Chapter VIII The Gamma function and Riemann's ζ-function

Contents

Introduction. The results in this chapter stem from early work by Euler and later studies by Gauss and Riemann. The Γ-function is defined and analyzed in § I. A major result is that $\frac{1}{\Gamma(z)}$ is an entire function. More precisely one has the Gauss representation

$$\frac{1}{\Gamma(z)} = z \cdot e^{\frac{\gamma}{z}} \cdot \prod_{m=1}^{\infty} (1 + \frac{z}{m}) \cdot e^{-\frac{z}{m}}$$

where γ is the Euler constant. Section II is devoted to Riemann's ζ-function defined by the Dirichlet series

(*) $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad : \quad \mathfrak{Re}\, s > 1$$

Euler proved that the ζ-function extends to the whole complex s-plane with a simple pole at $s = 1$ whose residue is one which gives the entire function

(**) $$\zeta(s) - \frac{1}{s-1}$$

In 1894 it was proved (independently) by Hadamard and de Vallé Poussin that the zeta-function has no zeros on $\mathfrak{Re}\, s = 1$, i.e.

(1) $$\zeta(1 + it) \neq 0 \quad \text{for all real} \quad t \neq 0$$

From (1) we shall deduce the Prime Number Theorem in § 4. We proceed to discuss properties of the ζ-function which are due to Riemann. His first major result is the functional equation:

(2) $$\zeta(1 - s) = \frac{2}{(2\pi)^s} \cdot \cos \frac{\pi}{2}\, s \cdot \Gamma(s) \cdot \zeta(s)$$

where the Γ-function appears with simple poles at $(0, -1, -2, \ldots)$. Hence this functional equation implies that

(3) $$\zeta(-2n) = 0 \quad : \quad n = 1, 2, \ldots$$

The *Riemann hypothesis* states that all other zeros belong to the critical line $\mathfrak{Re}\, s = 1/2$. Further comments about this conjecture are given in § 0.4

Remark about Riemann's work. An account of Riemann's original work appears in the article [xxx] by Siegel where a wealth of involved analytic formulas related to the ζ-function appear. So it was not by a mere guessing that Riemann arrived at his famous conjecture. Examples of his brilliant mastery establish various convergence formulas for the ζ-function which later were adopted by many authors. So even if we will prove a number of results about the ζ-function i this material in this chapter does not give a full tribute to Riemann's work. An example from Riemann's original studies of the zeta-function is the following inequality:

0.1 Theorem. *There exists a constant C such that*

(i) $$\max_{|s|=r} |\zeta(s)| \leq C \cdot \frac{\Gamma(r)}{(2\pi)^r} \quad : r \geq 2$$

Moreover, in the half-space $\mathfrak{Re}\, s > 0$ one has the equation

(ii) $$\zeta(z) = s \cdot \sin \frac{\pi s}{2} \cdot \frac{1}{\pi} \cdot \int_0^\infty \log \frac{e^{\pi x} - e^{-\pi x}}{2\pi x} \cdot \frac{dx}{x^{s+1}}$$

The fact that (*) gives (i-ii) is far from obvious and illustrates the depth in Riemann's work.

0.2 A class of Dirichlet series. Let $\mathcal{F}$ be the family of all non-decreasing increasing sequence of positive numbers $\lambda_1 \leq \lambda_2 \leq \ldots$ for which there exists some $\delta > 0$ such that

(i) $$\lambda_n \geq \delta \cdot n$$

hold for every n where δ can depend on the given sequence. General facts about entire functions of exponential type entail that (i) the Hadamard product

(ii) $$f(z) = \prod (1 + \frac{z^2}{\lambda_n^2})$$

is an entire function of exponential type, i.e. belongs to the class $\mathcal{E}$. Next, we construct the Dirichlet series

(iii) $$\Lambda(s) = \sum_{n=1}^\infty \frac{1}{\lambda_n^s}$$

By (i) this gives an analytic function in the half-plane $\mathfrak{Re}\, s > 1$. By a classic result known as the Mellin's inversion formula it follows that $\Lambda(s)$ is obtained from $f(z)$ by the equation

(iv) $$\Lambda(s) = s \cdot \sin \frac{\pi s}{2} \cdot \frac{1}{\pi} \cdot \int_0^\infty \log f(x) \cdot \frac{dx}{x^{s+1}}$$

Using this we prove the following conclusive result in § XX:

0.2.1 Theorem. *Every Λ-function obtained from a seqence in $\mathcal{F}$ extends to a meromorphic function in the complex s-plane.*

Example. By (ii) in Riemann's cited result above the assocaited f-function to the Direchlet series defining $\zeta(s)$ is equal to

$$f(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$$

0.2.2 An extremal property of $\zeta(s)$. In a lecture at Harvard University in 1949, Beurling proved that Riemann's zeta-function has a distinguished position in a class of functions defined by Dirichlet series. For each positive number k we denote by $\mathcal{C}_k$ the class of series $\Lambda(s)$ from (ii) with the properties:

(a) $$\Lambda(s) - \frac{1}{s-1} \quad \text{is entire}$$

(b) $$\Lambda(-2n) = 0 \quad \text{for all positive integers}$$

(c) $$\max_{|s|=r} |\Lambda(s)| \leq C \cdot \frac{\Gamma(r)}{(2\pi k)^r} \quad \text{hold for a constant} \quad C \quad \text{and all} \quad r \geq 2$$

Notice that the class $\mathcal{C}_k$ becomes more restrictive as k increases. We shall learn that $\zeta(s)$ satisfies (a-b) and (i) in Theorem 0.1 entails that the zeta-function belongs to $\mathcal{D}_1$. In § 5 we prove that for every $1/2 < k \leq 1$ the class $\mathcal{C}_k$ only consists of constants times the zeta-function while $\mathcal{C}_k = \emptyset$ if $k > 1$. This illustrates the special role of the ζ-function.

0.2.3 Beurling's closure theorem. In Theorem 7.1 we prove another result by Beurling which gives a *necessary and sufficient condition* for the validity of the Riemann hypothesis expressed by a certain L^2-closure on the interval $(0,1)$ generated by a specific family of functions. Theorem 7.1 is based upon a closure theorem in § 6 whose proof illustrates the efficiency of mixing functional analysis with analytic function theory. Let us remark that the study of distributions of primes and the Riemann hypothesis was one of the main issues in Beurling's research. His first extensive article on this subject from 1937 is entitled *Analyse de loi asymptotique de la distribution des nombres premier generalisés*. Even though this work does not settle the Riemann hypothesis the reader will find a number of interesting results concerned with Dirichlet series.

0.3. The distribution of prime numbers

A motivation to consider the ζ-function is Euler's product formula:

$$(*)\qquad \zeta(s)=\prod \frac{1}{1-p^{-s}} : \text{ product over all prime numbers } \geq 2$$

Indeed, (*) follows since every integer $n\geq 2$ can be factorised in a unique way as a product of prime numbers. Let us introduce the counting function:

$$\mathcal{N}(x)=\text{number of primes } \leq [x] \quad : [x]=\text{least integer } \leq x$$

Thus $\mathcal{N}(x)$ is the primitive of the discrete measure supported by $[2,+\infty)$ which assigns a unit point mass at every prime. Integration by parts gives:

$$\text{(i)}\qquad \log\ \zeta(s)=s\cdot\int_2^\infty \frac{\mathcal{N}(x)\cdot dx}{(x^s-1)x} \quad : s \text{ real and } >1$$

By Euler's result we can write

$$\text{(ii)}\qquad \zeta(s)=\frac{1}{s-1}+g(s) \quad : g(s) \text{ analytic in a disc } |s-1|<\delta$$

This gives the limit formula:

$$(**)\qquad \lim_{\epsilon\to 0}\ \frac{1}{\mathrm{Log}\,[\frac{1}{\epsilon}\,]}\cdot\int_2^\infty \frac{\mathcal{N}(x)\cdot dx}{x^{2+\epsilon}}=1$$

The Prime Number Theorem. Notice that

$$\lim_{\epsilon\to 0}\ \frac{1}{\log\,[\frac{1}{\epsilon}\,]}\cdot\int_2^\infty \frac{x\cdot dx}{x^{2+\epsilon}\cdot\log\, x}=1$$

which follows by the variable substitution $x\mapsto e^t$ and the observation that

$$\lim_{\epsilon\to 0}\ \frac{1}{\mathrm{Log}\,[\frac{1}{\epsilon}\,]}\cdot\int_1^\infty e^{-\epsilon t}\cdot\frac{dt}{t}=1$$

In view of (**) it is therefore no surprise that the following limit formula holds for $\mathcal{N}$.

0.3.1 Theorem. *One has*

$$\lim_{x\to\infty}\frac{\mathrm{Log}\,(x)\cdot\mathcal{N}(x)}{x}=1$$

Remark. This limit formula was known by heuristic considerations long before Riemann's study of the ζ-function. By elementary arithmetic one can show the prime number theorem holds under the *extra hypothesis* that $\mathcal{N}(x)$ has a "regular growth". But both Riemann and Gauss knew that $\mathcal{N}(x)$ does *not* increase in a regular way. Hence a solid proof of the prime number theorem was requested and it was finally established by Hadamard and de Valle Poussin. The detailed proof is given in § 4.

0.4. The Riemann Hypothesis.

The conjecture by Riemann states that the zeros of $\zeta(s)$ in the critical strip $0 < \mathfrak{Re}(s) < 1$ belong to the line $\mathfrak{Re}(s) = 1/2$. This line is special since the functional equation yields an entire function $\xi(s)$ defined by

$$(*) \qquad \xi(s) = \frac{s(s-1)}{2} \cdot \Gamma(\frac{s}{2}) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s)$$

Moreover, the ξ-function satisfies

$$\xi(s) = \xi(1-s) \quad : \quad \xi(s) = \bar{\xi}(\bar{s})$$

From this it follows that the function

$$t \mapsto \Gamma(\frac{1/2+it}{2}) \cdot \pi^{-\frac{1/2+it}{2}} \cdot \zeta(\frac{1}{2}+it)$$

is real-valued. Here $\Gamma(\frac{1/2+it}{2}) \cdot \pi^{-\frac{1/2+it}{2}} \neq 0$ for all real t and we can define the function

$$(**) \qquad \theta(t) = \arg\big[\,\Gamma(\frac{1/2+it}{2}) \cdot \pi^{-\frac{1/2+it}{2}}\,\big] \quad : \quad 0 \leq t \leq \infty$$

where we take $\theta(0) = 0$. So now we have the real-valued function

$$(***) \qquad X(t) = e^{i\theta(t)} \cdot \zeta(\frac{1}{2}+it)$$

Hence zeros on the critical line $\mathfrak{Re}(s) = \frac{1}{2}$ correspond to zeros of this real valued function. An asymptotic formula of $X(t)$ was established by Siegel based upon unpublished work by Riemann:

0.4.1 The Riemann-Siegel formula. In [Sie] the following asymptotic limit formula is proved:

$$(*) \qquad X(t) = \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{\cos\theta(t) - t \cdot \log n}{\sqrt{n}} + O(t^{-\frac{1}{2}})$$

Starting from this Julius Gram and Arvid Bäcklund established numerical results. Namely, from (*) one derives the asymptotic formula:

$$\theta(t) = \frac{t}{2} \cdot \big[\,\mathrm{Log}\frac{t}{2\pi} - \frac{1}{2}\,\big] - \frac{\pi}{8} + O(\frac{1}{t})$$

Consider the increasing sequence of real numbers $\{t_\nu\}$ for which

$$\theta(t_\nu) = (\nu - 1) \cdot \pi \quad : \quad \theta'(t_\nu) > 0$$

The Riemann-Siegel formula gives:

$$(1) \qquad \zeta(\frac{1}{2}+it_\nu) = 1 + \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{\cos(\,t_\nu \cdot \log n)}{\sqrt{n}} + O(t_\nu^{-\frac{1}{2}})$$

This suggests that $\zeta(\frac{1}{2}+it_\nu)$ in general is positive and that $X(t_{\nu-1})$ and $X(t_\nu)$ will have different sign which therefore gives a zero for the ζ-function in the interval $(t_{\nu-1} - t_\nu)$. The *Law of Gram* asserts that all zeros of $X(t)$ should appear in this fashion, i.e. one zero is produced in $(t_{\nu-1} - t_\nu)$ for every ν. Of course, this "law" was presented as an asymptotic formula only. A "weak asymptotic law" was confirmed in work by Hutchinson and Titchmarsh. But the situation is not so easy. For consider the actual zeros on the critical line:

$$0 < \gamma_1 < \gamma_2 < \ldots \quad : \quad \zeta(\frac{1}{2}+i\gamma_\nu) = 0$$

In an article from 1942, Atle Selberg proved that the γ-sequence increases in a certain *irregular* fashion.

