IV. Nevanlinna-Pick theory

Contents

Introduction.

In the unit disc D there exists the a metric defined by

$$\frac{|dz|}{1-|z|^2}$$

In a joint article from 1916, Lindelöf and Pick discovered that if $f(z) \in \mathcal{O}(D)$ has maximum norm ≤ 1, then the map $z \to f(z)$ does not increase the metric (*). This result turns out to be very useful and is applied in § 2 to give a proof of a theorem due Julia. In § 3 we prove some results due to Löwner about geometric properties of analytic mappings. Section 0 is devoted to an interpolation theorem due to Nevanlinna and Pick. We give a detailed proof since the result has a wide range of applications beyond analytic function theory in various optimization problems.

0. The Nevanlinna-Pick Interpolation Theorem

Let D be the open unit disc. Given an n-tuple of distinct points $z_1, \ldots, z_n$ in D and some n-tuple $w_1, \ldots, w_n$ of complex numbers we put:

$$(*) \qquad \rho(z(\cdot), w(\cdot)) = \min_{f \in \mathcal{O}(D)} |f|_D \quad : \quad f(z_\nu) = w_\nu \quad : 1 \leq \nu \leq n$$

Thus we seek to interpolate preassigned values at the points $\{z_k\}$ with an analytic function $f(z)$ whose maximum norm is minimal. The case $n = 1$ is trivial for then it is obvious that the constant function $f(z) = w_1$ minimizes (*) so $\rho(z_1, w_1) = |w_1|$ hold for all $\alpha_1 \in D$. If $n \geq 2$ there exists at least some $f \in \mathcal{O}(D)$ which gives a minimum. For let $\{f_\nu\}$ be a sequence of functions which solve the interpolation while their maximum norms tend to $\rho(\alpha(\cdot), w(\cdot))$. This is a normal family and hence we extract a subsequence which converges to a limit function f_* whose maximum norm is equal to $\rho(z(\cdot), w(\cdot))$. It turns out that the minimizing f is unique and of a special form. Before Theorem 1 is announced we introduce the class $\mathfrak{B}_{n-1}$ which consists of functions of the form:

$$(**) \qquad f(z) = e^{i\theta} \cdot \prod_{\nu=1}^{\nu=n-1} \frac{z-\alpha_\nu}{1-\bar{\alpha}_\nu \cdot z}$$

where $0 \leq \theta \leq 2\pi$ and $(\alpha_1, \ldots, \alpha_{n-1})$ is some $(n-1)$-tuple of points in D which are not necessarily distinct.

0.1. Theorem *For each pair of n-tuples $z(\cdot)$ and $w(\cdot)$ there exists a unique $f_* \in \mathfrak{B}_{n-1}$ and a positive real number ρ such that the $\rho \cdot f_*(z)$ minimizes the interpolation (*).*

Remark. With $\rho = \rho(z(\cdot), w(\cdot))$ the uniquenss means that if $g \in \mathcal{O}(D)$ is an arbitrary interpolating function which is $\neq f_*$ then $|g|_D > \rho$.

The proof of Theorem 0.1 requires several steps. First we shall establish a result about Blaschke products.

0.3 Proposition. *Let f be a function in $\mathfrak{B}_n$. For every $k(z) \in \mathcal{O}(D)$ with maximum norm $|k|_D = 1$ such that $f - k$ has at least n zeros counted with multiplicity in D, it follows that $f = k$.*

Proof. We argue by a contradiction. If $k \neq f$ we denote by $N(f-k:r)$ the number of zeros of $f-k$ in $|z|<r$ counted with multiplicities. The hypothesis gives some $r_* < 1$ such that

(ii) $$N(k-f, r_*) \geq n$$

Next, to each 0< $r<1$ we set

$$\mathfrak{m}(r) = \min_\theta |f(re^{i\theta}) - k(re^{i\theta})|$$

The hypothesis gives a sequence $\{r_\nu\}$ such hat $r_\nu \to 1$ and every $\mathfrak{m}(r_\nu) > 0$. Consider some $r_\nu > r_*$ and the analytic function

(iii) $$h_\nu = \epsilon_\nu f + \frac{1}{2}(f-k) \quad \text{where} \quad \epsilon_\nu = \frac{1}{4}\mathfrak{m}(r_\nu)$$

Since $|\frac{1}{2}(f-k)| > \epsilon_\nu \cdot |f|$ holds on the circle $|z| = r_\nu$ it follows from (i) and Rouche's theorem that we have:

(iv) $$N(h_\nu : r_\nu) = N(f-k : r_\nu) \geq n$$

At the same time we can write

(v) $$h_\nu = (1+\epsilon_\nu)f - \frac{1}{2}(k+f)$$

By assumption $|k|_D = |f|_D = 1$ and since f is a finite Blaschke product its absolute value tends uniformly to zero as $|z| \to 1$. So if r_ν is sufficiently close to 1 we get:

(vi) $$(1+\epsilon_\nu) \cdot |f(z)| > \frac{1}{2}|k(z) + f(z)| \quad : \quad |z| = r_\nu$$

Then another application of Rouche's theorem gives:

(vii) $$N(h_\nu : r) = N(f : r) \leq n-1$$

where the last inequality follows since $f \in \mathfrak{B}_n$. Now (vii) contradicts (iv) and hence we must have $k = f$ as requested.

