§ 0. Unbounded operators and their resolvent operators

Introduction. We expose some classic results where the major construction go back to pioneering work by Carl Neumann from 1879. Throughout X denotes a complex Banach space, i.e. a complex vector space equippped with a complete norm.

0.1 The class $\mathfrak{I}(X)$. It consist of bounded linear operators R on X with the property that R is injective and the range $R(X)$ is a dense subspace of X. Each such R gives a densely defined operator T whose domain of definition $\mathcal{D}(T)$ is the range of R. Namely, if $x \in R(X)$ the injectivity of R gives a unique vector $\xi \in X$ such that $R(\xi) = x$ and we set

(i) $$T(x) = \xi$$

It means that the composed operator $T \circ R = E$, where E is the identity operator on X, and the reader can check that

(ii) $$R \circ T(x) = x \quad : x \in \mathcal{D}(T)$$

Next, the bounded operator R has a finite operator norm $||R||$ and (i) entails that

(iii) $$||x||| \leq ||R|| \cdot ||T(x)||$$

Thus, with $c = ||R||^{-1}$ one has

(iv) $$||T(x)|| \geq c \cdot ||x||| \quad : x \in \mathcal{D}(T)$$

0.2 The graph $\Gamma(T)$. It is by definition the subset of $X \times X$ given by

$$\{(x, Tx) : x \in \mathcal{D}(T)\}$$

The construction of T gives

(0.2.1) $$\Gamma(T) = \{(Rx, x) \colon x \in X\}$$

Since R is a bounded linear operator the reader can check that the right hand side is closed in $X \times X$ and then conclude that $\Gamma(T)$ is closed. Next, the inequality (iv) shows that T is injective and since

$$T(Rx)) = x \quad : x \in X$$

the range of T is equal to X.

0.3 A converse result. Let T be a densely defined operator, where $\Gamma(T)$ is closed adn the range $T(\mathcal{D}(T))$ is dense in X, and finally (iv) holds for some constant $c > 0$. Then one has the equality

(0.3.1) $$T(\mathcal{D}(T)) = X$$

For if $y \in X$ the density of the range gives a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that

(0.3.2) $$\lim_{n \to \infty} ||T(x_n) - y|| = 0$$

Now (iv) gives

$$||x_n - x_m|| \leq c^{-1} \cdot ||T(x_n) - T(x_m)||$$

and (0.3.2) entails that $\{T(x_n)\}$ is a Cauchy sequence. Since X is a Banach space it follows that $\{x_n\}$ converges to a limit vector x and since $\Gamma(T)$ is closed it follows that (x, y) belongs to the graph, i.e. $x\mathcal{D}(T)$ and $T(x) = y$ which proves (0.3.1).

0.3.3 Exercise. Conclude from the above that there exists a unique bounded operator $R \in \mathfrak{I}(X)$ such that T is the attached operator from (0.1). Hence there is a 1-1 correspondence between bounded operators R in $\mathfrak{I}(X)$ and the family of densely defined operators T with a closed graph for which (iv) above holds for some $c > 0$.

0.4 The spectrum of densely defined operators.

Let T be a densely defined linear operator with a closed graph $\Gamma(T)$. Here (iv) above is not assumed. Each complex number λ gives the densely defined operator $\lambda \cdot E - T$. We say that λ is a resolvent value of T if $\lambda \cdot E - T$ is surjective and there exists a positive constant c such that

$$||\lambda \cdot x - T(x)|| \geq c \cdot ||x||$$

The set of resolvent values is denoted by $\rho(T)$. By Exercise (0.3.3) every $\lambda \in \rho(T)$ gives a unique bounded operator $R_T(\lambda) \in \mathcal{I}(X)$ where

$$(\lambda \cdot E - T) \circ R_T(\lambda)(x) = x \quad \& \quad \mathcal{D}(T) = \mathcal{D}(\lambda \cdot E - T)$$

In particular we see that the range of $R_T(\lambda)$ is equal to $\mathcal{D}(T)$ for every resolvent value.

0.4.1 Definition. *Tbe bounded operators $\{R_T(\lambda) : \lambda \in \rho(T)\}$ are called Neumann's resolvent operators of T, and the closed complement*

$$\sigma(T) = \mathbf{C} \setminus \rho(T)$$

is called the spectrum of T.

