Math 456/556: Networks and Combinatorics

Solutions to HW #4

Book problems:

2. The number of integers between 1 and 10,000 not divisible by 4, 6, 7, or 10 is

$$\begin{aligned} 10000 &- \lfloor\frac{10000}{4}\rfloor - \lfloor\frac{10000}{6}\rfloor - \lfloor\frac{10000}{7}\rfloor - \lfloor\frac{10000}{10}\rfloor \\ &+ \lfloor\frac{10000}{12}\rfloor + \lfloor\frac{10000}{28}\rfloor + \lfloor\frac{10000}{20}\rfloor + \lfloor\frac{10000}{42}\rfloor + \lfloor\frac{10000}{30}\rfloor + \lfloor\frac{10000}{72}\rfloor \\ &- \lfloor\frac{10000}{210}\rfloor - \lfloor\frac{10000}{140}\rfloor - \lfloor\frac{10000}{60}\rfloor - \lfloor\frac{10000}{84}\rfloor \\ &+ \lfloor\frac{10000}{420}\rfloor. \end{aligned}$$

5. Let $S := \{\infty \cdot a, 4 \cdot b, 5 \cdot c, 7 \cdot d\}$ and $S^* := \{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$. We wish to count the number of 10-combinations of S, which is the same as the number of 10-combinations of S^* that do not contain 5 copies of b, 6 copies of c, or 8 copies of d. By the inclusion-exclusion principle, we get

$$\binom{10+3}{3} - \binom{5+3}{3} - \binom{4+3}{3} - \binom{2+3}{3} = 185$$

(all double intersections are empty).

9. Let $y_1 = x_1 - 1$, $y_2 = x_2$, $y_3 = x_3 - 4$, $y_4 = x_4 - 2$. Then we are looking for non-negative integer solutions to the equation $y_1 + y_2 + y_3 + y_4 = 13$ with $y_1 \leq 5$, $y_2 \leq 7$, $y_3 \leq 4$, and $y_4 \leq 4$. Equivalently, we want to count 13-combinations of the multiset $\{5 \cdot a_1, 7 \cdot a_2, 4 \cdot a_3, 4 \cdot a_4\}$.

Though we do not have to do this, our task will be a little easier if we employ a trick that I mentioned in class: 13-combinations of this multiset are in bijection with 7-combinations (that is, we can think about what gets left out rather than what gets put in). Thus, we may replace the number 13 with 7.

Now let's use the inclusion-exclusion principle, following the strategy of problem 5. We get

$$\binom{7+3}{3} - \binom{1+3}{3} - \binom{2+3}{3} - \binom{2+3}{3} = 96.$$

(Without employing the trick we would have obtained the same number, but with many more terms in the sum.)

13. The number of permutations of $\{1, \ldots, 9\}$ with at least one odd integer fixed is

$$\binom{5}{1} \cdot 8! - \binom{5}{2} \cdot 7! + \binom{5}{3} \cdot 6! - \binom{5}{4} \cdot 5! + \binom{5}{5} \cdot 4! = 157{,}824.$$

15. See solutions in the back of the book.

21. To show that D_n is even if and only if n is odd, we will proceed by induction on n. It is clearly true when $n = 1$.

Recall that $D_n = nD_{n-1} + (-1)^n$ (Equation 6.8). If n is odd, then $n-1$ is even, so our inductive hypothesis says that D_{n-1} is odd. Then D_n is an odd times an odd plus an odd, which is even. On the other hand, if n is even, then our inductive hypothesis says that D_{n-1} is even. Then D_n is an even times an even plus an odd, which is odd. Either way, we win.

29. A subway has 6 stops and 10 passengers. Each passenger gets off at one of the stops, and at least one passenger gets off at each stop. In how many ways can this happen?

If we didn't have the restriction that at least one passenger gets off at each stop, the answer would be 6^{10} (each passenger must choose his or her stop). With the restriction, we have to use the inclusion-exclusion principle. We get

$$6^{10} - \binom{6}{1}5^{10} + \binom{6}{2}4^{10} - \binom{6}{3}3^{10} + \binom{6}{4}2^{10} - \binom{6}{5}.$$

32. Let ϕ be the Euler function. By the inclusion-exclusion principle, we have

$$\phi(n) = n - \sum_p \frac{n}{p} + \sum_{p_1,p_2} \frac{n}{p_1 p_2} - \ldots = n \prod_p \left(1 - \frac{1}{p}\right).$$

Additional problems:

4.1 How many integers between 1 and $729 = 3^6$ are neither squares nor cubes?

There are 27 squares, 9 cubes, and 3 sixth powers (i.e. numbers that are both squares and cubes). So the answer is $729 - 27 - 9 + 3 = 696$.

4.2 Fix a positive integer n.

(a) Given a random permutation of the set $\{1, \ldots, n\}$, show that the probability that exactly one element is fixed is very close to the probability that no elements are fixed. Which is more likely when $n = 100$?

The probability that no elements are fixed is $\frac{D_n}{n!} \approx \frac{1}{e}$, and the probability that exactly one element is fixed is $\frac{nD_{n-1}}{n!} = \frac{D_{n-1}}{(n-1)!} \approx \frac{1}{e}$. More precisely, we have $D_n = nD_{n-1} + (-1)^n$, so $\frac{D_n}{n!} = \frac{nD_{n-1}}{n!} + \frac{(-1)^n}{n!}$. So the probability that no elements are fixed is very slightly larger if n is even and very slightly smaller if n is odd.

4.3 Compute the number of integer solutions to the equation $x_1 + x_2 + x_3 = 17$ with $1 \leq x_1 \leq 6$, $0 \leq x_2 \leq 5$, and $2 \leq x_3 \leq 10$.

First let $y_1 = x_1 - 1$, $y_2 = x_2$, and $y_3 = x_3 - 2$, so that we are looking for solutions to the equation $y_1 + y_2 + y_3 = 14$ with $0 \leq y_1 \leq 5$, $0 \leq y_2 \leq 5$, and $0 \leq y_3 \leq 8$. Equivalently, we want to count 14-combinations of the multiset $\{5 \cdot a_1, 5 \cdot a_2, 8 \cdot a_3\}$. Using inclusion-exclusion, the answer is

$$\binom{14+2}{2} - \binom{8+2}{2} - \binom{8+2}{2} - \binom{5+2}{2} + \binom{2+2}{2} = 15.$$

Alternatively, we could observe that this is the same as counting 4-combinations of the same multiset (what gets left out rather than what goes in). Then we don't even need to use inclusion-exclusion; the answer is $\binom{4+2}{2} = 15$.