Math 456/556: Networks and Combinatorics

Partial Solutions to HW #5

Book problems:

2. We know that

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

thus

$$f_n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Since $\frac{1-\sqrt{5}}{2}$ has absolute value less than one, the absolute value of $-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ is less than or equal to $\frac{1}{\sqrt{5}}$, which is in turn less than $\frac{1}{2}$. Thus f_n is within $\frac{1}{2}$ of $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$, and is therefore the closest integer.

3. See hints in the back of the book.

13. Determine some generating functions.

(a) $\sum_{n=0}^{\infty} c^n x^n = \frac{1}{1-cx}$.

(b) $\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}$.

(d) $\sum_{n=0}^{\infty} \frac{1}{n!}x^n = e^x$.

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n = e^{-x}$.

15. Following the hint in the back of the book, we find that

$$\sum_{n=0}^{\infty} n^3 x^n = \frac{x}{(1-x)^2} + \frac{6x^2}{(1-x)^3} + \frac{6x^3}{(1-x)^4} = \frac{x(x^2+4x+1)}{(1-x)^4}.$$

16. The function $(1+x+x^2)(1+x^2+x^4+x^6)(1+x^2+x^4+\ldots)(x+x^2+x^3+\ldots)$ is the generating function for the sequence whose n^{th} term is the number of non-negative integer solutions to the equation

$$e_1 + e_2 + e_3 + e_4 = n$$

with e_1 at most 2, e_2 even and at most 6, e_3 even, and e_4 at least 1.

18. If h_n is the number of non-negative integer solutions to the equation

$$2e_1 + 5e_2 + e_3 + 7e_4 = n,$$

then

$$\sum_{n=0}^{\infty} h_n x^n = \frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1}{1-x}\frac{1}{1-x^7}.$$

19. We have

$$\sum_{n=0}^{\infty}\binom{n}{2}x^n = \sum_{n=0}^{\infty}\frac{n(n-1)}{2}x^n = \frac{x^2}{2}\left(\frac{d}{dx}\right)^2\sum_{n=0}^{\infty}x^n = \frac{x^2}{2}\left(\frac{d}{dx}\right)^2\frac{1}{1-x} = \frac{x^2}{(1-x)^3}.$$

33. The recursion $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$ tells us that

$$h_n = c_1 + c_2 3^n + c_3(-3)^n.$$

The initial conditions $h_0 = 0$, $h_1 = 1$, $h_2 = 2$ tell us that

$$c_1 = -\frac{1}{4}, \qquad c_2 = \frac{1}{3}, \qquad \text{and} \qquad c_3 = -\frac{1}{12}.$$

39. Let h_n be the number of ways to cover a $1 \times n$ board with monomioes and dominoes without using two consecutive dominoes. Every cover either starts with a monomino or a domino followed by a monomino, thus we have $h_n = h_{n-1}+h_{n-3}$. The initial conditions are $h_0 = 1$, $h_1 = 1$, and $h_2 = 2$.

Additional problems:

5.1 Compute h_n if

$$h_0 = 3, h_1 = -2, \text{ and } h_n = h_{n-1} + 12\,h_{n-2} \text{ for all } n \geq 2.$$

We have $h_n = c_1 4^n + c_2(-3)^n$, where $c_1 + c_2 = 3$ and $4c_1 - 3c_2 = -2$. Thus $c_1 = 1$ and $c_2 = 2$, so $h_n = 4^n + 2(-3)^n$.

5.2. Write down a rational generating function for each of the following sequences.

a) $h_n = \binom{100}{n}$.

$g(x) = (1+x)^{100}$.

b) $h_n =$ the number of non-negative integer solutions to the equation $e_1 + \ldots + e_{100} = n$.

$g(x) = (1-x)^{-100}$.

c) $h_n =$ the number of non-negative integer solutions to the equation $e_1 + 2e_2 + 3e_3 + 4e_4 = n$.

$g(x) = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}$.

d) $h_n =$ the number of n-combinations of apples, bananas, and oranges with a multiple of three apples, and odd number of oranges, and at most 17 bananas.

$g(x) = (1+x^3+x^6+\ldots)(x+x^3+x^5+\ldots)(1+x+\ldots+x^{17}) = \frac{1}{1-x^3}\frac{x}{1-x^2}\frac{1-x^{18}}{1-x}$.

e) h_n is the number that you computed in Problem 5.1.

We get the equation $(1-x-12x^2)g(x) = h_0+(h_0+h_1)x = 3+5x$, so $g(x) = \frac{3+5x}{1-x-12x^2} = \frac{3+5x}{(1-4x)(1+3x)}$.