A consequence. Let $z(\cdot)$ and $w(\cdot)$ be some pair of n-tuples and suppose there exists some $f_* \in \mathfrak{B}_{n-1}$ and some $\rho > 0$ such that $\rho \cdot f_*(z_k) = w_k$ hold for each k. Then the function $f = \rho \cdot f_*$ not only interpolates but it has also the minimal maximum norm, i.e. we have the equality $\rho = \rho(z(\cdot), w(\cdot))$. For suppose we have strict inequality $\rho(z(\cdot), w(\cdot)) < \rho$ which gives an interpolating function $k(z)$ with maximum norm $|k|_D < \rho$. Since $|\rho \cdot f| = \rho$ holds on $|z| = 1$ it follows by Rouche's theorem that f and $f-k$ has the same number of zeros i D. Now $f-k$ has at least n zeros while f has at most $n-1$ many zeros. This contradiction and proves that f is minimizing and Proposition 0.2 entails also that f is uniquely determined in $\mathfrak{B}_{n-1}$.

The results above show that there remains to establish an *existence result*. Namely, there remains to prove that for every given n-tuple $z(\cdot)$ in D and an arbitrary n-tuple $w(\cdot)$ of complex numbers, there exists a pair $f_* \in \mathfrak{B}_n$ and $\rho > 0$ such that $\rho \cdot f_*$ solves the interpolation. Moreover, if the pair has been found then we have the equality $\rho = \rho(z(\cdot), w(\cdot))$.

0.4 Induction over n. We shall prove the existence above by an induction over n. First we establish a result where Möbius transformations intervene. To each n-tuple $(z_1, \ldots, z_n)$ we denote for every $\rho > 0$ the family $I_\rho(z(\cdot))$ of all n-tuples $w(\cdot)$ such that $\rho(z(\cdot), w(\cdot)) = \rho$.

0.5 Exercise. For each point $a \in D$ we have the Möbius transformation

$$M_a(z) = \frac{z-a}{1-\bar{a}z}$$

If $f(z) \in \mathcal{O}(D)$ we get the new analytic function $f \circ M_a$ which has the same maximum norm as f. Use this to show that

$$I_\rho(z(\cdot)) = I_\rho(M_a(z(\cdot)))$$

hold for every n-tuple $z(\cdot)$. Thus, we can change the z-points via a Möbius transformation without affecting the I_ρ-sets.

0.6 The induction step. If $g \in \mathfrak{B}_{n-1}$ and $|a| < 1$ then the composed function $g \circ M_a$ again belongs to $\mathfrak{B}_{n-1}$. So by Exercise 0.5 it suffices to establish the existence of an interpolation $f \in \mathfrak{B}_{n-1}$ when $z_1 = 0$. So assume this and consider first the case $w_1 = 0$. Then we seek f of the form

$$f(z) = z \cdot g(z) \tag{1}$$

where g satisfies $g(z_k) = \frac{w_k}{z_k}$ when $2 \leq k \leq n-1$. By an an induction over n we can find $g = \rho \cdot g_*$ for some $g_* \in \mathfrak{B}_{n-2}$ and (1) entails that

$$f = \rho \cdot z \cdot g_*$$

where $z \cdot g_*$ belongs to $\mathfrak{B}_{n-1}$ which gives the requested interpolating function.

The case $w_1 \neq 0$. To begin with we find $f \in \mathcal{O}(D)$ with the maximum norm $\rho = \rho(z(\cdot), w(\cdot))$ such that $f(0) = w_1$ and $f(z_k) = w_k$ when $2 \leq k \leq n$. Put

$$\mu_k = \frac{w_k - w_1}{1 - \rho^{-2} \cdot \bar{w}_1 \cdot w_k} \tag{1}$$

We have also the analytic function

$$g(z) = \frac{f(z) - w_1}{1 - \rho^{-2} \cdot \bar{w}_1 \cdot f(z)}$$

Since $|f|_D = \rho$ we also have $|g|_D = \rho$ where $g(z_k) = \mu_k$ for each $k \geq 2$ while $g(0) = 0$. In particular

$$(0, \mu_2, \ldots, \mu_n) \in I_\gamma(0, z_2, \ldots, z_n) \quad \text{where} \quad \gamma \leq \rho$$

