Modeling the Motion of Orbiting Bodies

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Abstract

This paper constructs a model for an orbiting body. This model is of the orbiting body's position (relative to the object it's orbiting) as a function of time. I will construct this model using Newton's laws, given an orbiting body's instantaneous velocity, its position, and the masses of the two bodies. I will also show how to find the initial measurements needed to construct the model.

Introduction

In this paper I hope to construct a model of an orbiting body's[1] position as a function of time. This model is a simple model derived using Newtonian laws and relies on the velocity, position, and mass of the orbiting body to be known, as well as the mass of the orbited object. The model relies on the given measurements being known, and therefore fails if we cannot obtain these measurements. Therefore I will also discuss how to obtain these measurements given other, more easily obtainable, measurements.

The development of the Newtonian model will require us to solve a second order nonlinear differential equation. I will ignore the volumes of the orbiting object and orbited object, and thus also ignore the possibility of a collision. The model will also ignore any movement of the orbited object, which doesn't matter much since we only want the relative position of the orbiting object.

[1] The term orbital bodies only applies to celestial objects in gravitational orbit as far as this paper is concerned.

Newtonian Model

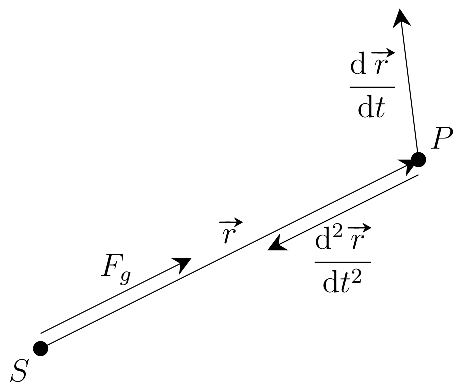


Figure 1: Orbiting Body Diagram

Suppose that m_s is the mass of an orbited body S, m_p is the mass of the corresponding orbiting body P, and $\vec{r}$ is the position vector of P relative to S (refer to Figure 1). The initial position of the orbiting body is

$$\vec{r}(0) = \vec{r}_0,$$

and the initial velocity is

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}(0) = \vec{v}_0.$$

We then find that the force exerted on the orbiting body is

$$m_p \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}.$$

Since this force is the opposing force to gravity ($\vec{F}_g$) predicted by Newton's third law, we find that

$$\begin{aligned}
m_p \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= -\vec{F}_g \\
m_p \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= -\frac{G m_s m_p}{|\vec{r}|^2} \frac{\vec{r}}{|\vec{r}|} \\
m_p \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= -\frac{G m_s m_p \vec{r}}{|\vec{r}|^3} \\
\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= -\frac{G m_s \vec{r}}{|\vec{r}|^3} \qquad (1)
\end{aligned}$$

Using the traditional physics notation where $dx/dt = \dot{x}$ and $d^2x/dt^2 = \ddot{x}$ equation (1) can be rewritten as

$$\ddot{\vec{r}} = -\frac{Gm_s\vec{r}}{|\vec{r}|^3}.$$

If we let $\vec{r} = \langle x, y\rangle$, then we can split equation (1) into to two second order ODEs that do not contain vectors.

$$\begin{aligned}
\ddot{x} &= -\frac{Gm_s x}{(x^2+y^2)^{3/2}} \\
\ddot{y} &= -\frac{Gm_s y}{(x^2+y^2)^{3/2}}
\end{aligned}$$

If we let $\dot{\vec{r}} = \left\langle v_x, v_y\right\rangle$, so that $v_x = \dot{x}$ and $v_y = \dot{y}$, then we get the following system of first order differential equations.

$$\dot{x} = v_x \tag{2}$$

$$\dot{v}_x = -\frac{Gm_s x}{(x^2+y^2)^{3/2}} \tag{3}$$

$$\dot{y} = v_y \tag{4}$$

$$\dot{v}_y = -\frac{Gm_s y}{(x^2+y^2)^{3/2}} \tag{5}$$

The initial conditions for this system are the components of $\vec{r}_0 = \langle x_0, y_0\rangle$, that is $x(0) = x_0$ and $y(0) = y_0$, and the components of $\vec{v}_0 = \left\langle v_{x0}, v_{y0}\right\rangle$, that is $v_x(0) = v_{x0}$ and $v_y(0) = v_{y0}$. This system can be solved using a numerical solver.

