Math 456/556: Networks and Combinatorics

Solutions to HW #6

Book problems:

5. Let G be a graph with n vertices. The possible degrees of vertices are $\{0, 1, \ldots, n-1\}$. We would like to show that there are two vertices with the same degree. Unfortunately, since the number of possible degrees is equal to the number of vertices, the pigeonhole principle cannot be directly applied. Instead, we break up the argument into two cases.

Case 1: There is a vertex of degree 0. In this case, there cannot be any vertex of degree $n-1$, so the possible degrees are $\{0, 1, \ldots, n-2\}$. By the pigeonhole principle, there are two vertices of the same degree.

Case 2: There is no vertex of degree 0. In this case, the possible degrees are $\{1, \ldots, n-1\}$. By the pigeonhole principle, there are two vertices of the same degree.

If G were a multigraph, then the degrees would be arbitrarily high, and the argument would fail. For example, consider the multigraph with vertices $\{a, b, c\}$ along with one edge from a to b and two edges from b to c. Then the degrees would be 1, 3, and 2.

14. Consider the dumbbell graph from page 415 of the textbook. The vertices a and e can be connected by a closed walk (just walk from a to e and then come back), but not by a closed trail, since any closed walk containing both a and e would be forced to cross the bridge twice.

20. Suppose that a graph G with n vertices is *not* connected. Then there is some $k \in \{1, \ldots, n-1\}$ such that G can be split into a graph with k vertices and a graph with $n-k$ vertices. Thus the most edges that G could possibly have is

$$\binom{k}{2} + \binom{n-k}{2} = \frac{(n-1)(n-2)}{2} - (k-1)(n-k-1) \leq \frac{(n-1)(n-2)}{2} < \frac{(n-1)(n-2)}{2} + 1.$$

This tells us that any graph with at least $\frac{(n-1)(n-2)}{2} + 1$ edges is connected. If we take the disjoint union of K_{n-1} and a point, we obtain a disconnected graph with exactly $\frac{(n-1)(n-2)}{2}$ edges.

29. The graph on the left has four vertices of degree 5, so it has no Eulerian trail (open or closed). The graph on the right has only vertices of even degree, so it has a closed Eulerian trail, but not an open one.

30. The complete graph K_n has a closed Eulerian trail if and only if n is odd. It has an open Eulerian trail if and only if $n = 2$.

49. There are $4 \times 6 = 24$ questions here, which is more than I want to write solutions to. I'd be happy to discuss any particular one that you would like to see.

68. The first and the last game are positive, and the middle one is neutral. Positivity of the first and the last can be shown by exhibiting pairs of disjoint spanning trees. The fact that the middle one is not not negative can be seen by adding an edge from u to v and exhibiting a pair of disjoint spanning trees. The fact that it is not positive can be seen by exhibiting a winning strategy for N if N plays first, starting with taking one of the two edges into u.

Additional problems:

6.1 Let G be the graph with vertices $\{1, \ldots, 101\}$ and an edge between i and j if and only if $i \times j$ is even. Determine whether G has

a) a closed Eulerian trail
b) an open Eulerian trail
c) an open Hamilton path
d) a Hamilton cycle.

Even vertices have degree 100 and odd vertices have degree 50. Since all degrees are even, G has a closed Eulerian trail but not an open one. The open path $1 - 2 - 3 - \ldots - 100 - 101$ is Hamilton. There is no Hamilton cycle, since we can never have two adjacent odd numbers and there are more odd numbers than even ones.

6.2 Let G be the graph consisting of the vertices and edges of a 17-dimensional cube. More precisely, G has 2^{17} vertices given by all possible ordered 17-tuples of zeros and ones. (For example, one of its vertices is $(0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0)$.) Two vertices are connected by an edge if and only if they differ in exactly one coordinate. Show that G is bipartite, and that it admits a perfect matching.

We can divide the vertex set into two parts according to whether the sum of the coordinates is even or odd; this shows that G is bipartite. The set of edges between vertices that differ only in the last coordinate is a perfect matching.