By the induction over n and the previous special case we find $f_* \in \mathfrak{B}_{n-2}$ such that the function

$$g_* = \gamma \cdot z \cdot f_*$$

takes the interpolating values $(0, \mu_2, \ldots, \mu_n)$. Next, there exists the analytic function

$$\phi(z) = \frac{g_*(z) + w_1}{1 + \rho^{-2} \cdot \bar{w}_1 \cdot g_*(z)}$$

If $\gamma < \rho$ we see that the maximum norm $|\phi|_D < \rho$. At the same time (1) entails that $\phi(0) = w_1$ and $\phi(z_k) = w_k$ for $k \geq 2$. Since ρ was the interpolation norm for the pair $(0, z_2, \ldots, z_n)$ and the n-tuple $w(\cdot)$ we have contradiction. Hence $\gamma = \rho$ holds and from the above it follows that

$$\rho \cdot \frac{z f_*(z) + w_1}{1 - \rho^{-2} \bar{w}_1 z f_*(z)}$$

takes the requested interpolation values $(w_1, \ldots, w_n)$ which gives the induction step over n.

Remark. Consider the case $n = 2$ in Theorem 0.1 where we assume that $w_1 \neq w_2$. Theorem 1 shows that there exists a unique triple

$$(a, \theta, \rho) \quad \text{where} \quad a \in D \quad : 0 \leq \theta \leq 2\pi \quad : \rho > 0$$

such that function

$$f(z) = \rho \cdot e^{i\theta} \cdot \frac{z-a}{1 - \bar{a} \cdot z}$$

solves the interpolation problem.

0.7 Exercise. With $n=2$ and $z_1=0$ while $z_2\neq 0$ is kept fixed we solve the interpolation for a given pair (w_1,w_2), The minimizing interpolation function is of the form

$$f(z)=\zeta\cdot\frac{z+\frac{w_1}{\zeta}}{1+\frac{\bar{w}_1\cdot z}{\zeta}}$$

for some $\zeta\neq 0$. So here ζ satisfies the equation

$$z_2\cdot\zeta+w_1=w_2+w_2\bar{w}_1\cdot z_2\cdot\frac{1}{\bar{\zeta}}$$

Writing $w_2=w_1+\gamma$ this amounts to solve the equation

(*) $$z_2\cdot|\zeta|^2=\gamma\cdot\bar{\zeta}+|w_1|^2\cdot z_2+\gamma\cdot\bar{w}_1\cdot z_2$$

We assume that $w_2\neq w_1$ which means that $\gamma\neq 0$ and that the minimizing function f is not reduced to a constant. Hence its maximum norm $|\zeta|$ must be $|w_1|$, Theorem 1 implies that under these conditions (*) has a unique solution ζ with absolute value $>|w_1|$. It is interesting to analyze how $|\zeta|$ depends on the triple z_2,w_1w_2. Dividing (*) with z_2 and regarding $\lambda=\gamma/z_2$ as a parameter which varies we are led to the equation

$$|\zeta|^2-\lambda\cdot\bar{\zeta}=|w_1|^2+\gamma\cdot\bar{w}_1$$

The reader is invited to analyze the behaviour of $|\zeta|$ with a special attention to the case when z_2 is close to the origin while γ stays fixed, So here the interpolating function f takes quite distinct values at the origin while z_2. So one expects that its maximum norm increases. Here is

0.8 A specific example. Let $\gamma=1$ and $z_2=\epsilon$ for some small positive ϵ while $w_1=a$ is real and positive. So the equation becomes

$$|\zeta|^2-\frac{\bar{\zeta}}{\epsilon}=a^2+a$$

The solution ζ is therefore real and we are led to the algebraic equation

$$s^2-\frac{s}{\epsilon}=a^2+a$$

Notice that we require that $|\zeta|>|w_1|=a$ so we seek the unique root s for which $s>a$ and it is given by

$$s=\frac{1}{2\epsilon}+\sqrt{a+a^2+4^{-1}\epsilon^{-2}}$$

With a kept fixed we obtain $|\zeta|\simeq\frac{1}{\epsilon}$ as $\epsilon\to 0$ which illustrates that the maximum norm of the interpolating function increases when $\epsilon\to 0$.

0.9 Interpolation constants. Let $E=(z_1,\ldots,z_n)$ be given. Each $f\in\mathfrak{B}_{n-1}$ has $n-1$ many roots counted with multiplicities in D. In particular f cannot vanish identically on E, i.e the maximum norm

$$|f|_E=\max|f(z_k)|>0$$

This leads us to define the number

$$\tau(E)=\min_{f\in\mathfrak{B}_{n-1}}|f|_E$$

We have also the interpolation number:

$$\mathtt{int}(E)=\max_{w(\cdot)}\rho(z(\cdot),w(\cdot))$$

with the maximum taken over all w-sequences with $|w_k|\leq 1$ for every k. With these notations one has the following result which is due to Beurling:

0.9 Theorem. *For every finite set E one has the equality*

$$(*) \qquad \tau(E) = \frac{1}{\mathfrak{int}(E)}$$

Moreover, a function $f \in \mathcal{B}_{n-1}$ which gives $|f|_E = \tau(E)$ is unique up to a constant and for such an extremal f one has $|f(\alpha_k)| = \tau(E)$ for every $1 \leq k \leq n$.

