XII. A system of infinite linear equations.

Introduction. The main issue in this section is the construction of a unique solution to the system

$$(*)\qquad \sum_{q\neq p}\frac{x_q}{p-q}=0$$

where (*) hold for all positive integers p and $\{x_q\}$ is a sequence of real numbers for which the series

$$(**)\qquad \sum_{q=1}^{\infty}\frac{x_q}{q}$$

is convergent. The fact that (*) has a non-trivial solution is far from evident. Before the study of (*) in Section 1 we discuss a general situation which was described by Carleman in his major lecture at the international congress in Zürich 1932. A homogeneous system of linear equations with an inifinite number of variables takes the form

$$(***)\qquad \sum_{q=1}^{\infty}c_{pq}x_q=a_p\quad:\quad p=1,2,\ldots$$

where $\{c_{pq}\}$ is a matrix with an infinite number of elements. A sequence $\{x_q\}$ of complex numbers is a solution to (***) if the sum in the left hand side converges for each p and has value a_p. Notice that one does not require that the series are absolutely convergent.

The generic case. To every $p\geq 1$ we have the linear form L_p defined on finite sequences $\{x_1,x_2,\ldots\}$, i.e. where $x_q=0$ when $q>>0$ by:

$$L_p(x_\bullet)=\sum_{q\geq 1}c_{pq}\cdot x_q$$

Follownig Carleman the generic case occurs when $c_{1q}\neq 0$ for every q and the linear forms $\{L_p\}_1^\infty$ are linearly independent. The last condition means that for every positive integer M there exists a strictly increasing sequence $q_1<\ldots<q_M$ so that the $M\times M$-matrix with elements $a_{p\nu}=c_{pq_\nu}$ has a non-zero determinant.

The R-functions. Assume that the system (***) is generic. Given a sequence $\{a_p\}$ we seek a solution $\{x_q\}$. The necessary and sufficient condition for its existence goes as follows: Consider n-tuples of positive integers $m_1,\ldots,m_n$ where $n\geq 2$. For every such n-tuple and $1\leq k\leq n-1$ we set

$$\mathcal{D}(k)=\{\nu:\ m_1+\ldots+m_k<\nu\leq m_1+\ldots+m_{k+1}\}$$

Next, with $M=m_1+\ldots+m_N$ we denote by $\mathcal{F}(m_1,\ldots,m_n)$ the family of sequences $(x_1,\ldots,x_M)$ such that the following inequalities hold for every $1\leq k\leq n-1$:

$$\Big|\sum_{q=1}^{q=\nu}c_{pq}x_q-a_q\Big|\leq\frac{1}{k}\quad:\ \nu\in\mathcal{D}(k)\quad\text{and}\quad 1\leq p\leq k$$

The generic assumption implies that the set $\mathcal{F}(m_1,\ldots,m_n)$ is non-empty provided that we start with a sufficiently large m_1 and for every such n-tuple we set

$$R(m_1,\ldots,m_n)=\min\sum_{\nu=1}^{m_1}x_\nu^2$$

where the minimum is taken over sequences $x_1,\ldots,x_{m_1}$ which give the starting terms of some sequence $x_1,\ldots,x_M\in\mathcal{F}(m_1,\ldots,m_n)$.

Theorem. *The necessary and sufficient condition in order that (***) has a least one solution is that there exists a constant K and an infinite sequence of positive integers $\mu_1,\mu_2,\ldots$ such that*

$$R(\mu_1,\mu_2,\ldots,\mu_r)\leq K\quad\text{hold for every } r$$

Remark. The reader may consult [Carleman] for further remarks about this result and also comments upon the more involved criterion for non-generic linear systems. From now on we study linear systems which arise as follows: Consider a rational function of two variables x, y:

$$f(x,y) = \frac{a_0(x) + a_1(x)y + \ldots a_n(x)y^n}{b_0(x) + b_1(x)y + \ldots b_m(x)y^m}$$

Here n and m are positive integers and $a_\nu(x)$ and $b_j(x)$ polynomials in x. No special assumptions are imposed on these polynomials except that $b_m(x)$ and $a_n(x)$ are not identically zero. For example, it is not neccesary that the degree of b_m is $\geq \deg(b_j)$ for all $0 \leq j \leq m-1$.

