

V. Uniqueness theorems for analytic functions.

0. *Introduction.*

A. *A sharp version of the Phragmén-Lindelöf theorem*

B. *Asymptotic series.*

C. *A uniqueness theorem for asymptotic series*

Introduction.

A sharp version of the Phragmén-Lindelöf theorem is proved in Theorem A.2. It is preceded by a differential inequality where Carleman's result in Theorem A.1 was inspired by earlier constructions due to Lindelöf. Asymptotic series are studied in section B where earlier work by Borel led Carleman to the general construction in Theorem B.1. The question of uniqueness is expressed via Theorem B.6 and is settled via solutions to a variational problem in Section C.

A. The Phragmén-Lindelöf theorem.

Let $f(z)$ be an entire function. To each $0 \leq \phi \leq 2\pi$ we set

$$(*) \quad \rho_f(\phi) = \max_r |f(re^{i\phi})|$$

The text-book *Le calcul des résidues* by Ernst Lindelöf contains examples of entire functions f where $\rho_f(\phi)$ is finite for all ϕ with the exception $\phi = 0$, i.e. only along the positive real axis the ρ -number fails to be bounded. An example is the entire function

$$f(z) = \frac{1}{z^2} \cdot \sum_{\nu=2}^{\infty} \frac{z^{\nu}}{(\log \nu)^{\nu}}$$

Here one verifies that there exists a constant k such that:

$$(**) \quad |f(re^{i\phi})| \leq \exp(e^{\frac{k}{|\phi| \cdot |2\pi - \phi|}})$$

It turns out that the example above is essentially sharp. Namely, assume that the ρ -number in (*) is finite for almost every ϕ . Then the ρ_f -function cannot be too small, unless f is reduced to a constant. Before Theorem A.1 is announced we introduce the non-negative function

$$(***) \quad \omega(\phi) = \log^+ [\log^+ \rho_f(\phi)]$$

Since we have taken a two-fold logarithm $\omega(\phi)$ is considerably smaller compared to the ρ -function.

A.1.Theorem. *For every non-constant entire function $f(z)$ one has*

$$\int_0^{2\pi} \omega(\phi) \cdot d\phi = +\infty$$

Proof. Assume that f is not a constant. Consider the maximum modulus function

$$M(r) = \max_{|z|=r} |f(z)|$$

By the ordinary Liouville theorem the M -function increases to infinity. So we may assume that $M(r) \geq 1$ when $r \geq r_*$ for some r_* . Put

$$(i) \quad v(r) = \log M(r) \quad : \quad U(z) = \log |f(z)|$$

Given $r \geq r_*$ we consider the domain

$$(ii) \quad \Omega_r = \{U > \frac{v(r)}{2}\} \cap \{|z| < r\}$$

Next, let ζ_r be some point on the circle $|z| = r$ where $|f(\zeta_r)| = M(r)$ where ζ_r always can be chosen so that there exist arbitrary small δ where $|f(\zeta_{r-\delta})| = M(r-\delta)$ and $\lim_{\delta \rightarrow 0} \zeta_{r-\delta} = \zeta_r$. Next, in Ω we get the connected component Ω_* whose boundary contains ζ_r . Put

$$(iii) \quad \gamma = \partial\Omega_* \cap \{|z| = r\}$$

Notice that

$$(iv) \quad U(z) \leq \frac{v(r)}{2} \quad : \quad z \in \partial\Omega_* \cap \{|z| < r\}$$

So if W is the harmonic function in the disc D_r with boundary values 1 on γ and 0 on $\{|z| = 1\} \setminus \gamma$ we have:

$$(v) \quad U(z) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(z) \leq 0 \quad : \quad z \in \partial\Omega_*$$

The maximum principle entails that (v) also holds in Ω_* . Hence there exist arbitrary small $\delta > 0$ such that

$$(vi) \quad v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot W(\zeta_{r-\delta}) \leq 0$$

