$\mathcal{D}$-module theory in dimension one

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The major result of the subsequent material gives a 1-1 correspondence between the class of multi-valued functions of the Nilsson class on the projective line $\mathbf{P}^1$ with a class of Fuchsian differential operators whose coefficients are polynomials of a single variable z. it means that such differential operators beliong to the 1-dimensional Weyl algebra $A_1(\mathbf{C})$. Roughly spekaing we sgall expoes

Introduction.

We shall expoes $\mathcal{D}$-module theory in dimension one where the focus is upon the construction of holomorphic differential operators with univalent coefficients which correspond to in general *multi-valued* analytic functions. As a first illustration of the philosophy which appears in $\mathcal{D}$-module we consider the multi-valued function $\log z$ which after analytic continuation around $z = 0$ adds $2\pi i$. It means that $\log z$ together with all its local branches generate 2-dimensional subspaces of germs of analytic functions in the punctured complex plane. It is easily seen that all branches are annihilated by the second order differential operator

$$Q = z\partial^2 + \partial \tag{*}$$

Conversely every holomorphic null solution to Q outside the origin comes from a constant plus a local branch of $\log z$. In this way we can identify the multi-valued log-function with the second order differential operator Q ! Another example is the multi-valued function z^α, where α is not an integer. It corresponds to the first order differential operator

$$z\partial - \alpha$$

More generally, let Ω be a connected open subset of $\mathbf{C}$. To a given point $z_0 \in \Omega$ there exists the family $M\mathcal{O}(z_0 : \Omega)$ which consists of gemrs of analytic functions f at x_0 which can be extended analytically along every curve γ in Ω which has x_0 as initial point why the terminal point can be arbitrary in Ω. Here we recall that Weierstrass gave the definition of analytic contiuations along curves. Given the pair (z_0, f) we can consider analytic continuations of f along *closed* curves γ which have z_0 both as initial and terminal point. The result after an analytic continuation along γ gives anew germ $T_\gamma(f) \in \mathcal{O}(z_0)$. One says that f has *finite determination* if the complex vector space generated by $\{T_\gamma(f)\}$ where γ run over all closed curves at z_0, is a finite dimensional complex vector space. It is denoted by $\mathcal{H}(f)(z_0)$ and its dimension is called the rank of the multi-valued function f. In §§ we prove that f and its local branches are recovered by a holomorphic differential operator with *single-valued* coefficients. More preciusely one has:

Theorem. *Let $f \in M\mathcal{O}(z_0)$ be a germ which produces a multi-vakued analytic funvtion with finite determination. Then, if m is the rank there exists a unique differential operator of the form*

$$Q(z,\partial) = (z - z_0)^\mu \cdot q_m(z) \cdot \partial^m + \sum_{\nu=0}^{\nu=m-1} q_\nu(z) \cdot \partial^\nu$$

where $q_0, \ldots, q_m$ belong to $\mathcal{O}(\Omega$ and this $(m+1)$-tuple of holomorphic funtions have no common zero in Ω.

Remark. The differential operator Q associated to f is rather special. The reason is that for every point $z_1 \in \Omega$ we find the m-dimensional subspace $\mathcal{H}(f)(z_1)$ generated by the germs $\{T_\gamma(f)\}$ in $\mathcal{O}(z_1)$ arising via analytic continuation along curves γ which start at z_0 and have z_1 as terminal point. Now the Q-kernel on $\mathcal{O}(z_1$ is equal to the m-dimensional vector space $\mathcal{H}(f)(z_1)$. We shall see in § xx that this puts certain restrictions upon Q for points z_1 which may appear as zeros of the holomorphic function $q_m(z)$ above.

A crucial local result. Much of the subsequent analysis relies upon the following result. Let $\mathcal{O}$ be the local ring of germs of analytic functions at the origin in the complex z-plane. We have also the ring $\mathcal{D}$ of germs of holomorphic differential operators, i.e. the elments are

$$Q(z.\partial) = \sum q_\nu(z)\partial^\nu$$

where the sum extends over a finite set of non-negative integers. The largest integer for which the germ $q_m(z) \neq 0$ in $\mathcal{O}$ gives the order of the differential operator.

Theorem. *Let $f_1, \ldots, f_m$ be a finite set of $\mathbf{C}$-linearly independent elements in $\mathcal{O}$. Then there exists a unique differential operator Q of the form*

$$Q(z, \partial) = z^k\partial^m + \sum_{\nu<m} q - \nu(z)\partial^\nu$$

Here k is a non-negative integer, and if $k = 0$ it is requested that at least some q_j has a non-zero constant term, i.e.$q_j(0) \neq 0$.

Example. Consider the two plynomials $f_1 = z$ and $f_2 = z^2$. With $m = 2$ we seek a differentialoperator

$$Q = A(z)\partial^2 + B(z)\partial + C(z)$$

which annihilates the two polynomials. Since Q annihilates z we first have

$$B(z) + C(z) \cdot z = 0$$

Next, we find that

$$2A(z) + 2B(z) \cdot z + C(z) \cdot z^2 = 0 \implies 2A(z) = C(z) \cdot z^2$$

We conclude that

$$Q = z^2\partial^2 - 2z\partial + 2$$

gives the requested differential operator in the theorem above. So this illustrates that the integer μ can be positive even in quite simple examples. Let us remark that a classic result due to Cauchy asserts that if $m \geq 1$ and

$$Q(z, \partial) = \partial^m + \sum_{\nu<m} q_\nu(z)\partial^\nu$$

where $\{q_\nu\}$ is an aribtrary m-tuple of germs, then the Q-kernel on $\mathcal{O}$ is m-dimensional. So differential operators of the form (xx) appear in Theorem xx. For pairs $1 \leq k \leq m$ the condition that Q given as in Theorem xx has an m-dimensional kernel on $\mathcal{O}$ is not so evident. In § xx we shall estaöbish necessasty and sufficient conditions in order that the Q-kernel on $\mathcal{O}$ is m-dimensional. The crucial step to attain this is a general *Index formula* due to Malgrange which is treated in § xx.

Operators of Fuchsian type. Let $Q = q_m(z)\partial^m + \ldots + q_0(z)$ be a germ of a differential operator of order $m \geq 1$. Here q_m can have a zero at $z = 0$ of some order k. Suppose that $q_m(z) = z^k \cdot \rho(z)$ with $\rho(0) \neq 0$. Replacing Q by the differential operator $\rho^{-1} \cdot Q$ does not affect the $\mathcal{O}$-kernel. Let us then assume that $q_m(z) = z^k$

for some $k \geq 1$ and Q is reduced in the sense that $q_j(0) \neq 0$ for at least some $0 \leq j \leq m-1$. Next, we introduce the Fuchisan operator

$$\nabla = x\partial$$

Exercise. Show that for every $\nu \geq 2$ there exist integers $c_0, \ldots, c_{\nu-1}$ such that

$$x^\nu \cdot \partial^\nu = \nabla^\nu + \sum_{j<\nu} c_j \cdot \nabla^\nu$$

The reader is also invited to calculate these integers for some values of ν such as 3 or 4. Next, with Q given as above where $q_m(z) = z^k$ we can write

$$z^{m-k} \cdot Q = z^m \cdot \partial^m + \sum_{\nu<m} x^{m-k} \cdot q_\nu(z) \cdot \partial^\nu$$

Using (*)nit is readly een that the right hand side can be expressed as

$$\nabla^m + \sum_{\nu<m} r_\nu(z) \cdot \nabla^\nu$$

where $\{r_\nu\}$ in general are germ of meromorphic functions at $z = 0$. If $r_\nu \in \mathcal{O}$ for every ν above we say that the given differentialoperator Q is of Fuchsian type.

Exercise. Let $Q = z^k\partial^m + \sum q_\nu(z(\cdot\partial^\nu$ where $k \geq 1$ and $q_j(0) \neq 0$ for at least some j. Show that Q is of Fuchsian type if and only if $k \leq m$ and

(*) $$m - k + \text{ord}(q_j) \geq j \quad : 0 \leq j \leq m-1$$

where $\text{ord}(q_j) = \kappa$ when q_j is a non-zero germ with a zero of multiplicity $\kappa \geq 0$.

Example. When $k = 1$ we ser that (*) hold for every choice of the m-tuple $\{q_j\}$. If $k \geq 2$ there can exist operators which are not of Fuchsian type. For example, an operator of the form

$$Q = z^2\partial^m + c_0\partial^{m1} + \sum_{\nu<m-1} q_\nu(z)\partial^\nu$$

is not of Fuchsian type when c_0 is a non-zero complex number. The family of differential operatod of Fuchsian type in $\mathcal{D}$ is denoted by $\text{Fuchs}(\mathcal{D})$.

The $\mathcal{O}$-kernel of Fuchsian operators. Consider a Fuchsian operator given in the form

$$Q = \nabla^m + \sum_{\nu<m} q_\nu(z) \cdot \nabla^\nu$$

So here $q_0, \ldots, q_{m-1}$ belong to $\mathcal{O}$. Now we regard the constant terms $\{q_j(0)\}$ and the polynomial

$$Q^*(s) = s^m + \sum q_\nu(0) \cdot s^\nu$$

of a single variable s. Using Malgrange's index formula one has the following conclusive result.