0.4.2 Theorem. *There exists an absolute constant $0 < C_* < 1$ such that for every positive integer r one has*

$$\limsup_{n\to\infty} \frac{\gamma_{n+r}-\gamma_n}{2\pi r}\cdot \operatorname{Log}\gamma_n > 1 + C_* \quad \text{and} \quad \liminf_{n\to\infty} \frac{\gamma_{n+r}-\gamma_n}{2\pi r}\cdot \operatorname{Log}\gamma_n < 1 - C_*$$

Let us finish by a citation from A. Selberg's lecture on the Zeta Function and the Riemann Hypothesis at the Scandinavian Congress in mathematics held at Copenhagen in 1946, where he gave some comments about the eventual validity of the Riemann Hypothesis:

In spite of the numerical evidence by which it is supported there are still reasons to regard the Riemann Hypothesis with suspicion. For in the range covered by calculations. the exceptions from Gram's Law are few and they are of the simplest kind, farther out more severe departures from Gram's law must occur and it seems likely that the irregularities in the variation of $\zeta(s)$ which should be necessary for producing zeros outside $\mathfrak{Re}(s) = \frac{1}{2}$, should be far more remote than the first exceptions from Gram's Law.

Note. Atle Selberg (191x-2007) received the Fields medal at the IMU-congress in 1950 for his outstanding contribution in number theory and deep studies of the ζ-function.

0.4.3 Hardy's inversion formula.

Another contribution in the study of zeros on the critical line was achieved by Hardy in [Har]. His method was to regard the function

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} \quad : \ x > 0$$

The construction of the Γ function gives the equality

$$(*) \qquad \frac{1}{n^s}\cdot\Gamma(\frac{s}{2})\cdot\pi^{-\frac{s}{2}} = \int_0^\infty e^{-n^2\pi x}\cdot x^{\frac{s}{2}}\cdot\frac{dx}{x}$$

for each positive integer n. A summation over n gives:

$$\zeta(s)\cdot\Gamma(\frac{s}{2})\cdot\pi^{-\frac{s}{2}} = \int_0^\infty \omega(x)\cdot x^{\frac{s}{2}}\cdot\frac{dx}{x} \quad : \mathfrak{Re}(s) > 1$$

Using Fourier's inversion formula and a shift of certain complex line integrals, Hardy established the following result:

0.4.4 Theorem. *When $\mathfrak{Re}\, x > 0$ one has the equality*

$$\omega(x) = \frac{1}{2\sqrt{x}} + \frac{1}{4\pi}\cdot\int_{-\infty}^{\infty} \zeta(1/2+it)\cdot\Gamma(1/4+it/2)\pi^{-it/2-1/4}x^{-1/4-it}\cdot dt$$

Remark. The right hand side yields a nicely convergent integral. The reason is that one has an exponential decay for the Γ-function, i.e. there are constants A and k such that

$$|\Gamma(1/4+it/2)| \leq A\cdot e^{-k|t|} \quad : \ -\infty < t < \infty$$

At the same time the ζ-function does not increase too fast. Namely, in § XX we show that there is constant B such that

$$|\zeta(1/2+it)| \leq B\cdot t^2 \quad : \ t \geq 1$$

We refer to Hardy's original work how the inversion formula is used to produce zeros of the zeta-function on the critical line. See also the text-book by Titchmarsch devoted Riemann's ζ-function where results from analytic function theory and Fourier analysis are used to the "bitter end" in the search for a positive answer to the Riemann Hypothesis.

Final remark. The literature about the Riemann hypothesis is extensive and there exist alternative conjectures, some of them even more general than Riemann's. We shall not enter into a discussion about this. But let recall that André Weil solved the Riemann hypothesis in characteristic p. This gives some support for the "optimistic point of view point " that Riemann's hypothesis is true. But until an answer is found the Riemann Hypothesis remains as an outstanding open problem in mathematics.

1. The Gamma function

The Γ-function has from the start a simple definition:

(*) $$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad : \mathfrak{Re}(z) > 0$$

With $z = x + iy$ the absolute value $|t^{z-1}| = t^{x-1}$ when t is real and positive. Moreover, $t^\alpha \cdot \log(t)$ is locally integrable on intervals $(0, t_*)$ with $t_* > 0$ and we have the exponential decay from e^{-t}. Hence (*) converges when $\mathfrak{Re}(z) > 0$ and gives a holomorphic function with the complex derivative

(0.1) $$\Gamma'(z) = \int_0^\infty e^{-t} \cdot \operatorname{Log}(t) \cdot t^{z-1} dt$$

Partial integration gives:

$$\Gamma(z) = e^{-t} \cdot \frac{t^z}{z} |_0^\infty + \frac{1}{z} \int_0^\infty e^{-t} t^z \, dt = \frac{1}{z} \cdot \Gamma(1+z)$$

Hence we have the equality

(0.2) $$z\Gamma(z) = \Gamma(z+1) \quad : \quad \mathfrak{Re}(z) > 0$$

Replacing z by $z + 1$ and so on we obtain

(0.3) $$z(z+1) \cdots (z+m)\Gamma(z) = \Gamma(z+m+1) \quad : \quad m = 0, 1, 2, \ldots$$

Since (0.3) hold for all non-negative integers, $\Gamma(z)$ extends to a meromorphic function defined in the whole complex plane and for every positive integer m one has:

(0.4) $$\Gamma(z) = \frac{1}{z(z+1)\ldots(z+m)} \cdot \Gamma(z+m) \quad : \quad \mathfrak{Re}(z) > -m$$

Here (0.4) shows that the poles of $\Gamma(z)$ are contained in the set of non-negative integers. We shall later prove that a simple pole exists for every such integer. Moreover, we will show that $\frac{1}{\Gamma(z)}]$ is an entire function and establish the functional equation:

(**) $$\frac{1}{\Gamma(z) \cdot \Gamma(1-z)} = \frac{\sin \pi z}{\pi}$$

1. The Gauss representation.

Consider the Hadamard product:

(1.1) $$H(z) = \prod_{m=1}^{\infty} (1 + \frac{z}{m}) e^{-\frac{z}{m}}$$

Here $H(z)$ is entire with simple zeros at negative integers and (0.4) gives the entire function

(1.2) $$zH(z) \cdot \Gamma(z) \in \mathcal{O}(\mathbf{C})$$

1.3 Theorem *Let γ be the Euler constant defined by*

$$\gamma = Lim_{n \to \infty} 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \operatorname{Log} n$$

Then one has

(**) $$zH(z) \cdot \Gamma(z) = e^{-\gamma z}$$

Proof. When $n \geq 2$ we consider the partial product

$$H_n(z) = \prod_{m=1}^{m=n} (1 + \frac{z}{m}) e^{-\frac{z}{m}}$$

A computation gives the identity

$$\frac{z(z+1)\cdots(z+n)}{n\,!\cdot n^z}=e^{z\cdot[1+\frac{1}{2}+\ldots+\frac{1}{n}-\operatorname{Log} n]}\cdot z\cdot H_n(z)$$

If $\mathfrak{Re}(z)>0$ the right hand side converges to the limit $e^{\gamma z}\cdot z\cdot H(z)$. Hence there exists the entire limit function

(i) $$G(z)=\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n\,!\cdot n^z}$$

It is clear that (**) holds in Theorem 1.3 if we have proved:

(ii) $$G(z)\cdot\Gamma(z)=1$$

To prove (ii) we regard the meromorphic function $\mathcal{G}=\frac{1}{G}$ so that

(iii) $$\mathcal{G}(z)=\lim_{n\to\infty}\frac{n\,!\cdot n^z}{z(z+1)\cdots(z+n)}$$

Let us put

(iv) $$\mathcal{G}_n(z)=\frac{n\,!\cdot n^z}{z(z+1)\cdots(z+n)}$$

Then we have

$$\mathcal{G}_n(z+1)=\frac{n\,!\cdot n^{1+z}}{(z+1)(z+2)\cdots(z+n+1)}=$$
$$\frac{n\,!\cdot n^z}{z(z+1)\cdots(z+n)}\cdot\frac{nz}{n+1+z}=z\mathcal{G}_n(z)\cdot\frac{n}{n+z+1}$$

The last quotient tends to z and we already know that $\mathcal{G}_n(z)\to\mathcal{G}(z)$. We conclude that the $\mathcal{G}$-function satisfies

(v) $$\mathcal{G}(z+1)=z\mathcal{G}(z)$$

Hence $\mathcal{G}$ satisfies the same functional equation as the Γ-function and there remains only to show that $\Gamma=\mathcal{G}$. Since we have two meromorphic functions it suffices that they are equal on the positive real axis. To show this we first regard complex derrivatives of $\operatorname{Log}\mathcal{G}$ which is holomorphic in the right half-plane. Since the limit in (iii) defines $\mathcal{G}$ in $\mathfrak{Re}(z)>0$ we have

(vi) $$\log\,\mathcal{G}(z)=\lim\,[z\cdot\log(n)+\sum_{\nu=1}^{\nu=n}\log\,\nu-\sum_{\nu=0}^{\nu=n}\log(z+\nu)\,]$$

In the right hand side we take the second order derivative which becomes

$$\sum_{\nu=0}^{\nu=n}\frac{1}{z+\nu)^2}$$

Passing to limit as $n\to\infty$ it follows that

(vii) $$\frac{d^2\,\log\,\mathcal{G}(z)}{dz^2}=\lim_{n\to\infty}-\sum_{\nu=0}^{n}\frac{1}{(z+\nu)^2}=-\sum_{\nu=0}^{\infty}\frac{1}{(z+\nu)^2}$$

Let us now regard the function defined for $x>0$ by

(vii) $$\delta(x)=\log\,\Gamma(x)-\log\,\mathcal{G}(x)$$

If we prove that $\delta(x)=0$ for all $x>0$ then (ii) follows by analyticity. To show that $\delta(x)=0$ we first notice that (v) gives:

(viii) $$\delta(x+1)=\delta(x)\quad:\quad x>0$$

Next, we shall regard the second derivative of δ. First, put

(ix) $$\psi(x)=\frac{d^2\,\log\,\Gamma(x)}{dx^2}=\frac{\Gamma(x)\Gamma''(x)-\Gamma'(x)^2}{\Gamma'(x)^2}$$

Next we have

(x) $$\Gamma'(x) = \int_0^\infty e^{-t} \cdot \log t \cdot t^{x-1} \cdot dt \quad : \quad \Gamma''(x) = \int_0^\infty e^{-t} (\log t)^2 \cdot t^{x-1} \cdot dt$$

From these two expressions the Cauchy Schwarz inequality gives

(xi) $$\Gamma(x)\Gamma''(x) - \Gamma'(x)^2 \geq 0$$

At the same time (vii) shows that the second order derivative of $\log \mathcal{G}(x)$ is < 0. We conclude that $\delta''(x) \geq 0$, i.e. the δ-function is strictly convex when $x > 0$. Next, the periodicity remains valid for the first order derivative, i.e.

