{\phi(t)}{(x-t-i\epsilon)^\alpha}\,dt=T_g(x)+$$

From this we conclude that the following hold when $0<x<1$:

$$G(x+i0)=\int_0^x \frac{\phi(t)}{(x-t)^\alpha}\,dt+e^{-\pi i\alpha}\cdot\int_x^1 \frac{\phi(t)}{(t-x)^\alpha}\,dt=T_g(x)+(e^{-\pi i\alpha}-1)\cdot\int_x^1 \frac{\phi(t)}{(t-x)^\alpha}\,dt$$

In the same way we find that

$$G(x-i0)=T_g(x)+(e^{\pi i\alpha}-1)\cdot\int_x^1\frac{\phi(t)}{(t-x)^\alpha}\,dt$$

Together (x-xx) give

$$e^{2\pi i\alpha}G(x+i0)-G(x-i0)=(e^{2\pi i\alpha}-1)T_g(x)$$

The case $T_g(x)=0$. Then (xx) gives

$$G(x-i0)=e^{2\pi i\alpha}G(x+i0)$$

where this equality holds almost everywhere on $\{0<x<1\}$. At the same time we notice that

$$G(x+i0)=G(x-i0)=T_g(x)\quad : x>1$$
$$e^{-\pi i\alpha}G(x+i0)=e^{\pi i\alpha}G(x-i0)=T_g(x)\quad : x<0$$

So if $T_g(x)=0$ it follows from the above that $G(z)$ exists as a multi-valued analytic function in $\mathbf{C}\setminus\{0,1\}$. Using (xx) we obtain a single-valued analytic funtion outside $\{0,1\}$ by

$$\Phi(z)=xxx\cdot G(z)$$

We are going to show that the vanishing of $T_g(x)$ implies that $\Phi(z)=0$. To prove this we study Φ locally around $z=1$ and $z=0$. Around $z=1$ it has a Laurent expansion

$$\Phi(z)=\sum_{n=-\infty}^{\infty}c_n(z-1)^n$$

Now a local analytic branch of z^α exista around $z=1$ which entails that $G(z)$ is locally of the form

$$G(z)=(z-1)^{-\frac{1+\alpha}{2}}\cdot\sum_{n=-\infty}^{\infty}a_n(z-1)^n$$

POINT. Priitive G-function is bounded+ Weierstrass to impliy that no negative a_m-occur and then Φ is meromorphic and even with a zero at $z=$. Same true at $z=0$ and this entire function is zero since it does not increase more than $|z|$. Finally classic Abel gives vanishing of ϕ.

15. An L^1-inequality for inverse Fourier transforms.

Theorem 15.1 below is due to Beurling in [Beurling]. Let $g(t)$ be a function defined on $t \geq 0$ where the inverse Fourier transform of $tg(t)$ is integrable, i.e. the function defined on the x-axis by

$$(*) \qquad f(x) = \int_0^\infty e^{itx} \cdot tg(t)\, dt$$

belongs to $L^1(\mathbf{R})$.

15.1 Theorem. *When $f \in L^1(\mathbf{R})$ it follows that $g(t)$ is integrable and one has the inequality*

$$\int_0^\infty |g(t)|\, dt \leq \frac{1}{2} \int_{-\infty}^\infty |f(x)|\, dx$$

Proof. Since (*) is taken over $t \geq 0$, $f(x)$ is the boundary value function of the analytic function defined in $\mathfrak{Im}(z) > 0$ by

$$(1) \qquad f(z) = \int_0^\infty e^{itz} \cdot tg(t) \cdot dt$$

We first prove the inequality in Theorem 15.1 when $f(z)$ is zero-free in the upper halfplane and consider the normalised situation where the L^1-integral of $|f(x)|$ is one. Now the complex square root of $f(z)$ exists in $\mathfrak{Im}(z) > 0$ and gives an analytic function $F(z)$ such that $F^2 = f$. Since $|F(x)|^2 = |f(x)|$ it follows that F belongs to the Hardy space $H^2(\mathbf{R})$ and Plancherel's theorem gives a function $h(t)$ on $t \geq 0$ where

$$(2) \qquad F(z) = \int_0^\infty e^{itz} \cdot h(t) dt$$

Parseval's equality gives

$$(*) \qquad 1 = \int |F(x)|^2\, dx = 2\pi \cdot \int_0^\infty |h(t)|^2\, dt$$

The Fourier transform of the convolution $h * h$ is equal to $F^2(x) = f(x)$. This gives

$$(**) \quad t \cdot g(t) = \int_0^t h(t-s) h(s)\, ds \implies |t \cdot g(t)| \leq H(t) = \int_0^t |h(t-s)| \cdot |h(s)|\, ds$$

Put

$$(4) \qquad F_*(x) = \int_0^\infty e^{itz} \cdot |h(t)|\, dt$$

Parseval's formula applied to the pair $|h|$ and F_* gives

$$\int |F^2(x)|\, dx = 2\pi \cdot \int_0^\infty |h^2(t)|\, dt$$

We conclude that the L^2-norm of F^* also is one and here

$$(5) \qquad F_*(x)^2 = \int_0^\infty e^{itz} \cdot H(t)\, dt$$

At this stage we use a result from XXX which shows that the function

$$\theta \mapsto \log \big[\int_0^\infty |F_*(re^{i\theta})|^2 \cdot dr \big]$$

is a convex function of θ where $-\pi \leq \theta \leq 0$. Apply this when $\theta = \pi/2$ with end-values 0 and π. This gives

$$\int_0^\infty |F_*(iy)|^2 \cdot dy \leq \sqrt{\int_{-\infty}^0 |F_*(x)|^2 \cdot dx} \cdot \sqrt{\int_0^\infty |F_*(x)|^2 \cdot dx}$$

Since $x \mapsto |F_*(x)|$ is an even function of x the equality in (*) entails that the product above is equal to one. Hence we obtain:

$$\int_0^\infty \bigl[\int_0^\infty e^{-ty} \cdot H(t) \cdot dt\bigr] \cdot dy \leq \frac{1}{2}$$

Integration by parts shows that the left hand side is equal to

$$\int_0^\infty \frac{H(t)}{t} \cdot dt$$

Finally, by (**) $|g(t)| \leq \frac{H(t)}{t}$ and hence the L^1-norm of g is bounded by $\frac{1}{2}$ as requested.

Removing zeros. If f is not zero-free we let $B(z)$ be the Blaschke product of its zeros and write

$$f = B(z)\phi(z)$$

Here ϕ is zero-free and we do not change the L^1-norm on the x-line since $|B(x)| = 1$ holds almost everywhere. Notice that we can write

$$f = \phi\bigl(\frac{1+B}{2}\bigr)^2 + \phi\bigl(\frac{1-B}{2}\bigr)^2 = F_1^2 - F_2^2$$

where F_1 and F_2 as above are zero-free in the Hardy space H^2. Since we have

$$\bigl|\frac{1+B}{2}\bigr|^2 + \bigl|\frac{1-B}{2}\bigr|^2 \leq 1$$

it follows that

$$|F_1|^2 + |F_2|^2 \leq |\phi|$$

Now the established zero-free case gives the inequality in Theorem 15.1

16. On functions with spectral gap

Introduction. A fore-runner to distribution theory appears in work by Beurling where spectral gaps of functions f on the real x-line are analyzed. We expose a result from a seminar by Beurling at at Uppsala University in March 1942.

16.1 Theorem. *Let $f(x)$ be a bounded and continuous function on the real x-line such that $\widehat{f}(\xi)$ is zero on $\{-1 \leq \xi \leq 1\}$ and*

$$f(x+h) - f(x) \leq h$$

hold for all $h > 0$ and every x. Then its maximum norm is at most π.

The proof is postponed until § 16.3. First we establish some preliminary results where an essential ingredient is the entire function:

$$2 \cdot H(z) = \big(2\sin\frac{z}{2}\big)^2 \cdot \big[\sum_{n=1}^{\infty} \frac{1}{(z-2\pi n)^2} - \sum_{n=0}^{\infty} \frac{1}{(z+2\pi n)^2} + \frac{1}{\pi z}\big]$$

A. Exercise. Verify the identity

$$(*) \qquad \frac{1}{\big(2\sin\frac{z}{2}\big)^2} = \sum_{-\infty}^{\infty} \frac{1}{(z-2\pi n)^2}$$

The hint is to consider the meromorphic function

$$\phi(z) = \frac{\cos z/2}{2\sin z/2}$$

It has simple poles at $\{2\pi n\}$ where n runs over all integers and we notice that

$$\psi(z) = \phi(z) - \sum_{-\infty}^{\infty} \frac{1}{z-2\pi n}$$

is entire. Hence the derivative $\psi'(z)$ is also entire. Since $\cos^2 z/2 + \sin^2 z/2 = 1$ we get:

$$\psi'(z) = -\frac{1}{\big(2\sin\frac{z}{2}\big)^2} + \sum_{-\infty}^{\infty} \frac{1}{(z-2\pi n)^2}$$

At the same time the reader may verify that the right hand side is bounded so this entire function must be identically zero which gives (*).

B. Exercise. Use (*) to show that if $x > 0$ then

$$2H(x) = 1 - \big(2\sin\frac{x}{2}\big)^2 \cdot \big[\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} - \frac{1}{\pi x}\big]$$

The θ-function. It is defined for all real x by:

$$\theta(x) = \frac{1}{2} \cdot \operatorname{sign}(x) - H(x)$$

where $\operatorname{sign}(x)$ is -1 if $x < 0$ and $+1$ if $x > 0$.

16.2 Proposition. *The θ-function is everywhere ≥ 0 and*

$$(*) \qquad \int_{-\infty}^{\infty} \theta(x) \cdot dx = \pi$$

Proof. Exercise B gives for every $x > 0$:

$$\theta(x) = \frac{1}{2}\big(2\sin\frac{x}{2}\big)^2 \cdot \big[\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} - \frac{1}{\pi x}\big]$$

Next, notice the two inequalities

(1) $$\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} \leq \int_0^{\infty} \frac{2dt}{(x+2\pi t)^2} = \frac{1}{\pi x}$$

(2) $$\sum_{n=1}^{\infty} \frac{2}{(x+2\pi n)^2} + \frac{1}{x^2} \geq \int_0^{\infty} \frac{2dt}{(x+2\pi t)^2} = \frac{1}{\pi x}$$

Here (2) entails that $\theta(x) \geq 0$ on $x > 0$ and (1) obviously implies that the integral

(3) $$\int_0^{\infty} \theta(x) \cdot dx < \infty$$

We leave as an exercise to the reader to verify the similar result for $x < 0$, i.e. that $\theta(x) \geq 0$ hold for $x < 0$ and that its integral over $(-\infty, 0)$ is finite. To establish the equality (*) in Proposition 16.2 we notice that the function

$$\text{sign}(x) + (2\sin\frac{x}{2})^2 \cdot \frac{1}{\pi x}$$

is odd so its integral over the real line is zero. The reader may also check the equation

$$\int_{-\infty} \frac{(\sin\frac{x}{2})^2}{(x-2\pi n)^2} \cdot dx = \int_{-\infty} \frac{(\sin\frac{x}{2})^2}{(x+2\pi n)^2} \cdot dx \quad \text{for every} \quad n \geq 1$$

and then verify the equality

$$\int_{-\infty}^{\infty} \theta(x) \cdot dx = \int_0^{\infty} \frac{(2\sin\frac{x}{2})^2}{x^2} \cdot dx$$

where residue calculus shows that the last integral is π.

16.3 Proof of Theorem 16.1

Let $\mathcal{H}$ be the Heaviside function which is one on $x > 0$ and zero on $\{(x \leq 0\}$. Recall that the distribution derivative $\partial_x(\mathcal{H}) = \delta_0$. Regarding f as a temperate distribution this gives the equation

(i) $$f = \partial_x(f * (\mathcal{H} - \frac{1}{2}))$$

As explained in § XX the Fourier transform $\widehat{H}(\xi)$ is supported by $[-1, 1]$. Since the support of $\widehat{f}$ is disjoint from $[-1, 1]$ it follows that $f * H = 0$ and hence the distribution derivative

(ii) $$\partial_x(f * H) = 0$$

Next, notice that

(iii) $$\frac{1}{2} \cdot \text{sign}(x) = \mathcal{H} - \frac{1}{2}$$

The construction of θ in (B) and (i-iii) therefore give

$$f = \partial_x(f * \theta)$$

This means that one has the equation

(iv) $$f(x) = \int_{-\infty}^{\infty} \theta(x-y) \cdot f'(y) \cdot dy$$

By assumption $f'(y) \leq 1$ for all y and since $\theta \geq 0$ the right hand side is bounded above by

$$\int_{-\infty}^{\infty} \theta(x-y) \cdot dy = \pi \implies f(x) \leq \pi$$

To get $f(x) \geq -\pi$ we consider the function $g(x) = -f(-x)$ which again is a bounded continuous function and the reader easily verifies that $g(x+h) - g(x) \leq h$ for all $h > 0$. Moreover, $\widehat{g}(\xi)$ is minus the complex conjugate of $\widehat{f}$ so g has the same spectral gap as f and just as above we get the upper bound $g(x) \leq \pi$ which entails that $f(x) \geq -\pi$ hold for all x. Hence its maximum norm is bounded by π which finishes the proof of Theorem 16.1

16.4 Question. Investigate if Theorem 16.1 is sharp, i.e. try to use the proof above in order to construct f whose maximum norm is close to π.

17. A theorem about limits

The space of complex-valued bounded and uniformly continuous functions on the real x-line is a Banach space $\mathcal{C}_*$ where we use the maximum norm over the whole line. A subspace arises as follows: On the ξ-line we have the space $\mathfrak{M}$ of complex Riesz measures γ with a finite total variation and to each γ we get the function

$$\mathcal{F}_\gamma(x) = \int_{-\infty}^{\infty} e^{ix\xi} \cdot d\gamma(\xi)$$

It is clear that $\mathcal{F}_\mu$ belongs to C_*. Denote by $\mathcal{A}$ the subspace of C_* given by $\mathcal{F}_\gamma$-functions as γ varies over $\mathfrak{M}$. Before we announce Theorem 17.1 below we recall the notion of weak-star limits in $\mathfrak{M}$. Let $\{\mu_n\}$ be a bounded sequence of Riesz measures, i.e. there exists a constant such that

$$||\mu_n|| \leq M$$

hold for all n. The sequence $\{\mu_n\}$ converges weakly to zero if

$$\lim_{n\to\infty} \int e^{ix\xi} \cdot d\mu_n(x) = 0$$

holds pointwise for every ξ.

17.1 Theorem. *A function $\psi \in C_*$ belongs to the closure of $\mathcal{A}$ if and only if*

$$(*) \qquad \lim_{n\to\infty} \int \psi(x) \cdot d\mu_n(x) = 0$$

whenever $\{\mu_n\}$ is a sequence in $\mathfrak{M}$ which converges weakly to zero.

The sufficiency part is easy. For suppose that ψ belongs to the closure of $\mathcal{A}$ and let $\{\mu_n\}$ converge weakly to zero. Since the total variations in this weakly convergent sequence of measures is uniformly bounded, it suffices to show that (*) holds when $\psi \in \mathcal{A}$. So let $\psi = \mathcal{F}_\gamma$ for some $\gamma \in \mathfrak{M}$. Since γ and μ_n both have a finite total variation it is clear that

$$\int \psi(x) \cdot d\mu_n(x) = \int \mathcal{F}_{\mu_n}(\xi) \cdot d\gamma(\xi)$$

Here $\{\mathcal{F}_{\mu_n}(\xi)\}$ is a sequence of uniformly bounded continuous functions on the real ξ-line which by assumption converges pointwise to zero. Since the Riesz measure γ has a finite total variation, the Borel-Riesz convergence result in [Measure] shows that (*) tends to zero with n.

Proof of necessity.

There remains to show that if $\psi \in C_*$ is outside the closure of $\mathcal{A}$, then there exists a sequence $\{\mu_n\}$ which converges weakly to zero while $\{\int \psi \cdot d\mu_n\}$ stay away from zero. To attain this we shall consider a class of variational problem and extract a certain sequence of measures which does the job.

