

BRA

A Non-Linear PDE-equation

Introduction. In the article *Über eine nichtlineare Randwertaufgabe bei der Gleichung $\Delta u = 0$* (Mathematisches Zeitschrift vol. 9 (1921), Carleman considered the following equation: Let Ω be a bounded domain in \mathbf{R}^3 with C^1 -boundary and \mathbf{R}^+ the non-negative real line where t is the coordinate. Let $F(t, p)$ be a real-valued and continuous function defined on $\mathbf{R}^+ \times \partial\Omega$. Assume that

$$(0.1) \quad t \mapsto F(t, p)$$

is strictly increasing for every $p \in \partial\Omega$ and that $F(0, p) \geq 0$. Moreover,

$$(0.2) \quad \lim_{u \rightarrow \infty} F(t, p) = +\infty$$

holds uniformly with respect to p . For a given point $q_* \in \Omega$ one seeks a function $u(x)$ which is harmonic in $\Omega \setminus \{q_*\}$ and at q_* it is locally $\frac{1}{|x - q_*|}$ plus a harmonic function. Moreover, it is requested that u extends to a continuous function on $\partial\Omega$ and that $u \geq 0$ in $\bar{\Omega}$. Finally, along the boundary the inner normal derivative $\partial u / \partial n$ satisfies the equation

$$(*) \quad \frac{\partial u}{\partial n}(p) = F(u(p), p) \quad : p \in \partial\Omega$$

Remark. The case when $F(t, p) = kt^4$ for some positive constant k means that we regard the Stefan-Boltzmann equation whose physical interpretation ensures that $(*)$ has a unique non-negative solution u .

Theorem. *For each F satisfying (0.1-0.2) the boundary value problem has a unique solution u .*

The strategy in the proof is to consider a family of boundary value problems where one for each $0 \leq h \leq 1$ seeks u_h to satisfy

$$(*) \quad \frac{\partial u_h}{\partial n}(p) = (1 - h)u_h + h \cdot F(u_h(p), p) \quad : p \in \partial\Omega$$

and u_h has the same pole as u above. Let us begin with

0.1 The case $h = 0$. Here we seek u_0 so that

$$(i) \quad \frac{\partial u_0}{\partial n}(p) = u_0$$

If $G(p)$ is the Greens' function with a pole at q_* we seek a harmonic function h in Ω such that

$$(ii) \quad u_0 = G - h$$

Since $G(p) = 0$ on $\partial\Omega$, the equation (i) holds if

$$(iii) \quad \frac{\partial h}{\partial n}(p) = h(p) + \frac{\partial G}{\partial n}(p) \quad : p \in \partial\Omega$$

This is a classic linear boundary value problem which has a unique solution h . See § xx for further details.

0.2 Properties of u_0 . The construction in (ii) entails that u_0 is superharmonic in Ω and therefore attains its minimum on the boundary. Say that

$$u_0(p_*) = \min_{p \in \bar{\Omega}} u_0(p)$$

It follows that $\frac{\partial u_0}{\partial n}(p_*) \geq 0$ and the equation (i) gives

$$u_0(p_*) \geq 0$$

Hence our unique solution u_0 is non-negative. We can say more. For consider the harmonic function h in (ii) which takes a maximum at some $p^* \in \partial\Omega$. Then $\frac{\partial h}{\partial n}(p^*) \leq 0$ so that (iii) gives

$$h(p^*) + \frac{\partial G}{\partial n}(p^*) \leq 0$$

Hence

$$\max_{p \in \partial\Omega} h(p) \leq -\frac{\partial G}{\partial n}(p^*)$$

which entails that

$$(0.2.1) \quad \min_{p \in \partial\Omega} u(p) = -\max_{p \in \partial\Omega} h(p) \geq \frac{\partial G}{\partial n}(p^*)$$

Here the function

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is continuous and positive on $\partial\Omega$ and if γ_* is the minimum value we conclude that

$$(0.2.2) \quad \min_{p \in \partial\Omega} u(p) \geq \gamma_*$$

Next, let h attain its minimum at some $p_* \in \partial\Omega$ which entails that $\frac{\partial h}{\partial n}(p_*) \geq 0$ and then (iii) gives

$$h(p_*) + \frac{\partial G}{\partial n}(p^*) \geq 0$$

It follows that

$$(0.2.3) \quad \max_{p \in \partial\Omega} u_0(p) = \min_{p \in \partial\Omega} h(p) = -h(p_*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^*$$

where

$$(0.2.4) \quad \gamma^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

So the unique solution u_0 in (i) satisfies

$$(0.2.5) \quad \gamma_* \leq u(p) \leq \gamma^* \quad : p \in \partial\Omega$$

where the positive constants γ_* and γ^* depend on the point $q_* \in \Omega$ and the given domain Ω .