(xii) $$\delta'(x+1) = \delta'(x) \quad : \quad x > 0$$

Finally, δ is convex the derivative $\delta'(x)$ is a non-increasing function and we notice that every non-increasing and 1-peridoic function on $x > 0$ is a constant. Hence $\delta'(x)$ is a constant which gives

(xiii) $$\delta(x) = ax + b \quad : a, b \text{ real constants}$$

Since $\delta(x+1) = \delta(x)$ it follows that $a = 0$. Hence $\delta(x) = b$ is a constant But $b = 0$ since it is clear that the functions Γ and $\mathcal{G}$ are equal at all positive integers. This proves that $\delta(x)$ is identically zero on $x > 0$ and the proof of Theorem 1.3 is finished.

1.4 Remark. The limit of products which defined $\mathcal{G}(z)$ was considered by Gauss. So one refers to $\mathcal{G}$ as the *Gauss representation* of the Γ-function. Theorem 1.3 is due to Weierstrass. The proof above using the δ-function was discovered by Erhard Schmidt. His method was later extended by Emil Artin who established a remarkable uniqueness property of the Γ-function in an article from 1931. More precisely he proved

1.5 Artin's Theorem *Let $f(z)$ be an entire function with $f(0) = 1$ satisfying:*

$$f(z+1) = f(z) \quad : : \quad \frac{d^2 \log f(x)}{dx^2} \geq 0 \quad : x > 0$$

Then $f(z) = \frac{1}{z \cdot \Gamma(z)}$.

We refer [Artin] for details of proof and a further discussions about the Γ-function.

2. A functional equation.

Consider the product

(2.1) $$\Gamma(z) \cdot \Gamma(1-z)$$

This is a meromorphic function with simple poles at all integers. Next, we have the entire function $\sin \pi z$ with simple zeros at all integers. Hence the function

$$F(z) = \Gamma(z) \cdot \Gamma(1-z) \cdot \sin \pi z$$

has no poles and is therefore entire. It turns out that this function is constant.

2.1 Theorem. One has the equality

$$\frac{1}{\Gamma(z) \cdot \Gamma(1-z)} = \frac{\sin \pi z}{\pi}$$

Proof. The Gauss representation from (iii) in the proof of Theorem 1.3 gives

$$\frac{1}{\Gamma(z)} = \lim_{n \to \infty} \frac{z(z+1) \cdots (z+n)}{n! \cdot n^z} = \lim_{n \to \infty} \frac{z(z+1) \cdot (\frac{z}{2}+1)) \cdots (\frac{z}{n}+1)}{n^z}$$

Similarly we obtain

$$\frac{1}{\Gamma(1-z)} = \lim_{n \to \infty} \frac{(1-z) \cdot (1-\frac{z}{2}) \cdots (1-\frac{z}{n})}{n^{-z}} \cdot \frac{n+1-z}{n}$$

Since $\lim_{n\to\infty} \frac{n+1-z}{n} = 1$ we conclude that

$$\frac{1}{\Gamma(z)\cdot\Gamma(1-z)} = \lim_{n\to\infty} z\cdot(1-z^2)\cdot(1-(\frac{z}{2})^2)\cdots(1-(\frac{z}{n})^2)$$

But the last term is the Hadarmard product for $\frac{\sin \pi z}{\pi}$ and Theorem 2.1 follows.

3. The integral formula.

Let z be a non-zero complex number such that $\mathfrak{Re}\, z < 1$. Keeping z fixed we define the complex powers s^{-z} for all s in $\mathbf{C} \setminus (-\infty, 0]$. More precisely, when the negative real axis is removed we have a unique polar form

(i) $$s = re^{i\theta} \quad -\pi < \theta < \pi$$

Then we can write

(ii) $$s^{-z} = r^{-z} \cdot e^{-i\theta z}$$

For every fixed z the function $s \mapsto s^{-z}$ is analytic in the simply connected domain $\Omega = \mathbf{C}\setminus(-\infty, 0]$ where it is equal to $e^{-z\cdot\log s}$ and (i) determines the value of $\log s$. Multiplying with e^s we get

(1) $$f(s) = s^{-z} \cdot e^s \in \mathcal{O}(\Omega)$$

With $s = \sigma + i\tau$ we get

(2) $$|f(\sigma + i\tau)| = |r|^{-x} \cdot e^{-\theta y} \cdot e^{\sigma} \quad : \quad z = x + iy \quad :$$

Hence $f(s)$ has exponential decay in the right half-plane $\mathfrak{Re}(s) < 0$. We profit upon this to construct two absolutely convergent integrals. Given $\epsilon > 0$ we consider the two half-lines

(3) $$\ell^*(\epsilon) = \{s = \sigma + i\epsilon \quad : \sigma \leq 0\} \quad : \quad \ell_*(\epsilon) = \{s = \sigma - i\epsilon \quad : \sigma \leq 0\}$$

Let $T_+(\epsilon) = \{s = \epsilon e^{i\theta} : \; -\pi/2 \leq \theta < \pi/2\}$ and put:

(4) $$\gamma(\epsilon) = \ell^*(\epsilon) \cup T_+(\epsilon) \cup \ell_*(\epsilon)$$

We choose the *negative* orientation along $\gamma(\epsilon)$ which means that we first integrate along $\ell^*(\epsilon)$ as σ increases from $-\infty$ to 0 and and so on. This gives:

(5) $$\int_{\gamma(\epsilon)} f(s)ds = \int_{-\infty}^{0} f(\sigma + i\epsilon)d\sigma - i\epsilon \cdot \int_{-\pi/2}^{\pi/2} f(\epsilon \cdot e^{i\theta})e^{i\theta} \cdot d\theta - \int_{-\infty}^{0} f(\sigma - i\epsilon)d\sigma$$

With $\mathfrak{Re}(z) = x < 1$ we see from (2) that

(6) $$\lim_{\epsilon\to 0} \epsilon \cdot \int_{-\pi/2}^{\pi/2} f(\epsilon \cdot e^{i\theta})e^{i\theta} \cdot d\theta = 0$$

Next, when $\sigma < 0$ we see that (ii) and (1) give

(7) $$\lim_{\epsilon\to 0} f(\sigma + i\epsilon) - f(\sigma - i\epsilon) = |\sigma|^{-x-iy}[e^{i\pi\cdot z} - e^{-i\pi\cdot z}]$$

Hence we have proved

3.1 Proposition. *One has*

$$\lim_{\epsilon\to 0} \int_{\gamma(\epsilon)} f(s)ds = 2i \cdot \sin(\pi z) \cdot \int_{-\infty}^{0} |\sigma|^{-x-iy} \cdot e^{\sigma} d\sigma$$

Dividing by $2\pi i$ and making the variable substitution $t = -\sigma$ we get

(8) $$\lim_{\epsilon\to 0} \frac{1}{2\pi i} \cdot \int_{\gamma(\epsilon)} f(s)ds = \frac{\sin(\pi z)}{\pi} \cdot \int_{0}^{\infty} t^{-x-iy} \cdot e^{-t}dt$$

The last integral is $\Gamma(1-z)$. Together with Theorem 2.1 we conclude the following:

3.2 Theorem. *When $\mathfrak{Re}(z) < 1$ one has the equality*

$$\frac{1}{\Gamma(z)} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{\gamma(\epsilon)} e^s \cdot s^{-z} ds$$

where $\gamma(\epsilon)$ has the negative orientation as described in (4).

3.3 Remark. In the right hand side we can change the contour γ_ϵ where convergence holds as long as the real part of s tends to $-\infty$ along the end-tails. For example, the integral representation holds for every $\epsilon > 0$, i.e. even for *large* ϵ. This flexible manner to represent $\frac{1}{\Gamma(z)}$ is used in many formulas where Γ-functions appear. An example are the integral formulas due to Barnes for hypergeometric functions.

II. Riemann's ζ-function

Introduction. The zeta-function is defined by the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad : \quad \mathfrak{Re}\, s > 1 \tag{0.1}$$

It is clear that $\zeta(s)$ is an analytic function in the half-space $\mathfrak{Re}(s) > 1$ whose complex derivative is found by termwise differentiation, i.e.

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^s} \quad : \quad \mathfrak{Re}\, s > 1 \tag{0.1}$$

In the subsequent sections we establish results about the ζ-functions concerned with growth properties and distribution of its zeros.

1. The meromorphic extension.

The fact that $\zeta(s)$ has a meromorphic extension with a simple pole a $s = 1$ goes back to Euler and is presented below.

1.1 Euler's summation formula. When $x \geq 1$ we let $[x]$ denote the largest integer which is $\leq x$. Set

$$P(x) = [x] - x + \frac{1}{2}\,, \quad x \geq 1$$

The differential dP is the counting function at positive integers. So if $\mathfrak{Re}\, s > 1$ we have

$$\zeta(s) - \int_1^{\infty} \frac{dx}{x^s} = \int_1^{\infty} \frac{dP(x)}{x^s} = \frac{P(x)}{x^s}\Big|_1^{\infty} - \int_1^{\infty} \frac{P(x)}{x^{s+1}} \cdot dx$$

Since $P(1) = \frac{1}{2}$ and $\int_1^{\infty} \frac{dx}{x^s} = \frac{1}{s-1}$ we obtain

1.2 Euler's integral formula. One has

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s \cdot \int_1^{\infty} \frac{P(x)}{x^{s+1}} \cdot dx$$

Remark. Since the function $P(x)$ is bounded the last integral extends to an analytic function in the half plane $\mathfrak{Re}\, s > 0$. So Euler's integral formula shows that the zeta-function extends to a meromorphic function in $\mathfrak{Re}\, s > 0$ with a simple pole at $s = 1$.

1.3 Further integral formulas. Notice that $P(x)$ is 1-periodic:

$$P(x+1) = P(x) \quad x \geq 1\,.$$

Moreover it has the Fourier series expansion

$$P(x) = \sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n\pi}$$

whose primitive function becomes

$$P_1(x) = -\sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{2n^2\pi^2}$$

A partial integration gives

$$s \cdot \int_1^{\infty} \frac{P(x)}{x^{s+1}} \cdot dx = s \cdot \frac{P_1(x)}{x^{s+1}}\Big|_1^{\infty} - s(s+1) \int_1^{\infty} \frac{P_1(x)}{x^{s+2}} \cdot dx$$

Next, one has the summation formula

$$P_1(1) = \sum_{n=1}^{\infty} \frac{1}{2n^2\pi^2} = \frac{1}{12}$$

It follows that

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - s(s+1) \cdot \int_1^{\infty} \frac{P_1(x)}{x^{s+2}} \cdot dx$$

The last integral is analytic when $\mathfrak{Re}\, s > -1$ which gives a further meromorphic extension of $\zeta(s)$. Repeating the process by taking the primitive function of P_1 one shows that $\zeta(s)$ extends to the whole complex plane with a simple pole at $s = 1$. Let us also notice that the boundedness of $P_1(x)$ and the formula above gives

1.4. Proposition. *Let $0 < \delta < 1$. Then there exists a constant $C(\delta)$ such that*

$$|\zeta(-1 + \delta + it)| \leq C(\delta) \cdot t^2 \,, \quad \text{for all } |t| \geq 1$$

2. Riemann's functional equation.

The next result consolidates Euler's results from section 1.

