Math 556 Homework 5

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5.1 Define a sequence by putting

$$h_0 = 3,\ h_1 = -2\ \text{, and } h_n = h_{n-1} + 12 \cdot h_{n-2}\ \text{, for all } n \geq 2$$

Compute h_n.

Assume $h_n = q^n$. Then we have,

$$\begin{aligned} q^n &= q^{n-1} + 12q^{n-2} \\ q^2 &= q + 12 \\ q^2 - q - 12 &= 0 \\ q = -3 &\text{ or } q = 4 \end{aligned}$$

So $h_n = -3^n$ and $h_n = 4^n$ satisfy the recurrence relation. Now we can find a linear combination that satisfies the initial conditions, $h_0 = 3$ and $h_1 = -2$.

$$h_n = c_1(-3^n) + c_2(4^n)$$

$$h_0 = 3 = c_1 + c_2$$

$$h_1 = -2 = -3c_1 + 4c_2$$

We can treat this as solving a system of linear equations.

$$c_1 = 3 - c_2$$

$$-2 = -3(3 - c_2) + 4c_2$$

$$-2 = -9 + 7c_2$$

$$c_2 = 1$$

$$c_1 = 2$$

Thus we finally have for all $n > 0$,

$$h_n = 2(-3^n) + 4^n$$

2.2 Write down a rational generating function for each of the following sequences.

a) $h_n = \binom{100}{n}$.

$$g(x) = \sum_{n=0}^{\infty} \binom{100}{n} x^n = (1+x)^{100}$$

b) $h_n =$ the number of non-negative integer solutions to the equation $e_1 + \cdots + e_{100} = n$.

$$\begin{aligned} g(x) &= \sum_{e_1,\cdots,e_{100}>0} x^{e_1+\cdots+e_{100}} \\ &= (\sum_{e_1=0}^{\infty} x^{e_1})\cdots(\sum_{e_{100}=0}^{\infty} x^{e_{100}}) \\ &= (\sum_{n=0}^{\infty} x^n)^{100} \\ &= (\frac{1}{1-x})^{100} \\ &= \frac{1}{(1-x)^{100}} \end{aligned}$$

c) $h_n =$ the number of non-negative integer solutions to the equation $e_1 + 2e_2 + 3e_3 + 4e_4 = n$.

$$g(x) = (\frac{1}{1-x})(\frac{1}{1-x^2})(\frac{1}{1-x^3})(\frac{1}{1-x^4})$$

d) $h_n =$ the number of n-combinations of apples, bananas, and oranges with a multiple of three apples, and odd number of oranges, and at most 17 bananas.

$$g(x) = (\frac{1}{1-x^3}) \cdot x(\frac{1}{1-x^2}) \cdot (\frac{1-x^{18}}{1-x})$$

e) h_n is the number that you computed in Problem 5.1.

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty}(2(-3^n) + 4^n) \cdot x^n \\ &= \sum_{n=0}^{\infty} 2(-3^n)x^n + 4^n x^n \\ &= \sum_{n=0}^{\infty} 2((-3x)^n) + (4x)^n \\ &= 2 \cdot \sum_{n=0}^{\infty}(-3x)^n + \sum_{n=0}^{\infty}(4x)^n \\ &= \frac{2}{1+3x} + \frac{1}{1-4x} \end{aligned}$$