0.5 Neumann's equation.

Let T be as in (0.4) and assume that $\rho(T) \neq \emptyset$. The result below is due to Neumann:

For each pair $\lambda \neq \mu$ in $\rho(T)$ the operators $R_T(\lambda)$ and $R_T(\mu)$ commute and

(0.5.1) $$R_T(\mu)R_T(\lambda) = \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu}$$

Proof. Notice that

$$(\mu \cdot E - T) \cdot \frac{R_T(\mu) - R_T(\lambda)}{\lambda - \mu} =$$

(i) $$\frac{E}{\lambda - \mu} - (\mu - \lambda) \cdot \frac{R_T(\lambda)}{\lambda - \mu} - (\lambda \cdot E - T) \cdot \frac{R_T(\lambda)}{\lambda - \mu} = R_T(\lambda)$$

Multiplying to the left by $R_T(\mu)$ gives (0.5.1) and proves at the same time that the resolvent operators commute.

0.6 The position of $\sigma(T)$.

Assume that $\rho(T) \neq \emptyset$. For a pair of resolvent values of T we can write Neumann's equation from (0.5.1) as:

(i) $$R_T(\lambda)(E + (\lambda - \mu)R_T(\mu)) = R_T(\mu)$$

Keeping μ fixed we conclude that $R_T(\lambda)$ exists if and only if $E + (\lambda - \mu)R_T(\mu))$ is invertible. This gives the set-theoretic equality

(0.6.1) $$\sigma(T) = \{\lambda : \frac{1}{\mu - \lambda} \in \sigma(R_T(\mu))\}$$

Hence one recovers $\sigma(T)$ via the spectrum of any fixed resolvent operator.

0.7 The Neumann series.

If $\lambda_0 \in \rho(T)$ we construct the operator valued series

(0.7.1) $$S(\zeta) = R_T(\lambda_0) + \sum_{n=1}^{\infty} (-1)^n \cdot \zeta^n \cdot R_T(\lambda_0)^{n+1}$$

It is clear that series converges in the Banach space of bounded linear operators when

(i) $$|\zeta| < \frac{1}{||R_T(\lambda_0)||}$$

Moreover we see that

(0.7.2) $$(\lambda_0 + \zeta - T) \cdot S(\zeta) = (\lambda_0 - T) \cdot S(\zeta) + \zeta S(\zeta) = E$$

where the last equality follows via the series expansion (0.7.1). Hence

$$S(\zeta) = R_T(\lambda_0 + \zeta)$$

give resolvent operators when ζ satisfy (i) above. This shows that the set $\rho(T)$ is open, and that the operator-valued function $\lambda \mapsto R_T(\lambda)$ is an analytic function of the complex variable λ in $\rho(T)$. If $\lambda \in \rho(T)$ we can pass to the limit as $\mu \to \lambda$ in Neumann's equation from (0.5.1) and conclude that the complex derivative is given by

$$\frac{d}{d\lambda}(R_T(\lambda) = -R_T^2(\lambda) \tag{0.7.3}$$

Thus, Neumann's resolvent operators satisfy a specifiic differential equation for every densely defined and closed operator T with a non-empty resolvent set.

0.7.4 Compact resolvent operators. A wellknown resuot about bounded linear operatos assers that if S is a compact operator on the Banach space X, then then $S \circ U$ and $U \circ S$ are compact for every bounded operator U. Apply this to Neumann's equation (0.5.1) and conclude that if one resolvent operator $R_T(\lambda_0)$ is compact, then all resolvent operators of T are compact.

0..8 Operational calculus.