Example: Halley's Comet

We can use Halley's Comet to demonstrate how to solve the system of equations (2) through (5). To do this must first know that the mass of the sun is 1.99×10^{30} kg, the comet's distance from the sun at its perihelion[2] is 8.78×10^{10} m, its speed at the perihelion is 5.46×10^{4} m/s, and the gravitational constant is $G = 6.67 \times 10^{-11}$ Nm²/kg². Note that these measurements where taken on February 9, 1986.

$$\begin{aligned}
\dot{x} &= v_x \\
\dot{v}_x &= -\frac{1.33 \times 10^{20} x}{(x^2+y^2)^{3/2}} \\
\dot{y} &= v_y
\end{aligned}$$

[2]The point at which the comet is closest to the sun.

$$\dot{v}_y = -\frac{1.33 \times 10[20] y}{(x^2 + y^2)^{3/2}}$$
$$x(0) = 8.78 \times 10^{10}$$
$$y(0) = 0$$
$$v_x(0) = 0$$
$$v_y(0) = 5.46 \times 10^4$$

Using Mathematica we can numerically solve this system, but doing so **will not** give us equations for x or y. Instead it only gives us approximate values, which are graphed below.

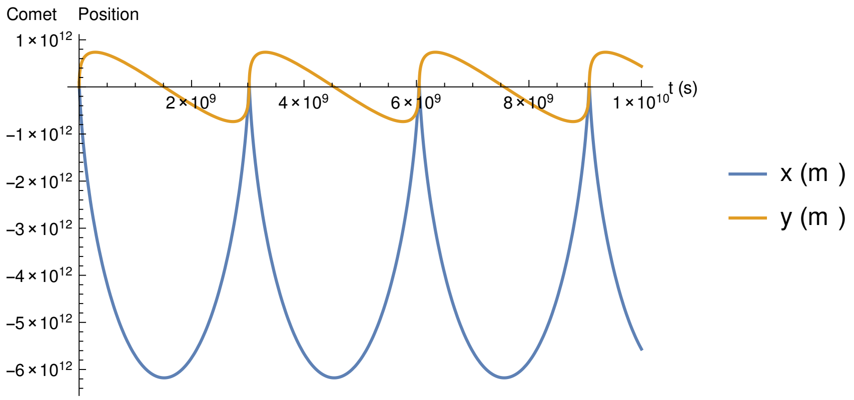


Figure 2: Halley's Comet's x and y Positions as a Function of Time

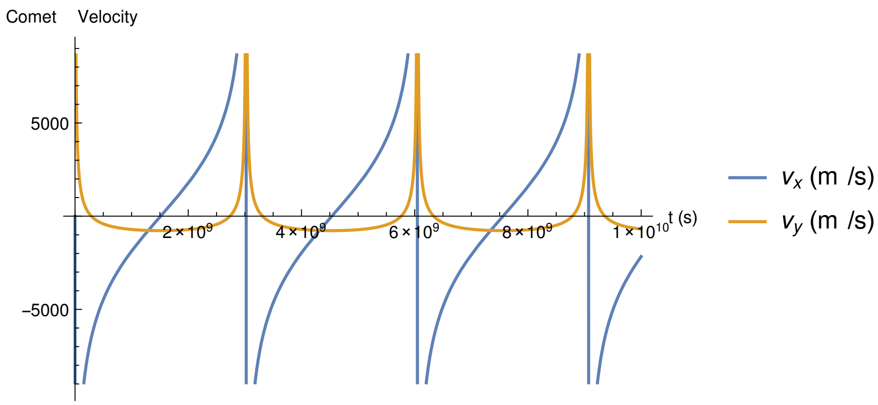


Figure 3: Halley's Comet's x and y Velocities as a Function of Time

If we plot the graph of the equations $x = x(t), y = y(t)$, then we can visualize the path Halley's Comet takes.

