Homogeneous distributions and the Mellin transform

Contents

Introduction. We establish some results where analytic function theory is used in connection with distributions and asymptotic expansions. Special attention in the first sections is given to homogeneous distributions in $\mathbf{R}^2$. while the last section exposes a famous result due to Mellin in [Mellin] which has a wide range of applications. A separate section is devoted to the Radon transform where the inversion formula has a wide range of applications but we shall not pursue the discussion any further. The reader may consult Helgason's text-book [Helgason] for the theory about radon transforms which include the case of higher dimension. In § 4 we establish an important result due to Mellin in [Mellin] and § 5 contains more advanced material which relies upon $\mathcal{D}$-modue theory. So the presentation is expositary in this section.

A. Polar distributions

In the (x,y)-plane we can take polar coordinates where $x = r\cdot\cos\theta$ and $y = r\cdot\sin\theta$. If $\phi(x,y)$ belongs to the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions we restrict ϕ to the circle of radius r which after a dilation is identified with unit circle T and obtain the θ-periodic function

(0.1) $$\theta \mapsto \phi_r(\theta) = \phi(r\cdot\cos\theta, r\cdot\sin\theta)$$

Let ν be a distribution on T. For each $r>0$ we can evaluate ν on the $C^\infty(T)$-function ϕ_r which yields a function:

(0.2) $$r \mapsto \nu(\phi_r) \quad : r > 0$$

A.0 Exercise. Show that (2) gives a C^∞-function defined on $\{r>0\}$. More precisely,verify that the first order derivative becomes:

$$\frac{d}{dr}(\nu(\phi_r)) = \nu(\cos\theta\cdot\partial_x(\phi)(r\cdot\cos\theta, r\cdot\sin\theta) + \sin\theta\cdot\partial_y(\phi)(r\cdot\cos\theta, r\cdot\sin\theta))$$

More generally, show that for each $m\geq 2$ the derivative of order m becomes:

(*) $$\frac{d^m}{dr^m}(\nu(\phi_r)) = \sum_{j=0}^{j=m}\binom{m}{j}\,\nu\big(\cos^j\theta\cdot\sin^{m-j}\theta\cdot\partial_x^j\partial_y^{m-j}(\phi)_r\big)$$

Show also that the function in (0.2) decreases rapidly as $r\to+\infty$. More precisely, for each positive integer N one has

(**) $$\lim_{r\to\infty} r^N\cdot\nu(\phi_r) = 0$$

A.1 The function V_λ. Using (**) above it follows that if λ is a complex number with $\mathfrak{Re}(\lambda) > -2$, then there exists the absolutely convergent integral

(1) $$V_\lambda(\phi) = \int_0^\infty r^{\lambda+1}\cdot\nu(\phi_r)\cdot dr$$

A.2 Exercise Show that V_λ is an analytic function of λ in the open half-plane $\mathfrak{Re}(\lambda) > -2$ and use a partial integration with respect to r to show that:

$$(\lambda+2)V_\lambda(\phi) = -\int_0^\infty r^{\lambda+2} \cdot \frac{d}{dr}[\nu(\phi_r)] \cdot dr$$

Continue this procedure and show that for every $N \geq 1$ one has:

$$(*) \qquad (\lambda+2)\cdots(\lambda+2+N) \cdot V_\lambda(\phi) = (-1)^{N+1} \int_0^\infty r^{\lambda+2+N} \cdot \frac{d^N}{dr^N}[\nu(\phi_r)] \cdot dr$$

Together with (*) in Exercis A.0 we can conclude that following:

A.3 Proposition. *For each $\phi \in \mathcal{S}$ it follows that $V_\lambda(\phi)$ extends to a meromorphic function in the whole complex λ-plane with at most simple poles at the integers $-2, -3, \ldots$.*

A.4 Polar distributions. As ϕ varies in $\mathcal{S}$ we obtain a distribution-valued function V_λ. If $\delta > 0$ and $\phi(x,y) \in \mathcal{S}$ is identically zero in the disc $\{x^2+y^2 < \delta^2\}$, then we only integrate (1) in A.1 when $r \geq \delta$ and we notice that the function

$$\lambda \mapsto \int_\delta^\infty r^{\lambda+1} \cdot \nu(\phi_r) \cdot dr$$

is an entire function of λ whose complex derivative is given by

$$\lambda \mapsto \int_\delta^\infty \log r \cdot r^{\lambda+1} \cdot \nu(\phi_r) \cdot dr$$

