Computable analysis with fast converging Cauchy sequences

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Two problems

A **practical** problem,

. . . and a **philosophical** one.

Floating point

$$\begin{pmatrix} 64919121 & -159018721 \\ 41869520.5 & -102558961 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution!

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 102558961 \\ 41869520.5 \end{pmatrix}$$

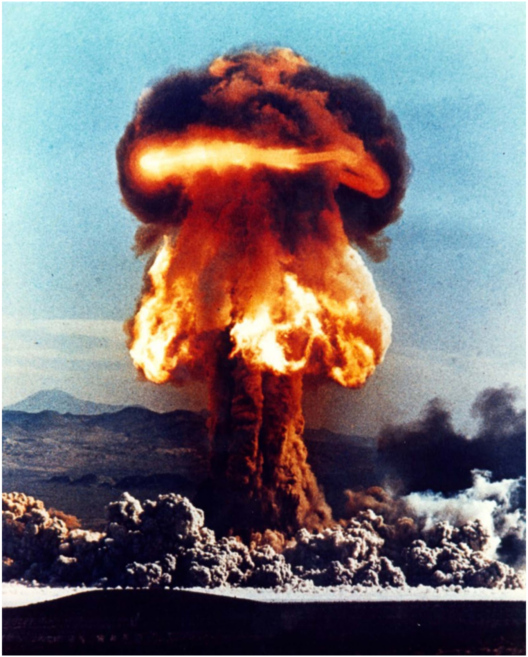
Solution. . . ?

Computed "solution"

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 102558961 \\ 41869520.5 \end{pmatrix}$$

Actual solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 205117922 \\ 83739041 \end{pmatrix}$$



(Mathematical) constructivism

Constructivism is the belief that to prove an object exists, we must provide a procedure to construct it

Constructivists reject the excluded middle ($A \vee \neg A$) and double negation elimination ($\neg\neg A \to A$) laws

Reaction against a perceived "lack of meaning" in modern mathematics

The pure mathematician is isolated... He suffers from an alienation which is seemingly inevitable; he has followed the gleam and it has led him out of this world.

—Bishop and Bridges, *Constructive Analysis*, 1987

Construction $\sim$ Computation

Since constructive proofs describe procedures, they can be written as computer programs

Called the "Curry–Howard correspondence" or "BHK interpretation"

So **constructive analysis** is also **computable**

Fast converging Cauchy sequences

Definition

Let $x : \mathbb{N} \to \mathbb{Q}$ be a sequence of rational numbers.
If for any natural numbers n and m,

$$|x_n - x_{n+m}| < \frac{1}{2^n} ,$$

then x is a **fast converging Cauchy sequence**.

Intuition: each element is at most $1/2^n$ away from the limit point.

Addition and subtraction

When we add two sequences pointwise, we also add the errors

So the error of the result is at most *double* the error of the inputs

Solution: shift the sequence by one step, halving the error to compensate

Addition and subtraction

Example

Let $x := (2, 1\frac{1}{2}, 1\frac{1}{4}, 1\frac{1}{8}, \dots)$.
Let $y := (3, 2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, \dots)$.
Adding pointwise, we get $z = (5, 4, 3\frac{1}{2}, 3\frac{1}{4}, \dots)$.
This is not fast converging because $|z_0 - z_2| = 1\frac{1}{2} \not< \frac{1}{2^0}$.
Shift z to get $z' = (4, 3\frac{1}{2}, 3\frac{1}{4}, \dots)$ which is fast converging.

Multiplication

With multiplication, the shift depends on the magnitude of the inputs

For example, if $x = 16$ then we need to shift by $\log_2 16 = 4$ steps

(Details in thesis)

Division?

We'll come back to that!

Equality

Definition

A pair of FCCS x and y are **equal** when for every natural number n,

$$|x_{n+1} - y_{n+1}| \leq \frac{1}{2^n} \; .$$

Note the use of $n + 1$ as the index!

Equality is undecidable

Theorem

The proposition

$$x = y \vee \neg(x = y)$$

is not derivable for arbitrary x and y.

Proof.

Given an arbitrary program P, define a FCCS x which converges to zero if and only if P does not terminate. By checking if $x = 0$, we can solve the Halting Problem. Contradiction. □

Inequality and apartness

Constructive analysis makes a distinction between *non-constructive* (not-equal) and *constructive* inequality (apart):

Not equal Not all elements are close enough

Apart We *know* the index after which subsequent elements will never come closer together

$\mathrm{A} \Rightarrow \mathrm{NE}$ always, but $\mathrm{NE} \Rightarrow \mathrm{A}$ requires **Markov's principle**.

Under classical logic, Markov's principle is trivial and so the two notions are equivalent.

Division

The reciprocal of x is only defined when $x \neq 0$

In fact, evidence of apartness from 0 gives a lower bound on the size of x

This bound tells us how much we need to shift x^{-1}

Exponentials and trigonometric functions

Use power series expansion

$$\exp(x) = \sum_{n=0}^{\infty} = 1 + x + \frac{x}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Observe that when $-1 \leq x \leq 1$, the numerator of each term is smaller than 1

Strategy: prove fast convergence on $[-1, 1]$, then use identity $\exp(2x) = \exp(x)^2$ to extend it to all of $\mathbb{Q}$

Can define sin and cos in a similar way

Question: what if x is a FCCS?

Cantor's diagonal argument

Theorem

Let $x : \mathbb{N} \to \mathrm{FCCS}$ *be a sequence of fast converging Cauchy sequences. Then we can construct a FCCS* $\mathrm{cantor}(x)$ *that is apart from every element in* x*.*

Analogous to argument on $\mathbb{R}$

Shows that the set of computable reals (FCCS) is **countable** but not **enumerable**

Thanks for listening!

https://github.com/lambda-fairy/plusone