0.1 Proposition *Let $b_m^{-1}(0)$ be the finite set of zeros of b_m. Let $\zeta_0 \in \mathbf{C} \setminus b_m^{-1}(0)$ be such that*

$$\sum_{j=0}^{j=m} b_j(\zeta_0) \cdot q^j = 0$$

holds for some finite set of positive integers, say $1 \leq q_1 < \ldots < q_k$. Then there exists $\delta > 0$ such that

$$\sum_{j=0}^{j=m} b_j(\zeta) q^j \neq 0 \quad : \text{for all positive integers } q \quad : \quad 0 < |\zeta - \zeta_0| < \delta$$

Exercise. Prove this result.

Next, consider the sequence of polynomials of the complex ζ-variable given by:

$$B_q(\zeta) = b_0(\zeta) + b_1(\zeta)q + \ldots + b_m(\zeta)q^m \quad : q = 1, 2, \ldots$$

0.2 Proposition *Put*

$$W = b_m^{-1}(0) \bigcup_{q \geq 1} B_q^{-1}(0)$$

Then W is a discrete subset of $\mathbf{C}$, i.e its intersection with any bounded disc is finite.

Exercise. Prove this result where a hint is to apply Rouche's theorem.

Next, put $W^* = W \cup a_n^{-1}(0)$, i.e. add the zeros of the polynomial a_n to W.

0.3 Proposition. *Let $\zeta_0 \in C \setminus W^*$ and suppose that $\{x_q\}$ is a sequence such that the series*

(i)
$$\sum_{q=1}^{\infty} f(\zeta_0, q) \cdot x_q$$

is convergent. Then the series

(ii)
$$\sum_{q=1}^{\infty} f(\zeta, q) \cdot x_q \quad \text{converges for every } \zeta \in C \setminus W^*$$

Moreover, the series sum is a meromorphic function of ζ whose poles are contained in W^*.

Remark. Proposition 0.3 gives a procedure to find solutions $\{x_q\}$ which is not the trivial null solution to a homogeneous system:

(*)
$$\sum_{q \neq p} f(p,q) \cdot x_q = 0 \quad : \quad p = 1, 2, \ldots$$

More precisely, assume that the rational function $f(x,y)$ is such that $f(p,q) \neq 0$ when p and q are distinct positive integers. To get a solution $\{x_q\}$ to (iii) it suffices to begin with to verify (i) in Proposition 0.3 for some ζ_0 and then also try to find $\{x_q\}$ so that the meromorphic function

(**)
$$\zeta \mapsto \sum_{q=1}^{\infty} x_q \cdot f(\zeta, q)$$

has zeros at every positive integer. Using this criterium for a solution we can show the following:

Theorem. *For every complex number $a \in \mathbf{C} \setminus (-\infty, 0\,]$ the system*

$$\sum_{q=1}^{\infty} \frac{x_q}{p+aq} = 0 \quad : \quad p = 1, 2, \ldots$$

has no non-trivial solution $\{x_q\}$.

There remains to analyze the case when a is real and < 0. In this case complete answers about possible when f is the rational function were established by K. Dagerholm in his Ph.D-thesis at Uppsala University from 1938 with Beurling as supervisor. The hardest case occurs when $a = 1$ which will be studied in the next section.

1. The Dagerholm series.

Let $\mathcal{F}$ be the family of all sequences of real numbers $x_1, x_2, \ldots$ such that the series

$$\text{(i)} \qquad \sum_{q=1}^{\infty} \frac{x_q}{q} < \infty$$

We only require that the series is convergent, i.e. it need not be absolutely convergent.

1.1 Theorem. *Up to a multiple with a real constant there exists a unique sequence $\{x_q\}$ in $\mathcal{F}$ such that*

$$\sum_{q \neq p} \frac{x_q}{p-q} = 0 \quad : \quad p = 1, 2, \ldots$$

The proof of uniquenss relies upon Jensen's formula and the solution to a specific Wiener-Hopf equation. We begin to describe the strategy of the proof. For each $\{x_q\} \in \mathcal{F}$ there exists the meromorphic function

$$\text{(ii)} \qquad h(z) = \sum_{q=1}^{\infty} \frac{x_q}{z-q}$$

To see that $h(z)$ is defined we notice that if s_* is the series sum in (i) then

$$\text{(iii)} \qquad h(z) + s_* = \sum_{q=1}^{\infty} x_q \cdot [\frac{1}{z-q} + \frac{1}{q}] = z \cdot \sum_{q=1}^{\infty} \frac{x_q}{q(z-q)}$$