Let T be a densely defined and closed operator on a Banach apace X where $\rho(T)$ is non-empty. To each pair (γ, f) where γ is a rectifiable Jordan arc contained in $\mathbf{C} \setminus \sigma(T)$ and $f \in C^0(\gamma)$, there exists the bounded linear operator

$$T_{(\gamma,f)} = \int_\gamma f(z) R_T(z)\, dz \tag{0.8.1}$$

The integral is calculated by a Riemann sum where the integrand has values in the Banach space of bounded linear operators on X. More precisely, let $s \mapsto z(s)$ be a parametrisation with respect to arc-length. If L is the arc-length of γ we get Riemann sums

$$\sum_{k=0}^{k=N-1} f(z(s_k)) \cdot (z(s_{k+1}) - z(s_k)) \cdot (s_{k+1} - s_k) \cdot R_T(z(s_k))$$

where $0 = s_0 < s_1 < \ldots s_N = L$ is a partition of $[0, L]$. These Riemann sums converge to a limit when $\{\max(s_{k+1} - s_k\} \to 0$ with respect to the operator norm and give the operator in (0.8.1). The triangle inequality entails that

$$||T_{(\gamma,f)}|| \leq L \cdot |f|_\gamma \cdot \max_{z \in \gamma} ||R_T(z)||$$

where $|f|_\gamma$ is the maximum norm of f on γ.

Exercise. Recall that Neumann's equation (0.5.1) implies that the operators $R_T(z_1)$ and $R_T(z_2)$ commute for all pairs $z_1, z-2$ on γ. Apply this to show that if g is another function in $C^0((\gamma)$ then the operators $T_{f,\gamma}$ and $T_{g,\gamma}$ commute. Moreover, for each $f \in C^0(\gamma)$ the reader can verify that since T has a closed graph, it follows implies that the range of $T_{f,\gamma}$ is contained in $\mathcal{D}(T)$ and one has

$$T_{f,\gamma} \circ T(x) = T \circ T_{f,\gamma}(x) \quad : x \in \mathcal{D}(T) \tag{0.8.2}$$

Next, let Ω be an open set of class $\mathcal{D}(C^1)$, i.e. $\partial\Omega$ is a finite union of closed differentiable Jordan curves. When $\partial\Omega \cap \sigma(T) = \emptyset$ we construct line integrals as in (0.8.1) for continuous functions on the boundary. Consider the algebra $\mathcal{A}(\Omega)$ of analytic functions in Ω which extend to be continuous on the closure. Each $f \in \mathcal{A}(\Omega)$ gives the operator

$$T_f = \int_{\partial\Omega} f(z) R_T(z)\, dz \tag{0.8.3}$$

0.8.4 Theorem. *The map $f \mapsto T_f$ is an algebra homomorphism from $\mathcal{A}(\Omega)$ into a commutative algebra of bounded linear operators on X denoted by $\mathcal{T}(\Omega)$.*

Proof. Let f,g be a pair in $\mathcal{A}(\Omega)$. To show that $T_{gf}=T_f\circ T_g$ we consider a slightly smaller open set $\Omega_*\subset\Omega$ which again is of class $\mathcal{D}(C^1)$ and each bounding Jordan curve of Ω_* is close to one boundary curve in $\partial\Omega$, and finally

$$(\Omega\setminus\Omega_*)\cap\sigma(T)=\emptyset$$

By Cauchy's theorem we can shift the integration to $\partial\Omega_*$ when we use g instead of f in (0.8.3). This gives

$$\text{(i)}\qquad T_g=\int_{\partial\Omega_*} g(z_*)R_T(z_*)\,dz_*$$

where we use z_* to indicate that integration takes place along $\partial\Omega_*$. Now

$$\text{(ii)}\qquad T_f\circ T_g=\iint_{\partial\Omega_*\times\partial\Omega} f(z)g(z_*)R_T(z)\circ R_T(z_*)\,dz_*dz$$

Neumann's equation (0.5.1) entails that the right hand side in (ii) becomes

$$\text{(iii)}\qquad \iint_{\partial\Omega_*\times\partial\Omega}\frac{f(z)g(z_*)R_T(z_*)}{z-z_*}\,dz_*dz+\iint_{\partial\Omega_*\times\partial\Omega}\frac{f(z)g(z_*)R_T(z)}{z-z_*}\,dz_*dz=A+B$$

Here A is evaluated by first integrating with respect to z and Cauchy's theorem gives

$$f(z_*)=\frac{1}{2\pi i}\cdot\iint_{\partial\Omega}\frac{f(z)}{z-z_*}\quad : z_*\in\partial\Omega_*\,dz$$

It follows that

$$A=\frac{1}{2\pi i}\cdot\iint_{\partial\Omega_*\times\partial\Omega} f(z_*)g(z_*)R_T(z_*)\,dz_*=T_{fg}$$

Next, B is evaluated when we first integrate with respect to z_*. Here

$$\iint_{\partial\Omega}\frac{g(z_*)}{z-z_*}\quad : z\in\partial\Omega$$

which entails that $B=0$ and Theorem 0.8.4 follows.

