Chapter VI: Conformal mappings.

Introduction.

In his disseration from 1851, Riemann announced the uniformisation theorem which applies to general Riemann surfaces. We give some comments about the general mapping theorem in § 0.6. The simplest version asserts that every simply connected open subset Ω of $\mathbf{C}$ which is not equal to the whole complex plane is conformal with the open unit disc D. More precisely, for every point $z_0 \in \Omega$ there exists a unique $f \in \mathcal{O}(\Omega)$ where $f(z_0) = 0$ and $f'(z_0)$ is real and positive such that

$$(*) \qquad f\colon \Omega \to D$$

is bijective and the complex derivative $f'(z) \neq 0$ for all $z \in \Omega$. Here f is the solution to a variational problem where one consider the family $\mathcal{C}(\Omega, z_0)$ of all analytic functions g in Ω for which $g(\Omega) \subset D$, $g(z_0) = 0$ and the derivative $g'(0)$ is real and non-negative. Then one seeks

$$\max_{g \in \mathcal{C}(\Omega, z_0)} g'(z_0)$$

The extremal function to this variational problem gives the conformal map (*). For an arbitrary $g \in \mathcal{O}(\Omega)$ with $g(\Omega) \subset D$ we can choose θ so that if $h = e^{i\theta} \cdot g$ then $h'(0)$ is real and positive and it turns out that it is majorized by $\leq f'(0)$. Since the absolute value of $g'(0)$ is equal to $h'(0)$ this means that

$$(**) \qquad |g'(z_0)| \leq f'(z_0)$$

for all $g \in \mathcal{O}(\Omega)$ such that $g(\Omega) \subset D$. The uniqueness in Riemann's mapping theorem entails that equality holds in (**) if and only if $g(z) = e^{i\theta} \cdot f(z)$ for some $0 \leq \theta \leq 2\pi$.

Remark. The mapping theorem above follows from the existence of conformal mappings between special Jordan domains. Namely, a bounded and simply connected domain Ω can be exhausted by an increasing sequence of Jordan domains, i.e. by a sequence of closed Jordan curves $\{\Gamma_\nu\}$ which all are contained in Ω and the Jordan domains $\{\Omega_\nu\}$ bounded by these curves form an increasing sequence of relatively compact regions of Ω. If $z_0 \in \Omega$ is given we can start with some Γ_1 so that z_0 lies in its interior. Suppose we have found Riemann's mapping function f_ν from Ω_ν into D where $f_\nu(z_0) = 0$. Now $\{f_\nu\}$ is a normal family of analytic functions and there exists $f \in \mathcal{O}(\Omega)$ such that $\{f_\nu\}$ converges uniformly to f on every compact subset of Ω. It follows easily via an application of Rouche's theorem that f is the mapping function for Ω. So the mapping theorem for simply connected domains is essentially a result about conformal mappings from Jordan domains to the unit disc. Moreover, during the exhaustion it suffices to consider Jordan curves which are piecewise linear. In this case Schwarz and Cristoffell independently gave a formula for the mapping function in 1868 which is exposed in § 5.

The case of unbounded domains.

If Ω is a simply connected domain in $\mathbf{C}$ which is not the whole plane then the closed complement $\mathbf{C} \setminus \Omega$ contains at least two points, say a and b. Now there exists the multi-valued root function

$$w(z) = \sqrt{\frac{z-a}{z-b}}$$

Since Ω is simply connected there exists a single-valued branch of w in Ω and in § 1:A we explain hos this gives the existence of a conformal mapping from Ω onto a simply connected subset of the unit disc. In this way Riemann's mapping theorem is reduced to the case when Ω from the start is bounded.

Adding the point at infinity. Consider the function:

$$f(z) = z + \frac{1}{z}$$

which is meromorphic with a simple pole at the origin. When $z = e^{i\theta}$ we have

$$f(e^{i\theta}) = 2 \cdot \cos\theta$$

The range as $0 \leq \theta \leq 2\pi$ is the real interval $[-2, 2]$. Introducing the point at infinity we set $\Sigma = \mathbf{C} \cup \{\infty\}$ and then $\Omega = \Sigma \setminus [-2, 2]$ is a simply connected domain on the Riemann sphere. If z_1, z_2 is a pair of distinct pints in the punctured open disc $\{0 < |z| < 1\}$ it is clear that $f(z_1) \neq f(z_1)$. If $w \in \mathbf{C} \setminus [-2, 2]$ there exists a unique z in this punctured disc such that

$$z + \frac{1}{z} = w$$

Indeed, z is given by

$$z = \frac{1}{2w} - \frac{1}{2w} \cdot \sqrt{w^2 - 4}$$

where the brancch of $\sqrt{w^2-4}$ outside $[-2, 2]$ is chosen so that it takes positive real values when w is real and with absolute value > 2. Now f yields a biholomorphic map between D and the simply connected domain Ω where $\zeta = \frac{1}{w}$ serves as local coordinate at the point at infinity. So here $f(0) = \infty$ and f is a conformal mapping in Riemann's sense. The restriction of f to T yields a double cover onto $[-2, 2]$ which reflects the geometric fact that this real segment as a boundary of the domain $\Sigma \setminus [-2, 2]$ has two sides, i.e. one can approach a point $-2 < x < 2$ from the above with $z = x + i\epsilon$ or from below by $z = x - i\epsilon$. The conformal mapping f is an example of a map from D onto a parallell slit region which will be treated in § 7. The example shows that Riemann's mapping theorem applies when D is replaced by other simply connected sets which in general appear on the Riemann sphere containing the point at infinity.

0.1 The equilibrium potential on plane curves. Riemann's mapping theorem for Jordan domains was predicted in electric engineering. Namely, let Γ be a closed Jordan curve of class C^1. Then one seeks a positive density function μ on Γ such that the logarithmic potential

$$(*) \qquad \int_\Gamma \log \frac{1}{|z-\zeta|} \cdot \mu(\zeta) \cdot |d\zeta|$$

is constant as z varies in Γ. The existence of μ is expected since μ corresponds to an equilibrium density for an electric field. It turns out that μ is found via the conformal mapping function f from the exterior domain bordered by Γ onto the exterior of the unit disc. In § 9 we prove that:

$$(**) \qquad \mu(\zeta) = \frac{1}{|f'(\zeta)|}$$

Moreover, the probability measure μ is unique when one requires that (*) is a constant on Γ.

0.2 Solution by Greens' functions. If Ω is a bounded Jordan domain and $z_0 \in \Omega$ there exists the unique harmonic function $H(z)$ in Ω with boundary function $\log |z - z_0|$. Since Ω is simply connected H has a unique harmonic conjugate V in Ω normalized so that $V(z_0) = 0$. Set

$$f(z) = (z - z_0)e^{-(H(z)+iV(z))}$$

Then f gives the conformal mapping from Ω onto D. We prove this result in Section XX.

0.3 Constructions of special conformal mappings. Solving a mixed boundary value problem in the unit disc D yield conformal mappings from D onto special image domains. Namely, let E and F be a a pair of closed subsets on the unit circe T where each set is a finite union of closed subintervals of T. We also assume that the sets are separated which means that T can be partioned into a pair of intervals J_1 and J_2 whose intersection is reduced to a pair of points while $E \subset J_1$ and $F \subset J_2$. Now there exists the harmonic function u in D whose normal derivative $\frac{\partial u}{\partial n}$ vanishes on $T \setminus E \cup F$ while $u = 1$ on E and zero on F. Let v be the harmonic conjugate of u. The Cauchy-Riemann equations entail that the boundary function $\theta \mapsto v(e^{i\theta})$ stays constant on intervals in $T \setminus E \cup F$ while it is strictly increasing on every open subinterval of E, respectively strictly decreasing on open subintervals of F. In § xx we shall learn that the analytic function $f = u + iv$ yields a conformal mappiing from D onto a domain $\mathcal{R}$ which consists of an open rectangle $\{0 < x < 1\} \times \{0 < y < h\}$ from which a finite number of horisontal spikes have been removed, i.e. closed vertical line segments which issue from one of the vertical sides and terminate at some point in the open rectangle. See figure § XX. The positive integer h depends on the pair E, F and is referred to as the Neumann-Schwarz number and denoted by $\mathcal{NS}(E, F)$. In § XX we expose Beurling's theory about extremal distances which gives a perspective upon these numbers and their relation to harmonic measures.

0.3.1 Comb domains. Let Ω be a simply connected open subset of the half-plane $\mathfrak{Re}\, z > 0$. It is called a comb domain if

$$z = x + iy \in \Omega \implies \{tx + iy : 0 < t \leq 1\} \subset \Omega \tag{1}$$

The set $\{x = 0\} \cap \partial\Omega$ is called its base and (1) means that Ω is starshaped with respect to the base which is a closed subset of the imaginary axis and in general can be unbounded. In § xx we construct conformal mappings from a simply connected domain onto comb domains with special properties. So one often employs constructions via solutions to mixed Neumann boundary value problems.

0.4 Koebe's function and the Bieberbach hypothesis.. Locally conformal mappings whose range are multiply connected domains and more generally Riemann surfaces leads to the uniformisation theorem which is described in § 0.6. It was proved in its full generality by Koebe in 1907. His proof employed a special conformal mapping defined by

$$k(z) = \frac{z}{(1-z)^2}$$

We leave it as an exercise to show that $k(z)$ is 1-1 in the open unit disc and hence gives a conformal mapping from D onto a simply connected set which is unbounded since $k(x) \to +\infty$ as x is real and tends to 1. The Taylor series of k is given by:

$$k(z) = z + 2z^2 + 3z^3 + \ldots$$

Koebe's k-function led to a conjecture posed by Bieberbach in 1916 which goes as follows: Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n \cdot z^n$$

give a conformal mapping from D onto some simply connected domain. Bieberbach conjectured that the coefficients $\{a_n\}$ must satisfy

$$|a_n| \leq n \quad : n = 2, 3, \ldots$$

The affirmative answer to the Bieberbach conjecture was established by de Branges in 1985. The interested reader can consult [Bieberbach-book] for background and history how the proof was finally finished. In addition to Bieberbach's article [Bieb:Acta] we refer to Chapter XX in [Krantz] for a detailed proof.

0.4.1 Baernstein's inequality. Koebe's function enjoys other extremal properties. The following result was proved in [Acta:A. Baerenstein] in 1977:

0.4.2 Theorem. *For every conformal mapping function $f(z)$ where $f(0) = 0$ and $f'(0) = 1$ one has the inequality*

$$\int_0^{2\pi} |f(re^{i\theta})| \cdot d\theta \leq \int_0^{2\pi} |k(re^{i\theta})| \cdot d\theta \quad \text{for each} \quad 0 < r < 1$$

where k is Koebe's function.

We shall not try to enter the proof of Baerenstein's result which requires an extensive excursion into variational techniques. See also the text-book *Univalent functions* by P.I. Duren which treats extremal problems in connection with conformal mappings.

0.5 Beurling's mapping theorem. Equation (**) in (0.1) suggests that if g is a conformal map from the unit disc D onto a simply connected domain Ω then there is a close interplay between the absolute value of $|g'(z)|$ and values of $g(z)$ when $z \to \partial D$. A result about this correspondence was proved in [Beurling] where the simply connected domain is not given in advance. Instead one starts from a positive and bounded continuous function $\Phi(w)$ defined in the whole complex plane such that $\text{Log}\,\frac{1}{|\Phi|}$ is subharmonic. Beurling proved that there exists a unique analytic function f in D with $f(0) = 0$ which yields a conformal map from D onto $f(D)$, and satisfies the limit formula:

(*)
$$\lim_{r\to 1} \max_{0\leq\theta\leq 2\pi} \left[\left|f'(re^{i\theta})\right| - \Phi(f(re^{i\theta}) \right] = 0$$

The proof in § 11 is quite involved. But it is utmost rewarding to pursue the details which give an instructive lessons how to solve problems in analytic function theory using calculus of variation.

0.6 The uniformisation theorem.

Let Ω be a bounded and connected domain but not simply connected. Then there does not exist a conformal mapping from Ω onto D. To compensate for this one considers *multi-valued* functions on Ω. If $z_0 \in \Omega$ we defined the family $M_\Omega(\mathcal{O})(z_0)$ in Chapter IV of germs of analytic functions at z_0 which extend to multi-valued functions on Ω. When f is such a germ and $z \in \Omega$ we denote by set $f^*(z)$ the set of values taken by all the local branches of f at z. We say that a germ $f \in M_\Omega(\mathcal{O})(z_0)$ gives a *multi-valued and locally conformal map* from Ω onto D if the following hold:

(i)
$$\widehat{f}(z_1) \cap \widehat{f}(z_2) = \emptyset \quad \text{for all pairs } z_1 \neq z_2 \text{ in } \Omega$$

(ii)
$$\bigcup_{z\in\Omega} \widehat{f}(z) = D$$

In § 1.B we prove that for each point z_0 in Ω there exists a unique germ f as above which is normalised so that $f'(z_0)$ is real and positive and $f(z_0) = 0$. Moreover, f solves an extremal problem. Namely, for every $g \in M_\Omega(\mathcal{O})(z_0)$ such that $g(0) = 0$ and $\cup_{z\in\Omega}\, \widehat{g}(z) \subset D$ one has:

(iii)
$$|g'(z_0)| \leq f'(z_0)$$

The uniqueness has several consequences. Namely identify the fundamental group $\pi_1(X)$ with homotopy classes of the family of closed curves at z_0. To each closed curve γ in this family we perform the analytic contonuation along γ and get the new germ $T_\gamma(f)$ at x_0. The monodromy theorem from Chapter IV entails that this germ only depends upon the homotopy class $\{\gamma\}$ and via the evaluation at x_0 one has a map from $\pi_1(X)$ into D defined by

$$\{\gamma\} \mapsto T_\gamma(z_0)$$

The uniqueness above entails that (*) is bijective which means that we have identified $\pi_1(X)$ with a subset of the open unit disc D. Using Möbious transformations we explain in § xx how this identifies $\pi_1(\Omega)$ with a group of Möbius transforms on D.

The existence of a multi-valued mapping above was established by Schwarz. But its full scope was put forward by Poincaré who used the result above to construct Fuchsian groups. For bounded and connected domains Ω with infinitely many holes, i.e. when its closed complement has inifintely many connected components the fundamental group is not finitely generated and the associated group of Möbius transforms is therefore hard to describe. For nice multiple connected domains Ω whose boundary consists of p disjoint and closed Jordan¨curves for some $p \geq 2$ the situation is more favourable. Here $\pi_1(\Omega)$ is free group generated by $p-1$ many closed loops and the associated group of Möbius transforms is a more manageable Fuchsian group in Poincaré's sense. However, in contrast to the simply connected case the equation about *conformal equivalence* between a pair of p-connected domains is a very subtle and leads to *moduli problems* on arbitrary Riemann surfaces. We refer to Chapter XX for a further account about this.

0.7 Picard's theorem.

In 1879 Picard found the affirmative answer to a question posed by Weierstrass. The result is that an entire function $f(z)$ whose range excludes two values must be a constant. Picard's proof used the modular function. A proof which does not use the modular function was found by Emile Borel in 1895. Inspired by Borel's methods, Shottky and Landau established extensions of Picard's theorem. For example, let a_0 and a_1 be two complex numbers where a_0 is $\neq 0$ and $\neq 1$. Then there exists a constant $C(a_0, a_1)$ which depends on this pair only such that the following hold:

If $f(z)$ is an analytic function in some disc D_R of radius R centered at the origin where the range $f(D_R)$ does not contain the two values 0 and 1 and the Taylor series at $z = 0$ starts with $a_0 + a_1 + \ldots$, i.e. $f(0) = a_0$ and $f'(0) = a_1$, then

$$R \leq C(a_0, a_1)\,.$$

Remark. Using the modular function the best possible constant $C(a_0, a_1)$ is determined in § 3.

0.8 Extension to the Boundary

Let Ω be bounded and simply connected set and f is a conformal map from Ω onto the open unit disc D. With no conditions on $\partial\Omega$ one cannot expect a continuous extension of f. But for those points p on $\partial\Omega$ which can be reached by a Jordan arc which except for p stays inside Ω there exist certain limits. Such boundary points are called *accessible* and denote this set by $\mathcal{A}(\partial\Omega)$. We will prove that $\mathcal{A}(\partial\Omega)$ is dense in $\partial\Omega$. Next, for each $p \in \mathcal{A}(\partial\Omega)$ we denote by $\mathcal{J}(p)$ the family of Jordan arcs J where p is an end-point while the remaining part of J stays in Ω. So if $t \mapsto \gamma(t)$ defines J then $\gamma(t) \in \Omega$ when $0 \leq t < 1$ and $\gamma_J(1) = p$. The image of $J \setminus \{p\}$ under f is a half-open Jordan arc in D defined by:

(i) $$t \mapsto \gamma^*(t) \quad : \quad f(\gamma(t)) = \gamma^*(t) \quad : 0 \leq t < 1$$

With these notations one has:

0.8.1 Koebe's limit theorem. *For every $J \in \mathcal{J}(p)$ there exists the limit*

$$\lim_{t \to 1} \gamma^*(t))$$

The limit above is a point on the unit circle which depends on the pair (p, J). It is denoted by $\mathcal{K}(J, p)$ and called the Koebe limit. The second major result is due to Lindelöf.

0.8.2 Lindelöf's separation theorem. *Let $p \neq q$ be distinct points on $\partial\Omega$. Then*

$$\mathcal{K}(J,p) \neq \mathcal{K}(J',q) \quad : \quad J \in \mathcal{J}(p) \quad : J' \in \mathcal{J}(q)$$

Next, let $p \in \partial\Omega$ and consider some $J \in \mathcal{J}(p)$. Let $\{J_\nu \in \mathcal{J}(q_\nu)\}$ be a sequence of Jordan arcs with end points $\{q_\nu\}$.

0.8.3 Definition. *We say that the sequence $\{(J_\nu, q_\nu)\}$ converges to (J,p) if $q_\nu \to p$ and for every $\epsilon > 0$ there exists some ν^* such that whenever $\nu \geq \nu^*$ there exists a Jordan arc γ_ν which is contained in the disc $D_\epsilon(p)$ and has endpoints on J and J_ν.*

0.8.4 Koebe's Continuity Theorem. *If $(J_\nu, q_\nu) \to (J,p)$ it follows that*

$$\lim_{\nu \to \infty} \mathcal{K}(J_\nu, q_\nu) = \mathcal{K}(J,p)$$

Next, consider a pair J_1, J_2 in $\mathcal{J}(p)$. The two Jordan arcs are *asymptotically linked* if there to every $\epsilon > 0$ exists a Jordan arc $\gamma_\epsilon \subset D_\epsilon(p)$ whose end-points belong to J_1 and J_2 respectively. Koebe's continuity theorem entails

0.8.5 Proposition. *Let J_1, J_2 be a pair in $\mathcal{J}(p)$ which are asymptotically linked. Then one has the equality:*

$$\mathcal{K}(J_1,p) = \mathcal{K}(J_2,p)$$

The results above show that if every pair of Jordan arcs in $\mathcal{J}(p)$ are asymptotically linked, then there exists a Koebe limit $\mathcal{K}_f(p)$ defined as the common value of $\mathcal{K}(J,p)$. Let $\mathcal{A}_*(\Omega)$ denote the set of such boundary points. Together with Lindelöf's separation theorem this yields an injective map from $\mathcal{A}_*(\Omega)$ into the unit circle T defined by

$$(*) \qquad p \mapsto \mathcal{K}_f(p)$$

0.8.6 The case of Jordan domains.

When Ω is a Jordan domain a classic result due to Camille Jordan asserts that every boundary point is accessible. Moreover, a sharpended version of Camille Jordan's theorem was proved by von Schoenflies in [vScH].

0.8.7 Schoenflies' theorem. *Let Ω be a Jordan domain. For each boundary point p the family of all Jordan arcs is in $\mathcal{J}(p)$ are asymptotically linked and if $J \in \mathcal{J}(p)$ and $\{(J_\nu, q_\nu)\}$ is a sequence such that $q_\nu \to p$, then the sequence (J_ν, q_ν) converges to (J,p) in the sense of Definition 0.8.3.*

0.8.8 Conclusion. Koebe's Continuity Lemma and Schoenflies' result show that when Ω is a Jordan domain then the Koebe-Lindelöf map (*) above is a continuous and bijective map from $\partial\Omega$ into T. Since both $\partial\Omega$ and T are closed Jordan curves this implies that $\mathcal{K}^*$ must be surjective. Finally, a continuous and bijective map between two compact sets has a continuous inverse. Hence the mapping function f yields a homeomorphism from T onto $\partial\Omega$ and as one easily sees f actually extends to a continuous map from the closed unit disc D onto the closed Jordan domain $\bar{\Omega}$. This result for Jordan domains is sometimes attributed to Caratheodory but all essential steps in the proof rely upon the results above due to Jordan, von Schoenfliess, Koebe and Lindelöf. Caratheodory's contribution is foremost his elegant proof of the uniformisation theorem for arbitrary connected domains and certain topological constructions adapted to the Koebe-Lindelöf map (*) for an arbitrary simply connected domain.

0.8.9 Smooth boundary points. Let Ω be a Jordan domain and f a conformal map from D onto Ω. So now we have a homeomorphism

$$e^{i\theta} \mapsto f(e^{i\theta}) \quad : \quad 0 \leq \theta \leq 2\pi$$

The closed Jordan curve $\partial\Omega$ has a parametrisation $t \mapsto \gamma(t)$ which may have extra regularity. A result in the pointwise differentiable case was presented by Lindelöf at the Scandinavian Congress of Mathematics held at Institute Mittag-Leffler in 1916. It goes as follows:

Let $p \in \partial\Omega$ and assume that $\partial\Omega$ has a tangent line at p. We can take $p = 1$ and after a rotation in the complex w-plane also assume that the vertical line $\mathfrak{Re}(w) = 1$ is tangent to $\partial\Omega$ at p. This means that we can choose a parametrisation $\gamma(t)$ where we may translate t and assume that γ is defined on some interval $[-A, A]$ with $\gamma(-A) = \gamma(A)$ while $p = \gamma(0)$ and the differentiable assumption means that there exists some positive real number a such that

$$(*) \qquad \gamma(t) = 1 + iat + \text{small ordo}\,(t) \quad : \; t \to 0$$

Let $g(z)$ be the conformal mapping from D onto Ω normalised so that $g(1) = 1$. When θ is close to zero we get a real-valued function $\beta(\theta)$ such that

$$(**) \qquad g(e^{i\theta}) = \gamma(\beta(\theta))$$

We already know that the β-function is continuous. But when (*) is added Lindelöf proved the following result:

0.8.10 Theorem. *The β-function has an ordinary derivative at $\theta = 0$.*

Moreover, Lindelöf proved that f is locally conformal up to the boundary at $z = 1$:

0.8.11 Theorem. *Assume (*) above. Then there exists a positive constant B such that the following limits exist and are equal:*

$$\text{(i)} \qquad \lim_{\theta \to 0} \frac{f(e^{i\theta}) - 1}{i\theta} = B$$

$$\text{(ii)} \qquad \lim_{s \to 0} \frac{1 - f(1 - se^{i\alpha})}{s} = B \cdot e^{i\alpha} \quad : \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

0.8.12 Remark. Caratheodory proved that the conformal property holds up to the boundary in the more general case when p is a corner point, i.e. when (*) is replaced by the weaker assumption that the γ-function has one-sided derivatives as t decreases or increases to 0. For a full account of various results due to Lindelöf and Caratheodory we refer to Lindelöf's article in [XX]. Here we are content to give a proof of Theorem 0.8.10 in § XXX. Regularity for higher order derivatives was studied by Painlavé in [Pain] (when???) who proved that if Ω is a Jordan domain whose boundary curve is of class C^∞, then a conformal map from Ω onto D extends to a C^∞-function from the closure of Ω onto the closed unit disc.

0.8.13 Properties of the inverse map f^{-1}.

Above we discussed the boundary behaviour of the conformal map from a bounded simply connected domain Ω onto the open unit disc. Let us instead consider the inverse function $\phi = f^{-1}$, i.e. now ϕ is a conformal map from D onto Ω. The Brothers Riesz theorem shows that ϕ has radial limits almost everywhere and. Actually the failure of radial limits is confined to a smaller set because the area integral

$$\iint_D |\phi'(z)|^2 \dot{d}xdy < \infty$$

We can therefore apply a result due to Beurling from Special Topics § XX which shows that radial limits exist outside a subset of T whose outer logarithmic capacity is zero. One can continue to analyze further regularity of ϕ. First we introduce the following:

0.8.14 Angular derivatives at the boundary. Consider some $e^{i\theta} \in T$ where the radial limit exists:

$$\lim_{r\to 1}\phi(re^{i\theta})=p \tag{1}$$

Results about Fatou limits for bounded analytic functions imply that ϕ has a limit in every Fatou sector having $e^{i\theta}$ as a corner point. Consider the argument of the difference quotients

$$z\mapsto \arg\big(\frac{\phi(e^{i\theta})-\phi(z)}{e^{i\theta}-z}\big) \tag{2}$$

If (2) has a limit when $z\to 1$ in the sense of Fatou we say that ϕ is conformal at $e^{i\theta}$. For example, if $\theta=0$ so that $e^{i\theta}=1$ we require that for every $0<\delta<\pi/2$ there exists the limit

$$\lim_{s\to 0}\arg\big[\,\frac{\phi(1)-\phi(1-s\cdot e^{i\alpha})}{s\cdot e^{i\alpha}}\,\big] \tag{3}$$

uniformly with respect to α provided that $\pi/2-\delta\leq\alpha\leq\pi/2-\delta$. We can also consider the complex differences without introducing their arguments and say that ϕ has an *angular derivative* at $e^{i\theta}$ when

$$z\mapsto \frac{\phi(e^{i\theta})-\phi(z)}{e^{i\theta}-z} \tag{4}$$

has a Fatou limit as $z\to e^{i\theta}$. The limit is then called the angular derivative of ϕ and denoted by $\phi'(e^{i\theta})$. If $\phi'(e^{i\theta})\neq 0$ it is clear that (2) also has a limit, i.e. the existence of a non-zero angular derivative implies that ϕ is conformal at $e^{i\theta}$. Results about angular derivatives are due to Ostrowski in his article [Ost] from 1937. It would take us to far to describe this in detail and instead we refer to the text-book [Marshall-Garnett where Chapter V.5 contains a detailed account of Ostrowski's results. Roughly speaking angular derivatives and conformality at boundary points are expressed via geometric conditions of Ω and a boundary point p which to begin with can be reached from a radial limit (1) above. In [loc.cit] the reader also finds an exposition of more recent results which give conditions for the existence of angular derivatives using conditions which rely upon various extremal metrics.

