Special Topics

Introduction.

Above are headlines for the sections. There detailed contents are listed in the next pages and further comments appear in the individual sections. The level of the material changes from fairly elementary facts to results whose proofs are quite demanding. The first three sections are essential for much of the subsequent material, especially facts about functions in the Jensen-Nevanlinna class which together with the Brothers Riesz Theorem from Section 1 are used to study Hardy spaces an they appear frequently in later sections as well. The topics treat both analytic function theory and harmonic analysis where the interplay lead to a powerful theory such as in Section 14 where Beurling-Wiener algebras are studied. A veritable high-light is the notion and properties of Carleson measures which are to establish the interpolation theorem for bounded analytic functions in section 12. Section 19-20 are a bit apart from analytic function theory but have been included since the methods are of interest. For example, symmetrizations is often used in analytic function theory and the proof in Section 20 illustrates one application of symmetrization methods and the non-linear solution to a PDE-equation in section 19 is proved via succesive analytic series which eventually reduce the proof of existence to solving a family of linear Neumann problems.

Guidance to the sections.

Below follows a more detailed list of material from some of the extensive sections. For the shorter sections we refer to the individual introductions.

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I. The disc algebra $A(D)$

Contents

Introduction.

Denote by $A(D)$ the subalgebra of continuous functions on the closed unit disc $\bar{D}$ which are analytic in the open disc. One refers to $A(D)$ as the *disc-algebra*. If $f \in A(D)$ we have the Poisson representation

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot f(e^{i\theta}) \cdot d\theta \quad : z \in D$$

Since the polynomials in z is a dense subalgebra of $A(D)$ it follows that a Riesz measure μ on T is $\perp$ to $A(D)$ if and only if

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = 0 \quad : \quad n = 0, 1, 2, \ldots \tag{0.1}$$

In Section 1 we will show that (*) implies that μ is absolutely continuous and deduce some facts about boundary values of analytic functions in the open disc. Section 2 is devoted to properties of the disc algebra. Theorem 3.1 in the last section shows that the disc algebra is maximal in a quite strong sense. The proof relies upon results from several complex variables and has been inserted to give the reader a perspective upon the relevance of analytic functions in several variables even for problems which from the start are formulated in $\mathbf{C}$.

1. Theorem of the Brothers Riesz

At the 4:th Scandinavian Congres held in Stockholm 1916, Friedrich and Marcel Riesz proved the following:

1.1 Theorem *Let $E \subset T$ be a closed null set. Then there exists $\phi \in A(D)$ such that $\phi(e^{i\theta}) = 1$ when $e^{i\theta} \in E$ while $|\phi(z)| < 1$ for every $z \in \bar{D} \setminus E$.*

Before the construction of such peak functions we draw a consequence.

1.2. Theorem *Let μ be a Riesz-measure on T such that*

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = \quad : \ n = 1, 2, \ldots$$

Then μ is absolutely continuous.

Proof. Assume the contrary. Then there exists a closed null set E in T such that

$$\int_E d\mu(\theta) \neq 0 \tag{i}$$

Theorem 1.1 gives $\phi \in A(D)$ which is a peak function for E. For each positive integer m we have $\phi^m \in A(D)$. The hypothesis in Theorem 1.2 and (0.1) give:

$$\int_0^{2\pi} \phi^m(e^{i\theta}) \cdot d\mu(\theta) = 0 \quad : m = 1, 2, \ldots \tag{ii}$$

Now we get a contraction since ϕ was a peak function for E. Namely, this implies that

$$\lim_{m \to \infty} \phi^m(e^{i\theta}) \to \chi_E$$

where χ_E is the characteristic function of E and the dominated convergence theorem applied to $L^1(\mu)$ would give $\int_E d\mu = 0$. But this was not the case by (i) above and this contradiction gives Theorem 1.2.

Proof of Theorem 1.1

Let $E \subset T$ be a closed null set and $\{(\alpha_\nu, \beta_\nu)\}$ is the family of open intervals in $T \setminus E$. Since $b_\nu - a_\nu \to 0$ as ν increases, we can choose a sufficiently spare sequence of positive numbers $\{p_\nu\}$ such that

$$\sum p_\nu(\beta_\nu - \alpha_\nu) < \infty \quad \text{and} \quad \lim_{\nu \to \infty} p_\nu = +\infty$$

To each ν we define a function $g_\nu(\theta)$ on the open interval (α_ν, β_ν) by

$$g_\nu(\theta) = \frac{p_\nu(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} : \quad : \ell_\nu = \frac{\beta_\nu - \alpha_\nu}{2} \quad : \gamma_\nu = \frac{\beta_\nu + \alpha_\nu}{2} \tag{1}$$

Next, for each ν a variable substitution gives:

$$\int_{\alpha_\nu}^{\beta_\nu} \frac{d\theta}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} = \int_0^1 \frac{ds}{\sqrt{\frac{1}{4} - (s - \frac{1}{2})^2}} = C \tag{2}$$

where C is a positive constant which the reader may compute. Next, (2) and the convergence of $\sum p_\nu(\beta_\nu - \alpha_\nu)$ imply the function

$$F(\theta) = \sum g_\nu(\theta) \tag{3}$$

has a finite L^1-norm. Here F is defined outside the null set E and since each single g_ν-function restrics to a real analytic function on (α_ν, β_ν) the same holds for F. Next, we notice that

$$\theta \mapsto \frac{(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\theta - \gamma_\nu)^2}} \geq 2 \quad \text{for all} \quad \alpha_\nu < \theta < \beta_\nu \tag{4}$$

In addition to this the reader can verify that

$$\frac{(\beta_\nu - \alpha_\nu)}{\sqrt{\ell_\nu^2 - (\alpha + s - \gamma_\nu)^2}} \geq \frac{\beta_\nu - \alpha_\nu}{\sqrt{s \cdot (\beta_\nu - \alpha_\nu - s)}} \quad : \quad 0 < s < \beta_\nu - \alpha_\nu \tag{5}$$

From (4-5) we can show that $F(\theta)$ gets large when we approach E. Namely, let N be an arbitrary positive integer. Then we find ν_* such that

$$\nu > \nu_* \implies p_\nu > N \tag{i}$$

Next, let $\delta > 0$ and consider the open set E_δ of points with distance $< \delta$ to E. If $\theta \in E_\delta$ we have $\alpha_\nu < \theta < \beta_\nu$ for some ν. If $\nu > \nu*$ then (i) and (4) give

$$F(\theta) > 2N \tag{ii}$$

Next, set

$$\gamma = \min_{1 \leq \nu \leq \nu_*} \rho_\nu \cdot \sqrt{\beta_\nu - \alpha_\nu} \tag{iii}$$

Let us now consider some $1 \leq \nu \leq \nu_*$ and a point $\theta \in E_\delta$. which belongs to (α_ν, β_ν). Since $E \cap (\alpha_\nu, \beta_\nu = \emptyset$ we see that

$$\theta - \alpha_\nu < \delta \quad \text{or} \quad \beta_\nu - \theta < \delta \tag{iv}$$

must hold. In both cases (4) gives:

$$g_\nu(\theta) \geq \frac{\rho_\nu \cdot \sqrt{(\beta - \nu - \alpha - \nu}}{\sqrt{\delta}} \geq \frac{\gamma}{\sqrt{\delta}} \tag{v}$$

With γ fixed we find a small δ such that the right hand side is $> N$ and together with (ii) it follows that

(vi) $$\theta \in E_\delta \setminus E \implies F(\theta) > N$$

The construction of ϕ. The Poisson kernel gives the harmonic function:

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2+\cos(\theta-t)} \cdot F(t)dt \quad : \quad re^{i\theta} \in D$$

Since $F \geq 0$ we have U it is ≥ 0 in D and by (vi) $U(z)$ increases to $+\infty$ as z approaces E. More precisely, the following companion to (vi) holds:

Sublemma To every positive integer N there exists $\delta > 0$ such that

$$U(z) > N \quad : \quad z \in D \cap E_\delta^*$$

where $E_\delta^* = \{z \in D : \; dist(z, E) < \delta\}$.

Now we construct the harmonic conjugate:

$$V(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} \frac{r \cdot \sin(\theta-t)}{1+r^2+\cos(\theta-t)} \cdot F(t)dt \quad : \quad re^{i\theta} \in D$$

We have no control for the limit behaviour of $V(re^{i\theta})$ as $r \to 1$ and $e^{i\theta} \in E$. But on the open intervals (α_ν, β_ν) where F restricts to a real analytic function there exists a limit function V^*:

$$\lim_{r\to 1} V(re^{i\theta}) = V^*(e^{i\theta})) \quad : \quad \alpha_\nu < \theta < \beta_\nu$$

Thus, V^* is a function defined on $T \setminus E$. Similarly, $U(re^{i\theta})$ has a limit function $U^*(e^{i\theta})$ defined on $T \setminus E$. Now we set

(*) $$\phi(z) = \frac{U(z) + iV(z)}{U(z) + 1 + iV(z)} \quad : \quad z \in D$$

This is an analytic function in D. Outside E we get the boundary value function

$$\lim_{r\to 1} \phi(re^{i\theta}) = \frac{U^*(e^{i\theta}) + iV^*(e^{i\theta})}{U^*(e^{i\theta}) + 1 + iV^*(e^{i\theta})}$$

The limit on E. Concerning the limit as $z \to E$ we have:

$$[1 - \phi(z)| = \frac{1}{|1 + U(z) + iV(z)|} \leq \frac{1}{1 + U(z)}$$

By the Sublemma the last term tends to zero as $z \to E$. We conclude that $\phi \in A(D)$ and here $\phi = 1$ on E while $|\phi(z)| < 1$ for al $z \in \bar{D} \setminus E$ which gives the requested peak function.

Remark. The proof of Theorem 1.1 above was constructive. There exist proofs using functional analysis and the Hilbert space $L^2(d\mu)$ attached to a Riesz measure on T. See the text-book [Koosis: p. 40-47] for such alternative proofs.

1.3 An application of Theorem 1.1

Let $f(z)$ be analytic in the open unit disc and assume there exists a constant M such that

$$\int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta \leq M \quad : \quad 0 < r < 1$$

Consider the family of measures on the unit circle defined by

$$\{\mu_r = f(re^{i\theta}) \cdot d\theta \; : \; r < 1\}$$

The uniform upper bound for their total variation implies by compactness in the weak topology that there exists a sequence $\{r_\nu\}$ with $r_\nu \to 1$ and a Riesz measure μ such that $\mu_{r_\nu} \to \mu$ holds *weakly*. In particular we have

$$\int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = \lim_{r_\nu \to 1} \int_0^{2\pi} e^{in\theta} f(r_\nu e^{i\theta}) \cdot d\theta$$

for every integer n. Since f is analytic the right hand side integrals vanish whenever $n \geq 1$ and hence μ is absolutely continuous by Theorem 1.2. So we have $\mu = f^*(\theta) d\theta$ for an L^1-function f^*. Now we construct the analytic function

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(\theta) \cdot e^{i\theta} d\theta}{e^{i\theta} - z}$$

When $z \in D$ is fixed the *weak* convergence applies to the θ-continuous function $\theta \mapsto \frac{e^{i\theta}}{e^{i\theta} - z}$ and hence

$$F(z) = \lim_{\nu \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r_\nu e^{i\theta}) e^{i\theta} d\theta}{e^{i\theta} - z}$$

At the same time, as soon as $|z| < r_\nu$ one has Cauchy's formula:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(r_\nu e^{i\theta}) \cdot r_\nu e^{i\theta} \cdot d\theta}{r_\nu \cdot e^{i\theta} - z}$$

Since this hold for every large ν we can pass to the limit and conclude that $F(z) = f(z)$ olds in D. Hence $f(z)$ is represented by the Cauchy kernel of the $L^1(T)$-function $f^*(\theta)$. At this stage we apply *Fatou's theorem* to conclude that

$$\lim_{r \to 1} f(re^{i\theta}) = f^*(\theta) \quad \text{holds almost everywhere}$$

Moreover, one has convergence in the L^1-norms:

$$\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta} - f^*(\theta)| = 0$$

Thus, thanks to Theorem 1.2 the $L^1(T)$- sequence defined by the functions $\theta \mapsto f(re^{i\theta})$ converges almost everywhere to a unique limit function $f^*(\theta) \in L^1(T)$.

1.4 Exercise. Show that for every Lebesgue point θ_0 of $f^*(\theta)$ there exists a radial limit:

$$\lim_{r \to 1} f(re^{i\theta_0}) = f^*(\theta_0)$$

1.5 Exercise. In general, let K be a compact subset of D and μ a Riesz measure supported by K which is $\perp$ to analytic polynomials, i.e.

$$\int z^n \cdot d\mu(z) = 0$$

hold for all $n \geq 0$. Use the existence of peaking functions in $A(D)$ to conclude that if $E \subset T$ is a null-set for linear Lebesgue measure $d\theta$, then E is a null-set for μ. In particular, if K contains a relatively open set given by an arc α on the unit circle, then the restriction of μ to α is absolutely continuous

2. Principal ideals in the disc algebra.

Let $A(D)$ be the disc algebra. The point $z = 1$ gives a maximal ideal in $A(D)$:

$$\mathfrak{m} = \{f \in A(D) \quad : f(1) = 0\}$$

Let $f \in A(D)$ be such that $f(z) \neq 0$ for all z in the closed disc except at the point $z = 1$. The question arises if the principal ideal generated by f is dense in $\mathfrak{m}$. This is not always true. A counterexample is given by the function

$$f(z) = e^{\frac{z+1}{z-1}}$$

Following the appendix in [Carleman: Note 3] we give a sufficient condition on f in order that its principal ideal is dense in $\mathfrak{m}$. Namely, since $f(z(\neq 0$ except when $z = 1$ there exists the analytic function

$$f^*(z) = \exp\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \log|\frac{1}{f(e^{i\theta})}| \cdot d\theta\}$$

We say that f has no logarithmic reside a $z = 1$ if $f = f*$ and now the following holds:

2.2 Theorem. *If f has no logarithmic residue then $A(D)f$ is dense in $\mathfrak{m}$.*

Proof. With $\delta > 0$ we choose a continuous function $\rho_\delta(\theta)$ on T which is equal to $\log|\frac{1}{f(e^{i\theta})|}$ outside the interval $(-\delta, \delta)$ while

(i) $$0 < \rho_\delta(\theta) < \log|\frac{1}{f(e^{i\theta})}| \quad : \; -\delta < \theta < \delta$$

Next, let $\phi \in \mathfrak{m}$ and set

(ii) $$\omega_\delta(z) = \phi(z) \cdot \exp\{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \rho_\delta(\theta) \cdot d\theta\}$$

It follows that

(iii) $$\big|\omega_\delta(z) \cdot f(z) - \phi(z)\big| = |f(z)| \cdot |\phi(z)| \cdot \big|1 - \exp\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot \big[\log\frac{1}{|f(e^{i\theta})|} - \rho_\delta(\theta)\big] \cdot d\theta\}|$$

Exercise. Show that the limit of the right hand side is zero when $\delta \to 0$ and conclude that ϕ belongs to the closure of the principal ideal generated by f.

2.6 Some facts about $A(D)$

The disc algebra $A(D)$ is a uniform algebra, where the spectral radius norm is equal to the maximum over the closed disc. By the maximum principle for analytic functions in D one has $|f|_D = |f|_T$. One therefore calls T the *Shilov boundary* of $A(D)$. A notable point is that $A(D)$ is a Dirichlet algebra which means that the linear space of real parts of functions restricted to T is a dense subspace of all real-valued and continuous functions on T. From XX we recall that if $\rho(\theta)$ is real-valued and continuous on T then $\rho = \mathfrak{Re}(f)$ on T for some $f \in A(D)$ if and only if the function

$$z \mapsto \int_0^{2\pi i} \frac{\mathfrak{Im}(ze^{-i\theta})}{|e^{i\theta}-z|^2} \cdot \rho(\theta)d\theta$$

extends to a continuous function on the closed disc. For example, every C^1-function on T belongs to $\mathfrak{Re}(A(D))$.

2.7 Wermer's maximality theorem. A result due to J. Wermer asserts that $A(D)$ is a maximal uniform algebra. It means that if $f \in C^0(T)$ is such that the closed subalgebra of $C^0(T)$ generated by f and z is not equal to $C^0(T)$, then f must belong to $A(D)$. Another way to phrase the result is that whenever $f \in C^0(T)$ is such that

$$\int_0^{2\pi} e^{ik\theta} \cdot f(e^{i\theta}) \cdot d\theta \neq 0$$

holds for at least one positive integer k, then $[z, f]_T = C^0(T)$.

Outline of the proof. Let $f \in C^0(T)$ and consider the uniform algebra $B = [z, f]_T$ on the unit circle. Now there exists the maximal ideal space $\mathfrak{M}_B$ whose points correspond to multiplicative functionals on B. If $p \in \mathfrak{M}_B$ and p^* is the corresponding multiplicative functional it is clear that there exists a unique point $z(p) \in D$ such that $p^*(f) = f(z(p))$ for every f in the subalgebra $A(D)$ of B. If $z(p) \in T$ holds for every p then the B-element z is invertible. But this means that B contains both $e^{i\theta}$ and $e^{-i\theta}$ and by Weierstrass theorem they already generate a dense subalgebra of $C^0(T)$. So if $B \neq C^0(T)$ there must exist at least one point $p \in \mathfrak{M}_B$ such that $z(p)$ stays in the open unit disc. In fact, *every point* $z_0 \in D$ is of the form $z(p)$ for some p for otherwise $\frac{1}{z-z_0}$ belongs to B and one verifies easily that the two functions on T given by $e^{i\theta}$ and $\frac{1}{e^{i\theta}-z_0}$ also generate a dense subalgebra of $C^0(T)$. There remains to consider the case when $p \mapsto z(p)$ sends $\mathfrak{M}_B$ onto the closed disc.

At this stage one employs a general result from uniform algebras. Namely, since every multiplicative functional has norm one it follows that that for every $p \in \mathfrak{M}_B$ there exists a probability measure μ_p on the unit circle such that

$$(*) \qquad p^*(g) = \int_T g(e^{i\theta}) \cdot d\mu_p(\theta) \quad \text{hold for all} \quad g \in B$$

Now we use that $A(D)$ is a Dirichlet algebra. Namely, (*) holds in particular for $A(D)$-functions and since μ_p is a real measure we conclude that it must be equal to the Poisson kernel of the point $z(p)$. This proves to begin with that the map $p \to z(p)$ is *bijective.* So for every $g \in B$ we get a continuous function on the closed unit disc defined by

$$g^*((z(p) = p^*(g)$$

But (*) above means that g^* is the harmonic extension to D of the boundary function g on T. Finally, since B is algebra one easily verifies that when every B-function is harmonic in D, then B consists of complex analytic functions only. This means precisely that $B = A(D)$. At this stage we conclude that when $B = [z, f]_T$ and $B \neq C^0(T)$ is assumed, then $f \in A(D)$ holds which is the assertion in Wermer's maximality theorem.

3. Relatively maximal algebras

Introduction. An extension of Wermer's maximality theorem was proved in [Björk] and goes as follows. Let K be a closed subset of $\bar{D}$ whose planar Lebesgue measure is zero. We also assume that K contains T and that $\bar{D} \setminus K$ is connected. Finally we assume that there exists some some open interval on T which does no belong to the closure of $K \setminus T$. In this situation the following holds:

3.1. Theorem. *Let $f \in C^0(K)$ be such that the uniform algebra $[z, f]_K \neq C^0(K)$. Then f extends from K to an analytic function in $D \setminus K$.*

Remark. The case when K is the union of T and a finite set of Jordan arcs where each arc has one end-point on T and the other in the open disc D is of special interest. If these Jordan arcs are not too fat, then f extends analytically across each arc which means that the restriction of f to T must belong to the disc-algebra. This case was a motivation for Theorem 3.1 since it is connected to the problem of finding conditions on a Jordan arc J in order that it is locally a removable singularity for continuous functions g which are analytic in open neighborhoods of J. The interested reader may consult [Björk:x] for a further discussion about this problem where comments are given by Harold Shapiro about the connection to between Theorem 3.1 and results by Privalov concerning analytic extensions across a Jordan arc.

Proof of Theorem 3.1. The proof will employ the *Local maximum Principle* by Rossi which is a powerful tool to study uniform algebras whose Shilov boundary is a proper subset of the maximal ideal space. Let us then start the proof. Set

$$B = [z, f]_K$$

Since $B \neq C^0(K)$ is assumed there exists a non-zero Riesz measure μ on K which annihilates B. Notice that μ can be complex-valued. Let π be the projection from $\mathfrak{M}_B$ into D which means that when z is regarded as an element in B then its Gelfand transform $\widehat{z}$ satisfies

$$\widehat{z}(p) = \pi(p) \quad : \ p \in \mathfrak{M}_B$$

As usual K is identified with a compact subset of $\mathfrak{M}_B$. If $e^{i\theta} \in T$ we use that it is a peak point for $A(D)$ and hence also for B. This entails that the fiber $\pi^{-1}(e^{i\theta})$ is reduced to the natural point $e^{i\theta} \in K$. Next, since we assume that K has planar measure zero we know from XX that the uniform algebra on K generated by rational functions with poles outside K is equal to $C^0(K)$. Since $z \in B$ and $B \neq C^0(K)$ it follows that $\pi^{-1}(D \setminus K) \neq \emptyset$. We are going to prove that the fiber above every point in $D \setminus K$ is reduced to a single point and for this purpose we define the following two analytic functions in the open set $D \setminus K$:

$$(*) \qquad W(z) = \int_K \frac{f(\zeta) \cdot d\mu(\zeta}{\zeta - z} \quad \text{and} \quad R(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z}$$

The main step in the proof is to show that if $z \in D \setminus K$ and $\xi \in \pi^{-1}(z)$ then the Gelfand transform $\widehat{f}$ satisfies:

$$(**) \qquad \widehat{f}(\xi) \cdot R(z) = W(z) \quad : \forall \ \xi \in \pi^{-1}(z)$$

Here $R(z)$ it cannot be identically zero in $D \setminus K$ for then the Riesz measure μ would be identically zero. If $R(z) \neq 0$ for some $z \in D \setminus K$ then (**) entails that the fiber $\pi^{-1}(z)$ is reduced to a single point. This hold for all points outside the eventual discrete zero-set of R and when a fiber $\pi^{-1}(z)$ is reduced to a single point the meromorphic function $\frac{W}{R}$ has a value taken by the continuous Gelfand transform of f at this unique fiber-point. This implies that $\frac{W}{R}$ is bounded outside the zeros of R and therefore analytic in the whole set $D \setminus K$. From this it follows easily that (**) implies that al fibers are reduced to single points and the analytic function $\frac{W}{R}$ in $D \setminus K$ is identified with the restriction of $\widehat{f}$ to this open set in the maximal ideal space of B. So there remains to give:

Proof of (**). Since μ annihilates the functions z^N and $z^N \cdot f(z)$ for every $N \geq 0$ we have

$$\int_K \frac{\bar{z} \cdot d\mu(\zeta)}{1 - \bar{z} \cdot \zeta} = \int_K \frac{\bar{z} \cdot f(\zeta) \cdot d\mu(\zeta)}{1 - \bar{z} \cdot \zeta} = 0 \quad \text{for every} \quad z \in D$$

Adding these zero-functions in (*) it follows that

$$(1) \qquad W(z) = \int_K \frac{(1 - |z|^2| \cdot f(\zeta) \cdot d\mu(\zeta)}{(\zeta - z)(1 - \bar{z}\zeta)} \quad \text{and} \quad R(z) = \int_K \frac{(1 - |z|^2 \cdot d\mu(\zeta)}{(\zeta - z)(1 - \bar{z}\zeta)}$$

The assumption that the closure of $K \setminus T$ does not contain T gives some open arc $\alpha = (\theta_0, \theta_1)$ on T which is disjoint from the closure of $K \setminus T$. The local version of the Brother's Riesz theorem from Exercise 1.5 implies that the restriction of μ to α is absolutely continuous. Hence, by Fatou's theorem there exist the two limits

$$(2) \qquad \lim_{r \to 1} W(re^{i\phi}) = W(e^{i\phi}) \quad : \lim_{r \to 1} R(re^{i\phi}) = R(e^{i\phi})$$

almost every on $\theta_0 < \phi < \theta_1$. Let us fix $\theta_0 < \phi_0 < \phi_1 < \theta_1$ where the radial limits in (2) exist for ϕ_0 and ϕ_1. Next, consider a point $z_0 \in D \setminus K$ and choose a closed Jordan curve Γ which is the union of the T-interval $[\phi_0, \phi_1]$ and a Jordan arc γ which is disjoint to the closure of $K \setminus T$ while z_0 belongs to the Jordan domain Ω bordered by Γ. We can always choose a nice arc Γ which is of class C^1 and hits T at $e^{i\phi_0}$ and $e^{i\phi_1}$ at right angles. Since Γ has a positive distance from $K \setminus T$ there exists $r_* < 1$ such that if $r_* < r < 1$ then the functions

$$(3) \qquad W_r(z) = W(rz) \quad : R_r(z) = R(rz)$$

are analytic in a neighborhood of the closure of Ω. Now we consider the set $\pi^{-1}(\Omega) = \Omega^*$ in $\mathcal{M}_B$ whose boundary in $\mathcal{M}_B$ is contained in $\pi^{-1}(\Gamma) = \Gamma^*$. If $Q(z)$ is an arbitrary polynomial the *Local Maximum Principle* gives

$$(4) \qquad |Q(z_0)| \cdot [\hat{g}(\xi) \cdot R_r(z_0) - W_r(z_0)| \leq \big|Q \cdot (\widehat{f} \cdot R - W_r)\big|_{\Gamma^*}$$

Recall that $\pi^{-1}(T)$ is a copy of T Identifying the subinterval $[\phi_0, \phi_1]$ with a closed subset of $\mathcal{M}_B$ we can write

$$(5) \qquad \Gamma^* = \gamma^* \cup [\phi_0, \phi_1] \quad : \gamma^* = \pi^{-1}(\Gamma \setminus (\phi_0, \phi_1))$$

Now (4) and the continuity of the Gelfand transform $\widehat{f}$ give a constant M which is independent of r such that the maximum norms

$$(6) \qquad \big|\widehat{f} \cdot R - W_r)\big|_{\Gamma^*} \leq M \quad : r_* < r < 1$$

Since $\widehat{f}(e^{i\theta}) = f(e^{i\theta})$ holds on T it follows from (2) that the maximum norms:

$$(7) \qquad \delta(r) = \big|\hat{g} \cdot R_r - W_r|_{[\phi_0, \phi_1]} = 0$$

tend to zero as $r \to 1$. Next, let $\epsilon > 0$. Runge's theorem gives a polynomial $Q(z)$ such that

$$(8) \qquad Q(z_0) = 1 \quad : |Q|_\gamma < \frac{\epsilon}{M}$$

When $\xi \in \pi^{-1}(z_0)$ it follows from (6) that

$$(9) \qquad |\widehat{f}(\xi)R(z_0) - W(z_0)| \leq \text{Max}\,(\epsilon, \big||Q|\big|_{[\phi_0, \phi_1]} \cdot \delta(r))$$

Passing to the limit as $r \to 1$ we use that $\delta(r) \to 0$ together with the obvious limit formulas $R_r(z_0) \to R(z_0)$ and $W_r(z_0) \to W(z_0)$, and conclude that that

$$(10) \qquad \big|\widehat{f}(\xi) \cdot R(z_0) - W(z_0)\big| \leq \epsilon$$

Since we can choose ϵ arbitrary small we get

$$(11) \qquad \widehat{f}(\xi) \cdot R(z_0) = W(z_0) \quad : \xi \in \pi^{-1}(z_0)$$

Since $z_0 \in D \setminus k$ was arbitrary we have proved (**) and as explained after (**) it follows that

$$(12) \qquad \pi^{-1}(D \setminus K) \simeq D \setminus K$$

3.2 The extension to K. At this stage we can easily finish the proof of Theorem 3.1. We have already found the analytic function $\widehat{f}(z)$ in $D \setminus K$ and it is clear that it extends to f on the free circular arc (θ_0, θ_1) of T. To see that $\widehat{f}$ extends to K and gives a continuous function on the whole closed unit disc we solve the Dirichlet problem for the continuous functions $\mathfrak{Re}\, f$ and $\mathfrak{Im}\, f$ on K and conclude that $\widehat{f}$ extends and moreover its boundary value function on K is equal to the restriction of f to K. The proof of Theorem 3.1 is therefore finished if we have shown the equality:

$$\mathcal{M}_B \simeq D$$

To see that this holds we put $U = \pi^{-1}(\mathcal{M}_B \setminus D)$ and notice that its boundary in $\mathcal{M}_B$ is contained in the closure of $K \setminus T$. Call, this compact set K_*. Since we have the free arc (ϕ_0, ϕ_1) and $D \setminus K$ is connected it follows that $\mathbf{C} \setminus K_*$ is connected, i.e. only the unbounded component exists. So by Mergelyan's Theorem polynomials in z generate a dense subalgebra of $C^0(K_*)$. But then the Local Maximum Principle implies that U must be empty and the proof of Theorem 3.1 is finished.

II. The Jensen-Nevanlinna class and Blaschke products.

Introduction.

If μ is a real Riesz measure on the unit circle there exist the harmonic function in the disc D defined by

$$H_\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\cdot d\mu(\theta) \tag{0.1}$$

For each $0<r<1$ one has the inequality

$$\int_0^{2\pi} \left|H_\mu(re^{i\theta})\right|\cdot d\theta \leq ||\mu|| \tag{0.2}$$

where $||\mu||$ is the total variation of μ. Moreover, there exists a weak limit, i.e.

$$\lim_{r\to 1}\int_0^{2\pi} g(\theta)\cdot H_\mu(re^{i\theta}) = \int_0^{2\pi} g(\theta)\cdot d\mu(\theta) \tag{0.3}$$

holds for every continuous function $g(\theta)$ on T. Conversely we proved in XX that if $H(z)$ is a harmonic function in D for which there exists a constant C such that

$$\int_0^{2\pi} \left|H(re^{i\theta})\right|\cdot d\theta \leq C \tag{0.4}$$

hold for all $r<1$, then there exists a unique Riesz measure μ on T where $H=H_\mu$. Hence there is a 1-1 correspondence between the space of harmonic functions in D satisfying (0.4) and the space of real Riesz measures on T. There also exist radial limits almost everywhere. More precisely, define the μ-primitive function

$$\psi(\theta) = \int_0^\theta d\mu(s)$$

Fatou's Theorem asserts that for each Riesz measure μ there exists a radial limit

$$H^*_\mu(\theta) = \lim_{r\to 1} H(re^{i\theta}) \tag{0.5}$$

for each θ where ψ has an ordinary derivative. Since ψ has a bounded variation this holds almost everywhere by Lebegue's Theorem in [Measure].

0.6 The case when μ is singular. If μ is singular the radial limit (0.5) is zero almost everywhere. If the singular measure μ is non-negative with total mass 2π we have $H_\mu(0)=1$ and the mean-value property for harmonic functions gives:

$$\int_0^{2\pi} H_\mu(re^{i\theta})\cdot d\theta = 1$$

for all $0<r<1$. At the same time the boundary function $H^*_\mu(\theta)$ is almost everywhere zero which means that no dominated convergence occurs.

0.7 Exercise. Let μ be singular with a Hahn-decomposition $\mu = \mu_+ - \mu_-$. Assume that the positive part $\mu_+(T) = a > 0$. Now there exists a closed null set E such that $\mu_+(E) \geq a - \epsilon$ while $\mu_-(E) = 0$. The last equation gives a small $\delta > 0$ such that if $E_{2\delta}$ is the open 2δ-neighborhood of E then $\mu_-(E_{2\delta}) < \epsilon$. Set

$$H_*(z) = \frac{1}{2\pi} \int_E \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu_+(\theta)$$

Since $\mu_+(E) \geq a - \epsilon$ we get

(ii) $$\int_0^{2\pi} H_*(re^{i\theta}) \cdot d\theta \geq a - \epsilon$$

Next, for each pair $\phi \in E_\delta$ and $e^{i\theta} \in T \setminus E_{2\delta}$ we have:

$$\frac{1 - r^2}{|e^{i\theta} - re^{i\phi}|^2} \leq \frac{2(1 - r)}{1 + r^2 - 2r\cos(\delta)}$$

So with

$$H_\delta(z) = \frac{1}{2\pi} \int_{T \setminus E_{2\delta}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot d\mu_(\theta)$$

it follows that

(iii) $$|H_\delta(re^{i\phi})| \leq \frac{1}{2\pi} \cdot \frac{2(1 - r)}{1 + r^2 - 2r\cos(\delta)} \cdot \int_{T \setminus E_{2\delta}} |d\mu(\theta)|$$

for each $\phi \in E_\delta$. Since H_* is constructed via the restriction of μ_+ to E, a similar reasoning gives:

(iv) $$|H_*(re^{i\phi})| \leq \frac{1}{2\pi} \frac{2(1 - r)}{1 + r^2 - 2r\cos(\delta)} \cdot \mu_+(E)$$

when $e^{i\phi} \in T \setminus E_\delta$. Next, by the constructions above we have

$$H = H_* + H_\delta + H_\nu$$

where ν is the measure given by the restriction of μ_+ to $E_{2\delta} \setminus E$ minus μ_- restricted to to $E_{2\delta}$. So by the above the total variation $||\nu|| \leq 2\epsilon$ which gives

$$\int_0^{2\pi} |H_\nu(re^{i\theta})| \cdot d\theta \leq 2\epsilon$$

Deduce from the above that one has an inequality

(*) $$\int_{E_\delta} H(re^{i\phi}) \cdot d\phi \geq a - \big[2\epsilon + \frac{1}{\pi} \frac{2(1 - r)}{1 + r^2 - 2r\cos(\delta)} \cdot ||\mu||\big]$$

Since E is a null-set this shows that mean-value integrals of H behave in an "irregular fashion" when $r \to 1$.

1. The Herglotz integral

Let μ be a real Riesz measure on the unit circle T. Set

(*) $$g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot d\mu(\theta)$$

This analytic function is called the Herglotz extension of the Riesz measure. Since μ is real it follows that

$$\mathfrak{Re}\, g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z[^2}{|e^{i\theta} - z|^2} \cdot d\mu(\theta) = H_\mu(z)$$

In particular the L^1-norms from (0.2) are uniformly bounded with respect to r when we integrate the absolute value of $\mathfrak{Re}\, g_\mu$. But the conjugate harmonic function representing $\mathfrak{Im}\, g_\mu$ does not satisfy (0.2) in general. However the following holds:

1.1 Theorem. *For almost every θ there exists a radial limit*

$$\lim_{r\to 1} g_\mu(re^{i\theta})$$

To prove Theorem 1.1 we shall use some tricks. The Hahn-decomposition $\mu = \mu_+ - \mu_-$ enables us to express g_μ as a difference $g_1 - g_2$ where g_1, g_2 both are Herglotz extensions of non-negative Riesz measures and hence $\mathfrak{Re}\, g_\nu(z) > 0$ in D. Let us now discuss analytic functions with a positive real part.

1.2 Exercise. Let $f \in \mathcal{O}(D)$ where $\mathfrak{Re}\, f(z) > 0$ and $\mathfrak{Im} f(0) = 0$. Set $f = u + iv$ which gives the analytic function

$$\phi(z) = \log(1 + u + iv)$$

Here

$$\mathfrak{Re}\, \phi = \log |1 + u + iv| = \frac{1}{2} \log[(1+u)^2 + v^2]$$

In particular $\mathfrak{Re}\, \phi > 0$ so this harmonic function has a radial limit almost everywhere. We also know that u has a radial limit almost everywhere and from this the reader may conclude that there almost everywhere exist finite radial limits

$$\lim_{r\to 1} v^2(re^{i\theta}) \tag{1}$$

In order to determine the sign of these radial limits we consider the analytic function

$$\psi = e^{-u-iv}$$

Since $u > 0$ we have $|\psi(z)| = e^{-u(z)} \leq 1$ and hence $\psi(z)$ is a bounded analytic function in D. The Brothers Riesz theorem shows that ψ has a radial limit almost everywhere. Finally, when we have a radial limit

$$\lim_{r\to 1} e^{-u(re^{i\theta})-iv(re^{i\theta})}$$

and in addition suppose that u has a radial limit, then it is clear that v has a radial limit too.

Proof of Theorem 1.1 By the Hahn-decomposition of μ the proof is reduced to the case $\mu \geq 0$ and Exercise 1.2 applies.

1.3 The case when μ is singular. When this holds the radial limits of $\mathfrak{Re}\, g_\mu$ are almost everywhere zero. With $v = \mathfrak{Im}\, g_\mu$ there remains to study the almost everywhere defined function

$$v^*(\theta) = \lim_{r\to 1} v(re^{i\theta})$$

It turns out that this Lebesgue-measurable function never is integrable when μ is singular. In fact, the Brothers Riesz theorem shows that if there exists a constant C such that

$$\int_0^{2\pi} |v(re^{i\theta})| \cdot d\theta \leq C$$

hold for all $r < 1$, then the analytic function g_μ belongs to the Hardy space and its radial limits give an L^1-function $g^*(\theta)$ on the unit circle which would entail that σ is equal to the absolutely continuous measure defined by g^*. Thus, for every singular measure μ one has

$$\lim_{r\to 1} \int_0^{2\pi} \bigl| \mathfrak{Im}\, g_\mu(re^{i\theta}) \bigr| \cdot d\theta = +\infty \tag{*}$$

1.4 Example. Take the case where μ is 2π times the Dirac measure at $\theta = 0$ which gives the analytic function

$$g(z) = \frac{1+z}{1-z}$$

It follows that

$$v(re^{i\theta}) = -2r \cdot \frac{\sin\theta}{1 + r^2 - 2r\cos\theta}$$

and radial limits exist except for $\theta = \pi/2$ or $-\pi/2$, i.e.

$$v^*(\theta) = -2 \cdot \frac{\sin\theta}{2 - 2\cos\theta}$$

when θ is $\neq \pi/2$ and $-\pi/2$. At the same time the reader may verify that $v^*(\theta)$ does not belong to $L^1(T)$ and that

$$\int_0^{2\pi} |v(re^{i\theta})| \cdot d\theta \simeq \log \frac{1}{1-r}$$

as $r \to 1$.

2. The Jensen-Nevanlinna class

Every Riesz measure μ on T gives the zero-free analytic function

(*) $$G_\mu(z) = e^{g_\mu(z)}$$

Here $\log |G_\mu(z)| = \mathfrak{Re}\, g_\mu(z)$ which gives the inequality

$$\log^+ |G_\mu(z)| \leq \big|\mathfrak{Re}\, g_\mu(z)\big|$$

Applying (0.2) we obtain:

(**) $$\int_0^{2\pi} \log^+ |G_\mu(re^{i\theta})| \cdot d\theta \leq ||\mu||$$

for each $r < 1$.

2.1 A converse. Let $F(z)$ be a zero-free analytic function in D where $F(0) = 1$. Assume that there exists a constant C such that

(i) $$\int_0^{2\pi} \log^+ |F(re^{i\theta})| \cdot d\theta \leq C$$

hold for each $r < 1$. The mean-value property applied to the harmonic function $H = \log |F|$ gives

(ii) $$\int_0^{2\pi} |H(re^{i\theta})| \cdot d\theta = 2 \cdot \int_0^{2\pi} \log^+ |F(re^{i\theta})| \cdot d\theta$$

Hence (i) entails that H satisfies (0.4) and now the reader can settle the following:

2.2 Exercise. Show that (i) above entails that there exists a Riesz measure μ such that $F = G_\mu$ where the normalisation $F(0) = 1$ gives $\mu(T) = 2\pi$.

2.3 Radial limits. Whenever g_μ has a radial limit for some θ it is clear that G_μ also has a radial limit in this direction. So Theorem 1.1 implies that there exists an almost everywhere defined boundary function

$$G^*_\mu(\theta) = \lim_{r \to 1} G_\mu(re^{i\theta})$$

The material above suggests the following:

2.4 Definition. *An analytic function f in D belongs to the Jensen-Nevanlinna class if there exists a constant C such that*

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \cdot d\theta \leq C$$

hold for all $r < 1$. The family of Jensen-Nevannlina functions is denoted by JN(D).

Above we described zero-free functions in JN(D). Now we shall study eventual zeros of functions in JN(D). Recall that if $f \in \mathcal{O}(D)$ where $f(0) = 1$ then Jensen's formula gives:

(*)
$$\sum_{|\alpha_\nu|<r} \text{Log}\,\frac{r}{|\alpha_\nu|} = \frac{1}{2\pi}\int_0^{2\pi} \log\,\big|f(re^{i\theta})\big| \cdot d\theta \quad : \quad 0 < r < 1$$

where the left hand side is the sum of zeros of f in the disc D_r.

A notation. If $f \in \mathcal{O}(D)$ and $r < 1$ we set

$$\mathcal{T}_f(r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \cdot d\theta$$

Since $\log\,\big|f(re^{i\theta})\big| \leq \log^+ |f|$ it follows that

$$\sum_{|\alpha_\nu|<r} \text{Log}\,\frac{r}{|\alpha_\nu|} \leq \mathcal{T}_f(r)$$

So if $f \in$ JN(D) we can pass to the limit as $r \to 1$ and conclude that the positive series

(**)
$$\sum \text{Log}\,\frac{1}{|\alpha_\nu|} < \infty$$

where the sum is taken over all zeros in D. Next, recall form XX that the positive series (**) converges if and only if

(***)
$$\sum (1 - |\alpha_\nu| < \infty$$

When (***) holds we say that the sequence $\{\alpha_\nu\}$ satisfies the Blaschke condition. Hence we have proved:

2.5 Theorem. *Let f be in JN(D). Then its zero set satisfies the Blaschke condition.*

3. Blaschke products.

Consider an infinite sequence $\{\alpha_\nu\}$ in D where $|\alpha_1| \leq |\alpha_2| \leq \ldots$ and the Blaschke condition holds. For every $N \geq 1$ we put:

$$B_N(z) = \prod_{\nu=1}^{\nu=N} \frac{|\alpha_\nu|}{\alpha_\nu} \cdot \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z}$$

We are going to prove that the sequence of analytic function $\{B_N\}$ converge in D to a limit function $B(z)$ expressed by the infinite product

(3.1)
$$B(z) = \prod_{\nu=1}^{\infty} \frac{|\alpha_\nu|}{\alpha_\nu} \cdot \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z}$$

To prove this we first analyze the individual factors. For each non-zero $\alpha \in D$ we set

$$B_\alpha(z) = \frac{|\alpha|}{\alpha} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}$$

Exercise. Show that

(i)
$$B_\alpha(z) = |\alpha| \cdot \frac{1 - z/\alpha}{1 - \bar{\alpha} z} = |\alpha| + \frac{|\alpha|^2 - 1}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha} \cdot z$$

and conclude that

(ii)
$$B_\alpha(z) - 1 = (|\alpha| - 1) \cdot [1 + \frac{|\alpha| + 1}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha} \cdot z]$$

Finally, use the triangle inequality to show the inequality

(iii) $$\max_{|z|=r} |B_\alpha(z) - 1| \leq (1 - |\alpha|) \cdot (1 + \frac{2r}{1-r}) = \frac{1+r}{1-r} \cdot (1 - |\alpha|)$$

The convergence of (3.1) From (iii) and general results about product series the requested convergence in (3.1) follows from the assumed Blaschke condition. In fact, when $|z| \leq r < 1$ stays in a compact disc the Blaschke condition and (iii) entail that

$$\sum_{\nu=1}^{\infty} \max_{|z|=r} |B_\alpha(z) - 1| < \infty$$

which implies that (3.1) converges uniformly on $|z| \leq r$ to an analytic function and since $r < 1$ is arbitrary we get a limit function $B(z) \in \mathcal{O}(D)$.

3.2 Exercise. The rate of convergence in $|z| \leq r$ can be described as follows: For each $N \geq 1$ we set

$$G_N(z) = \prod_{\nu=N+1}^{\infty} B_{\alpha_\nu}(z) \quad : \quad \Gamma_N = \sum_{\nu=N+1}^{\infty} 1 - |\alpha_\nu|$$

With $r < 1$ kept fixed we choose n so large that

$$\frac{1+r}{1-r} \cdot (1 - |\alpha_\nu|) \leq \frac{1}{2} \quad : \quad \nu > N$$

Show that this gives:

$$\max_{|z|=r} |G_N(z) - 1| \leq 8 \cdot \frac{1+r}{1-r} \cdot \Gamma_N$$

Since the Blaschke condition implies that $\Gamma_N \to 0$ as $N \to \infty$ this gives a control for the rate of convergence in $|z| \leq r$.

3.3 Radial limits of $B(z)$

When $z = e^{i\theta}$ the absolute value $|B_\alpha(e^{i\theta}| = 1$. So $B(z)$ is the product of analytic functions where every term has absolute value ≤ 1 and hence the maximum norm

$$\max_{z \in D} |B(z)| \leq 1$$

Since the analytic function $B(z)$ is bounded, Fatou's Theorem from Section XX gives an almost everywhere defined limit function

(1) $$B^*(e^{i\theta}) = \lim_{r \to 1} B(re^{i\theta})$$

where the radial convergence holds almost everywhere. Moreover, the Brothers Riesz theorem gives:

(2) $$\lim_{r \to 1} \int_0^{2\pi} |B^*(e^{i\theta}) - B(re^{i\theta})| d\theta = 0$$

3.4 Theorem. *The equality*

(*) $$|B^*(e^{i\theta})| = 1 \quad \text{holds almost everywhere}$$

Proof. Since $|B^*| \leq 1$ it is clear that (*) follows if we have proved that

(i) $$\int_0^{2\pi} |B^*(e^{i\theta})| \cdot d\theta = 1$$

Using (2) above and the triangle inequality we get (i) if we prove the limit formula

(ii) $$\lim_{r \to 1} \int_0^{2\pi} |B(re^{i\theta})| \cdot d\theta = 1$$

To show (ii) we will apply Jensen's formulas to $B(z)$ in discs $|z| \le r$. The convergent product which defines $B(z)$ gives

$$B(0) = \prod \log |\alpha_\nu|$$

Next, for $0 < r < 1$ Jensen's formula gives

$$\log B(0) = \sum_{\nu=1}^{\rho(r)} \log \frac{|\alpha_\nu|}{r} + \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta$$

where $\rho(r)$ is the largest ν for which $|\alpha_\nu| = r$. It follows that

$$\frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta \ge \sum_{\nu=1}^{\rho(r)} \log \frac{r}{|\alpha_\nu|} - \sum_{\nu=1}^{\infty} \log \frac{1}{|\alpha_\nu|} \tag{1}$$

Next, with $\epsilon > 0$ we find an integer N such that

$$\sum_{\nu=1}^{\nu=N} \log \frac{1}{|\alpha_\nu|} < \epsilon \tag{2}$$

Since $|\alpha_\nu| \to 1$ here exists r_* such that

$$r \ge r_* \implies \rho(r) \ge N \tag{3}$$

When (3) holds it follows from (1-2) that

$$\frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta \ge \sum_{\nu=1}^{\rho(r)} \log \frac{r}{|\alpha_\nu|} - \sum_{\nu=1}^{\rho(r_*)} \log \frac{1}{|\alpha_\nu|} - \epsilon \tag{4}$$

In the first sum every term is ≥ 1 so we get a better inequality when the sum is restricted to $\nu \le \rho(r_*)$, i.e. we have

$$\frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta \ge \sum_{\nu=1}^{\rho(r_*)} \log \frac{r}{\alpha_\nu|} - \sum_{\nu=1}^{\rho(r_*)} \log \frac{1}{\alpha_\nu|} - \epsilon \tag{5}$$

Here (5) hold for every $r_* < r < 1$ and a passing to the limit as $r \to 1$ where we only have a finite sum $1 \le \nu \le \rho(r_*)$ above we conclude that

$$\lim_{r \to 1} \frac{1}{2\pi} \int \int_0^{2\pi} \log |B(re^{i\theta})| \cdot d\theta > -\epsilon$$

Since $\epsilon > 0$ is arbitrary we have proved (ii) and hence also Theorem 3.4.

3.5 Division by Blaschke products.

Let $F \in \mathcal{O}(D)$ and assume that its zero set in D is a Blaschke sequence $\{\alpha_\nu\}$. Then we obtain the analytic function

$$G(z) = \frac{F(z)}{B(z)}$$

Here G has no zeros in D and we can construct the analytic function $\operatorname{Log} G(z)$. Set

$$\mathcal{I}_G^+(r) = \int_0^{2\pi} \log^+ |\, G(re^{i\theta})| \cdot d\theta$$

Since $\log^+ [ab| \le \log^+ |a| + \log^+ |b|$ for every pair of complex numbers we get:

$$\mathcal{I}_G^+(r) \le \mathcal{I}_F^+(r) + \int_0^{2\pi} \log^+ \frac{1}{|\, B(re^{i\theta})|} \cdot d\theta \tag{1}$$

The last nondecreasing function is $\leq \log^+ \frac{1}{|B(0)|}$ for every r which gives

$$(2) \qquad \mathcal{I}_G^+(r) \leq \mathcal{I}_F^+(r) + \log^+ \frac{1}{|B(0)|}$$

for every $r < 1$. When $F \in \mathrm{JN}(D)$ this implies that G also belongs to $\mathrm{JN}(D)$. Hence we have proved

3.6 Theorem. *For each $f \in JN(D)$ the function $\frac{f}{B_f}$ also belongs to $JN(D)$, where $B_f(z)$ is the Blaschke product formed by zeros of f outside the origin.*

3.7 Conclusion. Theorem 3.6 and the material in section 2 about zero-free Jensen-Nevanlinna functions give the following factorisation formula:

3.8 Theorem. *For each $f \in JN(D)$ there exists a unique real Riesz measure μ on T with $\mu(T) = 0$ such that*

$$f(z) = az^k \cdot B_f(z) \cdot e^{g_\mu(z)}$$

where $k \geq 0$ is the order of zero of f at $z = 0$ and $a \neq 0$ a constant. Moreover

$$\mu = \log|f(e^{i\theta})| \cdot d\theta + \sigma$$

where σ is the singular part of μ.

3.9 Outer factors. In Theorem 3.8 we get the analytic function

$$\mathfrak{O}_f(z) = e^{g_{\log|f|}(z)}$$

We refer to $\mathfrak{O}_f$ as the outer part of f.

3.10 A division result. Consider a pair f, h in $\mathrm{JN}(D)$ which gives the analytic function in D defined by

$$k(z) = \frac{\mathfrak{O}_h(z)}{\mathfrak{O}_f(z)}$$

By (2.3) there exists the almost everywhere defined quotient on T

$$k^*(\theta) = \frac{\mathfrak{O}_h^*(\theta)}{\mathfrak{O}_f^*(\theta)}$$

3.11 Theorem. *Assume that $k^* \in L^1(T)$. Then k^* belongs to the Hardy space $H^1(T)$.*

Proof. In D there exists the harmonic function

$$k(z) = \log|\mathfrak{O}_h(z)| - \log|O_f(z)|$$

The two harmonic functions in the right hasnd side have by definition boundary functions in $L^1(T)$ and Poisson's formula gives for each point $z = re^{i\theta}$:

$$\log|k(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)} \cdot \log|k^*(\phi)| \cdot d\phi$$

By the general mean-value inequality from (xx) the left hand side is majorized by:

$$\leq \log\Big[\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)} \cdot |k^*(\phi)| \cdot d\phi\Big]$$

Taking exponentials on both sides we get

$$|k(re^{i\theta})| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)} \cdot |k^*(\phi)| \cdot d\phi$$

Now we integrate both sides with respect to θ. Since

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)} \cdot d\theta = 1$$

for every ϕ, it follows that

$$\int_0^{2\pi} |k(re^{i\theta})| \cdot d\theta \leq \int_0^{2\pi} |k^*(e^{i\phi})| \cdot d\phi$$

This proves that the L^1-norms of $\theta \to k(re^{i\theta})$ are bounded which means that k belongs to $H^1(T)$. Moreover, by the Brother's Riesz theorem there exist radial limits almost everywhere so we have also the equality

$$\lim_{r\to 1} k(re^{i\theta}) = k^*(\theta)$$

almost everywhere. This proves that k^* is the boundary value function of the $H^1(T)$-function k.

3.12 Exercise. Show by a similar technique that if we instead assume that k^* is square-integrable, i.e. if $k^* \in L^2(T)$ then $k(z)$ belongs to the Hardy space $H^2(T)$.

3.13 Inner functions. If σ is a non-negative and singular measure on T we get the bounded analytic function

$$G_{-\sigma}(z) = e^{-g_\sigma(z)} \tag{1}$$

Keeping σ fixed we denote this function with f. Here

$$\lim_{r\to 1} |f(re^{i\theta})| = 1$$

holds almost everywhere. So the boundary function $f^*(\theta)$ has absolute value almost everywhere. The class of analytic functions obtained via (1) is denoted by $\mathfrak{I}_*(D)$ and are called zero-free inner functions. In general a bounded analytic function f in D whose boundary values have absolute value almost everywhere is called an inner function and this class is denoted by $\mathfrak{I}(D)$.

3.14 Exercise.. Use the factorisation in Theorem 3.8 to show that every $f \in \mathfrak{I}(D)$ is a product

$$f = B_f \cdot f_*$$

where f_* is a zero-free inner function.

3.15 The case of signed singular measures. Let $\mu = \mu_+ - \mu_-$ be a signed singular measure where $\mu_+ \neq 0$. We get the analytic function G_μ and from the above we know that it has radial limits almost everywhere and since μ is singular the boundary function G_μ^* has absolute value almost everywhere. Here the presence of μ_+ implies that the analytic function G_μ is unbounded. In fact, its maximum modules function

$$M(r) = \max_{|z|=r|} |G_\mu(z)|$$

has a quite rapid growth as $r \to 1$. Moreover one always has

$$\lim_{r\to 1} \int_0^{2\pi} |G_\mu(re^{i\theta})| \cdot d\theta = +\infty \tag{*}$$

in other words, G_μ-functions constructed by signed measures with non-zero negative part never belongs to $H^1(T)$.

3.16 Exercise. Prove (*) above using the divergence in (*) from 1.3.

4. Invariant subspaces of $H^2(T)$

The Hilbert space $L^2(T)$ of square integrable functions on T contains the closed subspace $H^2(T)$ whose elements are boundary values of analytic functions in D. If f $f \in H^2(T)$ it is expanded as

$$\sum_{n=0}^{\infty} a_n \cdot e^{in\theta}$$

and Parseval's theorem gives the equality

$$\sum_{n=0}^{\infty} |a_n|^2 = \frac{1}{2\pi} \cdot \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$$

Moreover, in D we get the analytic function $f(z) = \sum a_n z^n$ where radial limits

$$\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})$$

exist almost everywhere.in fact, this follows via the Brothers Riesz theorem and the inclusion $H^2(T) \subset H^1(T)$. We shall study subspaces of $H^2(T)$ which are invariant under multiplication by $e^{i\theta}$.

4.2 Definition. *A closed subspace V of $H^2(T)$ is called invariant if $e^{i\theta}V \subset V$.*

4.3 Theorem *Let V be an invariant subspace of $H^2(T)$. Then there exists $w(\theta) \in H^2(T)$ whose absolute value is one almost everywhere and*

$$V = H^2(T) \cdot w$$

Proof. First we show that that $e^{i\theta}V$ is a proper subspace of V. For an equality $e^{i\theta}V = V$ gives $e^{in\theta}V = V$ for every $n \geq 1$ which entails that if $0 \neq f \in V$ then $f = e^{in\theta} \cdot g_n$ for some $g_n \in H^2(T)$. This means that the Taylor series of f at $z = 0$ starts with order $\geq n$ which cannot hold for every n unless f is identically zero. So now $e^{i\theta}V$ is a proper closed subspace of V which gives some $0 \neq w \in V$ which is $\perp$ to $e^{i\theta}V$. It follows that

$$\langle w, e^{in\theta} \cdot w \rangle \int_0^{2\pi} w(e^{i\theta})\bar{w}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0$$

hold for every $n \geq 1$. Since $w \cdot \bar{w} = |w|^2$ is real-valued we conclude that this function is constant and we can normalize w so that $|w(\theta)| = 1$ holds almost everywhere. There remains to prove the equality

$$\text{(i)} \qquad V = H^2(T) \cdot w$$

Since $|w| = 1$ almost everywhere the right hand side is a closed subspace of V. If it is proper we find $0 \neq u \in V$ where $u \perp H^2(T)w$ which gives

$$\text{(ii)} \qquad \int_0^{2\pi} u(e^{i\theta})\bar{w}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0 \quad : \quad n \geq 0$$

Taking complex conjugates we get

$$\text{(iii)} \qquad \int_0^{2\pi} w(e^{i\theta})\bar{u}(e^{i\theta}) \cdot e^{in\theta} \cdot d\theta = 0 \quad : \quad n \geq 0$$

At the same time $w \perp e^{i\theta}V$ which entails that

$$\text{(iv)} \qquad \int_0^{2\pi} w(e^{i\theta})\bar{u}(e^{i\theta}) \cdot e^{-in\theta} \cdot d\theta = 0 \quad : \quad n \geq 1$$

Together (iiii-iv) imply that $w\bar{u}$ has vanishing Fourier coefficients and is therefore identically zero which gives $u = 0$ and proves that $V = H^2(T) \cdot w$ must hold.

4.4 Examples. Let $B(z)$ be a non-constant Blaschke product. Now $|B(e^{i\theta})| = 1$ holds almost everywhere and the presence of zeros of $B(z)$ in D show that $H^2(T) \cdot B$ is a proper and invariant subspace of $H^2(T)$. Next, let σ be a singular Riesz measure on T which is real and non-negative. We get the analytic function

$$f(z) = e^{-g_\sigma(z)}$$

Here

$$|f(z)| = e^{-H_\sigma(z)}$$

and since $\sigma \geq 0$ we have $H_\sigma(z) \geq 0$ and hence $|f(z)| \leq 1$. So f is a bounded analytic function in D and in particular it belongs to $H^2(T)$. Moreover we know from XX that the boundary function $f(e^{i\theta})$ has absolute value one almost everywhere. So $H^2(T) \cdot f$ is an invariant subspace of $H^2(T)$ and the question arises if it is proper or not. In contrast to the case for Blaschke functions B above this is not obvious since f has no zeros in D. However it turns out that one has

4.5 Theorem. *Let σ be a singular and non-negative Riesz measure which is not identically zero. Then $H^2(T) \cdot e^{-g_\mu}$ is a proper subspace of $H^2(T)$.*

Proof. Set $w(\theta) = e^{-g_\mu(e^{i\theta})}$. For the analytic function $w(z)$ in the disc its value at $z = 0$ becomes

$$w(0) = e^{-g_\mu(0} = e^{-\sigma(T)/2\pi}$$

Next, if $P(z)$ is a polynomial we have

$$\frac{1}{2\pi}\int_0^{2\pi} |P(\theta)w(\theta) - 1|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} |P(\theta)|^2 \cdot d\theta + 1 + 2\mathfrak{Re}\,[\int \frac{1}{2\pi}\int_0^{2\pi} P(\theta) \cdot w(\theta) \cdot d\theta]$$

By Cauchy's formula the last term becomes

$$2\mathfrak{Re}(P(0)w(0)) = 2w(0) \cdot \mathfrak{Re}(P(0))$$

By (i) we have $0 < w(0) < 1$ and if $||P||_2$ is the L^2-norm of P the right hand side majorizes

$$||P||_2^2 + 1 - 2w(0) \cdot |P(0)|$$

We have also the inequality

$$|P(0)| \leq ||P||_2$$

So if we set $\rho = ||P||_2$ then we have shown that

$$\frac{1}{2\pi}\int_0^{2\pi} |P(\theta)w(\theta) - 1|^2 d\theta \geq \rho^2 + 1 - 2w(0) \cdot \rho$$

Now we notice that the right hand side is $\geq 1 - w(0)^2$ for every ρ. Since P is an arbitrary polynomial we conclude that the L^2-distance of 1 to the subspace $H^2(T) \cdot e^{-g_\mu}$ is at least

$$(*) \qquad 1 - w(0)^2 = 1 - e^{-2\sigma(T)}$$

5. Beurling's closure theorem.

A zero-free function $f \in H^2(T)$ is of outer type when

$$f(z) = G_\mu(z)$$

where μ is the absolutely continuous Riesz measure $\log|f(e^{i\theta}|$. The following result is due to Beurling in [Beur]:

5.1 Theorem. *For every nonzero $f \in H^2(T)$ of outer type the closed invariant subspace generated by analytic polynomials $P(z)$ times f is equal to $H^2(T)$.*

Proof. If the density fails we find $0 \neq g \in H^2(T)$ such that

$$\text{(i)} \qquad \int_0^{2\pi} e^{in\theta} f(e^{i\theta}) \cdot \bar{g}(e^{i\theta}) \cdot d\theta = 0 \quad \text{for every} \quad n \geq 0$$

By Cauchy-Schwarz the product $f \cdot \bar{g}$ belongs to $L^1(T)$ and (i) implies that this function is of the form $e^{i\theta} \cdot h(\theta)$ where $h \in H^1(T)$. So on T we have almost everywhere:

(ii) $$\bar{g}(e^{i\theta}) = e^{i\theta} \cdot \frac{h(e^{i\theta})}{f(e^{i\theta})}$$

Now we take the outer factor $\mathfrak{O}_h$ whose absolute value is equal to $|k|$ almost everywhere on T. It follows that

(iii) $$|g^*(\theta)| = \frac{\mathfrak{O}_h^*(\theta)}{\mathfrak{O}_f^*(\theta)}$$

Since $g \in H^2(T)$ Exercise 3.12 shows that the quotient in (ii) is the boundary value of an analytic function in $H^2(T)$ which implies that the conjugate function $\bar{g}$ also belongs to $H^2(T)$. But then g must be a constant and this constant is zero because the factor $e^{i\theta}$ appears in (ii). So g must be zero which gives a contradiction and the requested density is proved.

