Math 456/556: Networks and Combinatorics

Solutions to HW #1

Book problems:

2. The number of orderings of a deck of cards with all suits together is $4! * (13!)^4$, given by first deciding how to order the four suits and then deciding the order of the ranks within each suit.

3. A poker hand can be dealt in P(52,5) = 52!/50! different ways. The number of different hands (not accounting for the order in which the cards are dealt) is $\binom{52}{5} = \frac{52!}{50!5!}$.

6. How many integers greater than 5400 have distinct digits, with the digits 2 and 7 not occurring?

Our number must have 4, 5, 6, 7, or 8 digits. We'll treat each one of these possibilities separately, and then add up the total.

- 8 digits: The multiplication principle tells us that we have $7 * 7 * 6 * 5 * 4 * 3 * 2 * 1$ possibilities. (The first number in this product is 7 rather than 8 because the first digit cannot be zero.)
- 7 digits: $7 * 7 * 6 * 5 * 4 * 3 * 2$ possibilities.
- 6 digits: $7 * 7 * 6 * 5 * 4 * 3$ possibilities.
- 5 digits: $7 * 7 * 6 * 5 * 4$ possibilities.
- 4 digits, with the first digit equal to 6, 8, or 9: $3 * 7 * 6 * 5$ possibilities.
- 4 digits, with the first digit equal to 5: $4 * 6 * 5$ possibilities. (The 4 comes from the fact that the second digit must be 4, 6, 8, or 9.)

Add them all up, and you get 94,830.

10. Given 10 men and 12 women, we need to choose a committee of 5, subject to the condition that it contains at least 2 women. (a) How many ways can this be done? (b) How about if one particular man and one particular woman refuse to serve together?

(a) Saying that the committee contains at least 2 women is the same as saying that it contains exactly 2, 3, 4, or 5 women. Let's analyze each case separately. If there are exactly 2 women, then we have $\binom{12}{2} * \binom{10}{3}$ possibilities. If there are exactly 3 women, then we have $\binom{12}{3} * \binom{10}{2}$ possibilities. If there are exactly 4 women, then we have $\binom{12}{4} * \binom{10}{1}$ possibilities. If there are exactly 5 women, then we have $\binom{12}{5} * \binom{10}{0}$ possibilities. Adding them up, we get 23,562.

(b) Let's count how many committees *do* have the two people in question serving together, and then subtract that from our answer to part (a). After putting those two on the committee, we need to choose an additional 3 members, with at least 1 woman. There can be 1, 2, or 3 women chosen, so we get

$$\binom{11}{1} * \binom{9}{2} + \binom{11}{2} * \binom{9}{1} + \binom{11}{3} * \binom{9}{0} = 1,056$$

possibilities. Subtracting, we get 23,562 - 1,056 = 22,506.

13. Dormitory A holds 25 students, Dormitory B holds 35 students, and Dormitory C holds 40 students. There are 100 students in the school. (a) How many ways are there to fill the dormitories? (b) What if 50 students are men, A is all-male, and B is all-female?

(a) First we choose who goes in A, then we choose which of the remaining students go in B; everyone else has to go in C. We get $\binom{100}{25} * \binom{75}{35}$ possibilities.

(b) First we choose which men go in A, then we choose which women go in B; everyone else has to go in C. We get $\binom{50}{25} * \binom{50}{35}$ possibilities.

15. At a party there are 15 men and 20 women. (a) How many ways are there to form 15 heterosexual couples? (b) How about 10 heterosexual couples?

(a) Each man needs to choose a woman, so we have P(20,15) = 20!/5! possibilities.

(b) First we choose the 10 men, and then each man chooses a woman, so we have

$$\binom{15}{10} * \mathrm{P}(20,10) = \frac{15!}{10!5!} * \frac{20!}{10!}$$

possibilities. Note that we also could have first chosen the 10 women, and then had each woman choose a man. This would give us

$$\binom{20}{10} * \mathrm{P}(15,10) = \frac{20!}{10!10!} * \frac{15!}{5!},$$

which is of course equal to the previous expression.

Additional problems:

1.1 How many integers greater than 76,000 have distinct digits, with the digits 1, 2, and 3 not occurring?

This problem is similar to problem 6 from the book. Our number must have 5, 6, or 7 digits. We'll treat each one of these possibilities separately, and then add up the total.

- 7 digits: The multiplication principle tells us that we have $6 * 6 * 5 * 4 * 3 * 2 * 1$ possibilities. (The first number in this product is 6 rather than 7 because the first digit cannot be zero.)
- 6 digits: $6 * 6 * 5 * 4 * 3 * 2$ possibilities.
- 5 digits, with the first digit equal to 8 or 9: $2 * 6 * 5 * 4 * 3$ possibilities.
- 5 digits, with the first digit equal to 7: $3 * 5 * 4 * 3$ possibilities. (The 3 comes from the fact that the second digit must be 6, 8, or 9.)

Add them all up, and you get 9,540.

1.2 Ten dishes are offered at the cafeteria, four of which have meat, and the remaining six of which are vegetables. You are allowed to choose any four dishes, and at least one of them has to be a vegetable.

a) How many ways can you do this?

There are $\binom{10}{4} = 210$ ways to choose four dishes, and $\binom{4}{4} = 1$ of them has no vegetables. So the answer is 210-1 = 209.

b) Suppose that you really hate the combination of meatloaf and okra (even though you are happy to eat either one individually). How many ways can you make your choice without taking both meatloaf and okra?

Of the 209 solutions to part (a), $\binom{8}{2} = 28$ of them contain both meatloaf and okra. So the answer is 209-28=181.