Theorem. *Let $0 \leq j_1 < \cdots < j_k$ be the distinct non-negative integers for which $Q^*(s) = 0$. Then the Q-kernel on $\mathcal{O}$ is k-dimensional. Moreover, to each j_ν there exists a unique holomorphic solution of the form $z^{j_\nu} \cdot \rho_\nu(z)$ with $\rho_\nu(0) \neq 0$.*

Remark. The theorem shows that the condition that the Q-kernel has maximal dimension m is rather restrictive. VCon sider as an example the differential operator

$$Q = (\nabla - 1)^2$$

so here $Q^*(s)$ has a zero when $s01$ and Theorem xx means that the Q-kernel on $\mathcal{O}$ is 1-dimensional. It is clear the the holomorphic function x is a solution. The reader may confirm that if

$$g(z) = c_0 + \sum_{\nu \geq 2} c_\nu \cdot z^\nu$$

is given in $\mathcal{O}$,then $Q(g) \neq 0$ unless g is identically zero. To see this the reader may check that

$$Q(g) = c_0 + \sum_{\nu \geq 2} (\nu - 1)^2 \cdot c_\nu \cdot z^\nu$$

A general result to be proved in § xx goes as follows: Add the point at infinity to get the projective line $\mathbf{P}^1$ and consider a finite set $\Sigma = (a_1, \ldots, a_m)$ of points which have been removed where one a-point may be ∞. To each positive integer K we denote by $\text{Nils}(K : \Sigma)$ the family of all Nilsson class functions of rank K defined in $\mathbf{P}^1 \setminus \Sigma$. In Theorem § xx we prove that to every such Nilsson class function F there exists a unique differential operator

$$Q_F(z, \partial) = q_K(z) \cdot \partial^K + \ldots + q_0(z)$$

whose coefficients $\{q_\nu(z)\}$ are polynomials with no common zero and the leading polynomial $q_K(z)$ is monic with the property that Q annihilates the local branches of F outside Σ. We shall also prove that Q_F is of a special type. More precisely it is locally of Fuchsian type at every point in $/\mathbf{P}^1$ where this local Fuchsian condition will be explained later on. Moreover, a converse reulst holds. Namely, let Q be an element of the Weyl algebra $A_1(\mathbf{C})$ of some order K which in additoon is locally Fuchsian at all points on the projctive line. Now there exists a finite subset Σ of $\mathbf{P}^1$ such that the local holomorphic solutions to the homgenenous equation $Q(f) = 0$ is a K-dimensional vector space in $\mathcal{O}(x)$ for every $x \in \mathbf{P}^1 \setminus \Sigma$, and with this choice of Σ one has $Q = Q_F$ for a unique Nilsson class functiuon F as above.

Historic comments The midern era in analytic function theory of one complex-variavle started in thr pioneering work by Niels Henrtk Abelo in his famous article from 1826 about trancendental functions, and thr basic facts about multi-valued functions are due to Riemann and Weierstrass. The idea to study differential systems in a general contexts in the complex domain started in the thesis by he ir Poincaré from 1879 and combined with some important discoverines due to Fuchs leads to the $\mathcal{D}$.module theory to be exposed in these lectures. of course these original results have been put in a consice form via more recent work. one sholud foremnost give credit to Bernard Malgrange whose aritcle in L'Ensignement des mathematiques" from 1968 has inspired the subsequent materal. aoart from t6his the results are "standard" to specialists in $\lceil$.module theory and we remark only the the present 1-dimensional studiy of differential systems in the complex domain can be regarded as an introduction to the higher dimensional case where the major results are due to Masaki Kashiwara.

Algebraic root functions. A basic example goes as follows: With a new variable y we consider an irreducible polynomial $P(x, y)$ in $\mathbf{C}[y]$ of the form

$$f(x, y) = y^e + q_1(x) y^{e-1} + \ldots + q_e(x)$$

For each freezed x this y-polynomial has e zeros counted with multiplicity, i.e.

$$f(x, y) = \prod_{\nu=1}^{\nu=e} (y - \alpha - \nu(x))$$

Since f is ireducible the discriminant polynomial

$$\delta(x) = \prod -\nu \neq j \, (\alpha_\nu(x) - \alpha_j(x))$$

is not identically zero. In $\mathbf{C} \setminus \sigma^{-1}(0)$ these e-tuple of distnict roots are in gebneral mylti-valued functions and they generate a Nilsson class function F of some rank

$K \leq e$. Now we get the unique Fuchsian operator Q_F. It appears that relatively few explicit computations of Q_F have been given in the literature for a given irrducible polynomial $f(x,y)$ as above. So here one encounters an involved elimination problem and I do not know to what extent computations can br performed via comuters to find Q_F for given specifiied polynomials $f(x,y)$. Wheter one referts to this as a problem in "pure a,gebra" or geometric analysis is a matter of taste. Personally I consider everytihng as a subject in mathematics since I detest to make any sort of distinction between the artificial walls between algebra, analysis and geometry.

The local analytic theory

The local theory starts from the ring $\mathcal{O}$ of germs of analytic functions at the origin whose elements are identified with convergent power series. On this ring we have the $\mathbf{C}$-linear derivation operator ∂. More generally one constructs differential operators with coefficients in $\mathcal{O}$ which gives a ring $\mathcal{D}$ whose elements are

$$Q = \sum q_\nu(x) \cdot \partial^\nu$$

where $\{q_\nu\}$ is a family in $\mathcal{O}$ and the sum extends over a finite set of non-negative integers. It turns out that $\mathcal{D}$ is a simple ring, i.e. it has no other two-sided ideals than the trivial zero ideal and the ring itself. Moreover, every left or right ideal can be generated by two elements and the global homological dimension is one. The last result means that every left or right ideal is projective as a $\mathcal{D}$-module. The proofs rely upon a study of the special $\mathcal{D}$-module $\mathcal{O}$ and the solution to systems of differential equations. More precisely, if $k \geq 1$ is a positive integer and $A(x)$ some $k \times k$-matrix whose elements belong to $\mathcal{O}$, then there exists a unique $k \times k$-matrix $\Phi(x)$ with elements in $\mathcal{O}$ such that the constant matrix $\Phi(0)$ is the identity matrix E_k and

$$(*) \qquad \frac{d\Phi}{dx} = A \cdot \Phi$$

holds in the ring of $k \times k$-matrices with elements in $\mathcal{O}$. Using this we prove a theorem in § 3 which goes as follows. Let m be a positive integer and H is an m-dimensional complex subspace of $\mathcal{O}$. Then there exists a unique $Q \in \mathcal{D}$ whose ∂-order is m and the leading coefficiebt of ∂^m some x-monomial, i.e.

$$Q = x^k \partial^m + \sum_{\nu=0}^{\nu=m-1} q_\nu(x) \cdot \partial^\nu$$

with the property that H is equal to the Q.kernel on $\mathcal{O}$, i.e.

$$H = \{f \in \mathcal{O} : \; Q(f) = 0\}$$

Hence there exists a 1-1 correspondence between finite dimensional subspaces of $\mathcal{O}$ and a class of differential operators in $\mathcal{D}$.

Example. Let $m = 2$ where H is generated by x and x^2. The reader may wheck that the associated second order differential operator becomes

$$x^2\partial^2 - 2x\partial + 2$$

Exercise. In general, let $1 \leq a_1 < \ldots < a_m$ be an $m$tuple of positive integers and H is the m-dimensional space generated by the monimials $\{x^{a_\nu}\}$. Try to find Q in this case.

A global result. Let Ω be a connected open set in $\mathbf{C}$. Following Weierstrass we consider a multi-valued analytic function ϕ in Ω whose set of local branches at every point $x_0 \in \Omega$ generartes an m-dimensional complex subspace of $\mathcal{O}(x_0)$. Wen it holds one says that ϕ produces a multi-valued function of finite determination. Fix a point $x_0 \in \Omega$ which to begin eith yields the unique $Q \in \mathcal{D}(x_0)$ of order m which as above annihilates all local branches of ϕ at x_0 and Q is of the form

$$(0.1) \qquad Q(x, \partial) = (x - x_0)^k \cdot \partial^k + q_{m-1}(x) \cdot \partial^{m-1} + \ldots + q_0(x)$$

Here $\{q_\nu\}$ are germs of analytic functions at x_0. However, we can consider the associated Q-operators every $x \in \Omega$ which annihilate the m-dimensional subspace of $\mathcal{O}(x)$ generted by the local brac nhes of ϕ. using these and abna,yticity the reader shold confirm that the q-functions in (*) extend to single-valued analytic functions in Ω and ther germ of q at an arbitrary point $\xi \in \Omega$ has an $\mathcal{O}(\xi)$-kernel which is equal to the m-dimensional space generated by the local bracnhes of ϕ at this point. So in this way a multi-valued analytic function with finte determination is recaptured via a holomorphic differential operator in the given domain Ω