Regarding the distribution-valued function V_λ this means that eventual poles consist of Dirac distributions at the origin. Let us first study if a pole can occur at -2. With $\lambda = -2+\zeta$ we have

$$\text{(i)} \qquad \zeta \cdot V_{-2+\zeta}(\phi) = -\int_0^\infty r^\zeta \cdot \frac{d}{dr}[\nu(\phi_r)] \cdot dr$$

Since $r^\zeta \to 1$ holds for each $r > 0$ as $\zeta \to 0$, the right hand side has the limit

$$\text{(ii)} \qquad \int_0^\infty \frac{d}{dr}[\nu(\phi_r)] \cdot dr = \nu(\phi_0) = \nu(1_T) \cdot \phi(0)$$

where 1_T is the identity function on T on which ν is evaluated. Hence V_λ has a pole at $\lambda = -2$ if and only if $\nu(1_T) \neq 0$ and in this case the polar distribution is $\nu(1_T)$ times the Dirac distribution δ_0.

A.5 Exercise Use the functional equation formula (*) in A.2 to show the following:

A.6 Proposition *For each $N \geq 1$ the polar distribution at $\lambda = -N-2$ is zero if and only if*

$$\nu(\cos^j\theta \cdot \sin^k\theta) = 0$$

hold for all pairs of non-negative integers j,k with $j+k=N$.

Remark. Thus, no pole occurs at $\lambda = -N-2$ if and only if ν vanishes on the $N+1$-dimensional subspace of $C^\infty(T)$ spanned by $\{\cos^j\theta \cdot \sin^{N-j}\theta) \quad : 0 \leq j \leq N\}$. Next, if a pole occurs we have a Laurent series:

$$V_{-N-2+z} = \frac{\gamma_N}{\zeta} + V_{-N-2} + \sum_{n=1}^\infty \rho_j \cdot \zeta^j$$

where γ_N is the polar distribution.

A.7 Exercise, Show that if a pole occurs then γ_N is the Dirac distribution given by:

$$(*) \qquad \gamma_N(\phi) = \frac{1}{N!} \cdot \sum_{j=0}^N \nu((\cos^j\theta \cdot \sin^{N-j}\theta) \cdot \partial_x^j \partial_y^{N-j}(\phi)(0)$$

B. Homogeneous distributions.

A distribution μ defined outside the origin in $\mathbf{R}^2$ is homogeneous of degree λ if

$$\mathcal{E}(\mu) = \lambda \cdot \mu \tag{*}$$

where $\mathcal{E} = x\partial_x + y\partial_y$ is the radial vector field. Denote by $\mathcal{S}^*(\lambda)$ the family of all λ-homogeneous distribution in $\mathbf{R}^2 \setminus \{0\}$.

B.1 Proposition. *$\mathcal{S}^*(\lambda)$ is in a 1-1 correspondence with $\mathfrak{Db}(T)$ when we for every distribution ν on T consider the restriction of V_λ to $\mathbf{R}^2 \setminus \{0\}$.*

B.2 Exercise. Prove this result. The hint is to verify that one has the equality

$$\mathcal{E}(V_\lambda) = \lambda \cdot V_\lambda$$

when one starts from an arbitrary distribution ν on T.

B.3 The space $\mathcal{S}^*[\lambda]$. This the space of tempered distributions on $\mathbf{R}^2$ which are everywhere homogeneous. So a tempered distribution μ belongs to $\mathcal{S}^*[\lambda]$ if and only if

$$\mathcal{E}(\mu) = \lambda \cdot \mu$$

where the equality holds in $\mathcal{S}^*$.

B.4 Example of distributions in $\mathcal{S}^*[\lambda]$. Let ν be a distribution on T and construct the meromorphic function V_λ. It is clear that

$$\mathcal{E}(V_\lambda) = \lambda \cdot V_\lambda \quad \text{holds when} \quad \mathfrak{Re}(\lambda) > -2 \tag{i}$$

Let λ_* be a complex number such that V_λ has no pole at λ_*. By analyticity it follows from (i) that the constant term V_{λ_*} satisfies

$$\mathcal{E}(V_{\lambda_*}) = \lambda_* \cdot V_{\lambda_*} \tag{ii}$$

Hence V_{λ_*} belongs to $\mathcal{S}^*[\lambda_*]$. By Proposition A.3 no poles occur when λ_* is outside the set $\{-2, -3, \ldots\}$ which gives the following:

B.5 Proposition. *For each λ_* outside the set $\{-2, -3, \ldots\}$ there exists a bijective map*

$$\nu \mapsto V_{\lambda_*}$$

from $\mathfrak{Db}(T)$ into $\mathcal{S}^*[\lambda_*]$.