Let $2r \cdot \ell$ be the total length of the intervals which belong to γ . By the general inequality from XX we have

$$(vii) \quad W(\zeta_{r-\delta}) \leq \frac{1}{2\pi} \int_{-\ell}^{\ell} \frac{r^2 - (r-\delta)^2}{r^2 - 2r(r-\delta)\cos\theta + (r-\delta)^2} d\theta$$

Let $h(r-\delta)$ denote the right hand side in (vii) which by (vi) gives us arbitrary small $\delta > 0$ such that

$$(viii) \quad v(r-\delta) - \frac{v(r)}{2} - \frac{v(r)}{2} \cdot h(r-\delta) \leq 0$$

Rewriting this inequality we obtain

$$(*) \quad \frac{v(r) - v(r-\delta)}{\delta} \geq \frac{v(r)}{2} \cdot \frac{1 - h(r-\delta)}{\delta}$$

Next, from the definition of the h -function one has the limit formula

$$(ix) \quad \lim_{\delta \rightarrow 0} \frac{1 - h(r-\delta)}{\delta} = \frac{1}{2\pi} \cdot \frac{\cos \ell}{\sin \ell}$$

Passing to the limit as $\delta \rightarrow 0$ in (vii) we get the differential inequality:

$$(**) \quad v'(r) \geq \frac{v(r)}{2\pi r} \cdot \frac{\cos \ell}{\sin \ell}$$

Next, put

$$\log r = s \quad \text{and} \quad \log \frac{v(r)}{2} = g(s)$$

By derivation rules we see that (**) gives

$$(***) \quad \frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos \ell}{\sin \ell}$$

Next, identifying γ with a subset of the periodic interval $0 \leq \phi \leq 2\pi$ it is clear that the definition of the ω -function gives the inclusion

$$(x) \quad \gamma \subset \{\omega(\phi) \geq g(s)\}$$

So if $\lambda(s)$ is the Lebesgue measure of the set $\{\omega(\phi) \geq g(s)\}$ then $\ell \leq \lambda(s)$ and (***) gives

$$(***) \quad \frac{dg}{ds} \geq \frac{1}{2\pi} \cdot \frac{\cos \lambda(s)}{\sin \lambda(s)}$$

Next, the inequality $\sin(t) \geq \frac{2}{\pi} \cdot t$ gives a positive constant k which is independent of s such that the following hold for sufficiently large s , i.e. to ensure that the corresponding r -value satisfies $M(r) \geq 1$:

$$(xi) \quad \frac{dg}{ds} \geq \frac{k}{\lambda(s)}$$

Hence, starting from some sufficiently large s_0 one has

$$(xii) \quad \int_{s_0}^s \lambda(s) \cdot dg(s) \geq k(s - s_0)$$

This inequality implies in particular that one has a divergent integral:

$$(xiii) \quad \int_{s_0}^{\infty} \lambda(s) \cdot dg(s) = +\infty$$

Finally, the general equality for distribution functions from XXX gives:

$$(xiiii) \quad \int_0^{2\pi} \omega(\phi) \cdot d\phi = \int_0^{\infty} \lambda(s) \cdot dg(s)$$

The last integral is $+\infty$ by (xiii) and the requested divergence for the integral of the ω -function follows.

Remark. At the end of the article [XXX] Carleman points out that the proof above gives a sharp version of the Phragmén- Lindelöf theorem. More precisely one has the following: Let $f(z)$ be analytic in a sector

$$S_\alpha = \{z = re^{i\phi} \quad : \quad -\alpha < \phi < \alpha\}$$

Define $\omega(\phi)$ as above when $-\alpha < \phi < \alpha$. With these notations one has:

A.2. Theorem. *Let f be bounded on the half-lines $\arg(z) = \alpha$ and $\arg(z) = -\alpha$ and assume also that*

$$\int_{-\alpha}^{\alpha} \omega(\phi) \cdot d\phi < \infty$$

Then $f(z)$ is bounded in the whole sector.

A.3. Exercise. Deduce Theorem A.2 from the preceeding results.

