Consider an analytic function $f(x,y)$ of the form

$$f(x,y) = y^e + q_{e-1}(x)y^{e-1} + \ldots + q_1(x)y + q_0(x)$$

where $e \geq 2$ and $\{q_\nu(x)\}$ are holographic functions in a disc $\{x| < r\}$ with $q_\nu(0) = 0$ for every ν. By the fundamental theorem of algebra we get a factorization

$$f(x,y) = \prod (y - \alpha_\nu(x))$$

The unordered e-tuple of roots give rise to multiple-valued functions in the punctured disc. Under analytic continuation they are permuted and when f is irreducible they matching each other. Follows that there exists an analytic function $A(\zeta)$ and with $x = \zeta^e$ the roots become

$$\alpha_\nu(x) = A(e^{2\pi i\nu/e} \cdot \zeta^e)$$

It follows that if $A(\zeta)$ has some order k one has

$$|\alpha_\nu(x)| = c \cdot |\zeta|^k (1 + O(|\zeta|))$$

So the absolute values are almost equal. Compare with the order of q_0 which is the product of the roots. Get

$$e \cdot k = ord(q_0) \cdot e$$

and hence we find exact formula. Complete expansion for A:

$$c_k \cdot \zeta^k + c_{k+1} \cdot \zeta^{k+1} + \ldots$$

Interest follows. roots must be separated. Sp e is not a common factor to coefficients. An optimal case occurs if k and e are relatively prime for then difference of roots like $|\zeta|^k$. and product over all distinct roots is estimated.

On complete intersections of two polynomials.

Jan-Erik Björk

1. Algebraic function fields

An algebraic function field K over $\mathbf{C}$ is an abstract field which contains $\mathbf{C}$ as a subfield with degree of trancendency equal to one and generated by a finite number of elements $k_1, \ldots, k_m$. When this holds we pick an arbitrary element $\xi \in K \setminus \mathbf{C}$, Since the complex filed is algebraically closed this yields a trancendental elnebnt and by the hypothesis the finite set of generators are algebraic over the field $\mathbf{C}(\xi)$ of rational functions in the ξ-variable. each $k \in K$ satisfies an equation

$$(*) \qquad k^m + q_{m-1}(\xi)k^{m-1} + \ldots + q_1(\xi)k + q_0(\xi) = 0$$

where the positive integer m can be chosen to be minimal. and $\{q_j(\xi))\}$ are ratiuonal functions. One refers to (*) as the minimal funvion satisfies by k. The field over $\mathbf{C}$ generated by ξ and k can then be identified wirth a vector space of dimension m over the field $\mathbf{C}(\xi)$. One can do this for each of the orighimal generators and conclude that K is a finite dimensional vector space over $\mathbf{C}(\xi)$.

Primitive elements. With ξ as above we set

$$m = \dim_{\mathbf{C}(\xi)} K$$

An elment $\eta \in K$ is primitive with respect tothe chosen trancendental elmenet if its minimal equation in (xx) has degree m whuch then entais that K is generated by the to elements $|xi$ and η. It turns out that primitive elements always exist and they are even found in a "generic way".

Elementary algebra teaches that when K is such a field and $\xi \in K \setminus \mathbf{C}$, then there exists $\eta \in K$ such that K is generated by η and the field $\mathbf{C}(\xi)$ whose elements are rational functions in ξ with complex coefficients. Moreover, η satisfies an equation

$$(*) \qquad \eta^m + r_{m-1}(\xi)\eta^{m-1} + \ldots + r_1(\xi)\eta + r_0(\xi) = 0$$

where $\{r_j(\xi)\}$ belong to $\mathbf{C}(\xi)$ and each $k \in K$ can be written as

$$(1) \qquad k = q_{m-1}(\xi)\eta^{m-1} + \ldots + q_1(\xi)\eta + q_0(\xi) = 0$$

where $\{q_j(\xi)\}$ is a unique m-tuple in $\mathbf{C}(\xi)$. So K is a vector space of dimension m over the subfield $\mathbf{C}(\xi)$. Moreover, the polynomial in the indeterminate variable t given by

$$(2) \qquad P(t,\xi) = t^m + r_{m-1}(\xi)t^{m-1} + \ldots + r_1(\xi)t + r_0(\xi)$$

is irreducible in the polynomial ring of t over the field $\mathbf{C}(\xi)$. This is expressed by saying that if $K_* = \mathbf{C}(z)$ is the standard field of rational functions in one variable, then every algebraic function field is isomorphic to a field

$$\frac{K_*[t]}{(P)}$$

where (P) denotes the principal ideal generated by an irreducible polynomial in $K_*[t]$. Here one has used that $K_*[t]$ is an euclidian ring which implies that this K_*-algebra is a unique factorisation domain and therefore gives a precise meaning in order that a polynomial $P(t)$ is irreducible.

Remark. When an algebraic function field K is given we can choose a trancendental element ξ in many ways. Once ξ is chosen, $\mathbf{C}(\xi)$ appears as a subfield and K becomes a finite dimensional vector pace over $\mathbf{C}(\xi)$. More precisely one has the equality

$$\dim_{\mathbf{C}(\xi)}(K) = \deg(\eta)$$

where η is a primitive element whose minimal equation from (*) has degree m. Let us remark that when a trancendental elment ξ is chosen in K, then there is a whole family of primitive η-generators. More precisely, every $\eta \in K$ whose minimal equation (*) has a degree which equals the dimension of K as a vector space over $\mathbf{C}(\xi)$ yields a primitive element in K with respect to the chosen trancendental ξ-element. Finally, two algebraic function fields K_1 and K_2 are isomorphic if there exists a $\mathbf{C}$-algebra isomorphism between them. With this kept in mind one has the following result predicted by Riemann and in the general version established by Weyl.

0.1 Theorem. *There is a 1-1 correspondence between the family of algebraic function fields in one variable and the family of compact Riemann surfaces.*

Remark about the proof. A pair of Riemann surfaces are identified when they are biholomorphic. Several steps are needed to prove Therorem 0.1. First Weyl's theorem shows that if X is a compact Riemann surface, then the field of meromorphic functions $\mathfrak{M}(X)$ is an algebraic function field. Next, let X_1, X_2 be a pair of biholomorphic Riemann surfaces and $\rho\colon X_1 \to X_2$ a biholomorphic mapping. Then it follows that $\mathfrak{M}(X_1) \simeq \mathfrak{M}(X_2)$. In fact, ρ yields an algebra isomorphism between these two algebraic function fields which sends $f \in \mathfrak{M}(X_2)$ to the meromorphic function on X_1 defined by

$$f^*(x_1) = f(\rho(x_1) \quad : \quad x_1 \in X_1$$

There remains to show that if X_1 and X_2 are two Riemann surfaces whose algebraic function fields $\mathfrak{M}(X_1)$ and $\mathfrak{M}(X_2)$ are isomorphic, then the Riemann surfaces X_1 and X_2 are biholomorphic. Moreover, we must show that for every algebraic function field K is isomorphic to $\mathfrak{M}(X)$ for a compact Riemann surface X.

To prove these results one considers an arbitrary algebraic function field K. A valuation map on K is an injective algebra homomorphism

$$(1) \qquad \rho\colon\ K \to \mathbf{C}\{t\}[t^{-1}]$$

where the right hand side is the field of germs of meromorphic functions at the origin with t regarded as a complex variable. In addition one requires that ρ is non-degenerated in the sense that there exists some $k \in K$ such that

$$(2) \qquad \rho(k) = t + \sum_{\nu=2}^{\infty} c_\nu \cdot t^\nu$$

i.e. this germ is holomorphic and its t-derivative is non-zero when $t = 0$. The ρ-map defines a valuation on K as follows: Each non-zero element k gives the meromorphic germ $\rho(k)$ and we find the unique integer $\rho_*(k)$ such that

$$\rho(k) = t^{\rho_*(k)} \cdot \phi(t)$$

where $\phi(t)$ is a unit in the local ring $\mathbf{C}\{t\}$, i.e its constant term is $\neq 0$. Let $\mathcal{V}(K)$ denote the family of all valuations maps on K. It turns out that $\mathcal{V}(K)$ corresponds to points in a compact Riemann surface X and that $K \simeq \mathfrak{M}(X)$ where each point $x \in X$ yields a valuation since there exists a local chart around X with a coordinate t so that every $f \in \mathfrak{M}$ has a series expansion at x expressed by a an element in $\rho_f(t) \in \mathbf{C}\{t\}[t^{-1}]$. By analyticity the map $f \to \rho_f(t)$ is injective which clarifies the1-1 correspondence between points on X and valuation maps on $\mathfrak{M}(X)$.

