20. A Non-Linear PDE-equation

Introduction. We expose Carleman's article *Über eine nichtlineare Randwertaufgabe bei der Gleichung* $\Delta u = 0$ (Mathematisches Zeitschrift vol. 9 (1921). Here is the equation to be considered: Let Ω be a bounded domain in $\mathbf{R}^3$ with C^1-boundary and $\mathbf{R}^+$ the non-negative real line where u is the coordinate. Let $F(u, p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$(0,1) \qquad u \mapsto F(u), p)$$

is strictly increasing for every $p \in \partial\Omega$ and that $F(0, p) \geq 0$. Moreover,

$$(0.2) \qquad \lim_{u \to \infty} F(u, p) = +\infty$$

holds uniformly with respect to p. For a given point $Q_* \in \Omega$ we seek a function $u(x)$ which is harmonic in $\Omega \setminus \{Q_*\}$ and at Q_* it is locally $\frac{1}{|x-Q_*|}$ plus a harmonic function and on $\partial\Omega$ the inner normal derivative $\partial u/\partial n$ satisfies the equation

$$(*) \qquad \frac{\partial u}{\partial n}(p) = F(u(p), p) \quad : p \in \partial\Omega$$

Finally it is also assumed that u extends to a continuous function on $\partial\Omega$.

Theorem. *For each F as above the boundary value problem has a unique solution.*

Remark. Apart from the general result the subsequent proof is interesting since it teaches how to handle a class of non-linear boundary value problems. The strategy in Carleman's proof is to consider the family of boundary value problems where we for each $0 \leq h \leq 1$ seek u_h to satisfy

$$(*) \qquad \frac{\partial u_h}{\partial n}(p) = (1-h)u_h + h \cdot F(u_h(p), p) \quad : p \in \partial\Omega$$

where u_h has the same pole as u above. Starting with $h = 0$ one has a classical linear Neumann problem where a unique solution exists. To proceed from $h = 0$ to $h = 1$ the idea is to use a "homotopy argument" where one first easily reduces the proof to the case when F is a real-analytic function of u. Then the subsequent proof will show that if we have found a solution u_{h_0} for some $0 \leq h_0 < 1$, we obtain solutions u_h when $h_0 < h < h_0 + \epsilon$ for sufficiently small ϵ by solving an infinite system of linear boundary value problems. It goes without saying that this method is restricted to favorable cases as above where the uniqueness and robust properties of these solutions are present as we show below. But it is instructive to see how one can employ analytic series to handle such cases.

Now we turn to the proof of the theorem and begin with preliminary results concerned with uniqueness and the reduction to the case when F is real-analytic.

A.0. Proof of uniqueness. Suppose that u_1 and u_2 are two solutions and notice that $u_2 - u_1$. is harmonic in Ω. If $u_1 \neq u_2$ we may without loss of generality we may assume that the maximum of $u_2 - u_1$ is > 0. The maximum is attained at some $p_* \in \partial\Omega$ and by the strict maximum principle for harmonic functions we have

$$(i) \qquad u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*)$$

for all $x \in \Omega$.With $v = u_2 - u_1$ we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p), p) - F(u_1(p), p)$$

Here (0.1) entails that $\frac{\partial v}{\partial n}(p_*) > 0$ and since we have an inner normal derivative this violates (i) which proves the uniqueness.

A.1 Montonic properties. Let F_1 and F_2 be two functions which both satisfy (0.1) and (0.2) where

$$F_1(u, p) \leq F_2(u, p)$$

hold for all $(u,p) \in \mathbf{R}^+ \times \partial\Omega$. If u_1, respectively u_2 solve (*) for F_1 and F_2 it follows that $u_2(q) \leq u_1(q)$ for all $q \in \Omega$. To see this we set $v = u_2 - u_1$ which is harmonic in Ω. If $p \in \partial\Omega$ we get

$$\frac{\partial v}{\partial n}(p) = F_2(u_2(p),p) - F_1(u_1(p),p) \geq 0 \tag{i}$$

Suppose that the maximum of v is > 0 and let the maximum be attained at some point p_*. Since (i) is an inner normal it follows that we must have $0 = \frac{\partial v}{\partial n}(p)$ which would entail that

$$F_2(u_2(p_*)p_*) > F_2(u_1(p_*),p_*) \geq F_1(u_1(p_*),p_*) \implies$$

and this contradicts the strict inequality $u_2(p_*) > u_1(p_*)$ since we have an increasing function in (0.1).

