Hardy-Littlewood's maximal function

Contents

Introduction. The results below are due to Hardy, Littlewood and Fatou. More recent work which foremost is due to Feffermann and Stein study the interplay between Hardy spaces and functions with a bounded mean oscillation. So let us first expose some results of this more advanced naure where details of proofs can be studied from the last chapter in the book [Koosis]. Let D be the unit disc. An L^1-function $u(z)$ in D is radially bounded if there exists a constant C such that

$$(*) \qquad \frac{1}{\pi}\cdot\iint_{S_h}|u(z)|\cdot dxdy\leq C\cdot h$$

for each sector

$$S_h=\{z\colon \theta-h/2<\arg z\theta+h/2\}$$

and every $h>0$. The least constant C for which (*) holds is denoted by $|u|^*$. Notice that $|u|^*$ in general is strictly larger than the L^1-norm over D which occurs when we take $h=\pi$ above. If u satisfies (*) we define a function P_u on the unit circle by

$$P_u(\theta)=\frac{1}{\pi}\cdot\iint_D\frac{1-|z|^2}{|e^{i\theta}-z|^2}\cdot u(z)\cdot dxdxy$$

With these notations one has

0.1 Theorem *There exists an absolute constant C such that*

$$|P_u|_{\text{BMO}}\leq C|u|^*$$

Thus, $u\mapsto P_u$ sends radially bounded $L^1(D)$-functions to BMO(T). The proof of Theorem 0.1 is relatively easy and relies upon the following:

Exercise. Show that when u is radially bounded and $H(z)$ is a harmonic function in D with continuous boundary values on T then

$$\iint_D H(z)\cdot u(z)\cdot dxdy=\int_0^{2\pi}H(e^{i\theta})\cdot P_u(\theta)\cdot d\theta$$

A result by Fefferman. Using the duality between the Hardy space $H^1(T)$ and BMO(T) the following converse result was proved by Fefferman:

0.2 Theorem. *Let $F(\theta)\in BMO(T)$. Then there exists a radially bounded $L^1(D)$-function u and some $s(\theta)\in H^\infty(T)$ such that*

$$F(\theta)=s(\theta)+P_u(\theta)$$

Now we turn to classic results where details of proofs are supplied.

1. The weak type estimate

Let $f(x)$ be a non-negative function on the real x-line with support in a finite interval $[0, A]$ for some $A > 0$. We assume that f is integrable, i.e.

$$\int_0^A f(x) \cdot dx < \infty$$

The forward maximal function of f is defined by

$$f^*(x) = \max_{h>0} \frac{1}{h} \int_x^{x+h} f(t) \cdot dt$$

It is clear that f^* is non-negative and supported by $[0, A]$. To each $\lambda > 0$ we get the set $\{f^* > \lambda\}$. We shall prove an upper bound for its measure.

1. Theorem *For each $\lambda > 0$ one has the inequality*

$$\mathbf{m}(\{f^* > \lambda\}) \leq \frac{1}{\lambda} \cdot \int_{\{f^*>\lambda\}} f(x) \cdot dx$$

Proof. Introduce the primitive function

$$F(x) = \int_0^x f(t) \cdot dt$$

With $\lambda > 0$ we have the continuous function $F(x) - \lambda$ and define the forward Riesz set by:

$$\mathcal{E}_\lambda = \{x \colon \exists\, y > x \quad \text{and} \quad F(y) - \lambda y > F(x) - \lambda y\}$$

Exercise. Show the equality

$$\mathcal{E}_\lambda = \{f^* > \lambda\}$$

Now $\mathcal{E}_\lambda$ is an open set and hence a disjoint union of intervals $\{(a_k, b_k)\}$. With these notations one has

Exercise. Show the following for each interval (a_k, b_k):

$$F(b_k) - \lambda \cdot b_k = \max_{a_k \leq x \leq b_k} F(x) - \lambda$$

In particular one has

$$\lambda(b_k - a_k) \leq F(b_k) - F(a_k)$$

This holds for each k and after a summation over the forward Riesz intervals the requested inequality in Theorem 1 follows.

2. An L^2-inequality. Using Theorem 1 we shall prove that

$$(*) \qquad \int_0^A f^*(x)^2 \cdot dx \leq \int_0^A f(x)^2 \cdot dx$$

We use general formulas for distribution functions which in particular give:

$$\int_0^A f^*(x)^2 \cdot dx = \int_0^\infty \lambda \cdot \mathbf{m}(\{f^* > \lambda\}) \cdot d\lambda$$

By Theorem 1 the last integral is majorised by

$$\int_0^\infty [\int_{\mathbf{m}(\{f^*>\lambda\}} f(x) \cdot dx\,] \cdot d\lambda) = \iint_{\{f^*(x)>\lambda\}} f(x) \cdot dx d\lambda =$$

$$\int_0^A [\int_0^{f^*(x)} d\lambda] \cdot f(x) \cdot dx = \int_0^A f^*(x) \cdot f(x) \cdot dx$$

