# Isomorphisms of Discriminant Algebras

## Owen Biesel and Alberto Gioia

### September 24, 2018

#### Abstract

For each natural number n, we define a category whose objects are discriminant algebras in rank n, i.e. functorial means of attaching to each rank-n algebra a quadratic algebra with the same discriminant. We show that the discriminant algebras defined in [2], [6], and [10] are all isomorphic in this category, and prove furthermore that in ranks  $n \leq 3$  discriminant algebras are unique up to unique isomorphism.

## Contents

1	Introduction and definitions	2
2	Properties of discriminant algebras 2.1 Existence	
	<ul><li>2.2 Description when 2 is a unit</li></ul>	
3	Uniqueness in low rank	6
	3.1 Proof for $n \le 1 \dots \dots \dots \dots \dots$	7
	3.2 A helpful lemma	7
	3.3 Proof for $n = 2 \dots \dots \dots \dots \dots$	Ö
	3.4 Proof for $n = 3 \dots \dots \dots \dots \dots \dots \dots \dots \dots$	11
4	Isomorphism with Rost's discriminant algebra	12
5	Isomorphism with Loos's discriminant algebra	15
$\mathbf{A}$	Miscellaneous Calculations	<b>2</b> 4
	A.1 Traces and norms in rank 3	24
	A.2 The existence of $\Sigma_F$	29
	A.3 A determinant identity	30

#### 1 Introduction and definitions

A discriminant algebra in rank n is an isomorphism- and base-change-preserving way of associating to each ring R and rank-n algebra A a rank-A algebra with the same discriminant bilinear form as A. Discriminant algebras have been constructed in rank 3 by Markus Rost [10], in arbitrary rank by Ottmar Loos in [6], and in rank  $\geq 2$  by the authors in [2]. The goal of this paper is to show that these three constructions are in some sense isomorphic; in this section we define the category of discriminant algebras in order to make the notion of isomorphism precise.

**Definition 1.1.** Form the following categories,  $\mathbf{Aff}$ ,  $\mathbf{Disc}$ , and  $\mathbf{Alg}_n$  for each natural number n, as follows:

- **Aff** is the category of affine schemes, the opposite of the category of commutative rings.
- **Disc** is the category of discriminant data. It has as objects triples (R, L, d), where R is a commutative ring, L is a locally free rank-1 R-module, and  $d: L^{\otimes 2} = L \otimes_R L \to R$  is an R-module homomorphism. A morphism from (R', L', d') to (R, L, d) consists of a ring homomorphism  $R \to R'$  and an R'-module isomorphism  $R' \otimes_R L \xrightarrow{\sim} L'$  such that the following square commutes:

$$R' \otimes_R (L^{\otimes 2}) \xrightarrow{\sim} L'^{\otimes_{R'} 2}$$

$$\downarrow^{\operatorname{id}_{R'} \otimes d} \qquad \qquad \downarrow^{d'}$$

$$R' \otimes_R R \xrightarrow{\sim} R'.$$

The forgetful functor  $\mathbf{Disc} \to \mathbf{Aff}: (R, L, d) \mapsto R$  makes  $\mathbf{Disc}$  a category fibered in groupoids (CFG) over  $\mathbf{Aff}$ .

•  $\mathbf{Alg}_n$  is the category of rank-n algebras. Its objects are pairs (R,A) with A an R-algebra that is locally free of rank n as an R-module. A morphism  $(R',A') \to (R,A)$  consists of a ring homomorphism  $R \to R'$  and an R'-algebra isomorphism  $R' \otimes_R A \xrightarrow{\sim} A'$ . The forgetful functor  $\mathbf{Alg}_n \to \mathbf{Aff}: (R,A) \mapsto R$  makes  $\mathbf{Alg}_n$  a CFG over  $\mathbf{Aff}$ .

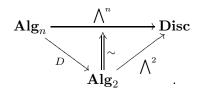
 $<sup>^{1}</sup>$ i.e. projective of constant rank n as an R-module

There is a map of CFGs  $\bigwedge^n$ :  $\mathbf{Alg}_n \to \mathbf{Disc}$  over  $\mathbf{Aff}$  sending a rank-n algebra to its discriminant data:  $(R, A) \mapsto (R, \bigwedge^n A, \delta_A)$ , where

$$\delta_A : \bigwedge^n A \otimes \bigwedge^n A \to R$$
  
:  $(a_1 \wedge \dots \wedge a_n) \otimes (b_1 \wedge \dots \wedge b_n) \mapsto \det(\operatorname{Tr}(a_i b_j))_{ij}$ 

is the discriminant bilinear form of A.

**Definition 1.2.** A discriminant algebra in rank n is a factorization of  $\bigwedge^n$ :  $\mathbf{Alg}_n \to \mathbf{Disc}$  via  $\mathbf{Alg}_2$ . More specifically, it consists of a morphism D of CFGs  $\mathbf{Alg}_n \to \mathbf{Alg}_2$  over  $\mathbf{Aff}$  together with a 2-isomorphism filling the following triangle of morphisms of CFGs:



In other words, to exhibit a discriminant algebra we must associate to every rank-n R-algebra A a rank-2 R-algebra D(R,A) together with functorial base-change isomorphisms  $R' \otimes_R D(R,A) \cong D(R',R' \otimes_R A)$ , and we must exhibit a natural isomorphism  $\bigwedge^2 D(R,A) \cong \bigwedge^n A$  that identifies the discriminant forms of A and D(R,A).

The category of discriminant algebras is then just the category of such factorizations of  $\bigwedge^n$  via  $\bigwedge^2$ . In particular, an isomorphism of discriminant algebras is a natural isomorphism of functors  $\mathbf{Alg}_n \to \mathbf{Alg}_2$  over  $\mathbf{Aff}$  making the resulting triangle of natural isomorphisms commute. In other words, an isomorphism between two discriminant algebras D and D' is an isomorphism  $D(R,A) \cong D'(R,A)$  that respects base change, and that also induces the composite isomorphism  $\bigwedge^2 D(R,A) \cong \bigwedge^n A \cong \bigwedge^2 D'(R,A)$  coming from the data of the two discriminant algebras.

## 2 Properties of discriminant algebras

#### 2.1 Existence

The first important fact about discriminant algebras is that they exist. We review the construction of one such morphism from [2]. Let R be a ring and A a rank-n R-algebra; the Ferrand homomorphism  $\Phi_{A/R}: (A^{\otimes_R n})^{S_n} \to R$ 

is the unique R-algebra homorphism such that for each R-algebra R' and element  $a \in A' := R' \otimes_R A$ , the composite homomorphism

$$(A'^{\otimes_{R'}n})^{S_n} \cong R' \otimes_R (A^{\otimes_R n})^{S_n} \to R' \otimes_R R \cong R'$$

sends  $a \otimes \cdots \otimes a$  to  $\operatorname{Nm}_{A'/R'}(a)$ . (This is the algebra homomorphism representing the multiplicative homogeneous degree-n polynomial law  $\operatorname{Nm}: A \to R$  in the sense of [9].) Then define

$$\Delta(R,A) := (A^{\otimes n})^{A_n} \bigotimes_{(A^{\otimes n})^{S_n}} R,$$

where the maps defining the tensor product are the inclusion  $(A^{\otimes n})^{S_n} \hookrightarrow (A^{\otimes n})^{A_n}$  and the Ferrand homomorphism  $(A^{\otimes n})^{S_n} \to R$ . By [2, Theorem 4.1], this is a quadratic R-algebra, and we have the following theorem:

**Theorem 2.1.** Let  $n \geq 2$  be a natural number. For each ring R and rank-n R-algebra A, let  $\Delta(R,A)$  be the rank-2 R-algebra defined above. This assignment constitutes a discriminant algebra  $\Delta: \mathbf{Alg}_n \to \mathbf{Alg}_2$ .

Proof. Theorem 4.1 of [2] tells us that  $\Delta(R,A)$  has rank 2 and is equipped with a discriminant-preserving isomorphism between its top exterior power and that of A. Theorem 6.1 there gives us a canonical isomorphism  $R' \otimes_R \Delta(R,A) \cong \Delta(R',R' \otimes_R A)$  for each R-algebra R'. Since this isomorphism is induced by the canonical isomorphism  $R' \otimes_R (A^{\otimes_R n}) \cong (R' \otimes_R A)^{\otimes_{R'} n}$ , it is easy to check that it is functorial with respect to multiple changes of base, and preserves the discriminant-preserving isomorphism between its top exterior power and A's.

Given any R-algebra A, it will be helpful to refer to the following elements of  $A^{\otimes n}$  and  $(A^{\otimes n})^{S_n}$ :

- For each  $a \in A$  and  $i \in \{1, ..., n\}$ , the element  $a^{(i)} \in A^{\otimes n}$  is the pure tensor  $1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1$ , with the a in the ith position.
- For each  $a \in A$  and  $k \in \{0, ..., n\}$ , the element  $e_k(a) \in (A^{\otimes n})^{\mathbb{S}_n}$  is the sum

$$e_k(a) := \sum_{\substack{i_1 < \dots < i_k \\ \in \{1,\dots,n\}}} \prod_{j=1}^k a^{(i_j)}.$$

If A is a rank-n R-algebra so that we have the Ferrand homomorphism  $\Phi_{A/R}: (A^{\otimes n})^{S_n} \to R$ , then the image of  $e_k(a)$  in R is the coefficient of  $(-1)^k \lambda^{n-k}$  in  $\operatorname{Nm}_{A[\lambda]/R[\lambda]}(\lambda - a)$ , the characteristic polynomial of a. This coefficient is denoted  $s_k(a)$ . In particular,  $s_n(a) = \operatorname{Nm}_{A/R}(a)$  and  $s_1(a) = \operatorname{Tr}_{A/R}(a)$ , and  $s_2: A \to R$  is called the quadratic trace and is used in Loos's construction of a discriminant algebra. See [2, Section 3] for more on  $\Phi_{A/R}$ .

It will also be helpful to note that  $\Delta(R, A)$  is generated as an R-module by  $\{1\} \cup \{\gamma(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A\}$ , where  $\gamma(a_1, \ldots, a_n)$  is the image under the map  $(A^{\otimes n})^{A_n} \to \Delta(A, R)$  of the element

$$\sum_{\sigma \in \mathcal{A}_n} \prod_{i=1}^n a_i^{(\sigma(i))}.$$

If two of the  $a_i$  are equal, or if we add  $\gamma(a_1, a_2, \ldots, a_n)$  to  $\gamma(a_2, a_1, \ldots, a_n)$ , then the result is the image of an  $S_n$ -invariant element of  $(A^{\otimes n})^{A_n}$  and is set equal to its image under  $\Phi_{A/R} \colon (A^{\otimes n})^{S_n} \to R$ . The discriminant-identifying isomorphism  $\bigwedge^2 \Delta(R, A) \to \bigwedge^n A$  sends elements of the form  $1 \wedge \gamma(a_1, \ldots, a_n)$  to  $a_1 \wedge \cdots \wedge a_n$ .

Note also that  $\gamma(a_1, a_2, \dots, a_n) - \gamma(a_2, a_1, \dots, a_n)$  is the image in  $\Delta(R, A)$  of the  $A_n$ -invariant tensor

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_i^{(\sigma(i))} \in (A^{\otimes n})^{A_n}.$$

#### 2.2 Description when 2 is a unit

We also know the following about every discriminant algebra operation in the case that 2 is invertible:

**Theorem 2.2.** Let  $D: \mathbf{Alg}_n \to \mathbf{Alg}_2$  be a discriminant algebra, and let R be a ring in which 2 is a unit. Then for each rank-n algebra A, there is a canonical isomorphism  $D(R,A) \cong R \oplus \bigwedge^n A$ , where the multiplication on the latter module has identity (1,0) and the product of  $\xi, \psi \in \bigwedge^n A$  is  $\delta_A(\xi,\psi)/4 \in R$ .

Proof. Since D(R, A) is a quadratic algebra over a ring in which 2 is a unit, by [2, Proposition 6.2] we have a canonical isomorphism  $D(R, A) \cong R \oplus \bigwedge^2 D(R, A)$  with unit (1,0) and multiplication  $\xi \cdot \psi = \delta_D(\xi, \psi)/4$ . Using the discriminant-identifying isomorphism  $\bigwedge^2 D \cong \bigwedge^n A$ , we obtain the desired description of D(R, A).

### 2.3 Description in the étale case

We also know what a discriminant algebra does to a connected ring R and finite étale algebra A.