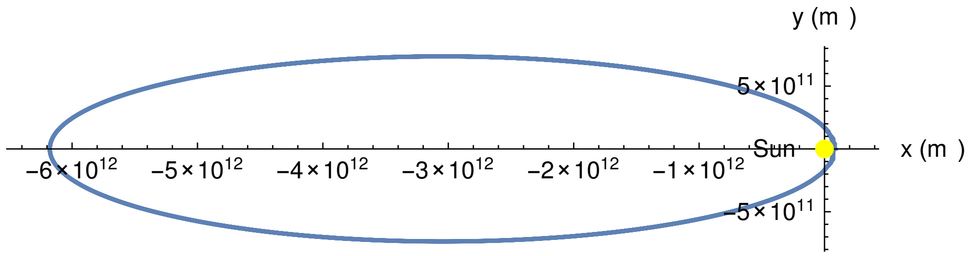


Figure 4: Halley's Comet's Path

Notice that Halley's Comet follows an elliptical path, as predicted by Kepler's third law. Noticing that $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$ we can also plot the speed of Halley's Comet as a function of time.

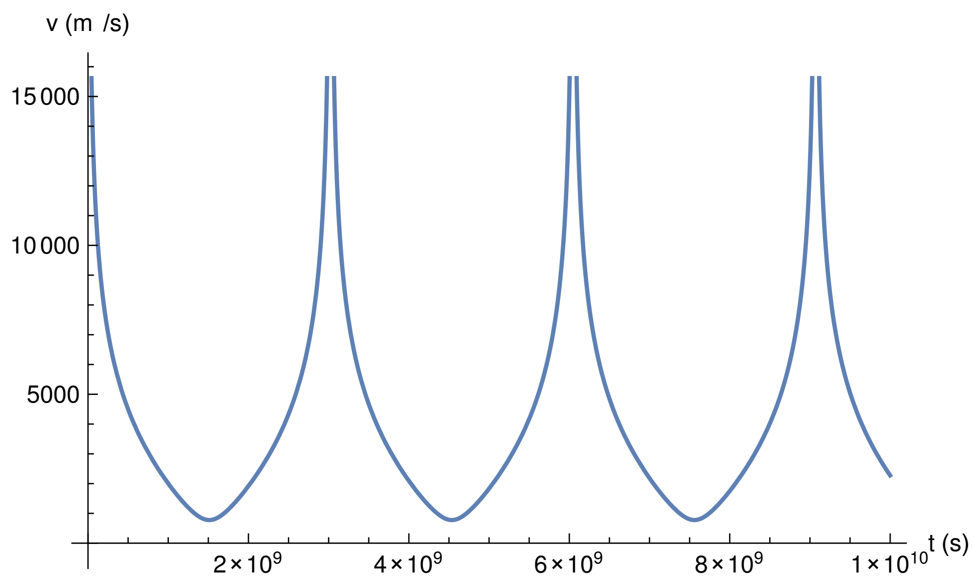


Figure 5: Halley's Comet's Speed

The troughs in the speed occur when the comet is at the aphelion[3], and the peaks occur at the perihelion. The source code for the images is appended to the end of this paper.

It should be noted that this model is not completely accurate. Aside from the fact that the models are only numerically based, this is because the gravitational effect of the planets, and other objects, in our solar system causes slight permutations in the orbit of Halley's comet.

[3]The point at which the comet is farthest to the sun.