A class of variational integrals. Let a, b, s be positive numbers and $q > 2$. With p chosen so that

$$\frac{1}{p} + \frac{1}{q} = 1$$

we have the space $L^p[-a, a]$ where $[-a, a]$ is an interval on the ξ-line. To each function $g(\xi) \in L^p[-a, a]$ we get the function

$$\mathcal{F}_g(x) = \int e^{ix\xi} \cdot g(\xi) \cdot d\xi$$

This gives a continuous function which is restricted to $[-b, b]$ and we set

$$||\psi - \mathcal{F}_g||_q^b = \Big[\int_{-b}^{b} [\psi(x) - \mathcal{F}_g(x)|^q \cdot dx\Big]^{1/q}$$

where the upper index b indicates that we compute a L^q-norm on the bounded interval $[-b,b]$. To each $g \in L^p[-a,a]$ we set

$$(*) \qquad \mathcal{J}(g;q,b,a,s) = ||\psi - \mathcal{F}_g||_q^b + ||g||_p$$

where the last term is the L^p-norm of g taken over $[-a,a]$. Since the Banach space $L^p[-a,a]$ is strictly convex one easily verifies:

17.2 Proposition. *The variational problem where $\mathcal{J}$ is minimized over g while a,b,s are fixed has a unique extremal solution.*

17.3 Exercise. Regarding infinitesmal variations via the classic device due to Euler and Lagrange, the reader can verify that there exists a unique extremal solution g which satisfies

$$(*) \qquad ||g||_p)^{1-p} \cdot \frac{|g(\xi)|^p}{g(\xi)} = M^{-1/p} \cdot \int_{-b}^{b} e^{i\xi x} \cdot \frac{|\psi(x) - \mathcal{F}_g(x)|^q}{|\psi(x) - \mathcal{F}_g(x)} \cdot dx$$

where we have put

$$M = \int_b^b |\psi - \mathcal{F}_g|^q \cdot dx$$

Consider the absolutely continuous measure on the x-line defined by the density

$$d\mu = M^{-1/p} \cdot \frac{\psi(x) - \mathcal{F}_g(x)|^q}{|\psi(x) - \mathcal{F}_g(x)} \quad : \quad -b \leq x \leq b$$

This gives

$$\int_{-b}^{b} |d\mu(x)| = M^{-1/p} \cdot \int_{-b}^{b} |\psi(x) - \mathcal{F}_g(x)|^{q-1} \cdot dx$$

Hölder's inequality applied to the pair of functions $|\psi(x) - \mathcal{F}_g(x)|^{q-1}$ and 1 on $[-b,b]$ gives the inequality below for the total variation:

$$(**) \qquad ||\mu|| \leq (2b)^{1/q}$$

17.4 Lemma. *The following two formulas hold:*

$$\int \psi \cdot d\mu = \mathcal{J}(g;q,b,a,s)$$

$$\int \mathcal{F}_g \cdot d\mu = s \cdot ||g||_p$$

Proof. To begin with we have

$$\int (\psi - \mathcal{F}_g) \cdot d\mu = M^{-1/p} \cdot \int_b^b |\psi - \mathcal{F}_g| \cdot dx = M^{1-1/p} \cdot M = M^{1/q}$$

Next Fubini's theorem gives

$$\int \mathcal{F}_g \cdot d\mu = \int_a^a [\int e^{ix\xi} \cdot \mu(x)] \cdot g(\xi) \cdot d\xi = s \cdot \int_a^a ||g||_p)^{1-p} \cdot \frac{|g(\xi)|^p}{g(\xi)} \cdot g(\xi) d\xi = s \cdot ||g||_p$$

This proves formula (2) and (1) follows via (i) and the equality

$$\mathcal{J}(g;q,b,a,s) = M + s \cdot ||g||_p$$

17.5 Passage to limits. Following [Beurling] we now consider certain limits where we first let $q \to +\infty$ and after $b \to \infty$, and finally use pairs $a = 2^m$ and $s = 2^{-m}$ where m will be large positive integers. To begin with, the measure μ depends on q,b,a,s and let us denote it by $\mu_q(b,a,s)$. The uniform bound (*) from Exercise

17.3 entails that while a, b, s are kept fixed, then there is a sequence $\{q_\nu\}$ which tends to $+\infty$ and give a weak limit measure

$$\mu_*(b,a,s) = \lim \mu_{q_\nu}(b,a,s)$$

The extremal g-functions depend on q and are denoted by g_q. Their L^q-norms remain bounded and passing to a subsequence we get an L^∞-function g_* on $[-a,a]$ where $g_{q_\nu} \to g_*$. Here g_* depends on b, a, s and is therefore indexed as $g_*(b,a,s)$. We have also a limit:

$$\lim_{\nu\to\infty} \mathcal{J}(g_g; q_\nu; b, a, s) = \mathcal{J}_*(g_*(a,b,s))$$

Moreover

$$\mathcal{J}_*(g_*(a,b,s)) = \max_{-b\leq x\leq b} |\psi(x) - \mathcal{F}_{g_*(b,a,s)}| + s \cdot \int_{-a}^{a} |g_*(b,a,s)(\xi)| \cdot d\xi$$

At this stage we use the hypothesis that ψ does not belong to the closure of $\mathcal{A}$ which entails that with a and s fixed, then here is a constant $\rho > 0$ such that

$$\liminf_{b\to\infty} \max_{-b\leq x\leq b} |\psi(x) - \mathcal{F}_{g_*(b,a,s)}| \geq \rho$$

At this stage proof is easily FINISHED.

18. Lindeberg's central limit theorem.

Introduction The classical version of the CLT (central limit theorem) was proved by De Moivre in 1733 and asserts that as n tends to infinity, the standardized binomial distribution tends to the normal distribution. Thus, let $\mathbf{B}$ be the two point random variable which takes the values +1 or -1 with probability 1/2. If $n \geq 2$ we denote by $\mathbf{B}_1, \ldots, \mathbf{B}_n$ an n-tuple of independent two points variables. This gives the random variable

$$(*) \qquad \chi_n = \frac{\mathbf{B_1} + \ldots + \mathbf{B_n}}{\sqrt{n}}$$

De Moivre's proof that χ_n tends to the normal distribution was direct in the sense that characteristic functions were not used. To see the idea we let $n = 2N$ be a large even integer. If $k \geq 1$ we denote by $\rho_N(k)$ the probability that the number of heads minus the number of tails is $\leq 2k$ after $2N$ many trials. The binomial formula gives

$$\rho_N(k) = \frac{1}{2} + 2^{-2N} \cdot \sum_{\nu=0}^{\nu=N+k} \binom{2N}{\nu}$$

The normal distribution is given by the increasing function

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-t/2} \cdot dt$$

With $k = \sqrt{N} \cdot a$ one wants the limit formula

$$(1) \qquad \lim_{N \to \infty} \rho_N(\sqrt{N} \cdot a) = \mathcal{N}(2a)$$

for each $a > 0$. This can be proved directly using Wallis' limit formula for products of sine-functions which gives Stirling's formula with a remainder term and therefore a quite sharp estimate for the rate of convergence. For readers familiar with Swedish a proof with good upper bounds for the rate of convergence is presented by Carleman in his outstanding text-book from 1926 for first year studies on university level which actually covers the more general where the equally distributed variables have a finite distribution. See also the section *Residue Calculus* for a proof of Wallis' formula.

A more general version of the CLT was formulated by Simon Laplace in his classic treatise on probability from 1812. His assertion was that sums of independent random variables which are suitably scaled so that the partial sums χ_n are random variables with mean value zero and variances σ_n which converge to a number σ, implies that the sequence χ_n converges to a normal distribution with variance σ, under the *hypothesis* that the individual random variables defining the sum variables χ_n give a *relatively insignificant contribution.* A rigorous proof of the CLT in the spirit of Laplace was given by Liapunoff in 1901. His result goes as follows: Let $W_1, W_2, \ldots$ be a sequence of independent random variables, each with mean value zero and variance σ_ν. Assume that there exists a constant M such that the *moment of order 3* is $\leq M$ for each W_ν and that the limit

$$\lim_{n \to \infty} \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\sqrt{n}} = \sigma^2$$

exists. Then that partial sums variables defined by

$$\chi_n = \frac{W_1 + \ldots + W_n}{\sqrt{n}}$$

converge to the normal distribution with variance σ. The conclusive version of the CLT was proved by Lindeberg in 1920. He weakened the conditions in Liapunoff's result by relaxing the hypothesis about finite moments of order 3 and proved the following:

1. Theorem *Let $W_1, W_2, \ldots$ be independent random variables with mean values zero. A sufficient condition that the sum variables*

$$\chi_n = \frac{W_1 + \ldots + W_n}{\sqrt{n}}$$

converge to a normal distribution with variance σ is that the following three conditions hold:

(1) $$\lim_{n\to\infty} \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\sqrt{n}} = \sigma^2$$

(2) There exists a constant M such that $\sigma_\nu \leq M$ for all ν

(3) $$\lim_{n\to\infty} \frac{\sum_{\nu=1}^{\nu=n} \int_{|x|>\delta\sqrt{n}} x^2 \cdot dW_\nu(x)}{n} = 0 \quad \text{hold for all} \quad \delta > 0$$

Remark. Condition (3) means that the individual random variables do not have fat tails.

Proof of Theorem 1

Let us first assume that each W_ν is a finite discrete random variable given by a finite sum $\sum p_j(\nu) \cdot \delta_{x_j(\nu)}$. Let $\mathcal{C}_\nu(\xi)$ be its characteristic function:

(*) $$\mathcal{C}_\nu(\xi) = \sum p_j(\nu) \cdot e^{ix_j(\nu)\xi}$$

We will show that

(**) $$\lim_{n\to\infty} \prod_{\nu=1}^{\nu=n} \mathcal{C}_\nu(\xi/\sqrt{n}) \to e^{-\xi^2\sigma^2/2}$$

holds uniformly when ξ varies in a compact interval $[-A, A]$. Keeping $A > 0$ fixed we take $\delta > 0$ such that

(1) $$A \cdot \delta \leq 1$$

Next, for each $n \geq 1$ we set

(2) $$\Psi_\nu(n) = \sum_{|x_j(\nu)|\geq\delta\sqrt{n}} p_j(\nu) x_j(\nu)^2$$

Notice that Lindeberg's 3rd condition gives

(3) $$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{\nu=1}^{\nu=n} \Psi_\nu(n) = 0$$

With the notations above the crucial step is the following:

2. Lemma. *For each ν one has the inequality*

(2.1) $$\left|\mathcal{C}_\nu(\xi/\sqrt{n}) - 1 + \frac{\xi^2}{2n} \cdot \sigma_\nu^2\right| \leq \frac{3\Psi_\nu}{\delta^2 \cdot n} + \frac{A^2\Psi_\nu}{n} + \frac{A^3 M\delta}{n}$$

We prove (2.1) in (xx) and show first how (2.1) gives (**) in a uniform way. Denote the right hand side in (2.1) by $\rho(\nu, \delta, n)$. Keeping δ fixed we find an integer N such that

(i) $$n \geq N \implies \frac{A^2 M^2}{2n} + \rho(\nu, \delta, n) \leq \frac{1}{2}$$

Next, for every complex number α with absolute value $\leq 1/2$ one has the inequality:

(ii) $$|\mathrm{Log}(1 + \alpha) - \alpha| \leq |\alpha|^2$$

Now $\sigma_\nu^2 \leq M^2$ for each ν and when $|\xi| \leq A$ it follows from (2.1) and (i-ii):

(iii) $\big|\log(\mathcal{C}_\nu(\xi/\sqrt{n}) + \frac{\xi^2}{2n} \cdot \sigma_\nu^2 \big| \leq \rho(\nu,\delta,n) + \big(\frac{A^2M^2}{n} + \rho(\nu,\delta,n)\big)^2 \quad : \ \nu = 1,2,\ldots$

Set

(iv) $$\rho^*(\nu,\delta,n) = \rho(\nu,\delta,n) + \big(\frac{A^2M^2}{n} + \rho(\nu,\delta,n)\big)^2$$

Taking a sum in (iii) we get

(v) $$\big|\sum_{\nu=1}^{\nu=n} \log(\mathcal{C}_\nu(\xi/\sqrt{n}) + \frac{\xi^2}{2} \cdot \frac{\sigma_1^2 + \ldots + \sigma_n^2}{n}\big| \leq \sum_{\nu=1}^{\nu=n} \rho^*(\nu,\delta,n)$$

We want a uniform limit:

(vi) $$\lim_{n\to\infty} \sum_{\nu=1}^{\nu=n} \log(\mathcal{C}_\nu(\xi/\sqrt{n}) = -\frac{\xi^2 \cdot \sigma^2}{2} \quad : \ -A \leq \xi \leq A$$

From (1) in Theorem 1 and (v) this follows if if

(vii) $$\lim_{n\to\infty} \sum_{\nu=1}^{\nu=n} \rho^*(\nu,\delta,n) = 0$$

To prove (vii) we take some $\epsilon > 0$ and choose δ so small that in addition to the previously imposed inequality $A \cdot \delta \leq 1$ we also have

(viii) $$A^3M\delta < \epsilon/2$$

With this choice of δ we get via the right hand side in (2.1):

$$\sum_{\nu=1}^{\nu=n} \rho(\nu,\delta,n) \leq (\frac{3}{\delta^2} + A) \cdot \frac{\Psi_1 + \ldots + \Psi_n}{n} + \epsilon/2$$

By (3) above Lemma 2.1 the first term in the right hand side above tends to zero with n and since ϵ can be made arbitrary small we get

(ix) $$\lim_{n\to\infty} \sum_{\nu=1}^{\nu=n} \rho(\nu,\delta,n) = 0$$

The verification that (ix) gives (viii) is left to the reader.

Remark. The proof above has shown that the rate of convergence with respect to n in (vi) only depends upon the conditions imposed in Lindeberg's theorem. From this the reader can conclude that the hypothesis that each single random variable has a finite discrete distribution is redundant. Hence Lemma 2.1 and the above proves Theorem 1.

Proof of Lemma 2.