**Theorem 2.3.** Let  $D: \mathbf{Alg}_n \to \mathbf{Alg}_2$  be a discriminant algebra with  $n \geq 2$ . Let R be a ring equipped with a geometric point  $R \to K$  and corresponding fundamental group  $\pi(R,K)$ , and let A be a rank-n étale R-algebra, with corresponding  $\pi(R,K)$ -set X. Then D(R,A) is also étale, and corresponds to the 2-element  $\pi(R,K)$ -set  $\mathrm{Or}(X) := \mathrm{Bij}(\{1,\ldots,n\},X)/\mathrm{A}_n$ .

*Proof.* A rank-n R-algebra A is étale if and only if its discriminant form  $\delta_A : \bigwedge^n A \times \bigwedge^n A \to R$  is nondegenerate in the sense of inducing an isomorphism  $\bigwedge^n A \to \operatorname{Hom}(\bigwedge^n A, R)$ . Since A and D(R, A) have isomorphic discriminant forms, if the former is étale then so is the latter. Thus D restricts to a functor  $\mathbf{\acute{E}t}_n \to \mathbf{\acute{E}t}_2$ , where  $\mathbf{\acute{E}t}_n$  is the full subcategory of  $\mathbf{Alg}_n$  consisting of pairs (R, A) with A étale over R.

Now  $\mathbf{E}\mathbf{t}_n$  is equivalent to the fibered category  $\mathbf{B}\mathbf{S}_n$  of  $\mathbf{S}_n$ -torsors, and we claim that there are only two morphisms of fibered categories  $\mathbf{B}\mathbf{S}_n \to \mathbf{B}\mathbf{S}_2$  up to 2-isomorphism. Indeed, such a morphism is determined up to 2-isomorphism by an object of the fiber category  $(\mathbf{B}\mathbf{S}_2)_{\mathbb{Z}}$  equipped with an action by  $\mathbf{S}_n$ ; the only such object is  $\mathbb{Z}^2$ , and the only such actions are the trivial action and the one coming from the sign homomorphism  $\mathbf{S}_n \to \mathbf{S}_2$  composed with the action of  $\mathbf{S}_2$  on the factors of  $\mathbb{Z}^2$ . The trivial action corresponds to the assignment  $(R,A) \mapsto (R,R^2)$ , which does not admit a discriminant-identifying isomorphism of top exterior powers. Thus every discriminant algebra operation restricts to the same homomorphism  $\mathbf{E}\mathbf{t}_n \to \mathbf{E}\mathbf{t}_2$ , which by [2, Theorem 4] is the one sending a rank-n algebra corresponding to X to the rank-2 algebra corresponding to  $\mathbf{Or}(X)$ .

## 3 Uniqueness in low rank

In this section we will prove the following uniqueness theorem:

**Theorem 3.1.** For ranks  $n \leq 3$ , there is a unique discriminant algebra up to unique isomorphism.

We already have existence from Theorem 2.1, so in the remainder of this section we will show that if D and D' are two discriminant algebras in rank  $n \leq 3$ , there is a unique isomorphism  $D \cong D'$ .

#### 3.1 Proof for n < 1

In ranks n=0 and n=1, for each ring R and rank-n R-algebra A there is a unique R-algebra isomorphism  $A\cong R^n$ , i.e. a unique morphism  $(R,A)\to (\mathbb{Z},\mathbb{Z}^n)$  in  $\mathbf{Alg}_n$ . Thus the fiber functor  $\mathbf{Alg}_n\to \mathbf{Aff}$  is an equivalence, and the category of discriminant algebras  $\mathbf{Alg}_n\to \mathbf{Alg}_2$  is equivalent to the category of splittings  $\mathbf{Aff}\to \mathbf{Alg}_2$  sending each ring R to a quadratic R-algebra with discriminant form isomorphic to multiplication  $R^{\otimes 2}\stackrel{\sim}{\longrightarrow} R$ . Such a splitting is determined up to unique isomorphism by where it sends  $\mathbb{Z}$ , and the only étale quadratic  $\mathbb{Z}$ -algebra up to isomorphism is  $\mathbb{Z}^2$ , so  $(R,A)\mapsto R^2$  is the unique discriminant algebra  $\mathbf{Alg}_n\to \mathbf{Alg}_2$  up to unique isomorphism.

### 3.2 A helpful lemma

In higher ranks, not every algebra can be uniquely expressed as the base change of a universal algebra defined over  $\mathbb{Z}$ . However, we can still show that over base rings similar to  $\mathbb{Z}$ , any two discriminant algebras must agree:

**Lemma 3.2.** Let R be a ring in which 2 is a unit or prime non-zerodivisor, and let A and B be quadratic R-algebras with an isomorphism  $\bigwedge^2 A \cong \bigwedge^2 B$  identifying their discriminant bilinear forms  $\delta_A$  and  $\delta_B$ . Then there is a unique R-algebra isomorphism  $A \cong B$  inducing the given isomorphism  $\bigwedge^2 A \cong \bigwedge^2 B$ .

*Proof.* Work locally, so that we can write  $A \cong R[x]/(x^2 - sx + t)$  and  $B \cong R[y]/(y^2 - uy + v)$ , with generators chosen so that the isomorphism  $\bigwedge^2 A \cong \bigwedge^2 B$  matches  $1 \wedge x$  with  $1 \wedge y$ . (The property of 2 being a unit or prime non-zerodivisor is preserved under localization.) Then we must show that there is a unique R-algebra isomorphism  $A \to B$  of the form  $x \mapsto y + c$  for some  $c \in R$ . If  $x \mapsto y + c$  is to be an isomorphism, we must have  $\operatorname{Tr}_A(x) = \operatorname{Tr}_B(y+c)$ , or s = u + 2c, so since 2 is a non-zerodivisor there is at most one such c.

For existence of a suitable c, note that the identification of discriminants gives us the equation  $s^2 - 4t = u^2 - 4v$  in R. Let k = s - u, so that

$$s^{2} - 4t = (s - k)^{2} - 4v$$
$$= s^{2} - 2sk + k^{2} - 4v$$
$$k^{2} = -4t + 2sk + 4v.$$

Then  $k^2$  is divisible by 2, so k must be as well because 2 is either prime or a unit. Then letting c be such that k=2c, we have obtained the desired c for which s=u+2c.

Now we show that  $x\mapsto y+c$  is in fact an R-algebra isomorphism. First, it is well-defined, since

$$(y+c)^{2} - s(y+c) + t = y^{2} + 2cy + c^{2} - sy - sc + t$$
$$= (uy - v) + 2cy + c^{2} - sy - sc + t$$
$$= (u - s + 2c)y + (c^{2} - sc + t - v).$$

Now s = u + 2c by construction, so the coefficient of y vanishes. But from the equation identifying discriminants, we also obtain

$$s^{2} - 4t = (s - 2c)^{2} - 4v$$
$$= s^{2} - 4sc + 4c^{2} - 4v$$
$$0 = 4(t - sc + c^{2} - v).$$

So since 2 is a non-zerodivisor, we have  $c^2 - sc + t - v = 0$  as well. Therefore the map  $A \to B \colon x \mapsto y + c$  is well-defined, and it is bijective since the R-module basis  $\{1, x\}$  for A is carried to an R-module basis  $\{1, y + c\}$  for B.

It is easy to find counterexamples to Lemma 3.2 if we drop the hypotheses on R—for example,  $\mathbb{F}_4$  and  $\mathbb{F}_2^2$  have isomorphic discriminants over  $\mathbb{F}_2$ —but note that it is not enough to assume that 2 is a non-zerodivisor:

**Example 3.3.** Let  $R = \mathbb{Z}[\sqrt{5}]$ , and define two quadratic R-algebras

$$A := R[x]/(x^2 - x - 1), \qquad B := R[y]/(y^2 - \sqrt{5}y).$$

The isomorphism  $\bigwedge^2 A \cong \bigwedge^2 B$  identifying basis elements  $1 \wedge x$  and  $1 \wedge y$  identifies the two discriminants:

$$\delta_A(1 \wedge x) = (-1)^2 - 4(1)(-1) = 5 = (-\sqrt{5})^2 - 4(1)(0) = \delta_B(1 \wedge y).$$

However, this isomorphism  $\bigwedge^2 A \cong \bigwedge^2 B$  is not induced by an algebra isomorphism  $A \cong B$ , because such an isomorphism would have to send  $x \mapsto y + c$  for some  $c \in R$ , and then we would have

$$0 = (y+c)^{2} - (y+c) - 1$$
  
=  $y^{2} + (2c-1)y + (c^{2} - c - 1)$   
=  $(2c - \sqrt{5} - 1)y + (c^{2} - c - 1)$ ,

but  $1 + \sqrt{5}$  is not divisible by 2 in R, so no such c can exist.

An immediate consequence of Lemma 3.2 is the main result of this subsection:

Corollary 3.4. Let n be a natural number, and let  $D, D' \colon \mathbf{Alg}_n \to \mathbf{Alg}_2$  be two discriminant algebras, and let  $(R, A) \in \mathbf{Alg}_n$  be such that 2 is a unit or prime non-zerodivisor in R. Then there is a unique isomorphism  $D(R, A) \cong D'(R, A)$  descending to the composite isomorphism  $\bigwedge^2 D(R, A) \cong \bigwedge^n A \cong \bigwedge^2 D'(R, A)$ .

#### **3.3 Proof for** n = 2

The proofs for n = 2 and n = 3 will both proceed as follows: first we identify a suitably universal case in which Corollary 3.4 applies, and then we use base changes to get a choice-free natural isomorphism  $D \cong D'$ .

For the case n = 2 we are showing that every endomorphism of  $\mathbf{Alg}_2$  over  $\mathbf{Disc}$ , as a morphism of CFGs over  $\mathbf{Aff}$ , is uniquely isomorphic to every other (e.g. the identity).

Indeed, let  $D: \mathbf{Alg}_2 \to \mathbf{Alg}_2$  be a discriminant algebra. Consider the quadratic algebra  $(R_1, A_1) \in \mathbf{Alg}_2$  given by

$$R_1 = \mathbb{Z}[s_1, t_1], \quad A_1 = R_1[x_1]/(x_1^2 - s_1x_1 + t_1).$$

Since  $2 \in R_1$  is a prime non-zerodivisor, Corollary 3.4 applies and we find that there is a unique isomorphism  $D(R_1, A_1) \cong A_1$  agreeing with the isomorphism we already have on their exterior powers.

Furthermore, morphisms to  $(R_1, A_1)$  in  $\mathbf{Alg}_2$  have a nice interpretation: if R is any ring and A is a quadratic R-algebra, then a map  $(R, A) \to (R_1, A_1)$  corresponds to a ring homomorphism  $R_1 \to R$  (i.e. a pair of elements  $s, t \in R$ ) and an isomorphism  $R \otimes_{R_1} A_1 \cong A$  (i.e. an R-algebra generator  $x \in A$  such that  $x^2 - sx + t = 0$ ). So each choice of R-algebra generator  $x \in A$  corresponds to a morphism  $f_x \colon (R, A) \to (R_1, A_1)$ .

Now given a choice of R-algebra generator x of A, we obtain an isomorphism  $D(R,A) \cong A$  via base change along  $f_x$  from the canonical isomorphism  $D(R_1,A_1) \cong A_1$ . However, this isomorphism  $D(R,A) \cong A$  may in principle depend on our choice of generator  $x \in A$ .

To show that there is no such dependence, we introduce a ring and algebra just slightly bigger than  $(R_1, A_1)$ . Define

$$R_2 = R_1[r_0, u_0, u_0^{-1}], \quad A_2 = R_2[x_1]/(x_1^2 - s_1x_1 + t_1).$$

We distinguish two separate algebra generators for  $A_2$  over  $R_2$ :  $x_1$  by itself does the job, and so does  $x_2 := u_0 x_1 + r_0$ , since  $u_0$  is a unit in  $R_2$ . We thus

obtain two morphisms  $(R_2, A_2) \rightrightarrows (R_1, A_1)$  corresponding to  $x_1$  and  $x_2$ ; call these morphisms  $\pi_1$  and  $\pi_2$ .

**Lemma 3.5.** The two morphisms  $\pi_1, \pi_2 \colon (R_2, A_2) \rightrightarrows (R_1, A_1)$  express  $(R_2, A_2)$  as the categorical product of  $(R_1, A_1)$  with itself in  $\mathbf{Alg}_2$ . In other words, given any two morphisms  $f, g \colon (R, A) \rightrightarrows (R_1, A_1)$ , they factor uniquely as a single map  $(f, g) \colon (R, A) \to (R_2, A_2)$  followed by  $\pi_1, \pi_2 \colon (R_2, A_2) \rightrightarrows (R_1, A_1)$ .