5.2 Szegö's theorem.

Let $w(\theta)$ be real-valued and non-negative function in $L^1(T)$ and denote by $\mathcal{P}_0$ the space of analytic polynomials $P(z)$ where $P(0) = 0$. Put

$$\rho(w) = \frac{1}{2\pi} \inf_{P \in \mathcal{P}_0} \int_0^{2\pi} \left|1 - P(e^{i\theta})\right| \cdot w(\theta) \cdot d\theta$$

5.3 Theorem. *One has the equality*

$$\rho(w) = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \log w(\theta) \cdot d\theta\right]$$

Proof. First we consider the case when $\log |w| \in L^1(T)$. Multiplying w with a positive constant we may assume that

(i) $$\int_0^{2\pi} \log w(\theta) \cdot d\theta = 0$$

Now we must show that $\rho(w) = 1$. To prove this we use that $\log w \in L^1(T)$ and construct the analytic function

$$f(z) = G_{\log w(z)}$$

So f is an outer function where on T one has

(ii) $$|f(e^{i\theta})| = e^{\log |w(\theta)|} = w(\theta)$$

Hence $f \in H^1(T)$ and (1) gives $f(0) = 1$. Let us now consider some $P(z) \in \mathcal{P}_0$ and set

$$F(z) = (1 - P(z))f(z)$$

Again $F(0) = 1$ and $F \in H^1(T)$ which gives the inequality

(iii) $$1 \leq \int_0^{2\pi} |F(e^{i\theta})| \cdot d\theta$$

By (ii) this means that

$$1 \leq \int_0^{2\pi} |1 - P(e^{i\theta})| \cdot w(\theta) \cdot d\theta$$

Since this hold for every $P \in \mathcal{P}_0$ we have proved the inequality

(iv) $$\rho(w) \geq 1$$

To prove the reverse inequality we apply Beurling's theorem to the outer function f. This gives a sequence of polynomials $\{Q_n(z)\}$ such that

(v) $$\lim_{n \to \infty} ||Q_n \cdot f - 1||_1 = 0$$

where we use the norm on $H^1(T)$. Since $f(0) = 1$ it follows that $Q_n(0) \to 1$ and we can normalize the approximating sequence so that $Q_n(0) = 1$ for every n and write $Q_n = 1 - P_n$ with $P_n \in \mathcal{P}_0$. Finally using (ii) we get

$$\lim_{n\to\infty} \int_0^{2\pi} |1 - P(e^{i\theta})| \cdot w(\theta) \cdot d\theta = 1$$

This gives $\rho(w) \geq 1$ and Szegö's theorem is proved for the case A above.

B. The case when $\log^+ \frac{1}{|w|}$ *is not integrable.* Here we must show that $\rho(w) = 0$ and the proof of this is left as an exercise to the reader.

6. The Helson-Szegö theorem

A trigonometric polynomial on the unit circle is of the form

$$P(\theta) = \sum a_n \cdot e^{in\theta}$$

where $\{a_n\}$ are complex numbers and only a finite family is $\neq 0$. The conjugation operator $\mathcal{C}$ is defined by

$$(*) \qquad \mathcal{C}(P) = i \cdot \sum_{n<0} a_n \cdot e^{in\theta} - i \cdot \sum_{n>0} a_n \cdot e^{in\theta}$$

Let $w(\theta)$ be a non-negative function in $L^1(T)$ and assume also that $\big|\log |w|\,\big| \in L^1(T)$.

6.1 Definition. *A w-function as above is of Helson-Szegö type if there exists a constant C such that*

$$(*) \qquad \int_0^{2\pi} |\mathcal{C}(P)(e^{i\theta})|^2 \cdot w(\theta) \cdot d\theta \leq C \cdot \int_0^{2\pi} |P(e^{i\theta})|^2 \cdot w(\theta) \cdot d\theta$$

hold for all trigonometric polynomials.

Notice that if (*) holds for some w then it holds for every function of the form $\rho \cdot w$ where $0 < c_0 \leq \rho(\theta) \leq c_1$ for some pair of positive constants. Or equivalently, with w replaced by $e^u \cdot w$ for some bounded function $u(\theta)$. With this in mind we announce the result below which is due to Helson and Szegö in [HS]:

6.2 Theorem. *A function $w(\theta)$ is of the Helson-Szegö type if and only if there exists a bounded function u and a function $v(\theta)$ for which the maximum norm of $|v|$ over T is < 1 and*

$$w(\theta) = e^{u(\theta)+v^*(\theta)}$$

where v^* is the harmonic conjugate of v.

The proof requires several steps. The first part is an exercise on norms on the Hilbert space $L^2(w)$ which is left to the reader.

Exercise. Show that w is of the Helson-Szegö type if and only if there exists a constant $\rho < 1$ such that

$$(*) \qquad \Big| \int_0^{2\pi} P(\theta) \cdot e^{-i\theta} \cdot Q(\theta) \cdot w(\theta) \cdot d\theta \,\Big| \leq \rho \cdot ||P||_w \cdot ||Q||_w$$

hold for all pairs P, Q in $\mathcal{P}_0$.

6.3 The outer function ϕ. We define the analytic function $\phi(z)$ by

$$\phi(z) = \exp\Big[\, \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot \log(\sqrt{w(\theta)}) \cdot d\theta \,\Big]$$

Since $\log \sqrt{w(\theta)} = \frac{1}{2} \cdot \log w(\theta)$ is in $L^1(T)$ it means that $\phi(z)$ is an outer function and on the unit circle we have the equality

$$(1) \qquad |\phi(\theta)|^2 = w(\theta)$$

Using (1) we find a real-valued function $\gamma(\theta)$ such that

$$w(\theta) = \phi^2(\theta) \cdot e^{i\gamma(\theta)} \tag{2}$$

Next, (1) implies that the weighted L^2-norm $||P||_w$ is equal to the standard L^2-norm of $\phi \cdot P$ on T. Hence (1) holds if and only if

$$\Big| \int_0^{2\pi} \phi(\theta) P(\theta) \cdot e^{-i\theta} \cdot \phi(\theta) Q(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||\phi \cdot P||_2 \cdot ||\phi \cdot Q||_2 \tag{3}$$

hold for all pairs P, Q in $\mathcal{P}_0$. Now we use that ϕ is outer which by Beurling's closure theorem means that $\mathcal{P}_0 \cdot \phi$ is dense in $H_0^2(T)$. Hence (3) is equivalent to

$$\Big| \int_0^{2\pi} F(\theta) \cdot e^{-i\theta} \cdot G(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||F||_2 \cdot ||G||_2 \tag{4}$$

for all pairs F, G in $H_0^2(T)$.

Next, in XX we prove that every $f \in H_0^1(T)$ admits a factorization $f = F \cdot G \cdot e^{-i\theta}$ for a pair F, G where $||f||_1 = ||F||_2 \cdot ||G||_2$. So (4) is equivalent to

$$\Big| \int_0^{2\pi} f(\theta) \cdot e^{i\gamma(\theta)} \cdot d\theta \Big| \leq \rho \cdot ||f||_1 \tag{5}$$

for each $f \in H_0^1(T)$. At this stage we use the duality between $H^\infty(T)$ and $H_0^1(T)$ from Section XX. It follows that (5) is equivalent to the following

6.4 Approximation condition. One has

$$\min_h ||e^{i\gamma(\theta)} - h(\theta)||_\infty = \rho$$

where the minimum is taken over h-functions in $H^\infty(T)$.

Final part of the proof.

Since $w \geq 0$ and > 0 outside a set of measure zero, the approximation condition is equivalent with the existence of some $h \in H^\infty(T)$ and some $\rho < 1$ such that

$$|w(\theta) - \phi^2(\theta) \cdot h(\theta)| \leq \rho \cdot w(\theta) \tag{*}$$

hold on T. It remains to show that (*) is equivalent to the existence of a pair u, v in Theorem 6.2. Let us begin with

Proof that (*) gives the pair u, v. Since $\log w$ is in $L^1(T)$ we have $w > 0$ almost everywhere and (*) entails that $\phi^2(\theta) \cdot h(\theta)$ stay in the sector

$$Z = \{z : \ -\pi/2 + \delta \leq \arg(z) \leq \pi/2 - \delta\}$$

where we have put $\delta = \arccos(\rho)$. This inclusion of the range of $\phi^2 \cdot h$ implies that it is outer. See XX above. Hence we can find a harmonic function V such that

$$\phi^2 \cdot h = e^{ia} \cdot e^{V + iV^*}$$

where a is some real constant. The inclusion of the range implies that

$$|a + V^*(\theta)| \leq \pi/2 - \delta$$

Next, define the harmonic function

$$v(\theta) = -(a + V^*(\theta))$$

It follows that

$$\phi^2(\theta) \cdot h(\theta) = e^{v(\theta) + iv^*(\theta) + c}$$

for some constant c. Finally, since $w = |\phi|^2$ we obtain

$$w(\theta) = e^{v(\theta)} \cdot \frac{e^a}{|h(\theta)|}$$

By (xx) above the last factor is bounded both below and above and hence e^u for some bounded function. Together with the bound (xx) for the harmonic conjugate of v we get the requested form for $w(\theta)$ in Theorem 6.2.

Proof that a pair (u, v) *gives (*).* Consider the special case when $w = e^v$ and

$$|v^*(\theta)| \leq \pi/2 - \epsilon$$

holds for some $\epsilon > 0$. It is clear that the corresponding ϕfunction obtained via (xx) above satisfies

$$\phi^2(\theta) = e^{v(\theta)+iv^*(\theta)}$$

This gives

$$e^{i\gamma(\theta)} = e^{-iv^*(\theta)}$$

and we notice that if we take the constant function $h(\theta) = \epsilon$ then the maximum norm

$$||e^{i\gamma(\theta)} - \epsilon||_\infty < 1$$

which proves that (*) holds.

III. A. Hardy-Littlewood's maximal function

Contents

Introduction. The results below are foremost due to Hardy, Littlewood and Fatou. Before we describe results about harmonic and analytic functions we expose some general facts about maximal functions from measure theory.

1. The weak type estimate

Let $f(x)$ be a non-negative function and integrable function on the real x-line with support in a finite interval $[0, A]$. The forward maximal function is defined for ever $x \geq 0$ by

$$f^*(x) = \max_{h>0} \frac{1}{h} \int_x^{x+h} f(t) \cdot dt$$

It is clear that f^* is non-negative and supported by $[0, A]$. To each $\lambda > 0$ we get the set $\{f^* > \lambda\}$.

1.1 Theorem *For each $\lambda > 0$ one has the inequality*

$$\mathbf{m}(\{f^* > \lambda\}) \leq \frac{1}{\lambda} \cdot \int_{\{f^*>\lambda\}} f(x) \cdot dx$$

Proof. Introduce the primitive function

$$F(x) = \int_0^x f(t) \cdot dt$$

With $\lambda > 0$ we have the continuous function $F(x) - \lambda$ and define the forward Riesz set by:

$$\mathcal{E}_\lambda = \{0 \leq x < A \colon \exists\, y > x \quad \text{and} \quad F(y) - \lambda y > F(x) - \lambda x\}$$

1.2 Exercise. Show the equality

$$\mathcal{E}_\lambda = \{f^* > \lambda\}$$

Now $\mathcal{E}_\lambda$ is an open set and hence a disjoint union of intervals $\{(a_k, b_k)\}$. With these notations one has

1.3 Exercise. Show the following for each interval (a_k, b_k):

$$F(b_k) - \lambda \cdot b_k = \max_{a_k \leq x \leq b_k} F(x) - \lambda$$

In particular one has

$$\lambda(b_k - a_k) \leq F(b_k) - F(a_k)$$

This holds for each k and after a summation over the forward Riesz intervals the requested inequality in Theorem 1.1 follows.

Using Theorem 1 can prove the following L^2-inequality.

1.4 Theorem. *One has*

$$||f^*||_2 \leq |f||_2$$

Proof. By the general formulas for distribution functions from XX we have:

$$\int_0^A f^*(x)^2 \cdot dx = \int_0^\infty \lambda \cdot \mathbf{m}(\{f^* > \lambda\}) \cdot d\lambda$$

By Theorem 1.1 the last integral is majorised by

$$\int_0^\infty \big[\int_{\mathbf{m}(\{f^*>\lambda\}} f(x) \cdot dx\,\big] \cdot d\lambda) = \iint_{\{f^*(x)>\lambda\}} f(x) \cdot dx d\lambda =$$

$$\int_0^A \big[\int_0^{f^*(x)} d\lambda\big] \cdot f(x) \cdot dx = \int_0^A f^*(x) \cdot f(x) \cdot dx$$

By the Cauchy-Schwarts in equality the last integral is majorised by the product of L^2-norms

$$||f^*||_2 \cdot |f||_2$$

Hence

$$||f^*||_2^2 = \int_0^A f^*(x)^2 \cdot dx \leq ||f^*||_2 \cdot |f||_2$$

and Theorem 1.4 follows after division with $||f^*||_2$.

1.5 Remark. In a similar way we get an L^2-inequality using the backward maximal function

$$f_*(x) = \max_{h>0} \frac{1}{h} \int_{x-h}^x f(t) \cdot dt$$

and also the full maximal function

$$f^{**}(x) = \max_{a,b} \frac{1}{a+b} \int_{x-a}^{x+b} |f(t)| \cdot dt \tag{**}$$

with the maximum taken over pairs $a, b > 0$. Then we get the L^2-inequality

$$||f^{**}||_2 \leq 2 \cdot |f||_2 \tag{1.6}$$

2. A study of harmonic functions.

Let $f(t)$ be complex-valued function on the real t-line such that

$$\int_{-\infty}^\infty \frac{|f(t)|}{1+t^2} \cdot dt < \infty$$

We also assume that

$$\max \frac{1}{b+a} \cdot \int_{-a}^b |f(t)| \cdot dt < \infty$$

where the maximum is taken over all pairs $a, b > 0$. Define the function $V(z) = V(x+iy)$ in the upper half-plane $y > 0$ by

$$V(z) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{y}{(x-t)^2 + y^2} \cdot f(t) \cdot dt$$

2.1 Exercise. Prove the inequality

$$|V(x+iy)| \leq \big(\frac{|x|}{y} + 2\big) \cdot f^{**}(0) \tag{1}$$

where

$$f^{**}(0) = \max_{a,b>0} \frac{1}{b-a}| \cdot \int_{-a}^b |f(t) \cdot dt$$

Next, define Fatou's maximal function on the real x-line by

$$V^*(x) = \max_{y \leq |s|} |V(x+iy)| \tag{3}$$

We also introduce the function $f^{**}(x)$ defined for each real x by:

$$f^{**}(x) = \max \frac{1}{b-a}| \cdot \int_{x-a}^{x+b} |f(t) \cdot dt$$

2.2 Exercise. Show the inequality

$$V^*(x) \leq 3 \cdot f^*(x)$$

Next, apply the inequality (1.6) which together with Exercise 2.2 give

(i) $$\int_{-\infty}^{\infty} V^*(x)^2 \cdot dx \leq 18 \cdot \int_{-\infty}^{\infty} f^2(x) \cdot dx$$

Since $V(x) = f(x)$ holds on the real line we conclude the following:

2.3 Theorem. *One has the inequality*

(*) $$||V^*||_2 \leq 3\sqrt{2} \cdot \sqrt{\int_{-\infty}^{\infty} V(x)^2 \cdot dx}$$

3. Application to analytic functions.

Let $f(z)$ be analytic in $\mathfrak{Im}(z) > 0$ and assume that there is a constant C such that

$$\int_{-\infty}^{\infty} \frac{|f(x+iy)|}{1+x^2} \cdot dx \leq C \quad \text{for all} \quad y > 0$$

It means that f belongs to the Hardy space H^1 in the upper half-plane U_+. We can divide out the zeros via a Blaschke product and write

$$f = B_f \cdot g$$

where g again belongs to H^1 and has no zeros in U_+. Then $\sqrt{g}$ is defined which gives a complex-valued harmonic function

$$V(z) = \sqrt{g(z)}$$

3.1 Exercise. Apply Theorem 2.3 to the V-function and use that $|f(z)| \leq |g(z)| \leq |V^2(z)|$ to show that

(1) $$\int_{-\infty}^{\infty} |f^*(x)| \cdot dx \leq 3\sqrt{2} \cdot \int_{-\infty}^{\infty} |f(x)| \cdot dx$$

where $f^*(x)$ is Fatou's maximal function for f defined for each real x by

$$f^*(x) = \max_{y \leq |s|} |f(x+iy)|/tag2$$

3.2 Exercise. Use the conformal map from U_+ to the unit disc D defined by

$$w = \frac{z-i}{z+i}$$

Explain how the previous result is translated when we start from an analytic function f in D for which the boundary value function $f(e^{i\theta})$ is in $L^1(T)$.

4. Hardy spaces and conformal maps

Let $g(z) = \sum a_n z^n$ be analytic in D and assume that there exists a constant C such that

$$\int_0^{2\pi} |g(re^{i\theta})| \cdot d\theta \leq C$$

for every $r < 1$. Thus, by the Brothers Riesz Theorem g belongs to the Hardy space $H^1(T)$. In D there exists a single-valued brach of $\log(1-z)$ whose imaginary part stays in $(-\pi/2, \pi/2)$ and with $z = re^{i\theta}$ we have

$$\mathfrak{Im}\,\log(1-z) = -\frac{1}{2i} \cdot \sum_{n=1}^{\infty} \frac{r^n}{n}(e^{in\theta} - e^{-in\theta})$$

4.1 Exercise. Deduce from the above that

$$(*) \qquad \int_0^{2\pi} \mathfrak{Im}\,\log(1-re^{i\theta}) \cdot g(re^{i\theta}) \cdot d\theta = -\pi i \cdot \sum_{n=1}^{\infty} \frac{a_n}{n} \cdot r^{2n}$$

The case when $\{b_n\}$ are real and ≥ 0. If this holds then (*) and the triangle inequality yield:

$$\pi \cdot \sum_{n=1}^{\infty} \frac{a_n}{n} \cdot r^{2n} \leq \frac{\pi}{2} \cdot \int_0^{2\pi} |g(re^{i\theta})| \cdot d\theta$$

So if we introduce the $H^1(T)$-norm

$$||g||_1 = \int_0^{2\pi} |g(e^{i\theta})| \cdot d\theta$$

it follows after a passage to the limit when $r \to 1$ that

$$(**) \qquad \sum_{n=1}^{\infty} \frac{b_n}{n} \leq \pi \cdot |g||_1$$

4.2 A study of conformal mappings. Let $\phi \colon D \to \Omega$ be a conformal mappng and assume that the complex derivative $\phi'(z)$ belongs to the Hardy space H^1 as above. Since $\phi' \neq 0$ in D there exists a single-valued analytic square-root:

$$\psi(z) = \sqrt{\phi'(z)}$$

Then $\psi \in H^2(T)$ so if

$$\psi(z) = \sum b_n z^n \implies \sum |b_n|^2 < \infty$$

Let us then consider the H^2-function

$$\Psi(z) = \sum |b_n| z^n$$

We get

$$\Psi^2(z) = \sum A_n z^n \quad \text{where} \quad A_n = \sum_{k=0}^{k=n} |b_k| \cdot |b_{n-k}|$$

and (**) in Exercise 4.1 gives:

$$(1) \qquad \sum_{n=1}^{\infty} \frac{A_n}{n} \leq \pi \cdot \int_0^{2\pi} |\Psi(e^{i\theta})|^2 \cdot d\theta$$

Next, consider the Taylor series

$$\phi'(z) = \sum a_n z^n \implies a_n = \sum_{k=0}^{k=n} b_k \cdot b_{n-k}$$

The triangle inequality gives $|a_n| \leq A_n$ for each n so (1) entails that

$$(2) \qquad \sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$$

Finally, consider the Taylor expansion of $\phi(z)$:

$$\phi(z) = \sum c_n z^n$$

Here

$$nc_n = a_{n-1} \quad : \quad n \geq 1$$

Then it is clear that (2) implies that the series $\sum |c_n| < \infty$. Hence we have proved the following result which is due to Hardy:

4.3 Theorem. *Let $\phi(z)$ be a conformal map such that ϕ' belongs to H^1. Then the Taylor series of ϕ is absolutely convergent.*

4.4 Exercise. Let Ω be a Jordan domain whose boundary curve $\Gamma = \partial\Omega$ has a finite arc-length. Let $\phi\colon D \to \Omega$ be the con formal mapping which by results from (xx) extends to a homeomorphism from the closed disc $\bar{D}$ onto $\bar{\Omega}$.' Let $\ell(\Gamma)$ be the arc-length of Γ. Show that the derivative $\phi'(z)$ belongs to the Hardy space and

$$\int_0^{2\pi} |\phi'(e^{i\theta})| \cdot d\theta \leq \ell(\Gamma)$$

From this it follows that the Taylor series of $\phi(z)$ is absolutely convergent.

A hint for the exercise. To each $n \geq 1$ we set $\epsilon = e^{2\pi i/n}$, i.e. the n:th root of the unity. Now ϕ yields a homeomorphism from T onto Γ. The definition of $\ell(\Gamma)$ gives the inequality below where we set $\epsilon^0 = 1$.

$$(1) \qquad \sum_{k=1}^{n} |\phi(\epsilon^k \cdot e^{i\theta}) - \phi(\epsilon^{k-1} \cdot e^{i\theta})| \leq \ell(\Gamma) \quad \text{for every} \quad 0 \leq \theta \leq 2\pi$$

Keeping n fixed we notice that the function

$$s_n(z) = \sum_{k=1}^{n} |\phi(\epsilon^k \cdot z) - \phi(\epsilon^{k-1} \cdot z)|$$

is subharmonic in D. So the maximum principle for subharmonic functions and (1) give

$$(2) \qquad \max_{\theta} s_n(re^{i\theta}) \leq \ell(\Gamma)$$

for each $r < 1$. Next, with $r < 1$ fixed the reader may verify the limit formula:

$$(3) \qquad \lim_{n\to\infty} s_n(r) = \int_0^{2\pi} |\phi'(re^{i\theta})| \cdot d\theta$$

Hence (2-3) give

$$\int_0^{2\pi} |\phi'(re^{i\theta})| \cdot d\theta \leq \ell(\Gamma)$$

Now the Brothers Riesz theorem implies that $\phi'(z)$ belongs to $H^1(T)$, i.e. the boundary value function $\phi'(e^{\theta})$ exists and belongs to $L^1(T)$.

III.B The Hardy space H^1

0. Introduction.

1. Zygmund's inequality

2. A weak type estimate.

3. Kolmogorov's inequality.

4. The dual space of $H^1(T)$

5. The class BMO

6. The dual of $\mathfrak{Re}\, H_0^1(T)$

7. Theorem of Gundy and Silver

8. The Hardy space on **R**.

9. BMO and radial norms on measures in D.

0. Introduction.

At several occasions we have met the situation where $F(z) \in \mathcal{O}(D)$ has bounded L^1-norms over circles of radius $r < 1$. The Brothers Riesz theorem in Section I shows that if there is a constant M such that

$$\int_0^{2\pi} |F(re^{i\theta})| d\theta \leq M \quad : \quad 0 < r < 1$$

then there exists an L^1-function $F(e^{i\theta})$ on the unit circle and

$$(*) \qquad \lim_{r \to 1} \int_0^{2\pi} |F(re^{i\theta}) - F(e^{i\theta}| \cdot d\theta = 0$$

The class of analytic functions F with boundary function in $L^1(T)$ is denoted by $H^1(T)$ and called the *Hardy space*. It is tempting to start with a real valued L^1-function $u(\theta)$ on the unit circle and apply the Herglotz integral formula which produces both the harmonic extension of u and its conjugate harmonic function by the equation:

$$(**) \qquad g_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot u(\theta) d\theta$$

It turns out that g_μ in general does not belong to $H^1(T)$, i.e. the condition that $u \in L^1(T)$ does not imply that $g_\mu \in H^1(T)$. Theorem 0.1 below is due to Zygmund and gives a necessary and sufficient condition for the inclusion $F \in L^1(T)$ when u is non-negative.

0.1 Theorem. *Let $u(\theta)$ be a non-negative L^1-function on T. Then $g_\mu(z) \in H^1(T)$ if and only if*

$$\int_0^{2\pi} u(\theta) \cdot \log^+ |u(\theta)| \cdot d\theta < \infty$$

Remark. That the condition is necessary is proved in § 1. The proof of sufficiency relies upon study of linear operators satisfying weak type estimates where a result due to Kolmogorov is the essential point. To profit upon Kolmogorov's result in section 3 we need a weak-type estimate for the harmonic conjugation functor which is proved in § 2.

0.2 The dual space of $H^1(T)$. On the unit circle the Banach space $C^0(T)$ of continuous complex-valued functions contains the closed subspace $A_*(D)$ which consists of those continuous function $f(e^{i\theta})$ on T which extend to analytic functions in the open disc $|z| < 1$ and vanish at $z = 0$. In Theorem 4.3 we prove that $H^1(T)$ is the dual of the quotient space

$$B = \frac{C^0(T)}{A_*(D)}$$

The proof uses the Brothers Riesz theorem. We shall also consider the subspace $H_0^1(T)$ of those functions in the Hardy space for which $f(0) = 0$. Here we find that

$$H_0^1(T) \simeq [\frac{C^0(T)}{A(D}]^* \tag{1}$$

Next, we seek the dual space of $H_0^1(T)$. Using the Brothers Riesz theorem one finds that

$$H_0^1(T)^* \simeq \frac{L^\infty(T)}{H^\infty(T)} \tag{2}$$

where $H^\infty(T)$ is the space of boundary values of bounded analytic functions in D.

0.3 The dual of $\mathfrak{Re}\, H_0^1(T)$. The real part determine functions in $H_0^1(T)$ which means that we can identify $H_0^1(T)$ with a real subspace of $L^1_{\mathbf{R}}(T)$ whose elements consist of those real-valued and integrable functions $u(\theta)$ for which the the Riesz transform also is integrable. Or equivalently, if we take the harmonic extension H_u then the harmonic conjugate has a boundary function in $L^1_{\mathbf{R}}(T)$ which we denote by u^*. The norm of such a u-function is defined as

$$||u|| = ||u||_1 + ||u^*||_1 \tag{*}$$

The norm in (3) is not equivalent to the L^1-norm so we cannot conclude that the dual space is reduced to real-valued functions in $L^\infty(T)$. To exhibit elements in the dual space we first consider some real-valued function $F(\theta)$ on T. Let H_F be its harmonic extension to D. For each $0 < r < 1$ we define the linear functional on $\mathfrak{Re}\, H_0^1(T)$ by:

$$u \mapsto \int_0^{2\pi} H_F(re^{i\theta}) \cdot u(\theta) \cdot d\theta \tag{**}$$

If the limit (*) exists for every u when $r \to 1$ and the absolute value of this limit is $\leq C \cdot ||u||$ for a constant C, then we have produced a continuous linear functional on $\mathfrak{Re}\, H_0^1(T)$. This leads to a description of the dual space which goes as follows. The definition of the norm in (*) and the Hahn-Banach theorem yields for each Λ in the dual space a pair (ϕ, ψ) in $L^\infty(T)$ such that when $f = u + iu^*$ is in $H_0^1(T)$ then

$$\Lambda(u + iu^*) = \int_0^{2\pi} u(\theta) \cdot \phi(\theta) \cdot d\theta + \int_0^{2\pi} u^*(\theta) \cdot \psi(\theta) \cdot d\theta$$

Let ψ^* be the harmonic conjugate of ψ which gives the analytic function $H_\psi + iH_\psi^*$ in D. Since $f = u + iu^*$ vanishes at $z = 0$ we get

$$\int_0^{2\pi} (u + iu^*)(\psi + i\psi^*) \cdot d\theta = 0$$

Regarding the imaginary part it follows that

$$\int_0^{2\pi} u^* \cdot \psi \cdot d\theta = -\int_0^{2\pi} u \cdot \psi^* \cdot d\theta$$

We conclude that Λ is expressed by

$$\Lambda(u) = \lim_{r \to 1} \int_0^{2\pi} H_F(re^{i\theta}) \cdot u(\theta) \cdot d\theta$$

where

$$F(\theta) = \phi(\theta) - \psi^*(\theta) \tag{***}$$

Above ψ^* is the harmonic conjugate of a bounded ψ-function where an arbitrary $\psi \in L^\infty(T)$ can be chosen. Next, recall from XXX that if $\psi \in L^\infty(T)$ then its conjugate ψ^* belongs to BMO(T). Hence (***) identifies the of $\mathfrak{Re}\, H_0^1(T)$ with a subspace of BMO(T). It turns out that one has equality. More precisely, Theorem 0.4 below which is due to C. Fefferman and E. Stein asserts that F yields such a continuous linear form if and only if F has a bounded mean oscillation in the sense of F. John and L. Nirenberg.

0.4 Theorem. *A real-valued L^1-function F on T yields a continuous linear functional on $H_0^1(T)$ as above if and only if $F \in BMO(T)$. Moreover, there exists an absolute constant C such that*

$$\Big|\int_0^{2\pi} H_F(re^{i\theta}) \cdot u(\theta) \cdot d\theta \Big| \leq C \cdot ||F||_{\text{BMO}} \cdot ||u||_1$$

for all $r < 1$ and $u \in H_0^1(T)$.

We refer to Section 6 for details of the proof which involves several steps where the essential step is to exhibit certain Carleson measures. The space of real-valued functions of bounded mean oscillation is denoted by BMO(T) and studied in Section 5 where Theorem 5.5 is an important result which clarifies many properties of functions in BMO(T).

0.5 The Hardy space on R. It consists of analytic functions $F(z)$ in the upper half-plane for which there exists a constant C such that

$$(*) \qquad \int_{-\infty}^{\infty} |F(x+i\epsilon) \cdot dx \leq C$$

hold for every $\epsilon > 0$. This space is denoted by $H^1(\mathbf{R})$. Let us remark that it differs from $H^1(\mathbf{T})$ even if we employ the conformal map

$$\text{(i)} \qquad w = \frac{z-i}{z+i}$$

onto the unit disc. More precisely, with $F(z)$ given in the upper half-plane we set

$$\text{(ii)} \qquad f(w) = F(\frac{i+iw}{1-w})$$

Then the reader can verify that

$$\text{(iii)} \qquad \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta = \int_{-\infty}^{\infty} \frac{|F(x)|}{1+x^2} \cdot dx$$

where $F(x)$ is the almost everywhere defined limit of F on the real x-line which by (*) identifies $F(x)$ with an element in $H^1(\mathbf{R})$. Since $\frac{1}{1+x^2}$ is bounded it follows that the right hand side is finite in (iii) and hence f belongs to $H^1(\mathbf{T})$. However, the map $F \to f$ is not bijective because the convergence in (iii) need not imply that (*) is finite. In other words, the Hardy space on the real line is more constrained and via $F \mapsto f$ it appears s a proper subspace of $H^1(\mathbf{T})$ and the corresponding norms are not equivalent. Sections 7 and 9 study $H^1(\mathbf{R})$ and at the end of § 9 we introduce Carleson norms on non-negative Riesz measures in $\mathfrak{Im}(z) > 0$ which will be used for interpolation of bounded analytic functions in Chapter XXX.

1. Zygmund's inequality

Let $u(\theta)$ be a non-negative real-valued function on T such that

$$(*) \qquad \frac{1}{2\pi}\int_0^{2\pi} u(\theta)d\theta = 1$$

Put

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\cdot u(\theta)d\theta$$

We write

$$F = u + iv$$

where u is the harmonic extension of $u(\theta)$ from T to D and v is the harmonic conjugate which by the Herglotz formula is normalised so that $v(0) = 0$ The sufficency part in Zygmund's theorem follows from the general inequality below:

1.1 Theorem. *When $u(\theta)$ is non-negative and (*) holds we have*

$$(*) \qquad \int_0^{2\pi} u(\theta)\cdot \log^+ |u(\theta)|\, d\theta \leq \frac{\pi}{2}\cdot \int_0^{2\pi} |v(\theta)|\cdot d\theta + \int_0^{2\pi} \log^+ |F(e^{i\theta}]|\, d\theta$$

Proof. Since $\mathfrak{Re}(F) > 0$ holds in D we can write

$$\text{(i)} \qquad \log\, F(z) = \log\,|F(z)| + i\gamma(z) \quad : \quad -\pi/2 < \gamma(z) < \pi/2$$

Set $G(z) = F(z)\cdot \log\, F(z)$. Since $F(0) = 1$ we have $G(0) = 0$ and the mean value formula for harmonic functions gives

$$\text{(iii)} \qquad \int_0^{2\pi} u(e^{i\theta})\cdot \log\,|F(e^{i\theta})|\, d\theta = \int_0^{2\pi} \gamma(e^{i\theta})\cdot v(e^{i\theta})\, d\theta$$

By (i) the absolute value of the right hand side is majorised by

$$\text{(iii)} \qquad \frac{\pi}{2}\cdot \int_0^{2\pi} |v(\theta)|\cdot d\theta$$

Now we use the decomposition

$$\log\,|F(e^{i\theta})| = \log^+\,|F(e^{i\theta})| + \log^+ \frac{1}{|F(e^{i\theta})|}$$

Then (ii) and (iii) give the inequality

$$\int_0^{2\pi} u(e^{i\theta})\cdot \log^+\,|F(e^{i\theta})|\, d\theta \leq$$

$$\text{(iv)} \qquad \frac{\pi}{2}\cdot \int_0^{2\pi} |v(\theta)|\, d\theta + \int_0^{2\pi} u(e^{i\theta})\cdot \log^+ \frac{1}{|F(e^{i\theta})|}\, d\theta$$

Since $\log^+ \frac{1}{|F(e^{i\theta})|} \neq 0$ entails that $|F| \leq 1$ and hence also $u \leq 1$, it follows that the last integral above is majorised by

$$\text{(v)} \qquad \int_0^{2\pi} \log^+ \frac{1}{|F(e^{i\theta})|}\, d\theta$$

Next, $\log\,|F(z)|$ is a harmonic function whose value at $z = 0$ is zero. So the mean-value formula for harmonic functions in D gives the equality:

$$\text{(vi)} \qquad \int_0^{2\pi} \log^+ \frac{1}{|F(e^{i\theta})|}\, d\theta = \int_0^{2\pi} \log^+\,|F(e^{i\theta})|\, d\theta$$

Using this and (iv) we get the requested inequality in Theorem 1.1 since we also have the trivial estimate

$$\text{(vii)} \qquad \log^+\, u(e^{i\theta}) \leq \log^+\,|F(e^{i\theta})|$$

2. The weak type estimate.

Let $u(\theta)$ be non-negative and denote by $v(\theta)$ its harmonic conjugate function which is obtained via Herglotz formula.If E is a subset of T we denote its linear Lebesgue measure by $\mathfrak{m}(E)$. With these notations the following weak-type estimate hods:

2.1 Theorem. *For each non-negative u-function on T with mean-value one the following holds:*

$$\mathfrak{m}(\{|v|) > \lambda\} \leq \frac{4\pi}{1+\lambda} \quad : \quad \lambda > 0$$

Proof. For a given $\lambda > 0$ we set

(1) $$\phi(z) = 1 + \frac{F(z) - \lambda}{F(z) + \lambda}$$

where $F(z)$ the analytic function constructed as in section 1. Here $F(0) = u(0) = 1$ which gives:

(2) $$\phi(0) = \frac{2}{\lambda + 1}$$

Next, since $\mathfrak{Re}\, F = u \geq 0$ it follows that

(3) $$|\frac{F(z) - \lambda}{F(z) + \lambda}| \leq 1$$

Hence (1) gives $\mathfrak{Re}(\phi) \geq 0$ and mean value formula for the harmonic function $\mathfrak{Re}(\phi)$ gives:

(4) $$\frac{4\pi}{1+\lambda} = \int_0^{2\pi} \mathfrak{Re}\, \phi(e^{i\theta}) \cdot d\theta \geq \int_{\mathfrak{Re}\, \phi \geq 1} \mathfrak{Re}\, \phi(e^{i\theta}) \cdot d\theta \geq \mathfrak{m}(\{\mathfrak{Re}\, \phi \geq 1\})$$

Rewriting the last inequality we get:

(5) $$\mathfrak{m}(\{\mathfrak{Re}\, \phi \geq 1\}) \leq \frac{4\pi}{1+\lambda}$$

Next, the construction of ϕ yields the following equality of sets:

(6) $$\{\mathfrak{Re}\, \phi(e^{i\theta}) \geq 1\} = \{\mathfrak{Re}\, \frac{F(e^{i\theta}) - \lambda}{F(e^{i\theta}) + \lambda} \geq 0\}$$

Finally, with $F(e^{i\theta}) = u(\theta) + iv(\theta)$ one has

$$\mathfrak{Re}\, [\, \frac{F(e^{i\theta}) - \lambda}{F(e^{i\theta}) + \lambda}\,] = \frac{u^2 + v^2 - \lambda^2}{(u+\lambda)^2 + v^2}$$

The right hand side is ≥ 0 when $|v| \geq \lambda$ which gives the set-theoretic inclusion:

(7) $$\{|v| > \lambda\} \subset \{\mathfrak{Re}\, \phi \geq 1\}).$$

Then (5) above gives Theorem 2.1.

3. Kolmogorov's inequality

3.1 Notations. Consider a measure space equipped with a probability measure μ. Let f be a complex-valued and μ-measurable function. For each $\lambda > 0$ we get the μ-measurable set $\{|f| > \lambda\}$ and then

$$\lambda \mapsto \mu(\{|f| > \lambda\})$$

is a decreasing function. Construct the differential function defined for every $\lambda > 0$ by:

(*) $$d\rho_f(\lambda) = \lim_{\delta \to 0} \frac{\mu(\{|f| > \lambda - \delta\}) - \mu(\{|f| > \lambda\})}{\delta}$$

For an arbitrary continuous function $Q(\lambda)$ defined when $\lambda \geq 0$ the formula in XX gives the equality:

(**) $$\int_0^\infty Q(|f|) d\mu = \int_0^\infty Q(\lambda) \cdot d\rho_f(\lambda)$$

Recall also from XX the formula

(***) $$\int_0^\infty \mu[\{|f|>\lambda\}]\cdot d\lambda = \int |f|\cdot d\mu$$

3.2 Operators of Weak type (1,1). Let γ be a probability measure on another sample space and T is some linear map from μ-measurable functions into γ-measurable functions.

3.3 Definition. *We say that T satisfies a weak-type estimate of type (1,1) if there is a constant K such that the inequality below holds for every $\lambda>0$:*

$$\gamma(\{|Tf|>\lambda\}\,)\leq \frac{K}{\lambda}\cdot\int |f|\cdot d\mu \quad \text{when} \quad f \ \text{is}\, \mu-\text{measurable}$$

We can also regard L^2-spaces. The operator T is L^2-continuous if there exists a constant K_2 such that one has the inequality

$$\int |T(f)|^2\cdot d\gamma \leq K_2^2\cdot\int |f|^2\cdot d\mu$$

Taking square roots it means that the L^2-norm is K_2.

3.4 Theorem. *Let T be a linear operator whose L^2-norm is 1 and with finite weak-type norm K. Then the following holds for each μ-measurable function f:*

$$\int |T(f)|\cdot d\gamma \leq 1+4\cdot\int |f|\cdot d\mu + 2K\cdot\int |f|\cdot \log^+|f|\cdot d\mu$$

Proof. When $\lambda>0$ we decompose f as follows:

(i) $$f=f_*+f^* \quad : f_*=\chi_{\{|f|\leq\lambda\}}\cdot f \quad : \quad f^*=\chi_{\{|f|>\lambda\}}\cdot f$$

For the lower f_*-function we use that T has L^2-norm ≤ 1 and get

(ii) $$\gamma[\{|Tf_*|>\lambda/2\}\,]\leq \frac{4}{\lambda^2}\int_0^\lambda s^2\cdot d\rho_f(s)$$

For Tf^* we apply the weak-type estimate which gives

(iii) $$\gamma[\{|Tf^*|>\lambda/2\}\,]\leq \frac{2K}{\lambda}\cdot\int_\lambda^\infty s\cdot d\rho_f(s)$$

where we used that $\int_\lambda^\infty s\cdot d\rho_f(s)$ is the L^1-norm of f^*. The set-theoretic inclusion

$$\{|Tf|>\lambda\}\subset\{|Tf_*|>\lambda/2\}\cup\{|Tf^*|>\lambda/2\}\implies$$

(iv) $$\gamma[\{|Tf|>\lambda\}\,]\leq \frac{4}{\lambda^2}\int_0^\lambda s^2\cdot d\rho_f(s)+\frac{2K}{\lambda}\cdot\int_\lambda^\infty sd\rho_f(s)$$

Next, since γ has total mass one the inequality:

(v) $$\int_0^\infty |Tf|\cdot d\gamma\leq 1+\int_{\{|Tf|>1\}}|Tf|\cdot d\gamma$$

Now (***) in (3.1) is applied to Tf and the measure γ which gives

$$\int_{\{|Tf|>1\}}|Tf|\cdot d\gamma=\int_1^\infty \gamma[\{|Tf|>\lambda\}\cdot d\lambda$$

By (iv) the last integral in (v) is majorised by:

(vi) $$4\cdot\int_1^\infty \big[\frac{1}{\lambda^2}\int_0^\lambda s^2\cdot d\rho_f(s)\big]\cdot d\lambda+2K\int_1^\infty \frac{1}{\lambda}\cdot\big[\int_\lambda^\infty sd\rho_f(s)\big]\cdot d\lambda$$

Next, from (XX) one has the equality:

(vii) $$\int_0^\infty \big[\frac{1}{\lambda^2}\int_0^\lambda s^2\cdot d\rho_f(s)\big]\cdot d\lambda=\int |f|\cdot d\mu$$

The left hand side is only smaller if the λ-integration starts at 1. It follows that the first term in (vi) above is majorised by $4 \cdot \int |f| \cdot d\mu$ and together with (v) we conclude that

$$\text{(viii)} \qquad \int_0^\infty |Tf| d\gamma \leq 1 + 4 \int |f| d\mu + 2K \int_1^\infty [\frac{1}{\lambda} \cdot \int_\lambda^\infty s d\rho_f(s)] \cdot d\lambda$$

Finally,

$$\int_1^\infty [\frac{1}{\lambda} \cdot \int_\lambda^\infty s d\rho_f(s)] \cdot d\lambda = \iint_{1 \leq \lambda \leq s} \frac{1}{\lambda} \cdot s\rho_f(s) ds = \int_1^\infty s \cdot \log s \cdot d\rho_f(s)$$

The last integral is equal to $\int f \cdot \log^+ |f| \cdot d\mu$ by the general formula XX. Inserting this in (viii) we get Theorem 3.4.

3.5. Final part of Theorem 0.1 There remains to show that if u is non-negative and if $u \cdot \log^+ u$ is integrable so is v. To prove this we use $d\mu = d\gamma = \frac{d\theta}{2\pi}$ on the unit circle. Theorem 2.1 which shows that the harmonic conjugation operator $T \colon u \mapsto v$ is of weak-type (1,1) and it is continuous on $L^2(T)$ by Parseval's formula. Hence Kolomogorv's Theorem gives $v \in L^1(T)$ which proves the necessity in Theorem 0.1.

Remark. Notice that Theorem 3.4 applies when we start from any real-valued function $u(\theta)$. So have the following supplement to Theorem 0.1.

3.6 Theorem. *There exists an absolute constant A such that*

$$\int_0^{2\pi} |v(\theta)| \, d\theta \leq A \cdot \Big[\int_0^{2\pi} |u(\theta)| \, d\theta + \int_0^{2\pi} |u(\theta)| \cdot \log^+ |u(\theta)| \, d\theta \Big]$$

4. The Dual space of $H^1(T)$

On the unit circle T we have the Banach space $L^1(T)$ where $H^1(T)$ is a closed subspace. Next, let $C^0(T)$ be the Banach space of continuous functions on T equipped with the maximum norm. It contains the closed subspace $A(D)$ whose functions can be extended as analytic functions in the open disc D. We have also the subspace $A_*(D)$ which consists of the functions in $A(D)$ whose analytic extensions are zero at the origin. As explained in XXX a continuous function f on T belongs to $A_*(D)$ if and only if

$$(*) \qquad \int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad n = 0, 1, \ldots$$

From (*) it follows that

$$(**) \qquad \int_0^{2\pi} g(e^{i\theta}) \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad f \in A_*(D) \quad \text{and} \quad g \in H^1(T)$$

Let us now regard the Banach space

$$B = \frac{C^0(T)}{A_*(D}$$

Riesz representation formula identifies the dual space of $C^0(T)$ with Riesz measures on T. Since B is a quotient space its dual space becomes

$$\text{(i)} \qquad B^* = \{ \mu \in M(T) \quad : \mu \perp A_*(D \}$$

Now a Riesz measure μ is $\perp A_*(D$ if and only if

$$\text{(ii)} \qquad \int_0^{2\pi} e^{in\theta} \cdot d\mu(\theta) = 0 \quad : \quad n = 1, 2 \ldots$$

The Brothers Riesz theorem means that (ii) holds if and only if μ is absolutely continuous, i.e. μ is given by some L^1-function f which satisfies:

$$\text{(iii)} \qquad \int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad n = 1, 2 \ldots$$

This is precisely the condition that $f \in H^1(T)$. Hence the whole discussion gives:

4.1 Theorem. *The Hardy space $H^1(T)$ is the dual of B.*

4.2 The dual of $H^1(T)$. Recall that $L^\infty(T)$ is the dual space of $L^1(T)$. So by a general formulafrom Appendix: Functional analysis we get:

$$H^1(T)^* = \frac{L^\infty(T)}{H^1(T)^\perp}$$

Next, an L^∞-function f is $\perp H^1(T)$ if and only if

$$\int_0^{2\pi} e^{in\theta} \cdot f(e^{i\theta}) \cdot d\theta = 0 \quad : \quad n = 0, 1, 2 \ldots$$

But this means precisely that f is the boundary value of an analytic function in D which vanishes at the origin. Let us identify $H^\infty(D)$ with a subalgebra of $L^\infty(T)$ which is denoted by $H^\infty(T)$. Then we also get the subspace $H_0^\infty(T)$ of those functions which are zero at the origin. Hence we have proved

Theorem 4.3 *The dual space of $H^1(T)$ is equal to the quotient space*

$$\frac{L^\infty(T)}{H_0^\infty(T)}$$

5. BMO

Introduction. Functions of bounded mean oscillation were introduced by F. John and L. Nirenberg in [J-N]. This class of Lebesgue measurable functions can be defined in $\mathbf{R^n}$ for every $n \geq 1$. Here we are content to study the case $n = 1$ and restrict the attention to periodic functions which is adapted to the class BMO on the unit circle. So let $F(x)$ be a locally integrable function on the real x-line which is 2π-periodic, i.e. $F(x + 2\pi) = F(x)$. If $J = (a, b)$ is an interval we get the mean value

$$F_J = \frac{1}{b-a} \cdot \int_a^b F(x)dx$$

To every interval J we put

$$|F|_J^* = \int_J |F(x) - F_J| \cdot dx$$

5.1 Definition. *The function F has a bounded mean oscillation if there exists a constant C such that $|F|_J \leq C$ for all intervals J. When this holds the smallest constant is denoted by $|F|_{BMO}$.*

5.2 The case $n \geq 2$. Even though these notes are devoted to complex analysis in dimension one, we cannot refrain from mentioning a result which illustrates how the class BMO enters in Fourier analysis. Let $F(x)$ be an L^1-function with compact support in $\mathbf{R^n}$. Assume that there exists a constant C such that the Fourier transform $\widehat{F}(\xi)$ satisfies the decay condition

$$(*) \qquad |\widehat{F}(\xi) \leq C \cdot (1 + |\xi|)^{-n} \quad : \quad \xi \in \mathbf{R^n}$$

This is not quite enough for $\widehat{F}$ to be integrable. So we cannot expect that (*) implies that $F(x)$ is a bounded function. However, its belongs to BM0 and more precisely one has:

5.3 Theorem. *To each $M > 0$ there exists a constant C_M such that if $F(x)$ has support in the ball $\{|x| \leq M\}$ then*

$$||F||_{\text{BMO}} \leq C_M \cdot \max_{\xi} \left[1 + |\xi|)^n \cdot |\widehat{F}(\xi)|\right]$$

For the proof we refer to [Björk] and [Sjölin] contains the improved result that F belongs to BMO under less restrictive conditions expressed by certain L^2-integrals of $\widehat{F}$ over dyadic grids.

5.4 The John-Nirenberg inequality. We turn to the main topic in this section and prove an inequality due to F. John and L. Nirenberg which is presented for the 1-dimensional periodic case. See [J-N] for higher dimensional results.

5.5 Theorem *Let $F(x)$ be a 2π-periodic function on the real x-line which belongs to BMO on T. For every interval J on $\mathbf{R}$ and every positive integer n one has*

$$\mathfrak{m}[\{x \in J : \left|F(x) - F_J\right| \geq 4n \cdot |F|_{\text{BMO}}\}] \leq 2^{-n} \cdot |J|$$

The proof requires several steps. To begin with we make some trivial observations. The BMO-norm of F is unaffected when we add a constant to F and also under a translation, i.e. when we regard $F_a(x) = F(x + a)$ for some real number a. Moreover, the BMO-norm is unchanged under dilations, i.e. when $t > 0$ and $F_t(x) = F(tx)$. Before we enter the proof we need a preliminary result.

5.6 Lemma. *Let F belong to BMO. Let $I \subset J$ be two intervals with the same mid-point. Then*

$$|F_J - F_I| \leq 2 \cdot [\text{Log}_2 \frac{|J|}{|I|} + 1] \cdot |F|_{\text{BMO}}$$

Exercise. Prove this result.

Proof of Theorem 5.5. Replacing F by cF for some positive constant we may assume that its BMO-norm is 1/2. and that $F_J = 0$. Moreover, by the invariance under dilations and translations we may assume that J is the unit interval. Thus, there remains to consider the set

(i) $$E_n = \{x \in [0,1] \ : \ F(x) > 2n\}$$

and show that

(ii) $$\mathfrak{m}(E_n) \leq 2^{-n}$$

Let us begin with the case $n = 1$. For every $x \in E_1$ which is a Lebesgue point for F we find the unique largest dyadic interval $J(x)$ such that

(iii) $$x \in J(x) \subset [0,1] \quad : \quad \frac{1}{\mathfrak{m}(J(x))} \int_{J(x)} F(t)dt > 1$$

Up to measure zero, i.e. ignoring the null set where F fails to have Lebesgue points, we have the inclusion

(iv) $$E_1 \subset \cup_{x \in E_1} J(x)$$

Next, suppose we have a *strict* inclusion $J(x) \subset J(y)$ for a pair of dyadic intervals in this family which means that $\mathfrak{m}(J(y)) > \mathfrak{m}(J(x))$. But this is impossible for then $x \in J(y)$ which contradicts the maximal choice of $J(x))$ as the dyadic interval of largest possible length containing x. Hence the family $\{J(x_\nu)\}$ consists of dyadic intervals which either are equal or disjoint. We can therefore pick a disjoint family where the corresponding x-points are denoted by x_ν^* and obtain the set-theoretic inclusion

(v) $$E_1 \subset \cup \ (J(x_\nu^*)$$

Next, put $\mathcal{E} = \cup J(x_\nu^*)$. Since the mean value of F over each $J(x_\nu^*)$ is ≥ 1 we obtain

$$\mathfrak{m}(\mathcal{E}) \leq \sum \int_{J(x_\nu^*)} F(x)dx = \int_{\mathcal{E}} F(x)dx \leq \int_{\mathcal{E}} |F(x)|dx \leq \int_0^1 |F(x)|dx \leq |F|_{\text{BMO}}$$

where the last inequality follows from the definition of the BMO-norm and the condition that the mean-value of F over the unit interval was zero. Since the BMO-norm of F was 1/2 the inclusion (v) gives:

$$(*) \qquad \mathfrak{m}(E_1) \leq \mathfrak{m}(\mathcal{E}) \leq 1/2$$

This proves the case $n = 1$ and we proceed by an induction over n. Let us first regard one of the dyadic intervals $J(x^*_\nu)$ from the family covering E_1. If 2^{-N} is the length of $J(x^*_\nu$ the dyadic exhaustion of $[0,1]$ gives a dyadic interval J' of length 2^{-N+1} which contain $J(x^*_\nu)$. The maximal choice of $J(x^*_\nu)$ gives:

$$\text{(vi)} \qquad \frac{1}{\mathfrak{m}(J')} \int_{J'} F(t)dt \leq 1$$

Apply Proposition XX to the pair $J(x^*_\nu)$ and J'. Since $|F|_{\text{BMO}} = 1/2$ is assumed and $\text{Log}_2(2) = 0$ we obtain

$$\text{(vii)} \qquad F_{J(x^*_\nu)} = \frac{1}{\mathfrak{m}(J(x^*_\nu))} \int_{J(x^*_\nu)} F(t)dt \leq 2$$

Let $n \geq 2$ and for every ν we set:

$$\text{(viii)} \qquad E_n(\nu) = E_n \cap J(x^*_\nu)$$

Since $F(x) > 2n$ holds on E_n we get

$$\text{(ix)} \qquad F(x) - F_{J(x^*_\nu)} > 2(n-1) \quad : \quad x \in E_n(\nu)$$

Hence we have the inclusion

$$\text{(x)} \qquad E_n(\nu) \subset W_n(\nu) = \{x \in J(x^*_\nu) \quad : \ F(x) - F_{J(x^*_\nu)} > 2(n-1)\}$$

By a change of scale we can use the interval $J(x^*_\nu)$ instead of the unit interval and by an induction assume that the inequality in Theorem 5.5 holds for $n-1$. It follows that the set in right hand side in (x) is estimated by:

$$\text{(xi)} \qquad \mathfrak{m}(W_n(\nu)) \leq 2^{-n+1} \cdot \mathfrak{m}(J(x^*_\nu))$$

The set-theoretic inclusion (x) therefore gives

$$\text{(xii)} \qquad \mathfrak{m}(E_n(\nu)) \leq 2^{-n+1} \cdot \mathfrak{m}(J(x^*_\nu))$$

Finally, since $E_n \subset E_1$ and we already have the inclusion (iv) we obtain

$$\mathfrak{m}(E_n) = \sum \mathfrak{m}(E_n(\nu)) \leq 2^{-n+1} \cdot \sum \mathfrak{m}(J(x^*_\nu)) = 2^{-n+1}\mathfrak{m}(\mathcal{E}) \leq 2^{-n+1} \cdot \frac{1}{2} = 2^{-n}$$

This proves the induction step and Theorem 5.5 follows.

5.7 An L^2-inequality

Let $F \in \text{BMO}(T)$ be given. Given some interval $J \subset T$ and $\lambda > 0$ we set

$$\mathfrak{m}_J(\lambda) = \{\theta \in J : \, |F(\theta) - F_J| > \lambda\}$$

Consider the integral

$$(*) \qquad I = \frac{1}{|J|} \cdot \int_0^\infty \lambda \cdot \mathfrak{m}_J(\lambda) \cdot d\lambda$$

Set $A = 4 \cdot ||F||_{\text{BMO}}$. Theorem 5.5. gives

$$I = \frac{1}{|J|} \cdot \sum_{n=0}^\infty \int_{nA}^{(n+1)A} \lambda \cdot \mathfrak{m}_J(\lambda) \cdot d\lambda \leq \frac{1}{|J|} \cdot \sum_{n=0}^\infty (n+1)A \cdot |J| \cdot \cdot 2^{-n} = C||F||_{\text{BMO}}$$

where $C = 4 \cdot \sum_{n=0}^{\infty} (n+1)\, 2^{-n}$ is an absolute constant. Next, by the general result in XX (*) is equal to

$$(**) \qquad \frac{1}{|J|} \cdot \int_J |F(x) - F_J|^2 \cdot dx$$

So by the above (**) is majorized by an absolute constant times the BMO-norm of F.

5.8 BMO and the Garsia norm.

Using the L^2-inequality in (5.7) an elegant description of BMO(T) was discovered by Garsia which we shall use in Section 6. First we give:

5.9 Definition. *To each real-valued $u \in L^1(T)$ we define a function in D by*

$$\mathcal{G}_u(z) = \frac{1}{8\pi^2} \cdot \iint \frac{(1-|z|^2)^2}{|e^{i\theta} - z|^2 \cdot |e^{i\phi} - z|^2} \cdot [u(\theta) - u(\phi)]^2 \cdot d\theta d\phi$$

If this function is bounded we set

$$(*) \qquad \mathcal{G}(u) = \max_{z \in D} \sqrt{\mathcal{G}_u(z)}$$

and say that u has a finite Garsia norm.

Remark. Notice that constant functions have zero-norm. So just as for BMO the $\mathcal{G}$-norm is defined on the quotient of functions modulu constants.

5.10 Exercise. Expanding the square $[u(\theta) - u(\phi)]^2$ the reader can verify that

$$(*) \qquad \mathcal{G}_u = H_{u^2} - H_u^2$$

where H_{u^2} is the harmonic extension of u^2. and H_u^2 the square of the harmonic extension H_u.

5.11 Theorem. *An L^1-function u has finite Garsia norm if and only if it belongs to BMO. Moreover, there exists a constant $C \geq 1$ such that*

$$\frac{1}{C} \cdot ||u||_{\text{BMO}} \leq \mathcal{G}(u) \leq C \cdot ||u||_{\text{BMO}}$$

5.12 Exercise. The reader is invited to prove this result using the previous facts about BMO and also straightforward properties of the Poisson kernel. if necessary, consult [Koosis p. xxx-xxx] for details.

5.13 The Garsia norm and Carleson measures. Let f be a real-valued continuous function on T. We get the two harmonic functions H_f and H_{f^2} and recall from (5.10) that

$$\mathcal{G}_f = H_{f^2} - (H_f)^2$$

In XX we introduced the family of Carleson sectors in D and now we prove an important inequality.

5.14 Theorem. *For every Carleson sector S_h with $0 < h < 1/2$ and each $f \in C^0(T)$ one has the inequality*

$$\frac{1}{h} \cdot \iint_{S_h} |z| \cdot \log \frac{1}{|z|} \cdot \left|\nabla(H_f)\right|^2 \cdot dxdy \leq 96 \cdot \mathcal{G}(f)^2$$

Proof. We use the conformal map where $z = \frac{\zeta - i}{\zeta + i}$. If $\phi(z)$ is a function in D we get the function $\phi^*(\zeta)$ in the upper half-plane where

$$\phi(\frac{\zeta - i}{\zeta + i}) = \phi^*(\zeta)$$

One easily verifies that

$$\text{(i)} \qquad (|z| \cdot \log \frac{1}{|z|})^*(\xi + i\eta) \leq 8 \cdot \eta$$

Set $w(\zeta) = H_f^*(\zeta)$. Then (i) implies that the double integral which appears in the Theorem 5.14 is majorised by

(ii) $$J^* = 8 \cdot \iint_{S_h^*} \eta \cdot |\nabla(w)|^2 \cdot d\xi d\eta$$

where S_h^* is the image of S_h under the conformal map and $|\nabla(w)|^2 = w_\xi^2 + w_\eta^2$. Next, from (*) in Exercise 5.10 we have

$$w^2 = H_{f^2}^* - \mathcal{G}_f^*$$

Since $H_{f^2}^*$ is harmonic we obtain

(iii) $$2 \cdot |\nabla(w)|^2 = \Delta(w^2) = -\Delta(\mathcal{G}_f^*)$$

where the first easy equality follows since w is harmonic. As explained by figure XX the set S_h^* is placed above an interval on the real ξ-line and and since the subsequent estimates are invariant under the center of this interval we therefore may take it as $\xi = 0$. Let us introduce the half-disc

$$D_{2h} = \{|\zeta| < 2h\} \cap \{\eta > 0\}$$

Then a figure shows that

(iv) $$S_h^* \subset D_{2h}$$

Next, consider the function $1 - \frac{|\zeta|}{2h}$ and notice that it is $\geq 1/4$ in D_{2h}. Recall from the above that

$$\Delta(\mathcal{G}_f^*) = -2 \cdot |\nabla(w)|^2 \leq 0$$

From the inclusion (iv) and taking the minus signs into the account it follows from (ii) that

(v) $$J^* \leq -16 \cdot \iint_{D_{2h}} \eta(1 - \frac{|\zeta|}{2h}) \cdot \Delta(\mathcal{G}_f^*) \cdot d\xi d\eta$$

Apply Green's formula to the pair $\mathcal{G}_f^*$ and $\rho = \eta(1 - \frac{|\zeta|}{2h})$. Here $\rho = 0$ on the boundary of D_{2h} and it is easily checked that the outer normal $\partial_n(\rho) \leq 0$. At the same time $\mathcal{G}_f^* \geq 0$ and from this it follows that (v) gives:

(iv) $$J^* \leq -16 \cdot \iint_{D_{2h}} \Delta(\eta(1 - \frac{|\zeta|}{2h})) \cdot \mathcal{G}_f^* \cdot d\xi d\eta$$

Next, using polar coordinates (r, ϕ) an easy computation gives

$$\Delta(\eta(1 - \frac{|\zeta|}{2h})) = -\frac{3}{2h} \cdot \sin\,\phi$$

It follows that

$$J^* \leq \frac{24}{h} \cdot \iint_{D_{2h}} \mathcal{G}_f^* \cdot \sin\,\phi \cdot r dr d\phi$$

Finally, by definition $\mathcal{G}(f)^2$ is the maximum norm of $\mathcal{G}_f$ in D which is $\geq$ the maximum norm of $\mathcal{G}_f^*$ in D_{2h}. So the last integral is majorised by

$$\frac{24\mathcal{G}(f)^2}{h} \cdot \iint_{D_{2h}} \sin\,\phi \cdot r dr d\phi = 96 \cdot \mathcal{G}(f)^2 \cdot h$$

After a division with h we get Theorem 5.14.

5.15 Remark. Since $|z| \geq 1/2$ holds in sectors S_h with $0 < h < 1/2$ we can remove the factor $|z|$ and hence Theorem 5.14 shows that if $\mathcal{G}_f(z)$ is bounded in D then we obtain a Carleson measure in D defined by

(*) $$\mu_f = \log\,\frac{1}{|z|} \cdot \left|\nabla(H_f)\right|^2$$

Moreover, its Carleson norm is estimated via Theorem 5.11 and Theorem 5.24 by an absolute constant times $|F|_{\text{BMO}}$.

6. The dual of $\mathfrak{Re}(H_0^1(T)$

By the observations before Theorem 0.4 there remains to prove that if $F \in \text{BMO}(T)$ then (*) holds in Theorem 0.4 for some constant C. To obtain this we need some preliminary results.

6.1 Some integral formulas. To simplify notations we set

$$\int_0^{2\pi} g(e^{i\theta}) \cdot d\theta = \int_T g \cdot d\theta$$

for integrals over the unit circle. Now follow some results which are left as exercises and proved by Green's formula.