Finally, by the Cauchy-Schwarts in equality the last integral is majorised by the product of L^2-norms

$$||f^*||_2 \cdot |f||_2$$

Hence

$$||f^*||_2^2 = \int_0^A f^*(x)^2 \cdot dx \leq ||f^*||_2 \cdot |f||_2$$

and after a division with $||f^*||_2$ we get

Theorem 2. *One has the inequality*

$$||f^*||_2 \leq |f||_2$$

Remark. In a similar way we get an L^2-inequality using the backward maximal function

$$f_*(x) = \max_{h>0} \frac{1}{h} \int_{x-h}^{x} f(t) \cdot dt$$

In general we define the full maximal function

$$f^{**}(x) = \max_{a,b} \frac{1}{a+b} \int_{x-a}^{x+b} |f(t)| \cdot dt$$

with the maximum taken over pairs $a, b > 0$. Then we get the L^2-inequality

$$||f^{**}||_2 \leq |f||_2$$

3. A study of harmonic functions.

Let $f(t)$ be complex-valued function on the real t-line such that

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1+t^2} \cdot dt < \infty$$

We also assume that

$$f^{**}(0) = \max \frac{1}{b+a}| \cdot \int_{-a}^{b} |f(t)| \cdot dt < \infty$$

where the maximum is taken over all pairs $a, b > 0$. Define the function $V(z) = V(x+iy)$ in the upper half-plane $y > 0$ by

$$V(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} \cdot f(t) \cdot dt$$

Exercise. Prove the inequality

(1) $$|V(x+iy)| \leq \big(\frac{|x|}{y} + 2\big) \dot{f}^{**}(0)$$

Next, define Fatou's maximal function on the real x-line by

(3) $$V^*(x) = \max_{y \leq |s|} |V(x+s+iy)|$$

Deduce via translations of x that (1) gives the inequality

$$V^*(x) \leq 3 \cdot f^*(x)$$

for all x where the maximal function $f^{**}(x)$ is defined for every x by

$$f^{**}(0) = \max \frac{1}{b-a}| \cdot \int_{x-a}^{x+b} |f(t) \cdot dt$$

Next, apply the Remark after Theorem 2 and conclude that

$$\int_{-\infty}^{\infty} V^*(x)^2 \cdot dx \leq 18 \cdot \int_{-\infty}^{\infty} f(x)^2 \cdot dx$$

Finally, by the construction $V(x) = f(x)$ it follows that

$$(*) \qquad ||V^*||_2 \leq 3\sqrt{2} \cdot \sqrt{\int_{-\infty}^{\infty} V(x)^2 \cdot dx}$$

4. Application to analytic functions.

Let $F(z)$ be analytic in $\mathfrak{Im}(z) > 0$ and assume that there is a constant C such that

$$\int_{-\infty}^{\infty} \frac{|F(x+iy)|}{1+x^2} \cdot dx \leq C \quad \text{for all} \quad y > 0$$

It means that F belongs to the Hardy space H^1 in the upper half-plane U_+. We can divide out the zeros via a Blaschke product and write

$$F = B \cdot G$$

where G again belongs to H^1 and has no zeros in U_+. Then $\sqrt{G}$ is defined which gives a complex-valued harmonic function

$$V(z) = \sqrt{G(z)}$$

Now (*) from (3) gives the inequality

$$(**) \qquad \int_{-\infty}^{\infty} |F^*(x)| \cdot dx \leq 3\sqrt{2} \cdot \int_{-\infty}^{\infty} |F(x)| \cdot dx$$

where $F^*(x)$ is Fatou's maximal function for F defined for each real x by

$$F^*(x) = \max_{y \leq |s|} |F(x + is + iy)|$$

Exercise. Use the conformal map from U_+ to the unit disc D defined by

$$w = \frac{z-i}{z+i}$$

Explain how the previous result is translated when we start from an analytic function f in D for which the boundary value function $f(e^{i\theta})$ is in $L^1(T)$.

5. Conformal maps and the Hardy space $H^1(T)$

Let $g(z) = \sum a_n z^n$ be analytic in D and assume that its boundary value function is integrable, i.e. there exists a constant C such that

$$\int_0^{2\pi} |g(re^{i\theta})| \cdot d\theta \leq C$$

for every $r < 1$. In D there exists a single-valued brach of $\log(1-z)$ whose imaginary part stays in $(-\pi/2, \pi/2)$ and with $z = re^{i\theta}$ we have

$$\mathfrak{Im}(\log(1-z)) = -\frac{1}{2i} \cdot \sum_{n=1}^{\infty} r^n (e^{in\theta} - e^{-in\theta})$$

Exercise. **1** Deduce from the above that

$$(\text{E.1}) \qquad \int_0^{2\pi} \mathfrak{Im}(\log(1 - re^{i\theta})) \cdot g(re^{i\theta}) \cdot d\theta = -\pi i \cdot \sum_{n=1}^{\infty} \frac{b_n}{n} \cdot r^{2n}$$

