Math 556 Homework 7

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7.1 Compute the chromatic number, the chromatic polynomial, and the number of 3-colorings of the complete bipartite graph $K_{2,3}$. Because $K_{2,3}$ is bipartite, we know $\chi(G) = 2$.

We can see all colorings fall into two cases. Either vertex 2 and 4 are the same color, or they are different. If 2 and 4 are the same color, we have k choices for the left partition, and $k-1$ choices for each vertex in the right partition.

Let $p(k)$ be the chromatic polynomial for $K_{2,3}$. If 2 and 4 are different colors, we have $k(k-1)$ choices for the left partition, and then $k-2$ choices for each vertex in the right partition. Thus we have

$$p(k) = k(k-1)^3 + k(k-1)(k-2)^2$$

To find the number of 3-colorings, we can calculate $p(3)$.

$$p(k) = k(k-1)^3 + k(k-1)(k-2)^2 = 3(2)^3 + 3(2)(1)^2 = 6 + 24 = 30$$

7.2 Let $p(k) = k(k-1)(k-2)(k-3)(k-4)(k-5)^2$ and $q(k) = k(k-1)(k-2)(k-3)(k-4)(k-6)^2$.

1. Is $p(k)$ the chromatic polynomial of any graph? If so, then find such a graph, and determine its chromatic number. If not, why not?

   Consider $r(k) = k(k-1)(k-2)(k-3)(k-4)$, the chromatic polynomial for K_5. Now add two additional vertices that are connected to each vertex of K_5, but not each other. Each vertex will have $k-5$ choices for color, thus giving $p(k) = k(k-1)(k-2)(k-3)(k-4)(k-5)^2$.

2. Same for $q(k)$.

   We see:

   $$q(5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot (-1)^2 = 5! = 120$$

   However, $q(6) = 0$. Then it is not possible for $q(k)$ to be the chromatic polynomial for a graph, because given 6 colors, we can color the graph in 120 ways using only 5 of the 6 colors. More generally, we see for any chromatic polynomial q,

   $$q(k) \leq q(k+1)$$

3. Answer parts (a) and (b) with graph replaced by planar graph.

   We know by the four color theorem that for any planar graph, $p(4) > 0$. However, we can easily see for both $p(k)$ and $q(k)$ that $p(4) = 0$ and $q(4) = 0$. So $p(k)$ and $q(k)$ are not the chromatic polynomials for any planar graphs.