B.6 The action by $\mathcal{E}$ on Dirac distributions. Using Dirac distributions at the origin we shall construct homogeneous distributions which do not arises via distributions from T as above. To begin with the complex vector space of all Dirac distributions is a direct sum of the subspaces

$$\text{Dirac}[m] = \oplus\, \mathbf{C} \cdot \partial_x^k \partial_y^j(\delta_0) \quad : \; j + k = m \tag{1}$$

where m are non-negative integers. Next, in the ring $\mathcal{D}$ of differential operators we have the identity

$$\mathcal{E} = \partial_x \cdot x + \partial_y \cdot y - 2$$

Since $x \cdot \delta_0 = y \cdot \delta_0 = 0$ it follows that

$$\mathcal{E}(\delta_0) = -2 \cdot \delta_0$$

In general the reader may verify by an induction over m that

$$\mathcal{E}(\gamma) = -(m+2) \cdot \gamma \quad \text{hold for all} \quad \gamma \in \text{Dirac}[m] \tag{2}$$

Hence Dirac$[-m-2]$ is a subspace of $S^*[-m-2]$ which has dimension $m+1$. In particular $S^*[-2]$ contains the 1-dimensional vector space generated by δ_0.

B.7 The description of $\mathcal{S}^*[-2-N]$

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Let m be a non-negative integer. Denote by $\mathfrak{Db}(T)[m+2]$ the set of distributions ν on T such that V_λ has no pole at $-2-m$. From B.4 it follows that one has an injective map

$$(*) \qquad \mathfrak{Db}(T)[m+2] \mapsto S^*[-2-m]$$

B.8 Exercise. Show that

$$S^*[-2-m] = \mathfrak{Db}(T)[m+2] \oplus \mathrm{Dirac}[m]$$

hold for every integer $m \geq 0$.

B.9 A converse to (*). Let m be a non-negative integer and consider a distribution ν on T. At $\lambda = -m-2$ we have the constant term V_{-m-2} of the Laurent expansion of V_λ. We have seen that the distribution V_{-m-2} is homogeneous of order $-m-2$ if no pole occurs. It turns out that the absence of a pole also is necessary in order that V_{-2-m} belongs to $S^*[-2-m]$. To show this we suppose that a pole is present which gives the Laurent series

$$V_{-2-m+\zeta} = \frac{\gamma}{\zeta} + V_{-2-m} + \sum_{j=1}^{\infty} \gamma_j \cdot \zeta^j$$

where γ now is a non-zero Dirac distribution which belongs to $\mathrm{Dirac}[-2-m]$. Now we have

$$\mathcal{E}(\frac{\gamma}{\zeta} + V_{-m-2} + \sum \rho_j \zeta^j) = (-m-2+\zeta) \cdot V_{-m-2+\zeta} =$$
$$(-m-2) \cdot \frac{\gamma}{\zeta} + \gamma + (-m-2)V_{-m-2} + \zeta \cdot V_{-2-m} + (-m-2+\zeta) \cdot \sum \rho_j \zeta^j$$

Identifying the constant term we get

$$(2) \qquad \mathcal{E}(V_{-m-2}) = \gamma - (m+2)V_{-2-m}$$

Hence the distribution V_{-m-2} fails to be homogeneous.

B.10 Example. Let $\nu = 1_T$ be the identity density on T. So here

$$V_\lambda(\phi) = \int_0^\infty [\int_0^{2\pi} \phi(r,\theta)\, d\theta\,]\, r^{\lambda+1} dr$$

Outside the origin we see that the distribution V_{-2} is given by the density function

$$f = \frac{1}{x^2+y^2}$$

Moreover, it is clear that V_λ has a pole when $\lambda = -2$ and whose polar distribution is δ_0. The conclusion is that the distribution V_{-2} is not homogeneous. Notice that we encounter another obstacle since the function f above is not locally integrable at the origin so it is not clear how to define the distribution V_{-2}. If a test-function ϕ is zero at the origin, then the integral

$$\iint_{\mathbf{R}^2} \frac{\phi(x,y)}{x^2+y^2} \cdot dxdy$$

is defined. So the action by V_{-2} is determined on a the hyperplane of all test-functions which are zero at the origin. There remains to evaluate $V_{-2}(\phi)$ when $\phi(0,0) \neq 0$.