Next, one studies the action by a single differential operator Q on holomorphic functions. If Q has order k it is in general not true that the leading coefficient $q_k(x)$ has a non-zero constant term. So up to a multiplication with an invertible element in $\mathcal{O}$ we are led to consider differential operators of the form

$$(0.1) \qquad Q(x, \partial) = x^m \cdot \partial^k + q_{k-1}(x) \cdot \partial^{k-1} + \ldots + q_0(x)$$

where m and k are positive integers. Now have the cokernel, i.e. the quotient space

$$\text{Coker}(Q) = \frac{\mathcal{O}}{Q(\mathcal{O})}$$

In § 4 we prove Malgrange's Index Formula expressed by the equality

$$(***) \qquad k - m = \dim(\text{Ker}(Q)) - \dim(\text{Coker}(Q))$$

The integer $k - m$ is denoted by $\chi(Q)$ and called the analytic index of Q. Formulas for the dimensions of the kernel and the cokernel are not so easily described in general and here one enters a very difficult subject dealing with Stokes Lines. The reader may consult Malgrange's text-book [Birkhäuser] for a deep studies of $\mathcal{D}$-module theory in dimension one. However, for a restricted class of differential operators one can read off these dimensions. Namely, let $\nabla = x\partial$ be the Fuchsian differential. To each $k \geq 1$ we denote by $F_k(\mathcal{D})$ the class of differential operators Q of the form

$$Q = \nabla^k + q_{k-1}(x) \cdot \nabla^{k-1} + \ldots + q_0(x)$$

where $\{q_\nu\}$ belong to $\mathcal{O}$. To each q-function we take the constant term and get the differential operator

$$Q^* = \nabla^k + q_{k-1}(0) \cdot \nabla^{k-1} + \ldots + q_0(0)$$

The fundamental theorem of algebra gives a factorisation:

$$Q^*(\nabla) = \prod_{\nu=1}^{\nu=k} (\nabla - \alpha_\nu)$$

Let $\rho(Q)$ be the number of distinct non-negative integers which appear in $\{\alpha_\nu\}$. In XX we prove the equality

$$\rho(Q) = \dim(\mathrm{Ker}(Q))$$

Fuchsian operators occur in § 5 to study Nilsson class functions in punctured discs. Recall from Chapter IV that Jordan's normal form for matrices describes Nilsson class functions in a punctured disc. From this we are able to classify locally defined $\mathcal{D}$-modules which arise via Nilsson class functions when we take some $Q \in F_k(\mathcal{D})$ and study Q-kernels at stalks of $\mathcal{O}(x)$ while $0 \neq x$ stays in a small punctured disc.

§ 1. The ring $\mathcal{D}$

Let $\mathcal{O}$ be the ring of germs of analytic function at the origin in $\mathbf{C}$ where x is the complex coordinate. Series expansions identify $\mathcal{O}$ with the ring $C\{x\}$ of convergent power series. If $f \in \mathcal{O}$ has a zero of some order $m \geq 1$ we can factor out x^m and write

$$f(x) = x^m(c_m + c_{m+1}x + \ldots) \quad : \quad c_m \neq 0$$

Then $g(x) = c_m + c_{m+1}x + \ldots$ has a non-zero constant term so $\frac{1}{g}$ is a an invertible germ of a holomorphic function. So each $\mathcal{O}$-element is the product of an invertible element and some monomial x^m. The non-invertible elements form a maximal ideal denoted by $\mathfrak{m}$. It is generated by x and $\mathcal{O}/\mathfrak{m}$ is the complex field.

1.1 The ring $\mathcal{D}$. The elements are differential operators with coefficients in $\mathcal{O}$:

$$Q(x, \partial) = q_0(x) + q_1(x)\partial + \ldots + q_m(x)\partial^m \quad : \quad q_0, \ldots, q_m \text{ belong to } \mathcal{O}$$

The largest degree m for which $q_m \neq 0$ is called the order of Q. Denote by $\mathcal{D}(m)$ the set of differential operators of order m at most which gives an increasing sequence

$$\mathcal{D}(0) \subset \mathcal{D}(1) \subset \ldots$$

The subring $\mathcal{D}(0)$ is isomorphic to $\mathcal{O}$ but the two rings play a different role. Namely, $\mathcal{D}(0)$ is a subring of $\mathcal{D}$ while $\mathcal{D}$ operates on $\mathcal{O}$ and in this way $\mathcal{O}$ is a left $\mathcal{D}$-module.

1.2 Product rules. Differential operators do not commute. Consider for example the first order operator ∂. If $g \in \mathcal{D}(0)$ we get the *product* $\partial \cdot g$ in the ring $\mathcal{D}$. By definition this is a first order differential operator defined by

$$f \mapsto \partial(gf) = g\partial(f) + \partial(g)f \quad : \quad f \in \mathcal{O}$$

We conclude that $\partial \cdot g = g\partial + \partial(g)$ holds in the ring $\mathcal{D}$. For example, the *commutator*

$$\partial \cdot x - x \cdot \partial = 1 \quad : \quad 1 = \text{identity in } \mathcal{D}$$

In general, if $g \in \mathcal{D}(0)$ and $m \geq 2$ then

(*) $$\partial^m \cdot g = \sum_{\nu=0}^{\nu=m} \binom{m}{\nu} \cdot g^{(m-\nu)} \cdot \partial^\nu \quad : \quad g^{(m-\nu)} = \partial^{m-\nu}(g)$$

1.3 The Fuchsian operator ∇. This is the first order operator $\nabla = x\partial$. If $Q(x, \partial)$ has some order k then we can write it in a unique way as

(*) $$Q(x, \nabla) = r_0(x) + r_1(x)\nabla + \ldots + r_k(x)\nabla^k$$

where the coefficients in general are meromorphic functions. For example

$$\partial^2 = x^{-2} \cdot \nabla^2 + x^{-1}\nabla$$

When Q is expanded as in (*) the leading coefficient $r_k(x) = x^m \cdot r_*(x)$ where $r_*(0) \neq 0$ and m is some integer. Then we obtain

(1) $$Q = r_*(x) \cdot x^m \cdot \big[\nabla^k + g_{k-1}(x)\nabla^{k-1} + \ldots + g_0(x)\big] : \; g_\nu(x) = x^{-m} \cdot \frac{r_k(x)}{r_*(x)}$$

If the g-functions are holomorphic at the origin we say that Q admits a *Fuchsian factorisation.* Notice that we allow $m < 0$ in (1). For example, the differential operator ∂^2 has a Fuchsian factorisation since

$$\partial^2 = x^{-2} \cdot \left[\nabla^2 + x \cdot \nabla\right]$$

An example of a differential operator which does not admit a Fuchsian filtration is:

$$(2) \qquad Q = x^2\partial + 1 = x \cdot \left[\nabla + \frac{1}{x}\right]$$

1.4 Left division in $\mathcal{D}$. Let Q be a differential operator of order $k \geq 1$. The highest coefficient $q_k(x)$ need not be invertible but we can write $q_k(x) = x^m g(x)$ where g is invertible. Identifying g^{-1} with an element of $\mathcal{D}(0)$ we get

$$g^{-1}(x) \cdot Q = x^m\partial^k + q_{k-1}(x)\partial^{k-1} + \ldots + q_0(x)$$

Thus, up to an invertible factor we can choose a monomial in x as the leading coefficient. Here $m = 0$ may occur and then Q equals ∂^k plus a lower order differential operator. Let Q be as above and $P = \sum p_\nu(x) \cdot \partial^\nu$ some differential oerator of order $\geq k$. We can perform a left division in the ring $\mathcal{D}$ and lower ∂-degrees to obtain a remainder.

Exercise. Let $Q = x^m\partial^k + q_{k-1}(x)\partial^{k-1} + \ldots + q_0(x)$ with $m \geq 1$. Show that for every $j \geq 1$ and $P \in \mathcal{D}(k+j)$ one can write

$$(*) \qquad x^{mj} \cdot P = S \cdot Q + R$$

where $R \in \mathcal{D}(k-1)$ and $S \in \mathcal{D}(j)$. Moreover, S and R are uniquely determined by the pair P and Q.

1.5 Generators of left ideals. Let L be a left ideal in $\mathcal{D}$. We find the unique smallest integer k such that L contains a non-zero differential operator of order k. Next, each such Q has a leading coefficient $q_k(x)$. The invertible factor $q_*(x)$ in the factorisation $q_k = x^m \cdot q_*$ where $q_*(0) \neq 0$ is also invertible in $\mathcal{D}$, i.e. as an element in $\mathcal{D}(0)$. So with k as above we also find the smallest $m \geq 0$ such that L contains a differential operator of the form

$$(1) \qquad Q = x^m\partial^k + q_{k-1}(x)\partial^{k-1} + \ldots + q_0(X)$$

By the minimal choice of m and k this operator is unique and is called the minimal operator of the *first kind* in L. It is denoted by Q_L to indicate its dependence upon the left ideal L. The minimal choice of k and left division in $\mathcal{D}$ imply that if $P \in L$ then there exists an integer $w \geq 0$ such that

$$(2) \qquad x^w \cdot P = S \cdot Q_L \quad : \quad S \in \mathcal{D}$$

The principal left ideal generated by Q_L is in general different from L. But we get the quotient module

$$M_L = \frac{L}{\mathcal{D} \cdot Q_L}$$

By (2) every element in M_L is annihilated by some power of the $\mathcal{D}(0)$-element x. This leads us to consider:

1.6 Torsion modules. A left $\mathcal{D}$-module M has torsion if every element in M is annihilated by some x-power. Notice that left division with x gives a direct sum decomposition

$$\mathcal{D} = \mathbf{C}[\partial] \oplus \mathcal{D} \cdot x \tag{1}$$

where $\mathbf{C}[\partial]$ is the ring of differential operators with constant coefficients. In the ring $\mathcal{D}$ we have:

$$x \cdot \partial^k = \partial^k \cdot x - k\partial^{k-1} \quad : \quad k \geq 1 \tag{2}$$

From this it follows by an induction that

$$x^{w+1} \cdot \partial^w \in \mathcal{D} \cdot x \quad \text{hold for every} \quad w \geq 1$$

Hence the left module $\mathcal{D}/\mathcal{D} \cdot x$ has torsion.