It is clear that the right hand side is a meromorphic function with poles confined to the set of positive integers. Hence we obtain the entire function:

$$H(z) = \frac{1}{\pi} \cdot \sin(\pi z) \cdot h(z)$$

1.2 Proposition. *The following hold for each positive integer:*

$$H(p) = (-1)^p \cdot x_p \quad : H'(p) = (-1)^q \cdot \sum_{q \neq p} \frac{x_q}{p-q} = 0$$

Proof. Let $p \geq 1$ be an integer. With ζ small we have

$$H(p+\zeta) = \frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) \cdot [\frac{x_p}{\zeta} + \sum_{q \neq p} \frac{x_q}{p+\zeta-q}]$$

A series expansion of the complex sine-function at πp gives

$$\frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) = [\zeta \cdot \cos(\pi p) + O(\zeta^3)] \cdot [\frac{x_p}{\zeta} + \sum_{q \neq p} \frac{x_q}{p+\zeta-q}]$$

Proposition 1.2 follow since $\cos \pi p = (-1)^p$.

Remark. Proposition 1.2 shows that $\{x_p\}$ solves the homogeneous system in Theorem 1.1 if the complex derivative of the entire H-function has zeros on all positive integers. This observation is the gateway towards the proof of Dagerholm's Theorem. But let us first establish the uniqueness.

2. Proof of uniqueness

Let $\{x_q\}$ be a sequence in $\mathcal{F}$. From the constructions in above it is clear that the meromorphic function $h(z)$ satisfies the following in the left half-plane $\mathfrak{Re}(z) \leq 0$:

(i) $$\lim_{x\to-\infty} h(x) = 0: \quad |h(x+iy)| \leq C_* \quad : x \leq 0$$

where C_* is a constant. Moreover, in the right half-plane there exists a constant C^* such that

(ii) $$|h(x+iy)| \leq C^* \cdot \frac{|x|}{1+|y|} \quad : |x-q| \geq \frac{1}{2} \text{ for all positive integers}$$

To $h(z)$ we get the entire function $H(z)$ and (i-ii) above give the two the estimates below in the right half-plane:

(iii) $$|H(x+iy)| \leq Ce^{\pi|y|} : x \leq 0 \quad : \quad |H(x+iy)| \leq C\frac{|x|}{1+|y|} \cdot e^{\pi|y|}$$

Moreover, the first limit formula in (i) gives

(iv) $$\lim_{x\to-\infty} H(x) = 0$$

It is easily seen that the same upper bounds hold for the entire function $H'(z)$ and a straightforward application of the Phragmén-Lindelöf theorem gives:

2.1 Proposition. *The complex derivative of $H(z)$ satisfies the growth condition:*

$$\lim_{r\to\infty} e^{-\pi r\cdot|\sin\phi|} \cdot |H'(re^{i\theta}| = 0 \quad : \text{holds uniformly when } 0 \leq \theta \leq 2\pi$$

Now we are prepared to prove the uniqueness part in Theorem 0.1. For suppose that we have two sequences $\{x_q\}$ and $\{x_q^*\}$ which both give solutions to (*) and are not equal up to a constant multiple of each other. The two sequences give entire functions H_1 and H_2. Since both are constructed via real sequences their Taylor coefficients are real and there exists a linear combination

$$G = aH_1 + bH_2$$

where a, b are real numbers and the complex derivative $G'(0) = 0$. The hypothesis that there exists two $\mathbf{R}$-linearly independent solutions to (*) leads to a contradiction once we have proved the following

2.2 Lemma *The entire function $G'(z)$ is identically zero.*

Proof. To simplify notations we set $g(z) = G'(z)$ and consider the series expansion

$$g(z) = a_p z^p + a_{p+1} z^{p+1} + \ldots$$

where a_p ia the first non-vanishing coefficient. Since $g(0) = G'(0) = 0$ we have $p \geq 1$ and since the two x-sequences both are solutions to (*), the second equation in Proposition 0.2 gives

(i) $$g(p) = 0 \quad : p = 1, 2, \ldots$$

Next, G is real-valued on the x-axis and since the H-functions are zero for every integer ≤ 0 the same holds for G. *Rolle's theorem* implies that for every $n \geq 1$ there exists

