Complex Analysis

Gilles Castel

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Contents

Les 1: Inleiding

di 12 feb 16:00

- Timur
- English
- Book: 'Complex Analysis', Stein, Shakarchi, vol. 2
- Exam is the most important part of the course. (Biggest focus is exercises)

Chapter 1

Complex numbers

Complex numbers

Definition 1 (Complex numbers)**.**

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}.$$

Addition and multiplication are defined as seen before:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$
$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

The complex numbers form a field.

Notation. $a + i0 = a$ and $0 + ib = ib$.

Definition 2. $z = a + ib$, then $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$

Definition 3. If $z = a + ib$, then $\overline{z} = a - ib$

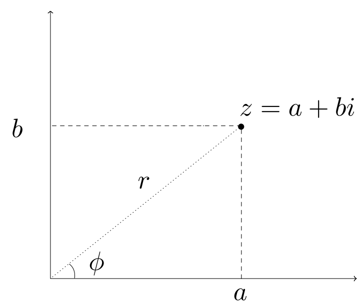

Figure 1.1: Complex plane

Definition 4. $r = |z|$, which is the absolute value of z, and $\phi = \arg z$ is the argument of z.

Note that the argument is not unique, therefore we define the principal value of arg:

Definition 5 (Principal argument)**.** The argument in $(-\pi, \pi]$.

Property.

- $|z|^2 = z\overline{z}$
- $|z|$ is a norm (triangle inequality, ...)
- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Note that we cannot compare complex numbers.

Another way. Riemann sphere. Complex numbers are a sphere $S^2 \setminus \{N\}$.

Definition 6 (Extended set of complex numbers)**.**

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = S^2.$$

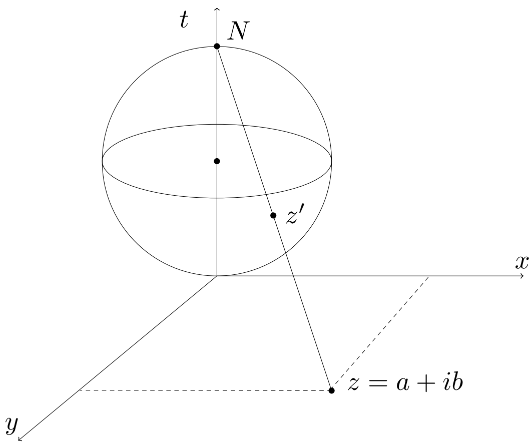

Figure 1.2: Threedimensional representation of $\mathbb{C}$: the Riemann sphere.

UOVT. Find an explicit formula for z'.

Convergence

Definition 7. We say that $z_n \to z$ if $|z_n - z| \to 0$.

Note. Suppose $z_n = a_n + ib_n$ and $z = a + ib$. Suppose $z_n \to z$, then $|z - z_n|^2 = |a_n - a|^2 + |b_n - b|^2 \to 0$, iff $|a_n - a| \to 0$ and same for b.

Proposition 1. $a_n + ib_n \to a + ib$ iff $a_n \to a$ and $b_n \to b$.

Theorem 1. Cauchy creterion z_n converges iff

$$\forall \varepsilon : \exists N : \forall n, m > N : |z_n - z_m| < \varepsilon.$$

Definition 8. $f(\zeta) \xrightarrow{\zeta \to z_0} z$ iff $|f(\zeta) - z| \xrightarrow{\zeta \to z_0} 0$.

Subsets of the complex plan

Let z_0 be a complex number and $r > 0$.[1]

Definition 9 (Open disk)**.** $D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$

Definition 10 (Closed disk)**.** $\overline{D_r(z_0)} = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$

Definition 11 (Circle)**.** $C_r(z_0) = \{z \mid |z - z_0| = r\}$

It's clear that $\overline{D_r(z_0)} = D_r(z_0) \cup C_r(z_0)$.

Suppose $\Omega \subset \mathbb{C}$ and $z_0 \in \Omega$.

Definition 12 (Interior point)**.** z is an interior point iff $\exists r > 0 : D_r(z_0) \subset \Omega$

Example. $\Omega = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1\}$. Then $\frac{1}{2} + 2i$ is an interior points, but i is not.

Definition 13 (Open)**.** A set Ω is called open when all points of Ω are interior points.

Example. $\Omega = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$

Definition 14 (Closed)**.** Ω is called closed if Ω^c is open.

Proposition 2. A point z_0 is called a limit point of Ω if there exists a sequence of points $z_n \in \Omega \setminus \{z_0\}$ such that $z_n \xrightarrow{n \to \infty} z_0$.

Example. $\Omega = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1\}$. Take $z_0 = 0$, and $z_n = \frac{1}{n} \to 0$, therefore, 0 is a limit point. Likewise, 1 and $\frac{1}{2}$ is a limit point.

Proposition 3. A set is called closed iff it contains all its limit points.

Proof. Ex. □

[1] Hence, we assume $r \in \mathbb{R}$

Definition 15 (Closure). $\overline{\Omega} = \Omega \cup \{$ limit points of $\Omega\}$

Definition 16 (Interior). $\mathring{\Omega}$ is the set of all interior points of Ω.

Definition 17 (Boundary). $\partial\Omega = \overline{\Omega} \setminus \mathring{\Omega}$

Definition 18 (Bounded set). Ω is called bounded if $\exists r > 0$ such that $\Omega \subset D_r(0)$.

Definition 19 (Open cover). A set $U_\alpha \subset \Omega$ is called an open cover if $\Omega \subset \bigcup_\alpha U_\alpha$, and U_α is open.

Definition 20 (Compact). Ω is called compact if (TFAE)

1. Ω is closed and bounded.
2. Every sequence $z_n \in \Omega$, there exists a subsequence z_{n_k} that converges in Ω.
3. Every open cover of Ω can be reduced to a finite subcover.

Proof. Exercise. □

Definition 21 (Connected). An open (closed) set Ω is called connected if it *cannot* be written as $\Omega = \Omega_1 \cup \Omega_2$ for $\Omega_1 \cap \Omega_2$ empty and Ω_1, Ω_2 open (closed). (Ω_1, Ω_2 not empty)

Note. We only consider open and closed sets in this definition

Definition 22 (Path-connected). A set Ω is called path connected if $\forall x, y \in \Omega : \exists \gamma : [0,1] \to \Omega$, such that $\gamma(0) = x$ and $\gamma(1)$, and γ continuous.

Proposition 4.

- Open path connected, iff it is open connected.
- Closed path connected, then it is closed connected.

Definition 23 (Region). Ω is called a region if Ω is open and connected.

Intermezzo.

$$\int_{-\infty}^{\infty} e^{-x^2} = \dots .$$

◇

Functions on $\mathbb{C}$

Definition 24. Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We call f continuous at $z_0 \in \mathbb{C}$ if for all sequences $z_n \to z_0$, $f(z_n) \to f(z_0)$.

Suppose $f(z) = f(x+iy) = f(x,y) = u(x,y) + iv(x,y)$. Now, $f(z_n) \to f(z)$ iff $|f(z_n) - f(z)| \to 0$. This is equivalent with the square going to 0. Therefore this is equivalent with

$$|u(x_n, y_n) - u(x,y)|^2 + |v(x_n, y_n) - v(x,y)|^2 \to 0,$$

which is again equivalent with

$$|u(x_n, y_n) - u(x,y)| \to 0 \quad \wedge \quad |v(x_n, y_n) - v(x,y)| \to 0.$$

Proposition 5. $f(z)$ is continuous at $z_0 = x_0 + iy_0$ iff u and v are continuous at (x_0, y_0).

Note. If f is continuous at z_0, then $g : z \to |f(z)|$ is continuous at z_0. (Composition of continuous functions)

Definition 25 (Maximum)**.** We say that f attains a maximum at $z_0 \in \Omega$ if

$$|f(z_0)| \geq |f(z)| \text{ for all } z \in \Omega.$$

Definition 26 (Minimum)**.** We say that f attains a minimum at $z_0 \in \Omega$ if

$$|f(z_0)| \leq |f(z)| \text{ for all } z \in \Omega.$$

Proposition 6. Let Ω be a compact set. Then a continuous function $f : \Omega \to \mathbb{C}$ attains its maximum and its minimum.

Example. Consider $f : \mathbb{C} \to \mathbb{C} : z \mapsto \operatorname{Re}(z)$. This is clearly continuous (proof using sequences) Take $\Omega_1 = D_1(0)$. and $\Omega_2 = \overline{D_1(0)}$. Then $\max_{z \in \Omega_2} \operatorname{Re} z = 1$, but $\max_{z \in \Omega_1} \operatorname{Re} z$ does not exists. The minimum is attained in 0, as we look at the modulus.

Definition 27. Let f_n be a sequence of functions from $\mathbb{C} \to \mathbb{C}$.

- $f_n \to f$ pointwise on Ω if for all $z \in \Omega$: $f_n(z) \to f(z)$
- $f_n \to f$ uniformly on Ω if $\forall \varepsilon > 0 : \exists N : \forall n > N : \forall z \in \Omega : |f_n(z) - f(z)| < \varepsilon$
- $f_n \to f$ uniformly on compact subsets of Ω, if for all compact subsets of Ω, $f_n \to f$ uniformly.

pointwise $\Leftarrow$ uniformly on compact subsets $\Leftarrow$ uniformly

Les 2: Holomorphic and power series

di 19 feb 16:00

Chapter 2

Power series

Opmerking (Examen)**.**

- Completely written.
- First part: 30 minutes, closed book. Either definition of theorem. E.g. Wat is closed set? You don't have to prove anything. $5 \times /1 = /5$
- Second part: Exercises. About 4 exercises increasing difficulty.

Example (Pointwise but not on compact subsets)**.** let

$$\Omega = \{z \mid \operatorname{Im} z = 0, 0 \leq \operatorname{Re} \leq 1\}.$$
$$f_n(z) = z^n$$
$$f(z) = \begin{cases} 0 & z \neq 1 \\ 1 & z = 1. \end{cases}$$

Clearly$f_n \to f$ pointwise. But Ω is compact and therefore f does not converge uniformly on compact subsets of Ω.

Example (Uniformly on compact subsets, not uniformly)**.**

$$\Omega = \{z \mid \operatorname{Im} z = 0, 0 \leq \operatorname{Re} z < 1\}.$$

and

$$f_n(z) = z^n, f(z) = 0.$$

UOVT

Remember:

$$\lim_{z \to z_0} f(z) = w \Leftrightarrow \lim_{z \to z_0} |f(z) - w| = 0.$$

We define the following:

$$\lim_{z \to \infty} f(z) = w \Leftrightarrow \forall \varepsilon : \exists R : \forall z : |z| > R \Rightarrow |f(z) - w| < \varepsilon.$$

Holomorphic functions

Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ a function.

Definition 28 (Holomorphic function, complex differentiable, regular)**.** Then f is holomorphic at z_0 if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Here $h \neq 0, z_0 + h \in \Omega$, i.e. h small enough such that $z_0 + h \in \Omega$.

If $\lim\limits_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists, then we denote it by $f'(z_0)$.

Note. In complex analysis, if a function is once derivable, it is infinitely differentiable

Definition 29 (Holomorphic)**.** f is holomorphic on Ω if f is holomorphic at every point of Ω.

Definition 30. If $S \subset \mathbb{C}$, not per se open, then we say that f is holomorphic on S if there exists an open set $\Omega \supset S$ and f is holomorphic on Ω. Note that f must be defined on S too!

Example. $f(z) = az$, $a \in \mathbb{C}_0$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(z_0)}{h} = a.$$

Therefore $f(z) = az$ is holomorphic at each point of $\mathbb{C}$.

Example. $f(z) = \overline{z}$. Then

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z)}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}.$$

Suppose this limit exists. Then it is equal to

$$\lim_{\mathbb{R} \ni h \to 0} \frac{\overline{h}}{h} = 1.$$

On the other hand, if this limit exists, then it is equal to

$$\lim_{\mathbb{C} \setminus \mathbb{R} \ni h \to 0} \frac{\overline{h}}{h} = -1.$$

Therefore f is not holomorphic at every point of $\mathbb{C}$.

Definition 31 (Entire function)**.** f is called entire if f is holomorphic at each point of $\mathbb{C}$.

Proposition 7. Let Ω be an open set.

- If f, g are holomorphic on Ω, then $f + g$ is holomorphic on Ω and $(f + g)' = f' + g'$.
- If f, g are holomorphic on Ω, then $f \cdot g$ is holomorphic on Ω and $(fg)' = f'g + fg'$.
- If f, g are holomorphic at $z_0 \in \Omega$, and $g(z_0) \neq 0$, then $\frac{f}{g}$ is holomorphic at z_0 and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- If $f : \Omega \to U$ (U open), holomorphic on Ω , $g : U \to \mathbb{C}$ holomorphic on U, then $g \circ f$ is holomorphic on Ω and $(g \circ f)' = g'(f(z))f'(z)$

Proof. See Real analysis. □

Corollary 1. Every polynomial is an entire function.

Note. f is holomorphic at z_0 if

$$\exists a : f(z_0 + h) - f(z_0) - ha = h\psi(h),$$

with $\lim\limits_{h\to 0} \psi(h) = 0$, i.e. $|h|\psi(h) = o(h)$.

$f(z) = f(x + iy) = u(x, y) + iv(x, y)$

Definition 32.

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).\end{aligned}$$

Theorem 2. Let f be a holomorphic function at z_0. *Then*

- $\frac{\partial}{\partial x}u = \frac{\partial}{\partial y}v$ and $\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}v$, Cauchy-Riemann equations.
- $\frac{\partial}{\partial \bar{z}}f = 0$
- $f'(z_0) = \frac{\partial}{\partial z}f = 2\frac{\partial}{\partial z}u$

Proof. Part 1:

$$\begin{aligned}f'(z_0) &= \lim_{h\to 0}\frac{f(z_0 + h) - f(z_0)}{h}\\ &= \lim_{\mathbb{R}\ni h\to 0}\frac{f(z_0 + h) - f(z_0)}{h}\\ &= \lim_{h\to 0}\frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}\\ &= \frac{\partial}{\partial x}f = \frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v.\end{aligned}$$

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\
&= \lim_{\mathbb{R}\ni h\to 0} \frac{f(z_0+ih)-f(z_0)}{ih} \\
&= \frac{1}{i} \lim_{\mathbb{R}\ni h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} \\
&= \frac{1}{i}\frac{\partial}{\partial y} f = \frac{1}{i}\left(\frac{\partial}{\partial y}u + i\frac{\partial}{\partial y}v\right).
\end{aligned}$$

Therefore

$$\frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v = \frac{1}{i}\frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v,$$

and thus

$$\begin{aligned}
\frac{\partial}{\partial x}u &= \frac{\partial}{\partial y}v \\
\frac{\partial}{\partial y}u &= -\frac{\partial}{\partial x}v.
\end{aligned}$$

Part 2:

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}f &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)(u+iv) \\
&= \frac{1}{2}\left[\frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v - \frac{1}{i}\frac{\partial}{\partial y}u - \frac{\partial}{\partial y}v\right] \\
&= 0,
\end{aligned}$$

using the CR equations.

Part 3:

$$\begin{aligned}
f'(z_0) &= \frac{1}{2}\left[\frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v + \frac{1}{i}\frac{\partial}{\partial y}u + \frac{1}{i}\frac{\partial}{\partial y}v\right] \\
&\overset{\text{CR}}{=} \frac{\partial}{\partial z}f \overset{\text{CR}}{=} 2\frac{\partial}{\partial z}u.
\end{aligned}$$

□

The reverse is not true. Sufficient that $\mathbb{R}^2 \to \mathbb{R}$ continuous differentiable. Is being differentiable enough? NO.

> **Theorem 3.** Let u and v be functions $\mathbb{R}^2 \to \mathbb{R}$ continuous differentiable at $z_0 = (x_0, y_0)$ and satisfy the Cauchy-Riemann equations. Then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic at $z_0 = x_0 + iy_0$.

Proof. u, v are continuously differentiable.

$$u(x_0+h_1, y_0+h_2) - u(x_0, y_0) = h_1 \frac{\partial}{\partial x} u + h_2 \frac{\partial}{\partial y} u + |h|\psi_1(|h|)$$
$$v(x_0+h_1, y_0+h_2) - v(x_0, y_0) = h_1 \frac{\partial}{\partial x} v + h_2 \frac{\partial}{\partial y} v + |h|\psi_2(|h|),$$

$\psi(|h|) \to 0$ as $h \to 0$.

