Taylor series and real-analytic functions.

The study of Taylor series of differentiable functions on the real line were investigated by Borel and Denjoy who established several results during the years 1910-1922. Among these we recall the following result. Consider a real-valued C^∞-function f on a bounded open interval (a,b) whose derivatives have have finite maximum norms. For each non-negative integer k we put

$$C_k(f) = \big(|f^{(k)}|\big)^{\frac{1}{k}} \tag{*}$$

where $|f^{(k)}|$ is the maximum of the k:th order derivative of f taken over (a,b). The question posed by Borel and Denjoy was to find conditions expressed by estimates on the sequence $\{C_k(f)\}$ which impliy that f cannot be flat at a point $x_0 \in (a,b)$, unless it is identically zero. To say that f is flat at x_0 means that

$$f^{(k)}(x_0) = 0 \quad : \quad k = 0,1,2,\ldots$$

Denjoy proved that if

$$\sum_{k=0}^{\infty} \frac{1}{C_k(f)} = +\infty \tag{**}$$

then f cannot be flat at a point $x_0 \in (a,b)$ unless it is identically zero. In 1923 Carleman established necessary and sufficient conditions for quasi.analyticity in his lectures at Sorbonnen which goes as follows:

Let $\mathcal{A} = \{\alpha_\nu\}$ be a non-decreasing sequence of positive real numbers. Denote by $\mathcal{C}_\mathcal{A}$ the family of all $f \in C^\infty[0,1]$ for which there exists a constants M and k which may depend on f such that

$$\max_{0\leq x\leq 1} |f^{(\nu)}(x)| \leq M \cdot k^\nu \cdot \alpha_\nu^\nu \quad : \quad \nu = 0,1,\ldots \tag{0.4.1}$$

One says that $\mathcal{C}_\mathcal{A}$ is a quasi-analytic class if every $f \in C_\mathcal{A}$ whose Taylor series is identically zero at $x=0$ vanishes identically on $[0,1]$. The following was proved by Carleman in 1922:

Theorem. *The class $C_\mathcal{A}$ is quasi-analytic if and only if*

$$\int_1^\infty \log\Big[\sum_{\nu=1}^{\infty} \frac{r^{2\nu}}{a_\nu^{2\nu}}\Big] \cdot \frac{dr}{r^2} = +\infty$$

We shall expose material from Carelan's wok where the major results appear in [Ca:xx.]. First we shall derive the result above by Denjoy via a remarkable theorem from Carleman's article [xxx].

An inequality for differentiable functions.

Let n be a positive integer and denote by $\mathcal{F}_n$ the family of n times continuously differentiable and real-valued functions f on the closed unit interval such that

$$f^{(k)}(0) = f^{(k)}(1) = 0 \quad : 0 \leq k \leq n-1 \tag{0.1}$$

with a normalized L^2-integral:

$$\int_0^1 f(t)^2 \, dt = 1 \tag{0.2}$$

Next, for each $1 \leq k \leq n$ we can consider the L^2-norm of the k:th order derivative, i.e. set

$$||f^{(k)}||_2) = \sqrt{\int_0^1 f^{(k)}(t)^2 \, dt}$$

Main Theorem. *For each $n \geq 1$ and every $f \in \mathcal{F}_n$ one has the inequality*

$$\sum_{k=1}^{k=n} \frac{1}{||f^{(k)}||_2)^{\frac{2}{k}}} \leq \pi \cdot e$$

where e is Neper's constant.

We shall first establish a general inequality of independent interest. Let $0 < b_1 < \ldots < b_n$ be a strictly increasing sequence of positive real numbers where $n \geq 1$ is some integer. Let $\phi(z)$ be an analytic function in the right half-plane $\mathfrak{Re} z > 0$ which in addition extends to a continuous function on the imaginary axis. Assume that its maximun norm over the right half-plane is ≤ 1 and in addition satisfies

$$|z|^k \cdot \phi(z) \leq b_k^k \quad : k = 1, \ldots, n \tag{1.1}$$

$$\phi(a) \geq e^{-a} \quad : \quad a > 0 \tag{1.2}$$

Here (1.2) means that the restriction of ϕ to the non-negative real axis is real-valued and satisfies the inequalities expressed by (1.2).