So the main burden is to prove the existence of an ample family of valuations on a given algebraic function field K and explain how these valuations fabricate points on a Riemann surface. This is done in the next sections using constrcutions by Puiseux from 1852, and put in a global form by Riemann a few years after.

1.1 Algebraic curves.

Recall that the polynomial ring $\mathbf{C}[x, y]$ in two variabes is a unique factorisation domain. Let $n \geq 2$ and consider an irreducible polynomial

$$(1) \qquad P(x, y) = y^n + q_1(x)y^{n-1} + \ldots + q_{n-1}(x)y + q_n(x)$$

To P corresponds the algebraic function field K whose elements are

$$(2) \qquad k = r_0(x) + r_1(x)y + \ldots + r_{n-1}(x)y^{n-1} \quad \text{where} \quad r_0, \ldots, r_{n-1} \in \mathbf{C}(x)$$

Next, we get the algebraic curve S in $\mathbf{C}^2$ defined by the zero-set $\{P = 0\}$. It has a closure in the projective space $\mathbf{P}^2$ whose homogeneous coordinates are $(\zeta_0, \zeta_1, \zeta_2)$ and points in the (x, y)-space are represented by $(1, x, y)$. The hyperplane at infinity is $\{\zeta_0 = 0\}$ and points $(x, y) \in S$ converge to this hyperplane when $|x|+|y| \to +\infty$ which yields the closure $\bar{S}$ in the compact manifold $\mathbf{P}^2$. The boundary $\partial S = \bar{S} \setminus S$ is a finite set of at most n points.

1.1.1 Example. Suppose that $\deg q_\nu \leq \nu$ hold for the q-polynomials in (1) and for each $1 \leq \nu \leq n$ we denote by c_ν the coefficient of x^ν in $q_\nu(x)$. Now

$$P^*(x,y) = y^n + c_1 x y^{n-1} + \ldots + c_{n-1} y x^{n-1} + c_n x^n$$

is a homogeneous polynomial and the fundamental theorem of algebra entails that

$$P^*(x,y) = \prod_{k=1}^{k=m} (y - \beta_k x)^{e_k}$$

where $\{\beta_k\}$ are distinct complex numbers and $e_1 + \ldots + e_m = n$.

1.1.2 Exercise. Show that ∂S consists of the m-tuple points $(0,1,\beta_1), \ldots, (0,1,\beta_m)$. The case where $\deg(q_\nu) < n$ for each ν is not excluded. Here $P^*(y) = y^n$ and ∂S is reduced to the single point $(0,1,0)$. The reason is that the conditions on the q-polynomials give a number $0 < a < 1$ and a constant C such that

$$|y| \leq C(1+|x|)^a$$

for all points on $\{P=0\}$. Then $(1,x,y) = (\frac{1}{x}, 1, \frac{y}{x})$ can only approach $(0,1,0)$ at infinity.

1.1.3 Regular points on S. Let us for a while restrict the attention to the affine curve S and consider the polynomial

$$P'_y(x,y) = n y^{n-1} + (n-1) q_1(x) y^{n-2} + \ldots + q_{n-1}(x)$$

1.1.4 The discriminant polynomial. By assumption P is irreducible in the polynomial ring $K[y]$ in the single variable y where k denotes the field $\mathbf{C}(x)$. Euclidian divisions give a unique pair $A(y), B(y)$ in $K[y]$ such that

$$A(y) \cdot P(x,y) + B(y) \cdot P'_y(x,y) = 1$$

where the degree of the y-polynomial A is at most $n-1$ and that of B at most $(n-2)$. Taking a common factor for the denominators in the K-coefficients of these two y-polynomials gives a unique monic polynomial $\delta(x)$ in $\mathbf{C}[x]$ such that

(i)
$$A_*(x,y)\dot{P}(x,y) + B_*(x,y) \cdot P'_y(x,y) = \delta(x)$$

where A_* and B_* now are polynomials in x and y. For example, one has

$$A_*(x,y) = a_{n-2}(x) y^{n-2} + \ldots a_1(x(y + a_0(x)$$

where the x-polynomials $\{a_k(x)\}$ have no common factor. We refer to $\delta(x)$ as the discriminant polynomial of P.

1.1.5 Root functions. For each fixed n the fundamental theorem of algebra yields an n-tuple of roots to the equation $P(y,x) = 0$ and we can write

(*)
$$P(y,x) = \prod_{k=1}^{k=n} (y - \alpha_k(x))$$

From (i) in (1.1.4) we see that the roots are all simple if and only if $\delta(x) \neq 0$. The zero-set $\{\delta(x) = 0\}$ is called the discriminant locus. Each point z_0 in the open and connected set $\mathbf{C} \setminus \delta^{-1}(0)$ consists of an unordered n-tuple of simple roots. As explained in § XX they give rise to germs of analytic functions of the complex variable z and extend to multi-valued analytic functions in $\mathbf{C} \setminus \delta^{-1}(0)$. By analyticity each new local branch is again a root. For example, start at some point $z_0 \in \mathbf{C} \setminus \delta^{-1}(0)$ and pick one root $\alpha_1(z_0)$ which to begin with gives an analytic function $\alpha_1(z)$ in a small open disc centered at z_0. It is now extended in the sense of Weierstrass and the multi-valued function produces a finite set of local branches at z_0. The fact that P from the start is irreducible entails that the local branches under all possible the analytic continuations of α_1

along closed curves in $\mathbf{C} \setminus \delta^{-1}(0)$ which start and finish at z_0 produce local branches of all the roots at z_0. The conclusion is that the set

$$S_* = S \setminus \delta^{-1}(0)$$

is connected and the projection $\pi(x, y) = x$ restricts to an n-sheeted covering map from S_* onto $\mathbf{C} \setminus \delta^{-1}(0)$. Moeover, since the root functions are analytic, S_* appears as a 1-dimensional complex submanifold of $\mathbf{C}^2 \setminus \delta^{-1}(0)$, and by continuity of the roots which appear in (*), the closure of S_* taken in $\mathbf{C}^2$ is equal to S.

Remark. The results above was the starting point when Riemann constructed Riemann surfaces, and later Weierstrass extended the construction to polynomials in y which may depend upon several x-variables. In this case the discrimant locus is a hypersurface in a multi-dimensional complex vector space so the properties of the algebraic hypersurface $P^{-1}(0)$ when $P = P(x_1, \ldots, x_n, y)$ is an irreducible polynomial of $n + 1$ variables with $n \geq 2$ is more involved and will not be treated here. Let us only mention that Riemann's constructions of curves were extended by Zariski to the case of surfaces, i.e. when $n = 2$. For arbitrary $n \geq 3$ a major result is the existence of desingularisation of algebraic hypersurfaces in every dimension which was established by Hironaka in a famous work from 1962 which settled a conjecture posed by Zariski in lectures at Tokyo in 1954.

2. Construction of local charts.

Let $p = (x_*, y_*)$ be a point in S. Consider the local ring $\mathcal{O} = \mathbf{C}\{x - x_*\}$ of germs of analytic functions in the complex x-variable at x_*. Now $P(x, y)$ is an element in the polynomial ring $\mathcal{O}[y]$ which has a unique factorisation

$$P(y,x) = q_*(x,y) \cdot \prod_{k=1}^{k=r} \phi_k(x,y) \tag{2.1}$$

where $q_*(x_*, y_*) \neq 0$ while $\{\phi_k\}$ are irreducible Weierstrass polynomials in y with coefficients $\mathcal{O}$. So here each ϕ_k is of the form

$$\phi_k(y,x) = y^{e_k} + \rho_{1,k}(x) \cdot y^{e_k - 1} + \ldots + \rho_{e_k,k}(x) \tag{2.2}$$

where the ρ-functions belong to $\mathcal{O}$ and vanish at $x = x_*$, and in a neighborhood of $p = (x_*, y_*)$ the curve S is defined by the union of the zero sets $\{\phi_\nu = 0\}$.