Initial Conditions

I have gone over the steps to find a function for an orbiting body's relative position as a function of time, given the mass of the body it's orbiting and its instantaneous position and velocity. However, in real life we would not know these values, and we must therefore devise ways to calculate them. The position vector can easily be calculated using basic trigonometry, so I will assume the we have found that value.

Mass of Orbited Object

Unfortunately, for general elliptical orbits the mass of the orbited object and the velocity of the orbiting object cannot be calculated without knowing the other. If we launch a satellite into a circular orbit around the orbited object, then the mass calculation becomes a lot more straightforward. If we let T be the period of the satellite's orbit and r be its radius of rotation, then

$$v = \frac{2\pi r}{T}.$$

If we let m be the mass of the satellite, then we can find the centripetal force acting on it

$$F_c = m\frac{v^2}{r}.$$

Since the centripetal force is the gravitational force, we find that

$$\begin{aligned}
F_c &= F_g \\
m\frac{v^2}{r} &= \frac{GMm}{r^2} \\
M &= \frac{v^2 r}{G} \\
M &= \left(\frac{2\pi r}{T}\right)^2 \frac{r}{G} \\
M &= \frac{4\pi^2 r^3}{GT^2} \qquad (6)
\end{aligned}$$

Equation (6) gives us the mass of the orbited object in terms of the satellite's orbital radius and orbital period, both of which can easily be measured.

As an example let's calculate the sun's mass. The Earth has a nearly circular orbit, with an eccentricity of about 0.017, so we can use it as our satellite. The Earth has a mean orbital radius of 1.496×10^{11} m, and an orbital period of 3.156×10^7 s. Plugging this data into equation (6) gives us a mass of 1.988×10^{30} kg, which is close to the actual value of 1.989×10^{30} kg.

Velocity of Orbiting Object

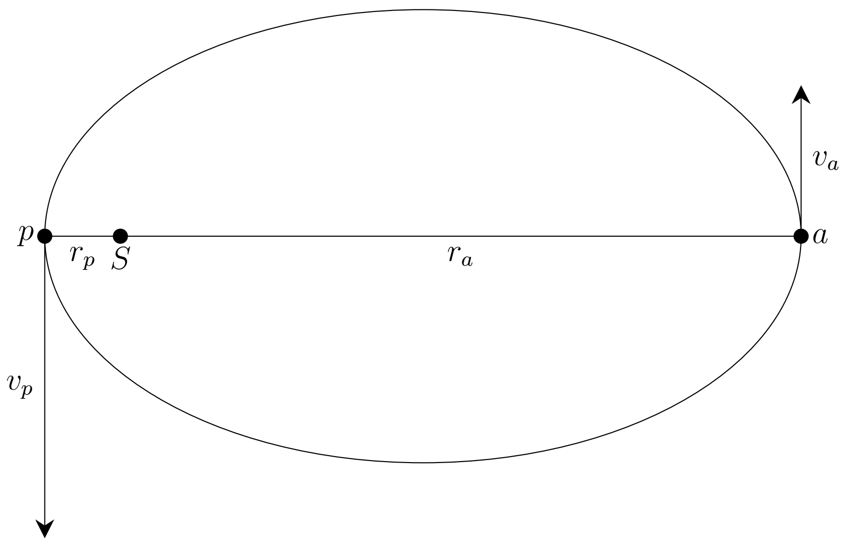


Figure 6: Path of an Orbiting Body

The velocity of the orbiting object is given by equation (7), which is known as the vis-viva equation.

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{7}$$

For this equation M is the mass of the orbited object, r is the distance from the orbited object to the orbiting object, and a is the semi-major of the orbiting objects path. It should be noted that by Kepler's first law all orbiting objects, in gravitational systems of two objects, follow elliptical paths. This is why it makes sense to include the semi-major in equation (7).