B. Asymptotic series.

Introduction. The notion of asymptotic series was expressed as follows by Poincaré:

Let $f(z)$ be complex-valued function defined in some subset E of \mathbb{C} and z_0 is a boundary point. We say that f has an asymptotic series expansion at z_0 if there exists a sequence of complex numbers c_0, c_1, \dots such that $\lim_{z \rightarrow z_0} f(z) = c_0$ and for each $n \geq 0$ one has:

$$(*) \quad \lim_{z \rightarrow z_0} (z - z_0)^{-n-1} [f(z) - (c_0 + c_1 + \dots + c_n z^n)] = c_{n+1}$$

where the limit is taken as z stay in E .

It is obvious that if f has an asymptotic expansion at z_0 then the sequence $\{c_n\}$ is unique. Constructions of functions which admit asymptotic expansions appear in Emile Borel's thesis *Sur quelques points de la théorie des fonctions* from 1895 and he proved for example that for every sequence of real numbers $\{c_n\}$ there exists a C^∞ -function $f(x)$ on the real line whose Taylor expansion at $x = 0$ is given by the sequence, i.e.

$$\frac{f^{(n)}(0)}{n!} = c_n \quad : \quad n = 0, 1, \dots$$

Following [Car: xx, page 29-31] we prove a complex version of Borel's result where D_+ denotes the open half-disc $\{\Re(z) > 0 \cap \{|z| < 1\}$.

B.1. Theorem. *To each sequence $\{c_n\}$ of complex numbers there exists a bounded analytic function $F(z)$ in D_+ which has an asymptotic series expansion at $z = 0$ given by $\{c_n\}$.*

Proof. It suffices to prove this when $c_0 = 0$. Let a_1, a_2, \dots be a sequence of positive real numbers such that $\sum a_n < \infty$. Given $\{c_n\}$ we construct a sequence of functions $P_1(z), P_2(z), \dots$ which are analytic in the half plane $\Re(z) > 0$ as follows: First

$$(i) \quad P_1(z) = c_1 z \left(1 - \frac{z}{z + \epsilon_1}\right) \quad : \quad \epsilon_1 = \frac{\alpha_1}{|c_1|} \implies$$

$$(ii) \quad |P_1(z)| = |c_1| \cdot \epsilon_1 \cdot \frac{|z|}{|z + \epsilon_1|} \leq \alpha_1 \quad : \quad \Re(z) \geq 0$$

Now $P_1(z)$ has a series expansion at $z = 0$:

$$(ii) \quad P_1(z) = \sum_{\nu=1}^{\infty} c_{\nu}^{(1)} \cdot z^{\nu}$$

Notice that the series converges in the disc $|z| < \epsilon_1$. Set

$$(iii) \quad P_2(z) = [c_2 - c_2^{(1)}] \cdot z^2 \cdot \left(1 - \frac{z}{z + \epsilon_2}\right) \quad : \quad |c_2 - c_2^{(1)}| \cdot \epsilon_2 \leq a_2$$

With such a careful choice of a small positive ϵ_2 we see that

$$(iii) \quad |P_2(z)| \leq a_2 \cdot |z| \quad : \quad \Re(z) \geq 0$$

Again we obtain a convergent series at $z = 0$:

$$(iv) \quad P_2(z) = P_1(z) = \sum_{\nu=2}^{\infty} c_{\nu}^{(2)} \cdot z^{\nu}$$

The inductive construction. Let $n \geq 3$ and suppose that P_1, \dots, P_{n-1} have been constructed where we for each $1 \leq k \leq n-1$ have a series expansion

$$(v) \quad P_k(z) = \sum_{\nu=k}^{\infty} c_{\nu}^{(k)} \cdot z^{\nu}$$

Then we define

$$P_n(z) = [c_n - (c_n^{(1)} + \dots + c_n^{(n-1)})] \cdot z^n \cdot \left(1 - \frac{z}{z + \epsilon_n}\right) \quad : \quad |c_n - (c_n^{(1)} + \dots + c_n^{(n-1)})| \cdot \epsilon_n \leq \alpha_n$$