2.3 Puiseux charts. Fix one ϕ-polynomial say ϕ_1. In a small punctured disc centered at x_* the y-polynomial $\phi_1(x, y)$ has e_1 many simple zeros which occur among roots of $P(x, y)$. As explained in Chapter 4 from my notes in analytic function theory, we can introduce a new complex variable ζ and find an analytic function $A_1(\zeta)$ in a disc of some radius $r_1 > 0$ centered at $\zeta = 0$ such that

$$q_1(x_* + \zeta^{e_1}, y_* + A_1(\zeta) = 0 \quad : |\zeta| < r_1 \tag{2.3.1}$$

Moreover, since q_1 was irreducible the Taylor series

$$A_1(\zeta) = \sum_{\nu=1}^{\infty} a_\nu \zeta^\nu$$

is such that the principal ideal in $\mathbf{Z}$ generated by those integers for which $a_\nu \neq 0$ does not contain any prime divisor of e_1. This entails that the map

$$\zeta \to (x_* + \zeta^{e_1}, y_* + A_1(\zeta)) \tag{2.3.2}$$

is bijective, i.e. the open ζ-disc can be identified with a subset of the given affine curve S where $\zeta = 0$ is mapped to $p = (x_*, y_*)$. In this subset of S we can wirte

$$x = x_* + \zeta^{e_1} \quad : \ y = y_* + A_1(\zeta) \tag{2.3.3}$$

This means that one can take ζ as a local coordinate and the image of the ζ-disc constitues a chart in the Riemann surface X attached to the curve S. One refers to a Puiseux chart. They can be constructed for each ϕ-function which appears in (2.1).

Remark. Conversely, let $A(\zeta)$ be an analytic germ of the ζ-variable with a Taylor series

$$A(\zeta) = \sum_{\nu=1}^{\infty} a_\nu \zeta^\nu$$

which converges in a disc $\{\zeta| < \delta\}$. Let $e \geq 2$ be an integer and assume that then integers ν for which $a_\nu \neq 0$ have no prime divisor in common with e. Set $x = \zeta^e$ and consider the map

$$\zeta \mapsto (\zeta^e, A(\zeta))$$

It sends $\zeta = 0$ to the origin in the (x, y)-space. In a small puntured disc centered at $\zeta = 0$.

Exercise. Show that the map above is injectivin a sufficently small punctured disc $\{0 < |\delta| < \delta_*\}$. Take as an example $e = 3$ and consider the case

$$A(\zeta) = \zeta^m(1 + c\zeta^k)$$

where $c \neq 0$ is a constant, m and k are both ≥ 2 and at least one of the two integers m and k do not have 3 as a prime factor.

Warning. If $A(\zeta)$ as above is an arbitraty holomorphic germ it is not always true that it comes from an equation in (2.3.1). Following the original studies by Heine around 1850, one says that the germ $A(\zeta)$ is of algebraic type if there exist polynomials $\{q_\nu(zeta)\}$ such that

$$q_m(\zeta)A(\zeta)^m + \ldots q_1(\zeta)A(\zeta) + q_0(\zeta) = 0$$

holds in a small disc. The questin arises if one can find conditions on the given Taylor coefficients $\{c_n\}$ in order that such an equation exists.

2.4 Riemann's construction.

To obtain the Riemann surface X associated to the curve S one must separate the Puiseux charts above. Riemann regarded these charts as disjoint. So if $r \geq 2$ then the Riemann surface X contains r *distinct* points above (x_*, y_*). In this way one gets a complex 1-dimensional manifold X and a map $\rho\colon X \to S$ which is bijective except for those points $p = (x_*, y_*)$ in S where more than one irreducible ϕ-function appears in (2.1) and local complex analytic cooordinates are found via the Puiseux series expansions above.

2.5 Example. Consider the irreducible polynomial

$$P(x,y) = y^4 - x^2(x+1)$$

At the point $p = (0,0)$ on t $S = P^{-1}(0)$ we get a factorisation

$$P = \phi_1 \cdot \phi_2$$

where we choose a local branch of $\sqrt{1+x}$ so that

$$\phi_1(x,y) = y^2 - x \cdot \sqrt{1+x} \quad : \phi_2(x,y) = y^2 + x \cdot \sqrt{1+x}$$

This gives two distinct Puiseux-Riemann charts. So the map $\rho\colon X \to S$ has an inverse fiber above the origin which contains two points. In our special case we notice that y serves as a local coordinate in each of these charts. Passing to the Riemann surfsce it means that y has a simple zero in each of the two Puiseux-Riemann charts. In addition one finds that y has a simple zero above $x = 1$.

2.6 The passage to infinity. There remains to construct charts around points which belong to ∂S. Consider the case when the irreducible polynomial $P(x,y)$ has the form

$$P(x,y) = y^m + q_1(x)y^{m-1} + \ldots + q_{m-1}(x)y + q_m(x)$$

where $m \geq 2$ and

$$\deg q_\nu \leq \nu$$

hold for every ν. It is clear that there is a constant C such that the absolute values of the roots satisfy

$$|\alpha(x)| \leq C(1 + |x|)$$

Next, each q_ν is $c_\nu x^\nu + \rho_\nu(x)$ where ρ_ν has degree $< \nu$ and we obtain the homogeneous polynomial

$$P^*(x,y) = y^m + c_{m-1}xy^{m-1} + \ldots + c_1 x^{m-1}y + c_0$$

which by the fundamental theorem of algebra can be written as

$$P^*(x,y) = \prod (y - a_\nu x)$$

where $\{a_\nu\}$ is an m-tuple of complex numbers. Suppose for example that $a_1 \neq 0$. Now one expects that if $|x|$ is large, then there exists a root $\alpha_1(x)$ which is asymptotically close to $a_1 x$ in the sense that

$$|\alpha_1(x) - a_1 x| \leq C|x|^\gamma$$

for some $\gamma < 1$ and a constant C. This is indeed true and to establish the result one perform a change of variables. Namely, $\mathbf{C}^2$ is identified with an open subset of $\mathbf{P}^2$ whose points are $(1, x, y)$ and when $x \neq 0$ they can be written as

$$(\frac{1}{x}, 1, \frac{y}{x})$$

With $\xi = \frac{1}{x}$ and $\eta = \frac{y}{x}$ we have coordinates in $\mathbf{P}^2$ outside the set of points where $x = 0$. Now

$$x^{-m}P(y,x) = \eta^m + \rho_1(\xi)\eta^{m-1} + \ldots + \rho_{m-1}(\xi)\eta + \rho_m(\xi)$$

where

$$\rho_\nu(\xi) = \xi^\nu \cdot q_\nu(1/\xi)$$

are polynomials in ξ. Here we seek η-rooots while $\xi \to 0$. Suppose for example that a_1 above is such that $a_\nu \neq a_1$ for the remaining a-numbers. This corresponds to the condition that $a-1$ is a simp,e root of the η-polynomial

$$q_*(\eta) = \eta^m + \rho_1(0)\eta^{m-1} + \ldots + \rho_{m-1}(0)\eta + \rho_m(0)$$

With $|eta = a_1 + s$ one is led to consider the function

$$\phi(\xi, s) = (a_1 + s)^m + \rho_1(\xi)(a_1 + s)^{m-1} + \ldots + \rho_{m-1}(\xi)(a_1 + s) + \rho_m(\xi)$$

which now has a simple zero $s = 0$ when $\xi = 0$. it follows that when $|xi$ is small then we find a unique simple root $s = s(\xi)$ where $s(\xi)$ is close to zero and by basic analytic function theory we get a series expansion

$$s(\xi) = c_1\xi + c_2\xi^2 + \ldots$$

It means that we get points in S defined for $\xi \neq 0$ by

$$(\xi, 1, a_1 + s(\xi)) = (1, \xi^{-1}, a_1/\xi + s(\xi)/\xi)$$

Retruring t the x-coordinates it means that we recover points of the form

$$(x, a_1x + xs_1(1/x))$$

Here

$$\lim_{x \to \infty} xs_1(1/x)) = c_1$$

So the asymptotic root $\alpha_1(x)$ deviattes from a_1x by a fixed constant as $|x| \to \infty$, i.e. we can even take $\gamma = 0$ in (xx).