2.1. Theorem *The ζ-function extends to a meromorphic function in the whole complex s-plane where it satisfies the functional equation*

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \cdot \cos(\frac{\pi s}{2}) \cdot \Gamma(s) \cdot \zeta(s) \quad : s \in \mathbf{C}$$

The proof requires several steps. To begin with we have the integral formula in Theorem 3.2 from § I:

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \cdot \int_{L_\epsilon} e^z \cdot z^{-s} dz \tag{1}$$

Notice that the role of z and s are interchanged in (1) above as compared to the formula in § 3 about the Γ-function. Since both sides in (1) are entire functions of s we can replace s by $1 - s$ and write

$$\frac{1}{\Gamma(1-s)} = \frac{1}{2\pi i} \cdot \int_{\gamma(\epsilon)} e^z \cdot z^s \cdot \frac{dz}{z} \tag{2}$$

If n is a positive integer the variable substitution $z \mapsto nz$ gives

$$\frac{1}{\Gamma(1-s)} = \frac{1}{2\pi i} \cdot n^s \cdot \int_{L_\epsilon} e^{nz} \cdot z^s \cdot \frac{dz}{z} \implies$$

$$n^{-s} \cdot \frac{1}{\Gamma(1-s)} = \frac{1}{2\pi i} \cdot \int_{L_\epsilon} e^{nz} \cdot z^s \cdot \frac{dz}{z} \tag{3}$$

Next, notice that

$$\sum_{n=1}^{\infty} e^{nz} = \frac{e^z}{1 - e^z} \quad : \mathfrak{Re}\, z < 0$$

Taking the sum over n in (3) we obtain

2.2 Proposition. *One has the equality*

$$\frac{\zeta(s)}{\Gamma(1-s)} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \cdot \int_{L_\epsilon} \frac{e^z}{1 - e^z} \cdot z^s \cdot \frac{dz}{z}$$

Proof of Theorem 2.1

Let $R = (2N+1)\pi$ where N is a positive integer. With $0 < \epsilon < 1$ we obtain a simply connected domain Ω bounded by the circle $|z| = R$ and the portion of of L_ϵ. See figure ! With s fixed we find a single valued branch of z^s in Ω and regard the analytic function

(ii) $$g_s(z) = \frac{e^z}{1-e^z} \cdot z^{s-1}$$

In Ω we encounter poles when $e^z = 1$, i.e. when $z = 2\pi i\nu$ when $1 \leq \nu \leq N$. Residue calculus gives - see figure !!:

(iii) $$\int_{-\pi+\delta}^{\pi-\delta} g_s(Re^{i\theta}) \cdot d\theta - \int_{L_\epsilon} g_s(z) \cdot dz = 2\pi i \cdot \sum_{\nu=1}^{\nu=N} \mathfrak{res}\left[g_s(2\pi i\nu) + g_s(-2\pi i\nu)\right]$$

The residue sum is easily found, i.e. one has

Sublemma The right hand side in (iii) becomes:

(*) $$2 \cdot \sum_{\nu=1}^{\nu=N} \frac{1}{\nu^{1-s}} \cdot (2\pi)^{s-1} \cdot \cos\frac{\pi}{2}(s-1)$$

So far we have allowed any s. Let us now specify s to be real and negative. A straightforward calculation which is left to the reader gives:

(iv) $$\lim_{N\to\infty} \int_{-\pi+\delta}^{\pi-\delta} g_s(Re^{i\theta}) \cdot d\theta = 0 : \quad s \text{ real and } < 0$$

Next, the definition of the ζ-function gives

(**) $$\zeta(1-s) = \lim_{N\to\infty} \sum_{\nu=1}^{\nu=N} \frac{1}{\nu^{1-s}} \quad : s < 0$$

Hence (iii-iv) together with Proposition 2.2, (*) and (**) give:

Sublemma. *When s is real and < 0 one has the equality:*

$$2 \cdot (2\pi)^{s-1} \cdot \zeta(1-s) \cdot \cos\frac{\pi}{2}(s-1) = \frac{\zeta(s)}{\Gamma(1-s)}$$

Final part of the proof . Notice that

(1) $$\cos\frac{\pi}{2}(s-1) = \sin\frac{\pi s}{2} = \frac{\sin \pi s}{2 \cdot \cos\frac{\pi s}{2}}$$

Hence the Sublemma gives

$$\frac{\zeta(s)}{\Gamma(1-s)} = (2\pi)^{s-1} \cdot \zeta(1-s) \cdot \sin \pi s \cdot \frac{1}{\cos\frac{\pi s}{2}} \implies$$

(1) $$\cos\frac{\pi s}{2} \cdot \zeta(s) = (2\pi)^{s-1} \cdot \zeta(1-s) \cdot \sin \pi s \cdot \Gamma(1-s)$$

At this stage we recall that

(2) $$\sin \pi s \cdot \Gamma(1-s) = \frac{\pi}{\Gamma(s)}$$

So (1-2) give together:

(3) $$\cos\frac{\pi s}{2} \cdot \zeta(s) = \frac{1}{2} \cdot (2\pi)^s \cdot \zeta(1-s) \cdot \frac{1}{\Gamma(s)}$$

Expressing $\zeta(1-s)$ alone we get the requested formula in Theorem 2.1 where analyticity gives equality for all s.

3. The asymptotic formula for $\mathcal{N}(T)$

Using the functional equation and the fact that the Γ-function has simple poles at all non-negative integers we will show that $\zeta(-2m) = 0$ for every positive integer m. Apart from these zeros it turns out that the remaining zeros of $\zeta(s)$ belong to the strip

$$\mathcal{S} = \{0 < \mathfrak{Re}\, s < 1\} \tag{0.1}$$

To find zeros in $\mathcal{S}$ we notice that the ζ-function is real when s is real and > 1. By analytic continuation it follows that

$$\zeta(s) = \bar{\zeta}(\bar{s}) \tag{0.2}$$

hold for all s. Hence zeros of ζ appear with conjugate pairs in $\mathcal{S}$ and it suffices to study the counting function

$$\mathcal{N}(T) = \text{number of zeros of } \zeta(s) \quad : \; s \in \mathcal{S} \cap \{\mathfrak{Im}\, s > 0\} \tag{0.3}$$

3.1 Riemann's asymptotic formula. *There exists a constant C_0 such that the following hold when $T \geq 1$:*

$$\mathcal{N}(T) = \frac{1}{2\pi} T \cdot \log T - \frac{1 + \log 2\pi}{2\pi} \cdot T + \rho(T) \cdot \log T \quad : \; |\rho(T)| \leq C_0$$

Remark. Riemann announced this asymptotic formula in [Rie]. It was later proved by von Mangoldt and here we shall present the elegant proof due to Bäcklund. Before we begin the proof of the asymptotic formula above we draw some conclusions from Riemann's functional equation. Let $n \geq 1$ be a positive integer and put $s = 2n + 1$. Here the cosine function has a simple zero, i.e. $\cos \pi n + \frac{\pi}{2} = 0$ and at the same time $\zeta(2n+1)$ and $\Gamma(2n+1)$ are real and positive. Hence Theorem 2.2 implies that the ζ-function has a simple zero at $1 - s = -2n$. So we have proved:

3.2 Proposition. *The ζ-function has simple zeros at all even negative integers.*

3.3 The entire ξ-function. Let us define the function

$$\xi(s) = \frac{s(s-1)}{2} \cdot \Gamma(\frac{s}{2}) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s)$$

3.4 Proposition. *The function $\xi(s)$ is entire and satisfies*

$$\xi(s) = \xi(1-s) \quad : \; \xi(s) = \bar{\xi}(\bar{s})$$

Proof. That $\xi(s)$ is entire is clear from the construction since the zeros of the ζ-function at even and negative integers compensate the simple poles of the Γ-function at negative integers. Moreover, the factor $s(s-1)$ takes care of the simple pole of $\zeta(s)$ at $s = 1$ and the pole of the Γ-function at $s = 0$. The equality $\xi(s) = \bar{\xi}(\bar{s})$ follows since the same conjugation property hold for the three factors defining $\xi(s)$ and the equality $\xi(s) = \xi(1-s)$ follows from Riemann's functional equation.

3.5. Zeros in the critical strip.

The construction of the ξ-function in 3.3. and the fact that $\Gamma(\frac{s}{2})$ has no zeros in the critical strip give:

3.6 Proposition. *The zero sets of ζ and ξ in the critical strip are equal.*

Now we shall count zeros of ξ. For this purpose we consider the rectangle

$$\square_T = \{s = \sigma + it \quad : \; -1/2 < \sigma < 3/2 : \; -T < t < T\} \quad : \; T \geq 1$$

Notice that $\square_T$ is symmetric around $\mathfrak{Re}(s) = 1/2$. We choose T so that $\xi(s)$ has no zeros occur when $\mathfrak{Im}(s) = T$. This gives

$$2 \cdot \mathcal{N}(T) = \frac{1}{2\pi i} \int_{\partial \square_T} \frac{\xi'(s) ds}{\xi(s)} \tag{*}$$

To estimate the line integral in (*) we recall that

(i) $$\xi(s) = \xi(1-s) \quad : \; \xi(s) = \bar{\xi}(\bar{s})$$

These two equations entail that the line integral over $\partial\square_T$ is *four times* the line integral over the the "quarter part" given by the union of the two lines

(ii) $$\ell_*(T) = \{2 + it \quad : \; 0 < t < T\} \quad : \; \ell^*(T) = \{\sigma + iT \quad : \; 1/2 < \sigma < 2\}$$

Next, the line integral is the real number $2 \cdot \mathcal{N}(T)$ and taking the factor 4 into the account we get:

3.7 Lemma. *One has*

$$\mathcal{N}(T) = \frac{1}{\pi i} \cdot \int_{\ell_*} \frac{\xi'(s)ds}{\xi(s)} + \frac{1}{\pi i} \cdot \int_{\ell^*} \frac{\xi'(s)ds}{\xi(s)}$$

Following Backlund we decompose the logarithmic derivative of $\xi(s)$ which gives a sum of five terms:

(iii) $$\cdot \frac{\xi'(s)ds}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} + \frac{1}{2} \cdot \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} - \frac{\operatorname{Log} \pi}{2} + \frac{\zeta'(s)}{\zeta(s)}$$

There remains to study the line integrals of each of these separate terms. The reader can verify that the contribution from the four first terms give the sum of the four first terms which appear in Riemann's asymptotic formula. There remains to investigate the contribution of:

(*) $$\frac{1}{\pi i} \cdot \int_{\ell_*(T)} \frac{\zeta'(s)ds}{\zeta(s)} + \frac{1}{\pi i} \cdot \int_{\ell^*(T)} \frac{\zeta'(s)ds}{\zeta(s)}$$

Here the requested estimate for the remainder term follows if we have proved

3.8 Lemma. *There exists a constant C such that the absolute value of (*) is bounded above by $C \cdot \log(T)$ for every $T \geq e$.*

Proof. First we consider the the line integral over the vertical line $\ell_*(T)$. To pursue the logarithmic derivative of ζ along ℓ_* which by assumption is $\neq 0$ we choose choose a branch of its Log-function and write

(1) $$\log\, \zeta(2 + it) = \log\, |\zeta(2 + it)| + i\phi(t)$$

where $\phi(t)$ is a real valued argument function which gives

(2) $$\frac{\zeta'(2 + it)}{\zeta(2 + it)} = \frac{1}{i} \frac{d}{dt} [\log\, \zeta(2 + it)\,] = \frac{1}{i} \frac{d}{dt} \,|\log\, \zeta(2 + it)|\,] + \frac{d\phi}{dt}$$

Along ℓ_* we have $ds = idt$ and since $\mathcal{N}(T)$ is real our sole concern is to study:

(3) $$\mathfrak{Re}\,[\frac{1}{\pi i} \cdot \int_{\ell^*} \frac{\zeta'(s)ds}{\zeta(s)}\,] = \frac{1}{\pi} \int_0^T \frac{d\phi}{dt} \cdot dt$$

To estimate (3) we shall need the following inequality

(4) $$\mathfrak{Re}[\,\zeta(2 + it)\,] > \frac{1}{3} \quad : \; t > 0$$

To verify (4) we use that $n^{it} = e^{it \log n}$ and get $\mathfrak{Re}\, n^{it} = \cos\,(t \cdot \log n)$. It follows that

(5) $$\mathfrak{Re}[\,\zeta(2 + it)\,] = 1 + \sum_{n=2}^{\infty} \frac{\cos\,(t \cdot \log n)}{n^2} \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} = 2 - \frac{\pi^2}{6} > \frac{1}{3}$$

which proves (4) and it shows that $\zeta(2+it)$ belongs to the right half-plane when $0 \leq t \leq T$.Hence the argument of the ϕ-function satisfies:

$$-\pi/2 < \phi(t) < \pi/2 \quad : \text{ along } \ell_*(T)$$

In particular we get

$$-1 < \frac{1}{\pi}\int_0^T \frac{d\phi}{dt}\cdot dt < 1$$

Thus, the contribution along $\ell_*(T)$ has absolute value < 1 for all T and is therefore harmless for the asymptotic estimate in Lemma 3.8.

