An optimization problem in the course Mathematical Economics

Let us first describe a model in continuous time. At time $t = 0$ one starts the Brownian motion $\mathcal{W}$ at $x = 0$. So $W(t)$ is normally distributed with variance $\sqrt{t}$ for every $t > 0$. Imagine a player who seeks to maximize a profit where the role is that the player can stop the random Brewnian path at any time value $0 < t \leq 1$ and get the reward $W(t)$. If the player never stops the game and $W(1) < 0$ no debt occurs, i.e. the net reward is zero in this case. It means that the player never can loose money in this game. The question arises how to maximize expected profit. The absence of "negative loss" at the terminal tim value $t = 1$ implies that the optimal strategy to maximize *expected profit* is to wait until $t = 1$ and receive the profit $W(1)$ if it happens to be positive. With this strategy the expected profit becomes

$$\frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty xe^{-x2/2}\, dx = \frac{1}{\sqrt{2\pi}}$$

A more involved problem arises if a discount factor is introduced. With $r > 0$ we suppose that the player gets the reward

$$e^{-rt} \cdot W(t)$$

if the game is stopped at a time value t. With $r > 0$ the previous strategoy is not optimal in order to maximize the expected profit. Denote the expected profit for a given $r > 0$ by $\mathcal{P}(r)$. It is clear that the function

$$r \mapsto \mathcal{P}(r)$$

decreases with r. One may ask if this function can be found in a "closed form". This is not known at present, i.e. so far mathmatics is not sufficiently developed to determine $\mathcal{P}(r)$ with an "analytic formula". The reason is that the $\mathcal{P}$-function appears to a solution of a free boundary value problem where certain non-linear differential PDE-equations appear whose solutions are not known. However, one can derive numerical solutions with high degree of precision using approximations of the continuous W-process with binomial trees. Here one follows a device introduced by Kolmogorov in 1930 which gives a backward solution to determine the discrete $\mathcal{P}$-functions. More precisely, let N be a large positive integer, say $N = 10^3$. In the discrete process which runs over N trials with heads or tails of equal probability, one moves at every step with the size plus or minus $\frac{1}{\sqrt{N}}$. After k many trials the player gets the reward

$$\rho_k(\nu) = e^{-rk/N} \cdot \frac{v}{\sqrt{N}} \tag{1}$$

where ν is the difference of the number of heads and tails after these k trials. Next, for each $0 \leq k \leq N$ we denote by $\Pi_k(\nu)$ the expected profit when the player has not terminated the game up to the k-th tossing with the coin and ν is the difference of number of heads and tails after these first k trials. If the player has continued the game until $k = N$ there is no longer a chice and we see that

$$\pi_N(\nu) = \frac{\nu}{\sqrt{N}} \cdot e^{-r} \quad : \nu \geq 1 \quad \& \quad \pi_N(\nu) = 0 : \ \nu \leq 0 \tag{2}$$

When $k < N$ the player can either stop the game and receive (1) or continue. The choice is determined by the equation

$$\pi_k(\nu) = \max\{\rho_k(\nu)\,;\,\frac{1}{2}(\pi_{k+1}(\nu-1) + \Pi_{k+1}(\nu+1)\} \tag{*}$$

One refers to (*) as Kolmogorov's backward equation. Since we know the function in (2) it is clear that an induction which starts with $K = N$ and then moves back determine the functions

$$\nu \mapsto \pi_k(\nu)$$

for every k. In particular we find the number $\pi_0(0)$ which is the expexted profit when the player starts the game. This number depends on r and N. Put

$$\mathcal{P}(N,r) = \pi_0(0)$$

with the π-function determined via (1-2) and (*). Now de Moivre's central limit theorem from 1730 implies that

$$\lim_{N\to\infty} \mathcal{P}_N(r) = \mathcal{P}(r)$$

2. Optimal strategy. If N is given and we have found the doubly indexed π-function, then the player knows how to maximize expected profit. More precisely, if the game has continued up to k trials and the number of heads minus tails is some integer ν then the player stops at this moment to pick the reward in (1) if and only if

$$\pi_k(\nu) = \rho_k(\nu) \tag{2.1}$$

Thus yields a "striking curve" of pairs (ν, k) for which (2.1) holds, With the aid of a computer one may also plot this curve for a given $r > 0$ and then let N increase, The "limit" of these strikting curves exists for a fixed r and turns out to be a decreasing and concave function

$$t \mapsto \phi_r(t)$$

So here $\phi_r(t)$ is the function where a player following a continuous Brownian motion stops the process at the first time value t_* for which $W(t_*) = \phi_r(t_*)$ while

$$t < t_* \implies W(t) < \phi_r(t)$$

Hence, using a computer one gets good numerical solutions to these ϕ-functios for every given $r > 0$.