The case of multie roots. Suppose that $a-1$ is a double root of P^*. In this case $s \mapsto \phi(0, s)$ has a double root at $s = 0$ and this time the pair of roots which arise for small non-zero $|xi$ have absolute vaölues

$$|s(\xi)| \leq C \cdot |\xi|^{1/2}$$

and returing to the affine (x, y)-coordinates we find a constant c such that

$$|x \cdot s(1/x)| \leq C \cdot |x|^{1/2}$$

and this time we can take $\gamma = 1/2$ in (xx). The case of a multple zero of P^* of arbitrary order is treated in a similar fashion. Let us remark that one always can take

$$\gamma = 1/m$$

to achieve asymptotic deviations from the "true roots" as $|x| \to \infty$ while one moves along the lines $y = a_\nu x$.

Exampes.

Consider the polynomial

$$P(x, y) = y^6 - x^3 - 1$$

When $|x|$ is large we have $|y| \simeq |x|^{1/2}$ which means that the x-coordinate tends faster to infinity than the y-coordinate. Here ∂S is reduced to the single point $p^* = (0, 1, 0)$. In $\mathbf{P}^2$ we have local coorfionstes (ζ, η) around p^* which corresponds to points (ζ, η). When $\zeta \neq 0$ we are outside the hyplerplane at infinity and have

$$x = \frac{1}{\zeta} \quad : y = \frac{\eta}{\zeta}$$

The equation $P = 0$ means that

$$\zeta^{-3} = \frac{\eta^6}{\zeta^6} - 1 \implies \zeta^3 + \zeta^6 = \eta^6$$

Exercise. Show that this gives three Puiseux-Riemann charts and in each chart we can take η as a local coordinate and that x regarded as a meromrphic function on the compact Riemann surface has a double pole in each of these charts. At the same time we notice that in the finite affine part the x-fuynction has six simple zeros which appear when y solves the equation $y^6 = 1$. So the number of poles counted with multiplicity is equal to the number of zeros of x as it should be. Show by similar ca.cukations that y has simple poles on the three Puiseux charts. In the affine part y has three simple zeros where the x-coordiantes are determined by the equation $x^3 = 1$. Let us then consider the function

$$\phi = \frac{y^2}{x}$$

From the above it is holomorphic and $\neq 0$ in the Puiseux charts. Passing to the affine part we notice that ϕ gets sim+le poles when $x = 0$ which occurs for 6 many y-values, i..e the number of poles is six. Zeros occur when x is a third root of unity. At these points y serves as a local coordinate so that ϕ has three many double zeros.

The 1-form $\omega = \frac{dy}{x}$. Since x has a double pole and y a simple pole in the charts around p^* we see that ω is holomorphic in these charts and also $\neq 0$. Next, in the finite part S we notice that $dy = 0$ can only occur when $x = 0$ an this occurs at six points $\{j_\nu, 0)\}$ where $j_\nu^6 = 1$. At each of these points$x$ is a local coordinste and y has a zero of order three. Hence the 1-form dy has a zero of order teo. We conclude that ω has six simple zeros and the Hurwitz-Riemann formula implies that X has genus four.

2.6.1 The curve $y^3 = x(x-1)^2$. Here $|y| \simeq |x|$ and this time ∂S contains three points:

$$p_k^* = (0, 1, e^{2\pi ik/3}) \quad : \; k = 0, 1, 2$$

We leave it to the reader to verify that around each p_k^* we get a single Puiseux-Riemann chart where both x and y have simple poles. Next, in the finite part x has a triple zero at $(0, 0)$ where y serves as a local coorfinste. At $(1, 0)$ we have a cusp-like sungularity where the Pusiuex-Riemann chart has a local coordinate ζ with

$$x = 1 + \zeta^3 \quad : \quad y = A(\zeta)$$

and $A(\zeta)$ has a zero of order two. Let us then consider the 1-form

$$\omega = \frac{dx}{y^2}$$

From the above it is holomorhic and $\neq 0$ at the p^*-points. At $(0, 0)$ x has a triple zero so dx has a double zero and since y has a simple zero we conclude that ω is holomorphic and $\neq 0$ at $(0, 0)$ At $(1, 0)$ the reader may verify that ω has a double pole. The result is that the divisor D for which $\mathcal{O}_D \cdot \omega = \Omega$ has degree -2 and it follows from the Riemann-Hurwitz formula that the genus of X is zero. Set

$$f = \frac{x-1}{y}$$

Then f is a meromorphic function and from the above we see that it is holomorphic and $\neq 0$ at the p^*-points while it has a simple pole at $(0, 0)$ and a simple zero at $(1, 0)$. Therefore $\mathfrak{M}(X)$ is reduced to the field $\mathbf{C}(f)$ which means that both x and y can be expressed in this field. That this holds can of course be verified directly. The reader may for example show that

$$x = \frac{1}{1 - f^3}$$

Remark. The example above illustrates that even if one may be "lucky" to discover the existence of a function like f, it is more systematic to proceed with a construction of charts in X and eventually discover f via the positions of poles and zeros of y and $x - 1$.

3. Intersection numbers.

A pair of different irreducible polynomials $P(x, y)$ and $Q(x, y)$ produce, after taking the projective closure of their affine zero-sets in $\mathbf{C}^2$, a pair of projective curves $\{P = 0\}$ and $\{Q = 0\}$ in $\mathbf{P}^2$. We are going to assign an intersection number which takes into the account eventual multiplicities. A "nice case" occurs if the points of intersection appear in the regular parts of the two curves and the gradient vectors of P and Q are linearly independent at every such point. Then one refers to a simple transversal intersection and the intersection number is the number of these transversal intersection points. In general a procedure to assign an integer to the pair P and Q is the following: Let X be the Riemann surface associated to the projective curve defined by P. Now Q is a meromorphic function on X and we can count its number of zeros with multiplicites which yields an integer denoted by

$$\mathbf{i}(P;Q)$$

Reversing the role, we start from the Riemann surface Y associated with $\{Q = 0\}$ and count the number of zeros for the meromorphic P-function on Y, which gives the integer $\mathbf{i}(Q; P)$. It turns out that one has the equality

$$\mathbf{i}(P;Q) = \mathbf{i}(P;Q) \tag{*}$$

The proof of (*) using Jacobi's residue for the pair P and Q is given in § 4. For the moment we admit (*). Keeping P fixed, the integer $\mathbf{i}(P;Q)$ is the degree of the meromorphic function Q on X. This entails that the intersection number is unchanged when Q is replaced by $Q - \beta$ for a complex number as long as $Q - \beta$ is not reduced to a constant function on X. A similar invariance hold when we repace P by $P - \alpha$. Hence intersection numbers enjoy a robust property in the sense that

$$\mathbf{i}(P;Q) = \mathbf{i}(P - \alpha; Q - \beta) \tag{3.1}$$

hold for every pair of complex numbers. In addition to (*) one has the result below which essentially goes back Abel and Jacobi, but in full generality was established by Riemann and later explained in a pure algebraic context by Brill and M. Noether (father of the eminent mathematician Emmy Noether).