Proof. We show that maps  $(R, A) \to (R_2, A_2)$  correspond to pairs of single generators for A as an R-algebra. Indeed, given any two generators x, y for A over R, they must each be a free R-module generator for the free rank-1 module A/R, so there is a unique unit  $u \in R^{\times}$  for which  $y \equiv ux$  modulo R. Then there is also a unique element  $r \in R$  such that y = ux + r. So the data of two single generators x and y for A is equivalent to the data of a single generator x along with an arbitrary element x and unit x. But this is exactly the data captured by a morphism  $(R, A) \to (R_2, A_2)$ .

Next we note that since  $R_2$  is the localization of a polynomial ring over  $\mathbb{Z}$ , Corollary 3.4 applies, and we obtain a unique isomorphism  $D(R_2, A_2) \cong A_2$  agreeing with the identification of their determinant bundles. In particular, the two base changes of the canonical isomorphism  $D(R_1, A_1) \cong A_1$  via the underlying ring homomorphisms of  $\pi_1$  and  $\pi_2$  give the same isomorphism  $D(R_2, A_2) \cong A_2$ .

Then given any two morphisms  $f,g:(R,A) \rightrightarrows (R_1,A_1)$  in  $\mathbf{Alg}_2$ , we may use each of f and g to define an isomorphism  $D(R,A) \cong A$ , by base changing the canonical isomorphism  $D(R_1,A_1) \cong A_1$  along the underlying ring homomorphisms of f and g. By the lemma, these base changes can also be accomplished by first base changing along  $\pi_1$  or  $\pi_2$  to an isomorphism  $D(R_2,A_2) \cong A_2$ , and then base changing along  $(f,g):(R,A) \to (R_2,A_2)$ . But then the resulting two isomorphisms  $D(R,A) \cong A$  are equal, since they are the base changes of the same isomorphism  $D(R_2,A_2) \cong A_2$ . Thus any two choices of R-algebra generator for A give rise to the same isomorphism between  $D(R,A) \cong A$ .

Then for an arbitrary (not necessarily monogenic) quadratic R-algebra A, we may define a choice-free isomorphism  $D(R,A) \xrightarrow{\sim} A$  by building it locally, and then gluing.

This completes the proof: We know that there is just one possible choice of isomorphism  $D(R_1, A_1) \cong A_1$ , and that for a monogenic quadratic R-algebra A, there is exactly one isomorphism  $D(R, A) \cong A$  that is compatible with the base changes from every  $(R, A) \to (R_1, A_1)$ . Furthermore, for a

general quadratic R-algebra A, there is exactly one isomorphism  $D(R, A) \cong A$  that is compatible with the base changes to every  $(R_r, A_r) \to (R, A)$  for which  $(R_r, A_r)$  is monogenic. Naturality of this isomorphism in general then follows from uniqueness.

#### **3.4** Proof for n = 3

We use the same approach for rank-3 algebras as for those of rank n=2: identify a cubic algebra over a simple enough ring that every cubic algebra is locally a base change of it. Not every cubic algebra is locally monogenic, but it does locally have a basis containing its multiplicative identity; in other words, if A is a cubic R-algebra then A/R is a locally free R-module of rank 2. (See, for example, [2, Lemma 2.3] for a proof that if A is a rank-R-algebra then A/R is locally free of rank R-1.)

We thus consider the structure of a cubic R-algebra A for which the R-module  $\dot{A} := A/R$  is free of rank 2 as an R-module. Each choice  $\{\dot{x},\dot{y}\}$  of basis for  $\dot{A}$  lifts uniquely to a basis  $\{1,x,y\}$  of A for which  $xy \in R$ . Then associativity requires that multiplication in A be of the following form, for some  $a,b,c,d \in R$ :

$$xy = bc$$

$$x^{2} = -bd + ax + by$$

$$y^{2} = -bc + cx + dy,$$
(1)

and conversely, any choice of  $a,b,c,d \in R$  gives rise to a cubic R-algebra structure on  $R\langle 1,x,y\rangle$  with multiplication defined by (1). (See the proof of Proposition 4.2 in [4], which is given over  $\mathbb Z$  but works over any base ring.)

So now define an object  $(R_1, A_1)$  of  $\mathbf{Alg}_3$  by  $R_1 = \mathbb{Z}[a_1, b_1, c_1, d_1]$  and

$$A_1 = R_1[x_1, y_1] / (x_1y_1 - (b_1c_1),$$

$$(x_1)^2 - (-b_1d_1 + a_1x_1 + b_1y_1),$$

$$(y_1)^2 - (-b_1c_1 + c_1x_1 + d_1y_1)).$$

Morphisms  $(R, A) \to (R_1, A_1)$  in  $\mathbf{Alg}_3$  correspond bijectively to choices of R-module basis for A/R. And since 2 is a prime non-zerodivisor of  $R_1$ , given any two discriminant algebras  $D, D' \colon \mathbf{Alg}_3 \to \mathbf{Alg}_2$  there is exactly one  $R_1$ -algebra isomorphism  $D(R_1, A_1) \cong D'(R_1, A_1)$  that induces the composite isomorphism  $\bigwedge^2 D(R_1, A_1) \cong \bigwedge^3 A_1 \cong \bigwedge^2 D'(R_1, A_1)$ . So for each R-module basis of A, we obtain by base change an isomorphism  $D(A, R) \cong D'(A, R)$ .

The proof now continues as in the n=2 case: we will prove that this isomorphism is independent of choice of basis by considering two bases  $\{\dot{x},\dot{y}\}$ 

and  $\{\dot{x}',\dot{y}'\}$  for  $\dot{A}$ . In this case there is a matrix  $\begin{pmatrix} e & f \\ a & h \end{pmatrix}$  in  $GL_2(R)$  such that

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.$$

Then we expand  $(R_1, A_1)$  to a new ring  $(R_2, A_2)$  by setting

$$R_2 = R_1 \left[ e_0, f_0, g_0, h_0, (e_0 h_0 - f_0 g_0)^{-1} \right]$$

and  $A_2 = R_2 \otimes_{R_1} A_1$ . We again have two distinguished maps from  $(R_2, A_2)$  to  $(R_1, A_1)$ , corresponding to the two  $R_2$ -module bases  $(\bar{x}_1, \bar{y}_1)$  and  $(\bar{x}_2, \bar{y}_2) := (e_0 \bar{x}_1 + f_0 \bar{y}_1, g_0 \bar{x}_1 + h_0 \bar{y}_1)$  for  $A_2/R_2$ . And once again,  $(R_2, A_2)$  is the categorical product of  $(R_1, A_1)$  with itself in  $\mathbf{Alg}_3$ , by a simple check that maps  $(R, A) \to (R_2, A_2)$  correspond to pairs of bases for  $\dot{A}$ . And since  $R_2$  is again a localization of a polynomial ring over  $\mathbb{Z}$ , Corollary 3.4 applies, and the same argument as in the n=2 case shows that if A/R is free then the isomorphism  $D(R, A) \cong D'(R, A)$  is independent of choice of basis. Then we can glue these isomorphisms to obtain a unique isomorphism  $D(R, A) \cong D'(R, A)$  for arbitrary A cubic over arbitrary R; the uniqueness guarantees naturality.

One might expect to be able to apply the same type of argument to arbitrary rank n, but it is not clear how to find the analogues of  $R_1$  and  $A_1$  for which 2 is a unit or prime non-zerodivisor in  $R_1$ , and for which every rank-n algebra is locally a base change of  $A_1$ . Melanie Matchett Wood's parameterization of quartic algebras [12] or a similar generalization of Manjul Bhargava's parametrizations of quintic  $\mathbb{Z}$ -algebras [1] may allow such an approach to work for n up to 5, but each higher rank would be a hard-won battle without a more general approach.

## 4 Isomorphism with Rost's discriminant algebra

As a consequence of Theorem 3.1, there is a canonical isomorphism between the discriminant algebra  $\Delta$  of Theorem 2.1 and any other for rank 3. In [10], Markus Rost defines such a discriminant algebra for rank 3, and we exhibit here the resulting isomorphism with  $\Delta$  as a simpler version of the analysis for Loos's discriminant algebra to follow.

We begin by reviewing the definition of Rost's discriminant algebra from [10].

**Definition 4.1.** Let R be a ring and let A be a rank-3 R-algebra. Define an R-module K(A) as follows: First let  $\dot{A}$  be the quotient R-module A/R.

Because A has rank 3, the function

$$A \to R: a \mapsto s_1(a^2) - s_2(a)$$

depends only on  $\dot{a}$ , the equivalence class of a modulo R, and thus defines a quadratic form  $q_A$  on  $\dot{A}$ .

Let  $C(q_A)$  be the Clifford algebra of  $q_A$ , the quotient of the tensor algebra  $\bigoplus_{n=0}^{\infty} \dot{A}^{\otimes n}$  by the two-sided ideal generated by elements of the form  $\dot{a} \otimes \dot{a} - q_A(\dot{a})$ . The Clifford algebra retains a  $\mathbb{Z}/2\mathbb{Z}$ -grading; we let  $K(A) = C_0(q_A)$  be the even-graded part.

Remark 4.2. Given any quadratic form q on a locally-free rank-2 R-module M, the even Clifford algebra  $C_0(q)$  is also locally free of rank 2: the construction commutes with localization, and if M has R-basis  $\{\theta_1, \theta_2\}$ , then  $C_0(q)$  has basis  $\{1, \theta_1 \otimes \theta_2\}$ , since  $\theta_2 \otimes \theta_1 = -\theta_1 \otimes \theta_2 + q(\theta_1 + \theta_2) - q(\theta_1) - q(\theta_2)$ . Thus we can present  $C_0(q)$  as the quotient of  $R \oplus M^{\otimes 2}$  by the R-submodule generated by elements of the form  $m \otimes m - q(m)$ . In particular, the ring homomorphism  $R \to C_0(q)$  fits into a short exact sequence  $0 \to R \to C_0(q) \to \bigwedge^2 M \to 0$  of R-modules, where the right-hand map sends each  $m_1 \otimes m_2 \mapsto m_1 \wedge m_2$ . Thus, in case A is a rank-3 R-algebra, we have (via the isomorphism  $\bigwedge^2 \dot{A} \to \bigwedge^3 A$  sending  $\dot{a} \wedge \dot{b} \mapsto 1 \wedge a \wedge b$ ) a short exact sequence of R-modules

$$0 \to R \to K(A) \to \bigwedge^3 A \to 0$$

and K(A) is a quadratic R-algebra.

**Lemma 4.3.** Let R be a ring and let A be a rank-3 R-algebra. There is a unique R-module homomorphism  $K(A) \to \Delta(A)$  sending 1 to 1 and  $\dot{a} \otimes \dot{b}$  to  $s_1(ab) - \gamma(1,b,a)$ . This homomorphism is a morphism of extensions of  $\bigwedge^3 A$  by R, and thus an R-module isomorphism.

Proof. Uniqueness is guaranteed because such elements generate K(A) as an R-module. As for existence, the map  $A \times A \to \Delta(A)$  sending (a,b) to  $s_1(ab) - \gamma(1,b,a)$  is R-bilinear, and sends elements of the form (a,r) to  $s_1(ar) - \gamma(1,r,a) = rs_1(a) - r\gamma(1,1,a) = 0$ , and similarly sends elements of the form  $(r,b) \mapsto 0$ . Thus this function extends and descends to an R-module homomorphism  $\dot{A} \otimes \dot{A} \to \Delta(A)$ . Together with the assignment  $1 \mapsto 1$ , we obtain an R-linear homomorphism  $R \oplus \dot{A}^{\otimes 2} \to \Delta(A)$ . This homomorphism sends elements of the form  $\dot{a} \otimes \dot{a}$  to  $s_1(a^2) - \gamma(1,a,a) = s_1(a^2) - s_2(a) = q(\dot{a})$ , so it descends to a homomorphism  $K(A) \to \Delta(A)$  of R-modules. As

suggested in the statement, this homomorphism fits into a map of short exact sequences:

$$0 \longrightarrow R \longrightarrow K(A) \longrightarrow \bigwedge^{3} A \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow R \longrightarrow \Delta(A) \longrightarrow \bigwedge^{3} A \longrightarrow 0$$

Commutativity of the left-hand square follows from the fact that  $K(A) \to \Delta(A)$  sends  $1 \mapsto 1$ ; commutativity of the right-hand square can be checked on a general element of the form  $\dot{a} \otimes \dot{b}$ , which is sent to  $1 \wedge a \wedge b$  in  $\bigwedge^3 A$ . Its image in  $\Delta(A)$  is  $s_1(ab) - \gamma(1,b,a)$ , which is sent to  $0 - 1 \wedge b \wedge a = 1 \wedge a \wedge b$  as desired. Thus this homomorphism is an R-module isomorphism  $K(A) \xrightarrow{\sim} \Delta(A)$ .