0.8.5 Spectral gap sets.

Let K be a compact subset of $\sigma(T)$ such that $\sigma(T)\setminus K$ is a closed set in $\mathbf{C}$. This implies that if V is an open neighborhood of K, then there exists a relatively compact subdomain $U\in\mathcal{D}(C^1)$ which contains K as a compact subset. To every such domain Ω we can apply Theorem 0.8.4. If $U_*\subset U$ for a pair of such domains we can restrict functions in $\mathcal{A}(U)$ to U_* which yields an algebra homomorphism $\mathcal{T}(U)\to\mathcal{T}(U_*)$. Next, denote by $\mathcal{O}(K)$ the algebra of germs of analytic functions on K. So each $f\in\mathcal{O}(K)$ comes from some analytic function in a domain U as above. The resulting operator $T_U(f)$ depends on the germ f only. In fact, this follows because if $f\in\mathcal{A}(U)$ and $U_*\subset U$ is a similar $\mathcal{D}(C^1)$-domain which again contains K, then Cauchy's vanishing theorem in analytic function theory is applied to $f(z)R_T(z)$ in $U\setminus\bar{U}_*$. It follows that

$$\int_{\partial U_*} f(z)R_T(z)\,dz=\int_{\partial U} f(z)R_T(z)\,dz$$

Hence there exists an algebra homorphism from $\mathcal{O}(K)$ into bounded linear operators on X whose image is denoted by $\mathcal{T}(K)$. The identity in $\mathcal{T}(K)$ is denoted by E_K and called the spectral projection operator attached to the compact set K in $\sigma(T)$. For every open set U surrounding K as above we have

$$E_K=\frac{1}{2\pi i}\cdot\int_{\partial U} z\cdot R_T(z)\,dz$$

0.8.6. The operator T_K.

Let K be a compact spectral gap set of T as in (0.8.5) and put

$$T_K = TE_K$$

This bounded linear operator is given by

$$\frac{1}{2\pi i}\cdot\int_{\partial U} z\cdot R_T(z)\,dz$$

where U is a domain as above containing K.

0.8.6.1 *Identify T_K with a densely defined operator on the space $E_K(X)$. Then one has the equality*

$$\sigma(T_K) = K$$

Proof. If λ_0 is outside K we can choose U so that λ_0 is outside $\bar{U}$ and construct the operator

$$S = \frac{1}{2\pi i}\cdot\int_{\partial U}\frac{1}{\lambda_0 - z}\cdot R_T(z)\,dz$$

The operational calculus gives

$$S(\lambda_0 E_K - T) = E_K$$

here E_K is the identity operator on $E_K(X)$ which shows that $\sigma(T_K)\subset K$.

0.8.7 Point spectra. Consider a spectral set reduced to a singleton set $\{\lambda_0\}$, i.e. λ_0 is an isolated point in $\sigma(T)$. The associated spectral projection is denoted by $E_T(\lambda_0)$ and from the above

$$E_T(\lambda_0) = \frac{1}{2\pi i}\cdot\int_{|\lambda-\lambda_0|=\epsilon} R(\lambda)\,d\lambda$$

for all sufficiently small ϵ. Here $R_T(\lambda)$ is an analytic function defined in some punctured disc $\{0 < \lambda - \lambda_0| < \delta\}$ with a Laurent expansion

$$R_T(\lambda) = \sum_{-\infty}^{\infty}(\lambda-\lambda_0)^k\cdot B_k$$

where $\{B_k\}$ are bounded linear operators obtained by residue formulas:

(i)
$$B_k = \frac{1}{2\pi i}\cdot\int_{|\lambda|=\epsilon}\frac{R_T(\lambda)}{(\lambda-\lambda_0)^{k+1}}\,d\lambda \quad : \epsilon < \delta$$