A. Exercise. For every C^2-function W in the closed unit disc with $W(0) = 0$ we have

$$\int_T W \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \Delta(W) \cdot dxdy \tag{1}$$

Next, if

$$W = |z| \cdot W_1 \tag{2}$$

B. Exercise. Let u, v is a pair of C^2-functions which both are harmonic in the open disc. Show that

$$\Delta(uv) = 2 \cdot \langle \nabla(u), \nabla(v) \rangle \tag{i}$$

and use (A) to prove the equality

$$\int_T uv \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \langle \nabla(u), \nabla(v) \rangle \cdot dxdy \tag{ii}$$

C. Exercise. Let $f = u + iv$ be analytic in D. Verify that

$$\Delta(|f|) = \frac{1}{|f|} \cdot |\nabla(u)|^2 \tag{i}$$

holds outside the zeros of f. Show also that if

$$f = z \cdot g \tag{ii}$$

where g is zero-free in D then

$$\int_T |f| \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \frac{|\nabla(u)|^2}{|f|} \cdot dxdy \tag{iii}$$

Finally, let f be as in (ii) and F a real-valued C^2-function in D. Show that

$$\frac{1}{2} \int_T u \cdot F \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \langle \nabla(u), \nabla H_F \rangle \cdot dxdy \tag{iii}$$

6.2 Proof of Theorem 0.4

Let $f \in H_0^1(T)$. Then one finds a Blaschke product B such that

$$f(z) = z \cdot B(z) \cdot g(z)$$

where g is zero-free in D. It follows that

$$2f = z(B+1) \cdot g + z(B-1) \cdot g = f_1 + f_2$$

where $||f_\nu||_1 \leq 2 \cdot ||f||_1$ hold for each ν. Using this trick we conclude that it suffices to estasblish Theorem 0.4 for $H^1(T)$-functions of the form $f(z) = z \cdot g(z)$ with a zero-free function g in D. We write $f = u + iv$ and for each real-valued C^2-function $F(\theta)$ on T we have by (iii) from Exercise C:

$$\frac{1}{2} \cdot \int_0^{2\pi} F(\theta) \cdot u(\theta) \cdot d\theta = \iint_D \log \frac{1}{|z|} \cdot \langle \nabla(u), \nabla(H_F) \rangle \cdot dxdy \tag{1}$$

Insert the factor $1 = \sqrt{|f|} \cdot \frac{1}{\sqrt{|f|}}$ and apply the Cauchy-Schwarz inequality which estimates the absolute value of (i) by

$$(2) \qquad J = \sqrt{\iint_D \log \frac{1}{|z|} \cdot \frac{|\nabla(u)|^2}{|f(z)|} \cdot dxdy} \cdot \sqrt{\iint_D \log \frac{1}{|z|} \cdot |\nabla(H_F)|^2 \cdot |f(z)| \cdot dxdy}$$

The equality (iii) in Exercise C shows the first factor is equal to $\sqrt{||f||_1}$. In the second factor appears the density function $\log \frac{1}{|z|} \cdot |\nabla(H_F)|^2$ in D.

Finally, by the Remark in (5.15) the Carleson norm of the density $\log \frac{1}{|z|} \cdot |\nabla(H_F)|^2$ is bounded by an absolute constant C times the BMO-norm of F. Together with the result in XXX in Section XXX we get an absolute constant C such that the last factor in (2) above is bounded by

$$(3) \qquad C \cdot |F|_{\mathrm{BMO}} \cdot \sqrt{||f||_1}$$

which finishes the proof of Theorem 0.4.

7. A theorem by Gundy Silver

Introduction. Let $U(x)$ be in $L^1(\mathbf{R})$ and construct its harmonic extension to the upper half plane:

$$U(x+iy) = \frac{1}{\pi} \cdot \int \frac{y}{(x-t)^2 + y^2} \cdot U(t) \cdot dt$$

The harmonic conjugate of $U(x+iy)$ is given by:

$$(0.1) \qquad V(x+iy) = \frac{y}{\pi} \int \frac{U(t) \cdot dt}{(x-t)^2 + y2}$$

Next, to each real x_0 the Fatou sector in the upper half-plane is defined by

$$(0.2) \qquad \{x+iy\} \quad \text{such that} \quad |x-x_0| \leq y\}$$

and the maximal function U^* over Fatou sectors is defined on the real x-axis by

$$(0.3) \qquad U^*(x_0) = \max |U(x+iy)| \colon \quad : |x-x_0| \leq y$$

In XXX we proved that if $V \in L^1(R)$ then $U^*(x) \in L^1(\mathbf{R})$ and there exists an absolute constant C_0 such that

$$(*) \qquad ||U^*||_1 \leq \int_{-\infty}^{\infty} \big(|U(x)| + |V(x)|\big) dx$$

A reverse inequality is due to Burkholder, Gundy and Silverstein.

Theorem 7.1. *One has the inequality*

$$\int |V(x)| dx \leq 4 \int U^*(x) dx$$

Remark. Hence U^* belongs to L^1 if and only if the boundary value function $V(x)$ belongs to L^1. The original proof in [BGS] used probabilistic methods. Here we give a proof based upon methods from [Feff-Stein]. Since we shall establish an *a priori estimate*, it suffices to assume that $U(x)$ from the start is a nice function. In particular we may assume that both $U(x+iy)$ and $V(x+iy)$ have rapid decay when $y \to +\infty$ in the upper half-plane. This assumption is used below to ensure that a certain complex line integral is zero.

Proof of Theorem 7.1

Given $\lambda > 0$ we put

$$J_\lambda = \{x \colon U^*(x) > \lambda\}$$

The closed complement $\mathbf{R} \setminus J_\lambda$ is denoted by E. Let $\{(a_\nu, b_\nu)\}$ be the disjoint intervals of J_λ. Construct the piecewise linear Γ-curve which stays on the real x-line on E while it follows the two sides of the triangle T_ν standing on (a_ν, b_ν) for each ν. So the corner point of T_ν in the upper half-plane is:

$$p_\nu = \frac{1}{2}(a_\nu + b_\nu)(1+i)$$

Set $\partial T = \Gamma \setminus E$ and notice that the construction of Fatou sectors gives

$$(1) \qquad U^*(x) \leq \lambda \quad : \quad x \in T$$

In $\mathfrak{Im}(z) > 0$ we have the analytic function $G(z) = (U + iV)^2$. By hypothesis U $y \mapsto G(x+iy)$ decreases quite rapidly which gives a vanishing complex line integral:

$$\int_\Gamma G(z)dz = 0$$

Now Γ is the union of E and the union of of the broken lines which give the two sides of the T_ν-triangles. Let ∂T denote the union of these broken lines. Since the complex differential $dz = dx + idy$ the real part of the complex line integral is zero which gives

$$(2) \qquad \int_E (U^2 - V^2) \cdot dx + \int_{\partial T} (U^2 - V^2) \cdot dx - 2 \cdot \int_{\partial T} U \cdot V dy$$

On the sides of the T-triangles the slope is plus or minus $\pi/4$ and hence $|dy| = |dx|$ where $|dx| = dx$ is positive. Hence the he inequality $2ab \leq a^2 + b^2$ for any pair of non-negative numbers gives:

$$(3) \qquad 2 \cdot \big| \int_{\partial T} UV dy \big| \leq \int_{\partial T} U^2 \cdot dx + \int_{\partial T} V^2 \cdot dx$$

Since (2) is zero we see that (3) and the triangle inequality give:

$$(4) \qquad \int_E V^2 \cdot dx \leq \int_E U^2 \cdot dx + 2 \cdot \int_{\partial T} U^2 \cdot dx$$

Next, put

$$V_\lambda^+ = \{x : |V(x)| > \lambda\}$$

Then (4) gives:

$$(5) \qquad \mathfrak{m}(V_\lambda^+ \cap E) \leq \frac{1}{\lambda^2} \cdot \int_E V^2 \cdot dx \leq \frac{1}{\lambda^2} \cdot \int_E U^2 \cdot dx + \frac{2}{\lambda^2} \cdot \int_{\partial T} U \cdot dx$$

Next, Since the integral $\int_{T_\nu} dx = (b_\nu - a_\nu)$ for each ν and (1) holds we have

$$(6) \qquad \frac{2}{\lambda^2} \cdot \int_{\partial T} U^2 \cdot dx \leq 2 \cdot \sum (b_\nu - a_\nu) = 2 \cdot \mathfrak{m}(J_\lambda)$$

Using the set-theoretic inclusion $V_\lambda^+ \subset (V_\lambda^+ \cap E_\lambda) \cup J_\lambda$ it follows after adding $\mathfrak{m}(J_\lambda)$ on both sides in (5):

$$(6) \qquad \mathfrak{m}(V_\lambda^+) \leq 3 \cdot \mathfrak{m}(J_\lambda) + \frac{1}{\lambda^2} \cdot \int_E U^2 \cdot dx$$

Finally, $U \leq U^*$ holds on E and since E is the complement of J_λ we have $E = \{x : U^*(x) \leq \lambda\}$. Now we apply general integral formulas which after integration over $\lambda \geq 0$ gives

$$\int |V(x) \cdot dx = 3 \cdot \int U^*(x) \cdot dx + \int_0^\infty \frac{1}{\lambda^2} \big[\int_{(U^* \leq \lambda} (U^*)^2 \cdot dx \big] \cdot d\lambda$$

By the integral formula from XX the last term is equal to $\int U^*(x) \cdot dx$ and Theorem 7.1 follows.

8. The Hardy space on R

Consider an analytic function $F(z)$ in the upper half-plane whose boundary value function $F(x)$ on the real line is integrable. This class of analytic functions in $\mathfrak{Im}\, z > 0$ is denoted by $H^1(\mathbf{R})$. To each such F we introduce the non-tangential maximal function

(*) $$F^*(x) = \max_{z \in \mathcal{F}(x)} |F(z)|$$

where $\mathcal{F}(x)$ is the Fatou sector of points $z = \xi + i\eta$ for which $|\xi - x| \leq \eta$. With these notations one has

8.1 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty}^{\infty} |F^*(x)| \cdot dx \leq C \cdot \int_{-\infty}^{\infty} |F(x)| \cdot dx$$

To prove this we shall first study harmonic functions and reduce the proof of Theorem 8.1 to a certain L^2-inequality. To begin with, let $u(x)$ is a real-valued function on the x-axis such that the integral

$$\int_{-\infty}^{\infty} \frac{|u(x)|}{1+x^2} \cdot dx < \infty$$

The harmonic extension to the upper half-plane becomes:

$$U(x+iy) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} \cdot u(t) \cdot dt$$

The non-tangential maximal function is defined by:

(*) $$U^*(x) = \max_{z \in \mathcal{F}(x)} |U(z)|$$

When $u(x)$ belongs to $L^2(\mathbf{R})$ it turns out that one there is an L^2-inequality.

8.2 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty}^{\infty} (U^*(x))^2 \cdot dx \leq \int_{-\infty}^{\infty} u^2(x) \cdot dx$$

for every L^2-function on the x-axis.

In 8.X below we show how Theorem 8.2 gives Theorem 8.1. The proof of Theorem 8.2 relies upon a point-wise estimate of U via the Hardy-Littlewood maximal function of u. Let us first consider a function $u(x)$ supported by $x \geq 0$ such that the function

$$t \mapsto \frac{1}{t} \int_0^t |u(x)| \cdot dx$$

is bounded on $(0, +\infty)$. Let $u^M(0)$ denote this supremum over t. Then one has

8.3 Proposition. *For each $z = x + iy$ in the upper half-plane one has*

$$|U(x+iy)| \leq (1 + \frac{|x|}{2y}) \cdot u^M(0)$$

Proof. Since the absolute values $|U(x+iy)|$ increase when u is replaced by $|u|$ we may assume that $u \geq 0$ from the start. Put

$$\Phi(t) = \int_0^t u(x) \cdot dx$$

which yields a primitive of u and a partial integration gives

$$U(x+iy) = \lim_{A \to \infty} \frac{1}{\pi} \cdot \Big| \frac{y}{(x-t)^2 + y^2} \cdot \Phi(t) \Big|_0^A + \lim_{A \to \infty} \frac{2}{\pi} \cdot \int_0^A \frac{y(t-x)}{((x-t)^2 + y^2)^2} \cdot \Phi(t) \cdot dt$$

With (x,y) kept fixed the finiteness of $u^M(0)$ entails that $t^{-2}\cdot\Phi(t)$ tends to zero with A and there remains

$$U(x+iy)=\frac{2}{\pi}\cdot\int_0^\infty \frac{y(t-x)}{((x-t)^2+y^2)^2}\cdot\Phi(t)\cdot dt$$

Now $\Phi(t)\leq u^M(0)\cdot t$ gives the inequality

$$U(x+iy)=\frac{2u^M(0)}{\pi}\cdot\int_0^\infty \frac{y(t-x)\cdot t}{((x-t)^2+y^2)^2}\cdot dt$$

To estimate the integrand we notice that it is equal to

$$\frac{y}{((x-t)^2+y^2)}+\frac{y(t-x)x}{((x-t)^2+y^2)^2}$$

The Cauchy-Schwarz inequality gives

$$|\frac{2y(t-x)x}{((x-t)^2+y^2)^2}|\leq\frac{|x|}{(x-t)^2+y^2}$$

It follows that

$$|U(x+iy)|\leq\frac{2u^M(0)}{\pi}\cdot\int_0^\infty \frac{y}{(x-t)^2+y^2)}+\frac{u^M(0)\cdot|x|}{\pi}\cdot\int_0^\infty \frac{1}{(x-t)^2+y^2}\cdot dt$$

The last sum of integrals is obviously majorised by $u^M(0)(1+\frac{|x|}{2y})$ and Proposition XX is proved.

8.4 General cae. If no constraint is imposed on the support of u it is written as u_1+u_2 where u_1 is supported by $x\leq 0$ and u_2 b $x\geq 0$. Here we consider the maximal function

$$u^M(0)=\max_t\frac{1}{2t}\int_{-t}^t|u(x)|\cdot dx$$

Exactly as above the reader may verify that

$$\text{(i)}\qquad |U(x,y)|\leq u^M(0)(2+\frac{|x|}{y})$$

In the Fatou sector at $x=0$ we have $x\leq|y|$ and hence (i) gives

$$U^*(0)\leq\leq 3\cdot u^M(0)$$

After a translation with respect to x a similar inequality holds. More precisely, put

$$u^M(x)=\max_t\frac{1}{2t}\int_{-t}^t|u(x+s)|\cdot ds$$

for every x, Then we have

$$U^*(x)\leq 3\dot{u}^M(x)$$

Now we apply the Hardy-Littlewood inequality from XX for the L^2-case and obtain the conclusive result:

8.5 Theorem. *There exists an absolute constant C such that*

$$\int_{-\infty>}^\infty U^*(x)^2\cdot dx\leq C\cdot\int_{-\infty>}^\infty u^2(x)\cdot dx$$

for every L^2-function u on the real line.

8.6 Proof of Theorem 8.1 We use a factorisation via Blaschke products which enable us to write

$$F(z)=B(z)\cdot g^2(z)$$

where $g(z)$ is a zero-free analytic function in the upper half-plane. Since $|B(z)| \leq 1$ holds in $\mathfrak{Im}(z) > 0$ we have trivially

$$F^*(x) \leq g^*(x)^2$$

On the real axis we have $|F(x)| = |g(x)|^2$ almost everywhere so the L^1-norm of F is equal to the L^2-norm of g. Next, with $g = U + iV$ we have a pair of harmonic functions and since $|g|^2 = U^2 + V^2$ we can apply Theorem 8.5 to each of these harmonic functions and at this stage we leave it to the reader to confirm the assertion in Theorem 8.1

8.7 Carleson measures

Let $F(z)$ be in the Hardy space $H^1(\mathbf{R})$. If $\lambda > 0$ we put

$$J_\lambda = \{F^*(x) > \lambda\}$$

We assume that the set is non-empty and hence this open set is a union of disjoint intervals $\{(a_k, b_k)\}$. To each interval we construct the triangle T_k with corners at the points a_k, b_k and $p_k = \frac{1}{2}(a_k + b_k) + \frac{i}{2}(b_k - a_k)$. Put

$$\Omega = \cup T_k$$

Exercise. Use the construction of Fatou sectors and the definition of F^* to show that

$$\{|F(z)| > \lambda\} \subset \Omega \tag{1}$$

Let us now consider a non-negative Riesz measure μ in the upper half-plane. For the moment we assume that μ has compact support and that $F(z)$ extends to a continuos function on the closed upper half-plane This is to ensure that various integrals exists but does not affect the final a priori inequality in Theorem X below. General formulas for distribution functions give:

$$\int |F(z)| \cdot d\mu(z) = \int_0^\infty \lambda \cdot \mu(\{|F(z)| > \lambda\}) \cdot d\lambda \tag{2}$$

To profit upon (1) we impose a certain norm on μ. To each x and every h we construct the triangle $T_x(a)$ standing on the interval $(x - a/2, x + a/2]$.

8.8 Definition. *The Carleson norm of μ is defined as smallest constant C such that*

$$\mu(T_x(a) \leq C \cdot a$$

hold for all pairs $x \in \mathbf{R}$ and $a > 0$ and is denoted by $\mathfrak{car}(\mu)$.

8.9 Application. Given μ with its Carleson norm the inclusion (1) gives

$$\mu(\{|F(z)| > \lambda\}) \leq \sum \mu(T_k) \leq \mathfrak{car}(\mu) \cdot \sum (b_k - a_k) \tag{i}$$

The last sum is the Lebesgue measure of $\{F^* > \lambda\}$ and hence the right hand side in (i) is estimated above by

$$\mathfrak{car}(\mu) \cdot \int_0^\infty \lambda \cdot \mathfrak{m}(\mu(\{F^* > \lambda\}) \cdot d\lambda = \mathfrak{car}(\mu) \cdot \int_{-\infty}^\infty F^*(x) \cdot dx \tag{ii}$$

Together with Theorem 8.1 we arrive at the conclusive result:

8.10 Theorem. *There exists an absolute constant C such that*

$$\int |F(z)| \cdot d\mu(z) \leq C \cdot \mathfrak{car}(\mu) \cdot \int_{-\infty}^\infty |F(x)| \cdot dx$$

hold for each $F \in H^1(\mathbf{R})$ and every non-negative Riesz measure μ in the upper half-plane.

9. BMO and radial norms of measures

Theorem 0.4 together with the preceeding description of the dual space of $\mathfrak{Re}\, H_0^1(T)$ implies that every BMO-function F can be written as a sum

(i) $$F = \phi + v^*$$

where ϕ is bounded and v^* is the harmonic conjugate of a bounded function. However, this decomposition is not unique. A *constructive* procedure to find a pair u, v in for a given BMO-function F was given by P. Jones in [Jones]. See also the article [Carleson] from 1976.

9.1 Radial norms on measures. Let D be the unit disc. An L^1-function $u(z)$ in D is radially bounded if there exists a constant C such that

(*) $$\frac{1}{\pi} \cdot \iint_{S_h} |u(z)| \cdot dxdy \leq C \cdot h$$

for each sector

$$S_h = \{z \colon \theta - h/2 < \arg z\theta + h/2\} \quad : \quad h > 0$$

The smallest C for which (*) holds is denoted by $|u|^*$. Notice that $|u|^*$ in general is strictly larger than the L^1-norm over D which occurs when we take $h = \pi$ above. If u satisfies (*) we define a function P_u on the unit circle by

$$P_u(\theta) = \frac{1}{\pi} \cdot \iint_D \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot u(z) \cdot dxdxy$$

With these notations Fefferman proved:

9.2 Theorem *There exists an absolute constant C such that*

$$|P_u|_{\text{BMO}} \leq C|u|^*$$

Thus, $u \mapsto P_u$ sends radially bounded $L^1(D)$-functions to BMO(T). The proof of Theorem 8.1 relies upon Theorem 0.4 and the following observation:

9.3 Exercise. Show that when u is radially bounded and $H(z)$ is a harmonic function in D with continuous boundary values on T then

$$\iint_D H(z) \cdot u(z) \cdot dxdy = \int_0^{2\pi} H(e^{i\theta}) \cdot P_u(\theta) \cdot d\theta$$

The following result is also due to Fefferman:

9.4 Theorem. *Let $F(\theta) \in BMO(T)$. Then there exists a radially bounded $L^1(D)$-function u and some $s(\theta) \in H^\infty(T)$ such that*

$$F(\theta) = s(\theta) + P_u(\theta)$$

For detailed proofs of the results above we refer to Chapter XX in [Koosis].

IV. Nevanlinna-Pick theory

Contents

Introduction.

In the unit disc D there exists the a metric defined by

$$\frac{|dz|}{1-|z|^2}$$

In a joint article from 1916, Lindelöf and Pick discovered that if $f(z) \in \mathcal{O}(D)$ has maximum norm ≤ 1, then the map $z \to f(z)$ does not increase the metric (*). This result turns out to be very useful and is applied in § 2 to give a proof of a theorem due Julia. In § 3 we prove some results due to Löwner about geometric properties of analytic mappings. Section 0 is devoted to an interpolation theorem due to Nevanlinna and Pick. We give a detailed proof since the result has a wide range of applications beyond analytic function theory in various optimization problems.

0. The Nevanlinna-Pick Interpolation Theorem

Let D be the open unit disc. Given an n-tuple of distinct points $z_1, \ldots, z_n$ in D and some n-tuple $w_1, \ldots, w_n$ of complex numbers we put:

(*) $$\rho(z(\cdot), w(\cdot)) = \min_{f \in \mathcal{O}(D)} |f|_D \quad : \quad f(z_\nu) = w_\nu \quad : 1 \leq \nu \leq n$$

Thus we seek to interpolate preassigned values at the points $\{z_k\}$ with an analytic function $f(z)$ whose maximum norm is minimal. The case $n = 1$ is trivial for then it is obvious that the constant function $f(z) = w_1$ minimizes (*) so $\rho(z_1, w_1) = |w_1|$ hold for all $\alpha_1 \in D$. If $n \geq 2$ there exists at least some $f \in \mathcal{O}(D)$ which gives a minimum. For let $\{f_\nu\}$ be a sequence of functions which solve the interpolation while their maximum norms tend to $\rho(\alpha(\cdot), w(\cdot))$. This is a normal family and hence we extract a subsequence which converges to a limit function f_* whose maximum norm is equal to $\rho(z(\cdot), w(\cdot))$. It turns out that the minimizing f is unique and of a special form. Before Theorem 1 is announced we introduce the class $\mathfrak{B}_{n-1}$ which consists of functions of the form:

(**) $$f(z) = e^{i\theta} \cdot \prod_{\nu=1}^{\nu=n-1} \frac{z - \alpha_\nu}{1 - \bar{\alpha}_\nu \cdot z}$$

where $0 \leq \theta \leq 2\pi$ and $(\alpha_1, \ldots, \alpha_{n-1})$ is some $(n-1)$-tuple of points in D which are not necessarily distinct.

0.1. Theorem *For each pair of n-tuples $z(\cdot)$ and $w(\cdot)$ there exists a unique $f_* \in \mathfrak{B}_{n-1}$ and a positive real number ρ such that the $\rho \cdot f_*(z)$ minimizes the interpolation (*).*

Remark. With $\rho = \rho(z(\cdot), w(\cdot))$ the uniquenss means that if $g \in \mathcal{O}(D)$ is an arbitrary interpolating function which is $\neq f_*$ then $|g|_D > \rho$.

The proof of Theorem 0.1 requires several steps. First we shall establish a result about Blaschke products.

0.3 Proposition. *Let f be a function in $\mathfrak{B}_n$. For every $k(z) \in \mathcal{O}(D)$ with maximum norm $|k|_D = 1$ such that $f - k$ has at least n zeros counted with multiplicity in D, it follows that $f = k$.*

Proof. We argue by a contradiction. If $k \neq f$ we denote by $N(f-k:r)$ the number of zeros of $f-k$ in $|z|<r$ counted with multiplicities. The hypothesis gives some $r_* < 1$ such that

$$\text{(ii)} \qquad N(k-f, r_*) \geq n$$

Next, to each 0$< r < 1$ we set

$$\mathfrak{m}(r) = \min_\theta |f(re^{i\theta}) - k(re^{i\theta})|$$

The hypothesis gives a sequence $\{r_\nu\}$ such hat $r_\nu \to 1$ and every $\mathfrak{m}(r_\nu) > 0$. Consider some $r_\nu > r_*$ and the analytic function

$$\text{(iii)} \qquad h_\nu = \epsilon_\nu f + \frac{1}{2}(f-k) \quad \text{where} \quad \epsilon_\nu = \frac{1}{4}\mathfrak{m}(r_\nu)$$

Since $|\frac{1}{2}(f-k)| > \epsilon_\nu \cdot |f|$ holds on the circle $|z| = r_\nu$ it follows from (i) and Rouche's theorem that we have:

$$\text{(iv)} \qquad N(h_\nu : r_\nu) = N(f-k : r_\nu) \geq n$$

At the same time we can write

$$\text{(v)} \qquad h_\nu = (1+\epsilon_\nu)f - \frac{1}{2}(k+f)$$

By assumption $|k|_D = |f|_D = 1$ and since f is a finite Blaschke product its absolute value tends uniformly to zero as $|z| \to 1$. So if r_ν is sufficiently close to 1 we get:

$$\text{(vi)} \qquad (1+\epsilon_\nu) \cdot |f(z)| > \frac{1}{2}|k(z) + f(z)| \quad : \quad |z| = r_\nu$$

Then another application of Rouche's theorem gives:

$$\text{(vii)} \qquad N(h_\nu : r) = N(f : r) \leq n-1$$

where the last inequality follows since $f \in \mathfrak{B}_n$. Now (vii) contradicts (iv) and hence we must have $k = f$ as requested.

A consequence. Let $z(\cdot)$ and $w(\cdot)$ be some pair of n-tuples and suppose there exists some $f_* \in \mathfrak{B}_{n-1}$ and some $\rho > 0$ such that $\rho \cdot f_*(z_k) = w_k$ hold for each k. Then the function $f = \rho \cdot f_*$ not only interpolates but it has also the minimal maximum norm, i.e. we have the equality $\rho = \rho(z(\cdot), w(\cdot))$. For suppose we have strict inequality $\rho(z(\cdot), w(\cdot)) < \rho$ which gives an interpolating function $k(z)$ with maximum norm $|k|_D < \rho$. Since $|\rho \cdot f| = \rho$ holds on $|z| = 1$ it follows by Rouche's theorem that f and $f-k$ has the same number of zeros i D. Now $f-k$ has at least n zeros while f has at most $n-1$ many zeros. This contradiction and proves that f is minimizing and Proposition 0.2 entails also that f is uniquely determined in $\mathfrak{B}_{n-1}$.

The results above show that there remains to establish an *existence result*. Namely, there remains to prove that for every given n-tuple $z(\cdot)$ in D and an arbitrary n-tuple $w(\cdot)$ of complex numbers, there exists a pair $f_* \in \mathfrak{B}_n$ and $\rho > 0$ such that $\rho \cdot f_*$ solves the interpolation. Moreover, if the pair has been found then we have the equality $\rho = \rho(z(\cdot), w(\cdot))$.

0.4 Induction over n. We shall prove the existence above by an induction over n. First we establish a result where Möbius transformations intervene. To each n-tuple $(z_1, \ldots, z_n)$ we denote for every $\rho > 0$ the family $I_\rho(z(\cdot))$ of all n-tuples $w(\cdot)$ such that $\rho(z(\cdot), w(\cdot)) = \rho$.

0.5 Exercise. For each point $a \in D$ we have the Möbius transformation

$$M_a(z) = \frac{z-a}{1-\bar{a}z}$$

If $f(z) \in \mathcal{O}(D)$ we get the new analytic function $f \circ M_a$ which has the same maximum norm as f. Use this to show that

$$I_\rho(z(\cdot)) = I_\rho(M_a(z(\cdot)))$$

hold for every n-tuple $z(\cdot)$. Thus, we can change the z-points via a Möbius transformation without affecting the I_ρ-sets.

0.6 The induction step. If $g \in \mathfrak{B}_{n-1}$ and $|a| < 1$ then the composed function $g \circ M_a$ again belongs to $\mathfrak{B}_{n-1}$. So by Exercise 0.5 it suffices to establish the existence of an interpolation $f \in \mathfrak{B}_{n-1}$ when $z_1 = 0$. So assume this and consider first the case $w_1 = 0$. Then we seek f of the form

$$f(z) = z \cdot g(z) \tag{1}$$

where g satisfies $g(z_k) = \frac{w_k}{z_k}$ when $2 \leq k \leq n-1$. By an an induction over n we can find $g = \rho \cdot g_*$ for some $g_* \in \mathfrak{B}_{n-2}$ and (1) entails that

$$f = \rho \cdot z \cdot g_*$$

where $z \cdot g_*$ belongs to $\mathfrak{B}_{n-1}$ which gives the requested interpolating function.

The case $w_1 \neq 0$. To begin with we find $f \in \mathcal{O}(D)$ with the maximum norm $\rho = \rho(z(\cdot), w(\cdot))$ such that $f(0) = w_1$ and $f(z_k) = w_k$ when $2 \leq k \leq n$. Put

$$\mu_k = \frac{w_k - w_1}{1 - \rho^{-2} \cdot \bar{w}_1 \cdot w_k} \tag{1}$$

We have also the analytic function

$$g(z) = \frac{f(z) - w_1}{1 - \rho^{-2} \cdot \bar{w}_1 \cdot f(z)}$$

Since $|f|_D = \rho$ we also have $|g|_D = \rho$ where $g(z_k) = \mu_k$ for each $k \geq 2$ while $g(0) = 0$. In particular

$$(0, \mu_2, \ldots, \mu_n) \in I_\gamma(0, z_2, \ldots, z_n) \quad \text{where} \quad \gamma \leq \rho$$

By the induction over n and the previous special case we find $f_* \in \mathfrak{B}_{n-2}$ such that the function

$$g_* = \gamma \cdot z \cdot f_*$$

takes the interpolating values $(0, \mu_2, \ldots, \mu_n)$. Next, there exists the analytic function

$$\phi(z) = \frac{g_*(z) + w_1}{1 + \rho^{-2} \cdot \bar{w}_1 \cdot g_*(z)}$$

If $\gamma < \rho$ we see that the maximum norm $|\phi|_D < \rho$. At the same time (1) entails that $\phi(0) = w_1$ and $\phi(z_k) = w_k$ for $k \geq 2$. Since ρ was the interpolation norm for the pair $(0, z_2, \ldots, z_n)$ and the n-tuple $w(\cdot)$ we have contradiction. Hence $\gamma = \rho$ holds and from the above it follows that

$$\rho \cdot \frac{z f_*(z) + w_1}{1 - \rho^{-2} \bar{w}_1 z f_*(z)}$$

takes the requested interpolation values $(w_1, \ldots, w_n)$ which gives the induction step over n.

Remark. Consider the case $n = 2$ in Theorem 0.1 where we assume that $w_1 \neq w_2$. Theorem 1 shows that there exists a unique triple

$$(a, \theta, \rho) \quad \text{where} \quad a \in D \quad : 0 \leq \theta \leq 2\pi \quad : \rho > 0$$

such that function

$$f(z) = \rho \cdot e^{i\theta} \cdot \frac{z - a}{1 - \bar{a} \cdot z}$$

solves the interpolation problem.

0.7 Exercise. With $n = 2$ and $z_1 = 0$ while $z_2 \neq 0$ is kept fixed we solve the interpolation for a given pair (w_1, w_2), The minimizing interpolation function is of the form

$$f(z) = \zeta \cdot \frac{z + \frac{w_1}{\zeta}}{1 + \frac{\bar{w}_1 \cdot z}{\zeta}}$$

for some $\zeta \neq 0$. So here ζ satisfies the equation

$$z_2 \cdot \zeta + w_1 = w_2 + w_2 \bar{w}_1 \cdot z_2 \cdot \frac{1}{\bar{\zeta}}$$

Writing $w_2 = w_1 + \gamma$ this amounts to solve the equation

$$(*) \qquad z_2 \cdot |\zeta|^2 = \gamma \cdot \bar{\zeta} + |w_1|^2 \cdot z_2 + \gamma \cdot \bar{w}_1 \cdot z_2$$

We assume that $w_2 \neq w_1$ which means that $\gamma \neq 0$ and that the minimizing function f is not reduced to a constant. Hence its maximum norm $|\zeta|$ must be $|w_1|$, Theorem 1 implies that under these conditions (*) has a unique solution ζ with absolute value $> |w_1|$. It is interesting to analyze how $|\zeta|$ depends on the triple $z_2, w_1 w_2$. Dividing (*) with z_2 and regarding $\lambda = \gamma / z_2$ as a parameter which varies we are led to the equation

$$|\zeta|^2 - \lambda \cdot \bar{\zeta} = |w_1|^2 + \gamma \cdot \bar{w}_1$$

The reader is invited to analyze the behaviour of $|\zeta|$ with a special attention to the case when z_2 is close to the origin while γ stays fixed, So here the interpolating function f takes quite distinct values at the origin while z_2. So one expects that its maximum norm increases. Here is

0.8 A specific example. Let $\gamma = 1$ and $z_2 = \epsilon$ for some small positive ϵ while $w_1 = a$ is real and positive. So the equation becomes

$$|\zeta|^2 - \frac{\bar{\zeta}}{\epsilon} = a^2 + a$$

The solution ζ is therefore real and we are led to the algebraic equation

$$s^2 - \frac{s}{\epsilon} = a^2 + a$$

Notice that we require that $|\zeta| > |w_1| = a$ so we seek the unique root s for which $s > a$ and it is given by

$$s = \frac{1}{2\epsilon} + \sqrt{a + a^2 + 4^{-1}\epsilon^{-2}}$$

With a kept fixed we obtain $|\zeta| \simeq \frac{1}{\epsilon}$ as $\epsilon \to 0$ which illustrates that the maximum norm of the interpolating function increases when $\epsilon \to 0$.

0.9 Interpolation constants. Let $E = (z_1, \ldots, z_n)$ be given. Each $f \in \mathfrak{B}_{n-1}$ has $n - 1$ many roots counted with multiplicities in D. In particular f cannot vanish identically on E, i.e the maximum norm

$$|f|_E = \max |f(z_k)| > 0$$

This leads us to define the number

$$\tau(E) = \min_{f \in \mathfrak{B}_{n-1}} |f|_E$$

We have also the interpolation number:

$$\mathtt{int}(E) = \max_{w(\cdot)} \rho(z(\cdot), w(\cdot))$$

with the maximum taken over all w-sequences with $|w_k| \leq 1$ for every k. With these notations one has the following result which is due to Beurling:

0.9 Theorem. *For every finite set E one has the equality*

(*) $$\tau(E) = \frac{1}{\mathfrak{int}(E)}$$

Moreover, a function $f \in \mathcal{B}_{n-1}$ which gives $|f|_E = \tau(E)$ is unique up to a constant and for such an extremal f one has $|f(\alpha_k)| = \tau(E)$ for every $1 \leq k \leq n$.

Proof. With n kept fixed the family of $\mathcal{B}_{n-1}$ enjoys normal properties in the sense of Montel so it follows that there exists at least some extremal $f \in \mathcal{B}_{n-1}$ such that $|f|_E = \tau(E)$. Now we prove that $|f(\alpha_k)| = \tau(E)$ for each k. For suppose strict inequality holds at some α-point which we can take to be α_1. Consider the Blaschke product

$$B(z) = \prod_{k=2}^{k=n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k \cdot z}$$

Rouche's theorem gives some $\delta > 0$ such that if $|\zeta| < \delta$ then the analytic function $f(z) + \zeta \cdot B(z)$ has $n-1$ zeros in D and we can therefore write

(1) $$f(z) + \zeta \cdot B(z) = \rho(\zeta) \cdot \psi_\zeta(z)$$

where the ζ-indexed ψ-functions belong to $\mathcal{B}_{n-1}$ and $\rho(\zeta)$ are complex numbers. Notice that

(2) $$f(\alpha_k) = \rho(\zeta) \cdot \psi_\zeta(\alpha_k)$$

hold when $2 \leq k \leq n$. Moreover, since $|f(\alpha_1)| < \tau(E)$ it is clear by continuity that if δ is sufficiently small then $|\psi_\zeta(\alpha_1)| < \tau(E)$ when $|\zeta| < \delta$. Since f is extremal we conclude from (2) that there exists $\delta > 0$ such that

(3) $$|\zeta| < \delta \implies |\rho(\zeta)| \geq 1$$

This gives a contradiction since the absolute value of the ρ-function cannot have a relative minimum at $\zeta = 0$ by the local complex expansion of this ρ-function in Chapter III:XX.

Uniqueness. Let f and g be two extremal functions so that $|f|_E = |g|_E = \tau(E)$ and suppose they are not identical. For each ζ where $|\zeta| < \delta$ for a sufficiently small δ we can write

$$1 - \zeta) \cdot f + \zeta \cdot g = \rho(\zeta) \cdot \psi_\zeta(z)$$

with $\psi_\zeta \in \mathcal{B}_{n-1}$. The triangle inequality gives

$$|1 - \zeta) \cdot f(\alpha_k) + \zeta \cdot g(\alpha_k)| \leq \tau(E)$$

for every k and since $|\psi_\zeta| \geq \tau(E)$ we get as above that $|\rho(\zeta(\geq 1$ whenever ζ is sufficiently close to zero. This contradicts again the complex expansion of this ρ-function from Chapter III.

The equality $\mathfrak{int}(E) = \frac{1}{\tau(E)}$. To begin with, let f be the unique extremal above which gives an n-tuple of points on the unit circle so that

$$f(\alpha_k) = \tau(E) \cdot e^{i\theta_k}$$

The Nevanlinna-Pick theorem shows that $\frac{f((z)}{\tau(E)}$ has smallest maximum norm over D when the n-tuple $\{w_k = e^{i\theta_k}\}$. This implies that

$$\mathfrak{int}(E) \geq \frac{1}{\tau(E)}$$

To prove the opposite inequality we consider some n-tuple $\{w_\bullet\}$ for which the interpolating function $g(z)$ has the maximum norm $|g|_D = \mathfrak{int}(E)$. Theorem 0.1 gives

$$g = \mathfrak{int}(E) \cdot f \quad \text{where} \quad f \in \mathcal{B}_{n-1}$$

This entails that

$$\tau(E) \leq |f|_E \leq \frac{1}{\mathfrak{int}(E)}$$

and the requested equality (*) in Theorem 0.9 follows.

1. The Lindelöf-Pick principle.

Introduction. The non-euclidian metric on D is defined by

(0.1) $$\frac{|dz|}{1-|z|^2} \quad : \ |z|<1$$

When D is equipped with this metric one gets a model of hyperbolic geometry in the sense of Bolyai and Lobatschevsky which led to an intense geometric study around 1890, foremost by F. Klein and H. Poincaré. We shall not enter a detailed discussion about the geometry since our main concern is to apply the metric (0.1) to derive inequalities for analytic functions. In a work from 1916, Lindelöf and Pick discovered that very analytic function $\phi(z)$ in the unit disc with maximum norm one at most decreases the metric (0.1). This result is called the Lindelöf-Pick principle and is proved in Theorem XX below. In section XX it is used to prove a result by Caratheodory and Julia concerned with the boundary behaviour of analytic functions.

1. Schwarz' inequality The non-euclidian distance between two point z_1 and z_2 in D will be denoted by

(1.1) $$\mathfrak{h}(z_1, z_2)$$

To grasp this distance function we first notice the equality:

(*) $$\mathfrak{h}(0,z) = \frac{1}{2}\cdot \operatorname{Log}\frac{1+|z|}{1-|z|}$$

Indeed, (*) follows since it is obvious from (0.1) that the geodesic curve from the origin to a point $z \in D$ is the ray from 0 to z. So with $|z| = r$ one computes

$$\int_0^r \frac{ds}{1-s^2}$$

which after integration gives (*). Next, with $a \in D$ we consider a Möbius transformation:

$$w = \frac{z-a}{1-\bar{a}\cdot z} \implies \frac{dw}{dz} = \frac{1-|a|^2}{(1-\bar{a}\cdot z)^2}$$

At the same time we notice that

$$1-|w|^2 = \frac{|1-\bar{a}z|^2 - |z-a|^2}{|1-\bar{a}\cdot z|^2} = (1-|a|^2)\cdot \frac{1-|z|^2}{|1-\bar{a}\cdot z|^2}$$

From this the reader may deduce that the Möbius transform preserves the $\mathfrak{h}$-metric.

1.1 Example. Take $z_1 = 1/2$ and $z_2 = e^{i\theta}/2$ with some $0<\theta<\pi$. Now

$$z \mapsto \frac{z-1/2}{1-z/4}$$

sends z_1 to the origin. It follows that

$$\mathfrak{h}(1/2, e^{i\theta}/2) = \frac{1}{2}\cdot \operatorname{Log}\frac{1+r}{1-r} \quad : \ r = \frac{2\cdot|e^{i\theta}-1|}{|2-e^{i\theta}|}$$

The following consequence of Schwarz inequality was discovered by G. Pick in 1915.

1.2 Theorem. *Let $\phi\colon D\to\Omega$ be a conformal map from the unit disc onto a simply connected domain contained in $|w|<1$. Then the non-euclidian metric decreases.*

Proof. Let $z_0 \in D$ and set $w_0 = \phi(z_0)$. The quotient

$$G(z) = \frac{\phi(z)-w_0}{1-\bar{w}_0\phi(z)} : \frac{z-z_0}{1-\bar{z}_0 z}$$

Since

$$\lim \frac{|z-z_0|}{|1-\bar{z}_0 z|} = 1 \quad \text{as} \quad |z| \to 1$$

we see that $|G(z)| \leq 1$ holds for all $z \in D$. With $z = z_0$ we have

$$G(z_0) = \phi'(z_0) \cdot \frac{1-|z_0|^2}{1-|\phi(z_0)^2|}$$

Since $z_0 \in D$ was arbitrary we get the differential inequality

$$\frac{|d\phi(z)|}{|1-\phi(z)|^2} \leq \frac{|dz|}{1-|z|^2}$$

and this is precisely the assertion in Pick's theorem.

The Lindelöf-Pick principle. Above ϕ was a conformal mapping. Since the $\mathfrak{h}$-metric is defined locally the inequality in Pick's theorem extends to analytic functions in D of absolute value < 1 and leads to the following general result:

1.3 Theorem *Let $\phi(z) \in \mathcal{O}(D)$ have maximum norm ≤ 1. Then ϕ decreases the $\mathfrak{h}$-metric.*

Remark. Thus, if we set $w = \phi(z)$ and z_1, z_2 is a pair in the unit disc D_z one has

(*) $$\mathfrak{h}(\phi(z_1), \phi(z_2)) \leq \mathfrak{h}(z_1), z_2)$$

1.4 The $\mathfrak{h}$-metric in half-spaces. Passing to the right half-plane U_+ where $\mathfrak{Re}(w) > 0$, the non-euclidian metric is obtained via the conformal map

$$z \mapsto w = \frac{1+z}{1-z}$$

From this it follows that

$$\frac{|dz|}{1-|z|^2} \mapsto 2 \cdot \frac{|w+1|^4 \cdot |dw|}{|w+1|^2 - |w-1|^2}$$

So with $w = \xi + i\eta$ the non-euclidian metric in the right half-plane becomes

(*) $$\frac{|w+1|^4 \cdot |dw|}{2\xi}$$

Next, the Lindelöf-Pick principle applies after a conformal mapping from D onto any other simply connected domain Ω where one then regards analytic functions $g \in \mathcal{O}(\Omega)$ such that $g(\Omega) \subset \Omega$.

1.5 Example. Let $\Phi(z) = u(x,y) + iv(x,y) \in \mathcal{O}(U^+)$ be such that its real part u is positive in U_+. The Lindelöf-Pick principle applies to Φ and using (*) in (1.4) one has the following result:

1.6 Proposition. *To every $k > 0$ there exists another constant k^* such that the following inequality holds for every pair of points $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$ in U_+:*

$$|\Phi(x_1 + iy_1)| \leq |v(x_0 + iy_0)| + k^* \cdot \frac{x_1 \cdot u(x_0, y_0)}{x_0} \quad : \ |y_1| < k \cdot x_1$$

1.7 Exercise. Try to prove this result. If necessary, consult the text-book [Nevanlinna: page 59-61] for a proof where it is also shown that for each $k > 0$ one can take

(*) $$k^* = 3 + 2(k+1)^2 \quad : \ \text{provided that } x_1 > x_0 \quad \text{and} \quad x_1 > |y_0|$$

2. A result by Julia.

Let $\phi \in \mathcal{O}(D)$ be such that $|\phi(z)| < 1$ when $z \in D$ and consider the boundary point $z = 1$.

2.1 Theorem. *For every $e^{i\theta}$ there exists the limit*

$$c(\theta) = \lim_{z \to 1} \frac{|e^{i\theta} - \phi(z)|}{|1 - z|} \quad : 0 \leq c(\theta) \leq +\infty \tag{1}$$

where the limit $z \to 1$ is taken in any Fatou sector at 1. Moreover, if θ is such that the limit $0 < c(\theta) < \infty$ then there exist the Fatou limits:

$$\phi'(z) \to c(\theta) \cdot e^{i\theta} \quad : \arg \frac{e^{i\theta} - \phi(z)}{1 - z} \to \theta \tag{2}$$

and the following inequality holds

$$\frac{1 - |\phi(z)|^2}{|e^{i\theta} - \phi(z)|^2} \geq \frac{1}{c} \cdot \frac{1 + |z|}{1 - |z|} \quad : z \in D \tag{3}$$

Remark. Of course, only the case when $c(\theta) < \infty$ is of interest. Notice that this finiteness only can occur for at most one θ-value. The theorem above was the starting point for an extensive study of boundary values of analytic functions in Julia's work [Ju] and has later led to a far-reaching study about Julia sets in complex dynamics. See [Carleson-Garnett] for this more recent and advanced theory in function theory. The reader may also consult Chapter IV in [Caratheodory] for an account of Julia's original theorem where some geometric interpretations appear.

Proof of Theorem 2.1

Applying the two conformal mappings

$$z \mapsto \frac{1 + z}{1 - z} \quad : w \mapsto \frac{e^{i\theta} + w}{e^{i\theta} - w}$$

we can work in the right half plane where $z = 1$ has been mapped into the point at infinity and ϕ has become an analytic function

$$\Phi(x + iy) = u(x + iy) + iv(x + iy) \quad : u(x, y) > 0 \text{ for all } (x, y) \in U_+$$

The crucial step in the proof is to show the result below:

Let $\Phi = u + iv$ be an arbitrary analytic map from U_+ to U_+ and assume that

$$\min_{x+iy \in U_+} \frac{u(x + iy)}{x} = 0 \tag{*}$$

Then it follows that

$$\lim_{x \to +\infty} \frac{u(x + iy)}{x} = 0 \quad : \text{holds uniformly inside any Fatou sector } |y| < kx \quad : k > 0$$

To prove this we take some $k > 0$ and for each $\epsilon > 0$ the hypothesis (*) gives a point $z_0 = x_0 + iy_0$ in U_+ such that

$$\frac{u(x_0, y_0)}{x_0} < \epsilon \tag{1}$$

Next, if $z = x + iy$ stays in the Fatou sector $|y| < k|x|$ and x_1 is large then Proposition 1.6 gives:

$$|\Phi(x + iy)| \leq |v(x_0 + iy_0)| + k^* \cdot \frac{x \cdot u(x_0 + iy_0)}{x_0} < |v(x_0 + iy_0)| + \epsilon \cdot k^* \cdot x$$

In particular we have

$$\frac{u(x + iy)}{x} < \frac{|v(x_0 + iy_0)|}{x} + \epsilon \cdot k^*$$

Since $\epsilon > 0$ can be chosen arbitrary small the conclusion after (*) follows.

Proof continued. Next, suppose that

$$(1) \qquad c = \min_{x+iy\in U_+} \frac{u(x+iy)}{x} > 0$$

is positive. The result above applies to $\Phi(z) - cz$ and hence $\frac{\Phi(z)}{z} \to c$ holds uniformly as $|z| \to \infty$ inside any Fatou sector $|y| < k|x|$. Moreover, this gives:

$$(2) \qquad \liminf_{x\to\infty} \frac{u(x,y)}{x} = c$$

Let us no consider the complex derivative of Φ assuming that (1) above holds for some $c > 0$.

Sublemma One has

$$\lim_{z\to\infty} \Phi'(z) = c$$

where this limit holds uniformly while z stays in any given Fatou sector.

Proof. Replacing Φ by $\Psi(z) = \Phi(z) - cz$ it suffices to show that

$$(i) \qquad \lim_{z} \Psi'(z) = c \quad : \text{uniformly when the limit is in a Fatou sector}$$

To show (i) we proceed as follows. Consider some $0 < p < 1$ and choose also some q so that $p < q < 1$. For every $r > 0$ we consider the disc

$$\Delta_r = \{|z - r| < q \cdot r\}$$

Since $q < 1$ this disc stays in a fixed Fatou sector for all large r and Cauchy's inequality gives

$$(ii) \qquad |\Psi'(z)| \leq \frac{qr}{2\pi} \int_0^{2\pi} \frac{|\Psi(r + qre^{i\theta})|}{|r + qre^{i\theta} - z|^2} \cdot d\theta \quad : \quad z \in \Delta_r$$

Next, if $\epsilon > 0$ Propostion 1.6 gives some large r^* such that

$$(iii) \qquad |\frac{\Psi(\zeta)}{\zeta}| < \epsilon \quad : |\zeta - r| = qr \quad : r \geq r^*$$

Hence, if $|z - r| \leq pr$, the Cauchy inequality from (ii) and a computation which is left to the reader gives:

$$(iii) \qquad |\Psi'(z)| \leq \epsilon \cdot \frac{q(1+q)}{(q-p)^2}$$

This proves that $\Psi'(z) \to 0$ holds uniformly when z stays in the sector

$$|\arg z| < \text{arc.sin}(p)$$

Above $p < 1$ is arbitrary which therefore gives the Caratheodory-Julia theorem after we have returned to the unit disc via a conformal map between D and U_+.

3. Some geometric results

3.1 A study of convex domains. Let Ω be a bounded convex domain and $p \in \Omega$ an interior point. The convexity implies that if we start at some boundary point $q_0 \in \partial\Omega$ where $q_0 - p$ is real and positive, then we obtain a function

$$(*) \qquad \phi \mapsto q(\phi) \quad : \arg[q(\phi) - p] = \phi \quad : q(\phi) \in \partial\Omega$$

where $q(2\pi) = q_0$ holds after one turn. The q-function is continuous and 1-1, i.e. a homeomorphism between the unit circle and $\partial\Omega$. Let $g(\phi)$ be a non-negative continuous function on T, i.e here $g(2\pi) = g(0)$. We get $g^* \in C^0(\partial\Omega)$ satisfying

$$g^*(q(\phi)) = g(\phi)$$

Starting from g^* we solve the Dirichlet problem and find the harmonic function G^* in Ω which extends g^*. With these notations we have

Theorem 3.2 *One has the inequality*

$$G^*(p) \leq \frac{1}{\pi} \int_0^{2\pi} g(\phi)d\phi$$

Remark. The inequality is of special interest when p approaches the boundary. Before Theorem 3.2 is proved we consider a general situation. Let W be any bounded Jordan domain and $p \in W$ an interior point. Let a, b be two points on ∂W. Denote by γ the Jordan subarc of ∂W which joins a and b. Let L be the line passing through these two points. Suppose that the two infinite half lines from a and b are outside W, i.e. $W \cap L$ is contained in the line segment (a, b). Now L cuts W into two halfs. Let W^* be one of these. Given a point $p \in W^*$ we shall find an upper bound for the harmonic measure $\mathfrak{m}_W(p; \gamma)$. After a rotation and a translation we may assume that $a = m$ and $b = -m$ for some $m > 0$, i.e. $[a, b]$ is an interval on the real axis and that W^* is contained in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$. Now $W \subset U$ and Carleman's principle from XX gives:

$$(1) \qquad \mathfrak{m}_W(p; \gamma) \leq \mathfrak{m}_{W^*}(p : [a.b]) \leq \mathfrak{m}_U^+(p : [a.b])$$

By the result in XXX the last term is equal to $\frac{1}{\pi} \cdot \alpha$ where α is the angle formed by $a - p$ and $b - p$.

Proof of Theorem 3.2. Consider a small arc $\gamma \subset \partial\Omega$ which by the parametrisation (*) above is defined by some ϕ-interval $\phi_* \leq \phi \leq \phi^*$. Let $\mathfrak{m}_\Omega(p : \gamma)$ be the harmonic measure at p with respect to this boundary arc. We can apply the inequality (1) and conclude that

$$\mathfrak{m}_\Omega(p : \gamma) \leq \phi^* - \phi_*$$

Now the Theorem 3.2 follows after an integration over $0 \leq \phi \leq 2\pi$ where we use that $G^*(p)$ is evaluated by the integral of g^* over $\partial\Omega$ with respect to the positive measure on $\partial\Omega$ defined by the harmonic measure at p.

3.3. On the range of analytic functions

Consider a domain $\Omega \in \mathcal{D}(C^1)$. Let $\phi \in \mathcal{O}(\Omega)$ and assume it extends to $C^0(\bar{\Omega})$. The ϕ-function is not supposed to be 1-1. We get the domain

$$W = \phi(\Omega)$$

Now the following may occur: There exists a subset Γ of $\partial\Omega$ given as a finite union of arcs $\{\gamma_\alpha\}$ such that the image set $\phi(\Gamma)$ gives the boundary ∂A of a domain $A \subset W$, i.e. here A is a relatively compact subset of the connected open set W. Put

$$\Omega_* = \{z \in \Omega \quad : \phi(z) \in W \setminus A\}$$

Here $A \subset \partial(W \setminus A)$ and we construct a harmonic measures as follows: If $z \in \Omega_*$ we have $\phi(z) \in W \setminus A$ and get the function

$$z \mapsto \mathfrak{m}_{W\setminus A}\,(\phi(z);\partial A) \quad : z \in \Omega_*$$

Since $w \mapsto \mathfrak{m}_{W\setminus A}(w;\partial A)$ is a harmonic function in $W \setminus A$ it follows that the function above is harmonic in Ω_*. Let us analyze its boundary values on $\partial\Omega_*$. If $z \in \Omega_*$ approaches Γ, then $\phi(z) \to A$ and hence

$$\lim_{z\to\Gamma} \mathfrak{m}_{W\setminus A}\,(\phi(z);\partial A) = 1$$

Let us now regard the harmonic measure function

$$z \mapsto \mathfrak{m}_{\Omega_*}(z:\Gamma)$$

By definition it has boundary value 1 along Γ and otherwise it is zero. Hence the maximum principle for harmonic functions gives:

3.4 Theorem. *In the situation above one has the inequality:*

$$\mathfrak{m}_{\Omega_*}(z:\Gamma) \leq \lim_{z\to\Gamma} \mathfrak{m}_{W\setminus A}\,(\phi(z);\partial A) \quad : \quad z \in \Omega_*$$

Application. Using Theorem 3.4 we prove a result due to Löwner. Let $w(z) \in \mathcal{O}(D)$ where $w(0) = 0$ and $|w(z)| < 1$. Suppose there exists an arc γ on the unit circle such that $w(z)$ extends continuously up to γ and that

$$|\gamma(e^{i\theta})| = 1 \quad : \quad e^{i\theta} \in \gamma$$

Consider the image $w(\gamma)$ which is an another arc on the unit cicle. With these notations Theorem 3.4 gives

3.5 Löwner's inequality. *The length of $w(\gamma)$ is $\geq$ the length of γ and equality can only hold if $w(z)$ from the start is $e^{i\alpha}z$ for some α.*

3.6 Remark. Actually Löwner proved a more precise result. Before it is announced we insert a preliminary remark. Given $w(z)$ and an arc $\gamma \subset T$ where $|w(z)| = 1$ one should expect that $|w(z)|$ must tend to 1 rather quick as $z \in D$ approaches γ. To put this in a precise form, Löwner proceeds as follows: Up to a rotation we may take

$$\gamma = \{e^{i\theta} \quad : -a < \theta < a\} \quad : 0 < a < \pi/2$$

Now we consider the family of circles K_λ passing the two end-points e^{ia} and e^{-ia} where $\lambda > 0$ expresses the angle of intersection beteen K_λ and the unit circle T.

The reader should draw a picture to see the situation where the constraint that the λ-numbers are chosen so that obtain a simple connected domain $\Omega_\lambda \subset D$ bordered by γ and a portion of K_λ. Next, regard the image set $w(\Omega_\lambda)$. On its boundary we find the arc $w(\gamma)$ which by the hypothesis that $|w| = 1$ on γ, is a sub-arc of T. At the same time we can start with the arc $w(\gamma)$ and take the circle K^*_λ which passes the end-points of $w(\gamma)$. This gives a domain Ω^*_λ bordered by $w(\gamma)$ and a subarc of the circle K^*_λ. With these notations the precise result by Löwner goes as follows:

3.7 Theorem. *For each λ as above one has the inclusion*

$$w(\Omega_\lambda) \subset \Omega^*_\lambda$$

3.8 Exercise. Deduce Theorem 3.7 from Theorem 3.5. The strategy is that if $w(z)$ is *outside* the set Ω_λ while $z \in \Omega^*_\lambda$, then the inequality for harmonic measures is violated. We leave it to the reader to discover this contradiction which gives Löwner's theorem. See also his article [Lö:1]: *Untersuchungen über schlichte konforme Abildungen* for details and further results.

V. Uniqueness theorems for analytic functions.

Introduction.

A sharp version of the Phragmén-Lindelöf theorem is proved in Theorem A.2. It is preceeded by a differential inequality where Carleman's result in Theorem A.1 was inspired by earlier constructions due to Lindelöf. Asymptotic series are studied in section B where earlier work by Borel led Carleman to the general construction in Theorem B.1. The question of uniqueness is expressed via Theorem B.6 and is settled via solutions to a variational problem in Section C.

A. The Phragmén-Lindelöf theorem.

Let $f(z)$ be an entire function. To each $0 \leq \phi \leq 2\pi$ we set

$$\rho_f(\phi) = \max_r |f(re^{i\phi})| \tag{*}$$

The text-book *Le calcus des residues* by Ernst Lindelöf contains examples of entire functions f where $\rho_f(\phi)$ is finite for all ϕ with the exception $\phi = 0$, i.e. only along the positive real axis the ρ-number fails to be bounded. An example is the entire function

$$f(z) = \frac{1}{z^2} \cdot \sum_{\nu=2}^{\infty} \frac{z^\nu}{\big(\log\, \nu\big)^\nu}$$

Here one verifies that that there exists a constant k such that:

$$|f(re^{i\phi})| \leq \exp(e^{\frac{k}{|\phi|\cdot|2\pi-\phi|}}\,) \tag{**}$$

It turns out that the example above is essentially sharp. Namely, assume that the ρ-number in (*) is finite for almost every ϕ. Then the ρ_f-function cannot be too small, unless f is reduced to a constant. Before Theorem A.1 is announced we introduce the non-negative function

$$\omega(\phi) = \log^+\big[\log^+ \rho_f(\phi)\,\big] \tag{***}$$

Since we have taken a two-fold logarithm $\omega(\phi)$ is considerably smaller compared to the ρ-function.

A.1.Theorem. *For every non-constant entire function $f(z)$ one has*

$$\int_0^{2\pi} \omega(\phi)\cdot d\phi = +\infty$$

Proof. Assume that f is not a constant. Consider the maximum modulus function

$$M(r) = \max_{|z|=r} |f(z)|$$

By the ordinary Liouville theorem the M-function increases to infinity. So we may assume that $M(r) \geq 1$ when $r \geq r_*$ for some r_*. Put

$$v(r) = \log M(r) \quad : \quad U(z) = \log|f(z)| \tag{i}$$

Given $r \geq r_*$ we consider the domain

$$\Omega_r = \{U > \frac{v(r)}{2}\} \cap \{|z| < r\} \tag{ii}$$

Next, let ζ_r be some point on the circle $|z| = r$ where $|f(\zeta_r)| = M(r)$ where ζ_r always can be chosen so that there exist arbitrary small δ where $|f(\zeta_{r-\delta})| = M(r-\delta)$ and $\lim_{\delta\to 0} \zeta_{r-\delta} = \zeta_r$. Next, in Ω we get the connected component Ω_* whose boundary contains ζ_r. Put

(iii) $$\gamma = \partial\Omega_* \cap \{|z| = r\}$$

Notice that

(iv) $$U(z) \leq \frac{v(r)}{2} \quad : \quad z \in \partial\Omega_* \cap \{|z| < r\}$$

So if W is the harmonic function in the disc D_r with boundary values 1 on γ and 0 on $\{|z| = 1\} \setminus \gamma$ we have:

(v) $$U(z) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(z) \leq 0 \quad : \quad z \in \partial\Omega_*$$

The maximum principle entails that (v) also holds in Ω_*. Hence there exist arbitrary small $\delta > 0$ such that

(vi) $$v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(\zeta_{r-\delta}) \leq 0$$

Let $2r \cdot \ell$ be the total length of the intervals which belong to γ. By the general inequality from XX we have

(vii) $$W(\zeta_{r-\delta}) \leq \frac{1}{2\pi} \int_{-\ell}^{\ell} \frac{r^2 - (r-\delta)^2}{r^2 - 2r(r-\delta)\cos\theta + (r-\delta)^2} d\theta$$

Let $h(r-\delta)$ denote the right hand side in (vii) which by (vi) gives us arbitrary small $\delta > 0$ such that

(viii) $$v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot h(r-\delta) \leq 0$$

Rewriting this inequality we obtain

(*) $$\frac{v(r) - v(r-\delta)}{\delta} \geq \frac{v(r)}{2} \cdot \frac{1 - h(r-\delta)}{\delta}$$

Next, from the definition of the h-function one has the limit formula

(ix) $$\lim_{\delta\to 0} \frac{1 - h(r-\delta)}{\delta} = \frac{1}{2\pi} \cdot \frac{\cos\ell}{\sin\ell}$$

Passing to the limit as $\delta \to 0$ in (vii) we get the differential inequality:

(**) $$v'(r) \geq \frac{v(r)}{2\pi r} \cdot \frac{\cos\ell}{\sin\ell}$$

Next, put

$$\log r = s \quad \text{and} \quad \log \frac{v(r)}{2} = g(s)$$

By derivation rules we see that (**) gives

(***) $$\frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos\ell}{\sin\ell}$$

Next, identifying γ with a subset of the periodic interval $0 \leq \phi \leq 2\pi$ it is clear that the definition of the ω-function gives the inclusion

(x) $$\gamma \subset \{\omega(\phi) \geq g(s)\}$$

So if $\lambda(s)$ is the Lebesgue measure of the set $\{\omega(\phi) \geq g(s)\}$ then $\ell \leq \lambda(s)$ and (***) gives

(****) $$\frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos\lambda(s)}{\sin\lambda(s)}$$

Next, the inequality $\sin(t) \geq \frac{2}{\pi} \cdot t$ gives a positive constant k which is independent of s such that the following hold for sufficiently large s, i.e. to ensure that the corresponding r-value satisfies $M(r) \geq 1$:

(xi) $$\frac{dg}{ds} \geq \frac{k}{\lambda(s)}$$

Hence, starting from some sufficiently large s_0 one has

(xii) $$\int_{s_0}^{s} \lambda(s) \cdot dg(s) \geq k(s - s_0)$$

This inequality implies in particular that one has a divergent integral:

(xiii) $$\int_{s_0}^{\infty} \lambda(s) \cdot dg(s) = +\infty$$

Finally, the general equality for distribution functions from XXX gives:

(xiiii) $$\int_{0}^{2\pi} \omega(\phi) \cdot d\phi = \int_{0}^{\infty} \lambda(s) \cdot dg(s)$$

The last integral is $+\infty$ by (xiii) and the requested divergence for the intergal of the ω-function follows.