3.9 The line integral over $\ell^*(T)$. Along $\ell^*(T)$ we have $ds = d\sigma$. So this time our concern is to estimate

(1) $$\frac{1}{\pi}\cdot\int_{1/2}^{2} \mathfrak{Im}\,[\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)}\,]\cdot d\sigma$$

By the result in XXX this amounts to find an upper bound for the zeros of the function

(2) $$\sigma \mapsto \mathfrak{Re}(\zeta(\sigma+iT) \quad : 1/2 < \sigma < 2$$

Since $\zeta(s) = \bar{\zeta}(\bar{s})$ this amounts to consider zeros of the function

(3) $$\sigma \mapsto \zeta(\sigma+iT) + \zeta(\sigma-iT) \quad : \ 1/2 < \sigma < 2$$

We seek an upper bound of zeros for large T. So from now on we assume that

$$T \geq \frac{7}{2}$$

To obtain an upper bound for the zeros in (3) we consider the analytic function of a new complex variable w:

(4) $$\phi_T(w) = \zeta(w+iT) + \zeta(w-iT) \quad : \ |w-3| \leq 5/2$$

Let $\frac{1}{2} < \sigma_1 \leq \ldots \leq \sigma_m < 2$ be the m-tuple which yield all the zeros counted with multiplicities in (3). They also give zeros of the analytic function ϕ in the disc $|w-3| \leq 5/2$ and by the general inequality from XXXX we have

(5) $$\sum_{\nu=1}^{m} \log\big[\frac{5}{2\sigma_\nu}\,\big] + \log|\phi_T(3)| \leq \frac{1}{2\pi}\cdot\int_0^{2\pi} \log\big|\phi_T(3+\frac{5}{2}e^{i\theta})\,\big|\cdot d\theta$$

By XX we have $|\phi_T(3)| \geq 1/3$ and since $1/2 \leq \sigma_\nu \leq 2$ hold we get

(6) $$m\cdot\log\frac{5}{4} \leq \log 3 + \max_{|w-3|=5/2}\,\log|\phi_T(w)|$$

Finally, Proposition xx gives the constant $C(1)$ such that

(7) $$\phi_T(w)| \leq 2\cdot C(1)\cdot T^2 \quad : \quad T \geq 7/2$$

Passing to $\operatorname{Log}|\phi_T(w)|$ we conclude that the m-number which counts the zeros is bounded above by an absolute constant times $\log T$ for all $T \geq 7/2$ which finishes the proof of Lemma 3.8.

4. The prime number theorem

The counting function for prime numbers is defined by

(i) $$\Pi(x) = \text{number of primes } \leq x$$

This is an increasing jump function where we for example have $\Pi(8) = \Pi(9) = \Pi(10) = 4$ and $\Pi(11) = 5$.

4.0 Exercise. Show that prime numbers are sufficiently sparse in order that

$$\lim_{x\to\infty} \frac{\Pi(x)}{x} = 0$$

4.1 Theorem *There exists the limit formula*

(*) $$\lim_{x\to\infty} \frac{\log x \cdot \Pi(x)}{x} = 1$$

To prove this we introduce the function defined for $x > 0$ by

$$\gamma(x) = \sum_{p \leq x} \log p$$

where the sum as indicated extends over all prime numbers $\leq x$. Next, let $d\Pi$ be the discrete measure which assigns a unit point mass at every prime number. A partial integration gives the equation:

(i) $$\gamma(x) = \int_2^x \log x \cdot d\Pi(x) = \log x \cdot \Pi(x) - \int_2^x \frac{\Pi(x)}{x} \cdot dx$$

Afer a division by x on both sides it follows from Exercise 0.4 that the limit formula (*) in Theorem 4.1. is equivalent to

(**) $$\lim_{x\to\infty} \frac{\gamma(x)}{x} = 1$$

There remains to prove (**) where a first step is the following:

Lemma 4.2. *The function* $\frac{\gamma(x)}{x}$ *is bounded.*

Proof. The idea is to use that if $N \geq 2$ is a positive integer then all terms in the binomial expansion of $(1+1)^{2N}$ are integers. In particular we have the integer

(i) $$\xi_N = \frac{(2N)\,!}{N\,! \cdot N\,!}$$

Let $q_1, \ldots, q_m$ be the distinct primes in $[N+1, 2N-1]$. By (i) each q_ν is a prime divisor of ξ_N. So we have trivially

$$q_1 \cdots q_m \leq \xi_N$$

Taking the logarithm the definition of the γ-function gives

(ii) $$\gamma(2N) - \psi(N) \leq \log \xi_N \leq N \cdot \log 2$$

where the reader may confirm the last inequality by a trivial calculation. Now we perform the usual trick using 2-powers, i.e. given $K \geq 2$ we apply (ii) with $N = 2^k \quad : 1 \leq k \leq K-1$. After a summation over k we get

$$\gamma(2^K - \psi(2) \leq \log 2 \cdot [1 + \ldots + 2^{K-1}] \leq \log 2 \cdot 2^K$$

Since this hold for all K and the γ-function is increasing we see that Lemma 4.2 holds.

Next, Lemma 4.2 and an elementary fact about convergent integrals give (**) if we have proved that the integral below exists:

(***) $$\int_2^\infty \frac{\gamma(x) - x}{x^2} \cdot dx$$

Notice that we only request that there exists a limit

$$\lim_{M\to\infty}\int_2^M \frac{\gamma(x)-x}{x^2}\cdot dx$$

whose the actual limit value is of no concern for us !

4.3 The Φ-function. To establish (***) we introduce the function

$$\Phi(z)=\sum \log p\cdot p^{-z}$$

where z is a complex variable. To begin with $\Phi(z)$ is defined in the half-plane $\mathfrak{Re}(z)>1$ and since $d\gamma$ is the discrete measure which assigns the mass $\log p$ at every prime number we have:

$$\Phi(z)=\int_1^\infty x^{-z}\cdot d\gamma(x)=z\cdot\int_1^\infty x^{-z-1}\cdot\gamma(x)\cdot dx \tag{1}$$

With $z=1+\zeta$ and $\mathfrak{Re}(\zeta)>0$ we can write (1) as

$$\frac{1}{1+\zeta}\cdot\Phi(1+\zeta)=\int_1^\infty x^{-\zeta-2}\cdot\gamma(x)\cdot dx \tag{2}$$

and the substitution $x\to e^t$ identifies the last integral by

$$\int_0^\infty e^{-\zeta t}\cdot e^{-t}\cdot\gamma(e^t)\cdot dy \tag{3}$$

Next, let us introduce the function

$$f(t)=e^{-t}\gamma(e^t)-1$$

Lemma 4.2 shows that f is a bounded function and the substitution $x\to e^t$ shows that the integral (***) converges if there exists the limit

$$\lim_{T\to\infty}\int_0^T f(t)\cdot dt \tag{****}$$

Hence there only remains to prove (****). For this purpose we introduce the Laplace transform

$$F(z)=\int_0^\infty e^{-zt}\cdot f(t)dt \tag{4}$$

Using the equality

$$\int_0^\infty e^{-zt}\cdot dt=\frac{1}{z}$$

we see that (2-3) above give

$$\frac{1}{1+z}\cdot\Phi(1+z)=F(z)-\frac{1}{z} \tag{5}$$

At this stage we can apply Ikehara's Tauberian Theorem which applied to the bounded function $f(t)$ (****) if $F(\zeta)$ extends to an analytic function in some open set Ω which contains the closed half-space $\mathfrak{Re}(z)\geq 0$. Hence, using (5) we conclude that the Prime Number theorem follows from the following:

4.4 Lemma *The function $\frac{1}{1+z}\cdot\Phi(1+z)+\frac{1}{z}$ extends to be analytic in an open set which contains $\mathfrak{Re}(z)\geq 0$.*

Proof. When $\mathfrak{Re}\,z<1$ Euler's product formula gives

$$\log\zeta(z)=\sum\log(1-p^{-z})$$

Passing to the logarithmic derivative we get

$$-\frac{\zeta'(z)}{\zeta(z)}=\sum\frac{\log p}{p^z-1}$$

The last sum is rewritten as

$$\sum \frac{\log p}{p^z} - \sum \frac{\log p}{p^z \cdot (p^z - 1)}$$

So the construction of Φ gives the equality

$$\Phi(z) = -\frac{\zeta'(z)}{\zeta(z)} - \sum \frac{\log p}{p^z \cdot (p^z - 1)}$$

With $z = 1 + w$ we can write

$$\Phi(1+w) - \frac{\zeta'(1+w)}{\zeta(1+w)} = -\sum \frac{\log p}{p^{1+w} \cdot (p^{1+w} - 1)} \tag{1}$$

It is clear the right hand side is an analytic function of w in the half-plane $\mathfrak{Re}(w) > -1/2$. Finally, since the ζ-function has a simple pole at $z = 1$ while $\zeta(1 + it) \neq 0$ for all real t we get Lemma 4.4.

5. A uniqueness result for the ζ-function

Introduction. In the introduction we defined a class $\mathcal{D}_k$ of Dirichlet series for every $k > 0$ in § 0.1. The proof of Beurling's result in (0.0.2) requires several steps and we describe some steps before the details of the proof start. Each Dirichlet series $\Lambda(s)$ in the family $\mathcal{D}_k$ gives an even and entire function of exponential type defined by an Hadamard product:

$$f(z) = \prod (1 + \frac{z^2}{\lambda_n^2})$$

Using Phragmén-Lindelöf inequalities together with properties of the Γ-function and Mellin's inversion formula, we shall prove that for each $\epsilon > 0$ there exists a constant C_ϵ such that the following hold for each real $x > 0$:

$$(*) \qquad \big|f(x) - ax^p \cdot e^{\pi x}\big| \leq C_\epsilon \cdot e^{\pi(1-2k+\epsilon \cdot x}$$

where

$$a = e^{2\Lambda(0)} \quad \text{and} \quad p = 2\Lambda'(0)$$

When $k > 1/2$ we can choose ϵ small so that $1 - k + \epsilon = -\delta$ for some $\delta > 0$ which means that $f(x) - ax^p \cdot e^{\pi x}$ has exponential decay as $x \to +\infty$. From this we shall deduce that $f(z)$ is of a special form and after deduce Theorem 0.1 via an inversion formula for Dirichlet series. First we shall establish a uniqueness result for entire functions in the class $\mathcal{E}$.