1.7 The simplicity of $\mathcal{D}/\mathcal{D} \cdot x$ By (1) in (1.6) a non-zero element in this left $\mathcal{D}$-module is represented by a differential operator $q(\partial) = c_k\partial^k + \ldots + c_0$ of some degree k. Then (2) above shows that

$$x^k \cdot q(\partial) + (-1)^{k-1} \cdot k! \cdot c_k \in \mathcal{D} \cdot x \tag{*}$$

It follows that the left $\mathcal{D}$-module generated by $q(\partial)$ is equal to the whole module $\mathcal{D}/\mathcal{D} \cdot x$ which proves that it is simple.

1.8 Theorem. *Every left $\mathcal{D}$-module M with torsion is isomorphic to a direct sum of $\mathcal{D}/\mathcal{D} \cdot x$.*

Proof. Set $M_0 = \{m \in M \quad : \ x \cdot m) = 0\}$. Let $\{m_\alpha\}$ be a basis in this complex vector space. By (1.7) the left ideal $\mathcal{D} \cdot x$ is maximal and it follows that $\mathcal{D} \cdot m_\alpha$ is isomorphic to the simple module $\mathcal{D}/\mathcal{D} \cdot x$ for each m_α. Next, we prove that these submodules is a direct sum. For suppose we have a relation

$$\sum p_\alpha(\partial) \cdot m_\alpha = 0 \tag{1}$$

where $\{p_\alpha(\partial)\}$ are differential operators with constant coefficients. If at least some $p_\alpha \neq 0$ we choose the largest integer w such that some operator has ∂-degree w. Denote by c_α the coefficient of ∂^w in every p_α. iIf we multiply (1) to the left with x^w then (*) from (1.7) gives:

$$0 = \sum c_\alpha \cdot m_\alpha = 0 \tag{2}$$

This contradicts that $\{m_\alpha\}$ are $\mathbf{C}$-linearly independent which gives the direct sum

$$M_* = \oplus \, \mathcal{D} \cdot m_\alpha \tag{3}$$

The proof is finished if we show that $M = M_*$. To prove this we consider some $m \in M$. Since m has torsion we find $w \geq 1$ such that $x^w \cdot m = 0$. If $w = 1$ we already have $m \in M_*$. If x^2m we get

$$0 = \partial(x^2m) = x^2\partial(m) + 2xm = x(x\partial(m) + 2m) = x(\partial(xm) + m) \tag{4}$$

This entails that $\partial(xm)+m$ belongs to M_0. We have also $xm \in M_*$ so $\partial(xm)$ belongs to M_* and hence (4) gives $m \in M_*$. In general, if $x^w m=0$ for some $w \geq 3$ the reader may finish the proof by an induction to conclude that $m \in M_*$.

2. The left $\mathcal{D}$-module $\mathcal{O}$.

The ring $\mathcal{D}$ operates on $\mathcal{O}$ and hence $\mathcal{O}$ is a left $\mathcal{D}$-module. The first observation is that this module is *simple*, i.e. if $0 \neq f \in \mathcal{O}$ then the $\mathcal{D}$-module generated by f is equal to $\mathcal{O}$. To see this, let m be the order of f. If $m=0$ we already have $\mathcal{D}(0)f=\mathcal{O}$. If $m \geq 1$ we notice that $\partial^m(f)$ is invertible in $\mathcal{O}$ and then $\mathcal{D}(0)\cdot\partial^m(f)=\mathcal{O}$. Next, the constant function 1 is a generator for the left $\mathcal{D}$-module $\mathcal{O}$. Its annihilating left ideal is obviously generated by ∂. Hence one has an isomorphism of left $\mathcal{D}$-modules:

$$\mathcal{O} \simeq \frac{\mathcal{D}}{\mathcal{D}\partial} \tag{1}$$

Notice that this reflects the direct sum decomposition

$$\mathcal{D}=\mathcal{D}\cdot\partial+\mathcal{D}[0] \tag{2}$$

Next, consider the space

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{O},\mathcal{O}) \tag{3}$$

We claim that (3) is reduced to the complex field. To see this we let ϕ be some left $\mathcal{D}$-linear map on $\mathcal{O}$. Since $\mathcal{D}(0) \subset \mathcal{D}$ this means in particular that ϕ commutes with multiplication by zero order differential operators and hence ϕ is given by the multiplication of some $g \in \mathcal{O}$, i.e.

$$\phi(f)=gf \quad : \quad f \in \mathcal{O}$$

Finally, ϕ also commutes with the $\mathcal{D}$-element ∂. Hence we must have

$$\partial(gf)=g\partial(f) \quad : \quad f \in \mathcal{O}$$

Now Leibniz' rule implies that g is a constant which gives the required equality

$$\mathbf{C}=\mathrm{Hom}_{\mathcal{D}}(\mathcal{O},\mathcal{O}) \tag{*}$$

2.1 The special role of $\mathcal{O}$. Let M be a left $\mathcal{D}$-module. Since $\mathcal{D}(0) \subset \mathcal{D}$ we can consider the underlying $\mathcal{D}(0)$-module. We say that M is finitely generated as a module over $\mathcal{D}(0)$ if there exists a finite set $m_1,\ldots,m_s$ such that

$$M=\mathcal{D}(0)m_1+\ldots\mathcal{D}(0)m_s$$

2.2 Theorem. *Let M be finitely generated as a module over $\mathcal{D}(0)$. Then M is isomorphic to a finite direct sum of the left $\mathcal{D}$-module $\mathcal{O}$.*

Proof TO BE GIVEN ... Solve the pfaffian system !

3. The duality theorem

Let H be a finite dimensional complex vector subspace of $\mathcal{O}$. In $\mathcal{D}$ we get the left annihilating ideal L_H:

(*) $$L_H = \{Q \in \mathcal{D} \quad : \ Q(H) = 0\}$$

3.1 Theorem. *For every finite dimensional subspace $H \subset \mathcal{O}$ one has the quality*

$$H = \mathrm{Ker}(L_H) = \{g \in \mathcal{O} \colon \quad L_H(g) = 0\}$$

Proof. Let H be a finite dimensional C-subspace of $\mathcal{O}$ of dimension $k \geq 1$ and choose a basis $f_1, \ldots, f_k$. We get a left $\mathcal{D}$-linear operator from $\mathcal{D}$ into $\mathcal{O}^k$ defined by

(1) $$Q \mapsto \ Q(f_1) \oplus \ldots \oplus Q(f_k)$$

By definition the kernel is L_H. Hence (1) yields an *injective* left $\mathcal{D}$-linear map from $\mathcal{D}/L_H$ into $\mathcal{O}^k$ which implies that $\mathcal{D}/L_H$ is finitely generated over $\mathcal{D}(0)$. So Theorem 2.2 gives an integer s such that

(2) $$\mathcal{D}/L_H \simeq \mathcal{O}^s$$

where the isomorphism holds in the category of left $\mathcal{D}$-modules. In particular (2) means that the underlying $\mathcal{D}(0)$-module $\mathcal{D}/L_H$ is free with rank s. The injetive map (1) entails that $s \leq k$. To see that $s = k$ holds we consider the space

(3) $$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/L_H, \mathcal{O})$$

By Theorem 2.3 and (2) above this is an contains the k-dimensional space H we must have $s = k$ which gives the requested equality $s = k$ and then it is clear that $H = \mathrm{Ker}(L_H)$ which finishes the proof of Theorem 3.1

4. The index formula an Fuchsian operators

Let Ω be a connected open subset of $\mathbf{C}$ and $k \geq 1$ a positive integer. Consider a differential operator $Q(x, \partial)$ of the form

(*) $$Q(x, \partial) = \partial^k + q_{k-1}(x) \cdot \partial^{k-1} + \ldots + q_0(x)$$

where $q_0, \ldots, q_{k-1}$ belong to $\mathcal{O}(\Omega)$. At each point $x_0 \in \Omega$ the 1-dimensional version of the Cauchy-Kovalevsky theorem shows that the Q-kernel on $\mathcal{O}(x_0)$ is a k-dimensional complex vector space. The reader may verify that Theorem 3.1 implies that the sheaf of $\mathcal{D}$-modules $\mathcal{D}/\mathcal{D} \cdot Q$ is isomorphic to the sheaf $\mathcal{O}^k$ in Ω. Using this we have:

4.1 Theorem. *The Q-kernel on $\mathcal{O}(\Omega)$ is a k-dimensional vector space and the map from this Q-kernel to $Ker_Q(\mathcal{O}(x_0))$ is bijective for every $x_0 \in \Omega$.*

Exercise. Supply the details of the proof of Theorem 4.1.