B.12 Fourier transforms.

The Fourier transform maps tempered distributions in the (x, y)-space to tempered distribution the (ξ, η)-space. By the laws from XX we the radial field $\mathcal{E} = x\partial_x + y\partial_y$ is sent into the first order differential operator

$$(i\partial_\xi) \cdot i\xi + (i\partial_\eta) \cdot i\eta = -\partial_\xi \cdot \xi - \partial_\eta \cdot \eta = -\xi\partial_\xi - 1 - -\eta\partial_\eta - 1 = -\mathcal{E}^* - 2$$

where $\mathcal{E}^*$ is the Euler field in the (ξ, η)-space. So if $\mu \in S^*[\lambda]$ is a homogeneous distribution in the (x, y)-space the equality $\mathcal{E}(\mu) = \lambda \cdot \mu$ entails that

$$-(\mathcal{E}^* + 2)(\widehat{\mu}) = \lambda \cdot \widehat{\mu} \implies \mathcal{E}^(\widehat{\mu}) = -(2 + \lambda)\widehat{\mu}$$

Hence the Fourier transform gives a bijective map between $S^*[\lambda]$ and the space öf hiomogeneous distributions in the (ξ, η)-space of degree $-2 - \lambda$.

Example. The Dirac measure δ_0 is homogenoues of degree -2 so here $\widehat{\delta_0}$ should be in $S^*[0]$ and this is indeed the fact since it is given by the constant density in the (x, ξ)-space. Notice also that the Fourier transform sends $S^*[-1]$ into itself.

B.13 The λ-maps on $\mathfrak{Db}(T)$. Let λ be a complex umber outside the set $\{-2, -3, \ldots\}$. To each $\nu \in \mathfrak{Db}(T)$ we get the distribution V_λ which belongs to $S^*[\lambda)$. It follows that the Fourier transform $\widehat{V_\lambda}$ belongs to $S^*[-\lambda - 2]$ and this gives a unique distribution ν^* on T such that

(*) $$\widehat{V_\lambda} = V^*_{-\lambda-2}$$

Keeping λ fixed this means that we get a bijective map from $\mathfrak{Db}(T)$ to itself defined by

$$\nu \mapsto \nu^*$$

where the rule is that (*) holds. We denote this map by $\mathcal{H}_\lambda$ and refer to this as Fourier's λ-map on $\mathfrak{Db}(T)$.

Exercise. Assume that λ is outside $\{-2, -3, \ldots\}$. Let m be a posiotove integer and $P_m(\xi, \eta)$ is a homogenous polynomial of degree m. Now we get a homogenous distribution in the (x, ξ)-space of degree $m + \lambda$ defined outside the origin by the density

(1) $$P_m(\xi, \eta) \cdot (\xi^2 + \eta^2)^{\frac{\lambda}{2}}$$

We seek a homogeneous distribution μ_λ in the (x, y)-space such that $\widehat{\mu}_\lambda$ is equal to (1). In the (x, y)-space we first consider the distribution

$$\gamma = (x^2 + y^2)^{-\frac{\lambda}{2}-1}$$

and notice that $\widehat{\gamma}$ is equal to the distribution $(\xi^2 + \eta^2)^{\frac{\lambda}{2}}$. Fourier's inversion formula entails that (1) is equal to

(2) $$i^{-m} \cdot P_m(\partial_x, \partial_y)(\gamma)$$

Example. Take $m = 1$ and $P_1(\xi, \eta) = \xi$.

C. The Radon transform

In the article [Radon] from 1917 Johann Radon established an inversion formula which recaptures a test-function $f(x,y)$ in $\mathbf{R}^2$ via integrals over affine lines in the (x,y)-plane. This family is parametrized by pairs (p,α), where $p \in \mathbf{R}$ and $0 \leq \alpha < \pi$ give the line $\ell(p,\alpha)$:

$$t \mapsto \big(p \cdot \cos\alpha - t \cdot \sin\alpha\,,\, p \cdot \sin\alpha + t \cdot \cos\alpha\big)$$

The Radon transform of f is a function of the pairs (α,p) defined by:

$$(*) \qquad R_\alpha(p) = \int_{\ell(p,\alpha)} f \cdot dt = \int_{-\infty}^{\infty} f\big(p \cdot \cos\alpha - t \cdot \sin\alpha\,,\, p \cdot \sin\alpha + t \cdot \cos\alpha\big) \cdot dt$$