Remark. See also the article *Angular derivatives and Lipschitz majorants* by D.E. Marshall, available on-line at http:// math.washington.edu/- marshall/personal. html.

0.10 Comments on other sections.

In section 4 we prove Koebe's *"One Quarter Theorem"* together with the *Area Theorem* and Koebe's *Verzerrungssatz.* § 5 is devoted to the construction of conformal maps from the unit disc onto convex polygons. In § 7 we construct conformal maps between multiple connected domains using certain harmonic functions and § 8 is devoted to the *Bergman kernel.*where we prove a result due to Carleman about the asymptotic behaviour of the kernel function with a sharp remained term for Jordan domains whose boundary curve is real-analytic. In § 10 we prove a result due to Koebe about conformal mappings between domains which are bordered by circles. In § 11 we prove the Riemann-Schwarz inequality for geodesic curves with respec to metrics with non-positive curvature.

I:A. Riemann's mapping theorem for simply connected domains

The family of simply connected domains were described by Schwarz in the article [Schwarz] from 1869.

The following are equivalent for an open subset Ω *of* $\mathbf{C}$:

(1) $$\mathbf{C} \setminus \Omega \text{ has no compact connected components}$$

(2) $$\text{For every closed curve } \gamma \text{ inside } \Omega \implies \mathfrak{w}_\gamma(a) = 0 \quad : a \in \mathbf{C} \setminus \Omega$$

(3) $$\text{Any } f \in \mathcal{O}(\Omega) \text{ has a primitive}$$

(4) $$\text{If } f \in \mathcal{O}(\Omega), \text{ has no zeros there exists } g \in \mathcal{O}(\Omega) \quad : f = e^g$$

(5) $$\text{If } f \in \mathcal{O}(\Omega), \text{ has no zeros there exists } h \in \mathcal{O}(\Omega) \quad : f = h^2$$

Remark. The reader should contemplate upon these conditions and prove they are equivalent. See also [Nah: page 151-153] which gives a detailed proof of the equivalence between (1-5). When Ω satisfies the equivalent conditions above we say that it is simply connected. The case $\Omega = \mathbf{C}$ is excluded. A punctured complex plane $\mathbf{C} \setminus \{a\}$ which arises when a single point is removed is obviously not simply connected. So when $\Omega \neq \mathbf{C}$ is simply connected then the boundary $\partial\Omega$ contains at least two points. This will be used below to establish Riemann's mapping theorem. Let us remark that the subsequent proof is due to Fejér and F. Riesz.

A.1 Reduction to bounded domains. Let $\Omega \neq \mathbf{C}$ be simply connected. Above we have seen that there exists two distinct points a and b in its closed complement. Consider the function

(i) $$w(z) = \sqrt{\frac{z-a}{z-b}}$$

From (5) in the list by Schwarz there exists a *single valued* branch of this root function which we denote by $w^*(z)$, i.e. here $w^* \in \mathcal{O}(\Omega)$. Let $z_0 \in \Omega$ and put $w^*(z_0) = c$. Since we have chosen a branch of the square root function it follow that $w^*(z)$ never attains the value $-c$. Moreover, since the complex derivative of w^* is $\neq 0$ there exists an open disc Δ centered at c which is disjoint from the disc $-\delta$ centered at $-c$, and an open neighborhood U of z_0 such that $w^* \colon U \to \delta$ is biholomorphic. It follows that the image $w^*(\Omega)$ has empty intersection with $-\Delta$. So if r is the radius of Δ we conclude that

$$|w^*(z) + c| \geq r \quad \text{for all} \quad z \in \Omega$$

We conclude that $\frac{1}{w(z)-c}$ yields a conformal map from Ω onto a bounded simply connected set U. Since the composition of two conformal maps is conformal there only remains only to prove Riemanns Mapping Theorem for U. So from now on we assume that Ω is bounded and proceed with the proof.

A.2 Proof of uniqueness. Let us prove that the mapping function is *unique* if it exists. For suppose that f and g are two conformal mappings from Ω onto D where $f(z_0) = g(z_0) = 0$ and both $f'(z_0)$ and $g'(z_0)$ are real and positive. Now there exists the inverse mapping function g^{-1} from D onto Ω and we set

$$\phi = f \circ g^{-1}$$

Then ϕ yields a conformal mapping of D onto itself where $\phi(0) = 0$. As explained in XX it follows that $\phi(z) = az$ for a constant a with $|a| = 1$. Here

$$a = \phi'(0) = \frac{f'(z_0)}{g'(z_0)}$$

Since both $f'(0)$ and $g'(0)$ are real and positive we get $a = 1$. So $\phi(z) = z$ is the identity map and $f = g$ follows.

A.3 Proof of Existence. Set

(i) $$\mathcal{F} = \{f \in \mathcal{O}(\Omega) \quad : \ f(z_0) = 0 \quad f(\Omega) \subset D \quad : \ f \text{ is 1-1}\}$$

Thus, each $f \in \mathcal{F}$ gives a conformal map from Ω into some open subset of D. There remains to find some f such that $f(\Omega) = D$. To attain this we put

(ii) $$M = \max_{f \in \mathcal{F}} |f'(z_0)|$$

Here M is finite since there exists $r > 0$ such that the disc $D_r(z_0) \subset \Omega$ and Schwarz' inequality gives $|f'(z_0)| \leq \frac{1}{r}$ for each $f \in \mathcal{F}$. Next, by the Montel Theorem in XXX, the family $\mathcal{F}$ is *normal* in $\mathcal{O}(\Omega)$. Hence we can find $f \in \mathcal{F}$ such that $|f'(z_0)| = M$. Multiplying f with some $e^{i\theta}$ we may assume that $f'(z_0) = M$. There remains to show that $f(\Omega) = D$. Assume the contrary, i.e. suppose there exists $a \in D \setminus f(\Omega)$. Put

(iii) $$\phi(z) = \frac{f(z) - a}{1 - \bar{a} f(z)}$$

Since a Möbius transform is a conformal map on D, it follows that $\phi(\Omega) \subset D$. Moreover, $\phi \neq 0$ and by (4) in the list by Schwarz there exists an analytic function $F(z)$ in Ω such that

(iv) $$F(z) = \text{Log}[\,\frac{f(z) - a}{1 - \bar{a} f(z)}] \in \mathcal{O}(\Omega) \quad : \quad \mathfrak{Re}(F(z)) < 0 \quad z \in \Omega$$

It is clear that F yields a conformal mapping from Ω into an open subset of the left half-plane $\mathfrak{Re}(w) < 0$. Next, consider the function

(v) $$G(z) = \frac{F(z) - F(z_0)}{F(z) + \bar{F}(z_0)}$$

Since $F(z)$ and $F(z_0)$ belong to the same half-plane $\mathfrak{Re}(w) < 0$ we see that the absolute value of G is < 1 for every $z \in \Omega$ and conclude that G yields a conformal mapping from Ω into D. Now $G(z_0) = 0$ and by the maximal property of M we get a contradiction if we have proved the strict inequality:

(vi) $$|G'(z_0)| > M$$

To prove (vi) we notice that

(vii) $$G'(z_0) = \frac{F'(z_0)}{F(z_0) + \bar{F}(z_0)}$$

Next, the construction of the Log-function F gives

(vii) $$F(z_0) + \bar{F}(z_0) = 2\mathfrak{Re}\, F(z_0) = 2\,\text{Log}\,|a|$$

Moreover, since $f(z_0) = 0$ a derivation of the Log-function F gives

(viii) $$F'(z_0) = f'(z_0)(1 - |a|^2) = M(1 - |a|^2)$$

Putting this together we obtain

(ix) $$G'(z_0) = M \cdot \frac{1 - |a|^2}{2\ \log\,|a|} \implies |G'(z_0)| = M \cdot \frac{1 - |a|^2}{2\ \log\,\frac{1}{|a|}}$$

Here $|a| < 1$. A trivial verification which is left to the reader shows that

$$1 - |a|^2) > 2\ \log\,\frac{1}{|a|} \quad : 0 < |a| < 1$$

Hence we have found $G \in \mathcal{F}$ with $|G'(z_0)| > M$. This is a contradiction and hence we must have $f(\Omega) = D$ which finishes the proof of Riemann's Mapping Theorem.

A.4 Other extremal properties.

Let Ω be simply connected and assume that $0 \in \Omega \subset D$. Let $f^*\colon \Omega \to D$ be the conformal mapping with $f^*(0) = 0$ and its derivative at 0 is real and positive. Then the following hold:

A.5 Theorem. *For every point $a \in \Omega$ one has:*

$$(*) \qquad [f^*(a)| = \max_f |f(a)| \quad : f \in \mathcal{O}(\Omega) \quad : f(0) = 0 \quad \text{and} \quad f(\Omega) \subset D$$

To prove (*) we use the inverse map $\phi\colon D \to \Omega$ which satisfies

$$\text{(i)} \qquad f^*(\phi(w) = w \quad : \quad w \in D$$

Next, consider some $f \in \mathcal{O}(\Omega)$ as above. Since $f(0) = 0$ and $\Omega \subset D$ we get $f \circ \phi \in \mathcal{O}(D)$. This analytic function in D is zero at the origin and has maximum norm ≤ 1. Hence Schwarz inequality gives:

$$\text{(ii)} \qquad |f(\phi(w))| \leq |w| \quad : w \in D$$

With $a \in \Omega$ we pick $w \in D$ so that $\phi(w) = a$ and then (i-ii) give

$$|f(a)| \leq |w| = |f^*(a)|$$

Since this hold for every f as above we get (*).

A.6 A result for Jordan domains. Assume that Ω is a Jordan domain bordered by a rectifiable closed Jordan curve Γ. Given $a \in \Omega$ we denote by $\mathcal{F}_a$ the family of analytic functions $g(z)$ in Ω such that $g(a) = 0$ and the complex derivative $g'(z)$ extends to a continuous function on Γ and

$$(*) \qquad \int_\Gamma |g'(z)| \cdot |dz = 2\pi$$

This means that the total length of the image curve $g(\Gamma)$ is 2π. Now one seeks

$$(**) \qquad \max_g |g'(a)| \quad : g \in \mathcal{F}_a$$

Theorem A.7 *The maximum in (**) is attained by the conformal map from Ω onto D which sends a to the origin.*

The proof is left as an exercise. The hint is to study the family $g \circ \phi$ where ϕ is the conformal map from D into Ω which sends the origin to a.

B. The mapping theorem for multiply connected sets.

Let Ω be an open and connected subset of $\mathbf{C} \setminus \{0, 1\}$. For each point $z_0 \in \Omega$ we have the family $M_\Omega(z_0)$ of germs of analytic functions at z_0 which extend to multi-valued functions in Ω. If $f \in M_\Omega\mathcal{O}(z_0)$ and $z \in \Omega$ we denote by $\widehat{f(z)}$ the set of values taken by all local branches of f, i.e. with the notations from § xx:

$$\widehat{f(z)} = \{T_\gamma(f)(z) : \gamma \in \mathcal{C}(z_0, z)\}$$

where $\mathcal{C}(z_0, z)$ is the family of curves with end-points at z_0 and z. We set

$$\mathcal{D}_f = \bigcup_{z \in \Omega} \widehat{f(z)}$$

and refer to this as the total range of f. Next, denote by $M_\Omega\mathcal{O}(z_0)^*$ the set of all $f \in M_\Omega\mathcal{O}(z_0)$ such that $f(z_0) = 0$ and

$$(*) \qquad \widehat{f(z_1)} \cap \widehat{f(z_2)} = \emptyset \quad : z_1 \neq z_2$$

Remark. In §§ we construcuted the modular function $\mathfrak{m}$ in $\mathbf{C} \setminus \{0, 1\}$ which by construction is multi-valued and satisfies the separation condition (*). If ϕ is a conformal maping from the upper half-plane onto D, then $\phi \circ \mathfrak{m}$ has total range D. Next, when $\Omega \subset \mathbf{C} \setminus \{0, 1\}$ we can restrict $\mathfrak{m}$ to Ω where it again becomes a multi-valued function and after a suitable Möbius transform on D the reader can conclude that the $M_\Omega\mathcal{O}((z_0)^*$ is non-empty for every $z_0 \in \Omega$.

The inverse function W_f. Let f be given in $M_\Omega\mathcal{O}(z_0)^*$ and put

$$\mathcal{D}_f = \bigcup_{z\in\Omega} \widehat{f(z)}$$

Notice that (*) implies that for each $w \in \mathcal{D}_f$ there exists a unique point $z(w) \in \Omega$ and a local branch $T_\gamma(f) \in \mathcal{O}(z(w))$ such that

$$\text{(i)} \qquad T_\gamma(f)(z(w)) = w$$

This gives a map

$$W_f \colon \mathcal{D}_f \to \Omega$$

B.0 Lemma. *W_f is an analytic function in $\mathcal{D}_f$.*

Proof. Let $w_0 \in \mathcal{D}_f$ which gives a pair (z_0, γ) such that (i) holds above. Now $T_\gamma(f)$ is analytic function in a small open disc U centered at $z(w_0)$. If $z \in U$ we have $T_\gamma(f)(z)|in\widehat{f(z)}$ and the separation condition (*) implies that $z \mapsto T_\gamma(f)(z)$ is 1-1 in U, i.e. the germ T_γ is locally conformal. In particular the range $T_\gamma(U)$ is an open set which contains w_0 and by the construction of W_f we see that its restriction to $T_\gamma(U)$ is the inverse of the conformal apping $T_\gamma \colon U \to T_\gamma(U)$ which proves Lemma B.0.

Now we announce the uniformisation theorem for Ω where D is the open disc.

B.1 Theorem. *There exists a unique $f \in M_\Omega\mathcal{O}(z_0)^*$ such that $f'(z_0)$ is real and positive and $\mathcal{D}_f = D$. Moreover one has the inequality:*

$$\text{(B.1.1)} \qquad |g'(z_0)| \leq f'(z_0)$$

for all germs $g \in M_\Omega\mathcal{O}(z_0)$ where $g(z_0) = 0$ and $\mathcal{D}_g \subset D$.

Proof of Theorem B.1

Let us first prove the inequality (B.1.1). First, the conditions on f give map $W_f \colon D \to \Omega$ and by Lemma B.0 we have $W_f \in \mathcal{O}(D)$. Next, with g as above the general result in § xx entials that we get an analytic function G in D such that

$$G(w) = g(W_f(w))$$

holds in a small disc $\{|w| < \delta\}$. Taking the complex derivative at $w = 0$ we get

$$\text{(i)} \qquad G'(0) = g'(z_0) \cdot W_f'(0)$$

At the same time

$$W_f(f(z)) = z$$

holds in a disc $\{|z - z_0| < \epsilon\}$ which entails that

$$\text{(ii)} \qquad W_f'(0) = \frac{1}{f'(z_0)}$$

Next, we have $G(0) = 0$ and since $\mathcal{D}_g \subset D$ holds we also have the inclusion $G(D) \subset D$, i.e. the maximum norm of G in D is ≤ 1. Schwarz's inequality applied to G gives $|G'(0)| \leq 1$ and then (i-ii) give (B.1.1).

The uniqueness of f. Suppose that f and g both belong to $M_\Omega\mathcal{O}(z_0)^*$ and $\mathcal{D}_f = \mathcal{D}_g = D$. Now we can apply (B.1.1) starting with f or g and first conclude that

$$\text{(iii)} \qquad |g'(z_0)| = |f'(z_0)|$$

Starting with f we get from the above the analytic function G and (iii) entials that $[G'(0) = 1$ which by Schwarz' reversed principle implies that $G(w) = e^{i\theta} \cdot w$ for some θ. Do of both $f'(0)$ and $g'(0)$ are real and positive we have $G(w) = w$ which implies that the two germs f and g are the same.

The existence of f. The multi-valued version of Montel's theorem in XXX shows that family $M_\Omega\mathcal{O}(z_0)^*$ is a normal. Hence we can solve a variational problem where the derivatives at z_0 are maximized and find $f \in M_\Omega\mathcal{O}(x_0)$ such that $f'(z_0)$ is real and positive and (B.1.1) holds. There remains to prove the equality

$$\mathcal{D}_f = D$$

Assume the contrary, i.e. suppose there exists some $a \in D \setminus \mathcal{D}_f$. Next, multi-valued functions compete in the variational problem and we consider the log-function

(i) $$F(z) = \log \frac{f-a}{1-\bar{a}\cdot f}$$

This germ at z_0 extends to a multi-valued analytic function in Ω. Moreover, since log-functions only add integer multiples of $2\pi i$ we have

$$\mathfrak{Re}\, F = \log |\frac{f-a}{1-\bar{a}\cdot f}| < 0$$

where the last inequality holds since $\mathcal{D}_f \subset d$. hence the essential range fF is contained in the half-space $\{\mathfrak{Re}\, w < 0\}$. Set Next, set

(ii) $$G = \frac{F - F(z_0)}{F + \bar{F}(z_0)}$$

From the above the essential range of G is contained in D so by the extremal property of f gives

(iii) $$|G'(z_0)| \leq f'(z_0)$$

Now a repetition of the proof in 1.1 leads to a contradiction, i.e. one verifies (i-ii) imples that $|G' * (x_0)| > |f'(x_0)|$ which contradicts (iii). Hence $\mathcal{D}_f = D$ and which completes the proof of existence in Theorem B.1.

II. Boundary behaviour

First we study arbitrary bounded analytic functions and establish some results due to Lindelöf and Koebe. Let D be the open unit disc and T the unit circle. Let $\{\omega_\nu\}$ and $\{\omega_\nu^*\}$ be two sequences in D which converge to different points p and q on the unit circle T and $\{\gamma_\nu\}$ is a sequence of Jordan arcs which are contained in D and connect ω_ν with ω_ν^* for each ν. Moreover, we assume that:

$$\lim_{\nu\to\infty}\,\min_{z\in\gamma_\nu}\,|z| = 1 \tag{*}$$

Thus, the joining Jordan arcs stay close to the unit circle T as ν increases.

2.1 Koebe's Lemma *Let f be a bounded analytic function in D such that the maximum norms of f on γ_ν tend to zero as $\nu \to \infty$. Then f must be identically zero.*

Proof. By continuity we may assume that the joining arcs γ_ν are polygons. After a rotation we may assume that $\omega_\nu \to 1$ and $\omega_\nu^* \to e^{i\theta^*}$ for some $0 < \theta^* < \pi$. Given ν we have $\omega_\nu = r_\nu e^{i\theta_\nu}$ and $\omega_\nu^* = r_\nu^* e^{i\theta_\nu^*}$. While drawing the Jordan arc γ_ν from ω_ν to ω_ν^* we encounter the last point $\xi_\nu \in \gamma_\nu$ whose argument is θ_ν and after the first point $\eta_\nu \in \gamma_\nu$ whose argument is θ_ν^*. Replace the pair ω_ν, ω_ν^* with ξ_ν, η_ν which then are joined by a simple polygon Γ_ν which except for its end-points stay in the circular sector where $0 < \arg(z) < \theta^*$. To each ν we therefore get a domain U_ν bordered by Γ_ν and the two rays from the origin to ξ_ν and η_ν. By the hypothesis the maximum norms

$$M_\nu = \max_{z\in\Gamma_\nu}\,|f(z)| \tag{i}$$

tend to zero as $\nu \to \infty$. At the same time the assumption in (*) above gives

$$\min_{z\in\Gamma_\nu}\; = \rho_\nu \quad \text{where } \rho_\nu \to 1$$

Since f is assumed to be bounded in D and without loss of generality we may assume that the maximum norm $|f|_D \leq 1$. Now we will finish the proof is using results from XXX applied to the subharmonic function $log\,|f|$. Namely, we have $\theta_\nu \to 0$ and $\theta_\nu^* \to \theta^*$ with $0 < \theta^* < \pi$ and may assume that $\rho_n u > 3/4$. So when ν is large we get

$$z_* = \frac{1}{2} \cdot e^{i\theta^*/2} \in U_\nu$$

Next, let U_ν^* be the circular sector bordered by the two rays which pass through ξ_ν and η_ν and put $\gamma_\nu^* = T \cap \partial U_v$. Let $\mathfrak{m}_{\gamma_\nu^*}^*(z_*)$ be the harmonic measure at z_* with respect to the boundary arc γ_ν^* of the circular sector. Then there is a fixed constant $a > 0$ such that

$$\mathfrak{m}_{\gamma_\nu^*}^*(z_*) \geq a \quad : \forall\, \nu$$

At the same time Carleman's majorant principle for harmonic measures gives:

$$\mathfrak{m}_{\gamma_\nu^*}^*(z_*) \leq \mathfrak{m}_{\Gamma_\nu^*}^*(z_*)$$

where the right and side is the harmonic measure at z_* in the domain U_ν. Regarding the subharmonic function $\log\,|f|$ we obtain

$$\log\,|f(z_*)| \leq a \cdot \log(M_\nu)$$

By (i) the right hand side tends to $-\infty$ and hence $f(z_*) = 0$. We can achieve a similar vanishing for points in a small disc centered at z_*. So by analyticity f is identically zero in D and Koebe's Lemma is proved.

2.2. Lindelöf's Theorem. *Let J_1 and J_2 be two Jordan arcs in D with a common end-point $p \in T$. Let $f \in \mathcal{O}(D)$ be bounded and assume that it has a limit along both J_1 and J_2, Then the two limit values are equal. Moreover, if two Jordan arcs are disjoint then $f(z)$ converges to the common limit value when z tends to p inside the domain bordered by J_1 and J_2.*

Proof of Lindelöf's theorem

Let a be the limit of f along J_1 and b the limit along J_2. If the curves J_1 and J_2 intersect at some sequence of points $\{z_\nu\}$ which tends to p, then we immediately get $a = b$. So we may assume that J_1 and J_2 do not intersect in the open disc D, i.e. their sole common point is p. After a rotation we may take $p = 1$. When $\delta > 0$ we consider the line L defined by $\mathfrak{Re}(z) = 1 - \delta$ where $\delta > 0$ is small. On J_1 we find the last point ξ which intersects L and similarly the last point η on J_2. We get the domain G_δ bordered by the portions $\Lambda_1 \subset J_1$ and $\Lambda_2 \subset J_2$ where $\mathfrak{Re}(z) > 1 - \delta$ and the line segment which joins η and ξ on L. Put

$$F(z) = (f(z) - a)(f(z) - b)$$

By the assumption on f the maximum norm of F tends to zero on both Λ_1 and Λ_2. At the same time F is a bounded analytic in the whole disc D and put $M = |F|_D$. Now we apply the reflection construction by Schwartz from XX and conclude that the maximum norm

$$|F|_{\partial G_\delta} \leq \sqrt{M} \cdot \sqrt{|F|_{\Lambda_1} + |F|_{\Lambda_2}}$$

Hence F converges to zero in G_δ as $\delta \to 0$. By the construction of F this means that f must tends to a or to b uniformly in G_δ as $\delta \to 0$. But then it is obvious that $a = b$ and at the same time we have proved that f converges to this common number in the domain bordered by J_1 and J_2 as we approach p. This finishes the proof of Lindelöf's theorem.

2.3 Proof of Theorem 0.6.1

Let $f \colon \Omega \to D$ be a conformal map. Let $p \in \partial\Omega$ and consider some Jordan arc $J \in \mathcal{J}(p)$. Put $w = f(z)$ so that $w \in D$. Let J be defined by $t \mapsto \gamma(t)$. In D we get the image arc

(ii) $$t \mapsto f(\gamma(t)) \quad : \quad 0 \leq t < 1$$

Denote it by J^*. We must prove that J^* tends to a point $e^{i\theta} \in T$, i.e.

(iii) $$\lim_{t \to 1} f(\gamma(t)) = e^{i\theta}$$

Assume the contrary. Then we obtain two sequences of points in D:

(iii) $$q_\nu = f(\gamma(t_\nu)) \quad : \quad s_\nu = f(\gamma(\tau_\nu)) \quad : \quad t_1 < \tau_1 < t_2 < \tau_2 \ldots$$

where t_ν and τ_ν both tend to 1 and a pair of *distinct* points q^*, s^* on T such that

(iv) $$q_\nu \to q^* \quad : \quad s_\nu \to s^*$$

Next, to each ν we get the image curve in D given by

(v) $$\gamma^*_\nu = f(\gamma[t_\nu, \tau_\nu])$$

which joints q_ν with s_ν. Now we regard the inverse function $g = f^{-1}$ which is a bounded analytic function in D. Here

(vi) $$\lim_{t \to 1} |g(f(\gamma(t)) - p| = 0$$

With $h(z) = g(z) - p$ it follows that

(vii) $$\lim_{\nu \to \infty} \max_{z \in \gamma^*_\nu} |h(z)| = 0$$

At the same time we notice that since both g and f are conformal we must have

(viii) $$\lim_{\nu \to \infty} \min_{|z|} z \in \gamma^*_\nu = 1$$

Since $q^* \neq s^*$ in (iv) Koebe's Lemma applied to h would entails that $h = 0$. This is a contradiction and Theorem 0.6.1. follows.