Another function arises for the owner of the Casino when many players are eager to try the game. Namely, each player will stop the game at a tandom time t_* determined as above. Now the owner of the Casino may be interested to find the distribution of these terminating time values which for every individual player is a random event. To solve this numerically with a given r one can first find a good numerical solution for the striking function $\phi_r$mabovce, i.e. in a discrete model it is picewise linear with N m any conrner points. Then the owner of the Casino - expecting that all players are as smart as Kolmogorov ! - can perfom a Monte Carlo simulation using this ϕ-function and find the distribution of the random time when players stop the game. The solution to this problem is of course of interest in a large

Casino where several players want to try their luck on "machines" at free disposal, but thr number of these machines is limited. If the enrtrance ticket for a game is precisly equal to $\mathcal{P}(r)$ this does not give any expected profit for the Casino when all palyers have learnt the lesson from Kolmogorov. One may then suppose that a "small" extra amount must be paid by every player to the casino which therefore makes a net profit in the long run, i.e. just as the extra zero in a roulette. While all this takes place, we notice that the casino may also choose the delay factor r in advance. Here one encounters a new optimisation problem related to the number of available machines where players try their luck. The point is of course that the fee for a play measured by $\mathcal{P}(r)$ plus a small quantity increases with r, and on the other hand the expected time before an individual player stops increases. So the Casino is confronted with optimization problem and it is not clear how to choose r in an optimal fashion.

Risky asssets

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Introduction.

We study *risky assets* where capital increases in a random way, ruled by "small independent changes" over small discrete time intervals. The CLT implies that the distribution of the risky asset at a later time T has a *log-normal* distribution. This can be used to compare the choice between a safe and a risky asset, given that the person has *risk aversion* measured by a strictly concave utility function. In Section 3 we describe how to optimize a portfolio with a part put in a safe asset and the rest in a risky asset.

1. The log-normal distribution

The *standard normal distribution* with mean-value zero and unit variance has the distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

Its frequency function becomes

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The normal distribution with mean value m and variance σ is denoted by $\mathbf{N}_\sigma(m)$. It has the distribution function $\Phi(\frac{x-m}{\sigma})$ and the frequency function

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}$$

1.1 The log-normal distribution For each pair m, σ we denote by $\mathcal{L}_\sigma(m)$ the random variable whose distribution is zero when $x \geq 0$ and if $x > 0$ it is defined by the increasing function

$$x \mapsto \Phi(\frac{\log(x) - m}{\sigma}) \tag{1}$$

Its frequency function becomes

$$\mathcal{L}'_\sigma(m)(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-(\log(x)-m)^2/2\sigma^2} \quad x > 0 \tag{2}$$

1.2 Moments of $\mathcal{L}_\sigma(m)$. For each $\alpha > 0$ the moment of order α is defined by

$$M_\alpha(\sigma,m)=\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty \frac{x^\alpha}{x}\cdot e^{-(\log(x)-m)^2/2\sigma^2}dx$$

1.3 Theorem *One has the formula* $M_\alpha(\sigma,m)=e^{\alpha m+\alpha^2\sigma^2/2}$.

Proof. The variable substitution $x\to e^t$ gives

$$M_\alpha(\sigma,m)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty e^{\alpha t}\cdot e^{-(t-m)^2/2\sigma^2}\cdot dt$$

To compute this integral we first use the substitution $t-m\to s$ and obtain

$$M_\alpha(\sigma,m)=\frac{1}{\sigma\sqrt{2\pi}}e^{\alpha m}\int_{-\infty}^\infty e^{\alpha s}\cdot e^{-s^2/2\sigma^2}\cdot dt$$

Using the substitution $s\to\sigma u$ and writing $\alpha\sigma u-u^2/2=-\frac{1}{2}(u-\alpha\sigma)^2-\alpha^2\sigma^2/2$ we get the asserted formula for $M_\alpha(\sigma,m)$.

1.4 The mean value and the central variance. Let ρ^2 be the central variance of $\mathcal{L}(\sigma,m)$. By definition it is the M_2-moment minus the square of the mean value. Hence Theorem 1.3. gives

(*) $$\rho^2=e^{2m+2\sigma^2}-e^{2m+\sigma^2}=e^{2m+\sigma^2}(e^{\sigma^2}-1)$$

With $\alpha=1$ we obtain the mean value

(**) $$e^{m+\sigma^2/2}$$

Notice that the mean value *increases* with σ.