Remark. At the end of the article [XXX] Carleman points out that the proof above gives a sharp version of the Phragmén- Lindelöf theorem. More precisely one has the following: Let $f(z)$ be analytic in a sector

$$S_\alpha = \{z = re^{i\phi} \quad : \quad -\alpha < \phi < \alpha\}$$

Define $\omega(\phi)$ as above when when $-\alpha < \phi < \phi$. With these notations one has:

A.2. Theorem. *Let f be bounded on the half-lines arg$(z) = \alpha$ and arg$(z) = -\alpha$ and assume also that*

$$\int_{-\alpha}^{\alpha} \omega(\phi) \cdot d\phi < \infty$$

Then $f(z)$ is bounded in the whole sector.

A.3. Exercise. Deduce Theorem A.2 from the preceeding results.

B. Asymptotic series.

Introduction. The notion of asymptotic series was expressed as follows by Poincaré:

Let $f(z)$ be complex-valued function defined in some subset E of $\mathbf{C}$ and z_0 is a boundary point. We say that f has an asymptotic series expansion at z_0 if there exists a sequence of complex numbers $c_0, c_1, \ldots$ such that $\lim_{z \to z_0} f(z) = c_0$ and for each $n \geq 0$ one has:

(*) $$\lim_{z \to z_0} (z - z_0)^{-n-1} \left[f(z) - (c_0 + c_1 + \ldots + c_n z^n) \right] = c_{n+1}$$

where the limit is taken as z stay in E.

It is obvious that if f has an asymptotic expansion at z_0 then the sequence $\{c_n\}$ is unique. Constructions of functions which admit asymptotic expansions appear in Emile Borel's thesis *Sur quelques points de la théorie des fonctions* from 1895 and he proved for example that for every sequence of real numbers $\{c_n\}$ there exists a C^∞-function $f(x)$ on the real line whose Taylor expansion at $x = 0$ is given by the sequence, i.e.

$$\frac{f^{(n)}(0)}{n\,!} = c_n \quad : \quad n = 0, 1, \ldots$$

Following [Car: xx, page 29-31] we prove a complex version of Borel's result where D_+ denotes the open half-disc $\{\mathfrak{Re}(z) > 0 \cap \{|z| < 1\}$.

B.1. Theorem. *To each sequence $\{c_n\}$ of complex numbers there exists a bounded analytic function $F(z)$ in D_+ which has an asymptotic series expansion at $z = 0$ given by $\{c_n\}$.*

Proof. It suffices to prove this when $c_0 = 0$. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $\sum a_\nu < \infty$. Given $\{c_n\}$ we construct a sequence of functions $P_1(z), P_2(z), \ldots$ which are analytic in the half plane $\mathfrak{Re}(z) > 0$ as follows: First

(i) $$P_1(z) = c_1 z(1 - \frac{z}{z + \epsilon_1}) \quad : \epsilon_1 = \frac{\alpha_1}{|c_1|} \implies$$

(ii) $$|P_1(z)| = |c_1| \cdot \epsilon_1 \cdot \frac{|z|}{|z + \epsilon_1} \leq \alpha_1 \quad : \quad \mathfrak{Re}(z) \geq 0$$

Now $P_1(z)$ has a series expansion at $z = 0$:

(ii) $$P_1(z) = \sum_{\nu=1}^{\infty} c_\nu^{(1)} \cdot z^\nu$$

Notice that the series converges in the disc $|z| < \epsilon_1$. Set

(iii) $$P_2(z) = \big[c_2 - c_2^{(1)}\big] \cdot z^2 \cdot \big(1 - \frac{z}{z + \epsilon_2}\big) \quad : |c_2 - c_2^{(1)}| \cdot \epsilon_2 \leq a_2$$

With such a careful choice of a small positive ϵ_2 we see that

(iii) $$|P_2(z)| \leq a_2 \cdot |z| \quad : \quad \mathfrak{Re}(z) \geq 0$$

Again we obtain a convergent series at $z = 0$:

(iv) $$P_2(z) = P_1(z) = \sum_{\nu=2}^{\infty} c_\nu^{(2)} \cdot z^\nu$$

The inductive construction. Let $n \geq 3$ and suppose that $P_1, \ldots, P_{n-1}$ have been constructed where we for each $1 \leq k \leq n-1$ have a series expansion

(v) $$P_k(z) = \sum_{\nu=k}^{\infty} c_\nu^{(k)} \cdot z^\nu$$

Then we define

$$P_n(z) = \big[c_n - (c_n^{(1)} + \ldots + c_n^{(n-1)}\big] \cdot z^n \cdot \big(1 - \frac{z}{z + \epsilon_n}\big) \quad : \quad \big|\, c_n - (c_n^{(1)} + \ldots + c_n^{(n-1)} \,\big| \cdot \epsilon_n \leq \alpha_n$$

So we obtain a new series at $z = 0$:

(vi) $$P_n(z) = \sum_{\nu=n}^{\infty} c_\nu^{(n)} \cdot z^\nu$$

Staying in the half-disc D_+, the inductive construction gives

$$\max_{z \in D_+} |P_n(z)| \leq \alpha_n \quad : \quad n = 1, 2, \ldots$$

Hence there exists a bounded analytic function in D_+ defined by

$$F(z) = P_1(z) + P_2(z) + \ldots$$

At this stage we leave as an exercise to the reader to verify that

$$\lim_{z \to 0} z^{-n-1} \cdot \big[\, F(z) - (c_1 z + \ldots + c_n z^n)\big] \, = c_{n+1}$$

B.2. Uniqueness of asymptotic expansions.

There exist functions whose asymptotic series is identically zero. Here is an example:

$$f(z) = e^{-\frac{1}{z^2}}$$

If $z = re^{i\theta}$ with $-\pi/8 \leq \theta \leq \pi/8$ we see that

$$|f(re^{i\theta})| = \exp\,(-\frac{\cos 2\theta}{r^2}) \leq \exp(-\frac{1}{\sqrt{2}\cdot r^2})$$

It follows that the asymptotic series at $z = 0$ is identically zero. Via a conformal map from the half-disc D_+ to the unit circle we are led to the following problem: Let $f(z)$ be analytic in the open unit disc D. Suppose that

$$(*) \qquad \lim_{z\to 1} \frac{f(z)}{(1-z)^n} = 0 \quad : \quad n = 1, 2, \ldots$$

We seek growth conditions on f in order that (*) implies that f is identically zero. An answer to this uniqueness problem was proved by Carleman in [Car]. Namely. consider a sequence of real positive numbers $A_1, A_2, \ldots$. To each $n \geq 1$ we put

$$(**) \qquad I_n = \exp\Big(\frac{1}{\pi}\int_1^\infty \log\big[\sum_{\nu=1}^{\nu=n} \frac{r^{2\nu}}{A_\nu^2}\big]\cdot dr\Big)$$

B.3. Definition. *Denote by $\mathfrak{B}$ the set of all sequences $\{A_n\}$ such that $\{I_n\}$ is bounded, i.e. there exists some K such that*

$$I_n \leq K \quad : \quad n = 1, 2 \ldots$$

In [Car: page 7-52] the following existence result is proved:

B.4. Theorem. *To each sequence $\{A_n\} \in \mathfrak{B}$ there exists an analytic function $f(z)$ in D which is not identically zero and satisfies:*

$$(1) \qquad \frac{|f(z)|}{|1-z|^n} \leq A_n \quad : \quad n = 1, 2, \ldots$$

while (*) holds.

A converse result. In [loc.cit] appears the converse to the result which ensures uniqueness of the asymoptotic expansion at $z = 1$.

B.5. Theorem. *Let $\{A_n\}$ be a sequence of positive numbers such that there exists an analytic function $f(z)$ in D which is not reduced to a constant and satisfies (*) and (1) in Theorem B.4. Then $\{A_n\} \in \mathfrak{B}$.*

Remark. The results above show that if $\{A_n\}$ is a sequence for which $\{I_n\}$ is unbounded then the asymptotic expansion at $z = 1$ is unique for every analytic function $f(z)$ satisfying (1) in Theorem B.4. The proofs of the two results above rely upon a varational problem which is presented below while the deduction after of the two cited results above are left to the reader who may find details in [Carleman].

C. A variational problem.

Let $n \geq 1$ and $a_0, a_1, \ldots, a_n$ some n-tuple of non-negative real numbers where $a_0 > 0$ is assumed. Let $\mathcal{O}(*)$ denote the family of analytic functions $f(z)$ in the unit disc satisfying $f(0) = 1$. Put

$$I(f) = \frac{1}{2\pi}\cdot\sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \int_0^{2\pi} \frac{|f(e^{i\theta})|^2}{|e^{i\theta}-1|^{2\nu}}\cdot d\theta \quad : \quad I_* = \min_{f\in\mathcal{O}(*)} I(f)$$

Remark. Above we have a variational problem. It turns out that there exists a unique function $f_*(z)$ which yields a minimum. To find f_* we shall use the rational function:

$$\Omega(z) = \sum_{\nu=0}^{\nu=n} a_\nu^2 \big[(1-z)(1-\frac{1}{z})\big]^{n-\nu}$$

Notice that

(i) $$\Omega(e^{i\theta}) = a_0^2 + \sum_{\nu=1}^{\nu=n} a_\nu^2 \cdot |e^{i\theta}-1|^{2n-2\nu}$$

In particular Ω is real and positive on the unit circle and by symmetry it has n zeros $\rho_1, \ldots, \rho_n$ in the unit disc and $\frac{1}{\rho_1}, \ldots, \frac{1}{\rho_n}$ are the zeros in the exterior disc which gives the factorization

(*) $$\Omega(z) = z^{-n} \cdot (-1)^n \cdot a_0^2 \cdot p_n(z) \cdot \prod (z - \frac{1}{\rho_\nu}) \quad : \ p_n(z) = (z-\rho_1)\cdots(z-\rho_n)$$

Next, for every $f \in \mathcal{O}(D)$ with $f(0) = 1$ we put

(ii) $$\phi(z) = \frac{f(z)}{(1-z)^n}$$

Then (i) gives the equality:

(iii) $$I(f) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |\phi(e^{i\theta})|^2 \cdot d\theta$$

We will use the last expression to prove

C.1 Theorem. *The variational problem has a unique solution whose minimum $I_*(n)$ is achieved by the function*

(i) $$f_*(z) = \frac{(1-z)^n}{\prod 1 - \rho_\nu \cdot z)}$$

Moreover,

(ii) $$I_*(n) = I(f_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}\big[\sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \frac{1}{(2 \cdot \sin\frac{\theta}{2})^{2\nu}}\big] \cdot d\theta$$

Proof By (iii) the variational problem is equivalent to seek the minimum of

(1) $$\min_\phi I(\phi) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |\phi(e^{i\theta})|^2 \cdot d\theta \quad : \phi(0) = 1$$

With f_* as in (i) from Theorem C.1 one has

(2) $$\phi_*(z) = \frac{1}{\prod 1 - \rho_\nu \cdot z)}$$

Now f_* is a unique minimizing function in Theorem C.1 if we have proved the strict inequality

(3) $$I(\phi_* + h) < I(\phi_*)$$

for every analytic function $h(z)$ in D such that $h(0) = 0$. To show (3) we notice that (1) can be replaced by a complex line integral over $|z| = 1$ which gives

$$I(\phi_* + h) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \Omega(z) \cdot |\phi_*(z) + h(z)|^2 \cdot \frac{dz}{z} =$$

(4) $$I(\phi_*) + I(h) + \frac{1}{2\pi i} \cdot \int_{|z|=1} \Omega(z) \cdot [\bar{\phi}_*(z) \cdot h(z) + \phi_*(z) \cdot \bar{h}(z)] \cdot \frac{dz}{z}$$

Since $I(h) > 0$ whenever $h \neq 0$ the requested strict inequality follows if we show that the last integral is zero. To prove this we notice that the construction of ϕ_* gives the equation

(5)
$$\Omega(e^{i\theta}) \cdot \bar{\phi}(e^{i\theta}) = (-1)^n \cdot a_0^2 \cdot \frac{1}{\prod (e^{i\theta} - \bar{\rho}_\nu)}$$

Now
$$k(z) = \frac{1}{\prod (z - \bar{\rho}_\nu)}$$

is analytic in D and since $h(0) = 0$ it follows that
$$\int_{|z|=1} k(z)h(z) \cdot \frac{dz}{z} = 0$$

This proves that the first term in the integral from (5) vanishes and the second is its complex conjugate since Ω was real on T. Hence f_* yields the unique minimizing function of the variational problem. The equality (ii) for $I(f_*)$ follows by a computation which is left to the reader.

VI. Lindelöf functions.

Introduction. For each real number $0 < a \leq 1$ there exists the entire function

$$Ea_(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+na)}$$

which for $a = 1$ gives the exponential function e^z. Growth properties of the E-functions were investigated in a series of articles by Mittag-Leffler between 1900-1904 using integral formulas for the entire function $\frac{1}{\Gamma}$. This inspired Phragmén to study entire functions $f(z)$ such that there are constants C and $0 < a < 1$ with:

$$\log|f(re^{i\theta})| \leq C \cdot (1 + |r|)^a) \quad : \, -\alpha < \theta < \alpha$$

for some $0 < \alpha < \pi/2$ while $f(z)| \leq C$ for all $z \in \mathbf{C} \setminus S$. When this holds we get the entire function

$$g(z) = \int_0^\infty f(sz) \cdot e^{-s} \cdot ds$$

If z is outside the sector S it is clear that $|g(z)|$ is bounded by $C \cdot \int_0^\infty e^{-s} ds = C$. When $z = re^{i\theta}$ is in the sector we still get a bound from (1) since $0 < a < 1$ and conclude that the entire function g is bounded and hence a constant. Since the Taylor coefficients of f are recaptured from g it follows that f must be constant. More general results of this kind were obtained in the joint article [PL] by Phragmén and Lindelöf from 1908 and led to what is called the Phragmén-Linedlöf principle. Here we discuss a continuation of [PL] from the article *Remarques sur la croissance de la fonction* $\zeta(s)$ where Lindelöf employed results in collaboration with Phragmén to investigate the growth of Riemann's ζ-function along vertical lines in the strip $0 < \mathfrak{Re}(z) < 1$. This leads to the study of various indicator functions attached to analytic functions. Here is the set-up: Consider a strip domain in the complex s-plane:

$$\Omega = \{s = \sigma + it \quad : t > 0 \quad \text{and} \quad 0 \leq a < \sigma < b\}$$

An analytic function $f(z)$ in Ω is of *finite type* if there exists some integer k, a constant C and some $t_0 > 0$ such that

$$|f(\sigma + it)| \leq C \cdot t^k \quad \text{hold for} \quad t \geq t_0$$

To every such f we define the Lindelöf function

(*) $$\mu_f(\sigma) = \limsup_{t \to \infty} \frac{\operatorname{Log}|f(\sigma + it)|}{\operatorname{Log} t}$$

Lindelöf and Phragmén proved that μ_f is a continuous and convex function on (a, b). No further restrictions occur on the μ-function because one has:

1. Theorem. *For every convex and continuous function $\mu(\sigma)$ defined in $[a, b]$ there exists an analytic function $f(z)$ without zeros in Ω such that $\mu_f = \mu$.*

2. Exercise. Prove this result using the Γ-function. First, to a pair of real numbers (ρ, α) we set

(i) $$f(s) = e^{-\frac{\pi i \cdot \rho s}{2}} \cdot \Gamma(\rho(s-a) + \frac{1}{2})$$

Use properties of the Γ-function to show that f has finite type in Ω and its indicator function becomes a linear function:

$$\mu_f(\sigma) = \rho \cdot (\sigma - a)$$

More generally one gets a function f where μ_f is piecewise linear by:

(ii) $$f = \sum_{k=1}^{k=m} c_k e^{-\frac{\pi i \cdot \rho_\nu s}{2}} \Gamma(\rho_k(s - a_k) + \frac{1}{2})$$

where $\{c_k\}$, $\{\rho_k\}$ and $\{a_k\}$ are m-tuples of real numbers. Finally, starting from an arbitrary convex curve we can choose some dense and enumerable set of enveloping tangents to this curve. Then an infinite series of the form above gives an analytic function $f(s)$ such that

$$\sigma \mapsto \mu_f(\sigma)$$

yields an arbitrarily given convex μ-function on (a,b).

1. A relation to harmonic functions.

Let $U(x,y)$ be a bounded harmonic function in the strip domain Ω and V its harmonic conjugate. Set

$$(*) \qquad f(s) = \exp\left[(\log(s) - \frac{\pi i}{2})(U(s) + iV(s))\right]$$

It is easily see that $f(z)$ has finite type in Ω. With $s = \sigma + it$ we have

$$|f(\sigma + it)| =$$

$$\exp(\frac{1}{2}\log(\sigma^2 + t^2) \cdot U(\sigma + it) \cdot \exp(-(\frac{\pi}{2} - \arg(\sigma + it)) \cdot V(\sigma + it))$$

It follows that

$$\frac{\log\,|f(\sigma + it)|}{t} = \frac{\log\,\sqrt{\sigma^2 + t^2} \cdot U(\sigma + it)]}{\log t} + \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t}$$

Exercise. With σ kept fixed one has

$$\arg(\sigma + it) = \tan\,\frac{t}{s}$$

which tends to $\pi/2$ as $t \to +\infty$. Next, $V(\sigma + it)$ is for large $t > 0$ up to a constant the primitive of

$$\int_1^t \frac{\partial V}{\partial u}(\sigma + iu) \cdot du$$

Here the partial derivative of V is equal to the partial derivative $\partial U/\partial\sigma(\sigma, u)$ taken along $\mathfrak{Re}\, s = \sigma$. Since U is bounded in the strip domain it follows from Harnack's inequalities that this partial derivative stays bounded when $1 \leq u \leq t$ by a constant which is independent of t. Putting this together the reader can verify that

$$(1) \qquad \lim_{t \to +\infty} \frac{(\arg(\sigma + it) - \frac{\pi i}{2}) \cdot V(\sigma + it)}{t} = 0$$

From (1) in the Exercise we obtain the equality

$$(*) \qquad \mu_f(\sigma) = \limsup_{t \to \infty} U(\sigma + it)$$

This suggests that we study growth properties of bounded harmonic functions in strip domains.

2. The M and the m-functions.

To a bounded harmonic function U in Ω we associate the maximum and the minimum functions:

$$M(\sigma) = \limsup_{t \to \infty} U(\sigma + it) \quad \text{and} \quad \liminf_{t \to \infty} U(\sigma + it)$$

2.1 Proposition. *$M(\sigma)$ is a convex function while $m(\sigma)$ is concave.*

We prove the convexity of $M(\sigma)$. The concavity of m follows when we replace U by $-U$. Consider a pair α, β with $a < \alpha < \beta < b$. Replacing U b $U + A + \cdot Bx$ for suitable constants A and B we may assume that $M(\alpha) = M(\beta) = 0$ and the requested convexity follows if we can show that

$$M(\sigma) \leq 0 \quad : \quad \alpha < \sigma < \beta$$

To see this we consider rectangles

$$\mathcal{R}[T_*, T^*] = \{\sigma + it \quad \alpha \leq \sigma \leq \beta \quad \text{and} \quad T_* \leq t \leq T^*\}$$

Let $\epsilon > 0$ and start with a large T_* so that

$$t \geq T_* \implies U(\alpha + it) \leq \epsilon$$

and similarly with α replaced by β. Next, we have a constant M such that $|U|_\Omega \leq M$. If $z = \sigma + it$ is an interior point of the rectangle above it follows by harmonic majorisation that

$$U(\sigma + it) \leq \epsilon + M \cdot \mathfrak{m}_z(J_* \cup J^*)$$

where the last term is the harmonic measure at z which evaluates the harmonic function in the rectangle at z with boundary values zero on the two verical lines of the rectangle which it is equal to 1 on the horizontal intervals $J^* = (\alpha, \beta) + iT^*$ and $J_* = (\alpha, \beta) + iT_*$

Exercise. Show (via the aid of figure that with $T^* = 2T_*$ one has

$$\lim_{T_* \to +\infty} \mathfrak{m}_{\sigma + 3iT_*/2}(J_* \cup J^*) = 0$$

where this limit is uniform when $\alpha \leq \sigma \leq \beta$. Since $\epsilon > 0$ is arbitrary in (xx) the reader can now conclude that $M(\sigma) \leq 0$ for every $\sigma \in (\alpha, \beta)$.

A special case. Suppose that we have the equalities

$$m(\alpha) = M(\alpha) \quad \text{and} \quad m(\beta) = M(\beta) \tag{1}$$

using rectangles as above and harmonic majorization the reader can verify that this implies that

$$m(\sigma) = M(\sigma) \quad : \quad \alpha < \sigma < \beta$$

This result is due to Hardy and Littlewood in [H-L].

The case when $M(\sigma) - m(\sigma)$ has a tangential zero. Put $\phi(\sigma) = M(\sigma) - m(\sigma)$ and suppose that this non-negative function in (a, b) has a zero at some $a < sigma_0 < b$ whose graph has a tangent at σ_0. This means that if:

$$h(r) = \max_{-r \leq |\sigma - \sigma_0| \leq r} \phi(\sigma)$$

then

$$\lim_{r \to 0} \frac{h(r)}{r} = 0 \tag{*}$$

Under this hypothesis the following result is proved in [Carleman].

2.2 Theorem. *When (*) holds we have*

$$m(\sigma) = M(\sigma) \quad \text{holds for all} \quad a < \sigma < b$$

The subsequent proof from [Carleman] was given at a lecture by Carleman in Copenhagen 1931 which has the merit that a similar reasoning can be applied in dimension ≥ 3. Adding some linear function to U we may assume that $M(\sigma_0) = m(\sigma_0) = 0$ which mans that

$$\limsup_{t \to \infty} U(\sigma_0, t) = 0 \tag{1}$$

Next, consider the function

$$\phi \colon t \mapsto \partial U / \partial \sigma(\sigma_0, t) \tag{1}$$

The assumption (*) and the result in XXX gives:

$$\lim_{t \to \infty} \partial U / \partial \sigma(\sigma_0, t) = 0 \tag{2}$$

Next, consider some $a < \sigma < b$ and let $\epsilon > 0$. By the result from XX there exist finite tuples of constants $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$ and some N-tuple $\{\tau_\nu\}$ which stays in a $[0, 1]$ such that

$$\left| U(\sigma, t) - \sum a_\nu \cdot U(\sigma_0, t_\nu + t) - \sum b_\nu \cdot \partial U / \partial \sigma(\sigma_0, t_\nu + t) \right| < \epsilon \quad \text{hold for all} \quad t \geq 1 \tag{5}$$

Since ϵ is arbitrary it follows from (1-2) that

$$\lim_{t \to \infty} U(\sigma, t) = 0 \tag{5}$$

for every $a < \sigma < b$ which obviously gives the requested equality in Theorem 2.2.

2.3. Integral indicator funtions.

Let $f(s)$ be an analytic function of finite order in the strip domain Ω and fix some $t_0 > 0$ which does not affect the subsequent constructions. For a pair (σ, p) where $a < \sigma < b$ and $p > 0$ we associate the set of of positive numbers χ such that the integral

$$(*) \qquad \int_{t_0}^{\infty} \frac{|f(\sigma + it)|^p}{t^\chi} \cdot dt < \infty$$

We get a critical smallest non-negative number $\chi_*(\sigma, p)$ such that (*) converges when $\chi > \chi_*(\sigma, p)$. In the case $p = 1$ a result due to Landau asserts that $\chi(\sigma, 1)$ determines the half-plane of the complex z-plane where the function

$$\gamma(z) = \int_{t_0}^{\infty} \frac{f(\sigma + it)}{t^z} \cdot dt$$

is analytic and $\sigma \mapsto \chi(\sigma, 1)$ is a convex function on (a, b). A more general convexity result holds when p also varies.

2.4 Theorem. *Define the ω-function by*:

$$\omega(\sigma, \eta) = \eta \cdot \chi(\sigma, \frac{1}{\eta}) \quad : a < \sigma < b \quad : \eta > 0$$

Then ω is a continuous and convex function of the two variables (σ, η) in the product set $(a, b) \times \mathbf{R}^+$.

2.5 Remark. Theorem 2.4 is proved using Hölder inequalities and factorisations of analytic functions which reduces the proof to the case when f has no zeros. The reader is invited to supply details of the proof or consult [Carleman].

3. Lindelöf estimates in the unit disc.

Let $f(z)$ be analytic in the open unit disc given by a power series

$$f(z) = \sum a_n \cdot z^n$$

We assume that the sequence $\{a_n\}$ has temperate growth, i.e. there exists some integer $N \geq 0$ and a constant K such that

$$|a_n| \leq K \cdot n^N \quad : \quad n = 1, 2, \ldots$$

In addition we assume that the sequence $\{a_n\}$ is not too small in the sense that

$$(*) \qquad \sum_{n=1}^{\infty} |a_n|^2 \cdot n^s = +\infty \quad : \quad \forall\, s > 0$$

Now there exists the smallest number $s_* \geq 0$ such that the Dirichlet series

$$\sum_{n=1}^{\infty} |a_n|^2 \cdot \frac{1}{n^s} < \infty, \quad \text{for all } s > s_*$$

To each $0 \leq \theta \leq 2\pi$ we set

$$(1) \qquad \chi(\theta) = \min_s \int_0^1 \left|f(re^{i\theta})\right| \cdot (1 - r)^{s-1} \cdot dr < \infty$$

$$(2) \qquad \mu(\theta) = \text{Lim.sup}_{r \to 1} \frac{\text{Log}\,|f(re^{i\theta})|}{\text{Log}\, \frac{1}{1-r}}$$

We shall study the two functions χ and μ. The first result is left as an exercise.

3.1. Theorem. *The inequality*

$$\chi(\theta) \leq \frac{s^*}{2}$$

holds almost everywhere, i.e. for all $0 \leq \theta \leq 2\pi$ *outside a null set on* $[0, 2\pi]$.

Hint. Use the formula

$$\frac{1}{2\pi} \cdot \int_{=}^{2\pi} |f(re^{i\theta})|^2 \cdot d\theta = \sum |a_n|^2$$

For the μ-function a corresponding result holds:

3.2. Theorem. *The inequality below holds almost everywhere.*

$$\mu(\theta) \leq \frac{s^*}{2}$$

Proof. Let $\epsilon > 0$ and introduce the function

$$\Phi(z) = \sum a_n \cdot \frac{\Gamma(n+1)}{\Gamma(n+1+\frac{s^*}{2}+\epsilon)} \cdot z^n = \sum c_n \cdot z^n$$

It is clear that the construction of s^* entails

$$\sum |c_n|^2 < \infty$$

Next, set $\Phi_0 = \Phi$ and define inductively the sequence $\Phi_0, \Phi_1, \ldots$ by

$$\Phi_\nu(z) = z^{\nu-1} \cdot \frac{d}{dz}\big[z^\nu \cdot \Phi_{\nu-1}(z)\big) \quad : \quad \nu = 1, 2, \ldots$$

3.3 Exercise. Show that for almost every $0 \leq \theta \leq 2\pi$ there exists a constant $K = K(\theta)$ such that

$$|\Phi_\nu(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{1-r)^\nu} \quad : 0 < r < 1$$

Next, with s^* and ϵ given we define the integers ν and ρ:

$$\nu = \big[\,\frac{s^*}{2} + \epsilon\,\big] + 1 \quad : \quad \rho = \frac{s^*}{2} + \epsilon - \big[\,\frac{s^*}{2} + \epsilon\,\big]$$

where the bracket term is the usual notation for the smallest integer $\geq \frac{s^*}{2} + 1$.

Exercise Show that with ν and ρ chosen as above one has

$$\Phi_\nu(z) = \sum a_n \cdot \frac{\Gamma(n+1+\nu)}{\Gamma(n+1+\rho-1)} \cdot z^n$$

and use this to show the inversion formula

$$(*) \qquad f(z) = \frac{1}{z^\nu \cdot \Gamma(1-\rho)} \cdot \int_0^z (z-\zeta)^{-\rho}\dot{\zeta}^{\nu+\rho-1} \cdot \Phi_\nu(\zeta) \cdot d\zeta$$

3.4 Exercise. Deduce from the above that for almost every θ there exists a constant $K(\theta)$ such that

$$(**) \qquad |f(re^{i\theta})| \leq K(\theta) \cdot \frac{1}{(1-r)^{\nu+\rho-1}}$$

Conclusion. From (**) and the construction of ν and ρ the reader can confirm Theorem 3. 2.

3.5 Example. Consider the function

$$f(z) = \sum_{n=1}^{\infty} z^{n^2}$$

Show that $s^* = \frac{1}{2}$ holds in this case. Hence Theorem B.2 shows that for each $\epsilon > 0$ one has

(E) $$\max_r (1-r)^{\frac{1}{4}+\epsilon} \cdot |f(re^{i\theta}| < \infty$$

for almost every θ.

3.6 Exercise. Use the inequality above to show the following: For a complex number $x + iy$ with $y > 0$ we set

$$q = e^{\pi ix - \pi y}$$

Define the function

$$\Theta(x+iy) = 1 + q + q^2 + \ldots$$

Show that when $\epsilon > 0$ then there exists a constant $K = K(\epsilon, x)$ for almost all x such that

$$y^{\frac{1}{4}+\epsilon} \cdot |\theta(x+iy)| \leq K \quad : \quad y > 0$$

8. Series and analytic functions.

Contents

1. A theorem by Kronecker.

Introduction. We seek necessary and sufficient condition in order that a sequence $c_0, c_1, c_2, \ldots$ of complex numbers yield the coefficients in the Taylor series at the origin of a rational function, i.e. that

$$(*) \qquad \sum c_\nu z^\nu = \frac{a_0 + a_1 z + \ldots + a_m z^m}{b_0 + b_1 z + \ldots + b_n z^n} \quad \text{where } b_0 \neq 0\,,$$

Here $A(z) = a_0 + a_1 z + \ldots + a_m z^m$ and $B(z) = b_0 + b_1 z + \ldots + b_n z^n$ are polynomials and we say that $\{c_\nu\}$is of *rational type* when (*) holds. A necessary condition for $\{c_\nu\}$ to be of rational type follows via euclidian division. Namely, let $f(z) = \sum c_\nu z^\nu$ and expand the product $f(z) \cdot B(z)$ into a power series. For each integer $M \geq n$ the coefficient of z^M becomes

$$(1) \qquad c_{M-n} b_n + c_{M-n+1} b_{n-1} + \ldots + c_M \cdot b_0 = 0$$

When (*) holds it follows that (1) is zero for every $M \geq m+1$. It means precisely that if λ is an integer which is $\geq \max(0, n-m+1)$ then

$$(2) \qquad c_\lambda b_n + c_{\lambda+1} b_{n-1} + \ldots + c_{\lambda+n} \cdot b_0 = 0$$

Here $(b_n, \ldots, b_0)$ is a fixed non-zero $(n+1)$-vector and hence (2) implies that the vectors

$$(c_\lambda + k, \ldots, c_{\lambda+n+k}) \quad : \quad 0 \leq k \leq n$$

are linearly dependent which entails that the determinant of the following matrix must be zero:

$$\begin{pmatrix} c_\lambda & \cdots & c_{\lambda+n} \\ c_{\lambda+1} & \cdots & c_{\lambda+1+n} \\ & & \\ & & \\ c_{\lambda+n} & \cdots & c_{\lambda+2n} \end{pmatrix}$$

Kronecker proved that the vanishing of similar matrices also yields a sufficient condition in order that $\{c_\nu\}$ is of rational type. More precisely, for each pair of integers $\lambda \geq 0$ and $\mu \geq 1$ we set

$$C_\lambda(\mu) = \det \begin{pmatrix} c_\lambda & \dots & c_{\lambda+\mu} \\ c_{\lambda+1} & \dots & c_{\lambda+1+\mu} \\ & & \\ & & \\ c_{\lambda+\mu} & \dots & c_{\lambda+2\mu} \end{pmatrix}$$

1.1 Theorem. *The sequence $\{c_\nu\}$ is of rational type if if there exist a pair of integers $\lambda_* \geq 0$ and $\mu_* \geq 1$ such that*

$$(*) \qquad C_\lambda(\mu_*) = 0 \quad \text{for all} \quad \lambda \geq \lambda_*$$

Proof. For each $\lambda \geq \lambda_*$ we consider the vectors

$$\text{(i)} \qquad \xi_\lambda = (c_\lambda, c_{\lambda+1} \dots, c_{\lambda+\mu_*})$$

If the family $\{\xi_\lambda\}_{\lambda_*}^\infty$ span $\mathbf{C}^{\mu_*+1}$ we find the smallest integer w_* for which there exist

$$\lambda_* \leq w_0 < \dots < w_{\mu_*-1} < w_{\mu_*} \quad \text{and} \quad \xi_{w_0}, \dots, \xi_{w_{\mu_*-1}}, \xi_{w_*} \quad \text{are linearly independent}$$

But this gives a contradiction because the vectors $\{\xi_{w_*-\mu_*}, \dots, \xi_{w_*}\}$ appear as row vectors in the matrix $C_{w_*-\mu_*}(\mu_*)$ whose determinant by hypothesis is zero because $w_* - \mu_* \geq \lambda_*$.

Notice that $w_* \geq M_*$ must hold and (ii) applied with $\lambda = w_* - M_*$ implies that ξ_{w_*} belongs to the linear hull of the vectors $\xi_{w-1}, \dots, \xi_{w-M_*}$. But this contradicts the minimal choice of w_*. Hence the linear hull of the vectors $\{\xi_\lambda\}_0^\infty$ must be a proper subspace of $\neq \mathbf{C}^{M_*+1}$. This gives a non-zero vector $(b_0, \dots, b_{M_*})$ such that

$$\text{(iv)} \qquad c_\lambda \cdot b_0 + \dots + c_{\lambda+M_*} \cdot b_{M_*} = 0 \quad \text{for all } \lambda \geq 0\,.$$

But these relations obviously imply that the sequence $\{c_\nu\}$ is of rational type and Kronecker's theorem is proved.

Sublemma. For each $\mu \geq 2$ and every $\lambda \geq 0$ one has the equality

$$C_\lambda(\mu) \cdot C_{\lambda+2}(\mu) - C_\lambda(\mu+1) \cdot C_{\lambda+2}(\mu+1) = C_{\lambda+1}(\mu) \cdot C_{\lambda+1}(\mu)\,.$$

Proof continued. Notice that the Kronecker matrix $\mathcal{K}_M = \mathcal{C}_0(M)$. Assume that there exists M_* such that

$$\text{(i)} \qquad \det \mathcal{K}_M = 0 \quad \text{for all } M \geq M_*$$

With the notations above (i) means that $C_0(\nu) = 0$ when $\nu \geq M_*$. With $\lambda = 0$ in the Sublemma we conclude that $C_1(\nu) = 0$ for all $\nu \geq M_*$. We can proceed by an induction over λ which gives:

$$\text{(ii)} \qquad C_\lambda(M_*) = 0 \quad \text{for all } \lambda \geq 0\,.$$

Let us then consider the $M_* + 1$-vectors

$$\xi_\lambda = (c_\lambda, c_{\lambda+1} \dots, c_{\lambda+M_*}) \quad : \lambda = 0, 1, \dots$$

The vanishing of the determinants in (i) means that the $(M_* + 1)$-tuple of vectors

$$\text{(iii)} \qquad \xi_\lambda, \xi_{\lambda+1} \dots, \xi_{\lambda+M_*}$$

are linearly dependent for every $\lambda \geq 0$. Suppose now that the family $\{\xi_\lambda\}_0^\infty$ span $\mathbf{C}^{M_*+1}$. Choose the *smallest* integer w_* for which there exist

$$0 \leq w_0 < \dots < w_{M_*-1} < w_* \quad \text{and} \quad \xi_{w_0}, \dots, \xi_{w_{M_*-1}}, \xi_{w_*} \quad \text{are linearly independent}$$

Notice that $w_* \geq M_*$ must hold and (ii) applied with $\lambda = w_* - M_*$ implies that ξ_{w_*} belongs to the linear hull of the vectors $\xi_{w-1}, \dots, \xi_{w-M_*}$. But this contradicts the minimal choice of w_*.

Hence the linear hull of the vectors $\{\xi_\lambda\}_0^\infty$ must be a proper subspace of $\neq \mathbf{C}^{M_*+1}$. This gives a non-zero vector $(b_0, \ldots, b_{M_*})$ such that

$$\text{(iv)} \qquad c_\lambda \cdot b_0 + \ldots + c_{\lambda+M_*} \cdot b_{M_*} = 0 \quad \text{for all } \lambda \geq 0\,.$$

But these relations obviously imply that the sequence $\{c_\nu\}$ is of rational type and Kronecker's theorem is proved.

Remark. Kronecker's theorem can be used to establish conditions in order that a meromorphic function is rational. One has for example the following result which is due to Polya in [Pol]:

1.2 Theorem. *Let $\{c_n\}$ be a sequence of integers. Suppose that the power series*

$$f(z) = \sum c_n \cdot z^n$$

converges in some open disc centered at the origin and that $f(z)$ extends to an analytic function in a simply connected domain Ω. whose mapping radius with respect to $z = 0$ is strictly greater than one Then $f(z)$ is a rational function.

Remark. For the definition and various results about the *mapping radius* of simply connected domains the reader may consult Chapter X in [Po-Szegö] where other results based upon Kronecker's theorem appear.

II. Newton polynomials and the disc algebra $A(D)$

Introduction. Let $A(D)$ be the disc algebra. If $f(z) \in A(D)$ then its Taylor series a $z = 0$ give the partial sum polynomials $\{s_n^f(z)\}$. Denote by $A^*(D)$ the unit ball, i.e. funtions f with maximum norm $|f|_D \leq 1$ and set

(*) $$\mathcal{M}_n = \max_{f \in A*(D)} |s_n^f|_D$$

We are going to determine these $\mathcal{M}$-numbers. In his text-books from 1666, Isaac Newton studied the funcion $\sqrt{1-z}$ whose series expansion becomes:

(1) $$\sqrt{1-z} = q_0 + q_1 z \ldots \quad : \quad q_n = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \cdots 2n}$$

Notice that these positive coefficients decrease, i.e.

(2) $$1 = q_0 > q_1 > q_2 > \ldots$$

To each $n \geq 1$ we get the Newton polynomial

(3) $$Q_n(z) = q_0 + q_1 z + \ldots + q_n z^n$$

By (2) and Kakeya's result from XXX, $Q_n(z)$ has no zeros in the closed unit disc. Put

(4) $$\mathcal{G}_n = 1 + q_1^2 + \ldots + q_n^2$$

1. Theorem. *For each integer $n \geq 1$ one has the equality $\mathcal{M}_n = \mathcal{G}_n$ and the maximum in (1) is attained by the $A^*(D)$-function*

(2) $$f_n^*(z) = \frac{z^n \cdot Q_n(\frac{1}{z})}{Q_n(z)}$$

2. Remark. Using *Stirling's formula* one can easily show that

$$\lim_{n \to \infty} \frac{\mathcal{G}_n}{\log\, n} = \frac{1}{\pi}$$

Before Theorem 1 is proved we need some preliminary observations about partial sum functions. Let $f \in A_*(D)$. Cauchy's formula gives

(i) $$s_n^f(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f(z)}{z^{n+1}} \cdot (1 + z + \ldots + z^n) \cdot dz \quad : \; n = 0, 1, \ldots$$

Since $\int_{|z|=1} f(z) z^k dz = 0$ for every $k \geq 0$ we see that if $Q(z)$ is any polynomial of the form

(ii) $$Q(z) = 1 + z + \ldots + z^n + q_{n+1} z^{n+1} + \ldots \implies$$

(iii) $$s_n^f(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f(z)}{z^{n+1}} \cdot Q(z) \cdot dx$$

Proof of Theorem 1. For each $n \geq 1$ the squared Newton polynomial $Q_n^2(z)$ satisfies (ii) above. So if $f \in A*(D)$ we have

(iv) $$s_n^f(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f(z)}{z^{n+1}} \cdot Q_n^2(z) \cdot dx$$

Since the maximum norm of $|f|_D \leq 1$, the triangle inequality gives:

(v) $$|s_n^f(1)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \left|Q_n(e^{i\theta})\right|^2 \cdot d\theta$$

By *Parseval's formula* the last integral is equal to $\mathcal{G}_n$. Hence (v) gives the inequality

(vi) $$\mathcal{M}_n \leq \mathcal{G}_n$$

Next, with n kept fixed we consider the function

$$f^*(z) = \frac{z^n \cdot Q_n(\frac{1}{z})}{Q_n(z)} \implies \tag{vii}$$

$$s_n^{f^*}(1) = \frac{1}{2\pi i} \cdot \int_{|z|=1} \frac{f_n^*(z)}{z^{n+1}} \cdot Q_n^2(z) \cdot dx \tag{viii}$$

where $\implies$ follows from (iii) above. Notice that

$$\frac{f_n^*(z)}{z^{n+1}} \cdot Q^2(z) = \frac{1}{z} Q_n(z) \cdot Q_n(\frac{1}{z}) \implies \tag{ix}$$

$$s_n^{f^*}(1) = \frac{1}{2\pi} \cdot \int_0^{2\pi} Q_n(e^{i\theta}) \cdot Q(e^{-i\theta}) \cdot d\theta = \frac{1}{2\pi} \cdot \int_0^{2\pi} \left|Q_n(e^{i\theta})\right|^2 \cdot d\theta = \mathcal{G}_n \tag{5}$$

Since $f^* \in A^*(D)$ we conclude that (vi) above is an equality and Theorem 1 is proved.

3. Convergence of Fourier series

Let $f(z) = \sum c_n z^n$ be in the disc algebra $A(D)$. With $c_n = a_n + ib_n$ and $f = u + iv$ we get series for the real and the imaginary part respectively:

$$u(e^{i\theta}) = \sum_{n=0}^{n=N} a_n \cdot \cos n\theta - \sum_{n=0}^{N} b_n \cdot \sin n\theta$$

$$v(e^{i\theta}) = \sum_{n=-N}^{n=N} a_n \cdot \sin n\theta + \sum_{n=-N}^{N} b_n \cdot \cos n\theta$$

Continuous boundary values of f certainly exist if

$$\sum |a_n| + |b_n| < \infty \tag{*}$$

We shall give a sufficient condition for the validity of (*) expressed by the modulos of continuity of u:

$$\omega_u(\delta) = \max |u(e^{i\theta}) - u(e^{i\phi})| \quad : \text{ maximum over pairs } \ |\theta - \phi| \leq \delta$$

1. Theorem. *The series (*) is convergent if*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot \omega_u(\frac{1}{n}) < \infty$$

Theorem 1 is due to S. Bernstein in [1]. One may ask for other convergence criteria. Denote by $\mathcal{F}$ the class of continuous functions $F(s)$ defined for $s \geq 0$ which are increasing and concave and $F(0) = 0$ while $F(s) > 0$ for every $s > 0$. Then the following result is in [Salem: P. 39]:

2. Theorem. *Let $F \in \mathcal{F}$ be such that*

$$\sum_{n=1}^{\infty} F\big(\frac{1}{n} \cdot \omega_u^2(\frac{1}{n})\big) < \infty \tag{*}$$

Then it follows that

$$\sum_{n=1}^{\infty} F(a_n^2 + b_n^2) < \infty \tag{**}$$

Notice that Bernstein's theorem is the case $F(s) = \sqrt{s}$. The proof of Theorem 2 relies upon a result due to La Vallée Poussin who established a *lower bound* for the modulus of continuity. Recall first the general L^2-equality:

$$2a_0^2 + \sum_{n \geq 1} a_n^2 + b_n^2 = \frac{1}{\pi} \int_0^{2\pi} u(e^{i\theta})^2 \cdot d\theta \tag{*}$$

Now we consider the tail sums

$$V_N = \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2)$$

3. Theorem. *For every real valued and continuous function u on the unit circle one has*

$$\omega_u^2(\frac{1}{N}) > \frac{1}{72} \cdot V_N \quad : \quad N = 1, 2, \ldots$$

Remark. See Vallée Poussin's text-book *Lecons sur l'approximation des fonctions d'une variable réelle* for the proof.

Proof of Theorem 2. Put $\rho_n^2 = a_n^2 + b_n^2$. Since $F(s)$ is concave and increasing we have for every $N \geq 1$:

$$\text{(i)} \qquad \frac{1}{N} \sum_{n=N+1}^{n=2N} F(\rho_n^2) \leq F(\frac{1}{N} \sum_{n=N+1}^{n=2N} \rho_n^2) < 72 \cdot F\big(\frac{1}{N}\omega^2(\frac{1}{N})\big)$$

where the last inequality follows from Theorem 3 above. Apply (i) with $N = 2^k$ as $k = 0, 1, \ldots$. Then (i) obviously gives the implication:

$$\text{(ii)} \qquad \sum_{k=0}^{\infty} 2^k \cdot F\big(2^{-k} \cdot \omega^2(2^{-k})\big) < \infty \implies \sum_{n=1}^{\infty} F(\rho_n^2) < \infty$$

Now we are almost done. Namely, the sequence $2^{-k} \cdot \omega^2(2^{-k})$ decreases with k and since F increases we have

$$\sum_{n=2^{k-1}+1}^{2^k} F(\frac{1}{n}\omega^2(\frac{1}{n})) \geq 2^k \cdot F(2^{-k} \cdot \omega^2(2^{-k})) \quad : \quad k = 1, 2, \ldots$$

Hence Salem's convergence condition (*) in Theorem 2 gives (ii) above and the proof is finished.

4. Harald Bohr's inequality.

Let $A^*(D)$ be the unit ball in $A(D)$. When $f \in A^*(D)$ and $0 < r < 1$ we set

$$(*) \qquad \mathfrak{M}_f(r) = \sum |a_n| r^n$$

The question arises for which r it holds that

$$(**) \qquad \mathfrak{M}_f(r) \leq 1 \quad : \forall\, f \in A^*(D)$$

1. Theorem. *One has (**) if and only if* $r \leq \frac{1}{3}$.

Proof. Given f in $A_*(D)$ we set

$$\text{(i)} \qquad \phi(z) = \frac{f(z) - a_0}{1 - \bar{a}_0 \cdot f(z)}$$

Then the maximum norm $|\phi|_D \leq 1$ and its derivative at $z = 0$ becomes

$$\text{(ii)} \qquad \phi'(0) = \frac{f'(0)}{1 - |a_0|^2}$$

Since $g(0) = 0$ we have $|g'(0)| \leq 1$ by the inequality of Schwarz. It follows that

$$\text{(iii)} \qquad |a_1| \leq (1 - |a_0|^2) \leq 2 \cdot [1 - |a_0|)$$

where the last inequality holds since $1 + |a_0| \leq 2$. Next, put $\rho = e^{2\pi i/n}$ for every $n \geq 2$ and regard the function

$$(*) \qquad F_n(z) = \frac{f(z) + f(\rho z) + \ldots + f(\rho^{n-1} z)}{n}$$

Since $1+1+\rho^{\nu}+\ldots+\rho^{\nu(n-1)}=0$ whenever ν is not a multiple of n we conclude that

(**) $$F_n(z)=a_0+a_nz^n+a_{2n}z^{2n}+\ldots$$

Notice that the maximum norm $|F_n(z)|_D\leq 1$. Now we regard the analytic function

$$g(z)=a_0+a_nz+a_{2n}z^2+\ldots$$

Since $|F_n(z)|_D\leq 1$ it is clear that we also have $|g|_D\leq 1$. Then (*) applied to g gives

(***) $$|a_n|\leq 2(1-|a_0|) \quad : \; n\geq 1$$

Armed with this we can finish the proof of Theorem 1. First we show

2. The inequality $\mathfrak{M}_f(\frac{1}{3})\leq 1$. From (***) we obtain

$$\mathfrak{M}_f(\frac{1}{3})=|a_0|+\sum_{n=1}^{\infty}3^{-n}|\cdot|a_n|\leq|a_0|+\sum_{n=1}^{\infty}3^{-n}\cdot 2(1-|a_0|)=1$$

There remains to prove that the upper bound $\frac{1}{3}$ is sharp in Theorem 1. To see this we take a real number $0<a<1$ and consider the Möbius function

$$f(z)=\frac{z-a}{1-az}=-a+(1-a^2)z+(a-a^2)z^2+(a^2-a^3)z^3+\ldots \implies$$

$$\mathfrak{M}_f(r)=+(1-a^2)r+(a-a^2)r?2+\ldots=a+\frac{(1-a^2)r}{1-ar}$$

The last term is ≤ 1 if and only if

$$a(1-ar)+(1-a^2)r\leq 1-ar \implies (1+a-2a^2)r\leq 1-a$$

With $a=1-s$ and $s>0$ small this gives

$$s+2s-2s^2)r\leq s \implies r\leq\frac{1}{3}+2s$$

Since s can be arbitrary small the upper bound $\frac{1}{3}$ in Theorem 1 is best possible.

5. A theorem by Fatou and M. Riesz.

Introduction. We prove a result due to Fatou and M. Riesz. See the article [M. Rie] from 1911. Let

(1) $$f(z)=\sum c_nz^n$$

be an analytic function in the open unit disc. We shall consider the situation when f extends analytically along some arc of the unit circle. For example, the analytic function $\frac{1}{1-z}$ extends analytically outside the boundary point $z=1$ and the series

$$\sum e^{in\theta}$$

converges for all $0<\theta<2\pi$. Let us now consider some $f\in\mathcal{O}(D)$ for which there exists some $0<\theta^*<\pi/2$ such that f extends to an analytic function in the union of D and the sector

(ii) $$S=\{z=re^{i\theta} \quad : \; 1\leq r<R \quad : \; -\theta^*<\theta<\theta^*\}$$

Moreover, we suppose that f extends to a continuous function on the closed union of $D\cup S$. See figure XXX. With these notations one has

1. Theorem. *Assume that $c_n\to 0$. Then the partial sums $\{s_n(e^{i\theta})\}$ converge uniformly to $f(e^{i\theta})$ one every compact interval of $(-\theta^*,\theta^*)$.*

Proof. To each $n \geq 1$ we consider the function

(i) $$g_n(z) = \frac{f(z) - (c_0 + c_1 z + \ldots + c_n z^n)}{z^{n+1}} \cdot (z + e^{i\theta^*})((z - e^{i\theta^*})$$

This is an analytic function in the domain $D \cup S$. Consider a closed circular interval $\ell = [-\theta_* \leq \theta \leq \theta_*]$ for some $0 < \theta_* < \theta^*$. It appears as a compact subset of $S \cup D$ and it is clear that the required uniform convergence of $\{s_n\}$ holds on ℓ if the g-functions converge uniformly to zero on ℓ,. In fact, this follows since the absolute values

$$|(z + e^{i\theta^*}| \cdot |z - e^{i\theta^*}| \geq (\theta^* - \theta_*)^2 \quad : \quad z \in \ell$$

To prove that the maximum norms $|g_n|_\ell \to 0$ it suffices by the maximum principle for analytic functions to show that g_n converges uniformly to zero on the boundary of the sector S which by the construction contains ℓ. Here ∂S contains the outer circular arc

(i) $$\Gamma = \{Re^{\imath\theta} \quad : \theta_* \leq \theta \leq \theta_*$$

In addition ∂S contains two rays. Let us regard the two pieces of ∂S given by

(ii) $$\Gamma_* = \{z = re^{i\theta^*} : 0 \leq r \leq 1\} \quad : \quad \Gamma^* = \{z = re^{i\theta^*} : 1 \leq r \leq R\}$$

There remains to estimate the maximum norms of g_n over each of these pieces of ∂S. Of course, in addition to (ii) we have the contribution when $z = re^{-i\theta^*}$ but by symmetry the subsequent estimates are valid here too. Before we establish the required estimates we introduce a notation. To each integer $m \geq 1$ we set:

(1) $$A_m = M + |c_0| + |c_1| \cdot R + \ldots + |c_m| \cdot R^m \quad : \epsilon_m = \max_{\nu > m} |c_\nu|$$

By hypothesis we have

(2) $$\lim_{m \to \infty} \epsilon_m = 0$$

2. The estimate of $|g_n|_\Gamma$. By assumption f extends continuously to the closure of S so the maximum norm $|f|_S = M$ is finite. If $z \in \Gamma$ we have $|z| = R$ and the triangle inequality gives for each pair $1 \leq M < N$:

(i) $$\big|f(z) - (c_0 + c_1 z + \ldots + c_n z^n)\big| \leq A_m + \epsilon_m (R^{m+1} + \ldots + R^n) \leq A_m + \epsilon_m \cdot \frac{R^{n+1}}{R-1}$$

With the constant

$$K = \max_{z \in \Gamma} |z - e^{i\theta^*}| \cdot |z + e^{i\theta^*}|$$

we therefore obtain

(ii) $$|g_n|_\Gamma \leq \frac{K}{R^{n+1}} \cdot \big(A_m + \epsilon_m \cdot \frac{R^{n+1}}{R-1}\big) = \frac{K \cdot A_m}{R^{n+1}} + \frac{K \cdot \epsilon_m}{R_1}$$

If $\delta > 0$ we use (2) above and find m so that $\epsilon_m < \delta$. Once m is fixed we use that $R > 1$ and hence $\frac{K \cdot A_m}{R^{n+1}} < \delta$ if n is large. Since $\delta > 0$ is arbitrary we conlcude that $|g_n|_\Gamma \to 0$ as required.

3. Estimate of $|g_n|_{\Gamma^*}$. With the same notations as above we consider some $z = re^{i\theta^*}$ with $1 < r < R$ and obtain:

$$\big|f(z) - (c_0 + c_1 z + \ldots + c_n z^n)\big| \leq$$

$$A_m + \epsilon_m (r^{m+1} + \ldots + r^n) \leq A_m + \epsilon_m \cdot \frac{r^{n+1}}{r-1} \implies$$

(i) $$|g_n(re^{i\theta}| \leq \big(A_m + \epsilon_m \cdot \frac{r^{n+1}}{r-1}\big) \cdot \frac{1}{r^{n+1}} \cdot |re^{i\theta^*} - e^{i\theta^*}| \cdot |re^{i\theta^*} - e^{-i\theta^*}|$$

Here $|re^{i\theta^*} - e^{i\theta^*}| = r - 1$ and $|re^{i\theta^*} - e^{-i\theta^*}| \leq 2R$ for all $1 \leq r \leq R$. So with $K = (R^2 - 1)$ we see that (i) gives

(ii) $$|g_n(re^{i\theta}| \leq \frac{K \cdot A_m}{r^{n+1}} \cdot (r-1) + R \cdot \epsilon_m$$

At this stage we use the obvious inequality for $r > 1$:

$$\frac{r-1}{r^{n+1}} < \frac{r-1}{r^{n+1}-1} = \frac{1}{1+r+\ldots+r^n} < \frac{1}{n} \implies$$

$$|g_n|_{\Gamma_*} \leq \frac{KA_m}{n} + 2R \cdot \epsilon_m$$

Since $\epsilon_m \to 0$ the reader concludes that $|g_n|_{\Gamma_*} \to 0$ as $n \to \infty$.

Estimate of $|g_n|_{\Gamma_*}$. With $x = re^{i\theta^*}$ and $0 < r < 1$ the triangle inequality gives

(i) $$\big|f(z) - (c_0 + c_1 z + \ldots + c_n z^n)\big| \leq |c_{n+1}| \cdot |z|^{n+1} + |c_{n+2}| \cdot |z|^{n+2} + \ldots$$

Recall that $\epsilon_n = \max_{\nu > n} |c_\nu|$. Hence (i) is majorized by

$$\epsilon_n \cdot \big(|z|^{n+1} + |z|^{n+2} + \ldots\big) = \epsilon_n \cdot \frac{|z|^{n+1}}{1-|z|}$$

Now $z = re^{i\theta^*}$ and we get as before with $K = \max|re^{i\theta^*} - e^{-i\theta^*}|$:

$$|g_n(z)| \leq \frac{\epsilon_n \cdot \frac{|z|^{n+1}}{1-|z|}}{|z|^{n-1}} \cdot (1-|z|) \cdot K = K \cdot \epsilon_n$$

Again, since ϵ_m can be chosen arbitrary small we conclude that $|g_n|_{\Gamma_*} \to 0$ as $n \to \infty$. This finishes the proof of the Theorem 1.

6. On Laplace transforms.

Let $f(t)$ be a bounded function defined on the real t-line. Consider its Laplace transform

$$L(z) = \int_0^\infty f(t)e^{-zt} \cdot dt$$

which is analytic in the open half-plane $\mathfrak{Re}\, z > 0$. *Assume* that there exists some open subset Ω of $\mathbf{C}$ which contains the closed half-plane $\mathfrak{Re}\, z \geq 0$ such that $L(z)$ extends to an analytic function in Ω. Under this assumption one has

1. Theorem. *There exists the limit*

$$\lim_{T \to \infty} \int_0^T f(t) \cdot dt$$

Moreover, the limit value is equal to $L(0)$ where L under its analytic extension to Ω has been evaluated at $z = 0$.

Proof. To each $T > 0$ we have the entire function

(i) $$L_T(z) = \int_0^T f(t)e^{-zt} \cdot dt$$

Theorem 1 amounts to prove that

(ii) $$\lim_{T \to \infty} L_T(0) = L(0)$$

To prove (ii) we consider certain complex line integrals. If $R > 0$ the assumption on L gives some $\delta > 0$ such that Ω contains the closed set given by the union of the half disc $\bar{D}_R^+ = \bar{D}_R \cap \mathfrak{Re}\, z \geq 0$ and the rectangle

$$\square = \{x + iy \quad : \quad -\delta \leq x \leq 0 \quad : \quad -R \leq y \leq R\}$$

Let Γ be the boundary of $\bar{D}_R^+ \cup \square$ and introduce the function

$$\text{(iii)} \qquad g(z) = \big[L(z) - L_T(z)\big] \cdot (1 + \frac{z^2}{R^2}) \cdot \frac{e^{zT}}{z}$$

Here $g(z)$ is a meromorphic function in Ω with a simple pole at $z = 0$ whose residue is $L(0) - L_T(0)$. Hence residue calculus applied to g and $\square$ gives:

$$\text{(iv)} \qquad L(0) - L_T(0) = \frac{1}{2\pi i} \cdot \int_\Gamma g(z) \cdot dz$$

Put $B = \max_t |f(t)|$ which for every $T > 0$ gives the inequality:

$$|L(z) - L_T(z)| = \Big|\int_T^\infty f(t) e^{-zt} dt\Big| \leq$$

$$\text{(v)} \qquad B \cdot \int_T^\infty \left|e^{-zT}\right| \cdot dt = B \cdot \int_T^\infty e^{-\mathfrak{Re}\, z \cdot T} \cdot dt = B \cdot \frac{e^{-\mathfrak{Re}(z)\dot{T}}}{\mathfrak{Re}(z)} \quad : \quad \mathfrak{Re}(z) > 0$$

Now we begin to estimate the line integral in (iv). Consider first the part of Γ given by the half circle ∂D_R^+. Here we notice that

$$\text{(vi)} \qquad |1 + \frac{R^2 e^{2i\theta}}{R^2}| = |1 + e^{2i\theta}| = 2 \cdot \cos\theta$$

Next, $\frac{dz}{z} = R \cdot d\theta$ holds during the integration on ∂D_R^+ and we also have

$$\frac{1}{\mathfrak{Re}(R \cdot e^{i\theta})} = \frac{1}{R \cdot \cos\theta}$$

Hence (v) and (vi) give

$$(*) \qquad \Big|\int_{\partial D^+ R} g(z) \cdot \frac{dz}{z}\Big| \leq 2 \cdot B \cdot \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{R \cdot \cos\theta} \cdot d\theta \leq 2 \cdot B \cdot \frac{\pi}{R}$$

There remains to estimate the integral over the part of Γ which belongs to $\partial\square$. Here we simply perform estimates for the two functions $L(z)$ and $L_T(z)$ separately. First, since $L_T(z)$ is entire we can just as well integrate over the half-circle D_R^- where $\mathfrak{Re}(z) < 0$. We notice that

$$|L_T(z)| \leq B \int_0^T e^{-\mathfrak{Re}\, z \cdot t} \cdot dt \leq B \cdot \frac{e^{-\mathfrak{Re}\, z \cdot T}}{|\mathfrak{Re}\, z|} \quad : \quad \mathfrak{Re}\, z < 0$$

Here $e^{-\mathfrak{Re}\, z \cdot T}$ is large when $z \in D_R^-$ but this factor is cancelled by the absolute value of e^{zT} which appears in the g-function. Hence we obtain

$$(**) \qquad \Big|\int_{D_R^-} L_T(z) \cdot (1 + \frac{z^2}{R^2}) \cdot e^{zT} \cdot \frac{dz}{z}\Big| \leq B \cdot \int_{\pi/2}^{3\pi/2} \frac{|1 + e^{2i\theta}|}{R \cdot |\cos\theta|} \cdot d\theta \leq \frac{2\pi \cdot B}{R}$$

Finally, consider the line integral along $\Gamma \cap \partial\square$ where the analytic function $L(z)$ appears. First we regard the line integral along the vertical line where $\mathfrak{Re}(z) = -\delta$ whose absolute value becomes:

$$\text{(vii)} \qquad \Big|\int_R^R L(-\delta + iy) \cdot (1 + \frac{(-\delta + iy)^2}{R^2}) \cdot e^{-\delta T} \cdot e^{iyT} \cdot \frac{i \cdot dy}{(-\delta + iy}\Big|$$

Notice that we have not imposed any growth condition Here $e^{-\delta T}$ appears and at the same time e^{iyT} has absolute value one. Hence (vii) is estimated by

$$(***) \qquad \max_{-R \leq y \leq R} \Big|\frac{L(-\delta + iy) \cdot (1 + \frac{(-\delta + iy)^2}{R^2})}{(-\delta + iy)}\Big| \cdot 2R \cdot e^{-\delta T} = M^*(R) \cdot e^{-\delta T}$$

where $M^*(R)$ depends on R only.