2.4 Proof of Theorem 0.6.2

Suppose that $\mathcal{K}_{J_1}(f) = \mathcal{K}_{J_2}(f) = z^*$ for some $z^* \in T$. In D we get the two Jordan arcs $\{J_\nu^* = f(J_\nu)\}$ which both tend to z^*. If $g = f^{-1}$ is the inverse function the limit of $g(z)$ along J_1^* is p and the limit along J_2^* is q. Now $p \neq q$ was assumed which contradicts Lindelöf's Theorem and Theorem 0.6.2 follows.

2.5 Proof of 0.6.4-0.6.5.

Both results are easy consequences of Koebe's Lemma using the same device as in the proof of Theorem 0.6.1 above. So we leave the details of the proofs of the announced results from 0.6.4 and Proposition 0.6.5 as exercises to the reader.

2.6 Accessible points·

Let Ω be bounded and simply connected while no further assumptions are imposed. Let $\phi\colon D \mapsto \Omega$ be a conformal mapping where z is the complex coordinate in D while $\zeta = \phi(z)$ denote points in Ω. Consider a accessible point $p \in \mathcal{A}(\partial\Omega)$ which gives a half-open Jordan arc γ_* in D such that

$$\lim_{t \to 1} \phi(z(t)) = p \tag{1}$$

Lindelöf's theorem applies to the bounded analytic function ϕ and hence (1) entails that ϕ has a non-tangential limit at p_*. In particular there exists the radial limit

$$p = \lim_{r \to 1} \phi(re^{i\theta}) \tag{2}$$

Conversely, let $e^{i\theta} \in T$ and assume that ϕ has a radial limit which yields a boundary point $p \in \partial\Omega$. Then it is clear that $p \in \mathcal{A}(\partial\Omega)$ and we have proved the following:

2.7 Theorem. *Let $\mathcal{R}(\phi)$ be the set of points on T where ϕ has a radial limit. Then $\mathcal{A}(\partial\Omega)$ is equal to the set of $p \in \partial\Omega$ such that*

$$p = \lim_{r \to 1} \phi(re^{i\theta}) \quad \text{for some} \quad e^{i\theta} \in \mathcal{R}(\phi)$$

Thus, $\mathcal{A}(\partial\Omega)$ is equal to the ϕ-image of all radial limit values. Recall from [Measure] that the subset of T where ϕ has a radial limit is dense Borel set. By Theorem 2.7 its image is equal to $\mathcal{A}(\partial\Omega)$ which to begin with implies that $\mathcal{A}(\partial\Omega)$ is a Borel subset of $\partial\Omega$.

2.8 Harmonic measures on $\partial\Omega$. When Ω is simply connected the condition in Theorem xx from Chapter V holds so Dirichlet's problem has a solution. Let $\phi\colon D \to \Omega$ be a conformal map where we let z be the variable in D and set $\zeta = \phi(z)$. Put $\zeta_* = \phi(0)$ and consider the harmonic measure $\mathfrak{m}_{z_*}$ on $\partial\Omega$. So when $f \in C^0(\partial\Omega)$ we have

$$H_f(\zeta_*) = \int f(\zeta) \cdot d\mathfrak{m}_{z_*}(\zeta) \tag{*}$$

if $r < 1$ the circle $|z| = r$ gives the image curve $\Gamma_r = \phi(|z| = r)$ which appears as a closed Jordan curve in Ω. When r is close to one the Jordan domain bounded by Γ_r contains z_* and we get the harmonic measure on Γ_r with respect to z_*. Restricting ϕ to the disc $|z| \leq r$ the transformation rule for harmonic measures under conformal mappings gives:

$$H_f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} H_f(\phi(re^{i\theta})) \cdot d\theta$$

Above $H_f \circ \phi$ is a bounded harmonic function in D. and has therefore radial limits almost everywhere which by dominated convergence gives:

$$H_f(\zeta_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} H_f(\phi(e^{i\theta})) \cdot d\theta$$

More precisely we have integrated the almost everywhere defined function $\theta \to H_f(\phi(e^{i\theta}))$ where $\phi(e^{i\theta})$ are points on $\mathcal{A}(\partial\Omega)$ Hence we have the equality

$$(**) \qquad H_f(\zeta_*) = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(\phi(e^{i\theta})) \cdot d\theta$$

Above $f \in C^0(\partial\Omega)$ is arbitrary so (**) gives a linear functional

$$f \mapsto \frac{1}{2\pi} \cdot \int_0^{2\pi} f(\phi(e^{i\theta})) \cdot d\theta$$

which evaluates H_f at ζ_*. The uniqueness of the harmonic measure this entails that $\mathfrak{m}_{z_*}$ is the Riesz measure representing (**). We can interpretate this in a direct measure theoretic sense. Namely, let $\mathcal{R}(\phi)$ denote the set of points $e^{i\theta}$ where $r \mapsto \phi(re^{i\theta})$ has a radial limit. Then we get the Borel measurable function from $\mathcal{R}(\phi)$ to $\mathcal{A}(\partial\Omega)$ given by

$$(1) \qquad e^{i\theta} \mapsto \phi(e^{i\theta})$$

and the discussion above shows that $\mathfrak{m}_{z_*}$ is the push-forward of $\frac{1}{2\pi} \cdot d\theta$

Remark. The result above implies $\mathcal{A}(\partial\Omega)$ carries all mass of the harmonic measure, i.e.

$$\mathfrak{m}_{\zeta_*}(\mathcal{A}(\partial\Omega)) = 1$$

More generally, for every Borel subset E of $\mathcal{A}(\partial\Omega)$ we have the equality

$$\mathfrak{m}_{\zeta_*}(E) = \frac{1}{2\pi} \int_{\phi^{-1}(E)} d\theta$$

where $\phi^{-1}(E)$ is a Borel set in $\mathcal{R}(\phi)$. Notice also that the results by Koebe and Lindelöf imply that the map (1) is injective.

2.9 A further remark. If $f \in C^0(\partial\Omega)$ we consider its restriction to $\mathcal{A}(\partial\Omega)$ and obtain the function f^* on $\mathcal{R}(\phi)$:

$$(1) \qquad f^*(e^{i\theta}) = h(\phi(e^{i\theta}))$$

Here f^* is defined almost everywhere so via the Poisson kernel it has a unique harmonic extension H^* to D. The previous material shows that

$$H^*(z^*) = H_f(\phi(z))$$

hold when $z \in D$. This gives an injective linear map from $C^0(\Omega)$ to a space of bounded harmonic functions in D. The description of the resulting range of this linear map is unclear without further assumptions on $\partial\Omega$.

3. Picard's Theorem

Introduction In 1879 E. Picard gave the affirmative answer to a question posed by Weierstrass and proved that an entire function $f(z)$ which excludes two values must be a constant. Picard's proof goes as follows: Without loss of generality we can assume that $f(z)$ never takes the values 0 or 1. Consider the *modular function* $\mathfrak{w}(z)$ which exists as a multi-valued function in $\mathbf{C} \setminus \{0,1\}$ with values in the upper half plane U_+. The composed function $g(z) = \mathfrak{w}((f(z))$ becomes a multi-valued function defined in the whole of $\mathbf{C}$. Here $\mathbf{C}$ is simply connected and hence g is single valued. So g is a entire function with values in U_*. But then g is a constant which gives a contradiction, for then f would also be a constant function.

3.1 The Landau-Schottky Theorem

In 1904 Shottky discovered a surprising consequence of the proof by Picard which led to a refined version of Picard's Theorem in joint work with E. Landau. First we give

0.2 Definition *Let $f(z) = a_0 + a_1 z + \ldots$ be an analytic function defined in a disc D_R of radius R centered at the origin. Then D_R is said to be $(0,1)$-excluding relative f if the range $f(D_R)$ does not contain the two points 0 and 1.*

0.3 Theorem *To each non-zero pair (a_0, a_1) there exists a constant $L(a_0, a_1)$ such that if D_R is $(0,1)$-excluding for some $f \in \mathcal{O}(D_R)$, then $R \leq L(a_0, a_1)$.*

Remark The point is of course that $L(a_0, a_1)$ does not depend on higher terms in the series expansion of f. The proof below is attributed to Caratheodory and gives a sharp estimate of $L(a_0, a_1)$ in (*) below.

Proof of Theorem 0.3. Suppose that a disc D_R is $(0,1)$-excluding with respect to f. Let $\mathfrak{w}$ be the modular function. As explained in § XX we get the analytic function $g(z) = \mathfrak{w}(f(z))$ is in D_R where $g(D_R)$ is contained in the upper half plane U. Set

$$\ell(z) = \frac{g(z) - g(0)}{g(z) - \bar{g}(0)}$$

Since $\mathfrak{Im}(g(z)) > 0$ we see that $\ell(z)$ has absolute value < 1 in D_R. Here $\ell(0) = 0$ and Schwarz' Lemma gives:

$$|\ell(z)| \leq \frac{|z|}{R} \quad : \; z \in D_R\,.$$

It follows that the derivative at the origin has absolute value $\leq \frac{1}{R}$. Here

(i) $$\ell'(0) = \frac{g'(0)}{g(0) - \bar{g}(0))} \quad \text{which gives} \quad |\frac{g'(0)}{g(0) - \bar{g}(0))}| \leq \frac{1}{R}$$

Next, notice that

(ii) $$g'(0) = \mathfrak{w}'(a_0) \cdot a_1 \quad \text{and} \quad g(0) = \mathfrak{w}(a_0)$$

Hence (i-ii) give;

(iii) $$R \leq 2 \cdot |\frac{\mathfrak{Im}(\mathfrak{w}(a_0)}{\mathfrak{w}'(a_0) \cdot a_1}|$$

This proves the inequality

(*) $$L(a_0, a_1) \leq 2 \cdot |\frac{\mathfrak{Im}(\mathfrak{w}(a_0)}{\mathfrak{w}'(a_0) \cdot a_1}|$$

Exercise. Show that (*) is sharp wherte a hint is to use the inverse modular function $\mathfrak{w}$.

0.4. The Schottky-Landau Theorem. Working a bit more using the $\mathfrak{w}$-function and various Green's functions, the following result was established by Landau and Schottky:

0.5 Theorem *For each pair (k,θ) where $k>0$ and $0<\theta<1$ there exists a constant $S(k,\theta)$ such that if $R>0$ and the open disc D_R is $(0,1)$ excluding for some $f\in\mathcal{O}(D_R)$, one has*

$$\max_{|z|\leq\theta R}|f(z)|\leq S_1(k,\theta)\quad:\quad|f(0)|\leq k$$

Remark This result applies locally since R can be small and in contrast to the previous case we only assume that f is analytic in the disc D_R. From Theorem 0.5 one can deduce the local version of Picard's Theorem, i.e. that an analytic function $f(z)$ with an isolated essential singularity at some point z_0 must take all values with at most one exception in arbitrary small punctured discs. For details of proof we refer to paragraph 2 in Ch.V from [Bieberbach] which contains further comments about upper bounds of the S-function. Of special interest is the asymptotic behavior as $k\to\infty$. For example, at the end of § 2 in [loc.cit], the following is proved:

0.6 Theorem *There exists an absolute constant* $\mathbf{B}$ *such that for any $R>0$ and any $f\in\mathcal{O}(D_R)$ where D_R is (0,1)- excluding for f, one has*

$$\max_{|z|<\theta R}|f(z)|\leq\exp[\frac{\mathbf{B}\cdot\log\big(|a_0|+2\big)}{1-\theta}\quad:\quad 0<\theta<1\quad:\ a_0=f(0)$$

0.7 Remark. Landau' text-book [Landau] contains a historic account of the Picard's Theorem and a proof of Theorem 0.5 which only uses elementary function theory, i.e avoiding the modular function. Let us also mention that Theorem 0.5 leads to certain a priori inequalities for conformal mappings. We announce one such result from Chapter 7 in [Landau]: Denote by $\mathcal{F}$ the family of all functions

$$f(z)=z+a_2z^2+\ldots$$

such that f gives a conformal map from the unit disc D onto some simply connected domain.

0.8 Theorem. *To each $0<r<1$ there exist constants $C_1(r)$ and $C_2(r)$ such that the following hold for every $f\in\mathcal{F}$:*

$$\max_{|z|\leq r}|f(z)|\leq C_1(r)\quad:\ \frac{1}{C_2(r)}\leq\max_{|z|\leq r}|f'(z)|\leq C_2(r)$$

Since the image domain $f(D)$ need not be bounded this *a priori* inequality is quite remarkable.

1. The method by Ahlfors.

Landau's Theorem can be viewed as a special case of a more general problem where one starts from some open and connected set Ω in the complex plane such that $\mathbf{C}\setminus\Omega$ contains at least two points. Given Ω we consider an analytic function $f(z)$ defined in some open disc D_R centered at the origin such that $f(D_R)\subset\Omega$. At the origin we have the Taylor expansion

$$f(z)=a_0+z+a_2z^2+\ldots\quad:\quad a_0\in\Omega$$

For simplicity we have normalised the situation so that $f'(0)=1$. Landau's theorem implies that there exists such an upper bound R^* which depends on a_0 and Ω only such that the analytic function $f(z)$ only can exist in a disc of radius $<R^*$. Ahlfors constructed certain subharmonic functions in Ω and used an extension of Schwartz inequality to obtain an upper bound for R^* which depends on the existence of a certain subharmonic function in Ω. We shall describe this result. Let w be the complex variable in Ω.

1.1 Definition. *Let $\gamma>0$. Denote by $SH_\gamma(\Omega)$ the class of subharmonic C^2-functions U in Ω satisfying*

$$\Delta(U)(w)\geq e^{\gamma\cdot U(w)}\quad:\quad w\in\Omega$$

Let us assume that 0 and 1 stay outside Ω. In this case one can show that the class $\mathrm{SH}_\gamma(\Omega)$ is non-empy when γ is small enough. Notice that the constraint upon u is more restricted as γ

increases and more relaxed when the open set Ω becomes smaller. Here follows Ahlfors' version of the Schwartz's inequality.

1.2 Theorem. *Let $U \in SH_\gamma(\Omega)$ and $f \in \mathcal{O}(D_R)$ for some $R > 0$. Then*

$$(*) \qquad e^{\gamma \cdot U(f(z))} \cdot |f'(z)|^2 \leq \frac{8 \cdot R^2}{(R^2 - |z|^2)^2} \quad : \quad z \in D_R$$

Proof. In D we consider the function

$$(1) \qquad \Phi(z) = e^{\gamma \cdot U(f(z))} \cdot |f'(z)|^2 \cdot (R^2 - |z|^2)^2$$

Since $\Phi = 0$ on the boundary $|z| = R$ it takes its maximum at some $a \in D_R$, i.e.

$$\Phi(a) = \max_{z \in D_R} \Phi(z)$$

Then $\log \Phi = \phi$ also takes its maximum at a and since the maximum value of Φ is positive we have $f'(a) \neq 0$. Now

$$(2) \qquad \phi(z) = \gamma \cdot U(f(z)) + \log |f'(z)|^2 + 2 \cdot \log (R^2 - |z|^2)$$

Since ϕ takes a maximum at a one has $\Delta(\phi)(a) \leq 0$. Moreover, $|f'(z)|^2$ is harmonic in a neighborhood of a which gives

$$(3) \qquad \gamma \cdot \Delta(U(f(z)) + 2 \cdot \Delta(\log (R^2 - |z|^2)) \leq 0 \quad \text{at the point} \quad z = a$$

Differential rules give:

$$(4) \qquad \Delta(U(f(a)) = \Delta U(f(a)) \cdot |f'(a)|^2$$

and an easy computation also gives

$$(5) \qquad 2 \cdot \Delta(\log (R^2 - |z|^2)) = -\frac{8R^2}{(R^2 - |z|^2)^2}$$

Hence (3-5) entail that

$$(6) \qquad \Delta U(f(a)) \cdot |f'(a)|^2 \leq \frac{8R^2}{(R^2 - |z|^2)^2}$$

Since $U \in \mathrm{SH}_\gamma(\Omega)$ we get

$$(7) \qquad e^{\gamma \cdot U(f(a))} \cdot |f'(a)|^2 \cdot (R^2 - |a|^2)^2 \leq 8R^2$$

Since $\Phi(z)$ attains its maximum at $z = a$ the inequality (7) gives (*) in Theorem 1.2.

1.3 Upper bound for R^*. Let $f(z) = a_0 + z + a_2 z^2 + \ldots$ and suppose that $f(D_R) \subset \Omega$ holds for some R. With $z = 0$ in Theorem 1.2 we must have

$$e^{\gamma \cdot U(a_0)} \leq \frac{8}{R^2} \implies R \leq \sqrt{8} \cdot e^{-\gamma U(a_0)/2}$$

Thus, the existence of some U function in $\mathrm{SH}_\gamma(\Omega)$ gives

$$R^*(a_0, \Omega) \leq \sqrt{8} \cdot e^{-\gamma \dot{U}(a_0)/2}$$

1.4 Remark. Theorem 1.2 illustrates the usefulness to construct subharmonic functions with certain extremal properties. The article [Ahlfors] studies extremal metrics which in addition to analytic functions also apply to *quasi-conformal mappings*. In this way Ahlfors extended Picard's Theorem to a set-up where quasi-conformal mappings appear. For an account about Ahlfors' theory the reader may consult the presentation talk by Caratheodory from the IMU congress at Oslo in 1936 when L. Alhfors received the Fields Prize for his contributions.

4. Some geometric results.

We study geometric properties of maps defined by analytic functions. Let Γ be an interval on the circle $|z| = r$. Suppose that $f(z)$ is analytic in some neighborhood of Γ and that $f'(z) \neq 0$ when $z \in \Gamma$. So if $z_0 \in \Gamma$ then f maps a small circle interval $\gamma \subset \Gamma$ centered at z_0 to a Jordan curve $f(\gamma)$. For each $z \in \gamma$ the curvature along γ at the point $f(z)$ is denoted by $\rho(z)$. With this notation one has:

4.1 Theorem *For each $z \in \gamma$ the following equality holds:*

$$\frac{1}{\rho(z)} = \frac{1 + \mathfrak{Re}[z \cdot \frac{f''(z)}{f'(z)}]}{r|f'(z)|}$$

Proof. The image curve $f(\gamma)$ has the parametrisation $\theta \mapsto f(re^{i\theta})$ defined for some θ-interval. Let ds be the arc-length measure along this curve. By the general result in XXX we have

$$\text{(i)} \qquad \frac{ds}{d\theta} = r \cdot |f'(re^{i\theta})|$$

Next, the θ-parametrisation of $f(T_r)$ gives the complex tangent vector defined by

$$\text{(ii)} \qquad v(\theta) = \lim_{\Delta|\theta \to 0} \frac{f(re^{i\theta + i\Delta\theta} - f(re^{i\theta})}{\Delta\theta} = izf'(z) \quad : \quad z = r^{i\theta}$$

Let $\alpha(\theta) = \arg(v(\theta)$. The definition curvature gives:

$$\text{(iii)} \qquad \frac{1}{\rho} = \frac{d\alpha}{ds} = \frac{d\alpha}{d\theta} \cdot \frac{d\theta}{ds}$$

So by (i) here remains to show that

$$\text{(iv)} \qquad \frac{d\alpha}{d\theta} = 1 + \mathfrak{Re}[z \cdot \frac{f''(z)}{f'(z)}]$$

To show (iv) we notice that $z = re^{i\theta}$ gives

$$\alpha(\theta) = \arg(izf'(z)) = \pi/2 + \theta + \mathfrak{Im}[\mathrm{Log}(f'(re^{i\theta}))]$$

It follows that

$$\frac{d\alpha}{d\theta} = 1 + \mathfrak{Im}\big(ire^{i\theta} \cdot \frac{f''(re^{i\theta})}{f''(re^{i\theta})}\big) = 1 + \mathfrak{Re}\big(re^{i\theta} \cdot \frac{f''(re^{i\theta})}{f''(re^{i\theta})}\big)$$

Since $z = re^{i\theta}$ we have (iv) and Theorem 3.1 is proved.

4.2 Convexity of image curves. Let $f(z)$ be analytic in some open neighborhood of a circle $T_r = \{|z| = r\}$. Notice that we do not require that f extends to an analytic function in the disc D_r. But we assume that f is 1-1 on T_r which gives the closed Jordan curve $f(T_r)$. Recall from analytic geometry that this curve is *strictly convex* if and only if the curvature is everywhere > 0. By Theorem 4.1 strict convexity therefore holds if and only if

$$\mathfrak{Re}\, z \cdot \frac{f''(z)}{f'(z)} > -1 \quad : \quad z \in T_r$$

4.3 Example. Let a be a real number different from 1 and -1. Consider the function

$$f(z) = z + \frac{a}{z}$$

It is analytic in a neighborhood of the unit circle T. We obtain

$$z \cdot \frac{f''(z)}{f'(z)} = \frac{2a}{a - z^2} = \frac{2a(a - z^2)}{|2a - z^2|^2}$$

So with $z = e^{i\theta}$ the real part becomes

$$\frac{2a^2 - 2a \cdot \cos(\theta)}{|2a - e^{i\theta}|^2}$$

Hence we have strict convexity if and only if

$$2a^2 + |2a - e^{2i\theta}|^2 > 2a \cdot \cos(\theta) \quad : \quad 0 \leq \theta \leq 2\pi$$

Remark. Here is a good occasion to use a computer and plot the Jordan curves when a varies and discover when the are convex or not via the criterion above.

A bound for convexity. Let f give a conformal map from the unit disc D onto a simply connected domain. To each $0 < r < 1$ the image of $|z| < r$ is a Jordan domain $\Omega - [r]$.

4.4 Theorem. *Set $r_* = 2 - \sqrt{3}$. Then r_* is the largest number such that $\Omega_f[r]$ are convex for all conformal mappings f.*

Proof. Given f and some $z_0 \in D$ we put

$$\phi(z) = B \cdot f(\frac{z + z_0}{1 + \bar{z}_0 \cdot z})$$

where B is determined so that the derivative $\phi'(0) = 1$ which gives

$$B \cdot f'(z_0) \cdot (1 - |z_0|^2) = 1 \tag{1}$$

Now $\phi(z) = f(z_0) + z + a_2 \cdot z^2 + \ldots$ is a conformal map on D and Theorem XX in XXX gives

$$|a_2| \leq 2 \tag{2}$$

At the same time a computation gives

$$\phi'(z) = B \cdot f'(\frac{z + z_0}{1 + \bar{z}_0 \cdot z}) \cdot \frac{1 - |z_0|^2}{(1 + \bar{z}_0 \cdot z)^2} \implies$$

$$\phi''(0) = B \cdot f''(z_0) \cdot \left[1 - |z_0|^2\right]^2 - B \cdot f'(z_0) \cdot (1 - |z_0|^2) \cdot 2 \cdot \bar{z}_0 =$$

$$\frac{f''(z_0)}{f'(z_0)} \cdot (1 - |z_0|^2) - 2 \cdot \bar{z}_0 \tag{3}$$

Now (2) gives $|\phi''(0)| \leq 4$ and after a multiplication with z_0 we get the inequality

$$\left| z_0 \cdot \frac{f''(z_0)}{f'(z_0)} \cdot (1 - |z_0|^2) - 2 \cdot |z_0|^2 \right| \leq 4 \cdot |z_0| \tag{4}$$

With $|z_0| = r < 1$ we see that (4) gives the inequality

$$\mathfrak{Re}\, z_0 \frac{f''(z_0)}{f'(z_0)} \geq \frac{-4r + 2r^2}{1 - r^2}$$

By (*) in 4.2 the image domain Ω_r is convex if the right hand side is > -1 Hence the critial value R_* is the smallest root of the equation $r^2 4r + 1 = 0$ which gives $r_* = 2 - \sqrt{3}$.

Exercise. Show that the bound r_* is sharp using the Koebe map from XX where we have the equality $|a_2| = 2$ for a normalised conformal map ϕ with $\phi'(0) = 1$.

4.5 Jensen's landscape surface.

Let $f(z) \in \mathcal{O}(\Omega)$ for some open set. Assume that $f \neq 0$ in Ω. To each $z \in \Omega$ we get the positive number $t = |f(z)|$. So with $z = x + iy$ we obtain a surface in the real (x, y, t)-space defined by

$$t = |f(x + iy)|$$

It is denoted by $\mathcal{J}_f$. To each $z = x + iy \in \Omega$ we get the point

$$p = (x, y, |f(x, y)|) \in \mathcal{J}_f$$

Let Π be the tangent plane to $\mathcal{J}_f$ at p. Let $\gamma(p)$ be the acute angle between Π and the (x, y)-plane. So here $0 < \gamma(p) < \frac{\pi}{2}$. With these notations one has;

$$\mathrm{tg}(\gamma(p)) = 2 \cdot |f(z)| \cdot |f'(z)|$$

Proof. Let n be the unit normal to Jensen's surface whose t-component becomes

$$n_t = \frac{1}{(\partial_x |f|)^2 + (\partial_y |f|)^2 + 1}$$

By elementary geometry we have

$$n_t \cos(\gamma(p)) = \frac{1}{(\partial_x |f|)^2 + (\partial_y |f|)^2 + 1}$$

At this stage the reader can finish the proof after calculating the partial derivatives of $|f|$ by expressing f as $u + iv$.