Proof. With n kept fixed the family of $\mathcal{B}_{n-1}$ enjoys normal properties in the sense of Montel so it follows that there exists at least some extremal $f \in \mathcal{B}_{n-1}$ such that $|f|_E = \tau(E)$. Now we prove that $|f(\alpha_k)| = \tau(E)$ for each k. For suppose strict inequality holds at some α-point which we can take to be α_1. Consider the Blaschke product

$$B(z) = \prod_{k=2}^{k=n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k \cdot z}$$

Rouche's theorem gives some $\delta > 0$ such that if $|\zeta| < \delta$ then the analytic function $f(z) + \zeta \cdot B(z)$ has $n-1$ zeros in D and we can therefore write

$$(1) \qquad f(z) + \zeta \cdot B(z) = \rho(\zeta) \cdot \psi_\zeta(z)$$

where the ζ-indexed ψ-functions belong to $\mathcal{B}_{n-1}$ and $\rho(\zeta)$ are complex numbers. Notice that

$$(2) \qquad f(\alpha_k) = \rho(\zeta) \cdot \psi_\zeta(\alpha_k)$$

hold when $2 \leq k \leq n$. Moreover, since $|f(\alpha_1)| < \tau(E)$ it is clear by continuity that if δ is sufficiently small then $|\psi_\zeta(\alpha_1)| < \tau(E)$ when $|\zeta| < \delta$. Since f is extremal we conclude from (2) that there exists $\delta > 0$ such that

$$(3) \qquad |\zeta| < \delta \implies |\rho(\zeta)| \geq 1$$

This gives a contradiction since the absolute value of the ρ-function cannot have a relative minimum at $\zeta = 0$ by the local complex expansion of this ρ-function in Chapter III:XX.

Uniqueness. Let f and g be two extremal functions so that $|f|_E = |g|_E = \tau(E)$ and suppose they are not identical. For each ζ where $|\zeta| < \delta$ for a sufficiently small δ we can write

$$1 - \zeta) \cdot f + \zeta \cdot g = \rho(\zeta) \cdot \psi_\zeta(z)$$

with $\psi_\zeta \in \mathcal{B}_{n-1}$. The triangle inequality gives

$$|1 - \zeta) \cdot f(\alpha_k) + \zeta \cdot g(\alpha_k)| \leq \tau(E)$$

for every k and since $|\psi_\zeta| \geq \tau(E)$ we get as above that $|\rho(\zeta(\geq 1$ whenever ζ is sufficiently close to zero. This contradicts again the complex expansion of this ρ-function from Chapter III.

The equality $\mathfrak{int}(E) = \frac{1}{\tau(E)}$. To begin with, let f be the unique extremal above which gives an n-tuple of points on the unit circle so that

$$f(\alpha_k) = \tau(E) \cdot e^{i\theta_k}$$

The Nevanlinna-Pick theorem shows that $\frac{f((z)}{\tau(E)}$ has smallest maximum norm over D when the n-tuple $\{w_k = e^{i\theta_k}\}$. This implies that

$$\mathfrak{int}(E) \geq \frac{1}{\tau(E)}$$

To prove the opposite inequality we consider some n-tuple $\{w_\bullet\}$ for which the interpolating function $g(z)$ has the maximum norm $|g|_D = \mathfrak{int}(E)$. Theorem 0.1 gives

$$g = \mathfrak{int}(E) \cdot f \quad \text{where} \quad f \in \mathcal{B}_{n-1}$$

This entails that

$$\tau(E) \leq |f|_E \leq \frac{1}{\mathfrak{int}(E)}$$

and the requested equality (*) in Theorem 0.9 follows.

1. The Lindelöf-Pick principle.

Introduction. The non-euclidian metric on D is defined by

$$\frac{|dz|}{1-|z|^2} \quad : \; |z|<1 \tag{0.1}$$

When D is equipped with this metric one gets a model of hyperbolic geometry in the sense of Bolyai and Lobatschevsky which led to an intense geometric study around 1890, foremost by F. Klein and H. Poincaré. We shall not enter a detailed discussion about the geometry since our main concern is to apply the metric (0.1) to derive inequalities for analytic functions. In a work from 1916, Lindelöf and Pick discovered that very analytic function $\phi(z)$ in the unit disc with maximum norm one at most decreases the metric (0.1). This result is called the Lindelöf-Pick principle and is proved in Theorem XX below. In section XX it is used to prove a result by Caratheodory and Julia concerned with the boundary behaviour of analytic functions.