Thus we have two quadratic R-algebras with a canonical isomorphism between their underlying modules. However, this module isomorphism is not generally an R-algebra isomorphism. In fact, the isomorphism  $\bigwedge^2 K(A) \cong K(A)/R \cong \bigwedge^3 A$  does not even identify the two discriminant forms on K(A) and  $\bigwedge^3 A$ . Hence, Rost continues the definition of his discriminant algebra by "shifting" the multiplication on K(A) so that the isomorphism  $\bigwedge^2 K(A) \cong \bigwedge^3 A$  identifies the discriminant forms; we review this process and then show that the resulting algebra is isomorphic to  $\Delta(A)$ .

**Definition 4.4.** Let R be a ring, let B be a quadratic R-algebra, and let  $\dot{B} = B/R$  be the quotient R-module. Given a bilinear form  $\varepsilon \colon \dot{B} \times \dot{B} \to R$ , we may define a new multiplication  $\star$  on B in the following way:

$$b_1 \star b_2 = b_1 b_2 - \varepsilon(\dot{b}_1, \dot{b}_2).$$

This new multiplication has the same two-sided identity element as the original multiplication on B, so it is automatically commutative and associative and defines a new R-algebra structure on the R-module B. This new R-algebra is called the shift of B by  $\varepsilon$  and denoted  $B + \varepsilon$ .

**Remark 4.5.** In [10], the shift of B by  $\varepsilon$  is defined so that the new product of  $b_1$  and  $b_2$  is  $b_1b_2 + \varepsilon(\dot{b}_1, \dot{b}_2)$ . We instead follow the usage by Loos in [6], which is defined so that traces, norms, and discriminants transform as follows:

$$\operatorname{Tr}_{B+\varepsilon}(b) = \operatorname{Tr}_{B}(b)$$

$$\operatorname{Nm}_{B+\varepsilon}(b) = \operatorname{Nm}_{B}(b) + \varepsilon(\dot{b}, \dot{b})$$

$$\delta_{B+\varepsilon}(1 \wedge b_{1}, 1 \wedge b_{2}) = \delta_{B}(1 \wedge b_{1}, 1 \wedge b_{2}) - 4\varepsilon(\dot{b}_{1}, \dot{b}_{2}).$$

**Definition 4.6.** Let R be a ring and let A be a rank-3 R-algebra. Then the discriminant bilinear form  $\delta_A$  on  $\bigwedge^3 A$  may be regarded as a bilinear form on K(A)/R. Then the Rost discriminant algebra of A is  $D(A) := K(A) - \delta_A$ , the shift of K(A) by  $-\delta_A$ .

Then the R-module isomorphism between K(A) and  $\Delta(A)$  becomes an R-algebra isomorphism between D(A) and  $\Delta(A)$ .

**Theorem 4.7.** Let R be a ring and let A be a cubic R-algebra. Then the R-module isomorphism  $D(A) = K(A) \to \Delta(A)$  sending 1 to 1 and  $\dot{a} \otimes \dot{b}$  to  $s_1(ab) - \gamma(1,b,a)$  is an R-algebra isomorphism  $D(A) \xrightarrow{\sim} \Delta(A)$ .

The proof proceeds by showing that for all  $a, b \in A$ , the traces and norms of  $\dot{a} \otimes \dot{b}$  in D(A) and  $s_1(ab) - \gamma(1, b, a)$  in  $\Delta(A)$  agree, which we verify in Appendix A. Then the fact that the R-module isomorphism  $D(A) \cong \Delta(A)$  is an algebra isomorphism is due to the following lemma, since elements of the form  $\dot{a} \otimes \dot{b}$  generate  $\dot{D}(A)$ :

**Lemma 4.8.** Let R be a ring and D, E two quadratic R-algebras. Let  $\phi: D \to E$  be an R-module isomorphism preserving unity. If there exists a subset  $\Omega \subseteq D$  whose image generates  $\dot{D}$  and on which  $\phi$  preserves traces and norms, then  $\phi$  is an algebra isomorphism.

*Proof.* By working locally, we may assume that D and E are of the form  $R[x]/(x^2-sx+m)$  and  $R[y]/(y^2-ty+n)$ , with the isomorphism  $\phi\colon D\to E$  sending  $x\mapsto y$ . Denote the set  $\Omega\subseteq D$  as  $\{a_i+b_ix\}_{i\in I}$ . Then we know that the ideal  $(b_i:i\in I)$  is the unit ideal of R, and that for all  $i\in I$ , we have

$$\operatorname{Tr}_D(a_i + b_i x) = \operatorname{Tr}_E(a_i + b_i y)$$
 and  $\operatorname{Nm}_D(a_i + b_i x) = \operatorname{Nm}_E(a_i + b_i y)$ .

The trace equation becomes  $2a_i + b_i s = 2a_i + b_i t$ , which implies s = t since the  $b_i$  generate the unit ideal. The norm equation becomes  $a_i^2 + a_i b_i s + b_i^2 m = a_i^2 + a_i b_i t + b_i^2 n$ , and since s = t this reduces to  $b_i^2 m = b_i^2 n$ . The  $b_i^2$  also generate the unit ideal, so m = n as well. Therefore  $\phi \colon R[x]/(x^2 - sx + m) \to R[y]/(y^2 - ty + n) : x \mapsto y$  is an algebra isomorphism.  $\square$ 

## 5 Isomorphism with Loos's discriminant algebra

In [6], Ottmar Loos uses Rost's shifting technique to construct a discriminant algebra for algebras of arbitrary finite rank n. We review his definition and show that it is also isomorphic to  $\Delta$ . We begin with the case of an arbitrary quadratic form on an even-rank module:

**Definition 5.1.** Let R be a ring, let M be a locally free rank-n R-module with n even, and let  $Q: M \to R$  be a quadratic form. Given a bilinear form f representing Q, we will define an R-algebra  $\mathfrak{D}_f(Q)$  with underlying R-module  $R \oplus \bigwedge^n M$ ; its elements (r,u) are denoted  $r \cdot 1_f + s_f(u)$ . By Lemma 4.8, the R-algebra structure is determined by the traces and norms of the  $s_f(u)$  with u decomposable as  $u = x_1 \wedge \cdots \wedge x_n$ ; these are defined to be

$$\operatorname{Tr}(s_f(u)) := \operatorname{Pf}\left(f(x_i, x_j) - f(x_j, x_i)\right)_{i,j=1}^n$$
  
$$\operatorname{Nm}(s_f(u)) := (-1)^{\frac{n}{2}+1} \operatorname{qdet}\left(f(x_i, x_j)\right)_{i,j=1}^n,$$

where the Pfaffian Pf(A) of an antisymmetric matrix A is the canonical square root of its determinant, and the quarter-determinant qdet(A) of an even-dimensional square matrix A is the canonical quarter of det( $A + A^{\top}$ ) +  $(-1)^{\frac{n}{2}+1}$ det( $A - A^{\top}$ ):

$$4q\det(A) = \det(A + A^{\top}) + (-1)^{\frac{n}{2} + 1} \det(A - A^{\top});$$

see [5] for details of these matrix constructions.

Loos's next step is to remove the dependence of  $\mathfrak{D}_f(Q)$  on f.

**Definition 5.2.** Let  $Q: M \to R$  be a quadratic form on a locally free rankn R-module M with n even, and let  $f,g: M \times M \to R$  both represent Q. Define a linear map  $\kappa_{fg}: \bigwedge^n M \to R$  as follows. Given two antisymmetric matrices A and A', let t be an indeterminate and define  $\Pi(t, A, A')$  to be the polynomial in t satisfying

$$Pf(A + tA') = Pf(A) + t\Pi(t, A, A').$$

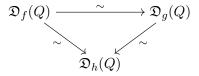
Then  $\kappa_{fg}: \bigwedge^n M \to R$  is defined by

$$\kappa_{fg}(x_1 \wedge \cdots \wedge x_n) = \Pi(-2, A, A'),$$

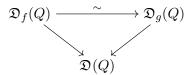
where  $A = (f(x_i, x_j) - f(x_j, x_i))_{i,j=1}^n$  and  $A' = (f(x_i, x_j) - g(x_i, x_j))_{i,j=1}^n$ . (Note that the latter is an antisymmetric matrix, because since f and g both represent Q we have  $f(x_i, x_j) + f(x_j, x_i) = Q(x_i + x_j) - Q(x_i) - Q(x_j) = g(x_i, x_j) + g(x_j, x_i)$ .)

<sup>&</sup>lt;sup>2</sup>We say that a bilinear form  $f: M \otimes M \to R$  represents a quadratic form  $Q: M \to R$  if for each  $m \in R$  we have  $Q(m) = f(m \otimes m)$ . Every quadratic form on a locally free finite-rank module is representable by some bilinear form; see Remark 5.3

Now define an R-module homomorphism  $\mathfrak{D}_f(Q) \to \mathfrak{D}_g(Q)$  sending  $1_f \mapsto 1_g$  and  $s_f(u) \mapsto s_g(u) + \kappa_{fg}(u) \cdot 1_g$ ; it is in fact an isomorphism of R-algebras (cf. [5, Theorem 2.3]). The family of isomorphisms constructed in this way is coherent in the sense that if  $h \colon M \times M \to R$  is another bilinear form representing Q, the resulting triangle of isomorphisms commutes (cf. [5, Theorem 1.6]):



Define an R-module  $\mathfrak{D}(Q)$  as follows: First form the direct sum of R-modules  $\bigoplus_f \mathfrak{D}_f(Q)$ , where f ranges over all bilinear forms representing Q. Then  $\mathfrak{D}(Q)$  is the quotient of this direct sum by the submodule generated by all differences of the forms  $1_f - 1_g$  and  $s_f(u) - (s_g(u) + \kappa_{fg}(u) \cdot 1_g)$ , so that each triangle of R-module homomorphisms of the following form commutes:



Then in fact each R-module homomorphism  $\mathfrak{D}_f(Q) \to \mathfrak{D}(Q)$  is an isomorphism, and we can transport each R-algebra structure on the  $\mathfrak{D}_f(Q)$  to an R-algebra structure on  $\mathfrak{D}(Q)$  that is independent of f. (In categorical terms, we have taken the colimit of a diagram whose shape category is contractible.) This quadratic R-algebra is called the  $Loos\ discriminant\ algebra$  of Q.

**Remark 5.3.** Note that the construction of  $\mathfrak{D}(Q)$  only makes sense if Q can be represented by some bilinear form f, so that we are not taking the colimit of an empty diagram. This is easy to see in case M is a free R-module: we can always represent a quadratic form  $Q: R^n \to R$  by the linear form  $f: R^n \times R^n \to R$  given by

$$f(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} B(\mathbf{e}_i, \mathbf{e}_j) & \text{if } i < j \\ Q(\mathbf{e}_i) & \text{if } i = j \\ 0 & \text{if } i > j, \end{cases}$$

where B(x,y) := Q(x+y) - Q(x) - Q(y) is the polar bilinear form of Q. Then if necessary, we could define  $\mathfrak{D}(Q)$  by working on patches where M is free and gluing together the resulting quadratic algebras. This is in fact what we would do to generalize  $\mathfrak{D}(Q)$  to the context of a quadratic form on a finite-rank vector bundle on a scheme, but in the affine context it is not necessary: every quadratic form on a finite projective module can already by represented by a bilinear form.

To see this, note that the module of bilinear forms on M is given by  $\operatorname{Hom}_R(M^{\otimes 2},R)$ , and the module of quadratic forms is  $\operatorname{Hom}_R((M^{\otimes 2})^{\operatorname{S}_2},R)$ . (For the latter, note that quadratic forms  $M\to R$  correspond to homogeneous degree-2 polynomial laws by [8, Prop. II.1 on p. 236], thus to R-linear functions  $\Gamma_R^2(M)\to R$  by [8, Thm. IV.1 on p. 266]. Since M is flat, we have  $\Gamma_R^2(M)\cong (M^{\otimes 2})^{\operatorname{S}_2}$  by [3, 5.5.2.5 on p. 123].) Since both  $M^{\otimes 2}$  and  $(M^{\otimes 2})^{\operatorname{S}_2}$  are finitely presented, both of these hom-module constructions commute with localization, so the homomorphism  $\operatorname{Hom}_R(M^{\otimes 2},R)\to \operatorname{Hom}_R((M^{\otimes 2})^{\operatorname{S}_2},R)$  is surjective because it is locally so.

Finally, Loos defines the discriminant algebra of an even-rank algebra as a shift of the discriminant algebra of its quadratic trace  $s_2$ .