(i) $$-n < \lambda_n < -n+1 \quad : g(\lambda_n) = 0$$

So if $\mathcal{N}$ is the counting function for the zeros of the entire g-function one has the inequality

(iii) $$\mathcal{N}(r) \geq [2r]$$

where $[2r]$ is the largest integer $\leq 2r$. Next, recall that a_p is the first non-zero term in the series expansion of g. Hence Jensen's formula gives:

$$(*) \qquad \log|a_p| + p\cdot\log\, r + \int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Log}\,|g(re^{i\theta}\,|\,\cdot d\theta$$

Proposition 2.1 applied to $g(z)$ gives:

$$(iv) \qquad \int_0^{2\pi} \mathrm{Log}\,|g(re^{i\theta}\,|\,\cdot d\theta = 2r - m(r) \quad \text{where } \lim_{r\to\infty} m(r) = +\infty$$

At this stage we get the contradiction as follows. First (iii) gives

$$\int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} \geq 2r - \mathrm{Log}(r) - 1$$

Now (*) and (iii) give the inequality

$$(vi) \qquad \log|a_p| + p\cdot\log\, r + 2r - 1 - \log r \leq 2r - m(r) \quad : \; r \geq 1$$

Here $p \geq 1$ which therefore would give:

$$\log|a_p| - 1 + m(r) \leq 0$$

But this is impossible since we have seen that $m(r) \to +\infty$.

3. Proof of existence

We start with a general construction. Let $\phi(z)$ be analytic in the unit disc D which extends to a continuous function on T except at the point $z = 1$. We also assume that there exists some $0 < \beta < 2$ and a constant C such that

$$(1) \qquad |\phi(\zeta)| \leq C|1-\zeta|^{-\beta}$$

This implies that the function

$$\theta \mapsto \theta\cdot\phi(e^{i\theta})$$

is integrable on the unit circle. Hence there exists the entire function

$$(2) \qquad f(z) = \frac{-i}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta z}\cdot\theta\cdot\phi(e^{i\theta})\cdot d\theta$$

Next, with $\epsilon > 0$ small we let γ_ϵ be the interval of the circle $|z-1| = \epsilon$ with end-points at the intersection with $|z| = 1$. So on γ_ϵ we have

$$z = 1 + \epsilon\cdot e^{i\theta} \quad : \; -\frac{\pi}{2} + \epsilon_* < \theta < \frac{\pi}{2} - \epsilon_*$$

where ϵ_* is small with ϵ. We obtain the entire function

$$F(z) = \frac{1}{2\pi}\int_\epsilon^\pi e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi}\int_{-\pi}^{-\epsilon} e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{e^{-z\cdot\mathrm{Log}\zeta}\cdot\phi(\zeta)d\zeta}{\zeta}$$

If $z = n$ is an integer we have

$$e^{-in\theta} = \zeta^{-n} \quad : \quad e^{-n\cdot\mathrm{Log}\zeta} = \zeta^{-n}$$

Hence we get

$$(*) \qquad F(n) = \frac{1}{2\pi i}\cdot\int_{\Gamma_\epsilon}\frac{\phi(\zeta)\cdot d\zeta}{\zeta^{n+1}}$$

where Γ_ϵ is the closed curve given as the union of γ_ϵ and the interval of T where $|\theta| \geq \epsilon$. Cauchy's formula applied to ϕ gives:

2.1 Proposition. *Let* $\phi(\zeta) = \sum c_n\zeta^n$. *Then*

$$F(n) = c_n \quad : \; n \geq 0 \quad \text{and} \quad F(n) = 0 \quad n \leq -1$$

Next, using (i) above we also have:

2.2 Proposition. *The complex derivative of F is equal to f.*

Proof. With $\epsilon > 0$ the derivative of the sum of first two terms from the construction of $F(z)$ above become

$$\text{(i)} \qquad \frac{1}{2\pi}\int_{|\theta|\geq\epsilon} -i\theta\cdot e^{-iz\theta}\phi(e^{i\theta})d\theta$$

In the last integral derivation with respect to z gives

$$\text{(ii)} \qquad -\frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{e^{-z\cdot\text{Log}\zeta}\cdot\phi(\zeta)d\zeta}{\zeta}$$

Now $\zeta = 1+\epsilon\cdot e^{i\theta}$ during the integration along γ_ϵ which gives:

$$|\text{Log}\,(1+\epsilon\cdot e^{i\theta})|\leq\epsilon$$

At the same time the circle interval γ_ϵ has length $\leq\epsilon$ and hence the growth condition (i) shows that the integral (iii) tends to zero when $\epsilon\to 0$. Finally, since we assumed that the function $\theta\mapsto\theta\cdot\phi(e^{i\theta})$ is absoutely integrable on T a passage to the limit as $\epsilon\to 0$ gives $F' = f$ as requested.