2. Risky assets

At time zero we start with a capital $K_0>0$. Given a time interval $[0,T]$ and a large positive integer N we consider the discrete time values $t_\nu=\frac{\nu T}{N}$. With fixed positive numbers μ,σ we assume that the capital varies in a random way under the rule

$$K_{\nu+1}=K_\nu\cdot(1+\frac{\mu T}{N}+\frac{\sigma\sqrt{T}}{\sqrt{N}}\cdot\mathbf{B}_\nu)\quad \mathbf{0\le\nu\le N-1}$$

Here $\mathbf{B}_0,\dots,\mathbf{B}_{N-1}$ are independent two point distributions, i.e. $\{\mathbf{B}_\nu\}$ are random variables which takes one of the values +1 or -1 with equal probability 1/2. With $\nu=N$ we arrive at time T and put

(1) $$K_T(N)=K_0\cdot\prod_{\nu=0}^{\nu=N-1}(1+\frac{\mu T}{N}+\frac{\sigma\sqrt{T}}{\sqrt{N}}\cdot\mathbf{B}_\nu)$$

This random variable has a sample space with 2^N possible events, each of which is given by a sequence of +1 or -1 taken by the independent random variables $\mathbf{B}_0,\dots,\mathbf{B}_{N-1}$. We assume that N is so large that $1+\frac{\mu T}{N}-\frac{\sigma\sqrt{T}}{\sqrt{N}}>0$. So $K_T(N)$ is the product of N independent and equally distributed random variables where each individual random variable can take two positive values.

2.1 A limit formula. Above we have defined random variables $K_T(N)$ for each N. When $N \to \infty$ the distributions of these random variables converge. Namely , let $F_N(x)$ be the distribution of $K_N(T)$. Then one has the limit formula:

$$\mathrm{Lim}_{N\to\infty} F_N(x) = K_0 \cdot \Phi(\frac{\log(x) - \mu T + \sigma^2 T/2}{\sigma\sqrt{T}}) \tag{1.2}$$

Remark. During this passage to the limit one uses the Taylor expansion up to order 3 of the log-function, i.e. that

$$\log(1+x) = 1 + x - x^2 + O(x^3)$$

The negative term $-x^2/2$ gives rise to the term $-\sigma^2 T/2$ in the log-normal limit distribution above. Using *Monte Carlo* simulations of the discrete random process one can "discover" the limit formuka abovce which goes back to work by De Moivre from 1730 by comparing experimental results with the analytically defined log-normal distribution. Thus, with the aid of the computer one becomes familiar - or rather *confident* - with the limit formula (1.2).

2.2 A general limit formula Above we only used two-point variables. Using Lindeberg's additive version of the CLT and taking exponential one extends the previous limit theorem. More precisely, consider the case where capital changes in a random way over the discrete time values via the rules:

$$K_{\nu+1} = K_\nu \cdot (1 + \frac{\mu_\nu T}{N} + \frac{\sqrt{T}}{\sqrt{N}} \cdot \chi_\nu) \quad 0 \leq \nu \leq N-1$$

Here $\mu_1, \mu_2, \ldots$ is a sequence of real numbers and $\chi_1, \chi_2, \ldots$ a sequence of discrete random variables, each with mean value equal to zero. Exactly as in (1) we construct the product

$$K_T(N) = K_0 \cdot \prod_{\nu=0}^{\nu=N-1} (1 + \frac{\mu_\nu T}{N} + \frac{\sqrt{T}}{\sqrt{N}} \cdot \chi_\nu)$$

Let σ_ν^2 be the variance of χ_ν and suppose the following two limits exist as $N \to \infty$:

$$\frac{\mu_1 + \ldots + \mu_N}{N} \to \mu \quad \frac{\sigma_1^2 + \ldots + \sigma_N^2}{N} \to \sigma^2$$

In addition we assume that the random variables χ_ν do not have too *fat tails*, i.e. the last condition in Lindenberg's theorem hold. Under these assumptions the distribution functions of $K_T(N)$ converge to a log-normal distribution given by (1. 2) above.