Koebe's One Quarter Theorem

The result below was established by Koebe in 1907. Let $f(z)$ be analytic in the unit disc D. Assume that $f(0) = 0$ and $f'(0) = 1$ and that f is 1-1, i.e. f gives a *conformal map* from D onto a domain Ω.

4.6 Theorem. *The image set $f(D)$ contains the open disc of radius $\frac{1}{4}$ centered at the origin, i.e. to any $|w| < 1/4$ there exists $z \in D$ so that $f(z) = w$.*

Proof. To begin with $f(D)$ certainly contains some open disc centered at the origin. This yields the existence of a positive number d defined by

$$d = \min |w[\quad : \quad w \in C \setminus f(D)$$

We find some $w^* C \setminus f(D)$ such that $|w^*| = d$. Next, since the image $f(D) = \Omega$ is *simply connected* the result in plane topology from XXX gives the existence of a simple curve Γ with a starting point at w^*, contained in $C \setminus f(D)$ and moving to the point at infinity. Removing Γ we also know from XXX that $C \setminus \Gamma$ is simply connected. This implies that there exists a *single-valued* branch of the root function $\sqrt{w-d}$ in $C \setminus \Gamma$ such that

$$\mathfrak{Re}(\sqrt{w-d}) > 0 \quad : \quad w \in C \setminus \Gamma$$

Set

$$g(w) = -4d \frac{\sqrt{w-d} - i\sqrt{d}}{\sqrt{w-d} + i\sqrt{d}}$$

Then we see that

$$g'(0) = 0 \quad : \quad |g(w)| < 4d \quad : \ w \in C \setminus \Gamma$$

Next, consider the composed function $H = g(f(z))$ which becomes analytic in D. Since $f(0 = 0$ and $f'(0) = 1$ we obtain

$$H'(0) = g'(0) f'(0) = 1 \quad : \quad |H(z)| < 4d \quad : \ z \in D$$

Since we also have $H(0) = g(0) = 0$, the last estimate and Schwarzs' inequality from XX give $|H'(0) \leq 4d$. Since we also have $H'(0) = 1$ we get $d \geq \frac{1}{4}$ as required.

4.7 The Area Theorem.

Consider a function

$$(*) \qquad w(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

where the negative Laurent series is convergent in the exterior disc $|z| > 1$. At the point at infinity the w-function has a simple pole since z appears in (*). Assume that (*) yields a conformal map from the exterior domain $D^* = \{|z| > 1\} \cup \{\infty\}$ onto a simply connected exterior domain which includes the point at infinity in the w-plane bordered by a closed Jordan arc Γ. Thus, the *bounded* Jordan domain bordered by Γ is outside the image of the map (*).

4.8 Theorem. *One has the inequality*

$$\sum_{n=1}^{\infty} n \cdot |a_n|^2 \leq 1$$

Proof. If $r > 1$ we denote by $\Gamma(r)$ the closed Jordan curve in the w-plane which is the image of the circle $|z| = r$. Let $J(r)$ be the area of the bounded Jordan domain in the w-plane which is bordered by $\Gamma(r)$. With $w = u + iv$ we recall from (xx) that Green's formula gives:

$$J(r) = \int_0^{2\pi} u(re^{i\theta}) \cdot \frac{dv(re^{i\theta})}{d\theta} \cdot d\theta \tag{i}$$

The Cauchy Riemann equations imply that (i) is equal to:

$$\int_0^{2\pi} \frac{w(re^{i\theta}) + \bar{w}(re^{i\theta})}{2} \cdot \frac{w'(re^{i\theta}) - \bar{w}'(re^{i\theta})}{2i} \cdot d\theta \tag{ii}$$

Now (ii) has the series expansion

$$\int_0^{2\pi} [\frac{re^{i\theta} + re^{-i\theta}}{2} + \sum_{n=1}^{\infty} \frac{a_n \cdot re^{-in\theta}) + \bar{a}_n \cdot (re^{in\theta})}{2 \cdot r^n}] \cdot$$
$$[\frac{re^{i\theta} + re^{-i\theta}}{2} - \sum_{n=1}^{\infty} \frac{na_n \cdot re^{in\theta}) + n\bar{a}_n \cdot (re^{in\theta})}{2 \cdot r^n}] \cdot d\theta$$

Since the integrals $\int_0^{2\pi} e^{ik\theta} \cdot d\theta = 0$ for all integers $k \neq 0$, a computation shows that the expression above becomes

$$\pi r^2 - \pi \cdot \sum_{n=1}^{\infty} \frac{n \cdot |a_n|^2}{r^{2n}} \tag{iii}$$

Here (iii) holds for every $r > 1$ and since it expresses the non-negative area $J(r)$ it is non-negative. Passing to the limit as $r \to 1$ we get the inequality in the theorem.

4.9 An extremal problem. Consider a double connected domain Ω where $\partial\Omega$ consists of two disjoint and closed Jordan curves Γ_* and Γ^*. Here Γ^* is the outer curve which borders the unbounded component of $\mathbf{C} \setminus \bar{\Omega}$. By XXX there exists a unique number $0 < \ell < 1$ such that Ω is conformally equivalent to the annulus $\mathcal{A}(\ell) = \{0 < \ell < |z| < 1\}$ and we set $\ell = \ell(\Omega)$. Let Ω^* be the Jordan domain bounded by the outer curve Γ^* and Ω_* is the Jordan domain bounded by Γ_*. Set

$$\rho_* = \text{Area}(\Omega_*) \quad \text{and} \quad \rho^* = \text{Area}(\Omega^*)$$

So if Ω is equal to $\mathcal{A}(\ell)$ or a translate of this annulus we have

$$\ell^2 = \frac{\rho_*}{\rho^*}$$

4.10 Theorem. *For every doubly connected domain Ω which is not a translate of an annulus one has strict inequality*

$$\ell(\Omega)^2 > \frac{\rho_*}{\rho^*}$$

Proof. Set $\ell = \ell(\Omega)$ and let $F\colon \mathcal{A}(\ell) \to \Omega$ be a conformal map. Now $F(z)$ has a Laurent series expansion

$$F(z) = \sum_{-\infty}^{\infty} c_n \cdot z^n \tag{1}$$

We get the area formulas

$$\rho_* = \frac{1}{\pi} \cdot \sum_{-\infty}^{\infty} n \cdot |c_n|^2 \cdot \ell^{2n} \quad \text{and} \quad \rho^* = \frac{1}{\pi} \cdot \sum_{-\infty}^{\infty} n \cdot |c_n|^2 \tag{2}$$

Since $\ell < 1$ we get :

$$\ell^2 \cdot \sum_{n=1}^{\infty} n \cdot |c_n|^2 \geq \sum_{n=1}^{\infty} n \cdot |c_n|^2 \cdot \ell^{2n} =$$

$$\pi \cdot \rho_* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2 \cdot \ell^{-2n} \geq \pi \cdot \rho_* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2 \tag{3}$$

Above strict inequality holds unless

$$c_n = 0 \quad \text{for all} \quad n \neq 1 \tag{4}$$

We have also the equality

$$\sum_{n=1}^{\infty} n \cdot |c_n|^2 = \pi \cdot \rho^* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2 \tag{5}$$

The inequality (3) and a division with $\sum_{n=1}^{\infty} n \cdot |c_n|^2$ gives

$$\ell^2 \geq \frac{\pi \cdot \rho_* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2}{\pi \cdot \rho^* - \sum_{n-1}^{-\infty} n \cdot |c_n|^2} \tag{6}$$

Since $\rho^* - \rho_* > 0$ it is obvious that the last term is $\geq \frac{\rho_*}{\rho^*}$ and Theorem 4.10 follows.

Remark. Theorem 4.10 appears in Carleman's article *Über eine Minimalproblem der mathematischen Physik* from 1917 where an application of Theorem 4.10 to *Zylinderkondensatoren* is described.

5. Schwarz-Christofell maps

Introduction. Following original constructions by H. Schwartz and Christoffell we shall find a conformal mapping from the unit disc onto a convex polygon Π whose corner points are denoted by $w_1, \ldots, w_N$ where $N \geq 3$. Performing a translation if necessary we may assume that the origin is an interior point of Π. Here Π is placed in the complex w-plane. The corner points $w_1, \ldots, w_N$ are arranged so that the boundary has a positive orientation - i.e. anti-clockwise. See figure XXX. At each corner point w_k we get the two line segments $\ell_*(k)$ and $\ell^*(k)$ where $\ell_*(k)$ joins w_{k-1} with w_k and $\ell^*(k)$ joins w_k with w_{k+1}. In the case $k = 1$ then $\ell_*(1)$ joins w_N with w_1 and $\ell^*(N)$ joins w_N with w_1. At each corner point we have the interior angle β_k where $0 < \beta_k < \pi$. A wellknown formula from euclidian geometry gives

$$\sum \beta_k = (N-2)\pi$$

The outer angles are defined by:

$$\alpha_k = \pi - \beta_k \quad \implies \quad \sum \alpha_k = 2\pi$$

See figure XXX for an illustration. Riemann's mapping theorem gives the unique analytic function $f(z)$ in D such that $f(0) = 0$ and $f'(0)$ is real and positive while f maps D conformally onto Π. Moreover, since Π is a Jordan domain we know that this conformal mapping. extends continuously up to the boundary and f yields a bi-continuous map from the unit circle T onto the boundary of Π. On T we get the points $z_1, \ldots, z_N$ which are mapped to the corner points of Π. Performing a rotation of Π in the w-plane we may assume that $z_1 = 1$ and if $2 \leq k \leq N$ we have

$$z_k = e^{i\theta_k} \quad \text{where} \quad 0 < \theta_2 < \ldots < \theta_N < 2\pi$$

When $k = 1$ we have $z_1 = 0$ so $\theta_1 = 0$. Put

$$a_k = \frac{\alpha_k}{\pi} \quad : \quad 1 \leq k \leq N$$

Hence (xx) entails that

$$\sum a_k = 2$$

To every $1 \leq k \leq n$ we get the analytic function in D defined by

(1) $$\phi_k(\zeta) = (1 - e^{-i\theta_k}\zeta)^{-a_k}$$

where single valued branches are chosen so that

$$-\pi/2 < \arg(\phi(\zeta)) < \pi/2$$

In particular $\phi_k(0) = 1$. With these notations one has:

5.1 Theorem. *The conformal map f from D onto Π is given by:*

$$f(z) = c_0 \cdot \int_0^z \frac{d\zeta}{(1 - e^{-i\theta_1}\zeta)^{\alpha_1} \cdots (1 - e^{-i\theta_N}\zeta)^{\alpha_N}} \quad : c_0 > 0 \text{ is a positive constant}$$

To prove this we analyze the function

$$\theta \mapsto f(e^{i\theta})$$

over θ-intervals which avoids the n-tuple $\{\theta_k\}$. When $\theta_k < \theta < \theta_{k+1}$ holds we have

$$f(e^{i\theta}) = c_0 \cdot \int_0^{e^{i\theta}} \frac{d\zeta}{(1 - e^{-i\theta_1}\zeta)^{a_1} \cdots (1 - e^{-i\theta_N}\zeta)^{a_N}}$$

It follows that the θ-derivative becomes

$$\frac{df}{d\theta} = c_0 \cdot i \cdot e^{i\theta} \cdot \frac{1}{\prod (1 - e^{-i\theta_k + i\theta})^{a_k}}$$

The argument of the right hand side becomes

(i) $$\frac{\pi}{2}+\theta-\sum a_k\cdot\arg\left(1-e^{-i\theta_k+i\theta}\right)$$

As explained in XX: Chapter I we have

$$\arg\left(1-e^{-i\theta_k+i\theta}\right)=\frac{\theta-\theta_k-\pi}{2}$$

for every k. Together with (*) above we conclude that

$$\arg\frac{df}{d\theta}=\frac{\pi}{2}+\theta+\sum a_k\cdot\frac{\theta_k+\pi-\theta}{2}=\frac{3\pi}{2}+\sum a_k\cdot\frac{\theta_k}{2}$$

where the last equality follows from XX. In particular the argument remains constant which means that f maps the circular arc $\{\theta_k<\theta<\theta_{k+1}\}$ onto a line segment. So this confirms that if f is s conformal mapping then it sends D onto a Jordan domain bounded by a piecewise linear boundary curve which we want to be equal to the given convex polygon Π.

Proof that f is a conformal mapping. To show this we investigate the conformal mapping $f_*(z)$ which is already predicted by Riemann's mapping theorem and we are going to show that the complex derivative of f_* is equal to the complex derivative

(1) $$f'(z)=c_0\cdot\phi_1(z)\cdot\phi_N(z)$$

To prove that $f'_*(z)$ is equal to the function in (1) we first consider a circular arc:

(1) $$\gamma_k=\{e^{i\theta}\quad:\theta_k<\theta<\theta_{k+1}\}$$

Now f_* maps this arc onto a line segment of the polygon Π. In particular $\arg(f'_*)$ is constant on γ_k and then Schwarz' reflection principle implies that the complex derivative f'_* extends analytically across γ_k. In fact, set

(2) $$\beta_k=\arg f'_*(e^{i\theta})\quad:\quad e^{i\theta}\in\gamma_k$$

In the exterior disc $|z|>1$ we define the analytic function

(3) $$g_k(z)=e^{2i\beta_k}\cdot\bar{f}'_*(\frac{1}{\bar{z}})$$

When $e^{i\theta}\in\gamma_k$ we obtain

(4) $$\arg g_k(e^{i\theta})=2\beta_k\arg f'_*(e^{i\theta})$$

It follows that $g_k=f'_*$ on γ_k and hence g_k yields the analytic extension of f'_* across γ_k where g_k is analytic in the exterior disc $\{|z|>1\}$. Next, for any other arc γ_ℓ we can return to the unit disc by performing an analytic extension of g_k across ℓ_ν. Moreover, we see that the new analytic function in d must be a constant times f'_*. This process can be continued, and the conclusion is that f'_* is a single-valued branch of a multi-valued analytic function F defined in $\mathbf{C}\setminus(e^{i\theta_1},\ldots,e^{i\theta_N})$. Moreover, by the explicit analytic continuations over the γ-segments the rank of the multi-valued function F is equal to one and hence the general result in XX shows that the single-valued branch f'_* is given in the form

(5) $$f'_*(z)=\prod_{k=1}^{k=N}(1-e^{-i\theta_k}z)^{\rho_k}\cdot H(z)$$

where $\rho_1,\ldots,\rho_N$ are complex numbers and $H(z)$ is an entire function. From (4) above the jump of $\arg f'_*$ at each point $e^{i\theta_k}$, via the local study in XX gives the equalities:

$$\rho_k=-a_k$$

There remains to prove that the entire function $H(z)$ is constant. To see this we consider the analytic extension of f_* across some γ_k as above which leads to the analytic function g_k in the exterior domain. By the construction g_k is bounded in this exterior disc and from this we see that the entire function $H(z)$ is bounded and hence reduced to a constant by Liouville's theorem. Hence we have proved that there exists a constant c such that

$$f'_* = c \cdot f'$$

Since both f and f'_* are zero at the origin we have proved that f is a constant times f'_* which shows that f gives the requested conformal mapping from D onto Π.

5.4 A local study.

With the notations as above we analyze the behaviour of the mapping function $f_*(z)$ as $z \to z_k$ for every k. We already know that f_* extends to a multi-valued function in a punctured disc centered at z_k and there remains to consider the situation locally for each given z_k. After a rotation we may take $z_k = w_k = 1$ and at the corner point w_k we have some angle α. Put

$$a = \frac{\alpha}{\pi}$$

Consider the analytic function $g(z)$ in D defined by

(i) $$\phi(z) = (z-1)^{1-a}$$

When $\theta > 0$ is small the complex argument

(ii) $$\arg(e^{i\theta} - 1) \simeq \frac{\pi}{2}$$

At the same time, when θ is small and negative we have

(iii) $$\arg(e^{i\theta} - 1) \simeq -\frac{\pi}{2} \quad : \theta < 0$$

Hence we have two limit formulas:

(iv) $$\lim_{\theta \to 0_+} \arg \phi(e^{i\theta}) = (1-\alpha)\frac{\pi}{2} \quad : \quad \lim_{\theta \to 0_-} \arg \phi(e^{i\theta}) = (\alpha - 1)\frac{\pi}{2}$$

From figure XX the two limit formulas reflect the jump discontinuity of the argument for the mapping function f at the corner point. From this we conclude that $f(z)$ close to $z = 1$ has a fractional series expansion

(v) $$f(z) = 1 + [\, c_1(x-1) + c_2(x-1) + \ldots]^{1-\alpha}$$

Passing to the complex derivative we get

(vi) $$f'(z) = [\, c_1(x-1) + c_2(x-1) + \ldots]^{-\alpha}$$

It follows that

(iv) $$\frac{f''(z)}{f'(z)} = \frac{-\alpha}{z-1} + \sum_{\nu=0}^{\infty} d_\nu \cdot (z-1)^\nu$$

where the last sum is an analytic function in some disc centered at $z = 1$. So in (iv) above we encounter a simple pole whose residue is determined by the angle α. Applying this to every corner point of the given polygon Π we have proved:

5.5 Lemma *Under analytic continuation of $f(z)$ it follows that $\frac{f''(z)}{f'(z)}$ becomes a single valued meromorphic function of the form:*

$$\phi(z) = \sum \frac{-\alpha_\nu}{z_\nu - z} + H(z) \quad : H(z) \text{ entire function}$$

There remains only to see that the entire function is zero. To show this we first notice that the analytic continuation of f is achived by reflections in boundary arcs of T and therefore remains bounded. In particular the entire function H is bounded and hence a constant. Finally, this constant must be zero for otherwise

$$\operatorname{Log} f(z) = -\sum \alpha_\nu \cdot \operatorname{Log}(z_\nu - z) + bz \quad : \ b \neq 0$$

and taking the exponential we see that $f'(z)$ increases too fast.

5.2 Determination of the z_k-numbers. The given polygon Π gives the α-numbers in the Schwarz-Christoffell formula. There remains to determine the points $\{z_k = e^{i\theta_k}\}$ on the unit circle which are mapped to corner points of the polygon. This amounts to compute the integrand in Theorem 5.1 to recover the lengths of the sides in Π. The determination of the N-tuple $z_1, \ldots, z_N$ when Π is prescribed is not easy. The reason is that one cannot solve these z_k-numbers step by step. One has to regard a system of N many *non-linear* equations where the z_k appear in a rather implicit way. However, computer programs offer algorithms which give numerical solutions for the z-numbers. See for example [XXX].

6. Privalov's theorem

Introduction. We shall prove a uniqueness result established by Privalov in 1917. Let $f(z) \in \mathcal{O}(D)$ where no conditions are imposed on $|f(z)|$, i.e. it may have arbitrary growth. There exist f which is not identically zero, and yet the radial limits are all zero, i.e.

$$\lim_{r \to 1} f(re^{i\theta}) = 0 \quad : \forall\, 0 \le \theta \le 2\pi$$

Remark. For the construction of such a function f we refer to Privalov's text-book [Pri]. See also [Bi:2.page 152-154] for the construction of an unbounded analytic function f in D which *never* has a radial limit.

To establish a uniqueness theorem for a general unbounded function one must therefore allow *non-tangential limits.* Following Privalov we describe how such non-tangential limit values should be defined.

The Jordan domain $\mathfrak{M}_E$ Let J be an interval in T with two end-points $\zeta_\nu = e^{i\theta_\nu}$ where $|\theta_1 - \theta_2| < \pi$. Then we obtain a curve linear triangle Δ_J constructed upon J as follows: Through ζ_1 we take the straight line ℓ_1 which has angle $\pi/4$ with T. Similarly we construct the line ℓ_2. The two lines intersect at a point $q \in D$. Then Δ_J is the domain bordered by J and the two line segments along the ℓ-lines from ζ_ν to q. The reader may illustrate this by a figure. Notice that the angle at q becomes $\pi/2$.

Next, let E be a closed subset of T. Now $T \setminus E$ is a disjoint union of open intervals J_ν. To each of them we get the domain Δ_ν. Put

$$\mathfrak{W}_E = D \setminus \cup \bar{\Delta}_\nu$$

We refer to $\mathfrak{W}_E$ as *Privalov's domain* attached to E. It is easily seen that it is a Jordan domain. In fact, to each $0 \le \theta \le 2\pi$ we consider the ray from the origin in the θ-direction and looking a a figure the reader discovers that there is a unique $0 < r(\theta) \le 1$ such that $re^{i\theta} \in \partial \mathfrak{W}_E$ while $re^{i\theta} \in \mathfrak{W}_E$ when $0 \le r < r(\theta)$. In this way we get a bi-continuous map from the periodic θ-interval onto $\partial \mathfrak{W}_E$. Notice also that $r(\theta) = 1$ precisely when $e^{i\theta} \in E$. By the construction the boundary of $\mathfrak{M}_E$ is piecewise linear inside D and where the total arc-length is $\sqrt{2} \cdot \sum |J_k \nu|$ as one sees from the construction of the Δ-triangles. When E has positive Lebesgue measure the remaining part of the simple and closed Jordan $\partial \mathfrak{W}_E$ has length $|E|$. Hence the boundary of $\mathfrak{M}_E$ is a *rectifiable* Jordan curve. Now we can announce Privalov's uniqueness theorem.

6.1 Theorem *Let $E \subset T$ be a closed set of positive measure. If $f \in \mathcal{O}(D)$ is such that*

$$\lim_{z \to E} f(z) = 0 \quad : \quad z \in \mathfrak{W}_E$$

Then f is identically zero.

Proof . By assumption we have a pointwise convergence to zero as $z \in \mathfrak{M}_E$ tends to E. By *Egoroff's theorem* in measure theory we can therefore find a closed subset E_* if E such that the limit is attained *uniformly* as $z \to E_*$ and at the same time E_* again has positive measure. Now we notice that $\mathfrak{W}_{E_*} \subset \mathfrak{W}_E$ and we restrict f to this Jordan domain. Let f_* denote this restricted function. By the uniform convergence it follows that f_* is a bounded analytic function in $\mathfrak{W}_{E_*}$. Next, consider a conformal map ϕ from D onto $\mathfrak{W}_{E_*}$ and put $g = f_* \circ g$. Then g is a bounded analytic function in D. Since E_* has positive Lebesgue measure and $\partial \mathfrak{W}_{E_*}$ is a rectifiable Jordan arc, it follows that the inverse set $\phi^{-1}(E_*)$ appears as a closed subset of T with positive Lebesgue measure. Hence g tends to zero on set of positive measure and is therefore identically zero, i.e. as a very special case of Fatou's result in XX. We conclude that $f = 0$ and Privalov's theorem is proved.

7. Maps between multiple connected domains.

Introduction. Let us begin the study with a doubly connected domain Ω whose outer boundary is the unit circle T and the inner boundary a closed Jordan curve γ contained in the unit disc. There exists the harmonic function ω with boundary value one on T and zero on γ, i.e. ω is the harmonic measure with respect to T. Set

$$b = \int_T \frac{\partial \omega}{\partial \mathbf{n}} \cdot ds$$

It is clear that $b > 0$ and we put:

$$\alpha = \frac{2\pi}{b}$$

The result in XXX shows that there exists an analytic function $f(z)$ in Ω defined by

$$f(z) = e^{\alpha \cdot \omega + iV}$$

where V is the locally defined harmonic conjugate of $\alpha \cdot \omega$. Since

$$|f(z)| = e^{\alpha \cdot \omega(z)}$$

and $\omega(z) = 0$ on γ we conclude that $|f| = 1$ on γ while $|f| = e^{\alpha}$ on T.

7.1. Proposition. *The function f maps from Ω conformally onto the annulus $1 < |z| < e^{\alpha}$.*

Proof. The maximum principle for analytic functions applied to f and $\frac{1}{f}$ give

$$1 < |f(z)| < e^{\alpha} \quad : \; z \in \Omega$$

Next, let w be a complex number in the annulus, i.e. $1 < |w| < e^{\alpha}$. By the general result in XX f is conformal if we prove the equality

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z) - w} \cdot dz = 1 \tag{1}$$

for every such w. To get (1) we use that $|f(z)| = 1 < |w|$ on the inner curve γ. So the result in XXX gives:

$$\int_{\gamma} \frac{f'(z)}{f(z) - w} \cdot dz = 0 \tag{2}$$

Next, on the unit circle T it follows from the general result in XX from Chapter 4 that

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z) - w} \cdot dz = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} \cdot dz = \tag{3}$$

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{dV}{d\theta}(e^{i\theta}) \cdot d\theta$$

Here V is the local harmonic conjugate to $\alpha\omega$ which means gives

$$\frac{dV}{d\theta} = \alpha \cdot \frac{\partial \omega}{\partial \mathbf{n}}$$

along T. Hence the choice of α shows that (3) has value one and together with (2) we conclude that (1) also has value one.