A.2. A bound for the maximum norm. Let u be a solution to (*) and M_u denotes the maximum norm of its restriction to $\partial\Omega$. Choose $p^* \in \partial\Omega$ such that

$$u(p^*) = M_u \tag{1}$$

Let G be the Green's function which has a pole at Q_* while $G = 0$ on $\partial\Omega$. Now

$$h = u - M_u - G$$

is a harmonic function in Ω. On the boundary we have $h \leq 0$ and $h(p^*) = 0$. So p^* is a maximum point for this harmonic function in the whole closed domain $\bar{\Omega}$. It follows that

$$\frac{\partial h}{\partial n}(p^*) \leq 0 \implies$$

$$F(u(p^*),p^*) = \frac{\partial u}{\partial n}(p^*) \leq \frac{\partial G}{\partial n}(p^*)$$

Set

$$A^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

Then we have

$$F(M_u,p^*) \leq A^* \tag{*}$$

Hence the assumption (0.2) for F this gives a robust estimate for the maximum norm M_u. Next, let m_u be the minimum of u on $\partial\Omega$ and consider the harmonic function

$$h = u - m_u - G$$

This time $h \geq 0$ on $\partial\Omega$ and if $u(p_*) = m_u$ we have $h(p_*) = 0$ so here p_* is a minimum for h. It follows that

$$\frac{\partial h}{\partial n}(p_*) \geq 0 \implies F(u(p_*),p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$A_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

$$F(m_u,p^*) \geq A_* \tag{**}$$

Remark. Above $0 < A_* < A^*$ are constants which are independent of F. Hence the maximum norms of solutions $u = u_F$ are controlled if the F-functions stay in a family where (0.2) holds uniformly.

B. The linear equation.

Let $f(p)$ and $W(p)$ be a pair of continuous functions on the boundary $\partial\Omega$ where W is positive, i.e. $W(p) > 0$ for every boundary point. The classical Neumann theorem asserts that there exists a unique function U which is harmonic in Ω, extends to a continuous function on the closed domain and its inner normal derivative satisfies:

(1) $$\partial U/\partial n(p) = W(p)\cdot U(p) + f(p) \quad p\in\partial\Omega$$

The uniqueness is a consequence of Green's formula. For suppose that U_1 and U_2 are two solutions to (1) and set $v = U_1 - U_2$. Since v is harmonic in Ω it follows that:

$$\iiint_\Omega |\nabla(v)|^2 dxdydz + \iint_{\partial\Omega} v\cdot \partial v/\partial n\cdot dS = 0$$

Here $\partial v/\partial n = W(p)v$ and since $W(p) > 0$ holds on $\partial\Omega$ we conclude that v must be identically zero. For the unique solution to (1) some estimates hold. Namely, set

$$M_U = \max_p U(p) \quad \text{and} \quad m_U = \min_p U(p)$$

Since U is harmonic in Ω the the maximum and the minimum are taken on the boundary. If $U(p^*) = M_U$ for some $p^*\in\partial\Omega$ we have $\partial U/\partial n(p^*) \leq 0$. Set

$$W_* = \min_p W(p)$$

By assumption $W_* > 0$ and we get

$$M_U\cdot W(p^*) + f(p^*) = \partial U/\partial n(p^*) \leq 0 \implies M_U \leq \frac{|f|_{\partial\Omega}}{W_*}$$

where $|f|_{\partial\Omega}$ is the maximum norm of f on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial\Omega}}{W_*}$$

Hence the following inequality holds for the the maximum norm $|U|_{\partial\Omega}$:

(*) $$|U|_{\partial\Omega} \leq \frac{|f|_{\partial\Omega}}{W_*}$$

B.1 Estimates for first order derivatives. Let $p\in\partial\Omega$ and denote by N the inner normal at p. Since $\partial\Omega$ is of class C^1 a sufficiently small line segment from p along N stays in Ω. So at points $q = p + \ell\cdot N$ we can take the directional derivative of U along N_p This gives a function

$$\ell \mapsto \partial U/\partial N(p+\ell\cdot N)$$

Since the boundary is C^1 these functions are defined on a fixed interval $0\leq \ell\leq \ell^*$ for all p. With these notations there exists a constant B such that

(**) $$\big|\partial U/\partial N(p+\ell\cdot N)\big| \leq B\cdot ||\partial U/\partial n||_{\partial\Omega} \quad : p\in\partial\Omega\ :\ 0\leq\ell\leq\ell^*$$

where the size of B is controlled by the maximum norm of f on $\partial\Omega$ and the positive constant W_* above.