1. Schwarz' inequality The non-euclidian distance between two point z_1 and z_2 in D will be denoted by

$$\mathfrak{h}(z_1, z_2) \tag{1.1}$$

To grasp this distance function we first notice the equality:

$$\mathfrak{h}(0,z) = \frac{1}{2}\cdot \operatorname{Log}\frac{1+|z|}{1-|z|} \tag{*}$$

Indeed, (*) follows since it is obvious from (0.1) that the geodesic curve from the origin to a point $z \in D$ is the ray from 0 to z. So with $|z|=r$ one computes

$$\int_0^r \frac{ds}{1-s^2}$$

which after integration gives (*). Next, with $a \in D$ we consider a Möbius transformation:

$$w = \frac{z-a}{1-\bar{a}\cdot z} \implies \frac{dw}{dz} = \frac{1-|a|^2}{(1-\bar{a}\cdot z)^2}$$

At the same time we notice that

$$1-|w|^2 = \frac{|1-\bar{a}z|^2-|z-a|^2}{|1-\bar{a}\cdot z|^2} = (1-|a|^2)\cdot\frac{1-|z|^2}{|1-\bar{a}\cdot z|^2}$$

From this the reader may deduce that the Möbius transform preserves the $\mathfrak{h}$-metric.

1.1 Example. Take $z_1 = 1/2$ and $z_2 = e^{i\theta}/2$ with some $0<\theta<\pi$. Now

$$z \mapsto \frac{z-1/2}{1-z/4}$$

sends z_1 to the origin. It follows that

$$\mathfrak{h}(1/2, e^{i\theta}/2) = \frac{1}{2}\cdot\operatorname{Log}\frac{1+r}{1-r} \quad : \; r = \frac{2\cdot|e^{i\theta}-1|}{|2-e^{i\theta}|}$$

The following consequence of Schwarz inequality was discovered by G. Pick in 1915.

1.2 Theorem. *Let $\phi\colon D \to \Omega$ be a conformal map from the unit disc onto a simply connected domain contained in $|w|<1$. Then the non-euclidian metric decreases.*

Proof. Let $z_0 \in D$ and set $w_0 = \phi(z_0)$. The quotient

$$G(z) = \frac{\phi(z)-w_0}{1-\bar{w}_0\phi(z)} : \frac{z-z_0}{1-\bar{z}_0 z}$$

Since

$$\lim \frac{|z - z_0|}{|1 - \bar{z}_0 z|} = 1 \quad \text{as} \quad |z| \to 1$$

we see that $|G(z)| \leq 1$ holds for all $z \in D$. With $z = z_0$ we have

$$G(z_0) = \phi'(z_0) \cdot \frac{1 - |z_0|^2}{1 - |\phi(z_0)^2|}$$

Since $z_0 \in D$ was arbitrary we get the differential inequality

$$\frac{|d\phi(z)|}{|1 - \phi(z)|^2} \leq \frac{|dz|}{1 - |z|^2}$$

and this is precisely the assertion in Pick's theorem.

The Lindelöf-Pick principle. Above ϕ was a conformal mapping. Since the $\mathfrak{h}$-metric is defined locally the inequality in Pick's theorem extends to analytic functions in D of absolute value < 1 and leads to the following general result:

1.3 Theorem *Let $\phi(z) \in \mathcal{O}(D)$ have maximum norm ≤ 1. Then ϕ decreases the $\mathfrak{h}$-metric.*

Remark. Thus, if we set $w = \phi(z)$ and z_1, z_2 is a pair in the unit disc D_z one has

(*) $$\mathfrak{h}(\phi(z_1), \phi(z_2)) \leq \mathfrak{h}(z_1), z_2)$$

1.4 The $\mathfrak{h}$-metric in half-spaces. Passing to the right half-plane U_+ where $\mathfrak{Re}(w) > 0$, the non-euclidian metric is obtained via the conformal map

$$z \mapsto w = \frac{1+z}{1-z}$$

From this it follows that

$$\frac{|dz|}{1 - |z|^2} \mapsto 2 \cdot \frac{|w+1|^4 \cdot |dw|}{|w+1|^2 - |w-1|^2}$$

So with $w = \xi + i\eta$ the non-euclidian metric in the right half-plane becomes

(*) $$\frac{|w+1|^4 \cdot |dw|}{2\xi}$$

Next, the Lindelöf-Pick principle applies after a conformal mapping from D onto any other simply connected domain Ω where one then regards analytic functions $g \in \mathcal{O}(\Omega)$ such that $g(\Omega) \subset \Omega$.