7.2 General case.

Consider a domain $\Omega \in \mathcal{D}(C^1)$ with p boundary curves $\gamma_1, \ldots, \gamma_p$ where $p \geq 3$. Let γ_p be the curve which borders the unbounded connected component of $\mathbf{C} \setminus \Omega$. To each $1 \leq j \leq p-1$ we have the harmonic function ω^j with boundary value 1 on γ_j and zero on the remaining boundary curves. This gives a $(p-1) \times (p-1)$-matrix $\mathbf{B}$ with elements

$$b_{\nu,j} = \int_{\gamma_\nu} \omega^j_{\mathbf{n}} \cdot ds \quad : \; 1 \leq j, \nu \leq p-1 \tag{1}$$

By the result in XXX this matrix is non-singular which gives a unique solution to the following system of inhomogeneous linear equations:

$$b_{11}\xi_1 + \ldots + b_{1,p-1} \cdot \xi_{p-1} = -2\pi$$
$$b_{\nu 1}\xi_1 + \ldots + b_{\nu,p-1} \cdot \xi_{p-1} = 0 \quad : 2 \leq \nu \leq p-1\,.$$

Set

$$H = \xi_1\omega_1 + \ldots + \xi_{p-1}\omega_{p-1}$$

This gives

(2) $$\int_{\gamma_1} H_{\mathbf{n}} \cdot ds = -2\pi \quad \text{and} \int_{\gamma_\nu} H_{\mathbf{n}} \cdot ds = 0 \quad : \ 2 \leq \nu \leq p-1\,.$$

Recall from XXX that the sum of the line integrals over all the boundary curves is zero and hence we have:

(*) $$\int_{\gamma_p} H_{\mathbf{n}} \cdot ds = 2\pi$$

Since the line integrals of $H_{\mathbf{n}}$ are integer multiples of 2π for every the boundary curve the locally defined harmonic conjugate V of H is determined up to integer multiples of 2π. Hence there exists the analytic function $f(z)$ in Ω defined by

$$f(z) = e^{H(z)+iV(z)}$$

Along each inner boundary curve γ_ν we get:

$$|f(z)| = e^{H(z)} = e^{\xi_\nu} \quad : \ 1 \leq \nu \leq p-1$$

and along the outer curve γ_p we have $|f(z)| = 1$. Moreover one has:

7.3 Theorem. *The function f yields a conformal map from Ω onto a domain Ω^*.*

Proof. The same reasoning as in the proof of Proposition 7.1 gives

$$\frac{1}{2\pi} \cdot \int_{\gamma_\nu} H_{\mathbf{n}} \cdot ds = \frac{1}{2\pi i} \int_{\gamma_\nu} \frac{f'(z)}{f(z)} \cdot dz \quad : \quad 1 \leq \nu \leq p$$

Moreover, from (2) we have

$$\int_{\gamma_\nu} \frac{f'(z)}{f(z)} \cdot dz = 0 \quad : \quad 2 \leq \nu \leq p-1$$

Since the absolute value of f along a curve γ_ν is the constant $e^{|xi_\nu}$ the general result in XX gives

(1) $$|w| \neq e^{\xi_\nu} \implies \int_{\gamma_\nu} \frac{f'(z)}{f(z)-w} \cdot dz = 0 \quad : \quad 2 \leq \nu \leq p-1$$

We have also

(2) $$\int_{\gamma_p} \frac{f'(z)}{f(z)-w} \cdot dz = 2\pi \quad \text{for each} \quad |w| < 1$$

(3) $$\int_{\gamma_1} \frac{f'(z)}{f(z)-w} \cdot dz = 0 \quad \text{for each} \quad |w| < e^{\xi_1}$$

while (2) is zero if $|w| > 1$ and similarly (3) is zero if $|w| > e^{\xi_1}$. Next, since f is a non-constant analytic function in Ω we find some w such that $f(z) = w$ has at least one solution where we may assume that $|w| \neq e^{\xi_\nu}$ for all ν and $|w| \neq 1$. Let $N(w)$ be the sum of zeros of $f(z) - w$ counted with multiplicities which by (1) give

$$2\pi i \cdot N(w) = \int_{\gamma_1} \frac{f'(z)}{f(z)-w} \cdot dz + \int_{\gamma_p} \frac{f'(z)}{f(z)-w} \cdot dz$$

From (2-3) we see that $N(w) \geq 1$ implies that $|w| < 1$ and $|w| > e^{\xi_1}$ and that the integer $N(w)$ cannot exceed one. Hence we have proved the inequality

$$e^{\xi_1} < 1$$

and we have also proved also that f restricts to a conformal map in Ω since $w \mapsto N(w)$ is the constant one.

7.4 The image domain $f(\Omega)$. Using (1-3) above one shows that f maps the outer curve γ_p onto the unit circle T and γ_1 onto the circle of radius e^{ξ_1}. Next, if $2 \leq \nu \leq p-1$ then f maps γ_ν onto an interval ℓ_ν of the circle $\{|z| = e^{\xi_\nu}\}$. See figure XX for an illustration of the image domain.

Exercise. Prove the assertions in 7.4. See also [page xx-xx] in [Ahlfors] for a discussion and a proof of (7.4).

Parallell slit regions Let Ω as above be a domain in the class $\mathcal{D})\mathbf{C}^1)$ with p many boundary curves. In Chapter V:A.6 we introduced the function $H(z,\zeta)$ defined in $\Omega \times \Omega$ where Theorem 6.1 in [ibid] shows the symmetry

$$H(z,\zeta) = H(\zeta, z)$$

for each pair of points in Ω. Moreover

$$H(z,\zeta) = \log |z - \zeta|$$

when $z \in \Omega$ and $\zeta \in \partial\Omega$. We can use H to construct special conformal mappings. Write $\zeta = \xi + i\eta$ and notice that the function

$$z \mapsto \frac{\partial}{\partial \xi} H(z, \zeta_0)$$

is a harmonic of the variable z in Ω. Keeping ζ_0 fixed we denote this function by $u(z)$. Next, to each inner boundary curve γ_ν we have the harmonic measure function ω_ν and as explained in Chapter V:A:xx we find real constants $a_1, \ldots, a_{p-1}$ such that if

$$U = u - (a_1 \cdot \omega_1 + \cdots + a_{p-1} \cdot \omega_{p-1})$$

then u has a single-valued harmonic conjugate V which gives the analytic function $f(z) = U + iV$ in Ω. Next, from (xx) we have the equality

$$u(z) = \mathfrak{Re}\, \frac{1}{z - \zeta_0} \quad : \quad z \in \partial\Omega$$

Let us put

$$p(z) = f(z) - \frac{1}{z - \zeta_0}$$

Then we see that $\mathfrak{Re}\, p$ is constant on each boundary curve.

Theorem. *The analytic function $p(z)$ yields a conformal mapping from Ω onto a a slit region whose complement consists of p vertical segments.*

Remark. Above p maps ζ_0 to the point at infinity. Its residue a ζ_0 is one and by this normalisation the vertical segments under a conformal map as above are determined except for a parallel translation.

Example.

8. The Bergman kernel.

Introduction. Let Ω be a bounded and connected open set in $\mathbf{C}$. Denote by $A^2(\Omega)$ the family of analytic functions $f(z)$ which are square integrable, i.e

(*) $$\iint_\Omega |f(z)|^2 \cdot dxdy < \infty$$

The space $A^2(\Omega)$ was introduced and studied by Stefan Bergman whose text-book [Bergman] used this Hilbert space to construct Riemann's mapping theorem via a kernel function. A merit of Bergman's approach is that the similar Hilbert spaces can be adapted in several variables, i.e. Bergman kernels exist when one regards square integrable analytic functions $f(z)$ defined in domains of $\mathbf{C}^n$ with $n \geq 2$. Here we only consider the 1-dimensional case. The starting point is the observation that if $f(z)$ is analytic in some open disc D_R of radius R centered at the origin, then the vanishing of the integrals

$$\int_0^{2\pi} e^{ik\theta} d\theta \quad : \quad k \neq 0$$

implies that with $f(z) = \sum a_n z^n$ one has the equality:

$$\iint_{D_R} |f(z)|^2 \cdot dxdy = \sum |a_n|^2 \cdot 2\pi \cdot \int_0^R r^{2n+1} dr = \sum |a_n|^2 \cdot 2\pi \cdot \frac{R^{2n+2}}{2n+2}$$

In particular

$$|f(0)|^2 = |a_0|^2 \leq \frac{1}{\pi R^2} \iint_{D_R} |f(z)|^2 \cdot dxdy$$

The square root of the double integral is by definition L^2-norm on $A^2(D_R)$. This gives:

(**) $$|f(0)| \leq \frac{1}{\sqrt{\pi} \cdot R} \cdot ||f||_2$$

Passing to a domain Ω we use that when $z_0 \in \Omega$ then the open disc of radius $\text{dist}(z_0, \partial\Omega)$ is contained in Ω. Since the area integral of $|f|^2$ taken over Ω is $\geq$ than that over $D_R(z_0)$ it follows that

(***) $$|f(z_0)| \leq \frac{1}{\sqrt{\pi} \cdot R} \cdot ||f||_2 \quad : \quad R = \text{dist}(z_0, \partial\Omega)$$

These local inequalities show that if $\{f_\nu\}$ is a Cauchy sequence with respect to the L^2-norm, then the maximum norms of $|f_\nu - f_j|$ over compact subsets of Ω tend to zero. Hence $\{f_\nu\}$ converges to an analytic function f_* in Ω and one easily verifies that $f_* \in A^2(\Omega)$ and that the L^2-norms of $f_\nu - f_*$ tend to zero. This proves that $A^2(\Omega)$ is a Hilbert space. Next, (***) shows that the linear functional on $f \mapsto f(z_0)$ is continuous on $A^2(\Omega)$ and the representation formula for elements in the dual of a Hilbert space gives a unique function $g(z) \in A^2(\Omega)$ such that

(i) $$f(z_0) = \iint_\Omega f(z)\bar{g}(z) \cdot dxdy \quad : \quad f \in A^2(\Omega)$$

To find an expression for the g-function we choose some orthonormal basis in $A^2(\Omega)$, i.e. a sequence $\{\phi_\nu(z)\}$ for which

(ii) $$\int_\Omega \phi_\nu(z)\bar{\phi}_j(z) \cdot dxdy = \text{Kronecker's delta function}$$

Now $g(z)$ has an expansion

(iii) $$g(z) = \sum c_k \cdot \phi_k(z)$$

Apply (i) with $f = \phi_k$. Then (ii) gives

(iv) $$\phi_k(z_0) = c_k$$

8.1. The kernel function.

Define the function $K(\zeta, z)$ on the product space $\Omega \times \Omega$ by

$$(1) \qquad K(\zeta, z) = \sum \phi_\nu(\zeta) \cdot \bar{\phi}_\nu(z)$$

The previous results give

8.2 Theorem. *For every $f \in A^2(\Omega)$ and each $\zeta \in \Omega$ one has*

$$(*) \qquad f(\zeta) = \iint f(z) \cdot K(\zeta, z) \cdot dxdy$$

Remark. The kernel function K is analytic in ζ for each fixed z while it is anti-analytic in z when ζ is fixed. A notable point is that the formula (*) does not depend on the chosen orthonormal basis in $A^2(\Omega)$, i.e. one has the equality

$$(*) \qquad K(\zeta, z) = \sum \psi_\nu(\zeta) \cdot \bar{\psi}_\nu(z)$$

for *every* orthonormal basis $\{\psi_\nu\}$ in $A^2(\Omega)$.

8.3 Transformation laws. Let $F\colon U \to \Omega$ be an analytic map. That is, $F \in \mathcal{O}(U)$ and the image $F(U) \subset \Omega$. For the moment we do not assume that F is injective and it may occur that its derivative is zero at some points in U. If $g \in \mathcal{O}(\Omega)$ we get $g^* = g \circ F \in \mathcal{O}(U)$. Recall that the Jacobian of the F-map is $|F'(z)|^2$. So if $w = u + iv$ is the complex coordinate in Ω then

$$(i) \qquad \iint_\Omega |g(w)|^2 \cdot dudv = \iint_U |g^*(z)|^2 \cdot |F'(z)|^2 \cdot dxdy$$

Hence there is a linear map from $A^2(\Omega)$ into $A^2(U)$ defined by:

$$\mathbf{T}\colon\ g(w) \mapsto F'(z) \cdot g^*(z)$$

Moreover, $\mathbf{T}$ is an isometry, i.e. the L^2-norms of g and of $\mathbf{T}(g)$ are the same. So if $\{\phi_\nu\}$ is an orthonormal basis in $A^2(\Omega)$ then $\{\mathbf{T}(\phi_\nu)\}$ is an orthonormal family in $A^2(U)$. The question arises when this orthonormal family is a basis in $A^2(U)$. From the construction of $\mathbf{T}$ via (ii) above we see that this holds if and only if the derivative $F'(z) \neq 0$ for all points in Ω and in addition F must be 1-1, i.e. $F(z)$ is a conformal map. Moreover, when F is conformal the Remark after Theorem 8.2 shows that the kernel function $K^*(\zeta, z)$ on U becomes:

$$(**) \qquad K^*(\zeta, z) = F'(\zeta) \cdot \bar{F}'(z) \cdot K(F(\zeta), F(z))$$

where K is the kernel function on Ω. We refer to (**) as the *transformation law* for Bergman's kernel function.

8.4 An extremal problem. Let Ω as be some bounded and connected domain. If $z_0 \in \Omega$ we put

$$(i) \qquad \lambda^*(z_0) = \max_g |g(z_0) \quad : \quad ||g||_2 = 1$$

To find $\lambda^*(z_0)$ and a maximizing g-function we choose an orthonormal basis and write

$$g(z) = \sum c_\nu \cdot \phi_\nu(z)$$

Now $||g||_2 = 1$ means that $\sum |c_\nu|^2 = 1$ so

$$\lambda^*(z_0) = \max \left|\sum c_\nu \cdot \phi_\nu(z_0)\right| \quad : \quad \sum |c_\nu|^2 = 1$$

By the Cauchy-Schwartz inequality the right hand side is majorised by:

$$\sum |\phi_\nu(z_0)|^2 \quad : \quad \text{for all sequences } \{c_\nu\} : \sum |c_\nu|^2 = 1$$

Moreover, equality holds if and only if there is a complex number ρ such that

$$c_\nu = \rho \cdot \bar{\phi}_\nu(z_0) \quad : \quad \nu = 1, 2, \ldots$$

Now ρ must be chosen so that

$$1 = \rho^2 \sum |\phi(z_0)|^2 = \rho^2 \cdot K(z_0.z_0) \implies \rho = \frac{1}{\sqrt{K(z_0, z_0)}}$$

At the same time we get

$$\lambda^*(z_0) = \rho \cdot K(z_0, z_0)$$

Hence we have proved:

8.5 Theorem. *For each $z_0 \in \Omega$ one has the equality*

$$\lambda^*(z_0) = \sqrt{K(z_0, z_0)}$$

Moreover, the g-function which maximizes (i) above is given by

$$g(z) = \frac{1}{\sqrt{K(z_0, z_0)}} \cdot K(z, z_0)$$

8.6 The simply connected case. Let Ω be bounded and simply connected. If $a \in \Omega$ we find the conformal mapping function $f_a \colon \Omega \to D$ using the Kernel function. The result is

8.7 Theorem. *The conformal map f_a is given by*

$$f_a(z) = \sqrt{\frac{\pi}{K(a, a)}} \cdot \int_a^z K(z, a)dz$$

Exercise. Prove this result. The hint is to use the various extremal properties satisfied by the mapping function f_a.

8.8 Orthogonal polynomials. Let Ω be a bounded simply connected domain. The Gram-Schmidt construction gives a special orthonormal basis in $A^2(\Omega)$ given by a sequence of polynomials $\{P_n(z)\}$ where P_n has degree n and

$$\iint P_k \cdot \bar{P}_m \cdot dxdy = \text{Kronecker's delta function}$$

One expects that these polynomials are related to a mapping function. We shall consider the case when Ω is a Jordan domain whose boundary curve Γ is *real-analytic*. Let ϕ be the conformal map ϕ from the *exterior domain* $\Omega^* = \Sigma \setminus \bar{\Omega}$ onto the exterior disc $|z| > 1$. Here ϕ is normalised so that it maps the point at infinity into itself. The inverse conformal mapping function ψ is defined in $|z| > 1$ and has a series expansion

$$(*) \qquad \psi(z) = \tau \cdot z + \tau_0 + \sum_{\nu=1}^{\infty} \tau_\nu \cdot \frac{1}{z^\nu}$$

where τ is a positive real number. The assumption that Γ is real-analytic gives some $\rho_1 < 1$ such that ψ extends to a conformal map from the exterior disc $|z| > \rho_1$ onto a domain whose compact complement is contained in Ω. It turns out that the polynomials $\{P_n\}$ are approximated by functions expressed by ϕ and is complex derivative on $\partial\Omega$.

8.9 Theorem. *There exists a constant C which depends upon Ω only such that to every $n \geq 1$ there is a function $\omega_n(z)$ defined in Ω^* and*

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \cdot \phi'(z) \cdot \phi(z)^n \cdot \big[1 + \omega_n(z)\big] \quad \text{holds on} \quad \partial\Omega$$

Finally the ω-functions satisfy

$$\max_{z \in \partial\Omega} |\omega_n(z)| \leq C \cdot \sqrt{n} \cdot \rho_1^n \quad : \quad n = 1, 2, \ldots$$

Remark. Theorem 8.10 stems from Faber's article *Über Tschebycheffsche Polynome* from 1920. The subsequent proof is taken from the article [Carleman]

Proof. Let $n \geq 2$ and consider the set of polynomials $Q(z)$ of degree n whose highest term is z^n. Keeping n fixed we set

$$I_*(n) = \min_Q I(Q) = \iint_\Omega |Q(z)|^2 \cdot dxdy$$

Given a polynomial Q as above we choose a primitive polynomial $R(z)$. So here

$$R(z) = \frac{z^{n+1}}{n+1} + b_n z^n + \ldots + b_0$$

1. Exercise. Use Green's formula to show that

$$I(Q) = \frac{1}{4} \int_{\partial\Omega} \partial_n(|R|^2) \cdot ds$$

where ds is the arc-length measure on $\partial\Omega$

Now we use the inverse conformal map $\psi(\zeta)$ and set

$$F(\zeta) = R(\psi(\zeta))$$

Then F is analytic in the exterior disc $|\zeta| > 1$ and (*) above Theorem 8.9 entails that F has the series expansion

$$\text{(i)} \qquad F(\zeta) = \tau^{n+1} \Big[\frac{\zeta^{n+1}}{n+1} + A_n \zeta^n + \ldots + A_1 \zeta + A_0 + \sum_{nu=1}^{\infty} \alpha_\nu \cdot \zeta^{-\nu}\Big]$$

2. Exercise. Use a variable substitution via ψ to show that the integral in Exercise 1 is equal to

$$\int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot d\theta$$

and use the series expansion (i) to show that this integral is equal to

$$\pi \cdot \tau^{2n+2} \cdot \Big[\frac{1}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 - \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2\Big]$$

An upper bound for I_*. The coefficients $A_1, \ldots, A_n$ are determined via Q. The reader may verify that there exists a polynomial $Q(z)$ with highest term z^n such that $A_1 = \ldots = A_n = 0$. It follows that

$$(*) \qquad I_* \leq \pi \cdot \tau^{2n+2} \cdot \Big[\frac{1}{n+1} - \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2\Big] \leq \pi \cdot \tau^{2n+2} \cdot \frac{1}{n+1}$$

A lower bound for I_*. The upper bound (*) did not use that $\partial\Omega$ is real-analytic, i.e. it is valid for every Jordan domain whose boundary curve is of class C^1. To get a lower bound we choose some $\rho_1 < \rho < 1$ and by assumption ψ maps the exterior disc $|\zeta| > \rho$ conformally to an exterior domain $U^* = \Sigma \setminus \bar{U}$ where U is a relatively compact Jordan domain inside Ω. Let Q be a polynomial for which $I(Q) = I_*$. Now $\Omega \setminus \bar{U}$ is contained in Ω so we have

(i) $$I_* > \iint \Omega \setminus \bar{U} \, |Q(z)|^2 \cdot dxdx$$

3. Exercise. Show that the integral in (i) is equal to

$$\int_{|\zeta|=1} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot d\theta - \int_{|\zeta|=\rho} \frac{d}{dr}(|F(e^{i\theta})|^2 \cdot \rho \cdot d\theta =$$

(ii) $$\pi \cdot \tau^{2n+2} \cdot [\frac{1-\rho^{2n+2}}{n+1} + \sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1)\,]$$

The last equality gives the lower bound

(**) $$I_* \geq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot (1-\rho^{2n+2})$$

Moreover the upper bound and (ii) give the inequality

$$\sum_{k=1}^{k=n} k \cdot |A_k|^2 \cdot (1-\rho^{2\nu}) + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \cdot (\frac{1}{\rho^{2\nu}} - 1) \,\leq\, \frac{\pi}{n+1} \cdot \rho^{2n+2}$$

Since $1-\rho^2 \leq 1-\rho^{2\nu}$ for every $\nu \geq 1$ it follows that

(***) $$\sum_{k=1}^{k=n} k \cdot |A_k|^2 + \sum_{nu=1}^{\infty} \nu \cdot |\alpha_\nu|^2 \leq \frac{\pi}{(1-\rho^2) \cdot n+1} \cdot \rho^{2n+2}$$

Conclusion. Recall that $F(\zeta) = R(\psi(\zeta)$ and $R' = Q$. So after a derivation we get

$$F'(\zeta) = \psi'(\zeta) \cdot Q(\psi(\zeta))$$

Hence the series expansion of $F(\zeta)$ gives

(i) $$Q(\psi(\zeta)) = \frac{\tau^{n+1}}{\psi'(\zeta)} \cdot \left[\zeta^n + \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}\,\right]$$

where the equality holds for $|\zeta| > \rho$. Put

$$\omega^*(\zeta) = \sum_{k=1}^{k=n} k \cdot A_k \zeta^{k-1} + \sum_{\nu=1}^{\infty} \nu \cdot \alpha_\nu \cdot \zeta^{-\nu-1}$$

When $|\zeta| = 1$ the triangle inequality gives

(ii) $$|\omega^*(\zeta)| \leq \sum_{k=1}^{k=n} k \cdot |A_k| + \sum_{\nu=1}^{\infty} \nu \cdot |\alpha_\nu|$$

4. Exercise. Notice that (***) holds for every $\rho > \rho_1$ and use this together with suitable Cauchy-Schwarz inequalities to show that (i) above gives a constant C which is independent of n such that

$$|\omega^*(\zeta)| \leq C \cdot \sqrt{n} \cdot \rho_1^{n+1}$$

Final part of the proof. Since ψ is the inverse of ϕ we have

$$\psi'(\phi(z)) \cdot Q(\psi(\phi(z = \frac{Q(z)}{\phi'(z)}$$

Then (iii) in the conclusion implies that if we define the function $\partial\Omega$ by

$$\omega_n(z) = \frac{\omega^*(\phi(z))}{\phi'(z)}$$

then we have

$$Q(z) = \tau^{n+1} \cdot \phi'(z) \cdot [\phi(z)^n + \omega_n(z)\,]$$

where Exercise 4 shows that $|\omega_n(z)|$ satisfies the estimate in Theorem 8.5. Finally, the Q-polynomial minimized th L^2-norm under the constraint that the leading term is z^n and for this variational problem he upper and the lower bounds in (*-**) above show that the minimum $I_*(n)$ satisfies

$$|I_*(n) - \frac{\pi}{n+1} \cdot \tau^{2n+2}| \leq \frac{\pi}{n+1} \cdot \tau^{2n+2} \cdot \rho^{2n+2}$$

So if we normalise Q so that its L^2-norm is one and hence gives the polynomial $P_n(z)$ then the factor τ^{n+1} is replaced by $\frac{\sqrt{n+1}}{\sqrt{\pi}}$ which finishes the proof of Theorem 8.10.

9. The electric equilibrium potential.

Introduction. Let Γ be a closed Jordan curve of class C^1. We seek a positive density function $\mu(s)$ where ds is the arc-length measure on Γ whose the logarithmic potential is constant on Γ. It turns out that μ is found from a conformal mapping. Namely, consider the *exterior domain* Ω^* which is bounded by Γ. So the Jordan domain DU bordered by Γ is the open complement of $\bar{\Omega}^*$. Adding the point at infinity we get the simply connected set $\Omega_\infty = \Omega^* \cup \infty$. Riemann's mapping theorem gives a conformal map f from Ω_∞ onto the exterior disc $|w| > 1$ which maps the point at infinity to the point at infinity . This means that there is some real number $a > 0$ and $f(z) \simeq az$ as $|z| \to \infty$. Now we shall prove that the function defined on Γ by

$$(*) \qquad z_* \mapsto \int_\Gamma \log\big(\frac{1}{|z - z_*|}\big) \cdot \frac{|dz|}{|f'(z)|}$$

is constant. To get this we set $w = f(z)$ and $\phi(w)$ is the inverse function. So here ϕ is analytic in $|w| > 1$ and the conformal map gives the arc-length formula

$$|dw| = \frac{|dz|}{|f'(z)|}$$

Hence (*) amounts to show that

$$(**) \qquad \theta_* \mapsto \int_0^{2\pi} \log\big(\frac{1}{|\phi(e^{i\theta}) - \phi(e^{i\theta_*})|}\big) \cdot d\theta$$

is a constant function of θ_*.To prove this we use that the image set $\phi(T) = \Gamma$ which enable us to define a function in U by:

$$H(z) = \int_0^{2\pi} \mathrm{Log}\big(\frac{1}{|\phi(e^{i\theta}) - z|}\big) \cdot d\theta$$

This is a harmonic function in U given by the real part of

$$z \mapsto \int_0^{2\pi} \mathrm{Log}\big(\frac{1}{\phi(e^{i\theta}) - z}\big) \cdot d\theta$$

Passing to the complex derivative we get

$$(1) \qquad H_x - iH_y = \int_0^{2\pi} \frac{d\theta}{\phi(e^{i\theta}) - z} \quad : \quad z \in D$$

With $z \in D$ kept fixed, the right hand side can be written as a complex line integral:

$$(2) \qquad \int_{|w|=1} \frac{1}{\phi(w) - z} \cdot \frac{dw}{iw}$$

Now the ϕ-image of $\{|w| > 1\}$ is contained in Ω^* so $z \in U$ stays outside this set. By Cauchy's theorem we can therefore shift the integration to circle $|w| = R$ with $R > 1$ and hence (2) becomes:

$$(3) \qquad \int_{|w|=R} \frac{1}{\phi(w) - z} \cdot \frac{dw}{iw}$$

Next, we have $\phi(w) \simeq \frac{w}{a}$ when $|w| \to \infty$. It follows that (3) tends to zero when $R \to +\infty$. This proves that (1) is identically zero as z varies on U and hence the harmonic function $H(z)$ is constant. Finally, $H(z)$ extends to a continuous function on the closed Jordan domain $\bar{U}$ and we conclude that the function from (**) is constant.