3.1 Theorem. *The intersection number in (*) is equal to the product of the degrees of the two polynomials.*

We prove this in § 4. But let us give the proof in a special case. Suppose that P has degree m and is of the form

$$P(x, y) = P_m(x, y) + P_*(x, y)$$

where P_* has degree $\leq m - 1$ and P_m is homogenous of degree m given as a product

$$P_m(x, y) = \prod_{\nu=1}^{\nu=m} (y - \alpha_\nu x)$$

where we assume that $\{\alpha_\nu\}$ is an m-tuple of distinct complex numbers. Let Q be of degree n and given as

$$Q(x, y) = Q_n(x, y) + Q_*(x, y) \quad : \ Q_n(x, y) = \prod_{j=1}^{j=n} (y - \beta_j x)$$

while Q_* has degree $\leq n - 1$. Assume that

$$\alpha_\nu \neq \beta_j \tag{i}$$

hold for all pairs ν, j. The construction of Puiseuex charts above $x = \infty$ on the Riemann surface X attached to P shows that one has an m-tuple of points

$$p_\nu = (0, 1, \alpha_\nu)$$

above $x = \infty$. Now (i) entails that the meromorphic function Q on X has poles at each of these points of order n. So the sum of poles counted with multiplicities is equal to nm. By general facts about meromorphic functons on a compact Riemann surface it follows that the number of zeros counted with multiplicities is equal to nm and the equality in Theorem 3.1 follows. Next follow some exampes which illustrate Theorem 3.1.

3.2 Example. Let $P(x,y) = x^2 + a - y$ and $Q(x,y) = y^2 - b - x^3$ where both where a and b are complex numbers. If S is the closure of $\{P = 0\}$ taken in $\mathbf{P}^2$ then ∂S is reduced to the single point $(0,0,1)$ while the projective closure of $\{Q = 0\}$ consists of $(0,1,0)$. Hence the curves only intersect in $\mathbf{C}^2$. On the Riemann surface X attached to $\{P = 0\}$ we see that the meromorphic Q-function becomes

$$(x^2 + a)^2 - b - x^3$$

The number of zeros is 4 for all pairs a, b. In the case $a = b = 0$ we get $x^3(x-1)$ which means that a zero of order three occurs at $(0,0)$ and geometrically one verifies that the two curves $\{P = 0\}$ and $\{Q = 0\}$ do not intersect transversally at the origin but have a contact of order three while a transveral intersection occurs when $x = 1$, i.e at the point $(1, 1-a)$ on X.

$$x^2 - a - y = 0 \quad : \quad y^2 - b - x^3 = 0$$

This gives

$$(x^2 + a)^2 = b + x^3$$

Let us reverse the role and with $b = 0$ we consider the Riemann surface Y attached to $\{y^2 - x^3\}$. At the origin we get the local coordinate ζ with

$$x = \zeta^2 \quad : \quad y = \zeta^3$$

Here $P = x^2 - y = \zeta^4 - \zeta^3$ has a zero of order three and at the point $(1,1)$ on Y one verifies that P has a simple zero, i.e. the total number of zeros of the meromorphic function P on Y is equal to four as predicted by Jacobi.

3.3 Example. Let $P(x,y) = y^2 - x^2 - 1$ while $Q(x,y) = y^2 - 2x^2 + L(x,y)$ where $L(x,y)$ is some linear polynomial. Here $\{P = 0\}$ and we notice that its boundary at infinity is reduced to the points $(0,1,1)$ and $(0,1-1)$ while those of $\{Q = 0\}$ are $(0,1,\sqrt{2})$ and $(0,1,-\sqrt{2})$ To compute the number of zeros of Q on the Riemann surface X we can equally well count the number of poles. At the two points $p_1^* = (0,1,1)$ and $p_2^* = (0,1,-1)$ we notice that the meromorphic function Q has poles of order two and it follows that the number of zeros counted with multiplicities is equal to four.

3.4 Example. Let P be as above but this time $Q(x,y) = (y-x)(y-\alpha x) + L(x,y)$ where α differs from one and -1. Again Q has a double pole at $(0,1,-1)$ but at $(0,1,1)$ we must analyze the pole in more detail. With ζ as a local coorniate on X at $(0,1,1)$ we have

$$x = \zeta^{-1} \quad : y = \zeta^{-1} \cdot \sqrt{1 + \zeta^2}$$

So with $L(x,y) = ax + by + c$ we get

$$Q = (1 - \sqrt{1+\zeta^2})\zeta^{-1} \cdot \zeta^{-1}(\sqrt{1+\zeta^2} - \alpha) + a\zeta^{-1} + b\zeta^{-1} \cdot \sqrt{1+\zeta^2} + c$$

It follows that the coefficient of ζ^{-1} becomes

(i) $$(1 - \alpha) + a + b$$

So if this number is $\neq 0$ then Q has a simple pole at $(0,1,1)$ and if it is zero no pole at all. This, if (i) is $\neq 0$ then the number of zeros is three and if (i)=0 then Q has two zeros on X.

3.5 Exercise. Find the equation which determines the three zeros of Q when (i) is $\neq 0$ and analyze under which conditions on the numbers α, a, b, c we get three transversal intersection points. Since the elimination to achieve these equations is rather cumbersome the material in the next section illustrates the efficiency of Jacobi's counting method for the number of intersection points.

4. Jacobi's residue

Introduction. The pioneering work by Jacobi was devotoed to the case of two complex variabes. An extensions to higher dimensions was given by Weil in the article *L'Integrale de Cauchy et les fonctions des plusieurs variables.* Here we follow Weil's methods applied to the case of a pair of polynomials in two variables which form a complete intersection. The subsequent material in gives in particular Theorem 3.1. In § 4.B we include extra material which goes beyond our present study of Riemann surfaces. It has been inserted since it illustrates calculus in several complex variables and show how currents appear in a natural context.

A. The construction of residues.

Let $P(x,y)$ and $Q(x,y)$ be a pair of polynomials. We do not assume that they are irreducible and they may even have multiple factors. But we suppose that they have no common factor in the unique factorisation domain $\mathbf{C}[x,y]$. This entails that the common zero set $\{P=0\}\cap\{Q=0\}$ is a finite subset of $\mathbf{C}^2$ and one says that the pair (P,Q) is a complete intersection. For a while we shall work close to the origin and choose some $r>0$ such that

$$\text{(A.1)} \qquad \min_{(x,y)\in B(r)} |P(x,y)|^2+|Q(x,y)|^2=\rho>0$$

where $B(r)=\{|x|^2+|y|^2\leq r^2\}$ is a closed ball centered at the origin and

$$\{P=0\}\cap\{Q=0\}\cap B(r)=\{(0,0)\}$$

With ρ and r kept fixed we consider pairs α,β such that $|\alpha|^2+|\beta|^2<\rho$ and in the common zero set $\{P=\alpha\}\cap\{Q=\beta\}$ we only pick points (x,y) which belong to $B(r)$. Recall from calculus that the real-analytic function $|P|^2$ only has a discrete set of critical values which entails that there exists some $\epsilon^*>0$ such that the real hypersurfaces $\{|P|^2=\epsilon\}$ are non-singular for every $0<\epsilon<\epsilon^*$. One can therefore perform integrals on these. With $x=u+iv$ and $y=\xi+i\eta$ we identify the 2-dimensional complex (x,y)-space with the 4-dimensional real space where (u,v,ξ,η) are coordinates. Calculus teaches how to integrate differential 3-forms ψ over the smooth hypersurfaces $\{|P|^2=\epsilon\}$ which are embeeded into the (u,v,ξ,η) and hence oriented in a natural way. If $0<\delta_*<\delta^*$ we set

$$\square(\epsilon;\delta_*,\delta^*)=\{|P|^2=\epsilon\}\cap\{\delta_*<|Q|^2=\delta^*\}$$

Stokes Theorem gives:

$$\text{(A.2)} \qquad \iiint_{\square(\epsilon;\delta_*,\delta^*)} d\phi=\iint_{\partial\square(\epsilon;\delta_*,\delta^*)}\phi$$

for every test-form ϕ of degree two. When $g(x,y)$ is a polynomial we apply this starting from the 2-form

$$\text{(A.3)} \qquad \phi=\frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}$$