1.5 Example. Let $\Phi(z) = u(x,y) + iv(x,y) \in \mathcal{O}(U^+)$ be such that its real part u is positive in U_+. The Lindelöf-Pick principle applies to Φ and using (*) in (1.4) one has the following result:

1.6 Proposition. *To every $k > 0$ there exists another constant k^* such that the following inequality holds for every pair of points $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$ in U_+:*

$$|\Phi(x_1 + iy_1)| \leq |v(x_0 + iy_0)| + k^* \cdot \frac{x_1 \cdot u(x_0, y_0)}{x_0} \quad : \ |y_1| < k \cdot x_1$$

1.7 Exercise. Try to prove this result. If necessary, consult the text-book [Nevanlinna: page 59-61] for a proof where it is also shown that for each $k > 0$ one can take

(*) $$k^* = 3 + 2(k+1)^2 \quad : \text{ provided that } x_1 > x_0 \quad \text{and} \quad x_1 > |y_0|$$

2. A result by Julia.

Let $\phi \in \mathcal{O}(D)$ be such that $|\phi(z)| < 1$ when $z \in D$ and consider the boundary point $z = 1$.

2.1 Theorem. *For every $e^{i\theta}$ there exists the limit*

$$(1) \qquad c(\theta) = \lim_{z \to 1} \frac{|e^{i\theta} - \phi(z)|}{|1 - z|} \quad : 0 \leq c(\theta) \leq +\infty$$

where the limit $z \to 1$ is taken in any Fatou sector at 1. Moreover, if θ is such that the limit $0 < c(\theta) < \infty$ then there exist the Fatou limits:

$$(2) \qquad \phi'(z) \to c(\theta) \cdot e^{i\theta} \quad : \arg \frac{e^{i\theta} - \phi(z)}{1 - z} \to \theta$$

and the following inequality holds

$$(3) \qquad \frac{1 - |\phi(z)|^2}{|e^{i\theta} - \phi(z)|^2} \geq \frac{1}{c} \cdot \frac{1 + |z|}{1 - |z|} \quad : z \in D$$

Remark. Of course, only the case when $c(\theta) < \infty$ is of interest. Notice that this finiteness only can occur for at most one θ-value. The theorem above was the starting point for an extensive study of boundary values of analytic functions in Julia's work [Ju] and has later led to a far-reaching study about Julia sets in complex dynamics. See [Carleson-Garnett] for this more recent and advanced theory in function theory. The reader may also consult Chapter IV in [Caratheodory] for an account of Julia's original theorem where some geometric interpretations appear.

Proof of Theorem 2.1

Applying the two conformal mappings

$$z \mapsto \frac{1 + z}{1 - z} \quad : w \mapsto \frac{e^{i\theta} + w}{e^{i\theta} - w}$$

we can work in the right half plane where $z = 1$ has been mapped into the point at infinity and ϕ has become an analytic function

$$\Phi(x + iy) = u(x + iy) + iv(x + iy) \quad : u(x, y) > 0 \text{ for all } (x, y) \in U_+$$

The crucial step in the proof is to show the result below:

Let $\Phi = u + iv$ be an arbitrary analytic map from U_+ to U_+ and assume that

$$(*) \qquad \min_{x+iy \in U_+} \frac{u(x + iy)}{x} = 0$$

Then it follows that

$$\lim_{x \to +\infty} \frac{u(x + iy)}{x} = 0 \quad : \text{holds uniformly inside any Fatou sector } |y| < kx \quad : k > 0$$

To prove this we take some $k > 0$ and for each $\epsilon > 0$ the hypothesis (*) gives a point $z_0 = x_0 + iy_0$ in U_+ such that

$$(1) \qquad \frac{u(x_0, y_0)}{x_0} < \epsilon$$

Next, if $z = x + iy$ stays in the Fatou sector $|y| < k|x|$ and x_1 is large then Proposition 1.6 gives:

$$|\Phi(x + iy)| \leq |v(x_0 + iy_0)| + k^* \cdot \frac{x \cdot u(x_0 + iy_0)}{x_0} < |v(x_0 + iy_0)| + \epsilon \cdot k^* \cdot x$$

In particular we have

$$\frac{u(x + iy)}{x} < \frac{|v(x_0 + iy_0)|}{x} + \epsilon \cdot k^*$$

Since $\epsilon > 0$ can be chosen arbitrary small the conclusion after (*) follows.

Proof continued. Next, suppose that

$$c = \min_{x+iy\in U_+} \frac{u(x+iy)}{x} > 0 \tag{1}$$

is positive. The result above applies to $\Phi(z) - cz$ and hence $\frac{\Phi(z)}{z} \to c$ holds uniformly as $|z| \to \infty$ inside any Fatou sector $|y| < k|x|$. Moreover, this gives:

$$\liminf_{x\to\infty} \frac{u(x,y)}{x} = c \tag{2}$$

Let us no consider the complex derivative of Φ assuming that (1) above holds for some $c > 0$.