For the integrals on the two intervals where $z = -s + iR$ and $z = -s - iR$ with $0 \leq s \leq \delta$ we also get a constant $M^{**}(R)$ which is independent of T while the sum of absolute values of the line integrals over these two lines is estimated by

$$(****) \qquad M^{**}(R) \cdot \int_0^\delta e^{-sT} \cdot ds = M^{**}(R) \cdot \frac{1 - e^{-\delta T}}{T}$$

Now the requested limit formula (ii) follows from the (*)-inequalities above. Namely, for a given $\epsilon > 0$ we first choose R so large that the sum of (*) and (**) is $\leq \epsilon/2$. With R kept fixed we can then choose T so large that (***) and (****) both are $\leq \epsilon/2$ which finishes the proof of Theorem 1.

7. The Lagrange series and the Kepler equation

Let $f(w)$ be an analytic function of the complex variable w defined in some disc of radius R centered at $w = 0$. We assume that $f(0) = 0$ and with another complex variable z we seek an analytic function $w = w(z)$ such that

$$(*) \qquad w(z) = z \cdot f(w(z))$$

We will use residue calculus and Rouche's theorem to find $w(z)$. Let z be fixed for a while and consider some $0 < r < R$ such that

$$\max_{|w|=r} |z \cdot f(w)| < r$$

This means that the analytic function $g(w) = z \cdot f(w)$ has absolute value $< |w|$ on the circle $|w| = r$. Rouche's theorem implies that the analytic function $w - zf(w)$ has a unique simple zero in the disc $|w| < r$. Moreover, by the formula in XX this zero is given by

$$w(z) = \frac{1}{2\pi i} \cdot \int_{|w|=r} \frac{1 - zf'(w)}{w - zf(w)} \cdot w \cdot dw$$

We can evaluate the integral using the series expansion

$$\frac{1}{1 - \frac{z^k \cdot f(w)^k}{w}} = 1 + \sum_{k=1}^\infty \frac{(zf(w))^k}{w^k}$$

More precisely, we see that $w(z)$ becomes

$$\frac{1}{2\pi i} \cdot \int_{|w|=r} \sum_{k=1}^\infty \frac{(z(f(w))^k}{w^k} - \frac{1}{2\pi i} \cdot \int_{|w|=r} \sum_{k=2}^\infty \frac{z^k \cdot f'(w) \cdot f(w))^{k-1}}{w^{k-1}}$$

If $k \geq 1$ residue calculus gives

$$\frac{1}{2\pi i} \cdot \int_{|w|=r} \frac{(z(f(w))^k}{w^k} - \cdot dw = z^k \cdot \frac{f^k)^{(k-1)}(0)}{(k-1)!}$$

Similarly we find

$$\frac{1}{2\pi i} \cdot \int_{|w|=r} \sum_{k=2}^\infty \frac{z^k \cdot f'(w) \cdot f(w))^{k-1}}{w^{k-1}} \cdot dw = z^k \cdot \frac{f' \cdot f)^{k-1})^{(k-2)}(0)}{(k-2)!}$$

Next, notice the equality

$$(f^k)^{(k-1)}(0) = k \cdot (f' \cdot f^{k-2})^{(k-2)}(0)$$

Since $\frac{1}{(k-1)!} - \frac{1}{k \cdot (k-2)!} = \frac{1}{k!}$ we conclude that one has the series formula:

$$(*) \qquad w(z) = \sum_{k=1}^\infty \frac{(f^k)^{(k-1)}(0)}{k!} \cdot z^k$$

Radius of convergence. The analytic function $w(z)$ has the expansion by the Lagrange series above. The determination of the radius of convergence depends on the given function $f(w)$. A *lower bound* for the radius of convergence is found by the use of Rouche's theorem above. Assume for simplicity that $f(w)$ is an entire function. If $r > 0$ is given we find the positive number $\rho(r)$ for which

$$\rho(r) \cdot \max_{|w|=r} |f(w)| = r$$

By (*) above and Rouche's theorem we have seen that the Lagrange series converges in the disc $|z| < r$. Here we have a *free choice* of r. But each time r is chosen we must take into the account the maximum of $f(w)$ on $|w| = r$. More precisely, put

$$M_f(r) = \max_{|w|=r} |f(w)|$$

Then the discussion above gives

Theorem. *The Lagrange series converges in the disc of the complex z-plane whose radius is*

$$\rho^* = \max_r \frac{r}{M_f(r)}$$

Example. In his far reaching studies of the motion of orbits of those planets which astronomers were able to watch before 1600, Kepler's work contains a study of the equation

$$(*) \qquad \zeta = a + z \cdot \sin \zeta$$

where $a > 0$ is a real constant. We shall determine the series expansion of $\zeta(z)$. Notice that if $w = \zeta - a$ then (*) becomes

$$w = z \cdot \sin(w + a)$$

So with the entire function $f(w) = \sin(w + a)$ we encounter the general case above and conclude that the series becomes

$$\zeta(z) = a + z \cdot \sin a + \sum_{k=2}^{\infty} \frac{z^k}{k!} \cdot \frac{d^{k-1}(\sin^k a)}{da^{k-1}}$$

Exercise. Let $r > 0$ and show that the series $\zeta(z)$ converges when

$$|z| \cdot \frac{e^r + e^{-r}}{2r} < 1 \implies |z| < \frac{2r}{e^r + e^{-r}}$$

and for z in this disc we get $|\zeta(z)| < r$. To obtain a largest possible disc we seek

$$\max_r \frac{2}{e^r + e^{-r}}$$

The reader is invited to calculate the maximum numerically and in this way find a *lower bound* for the radius of convergence of the Kepler series. In contrast to all "heroic computations" by Kepler carried out in the years 1600-1620 and the subsequent refined studies of series expansions by Lagrange around 1760 described above, today's student can use a computer to determine the radius of convergence numerically. This, it is an instructive exercise to determine numerically the radius of convergence of the Lagrange series for each real a. Here it is of course interesting to analyze hos the radius of convergence depends on a.

8. An example by Bernstein.

Let $n \geq 1$ and consider a polynomial $P(z) = a_0 + \ldots + a_n z^n$ of some degree n. We have the equality

$$\sum |a_n|^2 = \frac{1}{2\pi} \cdot \int_0^{2\pi} \left|P(e^{i\theta})\right|^2 \cdot d\theta$$

So if we consider the maximum norm over the unit disc D:

$$||P||_D = \max_\theta |P(e^{i\theta})|$$

then the Cauchy-Schwarz inequality gives

(*) $$\sum |a_k| \leq \sqrt{n+1} \cdot \sqrt{||P||_D}$$

It turns out that (*) is sharp, i.e for arbitrary large n we can find a polynomial $P_n(z)$ such that

(**) $$\frac{\left(\sum |a_k|\right)^2}{n+1} \simeq ||P_n||_D$$

The first example of this kind comes from a construction by S. Bernstein from 1914. He considered a prime number p of the form $4\mu+1$. For each integer $1 \leq k \leq p-1$ there exists the Legendre symbol $\binom{k}{p}$which is $+1$ if k is a quadratic remained to p and otherwise -1. Now we get the trigonometric cosine-polynomial

$$B_p(\theta) = \frac{2}{p^{\frac{3}{2}}} \cdot \sum_{k=1}^{p-1} (p-k) \binom{k}{p} \cdot \cos(k\theta) = \sum_{k=1}^{p-1} a_k^{[p)} \cdot \cos(k\theta)$$

Bernstein proved that

$$\max_\theta |B_p(\theta) \leq 1 \quad \text{and} \quad \sum_{k=1}^{p-1} a_k^{[p)} = \frac{p-1}{\sqrt{p}}$$

With $n = 4\mu$ we get the polynomial $Q_n(z)$ where

$$\mathfrak{Re}(Q_n(e^{i\theta}) = B_p(\theta) \quad \text{and} \quad \mathfrak{Im}(Q_n(0)) = 0$$

The maximum norm for $\mathfrak{Im}(P(e^{i\theta})$ is estimated above by the Exercise below. It follows that

$$Q_n(z) = \sum_{k=1}^{p-1} a_k^{[p)} \cdot z^k \quad \text{and} \quad |Q_n|_D \leq C \cdot \log p$$

where C is the absolute constant from Exercise XX below. So for this polynomial the left hand side in (**) becomes $\frac{(p-1)^2}{p^2}$ which is close to 1 when p is large. At the same time $\log p$ is considerably smaller than the degree $n = p-1$. So with

$$P_n(z) = \frac{1}{|Q_n|_D} \cdot Q_n(z)$$

we get a polynomial whose maximum norm is one while the left hand side gets close to one as p increases.

Exercise. Let $u(\theta) = \sum_{k=0}^n a_k \cdot \cos\theta + \sum_{k=1}^n b_k \cdot \sin\theta$ be a trigonometric polynomial of degree n where $\{a_k\}$ and $\{b_k\}$ are real. The conjugate trigonometric polynomial is defined by

$$v(\theta) = \sum_{k=1}^n \left[-b_k \cdot \cos\theta + a_k \cdot \sin\theta\right]$$

Show the integral formula

(1) $$v(\phi) = \frac{1}{\pi} \cdot \int_0^{2\pi} \frac{\sin\frac{n(\phi-\theta)}{2} \cdot \sin\frac{(n+1)(\phi-\theta)}{2}}{\sin\frac{(\phi-\theta)}{2}} \cdot u(\theta) \cdot d\theta$$

From (1) the reader should verify that if $M = \max_\theta |u(\theta)|$ then

$$|v(\phi)| \leq \frac{M}{\pi} \int_0^{2\pi} \left|\frac{\sin\frac{n(\phi-\theta)}{2}}{\sin\frac{(\phi-\theta)}{2}}\right| \cdot d\theta$$

Finally, show that there is an absolute constant C such that

$$\frac{1}{\pi}\cdot\int_0^{2\pi}\Big|\frac{\sin\frac{n(\phi-\theta)}{2}}{\sin\frac{(\phi-\theta)}{2}}\Big|\cdot d\theta\leq C\cdot\operatorname{Log} n\quad:\quad n\geq 2$$

Hence the maximum norm for the conjugate v-function satisfies

(*) $$\max_\theta\,|v(\theta)|\leq C\cdot\operatorname{Log} n\cdot\max_\theta\,|u(\theta)|$$

Remark. The inequality above was first demonstrated by Fekete in his article (Journal für mathematik 146).

9. Almost periodic functions and additive number theory.

Introduction. We expose a result presented by Beurling at a seminar at Uppsala University in April 1948. Let $2 \leq m_1 < m_2 < \ldots$ be a strictly increasing sequence of integers. Denote by S the even set given by the union of $\{m_\nu\}$ and $\{-m_\nu\}$. Assume that the additive group generated by the integers in S is equal to $\mathbf{Z}$ which means that the sequence $\{m_\nu\}$ has no common prime number ≥ 2 as factor. Next, consider some non-negative and even function ϕ defined on S. By hypothesis every integer n can be represented by a finite sum of integers from S where repetitions are allowed. Hence we can define the function $\mathfrak{p}_\phi$ on $\mathbf{Z}$ by

$$\mathfrak{p}_\phi(n) = \min \sum \phi(m_\nu) \quad \text{such that} \quad n = \sum m_\nu \tag{1}$$

where the minimum is taken over finite subsets of S. It is obvious that this function is even and subadditive:

$$\mathfrak{p}_\phi(n_1 + n_2) \leq \mathfrak{p}_\phi(n_1) + \mathfrak{p}(n_2)$$

In particular $\mathfrak{p}_\phi(n) = 0$ for all $n \neq 0$ if and only if $\mathfrak{p}_\phi(1) = 0$ and this vanishing holds if and only if for every $\delta > 0$ there exists a finite set $\{m_\nu\}$ in S such that

$$\sum m_\nu = 1 \quad \text{and} \quad \sum \phi(m_\nu) < \delta \tag{*}$$

We seek conditions on ϕ in order that (*) holds, or equivalently that $\mathfrak{p}_\phi(1) = 0$. To get such a criterion Beurling restricted the attention to a class of ϕ-functions satisfying the following extra condition. An even subset W of $\mathbf{Z}$ is called relatively dense if the additive group generated by W is equal to $\mathbf{Z}$.

9.1 Definition. *Given the even set S above we denote by $AP(S)$ the set of even functions ϕ defined on S such that for every $\epsilon > 0$ the set*

$$S_\epsilon(\phi) = \{m \in S : \ \phi(m) < \epsilon\}$$

is relatively dense.

The zig-zag function $\rho(x)$. Before Theorem 9.2 is announced we introduce the periodic ρ-function on the real x-line where

$$\rho(x) = |x| \quad : -1/2 \leq x \leq 1/2$$

and extended so that $\rho(x) = \rho(x+1)$ hold for every x.

9.2 Theorem. *For each $\phi \in AP(S)$ the necessary and sufficient condition in order that (*) holds is that*

$$\max_{m \in S} \ \rho(\alpha m) - \eta \cdot \phi(m) \geq 0 \tag{**}$$

hold for all pairs $0 < \alpha < 1$ and $\eta > 0$.

Before we enter the proof we recall some facts about almost periodic functions. A bounded complex-valued function f on $\mathbf{Z}$ is almost periodic if there to every $\epsilon > 0$ exists a relatively dense set W such that

$$\max_{n \in \mathbf{Z}} |f(n+w) - f(n)| < \epsilon \quad \text{for all} \quad w \in W$$

From this it follows easily that there exists the mean-value defined by

$$\mathcal{M}(f) = \lim_{b-a \to +\infty} \frac{f(a) + f(a+1) + \ldots f(b)}{b - a + 1}$$

Next, for each real number α the exponential function E_α defined by $E_\alpha(n) = e^{2\pi i \alpha n}$ is almost periodic on $\mathbf{Z}$. It follows that when f is almost periodic then there exists the function

$$\mathcal{C}_f(\alpha) = \mathcal{M}(E_\alpha \cdot f)$$

A result due to Harald Bohr asserts that if f is almost periodic and $f(1) \neq 0$ then the $\mathcal{C}_f$-function is not identically zero on $(0,1)$, i.e. there exists some $0 < \alpha < 1$ such that $\mathcal{C}_f(\alpha)$. For the proof of Theorem 9.2 we shall alos need the following:

9.3 Proposition *If ϕ belongs to $AP(S)$ it follows that $\mathfrak{p}_\phi$ is an almost periodic function on* $\mathbf{Z}$*.*

Exercise. Prove this assertion.

Proof of Theorem 9.2 Suppose first that $\mathfrak{p}_\phi(1) \neq 0$ which means that (*) has no solution for small δ. To show that the inequalitites (**) in Theorem 9.2 cannot hold for all pairs $\alpha.\eta$ we proceed as follows: Since $\mathfrak{p}_\phi$ by defintion is periodic it is in particular almost periodic and by a general formula for $\mathcal{M}$-functions attached to almost periodic functions we get for each integer $m \in S$:

$$|e^{2\pi i\alpha m} - 1| \cdot \mathcal{C}_{\mathfrak{p}_\phi}(\alpha)| \leq \max_n |\mathfrak{p}_\phi(n+m) - p(n)\big| \leq \mathfrak{p}_\phi(m)$$

where the last inequality follows since that $\mathfrak{p}_\phi$ is subadditive. Introducing the sine-function we get

$$2 \cdot |\sin(\pi\alpha m)| \cdot \mathcal{C}_{\mathfrak{p}_\phi}(\alpha)| \leq \mathfrak{p}_\phi(m) \leq \phi(m) \quad : \quad m \in S$$

Since $\mathfrak{p}_\phi(1) \neq 0$ is assumed we know from Bohr's theory that there exists some $0 < \alpha < 1$ such that $\mathcal{C}_{\mathfrak{p}_\phi}(\alpha) \neq 0$. At the same time the zig-zag function satisfies:

$$\rho(x) \leq \frac{\pi}{2} \cdot |\sin \pi \cdot x\,|$$

for every real x. Hence we get

$$\rho(\alpha \cdot m) \cdot \mathcal{C}(\alpha) \leq \frac{4}{\pi}\phi(m)$$

FINISH ....

10. Approximation theorems in complex domains

0. Introduction.

This chapter is devoted to results concerned with approximation by analytic functions due to Carleman, Lindelöf and Müntz.

A. Weierstrass approximation theorem.

A wellknown result due to Weierstrass asserts that if $f(x)$ is a complex valued continuous function on a bounded interval $[a, b]$ then it can be uniformly approximated by polynomials. It turns out that uniform approximations exist on the whole real line where the approximating functions are entire.

A.1 Theorem. *Let $f(x)$ be a continuous function on the real x-line. To every $\epsilon > 0$ there exists an entire function $G(z)$ such that*

$$\max_{x \in \mathbf{R}} \big|G(x) - f(x)\big| \leq \epsilon$$

An elementary proof using Cauchys integral formula only was given in [Carleman]. Here we extend Theorem A.1 to a more general situation. Let K be a closed null-set in $\mathbf{C}$ which in general is unbounded. If $0 < R < R^*$ we put

$$K[R, R^*] = K \cap \{R \leq |z| \leq R^*\}$$

If $R > 0$ we denote by $\bar{D}_R$ the disc of radius R and $K_R = K \cap \bar{D}_R$. With these notations one has:

A.2 Theorem. *Assume there exists a strictly increasing sequence $\{R_\nu\}$ where $R_\nu \to +\infty$ such that the sets*

$$\Omega_\nu = \mathbf{C} \setminus \bar{D}_{R_\nu} \cup K[R_\nu, R_{\nu+1}]$$

are connected for each $\nu \geq 1$ together with the set $\mathbf{C} \setminus K_{R_1}$. Then every continuous function on K can be uniformly approximated by entire functions.

To prove this result we first establish the following.

A. 3 Lemma. *Let $\nu \geq 1$ and ψ is a continuous function on $S = \bar{D}_{R_\nu} \cup K[R_\nu, R_{\nu+1}]$ where ψ is analytic in the open disc D_{R_ν}. Then ψ can be uniformly approximated on S by polynomials in z.*

Proof. If we have found a sequence of polynomials $\{p_k\}$ which approximate ψ uniformly on $S_* = \{|z| = R_\nu\} \cup K[R_\nu, R_{\nu+1}]$ then this sequence approximates ψ on S. In fact, this follows since ψ is analytic in the disc D_{R_ν} so by the maximum principle for analytic functions in a disc we have

$$||p_k - \psi||_S = ||p_k - \psi||_{S_*}$$

for each k. Next, if uniform approximation on S_* fails there exists a Riesz-measure μ supported by S_* which is $\perp$ to all analytic polynomials while

$$\int \psi \cdot d\mu \neq 0 \tag{1}$$

To see that this cannot occur we consider the Cauchy transform

$$\mathcal{C}(z) = \int \frac{d\mu(\zeta)}{z-\zeta}$$

Since $\int \zeta^n \cdot d\mu(\zeta) = 0$ for every $n \geq 0$ we see that $\mathcal{C}(z) = 0$ in the exterior disc $|z| > R_{\nu+1}$. The connectivity hypothesis implies that $\mathcal{C}(z) = 0$ in the whole open complement of S. Now K was a null set which means that the L^1_{loc}-function $\mathcal{C}(z)$ is zero in the exterior disc $|z| > R_\nu$ and hence its distribution derivative $\bar{\partial}(\mathcal{C}_\nu)$ also vanishes in this exterior disc. At the same time we have the equality

$$\bar{\partial}(C_\nu) = \mu$$

We conclude that the support of μ is confined to the circle $\{|z| = R_\nu\}$. But then (1) cannot hold since the restriction of ψ to this circle by assumption extends to be analytic in the disc D_{R_ν} and therefore can be uniformly approximated by polynomials on the circle.

Proof of Theorem A.2. Let $\epsilon > 0$ and $\{\alpha_\nu\}$ is a sequence of positive numbers such that $\sum \alpha_\nu < \epsilon$. Consider some $f \in C^0(K)$. Starting with the set K_{R_1} we use the assumption that its complement is connected and using Cauchy transforms as in Lemma A.3 one shows that the restriction of f to this compact set can be uniformly approximated by polynomials. So we find $P_1(z)$ such that

(i) $$||P_1 - f||_{K_{R_1}} < \alpha_1$$

From (i) one easily construct a continuous function ψ on $\bar{D}_{R_1} \cup K[R_1, R_2]$ such that $\psi = P_1$ holds in the disc $\bar{D}_{R_1}$ and the maximum norm

$$||\psi - f||_{K[R_1,R_2]} \leq \alpha_1$$

Lemma A.3 gives a polynomial P_2 such that

$$||P_2 - P_1||_{D_{R_1}} < \alpha_2 \quad \text{and} \quad ||P_2 - f||_{K[R_1,R_2]} \leq \alpha_1 + \alpha_2$$

Repeat the construction where Lemma A.3 is used as ν increases. This gives a sequence of polynomials $\{P_\nu\}$ such that

$$||P_\nu - P_{\nu-1}||_{D_{R_\nu}} < \alpha_\nu \quad \text{and} \quad ||P_\nu - f||_{K[R_{\nu-1},R_\nu]} < \alpha_1 + \ldots + \alpha_\nu$$

hold for all ν. From this it is easily seen that we obtain an entire function

$$P^*(z) = P_1(z) + \sum_{\nu=1}^{\infty} P_{\nu+1}(z) - P_\nu(z)$$

Finally the reader can check that the inequalities above imply that the maximum norm

$$||P^* - f||_K \leq \alpha_1 + \sum_{\nu=1}^{\infty} \alpha_\nu$$

Since the last sum is $\leq 2\epsilon$ and $\epsilon > 0$ was arbitrary we have proved Theorem A.3.

Exercise. Use similar methods as above to show that if $f(z)$ is analytic in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$ and has continuous boundary values on the real line, then f can be uniformly approximated by an entire function, i.e. to every $\epsilon > 0$ there exists an entire function $F(z)$ such that

$$\max_{z \in U^+} |F(z) - f(z)| \leq \epsilon$$

B. Polynomial approximation with bounds

Introduction. We begin with a result due to Lindelöf. Let U be a Jordan domain and set $\Gamma = \partial U$. For each $f(z) \in \mathcal{O}(U)$, Runge's theorem gives a sequence of polynomials $\{Q_\nu(z)\}$ which approximates f uniformly over each compact subset of U. If we impose some bound on f one may ask if an approximation exists where the Q-polynomials satisfy a similar bound as f. Lindelöf proved that bounds exist for many different norms on the given function f in the article *Sur un principe géneral de l'analyse et ses applications à la theorie de la représentation conforme* from

1915. Let us announce two results from [Lindelöf] of this nature. Let $p > 0$ and consider the H^p-space of analytic functions in U for which

$$(1) \qquad ||g||_p = \iint_U |g(z)|^p \cdot dxdy < \infty$$

B.1 Theorem *Let $p > 0$ and suppose that $f(z)$ has a finite H^p-norm. Then there exists a sequence of polynomials $\{Q_n(z)\}$ which converge uniformly to f in compact subsets of U while*

$$||Q_n||_p \leq ||f||_p \quad : \quad n = 1, 2, \ldots$$

A similar approximation holds when the H^p-norm is replaced by the maximum norm. Thus, if f is a bounded analytic function in U there exists a sequence of polynomials $\{Q_n\}$ which converge to f in every relatively compact subset of U and at the same time the maximum norms satisfy:

$$||Q_n||_U \leq ||f||_U \quad : \quad n = 1, 2, \ldots$$

To prove Theorem B.1 one constructs for each $n \geq 1$ a Jordan curve Γ_n which surrounds $\bar{U}$, i.e. its interior Jordan domain U_n contains $\bar{U}$ and for every point $p \in \Gamma_n$ the distance of p to Γ is $< 1/n$. It is trivial to see that such a family of Jordan curves exist where the domains $U_1, U_2, \ldots$ decrease. Next, fix a point $z_0 \in U$. There exists the unique conformal map ψ_n from U_n onto U such that

$$\psi_n(z_0) = z_0 \quad : \quad \psi_n'(z_0) \text{ is real and positive}$$

With these notations Lindelöf used the following lemma whose proof is left as an exercise:

B.2. Lemma *For each compact subset K of U the maximum norms $|\psi_n(z) - z|_K$ tend to zero as $n \to \infty$. Moreover, the complex derivatives $\psi_n'(z_0) \to 1$.*

Proof of Theorem B.1. To each n we set

$$F_n(z) = f(\psi_n(z)) \cdot \psi_n'(z)^{\frac{2}{p}}$$

By Lemma B.2 the sequence $\{F_n\}$ converges uniformly to f on compact subsets of U. Moreover, each $F_n \in \mathcal{O}(U_n)$ and it is clear that

$$\iint_U |f(z)|^p \cdot dxdy = \iint_{U_n} |F_n(z)|^p \cdot dxdy$$

hold for every n. To get the required polynomials $\{Q_n\}$ in Theorem B.1 for H^p-spaces it suffices to apply Runge's theorem for each single F_n. This detail of the proof is left to the reader. For maximum norms we use the functions

$$F_n(z) = f(\psi_n(z))$$

and after apply Runge's theorem in the domains $\{U_n\}$.

B.3 Remark. More delicate approximations by polynomials where other norms such as the modulus of continuity, were established later by Lindelöf and De Vallé-Poussin. We shall not pursue this any further. The reader can consult articles by De Vallé-Poussin which contain many interesting results concerned with approximation theorems.

C. Approximation by fractional powers

Here is the set-up in the article *Über die approximation analytischer funktionen* by Carleman from 1922. Let $0 < \lambda_1 < \lambda_2 < \ldots$ be a sequence of positive real numbers and Ω is a simply connected domain contained in the right half-space $\mathfrak{Re}(z) > 0$. Notice that the functions $q_\nu(z) = z^{\lambda_\nu}$ are analytic in the half-plane, i.e. with $z = re^{i\theta}$ and $-\pi/2 < \theta < \pi/2$ we have:

$$q_\nu(z) = r^{\lambda_\nu} \cdot e^{i\lambda_\nu \cdot \theta}$$

C.1 Definition. *We say that the sequence* $\Lambda = \{\lambda_\nu\}$ *is dense for approximation if there for each* $f \in \mathcal{O}(\Omega)$ *exists a sequence of functions of the form*

$$Q_N(z) = \sum_{\nu=1}^{N} c_\nu(N) \cdot q_\nu(z) \quad : \quad N = 1, 2, \ldots$$

which converges uniformly to f *on compact subsets of* Ω.

C.2 Theorem. *A sequence* Λ *is dense if*

$$(*) \qquad \limsup_{R\to\infty} \frac{\sum_R \frac{1}{\lambda_\nu}}{\operatorname{Log} R} > 0$$

where $\sum_R$ *means that we take the sum over all* $\lambda_\nu < R$.

Remark. Above condition (*) is the same for every simply connected domain Ω. Theorem C.2 gives a *sufficient* condition for an approximation. To get necessary condition one must specify the domain Ω and we shall not try to discuss this more involved problem. The proof of Theorem C.2 requires several steps, the crucial is the uniqueness theorem in C.4 while the proof of Theorem C.2 is postponed until C.5.

C.3 A uniqueness theorem.

Consider a closed Jordan curve Γ of class C^1 which is contained in $\mathfrak{Re}\, z > 0$. When $z = re^{i\theta}$ stays in the right half-plane we get an entire function of the complex variable λ defined by:

$$\lambda \mapsto z^\lambda = r^\lambda \cdot e^{i\theta\cdot\lambda}$$

We conclude that a real-valued and continuous function g on Γ gives an entire function of λ defined by:

$$G(\lambda) = \int_\Gamma g(z) \cdot z^\lambda \cdot \big|dz\big|_\Gamma$$

where $|dz|_\Gamma$ is the arc-length on Γ. With these notations one has

C.4 Theorem. *Assume that* Λ *satisfies the condition in Theorem C.2. Then, if* $G(\lambda_\nu) = 0$ *for every* ν *it follows that the* g*-function is identically zero.*

Proof. If we have shown that the G-function is identically zero then the reader may verify that $g = 0$. There remains to show that if $G(\lambda_\nu) = 0$ for every ν then $G = 0$. To attain this one first shows that there exist constants A, K and $0 < a < \frac{\pi}{2}$ such that:

$$\text{(i)} \qquad |G(\lambda)| \leq K \cdot e^{|\lambda|} \quad \text{and} \quad |G(is)| \leq K \cdot e^{|s|\cdot a} \quad : \lambda \in \mathbf{C} \,:\, s \in \mathbf{R}$$

The easy verification of (i) is left to the reader. Next, the first inequality in (i) means that G is an entire function of exponential type one. By assumption $G(\lambda_\nu) = 0$ for every ν. Now we can use Carleman's formula for analytic functions in a half-space from XXX to conclude that $G = 0$. Namely, set

$$\text{(ii)} \qquad U(r, \phi) = \log \big|G(re^{i\phi})\big|$$

Let $\{r_\nu e^{i\phi_\nu}\}$ be the zeros of G in $\mathfrak{Re}(z) > 0$ which by the hypothesis contains the set Λ. By Carleman's formula the following hold for each $R > 1$:

$$\sum_{1<r_\nu<R} \Big[\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\Big] \cdot \cos\theta_\nu = \frac{1}{\pi R} \cdot \int_{-\pi/2}^{\pi/2} U(R, \phi) \cdot \cos\phi \cdot d\phi +$$

$$\frac{1}{2\pi R} \cdot \int_1^R \big(\frac{1}{r^2} - \frac{1}{R^2}\big) \cdot \big[\, U(r, \pi/2) + U(r, -\pi/2) \,\big] \cdot dr + c_*(R)$$

where $c_*(R) \leq K$ holds for some constant which is independent of R. Finally, the set Λ satisfies (*) in Theorem C.2 and the sum over zeros in Carleman's formula above majorizes the sum extended over the real λ-numbers from Λ satisfying $1 < \lambda_\nu < R$. At this stage we leave it to the reader to verify that the second inequality in (i) above implies that G must be identically zero.

C.5 Proof of Theorem C.2

Denote by $\mathcal{O}^*(\Lambda)$ the linear space of analytic functions in the right half-plane given by finite $\mathbf{C}$-linear combinations of the fractional powers $\{z^{\lambda_\nu}\}$. To obtain uniform approximations over relatively compact subsets when Ω is a simply connected domain in $\mathfrak{Re}(z) > 0$, it suffices to regard a closed Jordan arc Γ which borders a Jordan domain U where U is a relatively compact subset of Ω. In particular Γ has a positive distance to the imaginary axis and there remains to show that when (*) holds in Theorem C.2, then an arbitrary analytic function $f(z)$ defined in some open neighborhood of $\bar{U}$ can be uniformly approximated by $\mathcal{O}^*(\Lambda)$-functions over a relatively compact subset U_* of U. To achieve this we shall use a trick which reduces the proof of uniform approximation to a problem concerned with L^2-approximation on Γ. To begin with we have

C.6 Lemma. *The uniqueness in Theorem C.4 implies that if V is a real-valued function on Γ then there exists a sequence $\{Q_n\}$ from the family $\mathcal{O}(\Lambda)$ such that*

$$\lim_{n\to\infty} \int_\Gamma |Q_n - V|^2 \cdot |dz| = 0$$

The proof of this result is left as an exercise.

C.7 A tricky construction. Let $f(z)$ be analytic in a neighborhood of the closed Jordan domain $\bar{U}$ bordered by Γ. Define a new analytic function

$$F(z) = \int_{z_*}^{z} \frac{f(\zeta)}{\zeta} \cdot d\zeta \tag{1}$$

where z_* is some point in $\bar{U}$ whose specific choice does not affect the subsequent discussion. We can write $F = V + iW$ where $V = \mathfrak{Re}(F)$. Lemma C.6 gives a sequence $\{Q_n\}$ which approximates V in the L^2-norm on Γ. Using this L^2-approximation we get

Lemma C.8 *Let U_0 be relatively compact in U. Then there exists a sequence of real numbers $\{\gamma_n\}$ such that*

$$\lim_{n\to\infty} \big||Q_n(z) - i\cdot\gamma_n - F(z)\big|_{U_0} = 0$$

Again we leave out the proof as an exercise. Next, taking complex derivatives Lemma C.8 implies that if U_* is even smaller, i.e. taken to be a relatively compact in U_0, then we a get uniform approximation of derivatives:

$$Q_n'(z) \to F'(z) = \frac{f(z)}{z}$$

This means that

$$z \cdot Q_n' \to f(z)$$

holds uniformly in U_*. Next, notice that

$$z \cdot \frac{d}{dz}(z^{\lambda_\nu}) = \lambda_\nu \cdot z^{\lambda_\nu}$$

hold for each ν. Hence $\{z \cdot Q_n'(z)\}$ again belong to the $\mathcal{O}(\Lambda)$-family. So we achieve the required uniform approximation of the given f function on U_* which completes the proof of Theorem C.2.

D. Theorem of Müntz

Introduction. Theorem D.1 below is due to Müntz. See his article *Über den Approximationssatz von Weierstrass* from 1914. The simplified version of the original proof below is given in [Car]. Here is the set up: Let $0 < \lambda_1 < \lambda_2 < \ldots$. To each ν we get the function x^{λ_ν} defined on the real unit interval $0 \leq x \leq 1$. We say that the sequence $\Lambda = \{\lambda_\nu\}$ is L^2-dense if the family $\{x^{\lambda_\nu}\}$ generate a dense linear subspace of the Hilbert space of square integrable functions on $[0,1]$.

D.1 Theorem. *The necessary and sufficient condition for Λ to be L^2-dense is that $\sum \frac{1}{\lambda_\nu}$ is convergent.*

D.2 Proof of necessity. If Λ is not L^2-dense there exists some $h(x) \in L^2[0,1]$ which is not identically zero while

$$\text{(1)} \qquad \int_0^1 h(x) \cdot x^{\lambda_\nu} \cdot dx = 0 \quad : \quad \nu = 1, 2, \ldots$$

Now consider the function

$$\text{(2)} \qquad \Phi(\lambda) = \int_0^1 h(x) \cdot x^{-i\lambda} \cdot dx$$

It is clear that Φ is analytic in the right half plane $\mathfrak{Im}\,\lambda > 0$. If $\lambda = s + it$ with $t > 0$ we have

$$|x^\lambda| = x^t \leq 1$$

for all $0 \leq x \leq 1$. From this and the Cauchy-Schwarz inequality we see that

$$\text{(3)} \qquad |\Phi(\lambda)| \leq ||h||_2 \quad : \lambda \in U_+$$

Hence Φ is a bounded analytic function in the upper half-plane. At the same time (1) means that the zero set of Φ contains the sequence $\{\lambda_\nu \cdot i\}$. By the integral formula formula we have seen in XX that this entails that

$$\text{(*)} \qquad \sum \frac{1}{\lambda_\nu} < \infty$$

which proves the necessity.

Proof of sufficiency. There remains to show that ifwe have the convergence in (*) above then there exists a non-zero h-function i $L^2[0,1]$ such that (1) above holds. To find h we first construct an analytic function Φ by

$$\text{(i)} \qquad \Phi(z) = \frac{\prod_{\nu=1}^\infty \left(1 - \frac{z}{\lambda_\nu}\right)}{\prod_{\nu=1}^\infty \left(1 + \frac{z}{\lambda_\nu}\right)} \cdot \frac{1}{(1+z)^2} \quad : \quad \mathfrak{Re}\, z > 0$$

Notice that $\Phi(z)$ is defined in the right half-plane since the series (*) is convergent. When $\mathfrak{Re}(z) \geq 0$ we notice that each quotient

$$\frac{1 - \frac{z}{\lambda_\nu}}{1 + \frac{z}{\lambda_\nu}}$$

has absolute value ≤ 1. It follows that

$$\text{(ii)} \qquad |\Phi(x+iy)| \leq \frac{1}{1 + x + iy|^2} = \frac{1}{(1+x)^2 + y^2}$$

In particular the function $y \mapsto \Phi(iy)$ belongs to L^2 on the real y-line. Now we set

$$\text{(ii)} \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ity} \cdot \Phi(iy) \cdot dy$$

using the inequality (ii) If $t < 0$ we can move the line integral of $e^{tz} \cdot \Phi(z)$ from the imaginary axis to a line $\mathfrak{Re}(z) = a$ for every $a > 0$ and it is clear that

$$\lim_{a \to +\infty} \int_{-\infty}^\infty e^{-at + ity} \cdot \Phi(a + iy) \cdot dy = 0$$

We conclude that $f(t) = 0$ when $t < 0$. Next, since $y \mapsto \Phi(iy)$ is an L^2-function it follows by Parseval's equality that

$$\int_0^\infty |f(t)|^2 \cdot dt < \infty$$

Moreover, for a fixed λ_ν we have

$$\int_0^\infty f(t) e^{-\lambda_\nu t} \cdot dt = \frac{1}{2\pi} \cdot \int_0^\infty \left[\int_{-\infty}^\infty e^{ity} \cdot \Phi(iy) \cdot dy \right] \cdot e^{-\lambda \nu t} \cdot dt =$$

$$\int_{-\infty}^\infty \frac{1}{iy - \lambda_\nu} \cdot \Phi(iy) \cdot dy$$

where the last equality follows when the repeated integral is reversed. By construction $\Phi(z)$ has a zero at λ_ν and therefore (xx) above remains true with Φ replaced by $\frac{\Phi(z)}{z-\lambda_\nu}$ which entails that

$$\int_0^\infty f(t)\cdot e^{-\lambda_\nu t}\cdot dt = 0$$

At this stage we obtain the requested h-function. Namely, since $t\mapsto e^{-t}$ identifies $(0,+\infty)$ with $(0,1)$ we get a function $h(x)$ on $(0,1)$ such that

$$h(e^{-t}) = e^{t/2}\cdot f(t)$$

The reader may verify that

$$\int_0^1 |h(x)|^2\cdot dx = \int_0^\infty |f(t)|^2\cdot dt$$

and hence h belongs to $L^2(0,1)$. Moreover, one verifies that the vanishing in (xx) above entails that

$$\int_0^1 h(x)\cdot x^{\lambda_\nu}\cdot dx = 0$$

Since this holds for every ν we have proved the sufficiency which therefore finishes the proof of Theorem XX.

D.2 Another density result

Density results using exponential functions are used in many applications. For example, equidistant sequences appear in the the *Sampling Theorem* by Shanning used in telecommunication engineering. When this result is expressed in analytic function theory via characteristic functions it can be formulated as follows:

D.3 Theorem. *Let $T>0$ and $g(t)$ is an L^2-function on the interval $[0,T]$ which is not identically zero and suppose that $a>0$ is such that*

$$\int_0^T e^{inat}\cdot g(t)\cdot dt = 0 \quad : \quad n\in\mathbf{Z}$$

Then we must have

$$a \geq \frac{2\pi}{T}$$

Remark This result is due to Fritz Carlson in his article [xx] from 1914. let us recall that Carlson and Carleman later became collegues at Stockholm University for more than two decades. Carlson's result was later improved by Titchmarsh and goes as follows:

D.4 Theorem. *Let $0<m_1<m_2<\ldots$ be an increasing sequence of positive real numbers such that*

(*) $$\limsup_{n\to\infty}\frac{n}{m_n} > 1$$

Then if $f(x)\in L^2(0,2\pi)$ and

(**) $$\int_0^{2\pi} e^{\pm i m_n x}\cdot f(x)\cdot dx = 0$$

hold for each n, it follows that $f=0$.

We give a proof below taken from the text-book [Paley-Wiener: page 84-85]. To begin with we notice that (**) implies that we also have vanishing integrals using $f(-x)$. Replacing f by $f(x)+f(-x)$ or $f)x)-f)-x)$ it suffices to prove the result when f is even or odd. Let us show that there cannot exist an even L^2-function f such that the integrals (**) vanish while

(i) $$\int_{-\pi}^{\pi} f(x)\cdot dx = 1$$

With f as above we set

(ii) $$\phi(z) = \int_{-\pi}^{\pi} e^{izt} \cdot f(t) \cdot dt$$

The entire function ϕ is even and (i) means that $\phi(0) = 1$. Moreover, ϕ is an entire function of exponential type and Cauchy-Schwarz inequality gives

(iii) $$|\phi(x+iy)| \leq ||f||_2 \cdot e^{\pi|y|}$$

for all $z = x + iy$. Next, Parseval's equality shows that the restriction of ϕ to the real x-line belongs to L^2 which by the Remark in (xx) implies that ϕ belong to the Carleman class $\mathcal{N}$. So Theorem § xx [Section $\mathcal{E}$] gives the existence of a limit

(iv) $$\lim_{R \to \infty} \frac{N_\phi(R)}{R} = A$$

The inequality (iiii) and the result in XXX entails that

(v) $$A \leq 2$$

Next, since the zeros of ϕ contains the even sequence $\{m_\nu\} \cup \{-m_\nu\}$ we have the inequality

(vi) $$N_\phi(m_n) \geq 2n$$

At the same time the limit formula (iv) gives:

(vii) $$A = \lim_{n \to \infty} \frac{N_\phi(m_n)}{m_n}$$

Finally, (vi) and the hypothesis (*) in Theorem D.4 give

(viii) $$\limsup_{n \to \infty} \frac{N_\phi(m_n)}{m_n} \geq 2 \cdot \limsup_{n \to \infty} \frac{n}{m_n} > 2$$

This contradicts (v) and we conclude that a non-zero function ϕ cannot exist which by the Laplace-Fourier formula (i) entails that a non-zero f cannot exist.

XI. Radial limit of functions with finite Dirichlet integral

We expose results from the article *Ensembles exceptionnels* by Beurling in [Beur] devoted to the study of functions $f(\theta)$ on the unit circle T whose harmonic extensions H_f to D have a finite Dirichlet integral. For such functions we shall prove that H_f has radial limits outside a set whose capacitiy is zero. A real-valued functions $f(\theta)$ on the unit circle T has a Fourier series:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos n\theta + \sum_{n=1}^{\infty} b_n \cdot \sin n\theta$$

We say that f belongs to the class $\mathcal{D}$ if

$$(*) \qquad \sum_{n=1}^{\infty} n(a_n^2 + b_n^2) < \infty$$

When the constant term $a_0 = 0$ the sum in (*) is denoted by $D(f)$ and is called the Dirichlet norm. Denote by $\mathcal{E}_f$ the set of all θ where the partial sums of the Fourier series of f does not converge.

0.1 Theorem. *For each $f \in \mathcal{D}$ the outer capacity of $\mathcal{E}_f$ is zero.*

Remark. Recall from XXX that if $E \subset T$ then its outer capacity is defined by

$$\text{Cap}^*(E) = \inf_{E \subset U} \text{Cap}(U)$$

with the infimum taken over open neighborhoods of E.

The proof of Theorem 0.1 has two essential ingredients. First, given some $f \in \mathcal{D}$ with constant term $a_0 = 0$ we obtain the harmonic function $f(r, \theta)$ defined in the open disc by

$$f(r,\theta) = \sum_{n=1}^{\infty} r^n(a_n \cdot \cos n\theta + b_n \cdot \sin n\theta)$$

We construct partial derivatives with respect to r and obtain:

$$(1) \qquad f'_r(r,\theta) = \sum_{n=1}^{\infty} n \cdot r^{n-1}(a_n \cdot \cos n\theta + b_n \cdot \sin n\theta)$$

Define the function $F(r, \theta)$ in D by

$$(2) \qquad F(r,\theta) = \int_0^r |f'_s(s,\theta)| \cdot ds$$

Thus, for each θ we integrate the absolute value of (1) along a ray from the origin. For every fixed θ $r \mapsto F(r, \theta)$ is non-decreasing and hence there exists a limit

$$(3) \qquad \lim_{r \to 1} F(r,\theta) = F^*(\theta)$$

The limit value can be finite or $+\infty$. It is clear that if (3) is finite then there exists the radial limit

$$(4) \qquad \lim_{r \to 1} f(r,\theta) = f^*(\theta)$$

Remark. For every θ such that the radial limit (4) exists, it follows that Fourier's partial sums converge to $f^*(\theta)$. In fact, this follows from Abel's theorem in [Series] since the inclusion $f \in \mathcal{D}$ entails that a_n and b_n both are small ordo of $\frac{1}{n}$. Hence we have:

Lemma *For every $\rho > 0$ one has the inclusion*

$$\mathcal{E}_f \subset \{F^*(\theta) > \rho\}$$

We conclude that Theorem 0.1 follows if the capacity of $\{F^* > \rho\}$ tends to zero as $\rho \to +\infty$. This follows from the result below.

0.2 Theorem. *Let $f \in \mathcal{D}$ where $a_0 = 0$ and $D(f) = 1$. Then*

$$\mathrm{Cap}(\{F^* > \rho\}) \leq e^{-\rho^2}$$

hold for every $\rho > 0$.

The essential step to get Theorem 0.2 relies upon the following inequality:

0.3 Theorem. *For each $f \in \mathcal{D}$ with $a_0 = 0$ one has $F^* \in \mathcal{D}$ and*

$$D(F^*) \leq D(f)$$

Once this is proved we can deduce Theorem 0.2. This is done in § 2 after we have proved Theorem 0.3 in § 1. Before we proceed to § 1 we shall need a result about logarithmic potentials. Let μ be a probability measure on T, i.e a non-negative Riesz measure of total mass one and put:

$$U_\mu(z) = \int \log \frac{1}{|z-\zeta|} \cdot d\mu(\zeta)$$

This is a harmonic function in $\{|z| < 1\}$ and passing to its radial limits as $r \to 1$ the energy integral is defined by:

(*) $$J(\mu) = \lim_{r \to 1} \int U_\mu(r,\theta) \cdot d\mu(\theta) = \int U_\mu(\theta) \cdot d\mu(\theta)$$

One says that μ has finite energy when (*) is finite. Assume that μ has finite energy. Using polar coordinates in D we have a series expansion:

$$U_\mu(r,\theta) = \sum \frac{r^n}{n}(h_n \cos n\theta + k_n \sin n\theta)$$

where $\{h_n\}$ and $\{k_n\}$ are real numbers. The energy integral $J(\mu)$ becomes the limit of the following expression as $r \to 1$:

(1) $$\int U_\mu(r,\phi) \cdot d\mu(\phi) = \iint \log \frac{1}{|1 - re^{i(\phi-\theta)}|} d\mu(\phi) \cdot d\mu(\theta)$$

To compute the right hand side we expand the complex Log-function:

$$\log \frac{1}{1 - re^{i(\phi-\theta)}} = \sum_{n=1}^{\infty} \frac{r^n}{n} \cdot e^{in(\phi-\theta)}$$

Taking real parts it follows that (1) is equal to

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cdot \cos n(\phi - \theta) \cdot d\mu(\phi) \cdot d\mu(\theta)$$

Now we use the trigonometric formula

$$\cos n(\phi - \theta) = \cos n\phi \cdot \cos n\theta + \sin n\phi \cdot \sin n\theta$$

Put

(2) $$h_n = \int \cos n\theta \cdot d\mu(\theta) \quad \text{and} \quad k_n = \int \sin n\theta \cdot d\mu(\theta)$$

Then we obtain

(3) $$J(\mu) = \sum \frac{1}{n}(h_n^2 + k_n^2)$$

Next, let $g(\theta) \in \mathcal{D}$ with Fourier coefficients $\{a_n\}$ and $\{b_n\}$ where $a_0 = 0$. Then we have

$$\int g \cdot d\mu = \sum a_n \cdot h_n + b_n \cdot k_n$$

and Cauchy-Schwarz inequality gives:

(4) $$[\int g \cdot d\mu]^2 \leq S(g) \cdot J(\mu)$$

From the above we obtain the following:

0.4 Theorem. *For each probability measure μ with finite energy and every function $g(\theta) \in \mathcal{D}$ which is lower semi-continuous one has the inequality*

$$\Big[\int g(\theta) \cdot d\mu(\theta)\Big]^2 \leq S(g) \cdot J(\mu)$$

Remark. Above the lower semi-continuity is imposed in order to ensure that the Borel integral of g with respect to μ is defined.

1. Proof of Theorem 0.3

To begin with one has

1.1 Lemma. *The function F is subharmonic in D.*

For each fixed $0 < \alpha < 1$ we define the function ϕ_α in d by

$$\phi_\alpha(x,y) = \frac{\partial}{\partial \alpha}\, f(\alpha x, \alpha y) = x \cdot f'_x(\alpha x, \alpha y) + y \cdot f'_y(\alpha x, \alpha y)$$

Now we notice that the function $f_\alpha(x,y) = f(\alpha x, \alpha y)$ is harmonic and (1) means that

$$\phi_\alpha = (x\partial_x + y\partial_y)(f_\alpha)$$

where $\mathfrak{e} = x\partial_x + y\partial_y$ is the Euler field. As explained in XX this first order operator satisfies the identity

$$\Delta \circ \mathfrak{e} = \Delta + \mathfrak{e} \cdot \Delta$$

in the ring of differential operators and then we conclude that ϕ_α is harmonic. Next, the absolute value of a harmonic function is subharmonic so $\{|\phi_\alpha|\}$ yield subharmonic functions and a change of variables gives:

$$F = \int_0^1 |\phi_\alpha| \cdot d\alpha$$

This shows that F is a Riemann integral of subharmonic functions which in compact subsets of D is uniformly approximated by finite sums

$$\frac{1}{N}\sum_{k=1}^{k=N} |\phi_{k/N}|$$

Lemma 1.1 follows since a convex sum of subharmonic functions again is subharmonic.

An inequality. Notice that the function $F(r,\theta)$ is continuous and its derivative with respect to r exists and equals $|f'_r(r,\theta)|$. But the partial derivative $\partial F/\partial\theta$ may have jump discontinuities along rays where the derivative f'_r has a zero. However, this cannot occur too often so when $0 < r < 1$ is fixed there exists the integral

$$I(r) = \int_0^{2\pi} (\frac{\partial F}{\partial \theta}(r,\theta))^2 \cdot d\theta$$

We have proved that F is subharmonic and from its definition it is clear that the partial derivative $\partial F/\partial r$ is non-negative. By the general result in Chapter V:B:xxx we obtain

1.2 Lemma. *The inequality below holds for each $0 < r < 1$:*

$$I(r) \leq r^2 \cdot \int_0^{2\pi} (\frac{\partial F}{\partial r}(r,\theta))^2 \cdot d\theta \tag{*}$$

1.3 Dirichlet integrals. Let $f \in \mathcal{S}$ with $a_0 = 0$. We construct the Dirichlet integral

$$\text{Dir}(f) = \frac{1}{\pi} \cdot \iint_D [(f'_x)^2 + (f'_y)^2] \cdot dxdy$$

Then one has the equality:

$$(*) \qquad \mathrm{Dir}(f) = D(f)$$

To see this we identify $f(r,\theta)$ with the real part of the analytic function

$$G(z) = \sum (a_n - i \cdot b_n) \cdot z^n$$

The Cauchy-Riemann equations give

$$\mathrm{Dir}(f) = \frac{1}{\pi} \cdot \iiint_D |G'(z)|^2 \cdot dxdy$$

Now the reader can verify that the double integral above is equal to $D(f)$. Notice that (*) identifies $\mathcal{D}$ with the space of real-valued functions on T whose harmonic extensions to D have a finite Dirichlet integral.

1.4 Exercise. Show that the Dirichlet integral of a function g of class C^2 in D also is given by the double integral

$$\text{(i)} \qquad \frac{1}{\pi} \cdot \int_0^1 \int_0^{2\pi} \big[\, r^2 \cdot (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2} \cdot (\frac{\partial g}{\partial \theta})^2 \,\big] \cdot r \cdot d\theta dr$$

Show also that if g is harmonic then

$$\text{(ii)} \qquad \mathrm{Dir}(g) = \frac{2}{\pi} \cdot \int_0^1 \int_0^{2\pi} (\frac{\partial g}{\partial r})^2 \cdot r \cdot d\theta dr$$

1.5 Proof of Theorem 0.3

Apply (i) in 1.4 with $g = F$ where the inequality in Lemma 1.2 and an integration with respect to r give

$$(1) \qquad \mathrm{Dir}(F) \leq \frac{2}{\pi} \cdot \int_0^1 \int_0^{2\pi} (\frac{\partial F}{\partial r})^2 \cdot r \cdot d\theta dr$$

Next, the construction of F gives the equality

$$(\frac{\partial F}{\partial r})^2 = (\frac{\partial f}{\partial r})^2$$

in the whole disc D. Then (1) and the equality (ii) applied to the harmonic function f give:

$$(2) \qquad \mathrm{Dir}(F) \leq \mathrm{Dir}(f) = D(f)$$

where the last equality used (*) in 1.3. Next, construct the harmonic extension of the boundary function $F^*(\theta)$ which we denote by H_F. Here we have the equations

$$(3) \qquad D(F^*) = D(H_F)$$

Next, recall that the Dirchlet integral is minimized when we take a harmonic extension which entails that

$$(4) \qquad \mathrm{Dir}(H_F) \leq \mathrm{Dir}(F)$$

Hence (2-4) give the requested inequality

$$D(F^*) \leq D(f)$$

2. Proof of Theorem 0.2

Let $\rho > 0$ and apply Theorem 0.4 to the function $g = F^*$ and the equilibrium distribution μ assigned to the set $E = \{F^* > \rho\}$. This gives

$$\rho^2 \leq \Big[\int F^* \cdot d\mu\Big]^2 \leq S(F^*) \cdot J(\mu) \tag{4}$$

Now $D(F^*) \leq D(f) = 1$ holds by Theorem 0.3 and hence we have:

$$\rho^2 \leq J(\mu) \tag{5}$$

Next, recall from XX that $J(\mu)$ is the the constant value $\gamma(E)$ of the potential function U_μ restricted to E. Hence (5) gives

$$e^{-\gamma(E)} \leq e^{-\rho^2} \tag{6}$$

By definition the left hand side is the capacity of E which proves Theorem 0.2.

An application

Let Ω be a simply connected domain which contains the origin in the complex ζ-plane and $\partial\Omega$ contains a relatively open set given by an interval ℓ situated on the line $\mathfrak{Re}\,\zeta = \rho$ for some $\rho > 0$. Consider the harmonic measure $\mathfrak{m}_0^\Omega(\ell)$. In other words, the value at the origin of the harmonic function in Ω which is 1 on ℓ and zero on $\partial\Omega \setminus \ell$. We shall find an upper bound for (*) in the family of simply connected domains which contain the origin and ℓ and at the same time has area π. To attain this we consider the conformal map ϕ from the unit disc onto Ω with $\phi(0) = 0$. The invariance of harmonic measures gives:

$$\mathfrak{m}_0^\Omega(\ell) = \mathfrak{m}_0^D(\alpha)$$

where α is the interval on T such that $\phi(\alpha) = \ell$. For an interval on the unit circle one has the equality

$$\text{Cap}(\alpha) = \sin\,\alpha/4$$

At the same time $\mathfrak{m}_0^D(\alpha) = \frac{\alpha}{2\pi}$ which entails that

$$\mathfrak{m}_0^\Omega(\ell) = \frac{2}{\pi}\,\text{arcsin Cap}(\alpha) \tag{1}$$

There remains to estimate last term above. Put $u = \mathfrak{Re}\,\phi$. The inclusion $\ell \subset \mathfrak{Re}\zeta = \rho$ means that $u = \rho$ on ℓ. So when ϕ is considered in the class $\mathcal{S}$ we have the inclusion

$$\alpha \subset \{|\phi| > \rho - \epsilon\}$$

for each $\epsilon > 0$. Next, since the area of $\phi(D) = \pi$ we have $S(u) = 1$ and Theorem 0.2 gives

$$\text{Cap}(\alpha) \leq e^{-\rho^2}$$

Hence we have proved the general inequality

$$\mathfrak{m}_0^\Omega(\ell) \leq \frac{2}{\pi} \cdot \text{arcsin}\; e^{-\rho^2} \tag{**}$$

Remark. There exists a special simply connected domain Ω for which equality holds in (**). See [Frostman: p. 39] : Potential theory.

XI. The Denjoy conjecture

Introduction. Let ρ be a positive integer and $f(z)$ is an entire function such that there exists some $0<\epsilon<1/2$ and a constant A_ϵ such that

$$|f(z)| \leq A_\epsilon \cdot e^{|z|^{\rho+\epsilon}} \tag{0.1}$$

hold for every z. Then we say that f has integral order $\leq \rho$. Next, the entire function f has an asymptotic value a if there exists a Jordan curve Γ parametrized by $t \mapsto \gamma(t)$ for $t \geq 0$ such that $|\gamma(t)| \to \infty$ as $t \to +\infty$ and

$$\lim_{t\to+\infty} f(\gamma(t)) = a \tag{0.2}$$

In 1920 Denjoy raised the conjecture that (0.1) implies that the entire function f has at most 2ρ many different asymptotic values. Examples show that this upper bound is sharp. The Denjoy conjecture was proved in 1930 by Ahlfors in [Ahl]. A few years later T. Carleman found an alternative proof based upon a certain differential inequality. Theorem A.3 below has applications beyond the proof of the Denjoy conjecture for estimates of harmonic measures. See [Ga-Marsh].

A. The differential inequality.

Let Ω be a connected open set in $\mathbf{C}$ whose intersection S_x between a vertical line $\{\mathfrak{Re}\, z = x\}$ is a bounded set on the real y-line for every x. When $S_x \neq \emptyset$ it is the disjoint union of open intervals $\{(a_\nu, b_\nu)\}$ and we set

$$\ell(x) = \max_\nu (b_\nu - a_\nu) \tag{*}$$

Next, let $u(x,y)$ be a positive harmonic function in Ω which extends to a continuous function on the closure $\bar\Omega$ with the boundary values identical to zero. Define the function ϕ by:

$$\phi(x) = \int_{S_x} u^2(x,y)\cdot dy \tag{1}$$

The Federer-Stokes theorem gives the following formula for the derivatives of ϕ:

$$\phi'(x) = 2\int_{S_x} u_x \cdot u(x,y)dy \tag{2}$$

$$\phi''(x) = 2\int_{S_x} u_{xx}\cdot u(x,y)dy + 2\int_{S_x} u_x^2 \cdot dy \tag{3}$$

Since $\Delta(u)=0$ when $u>0$ we have

$$2\int_{S_x} u_{xx}\cdot u(x,y)dy = -2\int_{S_x} u_{yy}\cdot u(x,y)dy = 2\int u_y^2 dy \tag{4}$$

The Cauchy-Schwarz inequality applied in (2) gives

$$\phi'(x)^2 \leq 4\cdot \int_{S_x} u_x^2 \cdot \int_{S_x} u^2(x,y)dy = 4\cdot\phi(x)\cdot\int_{S_x} u_x^2 dy \tag{5}$$

Hence (4) and (5) give:

$$\phi''(x) \geq 2\int_{S_x} u_y^2(x,y)\cdot dy + \frac{1}{2}\cdot\frac{\phi'^2(x)}{\phi(x)} \tag{6}$$

Next, since $u(x,y)=0$ at the end-points of all intervals of S_x, *Wirtinger's inequality* and the definition of $\ell(x)$ give:

$$\int_{S_x} u_y^2(x,y)\cdot dy \geq \frac{\pi^2}{\ell(x)^2}\cdot\phi(x) \tag{7}$$

Inserting (7) in (6) we have proved

A.1 Proposition *The ϕ-function satisfies the differential inequality*

$$\phi''(x) \geq \frac{2\pi^2}{\ell(x)^2} \cdot \phi(x) + \frac{\phi'^2(x)}{2\phi(x)}$$

Proof continued. The maximum principle for harmonic functions implies that the $\phi(x) > 0$ when $x > 0$ and hence there exists a ψ-function where $\phi(x) = e^{\psi(x)}$. It follows that

$$\phi' = \psi' e^{\psi} \quad \text{and} \quad \phi'' = \psi'' e^{\psi} + \psi'^2 e^{\psi}$$

Now Proposition A.1 gives

$$(*) \qquad \psi'' + \frac{\psi'^2}{2} \geq \frac{2\pi^2}{\ell(x)^2}$$

A.2 An integral inequality. From (*) we obtain

$$\frac{2\pi}{\ell(x)} \leq \sqrt{\psi'(x)^2 + 2\psi''(x)} \leq \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

Taking the integral we get

$$(**) \qquad 2\pi \cdot \int_0^x \frac{dt}{\ell(t)} \leq \psi(x) + \log\, \psi'(x) + O(1) \leq \psi(x) +, \psi'(x) + O(1)$$

where $O(1)$ is a remainder term which is bounded independent of x. Taking the integral once more we obtain:

A.3 Theorem. *The following inequality holds:*

$$2\pi \cdot \int_0^x \frac{x-s}{\ell(s)} \cdot ds \leq \int_0^x \psi(s) \cdot ds + \psi(x) + O(x)$$

where the remainder term $O(x)$ is bounded by Cx for a fixed constant.

B. Solution to the Denjoy conjecture

B.1 Theorem. *Let $f(z)$ be entire of some integral order $\rho \geq 1$. Then f has at most 2ρ many different asymptotic values.*

Proof. Suppose f has n different asymptotic values $a_1, \ldots, a_n$. To each a_ν there exists a Jordan arc Γ_ν as described in the introduction. Since the a-values are different the n-tuple of Γ-arcs are separated from each other when $|z|$ is large. So we can find some R such that the arcs are disjoint in the exterior disc $|z| > R$. We may also consider the tail of each arc, i.e. starting from the last point on Γ_ν which intersects the circle $|z| = R$. So now we have an n-tuple of disjoint Jordan curves in $|z| \geq R$ where each curve intersects $|z| = R$ at some point p_ν and after the curves moves to the point at infinity. See figure. Next, we take one of these curves, say Γ_1. Let D_R^* be the exterior disc $|\zeta| > R$. In the domain $\Omega = \mathbf{C} \setminus \Gamma_1 \cup D_R^*$ we can choose a single-valued branch of $\log \zeta$ and with $z = \log \zeta$ the image of Ω is a simply connected domain Ω^* where S_x for each x has length strictly less than 2π The images of the Γ-curves separate Ω^* into n many disjoint connected domains denoted by $D_1, \ldots, D_n$ where each D_ν is bordered by a pair of images of Γ-curves and a portion of the vertical line $x = \log\, R$.