**Definition 5.4.** Let R be a ring and let A be a rank-n R-algebra with n even, and let  $\mathfrak{D}(Q)$  be the Loos discriminant algebra of the quadratic form  $Q = s_2 \colon A \to R$ . Then the Loos discriminant algebra  $\mathrm{Dis}(A)$  of A over R is the shift of  $\mathfrak{D}(Q)$  by the bilinear form  $(-1)^{\frac{n}{2}+1} |n/4| \cdot \delta_A$ :

$$Dis(A) := \mathfrak{D}(Q) + (-1)^{\frac{n}{2}+1} \lfloor n/4 \rfloor \cdot \delta_A.$$

By [5, Theorem 2.3(c)] and [6, Lemma 3.6], this construction is a discriminant algebra in the sense of Definition 1.2. In the remainder of this section, we will exhibit an isomorphism of discriminant algebras Dis  $\cong \Delta$ .

**Remark 5.5.** Loos extends his definition of discriminant algebra to cover the odd-rank case by defining  $\mathrm{Dis}(A) := \mathrm{Dis}(R \times A)$  if A is an R-algebra of odd rank. By [2, Theorem 8.5], for such an algebra A we also have  $\Delta(A) \cong \Delta(R \times A)$ , so we obtain an isomorphism  $\mathrm{Dis} \cong \Delta$  in the odd-rank case from the isomorphism in the even-rank case.

**Definition 5.6.** Fix an even natural number n. Define a ring  $R_1$  as the polynomial ring  $\mathbb{Z}[f_{ij}:1\leq i< j\leq n]$  in  $\binom{n}{2}$  indeterminates. Let  $X_1$  be the polynomial  $R_1$ -algebra in n variables  $R_1[x_1,\ldots,x_n]$ . Then for each ring R and elements  $a_1,\ldots,a_n$  in an R-algebra A equipped with a bilinear form  $f\colon A\times A\to R$ , we obtain a ring homomorphism  $\psi_{(f,a_1,\ldots,a_n)}\colon R_1\to R$  sending each  $f_{ij}\mapsto f(a_i,a_j)$ . This makes R, and hence A, into an  $R_1$ -algebra, and we also obtain an  $R_1$ -algebra homomorphism  $\chi_{(f,a_1,\ldots,a_n)}\colon X_1\to A$  sending each  $x_i\mapsto a_i$ .

**Definition 5.7.** With  $R_1$  and  $X_1$  as in Definition 5.6, define the following  $S_n$ -invariant elements of  $X_1^{\otimes n}$ , the *n*th tensor power of  $X_1$  over  $R_1$ :

- For each  $i \in \{1, \ldots, n\}$ , define  $Q_i := e_2(x_i) = \sum_{k < \ell} x_i^{(k)} x_i^{(\ell)}$ .
- For each  $i, j \in \{1, ..., n\}$ , define  $B_{ij} := e_2(x_i + x_j) e_2(x_i) e_2(x_j) = \sum_{k \neq \ell} x_i^{(k)} x_j^{(\ell)}$ .

Now let R be a ring and A a rank-n R-algebra, let  $f: A \times A \to R$  be a bilinear form representing the quadratic form  $Q = s_2 \colon A \to R$ , and let B(a, a') = Q(a+a') - Q(a) - Q(a') = f(a,a') + f(a',a) be the polar form of Q. Choose elements  $a_1, \ldots, a_n \in A$ , and let  $\chi = \chi_{(f,a_1,\ldots,a_n)}$  be the homomorphism  $X_1 \to A$  of Definition 5.6. Then obtain the composite homomorphism

$$\Phi_{A/R} \circ \chi^{\otimes n} \colon (X_1^{\otimes n})^{S_n} \to (A^{\otimes n})^{S_n} \to R,$$

which sends  $Q_i \mapsto Q(a_i)$  and  $B_{ij} \mapsto B(a_i, a_j)$  for all  $i, j \in \{1, ..., n\}$ . Now if we also define elements  $F_{ij} \in (X_1^{\otimes n})^{S_n}$  for each  $i, j \in \{1, ..., n\}$  by

$$F_{ij} = \begin{cases} f_{ij} & \text{if } i < j \\ Q_i & \text{if } i = j \\ B_{ij} - f_{ji} & \text{if } i > j, \end{cases}$$

we find that

$$\phi_{A/R} \circ \chi^{\otimes n} \colon F_{ij} \mapsto \begin{cases} f(a_i, a_j) & \text{if } i < j \\ Q(a_i) = f(a_i, a_i) & \text{if } i = j \\ B(a_i, a_j) - f(a_j, a_i) = f(a_i, a_j) & \text{if } i > j. \end{cases}$$

So in all cases,  $F_{ij} \mapsto f(a_i, a_j)$ . We will denote by F the  $n \times n$ -matrix of elements of  $(X_1^{\otimes n})^{\mathbb{S}_n}$  whose i, jth entry is  $F_{ij}$ .

**Remark 5.8.** For each choice of representative f of Q and elements  $a_1, \ldots, a_n \in A$ , we may also use the homomorphism  $\chi = \chi_{f,a_1,\ldots,a_n} \colon X_1 \to A$  of Definition 5.7 to obtain a composite  $(X_1^{\otimes n})^{A_n} \to (A^{\otimes n})^{A_n} \to \Delta(A)$  that also sends each  $F_{ij}$  to  $f(a_i,a_j)$ . Thus we may construct elements of  $\Delta(A)$  as the images of  $A_n$ -invariant elements of  $X_1^{\otimes n}$ .

**Lemma 5.9.** Let  $R_1$  and its algebra  $X_1$  be as in Definition 5.6, and F the  $n \times n$ -matrix of Definition 5.7. There is a unique element  $\Sigma_F$  of  $(X_1^{\otimes n})^{A_n}$  such that

$$2\Sigma_F = \det(x_i^{(j)})_{i,j=1}^n + \Pr(F - F^{\top}).$$

We relegate the proof of this lemma to Appendix A.

**Definition 5.10.** Let R be a ring and A be a rank-n R-algebra with  $n \geq 2$  and even. Let f be a bilinear form representing the quadratic form  $Q = s_2 \colon A \to R$ , and let  $a_1, \ldots, a_n \in A$ . Let  $X_1$  be as in Definition 5.6, with the homomorphism  $\chi = \chi_{(f,a_1,\ldots,a_n)} \colon X_1 \to A$  of Definition 5.7 and the element  $\Sigma_F \in (X_1^{\otimes n})^{A_n}$  of Lemma 5.9. Define  $\sigma_f(a_1,\ldots,a_n) \in \Delta(A)$  as the image of  $\Sigma_F$  under the homomorphism  $(X_1^{\otimes n})^{A_n} \to (A^{\otimes n})^{A_n} \to \Delta(A)$  as in Remark 5.8.

**Remark 5.11.** By construction,  $2\sigma_f(a_1,\ldots,a_n)\in\Delta(A)$  is always equal to

$$\gamma(a_1, a_2, \dots, a_n) - \gamma(a_2, a_1, \dots, a_n) + Pf(f(a_i, a_j) - f(a_j, a_i))_{ij=1}^n$$

**Theorem 5.12.** Let R be a ring and let A be a rank-n R-algebra with  $n \geq 2$  and even, with the quadratic form  $Q = s_2 \colon A \to R$ . Then there is a unique R-module isomorphism  $\mathfrak{D}(Q) \to \Delta(A)$  sending 1 to 1 and  $s_f(a_1 \land \dots \land a_n)$  to  $\sigma_f(a_1, \dots, a_n)$  for each tuple  $(a_1, \dots, a_n) \in A^n$  and each bilinear representative f of Q.

Proof. Uniqueness is assured, since 1 and elements of the form  $s_f(a_1 \wedge \cdots \wedge a_n)$  generate  $\mathfrak{D}(Q)$ . To demonstrate existence, we show that for each f representing Q, the assignment  $1_f \mapsto 1$  and  $s_f(a_1 \wedge \cdots \wedge a_n) \mapsto \sigma_f(a_1, \ldots, a_n)$  extends to an R-module homomorphism  $\mathfrak{D}_f(Q) \to \Delta(A)$ , and then that for any pair of representatives f and g of Q, the homomorphisms  $\mathfrak{D}_f(Q) \to \Delta(A)$  and  $\mathfrak{D}_g(Q) \to \Delta(A)$  commute with the isomorphism  $\mathfrak{D}_f(Q) \to \mathfrak{D}_g(Q)$ . These amount to the following claims about the  $\sigma_f$ :

- 1. Multilinearity:  $\sigma_f(a_1, \ldots, a_n)$  is R-linear in each  $a_k$ .
- 2. Alternation:  $\sigma_f(a_1, \ldots, a_n) = 0$  if  $a_i = a_j$  for any  $i \neq j$ .
- 3. Compatibility:  $\sigma_f(a_1, \ldots, a_n) = \sigma_q(a_1, \ldots, a_n) + \kappa_{fq}(a_1 \wedge \cdots \wedge a_n)$ .

Proof of multilinearity. Let  $\chi = \chi_{(f,a_1,\ldots,a_n)} \colon X_1 \to A$  be the ring homomorphism of Definition 5.6. We know that under  $\phi_{A/R} \circ \chi^{\otimes n} \colon (X_1^{\otimes n})^{S_n} \to R$ , the images of  $f_{ij}$  and  $B_{ij}$  are  $f(a_i,a_j)$  and  $f(a_i,a_j) + f(a_j,a_i)$ , respectively, and hence vary R-linearly with  $a_i$  and  $a_j$ . Then for each  $k \in \{1,\ldots,n\}$ , since every term of  $\operatorname{Pf}(F-F^\top)$  contains exactly one factor of the form  $x_k^{(j)}$  for some j, or  $f_{ik}$  or  $f_{kj}$  for some i < k or j > k, the image of  $\operatorname{Pf}(F-F^\top)$  varies R-linearly with  $a_k$ . Similarly, every term in  $\det(x_i^{(j)})_{ij=1}^n$  contains exactly one factor of the form  $x_k^{(j)}$  for some j, and hence varies

R-linearly with  $a_k$ . The image of  $\Sigma_F$  under  $\chi$ , which is half of their sum, must then also vary R-linearly with each  $a_k$ . Thus  $\sigma_f$  extends to an R-linear map  $A^{\otimes n} \to \Delta(A)$ .

Proof of alternation. Suppose  $a_k = a_\ell$  for some pair  $k \neq \ell$ . Then the homomorphism  $X_1 \to A$  is unchanged under the automorphism of  $X_1$  transposing  $x_k$  and  $x_\ell$ ; therefore it factors through the quotient  $X_1' = X_1/(x_k - x_\ell)$ . But  $\det(x_i^{(j)})$  and  $\operatorname{Pf}(F - F^\top)$  both vanish under the quotient map  $X_1^{\otimes n} \to X_1'^{\otimes n}$ . Then  $\Sigma_F$  also maps to 0 in  $X_1'^{\otimes n}$ , and so  $\sigma_f(a_1, \ldots, a_n) = 0$ . Therefore  $\sigma_f$  descends to an R-linear map  $\bigwedge^n A \to \Delta(A)$ , and together with the assignment  $1_f \mapsto 1$  we obtain an R-linear map  $\mathfrak{D}_f(Q) \cong R \oplus \bigwedge^n A \to \Delta(A)$  sending each  $s_f(x_1 \wedge \cdots \wedge x_n)$  to  $\sigma_f(x_1, \ldots, x_n)$ .