9.1 A special class of curves

.

Let Γ be a closed Jordan curve whose curvature ρ is $\neq 0$ at all points. One may ask when the equilibrium density μ_Γ is proportional to a fractional power of ρ, i.e. when there exists some $\beta > 0$ and a positive constant k such that

$$(*) \qquad \mu_\Gamma(z)^\beta = k \cdot \rho(z) \quad : \quad z \in \Gamma$$

To find such example we consider for a given γ the conformal mapping $F(z)$ from the exterior disc $|z|>1$ onto the exterior domain bounded by Γ. The curvature formula from § 4:xx shows that if $f=F'$ is the complex derivative then

$$\frac{1}{\rho}=\frac{1}{|f(z)|}\cdot\mathfrak{Re}\,(z\cdot\frac{f'(z)}{f(z)}+1)\quad:\quad |z|=1$$

Hence (*) holds for those γ where the associated mapping function F satisfies

$$\mathfrak{Re}\,(z\cdot\frac{f'(z)}{f(z)}+1)=k\cdot|f(z)|^{\kappa}\quad:\quad |z|=1$$

for some pair of positive constants k,κ. We shall find all non-trivial conformal mappings where Γ is not a circle.

9.2 Theorem. *The non-trival class of conformal mappings for which (*) hold are those where the complex derivative is of the form*

$$f(z)=k(1+\frac{a}{z^m})^{\frac{2}{m}}$$

where $k>0$, $|a|<1$ and $m\geq 2$ is an integer.

The proof has two ingredients. First we show that every f as above is the complex derivative of a mapping function F for which the requested equation (*) is satisfied for a pair k,β. The more involved part of the proof is to show that no other cases exist.

The case $m=2$. With f as above the primitive function becomes

$$F(z)=k(z-\frac{a}{z})$$

By a scaling we may assume that $k=a=1$ so that

$$f(z)=1-\frac{1}{z^2}$$

Exercise. Show that F yields a bijective map from $|z|=1$ onto an ellipse $\mathcal{C}$ and that (xx) above holds and determine the constant κ.

The case $m\geq 3$**.** We may assume that $k=1$ and $0<a<1$ is real. In $|z|>1$ we have the analytic function

$$f(z)=(1+\frac{a}{z^m})^{\frac{2}{m}}$$

It extends to be analytic at ther point at infinity and from I:§ XX we have the newton expansion below where we set $\beta=\frac{2}{m}$:

$$f(z)=1+\sum_{k=1}^{\infty}\binom{\beta}{k}\cdot\frac{a^k}{z^{mk}}$$

The primitive function becomes:

$$F(z)=z-\sum_{k=1}^{\infty}\binom{\beta}{k}\cdot\frac{a^k}{mk-1}\cdot\frac{1}{z^{mk+1}}$$

Proposition. *The function F yields a conformal mapping from $\{|z|>1\}$ onto the exterior of a Jordan domain.*

Proof. It is clear that the complex derivative $f\neq 0$ in $|z|>1$ and there remains to show that F is 1-1. To show this we take a pair $z_1\neq z_2$ in $|z|>1$ and write

$$\frac{F(z_1)-F(z_2)}{z_1-z_2}=1+P(z_1,z_2)$$

A trivial computation which is left to the reader shows that

$$P(z_1,z_2)=\frac{1}{z_1z_2}\cdot\sum_{k=1}^{\infty}\frac{a^k}{km-1}\cdot\binom{\beta}{k}\cdot[\frac{1}{z_1^{km-2}}+\frac{1}{z_1^{km-3}}\cdot\frac{1}{z_2}+\ldots+\frac{1}{z_1}\cdot\frac{1}{z_2^{km-3}}\frac{1}{z_2^{km-2}}]$$

Since $m-1 \le km-1$ when $k \ge 2$ and each z_ν has abslute value ≥ 1, the triangle inequality gives:

$$|P(z_1, z_2)| \le \sum_{k=1}^{\infty} a^k \cdot \binom{\beta}{k} = (1+a)^\beta - 1$$

Finally, since $0 < a < 1$ the last term is < 1 and hence (i) is $\neq 0$ which means that $F(z_1) \neq F(z_2)$ as requested.

Conclusion. Proposition 9.3 gives the conformal mapping F and we get the Jordan curve $\Gamma = F(T)$. We leave it as an exercise to prove that (xx) holds for some κ and remark only that a computation gives

$$\kappa = XXX$$

Proof of necessity.

There remains to show that the examples above exhaust all possible cases. The idea is to introduce the analytic function in the disc $D = \{|z| < 1\}$ defined by

(i) $$g(z) = \bar{f}(\frac{1}{\bar{z}})$$

Since $f \neq 0$ in the exterior disc, it follows that g is zero-free in D and we know from § xx that f and hence also g extend to continuous functions on $|z| = 1$ where both remain zero-free. Notice that

(ii) $$g'(z) = -\frac{1}{z^2} \cdot \bar{f}'(z) \quad : \quad |z| = 1$$

Using (ii) it is easily seen that (**) above Theorem 9.1 entails that

(iii) $$1 + \frac{z}{2}[\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}] = k \cdot f(z)^\kappa \cdot g(z)^\kappa \quad : \quad |z| = 1$$

The case $\kappa = 0$. Then (iii) gives

$$\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} = \frac{2(k-1)}{z} \quad : \quad |z| = 1$$

It follows that

$$f(z) = A \cdot g(z) \cdot z^{2(k-1)}$$

holds for some constant A.

FINISH EASY....

The case $\kappa \neq 0$. Since f is zero-free we can consider the function $\phi = f^{-\kappa}$ and rules for derivatives show that (iii) gives the differential equality

(1) $$\frac{d}{dz}(\phi(z)) - \kappa(\frac{2}{z} - \frac{g'(z)}{g(z)}) \cdot \phi(z) = -\frac{2k\kappa}{z} \cdot g^{-\kappa}(z)$$

Since g is zero-free in D i follows that $g^{-\kappa}$ also is analytic in D and (1) can be regarded as a first order Fuchsia equation where we seek ϕ so that

$$\nabla(\phi(z)) - (2\kappa - A(z))\phi(z) = B(z)$$

Above we have put $\nabla = z\partial_z$ and

$$A(z) = \kappa z \cdot \frac{g'}{g} \quad : \quad B = -2k\kappa \cdot g^{-\kappa}(z)$$

In particular $A(0) = 0$ and provided that $m - 2\kappa \neq 0$ for all positive integers m, a formal recursion shows that the Fuchsian equation has a unique solution $\phi(z)$ in D where

$$\phi(0) = k \cdot g(0)^{-\kappa}$$

So when $m-2\kappa \neq 0$ for all positive integers m, it follows that $\phi(z)$ extends to an analytic function in D which means that $f^{-\kappa}$ extends from $|z|>1$ to an entire function. As $z \to \infty$ we know that $f(z)$ is bounded and non-zero so the entire function $f^{-\kappa}$ will be bounded and hence reduced to a constant, i.e. the derivative of the conformal mapping F is reduced to a constant which entails that $F(z)$ just is a linear function in z and then $\Gamma = f(T)$ is a circle.

There remains to consider the case when $\kappa = m/2$ for some positive integer m. Then the Fuchsian equation is not directly solved by a formal recursion. However, with $2\kappa = m$ the equation becomes

$$xxx$$

FINISH easy ....

10. Conformal maps of circular domains.

Introduction. In 1906 Koebe proved a result about conformal mappings between domains bordered by a finite set of circles. Let $p \geq 2$ and denote by $\mathcal{C}^*(p)$ the family of connected bounded domains Ω in $\mathbf{C}$ for which $\partial\Omega$ is the union of p many disjoint circles.

Theorem. *Let $f\colon \Omega \to U$ be a conformal map between two domains in $\mathcal{C}^*(p)$. Then $f(z)$ is a linear function, i.e. $f(z) = Az + B$ for some constants A and B.*

Koebe's original proof used reflections over the boundaries and results related to the uniformisation theorem. A more direct proof was given by Carleman in [Car] which we expose below. It teaches how to compute certain winding numbers in specific situations.

Let f be a mapping function as above. Let Ω be bordered by circles $C_1, \ldots, C_p$ where C_p is the outer circle. Similarly U is bordered by $C_1^*, \ldots, C_p^*$. Then f must map C_p to C_p^* and the remaining discs are arranged so that f maps C_ν onto C_ν^* for $1 \leq \nu \leq p-1$. Using a linear map of the outer discs we may assume from the start that $C_p = C_p^* = T$ where T is the unit circle. Moreover, after a suitable rotation we may also assume that the map $f\colon T \to T$ has at least 3 fixed points. There remains to show that when this holds, then $f(z) = z$ must be the identity map. To prove this we shall argue by a contradiction. Namely, if $f(z)$ is not the identity we get the non-constant function

$$\phi(z) = f(z) - z \tag{*}$$

Notations. To each $1 \leq \nu \leq p$ we denote by $n^{(\nu)}$ the number of zeros of ϕ counted with multiplicities which belong to C_ν. We also set

$$J_\nu = \frac{1}{2\pi i} \cdot \int_{C_\nu} \frac{\phi'(z)}{\phi(z)} \cdot dz \quad : \quad 1 \leq \nu \leq p$$

Now we will derive a contradiction using residue formulas to compute these J-numbers. For each $1 \leq \nu \leq p-1$ one encounters nine different case which are listed below:

Separate cases.

Let $1 \leq \nu \leq p-1$ be given. Then J_ν is found by the equations below depending on the positions of the two circles C_ν and C_ν^*:

Case 1: $C_\nu = C_\nu^*$. Here

$$J_\nu = 1 + \frac{1}{2} \sum n_k^{(\nu)}$$

Case 2: C_ν and C_ν^* are exterior to each other, i.e. C_ν^* is outside the closed disc bordered by C_ν. Then

$$J_\nu = 0$$

Case 3: C_ν inside the open disc bordered by C_ν^* or conversely C_ν^* inside the open disc bordered by C_ν. Then

$$J_\nu = 1$$

Case 4: $C_\nu \cap C_\nu^*$ consists of two points P, Q, none of which is a zero of ϕ. Then

$$J_\nu = 0, 1, 2 \text{ i.e. one of these numbers are attained}$$

Case 5: $C_\nu \cap C_\nu^*$ consists of two points P, Q where one of these two points is a zero of ϕ of some multiplicity e. Then

$$J_\nu = \frac{1+e}{2} \text{ or equal to } \frac{3+e}{2}$$

Case 6: $C_\nu \cap C_\nu^*$ consists of two points P, Q where both are zeros of ϕ with multiplicity e and f. Then

$$J_\nu = 1 + \frac{e+f}{2}$$

Case 7: C_ν and C_ν^* has a a common tangential point P which is not a zero of ϕ. See Figure XX for this case. Then

$$J_\nu = 0 \text{ or equal to } 1$$

Case 8: $C_\nu \cap C_\nu^*$ is reduced to a single point P and are otherwise external to each other and P is a zero of ϕ with multiplicity e. Then

$$J_\nu = \frac{1+e}{2}$$

Case 9: $C_\nu \cap C_\nu^*$ is reduced to a single point P and this time C_ν is contained in the closed disc bordered by C_ν^* or vice versa and p is a zero of ϕ of multiplicity e. Then

$$J_\nu = \frac{k+e}{2} + \quad : \quad k = 1, 2, 3$$

Exercise. Verify the nine formulas above. A hint is that the conformal mapping f restricts to bijective map from C_ν onto C_ν^* and preserves orientation.

Conclusion. For each $1 \leq \nu \leq p$ we have found that J-number is ≥ 0. Next, on the outer circle T we have

$$J_1 = \frac{1}{2\pi i} \cdot \int_T \frac{\phi'(z)}{\phi(z)} \cdot dz = 1 - \frac{1}{2} \cdot N_T(\phi)$$

where $N_T(\phi)$ is the number of zeros of ϕ on the unit circle counted with multiplicities. By the hypothesis $N_T(\phi) \geq 3$ Hence $J_1 \leq -\frac{1}{2}$ is strictly negative. Now we get a contradiction. Namely, let $\mathcal{N}_\Omega(\phi)$ be the number of zeros of ϕ in the open domain ϕ. Then we have the general formula:

$$(*) \qquad \mathcal{N}_\Omega(\phi) = J_1 - (J_2 + \ldots + J_p)$$

But this is a contradiction, i.e. the left hand side is a non-negative integer and by the above the right hand side is < 0. Hence the ϕ-function must be identically zero which proves Koebe's theorem.

Metrics with non-positive curvature and the Riemann-Schwarz inequalitiy.

Let D be the unit disc. Every conformal mapping of D onto itself is given by a Möbius map

$$S(z) = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}\cdot z} \tag{1}$$

The fact that (1) is a conformal mapping was demonstrated in Chapter 1 where one constructs the inverse map. Conversely, if S is an arbitrary conformal map and $S(0)=\alpha$ then the composed map

$$z \mapsto \frac{S(z)-\alpha}{1-\bar{\alpha}\cdot S(z)} \tag{2}$$

is again a conformal map which sends the origin into itself and Schwarz Lemma entails that (2) is given by $z \mapsto e^{i\theta}\cdot z$ for some $e^{i\theta}$ which implies that S is a Möbius map. Next, a circular arc α in D is called a curve of symmetry if intersects the unit circle T at right angles. In this family we include diameters , i.e. line segments which intersect T at a pair $e^{i\theta}$ and $-e^{-i\theta}$. Since every Möbius transform is conformal it preserves the family of curves of symmetry. In fact, each Möbius transformation S yields a bijective map on this family.

Definition. *A conformal rhomb in D consists of pair of curves of symmetry α, β which intersect at a right angle.*

0.1 Theorem. *For each conformal rhomb (α,β) and every subharmonic function u in D which extends to be continuous on the closed unit disc one has the inequality*

$$\Big(\int_\alpha e^{u(z)}\,|dz|\Big)^2 + \Big(\int_\beta e^{u(z)}\,|dz|\Big)^2 \leq \frac{1}{4}\cdot\Big(\int_T e^{u(z)}\,|dz|\Big)^2 \tag{*}$$

Theorem 0.1 can be applied to general connected domains Ω in $\mathbf{C}$-equipped with a λ-metric where $\lambda = e^u$ for a subharmonic and continuous function u in Ω. More precisely, for each rectifibale curve γ in Ω we set

$$\ell(\gamma) = \int_\gamma \lambda(z)\,|dz|$$

Let us then consider a pair of points a, b in Ω and the family $\mathcal{C}(a,b)$ of curves which have a, b as end-points.

0.2 Theorem. *For each pair of curves γ_1, γ_2 in $\mathcal{C}(a,b)$ and every point $p \in \gamma_1$ there exists a point $q \in \gamma_2$ and a pair of curves $\alpha \in \mathcal{C}(a,b)$ and $\beta \in \mathcal{C}(p,q)$ such that*

$$\ell(\alpha)^2 + \ell(\beta)^2 \leq \frac{1}{2}\cdot\big(\ell(\gamma_1)^2 + \ell(\gamma_2)^2\,\big) \tag{**}$$

We refer to this as the Riemann-Schwrz inequality. The proof requires several steps and is not finished until § 1.8. We begin with some special considerations.

1.1. A special class of Möbius transformations.

Let $0 < b < 1$ be real and set

$$S(z) = \frac{b-z}{1-bz} \quad : \quad 0 < b < 1$$

The composed map becomes

$$S^2(z) = \frac{b - \frac{b-z}{1-bz}}{1 - b\cdot\frac{b-z}{1-bz}} = z$$

where the last equality follows by a trivial calculation. Notice that S maps the real diameter $-1 < x < 1$ into itself. To be precise, $S(-1) = 1$ and after $x \mapsto S(x)$ is a decreasing function with $S(1) = -1$. Next, on the unit circle we find a point $e^{i\theta}$ such that

$$e^{-i\theta} = S(e^{i\theta})$$

A computation which is left to the reader shows that

$$\cos\theta = b$$

Let α be the circle which passes through $e^{i\theta}$ and $e^{-i\theta}$ and intersects T with right angles at these points. This holds when the center of the disc containing α belongs to the positive real x-axis and the reader can verify that α contains the real point $0 < x_0 < 1$ which solves the equation

$$x_0 = \frac{b - x_0}{1 - bx_0} \implies x_0 = \frac{1 - \sqrt{1 - b^2}}{b}$$

The reader may verify that S maps the circular arc α into itself, i.e. α is S-invariant.

1.2 Exercise. Draw some figures to illustrate how S maps circular arcs which are lines of symmetry and intersect T at a pair of points $e^{i\phi}, e^{-i\phi}$ when $0 < \phi < \theta$.

A special inequality Let $0 < \theta < \pi/2$ and α is the circular arc above. Then the following hold for every subharmonic function u in D which extends to be continuous on the closed disc:

$$\big(\int_{-1}^{1} \lambda(x)\, dx\big)^2 + \big(\int_\alpha \lambda(z)\, |dz|\big)^2 \leq \frac{1}{4} \cdot \big(\int_0^{2\pi} \lambda(e^{i\theta})\, d\theta\big)^2 \tag{1.3}$$

Proof. Put $\lambda = e^u$ so that $\log \lambda = u$ and set:

$$\lambda^* z) = \frac{1}{2} \cdot \big(\lambda(z) + \lambda(S(z)) \cdot |S'(z)|\,\big)$$

Recall from § XX that the positive λ^*-function also is of the form e^U for a subharmonic function U. Since S preserves α we have

$$\int_\alpha \lambda(z) \cdot |dz| = \int_\alpha \lambda^*(z) \cdot |dz|$$

and the same equality holds when we integrate on the real diameter $(-1, 1)$. Moreover, since S maps T onto itself we also get an equality for the integrals on T. Hence it suffices to prove (1.3) when λ is S-invariant, i.e. when $\lambda = \lambda \circ S$. From now on λ is S-invariant and with $\lambda = e^u$ we solve the Dirichlet problem for the boundary value function u and get

$$u = H + w$$

where H is the harmonic extension of $u|T$. Here w is subharmonic and vanishes on T so the maximum principle gives $w \leq 0$ in D. It follows that $\lambda \leq e^H$ and equality holds on T. Hence (1.3) holds for λ if we have proved (1.3) using the function e^H. To prove this we first notice that since λ is S-invariant the same holds for H. This entails that the normal derivatives

$$\frac{\partial H}{\partial y}(x, 0) = 0 \quad : \quad -1 < x < 1$$

and similarly the normal derivative of H along α vanishes. Let V be the conjugate harmonic function to H chosen so that $V(x_0)) = 0$ where α intersects the real diameter at x_0. Let Ψ be the primitive analytic function of e^{H+iV} where $\Psi(x_0) = 0$ which gives:

$$|\Psi'(z)| = |e^{H(z)+iV(z)}| = e^{H(z)} = \lambda(z) \tag{i}$$

The vanishing of normal derivatives of H along $(-1, 1)$ and α implies that the conjugate function V is constant on these two curves and by the normalisation it is identically zero. Using this the reader may verify the following

1.4 Lemma. *The analytic function Ψ maps $(-1, 1)$ onto a real line segment $(-A, A)$ and α to an imaginary line segment $(-iB, iB)$.*

1.5 Exercise. Deduce from Lemma 1.4 that one has the equations:

$$\int_{-1}^{1} \lambda(x)\, dx = 2A \quad : \quad \int_\alpha \lambda(z)\, |dz| = 2B \tag{1.5.1}$$

Notice also that

$$\int_0^{2\pi} \lambda(e^{i\theta})\, d\theta = \int_T |\Psi'(z)|\,|dz| \tag{1.5.2}$$

The right hand side in (1.5.2) measures the total arc-length of the image curve $\theta \mapsto \Psi(e^{i\theta})$. By the S-invariance this curve contains for four subarcs which join the vertices of the rectangle with corner points at $A, iB, -A, -iB$. Since the shortest distance between twe points is a straight line, Pythagoras' theorem gives the inequality

$$4 \cdot \sqrt{A^2 + B^2} \leq \int_T |\Psi'(z)|\,|dz| \tag{1.6}$$

Together (i) and (1.5.1-1.5.2) therefore give the requested inequality in (1.3).

1.7 The case of a general rhomb. Let us now consider a pair α, β of curves of symmetry which intersect at a right angle and $\lambda = e^u$ for some subharmonic function u. Up to a rotation we may assume that α intersects T at a pair $e^{i\theta}, e^{-i\theta}$ where $0 < \theta < \pi/2$. As explained in § XX there exists a Möbius transform S which maps the interval $(-\theta, \theta)$ to an interval of length π which implies that $S(\alpha)$ is a diameter. At the same time $S(\beta)$ is a curve of symmetry which intersect this diameter at a right angle since S is conformal. Put

$$\lambda^*(z) = \lambda(S^{-1}(z)) \cdot |S'(z)|^{-1}$$

Notice that $\log \lambda^*$ again is subharmonic. We have also

$$\int_\alpha \lambda(z)\,|dz| = \int_{S(\alpha)} \lambda^*(z)\,|dz|$$

with similar equations for integrals over β and the unit circle. Now (1.3) applies to λ^* and Theorem 0.1 follows.

1.8 Proof of Theorem 0.2. We are given a pair γ_1, γ_2 in $\mathcal{C}(a,b)$. Then $\gamma_2^{-1} \circ \gamma_1$ is a cklosed Jordan curve which borders a Jordan domain U contained in Ω. Let $\phi\colon D \to U$ be a conformal mapping which by Theorem XX extends to a homeomorphism from the closed unit disc onto $\bar{U}$. In DS we define the function λ^* by

$$\lambda^*(z) = \lambda(\phi(z)) \cdot |\phi'(z)|$$

Then Theorem 0.2 follows when we let α be the Jordan arc in $\mathcal{C}(a,b)$ for which $\phi^{-1}\alpha$ is the curve of symmetry which passes the points $\phi^{-1}(a)$ and $\phi^{-1}(b)$ on the unit circle. Finally β is chosen so that $\phi^{-1}\beta$ is the curve of symmetry which starts at the assgined point $\phi^{-1}(p)$ and intersects $\phi^{-1}\alpha$ at a right angle.

1.9 Existence of geodesic curves.

Let Ω be a connected domain in $\mathbf{C}$. it may have finite connectivity as well as infinite connectivity. let u be a continuos and strictly subharmonic function in Ω, i.e. $\Delta(u)$ is a positive function. With $\lambda = e^u$ we define the λ-distance between a pair of points a, b by

$$\ell(a,b) = \inf_{\gamma \in \mathcal{C}(a,b)} \ell(\gamma)$$

XXX. Beurling's conformal mapping theorem.

Introduction. Let D be the open unit disc $|z|<1$. Denote by $\mathcal{C}$ the family of conformal maps $w=f(z)$ which map D onto some simply connected domain Ω_f which contains the origin and satisfy:

$$f(0)=0 \quad \text{and} \quad f'(0) \text{ is real and positive}\,.$$

Riemann's mapping theorem asserts that for every simply connected subset Ω of $\mathbf{C}$ which is not equal to $\mathbf{C}$ there exists a unique $f\in\mathcal{C}$ such that $\Omega_f=\Omega$. We are going to construct a subfamily of $\mathcal{C}$. Consider a positive and bounded continuous function a function Φ defined in the whole complex w-plane.

0.1 Definition. *The set of all $f\in\mathcal{C}$ such that*

$$(*) \qquad \lim_{r\to 1}\ \max_{0\leq\theta\leq 2\pi}\big[|f'(re^{i\theta})|-\Phi(f(re^{i\theta}))\big]=0$$

is denoted by $\mathcal{C}_\Phi$.

Remark. Thus, when $f\in\mathcal{C}_\Phi$ then the difference of the absolute value $|f'(z)|$ and $\Phi(f(z))$ tends uniformly to zero as $|z|\to 1$. Let M be the upper bound of Φ. The maximum principle applied to the complex derivative $f'(z)$ gives

$$|f'(z)|\leq M \quad : \quad z\in D$$

Hence $f(z)$ is a continuous function in the open disc D whose Lipschitz norm is uniformly bounded by M. This implies that f extends to a continuous function in the closed disc, i.e. f belongs to the disc algebra $A(D)$.