Let $\zeta = \xi + i\eta$ be the complex coordinate in Ω^*. Here we get the analytic function $F(\zeta)$ where

$$F(\log(z)) = f(z)$$

We notice that F may have more growth than f. Indeed, we get

$$(1) \qquad |F(\xi + i\eta)| \leq \exp\bigl(e^{(\rho+\epsilon)\xi}\bigr)$$

With $u = \text{Log}^+ |F|$ it follows that

(2) $$u(\xi,\eta) \leq e^{(\rho+\epsilon)\xi}$$

Hence the ϕ-function constructed during the proof of Theorem A.3 satisfies

$$\phi(\xi) \leq e^{2(\rho+\epsilon)\xi}$$

It follows that the ψ-function satisfies

(3) $$\psi(\xi) = 2 \cdot (\rho+\epsilon)\xi + O(1)$$

Now we apply Theorem A.3 in each region D_ν where we have a function $\ell_\nu(\xi)$ constructed by (0) in section A. This gives the inequality

(4) $$2\pi \cdot \int_R^\xi \frac{\xi - s}{\ell_\nu(s)} \cdot ds \leq \int_R^\xi (\rho+\epsilon)s \cdot ds + (\rho+\epsilon)\xi + O(1) \quad : \quad 1 \leq \nu \leq n$$

Next, recall the elementary inequality which asserts that if $a_1, \ldots, a_n$ is an arbitrary n-tuple of positive numbers then

(5) $$\sum a_\nu \cdot \sum \frac{1}{a_\nu} \geq n^2$$

For each s we apply this to the n-tuple $\{\ell_\nu(s)\}$ where we also have

$$\sum \ell_\nu(s) \leq 2\pi$$

So a summation in (4) over $1 \leq \nu \leq n$ gives

(6) $$n \cdot \int_R^\xi (\xi - s) \cdot ds \leq \int_R^\xi (\rho+\epsilon)s \cdot ds + (\rho+\epsilon)\xi + O(1)$$

Another integration gives:

(7) $$n \cdot \frac{\xi^2}{2} \leq (\rho+\epsilon) \cdot \xi^2 + O(\xi)$$

This inequality can only hold for large ξ if $n \leq 2(\rho+\epsilon)$ and since $\epsilon < 1/2$ is assumed it follows that $n \leq 2\rho$ which finishes the proof of the Denjoy conjecture.

XII. A system of infinite linear equations.

Introduction. The main issue in this section is the construction of a unique solution to the system

$$\sum_{q\neq p}\frac{x_q}{p-q}=0 \tag{*}$$

where (*) hold for all positive integers p and $\{x_q\}$ is a sequence of real numbers for which the series

$$\sum_{q=1}^{\infty}\frac{x_q}{q} \tag{**}$$

is convergent. The fact that (*) ha a non-trivial solution is far from evident. Before the study of (*) in Section 1 we discuss a general situation described by Carleman in his major lecture at the international congress in Zürich 1932 where one of his topics was devoted to linear systems of equations in an infinite number of variables. A homogeneous system takes the form

$$\sum_{q=1}^{\infty} c_{pq}x_q=a_p \quad : \quad p=1,2,\dots \tag{***}$$

where $\{c_{pq}\}$ is a matrix with an infinite number of elements. A sequence $\{x_q\}$ of complex numbers is a solution to (***) if the sum in the left hand side converges for each p and has value a_p. Notice that one does not require that the series are absolutely convergent.

The generic case. To every $p\geq 1$ we have the linear form L_p defined on finite sequences $\{x_1,x_2,\dots\}$, i.e. where $x_q=0$ when $q>>0$ by:

$$L_p(x_\bullet)=\sum_{q\geq 1} c_{pq}\cdot x_q$$

The generic case occurs when $c_{1q}\neq 0$ for every q and the linear forms $\{L_p\}_1^\infty$ are linearly independent. The last condition means that for every positive integer M there exists a strictly increasing sequence $q_1<\dots<q_M$ so that the $M\times M$-matrix with elements $a_{p\nu}=c_{pq_\nu}$ has a non-zero determinant.

The R-functions. Assume that the system (***) is generic and let $\{a_p\}$ be a given sequence for which we seek a solution $\{x_q\}$. The necessary and sufficient condition for this existence goes as follows: Consider n-tuples of positive integers $m_1,\dots,m_n$ where $n\geq 2$. For every such n-tuple and $1\leq k\leq n-1$ we set

$$\mathcal{D}(k)=\{\nu\colon\ m_1+\dots+m_k<\nu\leq m_1+\dots+m_{k+1}\}$$

Next, with $M=m_1+\dots+m_N$ we denote by $\mathcal{F}(m_1,\dots,m_n)$ the family of sequences $(x_1,\dots,x_M)$ such that the following inequalities hold for every $1\leq k\leq n-1$:

$$\Big|\sum_{q=1}^{q=\nu} c_{pq}x_q-a_q\Big|\leq\frac{1}{k}\quad :\ \nu\in\mathcal{D}(k)\quad\text{and}\quad 1\leq p\leq k$$

The generic assumption implies that the set $\mathcal{F}(m_1,\dots,m_n)$ is non-empty provided that we start with a sufficiently large m_1 and for every such n-tuple we set

$$R(m_1,\dots,m_n)=\min\sum_{\nu=1}^{m_1} x_\nu^2$$

where the minimum is taken over sequences $x_1,\dots,x_{m_1}$ which give the starting terms of some sequence $x_1,\dots,x_M\in\mathcal{F}(m_1,\dots,m_n)$.

Theorem. *The necessary and sufficient condition in order that (***) has a least one solution is that there exists a constant K and an infinite sequence of positive integers $\mu_1,\mu_2,\dots$ such that*

$$R(\mu_1,\mu_2,\dots,\mu_r)\leq K\quad\text{hold for every } r$$

Remark. The reader may consult [Carleman] for further remarks about this result and also comments upon the more involved criterion for non-generic linear systems. From now on we study linear systems which arise as follows: Consider a rational function of two variables x, y:

$$f(x,y) = \frac{a_0(x) + a_1(x)y + \ldots a_n(x)y^n}{b_0(x) + b_1(x)y + \ldots b_m(x)y^m}$$

Here n and m are positive integers and $a_\nu(x)$ and $b_j(x)$ polynomials in x. No special assumptions are imposed on these polynomials except that $b_m(x)$ and $a_n(x)$ are not identically zero. For example, it is not neccesary that the degree of b_m is $\geq \deg(b_j)$ for all $0 \leq j \leq m-1$.

0.1 Proposition *Let $b_m^{-1}(0)$ be the finite set of zeros of b_m. Let $\zeta_0 \in \mathbf{C} \setminus b_m^{-1}(0)$ be such that*

$$\sum_{j=0}^{j=m} b_j(\zeta_0) \cdot q^j = 0$$

holds for some finite set of positive integers, say $1 \leq q_1 < \ldots < q_k$. Then there exists $\delta > 0$ such that

$$\sum_{j=0}^{j=m} b_j(\zeta) q^j \neq 0 \quad : \text{for all positive integers } q \quad : \quad 0 < |\zeta - \zeta_0| < \delta$$

Exercise. Prove this result.

Next, consider the sequence of polynomials of the complex ζ-variable given by:

$$B_q(\zeta) = b_0(\zeta) + b_1(\zeta)q + \ldots + b_m(\zeta)q^m \quad : q = 1, 2, \ldots$$

0.2 Proposition *Put*

$$W = b_m^{-1}(0) \bigcup_{q \geq 1} B_q^{-1}(0)$$

Then W is a discrete subset of $\mathbf{C}$, i.e its intersection with any bounded disc is finite.

Exercise. Prove this result where a hint is to apply Rouche's theorem.

Next, put $W^* = W \cup a_n^{-1}(0)$, i.e. add the zeros of the polynomial a_n to W.

0.3 Proposition. *Let $\zeta_0 \in C \setminus W^*$ and suppose that $\{x_q\}$ is a sequence such that the series*

(i) $$\sum_{q=1}^{\infty} f(\zeta_0, q) \cdot x_q$$

is convergent. Then the series

(ii) $$\sum_{q=1}^{\infty} f(\zeta, q) \cdot x_q \quad \text{converges for every } \zeta \in C \setminus W^*$$

Moreover, the series sum is a meromorphic function of ζ whose poles are contained in W^*.

Remark. Proposition 0.3 gives a procedure to find solutions $\{x_q\}$ which is not the trivial null solution to a homogeneous system:

(*) $$\sum_{q \neq p} f(p, q) \cdot x_q = 0 \quad : \quad p = 1, 2, \ldots$$

More precisely, assume that the rational function $f(x,y)$ is such that $f(p,q) \neq 0$ when p and q are distinct positive integers. To get a solution $\{x_q\}$ to (iii) it suffices to begin with to verify (i) in Proposition 0.3 for some ζ_0 and then also try to find $\{x_q\}$ so that the meromorphic function

(**) $$\zeta \mapsto \sum_{q=1}^{\infty} x_q \cdot f(\zeta, q)$$

has zeros at every positive integer. Using this criterium for a solution we can show the following:

(Theorem. *For every complex number $a \in \mathbf{C} \setminus (-\infty, 0]$ the system*

$$\sum_{q=1}^{\infty} \frac{x_q}{p+aq} = 0 \quad : \quad p = 1, 2, \ldots$$

has no non-trivial solution $\{x_q\}$.

There remains to analyze the case when a is real and < 0. In this case complete answers about possible when f is the rational function have been established by K. Dagerholm starting from his Ph.D-thesis at Uppsala University in 1938 with Beurling as supervisor. The hardest case occurs hen $a = 1$ which will be studied in the next section.

1. The Dagerholm series.

Let $\mathcal{F}$ be the family of all sequences of real numbers $x_1, x_2, \ldots$ such that the series

(i)
$$\sum_{q=1}^{\infty} \frac{x_q}{q} < \infty$$

We only require that the series is convergent, i.e. it need not be absolutely convergent.

1.1 Theorem. *Up to a multiple with a real constant there exists a unique sequence $\{x_q\}$ in $\mathcal{F}$ such that*

$$\sum_{q \neq p} \frac{x_q}{p-q} = 0 \quad : \quad p = 1, 2, \ldots$$

The proof of uniquenss relies upon Jensen's formula and the existence upon a solution to a specific Wiener-Hopf equation. We begin to describe the strategy to obtain Theorem 1.1.To begin with there exists a meromorphic function $h(z)$ defined by

(ii)
$$h(z) = \sum_{q=1}^{\infty} \frac{x_q}{z-q}$$

To see that $h(z)$ is defined we notice that if s_* is the series sum in (i) then

(iii)
$$h(z) + s_* = \sum_{q=1}^{\infty} x_q \cdot [\frac{1}{z-q} + \frac{1}{q}] = z \cdot \sum_{q=1}^{\infty} \frac{x_q}{q(z-q)}$$

It is clear that the right hand side is a meromorphic function with poles confined to the set of positive integers. Hence we obtain the entire function:

$$H(z) = \frac{1}{\pi} \cdot \sin(\pi z) \cdot h(z)$$

1.2 Proposition. *The following hold for each positive integer:*

(i)
$$H(p) = (-1)^p \cdot x_p$$

(ii)
$$H'(p) = (-1)^q \cdot \sum_{q \neq p} \frac{x_q}{p-q} = 0$$

Proof. Let $p \geq 1$ be an integer. With ζ small we have

$$H(p+\zeta) = \frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) \cdot [\frac{x_p}{\zeta} + \sum_{q \neq p} \frac{x_q}{p+\zeta-q}]$$

A series expansion of the complex sine-function at πp gives

$$\frac{1}{\pi} \cdot \sin(\,\pi p + \pi\zeta) = [\zeta \cdot \cos(\pi p) + O(\zeta^3)] \cdot [\frac{x_p}{\zeta} + \sum_{q \neq p} \frac{x_q}{p+\zeta-q}]$$

Proposition 1.2 follow since $\cos \pi p = (-1)^p$.

Remark. Proposition 1.2 shows that $\{x_p\}$ solves the homogeneous system in Theorem 1.1 if the complex derivative of the entire H-function has zeros on all positive integers. This observation is the gateway towards the proof of Dagerholm's Theorem First we establish the uniqueness part.

2. Proof of uniqueness

Let $\{x_q\}$ be a sequence in $\mathcal{F}$. From the constructions in above it is clear that the meromorphic function $h(z)$ satisfies the following in the left half-plane $\mathfrak{Re}(z) \leq 0$:

(i) $$\lim_{x \to -\infty} h(x) = 0 \colon \quad |h(x+iy)| \leq C_* \quad : \ x \leq 0$$

where C_* is a constant. Moreover, in the right half-plane there exists a constant C^* such that

(ii) $$|h(x+iy)| \leq C^* \cdot \frac{|x|}{1+|y|} \quad : \ |x-q| \geq \frac{1}{2} \text{ for all positive integers}$$

To $h(z)$ we get the entire function $H(z)$ and (i-ii) above give the two the estimates below in the right half-plane:

(iii) $$|f(x+iy)| \leq Ce^{\pi|y|} \ : \ x \leq 0 \quad : \quad |f(x+iy)| \leq C\frac{|x|}{1+|y|} \cdot e^{\pi|y|}$$

Moreover, the first limit formula in (i) gives

(iv) $$\lim_{x \to -\infty} f(x) = 0$$

It is easily seen that the same upper bounds hold for the entire function $H'(z)$ and a straightforward application of the Phragmén-Lindelöf theorem gives:

2.1 Proposition. *The complex derivative of $H(z)$ satisfies the growth condition:*

$$\lim_{r \to \infty} e^{-\pi r \cdot |\sin\phi|} \cdot |H'(re^{i\theta}| = 0 \quad : \text{holds uniformly when } 0 \leq \theta \leq 2\pi$$

Now we are prepared to prove the uniqueness part in Theorem 0.1. For suppose that we have two sequences $\{x_q\}$ and $\{x_q^*\}$ which both give solutions to (*) are not equal up to a constant multiple of each other. The two sequences give entire functions H_1 and H_2. Since both are constructed via real sequences their Taylor coefficients are real. We can choose a linear combination

$$G = aH_1 + bH_2 = G$$

where a, b are real numbers and the complex derivative $G'(0) = 0$. The hypothesis that there exists two $\mathbf{R}$-linearly independent solutions to (*) leads to a contradiction once we have proved the following

2.2 Lemma *The entire function $G'(z)$ is identically zero.*

Proof. To simplify notations we set $g(z) = G'(z)$. To show that $g = 0$ we first consider the series expansion

$$g(z) = a_p z^p + a_{p+1} z^{p+1} + \ldots$$

where a_p ia the first non-vanishing coefficient. Since $g(0) = G'(0) = 0$ we have $p \geq 1$ and since the two x-sequences both are solutions to (*) it follows from (ii) in Proposition 0.2 that

(i) $$g(p) = 0 \quad : \ p = 1, 2, \ldots$$

Next, the primitive function G is real-valued $G(x)$ on $x \leq 0$ and since the H-functions are zero for every integer ≤ 0 the same holds for G. Then *Rolle's theorem* implies that for every $n \geq 1$ there exists

(i) $$-n < \lambda_n < -n+1 \quad : \ g(\lambda_n) = 0$$

So if $\mathcal{N}$ is the counting function for the zeros of the entire g-function one has the inequality

(iii) $$\mathcal{N}(r) \geq [2r]$$

where $[2r]$ is the largest integer $\leq 2r$. Next, recall that a_p is the first non-zero term in the series expansion of g. Hence Jensen's formula gives:

$$(*)\qquad \log|a_p| + p\cdot\log\, r + \int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Log}\,|g(re^{i\theta}\,|\,\cdot d\theta$$

Proposition 2.1 applied to $g(z)$ gives:

$$(\text{iv})\qquad \int_0^{2\pi} \mathrm{Log}\,|g(re^{i\theta}\,|\,\cdot d\theta = 2r - m(r) \quad \text{where} \lim_{r\to\infty} m(r) = +\infty$$

At this stage we get the contradiction as follows. First (iii) gives

$$\int_0^r \frac{\mathcal{N}(t)\cdot dt}{t} \geq 2r - \mathrm{Log}(r) - 1$$

Now (*) and (iii) give the inequality

$$(\text{vi})\qquad \log|a_p| + p\cdot\log\, r + 2r - 1 - \log r \leq 2r - m(r) \quad : \; r\geq 1$$

Here $p\geq 1$ which therefore would give:

$$\log|a_p| - 1 + m(r) \leq 0$$

But this is impossible since we have seen that $m(r)\to +\infty$.

3. Proof of existence

We start with a general construction. Let $\phi(z)$ be analytic in the unit disc D which extends to a continuous function on T except at the point $z=1$. We also assume that there exists some $0<\beta<2$ and a constant C such that

$$(1)\qquad |\phi(\zeta)| \leq C|1-\zeta|^{-\beta}$$

This implies that the function

$$\theta\mapsto\theta\cdot\phi(e^{i\theta})$$

is integrable on the unit circle. Hence there exists the entire function

$$(2)\qquad f(z) = \frac{-i}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta z}\cdot\theta\cdot\phi(e^{i\theta})\cdot d\theta$$

Next, with $\epsilon>0$ small we let γ_ϵ be the interval of the circle $|z-1|=\epsilon$ with end-points at the intersection with $|z|=1$. So on γ_ϵ we have

$$z = 1+\epsilon\cdot e^{i\theta} \quad : \; -\frac{\pi}{2}+\epsilon_* < \theta < \frac{\pi}{2}-\epsilon_*$$

where ϵ_* is small with ϵ. We obtain the entire function

$$F(z) = \frac{1}{2\pi}\int_\epsilon^\pi e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi}\int_{-\pi}^{-\epsilon} e^{-i\theta\cdot z}\cdot\phi(e^{i\theta})d\theta + \frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{e^{-z\cdot\mathrm{Log}\zeta}\cdot\phi(\zeta)d\zeta}{\zeta}$$

If $z=n$ is an integer we have

$$e^{-in\theta} = \zeta^{-n} \quad : \quad e^{-n\cdot\mathrm{Log}\zeta} = \zeta^{-n}$$

Hence we get

$$(*)\qquad F(n) = \frac{1}{2\pi i}\cdot\int_{\Gamma_\epsilon}\frac{\phi(\zeta)\cdot d\zeta}{\zeta^{n+1}}$$

where Γ_ϵ is the closed curve given as the union of γ_ϵ and the interval of T where $|\theta|\geq\epsilon$. Cauchy's formula applied to ϕ gives:

2.1 Proposition. *Let* $\phi(\zeta) = \sum c_n \zeta^n$. *Then*

$$F(n) = c_n \quad : n \geq 0 \quad \text{and} \quad F(n) = 0 \quad n \leq -1$$

Next, using (i) above we also have:

2.2 Proposition. *The complex derivative of* F *is equal to* f.

Proof. With $\epsilon > 0$ the derivative of the sum of first two terms from the construction of $F(z)$ above become

(i) $$\frac{1}{2\pi} \int_{|\theta| \geq \epsilon} -i\theta \cdot e^{-iz\theta} \phi(e^{i\theta}) d\theta$$

In the last integral derivation with respect to z gives

(ii) $$-\frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{e^{-z \cdot \text{Log}\zeta} \cdot \phi(\zeta) d\zeta}{\zeta}$$

Now $\zeta = 1 + \epsilon \cdot e^{i\theta}$ during the integration along γ_ϵ which gives:

$$|\text{Log}\,(1 + \epsilon \cdot e^{i\theta})| \leq \epsilon$$

At the same time the circle interval γ_ϵ has length $\leq \epsilon$ and hence the growth condition (i) shows that the integral (iii) tends to zero when $\epsilon \to 0$. Finally, since we assumed that the function $\theta \mapsto \theta \cdot \phi(e^{i\theta})$ is absoutely integrable on T a passage to the limit as $\epsilon \to 0$ gives $F' = f$ as requested.

2.3 Conclusion. If n is a positive integer in Proposition 2.3 we have:

(**) $$F'(n) = \frac{-i}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \cdot \theta \cdot \phi(e^{i\theta}) \cdot d\theta$$

These integrals are zero for every $n \geq 1$ if and only if $\theta \cdot \phi(e^{i\theta})$ is the boundary value function of some $\psi(z)$ which is analytic in the exterior disc $|z| > 1$. In 2.X we will show that this is true for a specific ϕ-function satisfying the growth condition (1) above and in addition the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{c_n}{n}$$

converges.

2.4 How to deduce a solution $\{x_p\}$. Suppose we have found ϕ satisfying the conditions above which gives the entire function $F(z)$ whose derivatives are zero for all $n \geq 1$. Now we set

$$x_p = (-1)^p \cdot c_p$$

By (***) this sequence belongs to $\mathcal{F}$ and we construct the associated entire function $H(z)$. From (i) in Proposition 0.1 and Proposition 2.1 we get

$$H(p) = (-1)^p \cdot x_p = c_p = F(p)$$

In addition both H and F have zeros at all integers ≤ 0. Next, by the construction of F it is clear that this is an entire function of exponential type and by the above the entire function $G = H - F$ has zeros at all integers. We leave as an exercise to the reader to show that G must be identically zero. The hint is to use similar methods as in the proof of the uniqueness. It follows that

$$H'(q) = F'(q) = 0$$

for all $q \geq 1$. By (ii) in Proposition 0.2 this means precisely that $\{x_p\}$ is a solution to the requested equations in (*) which gives the existence in Dagerholm's Theorem.

2.5 The construction of ϕ.

There remains to find ϕ such that the conditions above hold. To obtain ϕ we start with the integrable function on T defined by:

$$u(\theta) = \frac{1}{2} \cdot \log \frac{1}{|\theta|} \quad : \ -\pi < \theta < \pi$$

We get the analytic function

$$g(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} - \zeta}{e^{i\theta} + \zeta} \cdot u(\theta) \cdot d\theta$$

In the exterior disc we find the analytic function

$$\psi(\zeta) = \exp \ -\bar{g}(\frac{1}{\bar{\zeta}})$$

Let us also put $\phi_*(\zeta) = e^{g(\zeta)}$. Now we have

$$\log |\phi(e^{i\theta})| = \mathfrak{Re}\, g(e^{i\theta}) = u(\theta)$$

In the same way we see that

$$\log |\psi(e^{i\theta})| = -\mathfrak{Re}\, g(e^{i\theta}) = -u(\theta)$$

Since $2u(\theta) = -\log |\theta|$ it follows that

$$\log |\theta| + \log |\phi_*(e^{i\theta})| = \log |\psi(e^{i\theta})|$$

Taking exponentials we obtain

$$|\theta| \cdot |\phi_*(e^{i\theta})| = |\psi(e^{i\theta})|$$

Exercise. Check also arguments and verify that we can remove absolute values in the last equality to attain

$$(*) \qquad |\theta| \cdot \phi_*(e^{i\theta}) = \psi(e^{i\theta})$$

Here (*) is not precisely what we want since our aim was to construct ϕ so that $\theta| \cdot \phi(e^{i\theta})$ is equal to the boundary value of an analytic function in $|z| > 1$. So in order to get rid of the absolute value for θ in (*) we modify ϕ_* as follows: Set

$$\rho(\theta) = \frac{\pi i}{2} \cdot \operatorname{sign} \theta \cdot e^{-i\theta} \quad : \ -\pi < \theta < \pi$$

Next, consider the two analytic functions in D, respectively in $|z| > 1$ defined by:

$$\phi_1(z) = \frac{1}{\sqrt{1 - z^2}} \quad \text{and} \quad \psi_1(z) = \frac{1}{\sqrt{1 - z^{-2}}}$$

Exercise. Show that one has the equality

$$\rho(\theta) = \frac{\phi_1(e^{i\theta})}{\psi_1(e^{i\theta})}$$

when $-\pi < \theta < \pi$ and $\theta \neq 0$.

The ϕ-function. it is defined by

$$\phi(z) = \frac{z}{\sqrt{1 - z^2}} \cdot \phi_*(z)$$

From (*) above and the construction of ρ it follows that

$$\theta \cdot \phi(e^{i\theta}) = \frac{\pi}{2} \cdot \psi_1(e^{i\theta}) \cdot \psi(e^{i\theta})$$

The right hand side is the boundary function of an analytic function in $|z| > 1$ and hence ϕ satisfies (**) from XX. Consider its Taylor expansion

$$\phi(z) = \sum c_n \cdot z^n$$

There remains to verify that the series (**) converges and that ϕ satisfies the growth condition in XX. To prove this we begin to analyze the function

(i) $$\phi_*(z) = e^{g(z)}$$

Rewrite the u function as a sum

(ii) $$u(\theta) = \frac{1}{2}\log\Big|\frac{1}{1-e^{i\theta}}\Big| + k(\theta) \quad \text{where} \quad k(\theta) = \frac{1}{2}\log\Big|\frac{1-e^{i\theta}}{\theta}\Big|$$

When θ is small we have an expansion

(iii) $$\frac{1-e^{i\theta}}{\theta} = -i + \theta/2 + \ldots$$

From this we conclude that the k-function is at least twice differentiable as a function of θ. So the Fourier coefficients in the expansion

(iv) $$k(e^{i\theta}) = \sum b_\nu e^{i\nu\theta}$$

have a good decay. For example, there is a constant C such that

(v) $$|b_\nu| \leq \frac{C}{\nu^2} \quad : \ \nu \neq 0$$

This implies that the analytic function

(vi) $$\mathcal{K}(z) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \cdot k(e^{i\theta}) \cdot d\theta$$

yields a bounded analytic function in the unit disc. Next, the construction of the g-function gives:

(vii) $$g(z) = \frac{1}{2} \cdot \log\frac{1}{1-z} + \sum b_\nu z^\nu \implies$$

$$\phi_*(z) = \frac{1}{\sqrt{1-z)}} \cdot e^{\mathcal{K}(z)}$$

We conclude that

$$\phi(z) = \frac{z}{1-z} \cdot \frac{1}{\sqrt{1+z)}} \cdot e^{\mathcal{K}(z)}$$

Since $\mathcal{K}(z)$ extends to a continuous function on the closed disc it follows that ϕ satisfies the growth condition (1) with $\beta = 1$. Moreover, the function $\theta \cdot \phi(e^{i\theta})$ belongs to $L^1(T)$ since $\frac{1}{\sqrt{1+e^{i\theta}}}$ is integrable. There remains only to prove:

Lemma. The series

$$\sum_{p=1}^{\infty} (-1)^n \cdot \frac{c_n}{n}$$

is convergent.

Proof. let us put

$$A(z) = \frac{z}{\sqrt{1+z)}} \cdot e^{\mathcal{K}(z)}$$

This gives

$$\phi(z) = \frac{A(1)}{1-z} + \frac{A(z)-A(1)}{1-z}$$

From (v) it follows that $K(z)$, and hence also $e^{K(z)}$ is differentiable at $z01$ which gives the existence of a constant C such that

$$|\frac{A(z)-1}{1-z}| \leq C \cdot \frac{1}{|\sqrt{1+z}|}$$

Here the function $\theta \mapsto \frac{1}{|\sqrt{1+e^{i\theta}}|}$ belongs to $L^p(T)$ for each $p < 2$ which by the inequality for L^p-norms between functions and their Fourier coefficients in XX for example implies that if $\{c_\nu^*\}$ give the Taylor series for $\frac{A(z)-1}{1-z}$ then

$$\sum |c_\nu^*|^3 < \infty$$

Now Hölder's inequality gives

$$\sum \frac{|c_\nu^*|}{\nu} \leq \big(\sum |c_\nu^*|^3\big)^{\frac{1}{3}} \cdot \big(\sum \nu^{-3/2}\big)^{\frac{2}{3}} < \infty \tag{8}$$

We conclude that the Taylor series for ϕ becomes

$$A(1) \cdot (1 + z + z^2 + \ldots) + \sum c_\nu^* z^\nu$$

Hence $c_n = A(1) + c_\nu^*$ and now Lemma xx follows since the alternating series $\sum (-1)^n \frac{1}{n}$ is convergent and we have the absolute convergence in XX above.

XIII. Uniform approximation by Blaschke products

Contents.

Introduction

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A bounded analytic function $f(z)$ in the unit disc D is an *inner function* if the boundary function on T has constant absolute value almost everywhere. The set of inner functions is denoted by $\mathcal{I}(D)$. The first result about inner functions is due to O. Frostman in [x] :

0.1 Theorem. *Let $w(z) \in \mathcal{I}(D)$. For every $\epsilon > 0$ there exists a real number c and a Blaschke product $B(z)$ such that*

$$\max_{z \in D} |w(z) - e^{ic}B(z)| < \epsilon$$

The next result is due to D. Marshall in [Mar] and asserts that the convex hull of Blaschke products is dense in the unit ball of $H^\infty(D)$.

0.2 Theorem. *Let $f(z) \in H^\infty(D)$ with $|f|_D \leq 1$. For each $\epsilon > 0$ there exists a finite family of Blaschke functions $B_1 \ldots, B_N$ and an N-tuple of positive real numbers $a_1, \ldots, a_N$ with $\sum a_\nu = 1$ such that*

$$\big|f(z) - \sum a_\nu \cdot B_\nu(z)\big|_D < \epsilon$$

Remark. Frostman's theorem reduces Theorem 0.2 to the assertion that the convex hull of inner functions is dense in $H^\infty(T)$. The first step in the proof of Theorem 0.2 is to use an approximation theorem which takes place in the Banach space $L^\infty(T)$. Notice that if w is an inner function then $\frac{1}{w} \in L^\infty(T)$. We can also take quotients $\frac{w_1}{w_2}$ and get new $L^\infty(T)$-functions. With this in mind the following result is due to Douglas and Rudin in [xx]

0.3 Theorem *The convex hull formed by functions $\frac{w_1}{w_2}$ from pairs in $\mathcal{I}(D)$ is dense in $L^\infty(T)$.*

We prove this result in § 5. Now we consider some $f \in H^\infty(T)$ with norm $|f|_\infty = 1$. Theorem 0.3 gives for each $\epsilon > 0$ an N-tuple of unimodular functions $\{\frac{w_1^\nu}{w_2^\nu}\}$ formed by inner functions such that

$$(*) \qquad \big|f - \sum a_\nu \cdot \frac{w_1^\nu}{w_2^\nu}\big|_\infty < \epsilon \quad : \quad a_1 + \ldots + a_N = 1 \;:\; a_\nu \geq 0$$

However, (*) does not give Theorem 0.2 since the individual quotients $\frac{w_1^\nu}{w_2^\nu}$ are not analytic in D. To proceed from (*) Marshall considered the inner function

$$\mathcal{J} = w_2^1 \cdots w_2^N$$

Then we can write

$$(**) \qquad \sum a_\nu \cdot \frac{w_1^\nu}{w_2^\nu} = \sum a_\nu \cdot \frac{W_\nu}{\mathcal{J}} \quad : W_\nu = w_1^\nu \cdot \prod_{j \neq \nu} w_2^j$$

The next step in Marshall's proof is to apply a theorem by Nevanlinna which implies that (*) gives the existence of another inner function W_* such that the function

$$(***) \qquad g = \sum a_\nu \cdot \frac{W_\nu}{\mathcal{J}} + \epsilon \cdot \frac{W_*}{\mathcal{J}} \quad \text{belongs to } H^\infty(T)$$

It follows that $\bigl|f - g\bigr| < 2\epsilon$ and to finish the proof of Theorem 0.2 there remains only to approximate the special $H^\infty(T)$-function g-function by a convex combination of inner functions. This step in the proof of Theorem 0.2 is given under the heading *Marshall's Lemma* in section 4.

Remark. Before we enter the proofs of Theorem 0.1 and 0.2 we expose some preliminaries about inner functions and Blaschke products based upon results from Chapter XX about the Jensen-Nevanlinna class of analytic functions in D.

1. Blaschke products and inner functions

.

A sequence $\{z_n\}$ of non-complex numbers in D arranged so that $0 < |z_1| \leq |z_2| \leq ...$ satisfies Blaschke's condition if

$$\sum_{n=1}^\infty (1 - |z_n|) < \infty$$

Blaschke's theorem asserts that the infinite product

$$(1) \qquad B(z) = \prod_{n=1}^\infty \frac{z - z_n}{1 - \bar{z}_n z} \cdot \frac{\bar{z}_n}{z_n}$$

converges in D and yields an analytic function such that

$$\lim_{r \to 1} |B(re^{i\theta})| = 1 \quad \text{holds almost everywhere } 0 \leq \theta \leq 2\pi$$

In particular every Blaschke product belongs to $\mathcal{I}(D)$. Next, let μ be a singular non-negative Riesz measure on the unit circle, i.e. μ carries all mass on a null set in the sense of Lebesgue. Then there exists the analytic function in D defined by

$$(2) \qquad G_{-\mu}(z) = \exp\bigl[-\frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{e^{i\theta} - z}{e^{i\theta} + z} \cdot d\mu(\theta)\bigr]$$

The Factorisation Theorem from XX gives:

1.1. Theorem *Every $w \in \mathcal{I}(D)$ is a unique product*

$$w(z) = e^{ic} \cdot B(z) \cdot G_{-\mu}(z) \quad : \quad c = \text{a real number}$$

where $B(z)$ is the Blascke product formed by the zeros of w in D and μ a singular and non-negative measure.

We will also need Theorem XXX from XXX which characterizes when an inner function is a Blaschke product.

1.2 Theorem. *An inner function $w(z)$ is of the form $e^c \cdot B(z)$ if and only if*

$$\limsup_{r \to 1} \int_0^{2\pi} \log\bigl|w(re^{i\theta})\bigr| \cdot d\theta = 0$$

2. Proof of Frostman's theorem

Let $0 < \rho < 1$ and γ is a complex number with $|\gamma| \leq 1$. Then one has the equality

$$(*) \qquad \frac{1}{2\pi} \int_0^{2\pi} \log\bigl|\frac{\gamma - \rho e^{i\theta}}{1 - \rho e^{-i\theta}\gamma}\bigr| \cdot d\theta = \max(\rho, \log|\gamma|)$$

The verification is left to the reader. Now we consider some $w \in \mathcal{I}(D)$. Apply (*) with $\gamma = w(re^{it})$ for pairs $0 < r < 1$ and $0 \leq t \leq 2\pi$. Integration with respect to t gives the equality:

$$(*)\qquad \frac{1}{2\pi}\int_0^{2\pi}\Big[\int_0^{2\pi}\log\Big|\frac{w(re^{it})-\rho e^{i\theta}}{1-\rho e^{-i\theta}w(re^{it})}\Big|\cdot d\theta\Big]dt=\int_0^{2\pi}\max\,[\rho,\log\,|w(re^{it})|\,]\cdot dt$$

Since $w\in\mathcal{I}(D)$ we have

$$\text{(i)}\qquad \lim_{r\to 1}\log\,|w(re^{it})|=0\quad:\quad\text{for almost all }\;t$$

Keeping $0<\rho<1$ fixed the function $t\mapsto\max\big[\rho,\log\,|w(re^{it}|\big]$ is bounded so by *dominated convergence* under the intergral sign we have:

$$\text{(ii)}\qquad \lim_{r\to 1}\int_0^{2\pi}\max\,[\rho,\log\,|w(re^{it}|)\,]\cdot dt=0$$

Replace this limit by the double integral which comes from the equality (*) and apply *Fubini's theorem* to interchange the order of integration. Hence (ii) gives:

$$\text{(iii)}\qquad \lim_{r\to 1}\frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi}\Big[\int_{t=0}^{t=2\pi}\log\Big|\frac{w(re^{it})-\rho e^{i\theta}}{1-\rho e^{-i\theta}w(re^{it})}\Big|\cdot dt\,\Big]d\theta=0$$

Now we use that the integrand

$$\text{(iv)}\qquad \log\Big|\frac{w(re^{it})-\rho e^{i\theta}}{1-\rho e^{-i\theta}w(re^{it})}\Big|\le 0\quad:\quad 0\le\theta,t\le 2\pi$$

Then (iii) and *Fatou's theorem* gives:

$$\text{(v)}\qquad \limsup_{r\to 1}\int_0^{2\pi}\log\Big|\frac{w(re^{it})-\rho e^{i\theta}}{1-\rho e^{-i\theta}w(re^{it})}\Big|\cdot dt\,]=0\quad:\text{almost everywhere for }\theta$$

At this stage the proof is almost finished. Namely, we notice that the functions

$$F_\theta(z)=\frac{w(z)-\rho e^{i\theta}}{1-\rho e^{-i\theta}w(z)}$$

belong to $\mathcal{I}(D)$ for every θ. Moreover, (v) gives a null set $\mathcal{N}$ in T such that

$$\text{(vi)}\qquad \limsup_{r\to 1}\int_0^{2\pi}\log|F_\theta(re^{it})|\,dt=0\quad:\quad\theta\in T\setminus\mathcal{N}$$

Finally, for every $\theta\in T\setminus\mathcal{N}$, Theorem 1.2 implies that $F_\theta(z)=e^{ic_\theta}B_\theta(z)$ for some Blaschke product B_θ and a constant c_θ. With such a choice of θ we have

$$\text{(vii)}\qquad |w(z)-e^{ic}\cdot B_\theta(z)|=\Big|\frac{\rho e^{i\theta}-\rho e^{-i\theta}(w(z))^2}{1-\rho e^{-i\theta}w(z)}\Big|\le\frac{2\rho}{1-\rho}$$

Here we can choose any $\rho<1$. So with $\epsilon>0$ we choose ρ so small that $\frac{2\rho}{1-\rho}<\epsilon$ and Frostman's Theorem follows.

3. A theorem by Nevanlinna

Theorem 3.1 below was proved by R. Nevanlinna in his article [Nev:xx.] from 1919. On the unit circle T we have the Banach space $L^\infty(T)$ which contains the closed subspace $H_0^\infty(T)$ of boundary values from bounded analytic functions $h(z)$ in D which vanish at $z=0$. Recall from XX that a bounded Lebesgue measurable function $f(e^{i\theta})$ belongs to $H_0^\infty(T)$ if and only if

$$\text{(0.1)}\qquad \int_0^{2\pi}e^{in\theta}\cdot f(e^{i\theta})d\theta=0\quad:\quad n=0,1,2,\ldots$$

Since $H_0^\infty(T)$ is a closed subset of $L^\infty(T)$ there exists the Banach space

$$\text{(0.2)}\qquad \mathcal{B}=\frac{L^\infty(T)}{H_0^\infty(T)}$$

For each $F \in L^\infty(T)$ we set:

$$\mathfrak{nev}(F) = \min ||F - h||_\infty \quad : \; h \in H_0^\infty(D) \tag{0.3}$$

We refer to $\mathfrak{nev}(F)$ as the Nevanlinna norm of F. Since $\mathfrak{nev}(F)$ is the norm in the quotient space $\mathcal{B}$ it is trivial that

$$\mathfrak{nev}(F) \leq |F|_\infty \tag{*}$$

When strict inequality holds in (*) one has the following:

3.1 Theorem *Let $F \in L^\infty(T)$ be such that $\mathfrak{nev}(F) < |F|_\infty$. Then there exists $h^* \in H^\infty(T)$ such that*

$$|F(e^{i\theta}) - h^*(e^{i\theta})| = |F|_\infty \quad : \quad \text{almost everywhere on } T$$

Proof. By a change of scale, i.e. replacing F by F times the inverse of $|F|_\infty$ we may assume that $|F|_\infty = 1$. Set

$$\mathcal{H}_F = \{h \in H^\infty(T) \quad : \; |F - h| \leq 1\} \tag{i}$$

The triangle inequality gives $|h|_\infty \leq 2$ for every $h \in \mathcal{H}_F$. So we have a uniform bound and hence $\mathcal{H}_F$ is a normal family of analytic functions in the unit disc D. By the general result from Ch. 3-XX there exists $h^* \in \mathcal{H}_F$ such that

$$h^*(0) = \max_{h \in \mathcal{H}_F} |h(0| \tag{ii}$$

Notice that $h^*(0) > 0$. In fact, since $\mathfrak{nev}(F) < 1$ is assumed there exists to begin with some $g \in H_0^\infty(T)$ with $|F - g|_\infty \leq 1 - \delta$. for some $\delta > 0$. Hence $\mathcal{H}_F$ contains the functions $g(z) + \delta$ which is $\neq 0$ at the origin.

Sublemma. One has the equality

$$|F - h^* - \phi|_\infty \geq 1 \quad : \quad \phi \in H_0^\infty(T)$$

Proof. Suppose there exists $\phi \in H_0^\infty(T)$ with $|F - h^* - \phi|_\infty = 1 - \delta$ for some $\delta > 0$. If $a > 0$ it follows that

$$|F - (1 + a)h^* - \phi| \leq 1 - \delta + a|h^*|_\infty \leq 1 - \delta + 2a$$

So $h_1 = (1 + \delta/2)h^* + \phi$ belongs to $\mathcal{H}_F$. Here $h_1(0) > h^*(0)$ which contradicts the maximality of $h^*(0)$ and the Sublemma follows.

Proof continued. Put $G = F - h^*$. The Sublemma means that the norm of the G-image in the Banach space $\mathcal{B}$ is at least 1. At the same time the $L^\infty(T)$-norm of G is one since $h^* \in \mathcal{H}_F$. It follows that the $\mathcal{B}$-norm of G is 1. Next, by the duality between $H_0^\infty(T)$ and $H^1(T)$ from XX G yields a linear functional on the Hardy space $H^1(T)$ of norm one. Hence there exists a sequence $\{\phi_n\}$ in $H^1(T)$ with L^1-norms equal to one such that:

$$\lim_{n \to \infty} \int_0^{2\pi} G \cdot \phi_n \cdot d\theta = 1 \tag{1}$$

Put $c_n = \phi_n(0)$. Since $h^* \in H^\infty(T)$ we notice that

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} h^* \cdot \phi_n \cdot d\theta = h^*(0) \cdot c_n \tag{2}$$

Sublemma 2. There exists some positive constant a such that

$$|c_n| \geq a \quad : \quad n = 1, 2, \ldots$$

Proof. Assume the contrary. If $c_n \to 0$ then (1) and (2) give

$$\lim_{n \to \infty} \frac{1}{2\pi} \cdot \int_0^{2\pi} F \cdot \phi_n \cdot d\theta = 1 \tag{3}$$

But this is impossible since $\mathfrak{nev}(F) < 1$. Indeed, this gives the existence of some $\psi \in H^\infty(T)$ with $[F - \psi|_\infty = 1 - \delta$ for some $\delta > 0$. At the same time $c_n \to 0$ implies that

$$\lim_{n\to\infty} \frac{1}{2\pi} \cdot \int_0^{2\pi} \psi \cdot \phi_n \cdot d\theta = 0$$

Now we get a contradiction, i.e. (3) cannot hold since

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} |F - \psi| \cdot \phi_n \cdot d\theta \leq |F - \psi|_\infty \cdot |\phi_n|_1 = 1 - \delta$$

Proof continued. Now we may assume that there is some $a > 0$ such that

(*) $$|\phi_n(0)| \geq a \quad : \quad n = 1, 2, \ldots$$

At the same time (1) above holds and we also know that $|G|_\infty = 1$. Using this we will show that

(**) $$|G(e^{i\theta})| = 1 \quad : \text{holds almost everywhere}$$

To prove (**) we use Jensen's inequality from XX which by (*) for every n gives:

(***) $$2\pi \cdot \operatorname{Log}|a| \leq \int_0^{2\pi} \operatorname{Log}\left|\phi_n(e^{i\theta})\right| \cdot d\theta$$

Suppose that (**) does not hold. This gives the existence of some $\rho < 1$ and a set E of positive Lesbesgue measure such that the maximum norm $|G|_E \leq \rho$. Since $|G|_\infty = 1$ it follows that

$$\int_0^{2\pi} G \cdot \phi_n \cdot d\theta \leq \rho \cdot \int_E |\phi_n| \cdot d\theta + \int_{T \setminus E} |\phi_n| \cdot d\theta$$

By (1) above the left hand side tends to one and since the L^1-norms of ϕ_n are all equal to one, we conclude that

(****) $$\lim_{n\to\infty} \int_E |\phi_n| \cdot d\theta = 0$$

But this contradicts (1) by the Nevanlinna-Jensen theory in XX. Namely, (****) first entails that

(1) $$\lim_{n\to\infty} \int_E \operatorname{Log}|\phi_n| \cdot d\theta = -\infty$$

At the same time ϕ_n have finite L^1-norms which by the material from XXX gives a constant C which is independent of n so that

$$\int_0^{2\pi} \operatorname{Log}^+ |\phi_n| \cdot d\theta \leq C$$

Then we see that (1) violates (***) and hence (**) must hold which finishes the proof of Nevanlinna's theorem.

Remark. Notice that the proof also shows how to find h^*, i.e. it is the extremal function in the family $\mathcal{H}_F$ which maximizes the value at $z = 0$.

3.2 A consequence of Nevanlinna's Theorem. Let $g \in H^\infty(T)$. If $\mathcal{J}$ is an inner function then $\frac{g}{\mathcal{J}}$ belongs to $L^\infty(T)$. But in general this quotient does not belong to $H^\infty(T)$. To compensate for this we study its deviation from $H^\infty(T)$ and obtain:

3.3 Theorem. *Let $0 < \epsilon < |g|_\infty$ be such that there exists $f \in H^\infty(T)$ of norm one for which*

(*) $$\left|f - \frac{g}{\mathcal{J}}\right|_\infty = \epsilon$$

Then there exists an inner function w such that

(**) $$\frac{\epsilon \cdot w + g}{\mathcal{J}} \in H^\infty(T)$$

Proof. Nevanlinna's Theorem gives some $h^* \in H^\infty(T)$ such that

$$|\frac{g}{\mathcal{J}} - h^*| = \epsilon \quad : \text{ almost everywhere}$$

Since the inner function $\mathcal{J}$ has absolute value one almost everywhere, it follows that the analytic function $g - h^*\mathcal{J}$ has absolute value ϵ almost everywhere on T. This this gives an inner function w such that

(iii) $$\mathcal{J} \cdot h^* - g = \epsilon \cdot w$$

After division with $\mathcal{J}$ we get (**) in Theorem 3.3

4. Marshall's Approximation Lemma

Consider a function $g \in H^\infty(T)$ expressed as

(*) $$g = \sum_{k=1}^{k=N} a_k \cdot \frac{w_k}{J}$$

where $w_1, \ldots, w_N$ and J are inner functions while $a_1, \ldots, a_N$ are some real numbers. Here we do not assume that they are ≥ 0 or that the sum is one. But we assume that the maximum norm $|g|_\infty < 1$.

4.1 Proposition. *For every $\epsilon > 0$ there exists a convex sum of inner functions V such that*

$$\big|g - V\big|_\infty < \epsilon$$

Proof. For the inner functions $\{w_k\}$ and J we have the equalities below on T:

(i) $$\frac{1}{w_k} = \bar{w}_k \quad : \frac{1}{J} = \bar{J}$$

It follows that almost everywhere on T:

(ii) $$\bar{g}(e^{i\theta}) = \sum_{k=1}^{k=N} a_k \cdot \frac{J(e^{i\theta})}{w_k(e^{i\theta})}$$

Next, $W = w_1 \cdots w_N$ is an inner function and (ii) implies that the product

(iii) $$W \cdot \bar{g} \in H^\infty(T)$$

Now we shall use (iii) to get another expression for g. By assumption $|g|_\infty \leq 1 - \delta$ for some $\delta > 0$. Since $\int_0^{2\pi} e^{i\nu t} dt = 0$ for every $\nu \geq 1$ we have the equality below for every complex number λ of absolute value < 1:

(iv) $$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\lambda e^{it} + g(z)}{1 + \lambda e^{it} \bar{g}(z)} \cdot dt$$

Next, for each $z \in D$ we can take $\lambda = W(z)$ and obtain

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{W(z)e^{it} + g(z)}{1 + e^{it} \cdot W(z) \cdot \bar{g}(z)} \cdot dt \quad : \quad z \in D$$

The family of functions

$$t \mapsto \Phi_t(z) = \frac{W(z)e^{it} + g(z)}{1 + e^{it} \cdot W(z) \cdot \bar{g}(z)}$$

is equi-continuous with respect to t since $|g|_\infty < 1 - \epsilon$ is assumed. So we can evaluate the integral which defines $g(z)$ by Riemann sums in a uniform manner, i.e. to every $\epsilon > 0$ there exists a finite set $0 \leq t_1 < \ldots t_M < 2\pi$ and positive numbers $\{b_\nu\}$ with $\sum b_\nu = 1$ such that

(v) $$\big|g(z) - \sum b_\nu \cdot \frac{W(z)e^{it_\nu} + g(z)}{1 + e^{it_\nu} W(z) \cdot \bar{g}(z)}\,\big| < \epsilon$$

Next, since $W(z)$ is inner we notice that the functions

$$\phi_\nu(z) = \frac{W(z)e^{it_\nu} + g(z)}{1 + e^{it_\nu} W(z) \cdot \bar{g}(z)} \quad : 1 \leq \nu \leq M$$

are all inner. Since $\epsilon > 0$ can be made arbitrary small it follows that $g(z)$ can be uniformly approximated by a convex combination of inner functions.

5. Density of unimodular functions

Let $\mathcal{U}(T)$ be the unimodular functions on T, i.e. L^∞-functions with absolute value one almost everywhere. Elementary geometry shows that the convex hull of $\mathcal{U}(T)$ is dense in the unit ball of $L^\infty(T)$, i.e. to every G with $|G|_\infty \leq 1$ there exist $g_1, \ldots, g_M$ in $\mathcal{U}(T)$ such that

(*) $$|G - \sum a_\nu \cdot g_\nu|_\infty < \epsilon \quad : \; a_1 + \ldots + a_M = 1$$

Since products of inner functions are inner, Theorem 0.3 follows if we show that every $g \in \mathcal{U}(T)$ can be uniformly approximated by convex combinations of quotients of inner functions. Again, since products of inner functions again are inner, it suffices to achieve this approximation for a generating set in the multiplicative group of unimodular functions on T. Hence it suffices to consider g-functions which only take two values. After a rotation we may assume that they are +1 and -1. So now we have a measurable set E where $g = 1$ on E and $g = -1$ on $T \setminus E$. Let K be some positive number. Solving the Dirichlet problem with a real valued function u which is K on E and and 0 on $T \setminus E$ we take the bounded analytic function

$$f(z) = e^{-(u+iv)}$$

where v is the harmonic conjugate of u. Here the absolute value $|f| = e^{-K}$ on E and $|f| = 1$ on $T \setminus E$. To be precise, equality holds almost everywhere in the sense of Lebesgue. Next, let $\epsilon > 0$. Let $\mathbf{C}^e$ denote the extended complex plane. By the general result from XX there exists for a suitable K a conformal map Φ from the annulus $1 < |z| < e$ to the doubly connected domain

$$\Omega = \mathbf{C}^e \setminus [-\epsilon, 0] \cup [\ell, \ell *]$$

where ℓ can be made large and $\ell^* \leq \ell + \epsilon$. Here $\Phi(z_0) = \infty$ for some $e^{-K} < |z_0| < 1$. Now we put

$$\Psi = \frac{i + \Phi \circ f}{i - \Phi \circ f}$$

This yields a meromorphic in D with poles at the points $a \in D$ for which $f(a) = z_0$. On T the composed function $\Phi \circ f$ takes values in the union of the real intervals $[-\epsilon, 0]$ and $[\ell, \ell^*]$ which implies that $|\Psi| = 1$ holds almost everywhere on T. With ϵ small we see that $\Psi \simeq 1$ holds on E and when ℓ is large we have $\Psi \simeq -1$ on $T \setminus E$. So the Ψ-function approximates the given unimodular function uniformly up to a small number of order ϵ. Now Ψ may have some poles and w take the Blaschke product B for this so that $B \cdot \Psi$ is analytic in D and since $|B| = 1$ holds almost everywhere this yields an inner function denoted by ψ. Now

$$\Psi = \frac{\psi}{B}$$

is a quotient of inner functions which gives the requested approximation.

6. Proof of Marshall's theorem

Consider some $f \in H^\infty(T)$ of norm 1. Theorem 0.3 gives a convex combination of quotients of inner functions such that:

$$\big| f - \sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{w_1^\nu}{w_2^\nu} \big|_\infty < \epsilon$$

Set

$$\mathcal{J} = \prod w_\nu^2$$

This is an inner function and we also get the inner functions

$$W_\nu = \frac{w_1^\nu}{w_2^\nu} \cdot \mathcal{J} \implies \big| f - \sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{W_\nu}{\mathcal{J}} \big|_\infty < \epsilon \tag{1}$$

Theorem 3.3 applies to the pair f and $g = \sum a_\nu W_\nu$ and gives an inner function W_* such that

$$\sum_{\nu=1}^{\nu=k} a_\nu \cdot \frac{W_\nu}{\mathcal{J}} + \epsilon \cdot \frac{W_*}{\mathcal{J}} \in H^\infty(T) \tag{2}$$

Let g be this analytic function. From (1) and the triangle inequality we have

$$|f - g|_\infty < 2 \cdot \epsilon \tag{3}$$

Finally, with $N = k + 1$ we can apply Marshall's lemma to the function $\frac{1}{1+\epsilon} \cdot g$ which is expressed by a convex combination (*) from (4). This gives a convex sum V formed by inner functions such that

$$|V - g|_\infty < 2 \cdot \epsilon \tag{4}$$

Since $\epsilon > 0$ is arbitrary we get Theorem 0.1 via (3) and (4) above.

6. Interpolation and solutions to the $\bar{\partial}$-equation.

Contents.

I. Carleson's Interpolation Theorem

Introduction. Let $U = \mathfrak{Im}(z) > 0$ be the upper half-plane. Denote by $\mathbf{c}_*$ the family of sequences of complex numbers $\{c_\nu\}$ where every $|c_\nu| \leq 1$. A sequence $z_\bullet = \{z_\nu\}$ in U has a finite interpolation norm if there exists a constant K such that for every sequence $\{c_\nu\}$ in $\mathbf{c}_*$ we can find an analytic function $f(z)$ in U where

$$(*) \qquad f(z_\nu) = c_\nu \quad : \quad \nu = 1, 2, \ldots \quad \text{and} \quad |f|_U \leq K$$

The least constant K above is denoted by $\mathfrak{int}(z_\bullet)$ and called the interpolation norm of $\{z_\nu\}$.

0.1 Theorem. *A sequence $z_\bullet$ is interpolating if*

$$\min_\nu \prod_{k \neq \nu} \left|\frac{z_\nu - z_k}{z_\nu - \bar{z}_k}\right| > 0$$

Moreover, if $\delta(z_\bullet)$ denotes the minimum above then

$$(2) \qquad \mathfrak{int}(z_\bullet) \leq \frac{4A}{\delta(z_\bullet)} \cdot \text{Log}\, \frac{1}{\delta(z_\bullet)}\,.$$

where A is an absolute constant.

Remark. Theorem 0.1 gives a sufficient condition in order that a sequence is interpolating. That the condition (1) also is *necessary* is easily verified. See Exercise XX below. In [Ca] the proof is carried out in the unit disc D where the companion to Theorem 0.1 is that a sequence $\{z_\nu\}$ is interpolating if and only

$$(3) \qquad \min_\nu \prod_{k \neq \nu} \frac{|z_\nu - z_k|}{|1 - \bar{z}_k \cdot z_\nu|} > 0$$

for every ν. After a conformal map one verifies that (1) and (3) give equivalent conditions. Here we prove Carleson's result in the upper half-plane since various constructions below become a bit more transparent as compared to the unit disc.

Let $\{z_\nu\}$ be a sequence with a positive δ-number. Since a set of analytic functions in U with a uniform upper bound for the maximum norm is a normal family it is sufficient to prove the requested interpolation by bounded functions for every finite subsequence. The Nevanlinna-Pick theorem assigns to each finite sequence $\{z_\nu\}$ and every sequence $\{c_\nu\}$ a unique interpolating analytic function $F(z)$ with smallest maximum norm. So Carleson's result gives a uniform bound in the Nevannlinna-Pick interpolation.

0.1 Carleson measures.

The main ingredient in the proof is to consider a certain class of non-negative measures in the upper half-plane $\mathfrak{Im}\, z > 0$. For every $h > 0$ we denote by $\mathfrak{square}(h)$ the family of squares of the form

$$\square = \{(x, y) \quad : \quad x_0 - h/2 < x < x_0 + h/2 \quad : 0 < y < h\} \quad : x_0 \in \mathbf{R}$$

0.2. Definition. *A non-negative measure μ in U is called a Carleson measure if there exists a constant Ksuch that*

$$\mu(\square) \leq K \cdot h \quad : \quad \square \in \mathfrak{square}(h) \quad : \ 0 < h < \infty$$

The least constant K is denoted by $\mathfrak{car}(\mu)$ and called the Carleson norm of μ.

An essential step during the proof Theorem 0.1 is the following inequality:

0.3 Theorem. *Let $\{z_\nu\}$ be a sequence with a positive δ-number. Then*

$$\mathfrak{car}\Big(\sum_{\nu=1}^{\nu=\infty} \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu} \Big) \leq 2 \cdot \text{Log}\, \frac{1}{\delta)z_\bullet)}$$

where $\{\delta_{z_\nu}\}$ denote Dirac measures.

Use of duality. Once Theorem 0.3 is established the interpolation theorem follows by a duality argument where the Hardy space $H^1(\mathbf{R})$ appears. Namely, to each $h \in H^1(\mathbf{R})$ we associate the maximal function h^* and the following inequality is proved in Section 2:

0.4 Theorem. *Let μ be a Carleson measure in the upper half-plane. Then*

$$\int_U |h(z)| \cdot d\mu(z) \leq \mathfrak{car}(\mu) \cdot ||h^*||_1 \quad : \ h \in H^1(\mathbf{R})$$

Armed with Theorem 0.3 and 0.4 the interpolation theorem is derived in section 3 below.

1. Proof of Theorem 0.3

First we establish an inequality which is attributed to L. Hörmander.

1.1 Lemma *Let $z_1, \ldots, z_N$ be a finite sequence in U and put $\delta = \delta(z_\bullet)$. Then*

$$\text{(i)} \qquad \sum_{\nu \neq k} \mathfrak{Im}(z_k) \cdot \frac{\mathfrak{Im}(z_\nu)}{|z_k - \bar{z}_\nu|^2} \leq \frac{1}{2} \cdot \text{Log}\, \frac{1}{\delta} \quad : \quad 1 \leq k \leq N$$

Proof. The left hand side as well as the δ-norm of the z-sequence are unchanged if we translate all points to $z_\nu + a$ where a is a real number. Similarly, the δ-norm and the left hand side in (i) are both unchanged when the sequence is replaced by $\{A \cdot z_\nu\}$ for some $A > 0$. To prove (i) for a fixed k which we may take $k = N$ and it suffices to consider the case when $z_N = i$. Put $z_\nu = a_\nu + ib_\nu$ when $1 \leq \nu \leq N - 1$. Then we must show

$$\text{(i)} \qquad \sum_{\nu=1}^{\nu=N-1} \frac{b_\nu}{(1 + b_\nu)^2 + a_\nu^2} \leq \frac{1}{2} \cdot \text{Log}\, \frac{1}{\delta}$$

Notice that

$$\text{(iii)} \qquad \frac{|i - \bar{z}_\nu|^2}{|i - z_\nu|^2} = \frac{(1 + b_\nu)^2 + a_\nu^2}{(1 - b_\nu)^2 + a_\nu^2}$$

Next, by inverting the δ-we have:

$$\text{(iii)} \qquad \prod_{\nu=1}^{\nu=N-1} \frac{(1 + b_\nu)^2 + a_\nu^2}{(1 - b_\nu)^2 + a_\nu^2} \leq \delta^{-2}$$

Passing to the Log-functions it follows that

(iv) $$\sum_{\nu=1}^{\nu=N-1} \log\Big[\,\frac{(1+b_\nu)^2+a_\nu^2}{(1-b_\nu)^2+a_\nu^2}\,\Big] \leq 2\cdot\log\,\frac{1}{\delta}$$

Next, for each ν we have the integral formula

$$\log\,\frac{(1+b_\nu)^2+a_\nu^2}{(1-b_\nu)^2+a_\nu^2} = \int_{-b_\nu}^{b_\nu} \frac{2(1+s)\cdot ds}{(1+s)^2+a_\nu^2}$$

Apply this with (a_ν, b_ν) and after a summation over ν the inequality (iv) gives (i) in Lemma 1.1.

Final part of the proof of Theorem 0.3

If $z_\bullet \in \mathcal{S}(\delta)$ and a is any real number then the translated sequence $z_\bullet + a = \{z_\nu + a\}$ also belongs to $\mathcal{S}(\delta)$. Since Theorem 0.3 asserts an a priori estimate we may assume that $\square$ is centered at $x = 0$, i.e.

$$\square = \{(x,y):\ -h/2 < x < h/2 \text{ and } 0 < y < h\}$$

There remains to estimate

(i) $$\sum_{z_\nu\in\square} \mathfrak{Im}\, z_\nu$$

Set

$$y^* = \max\,\{\mathfrak{Im}(z_\nu):\ z_\nu \in \square\}$$

Let k give the equality $y^* = \mathfrak{Im}(z_k)$. With $z_k = x_k + iy^*$ and $z_\nu = x_\nu + iy_\nu \in \square$ we have

$$|z_k - \bar{z}_\nu|^2 = (x_k - x_\nu)^2 + (y^* - y_\nu)^2 \leq h^2 + (y^*)^2 \implies$$

$$\frac{\mathfrak{Im}(z_k)}{|z_k - \bar{z}_\nu|^2} \geq \frac{y^*}{h^2 + (y^*)^2} \quad :\ \nu \neq k$$

Next, notice that

$$y^* \geq h/2 \implies \frac{y^*}{h^2+(y^*)^2} \geq \frac{1}{5h}\,.$$

Lemma 1.1. applied with $\nu = k$ gives therefore

(ii) $$\sum_{z_\nu\in\square} \mathfrak{Im}(z_\nu) \leq y^* + \frac{5h}{2}\cdot\text{Log}\,\frac{1}{\delta} \leq h\cdot\big(1 + \frac{5}{2}\cdot\text{Log}\,\frac{1}{\delta}\big)$$

So if $y^* \geq h/2$ we are done. Suppose now that $y^* < h/2$ and regard the cubes:

$$\square_1 = \{-h/2 < x < 0 \text{ and } 0 < y < h/2\} \quad \square_2 = \{0 < x < h/2 \text{ and } 0 < y < h/2\}$$

We want to estimate

$$S_1 + S_2 = \sum_{z_\nu\in\square_1} \mathfrak{Im}(z_\nu) + \sum_{z_\nu\in\square_2} \mathfrak{Im}(z_\nu)$$

We have also two sequences:

$$\{z_\nu:\ z_\nu \in \square_1\} \quad \text{and} \quad \{z_\nu:\ z_\nu \in \square_2\}$$

Since all factors defining the δ-norm are ≤ 1 these two smaller sequences both belong to $\mathcal{S}(\delta)$. Suppose that:

$$y_1^* = \max_{z_\nu\in\square_1}\ \mathfrak{Im}(z_\nu) \geq \frac{h}{4}$$

When this holds we obtain exactly as above:

$$S_1 \leq \frac{h}{2}\cdot 2\cdot\text{Log}\,\frac{1}{\delta}$$

If $y_1^* < \frac{h}{4}$ we continue to split the cube $\square_1$. In a similar fashion we treat the sequence which stays in $\square_2$. After a finite number of steps we get the required inequality in Theorem 0.3.