For a given ν we obtain

(1) $$\mathcal{C}_\nu(\xi/\sqrt{n}) = \Sigma_*\, p_j(\nu)e^{ix_j(\nu)\xi} + \Sigma^*\, p_j(\nu)e^{ix_j(\nu)\xi}$$

where Σ_* denotes summation over those j where $|x_j(\nu)| < \delta\sqrt{n}$ and Σ^* the sum when $|x_j(\nu)| \geq \delta\sqrt{n}$. We begin to study Σ_*. Since $|x_j(\nu)| \cdot \xi/\sqrt{n}| \leq \delta \cdot A \leq 1$ hold under Σ_*, the Taylor estimate in XX applied to each j gives:

$$\Sigma_*\, p_j(\nu)e^{ix_j(\nu)\xi} = \Sigma_*\, p_j(\nu) + \frac{i\xi}{\sqrt{n}}\Sigma_*\, p_j(\nu)x_j(\nu) - \frac{\xi^2}{2n} \cdot \Sigma_*\, p_j(\nu)x_j^2(\nu) + \text{error}(\nu)$$

where

(2) $$|\text{error}(\nu)| \leq \frac{|\xi|^3}{n\sqrt{n}} \cdot \Sigma_*\, p_j(\nu)|x_j(\nu)|^3 \leq \frac{A^3\delta\sqrt{n}}{n\sqrt{n}} \cdot \Sigma_*\, p_j(\nu)x_j(\nu)^2 \leq \frac{A^3M\delta}{n}$$

where the last inequality holds since M is the uniform upper bound of the variances of the random variables $\{W_\nu\}$. Next, we use that

$$(3) \qquad \Sigma_* \, p_j(\nu) = 1 - \sum^* p_j(\nu) \quad \text{and} \quad \sum_* p_j(\nu)x_j(\nu) = -\Sigma^* \, p_j(\nu)x_j(\nu)$$

where the last equality follows since the mean value of W_ν is zero. Now (2-3) and the triangle inequality give:

$$\big|\Sigma_* \, p_j(\nu)e^{ix_j(\nu)\xi} - 1 + \frac{\xi^2}{2n} \cdot \Sigma_* \; p_j(\nu)x_j^2(\nu)\big| \leq$$

$$(4) \qquad \Sigma^* \, p_j(\nu) + \frac{|\xi|}{\sqrt{n}} \cdot \Sigma^* \, p_j(\nu)|x_j(\nu)| + \frac{A^3 M\delta}{n}$$

Since $|x_j(\nu)| \geq \delta\sqrt{n}$ holds under Σ^* we obtain

$$(5) \qquad \Sigma^* \, p_j(\nu) \leq \frac{1}{\delta^2 \cdot n} \cdot \Sigma^* \, p_j(\nu) \cdot x_j^2(\nu) = \frac{\Psi_\nu}{\delta^2 \cdot n}$$

We have also

$$(6) \qquad \Sigma^* \, p_j(\nu) \cdot |x_j(\nu)| \leq \frac{1}{\delta \cdot \sqrt{n}} \cdot \Sigma^* \, p_j(\nu) \cdot x_j^2(\nu) = \frac{\Psi_\nu}{\delta \cdot \sqrt{n}}$$

From (5-6) sum in (4) is estimated above by

$$(7) \qquad \frac{\Psi_\nu}{\delta^2 \cdot n} + \frac{|\xi|}{\sqrt{n}} \frac{\Psi_\nu}{\delta \cdot \sqrt{n}} + \frac{A^3 M\delta}{n} \leq \frac{2\Psi_\nu}{\delta^2 \cdot n} + \frac{A^3 M\delta}{n}$$

where the last inequality follows since $|\xi| \leq A$ and $\delta \cdot A \leq 1$. Next

$$(8) \qquad \frac{\xi^2}{2n} \cdot \Sigma_* \, p_j(\nu)x_j^2(\nu) = \frac{\xi^2}{2n} \cdot \sigma_\nu^2 - \frac{\xi^2}{2n} \cdot \Sigma^* \, p_j(\nu)x_j^2(\nu) = \frac{\xi^2}{2n} \cdot \sigma_\nu^2 - \frac{\xi^2}{2n} \cdot \Psi_\nu$$

Now we estimate Σ^* from (1). Here

$$(9) \qquad \big|\Sigma^* \, p_j(\nu)e^{ix_j(\nu)\xi}\big| \leq \Sigma^* \, p_j(\nu) \leq \frac{\Psi_\nu}{\delta^2 \cdot n}$$

We have also

$$(10) \qquad \frac{\xi^2}{n} \cdot \Sigma^* p_j(\nu)x_j^2(\nu) \leq \frac{A^2 \cdot \Psi_\nu}{n}$$

Finally, (8-10) together with (4-7) and the triangle inequality give the requested estimate:

$$(11) \qquad \big|\mathcal{C}_\nu(\xi/\sqrt{n}) - 1 + \frac{\xi^2}{2n} \cdot \sigma_\nu^2\big| \leq \frac{3\Psi_\nu}{\delta^2 \cdot n} + \frac{A^2\Psi_\nu}{n} + \frac{A^3 M\delta}{n}$$

19. Poisson's summation formula.

Let μ be a distribution on the ξ-line with compact support in the open interval $(-\pi, \pi)$. Here $\mu = \widehat{\phi}$ where the tempered distribution ϕ on the x-line extends to an entire function of the complex variable $z = x + iy$. For example, if μ is a Riesz measure we have

$$\phi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\xi} \cdot d\mu(\xi) \tag{1}$$

It turns out that $\phi(x)$ is determined by its value taken on the set of integers. To see this we introduce the ordinary Fourier coefficients

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\xi} \cdot d\mu(\xi) \tag{2}$$

Now μ is recovered via Fourier's inversion formula on the periodic interval $(-\pi, \pi)$:

$$\mu = \sum c_n \cdot e^{in\xi} \tag{3}$$

At the same time (1) gives the equalities

$$c_n = \phi(-n)$$

for all integers n. This means that we formally can write

$$\phi(x) = \sum_{n \in \mathbf{Z}} \phi(-n) \cdot \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} e^{i(x+n)\xi} \cdot d\xi$$

Evaluating the integrals we obtain

$$\phi(x) = \sum_{n \in \mathbf{Z}} \phi(-n) \cdot \frac{\sin\, \pi(x+n)}{\pi} \tag{*}$$

Under the condition that μ is smooth, say a density given by a twice continuously differentiable function one has $\phi(-n) = O(|n|^{-2})$ and the series in the right hand side of (*) converges pointwise. Taking integrals we obtain:

$$\int_{-\infty}^{\infty} \phi(x) \cdot dx = \sum_{n \in \mathbf{Z}} \phi(-n) \cdot \int_{-\infty}^{\infty} \frac{\sin\, \pi(x+n)}{\pi}$$

Notice that

$$\int_{-\infty}^{\infty} \frac{\sin\, \pi(x+n)}{\pi} = 1 \quad \text{hold for every integer} \quad n$$

Hence we get the equality

$$\int_{-\infty}^{\infty} \phi(x) \cdot dx = \sum_{n \in \mathbf{Z}} \phi(-n) \tag{**}$$

This is no surprise since Fourier's non-periodic inversion formula gives

$$\int_{-\infty}^{\infty} \phi(x) \cdot dx = \mu(0) = \sum c_n = \sum \phi(-n)$$

Remark. One refers to (**) as Poisson's summation formula which was established by Poisson after Fourier had defined Fourier series in the periodic case.

19.1 Remark. If we relax the assumption and only assume that μ has compact support on the closed interval $[-\pi, \pi]$ then ϕ need not be determined by its restriction to $\mathbf{Z}$. A counter-example is when μ is the difference of the Dirac measures at π and $-\pi$ for then

$$\phi(x) = \frac{\sin \pi x}{i\pi}$$

and here the sine-function vanishes on the set of integers. A more refined question arises if we suppose that $\mu(\xi)$ is a continuous density where this continuous function is zero at the two end-points π and $-\pi$. Assume that and suppose also that the inverse Fourier transform ϕ vanishes on $\mathbf{Z}$. This gives the entire function

$$\psi(z) = \frac{\phi(z)}{\sin \pi z}$$

Since ψ is a quotient of two entire functions of exponential type Lindelöf's division theorem in § XX shows that ψ also belongs to the class $\mathcal{E}$. We can consider its restriction to the imaginary axis. With $z = iy$ and $y >> 0$ the absolute value $|\sin(\pi iy)| \simeq \frac{e^{\pi|y|}}{2}$. At the same time

$$|\phi(iy)| \leq \int_0^\pi e^{|y|\cdot\xi} \cdot |u(\xi)| \cdot d\xi$$

From this it is clear that ψ is bounded on the imaginary axis. So if ψ is not identically zero it follows from the Carleman formula in § xx that

$$\int_{-\infty}^\infty \log^+ \frac{1}{|\psi(iy)|} \cdot \frac{dy}{1+y^2} < \infty$$

From this one can analyze how fast $u(\xi)$ tends to zero as $\xi \to \pi$ and find conditions in order that ψ cannot be identically zero, i.e. when

$$\lim_{\xi \to \pi} u(\xi) = 0$$

holds sufficiently rapidly, then its Fourier transform is determined by its values on the integers. This leads to an involved analysis and we shall not try tenter a more detailed discussion.

19.2 A class of smoothing functions. In numerical analysis one often employs ϕ-functions via inverse Fourier transforms of even functions $\mu(\xi)$ which are supported by $[-\pi, \pi]$ and have derivatives up to some order k. In particular we can take such μ-functions where $\mu(0) = 0$ while the derivatives

$$\partial_\xi^j(\mu)(0) = 0 \quad : \quad 1 \leq j \leq k$$

At the end-points π and $-\pi$ we impose the condition that both μ and all derivatives up to order k are zero. Since μ is even we get:

$$\phi(x) = \frac{1}{\pi} \int_0^\pi \cos(x\xi) \cdot \mu(\xi) \cdot d\xi$$

Next, when μ is of class C^k we get a constant C such that decay condition:

$$|\phi(x)| \leq C \cdot (1 + |x|)^{-k} \tag{1}$$

holds on the x-line. But ϕ may have a rather slow decay. To see this we consider L^2-integrals. Parseval's equality gives for each $0 \leq p \leq k$ the equality:

$$\int_{-\infty}^\infty |x|^{2p} \cdot |\phi(x)|^2 \cdot dx = c_* \cdot \int_0^\pi |\mu^{(p)}(\xi)|^2 \cdot d\xi$$

In the right hand side we encounter L^2-integrals which appear in Carleman's inequality from § 8. Let us put

$$\gamma_p(\phi) = \Big| \int_{-\infty}^\infty |x|^{2p} \cdot |\phi(x) \cdot dx \Big|^{\frac{1}{2p}} \quad : \quad 0 \leq p \leq k$$

Then Theorem 1 from § xx gives the inequality

$$\sum_{p=1}^{p=k} \frac{1}{\gamma_p(\phi)} \leq C \tag{*}$$

for an absolute constant C. This means that it is not possible to obtain small L^2-integrals for all $1 \leq p \leq k$ while k increases. In other words, the *a priori inequality* (*) puts a constraint when one trues to exhibit ϕ-functions which have a good decay as $|x| \to \infty$ via (1) above and at the same time do not increase too much before this good decay begins to be uniformly effective.

Remark. One reason why it is of interest in numerical investigations to construct ϕ-functions as above stems from Poisson's formula which entails that via dilations where ϕ is replaced by $\delta \cdot \phi(x/\delta)$ for $\delta > 0$, it follows that certain discrete moment conditions hold up to order k which can be used to approximate functions $f(x)$ with bounded derivatives up to order k on the real x-line from its values on a discrete grid given by integer multiples of some positive number. We shall not pursue this any further since it would lead to an extensive discussion related to numerical analysis. The interested reader can consult [Zahedi: Chapter 3] for some theoretically interesting material about approximations by delta functions adapted for delicate numerical investigations.

20. The heat equation and integral formulas.

Introduction. The subsequent material stems from Beurling's lectures about quasi.analytic functions. See [Beurling: Collected work xx]. In § 0.X from the introduction we recalled some facts about the heat-equation. Here shall study a restricted class of solutions which is referred to as the logarithmic class in [ibid]. The major result appears in § xx which gives an integral fornula for such solutions to the heat equation. The construction of the kernel function for this integral formula requires considerable work which involves properties of analytic functions in the half-space $\mathfrak{Re}|, z > 0$ defined by

$$f(z) = \int_0^\infty t^{z-1}\, d\mu(t)$$

where μ is a non-negative Riesz measure. So we begin with a study of such functions and not until § 3.x we begin to consider the heat equation.

The class $\mathcal{P}$.

Let μ be a non-negative measure on the non-negative real t-line such that $a > -1$ gives

$$\int_0^\infty t^a \cdot |d\mu(t)| < \infty \tag{1}$$

This yields an analytic function $f(z)$ in the half-plane $\mathfrak{Re}\, z > 0$ defined by

$$f(z) = \int_0^\infty t^{z-1}\, d\mu(t) \tag{2}$$

The class of γ-functions which arise in this way is denoted by $\mathcal{P}$ and called positive definite analytic functions in the right half-plane.

Exercise. Show that if μ and ν is a pair of non-negative measures which satisfy (1) so does the convolution $\mu * \nu$. Conclude that a product of two $\mathcal{P}$-functions again belong to $\mathcal{P}$ and use this to show that when $f \in \mathcal{P}$ then the exponential sum below also belongs to $\mathcal{P}$:

$$e^{f(z)} = 1 + \sum_{n=1}^\infty \frac{\gamma(z)^n}{n!}$$

Now we construct a class of functions in $\mathcal{P}$:

20.1. Proposition. *Let μ be a positive measure on $\{t \geq 0\}$ such that*

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty$$

Then the analytic function below belongs to $\mathcal{P}$ for all pairs of real numbers a, B:

$$f(z) = \exp\big(Bz + (z-a)^2 \cdot \int_0^\infty \frac{d\mu(t)}{z+t}\big)$$

Proof. It is clear that f is analytic in the right half-plane. By the observations in the exercise about the class $\mathcal{P}$ and approximating μ weakly by finite sums of discrete point-masses it suffices to show that

$$\exp\big(Bz + (z-a)^2 \cdot \frac{b}{z+t}\big) \in \mathcal{P} \tag{i}$$

when b and t are positive real numbers. To get (i) we write

$$\frac{(z-a)^2}{z+t} = z + t + \frac{(t-a)^2}{z+t} - 2(a+t)z \tag{ii}$$

Since $e^{cz} \in \mathcal{P}$ for every real constant c and $\mathcal{P}$ is stable under products, there only remains to verify that

$$\text{(iii)} \qquad \exp\big(\frac{b(t-a)^2}{z+t}\big) \in \mathcal{P}$$

To prove (iii) we set $k = b(t-a)^2$ and notice that

$$\frac{k}{z+t} = k \cdot \int_0^t s^{z-1} \cdot s^t \cdot ds$$

Then (iii) follows after we have taken the exponential sum

$$\exp\big(\frac{k}{z+t}\big) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{k^n}{(z+t)^n}$$

20.2. The function $\gamma(z)$.

Let e be Neper's constant. Removing the negative interval $(-\infty, -e]$ from the complex plane we have a single valued branch of $\log(z+e)$ and get the analytic function

$$\gamma(z) = [\log(e+z)]^z$$

Exercise. Set

$$\mathcal{G}(t) = \arctan \frac{\pi}{\log(t-a)} \quad : t > e$$

Apply residue calculus to prove the equality

$$\frac{\log\log(z+e)}{z} = \frac{1}{e} + \frac{1}{\pi}\int_e^{\infty} \frac{\mathcal{G}(t)}{t(t+z)}\, dt + \int_{e-1}^{e} \frac{dt}{t(t+z)}$$

Deduce from this that

$$\text{(i)} \qquad \gamma(z+1/2) = \exp\big[A + Bz\frac{z^2}{\pi} \cdot \frac{1}{\pi}\int_{e-1/2}^{\infty} \frac{\Theta(t)(t-1/2)}{t+z)}\, dt\big]$$

where $\Theta(t) = \pi$ in the interval $[e-1/2, e+1/2]$ and after it is non-increasing and tends to zero as $t \to +\infty$.

Next, Euler's exponential formula for the Γ-function from the introduction in these notes, together with (i) above and Proposition 20.1 it follows that the function

$$(*) \qquad f(z) = \frac{\Gamma(z)}{\gamma(z+1/2)} \in \mathcal{P}$$

Remark. The expression of the non-negative measure μ for which

$$f(z) = \int_0^{\infty} t^{z-1}\, d\mu(t)$$

is rather involved. The reader is invited to find μ numerically with a computer.

20.3 The heat equation and integral formulas.

Consider the function $\gamma(z)$ above. With $a > 0$ real and positive we get the function

$$y \mapsto \gamma(a+iy)$$

Notice that

$$\lim_{y \to +\infty} \log(a+iy) = \frac{\pi}{2} \cdot i$$

Passing to the double log-function we have

$$\lim_{y \to +\infty} \log\, \log(a+iy) = \log\frac{\pi}{2} + \frac{\pi}{2} \cdot i$$

where we used that $\log i = \frac{\pi}{2} \cdot i$. It follows that the absolute value

$$|\gamma(a+iy)| \simeq e^{-\pi y/2} \quad : \ y \to +\infty$$

If y instead tends to $-\infty$ we use that $\log -i = -\frac{\pi}{2} \cdot i$ and find that

$$|\gamma(a+iy)| \simeq e^{-\pi |y|/2} \quad : \ y \to -\infty$$

Hence the integral

$$(*) \qquad \int_{-\infty}^{\infty} \gamma(a+iy) \cdot t^{-a-iy}\, dy$$

converges for every real and positive t.

3.1 Exercise. Show that (*) is independent of a and set

$$(3.1.1) \quad K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(a+iy) \cdot t^{-a-iy}\, dy = \frac{1}{2\pi i} \int_{\mathfrak{Re}\, z=a} t^{-z} \gamma(z)\, dz \quad : \ t > 0$$

It follows that

$$(3.1.2) \qquad K(t) \leq \inf_{x>0} t^{-x} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\gamma(x+iy)|\, dy$$

Show that (3.1.1)gives a function $\delta(t)$ which tends to zero as $t \to +\infty$ and

$$(3.1.3) \qquad K(t) \leq e^{-e^{t-\delta(t)}}$$

which means that $K(t)$ decreases very rapidly as $t \to +\infty$. Finally, show that Mellin's inversion formula gives:

$$(3.1.4) \qquad \gamma(z) = \int_0^{\infty} t^{z-1}\, K(t)\, dt$$

From § 2 we already know that γ is positive definite and hence $K(t)$ is real-valued and non-negative.