Exercise. Show that $R_T(\lambda)$ is meromorphic, i.e. that $B_k = 0$ hold when $k << 0$, if and only if there exists a constant C and some integer $M \geq 0$ such that the operator norms satisfy

(ii)
$$||R_T(\lambda)|| \leq C\cdot|\lambda-\lambda_0)^{-M}$$

Suppose now that (ii) holds and let R_T have a pole of some order $M\geq 1$ which gives an expansion

(iii)
$$R_T(\lambda) = \sum_1^M \frac{B_{-k}}{(\lambda-\lambda_0)^k} + \sum_0^{\infty}(\lambda-\lambda_0)^k\cdot B_k$$

Here $B_{-1} = E_T(\lambda_0)$ and if $M\geq 2$ the negative indexed operators satisfy

(iv)
$$B_{-k} = B_{-k}E_T(\lambda_0) \quad 2\leq k\leq M$$

In the case of a simple pole, i.e. when $M = 1$ the operational calculus gives

(v)
$$(\lambda_0 E - T)E_T(\lambda_0) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\cdot\int_{|\lambda-\lambda_0|=\epsilon}(\lambda_0-\lambda)R(\lambda)\,d\lambda = 0$$

which implies hat the range of the projection operator $E_T(\lambda_0)$ is equal to the kernel of $\lambda_0\cdot E - T$.

The case $M \geq 2$. Now one has a a non-decreasing family of subspaces

$$N_k(\lambda_0) = \{x\colon (\lambda_0 E - T)^k(x) = 0\} \quad : 1 \leq k \leq M \tag{0.8.8}$$

Let us analyzie the case when the range of $E_T(\lambda_0)$ has finite dimension. Here the operator $T(\lambda_0) = TE_T(\lambda_0)$ acts on this finite dimensional vector space and the B-matrices with negative indices can be expressed as in linear algebra via a Jordan decomposition of $T(\lambda_0)$. More precisely Jordan blocks of size > 1 may occur which occurs of the smallest positive integer m such that

$$(\lambda_0 E - T)^m(x) = \quad : x \in E_T(\lambda_0)(X)$$

is strictly larger than one. Moreover, $E - E_T(\lambda_0)$ is a projection operator and one has a direct sum decompostion

$$X = E_T(\lambda_0)(X) \oplus (E - E_T(\lambda_0))(X)$$

where $V = (E - E_T(\lambda_0))(X)$ is a closed subspace of X which is invariant under T and the reader should check that there exists some $c > 0$ such that

$$||\lambda_0 - Tx|| \geq ||x|| \quad x \in V \cap \mathcal{D}(T)$$

Remark. In applications it is often an important issue to decide when $E_T(\lambda_0)$ has a finite dimensional range for an isolated point in $\sigma(T)$. The Kakutani-Yosida theorem to be exposed in § xx is an example where this finite dimensionality will be established for certain operators T.

0.9 Adjoint operators and closed extensions.

Let T be densely defined. But for the monent we do not assune that its is closed. In the dual space X^* we have the family of vectors y for which there exists a constant $C(y)$ such that

$$|y(Tx)| \leq C(y) \cdot ||x|| \quad : x \in \mathcal{D}(T) \tag{0.9.1}$$

It is clear that the set of such y-vectors is a subspace of X^*. Moreover, when (i) holds the density of $\mathcal{D}(T)$ gives a unique vector $T^*(y)$ in X^* such that

$$y(Tx) = T^*(y)(x) \quad : x \in \mathcal{D}(T) \tag{0.9.2}$$

One refers to T^* as the adjoint operator of T whose domain of definition is denoted by $\mathcal{D}(T^*)$.

Exercise. Show that the graph of T^* is closed in $X^* \times X^*$.

Closed extensions. Let T be densely defined. There may exist closed operators S with the property that

$$\Gamma(T) \subset \Gamma(S) \tag{0.9.3}$$

When this holds we refer to S as a closed extension of T. Notice that the inclusion above is strict if and only if $\mathcal{D}(S)$ is strictly larger than $\mathcal{D}(T)$.