Therefore

$$\begin{aligned} f(z_0+h) - f(z_0) &= u(z_0+h) - u(z_0) + i(v(z_0+h) - v(z_0)) \\ &= h_1 \frac{\partial}{\partial x} u + h_2 \frac{\partial}{\partial y} u + ih_1 \frac{\partial}{\partial x} v + ih_2 \frac{\partial}{\partial y} v + |h|\psi(|h|) \\ &\overset{\text{CR}}{=} (h_1 + ih_2) \frac{\partial}{\partial z} f + |h|\psi(|h|). \end{aligned}$$

□

Power series

Definition 33. Power series is $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$.

Definition 34. Denote $S_N = \sum_{n=0}^{N} a_n z^n$. We say that $\sum_{n=0}^{\infty}$ converges if S_N converges when $N \to \infty$.

Definition 35. We say that a series converge absolutely if $\sum_{n=0}^{\infty} |a_n||z^n|$ converges.

Example. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ if $|z| < 1$.

Theorem 4. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series. Let

$$L = \lim_{n\to\infty} \sup k \geq n \sqrt[k]{|a_k|}.$$

- If $L = 0$ then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all $z \in \mathbb{C}$.
- If $L = \infty$, then $\sum_{n=0}^{\infty} a_n z^n$ diverges for all $z \in \mathbb{C} \setminus \{0\}$
- If $L < \infty$, then let $R = \frac{1}{L}$, then for all $z \in D_R(0)$, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. For all z outside the closure, the series diverges.

Proof.
- Let $z \in \mathbb{C}$. $|z| = r > 0$. Since $L = 0, \exists N : \forall n > N : \sqrt[n]{|a_n|} < \frac{1}{2r}$,
$$\sum_{n=N}^{\infty} |a_n||z^n| \leq \sum_{n=N}^{\infty} \frac{1}{(2r)^n} r^n = \sum_{n=N}^{\infty} \frac{1}{2^n} \xrightarrow{n\to\infty} 0.$$
- $L = \infty$, therefore, there exists a sequence a_{n_k} such that $|a_{n_k}| \xrightarrow{k\to\infty} \infty$. Therefore, for fixed z, $|a_{n_k} z^{n_k}| \to \infty$, so the series diverges.

- $R = \frac{1}{L}$. Let $|z| < R$. Then $\exists \delta : |z(1+\delta)| < R$. Let $\varepsilon = \frac{\delta}{R}$. $\exists N : \forall n > N$: $\sqrt[k]{|a_n|} \leq L + \varepsilon$, so $|a_n| \leq (L+\varepsilon)^n$ Therefore,

$$\sum_{n=N}^{\infty} |a_n z^n| \leq \sum_{n=N}^{\infty} (L+\varepsilon)^n |z|^n = \sum_{n=N}^{\infty} \left(\frac{1}{R} + \frac{\delta}{R}\right)^n |z|^n = \sum_{n=N}^{\infty} \frac{|(1+\delta)z^n|}{R^n} < \infty$$

Similarly, if $|z| > R$, the series diverges.

□

Let's agree that $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

Definition 36 (Radius of convergence, Hadamar's formula)**.** The radius of convergence is

$$R = \frac{1}{L} = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}.$$

Definition 37. $D_R(0)$ is called the disc of convergence.

Note. It can converge or diverge on the boundary (including converging at some points, ...)

Example. $\sum_{n=0}^{\infty} z^n$, then $R = 1$.

Example. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. $L = \limsup \sqrt[n]{\frac{1}{n!}} = 0$. Therefore this series converges absolutely at every point, it's e^z

Example. $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sin z$, $R = \infty$

Example. $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos z$, $R = \infty$

$$e^{iz} = \cos z + i \sin z \Rightarrow e^{i\pi} = -1.$$

Les 3: Integration

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Theorem 5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then $f(z)$ is holomorphic on its disc of convergence. Then $f'(z)$ has same radius of convergence and is given by

$$\sum_{n=1}^{\infty} a_n n z^{n-1}.$$

Proof. Let $g(z) = \sum_{n=1}^{\infty} a_n n z^{n-1}$. Since $\sqrt[n]{n} \to 1$ as $n \to \infty$, then

$$\lim_{n\to\infty} \sup_{k\geq n} \sqrt[n]{|a_n n|} = \lim_{n\to\infty} \sup_{k\geq n} \sqrt[n]{|a_n|}.$$

Therefore g and f have the same radius of convergence.

Let R be radius of convergence of f. Choose z such that $|z| < r < R$, let $h \in \mathbb{C}$ such that $|z+h| < r$.

We need to prove that

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| = 0.$$

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n \\ &= S_N(z) + E_N(z). \end{aligned}$$

Then the limit becomes

$$\begin{aligned} &\left|\frac{S_N(z+h)-S_N(z)}{h} - S_N' + S_N'(z) - g(z) + \frac{E_N(z+h)-E_N(z)}{h}\right| \\ &= \left|\frac{S_N(z+h)-S_N(z)}{h} - S_N'\right| + |S_N'(z) - g(z)| + \left|\frac{E_N(z+h)-E_N(z)}{h}\right| \\ &= I + II + III. \end{aligned}$$

- As S_N is a simple polynomial, we het that $I \xrightarrow{h\to 0} 0$.
- Since $S_N'(z) = \sum_{n=1}^{N} a_n n z^{n-1} \xrightarrow{n\to\infty} g(z)$, by definition of convergence of series.

Now let's look at

$$\begin{aligned} \left|\frac{E_N(z+h)-E_N(z)}{h}\right| &= \left|\frac{\sum_{n=N+1}^{\infty} a_n(z+h)^n - \sum_{n=N+1}^{\infty} a_n z^n}{h}\right| \\ &= \left|\frac{1}{h}\sum_{n=N+1}^{\infty} a_n((z+h)^n - z^n)\right|. \end{aligned}$$

Now note that

$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1}).$$

We get

$$\begin{aligned} &= \left|\frac{1}{h}\sum_{n=N+1}^{\infty} a_n h((z-h)^{n-1} + \cdots + z^{n-1})\right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n||(|z+h|^{n-1} + |z+h|^{n-2}|z| + \cdots + |z|^{n-1}) \\ &\leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1} \xrightarrow{N\to\infty} 0. \end{aligned}$$

As $r < R$, this series does converge. As this is the *tail* of a convergent series, $III \to 0$. Therefore: $f' = g$. $\square$

Corollary 2. Power series are infinitely differentiable!

Definition 38. A power series centered at $z_0 \in \mathbb{C}$ is

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

We can prove the above by chain rule and change of variables.

Definition 39. A function $f(z)$ is called analytic at $z_0 \in \mathbb{C}$ if $\exists r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D_r(z)$.

Corollary 3. If $f(z)$ analytic at z_0, then $f(z)$ is holomorphic at z_0.

Note. The converse: holomorphic implies analytic is also true, but its proof is complicated.

Chapter 3

Integration

Definition 40 (Smooth curve)**.** A smooth parametrized curve is a map (we call it the parametrisation) $z : [a, b] \to \mathbb{C}$ such that

- $\exists z'(t)$ such that $\forall t \in [a, b]$. Note that in a and b, we consider the one-sided derivative.
- $z'(t)$ is continuous
- $z'(t) \neq 0$

Note. Note that we distinguish between the curve itself: $z([a, b])$ and the curve $z : [a, b] \to \mathbb{C}$.

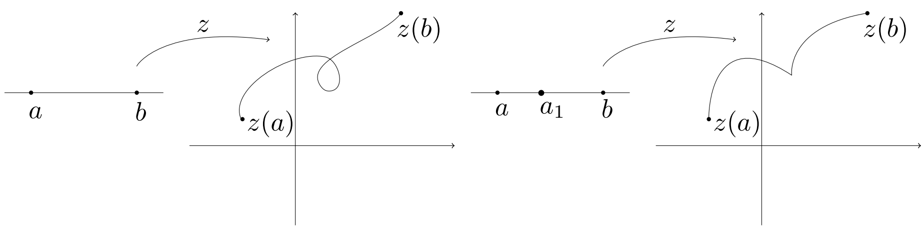

Figure 3.1: On the left: an example of a smooth curve, on the right: a piecewise smooth curve.

Definition 41 (Piecewise smooth curvve)**.** A parametrized piecewise smooth curve is a continuous map $z : [a, b] \to \mathbb{C}$ such that there exists $a = a_0 < a_1 < a_2 < \ldots < a_n = b$, such that on each $[a_i, a_{i+1}]$ we have a smooth parametrized curve.

Definition 42. We say that two smooth parametrized curves $z[: [a,b] \to \mathbb{C}$, $\tilde{z}[c,d] \to \mathbb{C}$ are equivalent if there exists a continuous differentiable bijection $t : [a,b] \to [c,d]$ such that

- $z(s) = \tilde{z}(t(s))$ (same image)
- $t'(s) > 0$ (orientation is the same)

Note. Two curves with different orientation are different!

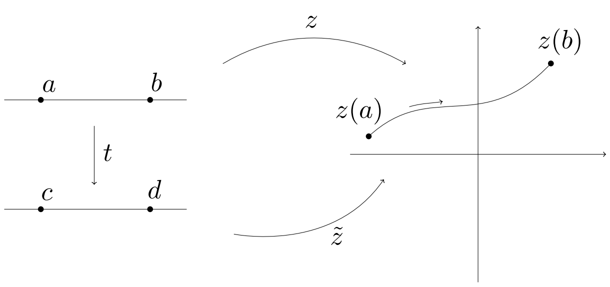

Figure 3.2: Equivalent curves

Definition 43 (Piecewise smooth equivalent parametrized curves)**.** These are defined the same.

Now we'll talk about non-parametrized curves.

Definition 44 (Curve)**.** A piecewise smooth curve the equivalence class of all parametrized piecewise smooth curve.

So a curve is a subset of the complex plane with an orientation. Usually we denote the curve with γ, and for one element of γ, we write z.

Definition 45 (Endpoints, starts, finishes, closed curve, simple curve)**.** If γ is a curve with parametrization $z : [a,b] \to \mathbb{C}$, then

- The points $z(a), z(b)$ are called *endpoints* of γ.
- We sayγ starts at $z(a)$ and *finishes* at $z(b)$.
- We say that γ is *closed* when $z(a) = z(b)$.
- We say that γ is *simple* if γ has no selfintersections, except endpoints.

For a curve γ, denote by γ^- a curve which differs from γ only in orientation. If $z : [a,b] \to \mathbb{C}$ is a parametrization of γ, then $z^- : [b,a] \to \mathbb{C} : z^-(s) = z(a+b-s)$ is a parametrization of γ^-.

If γ is a closed simple curve, then we call γ positive if γ has a counter-clockwise orientation.

Let γ be a smooth curve with parametrization $z : [a,b] \to \mathbb{C}$ and f be a function continuous on γ.

Definition 46. $\int_\gamma f(z)dz = \int_a^b f(z(s))z'(s)ds$

UOVT. This definition does not depend on the parametrization.

If γ is a piecewise smooth curve with parametrization $z : [a, b] \to \mathbb{C}$ such that z is smooth on $[a_i, a_{i+1}]$.

Definition 47. $\int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(z(s))z'(s)ds$

Definition 48. Length of γ is given by

$$\int_a^b |z'(s)|ds.$$

Proposition 8.

- $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$
- $\int_{\gamma^-} = -\int_\gamma$
- $|\int_\gamma f(z)dz| \leq \text{length}\, \gamma \sup_{z\in\gamma} |f(z)|$

Proof. First and second are trivial.

Third property. Let γ be smooth. Then

$$\begin{aligned} |\int_\gamma f(z)dz| &= |\int_a^b f(z(s))z'(s)ds| \\ &\leq |\int_a^b \left(\sup_{z\in\gamma} |f(z)|\right) z'(s)ds| \\ &\leq \text{length}\, \gamma \sup_{z\in\gamma} |f(z)|. \end{aligned}$$

□

Let $f(z)$ be a function on open set Ω.

Definition 49. A function $f(z)$ is called a primitive of $f(z)$ on Ω if $F(z)$ is holomorphic on Ω and $F'(z) = f(z)$.

Proposition 9. Let Ω be an open set and $\gamma \subset \Omega$ a curve. Suppose that f has a primitive F on Ω. Then $\int_\gamma f(z)dz = F(z(b)) - F(z(a))$.

Proof. Let γ be smooth, then $\int f(z)dz = \int_a^b f(z(s))z'(s)ds$.

$$\begin{aligned}\int f(z)dz &= \int_a^b f(z(s))z'(s)ds \\ &= \int_a^b F'(z(s))z'(s)ds \\ &= \int_{z(a)}^{z(b)} F'(s)ds \\ &= F(z(b)) - F(z(a)).\end{aligned}$$

□

If γ is piecewise smooth, then $\int_\gamma f(z)dz = \sum \int_\gamma \dots = F(z(b)) - F(z(a))$

Corollary 4. If Ω is open, γ is closed simple curve in Ω and f has a primitve on Ω, then $\int_\gamma f(z)dz = 0$.

UOVT. Prove that $(z) = \frac{1}{z}$ does not have a primitive on $\mathbb{C} \setminus \{0\}$

Theorem 6. Let Ω be a region (open and connected) and $f(z)$ be holomorphic on Ω, such that $f'(z) = 0$ for all $z \in \Omega$. Then f is constant.

Proof. Let $z_0, z \in \Omega$. Since Ω is connected, $\exists\gamma$ which connects z and 0. The f is the primitive of f'. Then

$$0 = \int_\gamma f'(z)dz = f(z_0) - f(z).$$

Therefore $f(z) = f(z_0)$ □

Chapter 4

Cauchy theorem

Theorem 7. Let γ be a closed simple curve and $f(z)$ is a function holomorphic in the interior of γ. Then $\int_\gamma f(z)dz = 0$.

Definition 50. Let $\Omega \subset \mathbb{C}$, then $\operatorname{diam}\Omega = \sup_{x,y\in\Omega} |x-y|$

UOVT. $\operatorname{diam}\Omega$ is finite iff Ω is bounded.

Theorem 8 (Goursat's theorem)**.**

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Les 4: Goursat's theorem

Theorem 9 (Goursat's theorem)**.** Let $\Omega \subset \mathbb{C}$ be open, f be holomorphic on Ω and $T \subset \Omega$ a triangle such that the interior of $T \subset \Omega$. (Note: triangle is a curve here!) Then $\int_T f(z)dz = 0$.

Proof. Denote by $T^{(0)} = T$, and let $\tilde{T}^{(0)}$ the interior of T (which is closed, as T is a curve) Denote by $d^{(0)} = \operatorname{diam}\tilde{T}^{(0)}$ and $p^{(0)}$ its perimeter of $T^{(0)}$. Construct small triangles T_1, T_2, T_3, T_4. Then

$$\int_{T_1} + \int_{T_2} + \int_{T_3} + \int_{T_4} f = \int_T f.$$

Therefore

$$\begin{aligned} |\int_T f| &\leq \sum \left| \int_{T_i} f \right| \\ &\leq 4 \left| \int T_j f \right| \quad \exists j. \end{aligned}$$

Denote by $T^{(1)} = T_j$. Then $p^{(1)}$ is $\frac{1}{2}p^{(0)}$. Then $d^{(1)}$ is $\frac{1}{2}d^{(0)}$.

Apply same procedure to $T^{(1)}$.

Continuing this procedure, we get ∞ triangles,

$$T^{(0)}, T^{(1)}, \ldots.$$

$$\left|\int_{T^{(0)}}\right| f \leq 4^n \left|\int_{T^{(n)}} f\right|.$$

The sequence of interiors of these triangles is a sequence of nested compact subsets in Ω. By the Lemma (geen notities van) about nested compact subsets, there exists a point in the interior.

