Math 456/556: Networks and Combinatorics

Solutions to HW #3

Book problems:

Solutions or hints for problems 6, 9, 15, and 16 can be found at the back of the book.

11. Let S be a set with n elements, including three distinguished elements a, b, and c. Let's count the number of subsets of S of size k that contain at least one of the three distinguished elements. On one hand, we can count all subsets of size k, and subtract the number that don't contain any of the three elements: this gives $\binom{n}{k} - \binom{n-3}{k}$. On the other hand, we could count the number of subsets that contain a, the number that contain b but not a, and the number that contain c but not a or b. Adding them up, we get $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$. Since we have counted the same thing in two different ways, the two answers that we got must be equal.

18. The binomial theorem tells us that

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Antidifferentiating both sides, we get

$$\frac{1}{n+1}(x+1)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} x^{k+1} + c.$$

The left-hand side is a polynomial with constant term $\frac{1}{n+1}$ and the right-hand side is a polynomial with constant term c, so we must have $c = \frac{1}{n+1}$. Evaluating at $x = -1$, we get

$$0 = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} (-1)^{k+1} + \frac{1}{n+1}.$$

Bringing the sum to the left-hand side, we have

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} (-1)^k = \frac{1}{n+1}.$$

19. We have

$$\sum_{k=0}^{n} k^2 = \sum_{k=0}^{n} \left[2\binom{k}{2} + \binom{k}{1} \right] = 2\sum_{k=0}^{n} \binom{k}{2} + \sum_{k=0}^{n} \binom{k}{1}.$$

By identity (5.19), this is equal to $2\binom{n+1}{3} + \binom{n+1}{2}$.

48. Let X be a poset with $|X| = mn+1$. Let r be the size of the largest chain and s the size of the largest antichain. By Theorem 5.6.1, X can be partitioned into r antichains. Since each antichain has size at most s, this means that $mn+1 = |X| \leq rs$. Thus it cannot be the case that $r \leq m$ and $s \leq n$.

50. Let $X = \{1, 2, \ldots, 12\}$ with the divisibility partial order.

(a) I claim that $\{1, 2, 4, 8\}$ is a maximal chain. To prove this, it is sufficient to exhibit a partition of X into 4 antichains. This can be done as follows:

$$X = \{1\} \sqcup \{2, 3, 5, 7, 11\} \sqcup \{4, 6, 9, 10\} \sqcup \{8, 12\}.$$

(b) I claim that $\{4, 6, 9, 5, 7, 11\}$ is a maximal antichain. To prove this, it is sufficient to exhibit a partition of X into 6 chains. This can be done as follows:

$$X = \{1, 2, 4, 8\} \sqcup \{3, 6, 12\} \sqcup \{9\} \sqcup \{5, 10\} \sqcup \{7\} \sqcup \{11\}.$$

Additional problems:

3.1(a) We have $\displaystyle\sum_{k=0}^{100} x^k \binom{100}{k} = (1+x)^{100}$. Plugging in $x = -2$, we get 1.

3.1(b) Antidifferentiating the expression in part (a), we have

$$\sum_{k=0}^{100} \frac{x^{k+1}}{k!} \binom{100}{k} = \frac{(1+x)^{101}}{101} - \frac{1}{101}.$$

The constant at the end can be determined by noting that the LHS has no constant term, so the same must be true of the RHS. Plugging in $x = -2$, we get $\frac{-2}{101}$.

3.2 Let X be the poset consisting of the elements $\{1, \ldots, 100\}$, ordered by divisibility. Show that X does not contain an antichain of size 51.

The poset X can be partitioned into 50 chains of the form $C_k = \{2^i k \mid i \geq 0, 2^i k \leq 100\}$, where k ranges over all odd positive integers less than 100.

3.3 What is the coefficient of $x^3 y^{10}$ in $(y - 2x)^{13}$?

The answer is $\binom{13}{3} \cdot (-2)^3 =$ -2,080.