5.A. On a uniqueness result in $\mathcal{E}$

.

Let a and δ be positive real numbers and p some real number. Consider an even entire function $f(z)$ of exponential type for which there exists a constant C such that

$$(*) \qquad \big|f(x) - ax^p \cdot e^{\pi x}\big| \leq Ce^{-\delta x} \quad : \ x \geq 1$$

We shall prove that f is of a special form. In the half-space $\mathfrak{Re}(z) > 0$ we have the analytic function

$$\text{(i)} \qquad h(z) = f(z) - az^p \cdot e^{\pi z}$$

where the branch of z^p is taken so that $x^p > 0$ when $z = x$ is real and > 0. Consider the domain

$$\Omega = \{z = x + iy \quad y > 0 \quad \text{and} \quad x > 1\}$$

Since $f \in \mathcal{E}$ there is a constant A such that $e^{-A|z|} \cdot f(z)$ is bounded which gives constants C and B such that

$$\text{(ii)} \qquad |h(1 + iy)| \leq C \cdot e^{By}$$

for all $y > 0$. At the same time (*) gives

$$\text{(iii)} \qquad |h(x)| \leq C \cdot e^{-\delta x}$$

The Phragmén-Lindelöf theorem applied to the quarter planer Ω therefore gives a constant C such that

$$\text{(iv)} \qquad |h(x + iy)| \leq Ce^{Ay - \delta x} \quad \text{for all} \quad x + iy \in \Omega$$

In exactly the same way one proves (iv) with y replaced by $-y$ in the quarter-plane where $x > 0$ and $y < 0$. Let us then consider the strip domain

$$S = \{x + iy \quad |y| \leq 1 \quad \text{and} \quad x > 1\}$$

Then we see that there is a constant such that

$$\text{(v)} \qquad h(x + iy)| \leq C \cdot e^{-\delta x} \quad : \quad x + iy \in S$$

If $n \geq 1$ we consider the complex derivative $h^{(n)}$ and with $x > 2$ Cauchy's inequality and (*) give a constant C_n such that

$$\text{(vi)} \qquad |h^{(n)}(x)| \leq C_n \cdot e^{-\delta x} \quad : x \geq 2$$

Next, consider the second order differential operator

$$L = x^2\partial_x^2 - 2p\cdot x\partial_x - \pi^2 x^2 + p(p+1)$$

The functions $x^p e^{\pi x}$ and $x^p e^{-\pi x}$ are solutions to the homogeneous equation $L = 0$ when $x > 0$ and hence

$$L(f) = L(h)$$

holds on $x > 0$. Now L also yields the holomorphic differential operator where ∂_x is replaced by ∂_z and here $g = L(f)$ is en entire function exponential type. Now (*) above and the estimates (vi) for $n = 0, 1, 2$ give a constant C such that

(vii) $$|g(x)| \leq C(1 + x^2)\cdot e^{-\delta|x|} \quad x > 0$$

Moreover, since f is even it follows that g is so and hence (vii) hold for all real x. Then the $\mathcal{E}$-function g is identically zero by the result in § XXX. Hence f satisfies the differential equation

(viii) $$L(f) = 0$$

The uniquenss for solutions of the in ODE-equation (viii) gives contants c_1, c_2 such that

$$f(x) = c_1 x^p e^{\pi x} + c_2 x^p e^{-\pi x} \quad x > 0$$

Since f is an even entire function it is clear that this entails that p must be an integer and if we moreover assume that $f(0) = 1$ then the reader may verify that we have $p = 0$ or $p = -1$ which yield corresponding f-functions

$$f_1(z) = \frac{e^{\pi z} + e^{-\pi z}}{2} \quad : \quad f_2(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$$

B. Dirichlet series and their transforms.

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be a non-decreasing sequence of positive real numbers in the family $\mathcal{F}$ (0.0.2). It is clear that the Dirichlet series

(1) $$\Lambda(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

is analytic in the half-space $\mathfrak{Re}\, s > 1$ and the results by Hadamard and Lindelöf in §§ XX give the entire function

(2) $$f(z) = \prod \left(1 + \frac{z^2}{\lambda_n^2}\right)$$

of exponential type.

B.1 Inversion formula. *One has the equation*

(B.1.) $$\int_0^\infty \log f(x)\cdot \frac{dx}{x^{s+1}} = \frac{\pi\cdot\Lambda(s)}{s\sin\frac{\pi s}{2}} \quad : \quad \mathfrak{Re}\, s > 1$$

Proof. When $0 < \mathfrak{Re}\, s < 2$ and $a > 0$ is real the reader may verify the equality:

(i) $$\int_0^\infty \log\left(1 + \frac{x^2}{a^2}\right)\cdot\frac{dx}{x^{s+1}} = \frac{1}{a^s}\cdot\frac{\pi}{\cdot\sin\frac{\pi s}{2}}$$

Apply (i) with $a = \lambda_n$ and then (B.1) follows after a summation over n.

B.2 Meromorphic extensions. The inversion formula (B.1) entails that Λ extends to a meromorphic function in the complex s-plane. To see this we notice that if $x > 0$ then the logartihmic derivative

(i) $$\frac{f'(x)}{f(x)} = \sum \frac{2x}{\lambda_n^2 + x^2}$$

Next, a partial integration gives

$$\text{(ii)} \qquad (s+1)\cdot\int_0^\infty \log\, f(x)\cdot\frac{dx}{x^{s+1}} = \int_0^\infty \frac{f'(x)}{f(x)}\cdot\frac{dx}{x^s}$$

Since $\sum \lambda_n^{-2}$ is convergent it is clear that the right hand side is analytic in the half-space $\mathfrak{Re}\, s > 0$ and by further integrations by parts the reader may verify that it extends to a meromorphic function in the s-plane. Together with the inversion for ua we conclude that $\Lambda(s)$ extends to a meromorphic function in the s-plane.

The case when $\Lambda \in \mathcal{D}_k$. Suppose this holds for some $k > 1/2$. In particular $\Lambda(-2n) = 0$ for every positive integer and then the right hand side in (B.1) is a meromorphic function whose poles are confined to $s = 0$ and $s = 1$. Denote this function with $\Phi(s)$. Now we shall estimate certain L^1-integrals.

B.3 Proposition. *There exists a constant C such that*

$$\int_{-\infty}^\infty |\Phi(-\sigma+it)|\cdot dt \le C\cdot\frac{\sigma^3\cdot\Gamma(\sigma)}{(2\pi k)^\sigma} \quad : \quad \sigma \ge 2$$

Proof. Consider the function

$$\psi_*(s) = (2\pi k)^s\cdot\Gamma(2-s)$$

The series expression for $\Lambda(s)$ gives a constant C such that

$$\text{(i)} \qquad \Lambda(3/2+it)| \le C \quad : \quad -\infty < t < +\infty$$

It follows that

$$\text{(ii)} \qquad |\frac{\Phi(3/2+it)}{\psi_*(3/2+it)}| \le \frac{C\pi}{|3/2+it|}\cdot\frac{1}{2\pi k)^{3/2}}\cdot\frac{1}{\sin\left(\pi(3/4+it/2)\cdot\Gamma(1/2-it)\right.}$$

The complex sine-function increases along this vertical line, i.e. there is a constant $c > 0$ such that

$$\text{(iii)} \qquad |\sin\left(\pi(3/4+it/2)\right| \ge c\cdot e^{\pi|t|/2}$$

At the same time the result in (§ xx) gives the lower bound

$$\text{(iv)} \qquad |\Gamma(1/2-it)| \ge \sqrt{\pi}\cdot e^{-\pi|t|/2}$$

From (iii-iv) we conclude that the function $\frac{\Phi}{\psi_*}$ is bounded on the line $\mathfrak{Re}(s) = 3/2$.

Sublemma 1. The function $\frac{\Phi}{\psi_*}$ is a bounded function in the domain

$$\Omega = \{\mathfrak{Re}(s) < 3/2\} \cap \{|s| > 2\}$$

Proof. Follows easily via the Phragmén-Lindelöf theorem and the bound above on $\mathfrak{Re}(s) = 3/2$.

Next, Sublemma 1 gives a constant C such that

$$\text{(v)} \qquad |\Phi(s)| \le C\cdot\left|(2\pi k)^s\cdot\Gamma(2-s)\right| \quad : \quad s \in \Omega$$

We shall also need an inequality for the Γ-function which asserts that there exists a constant C such that

$$\text{(vi)} \qquad \int_{-\infty}^\infty \left|\,\Gamma(\sigma+2+it)\,\right|\cdot dt \le C\cdot\sigma^3\cdot\Gamma(\sigma) \quad : \quad \sigma \ge 2$$

The verification of (vi) is left to the reader. Together (v) and (vi) give the inequality in Proposition B.3.

B.4 Mellin's inversion formula.

The integral inequality in Proposition B.3 enable us to apply the Fourier-Mellin inversion formula via (**) from XX. This gives

$$\log f(x) = \frac{1}{2\pi i} \cdot \int_{c-i\infty}^{c+i\infty} \Phi(s) \cdot x^s \cdot ds \quad : \quad 1 < c < 2$$

Using Proposition B.3 we can shift the contour the left and perform integrals over lines $\mathfrak{Re}\, s = -c$ where $c > 0$. During such a shift we pass the poles of Φ which appear at $s = 0$ and $s = 1$. Using condition (a) from (0.2) in the introduction the reader can deduce the integral formula:

$$(*) \qquad \log f(x) - \pi x - 2\phi(0) \cdot \log x - 2\phi'(0) = \frac{1}{2\pi i} \cdot \int_{-c-i\infty}^{-c+i\infty} \Phi(s) \cdot x^s \cdot ds \quad \text{for all} \quad c > 0$$

B.4.1 A clever estimate. To profit upon (*) we adapt the c-values when x are real and large. With $x \geq 2$ we take $c = x$ and notice that

$$|x^{(-x+it)}| = x^{-x}$$

Proposition B.3 and the triangle inequality show that the absolute value of the right hand side integral in (*) is majorized by

$$(**) \qquad 2\pi \cdot x^{-x} \cdot C \cdot \frac{x^3 \cdot \Gamma(x)}{(2\pi k)^x} \quad : \quad x \geq 2$$

B.4.2 Exercise. Recall that $\Gamma(N) = N!$ for positive integers. Use this and Stirling's formula to conclude that for every $\epsilon > 0$ there is a constant C_ϵ such that (**) is majorized by

$$C_\epsilon \cdot e^{-2\pi(k-\epsilon)x}$$

B.4.3 Consequences. With $x \geq 2$, $p = 2\phi(0)$ and $a = 2\phi'(0)$ we obtain from above:

$$\Big|\log f(x) - \pi x - p \cdot \log x - a\Big| \leq C_\epsilon \cdot e^{-2\pi(k-\epsilon)x}$$

Now (B.4.3) gives a constant C_ϵ^* such that

$$(\text{B.4.4}) \qquad |f(x) - e^a \cdot x^p \cdot e^{\pi x}| \leq C_\epsilon^* \cdot e^{\pi(1-2k+2\epsilon)\cdot x} \quad : \quad x \geq 2$$

Since $k > 1/2$ and ϵ can be arbitrary small we conclude the foillowing

B.4.5 Theorem. *When $\Lambda \in \mathcal{D}_k$ for some $k > 1/2$ there exists $\delta > 0$ and constants a, p such that*

$$|f(x) - e^a \cdot x^p \cdot e^{\pi x}| \leq C_\epsilon^* \cdot e^{-\delta \cdot x} \quad : \quad x \geq 2$$

STRUL to be deleted

B.2 A condition on f. Suppose there exist real constants a, p, k where $k > 1/2$ such that

$$f(x) = ax^p e^{\pi x} + O(e^{\pi(1+\epsilon-2k)x}$$

hold for each $\epsilon > 0$ as $x \to +\infty$. In other words, for every $\epsilon > 0$ there exists a constant C_ϵ such that

$$(*) \qquad |f(x) - ax^p e^{\pi x}| \leq C_\epsilon \cdot e^{\pi(1+\epsilon-2k)x} \quad \text{for all} \quad x \geq 1$$

Exercise. Show that (*) entails that

$$(**) \qquad \log f(x) = \log a + p \log x + \pi \cdot x + O(e^{\pi(1+\epsilon-2k)x} \quad \text{for all} \quad x \geq 1$$

where $\epsilon > 0$ can be arbitrary small.