Since f is holomorphic on Ω, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|).$$

$$\begin{aligned}
\left|\int_{T^{(n)}} f\right| &= \left|\int_{T^{(n)}} f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|)\right| \\
&\leq \left|\int_{T^{(n)}} f(z_0)\right| + \left|\int f'(z_0)(z - z_0)\right| + \left|\int o(|z - z_0|)\right|.
\end{aligned}$$

Now, the two first integrals are 0 as a constant and a linear function have a primitive.

$$\begin{aligned}
\left|\int_{T^{(n)}} f\right| &\leq \left|\int_{T^{(n)}} o(|z - z_0|)\right| \\
&\leq p^{(n)} \cdot \sup_{z \in T^{(n)}} o(|z - z_0|) \\
&\leq p^{(n)} \cdot \sup_{z \in T^{(n)}} (z - z_0)\psi(z - z_0) \\
&\leq p^{(n)} \cdot d^{(n)} \sup_{z \in T^{(n)}} \psi(z - z_0) \\
&\leq p^{(n)} \cdot d^{(n)} \varepsilon_n \quad \varepsilon_n \xrightarrow{n \to \infty} 0 \\
&\leq 4^n \frac{p^{(0)}}{2^n} \frac{d^{(0)}}{2^n} \varepsilon_n \\
&= p^{(0)} d^{(0)} \varepsilon_n \xrightarrow{n \to \infty} 0.
\end{aligned}$$

Therefore, we bounded the integral by something that goes to zero. As the integral on te left doesn't depend on n, it's 0. ☐

Corollary 5 (Cauchy theorem for rectangles)**.** Same thing for rectangles.

Proof. Split rectangle in triangles. ☐

Opmerking (Examen)**.** Alle namen van stellingen enzo kennen! Formuleer de stelling van Goursat.

Theorem 10. Let D be an open disc. If f is a holomorphic function on D, then f has a primitive on D.

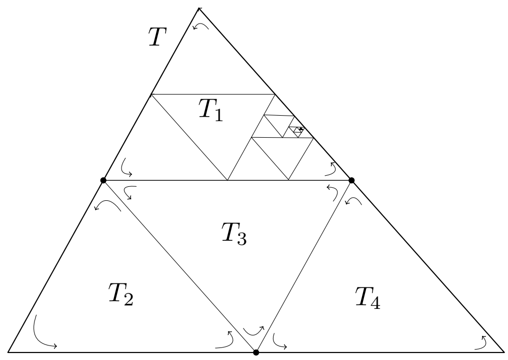
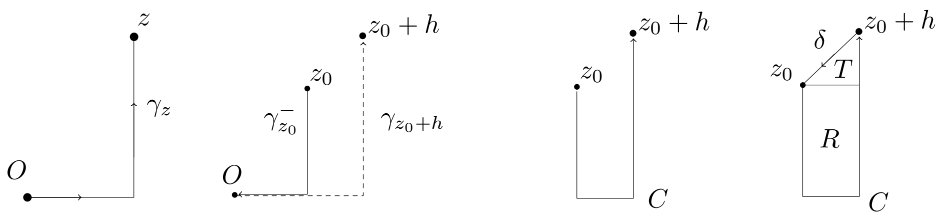


Figure 4.1: goursat

Proof. We can assume that D is centered at the origin. Let $z \in D$. Denote by γ_z the curve in the following figure.



This curve is uniquely defined by z. Denote by $F(z) = \int_{\gamma_z} f(z)dz$. We will prove that $F(z)$ is a primitive for $f(z)$ on D.

Let $z_0 \in D$, $h \in \mathbb{C}$ such that $z_0 + h \in D$

$$\begin{aligned} F(z_0+h) - F(z_0) &= \int_{\gamma_{z_0+h}} f - \int_{\gamma_{z_0}} f \\ &= \int_{\gamma_{z_0+h}} f + \int_{\gamma_{z_0}^-} f. \end{aligned}$$

Define δ, C, R and T as in the figure. Then

$$\int_C + \int_\delta = \int_R + \int_T = 0.$$

This implies

$$\int_C f = -\int_\delta f = \int_{\delta^-} f.$$

As f is holomorphic, f is continuous, or equivalently,$f(z) = f(z_0)+o(1)$. Therefore:

$$\begin{aligned}\int_{\delta^-} f(z_0) + o(1) &= \int_{\delta^-} f(z_0) + \int_{\delta^-} o(1)\\ &= f(z_0)h + \int_{\delta^-} o(1).\end{aligned}$$

So now, we have

$$\begin{aligned}F(z_0+h) - F(z_0) &= hf(z_0) + \int_{\delta^-} o(1)\\ |\frac{f(z_0+h) - F(z_0)}{h}| &= |f(z_0)| + \left|\frac{1}{h}\int_{\delta^-} o(1)\right|\\ &\leq f(z_0) + \frac{1}{h}\operatorname{length}(\delta)\sup\psi(z-z_0)\\ &= f(z_0) + \sup(\psi(z-z_0)) \xrightarrow{h\to 0} f(z_0).\end{aligned}$$

□

Note. We use that f is holomorphic by saying that $\int R$, $\int T = 0$.

Corollary 6. Let γ be a closed simple curve inside disc D If f is holomorphic on D,then $\int_\gamma f = 0$.

Remark. We can reformulate the theorem when we replace the disk with a rectangle, as the two curve segments are still inside. Note that when we take a different path, the integral is still the same (Figure 4.2): we can compare the areas that are created, the integral along these rectangles is 0.

'A theorem about existence of a primitive can be formulated and proved for every region Ω where every point z can be connected with fixed point z_0 by a polygonal line which consists of finite number of vertical and horizontal segments.'

This is not true in general: for example, the disk without a point. Then two paths can have different integrals because we cannot apply Goursat's theorem anymore. So add 'simply connected'

Remark. For example, this is correct for a keyhole.

Theorem 11 (Cauchy's integral formula)**.** Let Ω be an open subset of C, $D \subset \Omega$ is an open disk, f is holomorphic on Ω. Denoet by $C = \partial D$. Then

$$\forall z \in D : f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Proof. Let $z_0 \in D$. Denote by

$$F(z) = \frac{f(z)}{z - z_0}.$$

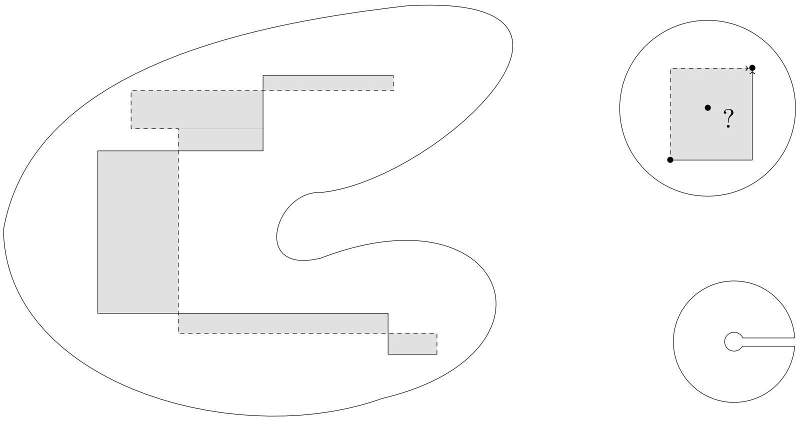


Figure 4.2: remark-cauchy

We know that $F(z)$ is holomorphicomorphic inside interior of $\Gamma_{\varepsilon,\delta}$. Therefore

$$\int_\Gamma F(z) = 0.$$

When $\delta \to 0$, the integrals along two sides of de corridor, will vanish.

Denote by C_ε the circle of radius ε.

As $\int_\Gamma F(z) = 0$, we get that

$$\lim_{\delta \to 0} \int_{\Gamma_{\varepsilon,\delta}} F(z) = 0,$$

therefore

$$\int_C F(z) + \int_{C_\varepsilon} = 0.$$

Therefore:

$$\int_C F(z) = \int_{C_\varepsilon^-} F(z)dz.$$

$$\begin{aligned}\int_{C_\varepsilon^-} F(z)dz &= \int \frac{f(z)}{z - z_0 A} \\ &= \int \frac{f(z) - f(z_0)}{z - z_0} + \int \frac{f(z_0)}{z - z_0}\end{aligned}$$

.

Now the first part $\to 0$, because as f is holomorphic, the difference quotient is bounded, and therefore, when $\varepsilon \to 0$, the integral $\to 0$ (using length, etc)

$$\begin{aligned}\int \frac{f(z_0)dz}{z - z_0} &= f(z_0) \int_0^{2\pi} \frac{i\varepsilon e^{ir}}{\varepsilon e^{ir}}\, dr \\ &= 2\pi i f(z_0).\end{aligned}$$

In conclusion:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}.$$

□

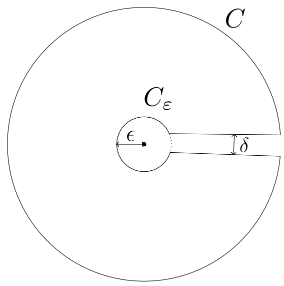


Figure 4.3: chauchy-integral-formula

Theorem 12. Let Ω be an open subset of $C \subset \Omega$ be a disc, $= \partial D$, f is holomorphic on Ω. Then f is infinitely many times differentiable on D and for $z \in D$:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Les 5: Cauchy integral formula and the fundamental theorem of algebra

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Proof of the Cauchy integral formula for derivatives

Proof. Induction on n.

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We've proved this previous lecture.

Step $n \to n+1$. Suppose for n,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n-1}} d\zeta.$$

Then

$$\begin{aligned}
\frac{f^{(n)}(z+h)-f^{(n)}(z)}{h} &= \frac{1}{h}\left(\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z-h)^{n-1}}d\zeta - \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n-1}}d\zeta\right) \\
&= \frac{1}{h}\frac{n!}{2\pi i}\int_C f(\zeta)\left(\frac{1}{(\zeta-z-h)^{n+1}} - \frac{1}{(\zeta-z)^{n+1}}\right).
\end{aligned}$$

Now we use $A^{n+1}-B^{n+1}=(A-B)(\dots)$

$$\begin{aligned}
&= \frac{1}{h}\frac{n!}{2\pi i}\int_C f(\zeta)\left(\frac{h}{(\zeta-z-h)(\zeta-z)}(\dots)\right) \\
&= \frac{n!}{2\pi i}\int_C f(\zeta)\left(\frac{1}{(\zeta-z-h)(\zeta-z)}(\dots)\right) \\
\xrightarrow{h\to 0} &= \frac{n!}{2\pi i}\int_C f(\zeta)\frac{1}{(\zeta-z)^2}\left(\frac{n+1}{(\zeta-z)^n}\right) \\
&= \frac{(n+1)!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta.
\end{aligned}$$

□

Corollary 7. Let $\Omega\subset\mathbb{C}$ be open, $z_0\in\Omega$, $R>0$ be such that $\overline{D_R(z_0)}\subset\Omega$. Denote by $D=\overline{D_R(z_0)}$, $C=\partial D$. If f is holomorphic on Ω, then

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{R^n}\sup_{z\in\mathbb{C}}|f(z)|.$$

Note that this is not the case in Real analysis, where we cannot afschat the derivative by the function itself.

Proof. We have

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\right| \\
&\le \frac{n!}{2\pi}\operatorname{length} C \sup_{\zeta\in\mathbb{C}}\left|\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\right| \\
&= \frac{n!}{2\pi}2\pi R\frac{1}{R^{n+1}}\sup_{\zeta\in\mathbb{C}}|f(\zeta)| \quad \text{as } C \text{ is the boundary of } D_R(z_0). \\
&= \frac{n!}{R^n}\sup_{\zeta\in\mathbb{C}}|f(\zeta)|.
\end{aligned}$$

□

Theorem 13 (Liouville)**.** Let f be an entire function. If $\exists N$ s.t. $\forall z$: $|f(z)|\le N$, then f is contant.

Proof. Let $z\in\mathbb{C}$. For all $R>0$, $D_R(z)\subset C$, and $C=\partial D$. By previous result,

$$|f'(z)| \le \frac{1}{R}\sup_{z\in C}|f(z)| \le \frac{N}{R} \xrightarrow{R\to\infty} 0.$$

However, $|f'(z)|$ does not depend on R, so $f'(z) = 0 \Rightarrow f \equiv$ a constant. □

Theorem 14 (Fundamental theorem of algebra)**.** Let $f(z)$ be a polynomial, $z_0 + a_1 z + a_2 z^2 + \ldots$. (not constant) Then there exists a z_0 such that $f(z_0)$.

Proof. We assume that $a_n \neq 0$.

$$\frac{f(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right).$$

$$g(z) := \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}.$$

Then $g(z) \xrightarrow{z\to\infty} 0$. Therefore $\exists R > 0 : \forall |z| > R : g(z) \leq \frac{|a_n|}{2}$. For all $|z| > R$:

$$\begin{aligned}\left|\frac{f(z)}{z^n}\right| &= |a_n + g(z)| \\ &> ||a_n| - |g(z)|| \\ &> \frac{|a_n|}{2}.\end{aligned}$$

Therefore: $\forall |z| > R$

$$|f(z)| \geq \frac{|a_n|}{2}|z|^n \geq \frac{|a_n|}{2}R^n.$$

Then for all $|z| > R$:

$$\frac{1}{|f(z)|} \leq \frac{2}{|a_n|R^n}.$$

Suppose $f(z)$ has no roots in $\mathbb{C}$. Therefore $\frac{1}{f(z)}$ is holomorphic on $\mathbb{C}$. Therefore $\frac{1}{f(z)}$ reaches a maximum on $\overline{D_R(0)}$, $\overline{D}$ is compact. Therefore $\exists K : \frac{1}{f(z)} \leq K$ for all $|z| < R$. Therefore for all $z \in \mathbb{C}$:

$$\frac{1}{|f(z)|} \leq \max\left\{K, \frac{2}{|a_n|R^n}\right\}.$$

By Liouville the function is constant. $\lightning$ □

Theorem 15. Let $\Omega \subset \mathbb{C}$ be open. $f(z)$ holomorphic on Ω. Then $f(z)$ is an analytic[a] function on Ω.

[a] Around each point there exists a point such that the power series converges.

We proved the converse. This is not true in Real Analysis.

Proof. Let $z_0 \in \Omega$. We have to prove that $\exists r > 0$ such that $\forall z \in D_r(z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Since Ω is open, $\exists R > 0$ such that $\overline{D_R(z_0)} \subset \Omega$. Denote by $D = D_R(z_0)$, $C = \partial D$. Then by Cauchy's integral formula

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - s} d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0 + z_0 - z)} d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 + \frac{z_0 - z}{\zeta - z_0}} d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} d\zeta. \end{aligned}$$

Now, z_0 is center of the disk and ζ is on the boundary of the disk. z is inside the disk. Therefore $|z - z_0| < |\zeta - z_0|$ and

$$\left| \frac{z - z_0}{\zeta - z_0} \right| \leq r < 1.$$

(Check what variables are moving! TODO) Now as $r < 1$, we can write it as a geometric progression:

$$\begin{aligned} &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \left(\frac{1}{\zeta - z_0} \right)^n d\zeta \right) \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \end{aligned}$$

As a power series converges uniformly on their disk of convergence. $\square$

Theorem 16. Let $\Omega \subset \mathbb{C}$ be a region, $f(z)$ a holomorphic function. Let $\{w_k\}$ a sequence of points in Ω such that

- $\{w_k\}$ are distinct
- $w_k \to z_0 \in \Omega$
- $f(w_k) = 0$ for all k

Then $f(z) = 0$ for all $z \in \Omega$.

Stronger conditions: suppose $f(z) = 0$ for all $z \in \ell$, ℓ a line inside Ω.

Remark. In real analysis, this is not the case!

Proof. By the previous theorem, $f(z)$ is analytic at z_0, i.e. $\exists r > 0$, $z \in D_r(z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Let us prove that $f(z) = 0$ for all z in some neighbourhood of z_0. Suppose not. Therefore, there exists a first N such that a_N is not 0. Then

$$\begin{aligned} f(z) &= (z - z_0)^N \sum_{n=N}^{\infty} a_n (z - z_0)^{n-N} \\ &= (z - z_0)^N \left(a_N + \sum_{n=N+1}^{\infty} a_n (z - z_0)^{n-N} \right) \\ g(z) &:= \sum_{n=N+1}^{\infty} a_n (z - z_0)^{n-N} \\ f(z) &= (z - z_0)^N (a_N + g(z)) \\ g(z) &\xrightarrow{z \to z_0} 0 \quad \text{because of the definition of } g \\ \text{In particular,} \, g(w_k) &\xrightarrow{k \to \infty} 0 \end{aligned}$$

And$f(z) = (z - z_0)^N (a_N + \underbrace{g(z)}_{\xrightarrow{z \to z_0} 0})$.