Thus, for a given α we take the mean value of f over an affine line which is $\perp$ to the vector $(\cos\alpha, \sin\alpha)$ and whose distance to the origin is $|p|$. We give an inversion formula which recaptures f from the Radon transform. To achieve this we construct the partial Fourier transform of R with respect to p, i.e. set

$$(1) \qquad \widehat{R}_\alpha(\tau) = \int e^{-i\tau p} \cdot R_\alpha(p) \cdot dp$$

Consider the linear map $(p,\tau) \mapsto (x,y)$ where

$$x = p \cdot \cos\alpha - t \cdot \sin\alpha \quad \text{and} \quad y = p \cdot \sin\alpha + t \cdot \cos\alpha \implies$$

$$(2) \qquad p = \cos(\alpha) \cdot x + \sin(\alpha) \cdot y$$

Since $\cos^2\alpha + \sin^2\alpha = 1$ this substitution gives $dpdt = dxdy$ and hence (2) entails that

$$(3) \qquad \widehat{R}_\alpha(\tau) = \int e^{-i\tau(x \cdot \cos\alpha + y \cdot \sin\alpha)} \cdot f(x,y) \cdot dxdy = \widehat{f}(\tau \cdot \cos\alpha\,,\, \tau \cdot \sin\alpha)$$

Next, Fourier's inversion formula applied to f gives:

$$f(x,y) = \frac{1}{(2\pi)^2} \cdot \int e^{i(x\xi + y\eta)} \cdot \widehat{f}(\xi,\eta) \cdot d\xi d\eta$$

Now we use the substitution $(\tau,\alpha) \mapsto (\xi,\eta)$ where

$$\xi = \cos(\alpha) \cdot \tau \quad \text{and} \quad \eta = \sin(\alpha) \cdot \tau$$

Here $d\xi d\eta = |\tau| \cdot d\tau d\alpha$ and then (3) gives the equality

$$(*) \qquad f(x,y) = \frac{1}{(2\pi)^2} \int_0^\pi \big[\int_{-\infty}^{\infty} e^{i\tau(x \cdot \cos\alpha + y \cdot \sin\alpha)} \cdot \widehat{R}_\alpha(\tau) \cdot |\tau| \cdot d\tau\big] \cdot d\alpha$$

To get an inversion formula where the partial Fourier transform $\widehat{R}_f(\alpha,\tau)$ does not appear we apply the Fouriers inversion formula in dimension one. Namely, for each $A > 0$ we set

$$(4) \qquad K_A(u) = \frac{1}{2\pi} \int_{-A}^{A} e^{i\tau u} \cdot |\tau| \cdot d\tau$$

This function admits a alternative description since we have

$$K_A(u) = \frac{1}{\pi} \int_0^A \tau \cdot \cos(\tau u) \cdot d\tau = \frac{1}{\pi} \cdot \frac{d}{du}\big(\int_0^A \sin(\tau u) \cdot d\tau\big) =$$

$$(5) \qquad \frac{1}{\pi} \cdot \frac{d}{du}\Big(\frac{1 - \cos(Au)}{u}\Big) = \frac{1}{\pi} \cdot \Big[A \cdot \frac{\sin Au}{u} - \frac{1 - \cos(Au)}{u^2}\Big]$$

Next, by the convolution formula for Fourier transforms the right hand side in (*) becomes

$$(6) \qquad \lim_{A \to \infty} \frac{1}{2\pi} \int_0^\pi \big[\int_{-\infty}^{\infty} R_\alpha(x \cdot \cos\alpha + y \cdot \sin\alpha - u) \cdot K_A(u) du\big] \cdot d\alpha$$

After the substitution $u \to \frac{s}{A}$ and applying (5) the limit in (*) becomes

$$(**) \qquad \lim_{A \to \infty} \int_0^\pi \big[\int_{-\infty}^{\infty} R_\alpha(x \cdot \cos\alpha + y \cdot \sin\alpha - \frac{s}{A}) \cdot \big(A \cdot \frac{\sin s}{s} - A \cdot \frac{1 - \cos s}{s^2}\big) \cdot ds\big] \cdot d\alpha$$