Sublemma One has

$$\lim_{z\to\infty} \Phi'(z) = c$$

where this limit holds uniformly while z stays in any given Fatou sector.

Proof. Replacing Φ by $\Psi(z) = \Phi(z) - cz$ it suffices to show that

$$\lim_{z} \Psi'(z) = c \quad : \text{uniformly when the limit is in a Fatou sector} \tag{i}$$

To show (i) we proceed as follows. Consider some $0 < p < 1$ and choose also some q so that $p < q < 1$. For every $r > 0$ we consider the disc

$$\Delta_r = \{|z - r| < q \cdot r\}$$

Since $q < 1$ this disc stays in a fixed Fatou sector for all large r and Cauchy's inequality gives

$$|\Psi'(z)| \leq \frac{qr}{2\pi} \int_0^{2\pi} \frac{|\Psi(r + qre^{i\theta})|}{|r + qre^{i\theta} - z|^2} \cdot d\theta \quad : \quad z \in \Delta_r \tag{ii}$$

Next, if $\epsilon > 0$ Propostion 1.6 gives some large r^* such that

$$|\frac{\Psi(\zeta)}{\zeta}| < \epsilon \quad : |\zeta - r| = qr \quad : \; r \geq r^* \tag{iii}$$

Hence, if $|z - r| \leq pr$, the Cauchy inequality from (ii) and a computation which is left to the reader gives:

$$|\Psi'(z)| \leq \epsilon \cdot \frac{q(1+q)}{(q-p)^2} \tag{iii}$$

This proves that $\Psi'(z) \to 0$ holds uniformly when z stays in the sector

$$|\arg z| < \text{arc.sin}(p)$$

Above $p < 1$ is arbitrary which therefore gives the Caratheodory-Julia theorem after we have returned to the unit disc via a conformal map between D and U_+.

3. Some geometric results

3.1 A study of convex domains. Let Ω be a bounded convex domain and $p \in \Omega$ an interior point. The convexity implies that if we start at some boundary point $q_0 \in \partial\Omega$ where $q_0 - p$ is real and positive, then we obtain a function

$$(*) \qquad \phi \mapsto q(\phi) \quad : \arg\left[q(\phi) - p\,\right] = \phi \quad : q(\phi) \in \partial\Omega$$

where $q(2\pi) = q_0$ holds after one turn. The q-function is continuous and 1-1, i.e. a homeomorphism between the unit circle and $\partial\Omega$. Let $g(\phi)$ be a non-negative continuous function on T, i.e here $g(2\pi) = g(0)$. We get $g^* \in C^0(\partial\Omega)$ satisfying

$$g^*(q(\phi)) = g(\phi)$$

Starting from g^* we solve the Dirichlet problem and find the harmonic function G^* in Ω which extends g^*. With these notations we have

Theorem 3.2 *One has the inequality*

$$G^*(p) \leq \frac{1}{\pi} \int_0^{2\pi} g(\phi) d\phi$$

Remark. The inequality is of special interest when p approaches the boundary. Before Theorem 3.2 is proved we consider a general situation. Let W be any bounded Jordan domain and $p \in W$ an interior point. Let a, b be two points on ∂W. Denote by γ the Jordan subarc of ∂W which joins a and b. Let L be the line passing through these two points. Suppose that the two infinite half lines from a and b are outside W, i.e. $W \cap L$ is contained in the line segment (a, b). Now L cuts W into two halfs. Let W^* be one of these. Given a point $p \in W^*$ we shall find an upper bound for the harmonic measure $\mathfrak{m}_W(p;\gamma)$. After a rotation and a translation we may assume that $a = m$ and $b = -m$ for some $m > 0$, i.e. $[a, b]$ is an interval on the real axis and that W^* is contained in the upper half plane $U^+ = \mathfrak{Im}(z) > 0$. Now $W \subset U$ and Carleman's principle from XX gives:

$$(1) \qquad \mathfrak{m}_W(p;\gamma) \leq \mathfrak{m}_{W^*}(p : [a.b]) \leq \mathfrak{m}_U^+(p : [a.b])$$

By the result in XXX the last term is equal to $\frac{1}{\pi} \cdot \alpha$ where α is the angle formed by $a - p$ and $b - p$.

Proof of Theorem 3.2. Consider a small arc $\gamma \subset \partial\Omega$ which by the parametrisation (*) above is defined by some ϕ-interval $\phi_* \leq \phi \leq \phi^*$. Let $\mathfrak{m}_\Omega(p : \gamma)$ be the harmonic measure at p with respect to this boundary arc. We can apply the inequality (1) and conclude that

$$\mathfrak{m}_\Omega(p : \gamma) \leq \phi^* - \phi_*$$

Now the Theorem 3.2 follows after an integration over $0 \leq \phi \leq 2\pi$ where we use that $G^*(p)$ is evaluated by the integral of g^* over $\partial\Omega$ with respect to the positive measure on $\partial\Omega$ defined by the harmonic measure at p.