To derive the vis-viva equation we must assume that the total energy in the system is constant, which gives us

$$\begin{aligned} \sum E &= k \\ K + U_g &= k \\ \frac{1}{2}mv^2 - \frac{GMm}{r} &= k \end{aligned} \tag{8}$$

If we let subscripts of a denote values at the aphelion and p denote values at the perihelion (see Figure 6), then we find that

$$\begin{aligned}
\frac{1}{2}mv_a^2 - \frac{GMm}{r_a} &= \frac{1}{2}mv_p^2 - \frac{GMm}{r_p} \\
\frac{1}{2}v_a^2 - \frac{GM}{r_a} &= \frac{1}{2}v_p^2 - \frac{GM}{r_p} \qquad (9)
\end{aligned}$$

Angular momentum is also conserved, so we find that

$$\begin{aligned}
L_a &= L_p \\
r_a v_a \sin\frac{\pi}{2} &= r_p v_p \sin\frac{\pi}{2} \\
v_p &= \frac{r_a}{r_p} v_a \qquad (10)
\end{aligned}$$

Substituting equation (10) into (9) gives us

$$\begin{aligned}
\frac{1}{2}v_a^2 - \frac{GM}{r_a} &= \frac{1}{2}\left(\frac{r_a}{r_p}v_a\right)^2 - \frac{GM}{r_p} \\
\frac{1}{2}v_a^2 - \frac{GM}{r_a} &= \frac{r_a^2 v_a^2}{2r_p^2} - \frac{GM}{r_p} \\
\frac{1}{2}v_a^2\left(1 - \frac{r_a^2}{r_p^2}\right) &= \frac{GM}{r_a} - \frac{GM}{r_p} \\
\frac{1}{2}v_a^2\left(\frac{r_p^2 - r_a^2}{r_p^2}\right) &= GM\left(\frac{r_p - r_a}{r_a r_p}\right) \\
\frac{1}{2}v_a^2 &= GM\left(\frac{r_p}{r_a(r_p + r_a)}\right)
\end{aligned}$$

Since $r_a + r_p$ gives us the length of the major of the path, we know that $2a = r_a + r_p$, and

$$\frac{1}{2}v_a^2 = GM\left(\frac{2a - r_a}{2ar_a}\right) \qquad (11)$$

Substituting equation (11) into the left hand side of equation (8) we get

$$GMm\left(\frac{2a - r_a}{2ar_a}\right) - \frac{GMm}{r_a} = GMm\left(-\frac{1}{2a}\right)$$

$$= -\frac{GMm}{2a}$$

Since the left hand side of equation (8) is constant, we find that for whatever position the orbiting object is in the following is true.

$$\begin{aligned}
\frac{1}{2}mv^2 - \frac{GMm}{r} &= -\frac{GMm}{2a} \\
\frac{1}{2}v^2 - \frac{GM}{r} &= -\frac{GM}{2a} \\
\frac{1}{2}v^2 &= GM\left(\frac{1}{r} - \frac{1}{2a}\right) \\
v^2 &= GM\left(\frac{2}{r} - \frac{1}{a}\right)
\end{aligned}$$

Thus we have derived the vis-viva equation.

As an example let's calculate the velocity of Halley's Comet at its perihelion. The distance between the comet and the sun at the perihelion is 8.78×10^{10} m, and the comet's semi-major is 2.663×10^{12} m. Using the vis-viva equation we find that $v^2 = 2.96 \times 10^9\ ^{\mathrm{m}^2}\!/_{\mathrm{s}^2}$, and $v = 5.44 \times 10^4\ ^{\mathrm{m}}\!/_{\mathrm{s}}$. This result agrees with the more accurate measurement from the Halley's Comet example.

References

[1] Eugene Butikov. *Relative motion of orbiting bodies*. research. St. Petersburg State University, St. Petersburg, Russia, n.d.

[2] Emily Davis. “Deriving Kepler’s Laws of Planetary Motion”. Presentation Slide.