1. Theorem. *Assume that Log $\frac{1}{\Phi(w)}$ is subharmonic. Then $\mathcal{C}_\Phi$ contains a unique function f^*.*

Remark. When $\Phi(w)=\Phi(|w|)$ is a radial function and $\Phi(\rho)=\rho$ holds for some $\rho>0$ then the function $f(z)=\rho\cdot z$ belongs to $\mathcal{C}_\Phi$. So for a radial Φ-function where different ρ-numbers exist uniqueness fails for the family $\mathcal{C}_\Phi$. The reader may verify that a radial function Φ for which $\Phi(\rho)=\rho$ has more than one solutions cannot satisfy the condition in Theorem 1. Next, let us give examples of Φ-functions which satisfy the condition in Theorem 1. Consider an arbitrary real-valued and non-negative L^1-function $\rho(t,s)$ which has compact support. Set

$$\Phi(w)=\exp\Big[\int \log\frac{1}{|w-t-is[}\cdot\rho(t,s)\cdot dtds\Big]$$

Here $\log\frac{1}{\Phi}$ is subharmonic and Theorem 1 gives a unique simply connected domain Ω such that the normalised conformal mapping function $f\colon D\to\Omega$ satisfies

$$|f'(e^{i\theta})|=\Phi(f(e^{i\theta})) \quad : \quad 0\leq\theta\leq 2\pi\,.$$

The proof of Theorem 1 requires several steps where notble point is that we also estblish some existence results under the sole assumption that Φ is continuous and positive.

1.1 The family $\mathcal{A}_\Phi$. Let Φbe continuous and positive. Denote by $\mathcal{A}_\Phi$ the subfamily of all $f(z)\in\mathcal{C}$ such that

$$(1) \qquad \limsup_{|z|\to 1}\ |f'(z)|-\Phi(f(z))\leq 0$$

Remark. By the definition of limes superior this means that for every $\epsilon>0$ there exists $\delta>0$ such that

$$|f'(z)|\leq\Phi(f(z))+\epsilon \quad : \text{for all } 1-\delta<|z|<1\,.$$

The maximal region $\Omega^*(\Phi)$. With Φ given we get a bounded open subset in the complex plane as follows:

$$\Omega^*(\Phi) = \cup f(D) \quad : \quad \text{union taken over all } f \in \mathcal{A}_\Phi \tag{1.2}$$

1.3 Theorem. *The maximal region* $\Omega^*(\Phi)$ *is simply connected. Moreover, the unique normalised conformal mapping for which* $f^*(D) = \Omega^*(\Phi)$ *belongs to* $\mathcal{C}_\Phi$.

1.4 The family $\mathcal{B}_\Phi$. It consists of all $f \in \mathcal{C}$ such that

$$\liminf_{|z|\to 1} \ |f'(z)| - \Phi(f(z)) \geq 0$$

To this family we assign minimal region

$$\Omega_*(\Phi) = \cap f(D) \quad \text{The intersection taken over all } f \in \mathcal{B}_\Phi \tag{1.5}$$

1.6 Theorem. *The set* $\Omega_*(\Phi)$ *is simply connected and the unique* $f_* \in \mathcal{C}$ *for which* $f_*(D) = \Omega_*(\Phi)$ *belongs to* $\mathcal{C}_\Phi$.

Remark. The constructions of the maximal and the minimal region give

$$\Omega_*(\Phi) \subset \Omega^*(\Phi) \tag{1}$$

This inclusion is in general strict as seen by the example when Φ is radial. But when $\operatorname{Log} \frac{1}{|\Phi|}$ is subharmonic the uniqueness in Theorem 1 asserts that one has the equality $\Omega_*(\Phi) = \Omega^*(\Phi)$.

Before we enter the proofs of Theorem 1.4 and 1.6 we show the uniqueness part in Theorem 1.

A. Proof of Theorem 1.

Let Φ be as in Theorem 1. Admitting Theorem 1.4 and 1.6 we get the two simply connected domains $\Omega^*(\Phi)$ and $\Omega_*(\Phi)$. Keeping Φ fixed we set $\Omega^* = \Omega^*(\Phi)$ and $\Omega_* = \Omega_*(\Phi)$. Let $f\colon D \to \Omega_*$ and $g\colon D \to \Omega^*$ be the corresponding conformal mappings. Since $\Omega_* \subset \Omega^*$ Riemann's mapping theorem gives an inequality for the first order derivative at $z = 0$:

$$f'(0) \leq g'(0) \tag{i}$$

Set

$$\Phi(w) = e^{U(w)}$$

where $U(w)$ by assumption is super-harmonic. Solving the Dirichlet problem with respect to the domain Ω^* we get the harmonic function U^* in Ω^* where

$$U^*(w) = U(w) \quad w \in \partial\Omega^*\,.$$

Similarly we find the harmonic function U_* in Ω_* such that

$$U_*(w) = U(w) \quad w \in \partial\Omega_*\,. \tag{*}$$

Next, since $g \in \mathcal{C}_\Phi$ we have the equality

$$\log|g'(z)| = U(g(z)) \quad |z| = 1 \tag{ii}$$

Now $\log|g'(z)|$ and $U^*(g(z))$ are harmonic in D and since g is normalised so that $g'(0)$ is real and positive it follows from (ii) that:

$$\log g'(0) = U^*(0)$$

In a similar way we find that

$$\log f'(0) = U_*(0)$$

Next, U is super-harmonic in Ω^* since and $\partial\Omega_*$ is a closed subset of $\bar{\Omega}^*$ one has:

$$U(w) \geq U^*(w) \quad w \in \partial\Omega_*$$

Then (*) entails that $U_* \geq U^*$ holds in Ω_*. In particular

$$\log f'(0) = U_*(0) \geq U^*(0) = \log g'(0)$$

Together with (i) we conclude that $f'(0) = g'(0)$and by the uniqueness in Riemann's mapping theorem it follos that $\Omega_* = \Omega^*$ which gives the uniqueness part in Theorem 1.

B. Proof of Theorem 2.

The first step in the proof is to construct a certain "union map" defined by a finite family $f_1, \ldots, f_n$ of functions $\mathcal{A}_\Phi$. Set

$$(*) \qquad S_\nu = f_\nu(D) \quad \text{and } S_* = \cup S_\nu$$

So above S_* is a union of Jordan domains which in general can intersect each other in a rather arbitrary fashion.

B.1 Definition. *The extended union denoted by $EU(S_*)$ is defined as follows: A point w belongs to the extended union if there exists some closed Jordan curve $\gamma \subset S_*$ and the Jordan domain bordered by γ contains w.*

Exercise. Verify that the extended union is simply connected.

B.2 Lemma *Let f_* be the unique normalised conformal map from D onto the extended union above. Then $f_* \in \mathcal{A}_\Phi$.*

Proof. First we reduce the proof to the case when all the functions $f_1, \ldots, f_n$ extend to be analytic in a neighborhood of the closed disc $\bar{D}$. In fact, with $r < 1$ we set $f_\nu^r(z) = f_\nu(rz)$ and get the image domains $S_\nu[r] = f_\nu^r(D) = f_\nu(D_r)$. Put $S_*[r] = \cup S_\nu[r]$ and construct its extended union which we denote by $S_{**}[r]$. Next, let $\epsilon > 0$ and consider the new function $\Psi(w) = \Phi(w) + \epsilon$. Let $f_*[r]$ be the conformal map from D onto $S_{**}[r]$. If Lemma B.2 has been proved for the n-tuple $\{f_\nu^r\}$ it follows by continuity that $f_*[r]$ belongs $\mathcal{A}_\Psi$ if r is sufficently close to one. Passing to the limit we see that $f_* = \lim_{r \to 1} f_*[r]$ and we get $f_* \in \mathcal{A}_\Psi$. Since $\epsilon > 0$ is arbitrary we get $f_* \in \mathcal{A}_\Phi$ as required.

After this preliminary reduction we consider the case when each f-function extends analytically to a neighborhood of the closed disc $|z| \leq 1$. Then each S_ν is a closed real analytic Jordan curve and the boundary of S_* is a finite union of real analytic arcs and some corner points. In particular we find the outer boundary which is a piecewise analytic and closed Jordan curve Γ and the extended union is the Jordan domain bordered by Γ. It is also clear that Γ is the union of some connected arcs $\gamma_1, \ldots, \gamma_N$ and a finite set of corner points and for each $1 \leq k \leq N$ there exists $1 \leq \nu(k) \leq n$ such that

$$\gamma_k \subset \partial S_{\nu(k)}$$

Denote by $\{F_\nu = f_\nu^{-1}\}$ and $F = f_*^{-1}$ the inverse functions and put:

$$G = \text{Log}\,\frac{1}{|F|} \quad : G_\nu = \text{Log}\,\frac{1}{|F_\nu|} \quad : 1 \leq \nu \leq n\,.$$

With $1 \leq \nu \leq n$ kept fixed we notice that G_ν and G are super-harmonic functions in S_ν and the difference

$$H = G - G_\nu$$

is superharmonic in S_ν. Next, consider a point $p \in \partial S_\nu$. Then $|F_\nu(p)| = 1$ and hence $G_\nu(p) = 0$. At the same time p belongs to ∂S_* or the interior of S_* so $|F(p)| \leq 1$ and hence $G(p) \geq 0$. This shows that $H \geq 0$ on ∂S_ν and by the minimum principle for harmonic functions we obtain:

$$\text{(i)} \qquad H(q) \geq 0 \quad \text{for all } q \in S_\nu$$

Let us then consider some boundary arc γ_k where $\gamma \subset \partial S_\nu$, i.e. here $\nu = \nu(k)$. Now $H = 0$ on γ_k and since (i) holds it follows that the *outer normal derivative*:

$$\text{(ii)} \qquad \frac{\partial H}{\partial n}(p) \leq 0 \quad p \in \gamma_k$$

Since $|F| = |F_\nu| = 1$ holds on γ_k and the gradient of H is parallell to the normal we also get:

$$\frac{\partial G}{\partial n}(p) = -|F'(w)| \quad \text{and} \quad \frac{\partial G_\nu}{\partial n}(p) = -|F'_\nu(w)| \quad : w \in \gamma_k$$

Hence (ii) above gives

$$\text{(iii)} \qquad |F'(w)| \geq |F'_\nu(w)| \quad \text{when} \quad w \in \gamma_k$$

Next, since $f_\nu \in \mathcal{A}_\Phi$ we have

$$\text{(iv)} \qquad |f'_\nu(F_\nu(w))| \leq \Phi(w)$$

and since F_ν is the inverse of f_ν we get

$$1 = f'_\nu(F_\nu(w)) \cdot F'_\nu(w)$$

Hence (iv) entails

$$\text{(v)} \qquad |F'_\nu(w)| \geq \frac{1}{\Phi(w)}$$

We conclude from (iii) that

$$\text{(vi)} \qquad |F'(w)| \geq \frac{1}{\Phi(w)} \quad : \ w \in \gamma_k$$

This holds for all the sub-arcs $\gamma_1, \ldots, \gamma_n$ and hence we have proved the inequality

$$(*) \qquad |F'(w)| \geq \frac{1}{\Phi(w)} \quad \text{for all} \quad w \in \Gamma$$

except at a finite number of corner points. To settle the situation at corner points we notice that Poisson's formula applied to the harmonic function $\log |f'_*(z)|$ in the unit disc gives

$$\text{(vii)} \qquad \log |f'_*(z)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot \log |f'(e^{i\theta})| \cdot d\theta \,.$$

Next, since F is the inverse of f_* we have

$$|f'_*(z)| \cdot |F'(f(z))| = 1 \quad \text{for all } |z| = 1 \,.$$

Hence (vi) gives

$$|f'(z)| \leq \Phi(f_*(z)) \quad \text{for all } |z| = 1 \,.$$

With $\Phi = e^U$ we therefore get

$$\log |f'_*(z)| \leq U(f(z)) \quad \text{for all } |z| = 1 \,.$$

From the Poisson integral (vii) it follows that

$$\log |f'_*(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot U(f_*(e^{i\theta})) \cdot d\theta \,. \quad z \in D$$

A passage to the limit. I addition to the obvious equi-continuity the passage to the limit requires some care which is exposed in [Beurling: Lemma 1, page 122]. Passing to the limit as $|z| \to 1$ the continuity of Φ implies that f_* belongs to $\mathcal{A}_\Phi$ which proves Lemma B.2.

B.3 The construction of Ω^*

By the uniform bound for Lipschitz norms the family $\mathcal{A}_\phi$ is equi-continuous. We can therefore find a denumerable dense subset $\{h_\nu\}$. It means that to every $f \in \mathcal{A}_\Phi$ and every $\epsilon > 0$ there exists some h_ν such that the maximum norm $|f - h_\nu|_D < \epsilon$. From this it is clear that

(i) $$\Omega^* = \cup\, h_\nu(D)$$

Next, to every $n \geq 2$ we have the n-tuple $h_1, \ldots, h_n$ Lemma B. 2 gives the function f_n where $f_n(D)$ is the extended union of $\{h_\nu(D)\}$. In particular

$$h_\nu(D) \subset f_n(D) \quad : 1 \leq \nu \leq n$$

The image domains $\{f_n(D)\}$ increase with n. and (i) gives

(ii) $$\Omega^* = \cup\, f_n(D)$$

Next, $\{f_n\}$ is a normal family of analytic functions and since their image domains increase there exists the limit function f^* and it belongs to $\mathcal{C}$ by the general results from Chapter VI. Morover, (ii) entails that

$$f^*(D) = \Omega^*$$

There remains to prove that

(*) $$f^* \in \mathcal{C}_\Phi$$

To establish (*) we shall need a relation between Φ and the maximal domain $\Omega^*(\Phi)$.

B.4 Proposition. *Let Ψ be a positive continuous function which is equal to Φ outside $\Omega^*(\Phi)$ while its restriction to $\Omega^*(\Phi)$ is arbitrary. Then one has the equality*

$$\Omega^*(\Phi) = \Omega^*(\Psi)$$

Proof. The assumption gives

(i) $$\Psi(w) = \Phi(w) \quad \text{for all } w \in \partial\Omega^*(\Phi)$$

It follows that $f^* \in \mathcal{A}_\Psi$. Hence the equality $f^*(D) = \Omega^*(\Phi)$ and the construction of the maximal domain $\Omega^*(\Psi)$ give the inclusion

(ii) $$\Omega^*(\Phi) \subset \Omega^*(\Psi)$$

Next, let $h^* \colon D \to \Omega^*(\Psi)$ be the conformal mapping function associated to Ψ. The equality (i) and the construction of the maximal region $\Omega^*(\Phi)$ gives $h^* \in \mathcal{A}_\phi$ and then

(iii) $$\Omega^*(\Psi) = h^*(D) \subset \Omega^*(\Phi)$$

Hence (ii-iii) give the requested equality in Proposition B.4.

B.5 A special choice of Ψ

.

Keeping Φ fixed we put $\Omega^*(\Phi) = \Omega^*$ to simplify the notations. We have the U-function such that

(B.5.i) $$\Phi(w) = e^{U(w)}$$

Here $U(w)$ is a continuous function on $\partial\Omega^*$ and solving the Dirichlet problem we obtain the function $U_*(w)$ where $U_* = U$ outside Ω^*, and in Ω^* the function U_* is the harmonic extension of the boundary function U restricted to $\partial\Omega^*$. Set

$$\Psi(w) = e^{U_*(w)}$$

Proposition B.4 gives

(B.5.ii) $$f^* \in \mathcal{A}_\Psi$$

Next, consider the function in D defined by:

$$V(z) = \log \Big| \frac{df^*(z)}{dz} \Big| - U_*(f(z))$$

From the above $V(z)$ is either identically zero in D or everywhere < 0. It is also clear that if $V = 0$ then $f^* \in \mathcal{C}_\Phi$ as required. So there remains only to prove:

B.6 Lemma. *The function $V(z)$ is identically zero in D.*

Proof. Assume the contrary. So now

(i) $$\Big| \frac{df^*(z)}{dz} \Big| < e^{U_*(z)} \quad \text{for all } z \in D\,.$$

Let $F(w)$ be the inverse of f so that :

$$F'(f^*(z)) \cdot \frac{df^*(z)}{dz} = 1 \quad z \in D$$

Then (i) gives:

(ii) $$\big|F'(w)\big| > e^{-U_*(w)} \quad w \in \Omega^*$$

Let $V(w)$ be the harmonic conjugate of $U_*(w)$ normalised so that $V(0) = 0$ and put

$$H(w) = \int_0^w e^{-U_*(\zeta)+iV(\zeta)} \cdot d\zeta\,.$$

Then (ii) gives

(iii) $$\big|H'(w)\big| < \big|F'(w)\big| \quad w \in \Omega^*$$

Since $H(w((0)) = F(w(0)) = 0$ it follows that

(iv) $$\inf_{w \in \Omega^*} \frac{|H(w)|}{|F(w)|} = r_0 < 1$$

Now $|F(w)| < 1$ in Ω^* and $|F(w)| \to 1$ as w approaches $\partial\Omega^*$. Hence (iv) entails the domain

(v) $$R_0 = \{w \in \Omega^* \quad : \ |H(w)| < r_0\}$$

has at least one boundary point w_* which also belongs to $\partial\Omega^*$. Next, the function $H(w)$ is analytic in R_0 and its derivative is everywhere $\neq 0$ while $|H(w)| = 1$ on ∂R_0. It follows that H gives a conformal map from R_0 onto the disc $|z| < r_0$. Let $h(z)$ be the inverse of this conformal mapping. Now we get the analytic function in D defined by

$$g(z) = h(r_0 z)$$

Next, let $|z| = 1$ and put $w = h(r_0 z)$. Then

(v) $$|g'(z)| = r_0 \cdot |h'(r_0 z)| = r_0 \cdot \frac{1}{|H'(g(z))|} = r_0 \cdot e^{U_*(g(z))} = r_0 \cdot \Psi(g(z)) < \Psi(g(z))$$

At the same time we have a common boundary point

$$w_* \in \partial\Omega^* \,\cap\, \partial g(D)$$

Since the g-function extends to a continuous function on $|z| \leq 1$ there exists a point $e^{i\theta}$ such that

$$g(e^{i\theta}) = w_*$$

Now we use that $r_0 < 1$ above. The continuity of Ψ gives the existence of $\epsilon > 0$ such that for any complex number a which belongs to the disc $|a - 1| < \epsilon$, it follows that the function

$$z \mapsto a \cdot g(z)$$

belongs to $\mathcal{A}_\Psi$. Finally, by Proposition B.4 the maximal region for the Ψ-function is equal to Ω^* and we conclude that

$$ag(e^{i\theta}) = aw_* \in \Omega^* \quad |a - 1| < \epsilon$$

This would mean that w_* is an *interior point* of Ω^* which contradicts that $w_* \in \partial\Omega_*$ and Lemma B.6 is proved.

C. Proof of Theorem 3.

First we have a companion to Lemma B.2. Namely, let $g_1, \ldots, g_n$ be a finite set in $\mathcal{B}_\Phi$. Set $S_\nu = g_\nu(D)$. Following [Beur: page 123] we give

C.1 Definition. *The reduced intersection of the family $\{S_\nu\}$ is defined as the set of these points w which can be joined with the origin by a Jordan arc γ contained in the intersection $\cap S_\nu$. The resulting domain is denoted by $RI\{S_\nu\}$.*

C.2 Proposition. *The domain $RI\{S_\nu\}$ is simply connected and if $g \in \mathcal{C}$ is the normalised conformal mapping onto this domain, then $g \in \mathcal{B}_\Phi$.*

The proof of this result can be carried out in a similar way as in the proof of Lemma B.2 so we leave out the details. Next, starting from a dense sequence $\{g_\nu\}$ in $\mathcal{B}_\Phi$ we find for each n the function $f_n \in \mathcal{B}_\Phi$ where

$$f_n(D) = RI\{S_\nu\} \quad : S_\nu = g_\nu(D) : \ 1 \leq \nu \leq n\,.$$

Here the simply connected domains $\{f_n(D)\}$ decrease and there exists the limit function $f_* \in \mathcal{C}$ where

$$f_*(D) = \Omega_*$$

There remains to prove

C.3 Proposition. *One has $f_* \in \mathcal{C}_\Phi$.*

Remark. Proposition C.3 requires a quite involved proof as comared to the case of maximal regions. The details are given in [Beur: page 127-130]. Let us just sketch the strategy in the proof. Put

$$m = \inf_{g \in \mathcal{B}_\Phi} g'(0)$$

Next, f_* belongs to $\mathcal{B}_\Phi$ because we have the trivial inclusion $\mathcal{C}_\Phi \subset \mathcal{B}_\Phi$ and using this Beurling proved that

$$m \geq \min_{w \in \Omega^*} \Phi(w)$$

Next, starting with Proposition C.2 above, Beurling introduces a normal family and proves that

$$f_*'(0) = m$$

Thus, f_* is a solution to an extremal problem which Beurling used to establish the inclusion $f_* \in \mathcal{C}_\Phi$. Let us remark that this part of the proof relies upon some some very interesting set-theoretic constructions where the family of regions of the *Schoenfliess' type* are introduced in [Beur: page 121]. So the whole proof involves topological investigations of independent interest.

11. Beurling's conformal mapping theorem.

Introduction. Let D be the open unit disc $|z|<1$. Denote by $\mathcal{C}$ the family of conformal maps $w=f(z)$ which map D onto some simply connected domain Ω_f which contains the origin and satisfy:

$$f(0)=0 \quad \text{and} \quad f'(0) \text{ is real and positive}\,.$$

Riemann's mapping theorem asserts that for every simply connected subset Ω of $\mathbf{C}$ which is not equal to $\mathbf{C}$ there exists a unique $f\in\mathcal{C}$ such that $\Omega_f=\Omega$. We are going to construct a subfamily of $\mathcal{C}$. Consider a positive and bounded continuous function a function Φ defined in the whole complex w-plane.

0.1 Definition. *The set of all $f\in\mathcal{C}$ such that*

$$(*) \qquad \lim_{r\to 1}\ \max_{0\le\theta\le 2\pi}\big[|f'(re^{i\theta})|-\Phi(f(re^{i\theta}))\big]=0$$

is denoted by $\mathcal{C}_\Phi$.

Remark. Thus, when $f\in\mathcal{C}_\Phi$ then the difference of the absolute value $|f'(z)|$ and $\Phi(f(z))$ tends uniformly to zero as $|z|\to 1$. Let M be the upper bound of Φ. The maximum principle applied to the complex derivative $f'(z)$ gives

$$|f'(z)|\le M \quad : \quad z\in D$$

Hence $f(z)$ is a continuous function in the open disc D whose Lipschitz norm is uniformly bounded by M. This implies that f extends to a continuous function in the closed disc, i.e. f belongs to the disc algebra $A(D)$. Notice also that (*) implies that the function $z\mapsto |f'(z)|$ extends to a continuous function on $\bar{D}$.

1. Theorem. *Assume that $Log\,\frac{1}{\Phi(w)}$ is subharmonic. Then $\mathcal{C}_\Phi$ contains a unique function f^*.*

Remark. In the special case when $\Phi(w)=\Phi(|w|)$ is a radial function we notice that for every $\rho>0$ such that $\Phi(\rho)=\rho$ it follows that the function $f(z)=\rho\cdot z$ belongs to $\mathcal{C}_\Phi$. So for a radial Φ-function where different ρ-numbers exist one does not have uniqueness. The reader may verify that a radial function Φ for which $\Phi(\rho)=\rho$ has several solutions cannot satisfy the condition in Theorem 1. Next, let us give examples of Φ-functions which satisfy the condition in Theorem 1. Consider an arbitrary real-valued and non-negative L^1-function $\rho(t,s)$ which has compact support. Set

$$\Phi(w)=\exp\Big[\int\log\frac{1}{|w-t-is[}\cdot\rho(t,s)\cdot dtds\Big]$$

Here $\log\frac{1}{\Phi}$ is subharmonic and Theorem 1 asserts that there exists a unique simply connected domain Ω which contains the origin such that the normalised conformal mapping function $f\colon D\to\Omega$ satisfies

$$|f'(e^{i\theta})|=\Phi(f(e^{i\theta})) \quad : \quad 0\le\theta\le 2\pi\,.$$

The proof of Theorem 1 relies upon some results where we only assume that the Φ-function is continuous and positive.

The family $\mathcal{A}_\Phi$. A conformal map $f(z)$ in $\mathcal{C}$ belongs to $\mathcal{A}_\Phi$ if

$$(1) \qquad \limsup_{|z|\to 1}\ |f'(z)|-\Phi(f(z))\le 0$$

Remark. By the definition of limes superior this means that for every $\epsilon>0$ there exists $\delta>0$ such that

$$|f'(z)|\le\Phi(f(z))+\epsilon \quad : \text{for all } 1-\delta<|z|<1\,.$$

The maximal region $\Omega^*(\Phi)$. With Φ given we define a bounded open subset in the w-plane as follows:

$$\Omega^*(\Phi) = \cup f(D) \quad : \quad \text{union taken over all } f \in \mathcal{A}_\Phi \,.$$

With these notations we have

2. Theorem. *The maximal region $\Omega^*(\Phi)$ is simply connected. Moreover, there exists a unique conformal map with $f^*(D) = \Omega^*(\Phi)$ which in addition belongs to $\mathcal{C}_\Phi$.*

The family $\mathcal{B}_\Phi$. It consists of all $f \in \mathcal{C}$ such that

$$\text{Lim.inf.}_{|z|\to 1} \, |f'(z)| - \Phi(f(z)) \geq 0$$

To this family we assign minimal region

$$\Omega_*(\Phi) = \cap f(D) \quad \text{The intersection taken over all } f \in \mathcal{B}_\Phi$$

3. Theorem. *The set $\Omega_*(\Phi)$ is simply connected and the unique $f_* \in \mathcal{C}$ for which $f_*(D) = \Omega_*(\Phi)$ belongs to $\mathcal{C}_\Phi$.*

The constructions of the maximal and the minimal region give

$$\Omega_*(\Phi) \subset \Omega^*(\Phi) \tag{1}$$

In general this inclusion it is strict as seen by the example when Φ is radial. But when $\text{Log}\,\frac{1}{|\Phi|}$ is subharmonic the uniqueness in Theorem 1 asserts that one has the equality $\Omega_*(\Phi) = \Omega^*(\Phi)$.