2.3 Remark The limit theorem above, where one starts from the discrete random variables $\{K_N(T)\}$, yields a *continuous version* if one regards the sequences μ_ν and σ_ν as values at the discrete time values t_ν of continuous functions $\mu(t)$ and $\sigma(t)$. The finite sample spaces of the random variables $K_N(T)$ can be "glued together" during the passage to the limit. This follows from constructions due to Norbert Wiener. In fact, Wiener showed that the sample space the whole stochastic process of a Brownian motion can be described by tossing "heads or tails" in a denumerable sequence, i.e. every outcome is given by as sequence of 0 or 1 which also can be identified with the binary series of a real number on the interval [01]. From this one constructs the so called Wiener measure of the Brownian motion. Wiener's construction has a *conceptual merit* but is not very useful for numerical investigations by computers where *Monte Carlo* simulations rely on discrete random walks.

3. Moments and mean values

Consider a person who owns the capital K_0 at time zero. During a time interval $[0, T]$ there are two alternatives for the growth of capital. First, there is a *safe asset* with constant rate of interest μ_*. Using this safe asset the capital at time T will be $K_0 \cdot e^{\mu_* T}$. Next, suppose there also exists a *risky asset* governed by the rules from section 2. The risky asset has a constant rate of interest μ while $\sigma > 0$ measures the volatility. With N large the risky asset changes over the discrete time values $\nu T/N$ as in Section 2.

If the capital K_0 is placed in the risky asset the outcome at time T is random ,i.e. the capital K_T at time $t = T$ is random. This amount of capital is evaluated via utility function U which as usual is increasing and concave. The *expected utility* of the risky asset at time T becomes

$$(*) \qquad \frac{K_0}{\sigma_T \cdot \sqrt{2\pi}} \int_0^\infty U(x) \cdot x^{-1} \cdot e^{-(\log x - m_T)^2/2\sigma_T^2} \cdot dx$$

where

$$\sigma_T = \sigma \cdot \sqrt{T} \quad \text{and} \quad m_T = \mu T - \sigma^2 T/2$$

With the substitution $x \to e^s$ the integral (*) becomes

$$(**) \qquad \frac{K_0}{\sigma_T \cdot \sqrt{2\pi}} \int_{-\infty}^\infty U(e^s) \cdot e^{-(s-m_T)^2/2\sigma_T^2} \cdot ds$$

The case $U(c) = c^\alpha$. in this case (**) becomes

$$\mathbf{M}_\alpha = \frac{K_0}{\sigma_T \cdot \sqrt{2\pi}} \int_{-\infty}^\infty e^{\alpha s} \cdot e^{-(s-m_T)^2/2\sigma_T^2} \cdot ds =$$

$$\frac{K_0 \cdot e^{\alpha m_T}}{\sigma_T \cdot \sqrt{2\pi}} \int_{-\infty}^\infty e^{\alpha \xi} \cdot e^{-\xi^2/2\sigma_T^2} \cdot d\xi = \frac{K_0 \cdot e^{\alpha m_T}}{\cdot \sqrt{2\pi}} \int_{-\infty}^\infty e^{\alpha \sigma_T \cdot \eta} \cdot e^{-\eta^2/2} \cdot d\eta =$$

$$K_0 \cdot e^{\alpha m_T} \cdot e^{\alpha^2 \sigma_T^2/2} = K_0 \cdot e^{\alpha \mu T} \cdot e^{\sigma^2 T(\alpha^2 - \alpha)/2}$$

Since $\alpha < 1$ this number is strictly smaller than $e^{\alpha \mu T}$. So in order that the risky asset has a mean value equal to a safe asset whose rate of interest is μ_* we must have $\mu > \mu_*$. More precisely, the person has *neutral preference* between the safe and the risky asset when

$$(***) \qquad \mu + (\alpha - 1)\sigma^2/2 = \mu_*$$

The case $U(c) = \log(1+c)$. With this utility function we see that (**) above gives the expected utility

$$\frac{K_0}{\sigma_T \cdot \sqrt{2\pi}} \int_{-\infty}^{\infty} \log(1+e^s) \cdot e^{-(s-m_T)^2/2\sigma_T^2} \cdot ds$$

Exercise. The integral above can only be computed numerically. Given the pair μ, σ one uses the computer to find μ_* in order neutral preference holds between the risky and the safe asset.