[3] Kyriacos Papadatos. “The Equations of Planetary Motion and Their Solution”. In: *The General Science Journal* (n.d.).

[4] Office of Public Information. *Comet Halley Summary*. N.A.S.A. URL: `http://er.jsc.nasa.gov/seh/halley.html`.

[5] Reymond Serway and John Jewett Jr. *Physics for Scientists and Engineers with Modern Physics*. Ninth. Brooks/Cole, 2014.

[6] David Surowski. *Kepler’s Laws of Planetary Motion and Newton’s Law of Universal Gravitation*. Retrieved from Surowski’s website.

[7] Wikipedia. *Wikipedia, The Free Encyclopedia*. 2015. URL: `http://en.wikipedia.org/`.

Source Code for Images

Halley’s Comet Images

In[1]:=
```
ms = 1.99*^30
```
Out[1]= 1.99×10^{30}

In[2]:=
```
pd = 8.78*^10
```
Out[2]= 8.78×10^{10}

In[3]:=
```
pv = 5.46*^4
```
Out[3]= 54600.

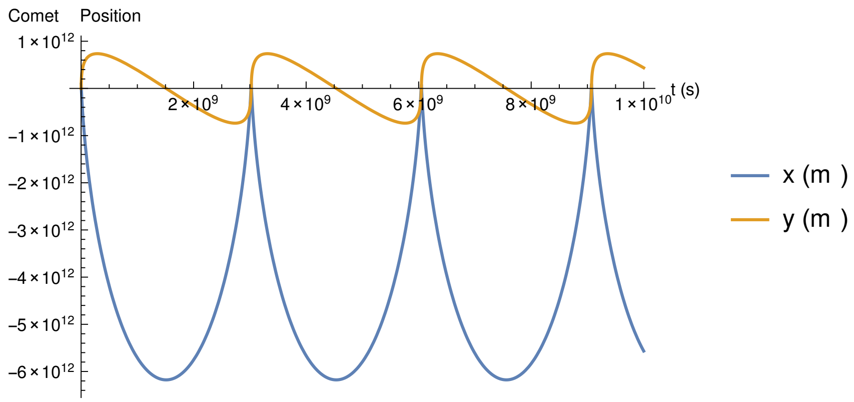
In[4]:=
```
g = 6.67*^-11
```
Out[4]= 6.67×10^{-11}

In[5]:=
```
ms *g
```
Out[5]= 1.32733×10^{20}

In[6]:=
```
{xSol, ySol, vxSol, vySol} = NDSolveValue[
    {x'[t] == vx[t], y'[t] == vy[t], vx'[t] == -(g ms x[t]) / (x[t]^2 + y[t]^2)^(3/2),
     vy'[t] == -(g ms y[t]) / (x[t]^2 + y[t]^2)^(3/2), x[0] == pd,
     y[0] == 0, vx[0] == 0, vy[0] == pv}, {x, y, vx, vy}, {t, 0, 1*^10}]
```
Out[6]=
```
{InterpolatingFunction[ Domain: {{0., 1.×10^10}} Output: scalar ],
 InterpolatingFunction[ Domain: {{0., 1.×10^10}} Output: scalar ],
 InterpolatingFunction[ Domain: {{0., 1.×10^10}} Output: scalar ],
 InterpolatingFunction[ Domain: {{0., 1.×10^10}} Output: scalar ]}
```

