II. Null solutions of PDE:s with constant coefficients.

Introduction. We expose material from the article *Null solutions to partial differential operators* [Arkiv för matematik. 1959] by Lars Hörmander. The Main Theorem to be announced below contains an instructive proof where complex line integrals taken over contours adapted to the complex zeros of the polynomial $P(\zeta)$ of n independent complex variables which corresponds to a PDE-operator $P(D)$ are used in order to get an ample family of null solutions, i.e. functions $u(x)$ for which $P(D)u = 0$. Hörmander employs Puiseux series constructed via embedded curves in the zeros of $P(\zeta)$ to get such u-functions supported by the half-space $\{x_n \geq 0\}$ in the case when the hyperplane $\{x_n = 0\}$ is characterstic to the differential operator $P(D)$. The remaining part of the proof of the Main Theorem is based upon the Paley-Wiener theorem and duality results from general distribution theory. Here one crucial point appears. Namely, thanks to constructions due to Gevrey, there exists test-functions whose higher order derivatives have a good control which entail that their Fourier transforms enjoy certain decay conditions. So the subsequent material offers an instructive mixture of algebra and analysis. For example, one ingredient employs a density result which goes back to Pusieeux which can be considered as a sharp version of the standard Nullstellen Satz. Namely. for every algebraic hypersurface $S = \{P(\zeta) = 0\}$ in the n-dimensional complex ζ-space there exists an ample family of curves of two independent complex contained in S with the property that if $g(\zeta)$ is an entire function which vanishes on all these curves then is is identically zero on S. This entails that

$$g(\zeta) = P(\zeta)h(\zeta)$$

for another entire function h. Moreover, when g is the Fourier-Laplace transform of a distribution μ with compact support, the Paley-Wiener theorem entails that $h = \widehat{\gamma}$ for another distribution whose compact support is contained in the convex hull of μ which is used during the final step in the proof of the Main Theorem.

Before we announce our Main Theorem we need some notations. Let $n \geq 2$ and in $\mathbf{R}^n$ we consider the hyperplane $H = \{x_n = 0\}$. Let $P(D)$ be a differential operator with constant coefficients. Here $D_k = -i \cdot \partial/\partial x_k$ and by Fourier's inversion formula

$$P(D)f(x) = (2\pi)^{-n} \cdot \int e^{i\langle x,\xi\rangle}\, \widehat{f}(\xi)\, d\xi$$

for test-functions $f(x)$. Let m be the order of $P(D)$ which means that

$$P(D) = \sum c_\alpha \cdot D^\alpha$$

where the sum is taken over multi-indices α for which $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq m$. The leading form is defined by

$$P_m(D) = \sum_{|\alpha|=m} c_\alpha \cdot D^\alpha$$

The hyperplane H is characteristic if $P_m(N) = 0$ where $N = (0, \ldots, 1)$, i.e. the term D_n^m does not appear in $P_m(D)$ with a non-zero coefficient. Put $H_+ = \{x_n > 0\}$ and

$$\mathcal{N}_+ = \{g \in C^\infty(H_+) : \; P(D)(g) = 0$$

Thus, we consider C^∞-functions in the open half-plane H_+ which are null solutions to $P(D)$ in this open half-plane. A smaller space is given by

$$\mathcal{N}_* = \{g \in C^\infty(\mathbf{R}^n) : \ P(D)(g) = 0 \text{ and } \mathrm{Supp}(g) \subset \overline{H_+}\}$$

Denote by $\mathcal{N}_*^\perp$ the family of distributions μ with compact support in H_+ which are zero on $\mathcal{N}_*$.

Main Theorem. *Every distribution μ in $\mathcal{N}_*^\perp$ is zero on $\mathcal{N}_+$*

The proof requires several steps. The crucial step is to construct functions in $\mathcal{N}_*$ and after prove that they give a dense subspace of $\mathcal{N}_+$. So we begin with:

1. A construction of null solutions.

Let ξ_0 be a real n-vector such that $P_m(\xi_0) \neq 0$ and ζ_0 some complex n-vector. Let s and t be independent complex variables and set

$$p(s,t) = P(s \cdot N + t\xi_0 + \zeta_0)$$

This gives a polynomial where the term t^m appears since $P_m(\xi_0) \neq 0$. At the same time s^m does not appear because $P_m(N) =$ is assumed. A classic result due to Pusieux from 1852 shows that there exists a positive integer p and a series

$$t(s) = s^{k/p} \cdot \sum_{j=0}^{\infty} c_j \cdot s^{-j/p} \tag{1.1}$$

where $0 \leq k < p$ which converges when $|s|$ is large, i.e. there exists some $M > 0$ such that

$$\sum_{j=0}^{\infty} |c_j| \cdot M^{-j/p} < \infty$$

Moreover,

$$P(s \cdot N + t(s)\xi_0 + \zeta_0) = 0 \quad : \ |s| \geq M \tag{1.2}$$

In the lower half-plane $\mathfrak{Im}(s) < 0$ we choose a single valued branch of $s^{1/p}$ where

$$s = |s| \cdot e^{i\phi} \implies s^{1/p} = |s|^{1/p} \cdot e^{i\phi/p} \quad : \quad -\pi < \phi < 0$$