So $\exists M : n \geq M \Rightarrow a_N + g(w_k) \neq 0$.

Then

$$0 = f(w_k) = \underbrace{(w_k - z_0)^N}_{\neq 0} \underbrace{(a_N + g(w_k))}_{\neq 0}.$$

$w_k \neq z_0$ if we look far enough.

This is a contradiction to the fact that $\exists a_N \neq 0 \Rightarrow f(z) = 0 \quad \forall z$ in some neighbourhood of z_0.

So we've proved the theorem for a neighbourhood of x_0. Denote by U the interior $\{z \in \Omega \mid f(z) = 0\}$. $U \neq \emptyset$, since $f(z)$ is zero in some neighbourhood of z_0.

- U is open, since it is the interior of a set.
- U is closed , let z_0 be a limit point of U. Therefore, $\exists$ a sequence of points $z_k \in U$ and $z_k \to z_0$. We know that $f(z_k) = 0$, because $z_k \in U$. By the same arguments of above, $f(z) = 0$.

Therefore $V = \Omega \setminus U$ is open and $\Omega = U \cup V$. As Ω is connected, therefore V should be empty (as U is not.) Therefore V is empty! □

Remark. As $\mathbb{C}$ is the only non empty clopen set, we've proved that $\mathbb{C} \subset \Omega$, as $U \subset \Omega$.

Corollary 8. Let Ω be a region and D be a disk in Ω. Let f, g bet holomorphic on Ω. If $f(z) = g(z)$ for all $z \in D$, then $f(z) = g(z)$ for all $z \in \Omega$.

Proof. Denote $F = f - g$. Let $D = D_r(z_0)$. Denote by $w_k = z_0 + \frac{r}{2k}$. Clearly, $w_k \to z_0$, w_k are distinct, $F(w_k) = 0$. Therefore, $F \equiv 0$ on Ω, therefore $f(z) = g(z)$ for all $z \in \Omega$. $\square$

Definition 51. Let Ω, Ξ be regions such that $\Xi \subset \Omega$. Let f be a holomorphic function on Ξ, F be a holomorphic function on Ω. If $F(z) = f(z)$ for all $z \in \Xi$, then F is called *the* analytic continuation of $f(z)$ from Ξ to Ω.

Remark. By the previous result, the analytic continuation is unique!

Les 6: Theorem of Morera and Schwarz reflection principle

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Theorem 17 (Morera)**.** Let D be a disk, and f be a continuous function on D. If for all $T \subset D$, $\int_T f = 0$, then f is holomorphic on D.

Proof. Let $z \in D$. Changing variables, we can assume that D is centred at the origin. Construct figure **??**, construct γ. Let $F(z) = \int_{\gamma_z} f(\zeta) d\zeta$. Using the same method as in the proof about the existence of a primitive on a disk, we can prove that $F(z)$ is a primitive for $f(z)$ on D.

This means that $F(z)$ is holomorphic on D (and $F'(z) = f(z)$) This implies that $F(z)$ is infinitely many times differentiable on D. Therefore, $f(z)$ is differentiable on D. $\square$

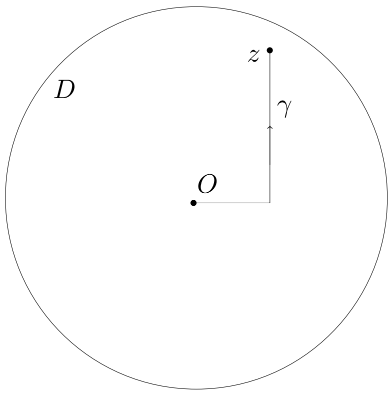

Figure 4.4: Proof of Morera.

Theorem 18 (About sequences of holomorphic functions)**.** Let $\Omega \subset \mathbb{C}$ be open, $\{f_n\}$ a sequence of holomorphic functions on Ω. If f_n converges to f uniformly on compact subsets of Ω, then f is holomorphic.

This result is not true in real analysis. (Every real continuous? function can be approximated by polynomials? But Weierstrass exists?)

Proof. Let $z_0 \in \Omega$. Since Ω is open, there exists $r > 0$ such that $\overline{D_r(z_0)} \subset \Omega$. Let $T \subset \overline{D_r(z_0)}$ be a triangle in this disk. Since f_n are holomorphic, $\int_T f_n(z)dz = 0$, by Goursat's theorem. Since, $f_n \to f$ uniformly on compact subsets, and therefore $f_n \to f$ on $\overline{D_r(z_0)}$, we have that[1]

$$\int_T f_n(z)dz \to \int_T f(z)dz = 0.$$

Therefore, by Morera, $f(z)$ is holomorphic on this disk, $\overline{D_r(z_0)}$. Therefore, $f(z)$ is holomorphic on Ω. □

Theorem 19 (About sequence of holomorphic functions and their derivatives)**.** Let $\Omega \subset \mathbb{C}$ be open, $\{f_n\}$ be a sequence of functions holomorphic on Ω. If $f_n \to f$ uniformly on compact subsets of Ω, then $\forall k \geq 0$,

$$f_n^{(k)} \to f^{(k)} \text{ uniformly on all compact subsets of } \Omega.$$

Proof. It is enough to prove the theorem only for $k = 1$. Let $A \subset \Omega$ a compact subset of Ω. Denote by $r = \inf_{x \in A, y \in \partial\Omega} |x - y|$ Since A is compact, $r > 0$ (ex). Denote by $R = \frac{r}{2}$.

$$A \subset \bigcup_{z \in A} D_R(z) \subset \Omega.$$

Let $F_n = f_n - f$. By evaluation of derivatives from the previous lecture, we have

$$|F_n'(z)| \leq \frac{1}{R} \sup_{w \in C_R(z)} |F_n(w)|.$$

Therefore

$$\begin{aligned} \sup_{z \in A} |F_n'(z)| &\leq \frac{1}{R} \sup_{z \in A, w \in C_R(z)} |F_n(w)| \\ &= \frac{1}{R} \sup_{z \in \bigcup_{\zeta \in A} D_r(\zeta)} |F_n(z)| \xrightarrow{n \to \infty} 0. \end{aligned}$$

since f_n converges uniformly on compact subsets of Ω, and $\overline{\bigcup_{\zeta \in A} D_r(\zeta)}$ is compact. As R is fixed,

$$\sup_{z \in A} |F_n'(z)| \to 0.$$

Since $F_n'(z) = f_n'(z) - f'(z)$, we have that

$$f_n'(z) \to f'(z).$$

uniformly on compact subsets of Ω. □

We can use this for power series, as this is a limit of a sequence.

[1] Uniform convergence implies dominated convergence

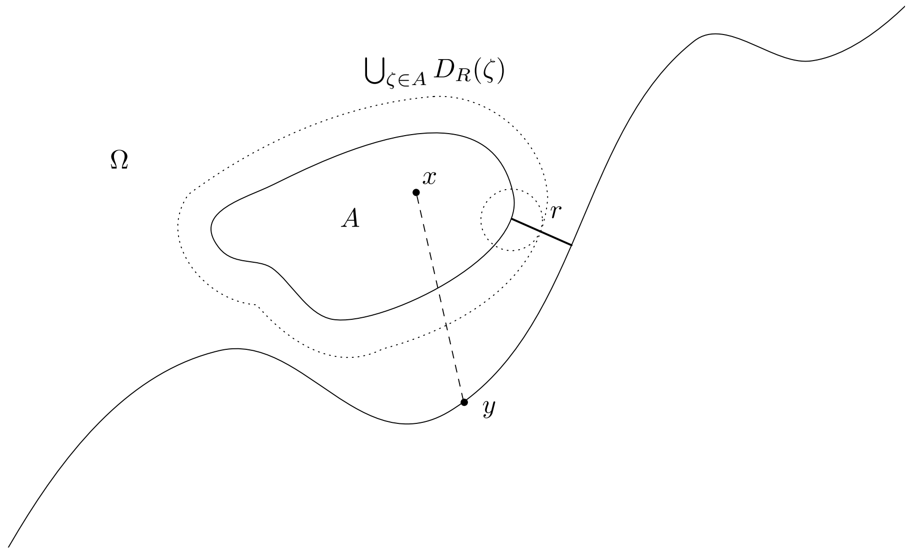

Figure 4.5: Proof of Theorem 7.

Definition 52. A $\Omega \subset \mathbb{C}$ is called symmetric iff

$$z \in \Omega \Leftrightarrow \overline{z} \in \Omega.$$

Denote by

$$\begin{aligned}\Omega^+ &= \Omega \cap \{z \mid \operatorname{Im} z > 0\}\\ \Omega^- &= \Omega \cap \{z \mid \operatorname{Im} z < 0\}\\ I &= \Omega \cap \mathbb{R}.\end{aligned}$$

Theorem 20 (Symmetry principle)**.** Let $\Omega \subset \mathbb{C}$, open and symmetric. Let $f^+(z)$ be a function holomorphic on Ω^+, and continuous on $\Omega^+ \cup I$. Let $f^-(z)$ be a function holomorphic on Ω^-, and continuous on $\Omega^- \cup I$. If $f^-|_I = f^+|_I$, then

$$f(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^+(z) = f^-(z) & z \in I \\ f^-(z) & z \in \Omega^-, \end{cases}$$

is holomorphic on Ω.

Proof. It is clear that $f(z)$ is holomorphic on $\Omega \setminus I$. We need to prove that $f(z)$ is holomorphic on I. Let $z_0 \in I$, $r > 0$ such that $D_r(z_0) \subset \Omega$.

Let T be a triangle in D There are multiple possibilities.

- $T \subset (D \cap \Omega^+) \cup (D \cap \Omega^-)$. Goursats handles this case.

- $T \subset D \cap (\Omega^+ \cup I)$ or $T \subset D \cap (\Omega^- \cup I)$ Denote by T_ε a smaller triangle. T_ε satisfies the first case. As $\int_{T_\varepsilon} \to \int_T$, since $f(z)$ is continuous, we get that $\int_T = 0$.
- Other case, split the triangle in T_1, T_2, T_3.

Now as $f(z)$ is holomorphic on $D_r(z_0)$, $f(z)$ is holomorphic on Ω. □

Note that we didn't really use the symmetry of the set.

Note that the analytical continuation is unique, f is the *only* holomorphic function on Ω.

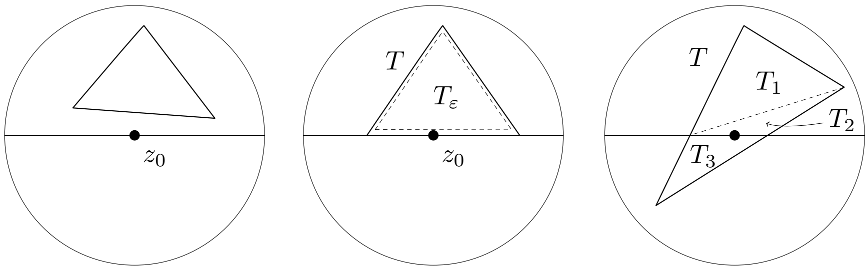

Figure 4.6: Symmetry principle

Theorem 21 (Schwarz reflection principle)**.** Let Ω be an open connected symmetric set. $f(z)$ is a holomorphic function on Ω^+, $f(z)$ is continuous on $\Omega^+ \cup I$, and for all $z \in \mathbb{R}$, $f(z) \in \mathbb{R}$. Then (z) can be analytically continued on Ω.

Proof. For $z \in \Omega^-$, define $g(z) = \overline{f(\overline{z})}$. Let us prove that $g(z)$ is holomorphic on Ω^-.

It is obvious that $g(z)$ is continuous on $\Omega^- \cup I$.

Let $z_0 \in \Omega^-$, then $\overline{z_0} \in \Omega^+$. Since $f(z)$ is holomorphic on Ω^+, $\exists r > 0, \forall z \in D_r(\overline{z_0})$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \overline{z_0})^n.$$

For $w \in \Omega^-$,

$$\begin{aligned} g(w) &= \overline{f(\overline{w})} \\ &= \overline{\sum_{n=0}^{\infty} a_n (\overline{w} - \overline{z_0})^n} \\ &= \sum_{n=0}^{\infty} \overline{a_n} (w - z_0)^n \end{aligned}$$

Therefore, g is analytic at z_0, therefore g is holomorphic at z_0.

$\forall z \in I, g(z) = \overline{f(\overline{z})} = \overline{f(z)} = f(z)$. Therefore

$$F(z) = \begin{cases} f(z) & z \in \Omega^+ \\ f(z) = g(z) & z \in I \\ g(z) & z \in \Omega^- \end{cases}$$

is holomorphic on Ω by symmetry principle. $\square$

Vraag. Is it possible to define $g(z) = f(\overline{z})$?

Chapter 5

Meromorphic functions

Definition 53 (Singular point)**.** A point z_0 is called a singular point of f if there exists $r > 0$ such that $f(z)$ is defined on $D_r(z_0) \setminus \{z_0\}$, and not defined at z_0.

Example. $f(z) = \frac{1}{z+1}$.

Example. $f : \mathbb{C}_0 \to \mathbb{C} : z \mapsto z$

Definition 54 (Zero)**.** A point z_0 is called a zero of a holomorphic function $f(z)$ if $f(z_0) = 0$.

Note. Zeros are isolated. Otherwise, we could construct a sequence of $f(z_n) = 0$, which would imply that $f \equiv 0$.

Theorem 22 (The behaviour of holomorphic functions around its zeros)**.** Let $\Omega \subset \mathbb{C}$ be open, $f(z) \not\equiv 0$ be holomorphic on Ω and z_0 be a zero of $f(z)$. Then $\exists r > 0$ and *unique* $n \geq 1$ such that for all $z \in D_r(z_0)$,

$$f(z) = (z - z_0)^n g(z),$$

where $g(z_0) \neq 0$ for all $z \in D$ and g holomorphic.

Proof. $f(z)$ is holomorphic at z_0, therefore f is analytic at z_0. Therefore, $\exists r > 0$ such that for all $z \in D_r(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Note that since $f(z_0) = 0$, $a_0 = 0$. Let n be a minimal number such that $a_N \neq 0$.

$$\begin{aligned} f(z) &= (z - z_0)^N \left(\sum_{n=N}^{\infty} a_n (z - z_0)^{n-N} \right) \\ &= (z - z_0)^N g(z). \end{aligned}$$

Then $g(z_0) = a_N \neq 0$. Therefore, $\exists R < r$, such that for all $z \in D_R(z_0)$, we get that $g(z_0) \neq 0$.

Now we prove that N is unique. Suppose

$$f(z) = (z - z_0)^{N_1} g_1(z) = (z - z_0)^{N_2} g_2(z)$$

and $N_2 > N_1$. Therefore $g_1(z) = (z - z_0)^{N_2 - N_1} g_2(z)$, which contradicts the fact that $g_1(z_0) \neq 0$. $\lightning$ $\square$

Definition 55 (Order, multiplicity)**.** The number n from the previous theorem is called the order or the multiplicity of the zero.

Example. Suppose $f(z) = z\sin(z)$, then 0 is a zero of multiplicity 2.

$$z\sin(z) = z(z + O(z^3)) = z^2 + O(z^4).$$

Definition 56 (Single zero.)**.** A zero of order one is called a single zero.

Les 7: Singular points and meromorphic functions

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Definition 57 (Removable singular point)**.** A point $z_0 \in \mathbb{C}$ is called a removable singular point of $f(z)$ if we can define the value $f(z_0)$ such that f is holomorphic in the neighbourhood of z_0.

Example. $f(z) = z^2$ on $D_5(\sqrt{2}) \setminus \{\sqrt{2}\}$. If we define $f(\sqrt{2}) = 2$, then f is holomorphic. Hence, $\sqrt{2}$ is a removable singular point.

Example. $f(z) = \frac{\sin z}{z}$ on $\mathbb{C}_0$. If we define $f(0) = 1$, we have a holomorphic function. Hence, 0 is a removable singular point.

Characterisation of removable singular points.

Lemma 1. Let $F(z, s) : \Omega \times [0, 1] \to \mathbb{C}$, where Ω open. Suppose

- $F(z, s)$ is holomorphic on Ω for all $s \in [0, 1]$.
- $F(z, s)$ is continuous on $\Omega \times [0, 1]$ (as a function of two variables)

Then $f(z) = \int_0^1 F(z, s)\, ds$ is holomorphic on Ω.

