Math 456/556: Networks and Combinatorics

Solutions to HW #2

Book problems:

Chapter 2

21. Since ADDRESSES has 9 letters, the number of 8-permutations is equal to the number of 9-permutations, which is $\frac{9!}{1!2!1!2!3!} = 15{,}120$.

28. (a) To walk to work, the secretary needs to make 17 "moves", and he needs to decide which 8 of those moves are north. Thus he has $\binom{17}{8} = 24{,}310$ choices. (b) Suppose that one of the blocks is impassable. The number of routes that include this block is equal to the number of routes from home to the beginning of the block times the number of routes from the end of the block to work: in our case, that's

$$\binom{7}{3} * \binom{9}{5} = 4{,}410.$$

Thus the number of passable routes is 24,310 - 4,410 = 19,900.

38. Let $y_1 = x_1 - 2$, $y_2 = x_2$, $y_3 = x_3 + 5$, and $y_4 = x_4 - 8$. Then we want to count non-negative integer solutions to the equation

$$(y_1 + 2) + y_2 + (y_3 - 5) + (y_4 + 8) = 30,$$

which is the same as

$$y_1 + y_2 + y_3 + y_4 = 25.$$

By Theorem 2.5.1, the answer is $\binom{25+4-1}{4-1} = \binom{28}{3} = 3{,}276$.

41. We have three choices for which child gets the orange. Then we give each of the two remaining children an apple. There are $\binom{10+3-1}{10} = 66$ ways to distribute the remaining 10 apples. Thus the answer is $3 * 66 = 198$.

43. r-combinations of this multiset are either r-combinations of the multiset with a_1 removed or $(r-1)$-combinations of that multiset, along with a_1. Thus we get

$$\binom{r + (k-1) - 1}{(k-1) - 1} + \binom{(r-1) + (k-1) - 1}{(k-1) - 1} = \binom{r+k-2}{k-2} + \binom{r+k-3}{k-2}.$$

56. The number of possible flushes is $4 * \binom{13}{5}$, where the 4 is the choice of suit, and the $\binom{13}{5}$ is the choice of ranks. Dividing by the total number of poker hands, the probability of drawing a flush is $4 * \binom{13}{5}/\binom{52}{5} \approx .002$.

57. There are 13 possible ranks for the pair and $\binom{12}{3}$ choices for the remaining 3 ranks. We also need to choose the two suits for the pair, and one suit for each of the remaining four ranks. Thus there are $13 * \binom{12}{3} * \binom{4}{2} * 4^3$ possible hands with exactly one pair. The probability of drawing such a hand is given by dividing by $\binom{52}{5}$; it comes out to about 0.423.

Chapter 3

9. The number of possible groups of people (counting the empty group) is $2^{10} = 1024$. The possible ages of a group range between 10 and 600, which means that there are more groups than ages. Thus there exist two different groups with the same total age. If these two groups have anyone in common, just throw those people out of both groups. Then we will have two disjoint groups with the same total age.

If we try to replace 10 by 9, we get $2^9 = 512$, which smaller than the 531 possible ages, so the above argument does not go through verbatim. But we can make some minor fixes. First, we can assume that each person has a different age. (If not, then there would be two one-person-groups of the same age, and we'd be done.) With this restriction, the largest possible age of a group is

$$60 + 59 + 58 + 57 + 56 + 55 + 54 + 53 + 52 = 504,$$

and the smallest is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45,$$

so the number of possible ages is $504 - 44 = 460$, which is less than 512. So indeed, the statement would still be true when 10 is replaced by 9. I'm not sure whether or not it is true if 10 is replaced by 8.

14. To make sure to get 12 of one kind, we need to pick for $4 * 11 + 1 = 45$ minutes.

18. We can partition the square into four smaller squares, each with diameter $\sqrt{2}$.

20. We want to show that $K_{17} \to K_3, K_3, K_3$. Tri-color the edges of K_{17}, and pick a vertex. By the strong pigeonhole principle, there is a set of 6 edges coming out of that vertex that all have the same color. Without loss of generality, assume that they are green. Consider the K_6 spanned by the other endpoints of these 6 edges. If any of those edges are green, then we have a green triangle. If not, then they are all blue and red. Since we know that every blue-and-red K_6 has a red triangle or a blue triangle, we're done.

Additional problems:

2.1 Compute the number of permutations of the letters BOOGIEWOOGIE.

The answer is $\frac{12!}{4!2!2!2!} = 2{,}494{,}800$.

2.2 Show that any 2-coloring of the edges of a K_8 contains at least 8 monochromatic triangles. Give an example of a specific 2-coloring with exactly 8 monochromatic triangles.

By reasoning that was given in class in the case of K_6, the number of monochromatic triangles in a given 2-coloring of K_8 is

$$\binom{8}{3} - \frac{1}{2}\sum_{i=1}^{8} r_i(7 - r_i),$$

where r_i is the number of red edges coming into the i^{th} vertex. The largest that $r_i(7 - r_i)$ can be is $3 \cdot 4 = 12$, so the entire expression is equal to at least $\binom{8}{3} - \frac{1}{2} \cdot 8 \cdot 12 = 8$. An example where this bound is achieved is obtained by putting 8 points around a circle, connecting each point to its two neighbors by red edges, and also connecting each point to its antipode by a red edge. All the other edges can be colored blue.

2.3 Suppose that I gave an quiz out of 20 points to a class of 10 students. Show that I can choose two disjoint groups of students such that each group has the same total score.

This is similar to problem 9 from Chapter 3. The number of possible groups is $2^{10} = 1024$, and the number of possible total scores of a group is 201. Since 201 is less than 1024, the pigeonhole principle applies to give us two different groups of students with the same total score. Getting rid of the overlap, we obtain two disjoint groups of students with the same total score.

Note that it's possible for one of these groups to be empty, if the other group consists entirely of people who received a score of zero. That's fine! With a little bit of extra work, you could also show that exist two disjoint, nonempty groups of students with the same total score, but you do not have to do this.

2.4 How many non-negative integral solutions of $x_1 + x_2 + x_3 \leq 17$ are there with $x_1 \geq 2$?

This is the same as the number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 = 17$ with $x_1 \geq 2$, which is the same as the number of non-negative integral solutions of $y_1 + x_2 + x_3 + x_4 = 15$ (here we are letting $y_1 = x_2 - 2$). The answer is $\binom{15+4-1}{4-1} = \binom{18}{3} = 816$.