Next, choose a number

$$1 - 1/p < \rho < 1$$

Now $(is)^\rho$ has a single valued branch for which

$$\mathfrak{Re}((is)^\rho) = \cos\frac{\rho\pi}{2} \cdot |s|^\rho \cdot \cos(\rho \cdot (\pi/2 + \phi)) \tag{1.3}$$

So if $\epsilon > 0$ we have

$$|e^{-\epsilon(is)^\rho}| = e^{-\epsilon \cdot \mathfrak{Re}(is)^\rho} = e^{-\epsilon \cdot |s|^\rho \cdot \cos(\rho(\pi/2+\phi))} \tag{1.4}$$

Since $\rho < 1$ we notice that

$$\cos(\rho(\pi/2 + \phi)) \geq \cos \rho\pi/2 = a$$

for all $-\pi < \phi < 0$ where a is positive constant a. It follows that

$$|e^{-\epsilon(is)^\rho}| \leq e^{-a\epsilon \cdot |s|^\rho} \tag{1.5}$$

for all s in the lower half-plane, arm also when s is real.

Let M be as above and denote by C_* the circle in the lower half-pane which consists of the two real intervals $(-\infty, -M)$ and $(M, +\infty)$ and the lower half-circle where

$|s| = M$. For each $x \in \mathbf{R}^n$ and every non-negative integer ν we get the complex line integral

$$(*) \qquad \int_{C_*} e^{i\langle x, sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

This integral is absolutely convergent. Namely, during the integration on the real interval $(-\infty, -M)$ or the real interval $[M, +\infty)$ we see that (1.5) gives estimates the absolute value of the integrand by

$$(1.6) \qquad |s|^{\nu/p} \cdot \cdot |e^{it(s)\langle x,\xi_0\rangle}| \cdot e^{-a\epsilon\cdot|s|^\rho}$$

Next, the Puiseux expansion for $t(s)$ entails that

$$|t(s) \leq A|s|^{1-1/p}$$

hold for some constant A. Since $\rho > 1 - 1/p$ It follows that (.6) is majorised by

$$(1.6) \qquad |s|^{\nu/p} \cdot e^{A\cdot|\langle x,\xi_0\rangle|\cdot|s|^{1-1/p}} \cdot e^{-a\epsilon\cdot|s|^\rho}$$

Since $\rho > 1 - 1/p$ we conclude that the line integral (*) converges absolutely for each positive integers ν.

Exercise. Show by Cauchy's theorem in analytic function theory that the line integral (*) does not depend on M as soon as it has been chosen so that the Puiseux series definiing $t(s)$ exists. The resulting value of (*) is therefore a function of x and ϵ and gives a function $u_\epsilon(x)$ defined for all x in $\mathbf{R}^n$. Moreover, the reader should check that when $\epsilon > 0$ kept fixed this yields a C^∞-function of x. In particular

$$(**) \quad P(D)(u_\epsilon)(x) = \int_{C_*} P(sN + t(s)\xi_0 + \zeta_0) \cdot e^{i\langle x, sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

Since $P(sN + t(s)\xi_0 + \zeta_0) = 0$ when $|s| \geq M$ we conclude that $P(D)(u_\epsilon) = 0$, i.e. u_ϵ is a null solution.

The inclusion Supp$(u) \subset \overline{H}_+$. In (*) we perform a line integral whose integrand is an analytic function in the lower half-plane. Using Cauchy's theorem the reader can check that for any $M^* > M$ we have

$$(**) \qquad u_\epsilon(x) = \int_{\mathfrak{Im}(s)=-M^*} e^{i\langle x, sN+t(s)\xi_0+\zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds$$

With $s = t - iM^*$ we have

$$|e^{i\langle x, sN\rangle}| = e^{M^*\langle x,N\rangle}$$

If $\langle x, N\rangle < 0$ this decreases exponentially to zero as $M^* \to +\infty$ and then the reader can check that the limit of (**) as $M^* \to +\infty$ is zero. This proves that the null solution u_ϵ is supported by the half-plane $\overline{H}_+$ and hence belongs to $\mathcal{N}_*$.

§ 2. A study of $\mathcal{N}_*^\perp$.

Consider a test-function ϕ with a compact support in H_+ such that $\phi(\mathcal{N}_*) = 0$. It gives the entire function in the n-dimensional complex ζ-space:

$$\Phi(\zeta) = \int e^{i\langle x,\zeta\rangle}\, \phi(x)\, dx \tag{2.0}$$

Using the convergence of the line integrals in (*) the reader should verify that Fubini's theorem gives the equation

$$\int u_\epsilon(x)\phi(x)\, dx = \int_{C_*} \Phi(sN + t(s)\xi_0 + \zeta_0) \cdot s^{\nu/p} \cdot e^{-\epsilon(is)^\rho}\, ds \tag{2.1}$$

Since $\phi(\mathcal{N}_*) = 0$ is assumed it follows that the last integral is zero for all non-negative integers ν and each $\epsilon > 0$.