2. Proof of Theorem 0.4

Let $h \in H^1(\mathbf{R})$ and recall that its maximal function is defined by

(i) $$h^*(t) = \max |h(x+iy)| \quad : \quad |x-t| < y$$

To each $\lambda > 0$ we consider the open subset on the real line defined by $\{h^* > \lambda\}$. It is some union of disjoint intervals $\{(a_j, b_j)\}$ and (i) gives the set-theoretic inclusion:

(ii) $$\{|h(x+iy)| > \lambda\} \subset \cup\, T_j \quad :$$

where T_j is the triangle side standing on the interval (a_j, b_j) as explained in XXX. (Hardy space). In particular we have the inclusion:

(iii) $$T_j \subset \square(a_j, b_j) = \{x+iy : |x - \frac{a_j + b_j}{2}| < b_j - a_j \ : \ 0 < y < b_j - a_j\}$$

See figure XXX. So if μ is a positive measure in U we obtain:

(iv) $$\mu(\{|h| > \lambda\}) \leq \sum \mu(T_j) \leq \sum \mu(\square(a_j, b_j))$$

If μ is a Carleson measure the right hand side is estimated by

(v) $$\mathfrak{car}(\mu) \cdot \sum (b_j - a_j) = \mathfrak{car}(\mu) \cdot \mathfrak{m}(\{h^* > \lambda\})$$

where $\mathfrak{m}$ refers to the 1-dimensional Lebesgue measure. Here (v) holds for every $\lambda > 0$. So by the general inequality for distribution functions from XXX we get:

$$\int_U |h| \cdot d\mu \leq \mathfrak{car}(\mu) \cdot ||h^*||_1$$

This finishes the proof of Theorem 0.4.

3. Proof of Theorem 0.1.

As explained in XX the Banach space $H^1(\mathbf{R})$ contains a dense subspace of functions $h(z)$ with polynomial decay at infinite, i.e. functions in the Hardy space for which

$$|h(z)| \leq C_N \cdot (1 + |z|)^{-N} \quad : \text{hold for some constant } C_N \quad : \ N = 1, 2, \ldots$$

This is used below to ensure that various integrals are defined where it suffices to to use "nice" functions while an *a priori* inequality is established. Consider a finite sequence $z_1, \ldots, z_N$ in U and a finite sequence $c_1, \ldots, c_N$ in $\mathbf{c}_*$. Newton's interpolation gives a unique polynomial $P(z)$ of degree $N-1$ such that:

(i) $$P(z_k) = c_k \quad : \quad 1 \leq k \leq N$$

Let $B(z)$ be the Blascke product associated to the z-sequence:

(ii) $$B(z) = \prod_{\nu=1}^{\nu=N} \frac{z - z_\nu}{z - \bar{z}_\nu}$$

Let $h \in H^1(\mathbf{R})$ have the polynomial decay $\geq N+2$. Residue calculus gives

(iii) $$\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx = \sum_{k=1}^{k=N} \frac{c_k}{B'(z_k)} \cdot h(z_k)$$

If k is fixed we have

(iv) $$\frac{1}{B'(z_k)} = \prod_{\nu \neq k} \frac{z_k - \bar{z}_\nu}{z_k - z_\nu} \cdot 2 \cdot \mathfrak{Im}(z_k)$$

It follows that

(v) $$|\frac{1}{B'(z_k)}| \leq \frac{2}{\delta(z_\bullet)} \cdot \mathfrak{Im}(z_k)$$

Since $\{c_\nu\} \in \mathbf{c}_*$ we see that (v) and the triangle inequality applied to (iii) give:

$$(*) \qquad \Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx \,\Big| \leq \frac{2}{\delta(z_\bullet)} \cdot \sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k)$$

Now Theorem 0.4 gives the inequality

$$(5) \qquad \sum_{k=1}^{k=N} |h(z_k)| \cdot \mathfrak{Im}(z_k) \leq \mathfrak{car}(\sum \mathfrak{Im}(z_\nu) \cdot \delta_{z_\nu} \cdot ||h^*||_1$$

Use of duality. Let us put

$$C_\delta = \frac{2}{\delta(z_\bullet)} \cdot 2 \cdot \log \frac{1}{\delta(z_\bullet)}$$

Then (5) and Theorem 0.3 Let C_δ be the constant from Theorem 0.3 together with (*) give

$$\Big|\int_{-\infty}^{\infty} \frac{P(x)}{B(x} \cdot h(x) \cdot dx \,\Big| \leq C \cdot ||h^*||_1$$

Next, the result in (Hardy Chapter) gives an absolute constant A such that

$$||h^*||_1 \leq A \cdot ||h||_1$$

Hence the densely defined linear functional

$$h \mapsto \int_{-\infty}^{\infty} \frac{P(x)}{B(x)} \cdot h(x) \cdot dx$$

has norm $\leq C \cdot A$. The Duality Theorem from XXX implies that if $\epsilon > 0$, then there exists some $G(z) \in \mathcal{O}(U)$ such that the maximum norm

$$(6) \qquad \Big|\frac{P(x)}{B(x)} - G(x)\Big|_U < A \cdot C_\delta + \epsilon$$

Since $B(x)$ is a Blaschke product we have $|B(x)| = 1$ almost everywhere and hence (6) gives:

$$\big|P(x) - B(x) \cdot G(x)\,\big| < A \cdot C_\delta + \epsilon$$

Now $f(z) = P(z) - B(z)G(z)$ is analytic in U and since $B(z_\nu) = 0$ for every ν we have

$$f(z_\nu) = P(z_\nu) = c_\nu$$

So the bounded analytic function $f(z)$ interpolates and since $\epsilon > 0$ can be arbitrary small and $c_1, \ldots, c_N$ was an arbitrary sequence in $\mathbf{c}_*$ we conclude that the interpolation norm of the finite sequence $z_1, \ldots, z_N$ is at most $A \cdot C_\delta$. Since this uniform bound holds for all N we get

$$\mathfrak{int}(z_\bullet) \leq A \cdot C_\delta$$

which finishes the proof of Theorem 0.1.

Exercise. Prove that (1) in the Interpolation Theorem is necessary.

II. Wolff's Theorem.

Introduction. The Pompieu formula solves the inhomogeneous $\bar\partial$-equation in the unit disc D. So if $h(z)$ is a C^∞-function defined in some open neighborhood of the closed disc there exists a C^∞-function v such that

(*) $$\bar\partial(v)(z)=h(z)\quad:\quad z\in D$$

We seek conditions in order that (*) has a solution v whose maximum norm over D is controlled by some extra properties of h. Conditions of this kind were imposed in [Wolff] which we begin to explain. To every C^∞-function h on $\bar D$ we define a pair of non-negative functions:

(**) $$\mu_h(z)=\text{Log}\,\frac{1}{|z|}\cdot|\partial(h)(z)|\quad:\quad \nu_h(z)=\text{Log}\,\frac{1}{|z|}\cdot|h(z)|^2$$

Wolff's condition is expressed in terms of Carleson norms on μ_h and ν_h. Before we announce Theorem 0.4 we recall the following.

0.1 Carleson measures in D. Consider the family of sector domains defined for all pairs $0<h<1$ and $0\leq\theta\leq 2\pi$ by:

$$S_h(\theta)=\{\,z=r\cdot e^{i\phi}\,:1-h<r<1:|e^{i\phi}-e^{i\theta}|\leq\frac{h}{2}\,\}$$

0.2 Definition. *A non-negative measure μ in D is called a Carleson measure if there exists a constant K such that*

$$\iint_{S_h(\theta)}d\mu\leq K\cdot h\quad:\quad 0<h>1\quad:\quad 0\leq\theta<2\pi$$

The least constant K is denoted by $\mathfrak{car}(\mu)$ and called the Carleson norm of μ.

0.3 An inequality. Exactly as in the upper half-plane there exists an absolute constant A such that the following holds for every pair of a Carleson measure μ in D and a function $f(z)$ in the Hardy space $H^1(T)$

$$\iint_D\big|f(z)\big|\cdot d\mu(z)\leq A\cdot\mathfrak{car}(\mu)\cdot||f||_1$$

0.4 Theorem. *Let A be as in (0.3). For every C^∞-function $h\in C^\infty(\bar D)$ the equation (*) has a C^∞-solution v_* where*

$$\max_\theta|v_*(e^{i\theta})|\leq 2\cdot A\cdot\mathfrak{car}(\mu_h)+2\cdot\sqrt{A\cdot\mathfrak{car}(\nu_h)}$$

For the proof we need an integral formula due to Jensen.

0.5 The Fourier-Jensen formula. *Let $F(z)$ be an analytic function in D with a simple zero at $z=0$ and otherwise it is $\neq 0$. Then one has the equality:*

(*) $$\iint_D\text{Log}\,\frac{1}{|z|}\cdot\frac{|F'(z)|^2}{|F(z)|}\cdot dxdy=\int_0^{2\pi}|F(e^{i\theta})|\cdot d\theta$$

To prove (*) we set $F(z)=z\cdot G(z)$ where the hypothesis means that G is zero-free so we can construct a square root function and write

(i) $$F(z)=z\cdot\Psi^2(z)\quad:\quad\Psi\in\mathcal{O}(D)$$

This implies that

$$\frac{|F'(z)|^2}{|F(z)|}=\frac{|\Psi(z)+2z\cdot\Psi'(z)|^2}{|z|}$$

Hence the left hand side in (*) becomes:

(ii) $$\iint_D\log\,\frac{1}{|z|}\cdot\big|\Psi(z)+2z\cdot\Psi'(z)\big|^2\cdot\frac{1}{|z|}\cdot dxdy$$

To evaluate this integral we consider the series expansion $\Psi(z) = \sum a_n z^n$. In polar coordinates the double integral becomes

$$\text{(iii)} \qquad \int_0^1 \int_0^{2\pi} \log \frac{1}{r} \cdot \left|\sum (2n+1) \cdot a_n \cdot r^n \cdot e^{in\theta}\right|^2 \cdot dr d\theta$$

Exercise. Show that (iii) is equal to

$$2\pi \cdot \sum |a_n|^2 = \int_0^{2\pi} |\Psi(e^{i\theta})|^2 \cdot d\theta = \int_0^{2\pi} |F(e^{i\theta})| \cdot d\theta$$

which gives the requested equality (*).

1. Proof of Theorem 0.4

The Pompieu formula gives a solution v to the $\bar{\partial}$-equation

$$\text{(i)} \qquad \bar{\partial}(v) = h$$

We get new solutions to (i) by $v_* = v - G$ when $G(z)$ are analytic functions in D. So in order to minimize the maximum norm of a solution to (i) we seek a bounded analytic function G_* such that

$$\text{(ii)} \qquad |v - G_*|_D = \min_G \; |v - G|_D \quad : \; G \in H^\infty(T)$$

Let m_* be the minimum value in (ii). To estimate m_* we use the duality between $H^\infty(T)$ and $H_0^1(T)$ where $H_0^1(T)$ is the space of functions $F(z)$ in the Hardy space $H^1(T)$ for which $F(0) = 0$. Denote by $S_*^(T)$ the set of functions $F \in H_0^1(T)$ such that

$$\text{(1)} \qquad \int_0^{2\pi} \left|F(e^{i\theta})\right| \cdot d\theta = 1$$

The duality result from (XX) gives:

$$\text{(2)} \qquad m_* = \max_F \left|\int_0^{2\pi} v(e^{i\theta}) \cdot F(e^{i\theta}) \cdot d\theta\right| \quad : \; F \in S_*(T)$$

Since $F(0) = 0$ Green's formula shows that (2) becomes

$$\text{(3)} \qquad \iint_D \operatorname{Log} \frac{1}{|z|} \cdot \Delta(vF) \cdot dxdy$$

Since $\Delta = \partial\bar{\partial}$ and v solves (i) while $\bar{\partial}(F) = 0$, we get

$$\text{(4)} \qquad \Delta(vF) = 4 \cdot \partial(hF) = 4 \cdot F \cdot \partial(h) + 4 \cdot h \cdot F'$$

Hence we have proved the following

1. Lemma. *One has the equality*

$$m_* = \max_F \left|\iint_D \operatorname{Log} \frac{1}{|z|} \cdot \left[F \cdot \partial(h) + h \cdot F'\right] \cdot dxdy\right| \quad : \quad F \in S_*^1(T)$$

To profit upon this expression for m_* we use the Jensen-Nevanlinna factorisation and reduce the estimate to the case when $F(z)$ has a simple zero at $z = 0$ while it is $\neq 0$ in the punctured disc $D \setminus \{0\}$. Thus, consider some F in $S_*(T)$. Since $F(0) = 0$ we there exists the Jensen-Nevanlinna factorisation:

$$\text{(i)} \qquad F(z) = z \cdot B(z) \cdot G(z)$$

where $B(z)$ is a Blaschke product and G has no zeros in D. Moreover, since $|B| = 1$ holds almost everywhere on T it follows that G belong to $S_*(T)$. Set:

(ii) $$F_1(z) = \frac{z}{2}\big(B(z)-1\big)G(z) \quad \text{and} \quad F_2(z) = \frac{z}{2}\big(B(z)+1\big)G(z)$$

It follows that $F = F_1 + F_2$ and since the maximum norms of $B(z)-1$ and $B(z)+1$ are at most 2 we have

(iii) $$||F_\nu||_1 \leq 1 \quad : \quad \nu = 1,2$$

From (ii) we see that F_1 and F_2 both have a simple zero at the origin and are otherwise $\neq 0$ in the punctured disc. Hence we can apply the Fourier-Jensen formula from (0.4) which gives

(iv) $$[\iint_D \text{Log}\,\frac{1}{|z|}\cdot\frac{|F'_\nu(z)|^2}{|F_\nu(z)|}\cdot dxdy = \int_0^{2\pi} |F_\nu(e^{i\theta})|\cdot d\theta \leq 1 \quad : \nu = 1,2$$

Final part of the proof. For each $\nu = 1,2$ we set

(1) $$V(F_\nu) = \iint_D \text{Log}\,\frac{1}{|z|}\cdot|\partial(h)|\cdot|F_1(z)|dxdy + \iint_D \text{Log}\,\frac{1}{|z|}\cdot|h(z)|\cdot|F'_1(z)|\cdot dxdy$$

By the triangle inequality the right hand side in Lemma 1 is $\leq V(F_1)+V(F_2)$. Let us for example estimate $V(F_1)$. By the inequality (0.3) the first integral in (1) is estimated by

(2) $$A\cdot\mathfrak{car}(\text{Log}\,\frac{1}{|z|}\cdot|\partial(h)|)\cdot||F_1||_1$$

Since $||F_1||_1 \leq 1$ the definition of μ_h means that (2) is majorised by

(*) $$A\cdot\mathfrak{car}(\mu_h)$$

To estimate the second integral in (1) we insert $\sqrt{|F_1|}$ as a factor and by the Cauchy-Schwartz inequality this second integral is estimated by the square root of

(3) $$[\iint_D \text{Log}\,\frac{1}{|z|}\cdot\frac{|F'_1(z)|^2}{|F_1(z)|}\cdot dxdy]\cdot[\iint_D \text{Log}\,\frac{1}{|z|}\cdot\,|h(z)|^2\cdot|F_1(z)|\cdot dxdy$$

In this product the first factor is given by the formula (iv) and is therefore $\leq ||F_1||_1 \leq 1$. Finally, by the definition of the Caelson norm the last factor is majorised by $A\cdot\mathfrak{car}(\nu_h)$. Taking the square root together with (*) above we have proved that

(4) $$V(F_1) \leq +A\cdot\mathfrak{car}(\mu_h) + \sqrt{A\cdot\mathfrak{car}(\nu_h)}$$

The same holds for F_2 and thanks to the factor 2 the requested inequality in Wolff's theorem follows.

III. A class of Carleson measures

Let $f(z)$ be a bounded analytic function in D and associate the non-negative measure in D by:

$$\mu_f = |f'(z)|^2\cdot\text{Log}\,\frac{1}{|z|}$$

3.1 Theorem. *There exists an absolute constant A_* such that*

$$\sqrt{\mathfrak{car}(\mu_f)} \leq A_*\cdot|f|_D \quad : \quad f \in H^\infty(D)$$

Proof. By the Heine-Borel Lemma it suffices to prove this for small sectors. Notice also that

$$\text{Log}\,\frac{1}{[z[} \simeq |1-z|$$

when z approaches the unit circle. By a conformal mapping the proof is therefore reduced to the case when we have a bounded analytic function $f(z)$ defined in a square

$$\square = \{z = x + iy \quad : \quad -1 < x < 1 \quad : 0 < y < 1\}$$

where it suffices to get an absolute constant such that

(i) $$\frac{1}{h}\int_{\square_h} y \cdot |f'(x+iy)|^2 \cdot dxdy \leq A \cdot |f|^2_{\square} \quad : 0 < h < \frac{1}{2}$$

Set $f = u + iv$ which gives $|f'(|^2 = u_x^2 + u_y^2$. The left hand side in (i) becomes:

(ii) $$\frac{1}{h}\int_{\square_h} y \cdot (u_x^2 + u_y^2) \cdot dxdy$$

It remains to find an absolute constant A such that (ii) is majorised by $A \cdot |u|^2_{\square}$. To achive this we replace $\square_h$ by the larger semi-disc

$$D_h = \{z = x + iy \quad : |z| < h \quad : y > 0\}$$

which only with increase the left hand side in (ii). Next, since $4h^2 - |z|^2 \geq 3h^2$ when $z \in D_h$ we get a larger contribution by integrating over the larger semi-disc D_{2h}. Hence it suffices to get an absolute constant A such that

(*) $$J(h) = \int_{D_{2h}} y(4h^2 - |z|^2) \cdot (u_x^2 + u_y^2) dxdy \leq A \cdot h^3 \cdot |u|^2_{\square}$$

To get A in (*) we use the equality

$$\Delta(u^2) = 2(u_x^2 + u_y^2)$$

Next, the function $g(x, y) = y(4h^2 - |z|^2)$ is zero on the boundary of D_{2h} and Green's formula gives

$$2 \cdot J(h) = \int_{D_{2h}} u^2 \cdot \Delta(y(4h^2 - |z|^2) \cdot dxdy - \int_{\partial D_{2h}} u^2 \cdot \partial_{\mathbf{n}}(y(4h^2 - |z|^2) \cdot ds$$

Notice that $\Delta(y(4h^2 - |z|^2) = -8y < 0$ in D_{2h} and an easy computation gives

$$-\int_{\partial D_{2h}} u^2 \cdot \partial_{\mathbf{n}}(y(4h^2 - |z|^2) \cdot ds =$$

$$\int_{-2h}^{2h} u^2(x, 0) \cdot (4h^2 - x^2) \cdot dx + \int_0^{\pi} u^2(2he^{i\theta}) \cdot \sin\theta \cdot \big[-4h^2 + 3 \cdot (2h)^2\big] \cdot h \cdot d\theta$$

Introducing the maximum norm $|u|_{\square}$ we conclude that

$$2 \cdot J(h) \leq |u|^2_{\square} \cdot \big[\int_{-2h}^{2h} (4h^2 - x^2) \cdot dx + \int_0^{\pi} \sin\theta \cdot \big[-4h^2 + 3 \cdot (2h)^2\big] \cdot h \cdot d\theta\big]$$

At this stage the reader can evaluate the requested constant A which estimates the last factor by $2A \cdot h^3$.

IV. Berndtsson's $\bar\partial$-solution

We announce an inequality due to Bo Berndtsson in [Bern] which has the merit that it is valid for a quite extensive family of domains in $\mathbf{C}$. Here is the set-up : Let $\mathcal{B}$ denote the family of bounded open sets Ω defined by

$$\Omega = \{\rho(z) < 0\}$$

where $\rho(z)$ is a real-valued C^2-function defined in some neighborhood of $\bar\Omega$ which satisfies

$$\Delta(\rho)(z) > 0 \; : \; z \in \Omega \quad : \quad \nabla(\rho)(z) \neq 0 \; z \in \partial\Omega$$

Next, let $\Omega \in \mathcal{B}$ be given together with a bounded and subharmonic function $\phi(z)$ in Ω. Denote by $\mathfrak{Bernt}(\Omega,\phi)$ the family of C^∞-functions $f(z)$ in Ω which satisfy:

$$(*) \qquad |f(z)| \leq -\rho(z)\cdot\Delta\,\phi(z) \quad : \quad z \in \Omega$$

4.1 Theorem. *To each $f \in \mathfrak{Bernt}(\Omega,\phi)$ the inhomogeneous equation*

$$(*) \qquad \bar\partial(u) = f$$

has a solution $u(z)$ which satisfies

$$(**) \qquad \max_{z\in\partial\Omega} \frac{\big|u(z)|\cdot e^{-\phi(z)/2}\big|}{|\nabla\rho(z)\big|} \leq \max_{z\in\Omega} \frac{\big|f(z)|\cdot e^{-\phi(z)/2}\big|}{|-\rho(z)\cdot\Delta\,\phi(z)+\nabla\rho(z)\big|}$$

Remark. A special case occurs when $\Omega = D$ and $\rho(z) = |z|^2 - 1$. Then (*) means that a function f in the Berndtsson class satisfies

$$|f(z)| \leq 2(1-|z|)\cdot\Delta(\phi)(z)$$

So here $|f|$ decays as $|z| \to 1$ and when $\Delta(\phi)$ is bounded this inequality estimates the Carleson norm of f. So in this situation Theorem 4.1 resembles Wolf's theorem. The solution u in Theorem 4.1 is found by solving an extremal problem in a Hilbert space. Namely, given the ϕ-function, Berndtsson considered the Hilbert space of functions in D which are square integrable with respect to $e^{-\phi}$, i.e. functions g for which

$$(1) \qquad \iint_D |g(z)|^2\cdot e^{-\phi(z)}\cdot dxdy < \infty$$

Now there exists the unique extremal solution u to the equation $\bar\partial(u) = f$ whose norm in $L^2(e^{-\phi})$ is minimal among all functions ψ in D satisfying $\bar\partial\psi = f$. In [Berndtsson] it is proved that his extremal solution u satisfies the inequality (**) in Theorem 4.1.

V. Hörmander's L^2-estimate

Let Ω be an open set in $\mathbf{C}$. If ϕ is a real-valued continuous and non-negative function we get the Hilbert space $\mathcal{H}_\phi$ whose elements are Lebesgue measurable functions f in Ω such that

$$(*) \qquad \int_\Omega |f|^2\cdot e^{-\phi}\cdot dxdy < \infty$$

The square root of (*) yields norm and is denoted by $||f||_{2,\phi}$. Let ψ be another real-valued continuous and non-negative function which gives the Hilbert space $\mathcal{H}_\psi$ where the norm of an element g is denoted by $||g||_{2,\psi}$. We are interested in the inhomogenous $\bar\partial$-equation, i.e. given $w \in \mathcal{H}_\psi$ we seek $f \in \mathcal{H}_\phi$ such that $\bar\partial(f) = w$. In addition we want to solve this equation with a bound for the L^2-norms. To attain this we impose

5.1 Hörmander's condition. The pair ϕ,ψ satisfies the Hörmander condition if there exists some positive constant c_0 such that the following pointwise inequality holds in Ω:

$$(*) \qquad \Delta(\psi) - 2\cdot|\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y \geq 2\cdot c_0^2\cdot e^{\psi(z)-\phi(z)}$$

where we have put $|\nabla(\psi]|^2 = \psi_x^2 + \psi_y^2$.

5.2 Theorem. *If (ϕ,ψ) satisfies (*) for some $c_0 > 0$ then the equation $\bar\partial(f) = w$ has a solution for every $w \in \mathcal{H}_\psi$ where*

$$||f||_\phi \leq \frac{1}{c_0}\cdot||w||_\psi$$

Proof. Since C_0^∞ is a dense subspace of $\mathcal{H}_\phi$ the linear operator T from $\mathcal{H}_\phi$ to $\mathcal{H}_\psi$ given by $T(f) = \bar\partial(f)$ is densely defined. Let w be in the domain of definition for the adjoint operator T^*. If $f \in C_0^\infty(\Omega)$ we get

$$(*) \qquad \langle T(f), w\rangle = \int \bar\partial(f) \cdot \bar w \cdot e^{-\psi} dxdy = -\int f \cdot \big[\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)\big] \cdot e^{-\psi} dxdy$$

Since ψ is real-valued it follows that $\bar\partial(\bar w) - \bar w \cdot \bar\partial(\psi)$ is the complex conjugate of $\partial(w) - w \cdot \partial(\psi)$ which gives

$$(**) \qquad T^*(w) = -\big[\partial(w) - w \cdot \partial(\psi)\big] \cdot e^{\phi-\psi}$$

Taking the squared L^2-norm we obtain

$$(1) \qquad ||T^*(w)||_\phi^2 = \int |\partial(w) - w \cdot \partial(\psi)|^2 \cdot e^{\phi-2\psi}$$

Expanding the integrand it follows that (1) is equal to

$$(2) \qquad \int \big[|\partial(w)|^2 + |w|^2 \cdot |\partial(\psi)|^2\big] \cdot e^{\phi-2\psi} - 2 \cdot \mathfrak{Re}\Big(\int \partial(w) \cdot \bar w \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}\Big)$$

In the last integral we perform a partial integration and conclude that the last term is the real part of

$$2 \cdot \int w \cdot [\partial(\bar w) \cdot \bar\partial(\psi) + \bar w \cdot \partial\bar\partial(\psi) - 2\bar w \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar w \cdot \bar\partial(\psi) \cdot \partial(\phi)] \cdot e^{\phi-2\psi}$$

Next, the Cauchy-Schwarz inequality shows that the absolute value of

$$2 \cdot \int w \cdot \partial(\bar w) \cdot \bar\partial(\psi) \cdot e^{\phi-2\psi}$$

is majorized by the left hand integral in (2). It follows that

$$(3) \qquad ||T^*(w)||_\phi^2 \geq 2 \cdot \mathfrak{Re} \int |w|^2 \cdot \big[\,\partial\bar\partial(\psi) - 2 \cdot \bar\partial(\psi) \cdot \partial(\psi) + \bar\partial(\psi) \cdot \partial(\phi)\,\big] \cdot e^{\phi-2\psi}$$

Since ϕ and ψ are real-valued the real part of the inner bracket above becomes

$$(4) \qquad \frac{1}{4}\big[\Delta(\psi) - 2 \cdot |\nabla(\psi)|^2 + \psi_x\phi_x + \psi_y\phi_y\,\big]$$

So when (4) majorizes $\frac{c_0^2}{2} \cdot e^{\psi-\phi}$ it follows that

$$(5) \qquad ||T^*(w)||_\phi^2 \geq c_0^2 \cdot \int |w|^2 \cdot e^{\psi-\phi} \cdot e^{\phi-2\psi} = c_0^2 \cdot ||w||_\psi^2$$

This lower bound implies that the norm of T is bounded by c_o and Theorem XX follows.

5.3 Remark. There exist different pairs (ϕ, ψ) for which Hörmander's condition (*) applies. and We refer to the article [Kiselman] by C. Kiselman for some specific applications of L^2-estimates in $\mathbf{C}$ applied to study carriers of Borel transforms. The full strength of L^2-estimate appears in dimension $n \geq 2$ where one works with *plurisubharmonic functions* and impose the condition that Ω is a strictly pesudo-convex set in $\mathbf{C}^n$ and solve inhomogeneous $\bar\partial$-equations for differential forms of bi-degree (p, q). As expected the proofs are more involved where various Hermitian forms appear. In addition to Hörmander's original article [Hörmander] we refer to his text-book [Hörmander] and Chapter XX in [Hömander XX-Vol 2] where certain bounds for $\bar\partial$-equations are established with certain relaxed assumptions which are used to settle the fundamental principle for over-determined systems of PDE-equations in the smooth case. Working on complex manifolds with various metric properties L^2-estimates is a powerful tool. For a wealth of results of this nature we refer to the notes by Demailly in [Dem].

VI. The Corona Theorem.

Introduction. In the unit disc D we have the Banach algebra $H^\infty(D)$ of bounded analytic functions. Let $\mathfrak{M}_\infty(D)$ denote its maximal ideal space. Via point evaluations in D we get a map

$$i\colon D \mapsto \mathfrak{M}_\infty(D)$$

The Corona problem asked if $i(D)$ is dense in $\mathfrak{M}_\infty(D)$. The affirmative answer to this question was found by Carleson. It is easily seen that the density of the i-image is equivalent to the following result which was proved in [Carleson]:

6.1 Theorem. *The ideal generated by a finite family $f_1,\ldots,f_n$ in $H^\infty(D)$ is equal to $H^\infty(D)$ if and only if there exists $\delta>0$ such that*

$$|f_1(z)|+\ldots+|f_n(z)|\geq\delta \quad \text{hold for all } \ z\in D$$

Remark. Just as in the proof of the Interpolation Theorem an essential ingredient of the proof relies upon Carleson measures. An alternative to Carleson's original proof was given by Wolff and relies upon his result for the inhomogeneous $\bar\partial$-equation from XX. The deduction from Wolff's Theorem in XX to the solution of the Corona problem is exposed a several places. Se for example Chapter XX in [Narasimhan] and also the article [Gamelin] by T.Gamelin. So here we refrain from giving further details. Let us only remark that one may consider related problems where the boundedness of the f-function is relaxed. For example, using L^2-estimates with weight functions one can solve a problem where $f_1,\ldots,f_n$ are analytic functions in D with moderate growth, i.e there is an integer m and a constant A such that

$$(1-|z|)^m\cdot|f_\nu(z)|\leq K$$

holds in D for every ν. Assume also that there is an integer m_* and $\delta>0$ such that

$$|f_1(z)|+\ldots+|f_n(z)|\geq\delta\cdot(1-|z|)^{m_*} \quad \text{hold for all } \ z\in D$$

Then one can show that there exists an n-tuple $g_1,\ldots g_n$ of analytic functions with moderate growth such that

$$g_1\dot{f}_1+\ldots+g_n\cdot f_n=1 \tag{1}$$

holds in D.

Question. Find the smallest possible number $\rho=\rho(m,m_*)$ such that (1) has a solution where the g-functions satisfy

$$|g_\nu(z(|\leq C\cdot(1-|z|)^{-\rho}$$

for some constant C. It appears that the best constant ρ is not known. However, upper bounds for ρ can be established using L^2-estimates for the $\bar\partial$-equation. But there remains to investigate how sharp such bounds are.

16. Entire functions of exponential type

Contents.

Introduction.

The class $\mathcal{E}$ of entire functions of exponential type is defined as follows:

0.1 Definition. *An entire function f belongs to $\mathcal{E}$ if and only if there exists constants A and C such that*

(*) $$|f(z)| \leq C \cdot e^{A|z|} \quad : \ z \in \mathbf{C}$$

We refer to the literature for studies of the more extensive class of entire functions with arbitrary finite order, i.e. those f where (*) is replaced by $|z|^\rho$ for some $\rho > 0$. The results in Sections A-B are foremost due to Hadamard and Lindelöf. The class $\mathcal{N}$ in § 3 was originally used by Carleman to prove certain approximation theorems related to moment problems. Our main concern deal with Tauberian theorems which are established in § D. Here the material is based upon Chapter V in [Paley-Wiener]. Let us describe some of the results to be proved in § D while we refer to sections A and B for elementary facts about $\mathcal{E}$. Consider a non-decreasing sequence $\{\lambda_\nu\}$ of positive real numbers such that the series

(1) $$\sum \lambda_\nu^{-2} < \infty$$

When this holds there exists the entire function given by a product series:

$$H(z) = \prod (1 - \frac{z^2}{\lambda_\nu^2})$$

Notice that the function $H(z)$ is even and positive on the imaginary axis. Consider the function defined for real y:

$$y \mapsto \frac{\log H(iy)}{y} = \frac{1}{y} \sum \log (1 + \frac{y^2}{\lambda \nu^2})$$

At the same time we have the intergals

$$J(R) = \int_{-R}^{R} \frac{\log |H(x)| \cdot dx}{x^2}$$

With these notations a major result in § D asserts the following:

0.1 Theorem. *The statements*

(i) $$\lim_{y \to \infty} \frac{\log H(iy)}{y} = \pi A y$$

and

(ii) $$\lim_{R \to \infty} J(R) = -\pi^2 A$$

hold for some $A \geq 0$ are completely equivalent

Example. Consider the case when $\{\lambda_\nu\}$ is the set of positive integers. Here one has the formula

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$$

Notice that

$$|\sin \pi i y| = \frac{e^{\pi y} - e^{-\pi y}}{2}$$

when $y > 0$. From this it follows that $A = 1$ holds in (i) above. At the same time we proved by residue calculus shows that if $f(z) = \frac{\sin \pi z}{\pi z}$ then

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\log |f(x)| \cdot dx}{x^2} = -\pi^2$$

which is in accordance with (ii) in Theorem 0.1. A special case occurs when the limit in (ii) is automatically satisfied by an integrability condition, i.e. when one has an absolutely convergent integral

$$(*) \qquad \int_{-\infty}^{\infty} \frac{\big|\log |H(x)|\,\big| \cdot dx}{x^2} < \infty$$

In this case the J-integrals converge and as a consequence there exists a limit in (i). It turns out that further conclusions can be made. Namely, the convergence of (*) implies that the sequence $\{\lambda_\nu\}$ has a regular growth which means that if $N(r)$ is the counting function which for every $r > 0$ counts the number of $\lambda\nu \leq r$, then there exists the limit

$$\lim_{R \to \infty} \frac{N(R)}{R} = A$$

with A determined via Theorem 0.1. We prove this in § D and remark that the integrability condition (*) is related to the study of the Carleman class in § C.

Remark. The general Tauberian theorems for entire functions of exponential type are due to Wiener. See his pioneering article [Wiener].

A. Growth of entire functions.

Let f be an arbitrary entire function. We shall associate certain functions which describe the growth and the number of its zeros in discs of radius R centered at the origin. We can write

$$f(z) = a z^m \cdot f_*(z)$$

where f_* is entire and $f_*(0) = 1$. The case when $f(0) = 1$ is therefore not so special and several formulas take a simpler form when this holds.

A.1 The functions $T_f(R)$ and $m_f(R)$. They are defined for every $R > 0$ by

$$\text{(i)} \qquad T_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}^+ \big|f(R(e^{i\theta})\big| \cdot d\theta$$

$$\text{(ii)} \qquad m_f(R) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \operatorname{Log}^+ \Big[\frac{1}{\big|f(R(e^{i\theta})\big|}\Big] \cdot d\theta$$

A.2 The maximum modulus function. It is defined by

$$M_f(R) = \max_{0 \leq \theta \leq 2\pi} |f(Re^{i\theta})|$$

A.3 The counting function $N_f(R)$. To each $R>0$ we count the number of zeros of f in the punctured disc $0<|z|<R$. This integer is denoted by $N_f(R)$, where multiple zeros are counted according to their multiplicities. Jensen's formula shows that if $f(0)=1$ then

$$(*) \qquad \int_0^R \frac{N_f(s)}{s}\cdot ds = \frac{1}{2\pi}\cdot\int_0^{2\pi} \operatorname{Log}\big|f(R(e^{i\theta})\big|\cdot d\theta = T_f(R)-m_f(R)$$

Since the left hand side always is ≥ 0 the inequality below holds under the hypothesis that $f(0)=1$:

$$(**) \qquad m_f(R)\leq T_f(R)$$

Next, since $N_f(R)$ is increasing we get

$$\log 2\cdot N_f(R)\leq \int_R^{2R}\frac{N_f(s)}{s}\cdot ds\leq T_f(2R) \implies$$

$$(***) \qquad N_f(R)\leq \frac{T_f(2R)}{\log 2}$$

A.4 Harnack's inequality. The function $\operatorname{Log}^+|f|$ is subharmonic which implies that whenever $0<r<R$ then one has

$$\operatorname{Log}^+|f(re^{i\alpha})|\leq \frac{1}{2\pi}\cdot\int_0^{2\pi}\frac{R+r}{R-r}\cdot\operatorname{Log}^+\big|f(R(e^{i\theta})\big|\cdot d\theta$$

It follows that

$$M_f(r)\leq \frac{R+r}{R-r}\cdot T_f(R)$$

In particular we can take $R=2r$ and conclude that

$$M_f(r)\leq 3\cdot T_f(2r) \quad \text{hold for every} \quad r>0$$

The last inequality gives:

A.5 Theorem. *An entire function f belongs to $\mathcal{E}$ if and only if there exists a constant A such that*

$$T_f(R)\leq A\cdot R$$

holds for every R.

A.6 A division theorem. Let f and g be in $\mathcal{E}$ and assume that $h=\frac{f}{g}$ is entire. Now

$$\text{(i)} \qquad \log^+|h|\leq \log^+|f|+\log^+|g|$$

In the case when $g(0)=1$ we apply (**) in A.3 and conclude that

$$T_h(R)\leq T_f(R)+T_g(R)$$

Hence Theorem A.5 implies that h belongs to $\mathcal{E}$. We leave it to the reader to verify that this conclusion holds in general, i.e. without any assumption on $g(0)$.

A.7 Hadamard products. Let $\{\alpha_\nu\}$ be a sequence of complex numbers arranged so that the absolute values are non-decreasing. The counting function of the sequence is denoted by $N_{\alpha(\bullet)}(R)$. Suppose that the counting function satisfies:

$$(*) \qquad N_{\alpha(\bullet)}(R)\leq A\cdot R \quad \text{for all} \quad R\geq 1$$

A.8 Theorem *When (*) holds the infinite product*

$$\prod (1-\frac{z}{\alpha_\nu})\cdot e^{\frac{z}{\alpha_\nu}}$$

converges for every z and gives an entire function to be denoted by $H_{\alpha(\bullet)}$ and called the Hadamard product of the α-sequence.

A.9 Exercise. Prove this theorem and show also that there exists a constant C which is independent of A such that the Hadamard product satisfies the growth condition:

$$\big|H_{\alpha(\bullet)}(z)\big| \leq C \cdot \exp\big[A \cdot |z| \cdot \log |z|\big] \quad \text{for all} \quad |z| \geq e$$

A.10 Lindelöf's condition. For a sequence $\{\alpha_\nu\}$ we define the Lindelöf function

$$L(R) = \sum_{|\alpha_\nu|<R} \frac{1}{\alpha_\nu}$$

We say that $\{\alpha_\nu\}$ is of Lindelöf type if there there exists a constant L^* such that

$$(**) \qquad \big|L(R)\big| \leq L^* \quad \text{hold for all} \quad R.$$

A.11 Theorem. *If the α-sequence satisfies (*) in A.8 and is of the Lindelöf type then there exists a constant C such that the maximum modules function of $H_{\alpha(\bullet)}$ satisfies*

$$M_{H_{\alpha(\bullet)}}(R) \leq C \cdot e^{AR}$$

and hence the Hadamard product belongs to $\mathcal{E}$.

A.12 Exercise. Prove this result. A hint is to study the products

$$\prod_{|\alpha_\nu|<2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}} \quad \text{and} \quad \prod_{|\alpha_\nu|\geq 2R} (1 - \frac{z}{\alpha_\nu})e^{\frac{z}{\alpha_\nu}}$$

separately for every $R \geq 1$. Try also to find an upper bound for C expressed by A and L^*.

A converse result. Le f belong to $\mathcal{L}$. Then it turns out that its set of zeros satisfies (**) in A.10 for a constant L^*. To prove this we shall use:

A.13 An integral formula. With $R > 0$ we put $g(z) = \frac{1}{z} - \frac{\bar{z}}{R^2}$. This is a is harmonic function in $\{0 < |z| > R\}$ and $g = 0$ on $|z| = R$. Apply Green's formula to g and $\text{Log}\,|f|$ on an annulus $\{\epsilon < |z| < R\}$. Let $f(z)$ be an entire function with $f(0) = 1$ and consider a pair $0 < \epsilon < R$ where f has not zeros in $|z| \leq \epsilon$.

A.14 Exercise. Show that

$$(*) \qquad \sum_{|\alpha_\nu|<R} \big[\frac{1}{\alpha_\nu} - \frac{\bar{\alpha}_\nu}{R^2}\big] = \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \text{Log}\,|f(Re^{i\theta})| \cdot e^{-i\theta} \cdot d\theta - f'(0)$$

where the sum is taken over zeros of f repeated with multiplicities in the disc $\{|z| < R\}$.

A.15 The case $f \in \mathcal{E}$. Asumme this. From XX we have seen that the counting function $N_f(R)$ is bounded by $C \cdot R$ for some constant C and this implies that the series

$$\sum |\alpha_\nu|^{-2} < \infty$$

To show that the Lindelöf function $L(R)$ is bounded it therefore sufficices to show that the function

$$R \mapsto \frac{1}{\pi \cdot R} \cdot \int_0^{2\pi} \text{Log}\,|f(Re^{i\theta})| \cdot e^{-i\theta} \cdot d\theta$$

is bounded and this follows from A.XX.

B. The factorisation theorem for $\mathcal{E}$

Consider some $f \in \mathcal{E}$. If f has a zero at the origin we can write

$$f(z) = az^m \cdot f_*(z) \quad \text{where} \quad f_*(0) = 1$$

It is clear that f_* again belongs to $\mathcal{E}$ and in this way we essentially reduce the study of $\mathcal{E}$-functions f to the case when $f(0)=1$. Above we proved that the set of zeros satisfies Lindelöf's condition and therefore the Hadamard product

$$H_f(z)=\prod(1-\frac{z}{\alpha_\nu})\cdot e^{\frac{z}{\alpha_\nu}}$$

taken over all zeros of f outside the origin belongs to $\mathcal{E}$. Now the quotient f/H_f is entire and we shall prove:

B.1 Theorem *Let $f\in\mathcal{E}$ where $f(0)=1$. Then there exists a complex number b such that*

$$f(z)=e^{bz}\cdot H_f(z)$$

Proof. The division in A.6 shows that the function

$$G=\frac{f}{H_f}$$

is entire and belongs to $\mathcal{E}$. By construction G is zero-free which gives the entire function $g=\log G$ for we have the inequality

$$|g(z)|\leq 1+\log^+|G(z)|\leq 1+C|z|$$

Since $G\in\mathcal{E}$ we see that $|g|$ increases at most like a linear function so by Liouvile's theorem it is a polynomial of degree 1. Since $f(0)=1$ we have $g(0)=0$ and hence $g(z)=bz$ for a complex number b and the formula in Theorem B.1 follows.

C. The Carleman class $\mathcal{N}$

Let $f\in\mathcal{E}$. On the real x-axis we have the non-negative function $\log^+|f(x)|$. If the integral

$$(*)\qquad \int_{-\infty}^{\infty}\frac{\log^+|f(x)|\cdot dx}{1+x^2}<\infty$$

we say that f belongs to the Carleman class denoted by $\mathcal{N}$. To study $\mathcal{N}$ the following integral formula plays an important role.

C.1 Integral formula in a half-plane. Let $g(z)$ be analytic in the half plane $\mathfrak{Im}(z)>0$. Assume that g extends continuously to the boundary $y=0$, i.e. to the real x-axis and that $g(0)=1$. Given a pair $0<\ell<R$ we consider the domain

$$\Omega_{\ell,R}=\{\ell^2<x^2+y^2<R^2\}\cap\{y>0\}$$

With $z=re^{i\theta}$ we have the harmonic function

$$v(r,\theta)=(\frac{1}{r}-\frac{r}{R^2})\sin\theta=\frac{y}{x^2+y^2}-\frac{y}{R^2}$$

Here $v=0$ on the upper half circle where $|z|=R$ and $y>0$ and the outer normal derivative along the x-axis becomes

$$\partial_n(v)=-\partial_y(v)=-\frac{1}{x^2}+\frac{1}{R^2}\quad:\quad x\neq 0$$

Let $\{\alpha_\nu\}$ be the zeros of g counted with multiplicites in the upper half-plane.Then Green's formula gives:

C.2 Proposition. *One has the formula*

$$2\pi\cdot\sum\frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha\nu|^2}-\frac{\mathfrak{Im}\,\alpha_\nu}{R^2}=$$

$$(*)\qquad \int_\ell^R(\frac{1}{R^2}-\frac{1}{x^2})\cdot\text{Log}\,|g(x)\cdot g(-x)|\cdot dx-\frac{2}{R}\int_0^\pi\sin(\theta)\cdot\text{Log}\,|g(Re^{i\theta})|\cdot d\theta+\chi(\ell)$$

where $\chi(\ell)$ is a contribution from line integrals along the half circle $|z|=\ell$ with $y>0$.

C.3 Exercise Prove via Green's theorem. Notice that the term $\chi(\ell)$ is independent of R so the formula can be used to study asymptotic behaviour as $R \to +\infty$.

Next, the family of analytic functions $g(z)$ in the upper half-plane is identified with $\mathcal{O}(D)$ using a conformal map, i.e. with a given g we get $g_* \in \mathcal{O}(D)$ where

$$g_*(\frac{z-i}{z+i}) = g(z)$$

holds when $\mathfrak{Im}(z) > 0$. When g extends to a continuous function on the real x-axis we have the equality As explained in XXX this gives the equality

$$(*) \qquad \int_0^{2\pi} \log^+ |g_*(e^{i\theta})| \cdot d\theta = 2 \cdot \int_{-\infty}^{\infty} \frac{\log^+ |g(x)| \cdot dx}{1+x^2}$$

This means that the last integral is finite if and only if g_* belongs to the Jensen-Nevanlinna class and in XX we proved that this entails that

$$\int_0^{2\pi} \log^+ \frac{1}{|g_*(e^{i\theta})|} \cdot d\theta < \infty$$

In particular we conclude that if an entire function f satisfies (*) above then it follows that

$$(**) \qquad \int_{-\infty}^{\infty} \log^+ \frac{1}{|f(x)|} \cdot \frac{dx}{1+x^2}$$

in other words, (*) entails that the absolute value $|\log|f(x)||$ is integrable with respect to the density $\frac{1}{1+x^2}$. Using (**) we can prove:

C.4 Theorem *Let $f \in \mathcal{N}$. Then*

$$\sum^* \mathfrak{Im} \frac{1}{\alpha_\nu} < \infty$$

where the sum is taken over all zeros of f which belong to the upper half-plane.

Proof. Since $f \in \mathcal{E}$ there exists a constant C such that $N_f(R) \leq C \cdot R$. If $R \geq 1$ it follows that

$$|R^{-2} \sum \bar{\alpha}_\nu| \leq R^{-2} \cdot R \cdot N_f(R) \leq C$$

where the sum is taken over zeros in $\Omega_{\ell,R}$ Next, since $\mathfrak{Im}\,\alpha_\nu > 0$ in this open set it follows that

$$\frac{\mathfrak{Im}\,\alpha_\nu}{|\alpha_\nu|^2} > 0$$

for every zero in the upper half-plane. In particular this holds for the zeros in $\Omega_{\ell,R}$ and passing to the limit as $R \to \infty$ it suffices to establish an upper bound in the right hand side of Proposition C.2 with $g = f$. The integral taken over the half-circle where $|z| = R$ is uniformly bounded with respect to R since $f \in \mathcal{E}$ and we have the inequality XX from A.XX. For the integral on the x-axis we therefore only need an upper bound. Since $R^{-2} - x^{-2} \leq 0$ during the integration it suffices to find a constant C such that

$$\int_\ell^R (\frac{1}{x^2} - \frac{1}{R^2}) \cdot \log^+ \frac{1}{|f(x) \cdot f(-x)|} \cdot dx \leq C \quad \text{hold for all} \quad R \geq 1$$

The reader may verify that such a constant C since (**) above holds.

C.5 A limit for the counting function. Using the Tauberian theorem which is proved in Section D one has the following:

C.6 Theorem *For each $f \in \mathcal{N}$ there exists the limit:*

$$\lim_{R \to \infty} \frac{N_f(R)}{R}$$

C.7 Remark. To prove this we first notice that if $f \in \mathcal{N}$ then the product $f(z) \cdot f(-z)$ also belongs to $\mathcal{N}$ and for this even function the counting function is twice that of f. Hence it suffices to prove Theorem C.6 when f is even. We may also assume that $f(0)=1$ and since $f \in \mathcal{E}$ it is given by a Hadamard product

$$f(z)=\prod^{*}(1-\frac{z^2}{\alpha_\nu^2}) \tag{1}$$

where $\prod^*$ indicates the we take the product of zeros whose real part is >0 and if they are purely imaginary they are of the form $b \cdot i$ with $b>0$. We can replace the zeros by their absolute values and construct

$$f_*(z)=\prod^{*}(1-\frac{z^2}{|\alpha_\nu|^2}) \tag{2}$$

If x is real we see that

$$|f_*[x)| \leq |f(x)| \tag{3}$$

We conclude that if f belongs to $\mathcal{N}$ so does f_*. At the same time their counting functions of zeros are equal. This reduces the proof of Theorem XX to the special case when f is even and the zeros are real. In the next section we study entire and even functions in $\mathcal{E}$ whose zeros are real and via a general Tauberian theorem deduce Theorem C.6 above.

D. Tauberian Theorems

To every non-decreasing and discrete sequence of positive real numbers $\{0<t_1 \leq t_2 \leq \ldots\}$ we associate the even sequence where we include $\{-t_\nu\}$. Assume that $\mathcal{N}_\Lambda(R) \leq C \cdot R$ for some constant. We get the entire function

$$f(z)=\prod(1-\frac{z^2}{t_\nu^2})$$

which by the results in Section A belongs to $\mathcal{E}$. If $R>0$ we set:

$$J_1(R)=\frac{\log f(iR)}{R} \quad \text{and} \quad J_2(R)=\int_{-R}^{R} \frac{\operatorname{Log}|f(x)|}{x^2} \cdot dx \tag{*}$$

D.1 Theorem. *There exists a limit*

$$\lim_{R\to\infty} \frac{N_f(R)}{R}=2A$$

if and only if at least one of the J-functions has a limit as $R \to \infty$. Moreover, when this holds one has the equalities:

$$\lim_{R\to\infty} J_1(R)=\frac{\pi \cdot A}{2} \quad \text{and} \quad \lim_{R\to\infty} J_2(R)=-\frac{\pi^2 \cdot A}{2}$$

To prove this we introduce the following:

D.2 The W-functions. On the positive real t-line we define the following functions:

$$W_0(t)=\frac{1}{t} \quad : \ t \geq 1 \quad \text{and} \quad W_0(t)=0 \quad \text{when} \ t<1 \tag{1}$$

$$W_1(t)=\frac{\operatorname{Log}(1+t^2)}{t} \tag{2}$$

$$W_2(t)=\int_0^t \frac{\operatorname{Log}|1-x^2|}{x^2} \cdot dx \tag{3}$$

Next, the real sequence $\Lambda=\{t_\nu\}$ gives a discrete measure on the positive real axis where one assigns a unit point mass at every t_ν. If repetitions occur, i.e. if some t-numbers are equal we add these unit point-masses. Let ρ denote the resulting discrete measure. The constructions of the J-functions obviously give:

(*) $$\frac{\mathcal{N}_\Lambda(R)}{R} = 2 \cdot \int_0^\infty W_0(R/t) \cdot \frac{d\rho(t)}{t}$$

(**) $$J_k(R) = \int_0^\infty W_k\big(\frac{R}{t}\big) \cdot \frac{d\rho(t)}{t} \quad : \, k = 1, 2$$

D.3 Exercise. Show that under the assumption that the function $\frac{\mathcal{N}_\Lambda(R)}{R}$ is bounded, it follows the three $\mathcal{W}$-functions belong to the $\mathcal{BW}$-algebra defined by the measure ρ as explained in XXX.

D.4 Fourier transforms. Recall that on $\{t > 0\}$ we have the Haar measure $\frac{dt}{t}$. We leave it to the reader to verify that all the W-functions above belong to $L^1(\mathbf{R}^+)$, i.e.

(i) $$\int_0^\infty \big|W_k(t)\big| \cdot \frac{dt}{t} < \infty \quad : k = 0, 1, 2$$

The Fourier transforms are defined by

(ii) $$\widehat{W}_k(s) = \int_0^\infty W_k(t) \cdot t^{-(is+1)} \cdot dt$$

We shall prefer to use the functions with reversed sign on s, i.e. set

(iii) $$\mathcal{F}\, W_k(s) = \int_0^\infty W_k(t) \cdot t^{is-1} \cdot dt$$

D.5 Proposition *One has the formulas*

(i) $$\mathcal{F}\, W_0(s) = \frac{1}{1 - is}$$

(ii) $$\mathcal{F}\, W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1 - is) \cdot (1 + e^{-\pi s})}$$

(iii) $$\mathcal{F}\, W_2(s) = \frac{2\pi}{(1 - is) \cdot (e^{\pi s/2} + e^{-\pi s/2})}$$

Proof. The equation (i) is easily verified. To prove (ii) we notice that a partial integration gives

$$\mathcal{F}\, W_1(s) = \frac{1}{is - 1} \cdot \int_0^\infty \frac{2 \cdot t^{is} \cdot dt}{1 + t^2}$$

To compute this integral we employ residue calculus where we consider the function

$$\phi(z) = \frac{z^{is}}{1 + z^2}$$

We perform line integrals over large half-circles where $z = Re^{i\theta}$ and $0 \leq \theta \leq \pi$. A reside occurs at $z = i$. Notice also that if $t > 0$ then

$$(-t)^{is} = t^{is} \cdot e^{-\pi s}$$

which gives

$$\mathcal{F}W_1(s) \frac{1}{1 - is) \cdot (1 + e^{-\pi s})} \cdot \lim_{R \to \infty} \int_{-R}^R \phi(t) \cdot dt$$

Here ϕ has a simple pole at $z = 1$ so by residue calculus the last integral becomes

$$-2\pi i \cdot (i)^{is} \cdot \frac{1}{2i} = -\pi \cdot e^{-\pi s/2}$$

Taking the minus sign into the account we conclude that

$$\mathcal{F}W_1(s) = \frac{\pi \cdot e^{-\pi s/2}}{(1-is)\cdot(1+e^{-\pi s})}$$

For (iii) a partial integration gives

$$\mathcal{F}\,W_2(s) = -\frac{1}{is}\cdot\int_0^\infty \log|1-t^2|\cdot t^{is-2}\cdot dt$$

The right hand side is computed in [§ X: Residue Calculus] which gives (iii).

D.6 Evaluations at $s=0$ From (i-iii) we find that

$$\mathcal{F}W_2(0) = \frac{\pi}{2} \quad \text{and} \quad \mathcal{F}W_2(0) = -\frac{\pi^2}{2}$$

Since we also have $\mathcal{F}_1(0)=1$ we apply the Tauberian Theorem for Beurling-Wiener algebras in § XX and read off the results in Theorem D.1.

D.8 Proof of Theorem D.1

The formulas for the Fourier transforms in Proposition D.5 show that each of them is $\neq 0$ on the whole real s-line. Hence we can apply the general result in XX to the discrete measure ρ since the $\mathcal{W}$-functions belong to the $\mathcal{BW}$-algebra from XXX. This implies that if one of the three limits in Theorem D.3 above exists, so do the other. To get the relation between the limit values we only have to evaluate the Fourier transform at $s=0$. From Proposition D.5 we see that

(**) $$\mathcal{F}W_0(0) = 1 \quad : \quad F\mathcal{W}_1(0) = 1 \quad \text{and} \quad F\mathcal{W}_2(0) = \pi$$

This gives the formulas in Theorem D.3 by the general result for $\mathcal{BW}$-algebras in XXX.

E. Application to measures with compact support.

Le μ be a Riesz measure on the real t-line with compact support in an interval $[-a,a]$ where we assume that both end-points belong to the support. The measure is in general complex-valued. Now we get the entire function

$$f(z) = \int_{-a}^{a} e^{-izt}\cdot d\mu(t)$$

Here f restricts to a bounded function on the real x-axis with maximum norm $\leq ||\mu||$. Hence f belongs to $\mathcal{N}$ which means that Theorem D.x holds.

E.1 Theorem. *One has the equality*

$$\lim_{R\to\infty}\frac{N_f(R)}{R} = \frac{a}{\pi}$$

E.2 Tauberian theorems with a remainder term Results which contain remainder terms were established by Beurling in 1936. An example from Beurling's results which involve remainder terms goes as follows: Let

$$f(z) = \prod(1-\frac{z^2}{t_\nu^2})$$

be an even and entire function of exponential type with real zeros as in section D.

E.1 Theorem. *Let $A>0$ and $0<a<1$ and assume that there exists a constant C_0 such that*

$$\Big|-\frac{1}{\pi^2}\cdot\int_0^R \frac{\log|f(x)|}{x^2}\cdot dx - A\,\Big| \leq C_0\cdot R^{-a}$$

hold for all $R\geq 1$. Then there is another constant C such that

$$\big|\,N_f(R) - R\big| \leq C_1\cdot R^{1-a/2}$$

Beurling's original manuscript which contains Theorem E.1 as well as other results dealing with remainder terms has remained unpublished. It was resumed with details of proofs in a Master's

Thesis at Stockholm University by F. Gülkan in 1994. As remarked by Beurling in his article [Beurling] proofs of results with remainders require the full force from the theory of Fourier integrals in addition to more direct use of analytic functions of exponential type. The interested reader should also consult articles by Beurlings former Ph.d student S. Lyttkens which prove various Tauberian theorems with remainder terms. See also work by T. Ganelius for closely related material.

XVI.. Beurling-Wiener algebras

Contents

Introduction.

The cornerstone in this section is Wiener's general Tauberian Theorem which we are going to apply to the class of Beurling-Wiener algebras where the ordinary convolution algebra $L^1(\mathbf{R})$ is replaced by various weight algebras which were introduced by Beurling in the article [Beurling: 1938]. The subsequent material relies upon [ibid] and on chapter XX in [Paley-Wiener]. Here follows the set-up in this section. Consider the Banach space $L^1(\mathbf{R})$ of Lebesgue measurable and absolutely integrable functions whose product is defined by convolutions:

$$f * g(x) = \int f(x-y)g(y)dy$$

A.1 The space $\mathcal{F}_0^\infty$. On the ξ-line we have the space C_0^∞ of infintely differentiable functions with compact support. Each $g(\xi) \in C_0^\infty$ yields an L^1-function on the real x-line defined by

$$(*) \qquad \mathcal{F}(g)(x) = \frac{1}{2\pi}\int e^{ix\xi}\, g(\xi)\cdot d\xi$$

The resulting subspace of L^1 is denoted by $\mathcal{F}_0^\infty$.

A.2 Beurling-Wiener algebras. A subalgebra B of L^1 is called a Beurling-Wiener algebra - for short a $\mathcal{BW}$-algebra - if the following two conditions hold:

Condition 1. B is equipped with a complete norm denoted by $||\cdot||_B$ such that

$$||f*g||_B \leq ||f||_B \cdot ||g||_B \quad : \quad f,g \in B \quad \text{and} \quad ||f||_1 \leq ||f||_B$$

Condition 2. $\mathcal{F}_0^\infty$ is a dense subalgebra of B.

A.3 Theorem *Let B be a $\mathcal{BW}$-algebra. For each multiplicative and continuous functional λ on B which is not identically zero there exists a unique $\xi \in \mathbf{R}$ such that*

$$\lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

Proof. Suppose that there exists some ξ such that

$$\text{(i)} \qquad \lambda(f) = 0 \implies \widehat{f}(\xi) = 0$$

This means that the linear form $f \mapsto \widehat{f}(\xi)$ has the same kernel as λ and hence there exists some constant c such that

$$\text{(ii)} \qquad \lambda(f) = c\cdot \hat{f}(\xi) \quad \text{for all } f \in B\,.$$

Since λ is multiplicative it follows that $c = c^n$ for every positive integer n and then $c = 1$. Next, since B contains $\mathcal{F}_0^\infty$ and test-functions on the ξ-line separate points, it is clear that ξ is uniquely determined. There remains to prove the existence of some ξ for which (i) holds.

To prove this we use the density of $\mathcal{F}_0^\infty$ in B which by the continuity of λ gives some $g \in \mathcal{F}_0^\infty$ such that $\lambda(g) \neq 0$. Let K be the compact support of the test-function $\widehat{g}(\xi)$ and suppose that (i) fails for each point $\xi \in K$. The density of $\mathcal{F}_0^\infty$ gives some $f_\xi \in \mathcal{F}_0^\infty$ such that

$$\text{(iii)} \qquad \widehat{f}(\xi) \neq 0 \quad \text{and } \lambda(f) = 0$$

Heine-Borel's Lemma yields a finite set of points $\xi_1, \ldots, \xi_N$ in K such that family $\{\widehat{f_{\xi_k}}\}$ have no common zero on K. To simplify notations we set $f_k = f_{\xi_k}$. The complex conjugates of $\{\widehat{f_k}\}$ are again test-functions. So for each k we find $h_k \in B$ such that $\widehat{h}_k$ is the s complex conjugate of $\widehat{f}_k$. Set

$$\phi(\xi) = \sum_{k=1}^{k=N} \widehat{h}_k(\xi) \cdot \widehat{f}_k(\xi)$$

This test-function is > 0 on the support of $\widehat{g}$ and hence there exists the test-function

$$\text{(iv)} \qquad Q(\xi) = \frac{\widehat{g}}{\phi}$$

By Condition 2, Q is the Fourier transform of some B-element q. Since $L^1(\mathbf{R})$-functions are uniquely determined by their Fourier transforms, it follows from (iv) that

$$\text{(v)} \qquad \sum_{k=1}^{k=N} q * h_k * f_k = g$$

Now we get a contradiction since $\lambda(f_k) = 0$ for each k while $\lambda(g) \neq 0$.

A.4 The algebra B_a.

Let $a > 0$ be a positive real number. Given a Beurling-Wiener algebra B we set

$$J_a = \{f \in B \quad : \widehat{f}(\xi) = 0 \text{ for all } -a \leq \xi \leq a\}$$

Condition 1 and the continuity of the Fourier transform on L^1-functions imply that J_a is a closed ideal in B. Hence we get the Banach algebra $\frac{B}{J_a}$ which we denote by B_a. Let $g \in \mathcal{F}_0^\infty$ be such that $\widehat{g}(\xi) = 1$ on $[-a, a]$. For every $f \in B$ it follows that $g * f - f$ belongs to J_a which means that the image of f in B_a is equal to the image of $g * f$. We conclude that the g-image yields an identity in the algebra B_a and hence B_a is a Banach algebra with a unit element.