The homotopy method. To proceed from $h = 0$ to $h = 1$ the idea is to use a "homotopy argument" which can be handled via precise estimates of solutions to Neumann's linear boundary value problem which are presented in § B. Thanks to this and some uniqueness properties in § A below, the reduction to the case when F is real-analytic is relatively easy. The crucial steps during the proof appear in § C where we carry out a "homotopy method" to get solutions in (*) as h increases from zero to one.

A.0. Proof of uniqueness.

Suppose that u_1 and u_2 are two solutions to the equation in the main theorem. Notice that $u_2 - u_1$ is harmonic in Ω . If $u_1 \neq u_2$ we may without loss of generality we may assume that the maximum of $u_2 - u_1$ is > 0 . The maximum is attained at some $p_* \in \partial\Omega$ and the strict maximum principle for harmonic functions gives:

$$(i) \quad u_2(x) - u_1(x) < u_2(p_*) - u_1(p_*)$$

for all $x \in \Omega$. With $v = u_2 - u_1$ we have

$$\frac{\partial v}{\partial n}(p) = F(u_2(p), p) - F(u_1(p), p)$$

Here (0.1) entails that $\frac{\partial v}{\partial n}(p_*) > 0$ and since we have an inner normal derivative this violates (i) which proves the uniqueness.

A.1 Monotonic properties.

Let F_1 and F_2 be two functions which both satisfy (0.1) and (0.2) where

$$F_1(u, p) \leq F_2(u, p)$$

hold for all $(u, p) \in \mathbf{R}^+ \times \partial\Omega$. If u_1 , respectively u_2 solve (*) for F_1 and F_2 it follows that $u_2(q) \leq u_1(q)$ for all $q \in \Omega$. To see this we set $v = u_2 - u_1$ which is harmonic in Ω . If $p \in \partial\Omega$ we get

$$(i) \quad \frac{\partial v}{\partial n}(p) = F_2(u_2(p), p) - F_1(u_1(p), p) \geq 0$$

Suppose that the maximum of v is > 0 and let the maximum be attained at some point p_* . Since (i) is an inner normal it follows that we must have $0 = \frac{\partial v}{\partial n}(p)$ which would entail that

$$F_2(u_2(p_*), p_*) > F_2(u_1(p_*), p_*) \geq F_1(u_1(p_*), p_*) \implies$$

and this contradicts the strict inequality $u_2(p_*) > u_1(p_*)$ since we have an increasing function in (0.1).

A.2. A bound for the maximum norm. Let G be the Green's function which has a pole at q_* while $G = 0$ on $\partial\Omega$. Then

$$p \mapsto \frac{\partial G}{\partial n}(p)$$

is a continuous and positive function on $\partial\Omega$. Set

$$m_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p) \quad : \quad m^* = \max_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

Next, let $0 \leq h \leq 1$ and suppose that u_h is a solution to (*). Put

$$(*) \quad m(h) = \min_{p \in \partial\Omega} u_h(p) \quad : \quad M(h) = \max_{p \in \partial\Omega} u_h(p)$$

To estimate these numbers we proceed as follows. Choose $p^* \in \partial\Omega$ such that

$$(1) \quad u_h(p^*) = M(h)$$

Now the function

$$H = u - G - M(h)$$

is harmonic function in Ω and non-negative on the boundary. Hence H is positive in Ω and since $H(p^*) = 0$ we have

$$\frac{\partial H}{\partial n}(p^*) \leq 0 \implies$$

which via the equation (*) give

$$(2) \quad (1 - h)M(h) + h \cdot F(M(h), p^*) \leq \frac{\partial G}{\partial n}(p^*) \leq \gamma^*$$