2.3 Conclusion. If n is a positive integer in Proposition 2.3 we have:

$$(**) \qquad F'(n) = \frac{-i}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\cdot\theta\cdot\phi(e^{i\theta})\cdot d\theta$$

These integrals are zero for every $n\geq 1$ if and only if $\theta\cdot\phi(e^{i\theta})$ is the boundary value function of some $\psi(z)$ which is analytic in the exterior disc $|z|>1$. In 2.X we will show that this is true for a specific ϕ-function satisfying the growth condition (1) above and in addition the series

$$\sum_{n=1}^{\infty}(-1)^n\cdot\frac{c_n}{n}$$

converges.

2.4 How to deduce a solution $\{x_p\}$. Suppose we have found ϕ satisfying the conditions above which gives the entire function $F(z)$ whose derivatives are zero for all $n\geq 1$. Now we set

$$x_p = (-1)^p\cdot c_p$$

By (***) this sequence belongs to $\mathcal{F}$ and we construct the associated entire function $H(z)$. From (i) in Proposition 0.1 and Proposition 2.1 we get

$$H(p) = (-1)^p\cdot x_p = c_p = F(p)$$

In addition both H and F have zeros at all integers ≤ 0. Next, by the construction of F it is clear that this is an entire function of exponential type and by the above the entire function $G = H - F$ has zeros at all integers. We leave as an exercise to the reader to show that G must be identically zero. The hint is to use similar methods as in the proof of the uniqueness. It follows that

$$H'(q) = F'(q) = 0$$

for all $q\geq 1$. By (ii) in Proposition 0.2 this means precisely that $\{x_p\}$ is a solution to the requested equations in (*) which gives the existence in Dagerholm's Theorem.

2.5 The construction of ϕ.

There remains to find ϕ such that the conditions above hold. To obtain ϕ we start with the integrable function on T defined by:

$$u(\theta) = \frac{1}{2}\cdot\log\frac{1}{|\theta|} \quad : \; -\pi<\theta<\pi$$

We get the analytic function

$$g(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} - \zeta}{e^{i\theta} + \zeta} \cdot u(\theta) \cdot d\theta$$

In the exterior disc we find the analytic function

$$\psi(\zeta) = \exp\ -\bar{g}(\frac{1}{\bar{\zeta}})$$

Let us also put $\phi_*(\zeta) = e^{g(\zeta)}$. Now we have

$$\log|\phi(e^{i\theta})| = \mathfrak{Re}\, g(e^{i\theta}) = u(\theta)$$

In the same way we see that

$$\log|\psi(e^{i\theta})| = -\mathfrak{Re}\, g(e^{i\theta}) = -u(\theta)$$

Since $2u(\theta) = -\log\,|\theta|$ it follows that

$$\log\,|\theta| + \log|\phi_*(e^{i\theta})| = \log|\psi(e^{i\theta})|$$

Taking exponentials we obtain

$$|\theta| \cdot |\phi_*(e^{i\theta})| = |\psi(e^{i\theta})|$$

Exercise. Check also arguments and verify that we can remove absolute values in the last equality to attain

(*) $$|\theta| \cdot \phi_*(e^{i\theta}) = \psi(e^{i\theta})$$

Here (*) is not precisely what we want since our aim was to construct ϕ so that $\theta| \cdot \phi(e^{i\theta})$ is equal to the boundary value of an analytic function in $|z| > 1$. So in order to get rid of the absolute value for θ in (*) we modify ϕ_* as follows: Set

$$\rho(\theta) = \frac{\pi i}{2} \cdot \operatorname{sign}\theta \cdot e^{-i\theta} \quad : \, -\pi < \theta < \pi$$

Next, consider the two analytic functions in D, respectively in $|z| > 1$ defined by:

$$\phi_1(z) = \frac{1}{\sqrt{1-z^2}} \quad \text{and} \quad \psi_1(z) = \frac{1}{\sqrt{1-z^{-2}}}$$

Exercise. Show that one has the equality

$$\rho(\theta) = \frac{\phi_1(e^{i\theta})}{\psi_1(e^{i\theta})}$$

when $-\pi < \theta < \pi$ and $\theta \neq 0$.