C. Proof of Theorem

Armed with the results above we can begin the proof of the Theorem. To begin with it suffices to prove the theorem when $F(u,p)$ is an analytic function with respect to u. For if we then take an arbitrary F-function satisfying (0.1) and (0.2), then F is uniformly approximated by a sequence $\{F_n\}$ of analytic functions and if $\{u_n\}$ are the unique solutions to $\{F_n\}$ then the estimates in (B) show that there exists a limit function $\lim_{n\to\infty} u_n = u$ where u solves (*) for the given F-function. So let us now assume that $u\mapsto F(u,p)$ is a real-analytic function on the positive real axis for each $p\in\partial\Omega$ where local power series converge uniformly with respect to p. In this situation there remains to prove the *existence* of a solution u to the PDE in (*) above Theorem 1. To attain this we proceed as follows.

C.1 The succesive solutions $\{u_h\}$. To each real number $0\leq h\leq 1$ we seek a solution u_h where

(1) $$\frac{\partial u_h}{\partial n}(p) = h\cdot F(u_h,p) + (1-h)\cdot u_h(p)$$

With $h = 0$ we have a linear equation

$$\frac{\partial u}{\partial n}(p) = u(p) \tag{2}$$

which is solved by the Green's function with a pole at Q_*. Next, suppose that $0 \leq h_0 < 1$ and that we have found the solution u_{h_0} in (1) above. Set $u_0 = u_{h_0}$ and with $h = h_0 + \alpha$ for some small $\alpha > 0$ we shall find u_h by a series

$$u_h = u_0 + \sum_{\nu=1}^{\infty} \alpha^\nu \cdot u_\nu \tag{3}$$

The pole at p_* occurs already in u_0. So $u_1, u_2, \ldots$ will be a sequence of harmonic functions in Ω. There remains to find this sequence so that u_h yields a solution to (1). We will show that this can be achieved when $h - h_0 = \alpha$ is sufficiently small. To begin with the results from (B) give positive constants $0 < c_1 < c_2$ such that

$$0 < c_1 \leq u_0(p) \leq c_2 \quad : p \in \partial\Omega \tag{4}$$

Now we use the analyticity of F with respect to u which enable us to write:

$$F(u_h(p), p) = F(u_0(p) + \sum_{k=1}^{\infty} c_k(p) \cdot \big[\sum_{\nu=1}^{\infty} \alpha^\nu u_\nu(p)\big]^k \tag{5}$$

where $\{c_k(p)\}$ are continuous functions on $\partial\Omega$ which appear in an expansion

$$F(u_0(p) + \xi, p) = F(u_0(p), p) + \sum_{k=1}^{\infty} c_k(p) \cdot \xi^k \tag{6}$$

In the last series expansion we notice that (4) and the hypothesis on F entail that the radius of convergence has a uniform bound below, i.e. there exists $\rho > 0$ which is independent of $p \in \partial\Omega$ and a constant K such that

$$\sum_{k=1}^{\infty} |c_k(p)| \cdot \rho^k \leq K \tag{7}$$

hold for all $p \in \partial\Omega$. Now the equation (1) is solved via a system of equations for the harmonic functions $\{u_\nu\}$ which are determined inductively while α-powers are identified. The linear α-term gives the equation

$$\frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1 - h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p) \tag{i}$$

For u_2 we find that

$$\frac{\partial u_2}{\partial n} = (1 - h_0)u_2 - u_1 + h_0 c_1(p) u_2 + c_1(p) u_1 + c_2(p) u_1^2 \tag{ii}$$

In general, for $\nu \geq 3$ one has

$$\frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \ldots, u_{\nu-1}, p) \tag{iii}$$

where $\{R_\nu\}$ are polynomials in the preeceding u-functions whose coefficients are continuous functions derived via the c-functions above. Here the function $c_1(p)$ is given by

$$c_1(p) = \frac{\partial F(u_0(p), p)}{\partial u}$$

which by the hypothesis on F is positive on $\partial\Omega$. Next, since u_0 is a solution we also have a pair of positive constants $0 < c_* < c^*$ such that

$$c_* < c_1(p) \leq c^* \quad : \quad p \in \partial\Omega \tag{iv}$$

Hence the function

$$W(p) = (1 - h_0) + h_0 \cdot c_1(p)$$

is positive on $\partial\Omega$. Now estimates for the linear inhomogeneous equations in (B) can be applied where the f-functions are the R-polynomials. Then (7) and a majorising positive series expressing maximum norms show that if α is sufficiently small then the series (3) converges and gives the requested solution for (1). Moreover, the positive α can be taken *independently* of h_0. So together with the uniqueness of solutions u_h whenever they exist, it follows that we can move from $h = 0$ until $h = 1$ where we get the requested solution u to the PDE in Theorem 1.

Remark. The reader may consult page 106 in [Carleman] where the existence of a uniform constant $\alpha > 0$ for which the series (3) converge for every h is demonstrated by an explicit majorant series.