Notice that ϕ is d-closed since we already have occupied the holomorphic 1-forms dx and dy while the rational function $\frac{g}{P\cdot Q}$ is holomorphic in a neighborhood of $\square(\epsilon;\delta_*,\delta^*)$. Hence (A.2) gives the equality

$$\text{(A.3)} \qquad \iint_{\sigma(\epsilon;\delta_*)}\frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}=\iint_{\sigma(\epsilon;\delta)}\frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}$$

where

$$\text{(A.4)} \qquad \sigma(\epsilon,\delta)=\{|P|^2=\epsilon\}\cap\{|Q|^2=\delta\}\quad:\ \epsilon,\delta>0$$

To be precise this is okay provided that the pair (ϵ,δ) are sufficiently small so that integration only takes place over small compact sets close to the origin in $\mathbf{C}^2$. One refers to the family $\{\sigma(\epsilon,\delta)\}$ as integration chains of degree two. In a similar fashion we can make a variation of ϵ and arrive at the following:

A.5. Proposition. *There exists a pair of positive numbers a, b such that the integrals*

$$\iint_{\sigma(\epsilon;\delta)} \frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)} \quad : 0<\epsilon<a:\ 0<\delta<b \tag{A.5.1}$$

are independent of ϵ, δ as long as the 2-chain $\sigma(\epsilon;\delta)$ stays in the ball $B(r)$.

A.6. Jacobi's residue. The common value in (A.5.1) is denoted by $\mathfrak{res}_{P,Q}(g)$ and called the Jacobi residue of g with respect to P and Q. Notice that Jacobi's residue depends on the ordering of P and Q because we started from the oriented real hypersurface $\{|P|^2=\epsilon\}$ which induces a positive orientation on the σ-chains which determines the sign of the integrals. If the role is changed so that we start with a hypersuface $|Q|^2=\delta$ then

$$\mathfrak{res}_{Q,P}(g) = -\mathfrak{res}_{P,Q}(g) \tag{A.6.1}$$

A.7 A continuity property. Keeping ϵ, δ fixed it is clear that

$$\lim_{(\alpha,\beta)\to(0,0} \iint_{\sigma(\epsilon;\delta)} \frac{g(x,y)\cdot dx\wedge dy}{(P(x,y)-\alpha)\cdot(Q(x,y)-\beta)} = \iint_{\sigma(\epsilon;\delta)} \frac{g(x,y)\cdot dx\wedge dy}{P(x,y)\cdot Q(x,y)}$$

This entails that

$$\lim_{(\alpha,\beta)\to(0,0} \mathfrak{res}_{P-\alpha,Q-\beta}(g) = \mathfrak{res}_{P,Q}(g) \tag{A.7.1}$$

A.8. The Jacobian.

For a pair P, Q as above we set

$$\mathcal{J}(x,y) = P'_x\cdot Q'_y - P'_y\cdot Q'_x \tag{A.8.1}$$

As explained in § XX the polynomial $\mathcal{J}$ is not identically zero.

Admissable pairs. A pair of complex numbers α and β which are close to zero is admissable if

$$\{P=\alpha\}\cap\{Q=\beta\}\cap\mathcal{J}^{-1}(0)=\emptyset$$

Exercise. Show that the family of non-admissable pairs is finite. A hint is that $\mathcal{J}$ is unchanged when the pair P, Q is replaced by $P-\alpha$ and $Q-\beta$.

A.8.2 Jacobi's residue formula. Let α, β be a pair of small complex numbers such that $\mathcal{J}\neq 0$ at the common zeros of $P-\alpha$ and $Q-\beta$ which are close to the origin in $\mathbf{C}^2$.

Exercise. Show by repeated use of Cauchy's residue formula that one has the equality

$$\mathfrak{res}_{P-\alpha,Q-\beta}(g) = (2\pi i)^2\cdot\sum\frac{g(p_k)}{\mathcal{J}(p_k)}$$

where the sum extends over the distinct points in $\{P=\alpha\}\cap\{Q=\beta\}$. which belong to $B(r)$.

A.8.3 A special case. Above we can take $g=\mathcal{J}$ in which case (2.1) is $-4\pi^2$ times an integer. The continuity in (A.7) shows that this integer is constant as α, β varies in the set of admissable pairs. Passing to the limit it follows that

$$\mathfrak{res}_{P,Q}(\mathcal{J}) = K$$

where K is the set of points when $\{P-\alpha\}$ and $\{Q-\beta\}$ have transversal intersections. The absolute value of K is called Jacobi's local intersection number and is denoted by $\mathbf{J}(P,Q)$.

B. Further results.

Above we defined the integers $\mathfrak{res}_{P,Q}(g)$ when g is a polynomial. Keeping P and Q fixed this gives an additive map

$$g\mapsto\mathfrak{res}_{P,Q}(g) \tag{*}$$

The kernel in (*) can be described in an algebraic fashion. Namely, let $\mathcal{O}_2 = \mathbf{C}\{x, y\}$ be the local ring of convergent power series in two variables. Thus, the elements are germs of analytic functions in x and y. Now P and Q are elements in $\mathcal{O}_2$ and generate an ideal denoted by (P, Q). Set

$$(**) \qquad \mathcal{A} = \frac{\mathcal{O}_2}{(P, Q)}$$

The assumption that the origin is an isolated point in the common seros of P and Q entails that the ideal (P, Q) contains a sufficiently high power of the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_2$. It follows that $\mathcal{A}$ is a local and finite dimensional complex algebra. In commutative algebra one refers to $\mathcal{A}$ as a local artinian ring. If g is a polynomial which belongs to the ideal (P, Q) it follows easily from Jacobi's residue formula in § 4 that $\mathfrak{res}_{P,Q}(g) = 0$. Less obvious is the following:

B.1 Theorem. *A polynomial g belongs to the ideal (P, Q) in $\mathcal{O}_2$ if and only if*

$$\mathfrak{res}_{P,Q}(h \cdot g) = 0 \quad \text{hold for all polynomials } h$$

B.2 Noetherian operators. Theorem B.1 entails that there exists a $\mathbf{C}$-linear form on the finite dimensional vector space $\mathcal{A}$ defined by

$$\bar{g} \mapsto \mathfrak{res}_{P,Q}(g)$$

where $\bar{g}$ is the image in $\mathcal{A}$ of a polynomial g. This linear functional can be expressed by a unique differential operator with constant coefficients. More precisely, we have the polynomial ring $\mathbf{C}[\partial_x, \partial_y]$ of differential operators with constant coefficients where ∂_x and ∂_y are the holomorphic first order operators defining partial derivatives with respect to x and y. Then there exists a unique differential operator $\mathcal{N}(\partial_x \partial_y) \in \mathbf{C}[\partial_x, \partial_y]$ such that

$$\mathfrak{res}_{P,Q}(g) = \mathcal{N}(\partial_x, \partial_y)(g)(0)$$

for every polynomial $g(x, y)$. Thus, in the right hand side one evaluates the polynomial $\mathcal{N}(g)$ at the origin. One refers to $\mathcal{N}$ as the noetherian operator attached to the pair P, Q. It has the special property that $\mathcal{N}(\phi) = 0$ for every ϕ in the ideal (P, Q). To avoid possible confusion we remark that these differential operators were introduced by Max Noether, i.e. not by his famous daughter Emmy whose name is attributed to the notion of noetherian rings as well many other deep results in algebra.