1.3 Theorem. *For each ϕ as above one has the inequality*

$$\sum_{k=1}^{k=n} \frac{1}{b_k} \leq \frac{e\pi}{2} \tag{1.3.1}$$

We prove Theorem 1.3 in § 2 and proceed to show how it gives the Main Theorem. We are given $f \in \mathcal{F}_n$ and put

$$\phi(z) = \int_0^1 e^{-zt} \cdot f(t)^2 \, dt$$

When $\mathfrak{Re}\, z \geq 0$ the absolute value $|e^{-zt}| \leq 1$ for all t on the unit interval and hence (0.2) implies that the maximum norm of ϕ is ≤ 1. Next, if $1 \leq k \leq n$ the vanishing in (0.1) and partial integration give

$$z^k \cdot \phi(z) = \sum_{\nu=0}^{\nu=k} \binom{k}{\nu} \int_0^1 f^{(\nu)}(t) \cdot f^{(k-\nu)}(t)(t) \, dt \tag{i}$$

The Cauchy-Schwarz inequality estimates the absolute value of the right hand side by

$$\sum_{\nu=0}^{\nu=k} \binom{k}{\nu} \cdot ||f^{(\nu)}||_2 \cdot ||f^{(k-\nu)}||_2 \tag{ii}$$

At this stage we use a wellknown resut from calculus which entails that

$$||f^{(\nu)}||_2 \leq ||f^{(k)}||_k \quad : 0 \leq \nu \leq k$$

and from this the reader can check that (ii) is majorised by $2^k \cdot ||f^{(\nu)}||_k^2$. Hence

(iii) $$|z|^k \cdot |\phi(z)| \leq 2^k \cdot (||f^{(k)}||_2)^2 \quad : \ k = 1, 2, \ldots$$

Put

(iv) $$b_k = 2 \cdot (||f^{(k)}||_2)^{\frac{2}{k}} \implies |z|^k \cdot |\phi(z)| \leq b_k^k$$

Next, if $a > 0$ we have

$$\phi(a) = \int_0^1 e^{-at} \cdot f(t)^2 \, dt \geq e^{-a} \cdot \int_0^1 f(t)^2 \, dt = e^{-a}$$

where the last equality holds by (0.2). Hence we can apply Theorem 1.3 to ϕ and conclude that

(v) $$\sum_{k=1}^{k=n} \frac{1}{b_k} \leq \frac{e\pi}{2}$$

Here the b-numbers are given by (iv) which gives the Main Theorem.

Proof of Theorem1.2

First we establish an inequality where condition (1.2) does not appear.

2 Theorem. *For each $\phi(z)$ which satisfies (1.1) and every real $a > 0$ one has the inequality*

(2.1) $$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq \log \frac{1}{\phi(a)}$$

Proof. On the imaginary axis we consider the intervals

(i) $$\ell_k = [e \cdot b_k, e \cdot eb_{k+1}] \quad : \ k = 1, \ldots, n-1 \quad \& \quad \ell_n = [eb_n, +\infty)$$

Since $\log e^{-1} = -1$ it is clear that (1.1) gives the following for each $1 \leq k \leq n$:

(ii) $$\log |\phi(iy)| \leq -k \quad : \quad y \in \ell_k$$

Taking the negative intervals $-\ell_k = [-e \cdot b_{k+1}, -e \cdot b_k]$ and $-\ell_n = (-\infty, -eb_n$ we also have

(iii) $$\log |\phi(iy)| \leq -k \quad : y \in -\ell_k$$

Moreover, since the maximum norm of ϕ is ≤ 1 one has

(iv) $$\log |\phi(iy)| \leq 0 \quad : \ -b_1 \leq y \leq b_1$$

Next, solving the Dirichlet problem we find the harmonic function u in the open right half-plane whose boundary values on $(-b_1, b_1)$ is zero ,while $u = -k$ in the the open intervals ℓ_k and $-\ell_k$ for every k. The principle of harmonic majorisation applied to the subharmonic function log $|\phi(z)|$ entails that

(v) $$\log |\phi(a) \leq u(a)$$

Now we evaluate $u(a)$ using Poisson's formula to represent harmonic functions in the right half-plane. For each $1 \leq k \leq n-1$ we denote by $\theta_a(k)$ the angle between the two vectors which join a to the end-points ieb_k and ieb_{k+1}. Computing the area of the triangle with corner points at a, ieb_k, ieb_{k+1} the reader may check that

(vi) $$\sqrt{a^2 + e^2 b_k^2} \cdot \sqrt{a^2 + e^2 b_{k+1}^2} \cdot \sin \theta_a(k) = a \cdot e \cdot (b_{k+1} - b_k)$$

Finally, let $\theta_a(n)$ be the angle between the vector which joins a with ieb_n and the vertical line $\{x = a$. The reader may check with the aid of a figure that

