## Riemann surfaces

A complex analytic manifold $X$ of dimension one is called a Riemann surface. If $X$ is non-compact one refers to an open Riemann surface, while $X$ is a closed Riemann surface if it is compact. The study of open, respectively closed Riemann surface leads to different results. When $X$ is closed the maximum principle for analytic functions entails that the space $\mathcal{O}(X)$ of globally defined holomorphic functions is reduced to constant function. For an open Riemann surface $X$ the situation is quite different. For example, let $\{p_\nu\}$ be a discrete sequence of points on $X$ snd $\{k_\nu\}$ a sequence of positive integers then there exists $f \in \mathcal{O}(X)$ which is zero-free outside the $p$-points, and for every $\nu$ it has a zero of order $k_\nu$ at $p_\nu$ Moreover the Cousin problem has a solution for every pair of open sets $U, V$ whose union is equal to $X$. It means that for every $\phi \in \mathcal{O}(U \cap V)$ there exist $a \in \mathcal{O}(U)$ and $b \in \mathcal{O}(V)$ such that the restriction of $a - b$ to $U \cap V$ is equal to $\phi$. When $X$ is a compact Riemann surface the obstruction to solutions of the Cousin problem for a pair of open sets $U, V$ whose union is $X$ turns put to be described by a $g$-dimensional complex vector space where $g$ is the genus of $X$. We treat the compact case in § x while open Riemann surfaces are studied in § xx. Closed Riemann surfaces arise via algebraic equations. Namely, let $\mathbf{C}[x, y]$ be the polynomial ring of two variables with complex coefficients. To each irreducible polynomial $p(x, y)$ we get the algebraic curve in $\mathbf{C}^2$ defined by $\{p = 0\}$ and let us denote it by $S$. Asume that $p$ is moniic with respect to $y$, i.e.

$$p(x, y) = y^m + q_{m-1}(x)y^{m-1} + \ldots + q_0(x)$$

where no special assumption is imposed on the $q$-polynomials except fopr the condition that $p(x, y)$ is an irredicible polynomial. Geberal residue calculus - which actually is valid for algebraic hypersurfaces in $\mathbf{C}^n$ for every $n \geq 2$ asserts that there exists a $\bar{\partial}$-closed current of bi-degree $(1, 0)$ define dfor test-forms $\psi^{0,1}$ in the 4-dimensional space attached to $\mathbf{C}^2$ and defined by

$$\psi^{0,1} \mapsto \int_S \frac{dx}{P'_y} \wedge \psi^{0,1}$$

Above $S$ is an algebraic curve in $\mathbf{C}^2$ and we remark that the $\bar{\partial}$-closed current excises in general, i.e. even when $S$ contains singular points. However, via a clöassic local contraction due to Puiseue there exists normalization of $S$ which yields a non-singular curve $\widehat{S}$ and a map $\rho \colon \widehat{S} \to S$. Passing to the proactive closure one can also construct local charts in the sense of Puiseux and in this way arrive at a closed Riemann surface $X$ for this one has a map $\rho \colon X \to \bar{S}$ which in many cases isda homoeomorphiusm but in certain examples it appears that for a finite set of points in $S$ the inverse finger $\rho^{-1}(s)$ is a finite set in $X$. However, in all cases the current (*) yields via a pull-back to $X$ s $\bar{\partial}$-closed differential form with therefore is called a holomorphic form. The whole space of holographic 1-forms on $X$ is denoted by $\Omega(X)$. We shall learn that it is a finite dimensional complex vector spec whose dimension is equalmto the genus of $X$. To find a basis for $\Omega(X)$ one must study under what conditions the special holomorophic 1-form extends to be holomorphic at points in $X$ which belong to $\rho^{-1}(\partial S)$., Here one encounters typical calculations during the study of closed Riemann surfaces.

**An example.** Consider the case

$$p(x, y) = y^2 - q(x)$$

where $q(x)$ is a polynomial on $x$ only and of some odd degree $2m + 1$ with $m \geq 1$. Here we shall learn that $y$ extends to a meromorphic function oin $X$ with a pole of order $2m + 1$ at the point at infinity, while $x$ has a pole of order 2. Dfrom this it follows that the meromorphic differential $dx$ has a pole of order 3 at infinity and at the same time $y^{-1}$ has a zero of order $2m + 1$. It follows that $\frac{dx}{y}$ remains holographic at the point at infinity. If $m \geq 2$ one gets more holomorphic 1-forms on $X$ using

$$\omega_\nu = x^\nu \cdot \frac{dx}{y}$$

where the condition is that

$$2\nu \leq 2m + 1 - 3 \implies \nu \leq m - 1$$

From this it follows that the $g = m$. Consider s an example the case $m = 2$ and we the polynomials $q(x)$ of the form

$$q(x) = \prod (x - \alpha_\nu)$$

where $\alpha_1, \ldots, \alpha_5$ are distinct complex numbers. So to every such $q$-polynomial we arrive at a closed Riemann surface of genus two. with we denote by $X(\alpha_1, \ldots, \alpha_5)$ Now one enuoviuner a typical modality problem, i.e decide when a pair of such closed Riemann surfaces are isomorphic. We shall find that the answer is highly non-trivial.