Next, set

$$\psi(s) = \int_0^\infty \log\, f(x) \cdot \frac{dx}{x^{s+1}}$$

Notice that f is an even entire function where the Hadamard product gives $f(0) = 1$ which implies that

$$\log f(x) \simeq x^2$$

when x is close to zero. Therefore it is only the behaviour of $f(x)$ when x is large which determines whether the ψ-function is nice or not in the half-space $\mathfrak{Re}\, s < 2$. Notice that

$$\int_0^\infty [\log a + p \log x + \pi \cdot x] \cdot \frac{dx}{x^{s+1}} = \frac{\log a}{s} + \frac{p}{s^2} + \frac{\pi}{s-1}$$

Moreover, we have $k > 1/2$ so when ϵ is small it follows that $1 + \epsilon - 2k = -\delta$ for some $\delta > 0$ and the function

$$g_\delta(s) = \int_0^\infty e^{-\pi\delta x} \cdot \frac{dx}{x^{s+1}}$$

is analytic in the half-space $\mathfrak{Re}\, s < 2$.

Exercise. Use the definition of the Γ-function to conclude that

$$|g_\delta(-\sigma + it)| \leq \frac{\Gamma(\sigma)}{(\pi\delta)^\sigma} \quad \text{hold for every} \quad \sigma > 0$$

Let us now add the assumption that

$$(*) \qquad \phi(-2n) = 0 \quad \text{hold for every positive integer}$$

When (*) holds we see that the right hand side in (B.1.2) is a meromorphic function whose poles in the half-plane $\mathfrak{Re} s < 2$ are confined to $s = 0$ and $s = 1$. Denote this function with $\Phi(s)$. Next, by assumption ϕ also belongs $\mathcal{D}_k$ for some $k > 1/2$ and using the growth condition (xx) from § XX we shall estimate certain L^1-integrals.

B.3 Proposition. *There exists a constant C such that*

$$\int_{-\infty}^\infty |\Phi(-\sigma + it)| \cdot dt \leq C \cdot \frac{\sigma^3 \cdot \Gamma(\sigma)}{(2\pi k)^\sigma} \quad : \quad \sigma \geq 2$$

Proof. Consider the function

$$\psi_*(s) = (2\pi k)^s \cdot \Gamma(2-s)$$

Next, the convergent series expression for the given Dirichlet series $\phi(s)$ gives a constant C such that

$$\text{(i)} \qquad \phi(3/2 + it)| \leq C \quad : \quad -\infty < t < +\infty$$

It follows that

(ii) $$\left|\frac{\Phi(3/2+it)}{\psi_*(3/2+it)}\right| \leq \frac{C\pi}{|3/2+it|}\cdot\frac{1}{2\pi k)^{3/2}}\cdot\frac{1}{\sin\left(\pi(3/4+it/2)\cdot\Gamma(1/2-it)\right.}$$

The complex sine-function increases along this vertical line,i.e. there is a constant $c > 0$ such that

(iii) $$|\sin\left(\pi(3/4+it/2)\right| \geq c\cdot e^{\pi|t|/2}$$

At the same time the result in (§ xx) gives the lower bound

(iv) $$|\Gamma(1/2-it)| \geq \sqrt{\pi}\cdot e^{-\pi|t|/2}$$

From (iii-iv) we conclude that the function $\frac{\Phi}{\psi_*}$ is bounded on the line $\mathfrak{Re}(s) = 3/2$.

Sublemma 1. The function $\frac{\Phi}{\psi_*}$ *is a bounded function in the domain*

$$\Omega = \{\mathfrak{Re}(s) < 3/2\} \cap \{|s| > 2\}$$

Proof. Follows easily via the Phragmén-Lindelöf theorem and the bound above on $\mathfrak{Re}(s) = 3/2$.

Next, Sublemma 1 gives a constant C such that

(v) $$|\Phi(s)| \leq C\cdot\left|(2\pi k)^s\cdot\Gamma(2-s)\right| \quad : \quad s\in\Omega$$

We shall also need an inequality for the Γ-function which asserts that there exists a constant C such that

(vi) $$\int_{-\infty}^{\infty}\left|\,\Gamma(\sigma+2+it)\,\right|\cdot dt \leq C\cdot\sigma^3\cdot\Gamma(\sigma) \quad : \quad \sigma\geq 2$$

The verification of (vi) is left to the reader. Together (v) and (vi) give the inequality in Proposition B.3.

B.4 Mellin's inversion formula.

The integral inequality in Proposition B.3 enable us to apply the Fourier-Mellin inversion formula via (**) from XX. This gives

$$\log f(x) = \frac{1}{2\pi i}\cdot\int_{c-i\infty}^{c+i\infty}\Phi(s)\cdot x^s\cdot ds \quad : \quad 1 < c < 2$$

Using Proposition B.3 we can shift the contour the left and perform integrals over lines $\mathfrak{Re}\,s = -c$ where $c > 0$. During such a shift we pass the poles of ψ which appear at $s = 0$ and $s = 1$ with residues described in §§ XX - at start - above. From this the reader can deduce the integral formula:

(*) $$\log f(x) - \pi x - 2\phi(0)\cdot\log x - 2\phi'(0) = \frac{1}{2\pi i}\cdot\int_{-c-i\infty}^{-c+i\infty}\Phi(s)\cdot x^s\cdot ds \quad \text{for all} \quad c > 0$$

B.6 A clever estimate. To profit upon (*) we shall adapt the c-values when x are real and large. More precisely, with $x \geq 2$ we take $c = x$ and notice that

$$|x^{(-x+it)}| = x^{-x}$$

Then Proposition B.4 and the triangle inequality show that the absolute value of the right hand side integral in (*) is majorized by

(**) $$2\pi\cdot x^{-x}\cdot C\cdot\frac{x^3\cdot\Gamma(x)}{(2\pi k)^x} \quad : \quad x\geq 2$$

B.7 Exercise. Recall that $\Gamma(N) = N!$ for positive integers. Use this and Stirling's formula to conclude that for every $\epsilon > 0$ there is a constant C_ϵ such that (**) is majorized by

$$C_\epsilon \cdot e^{-2\pi(k-\epsilon)x}$$

B.8 Consequences. With $x \geq 2$, $p = 2\phi(0)$ and $a = 2\phi'(0)$ we obtain from above:

$$\Big|\log f(x) - \pi x - p \cdot \log x - a\Big| \leq C_\epsilon \cdot e^{-2\pi(k-\epsilon)x}$$

B.9 Exercise. Deduce from this estimate that there is a constant C_ϵ^* such that

$$|f(x) - e^a \cdot x^p \cdot e^{\pi x}| \leq C_\epsilon^* \cdot e^{\pi(1-2k+2\epsilon)\cdot x} \quad : \quad x \geq 2$$

6. A theorem on functions defined by a semi-group

Let $f(x)$ be a complex-valued function in $L^2(0,1)$ which is not identically zero on any interval $(0,\delta)$ with $0<\delta<1$, i.e. for each $\delta>0$ one has

$$(*)\qquad \int_0^\delta |f(x)|\cdot dx>0$$

Next, for each $0<a<1$ we set

$$f_a(x)=f(ax)$$

We restrict each f_a to $(0,1)$ and denote by $\mathcal{C}_f$ the linear space generated by $\{f_a\}$ as $0<a<1$. Thus, a function in $\mathcal{C}_f$ is expressed as a finite sum

$$\sum c_k\cdot f_{a_k}(x)$$

where $\{c_k\}$ are complex numbers and $0<a_1<\ldots<a_n<1$ some finite tuple of points in $(0,1)$. Consider some $1<p<2$. The inclusion $L^2(0,1)\subset L^p(0,1)$ identifies $\mathcal{C}_f$ with a subspace of $L^p(0,1)$ and its closure in the Banach space $L^p(0,1)$. is denoted by $\mathcal{C}_f(p)$.

0.1 The function $F(s)$. It is defined by

$$(1)\qquad F(s)=\int_0^1 f(x)\cdot x^{s-1}\cdot ds$$

Here F is analytic in the half-plane $\mathfrak{Re}(s)>1/2$. Indeed, with $\sigma=\mathfrak{Re}(s)>1/2$ the Cauchy-Schwarz inequality gives

$$(2)\qquad |F(\sigma+it)|\leq\sqrt{\int_0^1 |f(x)|^2\cdot dx}\cdot\sqrt{\int_0^1 |x|^{2\sigma-2}\cdot dx}=||f||_2\cdot\sqrt{\frac{1}{2\sigma-1}}$$

6.1 Theorem *If there exists some $1<p<1$ such that $\mathcal{C}_f(p)$ is a proper subspace of $L^p[0,1]$, then $F(s)$ extends to a meromorphic function in the whole complex s-plane whose poles are confined to the open half-plane $\mathfrak{Re}(s)<1/2$. Moreover, for every pole λ the function $x^{-\lambda}$ belongs to $\mathcal{C}_f(p)$.*

Proof. Recall that $L^q(0,1)$ is the dual of $L^p(0,1)$ where $\frac{1}{q}=1-\frac{1}{p}$. The assumption that $\mathcal{C}_f(p)\neq L^p(0,1)$ gives a non-zero $k(x)\in L^q(0,1)$ such that

$$(1)\qquad \int_0^1 k(x)f(ax)\cdot dx=0\quad:\quad 0<a<1$$

To the k-function we associate the transform

$$(2)\qquad K(s)=\int_0^1 k(x)\cdot x^{-s}\cdot dx$$

Hölder's inequality implies that $K(s)$ is analytic in the half-plane $\mathfrak{Re}s<\frac{1}{p}$. Next, we define a function $g(\xi)$ for every real $\xi>1$ by

$$g(\xi)=\int_0^1 k(x)\cdot f(\xi x)\cdot dx$$

Hölder's inequality gives

$$|g(\xi)|\leq\Big[\int |k(x)|^q\cdot dx\Big]^{\frac{1}{q}}\cdot\Big[\int_0^{1/\xi} |f(\xi x)|^p\cdot dx\Big]^{\frac{1}{p}}$$

With $\xi>1$ we notice that the last factor after a variable substitution is equal to

$$||f||_p\cdot|\xi|^{-1/p}$$

Since the L^p-norm of f is majorized by its L^2-norm we conclude that

$$(3)\qquad |g(\xi)\leq||k||_q\cdot||f||_p\cdot\xi^{-1/p}\quad:\quad \xi>1$$

Next, put

$$G(s) = \int_1^\infty g(\xi) \cdot \xi^{s-1} \cdot d\xi \tag{4}$$

From (3) it follows that $G(s)$ is analytic in the half-space $\mathfrak{Re}\, s < 1/p$. Consider the strip domain:

$$\square = 1/2 < \mathfrak{Re}\, s < 1/p$$

Variable substitutions of double integrals show that the following holds in $\square$:

$$G(s) = F(s) \cdot K(s) \tag{*}$$

Conclusion. If follows from (*) that F extends to a meromorphic function in the whole s-plane. The inequality (2) from 0.1 which shows that no poles appear during the meromorphic continuation across $\mathfrak{Re}\, s = 1/2$. Hence F either is an entire function or else it has a non-empty set of poles where each pole λ has real part $< 1/2$. At this stage we are prepared to finish the proof of Theorem 6.1 and begin with:

Existence of at least one pole. There remains to prove that F has at least one pole. We prove this by a contradiction, i.e. suppose that F is an entire function and consider a real number $1/2 < \alpha < 1/p$. The construction of F shows that its restriction to the half-space $\mathfrak{Re}\, s \geq \alpha$ is bounded and it is also clear that

$$\lim_{\sigma \to +\infty} F(\sigma + it) = 0 \tag{i}$$

Next, in the half-space $\mathfrak{Re}\, s \leq \alpha$ we know that $F = \frac{G}{K}$ where G and K both are bounded and at the same time their quotient is analytic in this half-space. Moreover their constructions imply that

$$\lim_{\sigma \to -\infty} G(\sigma + it) = 0 \quad \text{and}| \quad K(\sigma + it) = 0$$

Next, the result by F. and R. Nevanlinna from XX gives some $M > 0$ and a real number c such that

$$F(\sigma + it)| \leq M \cdot e^{c(\sigma - \alpha)} \quad \text{holds when} \quad \sigma \leq \alpha \tag{iii}$$

If $c \geq 0$ we see that the entire function F is bounded and (i) implies that $F = 0$. But this is impossible since it entails that $f = 0$.