3.1 Maximising a portfolio

At time zero we suppose that the person can put a fraction of the initial capital in a risky asset and the rest in a safe asset. We assume that $U(c) = c^\alpha$. It turns out that such a mixture can lead to a *higher expected profit*, even in the case when the neutral equality (**) holds. To see this, let $0 < s < 1$ and suppose that the person initially puts sK_0 in the risky asset and $(1-s)K_0$ into the safe asset. Then capital changes over the discrete time values $t_\nu = \nu T/N$ by the rule:

$$K_{\nu+1} = K_\nu \Big(1 + \frac{(1-s)\mu_* T}{N} + \frac{s\mu T}{N} + \frac{s\sigma\sqrt{T}}{\sqrt{N}} \cdot \mathbf{B}_\nu\Big) \tag{1}$$

By De Moivre's limit formula $K_T(N)$ converges as $N \to \infty$ to a log normal distribution whose distribution function is

$$\Phi\Big(\frac{\log(x) - m^*}{\sigma^*}\Big) \quad m^* = T(1-s)\mu_* + s\mu - s^2\sigma^2/2) \quad \sigma^* = s\sigma\sqrt{T} \tag{2}$$

We can evaluate the expected utility of this portfolio using (*). It is denoted by $M_\alpha(s)$ and we obtain

$$M_\alpha(s) = e^{\alpha T((1-s)\mu_* + s\mu - \sigma^2 s^2/2) + T\alpha^2\sigma^2 s^2/2} \tag{3}$$

Now we choose s in order to maximise (3). This amounts to find the maximum of the second order s-polynomial

$$(1-s)\mu_* + s\mu - \sigma^2 s^2/2 + \alpha\sigma^2 s^2/2$$

Regarding the s-derivative we find that (3) is maximised when

$$\sigma^2(1-\alpha)s = \mu - \mu_*$$

3.2 The risk neutral case When (***) holds above we see that $s = \frac{1}{2}$ yields a maximum. Inserting this s-value in (3) a computation which is left to the reader shows that expected profit of the chosen portfolio becomes

$$e^{\alpha T\mu_* + T\alpha^2\sigma^2/8}$$

Since the factor $e^{T\alpha^2\sigma^2/8} > 1$ this means that the chosen portfolio yields a strictly larger expected profit compared to the risk neutral safe asset.

4. Infinite time horizon

When $T = \infty$ we can use "dynamic programming",i .e. the classic Hamilton-Jacboi variation introduced around 1840 which derives a differential equation for expected maximal utility. Let U be some utility function and seek:

$$(*) \qquad \max_c \int_0^\infty U(c(t))e^{-rt}dt$$

Here $c(t)$ is consumption taken from a capital ruled by the stochastic equation

$$(1) \qquad dK = \mu K - c(t) + \sigma K \cdot dW$$

where W is the standard Wiener process. At time zero one has $K(0) = K_0 > 0$. Moreover, μ and r are positive constants and we assume that $r > \mu$. Finally we impose the condition $K(t) \geq 0$ for all $t \geq 0$. This means that it is not possible to keep the consumption to high. To solve this stochastic optimisation problem we study the value function. To each initial capital $K > 0$ we denote by $V(K)$ the *expected value under optimal consumption*. If dt is a small initial time interval where $c(t) = c$ is kept constant, then Taylor's formula shows that the expected profit becomes $V(K)$ plus

$$(2) \quad U(c)dt - rV(K)dt + \mu K V'_K dt - cV'(K)dt + \sigma^2 K^2 V''(K)dt + \text{small ordo}(dt)$$

To attain maximal expected profit c is chosen to maximise $U(c) - cV'(K)$. This gives

$$(3) \qquad U'(c) = V'(K)$$

Since U is a utility function this determines $c = c(K)$ uniquely and now V satisfies the second order differential equation

$$(*) \qquad U(c(K)) - rV + \mu K V'(K) - c(K)V'(K) + \sigma^2 K^2 V''(K) = 0$$

The case $U(c) = \sqrt{c}$. In this case

$$c(K) = \frac{1}{4 \cdot V'(K)^2}$$

Hence the ODE (*) becomes

$$(**) \qquad \frac{1}{4V'(K)} - rV + \mu K V'(K) + \sigma^2 K^2 V''(K)/2 = 0$$

Here one can "guess" the solution to be of the form

$$(i) \qquad V(K,T) = a(K) \cdot \sqrt{K}$$

Indeed, we see that (**) will be satisfied when the a-function satisfies the linear ODE-equation

$$(ii) \qquad \frac{1}{2a} - ra + \frac{\mu a}{2} - \frac{\sigma^2 a}{8} = 0$$

Exercise. Perform numerical investigations where the parameters change. For example, analyze how the expected profit changes with respect to σ and r. In text-bboks examples of thisind are often chosen with a simple U-function such as c^α with

$0 < \alpha < 1$ which by the Hamilon-Jacobi metheod yields explicit solutions sitiub-vsle for examination problems. Today one should studt the general case and find numerical solutions where the U-functions is more general, i.e. strictly increasing and concave.