Proof. We don't prove this result. Thm. 5.4 from the book. $\square$

Theorem 23 (Riemann's theorem about removable singularities)**.** Let $f(z)$ be a function, holomorphic on $\overline{D_r(z_0)} \setminus \{z_0\}$ If $f(z)$ is bounded in the neighbourhood of z_0, then z_0 is a removable singular point.

Proof. Let us prove that for all $z \in D_r \setminus \{z_0\}$ we have that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} ds\zeta.$$

Let Γ be a closed simple curve depicted on Figure 5.1. The function $\frac{f(\zeta)}{\zeta - z}$ is holomorphic inside Γ. By Cauchy's theorem,

$$\int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Now, taking the limit of $\delta \to 0$, we have

$$\lim_{\delta \to 0} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Therefore,

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{C_0} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Therefore,

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_1^-} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{C01^-} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Now

$$\begin{aligned} \left| \int_{C_0^-} \frac{f(\zeta)}{\zeta - z} d\zeta \right| &\leq 2\pi\varepsilon_0 \cdot \sup_{\zeta \in C_0} \left| \frac{f(\zeta)}{\zeta - z} \right| \\ &\xrightarrow{\varepsilon_0 \to 0} 0. \end{aligned}$$

As $\zeta - z \not\to 0$, and $f(\zeta)$ is bounded.

Now, taking limits $\varepsilon_0 \to 0$,

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_1^-} \frac{f(\zeta)}{\zeta - s} d\zeta.$$

Now

$$\begin{aligned} \int_{C_1^-} \frac{f(\zeta)}{\zeta - z} d\zeta &= \int_{C_1^-} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + \int_{C_1^-} \frac{f(z)}{\zeta - z} d\zeta \\ &= \int_{C_1^-} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + \int_{C_1^-} \frac{f(z)}{\zeta - z} d\zeta \\ &= \int_{C_1^-} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + 2\pi i f(z). \end{aligned}$$

Therefore,

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z) + \int_{C_1^-} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta.$$

And now using sup:

$$\left| \int_{C_1^-} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq 2\pi\varepsilon_1 \sup_{\zeta \in C_1^-} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \xrightarrow{\varepsilon_1 \to 0} 0.$$

As f is holomorphic near z. Therefore,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

for all $D_r(z_0) \setminus \{z_0\}$.

By the previous Lemma, $\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$ is holomorphic on $D_r(z_0)$

Now defining $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$, we proved that z_0 is a removable singular point. □

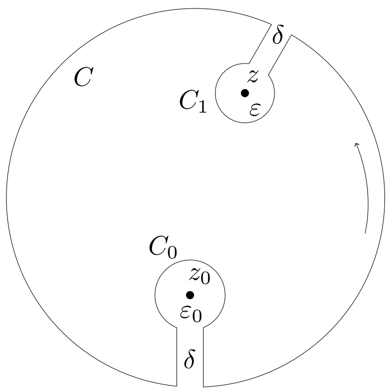

Figure 5.1: riemmans-theorem

Definition 58 (Pole)**.** z_0 is called a pole if the function

$$g(z) = \begin{cases} \frac{1}{f(z)} & z \neq z_0 \\ 0 & z = z_0 \end{cases},$$

is holomorphic.

Example. $f : \mathbb{C}_0 \to \mathbb{C} : \frac{1}{z}$.

The same theorem exists for poles:

Theorem 24 (Characterisation of poles)**.** Let $f(z)$ holomorphic on $D_r(z_0) \setminus \{z_0\}$. Then z_0 is a pole of $f(z)$ if and only if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Proof.

- If z_0 is a pole, then there exists a function

$$g(z) = \begin{cases} \frac{1}{z} & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

is holomorphic, by definition of a pole. As f is holomorphic, it's continuous, therefore, $g(z)$ is continuous. In particular $\lim_{z \to z_0} g(z) = \lim_{z \to z_0} \frac{1}{f(z)} = 0$. Therefore $\lim_{z \to z_0} |f(z)| = \infty$.

- Suppose $\lim_{z \to z_0} |f(z)| = \infty$. Therefore, in the neighbourhood of z_0, the function $\frac{1}{f(z)}$ is bounded. Therefore, z_0 is removable singularity of $\frac{1}{f(z)}$. The only value that's possible is 0. The defining of removable singularity gives

$$g(z) = \begin{cases} \frac{1}{f(z)} & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

is holomorphic.

□

We have three types of singularities:

- $|f(z)|$ is bounded $\Rightarrow$ Removable
- $|f(z)| \xrightarrow{z \to z_0} \infty \Rightarrow$ Pole
- $|f(z)| \xrightarrow{z \to z_0}$? doesn't exist

Definition 59 (Essential singularity)**.** Singularities that aren't removable or a pole.

Example. $e^{\frac{1}{z}}$ in 0. If we go to 0^- vs 0^+, we have different limits

Definition 60 (Meromorphic function)**.** A holomorphic function who's only singularities are poles.

Note. Poles are isolated, as zeros are isolated!

We have a similar theorem like 'The behaviour of holomorphic function around its zeros'.

Theorem 25 (Behaviour of a function near the pole)**.** Let z_0 be a pole of $f(z)$. Then $\exists r > 0$ such that for all $z \in D_r(z_0)$,

$$f(z) = (z - z_0)^{-n} g(z),$$

where n is a uniquely definiet positive integer and $g(z) \neq 0$ in this neighbourhood, and $g(z)$ is holomorphic.

Proof. Since z_0 is a pole of $f(z)$, z_0 is a zero of $\frac{1}{f(z)}$. By the theorem about behaviour of holomorphic functions in the neighbourhood of their zero's, we get that

$$\frac{1}{f(z)} = (z - z_0)^n h(z),$$

n unique, $h(z) \neq 0$ in the neighbourhood, holomorphic. Now $\frac{1}{h}$ is holomorphic and $\frac{1}{h} \neq 0$. Therefore

$$f(z) = (z - z_0)^{-n} \frac{1}{h(z)} = (z - z_0)^{-n} g(z).$$

□

Definition 61 (Multiplicity of a pole)**.** We call n the multiplicity of the pole.

Example. $f(z) = \frac{z}{(z+1)^2(z+2)^3}$. $z = -1$ is a pole of multiplicity 2,and $z = -2$ is a pole of multiplicity 3.

Theorem 26 (About power expansion of a function near a pole)**.** Suppose $f(z)$ has a pole at z_0, then in the neighbourhood of z_0, we can write

$$f(z) = \frac{A_{-n}}{(z - z_0)^n} + \frac{A_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{A_{-1}}{(z - z_0)} + G(z),$$

where n is the multiplicity of z_0 and $G(z)$ is holomorphic.

Proof. By the previous theorem, in the neighbourhood of z_0,

$$f(z) = (z - z_0)^{-n} g(z).$$

But $g(z)$ is holomorphic, and therefore analytic.

$$\begin{aligned} f(z) &= \frac{1}{(z - z_0)^{-n}} \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ &= \frac{a_0}{(z - z_0)^n} + \frac{a_1}{(z - z_0)^{n-1}} + \cdots + \frac{a_{n-1}}{z - z_0} + \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}. \end{aligned}$$

□

We define the following:

$$f(z) = \underbrace{\frac{A_{-n}}{(z - z_0)^n} + \frac{A_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{A_{-1}}{(z - z_0)}}_{\text{principle part of } f(z) \text{ near the pole } z_0} + \underbrace{G(z)}_{\text{holomorphic}}$$

Definition 62 (The residu of a function near the pole z_0)**.**

$$\operatorname*{res}_{z_0} f(z) = A_{-1}$$

As $\frac{1}{(z - z_0)^{>1}}$ has a primitive, and G is holomorphic, these parts become 0 when we're integration over a closed curve. Therefore, The most important part is A_{-1}.

Theorem 27 (Calculating residues using derivatives)**.** Let $f(z)$ have a pole z_0 of multiplicity n.

$$\operatorname*{res}_{z_0} f(z) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^n}{dz^n} \left[(z-z_0)^n f(z) \right].$$

Proof. Using the previous theorem,

$$(z-z_0)^n f(z) = A_{-n} + \cdots + A_{-2}(z-z_0)^{n-2} + A_{-1}(z-z_0)^{n-1} + G(z)(z-z_0)^n.$$

Taking $n-1$ derivatives, the first part won't survive, the A_{-1} part will stay, and the last part will $\frac{d^{n-1}}{dz^{n-1}} G(z)(z-z_0)^n \xrightarrow{z \to z_0} 0$ □

Theorem 28 (Calculating the residue using integrals)**.** Suppose $f(z)$ has a pole at z_0 of multiplicity n. Let $r > 0$ be such that on $D_r(z_0)$, $f(z)$ can be written as power series expansion. Let $C = \partial D_r(z_0)$. Then

$$\operatorname*{res}_{z_0} f(z) = \frac{1}{2\pi i} \int_C f(z) dz.$$

Proof.

$$\begin{aligned}\int_C f(z)dz &= \int_C \frac{A_{-n}}{(z-z_0)^n} + \frac{A_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{A_{-1}}{(z-z_0)} + G(z) \\ &= \int_C \frac{A_{-n}}{(z-z_0)^n} + \frac{A_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{A_{-1}}{(z-z_0)}\end{aligned}$$

Now, let's look at

$$\int_C \frac{A_{-k}}{(z-z_0)^k} dz.$$

Recall Cauchy integral formula:

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta.$$

Therefore

$$\int_C \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta = \frac{2\pi i}{k!} f^{(k)}(z).$$

Therefore, we find that

$$\int_C \frac{A_{-k}}{(z-z_0)^k} dz = 0 \quad k \neq 1,$$

as this is the derivative of a constant function!

Now, we're left with

$$\int_C \frac{A_{-1}}{(z-z_0)} dz = 2\pi i A_{-1} f(z).$$

Therefore

$$\int_C f(z) dz = 2\pi i \operatorname*{res}_{z_0} f(z).$$

□

Theorem 29 (Residue theorem)**.** Let $\Omega \subset \mathbb{C}$ be open. γ be a closed simple curve in Ω, such that the interior of $\gamma \subset \Omega$. Let $f(z)$ be a meromorphic function on Ω such that $f(z)$ has no poles *on* γ and $f(z)$ has poles $z_1, z_2, \dots, z_k$ in the interior of γ. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{n=1}^{k} \operatorname*{res}_{z_n} f.$$

Les 8: Argument and maximum modules principle

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Proof. Denote by ε, δ small enough numbers. Denote by $C_m = C_\varepsilon(z_m)$ and Γ is curve obtained from γ using the way from the picture. $f(z)$ is holomorphic inside Γ. Therefore, $\int_\Gamma f(z)dz = 0$ by Cauchy's theorem. Therefore,

$$\begin{aligned}
\lim_{\delta\to 0} \int_\Gamma f(z)dz &= 0 \\
\int_\gamma f(z)dz + \sum_{m=1}^{k} \int_{C_m} f(z)dz &= 0 \\
\int_\gamma f(z)dz &= \sum_{m=1}^{k} \int_{C_m^-} f(z)dz \\
&= \sum_{m=1}^{k} 2\pi i \operatorname*{res}_{z_m} f(z),
\end{aligned}$$

by the theorem about calculating residues by integrals. $\square$

Theorem 30. Let $\Omega \subset \mathbb{C}$ be open. Let γ be a closed simple curve inside Ω, such that interior of γ belongs to Ω. Let f be a meromorphic function such that f has no poles and zeros on γ. Then

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \\
&\quad= \begin{pmatrix}\text{number of zeros of } f \text{ inside} \\ \text{interior of } \gamma \text{ counted with} \\ \text{multiplicities}\end{pmatrix} - \begin{pmatrix}\text{number of poles of } f \text{ inside} \\ \text{the interior of } \gamma \text{ counted} \\ \text{with multiplicities}\end{pmatrix}.
\end{aligned}$$

Proof. If all the singularities of $\frac{f'}{f}$ are either zeros of f, or poles of f', which are poles of f, i.e. pole of $f' \Leftrightarrow$ pole of f:

$$\begin{aligned}
f(z) &= (z-z_0)^{-n} g(z) \\
f'(z) &= -n(z-z_0)^{-n-1} g(z) + (z-z_0)^n g'(z) \\
&= (z-z_0)^{n-1}(\dots).
\end{aligned}$$

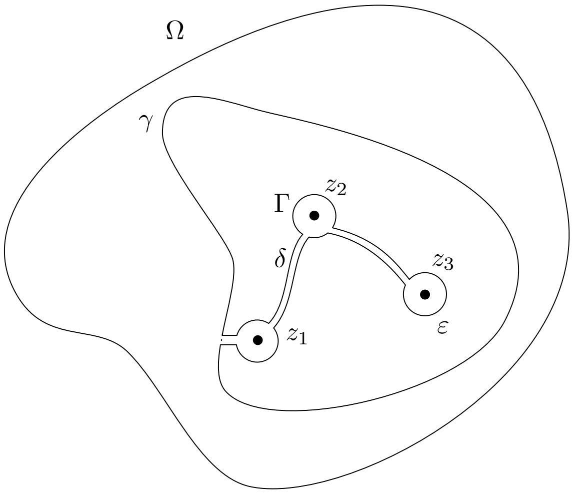


Figure 5.2: residue-theorem

Conversly,

$$\begin{aligned} f'(z) &= (z - z_0)^{-n} g(z) \\ f(z) &= ?. \end{aligned}$$

Eumm...

Restart!

What are singularities of $\frac{f'(z)}{f(z)}$?

- Zeros of $f(z)$,
- Poles of $f(z)$, as $f(z_0)$ is not defined in this point.
- Other? No! Since if z_0 is not a zero of $f(z)$, then there is a disk s.t. $f(z) \neq 0$. Therefore, for all z in the disk, $\frac{f'(z)}{f(z)}$ is holomorphic. Therefore, z_0 cannot be a singular point.

Therefore, all the singularities of $\frac{f'}{f}$ are either zeros of f, or poles of f.

Let z_0 be a zero of order n of $f(z)$. By the theorem about the behaviour of a holomorphic function near zero, for all z in a neighbourhood of z_0, we have

$$f(z) = (z - z_0)^n g(z).$$

where $g(z) \neq 0$ for all z in neighbourhood and $g(z)$ holomorphic.

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} \\ &= \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}.\end{aligned}$$

Therefore, z_0 is a simple pole of $\frac{f'}{f}$ and $\operatorname{res}_{z_0} \frac{f'(z)}{f(z)} = n$.

If z_0 is a pole of $f(z)$ of order n, then

$$f(z) = (z-z_0)^{-n} g(z)$$

for all z in neighbourhood of z_0 and $g \neq 0$ is holomorphic.

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-n(z-z_0)^{-n-1}g(z) + (z-z_0)^{-n} g'(z)}{(z-z_0)^{-n} g(z)} \\ &= \frac{-n}{z-z_0} + \frac{g'(z)}{g(z)}.\end{aligned}$$

Again, z_0 is a simple pole and $\operatorname{res}_{z_0} \frac{f'(z)}{f(z)} = -n$.

By the residue theorem,

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)} &= 2\pi i \sum_{m=1}^{k} \operatorname{res}_{z_k} \frac{f'(z)}{f(z)} \\ &= 2\pi i \left(\begin{array}{l}\text{number of zeros of } f \text{ inside} \\ \text{interior of } \gamma \text{ counted with} \\ \text{multiplicties}\end{array}\right) - \left(\begin{array}{l}\text{number of poles of } f \text{ inside} \\ \text{the interior of } \gamma \text{ counted} \\ \text{with multiplicities}\end{array}\right).\end{aligned}$$

□

Why is this theorem called the argument principle? See figure. When we go around a curve, the modulus of a function value at a point stays the same, but the argument can 'change'.

Example. Suppose $f(z) = z^6$. If we move from 1 to 60 degrees, $f(z)$ gets back to the same point, but the argument has changed.

Let γ be paramterizable, $z : [0,1] \to \mathbb{C}$. Then

$$\begin{aligned}\int_\gamma \frac{f'}{f} &= \int_0^1 \frac{f'(z(t))z'(t)}{f(z(t))} dt \\ &= \int_0^1 \frac{\frac{d}{dt}(f(z(t)))}{f(z(t))} dt \\ &= \int_0^1 \frac{r'(t)e^{i\theta(t)} + r(t)i\theta'(t)e^{i\theta(t)}}{r(t)e^{i\theta(t)}} dt \\ &= \int_0^1 \frac{r'(t)}{r(t)} dt + \int_0^1 i\theta'(t) dt\end{aligned}$$

.