Next, the hypothesis on F entails that

$$(3) \quad t \mapsto (1 - h)t + h \cdot F(t, p^*)$$

is a strictly increasing function for each fixed $0 \leq h \leq 1$ and the hypothesis (0.2) together with the inequality (2) above, give a positive constant A^* which is independent of h such that

$$(3) \quad M(h) \leq A^* \quad : \quad 0 \leq h \leq 1$$

Next, let $m(h)$ be the minimum of u_h on $\partial\Omega$ and this time we consider the harmonic function

$$H = u - m(h) - G$$

Here $H \geq 0$ on $\partial\Omega$ and if $u_h(p_*) = m(h)$ we have $H(p_*) = 0$ p_* is a minimum for H . It follows that

$$\frac{\partial H}{\partial n}(p_*) \geq 0 \implies F(u(p_*), p) = \frac{\partial u}{\partial n}(p_*) \geq \frac{\partial G}{\partial n}(p_*)$$

So with

$$\gamma_* = \min_{p \in \partial\Omega} \frac{\partial G}{\partial n}(p)$$

one has the inequality

$$(4) \quad F(m(h), p^*) \geq \gamma_*$$

Above γ_* is the constant from (xx) and the properties of F give a positive constant A_* such that

$$m(h) \geq A_*$$

Conclusion. Above $0 < A_* < A^*$ are constants which are independent of h . Hence the maxima and the minima of u_h stay in a fixed interval $[A_*, A^*]$ as soon as u_h exists.

B. The linear equation.

Let $f(p)$ and $W(p)$ be a pair of continuous functions on the boundary $\partial\Omega$ where W is positive, i.e. $W(p) > 0$ for every boundary point. Set

$$w_* = \min_p W(p)$$

So by the assumption on W we have $w_* > 0$. The classical Neumann theorem asserts that there exists a unique function U which is harmonic in Ω , extends to a continuous function on the closed domain and its inner normal derivative satisfies:

$$(1) \quad \partial U / \partial n(p) = W(p) \cdot U(p) + f(p) \quad p \in \partial\Omega$$

For the unique solution in (1) some estimates hold. Namely, set

$$M^* = \max_p U(p) \quad \text{and} \quad m_* = \min_p U(p)$$

Since U is harmonic in Ω the maximum and the minimum are both taken on the boundary. If $U(p^*) = M^*$ for some $p^* \in \partial\Omega$ we have $\partial U / \partial n(p^*) \leq 0$ which together with (1) entails that

$$M^* \cdot W(p^*) + f(p^*) \leq 0 \implies M^* \leq \frac{|f|_{\partial\Omega}}{w_*}$$

where $|f|_{\partial\Omega}$ is the maximum norm of f on the boundary. In the same way one verifies that

$$m_U \geq -\frac{|f|_{\partial\Omega}}{w_*}$$

Hence the following inequality holds for the the maximum norm $|U|_{\partial\Omega}$:

$$(B.0) \quad |U|_{\partial\Omega} \leq \frac{|f|_{\partial\Omega}}{w_*}$$

Notice that (B.0) and the equation (1) entails that Suppose that $W \in C^0(\partial\Omega)$ satisfies

$$w_* \leq W(p) \leq w^*$$

for a pair of positive constants. If $|f|_{\partial\Omega}$ is the maximum norm of f it follows from (B.0) that

$$|W(p) \cdot U(p) + f(p)| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega}$$

Hence the equation (1) gives

$$(B.1) \quad \max_{p \in \partial\Omega} \left| \frac{\partial U}{\partial n}(p) \right| \leq (1 + \frac{w^*}{w_*}) \cdot |f|_{\partial\Omega}$$

B.2 An estimate for first order derivatives. Let $p \in \partial\Omega$ and denote by N the inner normal at p . Since $\partial\Omega$ is of class C^1 a sufficiently small line segment from p along N stays in Ω . So for small positive ℓ we have points $q = p + \ell \cdot N$ in Ω and take the directional derivative of U along N_p . This gives a function

$$\ell \mapsto \partial U / \partial N(p + \ell \cdot N)$$

Since the boundary is C^1 these functions are defined on a fixed interval $0 \leq \ell \leq \ell^*$ for all boundary points p . A classic result which appears in *Der zweite Randwertaufgabe* gives a constant B such that

$$|\partial U / \partial N(p + \ell \cdot N)| \leq B \cdot \max_{p \in \partial\Omega} \left| \frac{\partial U}{\partial n}(p) \right|$$

hold for all $p \in \partial\Omega$ and $0 \leq \ell \leq \ell^*$.