B.2.1 The construction of Noetherian operators. They are obtained via the localised Weyl algebra $A_2(*)$ whose elements are differential operators in whose coefficients are rational functions with no pole at the origin. The crucial result is that if $\mathbf{C}[x, y]$ is identified with zero-order differential operators then the right ideal in $A_2(*)$ generated by P and Q yields a left module

$$\frac{A_2(*)}{A_2(*) \cdot P + A_2(*) \cdot Q}$$

which is isomorphic to m copies of the simple left A_2-module

$$\frac{A_2(*)}{A_2(*) \cdot x + A_2(*) \cdot y}$$

and m is the integer which gives the Jordan-Hölder length of the artinian local ring $\mathcal{A}$. We shall not enter a discussion about this and remark only that the result above belongs to basic material in $\mathcal{D}$-module theory. Moreover, one can assign Noetherian operators in higher dimension. The following result was proved by Palmodov in 1966, and fir a further account including a more algebraic construction I refer to my notes on residue calculus from 1996. One starts with a commutative field K of characteristic zero and the polynomial ring $K[x_1, \ldots, x_n]$ in n variables where n is some positive integer. Let $\mathfrak{q}$ be a primary ideal and $\mathfrak{p} = \sqrt{\mathfrak{q}}$ its prime radical. A differential operator Q in the Weyl algebra $A_n(K)$ is noetherian with respect to the primary ideal if one has the inclusion

$$Q(\mathfrak{q}) \subset \mathfrak{p}$$

Now one proves that there exists a finite set of noetherian operators $Q_1, \ldots, Q_e$ which determine when a polynomial q belongs to the primary ideal in the sense that

$$Q_\nu(q) \in \mathfrak{p} \implies q \in \mathfrak{q}$$

We remark that one can construct such a family of noetherian operators in a canonical fashion and the minimal number for which the implication above holds is the multiplicity of the primary ideal $\mathfrak{q}$ with respect to its prime radical.

B.3 The Gorenstein property. The local algebra $\mathcal{A}$ from (**) is special. Namely it is a local Gorenstein ring which means that the socle consists of elements $a \in \mathcal{A}$ which are annihilated by the maximal ideal $\mathfrak{m}$ in $\mathcal{O}_2$ is a 1-dimensional complex vector space. This is easily be proved via a diagram chasing in homological algebra where the assumption that the pair P, Q is a complete intersection gives an exact Koszul complex. A notable fact is that the image of $\mathcal{J}$ in $\mathcal{A}$ generates the 1-dimensional socle. This is a consequence of the following vanishing property of residue integrals:

$$\mathfrak{res}(g \cdot \mathcal{J}) = 0 \quad : \forall\, g \in \mathfrak{m}$$

The reader may notice that this is an immediate consequence of Jacobi's residue formula in § A.4.

B.4 The trace map. Let us introduce two new complex variables w and u. In the 4-dimensional complex (x, y, w, u)-space one has the non-singular analytic surface defined by the equation

$$S = \{w = P(x, y)\} \cap \{u = Q(x, y)\}$$

Let $g(x, y)$ be a polynomial. For every test-form $\psi^{0,2}$ of bi-degree $(0, 2)$ in the (x, y, w, u)-space we set

$$\iint_S g(x, y) \cdot dx \wedge \wedge \psi^{0,2}$$

This gives a current in $\mathbf{C}^4$ denoted by $g \cdot \square_S$ Consider the projection $\pi(x, y, w, u) = (w, u)$. The hypothesis that P and Q is a complete intersection entails that π restricts to a proper map on S and hence there exists a direct image current defined by

$$\phi^{0,2} \mapsto \iint_S g(x, y) \cdot dx \wedge \wedge \pi^*(\phi^{0,2}) \tag{1}$$

where $\pi^*(\phi^{0,2})$ is the pull-back of the test-form $\phi^{0,2}$ in the (w, u)-space. On S one has the equality

$$\pi^*(dw \wedge du) = \mathcal{J} \cdot dx \wedge dy$$

Next, in the (w, u)-space there exists the set of admissable points (w, u) for which $\mathcal{J}(x, y) \neq 0$ for all (x, y) with $P(x, y) = w$ and $Q(x, y) = u$. We notice that this is the same as the image set $\pi(S \cap \mathcal{J}^{-1}(0)$. Here $S \cap \mathcal{J}^{-1}(0)$ is an algebraic hypersurface in S and since π restrict to a proper mapping with finite fibers, it follows that the image set is a hypersurface in the (w, u)-space which we denote by Δ and refer to as a discriminant locus. The direct image current (1) can be described in the open complement of Δ. Namely, π restricts to an unramified covering map from $S \setminus \pi^{-1}(\Delta)$ onto the open complement of Δ in the (w, u)-space,. Let K be the number of points in every fiber which is given by an unordered k-tuple $p_k(w, u) = (x_k(w, u), y_k(w, u)\}$, Here the coordinates $\{x_k(w, u)\}$ and $\{y_k(w, u)\}$ are local branches of multi-valued analytic functions in the open complement of Δ. If $h(u, w)$ is a test-function in the (w, u)-space whose compact support does not intersect Δ and we take $\phi^{0,2} = h(w, u) \cdot dw \wedge du$ then one has the equality

$$\iint_S g(x, y) \cdot dx \wedge \wedge \pi^*(\phi^{0,2}) = \iint_{D^2} \mathfrak{Tr}(\frac{g}{\mathcal{J}})(w, u) \cdot h(w, u) \cdot dw \wedge du$$

where

$$\mathfrak{Tr}(\frac{g}{\mathcal{J}})(w, u) = \sum_{k=1}^{k=K} \frac{g(p_k(w, u)}{\mathcal{J}(p_k(w, u))} \tag{*}$$

We refer to (*) as a trace function of $\frac{g}{\mathcal{J}}$. Next, recall that the passage to direct image currents commute with differentials. Since $g(x,y)$ is holomorphic the current (1) is $\bar\partial$-closed. Indeed, Stokes theorem entails that

$$\iint_S g(x,y)\cdot dx\wedge\wedge dy\wedge\bar\partial(\psi^{0,1})=0$$

hold for every test-form $\psi^{0,1}$ with compact support in the (x,y,w,u)-space.

B.4.1 Conclusion. Denote by γ the direct image current from (1). By (*) its restriction to the open complement of Δ is the density expressed by the $(2,0)$-form

$$\mathfrak{Tr}(\frac{g}{\mathcal{J}})(w,u)\cdot dw\wedge du$$

Next, in (*) the trace function is constructed by a sum over fibers which entails that it extends to a meromorphic function in the (w,u)-space with eventual poles confined to Δ. Let G denote this meromorphic function so that

$$G(w,u)=\mathfrak{Tr}(\frac{g}{\mathcal{J}})(w,u) \tag{i}$$

holds when (w,u) are outside Δ. By the above the current γ is $\bar\partial$-closed, i.e. the $(2,0)$-current defined outside Δ by $G\cdot dw\wedge du$ can be extends via γ to a $\bar\partial$-closed current in the (w,u)-space. But this can only hold if the meromorphic function G has no poles at all. In fact, this follows from Hartogs' extension result in § XX. Hence we have proved:

B.4.2 Theorem. *For every polynomial $g(x,y)$ the trace function defined by (*) in the complement of Δ extends to a holomorphic function in the (w,u)-space.*

C. The local algebra $\mathbf{C}\{P,Q\}$

.

Given the pair P,Q there exists a subalgebra of $\mathcal{O}_2$ whose elements are germs of analytic functions which can be expanded into a power series in P and Q. More precisely, denote by $\mathbf{C}[P,Q]$ the set of entire functions of the form

$$\phi_n(x,y)=\sum c_{jk}\cdot P^j\cdot Q^k$$

where $\{c_{jk}\}$ is a finite set of doubly-indexed complex numbers. A germ g belongs to $\mathbf{C}\{P,Q\}$ if and only if there exists a small polydisc D^2 centered at the origin in $\mathbf{C}^2$ such that g is holomorphic in D^2 and there exists a sequence $\{\phi_n\}$ of (P,Q)-polynomials as above which converge uniformly to g in D^2. Concerning the algebra $\mathbf{C}\{P,Q\}$ a wellknown result in several complex variables shows that it is isomorphic to the local ring of convergent power series in two variables, i.e. the polynomials considered as germs in $\mathcal{O}_2$ are analytically independent. Moreover, $\mathcal{O}_2$ is a finitely generated module over its subring $\mathbf{C}\{P,Q\}$.