3.3. On the range of analytic functions

Consider a domain $\Omega \in \mathcal{D}(C^1)$. Let $\phi \in \mathcal{O}(\Omega)$ and assume it extends to $C^0(\bar{\Omega})$. The ϕ-function is not supposed to be 1-1. We get the domain

$$W = \phi(\Omega)$$

Now the following may occur: There exists a subset Γ of $\partial\Omega$ given as a finite union of arcs $\{\gamma_\alpha\}$ such that the image set $\phi(\Gamma)$ gives the boundary ∂A of a domain $A \subset W$, i.e. here A is a relatively compact subset of the connected open set W. Put

$$\Omega_* = \{z \in \Omega \quad : \phi(z) \in W \setminus A\}$$

Here $A \subset \partial(W \setminus A)$ and we construct a harmonic measures as follows: If $z \in \Omega_*$ we have $\phi(z) \in W \setminus A$ and get the function

$$z \mapsto \mathfrak{m}_{W\setminus A}\,(\phi(z); \partial A) \quad : z \in \Omega_*$$

Since $w \mapsto \mathfrak{m}_{W\setminus A}(w; \partial A)$ is a harmonic function in $W \setminus A$ it follows that the function above is harmonic in Ω_*. Let us analyze its boundary values on $\partial\Omega_*$. If $z \in \Omega_*$ approaches Γ, then $\phi(z) \to A$ and hence

$$\lim_{z\to\Gamma} \mathfrak{m}_{W\setminus A}\,(\phi(z); \partial A) = 1$$

Let us now regard the harmonic measure function

$$z \mapsto \mathfrak{m}_{\Omega_*}(z : \Gamma)$$

By definition it has boundary value 1 along Γ and otherwise it is zero. Hence the maximum principle for harmonic functions gives:

3.4 Theorem. *In the situation above one has the inequality:*

$$\mathfrak{m}_{\Omega_*}(z : \Gamma) \leq \lim_{z\to\Gamma} \mathfrak{m}_{W\setminus A}\,(\phi(z); \partial A) \quad : \quad z \in \Omega_*$$

Application. Using Theorem 3.4 we prove a result due to Löwner. Let $w(z) \in \mathcal{O}(D)$ where $w(0) = 0$ and $|w(z)| < 1$. Suppose there exists an arc γ on the unit circle such that $w(z)$ extends continuously up to γ and that

$$|\gamma(e^{i\theta})| = 1 \quad : \quad e^{i\theta} \in \gamma$$

Consider the image $w(\gamma)$ which is an another arc on the unit cicle. With these notations Theorem 3.4 gives

3.5 Löwner's inequality. *The length of $w(\gamma)$ is $\geq$ the length of γ and equality can only hold if $w(z)$ from the start is $e^{i\alpha}z$ for some α.*

3.6 Remark. Actually Löwner proved a more precise result. Before it is announced we insert a preliminary remark. Given $w(z)$ and an arc $\gamma \subset T$ where $|w(z)| = 1$ one should expect that $|w(z)|$ must tend to 1 rather quick as $z \in D$ approaches γ. To put this in a precise form, Löwner proceeds as follows: Up to a rotation we may take

$$\gamma = \{e^{i\theta} \quad : \; -a < \theta < a\} \quad : \; 0 < a < \pi/2$$

Now we consider the family of circles K_λ passing the two end-points e^{ia} and e^{-ia} where $\lambda > 0$ expresses the angle of intersection beteen K_λ and the unit circle T.

The reader should draw a picture to see the situation where the constraint that the λ-numbers are chosen so that obtain a simple connected domain $\Omega_\lambda \subset D$ bordered by γ and a portion of K_λ. Next, regard the image set $w(\Omega_\lambda)$. On its boundary we find the arc $w(\gamma)$ which by the hypothesis that $|w| = 1$ on γ, is a sub-arc of T. At the same time we can start with the arc $w(\gamma)$ and take the circle K_λ^* which passes the end-points of $w(\gamma)$. This gives a domain Ω_λ^* bordered by $w(\gamma)$ and a subarc of the circle K_λ^*. With these notations the precise result by Löwner goes as follows:

3.7 Theorem. *For each λ as above one has the inclusion*

$$w(\Omega_\lambda) \subset \Omega_\lambda^*$$

3.8 Exercise. Deduce Theorem 3.7 from Theorem 3.5. The strategy is that if $w(z)$ is *outside* the set Ω_λ while $z \in \Omega_\lambda^*$, then the inequality for harmonic measures is violated. We leave it to the reader to discover this contradiction which gives Löwner's theorem. See also his article [Lö:1]: *Untersuchungen über schlichte konforme Abildungen* for details and further results.