D. The Mellin transform

In many situations one encounters a function $J(\epsilon)$ which is defined for $\epsilon > 0$ and has an asymptotic expansion as $\epsilon \to 0$ by fractional powers which means that there exists a strictly increasing sequence of real numbers $0 < q_1 < q_2 \ldots$ with $q_N \to +\infty$ and constants $c_1, c_2, \ldots$ such that for every N there exists some $\delta > 0$ which in general depends upon N and:

$$(*) \qquad \lim_{\epsilon \to 0} \frac{J(\epsilon) - (c_1 e^{q_1} + \ldots + c_N e^{q_N})}{\epsilon^{q_N + \delta}} = 0$$

It is clear that the constants $\{c_k\}$ are uniquely determined by J and the q-numbers if (*) holds. We are only concerned with the behavior of J for small ϵ and may therefore assume that $J(\epsilon) = 0$ when $\epsilon > 1$.

The Mellin transform. Let $J(\epsilon)$ be some bounded and continuous function on $[0, 1]$ and zero if $\epsilon \geq 1$ and the integral

$$(1) \qquad \int_0^1 |J(\epsilon)| \cdot \frac{d\epsilon}{\epsilon} < \infty$$

When $\mathfrak{Re}(\lambda) > 0$ we set

$$(2) \qquad M(\lambda) = \lambda \cdot \int_0^1 J(\epsilon) \cdot \epsilon^{\lambda - 1} \cdot d\epsilon$$

It is clear that $M(\lambda)$ is an analytic function in the right half-plane $\mathfrak{Re}(\lambda) > 0$ which by (1) extends to a continuous function on the closed half-plane. Moreover, if we assume that j has an asymptotic expansion (*) it follows that $M(\lambda)$ extends to a meromorphic function in the whole complex λ-plane whose poles are contained in the set $\{-q_k\}$. In fact, this follows easily via (*) since

$$\lambda \int_0^1 \epsilon^q \cdot \epsilon^{\lambda - 1} \epsilon = \frac{1}{\lambda + q} \quad \text{for every} \quad q > 0$$

Along the imaginary axis we have

$$(3) \qquad M(is) = is \int_0^1 J(\epsilon) \cdot \epsilon^{is} \cdot \frac{d\epsilon}{\epsilon}$$

Apart from the factor i this is the Fourier transform of J on the multiplicative line $\mathbf{R}^+$. So by XXX one has the inversion formula

$$(4) \qquad J(\epsilon) = \lim_{R \to \infty} J_R(\epsilon) = \int_R^R \epsilon^{-is} \cdot \frac{M(is)}{is} \cdot ds$$

Mellin discovered a reverse process where one from the start only assumes that (1) holds after an asymptotic expansion (*) is derived when $M(\lambda)$ extends to a meromorphic function with simple poles confined to a set $\{-q_k\}$ of strictly negative real numbers which in addition satisfies certain growth conditions which we give below.

The Mellin conditions. Let $M(\lambda)$ be a meromorphic function in the complex λ-plane with simple poles at a strictly decreasing sequence of negative numbers $\{-q_\nu\}$ where $0 < q_1 < q_2 < \ldots$. We say that the meromorphic function satisfies the Mellin conditions when the following hold:

For every positive integer N there exists some $q_N < A < q_{N+1}$ such that the follwing two limit formulas hold:

$$(\text{i}) \qquad \lim_{R \to \infty} \frac{1}{R} \cdot \int_{-R}^R \big| M(-A + is) \big| \cdot ds = 0$$

$$(\text{ii}) \qquad \lim_{R \to \infty} \int_0^{-A} \big[\epsilon^{-t - iR} \cdot \frac{M(t + iR)}{t + iR} - \big[\epsilon^{-t + iR} \cdot \frac{M(t - iR)}{t - iR} \big] \cdot dt = 0$$

3. Theorem. *If the meromorphic function $M(\lambda)$ satisfies the Mellin conditions it follows that the J-function defined by*

$$J(\epsilon) = \lim_{R\to\infty} \int_{-R}^{R} \epsilon^{-is} \cdot \frac{M(is)}{is}\, ds$$

has an asymptotic expansion where the constants $\{c_k\}$ whis appear in (*) are given by:

$$c_k = \mathfrak{res}(M : q_k) \quad : \quad k = 1, 2, \ldots$$

Proof. With $\lambda = t + is$ we consider line integrals over rectangles

$$\Box_{R,A} = \{-A < t < 0\} \cap \{-R < s < R\} \tag{5}$$

where one for an arbitrary positive integer N choose A so that

$$q_N < A < q_{N+1} \tag{6}$$

With $\epsilon > 0$ kept fixed we have the analytic function in $\Box_{R,A}$ defined by