3.2 The W-function. Let $\Gamma(z)$ be the ordinary gamma-function and set

$$f(z) = \frac{\Gamma(z)}{\gamma(z+1/2)}$$

Show via Stirling's formula that if $-\pi/2 < \theta < \pi/2$ then

$$(3.2.1) \qquad \log\, f(re^{i\theta})| = \quad \text{Check (23) on page 422}$$

Conclude that there exists an analytic function in the half-plane $\mathfrak{Re}\, z > 0$ defined by

$$(3.2.2) \qquad W(z) = \frac{1}{2\pi i} \cdot \int_{\mathfrak{Re}\zeta=a} f(\zeta)\, z^{-\zeta} \quad : a > 0$$

Moreover, show that there exists some $\alpha > 0$ and $x_0 > 0$ such that

$$(3.2.3) \qquad x > x_0 \implies W(x) \leq e^{-\alpha x \log x}$$

Finally, apply the result in § 2 to conclude that $W(x) \geq 0$ on the non-negtive real x-line.

3.3 A harder exercise. Show that $W(z)$ extends to an entire function in the whole complex z-plane. The hint is to change the contour in (3.2.2) above when z moves into the left half-plane.

3.4 An integral equation. Using (3.1.4) and by studying the Mellin transform one has the equation

$$\int_0^\infty W(\frac{x}{t})\cdot\frac{K(t)}{\sqrt{t}}\,dt = e^{-x} \tag{3.4.1}$$

See page 422 for details of the proof.

3.5 The logarithmic class. Given a postive real numbers $A > 0$ we consider solutions $u(x,y)$ defined in a domain $\Omega = \{-\infty < x < +\infty\} \times \{-A < y < A\}$ to the heat equation

$$\partial_x^2(u) = \partial_y(u)$$

Denote by $\mathcal{L}$ the class of such solutions for which there exists a function $\rho(r)$ which tends to zero as $r \to +\infty$ and

$$|u(x,y)| \le e^{\rho(|x|)x^2\cdot\log|x|}$$

holds in Ω.

3.6 Theorem. *Each $u \in \mathcal{L}$ is determined by its values on the vertical line $\{y = 0\}$. Moreover there exists an integral representation of $u(x,y)$ for $0 < y < A$ via the function $x \mapsto u(x,0)$.*

To achieve this result we use the W-function above and define another function $V(x,y)$ by

$$V(x,y) = \frac{W(\frac{x^2}{4y})}{\sqrt{y}}$$

The estimate (3.2.3) entails that if u belongs to $\mathcal{L}$ then the convolution integral

$$\frac{1}{2\sqrt{\pi}}\cdot\int_{-\infty}^\infty V(x-\xi,y)\cdot u(\xi)\,d\xi$$

is absolutely convergent for each $y > 0$ and defines a function denoted by $v(x,y)$. With these notations the assetions in the theorem above follow from the follonig

3.7 An integral formula. With the notations as above one has

$$u(x,y) = \int_0^\infty v(x,yt)\cdot K(t)\,dt \tag{3.7.1}$$

where the integral in the right hand side converges absolutely for each $0 \le y < A$.

3.8 Remark. The proof of (3.7.1) is given in § xx below. It uses of course the assumption that u satisfies the heat equation and the reason why (3.7.1) holds stems from the equation

$$U(x,y) = \int_0^\infty W(\frac{x^2}{4yt})\cdot\frac{K(t)}{\sqrt{t}}\,dt$$

where U is the ordinary heat kernel

$$U(x,y) = e^{-x^2/y} \quad : y > 0$$

while $U(x,y) = 0$ when $y \le 0$.

21. Calculations of some Fourier transforms.

1. Inverse Fourier transforms of ξ_+^s. Let s be complex with $\mathfrak{Re}\, s > -1$. On the ξ-line we get the tempered distribution defined by

$$g \mapsto \int_0^\infty \xi^s \cdot g(\xi)\, d\xi \tag{i}$$

The inverse Foueier transform is the boundary value of an analytic function in $\mathfrak{Im}\, z > 0$. To find it we consider the analytic function in the upper half-plane defined by

$$\phi(z) = \frac{1}{2\pi} \cdot \int_0^\infty e^{iz\xi} \cdot \xi^s\, d\xi \tag{ii}$$

Taking the derivative with respect to z gives

$$\phi'(z) = \frac{i}{2\pi} \int_0^\infty e^{iz\xi} \cdot \xi^{s+1}\, d\xi$$

A partial integration with respect to ξ identifies the right hand side with

$$-\frac{i}{2\pi} \cdot \frac{1}{iz}(s+1) \int_0^\infty e^{iz\xi} \cdot \xi^s\, d\xi$$

It follows that

$$z \cdot \phi'(z) = -(s+1) \cdot \phi(z)$$

So if $\nabla = z \cdot \frac{\partial}{\partial z}$ is the Fuchsian differential operator then

$$\nabla(\phi) + (s+1)\phi = 0$$

holds in the upper half-plane U_+. The family of analytic functions in U_+ satisfying this equation are constants times z^{-s-1}. There remains to find the constant $c(s)$ such that ξ_+^s is the Fourier transform of the boundary value distribution

$$\mu_{-s-1} = c(s) \cdot (x + i0)^{-s-1}$$

To get $c(s)$ we take $z = i$ in (ii) which gives

$$\phi(i) = \frac{1}{2\pi} \int_0^\infty e^{-\xi} \cdot \xi^s\, d\xi = \frac{\Gamma(s+1)}{2\pi}$$

Hence we should take

$$c(s) \cdot i^{-s-1} = \frac{\Gamma(s+1)}{2\pi}$$

Here

$$i^{-s-1} = e^{-(s+1)\log i} = e^{-(s+1)\pi i} \implies c(s) = e^{(s+1)\pi i} \cdot \frac{\Gamma(s+1)}{2\pi} \tag{*}$$

Summing up we have proved the following:

21.1Theorem. *The distribution-valued function ξ_+^s extends to a meromorphic function the whole complex s-plane and one has*

$$\xi_+^s = e^{(s+1)\pi i} \cdot \frac{\Gamma(s+1)}{2\pi} \cdot \widehat{(x+i0)}^{-s-1}$$

21.2. Taylor extensions. The Taylor extension of the density ξ^{-1} on $(0, +\infty)$ is the distribution defined by:

$$g \mapsto \int_0^1 \frac{g(\xi) - g(0)}{\xi}\, d\xi + \int_1^\infty \frac{g(\xi)}{\xi}\, d\xi \tag{2.1}$$

where $g(\xi)$ belong to the Schwartz class on the ξ-line. To find the inverse Fourier transform we consider the analytic funtion in the upper half-plane

$$\phi(z)=\frac{1}{2\pi}\cdot\Big[\int_0^1\frac{e^{iz\xi}-1}{\xi}\,d\xi+\int_1^\infty\frac{e^{iz\xi}}{\xi}\,d\xi\,\Big] \tag{2.2}$$

Taking the complex derivative we get

$$\phi'(z)=\frac{i}{2\pi}\cdot\int_0^1 e^{iz\xi}\,d\xi+\int_1^\infty e^{iz\xi}\,d\xi\,\Big]=\frac{i}{2\pi}\cdot\int_0^\infty e^{iz\xi}\,d\xi \tag{2.3}$$

The right hand side is i times the inverse Fourier transform of the Heaviside function $H_+(\xi)$ on the ξ-line. So if μ is the boundary value distribution of ϕ one has

$$\widehat{\partial_x(\mu)}=i\cdot H_+(\xi) \tag{2.4}$$

The interchange formula under Fourier transforms gives

$$\widehat{\partial_x(\mu)}=i\xi\cdot\widehat{\mu}=i\cdot H_+(\xi)$$

Hence $\widehat{\mu}$ is expressed by the density $\frac{1}{\xi}$. There remains to determine the ϕ-function in (2.2). Performing a partial integration of the integral in the right hand side of (2.3) while $\mathfrak{Im}\,z>0$ gives:

$$\phi'(z)=-\frac{i}{2\pi}\cdot\frac{1}{iz}=-\frac{2\pi}{z} \tag{2.5}$$

In the upper half-plane the functions which satisfy this differential equations are of the form

$$-2\pi\cdot\log z+C$$

where C is some constant. The complex log-function has a boundary value distribution and hence the inverse Fourier transform of Taylor's extension of $\frac{1}{\xi}$ is equal to

$$\mu=-2\pi\log{(x+i0)}+C \tag{2.6}$$

where it remains to determine the constant C. To find C we take $z=i$ and notice that

$$\phi(i)=\phi(z)=\frac{1}{2\pi}\cdot\Big[\int_0^1\frac{e^{-\xi}-1}{\xi}\,d\xi+\int_1^\infty\frac{e^{-\xi}}{\xi}\,d\xi\,\Big]$$

is a real number. At the same time $\log i=\frac{\pi i}{2}$ is purely imaginary and must therefore be cancelled by C which gives:

$$\mu=-2\pi\log(x+i0)+2\pi\cdot\frac{\pi i}{2}+a$$

where a is the real number. To find a we consider the Γ-function. With $s>0$ and small one has

$$\Gamma(s)=\int_0^\infty e^{-\xi}\xi^{-1+s}\,d\xi=\int_0^1\frac{e^{-\xi}-1}{\xi^{1-s}}\,d\xi+\int_1^\infty e^{-\xi}\xi^{-1+s}\,d\xi+\int_0^1\xi^{-1+s}\,d\xi$$

The last integral is equal to s^{-1}. Hence $\Gamma(1+s)-\frac{1}{s}$ is equal to the sum of the first teo intergals. When $s\to 0$ it is clear that this sum converges to $2\pi a$ and which therefore is the constant term at $s=0$ of $\Gamma(1+s)-\frac{1}{s}$. The functional equation for the Γ-function from § XX Gamma entails that $a=0$. Hence we have proved

3. Theorem. *The inverse Fourier transform of Taylor's extension of ξ^{-1} from (2.1) is given by the distribution*

$$-2\pi\log{(x+i0)}+\pi^2 i$$

4. The distribution $\mathcal{T}\xi_+^{-2}$. It is the distribution defined by

$$g \mapsto \int_0^1 \frac{g(\xi) - g'(0)\cdot\xi - g(0)}{\xi^2}\, d\xi + \int_1^\infty \frac{g(\xi)}{\xi^2}\, d\xi \tag{4.1}$$

To find the inverse Fourier transform we consider the function

$$\psi(z) = \frac{1}{2\pi}\cdot \int_0^1 \frac{e^{iz\xi} - iz\xi - 1}{\xi^2}\, d\xi + \int_1^\infty \frac{e^{iz\xi}}{\xi^2}\, d\xi \tag{4.2}$$

It follows that

$$\psi'(z) = \frac{1}{2\pi}\cdot \int_0^1 \frac{i\xi\cdot e^{iz\xi} - i\xi}{\xi^2}\, d\xi + \int_1^\infty \frac{i\xi\cdot e^{iz\xi}}{\xi^2}\, d\xi \tag{4.3}$$

Hence we have

$$\begin{gathered}\psi'(z) = i\cdot\phi(z) = -2\pi i\cdot\log z - \pi^2 \implies \\ \psi(z) = -2\pi i\cdot z\log\, z + (2\pi i - \pi^2)z + C\end{gathered} \tag{4.4}$$

for some constant C As in § 3 we notice that $\psi(i)$ is real which implies that $-\pi^2 i + C$ must be real and hence

$$\psi(z) = -2\pi i\cdot z\log\, z + (2\pi i - \pi^2)z + \pi^2 i + a \quad : a \in \mathbf{R}$$

With $z = i$ we find that

$$\psi(i) = \pi^2 - 2\pi + a$$

Hence a is determined when we have evaluated the integral (4.2) with $z = i$.

Exercise. FIND a !

5. The general case. For every $n \geq 1$ we have the distribution $\mathcal{T}\xi_+^{-n}$ obtained via Taylor's extension of the density ξ^{-n} on $(0, +\infty)$ Its inverse Fourier transform μ_n is the boundary value of an analytic function in the upper half-plane. The following conclusive result holds:

Theorem. *For each positive integer n the inverse Fourier transform of $\mathcal{T}\xi_+^{-n}$ is the constant term at $s = -n$ of the meromorphic distribution-valued function*

$$s \mapsto e^{(s+1)\pi i}\cdot \frac{\Gamma(s+1)}{2\pi}\cdot (x + i0)^{-s-1}$$

Exercise. Prove this result.

22. Boundary values in two variables.

With two real variables (x_1, x_2) one considers the 2-dimensional complex space (z_1, z_2) with $z_k = x_k + iy_k$. In the real y-space we consider an open truncated cone:

$$\mathcal{K} = \{(y_1, y_2) \quad y_2 > M \cdot |y_1|\} \cap \{0 < y_1^2 + y_2^2 < \delta^2\}$$

where δ, M are positive constants. Put

$$\square = \{(x_1, x_2 \quad -1 < x_1, x_2 < 1\}$$

The open set $\mathcal{T} = \square + i \cdot \mathcal{K}$ in $\mathbf{C}^2$ is called a truncated tube. Let $f(z_1, z_2)$ be a bounded analytic function in $\mathcal{T}$. To each point $(y_1^*, y_2^*) \in \mathcal{K}$ the 1-dimensional results show that if $g(x_1, x_2)$ is a test-function in $\square$ then there exists a limit

$$(*) \qquad \lim_{\epsilon \to 0} \int_\square g(x_1, x_2) \cdot f(x_1 + i\epsilon y_1^*, x_2 + i\epsilon y_2^*) \cdot dx_1 dx_2$$

Moreover, (*) is independent of the point $y^* = (y_1^*, y_2^*)$ and we obtain a distribution in $\square$ denoted by $\mathbf{b}f$. Since f is bounded the resulting distribution has order zero, i.e. expressed by a Riesz measure μ_f in $\square$. The reader may notice the close interplay to the 1-dimensional case and we remark that the existence of a limit distribution can be established by regarding Fatou limits. Next, we can relax the assumption that f is bounded. Assume only that there is some integer $m \geq 1$ such that

$$(*) \qquad |f(x_1 + iy_1, x_2 + iy_2)| \leq C.(y_1^2 + y_2^2)^{-m/2}$$

hold for some constant C_m and all points in the tube. To get the distribution $\mathbf{b}f$ one employs small $\bar\partial$-extension of test-functions g. If $N \geq 1$ we set

$$G_N(x + iy) = \sum i^{k+j} \cdot \frac{\partial^{k+j}(g)}{\partial^k x_1 \partial^j x_2}(x) \cdot \frac{y_1^k \cdot y_2^j}{k! \cdot j!}$$

Using Stokes Theorem one verifies that when (*) above holds, then there exists a distribution $\mathbf{b}_f$ defined on test-functions $g(x)$ by

$$(**) \qquad \mathbf{b}_f(g) = \lim_{\epsilon \to 0} \int_\square G_m(x_1, x_2) \cdot f(x_1 + i\epsilon y_1^*, x_2 + i\epsilon y_2^*) \cdot dx_1 dx_2$$

where the limit does not depend upon the chosen point $y^* \in \mathcal{K}$. The conclusion is that if $\mathcal{O}(\mathcal{T})_{\text{temp}}$ denotes the space of all analytic functions $f(z)$ in the tube satisfying (**) above for some integer m, then $f \mapsto \mathbf{b}_f$ yields a linear map to a space of distributions in $\square$.

2.10 The Schwarz reflection in two variables. Let $\mathcal{K}^*$ be the opposed cone defined by

$$K^* = \{(y_1, y_2) \quad y_2 < -M \cdot |y_1|\} \cap \{0 < y_1^2 + y_2^2 < \delta 2\}$$

If $\phi(z)$ is a tempered analytic function in the corresponding truncated tube domain T^*) we obtain the distribution $\mathbf{b}_\phi$. Suppose we have an equality of distributions:

$$(1) \qquad \mathbf{b}_\phi = \mathbf{b}_f$$

Then there exists an analytic function $\Psi(z)$ defined in a complex neighborhood of the real square $\square$ whose restriction to $\mathcal{T}$ is f while $\Psi|\mathcal{T}^* = g$. A proof can be established by scrutinizing the 1-dimensional case carefully. But the efficient and easiest method is to consider the Fourier transforms of the two distributions $\mathbf{b}_f$ and $\mathbf{b}_\phi$ which entails that the common distribution in (1) has an empty analytic wave front set and is therefore defined by a real-analytic density on $\square$ which extends to be holomorphic in a small complex neighborhood and yields the complex analytic extension of the pair $f(z)$ and $\phi(z)$. Details about this procedure which extends to every any dimension $n \geq 2$ is given in [Björk; Chapter 8].