(vii) $$\sin \theta_a(n) = \frac{a}{\sqrt{a^2 + e^2 b_n^2}}$$

Poisson's formula gives

$$u(a) = -\frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \theta_a(k)$$

Together with (v) it follows that

$$\text{(viii)} \qquad \frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \theta_a(k) \leq \log \frac{1}{\phi(a)|}$$

The inequality $\sin t \leq t$ for every $t > 0$ implies that

$$\text{(ix)} \qquad \frac{2}{\pi} \cdot \sum_{k=1}^{k=n} k \cdot \sin\, \theta_a(k) \leq \log \frac{1}{\phi(a)|}$$

Next we use (vi-vii) to estimate $\{\sin\, \theta_a(k)\}$. When $1 \leq k \leq n-1$ we have from (vi)

$$e^2 \cdot b_k \cdot b_{k+1} \cdot \sqrt{1 + \frac{a^2}{e^2 b_k^2}} \cdot \sqrt{1 + \frac{a^2}{e^2 b_{k+1}^2}} \cdot \sin \theta_a(k) = a \cdot e \cdot (b_{k+1} - b_k) \implies$$

$$e \cdot (1 + \frac{a^2}{e^2 b_1^2}) \cdot \sin\, \theta_a(k) \leq a \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}})$$

where the last inequality follows since $b_k \geq b_1$ for every k. We conclude that the left hand side in (ix) majorizes

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n-1} k \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}}) + \frac{2}{\pi} \cdot n \cdot \sin \theta_a(n)$$

Finally, (vii) gives

$$\sin \theta_a(n) = \frac{a}{eb_n} \cdot \frac{1}{\sqrt{1 + \frac{a^2}{e^2 b_n^2}}} \geq \frac{a}{eb_n} \cdot \frac{1}{1 + \frac{a^2}{e^2 b_1^2}}$$

From this we conclude that the left hand side in (ix) majorizes

$$\frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot (\sum_{k=1}^{k=n-1} k \cdot (\frac{1}{b_k} - \frac{1}{b_{k+1}}) + n \cdot \frac{1}{b_n})$$

Abel's summation formula identifies the last term with $\sum_{k=1}^{k=n} \frac{1}{b_k}$. Hence we have proved the requested inequality

$$\text{(x)} \qquad \frac{2a}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq \log \frac{1}{\phi(a)}$$

2.3 A special case. Assume in addition to (1.1) that (1.2) holds.

$$\phi(a) \geq e^{-a} \implies \log \frac{1}{\phi(a)} \leq a \quad : a > 0$$

So after division with a we see that Theorem 1.2 gives

$$\text{(2.3.1)} \qquad \frac{2}{e\pi \cdot (1 + \frac{a^2}{e^2 b_1^2})} \cdot \sum_{k=1}^{k=n} \frac{1}{b_k} \leq 1$$

Passing to the limit as $a \to 0$ it follows that

$$\text{(2.3.2)} \qquad \sum_{k=1}^{k=n} \frac{1}{b_k} \leq \frac{e\pi}{2}$$

which proves Theorem 1.3.

Carleman's reconstruction theorem for real-analytic functions.

A real-valued C^∞-function f on the closed unit interval is real analytic if and only if there exist constant C and M such that

$$\max_{0\leq x\leq 1} |f^{(k)}(x)| \leq M\cdot k!\cdot C^k \quad : k=1,2,\ldots \tag{0.1}$$

The analyticity implies that f is determined by its derivatives at the origin. However, the Taylor series

$$\sum_{k\geq 0} f^{(k)}(0)\cdot \frac{x^k}{k!}$$

is in general only convergent for in a small interval $[0\leq x<\delta$. In 1921 Borel posed the question how on determines $f(x)$ on the whole interval from the sequence $\{f^{(k)}(0)\}$. An affirmative answer was given by Carleman in 1923 via solutions to a family of variational problems which goes as follows: Put $\alpha_k = f^{(k)}(0)$ for each $k\geq 0$. If N is a positive integer we denote by $\mathcal{H}_N$ the Hilbert space whose elements are $N-1$-times continuous differentiable functions g on $[0,1]$, and in addition $g^{(N)}$ is square integrable, i.e. it belongs to $L^2[0,1]$. In "contemporary mathematics" this means that H_N is a Sobolev space. But of course the notion of weak L^2-derivatives was perfectly well understood long before and for example used extensively in work by Weyl before 1910. Inside $\mathcal{H}_N$ we have the subspace $\mathcal{H}_N(f)$ which consists of functions g such that

$$g^{(k)}(0) = f^{(k)}(0) \quad : k=0,\ldots,N-1 \tag{0.2}$$

With these notations one regards the variational problem

$$\min_{g\in\mathcal{H}_N(f)} J_N(g) = \sum_{k=0}^{k=N} (\log(k+2))^{-2k}\cdot (k!)^{-2k}\cdot \int_0^1 g^{(k)}(x)^2\,dx \tag{0.3}$$