C.1 The algebraic trace. By the above the quotient field of $\mathcal{O}_2$ is a finite algebraic extension of the quotient field of $\mathbf{C}\{P,Q\}$. If m is the dimension we can choose an m-tuple $\phi_1,\dots,\phi_m$ in the quotient field of $\mathcal{O}_2$ whose images in $\mathcal{A}$ yields a basis for this finite algebraic extension. Let us then consider a polynomial g. For each $1\le\nu\le m$ we can write

$$g\cdot\phi_\nu=\sum_{j=1}^{j=m}\rho_{\nu,j}\cdot\phi_j$$

where $\{\rho_{\nu,j}\}$ belong to the quotient field of $\mathbf{C}\{P,Q\}$. As explained in §§ the trace defined by

$$\mathfrak{Tr}(g)=\sum\rho_{\nu,\nu}$$

does not depend upon the chosen ϕ-basis.

D. $\mathcal{D}$-module theory.

A gateway to compute intersection numbers in a complete intersection employs the 2-dimensional Weyl algebra whose elements are differential operators with coefficiets in $\mathbf{C}[x,y]$. Namely, let P,Q be a pair of polynomials in a compete interersection. They generate the left ideal

$$L = A_2 \cdot P + A_2 \cdot Q$$

Results in Chapter 1 from my text-book *Rings of Differential Operators*[North Holland. Math.lib.ser, 21: 1979] imply that

$$M = \frac{A_2}{L}$$

is a holonomic left A_2-module given as a finite sum

$$\sum k_\nu \cdot \mathcal{B}(p_\nu)$$

where $\{p_\nu\}$ is the finite set common zeros to P and Q in $\mathbf{C}^2$ and $\{k_\nu\}$ the local intersection numbers at these points. Finally, $\mathcal{B}(p_\nu)$ is the left A_2-module where one has taken the quotient of the left ideal generated by $x - x_\nu$ and $y - y_\nu$ with $p_\nu = (x_\nu, y_\nu)$. So in this way one recoveres the local ontersection numbers as well as the whole sum.

Durect images. A deeper insight about compete intersections using direct images of $\mathcal{D}$-modules. For their construction we refer to Chapter 2 in my text-book *Analytic $\mathcal{D}$-modules* [Kluwer. 1989]. Here one considers another copy of $\mathbf{C}^2$ with coordinates (ζ, η) and the polynomial map

$$\rho\colon (x,y) \mapsto (P(x,y), Q(x,y))$$

With notations from [ibid] there exists the direct image

$$\mathcal{M} = \rho_+(\mathcal{O})$$

where $\mathcal{O}$ is the sheaf of holomorphic functions in the (ζ, η)-space. General results from [ibid] entail that $\mathcal{M}$ is a regular holonomic $\mathcal{D}$-module in the (x,y)-space whose characteristic cycle recaptures the local intersection numbers from § D. A more delicate study arises when one considers the points in Sato's singular spectrum $\mathrm{SS}(\mathcal{M})$ which are outside the zero section of the cotangent bundle over $\mathbf{C}^2$. Again we refer to [ibid] for these standard concepts and notations in $\mathcal{D}$-module theory, The polynomial map ρ is special since it has finite fibers. To analyze situations at the points of ramification one employs micro-localisations, i.e. one regards the regular holonomic $\mathcal{E}$-module

$$\mathcal{E} \times_{\pi^{-1}(\mathcal{D}} \pi^{-1}(\mathcal{M})$$

Now one can begin to perform a micro-local study. Let us remark that during this investigation one should profit upon "modern sheaf theory" from the text-book by Kashiwara and Schapira which gives insight about singularites in the base manifold after the passage to micro-localisations of constructible sheaves. Here the fundamental notion of micro-support of sheaves is essential, and my opinoion is that this should be one of the first issues i when one teaches courses in general sheaf theory. since micro-loca considerations are needed in order to fully grasp the notion of singularities.

Summing up, a more comprehensive study of complete intersections with "generic parameters" arises via systematic use of $\mathcal{D}$-module theory which has the merit that various facts become clear, while they are not so easily understood if one restricts the attention to base manifolds only. Let us also remark that when P,Q is a complete intersection, then one can study more general direct images, i.e. replace $\mathcal{O}$ by an arbitrary regular holonomic $\mathcal{D}$-module and take its image under the direct image functor ρ_+. Now ρ_+ is an exact covariant functor from the abelian category of regular holonomic modules in the (x,y)-space to those in the (ζ, η) space. So this functor is attached to the given complete intersection and can be regarded as an "ultimate" object attached to the pair P,Q. It goes without saying that these constructions can be extended to cover the case of a complete interersection in any number of variables. One can also replace the affine space $\mathbf{C}^n$ by an arbitrary affine and non-singular algebraic variety V where the Weyl algebra is replaced by the ring $\mathcal{D}(V)$ of globally defined differential operators as explained in Chapter 3 from my book on rings of Differential Operators.

E. Residue calculus

Above we have considered Jacobi's residues when the denominators are holomorphic functions. More generally, if P and Q are in a complete intersection we can still define the integrals $J_g(\epsilon, \delta)$ from Proposition A.5.1 when $g(x, y)$ is a test-function with compact support close to the origin. But here the integral is no longer independent of the pair (ϵ, δ). This leads to a more involved situation which was originally studied by Miguel Herrera. He proved that there exists certain limits provided that one pays attention while ϵ and δ tend to zero. We shall not enter a detailed discussiona about multi-residue calculus but recall that an example discovered by Passare and Tsikh in [P-T] shows that the unrestricted limit of the integrals from Propostion A.5.1 does not exist in general. In fact, such examples are generic and one can even take one polynpomial to be x^M for suitable positive integer. The reader may consult my article [Björk:Abel Legacy] for further comments about the Passare-Tsikh example, as well about Passare sectors which describe when one can perform limits and arrive at the same residue current as that of Herrera and co-author Coleff. On the positive side there exists a remarkable result due to H. Samuelsson in his Ph.d-thesis at Chalmers University 2005. From an analytic point of view Theorem D.1 below is quite useful. Instead of taking "ugly residues" one employs regularisations. Namely, for each polynomial P there exists a smooth current

$$\frac{\bar{P}}{|P|^2 + \epsilon}$$

One can apply the $\bar{\partial}$-current and get a smooth $(0, 1)$-current

$$\rho_P(\epsilon) = \bar{\partial}(\frac{\bar{P}}{|P|^2 + \epsilon})$$

Similarly we construct smooth currents $\rho_\delta(Q)$. With these notations one has:

D.1 Theorem. *In the space of $(0, 2)$-currents there exists an unrestricted limit*

$$\gamma_{P,Q} = \lim_{(\epsilon,\delta)\to(0,0)} \rho_P(\epsilon) \wedge \rho_\delta(Q)$$

Moreover, this current is of the Coleff-Herrera type which means that

$$\gamma_{P,Q}(\phi^{2,0}) = 0$$

for every test-form $\phi^{2,0}$ given as $[\bar{x} \cdot g_1(x, y) + \bar{y} \cdot g_2(x, y)]\ddot{d} \wedge dy$ where g_1, g_2 is a pair of test-functions.

Final remark. It would bring us too far to discuss multi-valued residue theory in more detail. Above we have only exposed the so called absoute case in dimension two. Let is only say that in spite of many impressive results which have been achieved during the last four decades, there remain many open questions. Here one often regards problems related to analysis. To give an example, consider a complete intersection P and Q where the origin in $\mathbf{C}^2$ is the sole common zero. Now there exists the distribution valued function of two complex parameters s and t defined by

$$\mu_{s,t}(g) = \int_{\mathbf{C}^2} P^{-1} \cdot Q^{-1} \cdot |P|^s \cdot |Q|^t \cdot g\, dA$$

where dA is the Lebesgue measure in $\mathbf{C}^2$ and g are test-functions. $\mathcal{D}$-module theory entails that μ extends to a meromorphic distribution-valued function in the complex (s, t)-space. Poles occur and the complete intersection entails that polar distributions under this meromorphic extension are supported by the origin. To analyze all this leads to a quite involved study. We remark only that a number of results in connection with this have been obtained by Barlet, Sabbah and Yger who are among the invited speakers to the conference devoted to differential systems on the complex domain at Stockholm university in May 2010.