Next, let $m \geq 1$ and consider a differential operator of the form

(**) $$Q(x, \partial) = x^m \cdot \partial^k + q_{k-1}(x) \cdot \partial^{k-1} + \ldots + q_0(x)$$

where the coefficients $q_0, \ldots, q_{k-1}$ are analytic in some open disc D_R of radius R centered at $x = 0$. To each $0 < r \leq R$ we denote by $A^k(D_r)$ the algebra of k-times continuosly differentiable functions on the closed disc $\bar{D}_r$ which are analytic in D_r.

This is a Banach space equipped with the norm where one for each $f \in A^k(D_r)$ takes the maximum norm of the derivative of order k plus the maximum norm of the function f itself on $\bar{D}_r$. When $k=0$ we get the disc algebra $A(D_r)$ of continuous functions on $\bar{D}_r$ which are analytic in D_r and equipped with the maximum norm. Now $Q_* = q_0(x) + \ldots + q_{k-1}(x)\partial^{k-1}$ has order $k-1$. By the *Arzela Theorem* the linear operator

$$Q_* \colon\ A^k(D_r) \mapsto A(D_r)$$

is compact. Next, the linear operator

$$x^m\partial^k \colon\ A^k(D_r) \mapsto A(D_r)$$

has an m-dimensional kernel and its image space has co-dimension m, i.e. it is a Fredholm operator of index $k-m$. Now Q is the sum of $x^k\partial^m$ and the compact operator Q_*. Hence *Fredholm's Theorem* implies that Q has index $k-m$. That is the kernel and the co-image are both finite dimensional and

(1) $$\chi_r(Q) = \dim_{\mathbf{C}}(\mathrm{Ker}(Q)) - \dim_{\mathbf{C}}(\mathrm{Coim}(Q)) = k - m$$

Above (1) holds for every $0 < r < R$. At the same time Theorem 4.1 applies to the punctured disc $\Omega = D_R \setminus \{0\}$ and shows that the Q-kernel on $\mathcal{O}(\Omega)$ to $\mathcal{O}(x_0)$ is surjective for every $x_0 \in \Omega$.

4.2 The analytic index. Use the result above to conclude that for each pair $0 < r_* < r < R$ the map from the Q-kernel on $A(D_r)$ to $A(D_{r_*})$ is bijective and hence $r \mapsto \chi_r(Q)$ is constant as $0 < r < R$. Finally, show that when $r \to 0$ it follows that this χ-number is equal to

(2) $$\dim_{\mathbf{C}}(\mathrm{Ker}_Q(\mathcal{O})) - \dim_{\mathbf{C}}(\frac{\mathcal{O}}{Q(\mathcal{O})})$$

where $\mathcal{O} = \mathbf{C}\{x\}$ is the ring of germs of analytic functions at the origin. The integer in (2) is denoted by $\chi(Q)$ and called the analytic index of Q. So by (1) above we have the equality

(3) $$\chi(Q) = k - m$$

This is called the *local analytic index formula* for germs of differential operators.

4.3 The formal index. Let $\widehat{\mathcal{O}} = \mathbf{C}[[x]]$ be the local ring of formal power series. If $Q \in \mathcal{D}$ it operates naturally on $\widehat{\mathcal{O}}$. We shall prove that its kernel and cokernel are finite dimensional vector space and determine the index. Let Q be as in (*) above. Each non-zero $q_\nu(x)$ has some order m_ν, i.e. x^{m_ν} is the first x-term in its power series expansion and with $q_k = x^m$ we set

(1) $$\rho(Q) = \min\{\, \mathrm{ord}(q_\nu) - \nu \quad : \quad 0 \leq \nu \leq k\}$$

For those integers $1 \leq \nu \leq k$ such that $\mathrm{ord}(q_\nu) - \nu = \rho(Q)$ we write

$$q_\nu = c^*_\nu x^{m_\nu} + \text{higher order terms}$$

Now we regard the operator

(2) $$R = \sum\nolimits^{*} c^*_\nu x^{m_\nu}\partial^\nu$$

where the sum is taken over those ν for which $\mathrm{ord}(q_\nu) - \nu = \rho(Q)$. Let N be a positive integer which at least is so large that $N + \rho(Q) \geq 0$. Then it is clear that

$$\text{(3)} \qquad R(x^N) = \sum^{*} c_\nu \cdot N(N-1)\cdots(N-\nu+1) \cdot x^{N+\rho(Q)}$$

Next, for each $N \geq 1$ we denote by $\widehat{\mathfrak{m}}^N$ the maximal ideal in $\widehat{\mathcal{O}}$ generated by x^N. The definition of $\rho(Q)$ shows that

$$\text{(4)} \qquad (Q-R)(x^N) \subset \widehat{\mathfrak{m}}^{N+\rho(Q)+1}$$

Next, we find a large N^* such that

$$\text{(5)} \qquad N \geq N^* \implies \sum^{*} c^*_\nu \cdot N(N-1)\cdots(N-\nu+1) \neq 0$$

Exercise. Use(4-5) to show that if $N \geq N^*$ then the map

$$\text{(*)} \qquad Q \colon\ \hat{\mathfrak{m}}^N \mapsto \hat{\mathfrak{m}}^{N+\rho(Q)}$$

is bijective.

It is clear that the existence of an integer N^* such that (*) give bijective maps for all $N \geq N^*$ implies that both the kernel and the cokernel of Q under its action on $\widehat{\mathcal{O}}$ are finite dimensional complex vector spaces. Hence we can define the formal index number

$$\text{(**)} \qquad \widehat{\chi}(Q) = \dim_{\mathbf{C}}(\mathrm{Ker}_Q(\widehat{\mathcal{O}})) - \dim_{\mathbf{C}}\Big(\frac{\widehat{\mathcal{O}}}{Q(\widehat{\mathcal{O}})}\Big)$$

4.4 Theorem *The formal index of Q is equal to $\rho(Q)$.*

Proof. Choose N_* large so that the map (*) in Exercise 4.x3 is bijective. Put

$$V = \frac{\widehat{\mathcal{O}}}{\widehat{\mathfrak{m}}^N} \quad : \quad W = \frac{\widehat{\mathcal{O}}}{\widehat{\mathfrak{m}}^{N+\rho(Q)}}$$

Now Q induces a linear operator from V to W whose index by Linear Algebra is equal to $\rho(Q)$. At the same time we have the bijective map (*) which implies the Q-index on $\widehat{\mathcal{O}}$ is equal to $\rho(Q)$.

4.5 An inequality. For every $Q \in \mathcal{D}$ one has the inequality:

$$\text{(1)} \qquad \dim\Big[\frac{\widehat{\mathcal{O}}}{Q(\widehat{\mathcal{O}}}\Big] \leq \dim\Big[\frac{\mathcal{O}}{Q(\mathcal{O}}\Big]$$

To show this we set $p = \dim[\frac{\mathcal{O}}{Q(\mathcal{O}}]$ and choose a p-tuple $g_1, \ldots, g_p$ in $\mathcal{O}$ such that

$$\text{(i)} \qquad \mathcal{O} = Q(\mathcal{O}) + \sum Cg_\nu$$

Exercise 4.3 gives in particular the existence of a large integer M such that

$$\text{(ii)} \qquad \widehat{\mathfrak{m}}^M \subset Q(\widehat{\mathcal{O}})$$

Next , if $f \in \widehat{\mathcal{O}}$ we start its series expansion and write

$$f(x) = T_{M-1}(x) + \phi(x) \quad : \quad \phi \in \widehat{\mathfrak{m}}^M$$

while $T_{M-1}(x)$is a polynomial of degree $\leq M-1$. Here $T_{M-1}(x)$ belongs to $\mathcal{O}$ and (i) gives complex numbers $c_1, \ldots, c_p$ such that

$$T_{M-1} = \sum c_\nu \cdot g_\nu(x) \in Q(\mathcal{O})$$

At the same time the (ii) gives the inclusion $\phi \in Q(\widehat{\mathcal{O}})$ and hence $f(x) - \sum c_\nu \cdot g_\nu(x)$ belongs to $Q(\hat{\mathcal{O}})$. Since f is arbitrary we get

$$\sum \mathbf{C} \cdot g_\nu + Q(\widehat{\mathcal{O}}) = \widehat{\mathcal{O}}$$

and then inequality (1) follows.