2.9 Analytic wave front sets. The result in § 2.10 is often referred to as the Edge of the Wedge theorem. As indicated above the proof relies upon the notion of analytic wave front sets of distributions which has been introduced independently by Hörmander and Sato. In Sato's theory the construction of analytic wave front sets is expressed by their representations as boundary values of analytic functions. Hörmander considered decay properties of Fourier transforms in cones with varying directions. Apart from the article [Hö:xx] the reader may consult Chapter X in [Hörmander] or the material in [Björk:xx: page. xxx-xx] for an account. about analytic wave front sets from which one easily deduces Schwarz' reflection theorem in two variables.

23. The Radon transform

In the article [Radon] from 1917 Johann Radon established an inversion formula which recaptures a test-function $f(x,y)$ in $\mathbf{R}^2$ via integrals over lines in the (x,y)-plane parametrized by pairs (p,α), where $p\in\mathbf{R}$ and $0\leq\alpha<\pi$ give the line $\ell(p,\alpha)$:

$$t\mapsto\big(p\cdot\cos\alpha-t\cdot\sin\alpha\,,\,p\cdot\sin\alpha+t\cdot\cos\alpha\big)$$

The distance to the origin of a point on this line is equal to p^2+t^2 and the nearest point to the origin appears when $t=0$. With $z=x+iy$ we identify $\mathbf{R}^2$ with the complex plane. The Radon transform of a function f is defined for every $z\neq 0$ by

$$(*)\qquad \mathcal{R}_f(z)=|z|\cdot\int_{-\infty}^{\infty}f(z+izu)\,du$$

If $z=re^{i\alpha}$ we use the variable substitution $u\mapsto tr$ and the left hand side becomes

$$\mathcal{R}_f(z)=\int_{-\infty}^{\infty}f(z+ie^{i\alpha}\cdot t)\,dt$$

The unit vector $ie^{i\alpha}$ is $\perp$ to the vector z which means that $\mathcal{R}_f(z)$ evaluates the integral of f on the line which is $\perp$ to z and passes this point.

Since the absolute value $|1+iu|\geq 1$ for every real u, it is clear that if f has support in a disc of radius R centered at the origin then the same holds for the Radon transform. So $\mathcal{R}_f$ is a linear operator on the space of continuous functions with compact support.

Example. Let $f=\chi_D$ be the characteristic function of the unit disc. Then $\mathcal{R}_f(z)=\sqrt{1-|z|^2}$ when $|z|\leq 1$ and vanishes outside D.

When $z=x$ is real we notice that

$$(1)\qquad \mathcal{R}_f(x)=|x|\cdot\int_{-\infty}^{\infty}f(x+ixu)\,du=\int_{-\infty}^{\infty}f(x,s)\,ds$$

Consider the Fourier transform of $f(x,y)$ in the (ξ,η)-coordinates. From (1) we get

$$(2)\qquad \widehat{f}(\xi,0)=\int_0^{\infty}e^{-ix\xi}\,R_f(x)\,dx$$

When f has compact support its Fourier transform is a real-analytic function of (ξ,η) and hence $\widehat{f}(\xi,\eta)$ is determined by its restriction to $\{\eta=0\}$. So (2) shows that f is determined by the restriction of the Radon transform to the real line. In particular $\mathcal{R}$ is an injective operator.

1. Inversion formulas. Let us first consider the case when f is radial. So here $f(x,y)$ only depends on $r=\sqrt{x^2+y^2}$. When f is a radial function with compact support and of class C^2 we have

$$(i)\qquad \widehat{f}(\xi,0)=\int_{-\infty}^{\infty}e^{-ir\xi}\,f(r)\,dr$$

and Fourier's inversion formula for radial functions in § X gives

$$(ii)\qquad f(r)=\frac{1}{2\pi}\int e^{ir\xi}\,\widehat{f}(\xi,0)\cdot|\xi|\,d\xi$$

where the last integral is given as a limit:

$$(2)\qquad \lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}e^{ir\xi}\,\widehat{f}(\xi,0)\cdot|\xi|\,d\xi=\lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}e^{ir\xi}\cdot e^{-ix\xi}\,\mathcal{R}_f(x)\cdot|\xi|\,dxd\xi$$

For each $A > 0$ we define a function on the real u-line by

$$K_A(u) = \frac{1}{2\pi} \int_{-A}^{A} e^{iu\xi} \, |\xi| \, d\xi$$

Then (1-2) give

$$f(r) = \lim_{A \to \infty} \int K_A(r - x) \cdot \mathcal{R}_f(x) \, dx \tag{3}$$

Above $K(u)$ is an even function of u and a computation gives

$$K_A(u) = \frac{1}{\pi} \int_0^A \cos(u\tau) \, \tau \, d\tau = \frac{1}{\pi} \cdot \big(A \cdot \frac{\sin Au}{u} - \frac{1 - \cos Au}{u^2} \big) \tag{4}$$

Moreover, (3) can be written as:

$$f(r) = \lim_{A \to \infty} \int K_A(x) \cdot \mathcal{R}_f(r + x) \, dx \tag{5}$$

B. The general case. Above we treated radial functions. This covers the general case because the Radon transform commutes with rotations. More precisely, to each $e^{i\theta}$ we have the rotation operator sends a function $f(z)$ to

$$\mathfrak{r}_\theta(f)(z) = f(e^{i\theta} z)$$

Then it is clear from (*) that

$$\mathfrak{r}_\theta \circ \mathcal{R} = \mathcal{R} \circ \mathfrak{r}_\theta \quad \text{hold for all} \quad \theta \tag{**}$$

From (**) and (5) we obtain the general inversion formula:

1.1 Radon's Theorem. *For each $f \in C^0(\mathbf{R}^2)$ with compact support one has:*

$$f(x, y) = \lim_{A \to \infty} \frac{1}{2\pi} \int_0^\pi \big[\int_{-\infty}^{\infty} R_\alpha(x \cdot \cos \alpha + y \cdot \sin \alpha - u) \cdot K_A(u) du \big] \cdot d\alpha \tag{*}$$

Remark. The Radon transform has many applications. Especially in tomography. Here is a simple illustration: Let $f(x, y)$ be a positive function in some compact convex subset K of $\mathbf{R}^2$ which represents a distribution of mass. By evaluation integrals along different lines one gets information about the f-function and in this way Radon's inversion formula applies where an actual experiment in general consists of some finte family of such evaluations. Passing to the 3-dimensional case the radon transform of a given function $f(x, y, z)$ in $\mathbf{R}^3$ is defined in a similar fashion. More precisely, for each $0 \neq p \in \mathbf{R}^3$ we have the affine plane $\Pi^\perp(p)$ which passes through p and is $\perp$ to the vector p. Now

$$\mathcal{R}_f(p) = \int_{\Pi^\perp(p)} f(p + q) \, dA(q)$$

where $dA(q)$ is the area measure on $\Pi^\perp(p)$. Just as in the 2-dimensional case the $\mathcal{R}$-operator commutes with rotations and using this together with Fourier's inversion formula applied to f one gets Radon's inversion formula in $\mathbf{R}^3$ where one employs the the group of unitary transformations with determinant one and constructs a radial K-function which gives the inversion formula via a convolution. The reader is invited to carry out he details or consult text-books by Helgason which treat Radon transforms in great detail.

25. An extension of Runge's theorem.

Consider an open rectangle $\square = \{-a < x < a\} \times \{-b < y < b\}$ in the complex z-plane. Let $\omega(y)$ be a continuus function which is > 0 for $y \neq 0$ while $\omega(0) = 0$. Moreover, $y \to \omega(y)$ decreases as y tends to zero through positive or negative values. Denote by $C_\omega(\square)$ the space of complex-valued continuous functions $f(z)$ such that the product $\omega(y)f(z)$ is continuous on the closed rectangle $\overline{\square}$ and vanishes on the real interval $[-a, a]$. It becomes a Banach space under the norm:

$$||f|| = \max_{x+iy \in \square} \omega(y)|f(x+iy)|$$

Define

$$A_\omega(\square) = \{f \in C_\omega(\square) : \ f|\square \in \mathcal{O}(\square)\}$$

We have also the larger subspace $A_\omega^*(\square)$ of functions f in $C_\omega(\square)$ whose restrictions to the rectangles $\square_+$ and $\square_+$ both are analytic functions of z. It is clear that $A_\omega^*(\square)$ is a closed subspace of $C_\omega(\square)$, while the closedness of $A_\omega(\square)$ is not automatic. It turns out that properties of $A_\omega(\square)$ depend upon the ω-function.

Theorem. *Under the condition*

$$(*) \qquad \int_{-b}^{b} \log\log \omega(y)\, dy = -\infty$$

one has the equality $\overline{A_\omega^*(\square)} = A_\omega(\square)$. *If (*) is finite then* $A_\omega^*(\square)$ *is a closed proper subspace of* $A_\omega(\square)$.

The proof requires several steps and we begin with the first part, i.e. that (*) entails that $A_\omega^*(\square)$ is dense in $A_\omega(\square)$. To prove this we consider a Riesz measure μ supported by $\overline{\square}$ where $\omega(y) \cdot \mu$ is $\perp$ to $A_\omega^*(\square)$. It remains to show that

$$(1) \qquad \int f\, d\mu = 0 \quad : f \in A^*(\square$$

To achieve this we proceed as follows. For each complex number ζ we have $e^{i\zeta z} \in A_\omega^*(\square)$. It follows that

$$0 = \int_{y \geq 0} e^{i\zeta z} \cdot \omega(y)\, d\mu(z) + \int_{y<0} e^{i\zeta z} \cdot \omega(y)\, d\mu(z) = F_+(\zeta) + F_-(\zeta)$$

Here F_+ and F_- are entire functions of exponential type. If $\zeta = \xi$ is real and non-negative we have

$$|F_+(\xi)| \leq \int_{y \geq 0} e^{-\xi y} \cdot \omega(y)\, |d\mu(z)| \leq \max_{0 \leq y \leq b} \omega(y) \cdot ||\mu||$$

Hence $F_+(\xi)$ is bounded when $\xi \geq 0$ and in the same way $F_-(\xi)$ is bounded when $\xi \geq 0$. Since $F_+(\xi) + F_-(\xi) = 0$ on the whole real ξ-line we conclude that $F_+(\xi)$ is bounded. We have also

$$(\mathrm{i}) \qquad |F_+(\xi) \leq \int_{y \geq 0} e^{-\xi y} \cdot \omega(y)\, |d\mu(z)|$$

To profit upon (i) we consider the function

$$h(y) = -\log \omega(y) \quad : y > 0$$

Introduce the lower Legendre envelope defined for $\xi > 0$ by

$$k(\xi) = \min_{0 < y \leq b} h(y) + \xi y$$

Then we have

$$(\mathrm{ii}) \qquad e^{-\xi y} \cdot \omega(y) = e^{-\xi y + \log \omega(y)} \leq e^{-k(\xi)} \quad : 0 < y \leq b$$

Hence (i) gives

(iii) $$|F_+(\xi) \leq e^{-k(\xi)} \cdot ||\mu|| \quad : \xi > 0$$

At this stage we use the general result about Legendre envelopes from § xx where (*) in Theorem § xx entails that

(iv) $$\int_1^\infty \frac{k(\xi)}{\xi^2}\, d\xi = +\infty$$

Next, (ii) gives

$$\log^+ \frac{1}{|F_+(\xi)|} \geq k(\xi) - \log\, ||\mu||$$

hence (ii) entials that

(iv) $$\int_1^\infty \log^+ \frac{1}{|F_+(\xi)|} \cdot \frac{d\xi}{\xi^2} = +\infty$$

At the same time F_+ is an entire functionof exponential type which is bounded on the real axis and hence belongs to the Carleman class and the result in § xx shows that F_+ must be identically zero. This means that the restriction of $\omega(y) \cdot \mu$ to the closed rectangle $\overline{\square}_+$ is $\perp$ to functions of the form $z \mapsto e^{i\zeta z}$: $\zeta \in \mathbf{C}$. In the same way the restriction to the lower rectangle has this property. There remains to show that this gives (1). To achieve this we consider first the upper triangle where $z_1 = ib/2$ is a center. With $0 < \epsilon < 1$ we define

$$T_\epsilon(z) = -z_1 + (1-\epsilon)(z - z_1)$$

If $f \in A^*_\omega(\square)$ it follows that $f(T_\epsilon(z)$ is analytic in the closure of $\square_+$. By the standard Runge theorem it can be uniformly approximated by exponential polynomials $\{e^{iz\zeta}\}$ and the vanishing of F_+ entails that

(v) $$\int_{\square_+} f(T_\epsilon(z)) \cdot \omega(y)\, d\mu(z) = 0$$

Next, the reader may verify that there exists a constant C which is independent of ϵ such that

$$\omega(y) \cdot |f(T_\epsilon(z))| \leq C \cdot ||f||_\omega \quad : z \in \square_+$$

Now

$$\lim_{\epsilon \to 0} f(T_\epsilon(z)) = f(z)$$

holds in $\square_+$. By (xx) we can apply dominated convergence which we perform μ-integrals. Hence the vanishing integrals in (v) imply that

(vi) $$\int_{\square_+} f(z)) \cdot \omega(y)\, d\mu(z)$$

In the same way one proves that the integral over $\square_-$ is zero. Hence μ is $\perp$ to $A^*_\omega(\square)$ which finishes the proof of the density assertion in Theorem XX.

Special chapter: ODE-equations.

Ordinary differential equations are best treated by distributions. On the real x-line the space of distributions is denoted by $\mathfrak{Db}(\mathbf{R})$. Notice that we do not insist that the distriobutions are tempered, i.e. $\mathcal{S}^*$ appears as a proper subspace. As a first example we take the first order differential operator

$$\nabla = x \cdot \frac{\partial}{\partial x}$$

When $x \neq 0$ the equation $\nabla(f) = 0$ has solutions given by constant functions. To pass beyond $x = 0$ we take the Heaviside densities H^+ and H_-, where $H^+(x) = 1$ when $x > 0$ and zero if $x < 0$, while $H_- = 1 - H^+$. It turns out that these two linearly independent distributions on the x-line generate the vector space of all distribution solutions to the equation $\nabla(\mu) = 0$. See § xx below for details. On the other hand, if $f(x)$ is a C^1-function, i.e. continuously differentiable which satisfies $\nabla(f) = 0$, then it is clear that f must be a constant. So the set of distribution solutions is more extensive. Next, let s be a complex number which is not an integer. If $\mathfrak{s} > -1$ then x^s is integrable on intervals $(0, a)$ with $a > 0$ and there exists a distribution denoted by x_+^s acting as a linear functional on test-functions $\phi(x)$ by

$$x_+^s(\phi) = \int_0^\infty x^s \cdot \phi(x)\, dx$$

On the open interval $(0, +\infty)$ we notice that

$$(\nabla - s)(x^s) = 0 \quad : x > 0$$

It we apply $\nabla - s$ to the distribution x_+^s the construction of distribution derivatives means that $(\nabla - s)(x_+^s)$ acts on test-functions ϕ by

$$\text{(i)} \qquad \phi \mapsto \int_0^\infty x^s \cdot (-\partial(x\phi) - s\phi)\, dx$$

When $\mathfrak{Re}\, s > -1$ a partial integration shows that (i) is zero. Hence $(\nabla - s)(x_+^s) = 0$. It turns out that there exist more distributions μ such that $(\nabla - s)(\mu) = 0$. Namely, there exists the boundary value distribution $\mu = (x + i0)^s$ which also satisfies the equation $(\nabla - s)(\mu) = 0$. Here we recall that μ is defined on test-functions $\phi(x)$ by

$$\text{(ii)} \qquad \mu(\phi) = \lim_{\epsilon \to 0} \int (x + i\epsilon)^s \cdot \phi(x)\, dx$$

Hence we have found two linearly independent distribution solutions to the equation $\nabla - s(\mu) = 0$. It turns out that they give a basis for the null solutions which gives the dimension formula:

$$\dim_{\mathbf{C}} (\mathrm{Ker}_{\nabla - s}(\mathfrak{Db})) = 2$$

A special example. Take $Q = \nabla + 1$. Here two boundary value distributions $(x + i0)^{-1}$ and $(x - i0)^{-1}$ are null solutions. Let us also recall that the difference

$$(x - i0)^{-1} - (x + i0)^{-1} = \pi i \cdot \delta_0$$

So the Dirac measure at $x = 0$ is also a null solution which of course could have been verified directly. By the general result in § xx the space of null solutions is 2-dimensional so above we have found a basis.