Proof of compatibility. Let f and g be two bilinear forms representing Q. For each choice of  $a_1,\ldots,a_n\in A$ , we obtain two ring homomorphisms  $R_1\to R$  and two homomorphisms  $X_1\to A$ ; we may represent this situation by now making R into a  $R_2\coloneqq R_1\otimes_{\mathbb{Z}}R_1\cong \mathbb{Z}[f_{ij},g_{ij}:1\le i< j\le n]$ -algebra; then letting  $X_2\coloneqq R_2\otimes_{R_1}X_1\cong R_2[x_1,\ldots,x_n]$  we have an  $R_2$ -algebra homomorphism  $X_2\to A$  sending each  $x_i\mapsto a_i,\ f_{ij}\mapsto f(a_i,a_j),$  and  $g_{ij}\mapsto g(a_i,a_j)$ . Then we can build matrices F and G with entries in  $(X_2^{\otimes n})^{A_n}$ , and thus elements

$$\Sigma_F = \frac{1}{2} \left( \det(x_i^{(j)})_{i,j} + \operatorname{Pf}(F - F^{\top}) \right) \mapsto \sigma_f(a_1, \dots, a_n)$$
  
$$\Sigma_G = \frac{1}{2} \left( \det(x_i^{(j)})_{i,j} + \operatorname{Pf}(G - G^{\top}) \right) \mapsto \sigma_g(a_1, \dots, a_n)$$

of  $(X_2^{\otimes n})^{A_n}$ . Then  $\Sigma_F - \Sigma_G \mapsto \sigma_f(a_1, \ldots, a_n) - \sigma_g(a_1, \ldots, a_n)$ ; we will show that it also maps to  $\kappa_{fg}(a_1 \wedge \cdots \wedge a_n)$ . Indeed,  $\Sigma_F - \Sigma_G$  is a difference of Pfaffians:

$$\Sigma_F - \Sigma_G = \frac{1}{2} (\operatorname{Pf}(F - F^{\top}) - \operatorname{Pf}(G - G^{\top})).$$

Now F - G is an antisymmetric matrix whose ijth entry with i < j is  $f_{ij} - g_{ij}$ . Then  $(G - G^{\top}) = (F - F^{\top}) - 2(F - G)$ , so

$$\begin{aligned} \operatorname{Pf}(G - G^{\top}) &= \operatorname{Pf}\left((F - F^{\top}) - 2(F - G)\right) \\ &= \operatorname{Pf}(F - F^{\top}) - 2\Pi(-2, F - F^{\top}, F - G), \end{aligned}$$

so

$$\Sigma_F - \Sigma_G = \frac{1}{2} (\text{Pf}(F - F^{\top}) - \text{Pf}(F - F^{\top}) + 2\Pi(-2, F - F^{\top}, F - G))$$
$$= \Pi(-2, F - F^{\top}, F - G).$$

Now under the homomorphism  $X_2^{\otimes n} \to A^{\otimes n}$ , for each i < j the ijth entry of the matrix  $F - F^{\top}$  maps as  $F_{ij} - F_{ji} \mapsto f(a_i, a_j) - f(a_j, a_i)$ , and the ijth entry of F - G maps as  $F_{ij} - G_{ij} \mapsto f(a_i, a_j) - g(a_i, a_j)$ . Thus

$$\Sigma - \Sigma' \mapsto \Pi\left(-2, \left(f(a_i, a_j) - f(a_j, a_i)\right)_{i,j=1}^n, \left(f(a_i, a_j) - g(a_i, a_j)\right)_{i,j=1}^n\right)$$
$$= \kappa_{fg}(a_1 \wedge \dots \wedge a_n).$$

Thus the assignments  $1 \mapsto 1$  and  $s_f(a_1 \wedge \cdots \wedge a_n) \mapsto \sigma_f(a_1, \ldots, a_n)$  constitute a well-defined R-module homomorphism  $\mathfrak{D}(Q) \to \Delta(A)$ .

To show that this homomorphism is an R-module isomorphism, merely note that it fits into a map of short exact sequences

$$0 \longrightarrow R \longrightarrow \mathfrak{D}(Q) \longrightarrow \bigwedge^{n} A \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow R \longrightarrow \Delta(A) \longrightarrow \bigwedge^{n} A \longrightarrow 0$$

since it sends  $1 \mapsto 1$  and for each  $a_1, \ldots, a_n \in A$ , we have both  $s_f(a_1 \wedge \cdots \wedge a_n) \mapsto a_1 \wedge \cdots \wedge a_n$  and  $\sigma_f(a_1, \ldots, a_n) \mapsto a_1 \wedge \ldots a_n$ ; indeed, the difference between  $\Sigma_F$  and  $\gamma(x_1, \ldots, x_n)$  in  $X_1^{\otimes n}$  is  $S_n$ -invariant, hence  $\sigma_f(a_1, \ldots, a_n)$  and  $\gamma(a_1, \ldots, a_n)$  differ by an element of R in  $\Delta(A)$  and so have the same image in  $\Lambda^n A$ , namely  $a_1 \wedge \cdots \wedge a_n$ .

**Theorem 5.13.** Let R be a ring, and let A be a rank-n R-algebra with  $n \geq 2$  and even. The R-module isomorphism  $\mathrm{Dis}(A) \cong \mathfrak{D}(Q) \to \Delta(A)$  of Theorem 5.12 is an R-algebra isomorphism  $\mathrm{Dis}(A) \xrightarrow{\sim} \Delta(A)$ .

Proof. By Lemma 4.8, to show that the R-module isomorphism  $\mathrm{Dis}(A) \to \Delta(A)$  is an isomorphism of algebras, it is enough to check that it preserves traces and norms of elements of the form  $s_f(a_1 \wedge \cdots \wedge a_n)$ , since the images of these in  $\mathrm{Dis}(A)/R \cong \bigwedge^n A$  are a generating set. First we check that the traces of the two elements  $\sigma_f(a_1,\ldots,a_n) \in \Delta(A)$  and  $s_f(a_1 \wedge \cdots \wedge a_n) \in \mathrm{Dis}(A)$  agree. The trace of  $s_f(a_1 \wedge \cdots \wedge a_n)$  in  $\mathfrak{D}(Q)$  is defined to be  $\mathrm{Pf}(f(a_i,a_j)-f(a_j,a_i))_{i,j=1}^n$ , and this trace is preserved under the shift to  $\mathrm{Dis}(A)$ . To compute the trace of  $\sigma_f(a_1,\ldots,a_n)$ , we recall that the trace of an element of a quadratic algebra is the sum of it and its conjugate, i.e. the image of it under the standard involution. The standard involution on  $\Delta(A)$  descends from the action of a transposition on the tensor factors of  $(A^{\otimes n})^{A_n}$ , so we may compute the trace of  $\sigma_f(a_1,\ldots,a_n)$  as the image in

 $\Delta(A)$  of  $\Sigma_F + \tau(\Sigma_F)$ , where  $\tau \colon (X_1^{\otimes n})^{A_n} \to (X_1^{\otimes n})^{A_n}$  is the action of a transposition. This sum is

$$\Sigma_F + \tau(\Sigma_F) = \frac{1}{2} \left( \det(x_i^{(j)})_{i,j=1}^n + \operatorname{Pf}(F - F^\top) \right)$$
  
 
$$+ \frac{1}{2} \left( -\det(x_i^{(j)})_{i,j=1}^n + \operatorname{Pf}(F - F^\top) \right)$$
  
 
$$= \frac{1}{2} \cdot 2 \operatorname{Pf}(F - F^\top) = \operatorname{Pf}(F - F^\top).$$

Hence the trace of  $\sigma_f(a_1, \ldots, a_n)$  is the image of  $\operatorname{Pf}(F - F^{\top})$  in  $\Delta(A)$ , namely  $\operatorname{Pf}(f(a_i, a_j) - f(a_j, a_i))_{i,j=1}^n$  as desired.

Next, we calculate the norms of  $\sigma_f(a_1,\ldots,a_n)$  in  $\Delta(A)$  and of  $s_f(a_1 \wedge \cdots \wedge a_n)$  in Dis(A). The norm of  $\sigma_f(a_1,\ldots,a_n)$  may be calculated in the same manner as its trace, as the image of  $\Sigma_F \cdot \tau(\Sigma_F)$  under the homomorphism  $(X_1^{\otimes n})^{A_n} \to \Delta(A)$ . We obtain

$$\Sigma_{F} \cdot \tau(\Sigma_{F}) = \frac{1}{2} \left( \det(x_{i}^{(j)})_{i,j=1}^{n} + \operatorname{Pf}(F - F^{\top}) \right)$$

$$\cdot \frac{1}{2} \left( -\det(x_{i}^{(j)})_{i,j=1}^{n} + \operatorname{Pf}(F - F^{\top}) \right)$$

$$= \frac{1}{4} \left( -\left( \det(x_{i}^{(j)})_{i,j=1}^{n} \right)^{2} + \operatorname{Pf}(F - F^{\top})^{2} \right)$$

$$= \frac{1}{4} \left( -\det\left( \sum_{k=1}^{n} x_{i}^{(k)} x_{j}^{(k)} \right)_{i,j=1}^{n} + \det(F - F^{\top}) \right)$$

$$= \frac{1}{4} \left( -\det(e_{1}(x_{i}x_{j}))_{i,j=1}^{n} + \det(F - F^{\top}) \right),$$

and the norm of  $\sigma_f(a_1,\ldots,a_n)$  is therefore the image of this quantity in  $\Delta(A)$ .

On the other hand, the norm of  $s_f(a_1 \wedge \cdots \wedge a_n)$  in  $\mathfrak{D}(Q)$  is defined as

$$\operatorname{Nm}_{\mathfrak{D}(Q)}(s_f(a_1 \wedge \cdots \wedge a_n)) = (-1)^{\frac{n}{2}+1} \operatorname{qdet}(f(a_i, a_j))_{i,j=1}^n,$$

which is the image of

$$(-1)^{\frac{n}{2}+1}\operatorname{qdet}(F) = \frac{1}{4}\left((-1)^{\frac{n}{2}+1}\operatorname{det}(F+F^{\top}) + \operatorname{det}(F-F^{\top})\right)$$
$$= \frac{1}{4}\left((-1)^{\frac{n}{2}+1}\operatorname{det}(B_{ij})_{i,j=1}^{n} + \operatorname{det}(F-F^{\top})\right),$$

since  $F_{ij}+F_{ji}=B_{ij}$  for all  $i,j\in\{1,\ldots,n\}$ . Now the norm of  $s_f(a_1\wedge\cdots\wedge a_n)$  in  $\mathrm{Dis}(A)$  is  $(-1)^{\frac{n}{2}+1}\lfloor n/4\rfloor\,\delta_A(a_1\wedge\cdots\wedge a_n,a_1\wedge\cdots\wedge a_n)$  more than its norm in  $\mathfrak{D}(Q)$ , and is hence the image of

$$(-1)^{\frac{n}{2}+1} \left\lfloor \frac{n}{4} \right\rfloor \det \left( e_1(x_i x_j) \right)_{i,j=1}^n + \frac{1}{4} \left( (-1)^{\frac{n}{2}+1} \det \left( B_{ij} \right)_{i,j=1}^n + \det (F - F^\top) \right).$$

Now we claim that the following identity holds in the polynomial ring  $X_1^{\otimes n}$ ; it is proved as Lemma A.3.

$$\det(B_{ij})_{i,j=1}^{n} = (1-n)\det(e_1(x_ix_j))_{i,j=1}^{n}.$$

Then the norm of  $s_f(a_1 \wedge \cdots \wedge a_n)$  in Dis(A) is the image of

$$\frac{1}{4} \left( (-1)^{\frac{n}{2}+1} \left( 4 \left\lfloor \frac{n}{4} \right\rfloor \det \left( e_1(x_i x_j) \right)_{i,j=1}^n + \det(B_{ij})_{i,j=1}^n \right) + \det(F - F^\top) \right) 
= \frac{1}{4} \left( (-1)^{\frac{n}{2}+1} \left( 4 \left\lfloor \frac{n}{4} \right\rfloor + (1-n) \right) \det \left( e_1(x_i x_j) \right)_{i,j=1}^n + \det(F - F^\top) \right),$$

and since  $4 \lfloor n/4 \rfloor = n - 1 + (-1)^{\frac{n}{2}}$  for n even, this simplifies to

$$\frac{1}{4} \left( -\det(e_1(x_i x_j))_{i,j=1}^n + \det(F - F^\top) \right),\,$$

whose image in R we have already shown to be the norm of  $\sigma_f(a_1, \ldots, a_n)$  in  $\Delta(A)$ , as desired.

Therefore the trace and norm of  $s_f(a_1 \wedge \cdots \wedge a_n)$  in  $\mathrm{Dis}(A)$  and of  $\sigma_f(a_1, \ldots, a_n)$  in  $\Delta(A)$  agree, so the R-module isomorphism  $\mathrm{Dis}(A) \to \Delta(A)$  sending  $1 \mapsto 1$  and  $s_f(a_1 \wedge \cdots \wedge a_n) \mapsto \sigma_f(a_1, \ldots, a_n)$  is an R-algebra isomorphism, as desired.

### A Miscellaneous Calculations

In this appendix, we perform various computations used to support claims made in the main text.

#### A.1 Traces and norms in rank 3

First, we show that the module isomorphism  $D(A) \to \Delta(A)$  of Theorem 4.7 preserves traces and norms by means of the following identity:

**Lemma A.1.** Let A be an R-algebra of rank 3 with elements  $a, b \in A$ . Then

$$s_1(a)s_1(b) = s_1(ab) + s_2(a+b) - s_2(a) - s_2(b)$$

$$s_1(a)^2 = s_1(a^2) + 2s_2(a)$$

$$s_1(a)s_1(ab)s_1(b) = s_1(a^2b)s_1(b) + s_1(a)s_1(ab^2) - s_1(a^2b^2) + s_2(ab) + s_2(a)s_2(b).$$

*Proof.* It is possible to prove these with the expressions for  $s_k(a+b)$  shown by Christophe Reutenauer and Marcel-Paul Schützenberger in [7], but we present here a different argument.