2.2 Another vanishing integral. In the upper half-plane $\mathfrak{Im}(s) > 0$ we can also choose single-valued branches of $s^{1/p}$ and $(-is)^\rho$, where the last branch is chosen so that the value is $a^\rho > 0$ when $s = ai$ for $a > 0$. Then we construct the contour C^* given by the real intervals $(\infty, -M)$ and $(M, +\infty)$ together with the upper half circle of radius M, which for each non-negative integer ν gives the function

$$v_\epsilon(x) = \int_{C^*} e^{i\langle x, sN + t(s)\xi_0 + \zeta_0\rangle} \cdot s^{\nu/p} \cdot e^{-\epsilon(-is)^\rho}\, ds \tag{*}$$

Exactly as in § 1 one verifies that this gives a C^∞-function of x supported by the right half space $\{x_n \le 0\}$. Since ϕ has compact support in H_+ it follows that

$$0 = \int v_\epsilon(x)\phi(x)\, dx = \int_{C^*} \Phi(sN + t(s)\xi_0 + \zeta_0) \cdot s^{\nu/p} \cdot e^{-\epsilon(-is)^\rho}\, ds \tag{2.2.1}$$

2.3 The limit as $\epsilon \to 0$. In (2.2.1) we have vanshing integrals for each $\epsilon > 0$. If the test-function $\phi(x)$ belongs to a suitable Gevrey class with more regularity than an arbitrary test-function, then the entire function $\Phi(\zeta)$ enjoys a decay condition which enable us to pass to the limit as $\epsilon \to 0$ in (2.2.1). To find a sufficient decay condition we set $\zeta = \xi + i\eta$, and with M kept fixed we study the function

$$s \mapsto \Phi(sN + t(s)\xi_0 + \zeta_0)$$

We already know that there is a constant C such that $|t(s)| \le C|s|^{1-1/p}$ when $|s| \ge M$. Since ξ_0 and ζ_0 are fixed this gives a constant C_1 such that

$$|\mathfrak{Im}(sN + t(s)\xi_0 + \zeta_0)| \le C_1(1 + |s|)^{1-1/p} \tag{2.3.1}$$

At the same time we have the unit vector N and get a positive constant C_2 such that

$$|\mathfrak{Re}(sN + t(s)\xi_0 + \zeta_0)| \ge C_1(1 + |s|) \tag{2.3.2}$$

when $|s|$ is large. Suppose now that the test-function ϕ has been chosen so that

$$|\Phi(\xi + i\eta) \le C \cdot e^{A|\eta|) - B|\xi|^b} \tag{2.3.3}$$

hold for some constants C, A, B, a where $b < 1$. From (2.3.1-2.3.2) this gives with other positive constants

$$|\Phi(sN + t(s)\xi_0 + \zeta_0) \le C_1 e^{A_1|s|^{1-1/p} - B_1|s|^b} \tag{2.3.4}$$

With ρ chosen as in § 1 where the equality (1.3) is used, it follows that as sson as

$$a > \rho$$

then we get absolutely convergent integrals

$$\int_{|s|\geq M} |\Phi(sN + t(s)\xi_0 + \zeta_0) \cdot |s|^w| \, ds < \infty$$

for every positive integer w. This enable us to pass to the limit in (2.2) and conclude that

$$\int_{C^*} \Phi(sN + t(s)\xi_0 + \zeta_0) \cdot s^{\nu/p} \, ds = 0 \tag{2.3.5}$$

for every non-negative integer ν. In the same fashion we find vanishing integrals with C^* replaced by C_*. The vanishing of these integrals for all $\nu \geq 0$ entails by the classic result due to Puiseux that that $\frac{\Phi}{P}$ is an entire function. Then a division theorem with bounds due to Lindelöf, together with the Paley-Wiener theorem imply that the entire quotient

$$\frac{\Phi}{P} = \Psi \tag{i}$$

where Ψ is given as in (2.0) for some test-function ψ supported by the convex hull of the support of ϕ. Moreover, (i) entails that

$$P(-D)(\psi) = \phi$$

and then it is obvious that ϕ annihilates $\mathcal{N}_+$. Hence we have proved the implication in Theorem 0 for distributions which are defined by test-functions ϕ whose associated entire Φ-function satisfies (2.3.3) with some $a > 1 - 1/p$. But this finishes the proof of the Main Theorem. Namely, fix a as above and put

$$\delta = 1/a$$

Now $\delta > 1$ which by a classic construction due to Gevrey enable us to construct an ample family of test-functions ϕ for which (2.3.3) hold and at the same time this family is weak-star dense in the space of distributions with compact support in H_+ which gives the Main Theorem. For details about this density the reader can consult Hörmander's article or his text-book [Hö:xx] if necessary. See also the article [Bj] by Göran Björck which offers a very detailed study of distributions arising from Gevrey classes.