In § xx we consider general differential operators with polynomial coefficients

$$P(x, \partial) = p_m(x)\partial^m + \ldots + p_0(x)$$

Under the assumption that the real zeros of the leading polynomial $p_m(x)$ are simple and consists of some k-tuple $\{a_1 < \ldots a_k\}$ we show in § xx that the P-kernel on $\mathfrak{Db}$ has dimension $k + m$.

0.1 A first order ODE-equation. Let p and q be a pair of polynomials and set

$$Q = q(x) \cdot \partial - p(x)$$

Assume that q is a monic polynomial of some degree $k \geq 2$ whose zeros are real and simple and arranged in strictly increasing order $\{a_1 < a_2 < \ldots < a_k\}$. The polynomial p is such that $p(a_\nu) \neq 0$ for every ν and in general it has complex coefficients and no condition is imposed upon its degree. Now we seek distributions μ on the x-line such that $Q(\mu) = 0$. One such solution is found as the boundary value of the analytic function defined in the upper half-plane by

$$f(z) = e^{\int_i^z \frac{p(\zeta)}{q(\zeta)}\, d\zeta}$$

Tos see this we notice that when $\mathfrak{Im} z > 0$ it is evident that the complex derivative

$$\text{(i)} \qquad \frac{\partial f}{\partial z} = \frac{p(z)}{q(z)}$$

Since the passage to boundary value distributions commute with derivations it follows that the boundary value distribution $f(x + i0)$ is a null solution to Q. Less obvious is that each simple and real zero a_ν of q yields a null solution μ_ν supported by the half-line $[a_\nu. + \infty)$.This is a consequence of general results in § xx. Let us remark that without using boundary values of analytic functions it is not easy to discover all this.

0.2 The equation $\nabla^2(\mu) = 0$. The Fuchsian operator is defined by $\nabla = x\partial$. It turns out that the space of distributions μ satisfying $\nabla^2(\mu)$ is a 4-dimensional vector space. One solution is the Heavisde function H_+ defined by the density 1 if $x > 0$ and zero if $x \leq 0$. Here

$$\partial(H_+)(g) = -\int_0^\infty g'(x)\, dx = g(0) \quad : g \in C_0^\infty(\mathbf{R})$$

This means that the distribution derivative $\partial(H_+) = \delta_0$ and since $x \cdot \delta_0 = 0$ we have $\nabla(H_+) = 0$. Next, on $\{x > 0\}$ we see that the density $\log x$ satisfies $\nabla^2(\log x) = 0$. It is tempting to extend the locally integrable function $\log x$ on the positive half-line to $\mathbf{R}$ by setting the value zero if $x \leq 0$. Denote the resulting distribution by $\log_+ x$. Now

$$\nabla(\log_+ x)(g) = -\int_0^\infty -\partial(xg) \cdot \log x\, dx = \int_0^\infty xg \cdot \frac{1}{x}\, dx = \int_0^\infty g dx$$

Hence $\nabla(\log_+ x) = H_+$ Since $\nabla(H_+) = 0$ we get $\nabla^2(\log_+ x) = 0$. Hence we have found two linearly independent null solutions given by the pair $(H_+, \log_+ x)$ which are supported by $x \geq 0$. In addition we find two other null solutions. The first is the constant density 1, The second is the boundary v alue distribution $\log(x + i0)$. By the gneral resu,t in § xx the space of null solutions is 4-dimensional so above we have found a basis for these.

0.3. Higher order Fuchsian equations. Let $m \geq 2$ and consider an operator of the form

$$Q = \nabla^m + q_{m-1}(x)\nabla^{m-1} + \ldots + q_1(x)\nabla + q_0(x)$$

where $\{q_\nu(x)\}$ are polynomials. With $\{c_\nu = q_\nu(0)\}$ we associate the polynomial

$$Q^*(s) = s^m + c_{m-1}s^{m-1} + \ldots + c_1 s + c_0$$

Under the assumption that $Q^*(k) \neq 0$ for all non-negative integers the solution space $\mathcal{S} = \{\mu\colon\, Q(\mu) = 0\}$ has dimension $2m$ and a basis is found as follows: In the upper half-plane the Picard-Fuchs theory about holomorphic differential equations entails that there exists an m-tuple of linearly independent analytic functions $\{\phi_\nu(z)\}$ which solve $Q(z, \partial_z)(\phi_\nu) = 0$. Similarly one finds an m-tuple $\{\psi_\nu\}$ of linearly independent analytic functions in the lower half-plane. The boundary value distributions $\{\phi_\nu(x+i0)\}$ and $\{\psi_\nu(x-i0)\}$ belong to $\mathcal{S}$ and are linearly indepedent. For if $\sum c_\nu\phi_\nu(x+i0)+\sum d_\nu\psi_\nu(x-i0)=0$ where at least some c_ν or d_ν is $\neq 0$ then

(i) $$\phi_*(x+i0)=\psi_*(x-i0)=0$$

where $\phi_*=\sum c_\nu\phi(x+i0)\neq 0$ and $\psi_*=-\sum d_\nu\psi(x-i0)\neq 0$. Now (i) cannot hold. The reason is that the assumption about $Q^*(s)$ entails that the equation $Q(z,\partial_z)(f)=0$ has no holomorphic solutions at $z=0$. This fact stems from local $\mathcal{D}$-module theory and is exposed in § xx. Now the reflection principle for analytic functions entails that the analytic wave front sets of the distributions ϕ_* and ψ_* both are non-empty. On the other hand the material in § xx shows that these non-empty wave fronts have opposed directions and hence the equality (i) cannot hold. This proves that $\mathcal{S}$ is at least $2m$-dimensional and by the general results in § xx we have equality. So above we have constrcuted a basis for the null solutions.

§ 1. Fundamental solutions to ODE-equations with constant coefficients

We consider differential operators with constant coefficients acting on the real x-line. To simplify the passage to Fourier transform we introduce the first order operator

$$D=\frac{1}{i}\cdot\frac{d}{dx}$$

If $P(\xi)$ is a polynomial of the ξ-variable and μ is a tempered distributionon the x-line this gives the equality:

(*) $$\widehat{P(D)\mu}(\xi)=P(\xi)\cdot\widehat{\mu}(\xi)$$

By a tempered fundamental solution to $P(D)$ we mean a distribution $\mu\in\mathcal{S}^*$ such that

$$P(D)\mu=\delta_0$$

where δ_0 is the Dirac measure at $x=0$. Since the Fourier transform of δ_0 is the identity on the ξ-line the Fourier transform of a fundamental solution satisfies

$$P(\xi)\cdot\widehat{\mu}(\xi)=1$$

When $P(\xi)$ has no zero on the real ξ-line there exists a fundamental solution given as the inverse Fourier transform of the smooth density $P(\xi)^{-1}$. If $P(\xi)$ has some real zeros we can write

$$P(\xi)=Q(\xi)\cdot R(\xi)$$

where R has real zeros and the zeros of Q are all non-real. The factorisation is unique when we choose constants so that $Q(\xi)$ is a monic polynomial. The case $\deg Q=0$ is not excluded, i.e. this holds when all zeros of $P(\xi)$ are real. But in general one has a mixed case where $n=\deg P$ and $1\leq\deg Q\leq n-1$.

1. The case $\deg Q=0$. When all zeros of $P(\xi)$ real there exists the boundary value distribution on the ξ-line defined by

(1.1) $$\gamma=\frac{1}{P(\xi-i0)}$$

By the general results from § XX its inverse Fourier transform is supported by the half-line $\{x\geq 0\}$. Let μ_+ denote this distribution. Then

$$\widehat{P(D)\mu_+}=P\xi)\cdot\gamma=1$$

and hence μ_+ is a fundamental solution.

2. The mixed case. If $P = Q \cdot R$ where $1 \leq \deg Q \leq n-1$ we proceed as follows. First one has a bijective map on the space of tempeted distributions on the ξ-line defined by

$$\gamma \mapsto Q(\xi)^{-1} \cdot \gamma$$

Fourier's inversion fomrula gives a bijective linear operator T_Q on the space of tempered distribtutions on the x-line such that

$$\widehat{T_Q(\mu)} = Q(\xi)^{-1} \cdot \widehat{\mu}$$

So if μ is a temepred distribution we get

$$P(D)(T_Q(\mu)) = R(D)(\mu) \tag{2.1}$$

The zeros of $R(\xi)$ are real which gives the fundamental solution ν_+ to $R(D)$ and now

$$\mu = T_Q(\nu_+) \tag{2.2}$$

yields a fundamental solution to $P(D)$. In this way we have constructed a fundamental solution in a canonical fashion. In contrast to the real case where $\deg Q = 0$ the distribution μ above is in general not supported by the half-line $\{x \geq 0\}$. We give examples in § XX below.

3. The determination of μ_+. Consider the case when $\deg Q = 0$ so that the fundamental solution μ_+ is the inverse Fourier transform of (xx) above. Let us for the moment assume that the real zeros of $P(\xi)$ are all simple and given by an n-tuple $\{\alpha_k\}$. Define the distribution ρ on the real x-line by the density

$$\rho(x) = \sum \frac{1}{P'(\alpha_k)} \cdot e^{i\alpha_k x} \quad : x \geq 0$$

while $\rho(x) = 0$ when $x < 0$. It is clear that the distribution $P(D)\rho$ vanishes when $x \neq 0$, i.e. supported by the singleton set $\{x = 0\}$. Newton's formula from § xx gives

$$\sum_{k=1}^{n} \frac{1}{P'(\alpha_k)} \cdot \alpha_k^m = 0 \quad : \quad 0 \leq m \leq n-2$$

This entails that the derivatives up to order $n-2$ of ρ vanish at $x = 0$. Using this we show that ρ up to a constant gives a fundamental solution to $P(D)$. For consider a test-function $f(x)$ and let $P^*(D)$ be the adjoint of $P(D)$. The vanishing of the derivatives of ρ at $x = 0$ above gives after partial integration

$$\int \rho(x) \cdot P^*(D)(f)(x)\,dx = (-1)^{xx} \cdot \rho^{(n-1)}(0) \cdot f(0)$$

4. Conclusion. The fundamental solution μ_+ supported by $x \geq 0$ is given by the density

$$\mu_+(x) = \frac{n}{xx} \cdot \rho(x) = \sum \frac{1}{P'(\alpha_k)} \cdot e^{i\alpha_k x}$$

5. Example. Consider $P(D) = D^2 - 1$ so that $P(\xi) = \xi^2 - 1$. Here 1 and -1 are the simple zeros and (xx) gives

$$\mu_+(x) = XXX \cdot \sum \frac{1}{-2} \cdot e^{-ix} + \frac{1}{2} \cdot e^{ix} = -\sin x \quad : \ x \geq 0$$

6. An example in the mixed case. Let $P(D) = (D^2+1)(D-a)$ where a is some real number $\neq 0$. So here $Q(\xi) = \xi^2 + 1$ and the fundamental solution from § 2 becomes

$$\mu = T_Q(\nu_+) \tag{6.1}$$

where ν_+ is the inverse Fourier transform derived from the linear polynomial. $R(\xi) = \xi - a$. This gives

$$\nu_+(x) = -e^{iax} \quad : x \geq 0 \tag{6.2}$$

7. The expression of μ. By the above μ is the convolution of ν_+ and the continuous density

$$\phi(x) = \frac{1}{2\pi} \int \frac{e^{ix\xi}}{1+\xi^2}\, d\xi$$

We leave it to the reader to verify that

$$\phi(x) = \frac{1}{2} \cdot e^{-|x|}$$

Hence

$$\mu(x) = -\frac{1}{2} \cdot \int_0^\infty e^{-[x-y|} \cdot e^{-aiy}\, dy$$

The reader is invited to analyze this function using a computer to plot this function with different choice of a.

2.0. ODE-equations on the real line

0.1.2 Higher order equations. In § xx we consider a differential operator $Q(x,\partial)$ of order $m \geq 2$ with polynomial coefficients:

$$Q(x,\partial) = q_m(x)\partial^m + \ldots + q_1(x)\partial + q_0(x)$$

Notice that we get a holomorphic differential operator when we pass to the complex variable $z = x + iy$ and set

$$Q(z,\partial_z) = q_m(z)\partial_z^m + \ldots + q_1(z)\partial_z + q_0(z)$$

Assume that the polynomials $\{q_\nu(x)\}$ have no common zero in the complex plane and that the leading polynomial q_m has real and simple zeros $\{a_1 < \ldots < a_k\}$ for some positive integer k. Since $q_m(z) \neq 0$ when z is outside the real axis it is easily shown that in the upper half-plane $U_+ = \mathfrak{Im}\, z > 0$ there exists an m-dimensional subspace of $\mathcal{O}(U_+)$ whose functions $f(z)$ satisfy $Q(z,\partial_z)(f) = 0$. Let $\{f_1,\ldots,f_m\}$ be the basis for the solutions in $\mathcal{O}(U_+)$. To each zero a_ν we consider small open disc $D = \{|z - a_\nu| < r\}$ with r so small that $|a_j - a_\nu| \geq r$ for all $j \neq \nu$. Here two case can occur: The first is that the restriction of every f_ν to the half-disc D_+ extends to be analytic in D and then a_ν is called a negligable singular point for the ODE-equation. If some f-function fails to extend. Malgrange's index formula from § xx shows that there exists an $m-1$-dimensional subspace of these f-functions whose restrictions extend to be holomorphic in D. Using this local index formula we show in § xx that if k_* is the number of zeros of q_m which are not negligable then the space of distributions μ defined on the whole real line such that $Q(\mu) = 0$ is a vector space of dimension $m + k_*$.

To grasp the notion of distributions it is natural to start with a study of distribution solutions to ordinary differential operators which leads to more systematic results as compared to studies before distribution theory was established. An example is the confluent hypergeometric function which arises as a solution to a differential operator of the form

$$P = x\partial^2 + (\gamma - x)\partial - a$$

where γ is a non-zero complex number while a is arbitrary. In the classic literature one solves this equation via the Laplace method which involves a rather cumbersome use of residue calculus. More information about the operator P arises when one determines its kernel on the space of distributions on the real x-line. The result in Theorem 0.0.1 below shows that this P-kernel is a 3-dimensional subspace of $\mathfrak{Db}$. Moreover, there exists a fundamental solution supported by the half-line $\{x \geq 0\}$.

Let us now discuss the general situation where one regards a differential operator with polynomial coefficients

$$(*) \qquad P(x,\partial) = q_m(x)\cdot\partial^m + q_{m-1}(x)\partial^{m-1} + \ldots + q_0(x)$$

where $m \geq 1$ and $q_0(x),\ldots,q_m(x)$ are polynomials which in general have complex coefficients. Let $\mathfrak{Db}$ be the space of distributions on the real x-line. A first question is to determine the P-kernel, i.e. one seeks all distributions μ such that $P(\mu) = 0$. Following material from the thesis by Ismael (xxx - University of xxx) we expose some general facts. The reader may postpone the subsequent discussion until later since we shall use notions such as analytic wave front sets and boundary value distributions whose contructions are given later on. But in anny case it is instructive to pursue the results below and the reader who has learnt the details in the examples 0.0.4-0.0.5 has begun to master distribution theory.