4.6 A comparison theorem. Let Q be a differential operator. With $Q = x^m \cdot \partial^k + \sum q_\nu \partial^\nu$ the integer $m-k$ is competing to achieve $\rho(Q)$. So by the two index formulas one always have the inequality

(*) $$\chi(Q) \leq \widehat{\chi}(Q)$$

At the same time, every analytic germ g in the Q-kernel is also a formal solution. So we have the inequality

(**) $$\dim_{\mathbf{C}}\big(\ker_Q(\mathcal{O})\big) \leq \dim_{\mathbf{C}}\big(\ker_Q(\widehat{\mathcal{O}})\big)$$

Together (*) and (**) give

4.7 Theorem *The equality $\chi(Q) = \widehat{\chi}(Q)$ holds if and only if*

$$\dim[\ker_Q(\mathcal{O})] = \dim[\ker_Q(\widehat{\mathcal{O}})] \quad \text{and} \quad \dim[\frac{\widehat{\mathcal{O}}}{Q(\widehat{\mathcal{O}}}] = \dim[\frac{\mathcal{O}}{Q(\mathcal{O}}]$$

Remark When the two equalities above hold we see that Q yields a *bijective* operator on the quotient space $\widehat{\mathcal{O}}/\mathcal{O}$. For example, if $g \in \mathcal{O}$ and there exists $\phi \in \widehat{\mathcal{O}}$ such that $Q(\phi) = g$ then we can find $h \in \mathcal{O}$ so that $Q(h) = g$.

4.8 Fuchsian operators.

Let $Q = x^m \partial^k + \sum q_\nu(x) \partial^\nu$ be given in $\mathcal{D}$. We have already seen that it can be written in a unique way as

(1) $$Q = x^{m-k} \cdot [\nabla^m + \sum g_\nu(x) \cdot \nabla^\nu] \quad : \quad g_\nu \in \mathcal{O}[x^{-1}]$$

where $\{g_\nu\}$ in general are germs of meromorphic functions. Using the two index formulas the reader may verify the following:

4.9 Theorem. *The equality $\widehat{\chi}(Q) = \chi(Q)$ holds if and only if the g-functions in (1) above all belong to $\mathcal{O}$.*

4.10 The class $F(\mathcal{D})$. A differential operator Q of some order k is of Fuchsian type if it can be written as

$$Q = r(x) \cdot [\nabla^k + \sum_{\nu=0}^{k-1} g_\nu(x) \cdot \nabla^\nu]$$

where $g_0, \ldots, g_{k-1}$ belong to $\mathcal{O}$ and $r(x)$ is a germ of a meromorphic function at $x = 0$. The class of such differential operators os denoted by $F(\mathcal{D})$.

4.11 Example. Let $k \geq 1$. The reader may verify the equality

(1) $$x^k \partial^k = \nabla(\nabla - 1) \cdots (\nabla - k + 1)$$

The right hand side is a ∇-polynomial with constant coefficients and hence

$$\partial^k = x^{-k} \cdot [\nabla^k + c_{k-1}\nabla^{k-1} + \ldots + c_0] \implies \partial^k \in F(\mathcal{D})$$

4.12 The $\mathcal{O}$-kernel of $F(\mathcal{D})$-operators. If $Q \in F(\mathcal{D})$ the $r(x)$-function in (x) does not affect the Q-kernel on $\mathcal{O}$. Let us therefore consider some Q where $r(x) = 1$. For the g-functions we can pick the constant terms and write $g_\nu(x) = c_\nu + xh_\nu(x)$ with $\{h_\nu\}$ in $\mathcal{O}$. Hence we can write

(1) $$Q = \nabla^k + c_{k-1}\nabla^{k-1} + \ldots + c_0 + x \cdot \sum_{\nu=0}^{\nu=k} h_\nu(x) \cdot \nabla^\nu$$

Next, by the fundamental theorem of algebra we can write

(2) $$\nabla^k + c_{k-1}\nabla^{k-1} + \ldots + c_0 = \prod (\nabla - \alpha_\nu)$$

where $\alpha_1, \ldots, \alpha_k$ is a k-tuple of complex numbers. Notice that multiple roots can appear.

4.13 Theorem. *Let Q be as above and let s be the number of distinct non-negative integers which appear in the k-tuple of α-roots. Then*

$$s = \dim_{\mathbf{C}}(\mathrm{Ker}_Q(\mathcal{O}))$$

The proof relies upon the following:

4.14 Exercise. Let $H \subset \mathcal{O} =$ be a finite dimensional subspace of some dimension s. Then there exists a unique s-tuple of integers $0 \leq m_1 \ldots < m_s$ such that a basis of the k-dimensional vector apace H is given by a s-tuple in $\mathcal{O}$ of the form

$$g_\nu(x) = x^{m_\nu} + \rho_\nu(x) \quad : 1 \leq \nu \leq s$$

where the ρ-functions have expansions:

$$\rho_\nu(x) = \sum_{j > m_\nu}^{*} c_j(\nu) \cdot x^j$$

where $\sum^*$ indicates that $c_j(\nu) = 0$ whenever $j = m_i$ for some $i > \nu$.

Proof of Theorem 4.13 Let $0 \leq m_1 < \ldots < m_s$ be the distinct integers of the α-tuple and set

(i) $$Q^* = \prod (\nabla - \alpha_\nu)$$

For each integer $k \geq 0$ we notice that

(ii) $$Q^*(x^k) = \prod (k - \alpha_\nu) \cdot x^k$$

At the same time the differential operator $Q_* = x \cdot \sum_{\nu=0}^{\nu=k} h_\nu(x) \cdot \nabla^\nu$ sends x^k into $\mathfrak{m}^{k+1}$. Using this the reader may verify that the Q-kernel on $\mathcal{O}$ is s-dimensional and a basis H is given by some s-tuple as in the exercise.

4.15 Example. Consider a differential operator

$$Q = \partial^k + q_{k-1}(x)\partial^{k-1} + \ldots + q_0(x)$$

where $\{q_\nu\}$ belong to $\mathcal{O}$. From 4.11 we see that $Q = x^{-k} \cdot [Q^* + Q_*]$ where

$$Q^* = \prod_{\nu=1}^{\nu=k-1} (\nabla - \nu) \quad \text{and} \quad Q_* = x \cdot \sum_{\nu=0}^{\nu=k} h_\nu(x) \cdot \nabla^\nu$$

From this we see that Theorem 4.13 means that the Q-kernel is k-dimensional where a basis is given by an m-tuple of functions of the form

$$g_\nu(x) = x^\nu + \rho_\nu(x) \quad : \quad \rho_\nu \in \mathfrak{m}^k$$

This reflects the Cauchy-Kovalevsky theorem applied to the special Q-operator above.

4.16 Remark. When multiple zeros with non-negative integers occur we notice that Theorem 4.13 means that the $\mathcal{O}$-kernel is reduced. Consider as an example the operator

$$Q = \nabla^2$$

It is easily see that the Q-kernel is reduced to constant functions, i.e. the kernel is one-dimensional as asserted by Theorem 4.13.

5. Nilsson class functions

Let $\dot{D}$ be a punctured disc $0 < |x| < R$ for some $R > 0$. In XXX we described all Nilsson classes in $\dot{D}$. For a given pair $\lambda \in \mathbf{C}$ and a positive integer k we consider the special Nilsson class function of rank k in the punctured disc which is generated by a local branch of the multi-valued function

(*) $$G(x) = g(x) \cdot x^\lambda \cdot [\log(x)]^{k-1}$$

where $g \in \mathcal{O}(D)$ and $g(0) = 1$. Let us determine the left annihilating ideal in $\mathcal{D}$ of such a Nilsson class function. First we consider the case when $g = 1$. Applying the Fuchsian ∇-operator we have for every non-negative integer ν:

(1) $$\nabla(x^\lambda \cdot [\mathrm{Log}(x)]^\nu) = \lambda \cdot x^\lambda \cdot [\mathrm{Log}(x)]^\nu + \nu \cdot x^\lambda \cdot [\mathrm{Log}(x)]^{\nu-1}$$

Set

$$(2) \qquad Q = (\nabla - \lambda)^k$$

Then we see that this k:th order differential operator annihilates the Nilsson class function g. Moreover, from Theorem 1.6 the Q-kernel on $\mathcal{O}(x)$ for points x in the punctured disc $\{0 < |x| < R\}$ is equal to the k-dimensional subspace of $\mathcal{O}(x)$ generated by the local branches of G at x.

Next, if g is not constant we may shrink D if necessary and assume that $g \neq 0$ in the whole disc $\{|x| < R\}$. Now the annihilating left ideal of G becomes:

$$(3) \qquad L_G = \{P \in \mathcal{D} \quad : \ P \cdot g \in \mathcal{D} \cdot Q\}$$

where g has been identified with a zero-order differential operator. Since g is invertible we conclude that L_G is the left principal ideal generated by the differential operator $Q \cdot g^{-1}$. With $g(0) = 1$ the product rules in give

$$(4) \qquad Q \cdot g^{-1} = g^{-1} \cdot [Q + xR(x, \nabla)]$$

where

$$R(x, \nabla) = r_{k-1}(x) \cdot \nabla^{k-1} + \ldots + r_0(x) \quad : \quad r_\nu \in \mathcal{O}$$

Notice that $Q+xR(x, \nabla)$ belongs to $F(\mathcal{D})$. Hence the indecomposable Nilsson class function G from (*) is annihilated by a Fuchsian operator Q of the form

$$(**) \qquad Q_G = (\nabla - \lambda)^k + xR(x, \nabla)$$

where R has order $k-1$ at most.

