Math 456/556: Networks and Combinatorics

Solutions to HW #7

Book problems:

5. The first graph is bipartite with at least one edge, so its chromatic number is 2. The second graph contains a K_3, which tells us that its chromatic number is at least 3. It is easy to exhibit a 3-coloring, so its chromatic number is exactly 3. The last graph contains a K_4, which tells us that its chromatic number is at least 4. On the other hand, it is a proper subgraph of K_5, which tells us that its chromatic number is less than 5. Thus its chromatic number is equal to 4.

11. Let G be the graph obtained from K_n by removing a single edge. Equation (12.2) applied to K_n tells us that $p_{K_n}(k) = p_G(k) - p_{K_{n-1}}(k)$, so

$$p_G(k) = p_{K_n}(k) + p_{K_{n-1}}(k) = [k]_n + [k]_{n-1}.$$

14. We will show that $p_{C_n} = (k-1)^n + (-1)^n(k-1)$ by induction on n. It doesn't make sense to think about C_n for $n < 3$, so we'll start with $n = 3$. We have

$$p_{C_3}(k) = k(k-1)(k-2) = (k-1)[k(k-2)] = (k-1)[(k-1)^2 - 1] = (k-1)^3 - (k-1).$$

Now assume that the formula holds for C_n. By Equation (12.2), we have

$$p_{C_{n+1}}(k) = p_{T_{n+1}}(k) - p_{C_n}(k),$$

where T_{n+1} is a tree of order $n+1$. By Theorem 12.1.7 and our inductive hypothesis,

$$p_{C_{n+1}}(k) = k(k-1)^n - (k-1)^n - (-1)^n(k-1) = (k-1)(k-1)^n - (-1)^n(k-1) = (k-1)^{n+1} + (-1)^{n+1}(k-1).$$

This completes the proof.

16. We have $k^3 - 4k^3 + 3k^2 = k^2(k-1)(k-3)$. If this were the chromatic polynomial of G, then G would have -4 2-colorings, which is ridiculous.

18. The chromatic polynomial of the graph given by the edges of a cube is

$$k^8 - 11k^7 + 52k^6 - 139k^5 + 234k^4 - 262k^3 + 186k^2 - 61k.$$

21. A plane is divided into regions by straight lines $L_1, \ldots, L_n$. Let G be the graph whose vertices correspond to regions, with an edge in between adjacent regions. We will show that G is bipartite, and therefore that it admits a 2-coloring.

For each line, choose a positive side and a negative side. Given any region R, let ℓ_R be the number of lines for which R lies on the positive side. If R is adjacent to R', then ℓ_R and $\ell_{R'}$ differ by 1. Thus our graph is bipartite, where the vertices are partitioned into those R for which ℓ_R is even and those R for which ℓ_R is odd.

26. Let G be a planar graph in which every vertex has the same degree k. By Theorem 12.2.2, there is at least one vertex of degree at most 5, thus $k \leq 5$.

7.1 Compute the chromatic number, the chromatic polynomial, and the number of 3-colorings of the complete bipartite graph $K_{2,3}$.

Since $K_{2,3}$ is bipartite (with at least one edge), its chromatic number is 2. The chromatic polynomial can be computed by repeated application of the deletion-contraction algorithm. We get $p_{K_{2,3}}(t) = t^5 - 6t^4 + 15t^3 - 17t^2 + 7t$. The number of 3-colorings is $p_{K_{2,3}}(3) = 30$.

7.2 Let $p(k) = k(k-1)(k-2)(k-3)(k-4)(k-5)^2$ and $q(k) = k(k-1)(k-2)(k-3)(k-4)(k-6)^2$.

a) Is $p(k)$ the chromatic polynomial of any graph? If so, then find such a graph, and determine its chromatic number. If not, why not?

Yes, $p(k)$ is the chromatic polynomial of the graph obtained from K_7 by deleting a single edge (see problem 5 from the book). Its chromatic number is 6 (the smallest positive integer we can plug into $p(k)$ without getting zero).

b) Same for $q(k)$.

No, because $q(5) \neq 0 = q(6)$. Any graph that admits a 5-coloring also admits a 6-coloring!

c) Answer parts (a) and (b) with “graph” replaced by “planar graph”.

No, because $p(4) = 0 = q(4)$.