The case when $\{b_n\}$ are real and ≥ 0. If this holds then (E.1) and the triangle inequality yield:

$$\pi \sum_{n=1}^{\infty} \frac{b_n}{n} \cdot r^{2n} \leq \frac{\pi}{2} \cdot \int_0^{2\pi} |g(re^{i\theta})| \cdot d\theta$$

So if we introduce the $H^1(T)$-norm

$$||g||_1 = \int_0^{2\pi} |g(e^{i\theta})| \cdot d\theta$$

it follows after a passage to the limit when $r \to 1$ that

$$(*) \qquad \sum_{n=1}^{\infty} \frac{b_n}{n} \leq \pi \cdot |g||_1$$

Application to conformal mappings. Let $\phi \colon D \to \Omega$ be a conformal mappng and assume that the complex derivative $\phi'(z)$ belongs to the Hardy space H^1 as above. Since $\phi' \neq 0$ in D there exists a single-valued analytic square-root:

$$\psi(z) = \sqrt{\phi'(z)}$$

Now ψ belongs to the Hardy space H^2 so if

$$\psi(z) = \sum b_n z^n \implies \sum |b_n|^2 < \infty$$

Let us then consider the H^2-function

$$\Psi(z) = \sum |b_n| z^n$$

We get

$$\Psi^2(z) = \sum A_n z^n \quad \text{where} \quad A_n = \sum_{k=0}^{k=n} |b_k| \cdot |b_{n-k}|$$

and (*) gives:

$$(1) \qquad \sum_{n=1}^{\infty} \frac{A_n}{n} \leq \pi \cdot \int_0^{2\pi} |\Psi(e^{i\theta})|^2 \cdot d\theta$$

Next, consider the Taylor series

$$\phi'(z) = \sum a_n z^n \implies a_n = \sum_{k=0}^{k=n} b_k \cdot b_{n-k}$$

The triangle inequality gives $|a_n| \leq A_n$ for each n so (1) entails that

$$(2) \qquad \sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$$

Finally, consider the Taylor expansion of $\phi(z)$:

$$\phi(z) = \sum c_n z^n$$

Here

$$nc_n = a_{n-1} \quad : \quad n \geq 1$$

Then it is clear that (2) implies that the series $\sum |c_n| < \infty$. Hence we have proved the following result which is due to Hardy:

5. Theorem. *Let $\phi(z)$ be a conformal map such that ϕ' belongs to H^1. Then the Taylor series of ϕ is absolutely convergent.*

Exercise. Let Ω be a Jordan domain whose boundary curve $\Gamma = \partial\Omega$ has a finite arc-length. Let $\phi \colon D \to \Omega$ be the con formal mapping which by results from (xx) extends to a homeomorphism from the closed disc $\bar{D}$ onto $\bar{\Omega}$.' Let $\ell(\Gamma)$ be the arc-length of Γ. Show that the derivative $\phi'(z)$ belongs to the Hardy space and

$$\int_0^{2\pi} |\phi'(e^{i\theta})| \cdot d\theta \leq \ell(\Gamma)$$

From this it follows that the Taylor series of $\phi(z)$ is absolutely convergent.

A hint for the exercise. To each $n \geq 1$ we set $\epsilon = e^{2\pi i/n}$, i.e. the n:th root of the unity. Now ϕ yields a homeomorphism from T onto Γ. The definition of $\ell(\Gamma)$ gives the inequality below where we set $\epsilon^0 = 1$.

$$(1) \qquad \sum_{k=1}^{n} |\phi(\epsilon^k \cdot e^{i\theta}) - \phi(\epsilon^{k-1} \cdot e^{i\theta})| \leq \ell(\Gamma) \quad \text{for every} \quad 0 \leq \theta \leq 2\pi$$

Keeping n fixed we notice that the function

$$s_n(z) = \sum_{k=1}^{n} |\phi(\epsilon^k \cdot z) - \phi(\epsilon^{k-1} \cdot z)|$$

is subharmonic in D. So the maximum principle for subharmonic functions and (1) give

$$(2) \qquad \max_\theta s_n(re^{i\theta}) \leq \ell(\Gamma)$$

for each $r < 1$. Next, with $r < 1$ fixed the reader may verify the limit formula:

$$(3) \qquad \lim_{n\to\infty} s_n(r) = \int_0^{2\pi} |\phi'(re^{i\theta})| \cdot d\theta$$

Hence (2-3) give

$$\int_0^{2\pi} |\phi'(re^{i\theta})| \cdot d\theta \leq \ell(\Gamma)$$

Now the Brothers Riesz theorem implies that $\phi'(z)$ belongs to $H^1(T)$, i.e. the boundary value function $\phi'(e^\theta)$ exists and belongs to $L^1(T)$.