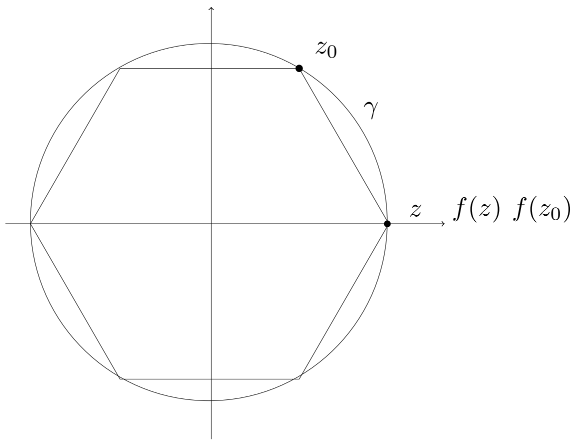


Figure 5.3: argument-principle

The first integral is 0, as the radius in $t = 0$ is equal to the radius in $t = 1$.

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)} dz &= 0 + i(\theta(1) - \theta(0)) \\ &= i(\theta(1) - \theta(0)) \\ &= 2\pi i \left(\begin{pmatrix} \text{number of zeros of } f \text{ inside} \\ \text{interior of } \gamma \text{ counted with} \\ \text{multiplicties} \end{pmatrix} - \begin{pmatrix} \text{number of poles of } f \text{ inside} \\ \text{the interior of } \gamma \text{ counted} \\ \text{with multiplicities} \end{pmatrix} \right).\end{aligned}$$

Theorem 31 (Rouché's theorem)**.** Let C be a circle in Ω. Let f, g two holomorphic functions on Ω s.t. $|f(z)| > |g(z)|$ for all $z \in C$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros *inside* the interior of $\mathbb{C}$.

`https://www.wikiwand.com/en/Rouch%C3%A9%27s_theorem`

Proof. For $t \in [0, 1]$, denote by

$$f_t(z) = f(z) + tg(z).$$

Since $|f(z)| > |g(z)|$ for all $z \in C$, the function

$$\frac{f_t'(z)}{f_t(z)}$$

is continuous on $[0, 1] \times C$.

$$\frac{f_t'(z)}{f_t(z)} = \frac{f'(z) + tg'(z)}{f(z) + tg(z)}.$$

Therefore,

$$\int_C \frac{f_t'(z)}{f_t(z)} dz$$

is continuous on $[0,1]$. Then

$$\int_C \frac{f_t'(z)}{f_t(z)}dz = 2\pi i \begin{pmatrix}\text{number of zeros of } f \text{ inside} \\ \text{interior of } \gamma \text{ counted with} \\ \text{multiplicties}\end{pmatrix}$$

As this is a 'discrete' continuous function, it is constant. Therefore f_0 and f_1 have the same number of zeros. □

Definition 63 (Open map)**.** A map $f : \mathbb{C} \to \mathbb{C}$ is called open iff for all $U \subset \tau$, $f(U) \in \tau$.

Theorem 32 (Open mapping theorem)**.** Let $\Omega \subset \mathbb{C}$ be open and f holomorphic on Ω. If $f \not\equiv$ constant, then $f(z)$ is open.

Proof. Let $z_0 \in A \subset \Omega$ be open. We want to prove that $f(A) \ni w_0$ is open. Let $g(z) = f(z) - w_0$. Then there exists δ such that for all $z \in \overline{D_\delta(z_0)} \setminus \{z_0\}$: $g(z) \neq 0$. This is because z_0 is a zero of $g(z)$ and zeros are isolated. Denote by $C = \partial D_\delta(z_0)$. Since $g(z) \neq 0$ for all $z \in C$, and C is compact, $|g(z)| > \varepsilon$ for all $z \in C$.

Now we use Rouche's theorem. Let w we such that $|w - w_0| < \varepsilon$, i.e. $w \in D(w_0, \varepsilon)$. Denote by $F(z) = f(z) - w$, $G(z) = w_0 - w$. Then for all $z \in C$,

$$|F(z)| = |g(z)| > \varepsilon > |w_0 - w| = |G(z)|.$$

Therefore, $F(z)$ and $F(z) + G(z)$ have the same number of zeros, inside the interior of C. Now, $F(z) = f(z) - w_0$ has at least one zero. And $F(z) + G(z) = f(z) - w_0 + w_0 - w = f(z) - w$. This also has a zero. Therefore $\exists z : f(z) = w$. Therefore, all the elements in the neighbourhood around w_0 are contained in the image. □

Theorem 33 (Maximum modules principle)**.** Let $\Omega \subset \mathbb{C}$ be open, $f(z) \not\equiv$ constant, be holomorphic on Ω. Then $f(z)$ cannot attain its maximum (in absolute value) inside Ω.

Proof. By contradiction. Suppose $\exists z_0 \in \Omega$ s.t. $f(z)$ attains its maximum at z_0. Since Ω is open, $\exists r > 0$ s.t. $D_r(z_0) \subset \Omega$. By the open mapping theorem, $f(D_r(z_0))$ is also open. Therefore, $D_\varepsilon(f(z_0)) \subset f(D_r(z_0))$. This clearly contradicts $f(z_0)$ being the maximum. □

Minimum: consider $f : D_1(0) \to \mathbb{C} : z \mapsto z^2$.

Intermezzo (Logarithms)**.** $z : \log a \Leftrightarrow e^z = a$. Therefore,

$$\log a = \log|a| + i(\arg(a) + 2\pi k).$$

Interesting fact: $\log(ab) \neq \log a + \log b$. ◇

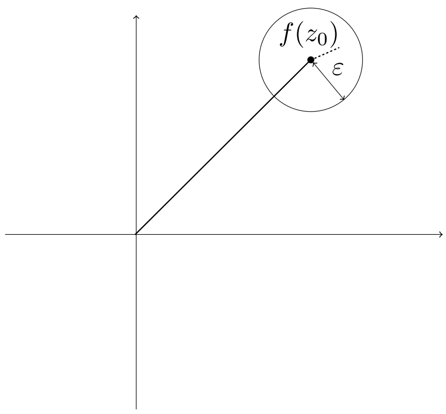


Figure 5.4: maximum-modules-principle

Chapter 6

Homotopies

Les 9: Homotopies and simply connected domains

di 23 apr 16:05

Let $\Omega \subset \mathbb{C}$ be open, $\gamma_0, \gamma_1[a, b] \to \Omega$ such that $\gamma_0(a) = \gamma_1(a) = \alpha$, $\gamma_0(b) = \gamma_1(b) = \beta$.

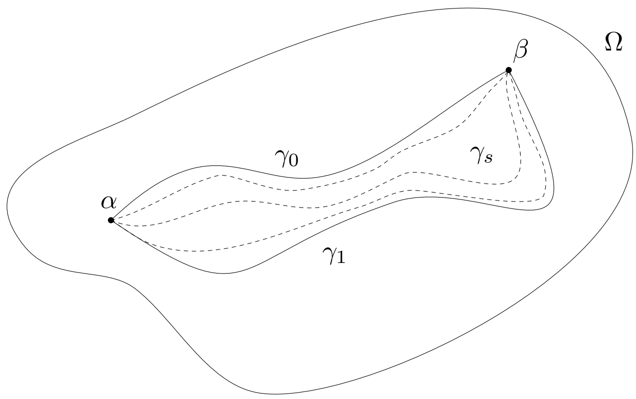


Figure 6.1: Homotopies

Definition 64. γ_0, γ_1 are called homotopic in Ω if there exists a map $F(s, t) = \gamma_s(t)$ defined on $[0, 1] \times [a, b]$ such that

- $\forall s \in [0, 1] : \gamma_s(a) = \alpha, \gamma_s(b) = \beta$.
- $\forall (s, t) \in [0, 1] \times [a, b] : \gamma_s(t) \in \Omega$.
- $F(s, t)$ is continuous on $[0, 1] \times [a, b]$.

Note. Note that this depends on Ω.

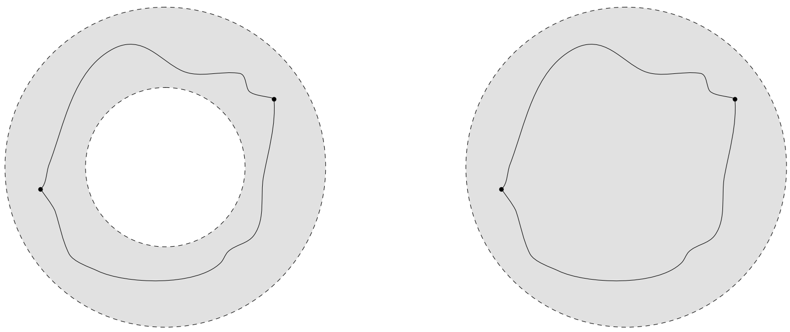


Figure 6.2: Dependence of Ω

Theorem 34 (about integrals along homotopic curves)**.** Let $\Omega \subset \mathbb{C}$ be open, γ_0, γ_1 homotopic curves in Ω, f be holomorphic on Ω. Then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Proof. Let $F(s,t)$ be a homotopy between γ_0, γ_1, defined on $[0,1] \times [a,b]$. Denote by $K = F([0,1] \times [a,b])$. Since $[0,1] \times [a,b]$ is compact, and F is continuous, K is also compact.

Since K is compact, there exisst $\varepsilon > 0$ s.t.

$$\forall z \in K : B_{3\varepsilon}(z) \subset \Omega.$$

Indeed: take $\varepsilon = d(K, \partial\Omega)/6 > 0$, which works for every $z \in K$, it doesn't depend on z.

Since $F(s,t)$ is continuous on $[0,1] \times [a,b]$ which is compact, it is also uniformly continuous. Since $F(s,t)$ is uniformly continuous on $[0,1] \times [a,b]$, $\exists \delta$ such that

$$|s_1 - s_2| < \delta \Rightarrow \sup_{t \in [a,b]} |F(s_1,t) - F(s_2,t)| < \varepsilon.$$

in other words:

$$\sup_{t \in [a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \varepsilon,$$

i.e. the curves are very close. δ depends only on F and ε.

Since $\sup < \varepsilon$, we can cover these curves by a finite number of balls $D_1, \ldots, D_n$ of radius 2ε. Such that $D_i \cap D_{i+1}$ intersects both $\gamma_{s_1}, \gamma_{s_2}$ Finite because ε is fixed.

Construction Let $\alpha = z_0 = w_0$. Now choose points $z_i = \gamma_{s_1} \in \gamma_{s_1} \cap D_{i-1} \cap D_i$ (n points), and $w_i = \gamma_{s_1} \in \gamma_{s_2} \cap D_{i-1} \cap D_i$ (n points) Now define $\beta = z_{n+1} = \omega_{n+1}$ Denote by $\gamma_{s_1,k}$ a curve on γ_{s_1} between z_k and z_{k+1}.

Since f is holomorphic on Ω, f is holomorphic on all D_k, and therefore f has a primitive F_k on D_k. Now, look at $F_k - F_{K+1}$ on $D_k \cap D_{k+1}$. Now

$$(F_k - F_{k-1})' = f - f = 0,$$

and therefore $F_k - F_{k+1}$ is a constant on $D_k \cap D_{k-1}$. But we already have some points.

$$F_k(z_{k+1}) - F_{k+1}(z_{k+1}) = F_k(w_{k+1}) - F_{k+1}(w_{k+1}).$$

Therefore

$$F_k(z_{k+1}) - F_k(w_{k+1}) = F_{k+1}(z_{k+1}) - F_{k+1}(w_{k+1}).$$

Now, the proof starts.

$$\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f = \sum_{k=0}^{n} \left(\int_{\gamma_{s_1,k}} f - \int_{\gamma_{s_2,k}} f \right).$$

But we now that in these balls, there is a primitive!

$$\begin{aligned}
&= \sum_{k=0}^{n} \left(F_k(z_{k+1}) - F_k(z_k) - (F_k(w_{k+1}) - F_k(w_k))\right) \\
&= \sum_{k=0}^{n} \left(F_k(z_{k+1}) - F_k(w_{k+1}) - (F_k(z_k) - F_k(w_k))\right) \\
&= 0,
\end{aligned}$$

by using the previous property.

So we proved that as soon as $s_1 - s_2| < \delta$, then

$$\int_{\gamma_{s_1}} f = \int_{\gamma_{s_2}} f.$$

Now, choose N such that $\frac{1}{N} < \delta$. Then

$$\int_{\gamma_0} f = \int_{\gamma_{\frac{1}{N}}} f = \int_{\gamma_{\frac{2}{N}}} f = \int_{\gamma_{\frac{3}{N}}} f = \int_{\gamma_{\frac{N}{N}}} f = \int_{\gamma_1} f.$$

□

Note. Why can't we just use Cauchy's theorem? We only proved that for toy contours (with paths that are made up of a finite number of straight subpaths).

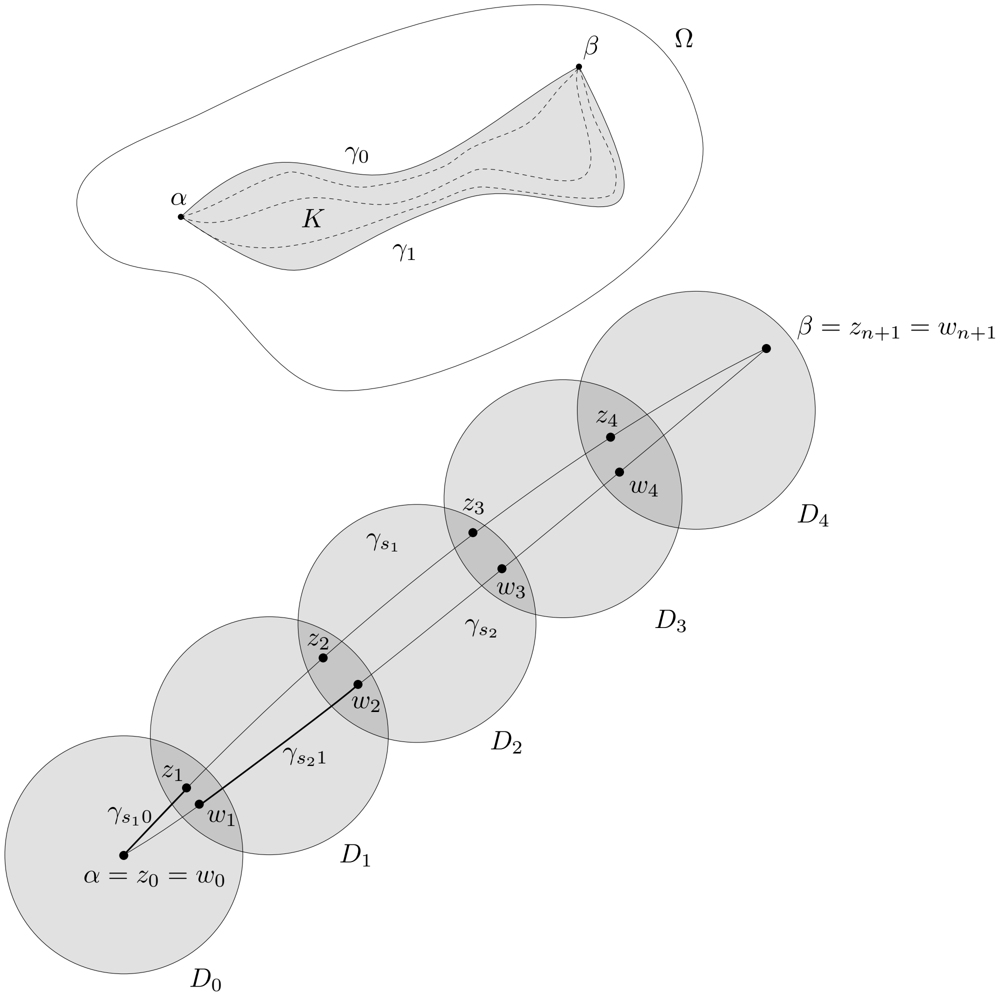


Figure 6.3: proof of homotopic curves

Simply connected domains

Definition 65. Let $\Omega \subset \mathbb{C}$ be open. Ω is simply connect connected if for all $\gamma_0, \gamma_1 : [a, b] \to \Omega$ such that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$, γ_0 is homotopic to γ_1 in Ω.

