Math 556 Homework 6

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6.1 Let G be the graph with vertices $\{1, \dots, 101\}$ and an edge between i and j if and only if $i \cdot j$ is even. Determine whether G has

a) a closed Eulerian trail

No. We know G has a closed Eulerian trail if and only if the degree of each vertex is even. There are 50 even vertices $\{2, 4, \dots, 100\}$, and even vertices are only connected to even vertices, $(e \cdot o = o)$. Therefore the $deg(v) = 49$ for all even v.

b) an open Eulerian trail

No. We know G has an open Eulerian trail if and only if there are exactly two vertices u and v of odd degree. However we saw $deg(v) = 49$ for all 50 even v.

c) an open Hamilton path

Yes. We can see for $i \in \{1, \dots, 101\}$, i is always connected to $i + 1$.

- i even $(2k)$:
$$(2k)(2k+1) = 4k^2 + 2k = 2(k^2 + k)$$
- i odd $(2k + 1)$:
$$(2k+1)((2k+1)+1) = (2k+1)(2k+2) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$$

So the path $(1, 2, \dots, 101)$ is an open Hamilton path.

d) a Hamilton cycle

No. This stems from the fact that we have more odd vertices than even, and no odd vertices are connected $(o \cdot o = o)$. Consider any permutation of the vertices. Because we have 50 even vertices and 51 odd vertices, any consecutive even vertices $\{\dots, e, e, \dots\}$ requires that at some point in the permutation there are consecutive odd vertices, which means there is not a path that represents this permutation. So any hamiltonian path looks like the following:

$$\{o, e, o, e, \dots, e, o\}$$

However we can see this will still not form a cycle, because there is no edge between the first and last odd vertices by $(o \cdot o = o)$.

6.2 Let G be the graph consisting of the vertices and edges of a 17-dimensional cube. More precisely, G has 2^{17} vertices given by all possible ordered 17-tuples of zeros and ones. (For example, one of its vertices is $(0,0,0,1,0,1,1,0,1,0,0,1,1,1,1,1,0)$.) Two vertices are connected by an edge if and only if they differ in exactly one coordinate. Show that G is bipartite, and that it admits a perfect matching.

- G is bipartite.

  Let the first partition be those vertices where the sum of the zeros and ones is even. For example, the sum for the vertex $(0,0,0,1,0,1,1,0,1,0,0,1,1,1,1,1,1)$ is 10, thus even.

  Let the second partition be those vertices where the sum of the zeros and ones is odd. For example, the sum for the vertex $(0,0,0,1,0,1,1,0,1,0,0,1,1,1,1,1,0)$ is 9, thus odd.

  We can see this partition forms a bipartite graph. If the sum for two different vertices differs by at least two, then they must differ in more than one coordinate. For example:

  $$(\ldots 0,\ldots,0,\ldots) \to (\ldots,1,\ldots,1,\ldots)$$

  Similarly, if the sum for two different vertices is the same, they must differ in at least two coordinates. For example:

  $$(\ldots 0,\ldots,1,\ldots) \to (\ldots,1,\ldots,0,\ldots)$$

  Therefore there are no edges between two vertices in the same partition, and G is bipartite.

- G admits a perfect matching.

  Consider a binary number representation of each vertex and order the vertices by binary value.

  $$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \to 00000000000000000$$
  $$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1) \to 00000000000000001$$
  $$\ldots$$
  $$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0) \to 11111111111111110$$
  $$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1) \to 11111111111111111$$

  We can see for each even binary number b, $b+1$ differs by exactly one coordinate, the 2^0 bit. Therefore match each even b with $b+1$, or $(2k, 2k+1)$.