For each  $x \in A$ ,  $s_k(x)$  is the image under  $\Phi_{A/R}: (A^{\otimes 3})^{S_3} \to R$  of the element  $e_k(x) \in (A^{\otimes 3})^{S_3}$ . Thus it suffices to check that the corresponding identities in the  $e_k(x)$  hold in  $A^{\otimes 3}$ .

For example,

$$e_1(a)e_1(b) = (a \otimes 1 \otimes 1 + \dots)(b \otimes 1 \otimes 1 + \dots)$$
  
=  $(ab \otimes 1 \otimes 1 + \dots) + (a \otimes b \otimes 1 + \dots),$ 

where  $+ \dots$  denotes the sum over all ways of forming a pure tensor out of the same tensor factors (thus the first three parenthesized expressions contain 3 terms each, and the fourth contains 6). Now  $(ab \otimes 1 \otimes 1 + \dots) = e_1(ab)$ , and the remainder is also the difference between  $e_2(a+b) = (a+b) \otimes (a+b) \otimes 1 + \dots$  and  $e_2(a) + e_2(b) = (a \otimes a \otimes 1 + \dots) + (b \otimes b \otimes 1 + \dots)$ . So  $e_1(a)e_1(b) = e_1(ab) + e_2(a+b) - e_2(a) - e_2(b)$ .

The second identity follows from the first by setting a = b.

We handle the third identity similarly to the first. Expanding out  $e_1(a)e_1(ab)e_1(b)$  gives

$$e_{1}(a)e_{1}(ab)e_{1}(b) = (a \otimes 1 \otimes 1 + \dots)(ab \otimes 1 \otimes 1 + \dots)(b \otimes 1 \otimes 1 + \dots)$$

$$= (a^{2}b^{2} \otimes 1 \otimes 1 + \dots)$$

$$+ (a^{2}b \otimes b \otimes 1 + \dots)$$

$$+ (a \otimes ab^{2} \otimes 1 + \dots)$$

$$+ 2(ab \otimes ab \otimes 1 + \dots)$$

$$+ (a \otimes ab \otimes b + \dots)$$

$$= e_{1}(a^{2}b^{2}) + (e_{1}(a^{2}b)e_{1}(b) - e_{1}(a^{2}b^{2}))$$

$$+ (e_{1}(a)e_{1}(ab^{2}) - e_{1}(a^{2}b^{2})) + 2(e_{2}(ab))$$

$$+ (a \otimes ab \otimes b + \dots).$$

That last term also appears in the expansion of  $e_2(a)e_2(b)$ :

$$e_2(a)e_2(b) = (a \otimes a \otimes 1 + \dots)(b \otimes b \otimes 1 + \dots)$$
$$= (ab \otimes ab \otimes 1 + \dots) + (a \otimes ab \otimes b + \dots)$$
$$= e_2(ab) + (a \otimes ab \otimes b + \dots).$$

Hence 
$$e_1(a)e_1(ab)e_1(b) = e_1(a^2b)e_1(b) + e_1(a)e_1(ab^2) - e_1(a^2b^2) + e_2(ab) + e_2(a)e_2(b)$$
.

We will also use the following fact:

**Lemma A.2.** Let R be a ring and D a quadratic R-algebra. If  $\tau: D \to D$  is an R-module homomorphism with the properties that

1. 
$$\tau(1) = 1$$
, and

2. 
$$a \cdot \tau(a) \in R$$
 for all  $a \in D$ ,

then  $\tau$  is an R-algebra automorphism of D, and for all  $a \in D$ , we have  $\operatorname{Tr}_D(a) = a + \tau(a)$  and  $\operatorname{Nm}_D(a) = a \cdot \tau(a)$ .

*Proof.* See [11, Lemmas 2.9 and 2.13]. 
$$\square$$

For example, if A is an R-algebra of rank  $n \geq 2$ , then the automorphism of  $A^{\otimes n}$  given by interchanging the first two tensor factors induces an involution on  $\Delta(A)$  satisfying the hypotheses of Lemma A.2, so we can deduce (for example) that

$$\operatorname{Tr}_{\Delta(A)}(\gamma(a_1,\ldots,a_n)) = \gamma(a_1,a_2,\ldots,a_n) + \gamma(a_2,a_1,\ldots,a_n).$$

Similarly, given any quadratic form q on a locally-free rank-2 module M, the even Clifford algebra  $C_0(q) \cong R \oplus M^{\otimes 2}/(m \otimes m - q(m) : m \in M)$  has an involution sending  $1 \mapsto 1$  and elements of the form  $m_1 \otimes m_2$  to  $m_2 \otimes m_1$ . This involution satisfies the hypotheses of Lemma A.2, so the norm (for example) of  $m_1 \otimes m_2$  is

$$\operatorname{Nm}_{C_0(q)}(m_1 \otimes m_2) = (m_1 \otimes m_2)(m_2 \otimes m_1)$$
$$= m_1 \otimes m_2 \otimes m_2 \otimes m_1$$
$$= (m_1 \otimes m_1) \cdot q(m_2) = q(m_1)q(m_2).$$

Now we in a position to prove Theorem 4.7, that the module isomorphism  $\Delta(A) \cong K(A)$  of Lemma 4.3 becomes an algebra isomorphism when we shift the multiplication of K(A) by  $-\delta_A$ .

Proof of Theorem 4.7. By Lemma 4.8, it is enough to show that the Rmodule isomorphism  $D(A) \to \Delta(A)$  preserves 1 (which is immediate), as
well as traces and norms of elements of the form  $\dot{a} \otimes \dot{b}$ . Given such an

element, its trace in D(A) is

$$\operatorname{Tr}_{D(A)}(\dot{a} \otimes \dot{b}) = \dot{a} \otimes \dot{b} + \dot{b} \otimes \dot{a}$$

$$= (\dot{a} + \dot{b}) \otimes (\dot{a} + \dot{b}) - \dot{a} \otimes \dot{a} - \dot{b} \otimes \dot{b}$$

$$= q(\dot{a} + \dot{b}) - q(\dot{a}) - q(\dot{b})$$

$$= s_1((a+b)^2) - s_2(a+b) - s_1(a^2) + s_2(a) - s_1(b^2) + s_2(b)$$

$$= s_1(a^2 + 2ab + b^2) - s_1(a^2) - s_1(b^2) - (s_2(a+b) - s_2(a) - s_2(b))$$

$$= s_1(2ab) - (s_1(a)s_1(b) - s_1(ab))$$

$$= 3s_1(ab) - s_1(a)s_1(b).$$

On the other hand, its image  $s_1(ab) - \gamma(1, b, a)$  in  $\Delta(A)$  has trace

$$\operatorname{Tr}_{\Delta(A)}(s_1(ab) - \gamma(1, b, a)) = 2s_1(ab) - \gamma(1, b, a) - \gamma(1, a, b)$$
$$= 2s_1(ab) - (s_1(a)s_1(b) - s_1(ab))$$
$$= 3s_1(ab) - s_1(a)s_1(b),$$

so the isomorphism  $D(A) \to \Delta(A)$  preserves the traces of elements of the form  $\dot{a} \otimes \dot{b}$ .

Now the norms of K(A) and D(A) differ; the equation  $\operatorname{Nm}_{D(A)}(\dot{a} \otimes \dot{b}) = \operatorname{Nm}_{\Delta(A)}(s_1(ab) - \gamma(1,b,a))$  expands to

$$\operatorname{Nm}_{K(A)}(\dot{a} \otimes \dot{b}) - \delta_A(1 \wedge a \wedge b, 1 \wedge a \wedge b) = \operatorname{Nm}_{\Delta(A)}(s_1(ab) - \gamma(1, b, a))$$
$$= s_1(ab)^2 - s_1(ab)\operatorname{Tr}_{\Delta(A)}(\gamma(1, b, a)) + \operatorname{Nm}_{\Delta(A)}(\gamma(1, b, a)).$$

We compute each of these terms in turn. Now the norm of  $\dot{a} \otimes \dot{b}$  in K(A) is

$$\operatorname{Nm}_{K(A)}(\dot{a} \otimes \dot{b}) = (\dot{a} \otimes \dot{b})(\dot{b} \otimes \dot{a}) = \dot{a} \otimes \dot{b} \otimes \dot{b} \otimes \dot{a} = q(\dot{b})(\dot{a} \otimes \dot{a}) = q(\dot{a})q(\dot{b})$$

$$= (s_1(a^2) - s_2(a))(s_1(b^2) - s_2(b))$$

$$= s_1(a^2)s_1(b^2) - s_2(a)s_1(b^2) - s_1(a^2)s_2(b) + s_2(a)s_2(b).$$

Next, we expand  $\delta_A(1 \wedge a \wedge b, 1 \wedge a \wedge b)$  as

$$\det \begin{pmatrix} 3 & s_1(a) & s_1(b) \\ s_1(a) & s_1(a^2) & s_1(ab) \\ s_1(b) & s_1(ab) & s_1(b^2) \end{pmatrix} = \begin{pmatrix} 3s_1(a^2)s_1(b^2) - 3s_1(ab)^2 \\ - s_1(a)^2s_1(b^2) - s_1(a^2)s_1(b)^2 \\ + 2s_1(a)s_1(ab)s_1(b). \end{pmatrix}$$

We can use Lemma A.1 to expand  $s_1(a)s_1(ab)s_1(b)$  and terms of the form

 $s_1(\cdot)^2$ , and then simplify:

$$= 3s_1(a^2)s_1(b^2) - 3(s_1(a^2b^2) + 2s_2(ab)) - (s_1(a^2) + 2s_2(a))s_1(b^2) - s_1(a^2)(s_1(b^2) + 2s_2(b)) + 2(s_1(a^2b)s_1(b) + s_1(a)s_1(ab^2) - s_1(a^2b^2) + s_2(ab) + s_2(a)s_2(b)),$$
  
$$= s_1(a^2)s_1(b^2) - 5s_1(a^2b^2) - 4s_2(ab) - 2s_2(a)s_1(b^2) - 2s_1(a^2)s_2(b) + 2s_1(a^2b)s_1(b) + 2s_1(a)s_1(ab^2) + 2s_2(a)s_2(b).$$

Hence the difference  $\operatorname{Nm}_{K(A)}(\dot{a}\otimes\dot{b})-\delta_A(1\wedge a\wedge b,1\wedge a\wedge b)$  is

$$5s_1(a^2b^2) + 4s_2(ab) + s_2(a)s_1(b^2) + s_1(a^2)s_2(b)$$
$$-2s_1(a^2b)s_1(b) - 2s_1(a)s_1(ab^2) - s_2(a)s_2(b).$$

Next we expand  $s_1(ab)^2 - s_1(ab)\operatorname{Tr}_{\Delta(A)}(\gamma(1,b,a)) + \operatorname{Nm}_{\Delta(A)}(\gamma(1,b,a))$ . We computed the trace of  $\gamma(1,b,a)$  above as  $\gamma(1,b,a)+\gamma(1,a,b)=s_1(a)s_1(b)-s_1(ab)$ . Therefore the quantity  $s_1(ab)^2-s_1(ab)\operatorname{Tr}_{\Delta(A)}(\gamma(1,b,a))$  is

$$s_1(ab)^2 - s_1(ab)(s_1(a)s_1(b) - s_1(ab)) = 2s_1(ab)^2 - s_1(a)s_1(ab)s_1(b)$$

$$= 2(s_1(a^2b^2) + 2s_2(ab)) - (s_1(a^2b)s_1(b) + s_1(a)s_1(ab^2)$$

$$- s_1(a^2b^2) + s_2(ab) + s_2(a)s_2(b))$$

$$= 3s_1(a^2b^2) + 3s_2(ab) - s_2(a)s_2(b) - s_1(a^2b)s_1(b) - s_1(a)s_1(ab^2).$$

Finally, the norm of  $\gamma(1, b, a)$  may similarly be computed as

$$\begin{aligned} \operatorname{Nm}(\gamma(1,b,a)) &= \gamma(1,b,a)\gamma(1,a,b) \\ &= \gamma(1,ab,ab) + \gamma(b,b,a^2) + \gamma(a,b^2,a) \\ &= s_2(ab) + s_1(a^2)s_2(b) - s_1(a^2b)s_1(b) + s_1(a^2b^2) \\ &+ s_2(a)s_1(b^2) - s_1(a)s_1(ab^2) + s_1(a^2b^2) \\ &= 2s_1(a^2b^2) + s_2(ab) - s_1(a^2b)s_1(b) - s_1(a)s_1(ab^2) \\ &+ s_1(a^2)s_2(b) + s_2(a)s_1(b^2). \end{aligned}$$

So all together we have

$$\operatorname{Nm}_{\Delta(A)}\big(s_1(ab) - \gamma(1,b,a)\big) = 5s_1(a^2b^2) + 4s_2(ab) - s_2(a)s_2(b) - 2s_1(a^2b)s_1(b) - 2s_1(a)s_1(ab^2) + s_1(a^2)s_2(b) + s_2(a)s_1(b^2).$$

So the isomorphism  $D(A) \to \Delta(A)$  preserves the norm and trace of elements of the form  $\dot{a} \otimes \dot{b}$ , and it is then an isomorphisms of R-algebras, as we wanted to show.