Example. If Ω is convex, then Ω is simply connected.

$$\gamma_s(t) = \gamma_0(t)(1 - s) + \gamma_1(t)s.$$

For fixed t, this is an equation of a line.

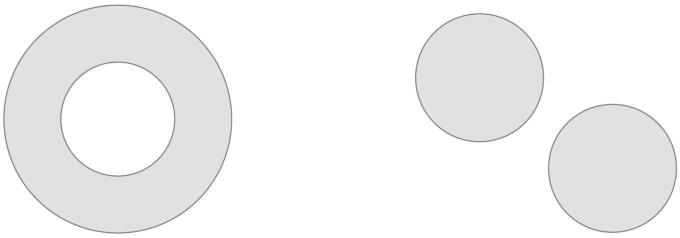
Example. Toy contours are simply connected. (Difficult exercise)

Theorem 35 (About existence of primitives in simply connected domains)**.** Let $\Omega \subset C$ be open, connected, and simply connected. f is holomorphic on Ω. Then f has a primitive.

Proof. Fix $z_0 \in \Omega$. For $z \in \Omega$ denote by γ_z a curve connecting z_0 and z. This exists because Ω is connected. (Note that in open sets, connected is path connected) Let $F(z) = \int_{\gamma_z} f(\zeta)d\zeta$. The proof that $F(z)$ is a primitive of $f(z)$, is completely the same as the proof of existence of primitives in a disk.

$$F(z+h) - F(z) = \int_\eta f(\zeta)d\zeta,$$

where η is a line segment between z and $z+h$. Now, the prove is almost the same! $\square$



Connected Simply connected

Figure 6.4: simply connected not connected

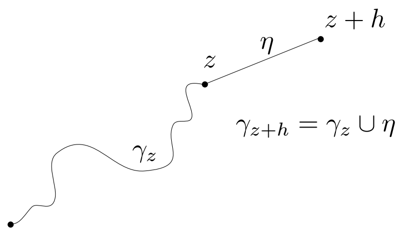


Figure 6.5: Theorem about existence of primitives in simply connected domains

Complex logarithm

Wat is $\log a$ when $a \in \mathbb{C}$? We want $z = \log a$ if $e^z = a$.

$$a = re^{i\theta} = e^{\log r} e^{i\theta} = e^{\log r + i\theta}.$$

Therefore

$$z = \log r + i\theta + 2\pi i k, \quad k \in \mathbb{Z}.$$

If $e^z = a$, then $z = \log |a| + i \arg(a) + 2\pi i k$.

Theorem 36 (About correct, good definition of logarithm)**.** Let $\Omega \in \mathbb{C}$, which is open, connected, simply connected, and $0 \notin \Omega$ and $\in \Omega$. Then there exists a function $F(z) = \log_\Omega z$ such that

- $F(z)$ is holomorphic on Ω
- $e^{F(z)} = z$
- $F(x) = \log x$, if $x \in U(1)$, i.e. 'close to 1'

Proof. Since $0 \in \Omega$, $\frac{1}{z} = f(z)$ is holomorphic on Ω. For $z \in \Omega$ denote by γ_z a curve which connects z and 1. Let

$$F(z) = \int_{\gamma_z} f(\zeta) d\zeta = \int_{\gamma_z} \frac{1}{\zeta} d\zeta$$

By theorem about existence of primitives in simply connected domains, $F(z)$ is the primitive of $\frac{1}{z}$. As Ω is simply connected, $F(z)$ doesn't depend on γ_z. Therefore $F(z)$ is holomorphic: we proved (1).

$$\begin{aligned}(ze^{-F(z)})' &= e^{-F(z)} - zF'(z)e^{-F(z)} \\ &= e^{-F(z)} - \frac{z}{z} e^{-F(z)} = 0.\end{aligned}$$

Therefore, $ze^{-F(z)} = \text{constant} = 1e^{-F(1)} = 1$, looking at the definition of $F(z)$. Therefore $e^{F(z)} = z$.

Third part. Now, what is close to 1? x such that then line between 1 and x lies in Ω. Let γ_x be this line. Therefore $F(x) = \int_{\gamma_x} \frac{1}{\zeta} d\zeta = \int_1^x \frac{1}{s} ds = \log x$. $\square$

Restricting a domain where log is defined is called 'branching'. For example, let $\Omega = \mathbb{C} \setminus \{z \mid \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$ Then $\log_\Omega$ is called the principal branch of logarithm and is defined by $\log(z)$.

Proposition 10. If $z \in \Omega = \mathbb{C} \setminus \mathbb{R}^-$, then

$$\log z = \log |z| + i \arg z,$$

where $\arg(z) \in (-\pi, \pi)$.

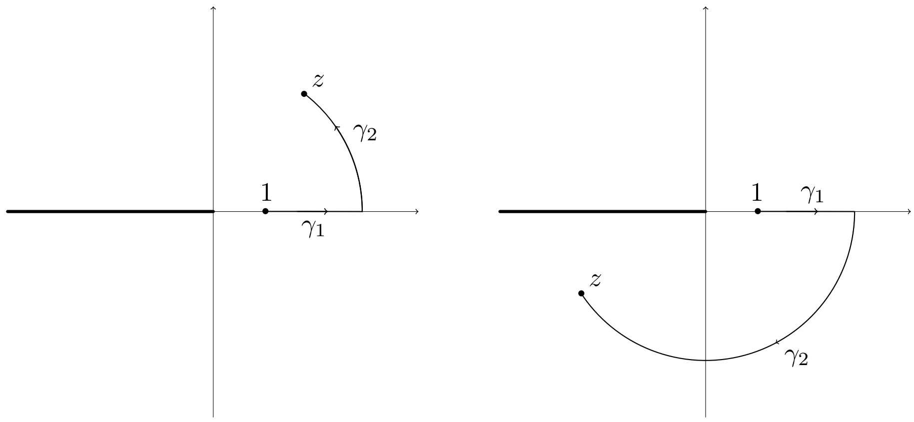


Figure 6.6: logarithm branching

Proof. Let γ_z as on the figure. 1 to $|z|$ and then by circle from $|z|$ to z. Then

$$\begin{aligned}\log z &= \int_{\gamma_z} \frac{1}{\zeta} d\zeta \\ &= \int_1 + \int_2 \\ &= \int_1^{|z|} \frac{1}{s} ds + \int_0^{\arg z} \frac{1}{|z|e^{it}} i|z|e^{it} dt \\ &= \log|z| + i \arg z.\end{aligned}$$

□

Note. $\log(z_1 z_2) \neq \log z_1 + \log z_2$. For example: $z_1 = z_2 = e^{\frac{2\pi i}{3}}$. And therefore, $\log z_i = \frac{2\pi i}{3}$, but $\log(z_1 z_2) = -\frac{\pi i}{3}$.

Definition 66. $z_1^{z_2} = e^{z_2 \log z_1}$.

UOVT. prove that

$$(z^{\frac{1}{n}})^n = z$$

Les 10: Conformal mappings

di 30 apr 16:01

Chapter 7

Conformal mappings

Main question: $U, V \subset \mathbb{C}$, $\exists$ bijective holomorphic function $f : U \to V$? Answer: can be done almost always!

Example. $f : \mathbb{C} \to D(0,1)$ cannot be bijective and holomorphic, as it is bounded, hence constant.

Definition 67 (Conformal mappings)**.** $f : U \to V$ is called conformal if f is holomorphic on U

Proposition 11. Let $f : U \to V$ be a conformal map. Then $\forall z \in U$: $f'(z) \neq 0$ and $f^{-1} : V \to U$ is also holomorphic.

Proof. By contrary. Suppose $f'(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Since f is holomorphic on U, $\exists \varepsilon_1 > 0$ such that for all $z \in D_\varepsilon(z_0)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Now, a_0 is $f(z_0)$ and a_1 is $f'(z_0) = 0$ Therefore, $\forall z \in D$,

$$f(z) - f(z_0) = a(z - z_0)^k + G(z),$$

where $k \geq 2$ and $G(z)$ is holomorphic which converges to 0 as $(z - z_0)^{k+1}$ when $z \to z_0$. a is the first a_k such that this holds. Since $f'(z_0) = 0$, $\exists \varepsilon_2 \leq \varepsilon_1$ such that $\forall z \in D_{\varepsilon_2}(z_0)$ such that $z \neq z_0$, $f'(z) \neq 0$, as zeros of holomorphic functions must be isolated. In the neighbourhood of z_0, there is only one zero.

Let w be small enough.

$$f(z) - f(z_0) - w = \underbrace{a_k(z - z_0)^k}_{F(z)} - w + G(z).$$

Since $G(z)$ converges to 0 as $(z - z_0)^{k+1}$ when $z \to z_0$ and $F(z)$ converges to $-w$, when $z \to z_0$, $\exists \varepsilon_3 < \varepsilon_2$ such that for all $z \in D_{\varepsilon_3}(z_0)$:

$$|F(z)| > |G(z)|.$$

Therefore, by Rouché's theorem, $F(z)$ and $F(z)+G(z)$ have the same number of roots inside $D_{\varepsilon_3}(z_0)$. $F(z)$has at least 2 roots (fundamental theorem of algebra). Therefore $F(z)+G(z)$ has at least 2 roots. Therefore, $f(z)-f(z_0)-w$ has at least two roots. Can these roots be the same? No, as $f'(z) \neq 0$ for $z \in D_{\varepsilon_2}(z_0)$. As f is bijective, this is a contradiction! ↯

ε_1 We can express f as a power series

ε_2 Derivative is not 0

ε_3 $|F(z)| > |G(z)|$

Vraag. Why does $f'(z)$ imply that the roots are distinct? A: When $f(z)$ has a root z_1 of order k, then

$$f^{(k)}(z_1) = 0.$$

Second part. Let $g(z) = f^{-1}(z)$, which exists because of the bijectivity. Substitute $w = f(z)$, $w_0 = f(w_0)$.

$$\begin{aligned}
\frac{g(w)-g(w_0)}{w-w_0} &= \frac{g(f(z))-g(f(z_0))}{f(z)-f(z_0)} \\
&= \frac{z-z_0}{f(z)-f(z_0)} \\
&= \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}} \\
&\xrightarrow{z\to z_0} f'(z_0).
\end{aligned}$$

Therefore, $g(z)$ is holomorphic and $g'(w) = \frac{1}{f'(g(w))}$. □

Not that conformal mappings $\rightsquigarrow$ equivalence relation.

Corollary 9. *U, V are conformally equivalent iff there exists holomorphic functions $F : U \to V$ and $G : V \to U$ such that $F(G(z)) = z$ and $G(F(z)) = z$.*

Note. If f is defined on the boundary. If $f : U \to V$ is conformal, then $f(\partial U) = \partial V$, because of the open mappings theorem.

Note. Plotting complex functions.

Important examples of conformal mappings

Definition 68 (Mobius transformations)**.** $f(z) = \frac{az+b}{cz+d}$ is called a Möbius transformation. If $ad-bc = 0$, then $f(z)$ is constant.

Property. A Möbius transformation is holomorphic everywhere, except at at most 1 point.

We think about $f(z)$ as a map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ taking $f(\infty) = \frac{a}{c}$. and $f\left(-\frac{d}{c}\right) = \infty$

Examples

- Translations $f(z) = z + a$

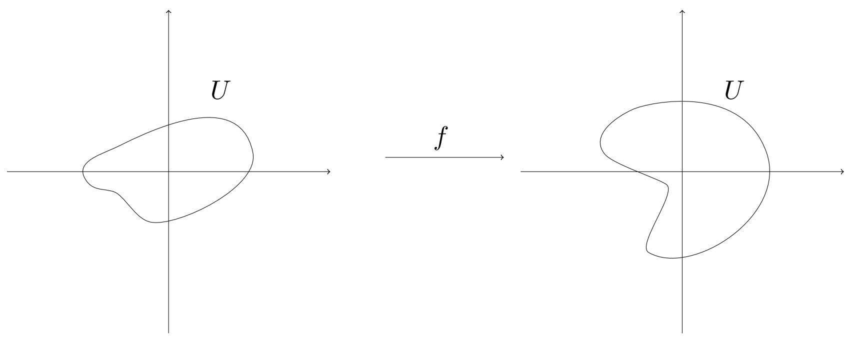


Figure 7.1: plotting complex functions

- Homothety $f(z) = az$
- Rotation $f(z) = e^{i\theta}z$
- Inversion.

Proposition 12 (Fundamental properties of Mobiustransformations)**.**

1. composition of Möbius transformations are Möbius transformations.
2. Every Möbius transformation can be expressed as a composition of transformations, homothety, rotations and inversions.
3. Every Möbius transformation maps lines and circles to lines and circles.
4. If a Möbius transformation maps four distinct points z_1, z_2, z_3, z_4 to another four distinct points w_1, w_2, w_3, w_4 then
$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_2 - w_3)(w_1 - w_4)}.$$
We call this cross ratio preserving.
5. If z_1, z_2, z_3 are distinct points, and w_1, w_2, w_3 are also distinct, then there exists a Mobius transformation f such that $f(z_k) = w_k$.

Proof. 1. Calculations

2. Let $f(z) = \frac{az+b}{cz+d}$ for $ad - bc \neq 0$.
   - If $c = 0$, then $f(z) = \frac{a}{c}z + \frac{b}{d}$ (easy)
   - If $c \neq 0$, then denote by
   $$f_1(z) = z + \frac{d}{c}, \quad f_2(z) = \frac{1}{z}, \quad f_3(z) = \frac{bc - ad}{c^2}z, \quad f_4(z) = z + \frac{a}{c}.$$
   Then $f(z) = f_4(f_3(f_2(f_1(z))))$.

3. It follows from 2. (not difficult)
4. Direct calculations.
5. Take in 4, $w_4 = f(z)$ and $z_4 = z$.

$$\frac{(z_1 - z_3)(z_2 - z)}{(z_2 - z_3)(z_1 - z)} = \frac{(w_1 - w_3)(w_2 - f(z))}{(w_2 - w_3)(w_1 - f(z))}.$$

From this, we can express $f(z)$ in terms of z.

□

Let $\mathbb{H} := \{z \mid \operatorname{Im} z > 0\}$ and $\mathbb{D} = \{z \mid |z| < 1\}$.

Theorem 37 (Conformal mapping $\mathbb{H} \to \mathbb{D}$)**.** Let $F(z) = \frac{i-z}{i+z}$ and $G(w) = i\frac{1-z}{1+z}$. Then $F : \mathbb{H} \to \mathbb{D}$ is a conformal such that $F(G(z)) = z$, G is conformal and $G(F(w)) = w$.

Proof. F is holomorphic on $\mathbb{H}$ and G is holomorphic on $\mathbb{D}$. Now we prove that $F(\mathbb{H}) = \mathbb{D}$. Let $z \in \mathbb{H}$. It's clear that $|z - i| < |z - (-i)|$. Therefore $\frac{|z-i|}{|z+i|} < 1$, which means that $|F(z)| < 1$.

Now, the opposite direction. Take $w \in \mathbb{D}$. Therefore $w = u + iv$ such that $u^2 + v^2 < 1$. Then $\operatorname{Im} G(w) = \operatorname{Im}\left(i\frac{1-w}{1+w}\right) = \operatorname{Re}\left(\frac{1-w}{1+w}\right)$. Therefore

$$\begin{aligned}
\operatorname{Im} G(w) &= \operatorname{Re}\frac{1-u-iv}{1+u+iv} \\
&= \operatorname{Re}\frac{(1-u-iv)(1+u-iv)}{(1+u+iv)(1+u-iv)} \\
&= \operatorname{Re}\frac{(1-iv)^2-u^2}{(1+u)^2+v^2} \\
&= \operatorname{Re}\frac{1-2iv-v^2-u^2}{(1+u)^2+v^2} \\
&= \frac{1-v^2-u^2}{(1+u)^2+v^2} > 0,
\end{aligned}$$

as $v^2 + u^2 < 1$. Therefore G maps $\mathbb{D} \to \mathbb{H}$.