So we obtain a new series at $z = 0$:

$$(vi) \quad P_n(z) = \sum_{\nu=n}^{\infty} c_{\nu}^{(n)} \cdot z^{\nu}$$

Staying in the half-disc D_+ , the inductive construction gives

$$\max_{z \in D_+} |P_n(z)| \leq \alpha_n \quad : \quad n = 1, 2, \dots$$

Hence there exists a bounded analytic function in D_+ defined by

$$F(z) = P_1(z) + P_2(z) + \dots$$

At this stage we leave as an exercise to the reader to verify that

$$\lim_{z \rightarrow 0} z^{-n-1} \cdot [F(z) - (c_1 z + \dots + c_n z^n)] = c_{n+1}$$

B.2. Uniqueness of asymptotic expansions.

There exist functions whose asymptotic series is identically zero. Here is an example:

$$f(z) = e^{-\frac{1}{z^2}}$$

If $z = re^{i\theta}$ with $-\pi/8 \leq \theta \leq \pi/8$ we see that

$$|f(re^{i\theta})| = \exp\left(-\frac{\cos 2\theta}{r^2}\right) \leq \exp\left(-\frac{1}{\sqrt{2} \cdot r^2}\right)$$

It follows that the asymptotic series at $z = 0$ is identically zero. Via a conformal map from the half-disc D_+ to the unit circle we are led to the following problem: Let $f(z)$ be analytic in the open unit disc D . Suppose that

$$(*) \quad \lim_{z \rightarrow 1} \frac{f(z)}{(1-z)^n} = 0 \quad : \quad n = 1, 2, \dots$$

We seek growth conditions on f in order that $(*)$ implies that f is identically zero. An answer to this uniqueness problem was proved by Carleman in [Car]. Namely. consider a sequence of real positive numbers A_1, A_2, \dots . To each $n \geq 1$ we put

$$(**) \quad I_n = \exp\left(\frac{1}{\pi} \int_1^\infty \log \left[\sum_{\nu=1}^{\nu=n} \frac{r^{2\nu}}{A_\nu^2} \right] \cdot dr\right)$$

B.3. Definition. Denote by \mathfrak{B} the set of all sequences $\{A_n\}$ such that $\{I_n\}$ is bounded, i.e. there exists some K such that

$$I_n \leq K \quad : \quad n = 1, 2, \dots$$

In [Car: page 7-52] the following existence result is proved:

B.4. Theorem. To each sequence $\{A_n\} \in \mathfrak{B}$ there exists an analytic function $f(z)$ in D which is not identically zero and satisfies:

$$(1) \quad \frac{|f(z)|}{|1-z|^n} \leq A_n \quad : \quad n = 1, 2, \dots$$

while $(*)$ holds.

A converse result. In [loc.cit] appears the converse to the result which ensures uniqueness of the asymptotic expansion at $z = 1$.

B.5. Theorem. Let $\{A_n\}$ be a sequence of positive numbers such that there exists an analytic function $f(z)$ in D which is not reduced to a constant and satisfies $(*)$ and (1) in Theorem B.4. Then $\{A_n\} \in \mathfrak{B}$.

Remark. The results above show that if $\{A_n\}$ is a sequence for which $\{I_n\}$ is unbounded then the asymptotic expansion at $z = 1$ is unique for every analytic function $f(z)$ satisfying (1) in Theorem B.4. The proofs of the two results above rely upon a variational problem which is presented below while the deduction after of the two cited results above are left to the reader who may find details in [Carleman].

C. A variational problem.

Let $n \geq 1$ and a_0, a_1, \dots, a_n some n -tuple of non-negative real numbers where $a_0 > 0$ is assumed. Let $\mathcal{O}(*)$ denote the family of analytic functions $f(z)$ in the unit disc satisfying $f(0) = 1$. Put

$$I(f) = \frac{1}{2\pi} \cdot \sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \int_0^{2\pi} \frac{|f(e^{i\theta})|^2}{|e^{i\theta} - 1|^{2\nu}} \cdot d\theta \quad : \quad I_* = \min_{f \in \mathcal{O}(*)} I(f)$$