$$\lambda \mapsto \epsilon^{-\lambda} \cdot \frac{M(\lambda)}{\lambda} \tag{6}$$

Cauchy's residue formula gives the equality:

$$2\pi i \cdot J_R(\epsilon) = 2\pi i \cdot \sum_{k=1}^{k=N} q_k^{-1} \cdot \mathfrak{res}(M(\lambda) : q_k) \cdot \epsilon^{q_k} + I_1(R, A) + I_2(R, A) \tag{i}$$

where

$$I_1(R) = \int_0^{-A} \big[\epsilon^{-t-iR} \cdot \frac{M(t+iR)}{t+iR} - \big[\epsilon^{-t+iR} \cdot \frac{M(t-iR)}{t-iR}\big] \cdot dt \tag{ii}$$

$$I_2(R) = -\int_{-R}^{R} \epsilon^{A-is} \cdot \frac{M(-A+is)}{-A+is} \cdot ds \tag{iii}$$

The triangle inequality gives

$$|I_2(R)| \leq \epsilon^A \cdot \int_{-R}^{R} \big|\frac{M(-A+is)}{-A+iRs}\big| \cdot ds \leq \epsilon^A \cdot \frac{1}{R} \cdot \int_{-R}^{R} \big|M(-A+is)\big| \cdot ds \tag{3.1}$$

Above we have $A > q_N$ and hence ϵ^A gives an admissable error for an asymptotic expansion up to order N and Theorem § XX follows.

4. The case of multiple roots. Keeping Meelin's conditions one can relax the hypothesis that the poles of $M(\lambda)$ are simple and obtain an asymptotic expansion of the J-function where the terms $\{c_k\epsilon^{q_k}\}$ in (*) are replaced by finite sums of the form

$$\sum_{\nu=0}^{m_k-1} c_{k,\nu} \cdot (\log \epsilon)^\nu \cdot \epsilon^{q_k} \tag{1}$$

where m_k is the multiplicity of the pole of $M(\lambda)$ at q_k. To see this we notice that while Cauchy's residue formula was applied during the proof above the terms $2\pi i \cdot q_k^{-1}\mathfrak{res}(M(\lambda) : q_k)$ are replaced by

$$2\pi i \cdot \mathfrak{res}(\epsilon^{-\lambda} M(\lambda) : q_k) \tag{2}$$

For a given k we set $\lambda = -q_k + \zeta$ and here the residue is found via the expansion

$$\epsilon^{q_k+\zeta} = \epsilon^{q_k} \cdot \big[1 + \sum_{\nu=1}^{\infty} (\log \epsilon)^\nu \cdot \zeta^\nu\big]$$

Example. Suppose that $M(\lambda)$ has a double pole at $-q_k$ with a local Laurent expansion

$$M(-q_k+\zeta)=\frac{c_k}{\zeta^2}+a_0+a_1\zeta+\ldots$$

In this case the residue in (2) becomes

$$\mathfrak{res}(\epsilon^{-\lambda}M(\lambda):q_k)=c_k\cdot\epsilon^{q_k}\cdot\log\epsilon \tag{3}$$

Remark. To appreciate Mellin's result one should consider various examples where the point is that other kind of methods to begin with prove that the M-function has a "nice" meromorphic extension as above. Quite extensive classes of situations where this applies are derived via $\mathcal{D}$-module theory. The reader may also consult the article [Barlet-Maire] where Mellin's result is extended to give complex expansions, i.e here a J-function is defined in a punctured complex disc where asymptotic expansions appear when a complex variable ζ tends to zero instead of taking limits as in (*) over positive real ϵ.

Let us also point out that one can consider aymptotic expansions of distribution-valued functions. Consider for example a real-valued polynomial $P(x,y)$ of two variables. By Sard's Lemma there exists some $s_*>0$ such that P has non-critical values in $(0,s_*)$, i.e. for every $0<s<s_*$ the real hypersurface $\{P=s\}$ is a non-singular. It consists in general of a union of a (possibly empty) finite family of closed bounded curves and some union of simple curves which are unbounded. The reader should contemplate upon examples such as $P(y,x)=y^2-x^3-2x$. In addition to "naive geometric pictures" one can study distribution-valued functions. Namely, for each $\phi(x,y)$ in the Schwartz class we set