C. Proof of Theorem when $t \mapsto F(t, p)$ is analytic.

Assume that $t \mapsto F(t, p)$ is a real-analytic function on the positive real axis for each $p \in \partial\Omega$ where local power series converge uniformly with respect to p . In this situation we shall prove the *existence* of a solution u in the Theorem. To attain this we proceed as follows. To each real number $0 \leq h \leq 1$ we seek a solution u_h where

$$(1) \quad \frac{\partial u_h}{\partial n}(p) = h \cdot F(u_h, p) + (1 - h) \cdot u_h(p)$$

When $h = 0$ we found the solution u_0 in § xx. Next, suppose that $0 \leq h_0 < 1$ and that we have found the solution u_{h_0} to (1). By the result in § B there exists a pair of positive constants $A_* < A^*$ such that

$$(*) \quad A_* \leq u_{h_0}(p) \leq A^*$$

which are independent of h_0 and of p .

Set $u_0 = u_{h_0}$ and with $h = h_0 + \alpha$ for some small $\alpha > 0$ we shall find u_h by a series

$$(2) \quad u_h = u_{h_0} + \sum_{\nu=1}^{\infty} \alpha^\nu \cdot u_\nu$$

The pole at q_* occurs already in u_0 . So u_1, u_2, \dots is a sequence of harmonic functions in Ω and there remains to find them so that u_h solves (1). We will show that this can be achieved when α is sufficiently small. Keeping h_0 fixed we set

$$u_0 = u_{h_0}$$

The analyticity of F with respect to t gives for every $p \in \partial\Omega$ a series expansion

$$(3) \quad F(u_0(p) + \alpha, p) = F(u_0(p), p) + \sum_{k=1}^{\infty} c_k(p) \cdot \alpha^k$$

where $\{c_k(p)\}$ are continuous functions on $\partial\Omega$. Here (*) and the hypothesis on F entail that the radius of convergence has a uniform bound below, i.e. there exists $\rho > 0$ which is independent of $p \in \partial\Omega$ and a constant K such that

$$(4) \quad \sum_{k=1}^{\infty} |c_k(p)| \cdot \rho^k \leq K$$

Now the equation (1) can be solved via a system of equations where the harmonic functions $\{u_\nu\}$ are determined inductively while α -powers are identified. The linear α -term gives the equation

$$(i) \quad \frac{\partial u_1}{\partial n} = F(u_0(p), p) - u_0(p) + (1 - h_0)u_1 + h_0 \cdot c_1(p) \cdot u_1(p)$$

For u_2 we find that

$$(ii) \quad \frac{\partial u_2}{\partial n} = (1 - h_0)u_2 - u_1 + h_0 c_1(p)u_2 + c_1(p)u_1 + c_2(p)u_1^2$$

In general we have

$$(iii) \quad \frac{\partial u_\nu}{\partial n} = (1 - h_0 + h_0 \cdot c_1(p)) \cdot u_\nu + R_\nu(u_0, \dots, u_{\nu-1}, p) \quad : \nu \geq 1$$

where $\{R_\nu\}$ are polynomials in the preceding u -functions whose coefficients are continuous functions obtained from the c -functions. The function $c_1(p)$ is given by

$$c_1(p) = \frac{\partial F}{\partial t}(u_0(p), p)$$

which by the hypothesis on F is a positive continuous function on $\partial\Omega$. It follows that the function

$$(iv) \quad W(p) = (1 - h_0) + h_0 \cdot c_1(p)$$

also is positive on $\partial\Omega$ and in the recursion above we have

$$(v) \quad \frac{\partial u_\nu}{\partial n} = W(p) \cdot u_\nu(p) + R_\nu(u_0, \dots, u_{\nu-1}, p) \quad : \nu = 1, 2, \dots$$

Above we encounter linear equations exactly as in (B.0) where the f -functions are the R -polynomials. Put

$$w_* = \min_{p \in \partial\Omega} W(p)$$

From § B.XX we get

$$(vi) \quad |u_\nu|_{\partial\Omega} \leq w_*^{-1} \cdot |R_\nu(u_0, \dots, u_{\nu-1}, p)|_{\partial\Omega}$$

Finally, (vi) and a majorising positive series expressing maximum norms imply that if α is sufficiently small then the series (2) converges and gives the requested solution for (1). Moreover, α can be taken *independently* of h_0 . Together with the established uniqueness of solutions u_h whenever they exist, it follows that we can move from $h = 0$ until $h = 1$ and arrive at the requested solution in Theorem 1.

Remark. The reader may consult page 106 in [Carleman] where the existence of a uniform constant $\alpha > 0$ for which the series (2) converge for every h is demonstrated by an explicit majorant series.