Exercise. Use the density of $\mathcal{D}(T)$ to show the equality

$$T^* = S^* \tag{0.9.4}$$

for every closed extension S of T.

The case when $\mathcal{D}(T^*)$ is dense. Let T be densely defined and assume that its adjoint has a dense domain of definition. In this situation the following holds:

0.9.5 Theorem. *If $\mathcal{D}(T^*)$ is dense there exists a closed operator $\widehat{T}$ whose graph is the closure of $\Gamma(T)$.*

Proof. Consider the graph $\Gamma(T)$ and let $\{x_n\}$ and $\{\xi_n\}$ be two sequences in $\mathcal{D}(T)$ which both converge to a point $p \in X$ while $T(x_n) \to y_1$ and $T(\xi_n) \to y_2$ hold for some pair y_1, y_2. We must sahow that $y_1 = y_2$. To achieve this we take some $x^* \in \mathcal{D}(T^*)$ which gives

$$x^*(y_1) = \lim x^*(Tx_n) = \lim T^*(x^*)(x_n) = T^*(x^*)(p)$$

In the same way we get $x^*(y_2) = T^*(x^*)(p)$. Now the density of $\mathcal{D}(T^*)$ gives $y_1 = y_2$ which proves that the closure of $\Gamma(T)$ is a graphic subset of $X \times X$ and therefore gives the closed operator $\widehat{T}$ for which

$$\Gamma(\widehat{T}) = \overline{\Gamma(T)}$$

0.9.6 The case when X is reflexive. Assume that X is equal to its bidual X^{**} and let T be densely defined and closed. Suppose in addition that T^* also is densely defined. Then we can construct the adjoint of T^* denoted by T^{**}. Since X is reflexive we can regard T^{**} as a closed and densely defined operator on X. If $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$ we have the vector $\widehat{x} \in X^{**}$ and

$$\widehat{x}(T^*(y)) = T^*(y)(x) = y(T(x))$$

From this it is clear that $\widehat{x} \in \mathcal{D}(T^{**})$ and one has the equality

$$T^{**}(\widehat{x}) = T(x)$$

Hence the graph of T is contained in that of T^{**}, i.e. T^{**} is a closed extension of T.

0.9.7 The spectrum of T^*. Let X and T be as in (0.9.6). Then one has the inclusion

(*) $$\rho(T) \subset \rho(T^*)$$

Proof. By translations it suffices to show that if the origin beongs to $\rho(T)$ then it also belongs to $\rho(T^*)$. So now the resolvent $R_T(0)$ exists which means that T is surjective and there is a constant $c > 0$ such that

(i) $$||x|| \leq c^{-1} \cdot ||Tx|| \quad : x \in \mathcal{D}(T)$$

Consider some $y \in \mathcal{D}(T^*)$ of unit norm. Since T is surjective we find $x \in \mathcal{D}(T)$ with $||Tx|| = 1$ and

(ii) $$|y(Tx)| \geq 1/2$$

Now

(iii) $$y(Tx) = T^*(y)(x)$$

and from (i) we have

(iv) $$||x|| \leq c^{-1} \cdot ||Tx|| = c^{-1}$$

Then (ii) and (iv) entail that

$$||T^*(y)|| \geq c/2$$

This proves that

(v) $$||T^*(y)|| \geq c/2 \cdot ||y|| \quad : y \in \mathcal{D}(T^*)$$

Hence the origin belongs to $\rho(T^*)$ if we prove that T^* has a dense range. If the density fails there exists a non-zero linear functional $\xi \in X^{**}$ such that

$$\xi(T^*(y)) = 0 \quad : y \in \mathcal{D}(T^*)$$

Since X is reflexive we have $\xi = \mathfrak{i}_X(x)$ for some vector x and obtain

$$y(Tx) = 0 \quad : y \in \mathcal{D}(T^*)$$

The density of $\mathcal{D}(T^*)$ gives $Tx = 0$ which contradicts the hypothesis that T is injective and (*) follows.

0.9.8 The case when X is a Hilbert space. In § xx we prove that when X is a Hilbert space where T and T^* as above are both closed and densely defined, then one has the equality

(**) $$\sigma(T) \subset \sigma(T^*)$$