The ϕ-function. it is defined by

$$\phi(z) = \frac{z}{\sqrt{1-z^2}} \cdot \phi_*(z)$$

From (*) above and the construction of ρ it follows that

$$\theta \cdot \phi(e^{i\theta}) = \frac{\pi}{2} \cdot \psi_1(e^{i\theta}) \cdot \psi(e^{i\theta})$$

The right hand side is the boundary function of an analytic function in $|z| > 1$ and hence ϕ satisfies (**) from XX. Consider its Taylor expansion

$$\phi(z) = \sum c_n \cdot z^n$$

There remains to verify that the series (**) converges and that ϕ satisfies the growth condition in XX. To prove this we begin to analyze the function

(i) $$\phi_*(z)=e^{g(z)}$$

Rewrite the u function as a sum

(ii) $$u(\theta)=\frac{1}{2}\log\Big|\frac{1}{1-e^{i\theta}}\Big|+k(\theta)\quad\text{where}\quad k(\theta)=\frac{1}{2}\log\Big|\frac{1-e^{i\theta}}{\theta}\Big|$$

When θ is small we have an expansion

(iii) $$\frac{1-e^{i\theta}}{\theta}=-i+\theta/2+\ldots$$

From this we conclude that the k-function is at least twice differentiable as a function of θ. So the Fourier coefficients in the expansion

(iv) $$k(e^{i\theta})=\sum b_\nu e^{i\nu\theta}$$

have a good decay. For example, there is a constant C such that

(v) $$|b_\nu|\leq\frac{C}{\nu^2}\quad:\ \nu\neq 0$$

This implies that the analytic function

(vi) $$\mathcal{K}(z)=\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\cdot k(e^{i\theta})\cdot d\theta$$

yields a bounded analytic function in the unit disc. Next, the construction of the g-function gives:

(vii) $$g(z)=\frac{1}{2}\cdot\log\frac{1}{1-z}+\sum b_\nu z^\nu\implies$$
$$\phi_*(z)=\frac{1}{\sqrt{1-z)}}\cdot e^{\mathcal{K}(z)}$$

We conclude that

$$\phi(z)=\frac{z}{1-z}\cdot\frac{1}{\sqrt{1+z)}}\cdot e^{\mathcal{K}(z)}$$

Since $\mathcal{K}(z)$ extends to a continuous function on the closed disc it follows that ϕ satisfies the growth condition (1) with $\beta=1$. Moreover, the function $\theta\cdot\phi(e^{i\theta})$ belongs to $L^1(T)$ since $\frac{1}{\sqrt{1+e^{i\theta}}}$ is integrable. There remains only to prove:

Lemma. The series

$$\sum_{p=1}^{\infty}(-1)^n\cdot\frac{c_n}{n}$$

is convergent.

Proof. let us put

$$A(z)=\frac{z}{\sqrt{1+z)}}\cdot e^{\mathcal{K}(z)}$$

This gives

$$\phi(z)=\frac{A(1)}{1-z}+\frac{A(z)-A(1)}{1-z}$$

From (v) it follows that $K(z)$, and hence also $e^{K(z)}$ is differentiable at $z01$ which gives the existence of a constant C such that

$$|\frac{A(z)-1}{1-z}| \leq C \cdot \frac{1}{|\sqrt{1+z}|}$$

Here the function $\theta \mapsto \frac{1}{|\sqrt{1+e^{i\theta}}|}$ belongs to $L^p(T)$ for each $p < 2$ which by the inequality for L^p-norms between functions and their Fourier coefficients in XX for example implies that if $\{c^*_\nu\}$ give the Taylor series for $\frac{A(z)-1}{1-z}$ then

$$\sum |c^*_\nu|^3 < \infty$$

Now Hölder's inequality gives

$$\sum \frac{|c^*_\nu|}{\nu} \leq \big(\sum |c^*_\nu|^3\big)^{\frac{1}{3}} \cdot \big(\sum \nu^{-3/2}\big)^{\frac{2}{3}} < \infty \tag{8}$$

We conclude that the Taylor series for ϕ becomes

$$A(1) \cdot (1 + z + z^2 + \ldots) + \sum c^*_\nu z^\nu$$

Hence $c_n = A(1) + c^*_\nu$ and now Lemma xx follows since the alternating series $\sum (-1)^n \frac{1}{n}$ is convergent and we have the absolute convergence in XX above.