The local Fuchsian condition. We shall restrict the study to operators P which are *locally Fuchsian* at every real zero of the leading polynomial $q_m(x)$. This means the following: Let a be a real zero of $q_m(x)$ with some multiplicity $e \geq 1$ so that $q_m(x) = q(x)(x-a)^e$ where the polynomial q is $\neq 0$ at a. Then we can write

$$P(x,\partial) = q(x) \cdot [(x-a)^e \partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_0(x)]$$

where $\{r_\nu = \frac{p_\nu}{q}\}$ are rational functions with no pole at a and therefore define analytic functions in a neighbrohood of a. Hence

$$P_*(x,\partial) = (x-a)^e \partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_0(x)$$

can be identified with a germ of a differential operator with coefficients in the local ring $\mathcal{O}(a)$ of germs of analytic functions at a. The ring $\mathcal{D}$ of such germs of differential operatoes is studied in § x where we define the subfamily of Fuchsian operators. For example, if $a = 0$ then a Fuchsian operator in $\mathcal{D}$ is of the form

$$\rho(x) \cdot [\nabla^m + g_{m-1}(x)\nabla^{m-1} + \ldots + g_0(x)]$$

where $\rho, g_{m-1}, \ldots, g_0$ belong to $\mathcal{O}$ and $\nabla = x\partial$ is the first order Fuchsiuan operator.

From the above we can announce the following conclusive result:

0.0.1 Theorem *Let $P(x,\partial)$ in (*) above be locally Fuchsian at the real zeros of p_m. Then $Ker_P(\mathfrak{Db})$ is a complex vector space of dimension is equal to $m + e_1 + \ldots + e_k$ where $\{e_\nu\}$ are the multiplicities at the real zeros of p_m. Moreover, for each real zero of p_m there exists a distribution μ supported by $\{x \geq a\}$ such that $P(\mu) = \delta_a$.*

Remark. The crucial part in the proof of Theorem 0.1 relies upon constructions of boundary values of analytic functions. Moreover the following supplement to Theorem 0.0.1 hold. For each real zero a of $p_m(x)$ with some multiplicity e there exists a distinguished e-dimensional subspace V_a of $\text{Ker}_P(\mathfrak{Db})$ which consists of distributions μ supported by the closed half-line $[a, +\infty]$ whose analytic wave front sets satisfy the following: First it contains the whole fiber above a and the remaining part of the analytic wave front set is either empty or a union of half-lines above some of the real zeros of p_m which are $> a$. Moreover, one has a direct sum decomposition

$$(**) \qquad \text{Ker}_P(\mathfrak{Db}) = \mathcal{F}_+ \oplus V_{a_\nu}$$

where the last direct sum is taken over the real zeros of p_m, and $\mathcal{F}_+$ is an m-dimensional subspace of $\mathfrak{Db}$ with a basis given by an m-tuple of boundary value distributions $\{\phi_k(x + i0)\}$. Here $\{\phi_k(z)\}$ are analytic functions in a strip domain $U = \{-\infty < x < +\infty\} \times \{0 < y < A\}$ with $A > 0$ chosen so that the complex polynomial $p_m(z)$ is zero-free in this domain and each $\phi_k(z)$ satisfies the homogeneous equation $P(z,\partial)(\phi) = 0$ in U.

Example. Consider the first order differential operator

$$P = x\partial + 1$$

Outside $x = 0$ the density x^{-1} is a solution. In §§ we shall learn hos to construct the Euler distribution x_+^{-1} which is suppored by $[0, +\infty)$ and find that

$$P(x_+^{-1}) = \delta_0$$

The 1-dimensional $\mathcal{F}_+$-space in (**) is generated by the boundary value distribution $(x + i0)^{-1}$.

0.0.2 Tempered solutions. The P-kernel in Theorem 0.0.1 need not consust of tempered distributions. The reason is that we have not imposed the condition that P is locally Fuchsian at infinity. So if $\mathcal{S}^*$ denotes the space of tempered distributions, then $\text{Ker}_P(\mathcal{S}^*)$ can have strictly smaller dimension than $m + k$ and the determination of the tempered solution space leads to a more involved analysis. Already the case $P = \partial - 1$ illustrates the situation. Here the P-kernel on $\mathfrak{Db}$ is the 1-dimensional space given by the exponential density e^x which is not tempered so the P-kernel on $\mathcal{S}^*$ is reduced to zero. During the search for tempered fundamental solutions to P supported by half-lines $\{x \geq a\}$ one can use a result due to Poincaré under the extra assumption that $\deg p_k \leq \deg p_m$ hold for every $0 \leq k < m$. For in this case there are series expansions when x is large and positive:

$$\frac{p_k(x)}{p_m(x)} = c_k + \sum_{\nu=1}^{\infty} c_{k\nu} x^{-\nu} \quad : 0 \leq k \leq m-1$$

The leading coeficients $c_0, \ldots, c_{m-1}$ give a monic polynomial

$$\phi(\alpha) = \alpha^m + c_{m-1}\alpha^{m-1} + \ldots c_0$$

Let us also chose $A > 0$ so large that the leading polynomial p_m has no real zeros on $[A, +\infty]$. This gives an m-dimensional space of null solutions where a basis consists of real-analytic densities $u_1(x), \ldots, u_m(x))$ on this interval.

0.0.3 Poincaré's theorem. *Suppose that ϕ has simple simple zeros $\alpha_1, \ldots, \alpha_m$. Then, with A as above one can arrange the u-basis so that*

$$u_k(x) = e^{\alpha_k x} \cdot g_k(x)$$

and there exists a non-negative integer w and a constant C such that

$$|g_k(x) \leq C \cdot (1+x)^w : 1 \leq k \leq m$$

hold for all $x \geq A$.

So for indices k such that $\mathfrak{Re}\, \alpha_k \leq 0$, it follows that $u_k(x)$ has tempered growth as $x \to +\infty$. In particular Poincaré's result entails that if the real parts are all ≤ 0, then the fundamental solutions from Theorem 0.0.1 are all tempered.

0.0.4 Example. Consider the operator

$$P = x\partial^2 - x\partial - B$$

where $B > 0$. In this case

$$\phi(\alpha) = \alpha^2 - \alpha = \alpha(\alpha - 1)$$

so one of the u-solutions above increse exponentially while the other has tempered growth as $x \to +\infty$. It is easily seen that the there exists an entire solution

(i) $$f(x) = x + c_2 x^2 + \ldots$$

such that $P(f) = 0$, whose coefficients are found by the recursive formulas

$$k(k-1)c_k = (k - 1 + B(c_{k-1} \quad : k \geq 2$$

Hence $\{c_k\}$ are positive and it is clear that f has exponential growth as $x \to +\infty$. In addition we have a solution on $x > 0$ of the form

$$g(x) = f(x) \cdot \log x + a(x)$$

In §§ we explain that $P(g_+) = a \cdot \delta_0$ hold for a non-zero constant while $P(f_+) = 0$. Next, let et u_1 be the tempered solution and u_2 the non-tempered solution in Poincaré's theorem on the half-line $x > 0$. There are constants c_1, c_2 such that

$$f(x) = c_1 u_1(x) + c_2 u_2(x)$$

Here $c_2 \neq 0$ because f increases exponentially on $(0, +\infty)$. At the same time

$$g(x) = d_1 u_1(x) + d_2 u_2(x)$$

hold for some constants d_1, d_2. Set

$$\gamma = g_+ - \frac{d_2}{c_2} \cdot f_+$$

From the above γ has tempered growth as $x \to +\infty$ and $P(\gamma) = a \cdot \delta_0$ with $a \neq 0$. Hence $\mu = a^{-1} \cdot \gamma$ yields a tempered fundamental solution supported by $\{x \geq 0\}$.

In § xx we give further examples of tempered fundamental solutions.

0.5 Another example. Here we take

$$P = \nabla^2 + q(x) \tag{0.3.1}$$

where $q(x)$ is a polynomial such that $q(0) = -1$ and $q'(0) = 0$. For example, if $q(x) = x^2 - 1$ we encounter a wellknown Bessel operator. It is easily seen that there exists a unique entire solution $f(x)$ which satisfies $P(f) = 0$ with a series expansion

$$f(x) = x + c_3x^3 + \ldots$$

Moreover, one verifies easily that there exists another entire function $g(x)$ with $g(0) = 0$ such that the multi-valued function

$$\phi(z) = f(z) \cdot \log z + g(z) \tag{i}$$

satisfies $P(\phi) = 0$. Theorem 0.0.1 predics that the P-kernel on $\mathfrak{Db}$ is 4-dimensional. To begin with f restricts to a real analytic densitiy on the x-line and gives a null solution. A second solution is obtained by the boundary value distribution

$$\gamma = \phi(x + i0) = f(x) \cdot \log(x + i0) + g(x)$$

Together they give a basis in the 2-dimensional space $\mathcal{F}_+$ from (*) in the remark after Theorem 0.0.1. There remains to find two linearly independent distributions in V_0 since the leading polynomial of P has a double zero at $x = 0$. To attain such a pair we first consider the boundary value distribution

$$\gamma_* = f(x) \cdot \log(x - i0) + g(x)$$

which also is a null solution. Here the multi-valuedness of the complex log-function entails that

$$\gamma - \gamma_* = 2\pi i \cdot f(x) \cdot H_-(x)$$

where $H_-(x)$ is the Heaviside distribution supported by the negative half-line. Then

$$\gamma^* = \gamma - \gamma_* - 2\pi i \cdot f(x)$$

is a null solution supportec by the half-line $x \geq 0$ and hence belongs to the 2-dimensional space V_0. A second null solution in V_0 is given by the Dirac measure δ_0. To see that δ_0. is a null solution for P we recall that in the non-commutative ring of differential operators one has the equality $\nabla = \partial x \circ x - 1$. Since $x \cdot \delta_0 = 0$ we get the distribution equation

$$\nabla(\delta_0) = -\delta_0 \implies \nabla^2(\delta_0) = \delta_0$$

Since $q(0) = -1$ is assumed in (0.3.1) it follows that $P(\delta_0) = 0$. Hence we have found four linearly independent null solutions $f_+.f_-, \gamma^*, \delta_0$ in accordance with Theolrem 0.0.1.

The fundamental solution. A fundamental solution μ supported by $x \geq 0$ is found as follows: From (i) we have the real-analytic density $\phi(x)$ on the open half-line $\{x > 0\}$ which gives the distribution ϕ_+ supported by $\{x \geq 0\}$ defined by

$$\phi_+ = f(x) \cdot (\log x) \cdot H_+ g(x) \cdot H_+$$

In § xx we shall explain that

$$\nabla(\log x \cdot H_+) = \delta_0$$

and from this deduce that

$$P(\phi_+) = -\delta_0$$

Hence $\mu = -\phi_+$ gives the requested fundamental solution.

0.2 PDE-equations with constant coefficients.

The study of PDE-equations with contant coefficients in $\mathbf{R}^n$ for arbitrary $n \geq 2$ is a rich subject. The interested reader may consult Chapter xx in [Hörmander:Vol 2] for an extensive study of PDE-equations with constant coefficients. Here we shall give a construction from Hörmander's article [xxx] which illustrates how analytic function theory can be used with PDE-theory. Fourier's inversion formula for an arbitrary $n \geq 1$ asserts the following: Let $f(x) = f(x_1, \ldots, x_n)$ be a C^∞-function which is rapidly decreasing as $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ tends to $+\infty$. Then

$$(*) \qquad f(x) = \frac{1}{(2\pi)^n} \cdot \int e^{i\langle x,\xi\rangle} \cdot \widehat{f}(\xi)\, d\xi \quad \text{where} \quad \widehat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} \cdot f(x)\, dx$$

The inversion inversion formula (*) entails that the Fourier transform of the partial derivative $\frac{\partial f}{\partial x_j}(x)$ is equal to $i\xi_j \cdot \widehat{f}(\xi)$. In PDE-theory one introduces the first order differential operators

$$D_j = -i \cdot \frac{\partial}{\partial x_j} \quad : 1 \leq j \leq n$$

When $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index we get the higher order differential operator

$$D^\alpha = D_1^{\alpha_1} \cdots D_1^{\alpha_n}$$

We can take polynomials of these and get differential operatators with constant coefficients

$$P(D) = \sum c_\alpha \cdot D^\alpha$$

Fourier's inversion formula gives

$$(**) \qquad P(D)(f)(x) = \frac{1}{2\pi^n} \cdot \int e^{i\langle x,\xi\rangle} \cdot P(\xi) \cdot \widehat{f}(\xi)\, d\xi$$

Thus, applying a differential operator with constant coefficients to f corresponds to the product of its Fourier transform with the polynomial $P(\xi)$ and (**) can be used to construct solutions of the homogeneous equation $P(D)(f) = 0$. Following [Hörmander] we construct distributions μ such that $P(D)(\mu) = 0$ for a suitable class of PDE-operators. Let $\phi_1(s), \ldots, \phi_n(s)$ be some n-tuple of analytic functions of the complex variable s which extend to continuous functions on the boundary of the domain

$$\Omega = \{\mathfrak{Im}(s) < 0\} \cap \{|s| > M\}$$

where M is some positive number. Assume that the ϕ-functions satisfy the growth conditions

$$\text{(i)} \qquad |\phi_k(s)| \leq C|s|^a$$

for a constant C and some $0 < a < 1$. If $a < \rho < 1$ there exists the analytic function in Ω defined by

$$\psi(s) = e^{-(is)^\rho}$$

As explained in § xx one has the estimate

$$|\psi(s)| \leq e^{-\cos\frac{\pi\rho}{2} \cdot |s|^\rho} \quad : \mathfrak{Im}\, s \leq 0$$

The inequality $a < \rho$ and (i) entail that the functions

$$s \mapsto e^{a_1 \cdot \phi_1(s) + \cdots + a_n \phi_n(s)} \cdot \psi(s)$$

decrease like $e^{-\cos\frac{\pi\rho}{2} \cdot |s|^\rho}$ in Ω.

Exercise. Verify that the complex line integrals below converge absolutely for every s-polynomial $Q(s)$ and every n-tuple of real numbers $x_1, \ldots, x_n$:

$$\text{(ii)} \qquad \frac{1}{(2\pi)^n} \cdot \int_{\partial\Omega} e^{x_1\phi_1(s) + \ldots + x_n\phi_n(s)} \cdot e^{ix_n s} \cdot Q(s) \cdot \psi(s)\, ds$$

and show that when x varies in $\mathbf{R}^n$ this gives a C^∞-function $f(x)$. If $1 \leq j \leq n-1$ one has for example

$$\frac{\partial f}{\partial x_j} = \frac{1}{(2\pi)^n} \cdot \int_{\partial\Omega} \phi_j(s) \cdot e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_ns} \cdot Q(s) \cdot \psi(s)\, ds$$

Less obvious is that the C^∞-function $f(x)$ is supported by the half-space $\{x_n \geq 0\}$. To prove it one uses the analyticity of the integrand as a function of s which enable us to shift the contour of integration so that (ii) is unchanged while we integrate on a horisontal line $\mathfrak{Im}\, s = -N$ for every $N > M$. With $s = u - iN$ we have

$$|e^{ix_n \cdot s}| = e^{N \cdot x_n}$$

If $x_n < 0$ this term tends to zero as $N \to +\infty$ and from this the resder should confirm that the C^∞-function $f(x)$ is identically zero in $\{x_n < 0\}$.

Suppose now that we are given a PDE-operator $P(D)$ and the ϕ-functions are chosen so that

$$s \mapsto P(\phi_1(s) \ldots \phi_{n-1}(s), \phi_n(s) + s) = 0 \quad : s \in \Omega$$

Then it is clear that

$$P(D)(e^{x_1\phi_1(s)+\ldots+x_n\phi_n(s)} \cdot e^{ix_ns}) = 0$$

hold for all $x \in \mathbf{R}^n$ and $s \in \Omega$. Hence $P(D)(f) = 0$ where f is a C^∞-function supported by the half-space $\{x_n \geq 0\}$. In § xx we will show that the construction of solutions as above is not so special for PDE-operators P such that the hyperplane $\{x_n = 0\}$ is non-characteristic.