### A.2 The existence of $\Sigma_F$

Next we prove Lemma 5.9, that the element  $\det(x_i^{(j)})_{i,j} + \operatorname{Pf}(F - F^{\top})$  of  $X_1^{\otimes n}$  is a multiple of 2 by an element of  $(X_1^{\otimes n})^{A_n}$ .

Proof of Lemma 5.9. A permutation of the  $x_i$  permutes the rows of the matrix  $(x_i^{(j)})_{i,j=1}^n$ , so its determinant is  $A_n$ -invariant. And since the entries of  $F - F^{\top}$  are  $S_n$ -invariant, so is its Pfaffian  $\operatorname{Pf}(F - F^{\top})$ . Thus we need only show that the sum  $\det(x_i^{(j)})_{i,j=1}^n + \operatorname{Pf}(F - F^{\top})$  is a multiple of 2 in  $X_1^{\otimes n}$ , i.e. that it is sent to 0 in  $X_1^{\otimes n}/(2) \cong (X_1/(2))^{\otimes n}$ . Now consider  $\operatorname{Pf}(F - F^{\top})$  modulo 2. Since  $F_{ij} - F_{ji} = 2f_{ij} - B_{ij} \equiv B_{ij}$  for i < j, the Pfaffian  $\operatorname{Pf}(F - F^{\top})$  is congruent modulo 2 to the Pfaffian of the alternating  $n \times n$  matrix  $B_0$  whose ijth entry is  $B_{ij}$  if i < j and  $-B_{ij}$  if i > j. Working in  $X_1^{\otimes n}/(2)$ , we can expand this Pfaffian as

$$Pf(B_0) = \sum_{S \in \binom{n}{2,...,2}} \prod_{(i,j) \in S} B_{ij} = \sum_{S \in \binom{n}{2,...,2}} \prod_{(i,j) \in S} \sum_{k \neq \ell} x_i^{(k)} x_j^{(\ell)},$$

where the outer sum is over all partitions S of  $\{1, \ldots, n\}$  into n/2 pairs  $\{i, j\}$  with  $i \neq j$ . Expanding this sum, we obtain a sum of products of the form  $x_1^{(k_1)} \ldots x_n^{(k_n)}$ ; we claim that such a monomial appears an odd number of times in the sum if and only if the  $k_i$  are all distinct.

To wit, suppose that  $k_i \neq k_j$  for all  $i \neq j$ ; then for each partition S of  $\{1,\ldots,n\}$  into ordered pairs, the monomial  $x_1^{(k_1)}\ldots x_n^{(k_n)}$  appears exactly once in  $\prod_{(i,j)\in S}\sum_{k\neq \ell}x_i^{(k)}x_j^{(\ell)}$ , namely by setting  $k=k_i$  and  $\ell=k_j$  for each  $\{i,j\}\in S$ . There are  $(n-1)!!=(n-1)(n-3)\ldots 5\cdot 3\cdot 1$  such partitions S, so the monomial  $x_1^{(k_1)}\ldots x_n^{(k_n)}$  appears an odd number of times in the expansion of  $\mathrm{Pf}(B_0)$ .

On the other hand, suppose that  $k_{i_1} = k_{i_2}$  for some  $i_1 \neq i_2$  in  $\{1, \ldots, n\}$ . We will produce a fixed-point-free involution  $\tau$  on the subset of partitions S for which  $x_1^{(k_1)} \ldots x_n^{(k_n)}$  appears in the expansion of  $\prod_{\{i,j\} \in S} \sum_{k \neq \ell} x_i^{(k)} x_j^{(\ell)}$ . If S is such a partition, then  $k_i \neq k_j$  for each  $\{i,j\} \in S$ . In particular,  $\{i_1,i_2\}$  is not in S; let  $j_1$  and  $j_2$  be such that  $\{i_1,j_1\}$  and  $\{i_2,j_2\}$  are elements of S. Then set  $\tau(S)$  to be the partition of  $\{1,\ldots,n\}$  differing from S only in that it contains  $\{i_1,j_2\}$  and  $\{i_2,j_1\}$  instead of  $\{i_1,j_1\}$  and  $\{i_2,j_2\}$ . The monomial  $x_1^{(k_1)} \ldots x_n^{(k_n)}$  again appears in  $\tau(S)$  since  $k_{i_1} = k_{i_2} \neq k_{j_2}$  and  $k_{i_2} = k_{i_1} \neq k_{j_1}$ ; furthermore  $\tau(S)$  never equals S and  $\tau(\tau(S))$  always equals S. Thus there is an even number of such partitions S, and the monomial  $x_1^{(k_1)} \ldots x_n^{(k_n)}$  appears an even number of times in the expansion of  $\mathrm{Pf}(B_0)$ .

We have shown that in  $X_1^{\otimes n}/(2)$ ,

$$Pf(F - F^{\top}) = Pf(B_0) = \sum_{\sigma \in S_n} x_1^{(\sigma(1))} \dots x_n^{(\sigma(n))} = \det(x_i^{(j)})_{i,j=1}^n,$$

since all other terms in the sum appear an even number of times, and hence vanish. Thus the sum  $\det(x_i^{(j)})_{i,j=1}^n + \operatorname{Pf}(F - F^\top)$  vanishes modulo 2, so the sum is a multiple of 2 in  $X_1^{\otimes n}$ .

### A.3 A determinant identity

Lastly, we prove the claim in the proof of 5.13 that  $\det(B_{ij})_{i,j=1}^n = (1-n)\det(e_1(x_ix_j))_{i,j=1}^n$  for n even.

**Lemma A.3.** Let n be a natural number, and consider the following two matrices over the polynomial ring  $\mathbb{Z}[x_1,\ldots,x_n]^{\otimes n}$ : Let B be the  $n\times n$  matrix whose ijth entry is  $B_{ij}=e_1(x_i)e_1(x_j)-e_1(x_ix_j)$ , and let D be the  $n\times n$  matrix whose ijth entry is  $D_{ij}=e_1(x_ix_j)$ . Then

$$\det(B) = (-1)^n (1-n)\det(D).$$

*Proof.* First, note that if n=0 the identity becomes the tautology 1=1, so assume  $n \geq 1$ . Let v be the column vector whose ith entry is  $e_1(x_i)$ ; then  $B = vv^{\top} - D$ . Thus we need only prove that  $\det(D - vv^{\top}) = (1-n)\det(D)$ ; we will prove this in the larger ring  $\mathbb{Q}[x_1, \ldots, x_n]^{\otimes n}$ . Note that since  $vv^{\top}$  is a rank-1 matrix, the polynomial

$$p(\lambda) = \det(D - \lambda v v^{\top})$$

has constant term  $\det(D)$  and all terms of degree at least 2 vanishing; this can be checked by changing to a basis in which  $vv^{\top}$  is in row-echelon form, or more elementarily by showing that all contributions to the determinant involving at least two factors of  $\lambda$  must cancel. Writing  $p(\lambda) = \det(D) - \lambda a$ , we would like to show that  $a = n \det(D)$  so that  $p(1) = (1 - n)\det(D)$ . To prove this, we will show that p(1/n) = 0, i.e. that  $D - \frac{1}{n}vv^{\top}$  is singular.

Let A be the  $n \times \binom{n}{2}$  matrix with rows indexed by elements of  $\{1, \ldots, n\}$  and columns indexed by pairs  $(k, \ell)$  with  $1 \leq k < \ell \leq n$ , and whose  $(i, (k, \ell))$ th entry is  $A_{i,(k,\ell)} = x_i^{(k)} - x_i^{(\ell)}$ . Then we claim that  $D - \frac{1}{n}vv^{\top} = \frac{1}{n}AA^{\top}$ .

Indeed, we have

$$(AA^{\top})_{ij} = \sum_{k<\ell} \left( x_i^{(k)} - x_i^{(\ell)} \right) \left( x_j^{(k)} - x_j^{(\ell)} \right)$$

$$= \sum_{k<\ell} \left( x_i^{(k)} x_j^{(k)} + x_i^{(\ell)} x_j^{(\ell)} \right) - \sum_{k<\ell} \left( x_i^{(k)} x_j^{(\ell)} + x_i^{(\ell)} x_j^{(k)} \right)$$

$$= \sum_{k\neq\ell} x_i^{(k)} x_j^{(k)} - \sum_{k\neq\ell} x_i^{(k)} x_j^{(\ell)}$$

$$= (n-1) \sum_k x_i^{(k)} x_j^{(k)} - \sum_{k\neq\ell} x_i^{(k)} x_j^{(\ell)}$$

$$= n \sum_k x_i^{(k)} x_j^{(k)} - \sum_k x_i^{(k)} x_j^{(\ell)}$$

$$= n \sum_k x_i^{(k)} x_j^{(k)} - \sum_k x_i^{(k)} \sum_{\ell} x_j^{(\ell)}$$

$$= n e_1(x_i x_j) - e_1(x_i) e_1(x_j)$$

$$= n D_{ij} - v_i v_j.$$

Thus it remains only to show that A has rank less than n; then  $D - \frac{1}{n}vv^{\top}$  will also have rank less than n. So choose any n distinct columns of A, and we will show that they are linearly dependent. Indeed, such a choice amounts to choosing n distinct pairs of distinct elements from  $\{1, \ldots, n\}$ , which is the data of a graph with n edges and n vertices. Such a graph must contain a cycle, since a tree with n-1 edges already spans n vertices. Then our collection of n columns of n contains a subset of the form (up to changing the overall sign of some columns)

$$\begin{pmatrix} x_1^{(k_1)} - x_1^{(k_2)} \\ \vdots \\ x_n^{(k_1)} - x_n^{(k_2)} \end{pmatrix}, \begin{pmatrix} x_1^{(k_2)} - x_1^{(k_3)} \\ \vdots \\ x_n^{(k_2)} - x_n^{(k_3)} \end{pmatrix}, \dots, \begin{pmatrix} x_1^{(k_\ell)} - x_1^{(k_1)} \\ \vdots \\ x_n^{(k_\ell)} - x_n^{(k_1)} \end{pmatrix},$$

and the sum of these is 0. Thus every  $n \times n$  minor of A vanishes, so A has rank at most n-1, and so does  $D-\frac{1}{n}vv^{\top}$ . Then  $\det(D-\frac{1}{n}vv^{\top})=0$ , so  $\det(D-vv^{\top})=(1-n)\det(D)$  and  $\det(B)=(-1)^n(1-n)\det(D)$ .

## References

[1] Bhargava, M. Higher composition laws. III. The parametrization of quartic rings. Ann. of Math. (2) 159, 3 (2004), 1329–1360.

- [2] BIESEL, O., AND GIOIA, A. A new discriminant algebra construction. *ArXiv e-prints* (Mar. 2015).
- [3] Deligne, P. Cohomologie a supports propres. Springer, 1973.
- [4] GAN, W. T., GROSS, B., SAVIN, G., ET AL. Fourier coefficients of modular forms on g2. *Duke Mathematical Journal* 115, 1 (2002), 105.
- [5] Loos, O. Discriminant algebras and adjoints of quadratic forms. Beiträge Algebra Geom. 38, 1 (1997), 33–72.
- [6] Loos, O. Discriminant algebras of finite rank algebras and quadratic trace modules. *Math. Z. 257*, 3 (2007), 467–523.
- [7] REUTENAUER, C., AND SCHÜTZENBERGER, M.-P. A formula for the determinant of a sum of matrices. *Lett. Math. Phys.* 13, 4 (1987), 299–302.
- [8] Roby, N. Lois polynomes et lois formelles en théorie des modules. Ann. Sci. École Norm. Sup. (3) 80 (1963), 213–348.
- [9] Roby, N. Lois polynômes multiplicatives universelles. C. R. Acad. Sci. Paris Sér. A-B 290, 19 (1980), A869–A871.
- [10] ROST, M. The discriminant algebra of a cubic algebra. Available at http://www.math.uni-bielefeld.de/~rost/data/cub-disc.pdf, 2002.
- [11] Voight, J. Rings of low rank with a standard involution. *Illinois Journal of Mathematics* 55, 3 (2011), 1135–1154.
- [12] WOOD, M. Parametrizing quartic algebras over an arbitrary base. Algebra & Number Theory 5, 8 (2012), 1069–1094.