Math 556 Homework 2

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2.1 Compute the number of permutations of the letters BOOGIEWOOGIE.

BOOGIEWOOGIE has 12 letters, but we need to account for repeated letters.

$$\frac{12!}{1!4!2!2!2!1!} = \frac{12!}{4!(2!)^3} = 2494800$$

2.2 Show that any 2-coloring of the edges of a K_8 contains at least 8 monochromatic triangles. Give an example of a specific 2-coloring with exactly 8 monochromatic triangles. (To make your picture clear, just draw the red edges. Any edge that you dont draw will be assumed to be blue.)

Label the 8 vertices of K_8 as $v_1, \cdots, v_8$. Let r_i be the number of red edges connected to v_i and let b_i be the number of blue edges connected to v_i. Then we know:

$$\begin{aligned} \#\text{ monochromatic triangles} &= \#\text{ total triangles} - \#\text{ pied triangles} \\ &= \binom{8}{3} - \frac{1}{2} \cdot \sum_{i=1}^{8} r_i \cdot b_i \end{aligned}$$

To find the minimum number of monochromatic triangles, we need to maximize $\sum_{i=1}^{8} r_i \cdot b_i$. This occurs when we have either 3 blue and 4 red edges or 4 red and 3 blue edges at each vertex. Therefore we have:

$$
\begin{aligned}
\text{min \# monochromatic triangles} &= \binom{8}{3} - \frac{1}{2} \cdot \sum_{i=1}^{8} r_i \cdot b_i \\
&= \binom{8}{3} - \frac{1}{2} \cdot \sum_{i=1}^{8} 4 \cdot 3 \\
&= 56 - \frac{1}{2} \cdot 8 \cdot 12 \\
&= 56 - 48 \\
&= 8
\end{aligned}
$$

We can give an example of this by assigning the odd-labeled vertices 3 red edges and 4 blue edges, and by assigning the even-labeled vertices 4 red edges and 3 blue edges.

2.3 Suppose that I gave a quiz out of 20 points to a class of 10 students. Show that I can choose two disjoint groups of students such that each group has the same total score.

Consider all possible subsets of the class and the total sum of their test scores.

- $2^{10} = 1024$ different subsets (where the elements of the sets are the student's scores)
- minimum total score of 0.
- maximum total score of $10 \cdot 20 = 200$ (whole class)

Therefore by the Pigeonhole Principle, two of the subsets must have the same score (1023 pigeons, 201 holes). Call these subsets A and B, and let $A \cap B = C$. Let:

$$\Sigma x \ \forall x \in A = \Sigma x \ \forall x \in B = k$$
$$\Sigma x \ \forall x \in C = l$$

Then we can see:

$$\Sigma x \ \forall x \in A \backslash C = k - l$$
$$\Sigma x \ \forall x \in B \backslash C = k - l$$

Therefore we have two disjoint subsets, $A \backslash C$ and $B \backslash C$ with the same total score, $k - l$.

2.4 How many non-negative integral solutions of $x_1 + x_2 + x_3 \leq 17$ are there with $x_1 \geq 2$?

Let

$$\begin{aligned} y_1 &= x_1 - 2 & \qquad x_1 &= y_1 + 2 \\ y_2 &= x_2 & x_2 &= y_2 \\ y_3 &= x_3 & x_3 &= y_3 \end{aligned}$$

So we have

$$(y_1 + 2) + y_2 + y_3 \leq 17$$
$$y_1 + y_2 + y_3 \leq 15$$

Using the formula for r-combinations, we get:

$$\binom{r + k - 1}{r} = \binom{15 + 3 - 1}{15} = \binom{17}{15} = 136$$