0.1 The distributions x_+^s

If s is a complex number where $\mathfrak{Re}\, s > -1$ the function defined by x^s for $x > 0$ and zero on the half-line $x \leq 0$ is locally integrable and defines a distribution denoted by x_+^s acting on test-functions g by

$$x_+^s(g) = \int_0^\infty x^s \cdot g(x)\, dx$$

The distribution-valued function $s \mapsto x_+^s$ is analytic in $\mathfrak{Re}\, s > -1$. Indeed if $x < 0$ we have $\frac{d}{ds}(x^s = \log x \cdot x^s$ which entails that the complex derivative of x_+^s is the distribution defined by

$$g \mapsto \int_0^\infty \log x \cdot x^s \cdot g(x)\, dx$$

It turns out that x_+^s extends to a meromorphic distribution-valued function in the whole s-plane. To prove this we perform a partial integration which gives

$$(0.0.1) \qquad x_+^{s+1}(g') = \int_0^\infty x^{s+1} \cdot g(x)\, dx = -(s+1) \cdot \int_0^\infty x^s \cdot g(x)\, dx$$

By the construction of distribution derivatives this means that

$$\frac{d}{dx}(x_+^s + 1) = (s+1) \cdot x_+^s$$

Euler's functional equation. Set $\partial = \frac{d}{dx}$. We can iterate (0.0.1) which for every positive integer m gives

$$(0.0.2) \qquad (s+1) \cdots (s+m) x_+^s = \partial^m(x_+^{s+m})$$

We refer to (0.0.2) as Euler's functional equation. It entails that the distribution-valued function x_+^s extends to a meromorphic function with at most simple poles at negative integers. Let us investigate the situation close to a negative integer. With $s = -m + t$ and t small one has

$$t(t-1) \cdot (t - m + 1) x_+^{-m+t} = \partial^m(x_+^t)$$

When $x > 0$ one has the expansion

$$x^t = 1 + t \log x + \frac{t^2}{2} \cdot (\log x)^2 + \frac{t^3}{3!} \cdot (\log x)^3 + \ldots$$

From this we obtain a series expansion

$$x_+^{-m+t} = t^{-1} \cdot \rho_m + \gamma_0 + t\gamma_1 + \ldots$$

where ρ_m and $\{\gamma_\nu\}$ are distributions. In particular the reader may verify that

$$(-1)^{m-1}(m-1)! \cdot \rho_m = \partial^m(H_+)$$

Let us then consider the constant term γ_0. The linear t-term in the expansion of $t(t-1) \cdot (t - m + 1) x_+^{-m+t}$ becomes

$$(-1)^{m-1}(m-1)! \cdot \gamma_0 + \frac{m(m-1)}{2} \cdot \rho_m$$

If $x > 0$ we notice that

$$\partial^m(\log x) = (-1)^{m-1} \cdot (m-1)! \cdot x^{-m}$$

From the above γ_0 restricts to the density x^{-m} when $x > 0$. At the same time γ_0 is a distribution defined on the whole x–line supported by $\{x \geq 0\}$. We set

$$(*) \qquad x_+^{-m} = \gamma_0$$

and refer to this as Euler's extension of the density x^{-m} which from the start is defined on $\{x > 0\}$. So in (*) we have found distributions for every positive integer m.

0.1.2 Further formulas. With $s = -1 + z$ where z is a small non-zero complex number one has

(i) $$z \cdot x_+^{-1+z} = \partial(x_+^z)$$

Next, if $x > 0$ we have

$$x^z = e^{z \log x} = 1 + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!} \cdot z^k$$

Introducing the Heaviside distribution H_+ which is 1 on $x \geq 0$ and zero on $x < 0$ this means that

(ii) $$\partial(x_+^z) = \partial(H_+) + \sum_{k=1}^{\infty} \partial(\frac{(\log x)^k}{k!} \cdot H_+(x))z^k$$

From this we get a Laurent expansion of x_+^s at $s = -1$. The crucial point is that the distribution derivative

(iii) $$\partial(H_+) = \delta_0$$

where δ_0 is the Dirac distribution at $x = 0$. It follows that

(iv) $$x_+^{-1+z} = z^{-1} \cdot \delta_0 + \sum_{k=1}^{\infty} \partial(\frac{(\log x)^k}{k!} \cdot H_+(x))z^{k-1}$$

In particular the constant term becomes

(v) $$\partial(\log x \cdot H_+(x))$$

To find this distribution we take a test-function g and a partial integration gives

$$-\int_0^{\infty} (\log x \cdot g'(x)\, dx = \int_0^1 \frac{g(x) - g(0)}{x}\, dx + \int_1^{\infty} \frac{g(x)}{x}\, dx$$

From this we conclude that the distribution x_+^{-1} is defined on test-functions by the formula:

$$x_+^{-1}(g) = \frac{(-1)^{m-1}}{(m-1)!} \cdot \int_0^1 \frac{g(x) - g(0)}{x}\, dx + \int_1^{\infty} \frac{g(x)}{x}\, dx$$

0.1.3 Exercise. For each test-function g and integer $m \geq 2$ we set

$$T_{m-1}(g)(x) = g(0) + g'(0)x \ldots \frac{g^{(m-1)}(0)}{(m-1)!} \cdot x^{m-1}$$

Show from the above via partial integrations that

$$x_+^{-m}(g) = \frac{(-1)^{m-1}}{(m-1)!} \cdot \int_0^1 \frac{g(x) - T_{m-1}(g)(x)}{x^m}\, dx + \int_1^{\infty} \frac{g(x)}{x^m}\, dx$$

0.1.4 The distributions $(x + i0)^\lambda$ and $(x - i0)^\lambda$. In the upper half plane there exists the single valued branch of $\log z$ whose argument stays in $(0, \pi)$ and for every complex number λ we have

$$z^\lambda = e^{\lambda \cdot \log z}$$

In § 3 we shall learn how to contruct boundary value distributions of analytic functions defined in strip domains above or below the real x-line. In particular there exists the distribution $(x + i0)^\lambda$ defined on test-functions $g(x)$ by the limit formula

$$\lim_{\epsilon \to 0} \int (x + i\epsilon)^\lambda \cdot g(x)\, dx$$

Notice that this limit exists for all complex λ, i.e even when the real part becomes very negative. In the same way we have the single valued branch of $\log z$ in the lower half-plane whose argument stays in $(-\pi, 0)$ and construct the distribution $(x - i0)^\lambda$ defined by

$$(x - i0)^\lambda(g) = \lim_{\epsilon \to 0} \int (x - i\epsilon)^\lambda \cdot g(x)\, dx$$

Since $\lambda \mapsto e^{\lambda \cdot \log z}$ are entire in λ, we get two entire distribution valued functions by $(x - i0)^\lambda$ and $(x - i0)^\lambda$. Regarding the choice of branches for the log-functions we see that

$$(x - i0)^\lambda = e^{-2\pi i\lambda} \cdot (x + i0)^\lambda \quad : x < 0$$

At the same time

$$(x + i0)^\lambda = (x - i0)^\lambda = x^\lambda \quad : x > 0$$

From this we see that the distribution

$$(x + i0)^\lambda - e^{2\pi i\lambda} \cdot (x - i0)^\lambda$$

is supported by $x \geq 0$ and expressed by the density $(1 - e^{2\pi i\lambda}) \cdot x^\lambda$. The conclusion is that one has the equation

$$\mu_\lambda = \frac{(x + i0)^\lambda - e^{2\pi i\lambda} \cdot (x - i0)^\lambda}{1 - e^{2\pi i\lambda}}$$

Remark. The equation (xx) is more involved compared to the previous description of the meromorphic μ-function found via Euler's functional equation. But (xx) has the merit that the denominator is an entire distribution valued function and when one passes to Fourier transforms it turns out that (xx) is quite useful.

Principal value integrals. If $g(x)$ is a test-function there exists a limit

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{g(x)}{x}\, dx$$

This yields a distribution denoted by $\mathrm{VP}(x^{-1})$. Outside $\{x = 0\}$ it is given bt the density x^{-1} where it agrees with $(x + i0)^{-1}$ and hence the difference

$$\mu = \mathrm{VP}(x^{-1}) - (x + i0)^{\cdot -1}$$

is supported by $\{x = 0\}$.

Exercise. Notoice that

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{1}{x + i\epsilon}\, dx = \log(1 + i\epsilon) - \log(-1 + i\epsilon) = -\pi i$$

and use this to show that

$$\mu = -\pi i \cdot \delta_0$$

0.1 The distributions x_+^s

If s is a complex number where $\mathfrak{Res} > -1$ the function defined by x^s for $x > 0$ and zero on the half-line $x \leq 0$ is locally integrable and defines a distribution denoted by x_+^s acting on test-functions g by

$$x_+^s(g) = \int_0^\infty x^s \cdot g(x)\, dx$$

The distribution-valued function $s \mapsto x_+^s$ is analytic in $\mathfrak{Res} > -1$. Indeed if $x < 0$ we have $\frac{d}{ds}(x^s = \log x \cdot x^s$ which entails that the complex derivative of x_+^s is the distribution defined by

$$g \mapsto \int_0^\infty \log x \cdot x^s \cdot g(x)\, dx$$

It turns out that x_+^s extends to a meromorphic distribution-valued function in the whole s-plane. To prove this we perform a partial integration which gives

$$x_+^{s+1}(g') = \int_0^\infty x^{s+1} \cdot g(x)\, dx = -(s+1) \cdot \int_0^\infty x^s \cdot g(x)\, dx \tag{0.0.1}$$

By the construction of distribution derivatives this means that

$$\frac{d}{dx}(x_+^s + 1) = (s+1) \cdot x_+^s$$

Euler's functional equation. Set $\partial = \frac{d}{dx}$. We can iterate (0.0.1) which for every positive integer m gives

$$(s+1) \cdots (s+m) x_+^s = \partial^m(x_+^{s+m}) \tag{0.0.2}$$

We refer to (0.0.2) as Euler's functional equation. It entails that the distribution-valued function x_+^s extends to a meromorphic function with at most simple poles at negative integers. Let us investigate the situation close to a negative integer. With $s = -m + t$ and t small one has

$$t(t-1) \cdot (t - m + 1) x_+^{-m+t} = \partial^m(x_+^t)$$

When $x > 0$ one has the expansion

$$x^t = 1 + t \log x + \frac{t^2}{2} \cdot (\log x)^2 + \frac{t^3}{3!} \cdot (\log x)^3 + \ldots$$

From this we obtain a series expansion

$$x_+^{-m+t} = t^{-1} \cdot \rho_m + \gamma_0 + t\gamma_1 + \ldots$$

where ρ_m and $\{\gamma_\nu\}$ are distributions. In particular the reader may verify that

$$(-1)^{m-1}(m-1)! \cdot \rho_m = \partial^m(H_+)$$

Let us then consider the constant term γ_0. The linear t-term in the expansion of $t(t-1) \cdot (t - m + 1) x_+^{-m+t}$ becomes

$$(-1)^{m-1}(m-1)! \cdot \gamma_0 + \frac{m(m-1)}{2} \cdot \rho_m$$

If $x > 0$ we notice that

$$\partial^m(\log x) = (-1)^{m-1} \cdot (m-1)! \cdot x^{-m}$$

From the above γ_0 restricts to the density x^{-m} when $x > 0$. At the same time γ_0 is a distribution defined on the whole x–line supported by $\{x \geq 0\}$. We set

$$x_+^{-m} = \gamma_0 \tag{*}$$

and refer to this as Euler's extension of the density x^{-m} which from the start is defined on $\{x > 0\}$. So in (*) we have found distributions for every positive integer m.

0.1.2 Further formulas. With $s=-1+z$ where z is a small non-zero complex number one has

$$\text{(i)}\qquad z\cdot x_+^{-1+z}=\partial(x_+^z)$$

Next, if $x>0$ we have

$$x^z=e^{z\log x}=1+\sum_{k=1}^\infty \frac{(\log x)^k}{k!}\cdot z^k$$

Introducing the Heaviside distribution H_+ which is 1 on $x\geq 0$ and zero on $x<0$ this means that

$$\text{(ii)}\qquad \partial(x_+^z)=\partial(H_+)+\sum_{k=1}^\infty \partial(\frac{(\log x)^k}{k!}\cdot H_+(x))z^k$$

From this we get a Laurent expansion of x_+^s at $s=-1$. The crucial point is that the distribution derivative

$$\text{(iii)}\qquad \partial(H_+)=\delta_0$$

where δ_0 is the Dirac distribution at $x=0$. It follows that

$$\text{(iv)}\qquad x_+^{-1+z}=z^{-1}\cdot\delta_0+\sum_{k=1}^\infty \partial(\frac{(\log x)^k}{k!}\cdot H_+(x))z^{k-1}$$

In particular the constant term becomes

$$\text{(v)}\qquad \partial(\log x\cdot H_+(x))$$

To find this distribution we take a test-function g and a partial integration gives

$$-\int_0^\infty(\log x\cdot g'(x)\,dx=\int_0^1\frac{g(x)-g(0)}{x}\,dx+\int_1^\infty\frac{g(x)}{x}\,dx$$

From this we conclude that the distribution x_+^{-1} is defined on test-functions by the formula:

$$x_+^{-1}(g)=\frac{(-1)^{m-1}}{(m-1)!}\cdot\int_0^1\frac{g(x)-g(0)}{x}\,dx+\int_1^\infty\frac{g(x)}{x}\,dx$$

0.1.3 Exercise. For each test-function g and integer $m\geq 2$ we set

$$T_{m-1}(g)(x)=g(0)+g'(0)x\ldots\frac{g^{(m-1)}(0)}{(m-1)!}\cdot x^{m-1}$$

Show from the above via partial integrations that

$$x_+^{-m}(g)=\frac{(-1)^{m-1}}{(m-1)!}\cdot\int_0^1\frac{g(x)-T_{m-1}(g)(x)}{x^m}\,dx+\int_1^\infty\frac{g(x)}{x^m}\,dx$$

0.1.4 The distributions $(x+i0)^\lambda$ and $(x-i0)^\lambda$. In the upper half plane there exists the single valued branch of $\log z$ whose argument stays in $(0,\pi)$ and for every complex number λ we have

$$z^\lambda=e^{\lambda\cdot\log z}$$

In § 3 we shall learn how to contruct boundary value distributions of analytic functions defined in strip domains above or below the real x-line. In particular there exists the distribution $(x+i0)^\lambda$ defined on test-functions $g(x)$ by the limit formula

$$\lim_{\epsilon\to 0}\int(x+i\epsilon)^\lambda\cdot g(x)\,dx$$

Notice that this limit exists for all complex λ, i.e even when the real part becomes very negative. In the same way we have the single valued branch of $\log z$ in the lower half-plane whose argument stays in $(-\pi, 0)$ and construct the distribution $(x - i0)^\lambda$ defined by

$$(x - i0)^\lambda(g) = \lim_{\epsilon \to 0} \int (x - i\epsilon)^\lambda \cdot g(x)\, dx$$

Since $\lambda \mapsto e^{\lambda \cdot \log z}$ are entire in λ, we get two entire distribution valued functions by $(x - i0)^\lambda$ and $(x - i0)^\lambda$. Regarding the choice of branches for the log-functions we see that

$$(x - i0)^\lambda = e^{-2\pi i\lambda} \cdot (x + i0)^\lambda \quad : x < 0$$

At the same time

$$(x + i0)^\lambda = (x - i0)^\lambda = x^\lambda \quad : x > 0$$

From this we see that the distribution

$$(x + i0)^\lambda - e^{2\pi i\lambda} \cdot (x - i0)^\lambda$$

is supported by $x \geq 0$ and expressed by the density $(1 - e^{2\pi i\lambda}) \cdot x^\lambda$. The conclusion is that one has the equation

$$\mu_\lambda = \frac{(x + i0)^\lambda - e^{2\pi i\lambda} \cdot (x - i0)^\lambda}{1 - e^{2\pi i\lambda}}$$

Remark. The equation (xx) is more involved compared to the previous description of the meromorphic μ-function found via Euler's functional equation. But (xx) has the merit that the denominator is an entire distribution valued function and when one passes to Fourier transforms it turns out that (xx) is quite useful.

Principal value integrals. If $g(x)$ is a test-function there exists a limit

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{g(x)}{x}\, dx$$

This yields a distribution denoted by $\mathrm{VP}(x^{-1})$. Outside $\{x = 0\}$ it is given bt the density x^{-1} where it agrees with $(x + i0)^{-1}$ and hence the difference

$$\mu = \mathrm{VP}(x^{-1}) - (x + i0)^{\cdot -1}$$

is supported by $\{x = 0\}$.

Exercise. Notoice that

$$\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{1}{x + i\epsilon}\, dx = \log(1 + i\epsilon) - \log(-1 + i\epsilon) = -\pi i$$

and use this to show that

$$\mu = -\pi i \cdot \delta_0$$