Elementary Hilbert space methods yield a unique minimzing function denoted by f_N. These successive solutions give a sequence $\{f_N\}$ where each f_N has at least $N-1$ continuous derivatives. Less obvious is the following:

Main Theorem. *For each real-analytic function f the sequence $\{f_N\}$ converges uniformly together with all derivatives to f, i.e. for every $m\geq 0$ it holds that*

$$\lim_{N\to\infty} |f_N^{(m)} - f^{(m)}|_{0,1} = 0$$

Remark. Since every f_N is determined by derivative of f up to order $N-1$ at $x=0$ it means that one has a reconstruction of the real-analytic function f via these derivatives.

Proof of the Main Theorem

For each N we denote by $J_*(N)$ the minimum in the variational problem (0.3). Among the competing functions we can choose f and hence

$$J_*(N)\leq J_N(f)$$

Now there exist constants C and M from (0.1) which entails that

$$J_N(f)\leq M\cdot\sum_{k=0}^{N} (\log(k+2))^{-k} C^{2k}$$

Since $\log(k+2)$ tends to $+\infty$, it is clear that the series

$$\sum_{k=0}^{\infty} (\log(k+2))^{-k}\cdot C^{2k} < \infty$$

We conclude that there exists a constant J_* such that

$$J_*(N)\leq J_* \quad : N=1,2,\ldots \tag{i}$$

So if m is some positive integer and $N \geq m$ we have

(ii) $$\sum_{k=0}^{k=m} (\log(k+2))^{-2} \int_0^1 f_N^{(k)}(x)^2\, dx \leq J_*$$

Now we recall the classic resut due to Arzela-Ascoli which implies that bounded sets in H_m give relatively compact subsets of $C^{m-1}[0,1]$. Since (ii) hold for each m, it follows by a standard diagonal procedure which is left to the reader that we can find a subsequence $\{g_\nu = f_{N_\nu}\}$ such that the sequence of derivatives $\{g_\nu^{(m)}\}$ converge uniformly for every m, i.e $g_\nu \to g_*$ holds in the space $C^\infty[0,1]$. Next, by (0.2) we have for each fixed integer $k \geq 0$:

$$f^{(k)}(0) = f_N^{(k)}(0) \quad : N \geq k+1$$

From this it follows that

(iii) $$f^{(k)}(0) = g_*^{(k}(0) \quad : k = 0, 1, 2 \ldots$$

Hence the C^∞-function

$$\phi = f - g_*$$

is flat at $x = 0$. Next, for a fixed integer k the uniform bound in (ii) gives

(iv) $$\int_0^1 \phi^{(k)}(x)^2\, dx \leq J_* \cdot (\log(k+2))^{2k} \cdot (k!)^2$$

Moreover, for each $0 < x \leq 1$ the Cauchy-Schwartz inequality gives

$$\phi^{(k)}(x) = \int_0^x \phi^{(k+1)}(t)\, dt \leq \sqrt{\int_0^1 \phi^{(k)}(x)^2\, dx}$$

and since (iv) hold for every k it follows that

$$\max_x |\phi^{(k)}(x)| \leq J_* \cdot (\log(k+2))^k \cdot k!$$

Since $k! \leq k^k$ this entails that

$$\mathcal{C}_k(\phi) \leq J_*^{\frac{1}{k}} \cdot k \cdot (\log(k+2))$$

Since the series $\sum_{k=1}^\infty \frac{1}{k \log k}$ is divergent we conclude that

$$\sum_{k=1}^\infty \frac{1}{\mathcal{C}_k(\phi)} = +\infty$$

Hence Denjoy's result in xxx proves that ϕ is identically zero. which means that

(*) $$\lim_{k \to \infty} f_{N_k} = f$$

where the convergence holds in the space $C^\infty[0,1]$. Finally, the reader may check that (*) holds for an arbitrary convergent subsequence which by the previous compactness by Arzela-Ascoli entails that the whole sequence $\{f_N\}$ converges to f. This finishes the proof of the Main Theorem.