5.1 Uniqueness. Let $P = \nabla^k + p_{k-1}(x) \cdot \nabla^{k-1} + \ldots + p_0(x)$ is be a differential operator where each $p_\nu \in \mathcal{O}$ and $P(G) =$. Then one has the equality

$$P = Q$$

5.2 Exercise. Prove this uniqueness. The hint it is that the multi-valued behaviour of $\log x$ entails that if $g_0, \ldots, g_k$ is an arbitrary k-tuple in $\mathcal{O}$ then the multi-valued function

$$g_0(x) + g_1(x) \cdot \log x + \ldots + g_k(x) \cdot (\log x)^k$$

is not identically zero, unless all g-functions are identically zero. Use this together with the observation that

$$\nabla([\log x]^j) = j \cdot [\log x]^{j-1}$$

hold for each $j \geq 1$.

5.3 The general case

Above we considered special Nilsson class functions. In general, let m be some positive integer and F is a Nilsson class function of tank m defined in a punctured disc $\{0 < |z| < R\}$ for some $R > 0$ which in general can be small. In $\mathcal{D}$ we get the left ideal L_F of germs of differential operators $Q(x, \partial)$ which annihilate F. To be precise, if $Q = \sum q_\nu(x) \cdot \partial^\nu$ where $\{q_\nu\}$ are analytic in a disc $\{|z| < r\}$ with $r < R$ then we require that Q annihilates F in the punctured disc of radius r.

5.4 Theorem. *For each Nilsson class function F as above the left ideal L_F is non-zero and its minimal operator Q is of Fuchsian type.*

The unique differential operator in Theorem 5.4 depends on F and is denoted by Q_F.

6. The global algebraic case.

Let A_1 be the Weyl algebra whose elements are differential operators in $\mathbf{C}$ with polynomial coefficients. These differential operators are sections over $\mathbf{C}$ with values in the sheaf of holomorphic differential operators on $\mathbf{C}$. Adding the point at infinity we get the projective line $\mathbf{P}^1$ which is a compact complex manifold. Let $\Sigma = \{a_1, \ldots, a_m\}$ be a finite subset of $\mathbf{P}^1$ where the a-points contain the point at infinity. For each integer $K \geq 1$ we get the family of Nilsson class functions of rank K defined in $\mathbf{P}^1 \setminus \Sigma$ and we denote this family by $\text{Nils}_K(\mathbf{P}^1 \setminus \Sigma)$. With these notations the following holds:
6.1 Theorem. *For each $F \in Nils_K(\mathbf{P}^1 \setminus \Sigma)$ there exists a unique $Q(x, \partial)$ in A_1 of order K such that $Q(F) = 0$ where the polynomial coefficients $\{q_\nu\}$ have no common zero in $\mathbf{C}$ and $q_K(x)$ is a monic polynomial Moreover, Q is locally Fuchsian at each a-point including ∞.*

6.2 Remark. The proof follows easily via Theorem 5.4 above. One has also a converse result. Namely, assume that $Q \in A_1$ has degree K where q_K is monic and $\{q_\nu\}$ have no common zeros and the germ of Q at every point $x_0 \in \mathbf{P}^1 \setminus \Sigma$ has an K-dimensional kernel in $\mathcal{O}(x_0)$. Assume in addition that the germ of Q a every Σ-point is of Fuchsian type. Then $Q = Q_F$ for a unique F in $\text{Nils}_K(\mathbf{P}^1 \setminus \Sigma)$. Hence there exists a 1-1 correspondence between $\text{Nils}_K(\mathbf{P}^1 \setminus \Sigma)$ and A_1-elements satisfying the conditions above.

6.3 Condition at ∞. Here $w = 1/z$ is a new coordinate which means that $\partial_z \mapsto -w^2\partial_w$. Let F be given in $\text{Nils}_m(\mathbf{P}^1 \setminus \Sigma)$. We find the unique Q_F in A_1. In general Q_F does not extend to a globally defined differential operators on $\mathbf{P}^1$. An example occurs if we start with the Nilsson class function given by the polynomial $F(x) = x^2 + 1$. Here $\Sigma = \{\infty\}$ and the annihilating differential operator is

$$Q_F = (x^2 + 1)\partial - 2x$$

At ∞ it becomes $-(1 + w^2)\partial_w - 2 \cdot w^{-1}$ which is a germ of $\mathcal{D}$ at this point of the complex manifold $\mathbf{P}^1$. However, by a general result $\mathbf{P}^1$ is affine for $\mathcal{D}$-modules so the coherent left ideal $\mathcal{L}$ which annihilates P is generated by global sections. In our specific example we notice that the following second order differential operators annhilate F:

$$Q_1 = (1 + x^2)\partial^2 - 2 \quad \text{and} \quad Q_2 = x(1 + x^2)\partial^2 - (1 + x^2)\partial$$

Here

$$xQ_1 - Q_2 = Q$$

so Q belongs to the left ideal generated by these two global sections of $\mathcal{L}_F$. In general we have a similar result, i.e. let $F \in \text{Nils}_m(\mathbf{P}^1 \setminus \Sigma)$ be given. We find Q_F and we also have the algebra the left ideal

$$L_F^* = \Gamma(\mathbf{P}^1, \mathcal{L}_F)$$

of global sections in $\mathcal{L}_F$. This is a left ideal in the ring $\mathcal{D}(\mathbf{P}^1)$ of globally defined differential operators on the projective line. By a general result the left ideal generated by L_F^* in A_1 contains Q_F. The example above illustrates his.

6.4 Another example.

Consider the differential operator:

$$Q = \nabla^2 + x$$

This corresponds to the case $k = 2$ and $\lambda = 0$ so at a first sight one may expect that $Q = Q_G$ where G is generated by $g(x) \cdot \log x$ for some g-function. However, this is not the case. To see this we notice that

$$Q(g(x) \cdot \log x)) = 2 \cdot \nabla(g) + \nabla^2(g) \cdot \log x + xg \cdot \log x$$

Since $\log x$ is multi-valued this would entail that

$$2 \cdot \nabla(g) = 0 \quad \text{and} \quad \nabla^2(g) + xg = 0$$

But then we see that g must be the zero function. However, let us instead seek a Nilsson class function G in the punctured disc generated by a local branch of the form

$$G = g \cdot \log x + \phi$$

such that $Q(G) = 0$ and the pair g, ϕ belong to $\mathcal{O}$. Here we have

(i) $$Q(G) = Q(g) \cdot \log x + 2\nabla(g) + Q(\phi)$$

Hence the pair g, ϕ should satisfy

$$Q(g) = \nabla^2(g) + xg = 0$$

Exercise. Verify that the Q-kernel on $\mathcal{O}$ is one-dimensional and contains a unique function of the form:

$$g(x) = 1 + c_1 x + c_2 x^2 + \ldots$$

Show also that the inhomogeneous equation

$$Q(\phi) = -2 \cdot \nabla(g)$$

has a unique solution ϕ which belongs to $\mathfrak{m}$.

Distribution solutions.

Let Q be a a differential operator of the form

$$Q(z,\partial) = z^k(z)\cdot\partial^m + q_{m-1}(z)\partial^{m-1} + \ldots + q_0(z)$$

where at least some $q_\nu(0)\neq 0$. We also assume that Q is Fuchsian and stay in a small disc D of radius δ where the q-functions are analytic. With $z = x+iy$ we can restrict Q to the real line where it operates on the space of distributions. Denote by $\mathfrak{Db}(0)$ the space of germs of distributions at $x = 0$.

1. Theorem. *The Q-kernel on $\mathfrak{Db}(0)$ is a vector space with dimension $m+k$.*

To prove the equality we use the sheaf-theoretic formulas in § xx. This, with $\mathcal{M} = \mathcal{D}/\mathcal{D}Q$ we have the sheaf complex $Sol(\mathcal{M})$. Next, consider the closed subset of D defined by the non-negative real interval $K = \{x\geq 0\}$. by the formula in § xx we have

$$H^1_{(K]}(\mathcal{O}) = \mathfrak{Db}(+)$$

where the right hand side is the sheaf of distributions supported by K. Since Q is Fuchsian the general result in xx gives the sheaf theoretic equality

$$\mathbf{R}_K(Sol(\mathcal{M})) = \mathbf{R}Hom_{\mathcal{D}}(\mathcal{D}/\mathcal{D}Q, \mathfrak{Db}(+))$$

Let $\mathcal{N}$ be the q-kernel on $\mathfrak{Db}(0)$. Consider the open and simply connected domain

$$D_* = D\setminus[0,\delta)$$

The Q-kernel on $\mathcal{O}(D_*)$ is an m-dimensional complex vector space and let $\psi_1,\ldots,\psi_m$ be a basis. To each ν we get the Euler distribution $\psi_\nu(+)$ supported by $0\leq x<\delta$. Since the γ-functions are linearly indeendent the same hold by analyticity for the Euler distributions. Denote by $\Psi(+)$ the m-diumensional vector space gererated by these Euler distributions. Next, let $\mathfrak{Dir}$ be the space of Dirac distributions. Notice that

$$Q(\psi_\nu(+))\in\mathfrak{Dir}$$

hold for each ν. Hence we obtain a $\mathbf{C}$-linear map

$$\rho\colon\Psi(+)\xrightarrow{Q}\frac{\mathfrak{Dir}}{Q(\mathfrak{Dir})}$$

Let s be the dimension of the ρ-kernel. We can arrange the indices so that $\psi_1(+),\ldots\psi_s(+)$ is a basis. To each $1|leq\nu\leq s$ we find some Dirac distribution ξ_ν such that

$$Q(\psi_\nu(+)) = Q(\xi_\nu)$$

This entails that the s-tuple $\{\mu_\nu = \psi_\nu(+)) - \xi_\nu\}$ belongs to $\mathcal{N}$.