The case $c < 0$. When this holds we set $a = e^c$ so that $0 < a < 1$ and define

$$F_1(s) = \int_0^1 f(ax) x^{s-1} \cdot ds \tag{iv}$$

Here a variable substitution gives

$$F_1(s) = a^s \big(F(s) - \int_a^1 f(x) x^{s-1} \cdot ds \big) \tag{v}$$

It follows that the entire function $F_1(s)$ is bounded so by Liouville's theorem it is identically zero. Si by (iv) the the transform of the function $f_a(x) = f(ax)$ is identically zero. This means precisely that f vanishes on the interval $[0, a]$. But this was excluded by condition (*) above Theorem 6.1. which shows that F cannot be an entire function.

The case at pole Suppose that F as a pole at some λ with real part $< 1/2$. Since G is analytic in $\mathfrak{Re}(s) < 1/2$ the equality (*) implies that λ is a zero of K. Notice also that the presence of the pole of F at λ is *independent* of the chosen L^q-function k which is $\perp$ to $\mathcal{C}_f$. Hence the following implication holds:

$$k \perp \mathcal{C}_f(p) \implies K(\lambda) = \int_0^1 k(x) x^{-\lambda} \cdot dx = 0$$

The Hahn-Banach theorem entails that the L^p-function $x^{-\lambda}$ belongs to $\mathcal{C}_h(p)$ which proves the last claim in Theorem 6.1

7. Beurlings criterion for the Riemann hypothesis

Let $\rho(x)$ denote the 1-periodic function on the positive real x-line with $\rho(x) = x$ if $0 < x < 1$. So if $\{x\}$ is the integral part of x then

$$\rho(x) = x - \{x\}$$

To each $0 < \theta < 1$ we get the function

$$\rho_\theta(x) = \rho(\theta/x)$$

whose restriction to $(0,1)$ gives a non-negative function with jump-discontinuites at the discrete set of x-values where θ/x is an integer. Denote by $\mathcal{D}$ the linear space of functions on $(0,1)$ of the form

$$f(x) = \sum c_\nu \cdot \rho(\theta_\nu/x)$$

where $0 < \theta_1 < \ldots < \theta_N < 1$ is a finite set and $\{c_\nu\}$ complex numbers such that

$$\sum c_\nu \cdot \theta_\nu = 0$$

7.1 Theorem. *The Riemann hypothesis is valid if and only if the identity function 1 belongs to the closure of $\mathcal{D}$ in $L^2(0,1)$.*

The proof will use the following formula:

7.2 Proposition. *For each $0 < \theta < 1$ one has the equality*

$$(*) \qquad \int_0^1 \rho(\theta/x)x^{s-1} \cdot dx = \frac{\theta}{s-1} - \frac{\theta^s \cdot \zeta(s)}{s} \quad \text{when} \quad \mathfrak{Re}\, s > 1$$

Proof. The variable substitutions $x \to \theta \cdot y$ and $y \to 1/u$ identifies the left hand side with

$$\text{(i)} \qquad \theta^s \cdot \int_0^{1\theta} \rho(1/y) \cdot y^{s-1} \cdot dy = \theta^s \cdot \int_\theta^\infty \rho(u) \cdot u^{-s-1} \cdot du$$

Next, since ρ is periodic we have

$$\text{(ii)} \qquad \int_1^\infty \rho(u) \cdot u^{-s-1} \cdot du = \sum_{n=1}^\infty \int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du$$

An integration by parts gives for each $n \geq 1$:

$$\int_0^1 \frac{u}{(u+n)^{s+1}} \cdot du = -\frac{1}{s}(n+1)^{-s} + \frac{1}{s}\int_0^1 \frac{du}{(n+u)^s}$$

After a summation over n we see that (ii) becomes

$$-\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s}\int_1^\infty u^{-s} \cdot du = -\frac{\zeta(s)}{s} + \frac{1}{s} + \frac{1}{s(s-1)} = -\frac{\zeta(s)}{s} + \frac{1}{s-1}$$

It follows that the left hand side in (*) is equal to

$$\theta^s \cdot [\int_\theta^1 u \cdot u^{-s-1} \cdot du - \frac{\zeta(s)}{s} + \frac{1}{s-1}] =$$

$$\theta^s \cdot [\frac{\theta^{-s+1} - 1}{s-1} - \frac{\zeta(s)}{s} + \frac{1}{s-1}] = \frac{\theta}{s-1} - \frac{\theta^s \cdot \zeta(s)}{s}$$

Now we are prepared to begin the proof of Theorem 7.1 and begin with the sufficiency.

2. The case when 1 *in the* L^2*-closure of* $\mathcal{D}$.

If $\epsilon > 0$ this assumption gives some $f \in \mathcal{D}$ such that the L^2-norm of $1+f$ is $< \epsilon$. Since $\sum c_\nu \cdot \theta_\nu = 0$, Proposition 7.2 gives:

$$\int_0^1 (1+f(x)) \cdot x^{s-1} \cdot dx = \frac{1}{s} - \frac{\zeta(s)}{s} \cdot \sum c_\nu \cdot \theta_\nu^s$$

With $s = \sigma + it$ and $\sigma > 1/2$ we have x^{s-1} in L^2 and Cauchy-Schwarz inequality gives:

$$\Big|\int_0^1 (1+f(x)) \cdot x^{s-1} \cdot dx \Big| \leq ||f||_2 \cdot \sqrt{\int_0^1 x^{2\sigma-2} \cdot dx} = ||f||_2 \cdot \frac{1}{\sqrt{2\sigma - 1}}$$

Hence we obtain

$$\Big|\frac{1}{s} - \frac{\zeta(s)}{s} \cdot \sum c_\nu \cdot \theta_\nu^s\Big| \leq \epsilon \cdot \frac{1}{\sqrt{2\sigma-1}} \quad : \quad \sigma > 1/2$$

If $\zeta(s_*) = 0$ holds for some $s_* = \sigma_* + it_*$ with $\sigma_* > 1/2$, the left hand side is reduced to $\frac{1}{|s_*|}$. Since we can find f as above for every $\epsilon > 0$ it would follow that

$$\frac{1}{|s_*|} \leq \epsilon \cdot \frac{1}{\sqrt{2\sigma_* - 1}} \quad \text{for every} \quad \epsilon > 0$$

But this is clearly impossible so if 1 belongs to the L^2-closure of $\mathcal{D}$ then the Riemann-Hypothesis holds.

3. Proof of necessity.

There remains to show that if 1 is outside the L^2-closure of $\mathcal{D}$ then the ζ-function has a zero in the half-plane $\mathfrak{Re}\, s > 1/2$. To show this we introduce a family of linear operators $\{T_a\}$ as follows: If $0 < a < 1$ and $g(x)$ is a function on $(0,1)$ we set

$$T_a(g)(x) = g(x/a) \quad : \quad 0 < x < a$$

while $T_a(g) = 0$ when $x \geq a$.

Exercise. Show that each T_a maps $\mathcal{D}$ into itself and one has the inequality

$$||T_a(f)||_2 \leq ||f||_2$$

Since 1 is outside the L^2-closure of $\mathcal{D}$ its orthogonal complement in the Hilbert space is $\neq 0$ which gives a non-zero $g \in L^2(0,1)$ such that

$$(*) \qquad \int_0^1 f(x) \cdot g(x) \cdot dx = 0 \quad : \quad f \in \mathcal{D}$$

Since $\mathcal{D}$ is invariant under the T-operators it follows that if $0 < a < 1$ then we also have

$$(1) \qquad 0 = \int_0^a f(x/a) \cdot g(x) \cdot dx = a \cdot \int_0^1 f(x) \cdot g(ax) \cdot dx$$

At this stage we apply Theorem 6.1. To begin with we show that the g-function satisfies (*) in Theorem 6.1. For suppose that $g = 0$ on some interval $(0,a)$ with $a > 0$. Choose some b where

$$a < b < \min(1, 2a)$$

Now $\mathcal{D}$ contains the function $f(x) = b\rho(x/a) - a\rho(x/b)$. The reader may verify that $f(x) = 0$ for $x > b$ and is equal to the constant a on (a,b). With (1) applied to f we therefore get

$$\int_a^b g(x) \cdot dx = 0$$

This means that the primitive function

$$G(x) = \int_0^x g(u) \cdot du$$

has a vanishing derivative on the interval (a,b). The derivative is also zero on $(0,a)$ where $g=0$. We conclude that $G=0$ on the interval $(0,b)$ so the L^2-function g is almost everywhere a fixed constant on this interval. But this constant is zero since $g=0$ on $(0,a)$. Hence we have shown that $g=0$ on the whole interval $(0,b)$. We can repeat this with a replaced by b and conclude that g also is zero on the interval

$$0<x<\min(1,2b)=\min(1,4a)$$

After a finite number of steps $2^m a\geq 1$ and hence g would be identically zero on $(0,1)$ which is not the case. Hence Theorem 6.1 applies to g and gives some λ_* with $\mathfrak{Re}\,\lambda_*<1/2$ such that $x^{-\lambda_*}$ belongs to $\mathcal{C}_g(p)$ for every $p<2$. Next, for each $\theta>0$ we get the $\mathcal{D}$-function

$$x\mapsto\rho(1/x)-\frac{1}{\theta}\cdot\rho(\theta/x)$$

Since (1) holds for all $0<a<1$ it follows that

$$\int_0^1[\rho(1/x)-\frac{1}{\theta}\cdot\rho(\theta/x)]\cdot x^{-\lambda_*}\cdot dx=0 \tag{2}$$

Put $s_*=1-\lambda_*$. The formula in Proposition 7.2 shows that the vanishing in (2) gives

$$\frac{\theta^{s_*}-1}{s_*}\cdot\zeta(s_*)=0$$

This hold for every $0<\theta<1$ and we can choose θ so that $\theta^{s_*}-1\neq 0$ which would give a zero $\zeta(s_*)=0$ where $\mathfrak{Re}(s_*)=1-\mathfrak{Re}(\lambda_*)>1/2$.