Remark. Above we have a variational problem. It turns out that there exists a unique function $f_*(z)$ which yields a minimum. To find f_* we shall use the rational function:

$$\Omega(z) = \sum_{\nu=0}^{\nu=n} a_\nu^2 \left[(1-z) \left(1 - \frac{1}{z} \right) \right]^{n-\nu}$$

Notice that

$$(i) \quad \Omega(e^{i\theta}) = a_0^2 + \sum_{\nu=1}^{\nu=n} a_\nu^2 \cdot |e^{i\theta} - 1|^{2n-2\nu}$$

In particular Ω is real and positive on the unit circle and by symmetry it has n zeros ρ_1, \dots, ρ_n in the unit disc and $\frac{1}{\rho_1}, \dots, \frac{1}{\rho_n}$ are the zeros in the exterior disc which gives the factorization

$$(*) \quad \Omega(z) = z^{-n} \cdot (-1)^n \cdot a_0^2 \cdot p_n(z) \cdot \prod (z - \frac{1}{\rho_\nu}) \quad : p_n(z) = (z - \rho_1) \cdots (z - \rho_n)$$

Next, for every $f \in \mathcal{O}(D)$ with $f(0) = 1$ we put

$$(ii) \quad \phi(z) = \frac{f(z)}{(1-z)^n}$$

Then (i) gives the equality:

$$(iii) \quad I(f) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |\phi(e^{i\theta})|^2 \cdot d\theta$$

We will use the last expression to prove

C.1 Theorem. *The variational problem has a unique solution whose minimum $I_*(n)$ is achieved by the function*

$$(i) \quad f_*(z) = \frac{(1-z)^n}{\prod (1 - \rho_\nu \cdot z)}$$

Moreover,

$$(ii) \quad I_*(n) = I(f_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \text{Log} \left[\sum_{\nu=0}^{\nu=n} a_\nu^2 \cdot \frac{1}{(2 \cdot \sin \frac{\theta}{2})^{2\nu}} \right] \cdot d\theta$$

Proof By (iii) the variational problem is equivalent to seek the minimum of

$$(1) \quad \min_{\phi} I(\phi) = \frac{1}{2\pi} \cdot \int_0^{2\pi} \Omega(e^{i\theta}) \cdot |\phi(e^{i\theta})|^2 \cdot d\theta \quad : \phi(0) = 1$$

With f_* as in (i) from Theorem C.1 one has

$$(2) \quad \phi_*(z) = \frac{1}{\prod (1 - \rho_\nu \cdot z)}$$

Now f_* is a unique minimizing function in Theorem C.1 if we have proved the strict inequality

$$(3) \quad I(\phi_* + h) < I(\phi_*)$$

for every analytic function $h(z)$ in D such that $h(0) = 0$. To show (3) we notice that (1) can be replaced by a complex line integral over $|z| = 1$ which gives

$$(4) \quad I(\phi_*) + I(h) + \frac{1}{2\pi i} \cdot \int_{|z|=1} \Omega(z) \cdot [\bar{\phi}_*(z) \cdot h(z) + \phi_*(z) \cdot \bar{h}(z)] \cdot \frac{dz}{z} =$$

Since $I(h) > 0$ whenever $h \neq 0$ the requested strict inequality follows if we show that the last integral is zero. To prove this we notice that the construction of ϕ_* gives the equation

$$(5) \quad \Omega(e^{i\theta}) \cdot \bar{\phi}(e^{i\theta}) = (-1)^n \cdot a_0^2 \cdot \frac{1}{\prod (e^{i\theta} - \bar{\rho}_\nu)}$$

Now

$$k(z) = \frac{1}{\prod (z - \bar{\rho}_\nu)}$$

is analytic in D and since $h(0) = 0$ it follows that

$$\int_{|z|=1} k(z)h(z) \cdot \frac{dz}{z} = 0$$

This proves that the first term in the integral from (5) vanishes and the second is its complex conjugate since Ω was real on T . Hence f_* yields the unique minimizing function of the variational problem. The equality (ii) for $I(f_*)$ follows by a computation which is left to the reader.