A theorem by Kjellberg.

Introduction. We expose an article by Bo Kjellberg - former Ph.D-student of Beurling - which deals with a comparison between integrals and certain sums of analytic functions of exponential type in a half-space. Here is the situation: Let $f(z)$ be analytic in the closed half-spae $\mathfrak{Re}\, z \geq 0$. Assume that the function $x \mapsto |f(x)|$ is bounded on the real x-axis and there exists a positive real number c such that

$$\limsup_{|z|\to\infty} \frac{\log |f(z)|}{|z|} = c \tag{1}$$

Next, let $\phi(t)$ be a C^2-functionon $t \geq$ with $\phi(0) = 0$ for which

$$\phi'(t) \geq 0 \quad : \quad t \cdot \phi''(t) + \phi'(t) \geq 0$$

Thus, ϕ is non-decreasing and the last inequality means that it is convex as a function of $\log t$.

Gap sequences. A strictly increasing sequence $\{\lambda_n\}$ has gaps of order $\geq \delta$ if

$$\lambda_{n+1} - \lambda_n \geq 2 \cdot \delta \quad : \, n = 1, 2, \ldots$$

0.1 Theorem. *Let f and ϕ be as above. Then, for every $\delta > 0$ and each λ-sequence with gaps of order $\geq \delta$ one has the implication*

$$\int_0^\infty \phi(|f(x)|)\, dx < \infty \implies \sum_{n=1}^\infty \phi(e^{-\delta \cdot c} \cdot |f(\lambda_n)|) < \infty \tag{*}$$

Remark. The result above gave an affirmative anser to a question posed by Boas in the article *Inequalities between series and integrals involving entire functions.* The proof of Theorem 0.1 has two ingredients. First one has a uniqueness result which goes as follows:

0.2 Theorem. *Let f and ϕ be as above where (1) holds for f and the integral in the left hand side of Theorem 0.1 is finite. Then, if*

$$\int_0^{1/2} \frac{\phi(t)}{t \cdot \log \frac{1}{t}}\, dt = +\infty$$

it follows that f is identically zero.

Remark. We shall first prove Theorem 0.2 and notice that it then is suffcient to prove Theorem 0.1 under the *additional assumption* that the integral in Theorem 0.2 is finite. In both theorems the convexity of ϕ with respect to $\log t$ will be used. More precisely we first notice that if $H(x, y)$ is a harmonic function defined in some open set in $\mathbf{C}$ then

$$u(x, y) = \phi(e^{H(x,y)})$$

is subharmonic. Indeed , this follows since

$$u_x = H_x \cdot e^H \cdot \phi'(e^H) \implies u_{xx} = H_{xx} e^H \cdot \phi'(e^H) + H_x^2 e^H (e^H \cdot \phi''(e^H) + \phi'(e^H))$$

A similar equation holds for u_{yy} and adding the result we use that $\Delta(H) = 0$ and see that (2) above entails that $\Delta(u) \geq 0$. We shall need the following prelimninary result:

1.1 Proposition. *Let $u(x)$ be a continuous and non-negtive function on $x \geq 0$ where*

$$\limsup_{x\to+\infty} u(x) = +\infty$$

and g is a stricty increasing function on $x > 0$. Then the implication below holds:

$$\int_0^\infty g(u(x))\, dx < \infty \quad \text{and} \quad \int_1^\infty \frac{g(u)}{u}\, du = +\infty \implies \int_1^\infty \frac{u(x)}{x^3}\, dx = +\infty$$

Proof. Let $v(x)$ be the non-decreasing funftion which is equi-distributed with u as explained in §§. Then we have

$$\int_1^\infty \frac{v(x)}{x^3}\,dx \le \int_1^\infty \frac{u(x)}{x^3}\,dx$$

while the left and side integrals are unchanged when u is replaced by v. Hence it suffices to prove the result for v, i.e. from now on we assume that u is non-decreasing. If $0 < a < A$ a partial integration gives

$$\int_a^A g(u(x))\,dx = \quad \text{and}$$

Easy finish ....