$$J_\phi(s)=\int_{\{P=s\}}\phi\cdot dx$$

where one has taken a sum of line integrals over the family of curves which constitute $\{P=s\}$ and they have been oriented in a natural fashion. Now one attains an asymptotic expnasion which gives a far more detailed description of the "naive geometric pictures". Namely, one finds that the $\mathcal{S}^*$-valued J-function has an asymptoptic expansion (*) whose coefficients are tempered distributions. That such asymptotic expansions exist was shown by Marcel Ruesz in special cases and used to construct fundamenntal solutions. See also the text-book series [Gelfand-et.al]. In a more general set-up where $P(x,y)$ is an arbitrary polynomial and even can depend upon more than two variables, the existence of an aymptotic expnbasion was established by Nils Nilsson in the impressive article [Nilsson]. Later work has employed Hironka's desingularisation which and far-rechaing studies of asymptotic expansions as above occur in $\mathcal{D}$-module theory where one foremost should mention contributions by Barlet. Here it would bring us to far to expose this theory in detail. For a general account about $\mathcal{D}$-module theory and how it is used to construct asymptotic expansions I refer to my article [Björk] in [Abel Legacy].

E. The family $\int P_+^{2\lambda}$

Let $P(x,y)$ be a real-valued polynomial in $\mathbf{R}^2$. When $\mathfrak{Re}(\lambda)>0$ it is clear that we obtain a tempered distribution defined by

$$\phi\mapsto\int_{\{P>0\}}P(x,y)^{2\lambda}\cdot\phi(x,y)\cdot dxdy \tag{*}$$

Moreover, this gives a distribution-valued holomorphic function in the right half-plane. More generally we can consider a connected component Ω of the set $\{P>0\}$ and obtain the $\mathcal{S}^*$-valued function defined by:

$$\phi\mapsto\int_{\Omega}P(x,y)^{2\lambda}\cdot\phi(x,y)\cdot dxdy \tag{**}$$

It turns out that both (*) and (**) extend to meromorphic distribution valued functions in the whole complex λ-plane and there exists a finite set of positive rational numbers $\{q_\nu\}$ such that the poles are contained in the set

$$\cup_{k=1}^m \mathcal{A}_k = \{-q_k - n \quad : n = 0, 1, 2 \ldots\}$$

Moreover, these meromorphic extensions satisfy the Mellin condtiions which can be used to recover the asymptotic expansions which were described in § XX.

The functional equation. Using algebraic properties of the Weyl algebra of differential operators with polynomial coefficients, Joseph Bernstein gave a remarble simple proof of the mere existence of the meromorphic extension of the distribution-valued function in (*) above which we denote by $\mathcal{P}_+^\lambda$. More precisely, the meromorphic extension is achieved by a functional equation. Namely, the meromorphic extension in (*) follows from the existence of a non-zero polynomial $b(\lambda)$ in $\mathbf{C}[\lambda]$ such that

$$b(\lambda) \cdot \int_{\{P>0\}} P(x,y)^{2\lambda} \cdot \phi(x,y) \cdot dxdy =$$

$$(*) \qquad \sum \lambda^k \cdot \int_{\{P>0\}} P(x,y)^{2\lambda+1} \cdot Q_k(\phi)(x,y) \cdot dxdy$$

hold for every ϕ in $\mathcal{S}(\mathbf{R}^2)$ where $\{Q_k\}$ is a finite set of differential operators indexed by non-negative integers which belong to the Weyl algebra A_2, i.e., they are globally definied differential operators wirh polynomial coefficients. Above one can choose $b(\lambda)$ of smallest possible degree and it is then referred to as the *Bernstein-Sato polynomial* of P. We remark that the tribute to M. Sato stems from his early discoveries of functional equations as above. The fact that the roots of the b-function always are strictly negative rational numbers was established by Masaki Kashiwarin in the article [Kashiwara] from 1975. In addition to (*), Kashiwara also estalnished a second functional equation which implies that Mellin's two conditions are valid. More precisely, given the polynomial P as above, it is proved in [loc.cit] that there exists a positive integer m and a finite set $\{Q_0, \ldots, Q_{m-1}\}$ in A_2 such that

$$(**) \qquad \lambda^m \cdot \mathcal{P}_\lambda(\phi) = \sum_{k=0}^{m-1} \lambda^k \cdot \int_{\{P>0\}} P(x,y)^{2\lambda} \cdot Q_k(\phi)(x,y) \cdot dxdy$$