Remark about the proofs. Following Beurling's article [Beur] we shall give the details of the proof of Theorem 2. Concerning Theorem 3 it is substantially harder and for this part of the proof we refer to [Beur: p. 127-130] for details, Before we enter the proof of Theorem 2 we show how Theorem 2 and 3 together give the uniqueness in Theorem 1.

A. Proof of Theorem 1.

Let Φ be as in Theorem 1. Admitting Theorem 2 and 3 we get the two simply connected domains $\Omega^*(\Phi)$ and $\Omega_*(\Phi)$. Keeping Φ fixed we set $\Omega^* = \Omega^*(\Phi)$ and $\Omega_* = \Omega_*(\Phi)$. Since $\Omega_* \subset \Omega^*$ Riemann's mapping theorem gives an inequality for the first order derivative at $z = 0$:

$$f_*'(0) \leq (f^*)'(0) \tag{i}$$

Next, we can write

$$\Phi(w) = e^{U(w)}$$

where $U(w)$ by assumption is super-harmonic. We can solve the Dirichlet problem with respect to the domain Ω^*. This gives the harmonic function U^* in Ω^* where

$$U^*(w) = U(w) \quad w \in \partial\Omega^* \,.$$

Similarly we find the harmonic function U_* in Ω_* such that

$$U_*(w) = U(w) \quad w \in \partial\Omega_* \,. \tag{*}$$

Next, since $f^* \in \mathcal{C}_\Phi$ we have the equality

$$\log|(f^*)'(z)| = U(f(z)) \quad |z| = 1 \tag{ii}$$

Now $\log|(f^*)'(z)|$ and $U^*(f(z))$ are harmonic in D and (ii) gives:

$$\log{(f^*)'(0)} = U^*(0)$$

In a similar way we find that

$$\log f_*'(0) = U_*(0)$$

Since U is super-harmonic in Ω^* and $\partial\Omega_*$ is a closed subset of $\bar{\Omega}^*$ we get:

$$U(w) \geq U^*(w) \quad w \in \partial\Omega_*$$

From (*) it therefore follows that $U_* \geq U^*$ holds in Ω_*. So in particular

$$\log f_*'(0) = U_*(0) \geq U^*(0) = \log (f^*)'(0)$$

Together with (i) we conclude that $f_*(0) = (f^*)'(0)$. Finally, the uniqueness in Riemann's mapping theorem gives $\Omega_* = \Omega^*$ and hence that $f_* = f^*$ which proves Theorem 1.

B. Proof of Theorem 2.

The first step in the proof is to construct a certain "union map" defined by a finite family $f_1, \ldots, f_n$ of functions $\mathcal{A}_\Phi$. Set

(*) $$S_\nu = f_\nu(D) \quad \text{and } S_* = \cup S_\nu$$

So above S_* is a union of Jordan domains which in general can intersect each other in a rather arbitrary fashion.

B.1 Definition. *The extended union denoted by $EU(S_*)$ is defined as follows: A point w belongs to the extended union if there exists some closed Jordan curve γ which contains w in its interior domain while $\gamma \subset S_*$.*

Exercise. Verify that the extended union is *simply connected.*

B.2 Lemma *Let f_* be the unique normalised conformal map from D onto the extended union above. Then $f_* \in \mathcal{A}_\Phi$.*

Proof. First we reduce the proof to the case when all the functions $f_1, \ldots, f_n$ extend to be analytic in a neighborhood of the closed disc $\bar{D}$. In fact, with $r < 1$ we set $f_\nu^r(z) = f_\nu(rz)$ and get the image domains $S_\nu[r] = f_\nu^r(D) = f_\nu(D_r)$. Put $S_*[r] = \cup S_\nu[r]$ and construct its extended union which we denote by $S_{**}[r]$. Next, let $\epsilon > 0$ and consider the new function $\Psi(w) = \Phi(w) + \epsilon$. Let $f_*[r]$ be the conformal map from D onto $S_{**}[r]$. If Lemma B.2 has been proved for the n-tuple $\{f_\nu^r\}$ it follows by continuity that $f_*[r]$ belongs $\mathcal{A}_\Psi$ if r is sufficently close to one. Passing to the limit we see that $f_* = \lim_{r \to 1} f_*[r]$ and we get $f_* \in \mathcal{A}_\Psi$. Since $\epsilon > 0$ is arbitrary we get $f_* \in \mathcal{A}_\Phi$ as required.

After this preliminary reduction we consider the case when each f-function extends analytically to a neighborhood of the closed disc $|z| \leq 1$. Then each S_ν is a closed real analytic Jordan curve and the boundary of S_* is a finite union of real analytic arcs and some corner points. In particular we find the outer boundary which is a piecewise analytic and closed Jordan curve Γ and the extended union is the Jordan domain bordered by Γ. It is also clear that Γ is the union of some connected arcs $\gamma_1, \ldots, \gamma_N$ and a finite set of corner points and for each $1 \leq k \leq N$ there exists $1 \leq \nu(k) \leq n$ such that

$$\gamma_k \subset \partial S_{\nu(k)}$$

Denote by $\{F_\nu = f_\nu^{-1}\}$ and $F = f_*^{-1}$ the inverse functions and put:

$$G = \operatorname{Log} \frac{1}{|F|} \quad : G_\nu = \operatorname{Log} \frac{1}{|F_\nu|} \quad : 1 \leq \nu \leq n\,.$$

With $1 \leq \nu \leq n$ kept fixed we notice that G_ν and G are super-harmonic functions in S_ν and the difference

$$H = G - G_\nu$$

is superharmonic in S_ν. Next, consider a point $p \in \partial S_\nu$. Then $|F_\nu(p)| = 1$ and hence $G_\nu(p) = 0$. At the same time p belongs to ∂S_* or the interior of S_* so $|F(p)| \leq 1$ and hence $G(p) \geq 0$. This shows that $H \geq 0$ on ∂S_ν and by the minimum principle for harmonic functions we obtain:

(i) $$H(q) \geq 0 \quad \text{for all } q \in S_\nu$$

Let us then consider some boundary arc γ_k where $\gamma \subset \partial S_\nu$, i.e. here $\nu = \nu(k)$. Now $H = 0$ on γ_k and since (i) holds it follows that the *outer normal derivative*:

(ii) $$\frac{\partial H}{\partial n}(p) \leq 0 \quad p \in \gamma_k$$

Since $|F| = |F_\nu| = 1$ holds on γ_k and the gradient of H is parallell to the normal we also get:

$$\frac{\partial G}{\partial n}(p) = -|F'(w)| \quad \text{and} \quad \frac{\partial G_\nu}{\partial n}(p) = -|F'_\nu(w)| \quad : w \in \gamma_k$$

Hence (ii) above gives

(iii) $$|F'(w)| \geq |F'_\nu(w)| \quad \text{when} \quad w \in \gamma_k$$

Next, since $f_\nu \in \mathcal{A}_\Phi$ we have

(iv) $$|f'_\nu(F_\nu(w))| \leq \Phi(w)$$

and since F_ν is the inverse of f_ν we get

$$1 = f'_\nu(F_\nu(w)) \cdot F'_\nu(w)$$

Hence (iv) entails

(v) $$|F'_\nu(w)| \geq \frac{1}{\Phi(w)}$$

We conclude from (iii) that

(vi) $$|F'(w)| \geq \frac{1}{\Phi(w)} \quad : \; w \in \gamma_k$$

This holds for all the sub-arcs $\gamma_1, \ldots, \gamma_n$ and hence we have proved the inequality

(*) $$|F'(w)| \geq \frac{1}{\Phi(w)} \quad \text{for all} \quad w \in \Gamma$$

except at a finite number of corner points. To settle the situation at corner points we notice that Poisson's formula applied to the harmonic function $\log |f'_*(z)|$ in the unit disc gives

(vii) $$\log |f'_*(z)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot \log |f'(e^{i\theta})| \cdot d\theta \,.$$

Next, since F is the inverse of f_* we have

$$|f'_*(z)| \cdot |F'(f(z))| = 1 \quad \text{for all } |z| = 1 \,.$$

Hence (vi) gives

$$|f'(z)| \leq \Phi(f_*(z)) \quad \text{for all } |z| = 1 \,.$$

With $\Phi = e^U$ we therefore get

$$\log |f'_*(z)| \leq U(f(z)) \quad \text{for all } |z| = 1 \,.$$

From the Poisson integral (vii) it follows that

$$\log |f'_*(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \cdot U(f_*(e^{i\theta})) \cdot d\theta \,. \quad z \in D$$

A passage to the limit. I addition to the obvious equi-continuity the passage to the limit requires some care which is exposed in [Beurling: Lemma 1, page 122]. Passing to the limit as $|z| \to 1$ the continuity of Φ implies that f_* belongs to $\mathcal{A}_\Phi$ which proves Lemma B.2.

B.3 The construction of f^*

By the uniform bound for Lipschitz norms the family $\mathcal{A}_\phi$ is equi-continuous. We can therefore find a denumerable dense subset $\{h_\nu\}$. It means that to every $f \in \mathcal{A}_\Phi$ and every $\epsilon > 0$ there exists some h_ν such that the maximum norm $|f - h_\nu|_D < \epsilon$. It follows that

$$\Omega^* = \cup\, h_\nu(D)$$

Next, to every $n \geq 2$ we have the n-tuple $h_1, \ldots, h_n$ and by Lemma B. 2 we construct the function f_n where we have the inclusions

$$h_\nu(D) \subset f_n(D) \quad : 1 \leq \nu \leq n$$

Moreover, the image domains $\{f_n(D)\}$ increase with n. So (i) above gives

(*) $$\Omega^* = \cup\, f_n(D)$$

Next, $\{f_n\}$ is a normal family of analytic functions and since their image domains increase it follows there exists the limit function f^* which belongs to $\mathcal{C}$ and (*) above gives the equality $f^*(D) = \Omega^*$. There remains to prove that f^* also belongs to $\mathcal{C}_\Phi$. To get the inclusion

$$f^* \in \mathcal{C}_\Phi$$

we establish a relation between Φ and the maximal domain $\Omega^*(\Phi)$.

B.4 Proposition. *Let Ψ be a positive continuous function which is equal to Φ outside $\Omega^*(\Phi)$ while its restriction to $\Omega^*(\Phi)$ is arbitrary. Then one has the equality*

$$\Omega^*(\Phi) = \Omega^*(\Psi)$$

Proof. The assumption gives

(i) $$\Psi(w) = \Phi(w) \quad \text{for all } w \in \partial\Omega^*(\Phi)$$

It follows that $f^* \in \mathcal{A}_\Psi$. Hence the equality $f^*(D) = \Omega^*(\Phi)$ and the construction of $\Omega^*(\Psi)$ give the inclusion

(ii) $$\Omega^*(\Phi) \subset \Omega^*(\Psi)$$

Next, let h^* be the mapping function associated to Ψ. By the construction of the maximal region $\Omega^*(\Phi$ we get:

(iv) $$\Omega^*(\Psi) = h^*(D) \subset \Omega^*(\Phi)$$

Hence (iii) and (iv) give the requested equality

(v) $$\Omega^*(\Psi) = \Omega^*(\Phi)$$

B.5 A special choice of Ψ

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Keeping Φ fixed we put $\Omega^*(\Phi) = \Omega^*$ to simplify the notations. We have the U-function such that

(vi) $$\Phi(w) = e^{-U(w)}$$

Now $U(w)$ is a continuous function on $\partial\Omega^*$ and solving the Dirichlet problem we obtain the function $U_*(w)$ where $U_* = U$ outside Ω^*, and in Ω^* the function U_* is the harmonic extension of the boundary function U restricted to $\partial\Omega^*$. Set

$$\Psi(w) = e^{U_*(w)}$$

Proposition B.4 gives

(i) $$f^* \in \mathcal{A}_\Psi$$

Next, consider the function in D defined by:

$$V(z) = \log\left|\frac{df^*(z)}{dz}\right| - U_*(f(z))$$

From (i) it follows that $V(z)$ either is identically zero in D or everywhere < 0. It is also clear that if $V = 0$ then $f^* \in \mathcal{C}_\Phi$ as required. So there remains only to prove:

B.6 Lemma. *The function $V(z)$ is identically zero in D.*

Proof. Assume the contrary. So now

(i) $$\Big|\frac{df^*(z)}{dz}\Big| < e^{U_*(z)} \quad \text{for all } z \in D\,.$$

Let $F(w)$ be the inverse of f so that :

$$F'(f^*(z)) \cdot \frac{df^*(z)}{dz} = 1 \quad z \in D$$

Then (i) gives:

(ii) $$\big|F'(w)\big| > e^{-U_*(w)} \quad w \in \Omega^*$$

Next, let $V(w)$ be the harmonic conjugate of $U_*(w)$ normalised so that $V(0) = 0$ and set

$$H(w) = \int_0^w e^{-U_*(\zeta)+iV(\zeta)} \cdot d\zeta\,.$$

Then (ii) gives

(iii) $$\big|H'(w)\big| < \big|F'(w)\big| \quad w \in \Omega^*$$

Since $H(w((0)) = F(w(0)) = 0$ it follows that

(iv) $$\inf_{w\in\Omega^*} \frac{|H(w)|}{|F(w)|} = r_0 < 1$$

Since $|F(w)| < 1$ in Ω^* while $|F(w)| \to 1$ as w approaches $\partial\Omega^*$ we see that (iv) entails the domain

(v) $$R_0 = \{w \in \Omega^* \quad : \ |H(w)| < r_0\}$$

has at least one boundary point w_* which also belongs to $\partial\Omega^*$. Next, the function $H(w)$ is analytic in R_0 and its derivative is everywhere $\neq 0$ while $|H(w)| = 1$ on ∂R_0. It follows that H gives a conformal map from R_0 onto the disc $|z| < r_0$. Let $h(z)$ be the inverse of this conformal mapping. Now we get the analytic function in D defined by

$$g(z) = h(r_0 z)$$

Next, let $|z| = 1$ and put $w = h(r_0 z)$. Then

(v) $$|g'(z)| = r_0 \cdot |h'(r_0 z)| = r_0 \cdot \frac{1}{|H'(g(z))|} = r_0 \cdot e^{U_*(g(z))} = r_0 \cdot \Psi(g(z)) < \Psi(g(z))$$

At the same time we have a common boundary point

$$w_* \in \partial\Omega^* \cap \partial g(D)$$

Since the g-function extends to a continuous function on $|z| \leq 1$ there exists a point $e^{i\theta}$ such that

$$g(e^{i\theta}) = w_*$$

Now we use that $r_0 < 1$ above. The continuity of the Ψ gives some $\epsilon > 0$ such that for any complex number a which belongs to the disc $|a - 1| < \epsilon$, it follows that the function

$$z \mapsto a \cdot g(z)$$

belongs to $\mathcal{A}_\Psi$. Finally, by Proposition B.4 the maximal region for the Ψ-function is equal to Ω^* and we conclude that

$$ag(e^{i\theta}) = aw_* \in \Omega^* \quad |a - 1| < \epsilon$$

This would mean that w_* is an *interior point* of Ω^* which contradicts that $w_* \in \partial\Omega_*$. Hence Lemma B.6 is proved.

C. Proof of Theorem 3.

First we have a companion to Lemma B.2. Namely, let $g_1, \ldots, g_n$ be a finite set in $\mathcal{B}_\Phi$. Set $S_\nu = g_\nu(D)$. Following [Beur: page 123] we give

C.1 Definition. *The reduced intersection of the family $\{S_\nu\}$ is defined as the set of these points w which can be joined with the origin by a Jordan arc γ contained in the intersection $\cap S_\nu$. The resulting domain is denoted by $RI\{S_\nu\}$.*

C.2 Proposition. *The domain $RI\{S_\nu\}$ is simply connected and if $g \in \mathcal{C}$ is the normalised conformal mapping onto this domain, then $g \in \mathcal{B}_\Phi$.*

The proof of this result can be carried out in a similar way as in the proof of Lemma B.2 so we leave out the details. Next, starting from a dense sequence $\{g_\nu\}$ in $\mathcal{B}_\Phi$ we find for each n the function $f_n \in \mathcal{B}_\Phi$ where

$$f_n(D) = RI\{S_\nu\} \quad : \, S_\nu = g_\nu(D) : \, 1 \leq \nu \leq n \, .$$

Here the simply connected domains $\{f_n(D)\}$ decrease and there exists the limit function $f_* \in \mathcal{C}$ where

$$f_*(D) = \Omega_*$$

There remains to prove

C.3 Proposition. *One has $f_* \in \mathcal{C}_\Phi$.*

Remark. Proposition C.3 requires a quite involved proof and is given in [Beur: page 127-130]. We shall not try to present all the details and just sketch the strategy in the proof. Put

$$m = \inf_{g \in \mathcal{B}_\Phi} g'(0)$$

Next, f^* belongs to $\mathcal{B}_\Phi$ because we have the trivial inclusion $\mathcal{C}_\Phi \subset \mathcal{B}_\Phi$ and using this Beurling proved that

$$m \geq \min_{w \in \Omega^*} \Phi(w)$$

Next, starting with Proposition C.2 above, Beurling introduces a normal family and proves that

$$f_*'(0) = m$$

Thus, f_* is a solution to an extremal problem. This is used in the final part of Beurling's proof to establish the inclusion $f_* \in \mathcal{C}_\Phi$. Let us remark that this part of the proof relies upon some subtle set-theoretic constructions where the family of regions of the *Schoenfliess' type* are introduced in [Beur: page 121]. The whole analysis involves topological investigations of independent interest.

Conformal mapping onto comb domain

Introduction. Let Ω be a simply connected open set in $\mathbf{C}$ with finite area and ω an open Jordan arc which appears as a relatively open subset of $\partial\Omega$. We say that ω has a convex position in Ω if the following hold: For each pair of points $z \in \Omega$ and $a \in \gamma$ the half-open line segment $[z, a)$ is contained in Ω. When this holds we define a function ω_* in Ω by

$$\omega_*(z) = \inf_{\gamma} \int_{\gamma} |dz|$$

where the infimum is taken over rectifiable Jordan arcs from z with an end-point on ω.

Exercise*Show that if γ has a convex position in Ω then the function ω_* is subharmonic.*

Comb domains. XXXX

Theorem. *Given the pair Ω, ω there exists a comb domain D and a conformal mapping $f \colon \Omega \to D$ such that $f(\gamma)$ is the base of D and the following two inequalities hold:*

$$\text{Area}(D) \leq \text{Area}(\Omega) \quad : \ \gamma_*(z) \leq \mathfrak{Re}(f(z)) \quad : \ z \in \Omega$$

The conformal mapping κ. Since ω is an aric in $\partial\Omega$ there exists a conformal mapping κ from Ω onto a domain B in the right half-plane such that $\kappa(\omega)$ is the imaginary axis, i.e. the line $\{x = 0\}$. In B we get the subharmonic function

$$u_0^* \circ \kappa = \omega_*$$

We are going to construct a conformal map $\chi \colon B \to D$ where D such that the composed conformal mapping $f = \chi \circ \kappa$ gives (*) in the theorem. To achieve this we shall perform certain constructions in the strip domain B. To begin with we have the subharmonic function u_0^* in B where a notble fact is that

$$\text{Area}(\Omega) = D(u_0^*)$$

Same as the dirichlet integral !!!!!

1. A study of B-domains.

Denote by $\mathcal{B}$ the class of simply connected domains contained in the strip $\{0 < \mathfrak{Re}\, z < 1\}$. If $B \in \mathcal{B}$ a portion of its boundary appears on the imaginary axis, i.e. we get the closed set

$$B_* = \{y \colon \ iy \in \partial B\}$$

This set may be infinite and is in general not connected.

1.1 Proposition. *Let $u(x, y)$ be a harmonic function in B which extends to a C^1-function on its closure. Suppose also that $u(0, y) = 0$ for all $iy \in \partial B$ and that the Dirichlet integral*

$$D(u) = \iint_B (u_x^2 + u_y^2)\, dxdy < \infty$$

Then one has the equation

$$D(u) = 2 \cdot \iint_B u_x^2 \, dxdy - \int_{B_*} u_x^2(0, y)\, dy$$

Proof. In general consider a C^2-function $f(x, y)$ and set

$$I(x) = \int_{-\infty}^{\infty} [\, f_x^2(x, y) - f_y^2(x, y)]\, dy$$

The derivative becomes

$$\frac{dI}{dx} = 2 \cdot \int_{-\infty}^{\infty} [f_x f_{xx} - f_y f_{xy}\, dy$$

Next, a partial integration gives

$$-\int_{-\infty}^{\infty} f_y f_{xy}\,dy = -f_y f_x|xx + -\int_{-\infty}^{\infty} f_x \cdot f_{yy}\,dy$$

Hence we obtain

$$\frac{dI}{dx} = 2\cdot\int_{-\infty}^{\infty} f_x \cdot \Delta f\,dy$$

Taking the integral over $0 \leq x \leq 1$ gives

$$I(1) - I(0) = \iint_B f_x \cdot \Delta f\,dxdy$$

A special case. Let $f = u$ where u is harmonic and $u(0,y) = 0$ for all y. This implies that $u_y(0,y) = 0$ for all y which shows that

$$I(0) = \int_{-\infty}^{\infty} u_x^2(0,y)\,dy$$

At the same time (x) and $\Delta u = 0$ show that the function $x \to I(x)$ is constant which gives

$$I(0) = \int_0^1 I(x)\,dx = \iint_B (u_x^2 - u_y^2)\,dxdy$$

From (xx) it follows that the Dirichlet integral

$$(*) \qquad D(u) = \iint (u_x^2 + u_y^2)\,dxdy = 2\cdot\iint u_x^2\,dxdy - \int_{-\infty}^{\infty} u_x^2(0,y)\,dy$$

Exercise. Let ϕ be subharmonic in B where $u^*(0,y) = 0$ for all y. Use that $\Delta u^* \geq 0$ and the same proof as above to deduce that

$$D(\phi) \geq 2\cdot\iint \phi_x^2\,dxdy - \int_{-\infty}^{\infty} \phi_x^2(0,y)\,dy$$

Exrercise In general, let $\phi(x,y)$ be a subharmonic function of class C^2 in B which extends continuously to $\bar{B}$. Define the function Φ by

$$\Phi(x,y)\int_0^x |\phi_x(t,y)|\,dt$$

Show that Φ is subharmonic and notice that

$$\Phi_x^2 = \phi_x^2$$

Conclude from (*) and Exericce xx that one gets the inequality

$$D(\Phi) \geq D(\phi)$$

Next, let ϕ be a subharmonic function in B with a finite Dirichlet integral. Assume that $\phi(0,y) = 0$ for all y while $y \to \phi(1,y)$ is a continuous function. Solving the Dirichlet problem we find the harmonic function U_ϕ which extends ϕ. By Dirchlet's principle in § xx we have

$$D(U_\phi) \leq D(\phi)$$

A recursion. Start with a subharmonic function u_0^* in B where $u_0^*(0,y) =$. Denote by u_1 the harmonic function which solves the Dirichlet problem with u_0^* as boundary value. Then we construct the subharmonic function

$$u_1^*(x,y) = \int_0^x |(u_0^*)_x|(t,y)\,dt$$

In the next stage u_2 is the harmonic function with u_1^* s boundary function. Inductively we get a sequence of such u and u^*-functions. Here

$$u_n^* \leq u_{n+1} \leq u_{n+1}^* \leq \ldots$$

hold for all n. At the same time (xx) above entails that the Dirichlet integrals decrease:

$$D(u_0^*) \geq D(u_1) \geq D(u_1^*) \geq D(u_2) \geq \dots$$

These two reversed sequences of inequalities and Harnacks Theorem ential that there exists a limit function

$$U = \lim\, u_n = \lim u_n^*$$

Here U is harmonic in B and by construction the partial x-derivatives of the u^*-functions are all ≥ 0 and hence $U_x \geq 0$ in B. At the same time $U(0,y) = 0$ for all y. Now Theorem xx implies that if V is the conjugate harmonic function of U then $f = U + iV$ gives a conformal mapping from B onto a comb domain Ω. Moreover the dewcrsing sequece of Dirichlet integrals entail that

$$\text{Area}(\Omega) = D(U) \leq D(u_0^*)$$

LOGIC: We have started with a domain D with a portion on boundary convex and and first find $\kappa\colon D \to B$ sends portion to the y-axis and u_0^* becomes a subharmonic function in B. Starting from u_0^* we perform the recursion above¨ and arrive at U which eith $f = U + iV$ gives a conformal map B to comb domain. The composed map $f \circ \kappa$ sends the given domin onte the comb domsin where area now shrinks why one has a stretched in the other direction:

$$[z, \alpha] \leq \mathfrak{Re}\, f(z) \quad zindomain$$

Let $0 < b < 1$ and consider the Möbius transform

$$T_b(z) = \frac{b - z}{1 - bz}$$

Notice that the composed map

$$T_b \circ T_b(z) \frac{b - \frac{b-z}{1-bz}}{1 - b \cdot \frac{b-z}{1-bz}} = z$$

Next, if $w \in \mathbf{C}$ we have $w = T_b(w)$ if

$$w = \frac{b - w}{1 - bw} \implies 2w = b + bw^2$$

Thus can be rewritten as

$$(w - \frac{1}{b})^2 = \frac{1}{b^2} - 1$$

Thus, the fixed points of T_b consists of the circle of radius $\sqrt{b^{-2} - 1}$ centered at $\frac{1}{b}$. By a figure we see that it intersects the unit disc $|w| = 1$ for all $0 < b < 1$ in a circular arc γ_b intersects the real axis at the point

$$u = \frac{1 - \sqrt{1 - b^2}}{b}$$