1.2 Majorisations in quarter-planes. We are given the constnst c which implies that if $c' > c$ then there is a constant B such that the inequality below holds in the right half-plane:

$$|f(x+iy)| \le B\dot{e}^{c'|y|}$$

Let us then choose $c^* > c'$ snd set

$$\psi(z) = \phi(e^{ic^*z} \cdot f(z)|)$$

By (xx) ψ is a subharmonic function in the quarter plane $U_* = \{x > 0\} \times \{y > 0\}$ and here

$$\psi(x+iy) = \phi(e^{-c^*\cdot y} \cdot |f(x+iy)|)$$

The subharmonicity of ψ entails that

$$\psi(z) \le h_1(z) + h_2(z)$$

where $h_1(z)$ is the harmonic extension of the boundary value function which is $\phi|f(x)|$ on the positive real axis and zero on the positive y-axis, while $h_2(x,0) = 0$ and

$$h_2(0,y) = \phi(e^{-c^*\cdot y} \cdot |f(iy)|)$$

Since $c^* > c'$ it follows from (xx) that $h_2(0,y)$ is bounded and tends to zero as $y \to \infty$ The growth properties of the two h-functions on the boundsry of U_* entail that both are represented as in § XX. Thus, one has

$$h_1(x+iy) = \frac{1}{\pi}\int_0^\infty y\cdot\psi(\xi)\cdot\Big[\frac{1}{(\xi-x)^2+y^2} - \frac{1}{(\xi+x)^2+y^2}\Big]\,d\xi$$

$$h_2(x+iy) = \frac{1}{\pi}\int_0^\infty x\cdot\psi(i\eta)\cdot\Big[\frac{1}{(\xi-x)^2+y^2} - \frac{1}{(\xi+x)^2+y^2}\Big]\,d\eta$$

From (i) we obtain

$$(1)\qquad \int_0^\infty h_1(x+iy)\,dx = \frac{2}{\pi}\int_0^\infty \psi(\xi)\cdot\arctan\frac{\xi}{y}\,d\xi \le \int_0^\infty \psi(\xi)\,d\xi$$

For h_2 we obtain

$$(2)\qquad \int_0^\infty h_1(x+iy)\,dx = \frac{1}{\pi}\int_0^\infty \psi(i\eta)\cdot\log\Big|\frac{\eta+y}{\eta-y}\Big|\,d\eta \le C + A\cdot\int_{2\delta}^\infty \frac{\psi(i\eta)}{\eta}\,d\eta$$

where A and C are two constants which the reader may derive from (x-x) above, Let us now estaimte the last integral in (2). Since ϕ is non-decreasing we see that (xx) and (xx) give have

$$\psi(i\eta) \le \phi(B\cdot e^{(c'-c^*)\eta})$$

Consider the variable substitution

$$t = B\cdot e^{(c'-c^*)\eta)} \implies \frac{dt}{t} = -Bc^*\cdot d\eta$$

Notice that it gives

$$\eta = \frac{1}{c^*-c'}\cdot\log\frac{B}{t}$$

It follows that
$$\int_{2\delta}^{\infty} \frac{\phi(B\cdot e^{(c'-c^*)\eta}}{\eta})\,d\eta = (c^*-c')\cdot \int_{t_0}^{\infty} \frac{\phi(t)}{t\cdot \log\frac{B}{t}}\,dt$$

Conclusion. Under the hypothesis that the integral (**) in Theorem 0.2 converges and integral in the left hand side in Theorem 0.1 also converges, it follows that for every $\delta > 0$ there exis constants C_1, C_2 which may depend upon δ such that
$$\int_{C_1}^{\infty} (h_1(x+iy) + h_2(x+iy))\,dy \leq C_2 \quad : \quad 0 \leq y \leq \delta$$
Together with the majorization in (xx) it follows that with another constant C_3 we have
$$\iint_{\square_\delta} \psi(x+iy)\,dxdy \leq C_3$$
where $\square_\delta = \{0 < x < \infty\} \times \{0 < y < \delta\}$.

NOW easy to finish the proof of theorems....