Next. consider a distribution $\mu\in\mathcal{N}$. Its restriction to $(-\delta,0)$ is a linear combination of the ψ-functions which entials that there exists a unique m-tuple of complex numbers such that the distribution

$$\mu^* = \mu - \sum c_\nu\cdot\psi_\nu(x+i0))$$

is supported by $0 \leq x < \delta$. Since the boundary value distributions $\psi_\nu(x+i0)$ belong to $\mathcal{N}$ the same is true for μ^*. Next, on $(0,\delta)$ we find another unique m-tuple of complex numbers such that

$$\mu^* = \sum d_\nu \cdot \psi_\nu(x+i0))$$

$$\gamma_\nu = \psi_\nu(x+i0) - \psi_\nu(x-i0)$$

Notice that $\gamma_\nu = 0$ if and only if ψ_ν extends to a holomorphic function in D. So if s is the dimension of the Q-kernel on $\mathcal{O}$, then the γ-distributions generate a vector space of dimension $m-s$ and we can arrange the indices so that $\gamma_1, \ldots, \gamma_{m-s}$ is a basis while the remaining γ-distributions in (xx) are zero. Since Q commnutes with the passage to boundary value distributions each γ_ν belongs to $\mathcal{N}$. Next, Malgrange's index formula gives

$$s = m - k + \dim[\frac{\mathcal{O}}{Q(\mathcal{O})}]$$

Let $\widehat{\mathcal{O}}$ be the local ring of formal power series. Since Q is of fuchsian type we know from §§ that

$$\frac{\mathcal{O}}{Q(\mathcal{O})} \simeq \frac{\widehat{\mathcal{O}}}{Q(\widehat{\mathcal{O}})}$$

Next, let $\mathfrak{Dir}$ be the space of Dirac distributions from § xx. The duality result in § xx entails that (x) is the dual of the Q-kernel on $\mathfrak{Dir}$. Hnece we arrive at the equation

$$s = m - k + \dim(\mathrm{Ker}_Q(\mathfrak{Dir}))$$

It can also be expressed by

$$k = m - s + \dim(\mathrm{Ker}_Q(\mathfrak{Dir}))$$

Global results.

The local duality expressed by (**) above associates a differential operator to a multi-valued analytic function with finite determination. Namely, let Ω be an arbitrary connected and open subset of $\mathbf{C}$. For each positive integer K we have the class $M\mathcal{O}(\Omega)_K$ of multi-valued functions of rank K as described in Chapter IV. Let G be an element in this class which means that the stalk $G(x_0)$ is a K-dimensional subspace of $\mathcal{O}(x_0)$ for every $x_0 \in \Omega$. Now (**) gives a differential operator

$$Q = \sum_{\nu=0}^{\nu=K} q_\nu(x) \cdot \partial^\nu$$

where $\{q_\nu\}$ are holomorphic functions in Ω with no common zero and $Q(G) = 0$. Moreover, Q is unique in the ring $\mathcal{D}(\Omega)$ up to a left multiplication with a zero-free analytic function in Ω. Hence this $\mathcal{D}$-module result identifies $M\mathcal{O}(\Omega)_K$ with a family of differential operators in $\mathcal{D}(\Omega)$ whose q-coefficients *single-valued* in Ω. A case of special interest occurs when

$$\Omega = \mathbf{P}^1 \setminus \{a_1, \ldots, a_m\})$$

Here $a_1, \ldots, a_m$ is a finite subset of the projective line and for simplicity we assume that the point at inifinity occurs in this m-tuple where $m \geq 2$. For each $K \geq 1$ we have the family of Nilsson class functions in Ω of rank K. When G belongs to $\mathrm{Nils}_K(\mathbf{P}^1 \setminus \{a_1, \ldots, a_m\})$ it turns out that the corresponding differential operator Q_G has polynomial coefficients, i.e. it is an element of the Weyl algebra $A_1\mathbf{C})$ and $Q = Q_G$ is unique when we require that the polynomials $\{q_\nu\}$ have no common zeros in $\mathbf{C}$ and the leading polynomial $q_K(x)$ is monic. In § XX we expose ring theoretic properties of $A_1\mathbf{C})$ and discuss the range of $G \mapsto Q_G$ when G varies in $\mathrm{Nils}_K(\mathbf{P}^1 \setminus \{a_1, \ldots, a_m\})$.

Example. Let $x_1, \ldots, x_m$ be a finite set of points in $\mathbf{C}$ and $\alpha_1, \ldots, \alpha_m$ some m-tuple of complex numbers where none if them is an integer. We get the Nilsson class function of rank one given by

$$G(x) = \prod (x - x_\nu)^{\alpha_\nu}$$

Here we have

$$\partial(G) = \sum \frac{\alpha_\nu}{x - x_\nu} \cdot G$$

So Q_G is the first order differential operator given by

$$Q_G = \prod (x - x_\nu) \cdot \partial + q_0(x) \quad : \quad q_0(x) = -\sum \alpha_\nu \cdot \prod_{j \neq \nu} (x - x_j)$$

0.2 An integral formula for solutions to ODE:s of the Laplace type. Let us finish the introduction with a classic example which illustrates the efficiency of adopting a $\mathcal{D}$-module theoretic point of view. Let $n \geq 2$ and consider a holomorphic differential operators of the form

$$Q = z\partial^n + (a_{n-1} + b_{n-1} \cdot z)\partial^{n-1} + \ldots + (a_1 + b_1 \cdot z)\partial + a_0$$

Here $\{a_\nu\}$ and $\{b_\nu\}$ are complex numbers. Malgrange's index formula shows that there exist $n-1$ or eventually n many $\mathbf{C}$-linearly independent holomorphic functions defined close to $z=0$ which solve the homogeneous equation $Q(f)=0$. Every such solution f extends to a in general multi-valued function in the punctured complex plane. Indeed, this follows since z is the sole factor for the highest ∂-monomial in Q. It turns out that the solutions admit integral representation formulas. More precisely, consider some pair of n-tuples of complex numbers $\{w_\nu\}$ and $\lambda_\nu\}$. As explained in XX we obtain an analytic function $F(z)$ which to begin with is defined in a half-plane $\mathfrak{Re}\, z > A$ where $A = \max\{|w_\nu|\}$ and

$$(*) \qquad F(z) = \int_C e^{wz} \cdot \prod (w - w_\nu)^{\lambda_\nu} \cdot dw$$

Here C is a curve which starts at $-\infty$ and surrounds the α-points before it returns to $-\infty$ in the negative real direction in the complex w-plane. See XX for a figure and details. Partial integration gives:

$$z \cdot F(z) = -\int_C e^{wz} \cdot \partial_w \big(\prod (w - a_\nu)^{\lambda_\nu}\big) \cdot dw$$

At the same time

$$\partial_z^k(F) = \int_C e^{wz} \cdot w^k \cdot \prod (w - a_\nu)^{\lambda_\nu} \cdot dw$$

This suggests that we assign a differential operator $Q^*(w, \partial_w)$ to the Q-operator above by

$$Q^* = -\partial_w \cdot (w^n + \sum b_j \cdot w^j) + \sum a_\nu \cdot w^\nu$$

This is a first order differential operator which can be rewritten as

$$-(w^n + \sum b_j \cdot w^j) \cdot \partial_w + \sum a_\nu \cdot w^\nu - (nw^{n-1} + \sum jb_j w^{j-1})$$

Put

$$q(w) = w^n + \sum b_j \cdot w^j \quad \text{and} \quad p(w) = \sum a_\nu \cdot w^\nu - (nw^{n-1} + \sum jb_j w^{j-1})$$

Now we seek functions $\phi(w)$ whose logarithmic derivative satisfies

$$\frac{\phi'(w)}{\phi(w)} = \frac{p(w)}{q(w)}$$

Assume that the q-polynomial has simple zeros $w_1, \ldots, w_n$. Then we get a fractional decomposition

$$\frac{p(w)}{q(w)} = \sum \frac{\lambda_\nu}{w - w_\nu}$$

It is clear that

$$\prod (w - w_\nu)^{\lambda\nu}$$

is a solution to the homogenous Q^*-equation. So with $\{w_\nu\}$ and $\{\lambda_\nu\}$ determined in this way we conclude that the function $F(z)$ in (*) solves $Q(F)=0$. This process can be reversed if one keeps control of suitable branches and proves the Laplace integral formula expressed by (*) for solutions to the original equation $Q(f) = 0$. Many specific cases occur. One example is the confluent hypergeometric function which arises with $n = 2$.