A.5 Theorem. *The Gelfand space of B_a is equal to the compact interval $[-a, a]$.*

A.6 Exercise. Prove this using Theorem A.3

A.7. Examples of $\mathcal{B}W$-algebras

Let B be the space of all continuous functions $f(x)$ on the real x-line such that the positive series below is convergent:

$$(*) \qquad \sum_{-\infty}^{\infty} ||f||_{[\nu,\nu+1]}$$

where $||f||_{[\nu,\nu+1]}$ is the maximum norm of f on the closed interval $[\nu.\nu+1]$ and the sum extends over all integers. The norm on B-elements is defined by the sum of the series above. It is obvious that this norm dominates the L^1-norm. Moreover, one easily verifies that

$$\text{(i)} \qquad ||f * g||_B \leq ||f||| \cdot ||g||_B$$

for pairs in B. Hence B satisfies Condition 1 from B.

Exercise. Show that the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on the real x-line is a dense subalgebra of B.

Next, since $\mathcal{F}_0^\infty \subset \mathcal{S}$ we have the inclusion

$$\text{(ii)} \qquad \mathcal{F}_0^\infty \subset B$$

There remains to see why $\mathcal{F}_0^\infty$ is dense in B. To prove this we construct some special functions on the x-line whose Fourier transforms have compact support. If $b>0$ we set

$$f_b(x)=\frac{1}{2\pi}\int_{-b}^{b} e^{ix\xi}\cdot(1-\frac{|\xi|}{b})\cdot d\xi$$

By Fourier's inversion formula this means that

$$\widehat{f}_b(\xi)=1-\frac{|\xi|}{b}\qquad -b\leq\xi\leq b\quad\text{and zero if } |\xi|>b$$

A computation which is left to the reader gives

$$f_b(x)=\frac{1}{\pi}\cdot\frac{1-\cos bx}{bx^2}$$

From this expression it is clear that $f_b(x)$ belongs to B and we leave it to the reader to verify that

(iii) $$\lim_{b\to+\infty}||f_b*g-g||_B=0\quad\text{for all } g\in B$$

Next, the functions $\widehat{f}_b(\xi)$ have compact support but they are not smooth, i.e. they do not belong to $\mathcal{F}_0^\infty$. However, we can perform a smoothing of these functions as follows: Let $\phi(\xi)$ be an even and non-negative C_0^∞-function with support in $-1\leq\xi\leq 1$ such that the integral

$$\int\phi(\xi)\cdot d\xi=1$$

With $\delta>0$ we set $\phi_\delta(\xi)=\frac{1}{\delta}\cdot\phi(\xi/\delta)$ and for each pair δ,b we get the test-function on the ξ-line defined by

$$\psi_{\delta,b}(\xi)=\int_{-b}^{b}\phi_\delta(\xi-\eta)\cdot(1-\frac{|\eta|}{b})\cdot d\eta$$

The inverse Fourier transforms

$$f_{\delta,b}(x)=\frac{1}{2\pi}\int e^{ix\xi}\cdot\psi_{\delta,b}(\xi)\cdot d\xi$$

yield functions in $\mathcal{F}_0^\infty$ for all pairs δ,b. Next, if $g\in B$ then the Fourier transform of the B-element $f_{\delta,b}*g$ is equal to the *convolution* of $\phi_\delta(\xi)$ and the Fourier transform of f_b*g. This implies that

$$f_{\delta,b}*g\in\mathcal{F}_0^\infty\,.$$

At this stage we leave it to the reader to verify that

$$\lim_{(\delta,b)\to(0,0)} f_{\delta,b}*g=g$$

holds for every $g\in B$. Hence the required density of $\mathcal{F}_0^\infty$ is proved and B is a Beurling-Winer algebra.

A.8 Adding discrete measures

Let $M_d(\mathbf{R})$ be the Banach algebra of discrete measures of finite total variation, i.e. measures of the form

$$\mu=\sum c_\nu\cdot\delta_{x_\nu}\quad:\ ||\mu||=\sum|c_\nu|<\infty$$

As explained in XX the Gelfand space is the compact Bohr group $\mathfrak{B}$, where the real ξ-line via the Fourier transform appears as a dense subset. Now we adjoin some $\mathcal{BW}$-algebra B and obtain a Banach algebra B_d which consists of measures of the form

$$f+\mu\quad:\quad f\in B\text{ and }\mu\in M_d(\mathbf{R})$$

where the norm of $f+\mu$ is the sum of the B-norm of f and the total variation of μ. Since B is a subspace of L^1 one easuly checks that this yields a complete norm. next, by condition (2) in A.2 it follows that if $f\in b$ and $\mu\in M_d(\mathbf{R})$ then the convolution $f*\mu$ belongs to B. This means that B appears as a closed ideal in B_d.

A.9 The Gelfand space $\mathcal{M}_{B_d}$. Let λ is a multiplicative functional on B_d which is not identically zero on B. Theorem A.3 gives a unique ξ such that

$$\text{(i)} \qquad \lambda(f) = \widehat{f}(\xi) \quad : \quad f \in B$$

If a is a real number then $\delta_a * f$ has the Fourier transform becomes $e^{ia\xi} \cdot \widehat{f}(\xi)$. It follows that

$$\text{(ii)} \qquad \lambda(\delta_a) \cdot \widehat{f}(\xi) = \lambda(\delta_a * f) = e^{-ia\xi} \cdot \widehat{f}(\xi)$$

We conclude that $\lambda(\delta_a) = e^{-ia\xi}$ and hence the restriction of λ is the evaluation of the Fourier transform at ξ on the whole algebra B_d. In this way the real ξ-line is embedded in $\mathcal{M}_B$ where a point $\lambda \in \mathcal{M}_B$ belongs to this subset if and only if $\lambda(f) \neq 0$ for some $f \in B$. The construction of the Gelfand topology shows that this copy of the real ξ-line appears as an *open* subset of $\mathcal{M}_{B_d}$ denoted by $\mathbf{R}_\xi$.

A.10 The set $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$. If λ belongs to this closed subset it is identically zero on the ideal B and its restriction to $M_d(\mathbf{R})$ corresponds to a point γ in the Bohr group $\mathfrak{B}$. Conversely, every point in $\mathfrak{B}$ yields a $\lambda \in \mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ since the quotient algebra

$$\frac{B_d}{B} \simeq M_d(\mathbf{R})$$

Hence we have the set-theoretic equality

$$(*) \qquad \mathcal{M}_{B_d} = \mathbf{R}_\xi \cup \mathfrak{B}$$

A.11 Proposition. *The open subset $\mathbf{R}_\xi$ is dense in $\mathcal{M}_B$.*

Proof. Let λ be a point in $\mathcal{M}_{B_d} \setminus \mathbf{R}_\xi$ which therefore corresponds to a point $\gamma \in \mathfrak{B}$. By the result in XX we know that for every finite set $\mu_1, \ldots, \mu_N$ of discrete measures, there exists a sequence $\{\xi_\nu\}$ such that

$$\lim_{\nu \to \infty} \widehat{\mu}_i(\xi_\nu) = \gamma(\mu_i) \quad \text{and } |\xi_\nu| \to \infty$$

At the same time the Riemann-Lebesgue Lemma entails that

$$\lim_{\nu \to \infty} \widehat{f}(\xi_\nu) = 0$$

for every $f \in B$. Hence the construction of the Gelfand topology on $\mathcal{M}_{B_d}$ gives the requested density in Proposition A.11

A.12 An inversion formula. Let $f \in B$ and μ is some discrete measure. Suppose that there exists $\delta > 0$ such that the Fourier transform of $f + \mu$ has absolute value $\geq \delta$ for all ξ. Proposition A.11 implies that its Gelfand transform has no zeros and hence this B_d-element is invertible, i.e. there exist $g \in B$ and a discrete measure γ such that

$$\text{(i)} \qquad \delta_0 = (f + \mu) * (g + \gamma)$$

Notice that the right hand side becomes

$$f * g + f * \gamma + g * \mu + \mu * \gamma$$

Here $f * g + f * \gamma + g * \mu$ belongs to B while $\mu * \gamma$ is a discrete measure. So (i) implies that γ must be the inverse of μ in $M_d(\mathbf{R})$ and hence (i) also gives the equality:

$$\text{(ii)} \qquad f * g + f * \mu^{-1} + g * \mu = 0$$

B. A Tauberian Theorem.

Consider the Banach algebra B above. The dual space B^* consists of Riesz measures μ on the real line for which there exists a constant A such that

$$\int_\nu^{\nu+1} |d\mu(x)| \leq A \quad \text{for all integers } \nu\,.$$

The smallest A above is the norm of μ in B^* and duality is expressed by:

$$\mu(f) = \int f(x) \cdot d\mu(x) \quad : \quad f \in B \text{ and } \mu \in B^*$$

Let $f \in B$ be such that $\widehat{f}(\xi) \neq 0$ for all ξ. For each $a > 0$ it follows from Theorem A.5 that the f-image in B_a generates the whole algebra. Since this hold for every $a > 0$ it follows that each $\phi \in \mathcal{F}_0^\infty$ belongs to the principal ideal generated by f in B, i.e. there exists some $g \in B$ such that

$$(*) \qquad \phi = g * f$$

Since $\mathcal{F}_0^\infty$ is dense in B we conclude that $B \cdot f$ is dense in B. Using this density we have:

B.1 Theorem *Let $\mu \in B^*$ be such that*

$$\lim_{y \to +\infty} \int f(y-x) \cdot d\mu(x) = A \text{ exists.}$$

Then, for each $g \in B$ it follows that

$$\lim_{y \to +\infty} \int g(y-x) \cdot d\mu(x) = B \quad \text{where} \quad B = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

Proof. Let $g \in B$. If $\epsilon > 0$ we find $h_\epsilon \in B$ such that $||g - f * h_\epsilon||_B < \epsilon$. When $y > 0$ we get:

$$\text{(i)} \qquad \int (f * h_\epsilon)(y-x) \cdot d\mu(x) =$$

$$\int \big[f(y-s-x) h_\epsilon(s) \cdot ds \big] \cdot d\mu(x) = \int h_\epsilon(s) \cdot \big[\int f(y-s-x) \mu(x) \big] \cdot ds$$

By the hypothesis the inner integral converges to A when $y \to +\infty$ every fixed s. Since h belongs to B it follows easily that the limit of (i) when $y \to +\infty$ is equal to

$$\text{(ii)} \qquad A \cdot \int h_\epsilon(s) \cdot ds = A \cdot \widehat{h}_\epsilon(0)$$

Next, since the B-norm is stronger than the L^1-norm it follows that

$$\text{(iii)} \qquad |\widehat{g}(0) - \hat{h}_\epsilon(0) \cdot \widehat{f}(0)| < \epsilon$$

Moreover, since the B-norm is invariant under translations we have

$$\text{(iv)} \qquad \Big| \int g(y-x) d\mu(x) - \int (f * h_\epsilon)(y-x) \cdot d\mu(x) \Big| \leq \epsilon \cdot ||\mu|| \quad \text{for all } y$$

where $||\mu||$ is the norm of μ in the dual space B^*. Notice also that (iii) gives:

$$\lim_{\epsilon \to} \hat{h}_\epsilon(0) = \frac{\hat{g}(0)}{\hat{f}(0)}$$

Finally, since $\epsilon > 0$ is arbitrary we see that the limit formula for (i) when $y \to +\infty$ expressed by (ii) and (iv) above together imply that

$$\lim_{y \to +\infty} \int g(y-x) d\mu(x) = A \cdot \frac{\hat{g}(0)}{\hat{f}(0)}$$

This finishes the proof of Theorem A.9

B.2 The multiplicative version

Let $\mathbf{R}^+$ be the multiplicative group of positive real numbers. To each function $f(t)$ on $\mathbf{R}^+$ we get the function $E_f(x) = f(e^x)$ on the real x-line. Since $dt = e^x dx$ under the exponential map we have

$$\int_0^\infty f(t) \frac{dt}{t} = \int_{-\infty}^\infty E_f(x) dx$$

provided that f is integrable. On $\mathbf{R}^+$ we get the convolution algebra $L^1(\mathbf{R}^+)$ where

$$f * g(t) = \int_0^\infty f(\frac{t}{s}) \cdot g(s) \cdot \frac{ds}{s}$$

This convolution commutes with the E map from $L^1(\mathbf{R}^+$ into $L^1(\mathbf{R}^1)$, i.e.

$$E_{f*g} = E_f * E_g$$

Next, recall that the Fourier transform on $L^1(\mathbf{R}^+)$ is defined by

$$\widehat{f}(\xi) = \int_0^\infty t^{-i\xi} \cdot f(t) \cdot \frac{dt}{t}$$

B.3 The Banach algebra B_m. The companion to B on $\mathbf{R}^+$ consists of continuous functions $f(t)$ for which

$$\sum ||f||_{[2^\nu, 2^{\nu+1}]} < \infty$$

where the is taken over all integers. Notice that with $\nu < 0$ one takes small intervals approaching $t = 0$. Just as in Theorem A.9 we obtain a Tauberian Theorem for functions $f \in B_m$ whose Fourier transform is everywhere $\neq 0$. Here we the dual space B_m^* consists of Riesz measures μ on $\mathbf{R}^+$ for which there exists a constant C such that

$$\int_{2^m}^{2^{m+1}} |d\mu(t)| \leq C$$

for all integers m.

C. Ikehara's theorem.

Let ν be a non-negative Riess measure supported on $[1, +\infty)$ and assume that

$$\int_1^\infty x^{-1-\delta} \cdot d\nu(x) < \infty \quad \text{for all } \delta > 0$$

When this holds we obtain an analytic function $f(s)$ of the complex variable s defined in the right half plane $\mathfrak{Re}(s) > 1$ by

$$f(s) = \int_1^\infty x^{-s} \cdot d\nu(x)$$

D.1 Theorem. *Assume that there exists a constant A and a continuous function $G(u)$ defined on the real u-line such that*

$$(*) \qquad \lim_{\epsilon \to 0} \big[f(1+\epsilon+iu) - \frac{A}{1+\epsilon+iu} \big] = G(u)$$

where the limit holds uniformly on every bounded interval $-b \leq u \leq b$. Then

$$(**) \qquad \lim_{x \to +\infty} \frac{1}{x} \int_1^x d\nu(t) = A$$

We shall prove a sharper version of Ikehara's result where the assumption on $G(u)$ is relaxed. Namely, replace (*) by the weaker assumption that there exists a locally integrable function $G(u)$ such that

$$(***) \qquad \lim_{\epsilon \to 0} \int_{-b}^{b} \big| f(1+\epsilon+iu) - \frac{A}{1+\epsilon+iu} - G(u) \big| \cdot du = 0 \quad \text{holds for each } b > 0$$

Proof that (***) gives (**). To show this implication we use some variable substitutions. With $x \mapsto e^\xi$ we can write

$$f(s) = \int_0^\infty e^{-\xi s} \cdot d\nu(e^\xi)$$

Next, define the function measure μ on the non-negative real ξ-line by

(1) $$d\mu(\xi) = e^{-\xi} \cdot d\nu(e^{\xi}) - A \cdot d\xi \quad : \ \xi \geq 0$$

Then we see that

(2) $$f(s) - \frac{A}{s-1} = \int_0^\infty e^{(1-s)\xi} d\mu(\xi)$$

It is clear that (**) holds if and only if

(3) $$\lim_{\eta \to \infty} \int_0^\eta e^{-\eta+\xi} \cdot d\mu(\xi) = 0$$

A reformulation of Ikehara's theorem. From the observations above we can restate the sharp version of Ikehara's theorem. Let ν^* be a non-negative measure on $0 \leq \xi < +\infty$ such that

(1) $$\int_0^\infty e^{-\delta \cdot \xi} \cdot d\nu^*(\xi) < \infty \quad \text{for all } \delta > 0$$

Next, let $A > 0$ be some positive constant and put $d\mu(\xi) = d\nu^*(\xi) - A \cdot d\xi$. Then (1) gives the analytic function $g(s)$ defined in $\mathfrak{Re}(s) > 0$ by

$$g(s) = \int_0^\infty e^{-s \cdot \xi} \cdot d\mu(\xi)$$

D.2. Definition. *We say that the measure μ is of the Ikehara type if there exists a locally integrable function $G(u)$ defined on the real u-line such that*

$$\lim_{\epsilon \to 0} \int_{-b}^{b} \big|g(\epsilon + iu) - G(u)\big| \cdot du = 0 \quad \text{holds for each } b > 0$$

D.3. The space $\mathcal{W}$. Let $\mathcal{W}$ be the space of continuous functions $\rho(\xi)$ defined on $\xi \geq 0$ which satisfy:

$$\sum_{n \geq 0} ||\rho||_n < \infty \quad \text{where } ||\rho||_n = \max_{n \leq u \leq n+1} |\rho(u)|$$

The dual space $\mathcal{W}^*$ consists of Riesz measures γ on $[0, +\infty)$ such that

$$\max_{n \geq 0} \int_n^{n+1} |d\gamma(\xi)| < \infty$$

With these notations we have

D.4. Theorem. *Let ν^* be a non-negative meausure on $[0, +\infty)$ and $A \geq 0$ some constant such that the measure $\mu = \nu^* - A \cdot d\xi$ is of Ikehara type. Then $\mu \in \mathcal{W}^*$ and for every function $\rho \in \mathcal{W}$ one has*

$$\lim_{\eta \to +\infty} \int_0^\eta \rho(\eta - \xi) \cdot d\mu(\xi) = 0$$

Exercise. Use the material above to show that Theorem D. 4 gives the sharp version of Ikehara's theorem. The hint is to use the function $\rho(s) = e^{-s}$ above.

Proof of Theorem D.4.

Let $b > 0$ and define the function $\omega(u)$ by

(i) $$\omega(u) = 1 - \frac{|u|}{b}, \quad -b \leq u \leq b \quad \text{and } \omega(u) = 0 \text{ outside this interval}$$

Set

(ii) $$J_b(\epsilon, \eta) = \int_{-b}^{b} e^{i\eta u} \cdot g(\epsilon + iu) \cdot \omega(u) \cdot du$$

From Definition 2 we have the L^1_{loc}-function $G(u)$ and since $\omega(u)$ is a continuous function on the compact interval $[-b,b]$ we have

(iii) $$\lim_{\epsilon\to 0} J_b(\epsilon,\eta)=J_b(0,\eta)=\int_{-b}^{b} e^{i\eta u}\cdot G(u)\cdot\omega(u)\cdot du$$

With b kept fixed the right hand side is a Fourier transform of an L^1-function. So the Riemann-Lebesgue theorem gives:

(iv) $$\lim_{\eta\to+\infty} J_b(0,\eta)=0$$

Moreover, the triangle inequality gives the inequality:

(v) $$|J_b(0,\eta)|\leq\int_{-b}^{b}\bigl|G(u)\bigr|\cdot du$$

Some integral formulas. From the above it is clear that

(1) $$J_b(\epsilon,\eta)=\int_0^\infty\bigl[\int_{-b}^{b} e^{i\eta u-i\xi u}\cdot\omega(u)\cdot du\bigr]\cdot e^{-\epsilon\cdot\xi}\cdot d\mu(\xi)$$

Next, notice that

(2) $$\int_{-b}^{b} e^{i\eta u-i\xi u}\cdot\omega(u)\cdot du=2\cdot\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}$$

Hence we obtain

(3) $$J_b(\epsilon,\eta)=2\cdot\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\mu(\xi)$$

From (iii) above it follows that (3) has a limit as $\epsilon\to 0$ which is equal to the integral in the right hand side in (iii) which is denoted by $J_b(0,\eta)$. Next, it is easily seen that there exists the limit

(4) $$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot Ad\xi=2\pi\cdot A$$

Hence (3-4) imply that there exists the limit

(5) $$\lim_{\epsilon\to 0} 2\cdot\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot e^{-\epsilon\xi}\cdot d\nu^*(\xi)=J_b(0,\eta)+2\pi\cdot A$$

Next, the measure $\nu^*\geq 0$ and the function $\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\geq 0$ for all ξ. So the existence of a finite limit in (5) entails that there exists the convergent integral

(6) $$\int_0^\infty\frac{1-\cos b(\eta-\xi)}{b(\eta-\xi)^2}\cdot d\nu^*(\xi)=J_b(0,\eta)+2\pi\cdot A$$

Proof that $\mu\in\mathcal{W}^*$. Since $A\cdot d\xi$ obviously belongs to $\mathcal{W}^*$ it suffices to show that $\nu^*\in\mathcal{W}^*$. To prove this we consider some integer $n\geq 0$ and with $b=1$ it is clear that (6) gives

$$\bigl|\int_n^{n+1}\frac{1-\cos(\eta-\xi)}{(\eta-\xi)^2}\cdot d\nu^*(\xi)\bigr|\leq|J_1(0,\eta)|+2\pi=\int_{-1}^{1}|G(u)|\cdot du+2\pi\cdot A$$

Apply this with $\eta=n+1+\pi/2$ and notice that

$$\frac{1-\cos(n+1+\pi/2-\xi)}{(n+1+\pi/2-\xi)^2}\geq a\quad\text{for all } n\leq\xi\leq n+1$$

This gives a constant K such that

$$\int_n^{n+1} d\nu^*(\xi)\leq K\quad n=0,1,\ldots$$

Final part of the proof. We have proved that $\mu \in \mathcal{W}^*$. Moreover, from (iv) above and the integral formula (6) we get

$$(*) \qquad \lim_{\eta \to +\infty} \int_0^\infty \frac{1 - \cos b(\eta - \xi)}{b(\eta - \xi)^2} \cdot d\mu(\xi) = 0 \quad \text{for all } b > 0$$

Next, for each fixed $b > 0$ we notice that the function

$$\rho_b(\xi) = 2 \cdot \frac{1 - \cos(b\xi)}{b \cdot \xi^2}$$

belongs to $\mathcal{W}$ and its Fourier is $\omega_b(u)$. Here $\omega_b(u) \neq 0$ when $-b < u < b$. So the family of these ω-functions have no common zero on the real u-line. By the Remark in XX this means that the linear subspace of $\mathcal{W}$ generated by the translates of all ρ_b -functions with arbitrary large b is dense in $\mathcal{W}$. Hence (*) above implies that we get a zero limit as $\eta \to +\infty$ for every function $\rho \in \mathcal{W}$. But this is precisely the assertion in Theorem 4.

E. The algebra $L^1(\mathbf{R}^+)$

Consider the family of L^1-functions on the real x-line which are supported by the half-line $x \geq 0$. This yields a closed subalgebra of $L^1(\mathbf{R})$ denoted by $L^1(\mathbf{R}^+)$. Indeed, if $f(x)$ and $g(x)$ are two such functions in $L^1(\mathbf{R}^+)$. the support of the convolution $g * f$ stays in $[0, +\infty)$. Adding the unit point mass δ_0 we obtain a commutative Banach algebra

$$B = \mathbf{C} \cdot \delta_0 + L^1(\mathbf{R}^+)$$

E. 1. The Gelfand space $\mathfrak{M}_B$. To obtain this space we consider some $f(x) \in L^1(\mathbf{R}^+)$ and set:

$$\widehat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot f(x) \cdot dx, \quad \text{where} \quad \mathfrak{Im}(\zeta) \geq 0$$

With $\zeta = \xi + i\eta$ we get

$$|\widehat{f}(\xi + i\eta)| \leq \int_0^\infty |e^{i\xi x - \eta x}| \cdot |f(x)| \cdot dx = \int_0^\infty |e^{-\eta x} \cdot |f(x)| \cdot dx \leq ||f||_1$$

We conclude that for every point $\zeta = \xi + i\eta$ in the closed upper half-plane corresponds to a point in $\zeta^* \in \mathfrak{M}_B$ defined by

$$\zeta^*(f) = \widehat{f}(\zeta) \quad \text{and} \quad \zeta^*(\delta_0) = 1$$

In addition to this $L^1(\mathbf{R}^+)$ is a maximal ideal in B and there is the special point $\zeta^\infty \in \mathfrak{M}_B$ such that

$$\zeta^\infty(f) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

E.2. Theorem. *The Gelfand space $\mathfrak{M}_B$ can be identified with the union of ζ^∞ and the closed upper half-plane.*

Remark. The theorem asserts that every multiplicative functional on B is either ζ^∞ or determined by a point $\zeta = \xi + i\eta$ where $\eta \geq 0$. Concerning the topological identification ζ^∞ corresponds to the one point compactification of the closed upper half-plane. Thus, whenever $\{\zeta_\nu\}$ is a sequence in $\mathfrak{Im}(\zeta) \geq 0$ such that $|\zeta_\nu| \to \infty$ then $\{z_\nu^*\}$ converges to ζ^* in $\mathfrak{M}_B$. In fact, this follows via the Riemann-Lebegue Lemma which gives

$$\lim_{|\zeta| \to \infty} \widehat{f}(\zeta) = 0 \quad \text{for all } f \in L^1(\mathbf{R}^+)$$

By the general result in XX Theorem 2 holds if we have proved if the following:

E.3. Proposition. *Let $\{g_\nu = \alpha_\nu \cdot \delta_0 + f_\nu\}_1^k$ be a finite family in B such that the k-tuple $\{\hat{g}_\nu\}$ has no common zero in $\bar{U}_+ \cup \{\infty\}$. Then the ideal in B generated by this k-tuple is equal to B.*

The proof requires some preliminary constructions. We use the conformal map from the upper half-plane onto the unit disc defined by

$$w = \frac{\zeta - i}{\zeta + i}$$

So here w is the complex coordinate in D. Next, consider the disc algebra $A(D)$. Via the conformal map each transform $\widehat{f}(\zeta)$ of a function $f \in L^1(\mathbf{R}^+)$ yields an element of $A(D)$ defined by:

$$F(w) = \widehat{f}(\frac{i + iw}{1 - w})$$

It is clear that $F(w) \in A(D)$. Moreover, we notice that

$$w \to 1 \implies |\frac{i + iw)}{1 - w}| \to \infty$$

It follows that the $A(D)$-function $F(w)$ is zero at $w = 1$ and we can conlcude:

E.4. Lemma. *By $f \mapsto F$ we have an algebra homomorphism from $L^1(\mathbf{R}^+)$ to functions in $A(D)$ which are zero at $w = 1$.*

Next, let $\mathcal{H}$ denote the algebra homomorphism in Lemma 4 and consider the function $1 - w$ in $A(D)$. We claim this it belongs to the image under $\mathcal{H}$. To see this we consider the function

$$f(x) = e^{-x} \quad x \geq 0 \quad \text{and } f(x) = 0 \quad \text{when } x < 0$$

Then we see that

$$\hat{f}(\zeta) = \int_0^\infty e^{i\zeta x} \cdot e^{-x} \cdot dx = \frac{1}{1 - i\zeta}$$

It follows that

$$F(w) = \frac{1}{1 - i(\frac{iw+i}{1-w})} = \frac{1 - w}{1 - w + w + 1} = \frac{1 - w}{2}$$

Using $2f$ we conclude that $1 - w$ belongs to the $\mathcal{H}$-image. Next, the identity element δ_0 is mapped to the constant function on D. So via $\mathcal{H}$ we have an algebra homomorphism from B into a subalgebra of $A(D)$ which contains $1 - w$ and the identity function and hence all w-polynomials. Returning to the special B-element f we notice that the convolution

$$f * f(x) = x \cdot e^{-x}$$

We can continue and conclude that the subalgebra of B generated by f and δ_0 contains L^1-functions of the form $p(x) \cdot e^{-x}$ where $p(x)$ are polynomials.

E.5. Exercise. Prove that the linear space $\mathbf{C}[x] \cdot e^{-x}$ is a dense subspace of $L^1(\mathbf{R}^+)$.

From the result in the exercise it follows that the polynomial algebra $\mathbf{C}[w]$ appears as a dense subalgebra of $\mathcal{H}(B)$ when it is equipped with the B-norm. At this stage we are prepared to give:

Proof of Proposition E.3. In $A(D)$ we have the functions $\{G_\nu = \mathcal{H}(g_\nu)\}$. By assumption $\{G_\nu\}$ have no common zero in the closed disc D. Since D is the maximal ideal space of the disc algebra and $\mathbf{C}[w]$ a dense subalgebra, it follows that for every $\epsilon > 0$ there exist polynomials $\{p_\nu(w)\}$ such that the maximum norm

(1) $$|p_1 \cdot G_1 + \ldots + p_k \cdot G_k - 1|_D < \epsilon|$$

where 1 is the identity function. Now $p_\nu = \mathcal{H}(\phi_\nu)$ for B-elements $\{\phi_\nu\}$. So in B we get the element

(2) $$\psi = \phi_1 g_1 + \ldots + \phi_k \cdot g_k$$

Moreover we have $|\mathcal{H}(\psi) - 1|_D < \epsilon$ and here we can choose $\epsilon < 1/4$ and by the previous identifications it follows that

(3) $$|\widehat{\psi}(\xi)| \geq 1/4 \quad \text{for all} \quad -\infty < \xi < \infty$$

The proof of Proposition E.3 is finished if we can show that (3) entails that the B-element ψ is invertible. Multiplying ψ with a non-zero scalar we may assume that

$$\psi = \delta_0 - g \quad : \quad g \in L^1(\mathbf{R}^+)$$

and the Fourier transform $\widehat{\psi}(\xi)$ satisfies

$$|\widehat{\psi}(\xi) - 1| \leq 1/2$$

for all ξ. It means that $|\widehat{g}(\xi)| \leq 1/2$. The spectral radius formula applied to L^1-functions shows that if N is a sufficiently large integer then

$$||g^{(N)}||_1 \leq (3/4)^N \tag{4}$$

where $g^{(N)}$ is the N-fold convolution of g. Now we have

$$(1 + g + \ldots + g^{N_1}) \cdot \psi = 1 - g^{(N)} \tag{5}$$

By (4) the norm of the B-element $g^{(N)}$ is strictly less than one and hence the right hand side is invertible where the inverse is given by a Neumann series, i.e. with $g_* = g^{(N)}$ the inverse is

$$\delta_0 + \sum_{\nu=1}^{\infty} g_*^{\nu}$$

Since convolutions of $L^1(\mathbf{R}^+)$-functions still are supported by $x \geq 0$, it follows from the above that ψ is invertible in B and Proposition E.3 is proved.

14. Sets of harmonic measure zero

0. Introduction

A. Myhrberg's theorem

B. Equilibrium distributions and Robin's constant.

C. Cartan's theorem

D. Cantor sets.

Introduction.

The study of harmonic measures and other areas in potential theory goes back to a problem raised by G. Robin in the article [Rob] from 1886 which had physical background in electric engineering. The problem is: *Let E be a compact set in $\mathbf{C}$. Find a probability measure μ on E such that the function*

$$(*)\qquad U_\mu(z) = \int_E \log \frac{1}{[z-\zeta|} \cdot d\mu(\zeta) \quad : z \in \mathbf{C} \setminus E$$

takes constant boundary values on E. For every probability measure μ on E, i.e. a non-negative Riesz measure of unit mass supported by E, the integral (*) is defined for points in E where the value can be finite or infinite. To be precise, if $z_* \in E$ is fixed and n is a positive integer we put $E_n = E \setminus \{|z - z_*| \geq 1/n\}$. Since $\log \frac{1}{|z_*-\zeta|} \geq 0$ when $|\zeta - z_*| \leq 1$ it follows

$$n \mapsto \int_{E_n} \log \frac{1}{|z_* - \zeta|} \cdot d\mu(\zeta)$$

is increasing and by definition the integral (*) taken on E with $z = z_*$ is equal to the limit if (1) which therefore is finite or $+\infty$. Using the limits above to compute $U_\mu(z)$ at points in E it follows that the function

$$z \mapsto U_\mu(z)$$

is superharmonic function and always is harmonic in the open complement of E. One refers to $U_\mu(z)$ as the logarithmic potential of μ.

Energy integrals. We can integrate U_μ with respect to μ, i.e. there exists a well defined Borel integral:

$$J(\mu) = \int U_\mu(z) \cdot d\mu(z)$$

We refer to $J(\mu)$ as the energy integral of μ. Notice that the energy also is expressed by a double integral:

$$J(\mu) = \iiint \log, \frac{1}{|z-\zeta|} \cdot d\mu(\zeta) \cdot d\mu(z)$$

We shall foremost study the case when E is compact and totally disconnected. So Perron's criterium for a solution to the Dirichlet problem fails. But it is still meaningful to speak about harmonic functions defined in open complementary sets to E. Notice also that if $z_* \in E$ then we can construct arbitrary small Jordan domains U which where z_* is an interior point while the closed Jordan curve ∂U has empty intersection with E. This leads to the following:

0.1 Definition. *A compact and totally disconnected set E is a removable singularity for bounded harmonic functions if every bounded harmonic function H in $U \setminus E$ for a pair (z_*, U) as above extends to a harmonic function in the whole Jordan domain U.*

Remark. This is a local condition at each point in E and the extensions should hold at all points in E. The following result is due to Myhrberg.

0.2 Theorem. *A compact and totally disconnected set E is a removable singularity for bounded harmonic functions if and only if there exists a probability measure μ on E such that $U_\mu(z) = +\infty$ for every $z \in E$.*

We prove this result in § A. When the two equivalent conditions hold in Theorem 0.2 we say that E is a harmonic null-set. Denote by $\mathcal{N}_{\text{harm}}$ the family of totally disconnected sets harmonic null-sets. One may ask for metric conditions in order that a given compact and totally disconnected set E belongs to $\mathcal{N}_{\text{harm}}$. To analyze this we introduce h-measures where $h(r)$ is a continuous and non-decreasing function defined for $r > 0$ and $h(0) = 0$. If F is a compact set we consider open coverings of F by discs and define its outer h-measure by

$$h^*(F) = \min \sum h(r_\nu)$$

where the minimum is taken over coverings of F by open discs $\{D_\nu\}$ of radius $\{r_\nu\}$. The family of compact sets whose outer h-measure is zero is denoted by $\mathcal{N}(h)$. The case $h(r) = r^2$ means precisely that F has planar Lebesgue measure zero. If $h(r)$ tends more slowly to zero as $r \to 0$ we get a more restrictive class, i.e. then $\mathcal{N}(h)$ consists of sets which are more thin than sets with planar Lebesgue measure zero. In the case when

$$h(r) = \frac{1}{\log \frac{1}{r}}$$

we say that a compact set in $\mathcal{N}(h)$ has logarithmic capacity zero . The first major result about harmonic null-sets was proved by Lindeberg in 1918 and goes as follows:

0.3 Theorem. *Let E be a compact set whose logarithmic measure zero. Then E has harmonic measure zero.*

A result which gives a necessary metric condition for a set E to be a harmonic null-set was proved by Henri Cartan in [Cartan]. First we give:

0.4 Definition *Let $\mathfrak{H}_*$ denote the class of non-decreasing and continuous function $h(r)$ satisfying*

$$\int_0^1 \frac{h(r)}{r} \cdot dr < \infty$$

0.5 Theorem. *For every $E \in \mathcal{N}_{harm}$ it follows that*

$$E \in \mathcal{N}(h) = 0 \quad : \forall\, h \in \mathfrak{H}_*$$

0.6 Remark. Cartan's result is close to Lindeberg's sufficiency result. Namely, if $\eta > 0$ we set

$$h(r) = \frac{1}{[\text{Log}\, \frac{1}{r}]^{1+\eta}}$$

It is clear that $h \in \mathfrak{H}_*$ and hence $h^*(E) = 0$ for every $E \in \mathcal{N}_{\text{harm}}$. With η small this comes close to say that the logarithmic capacity of E is zero. However, Cartan's Theorem does not give sufficient conditions in order that a compact and totally disconnected set E has harmonic measure zero. The search for other metric conditions which are either necessary or sufficient in order that a compact set has harmonic measure zero is unclear and one should perhaps not expect too much. The text-book on *Exceptional Sets* by Carleson contains examples which illustrate the difficulty to get definite answers. However, a metric criterion for harmonic null-sets exists if E is a Cantor set on a line. See XXX below.

0.7 Transfinite diameters and the logarithmic capacity. Let E be a compact set which is assumed to be infinite. Here we do not assume that E is totally disconnected. To each n-tuple of distinct points $z_1, \ldots, z_n$ we put:

$$L_n(z_\bullet) = \frac{1}{n(n-1)} \cdot \sum_{k \neq j} \log \frac{1}{|z_j - z_k|}$$

Then we define the number

$$\mathcal{L}_n(E) = \min\, L_n(z_\bullet)$$

where the minimum is taken over all n-tuples in E. Since $\log \frac{1}{r}$ is large when $r \simeq 0$ this means intuitively that we tryo to choose separated n-tuples in order to minimize the L_n-function. Notice for example that when $n = 2$ then the minium is achieved for a pair of points in E whose distance is maximal, i.e. $\mathcal{L}_2$ is the diameter of E. As n increases one has

0.8 Proposition. *The sequence $\{\mathcal{L}_n\}$ is non-decreasing.*

Proof. Let $z_1, \ldots, z_{n+1}$ minimize the L_{n+1}-function which gives

$$\mathcal{L}_{n+1}(E) = \frac{1}{n(n+1)} \cdot \sum_{k \neq j}^{(1)} \log \frac{1}{|z_j - z_k|} + \frac{2}{n(n+1)} \cdot \sum_{k=2}^{k=n+1} \log \frac{1}{z_1 - z_k|}$$

where (1) above the sum above means that we only consider pairs k, j which both are ≥ 2. Since $z_2, \ldots, z_{n+1}$ is an n-tuple we get the inequality

$$\mathcal{L}_{n+1}(E) \geq \frac{1}{n(n+1)} \cdot n(n-1) \cdot \mathcal{L}_n(E) + \frac{2}{n(n+1)} \cdot \sum_{k=2}^{k=n+1} \log \frac{1}{z_1 - z_k|}$$

The same inequaliiy holds when when we instead of z_1 delete some z_j for $2 \leq j \leq n+1$ and taking the sum of the resulting inequalities we obtain

$$(n+1)\mathcal{L}_{n+1}(E) \geq \frac{1}{n} \cdot n(n-1) \cdot \mathcal{L}_n(E) + \frac{2}{n(n+1)} \cdot \sum_{k \neq j} \log \frac{1}{|z_j - z_k|}$$

The last term is $2 \cdot \mathcal{L}_{n+1}$ which gives:

$$(n-1) \cdot \mathcal{L}_{n+1}(E) \geq \frac{1}{n} \cdot n(n-1) \cdot \mathcal{L}_n(E) = (n-1)\mathcal{L}_n(E)$$

A division by $n - 1$ gives the requested inequality.

0.8 Definition. *Put*

$$\mathfrak{D}(E) = \lim_{n \to \infty} e^{-\mathcal{L}_n(E)}$$

This non-negative number is called the transfinite diameter of E.

Remark. The definition means that $\mathfrak{D}(E) = 0$ if and only if $\mathcal{L}_n(E)$ tends to $+\infty$ as n increases. Intuitively this means that we are not able to choose large tuples in E separated enough to keep the sum of the log-terms bounded. Another number is associated to E is defined by:

$$\mathcal{J}_*(E) = \min_{\mu} J(\mu)$$

where the minimum is taken over all probability measures in E.

0.9 Definition. *The logarithmic capacity of E is defined by:*

$$\mathrm{Cap}(E) = e^{-J_*(E)}$$

0.10 Theorem. *For each compact set E one has the equality*

$$\mathrm{Cap}(E) = \mathfrak{D}(E)$$

Proof. First, let $n \geq 2$ and $z_1^*, \ldots, z_n^*$ is some n-tuple where $L_n(z_\bullet) = \mathcal{L}_n(E)$. Now we have the probability measure

$$\mu = \frac{1}{n} \cdot \sum_{k=1}^{k=n} \delta_{z_k}$$

It is clear that the energy

$$J(\mu) = \frac{n(n-1)}{n^2} \cdot L_n(z_\bullet)$$

Hence we have the inequality

$$\mathcal{J}_*(E) \leq \frac{n(n-1)}{n^2} \cdot \mathcal{L}_n(E)$$

Since $\frac{n(n-1)}{n^2}$ tends to one as $n \to \infty$ a passage to the limit gives:

$$\mathcal{J}_*(E) \leq \lim_{n \to \infty} \mathcal{L}_n(E)$$

Taking exponentials and recalling the negative signs in Definition x and x we conclude that

$$\text{(i)} \qquad \mathfrak{D}(E) \leq \text{Cap}(E)$$

The opposite inequality follows since we can approximate probability measures on E by discrete measures. Leave this as a TRIVIAL EXERCISE,

A. Proof of Myhrberg's theorem

When E is totally disconnected we can surrender E by open sets Ω such that Dirichlet's problem has a solution in the exterior domain $\mathbf{C} \setminus \bar{\Omega}$. After a suitable passage to the limit as these domains shrink to E we obtain a special measure supported by E. To obtain this we employ a construction which was introduced in the present context by De Vallé Poussin.

A.1 Nested coverings Let E as above be a totally disconnected and compact set and consider some $z_* \in E$. Choose a small Jordan domain U which contains z_* while $\partial U \cap E = \emptyset$. In particular can take U to be contained in the disc of radius $1/2$ centered at z_* which gives

$$|z_1 - z_2| < 1$$

for each pair z_1, z_2 in $E \cap U$. To compensate for the failure of solving Dirichlet's problem we construct a sequence of open sets $\{V_N\}$ as follows. For each positive integer N one has the dyadic grid $\mathcal{D}_N$ of open squares whose sides are 2^{-N}. We get the finite family $\mathcal{D}_N(E)$ of dyadic squares in $\mathcal{D}_N$ which have a non-empty intersection with E. The union of this finite family of open squares gives an open neighborhood V_N of E. We consider only large N so that 2^{-N} is strictly larger than the distance of E to ∂U. Let Ω_N^* be the exterior connected component of $D \setminus \bar{V}_N$ whose closure contains ∂U.

A.2 Exercise. Show by a figure that Ω_N^* is a doubly connected domain whose boundary is the disjoint union of ∂U and a closed Jordan curve Γ_N formed by line segments from squares in the dyadic grid. Notice also that $\{\Omega_N^*\}$ form an increasing sequence of open sets where Γ_N appears as a compact subset of Ω_{N+1}^* for each N and conclude that

$$\cup \, \Omega_N^* = D \setminus E$$

Next, fix some point $z_0 \in D \setminus E$ and from now on N are so large that $z_0 \in \Omega_N^*$ hold. The Dirichlet problem has a solution in each domain Ω_N^*. This gives a unique pair of non-negative measures μ_N, γ_N where μ_N is supported by Γ_N and ρ_N by ∂U such that

$$(*) \qquad h(z_0) = \int_{\Gamma_N} h(\zeta) \cdot d\mu_N(\zeta) + \int_{\partial U} h(\zeta) \cdot d\rho_N(\zeta)$$

hold for every h-function which is harmonic in Ω_N^* with continuous boundary values. In particular we let h be the harmonic measure function $\mathfrak{m}_N$ with respect to Γ_N, i.e. it is 1 on Γ_N and zero on ∂U. Then

$$(**) \qquad \mathfrak{m}_N(z_0) = \int_{\Gamma_N} d\mu_N(\zeta) = ||\mu_N||$$

Since $\Gamma_N \subset \Omega_{N+1}^*$ we have $\mathfrak{m}_{N+1} \leq \mathfrak{m}_N$ in Ω_N^* which implies that

$$||\mu_{N+1}|| \leq ||\mu_N||$$

for each N. Hence there exists the limit

$$(***) \qquad \alpha = \lim_{N \to \infty} ||\mu_N||$$

A.3 The case $\alpha = 0$. When this holds the mass of ρ_N tends to one and since (*) in particular hold for h-functions which are harmonic in the whole set U with continuous boundary values on ∂U the reader may verify:

A.4 Proposition. *If $\alpha = 0$ the sequence $\{\rho_N\}$ converges weakly to the representing measure $m(z_0)$ for which*

$$H(z_0) = \int_{\partial U} H(\zeta) \cdot dm(z_0, \zeta)$$

when H is harmonic in U and continuous on $\bar{U}$.

Keeping the case A.3 we can apply the result in Exercise A.4 and obtain the following crucial result.

A.5 Proposition. *When A.3 holds there exists a pair of positive numbers $0 < a < A$ such that*

$$a \leq \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) \leq A$$

hold for all $w \in E$ and every N.

A.6 Exercise. Prove this result. Here is a hint in the case when U is the unit disc so that $\partial U = T$ is the unit circle. If $w \in E$ we have

$$|e^{i\theta} - w| = |1 - \bar{w} \cdot e^{i\theta}|$$

This gives the equality

$$\text{(i)} \qquad \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta) = \int_T \log \frac{1}{|1 - \bar{w}\zeta|} \cdot d\rho_N(\zeta)$$

Next, in D we have the harmonic function $H(z) = \log \frac{1}{|1 - \bar{w}z|}$ in D and Exercise A.4 together with (i) give

$$\text{(ii)} \qquad \log \frac{1}{|1 - \bar{w}z_0|} = \lim_{N \to \infty} \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta)$$

At the same time (**) applied with $h(z) = \log \frac{1}{|z - w|}$ gives

$$\text{(iii)} \qquad \log \frac{1}{|z_0 - w|} = \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) + \int_T \log \frac{1}{|\zeta - w|} \cdot d\rho_N(\zeta)$$

By (ii) the last integral converges to $\log \frac{1}{1 - \bar{w}z_0|}$ which entails that

$$\lim_{N \to \infty} \int_{\Gamma_N} \log \frac{1}{|\zeta - w|} \cdot d\mu_N(\zeta) = \log \frac{1 - \bar{w}z_0|}{|z_0 - w|}$$

Since $|z_0| < 1$ and there is some $r < 1$ such that $|w| \leq r$ for every $w \in E$ the last term is between a and A for a pair of positive numbers which proves Proposition A.5 in the case when $U = D$. For a general Jordan domain the reader can deduce the requested result using a conformal map or suitable Green's functions in U adapted to the point z_0.

A.7 The limit measure μ_*. Keeping A.3 we have the probability measures

$$\nu_N = \frac{1}{||\mu_N||} \cdot \mu_N$$

We can extract a subsequence which converges weakly to a probability measure μ which by (xx) is supported by E. For this limit measure the following holds:

A.8 Proposition. *The potential function U_μ is $+\infty$ on E except possible at a finite set of points.*

Proof. GIVE IT is easy ...

A.9 Final part of the proof of Theorem 0.2. Suppose first that E is a removable singularity for bounded harmonic functions. Working locally as above around some $z_* \in E$ and a Jordan domain U, we obtain for each N the harmonic measure function $\mathfrak{m}_N$ in Ω_N^*. They take values in $(0,1)$ and passing to a subsequence we obtain a bounded harmonic limit function $\mathfrak{m}_*$ defined in $U \setminus E$. By construction $\mathfrak{m}_* = 0$ on ∂U so if it extends to a harmonic function in U it must be identically zero. This entails that

$$\lim_{N \to \infty} \mathfrak{m}_N(z_0) = 0$$

Now we recall that $\mathfrak{m}_N(z_0) = ||\mu_N||$ and conclude that A.3 holds which by Theorem XX gives the existence of a probability measure μ on E such that $U_\mu(w) = +\infty$ for all $w \in E \cap U$.

The converse. Assume that there exists μ on E for which $U_\mu(w) = +\infty$ for all $w \in E$. Given a large positive constant we consider the open subset of U defined by

$$\{z \in U \quad : \quad U_\mu(z) > C\}$$

By the hypothesis this is an open neighborhood of $E \cap U$ so when N is large we have $U_\mu \geq C$ on Γ_N. Next, U_μ restricts to a continuous function on ∂U and we find its harmonic extension $H(z)$ to the whole Jordan domain U. Now $U_\mu - H$ is harmonic in Ω_N^* and if C_* is the maximum of H on Γ_N it follows that $U_\mu - H \geq C - C_*$ on Γ_n. This gives the inequality

$$\mathfrak{m}_N(z_0) \leq \frac{1}{C - C_*} \cdot [U_\mu(z_0) - H(z_0)]$$

Here we can start with arbitrary large C and get (xx) for sufficiently large N which entails that

$$\lim_{N \to \infty} \mathfrak{m}_N(z_0) = 0$$

The conclusion is valid for any starting point $z_0 \in U \setminus E$ and let us consider a bounded harmonic function h in $U \setminus E$ with continuous boundary values on ∂U. It can be restricted to Ω_N^* for every N. In A.2 we could have started with an arbitrary point $z_0 \in U \setminus E$ and by (*) from A.2 we have

$$h(z_0) =$$

Passing to the limit as $N \to \infty$ it follows from (x) and the discussion in (xx) that

$$h(z_0) = \int_{\partial U} h(\zeta) \cdot dm(z_0, \zeta)$$

Since this hold for every z_0 in $U \setminus E$ we get the requested harmonic extension H given by

$$H(z) = \int_{\partial U} h(\zeta) \cdot dm(z, \zeta) \quad : \quad z \in U$$

B. Equilibrium distributions and Robin's constant.

Let E be a compact set in $\mathbf{C}$. To each probability measure μ supported by E we get the potential function

$$U_\mu(z) = \int \log \frac{1}{|z - \zeta|} \cdot d\mu(\zeta)$$

We are going to construct a special μ for which U_μ either is identically $+\infty$ or else takes a constant value almost everywhere on E with respect to μ. First we carry out the construction in the special case when E is a finite union of pairwise disjoint and closed Jordan domains $U_1, \ldots, U_m$ for some $m \geq 1$. We also assume that each Jordan curve ∂U_k is of class C^1. When this holds we get the connected exterior domain

$$\Omega^* = \mathbf{C} \cup \{\infty\} \setminus \cup \bar{U}_k$$

Here we can solve Dirichlet's problem. in particular we obtain the unique probability measure μ on $\partial \omega^*$ such that

$$H(\infty) = \int H \cdot d\mu$$

for every harmonic function H in Ω^* with continuous boundary values. If z_1 and z_2 are two points in $/cup\, U_k$ which may or may not belong to the same Jordan domain then we notice that the function

$$H(z) = \log|z - z_1| - \log|z - z_2|$$

is harmonic i Ω^*. Moreover, as $|z| \to \infty$ we notice that

$$H(z) = \log|1 - \frac{z_1}{z}| - \log|1 - \frac{z_2}{z}|$$

and in the limit we have $H(\infty) = 0$. Since $\log r = -\log \frac{1}{r}$ for each $r > 0$ it follows that

$$U_\mu(z_1) = U_\mu(z_2)$$

Hence the function $z \mapsto U_\mu(z)$ is constant in the interior of E Since the boundary curves $\{\partial U_k\}$ are C^1 it follows that U_μ extends to a continuous function with constant value on the whole set E. Of course, U_μ is also continuous outside E where it is harmonic. In fact, we conclude that U_μ is a globally defined and continuous super-harmonic function in $\mathbf{C}$. The measure μ is called the equilibrium distribution of E.

Remark. If E is contained in the unit disc it is clear that the constant value of U_μ is positive. On the other hand, let $R > 1$ and E is the disc $|z \leq R$. here μ is the measure $\frac{1}{\pi} \cdot d\theta$ on the circle of radius R and we find that the constant value is $-\log R$.

Notation. If a is the constant value of U_μ we set

$$\text{cap}(E) = e^{-a}$$

and refer to this as the capacity of E. For example, if E is the disc $|z| \leq r$ where r is small we see that the capacity becomes r.

The general case. Now E is an arbitrary compact set. To construct a special probability measure μ_E we use a similar construction as in section A. Thus, for $N \geq 1$ we get the family of cubes in $\mathcal{D}_N$ which have a non-empty intersection with E and then we construct the outer boundary curves of thus set which borders a connected exterior domain Ω_N^* whose boundary now will be a union of closed and piecewise linear Jordan curves where two of these may interest at corner points. We solve the Dirichlet problem and exactly as above we find the equilibrium measure μ_N supported by $\partial\Omega_N^*$.

C. Cartan's theorem

We shall actually establish an inequality in Theorem C.1 below which has independent interest since it applies to compact sets E which are not necessarily harmonic null sets. Consider a pair (h, μ) where μ is a probability measure with compact support in a compact set E of $\mathbf{C}$ with planar Lebesgue measure zero while $h \in \mathfrak{H}_*$. To each point $a \in E$ and every $r > 0$ we have the open disc $D_r(a)$ centered at a and can regard its μ-mass. This gives anon-decreasing function

$$r \mapsto \mu(D_r(a)) \quad : \; r > 0$$

Put

$$\mathcal{U}^* = \{a \in E \quad : \exists\, r > 0 \quad : \; \mu(D_r(a)) > h(r)\} \tag{1}$$

We assume that the pair (h, μ) is such that this set is non-empty. Since μ is a Riesz measure one has the limit formula

$$\lim_{\rho \to r} \mu(D_\rho(a) = \mu(D_r(a)$$

for each $r > 0$ where the limit is taken as ρ increases to r. From this it is obvious that $\mathcal{U}^*$ is a relatively open subset of E and in the closed complement we have

$$a \in E \setminus \mathcal{U}^* \implies \mu(D_r(a)) \leq h(r) \quad : \; \forall\, r > 0 \tag{2}$$

Now the size of $\mathcal{U}^*$ is controlled as follows:

C.1 Cartan's Covering Lemma. *There exists a sequence $\{a_\nu\}$ in E and a sequence of positive numbers $\{r_\nu\}$ such that the following hold:*

$$\mathcal{U}^* \subset \cup \bar{D}_{r_\nu}(a_\nu) \quad \text{and} \quad \sum h(r_\nu) \leq 6$$

Moreover, for each $z \in \mathbf{C}$ at most five discs from the family $\{D_{r_\nu}(a_\nu)\}$ contains z.

Proof We may assume that $\mathcal{U}^* \neq \emptyset$. Set

(1) $$\lambda_1^*(r) = \max_{a \in E} \mu(D_r(a))$$

Since the functions $r \mapsto \mu(D_r(a))$ are lower semi-continuous for each a, it follows that the maximum function $\lambda_1^*(r)$ also is lower semi-continuous. Hence the set $\{r\colon\ \lambda_1^*(r) > h(r)\}$ is open and we find its least upper bound r_1^*. Thus,

(2) $$\lambda_1^*(r_1^*) = h(r_1^*) \quad : \lambda_1^*(r) < h(r) \ \text{ for all } \ r > r_1^*$$

Pick $a_1 \in E$ so that

(3) $$\lambda_1^*(r_1^*) < \mu(D_{r_1^*}(a_1)) + 1/2$$

Next, set $E_1 = E \setminus D_{r_1^*}(a_1)$ and define

$$\lambda_2^*(r) = \max_{a \in E_1} \mu(D_r(a))$$

If $\lambda_2^*(r) \leq h(r)$ for every r we stop the process. Otherwise we find the unique largest r_2^* such that

$$\lambda_2^*(r_2^*) = h(r_2^*)$$

Notice that $r_2^* \leq r_1^*$ holds since h is non-decreasing while it is obvious that $\lambda_2^* \leq \lambda_1^*$. This time we pick $a_2 \in E$ so that

$$\lambda_2^*(r_2^*) < \mu(D_{r_2^*}(a_2)) + 2^{-2}$$

Put $E_2 = E_1 \setminus D_{r_2^*}$ and continue as above, i.e. inductively we get E_n and set

$$\lambda_{n+1}(r) = \max_{a \in E_n} \mu(D_r(a))$$

The process continues if we have found r_{n+1}^* so that $\lambda_{n+1}(r_{n+1}^*) = h(r_{n+1}^*)$, then we pick $a_{n+1} \in E_n$ where

(4) $$\lambda_{n+1}(r_{n+1}^*) \leq \mu(D_{r_{n+1}^*}(a_{n+1})) + 2^{-n-1}$$

In this way we get the sequence $r_1^* \geq r_2^* \geq \ldots$ and a family of discs $\{D_{r_\nu^*}(a_\nu)\}$. To simplify notations we set

$$D_\nu^* = D_{r_\nu^*}(a_\nu)$$

Sublemma Every point $a \in E$ belongs to at most five many D^*-discs.

Proof. If some a belongs to six discs then elementary geometry gives a pair a_k, a_ν such that the angle between the lines $[a, a_k]$ and $[a.a_\nu]$ is $< \pi/3$. Suppose that for example that $|a - a_k| \geq [a - a_\nu|$. Euclidian geometry gives

$$|a_k - a_\nu| < |a - a_k|$$

But this is impossible. For say that $k < \nu$. Now the disc D_k^* was removed and a_ν is picked from the subset E_ν of E_k while $E_k \cap \Delta_k = \emptyset$.

Proof continued. The Sublemma implies that

(5) $$\sum \mu(D_\nu^*) \leq 5 \cdot \mu(E) = 5$$

The convergence of (5) and (4) imply that $\lim_{\nu \to \infty} r_\nu^* = 0$. From this it follows that

(6) $$\mathcal{U}^* \subset \cup \bar{D}_{r_\nu}(a_\nu)$$

Finally we have

(7) $$\sum h(r_\nu^*) = \sum \lambda_\nu^*(r_\nu^*) \leq \sum [\mu(D_\nu^*) + 2^{-\nu}] \leq 5 \cdot \mu(E) + \sum 2^{-\nu} = 6$$

This completes the proof of Cartan' s Covering Lemma.

The family $\mathcal{G}_h$. Let $g(r)$ be a positive function defined on $(0, +\infty)$ which satisfies:

$$\lim_{r \to 0} g(r) = +\infty$$

In this family we get those g-functions for which

$$(*) \qquad \int_0^1 g(r) \cdot dh(r) < \infty$$

This family is denoted by $\mathcal{G}_h$. With this notation we have:

C.2 Lemma *For each $g \in \mathcal{G}_h$ and every point $a \in E \setminus \mathcal{U}^*$ one has*

$$\int_E g(|z-a|) d\mu(z) \leq \int_0^\rho g(r) dh(r) \quad \text{where } \; h(\rho) = 1$$

Proof. Since a is outside $\mathcal{U}^*$ we have

$$\mu(D_r(a)) \leq h(r)$$

for every $r > 0$. Moreover, we recall that μ has total mass one and now the reader can verify the inequality in Lemma C.2 b using a partial integration.

C.3 A special choice of g. Let us take

$$g(r) = \text{Log}\,\frac{1}{r} \quad : \; 0 < r < 1 \quad : \quad g(r) = 0 \quad : \; r \geq 1$$

This g-function belongs to $\mathcal{G}_h$ by the condition on h-functions in Cartan's theorem. Next, for every $\lambda > 1$ we get the function $h_\lambda = \lambda \cdot h$ in $\mathfrak{H}_*$ and set:

$$E \setminus \mathcal{U}^*(\lambda) = \{a \in E \quad : \; \mu(D_r(a)) \leq \lambda \cdot h(r) \; : \; \forall \; r > 0\}$$

Proposition XX(measure general) applied with h_λ gives:

$$(1) \qquad \int_E g(|z-a|) d\mu(z) \leq \lambda \cdot \int_0^{\rho/\lambda} g(r) dh(r) \quad : \; a \in E \setminus \mathcal{U}^*(\lambda)$$

A partial integration shows that the right hand side in (1) becomes

$$g(\rho) + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r}$$

Hence we have the inequality

$$(2) \qquad \int g(|z-a|) \cdot d\mu(z) \leq g(\rho) + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r} \quad : \quad a \in E \setminus \mathcal{U}^*(\lambda)$$

In addition to this, the Covering Lemma gives an inclusion

$$(3) \qquad \mathcal{U}^*(\lambda) \subset \cup \, D_{r_\nu}(a_\nu) \quad \text{where} \sum \; h_\lambda(r_\nu) < 6$$

Since $h_\lambda = \lambda \cdot h$ this means that the outer h-measure

$$(4) \qquad h^*(\mathcal{U}^*(\lambda)) \leq \frac{6}{\lambda}$$

Hence we have proved the following where we recall that $g(r) = \text{Log}\,\frac{1}{r}$:

C.4 Theorem. *For every triple (E, μ, h), where μ is a probability measure supported by E and $h \in \mathfrak{H}_*$, and any $\lambda > 1$ there exists a relatively open subset $\mathcal{U}^*(\lambda) \subset E$ such that the following two inequalities hold:*

$$\text{(i)} \qquad \int_E \log \frac{1}{|z-a|} \cdot d\mu(z) \leq \log \frac{1}{\rho} + \lambda \cdot \int_0^\rho \frac{h(r)dr}{r} \quad : \quad a \in E \setminus \mathcal{U}_*(\lambda)$$

$$\text{(ii)} \qquad h^*(U_*(\lambda)) < \frac{6}{\lambda}$$

C.11 Proof of Theorem 0.5. Let $E \in \mathcal{N}_{\text{harm}}$ which by Theorem 0.2 gives a probability measure μ supported by E such that that the left hand side in (i) is $+\infty$ for every $a \in E$. It follows that the set $E \setminus \mathcal{U}^*(\lambda)$ is empty for every $\lambda > 1$. With a fixed λ we apply Cartan's covering Lemma and since $E = \mathcal{U}^*(\lambda)$ it follows that

$$h^*(E) \leq \frac{6}{\lambda}$$

Here $\lambda > 1$ is arbitrary which gives $h^*(E) = 0$ as required and of Cartan's theorem follows.

D. Cantor sets.

We construct a family of closed subsets of $[0, 1]$ as follows. Let $1 < p_1 < p_2 < \ldots$ be some strictly increasing sequence of real numbers such that the products $\{p_1 \cdots p_n\}$ tend to $+\infty$ as n increases. Then we can construct a decreasing sequence of closed sets $E_1, E_2, \ldots$ where each E_n is the union of 2^n-many closed intervals with equal length

$$\ell_n = 2^{-n} \cdot \frac{1}{p_1 \cdots p_n}$$

D.1 The construction. First E_1 is any closed interval $[a_1, b_1]$ with

$$b_1 - a_1 = \frac{1}{2p_1}$$

Inside this closed interval we pick two pairwise disjoint closed interval both of length ℓ_2 and let E_2 be their union. In the next step we pick a pair of closed intervals both of length ℓ_3 from each of the two intervals in E_2. Their union gives the set E_3 and we continue in the same way for every n and arrive at a closed set

$$\mathcal{E} = \cap E_n$$

We refer to $\mathcal{E}$ as a Cantor set. The construction is flexible since we do not impose any condition on specific positions while we at stage n pick pairs of intervals of length ℓ_{n+1} from each of the 2^n many intervals of E_n. Thus, for a given p-sequence we obtain a whole family of Cantor sets denoted by $\text{Cantor}(p_\bullet)$. The next result gives a condition for such Cantor sets to have harmonic measure zero.

D.2 Theorem. *The following are equivalent for an arbitrary sequence* $p_\bullet$ *as above*:

$$\text{Cantor}(p_\bullet) \subset \mathcal{N}_{\text{harm}} \text{ holds if and only if } \sum_{\nu=1}^{\infty} \frac{\text{Log}\, p_\nu}{2^\nu} = +\infty$$

The proof uses the explicit formulas for Robin constants of intervals on the real line. For the detailed proof we refer to page xx-xx in [Nevanlinna].