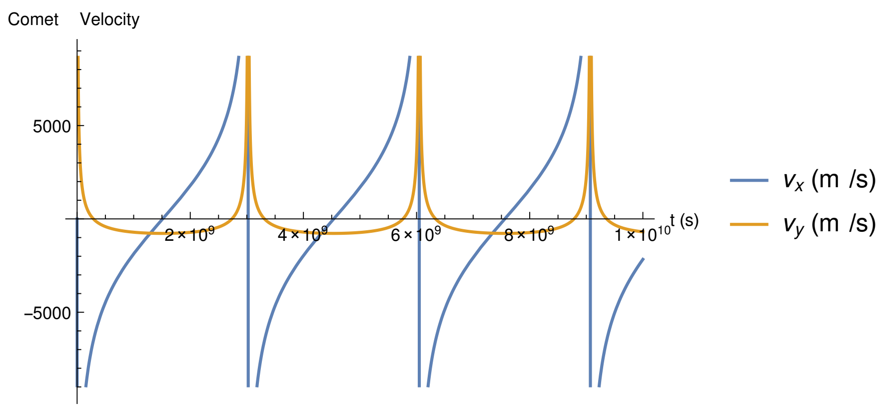
In[7]:=
```
posPlt = Plot[{xSol[t], ySol[t]}, {t, 0, 1*^10},
   AxesLabel → {"t  (s)", "Comet   Position"},
   PlotLegends → {"x  (m )", "y  (m )"}]
```

Out[7]=



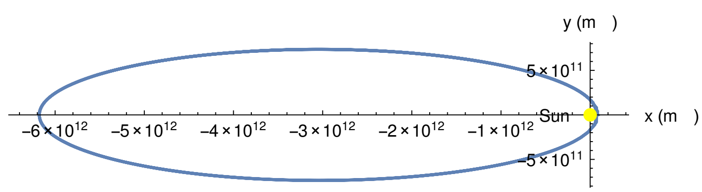
In[8]:=
```
velPlt = Plot[{vxSol[t], vySol[t]}, {t, 0, 1*^10},
   AxesLabel → {"t  (s)", "Comet   Velocity"},
   PlotLegends → {"vx  (m /s)", "vy  (m /s)"}]
```

Out[8]=



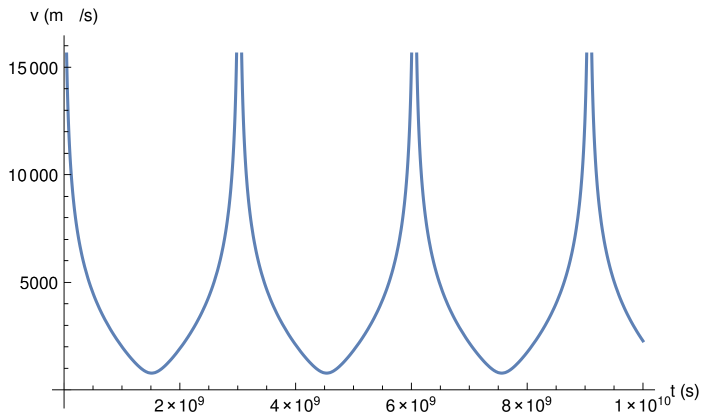
In[9]:=
```
xyPlt = ParametricPlot [{xSol[t], ySol[t]}, {t, 0, 1*^10},
   AxesLabel → {"x  (m )", "y  (m )"},
   Epilog→
    {Yellow, PointSize[Large], Point[{0, 0}], Black, Text["Sun", {-4*^11, 0}]}]
```

Out[9]=



In[10]:=
```
vxvyPlt = Plot[Sqrt[vxSol[t]^2 + vySol[t]^2], {t, 0, 1*^10},
  AxesLabel → {"t  (s)", "v  (m /s)"}]
```

Out[10]=



In[11]:=
```
Export["posPlt.pdf", posPlt]
```

Out[11]= posPlt.pdf

In[12]:=
```
Export["velPlt.pdf", velPlt]
```

Out[12]= velPlt.pdf

In[13]:=
```
Export["xyPlt.pdf", xyPlt]
```

Out[13]= xyPlt.pdf

In[14]:=
```
Export["vxvyPlt.pdf", vxvyPlt]
```

Out[14]= vxvyPlt.pdf

In[15]:=
```
Export["halley_comet .pdf", EvaluationNotebook[]]
```