Now we only need to prove that $F(G(w)) = w$ and $G(F(z)) = z$. □

Theorem 38 (Schwarz lemma)**.** Let : $\mathbb{D} \to \mathbb{D}$ be a conformal[a] map such that $f(0) = 0$. Then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
2. If $z_0 \in \mathbb{D}$ such that $|f(z_0)| = |z_0|$, then f is a rotation.
3. $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then f is a rotation.

[a]Doesn't have to be conformal, h. . . is enough

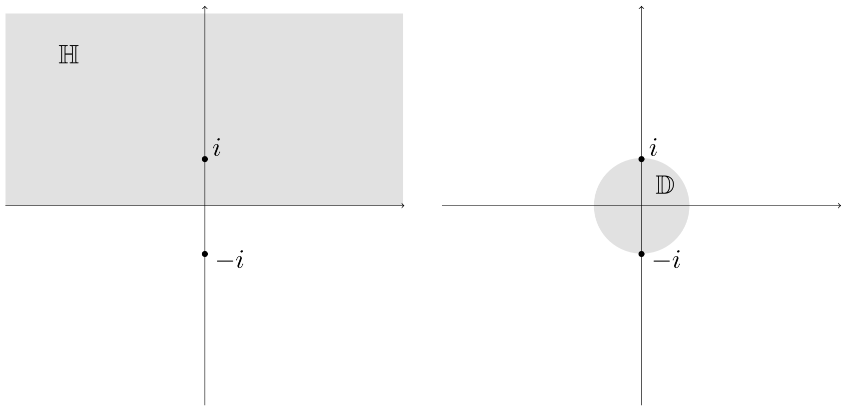


Figure 7.2: Conformal mapping from $\mathbb{H} \to \mathbb{D}$

Proof.

1. $f(z)$ is holomorphic on $\mathbb{D}$. Therefore,
$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$
for all $z \in \mathbb{D}$. As $f(0) = 0$, $a_0 = 0$. Therefore, $\frac{f(z)}{z}$ is holomorphic on $\mathbb{D}$.
Let z be a point from D and $|z| = r < 1$.
$$\left|\frac{f(z)}{z}\right| \leq \frac{|f(z)|}{r} \leq \frac{1}{r} \quad \text{as } f(z) \in \mathbb{D}.$$
Since $\frac{f(z)}{z}$ is holomorphic on $\mathbb{D}$, we can apply the maxim modules principle, which says that if a function has a maximum, it must be on the boundary. Therefore $\frac{f(z)}{z}$ cannot obtain maximum *inside* $\mathbb{D}$. Therefore $\left|\frac{f(z)}{z}\right| \leq 1$, which means that $|f(z)| \leq |z|$.

2. The function $g(z) = \frac{f(z)}{z}$ is holomorphic on $\mathbb{D}$. Since $|f(z_0)| = |z_0|$, we have that $|g(z_0)| = 1$, per definition of g. Since $|g(z)| \leq 1$ (by point 1), by maximum modules principle, $g(z) = c$, a constant. Therefore $\frac{f(z)}{z} = c \Rightarrow f(z) = cz$. Now we have to prove that $|c| = 1$. $|f(z)| = |c||z|$, in particular $|f(z_0)| = |c||z_0|$, which implies that $|c| = 1$, as $|f(z_0)| = |z_0|$.

3. By definition
$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} g(z) = g(0).$$
From point 2, we proved that $|g(z)| \leq 1$. Therefore, $|f'(0)| \leq 1$.
If $|f'(0)| = 1$, $|g(0)| = 1$, therefore by 2, f is a rotation.

□

Definition 69. Let $\Omega \subset \mathbb{C}$ be open. A conformal map $f : \Omega \to \Omega$ is called an automorphism of Ω.

$$\operatorname{Aut}\Omega = \{f : f \text{ is an automorphism of } \Omega\}$$

is a group.

di 07 mei 16:02

Les 11: Automorphism and conformal maps

Question: $\operatorname{Aut}\mathbb{D}$?

- Identity
- Rotations: $r_\theta : z \mapsto ze^{i\theta}$
- Let $\alpha \in \mathbb{D}$. $\psi_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha}z}$.

Proof. Note that ψ_α is holomorphic on $\mathbb{D}$, as $|\overline{\alpha}| < 1$ and $|z| < 1$, therefore the singular point of ψ_α is not in $\mathbb{D}$.

Now we prove that $\psi_\alpha(\mathbb{D}) = \mathbb{D}$. Let $|z| = 1$, therefore $z = e^{i\theta}$.

$$\begin{aligned}
\psi_\alpha(z) &= \frac{\alpha - e^{i\theta}}{1 - \overline{\alpha}e^{i\theta}} \\
&= \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \overline{\alpha})} \\
&= e^{i\theta}\frac{\alpha - e^{i\theta}}{-\overline{\alpha - e^{i\theta}}} \\
|\psi_\alpha(z)| &= 1\frac{|\alpha - e^{i\theta}|}{|\alpha - e^{i\theta}|} = 1.
\end{aligned}$$

Since ψ_α is holomorphic on $\mathbb{D}$, by maximum modules principle, it attains it maximum on the boundary. Therefore:

$$|\psi_\alpha(z)| \leq 1 \quad \forall z \in \mathbb{D}.$$

□

Also note that

$$\begin{pmatrix} \alpha & -1 \\ 1 & -\overline{\alpha} \end{pmatrix}^2 \sim I.$$

And therefore $\psi_\alpha = \psi_\alpha^{-1}$

Note that $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$

Blaschke Factors

Theorem 39. Let $f \in \operatorname{Aut}\mathbb{D}$. Then $\exists \theta \in \mathbb{R}$, $\alpha \in \mathbb{D}$ such that

$$f(z) = e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha}z}.$$

Proof. Since $f \in \operatorname{Aut}\mathbb{D}$, there exists an unique α such that $(\alpha) = 0$. Denote by $g = f \circ \psi_\alpha$. g is an automorphism of $\mathbb{D}$ such that

$$g(0) = f(\psi_\alpha(0)) = f(\alpha) = 0.$$

By Schwarz lemma: $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. As $g \in \operatorname{Aut}\mathbb{D}$, g^{-1} is also an automorphism. Moreover $g^{-1}(0) = 0$. Therefore, once again by Schwarz lemma, we have that $|g^{-1}(w)| \leq |w|$. As this happens for all $w \in \mathbb{D}$, we can substitute $g(z)$ for w. Now,

$$|g^{-1}(w)| \leq |w| \Rightarrow |z| \leq |g(z)|.$$

Therefore $|g(z)| = |z| = 1$. As this is true for all points, g is a rotation (By Schwarz lemma again) Therefore $g(z) = e^{i\theta} z$. But $g(z)$ is also $f(\psi_\alpha(z))$. Therefore $g(z) = f(\psi_\alpha(\psi_\alpha(w))$, so

$$f(w) = e^{i\theta}\psi_\alpha(w) \quad \forall w \in \mathbb{D}.$$

□

Corollary 10. Let $f \in \operatorname{Aut}\mathbb{D}$ such that $f(0) = 0$, f is a rotation

Note. If $\alpha, \beta \in \mathbb{D}$, then $\exists f \in \operatorname{Aut}\mathbb{D}$ such that $f(\alpha) = \beta$. Take $f = \psi_\beta \circ \psi_\alpha$

Opmerking (Examen)**.** How to find $\operatorname{Aut}\Omega$ for $\Omega \neq \mathbb{D}$? (Maybe, maybe not on the exam)

- Find a conformal map $f : \Omega \to \mathbb{D}$ (Difficult part)
- If ϕ is an automorphism of Ω, then $f \circ \phi \circ f^{-1}$ is an automorphism of $\mathbb{D}$. Therefore, all automorphisms of Ω can be written as $\phi = f^{-1} \circ g \circ f$.

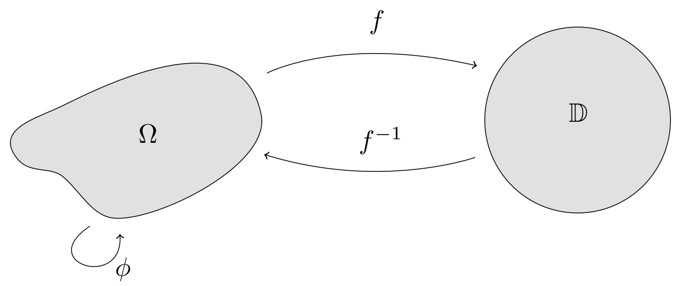


Figure 7.3: how to find automorphisms

Example. $F(z) = \frac{i-z}{i+z}$, $G(z) = i\frac{1-z}{1+z}$. We proved that $F : \mathbb{H} \to \mathbb{D}$ and $G = F^{-1}$. Therefore, an arbitrary automorphisms:

Opmerking. Op het examen moet je het niet volledig uitwerken.

Note. Als het connected en simply connected is en niet volledig $\mathbb{C}$ dan kannet gemapt worden naar $\mathbb{D}$.

Main question: Let $U, V \subset \mathbb{C}$ be open. When $\exists$ a conformal map $f : U \to V$.

Main question 2: Let $U \subset \mathbb{C}$ be open. When $\exists$ a conformal map $f : U \to \mathbb{D}$.

Note. If $f : U \to \mathbb{D}$ and $g : V \to \mathbb{D}$ conformal, then $g \circ f^{-1} : U \to V$ is conformal map.

Let's start solving these questions.

Definition 70 (Normal)**.** Let $\Omega \subset \mathbb{C}$ be open. Let $\mathcal{F}$ be some set of holomorphic functions on Ω. $\mathcal{F}$ is called normal if for every $\{f_n\} \subset \mathcal{F}$, there exists a subsequence which converges uniformly on compact subsets of Ω.

Note. The limit function is not necessarily an element of $\mathcal{F}$.

Definition 71 (Uniformly bounded)**.** $\mathcal{F}$ is called uniformly bounded on compact subsets of Ω if for all compact subset $K \subset \Omega$, $\exists B \geq 0$ such that

$$|f(z)| \leq B \quad \forall z \in K \quad \forall f \in \mathcal{F}.$$

Definition 72 (Equicontinuous)**.** $\mathcal{F}$ is called equicontinuous on a compact set $K \subset \Omega$ if $\forall \varepsilon > 0 : \exists \delta > 0$ such that

$$|z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon \quad \forall f \in \mathcal{F}, \quad \forall z, w \in K?.$$

Example. Let $\mathcal{F} = \{z^n \mid n \in \mathbb{N}\}$ Not equicontinuous. Let $z = 1$, $w \in [0, 1)$.

$$|f_n(z) - f_n(w)| = |1 - w^n| \xrightarrow{n \to \infty} 1,$$

so this cannot be small true for all n.

Definition 73 (Exhaustion by compact sets)**.** Let $\Omega \subset \mathbb{C}$ be open. A family $\{K_n\}$ of compact subsets of Ω, is called an exhaustion if

- $K_n \subset \mathring{K}_{n+1}$
- For all compact $K \subset \Omega$, $\exists n$ such that $K \subset K_n$.

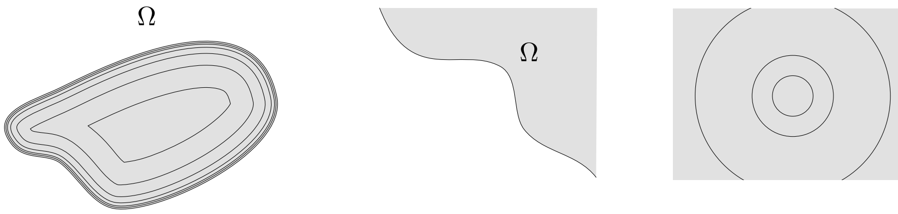
Note. In particular, $\Omega = \bigcup_{i \in 1}^{\infty} K_n$. Take $z \in \Omega$, z is compact, ...

Lemma 2. Let $\Omega \subset \mathbb{C}$ be open. Then $\exists$ an exhaustion of Ω.

Proof. Multiple cases

- Ω is bounded. Take $K_n = \{z \in \Omega \mid d(z, \partial\Omega) \geq \frac{1}{n}\}$
- Ω is unbounded and $\partial\Omega \neq \emptyset$ Take $K_n = \{z \in \Omega \mid d(z, \partial\Omega) \geq \frac{1}{n}\} \cap \overline{B_n(0)}$
- Ω is unbounded and $\partial\Omega = \emptyset$. Take $K_n = \overline{B_n(0)}$.

$\square$



Ω is bounded | Ω is unbounded $\partial\Omega \neq \emptyset$ | $\Omega = \mathbb{C}$

Figure 7.4: Exhaustion by compact sets

Theorem 40 (Montel's theorem)**.** Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{F}$ be a set of holomorphic functions on Ω. If $\mathcal{F}$ is uniformly bounded on compact subsets of Ω, then

- $\mathcal{F}$ is equicontinuous on all compact subsets of Ω.
- $\mathcal{F}$ is normal.

Proof. First part.

Let $K \subset \Omega$ be compact. Since $\mathcal{F}$ is uniformly bounded on compact subsets of Ω, then $\exists B \geq 0$ such that $|f(z)| \leq B$ for all $z \in K$ and all $f \in \mathcal{F}$.

Let $r > 0$ be such that $D_{3r}(z) \subset \Omega$ for all $z \in K$. Just take $r = d(K, \partial\Omega)/6 > 0$. Let $z, w \in K$ such that $|z - w| \leq r$. Let $\gamma = \partial D_{2r}(z)$ Since $|z - w| \leq r$, $z, w \in D_{2r}(z)$. By Chauchy's integral formula, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta \\ f(w) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - w} \, d\zeta. \end{aligned}$$

Then we have

$$\begin{aligned}
|f(z)-f(w)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-w}\,d\zeta\right| \\
&= \left|\frac{1}{2\pi i}\int_\gamma f(\zeta)\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-w}\right)d\zeta\right| \\
&\le \frac{1}{2\pi}\operatorname{length}\gamma\cdot\sup_{\zeta\in\gamma}\left|f(\zeta)\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-w}\right)\right| \\
&\le \frac{1}{2\pi}2\pi(2r)B\sup_{\zeta\in\gamma}\left|\frac{1}{\zeta-z}-\frac{1}{\zeta-w}\right| \\
&= 2Br\sup_{\zeta\in\gamma}\left|\frac{z-w}{(\zeta-z)(\zeta-w)}\right| \\
&\le 2Br|z-w|\sup_{\zeta\in\gamma}\left|\frac{1}{(\zeta-z)(\zeta-w)}\right|.
\end{aligned}$$

Looking at the picture, we get

$$|\zeta-z|=2r \qquad |\zeta-w|\ge r.$$

Using this, we get

$$|f(z)-f(w)|\le 2Br|z-w|\frac{1}{2r^2}=\frac{B}{r}|z-w|.$$

This concludes the proof of equicontinuity. Let $\varepsilon>0$. Choose $\delta=\min(r,\frac{B}{r}\varepsilon)$ If $|z-w|<\delta$, then $|f(z)-f(w)|<\varepsilon$. □

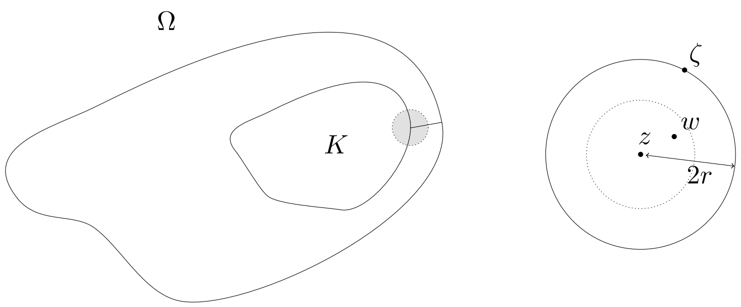


Figure 7.5: montel theorem proof

Les 12: Proof of Montel's theorem

di 14 mei 16:05

Proof. Continuation of the proof of Montel's theorem Let $\{f_n\}$ be a sequence of functions from $\mathcal{F}$, $K\subset\Omega$, compact. We'll prove that there exists a subsequence that converges. Let $\{w_1,w_2,\ldots\}$ be a dense subset of Ω, for example $\Omega\cap(\mathbb{Q}+i\mathbb{Q})$.

There exists a subsequence $\{f_{1,n}\} = f_{1,1}, f_{12}, f_{13}, \dots$ of $\{f_n\}$ such that $f_{1n}(w_1)$ converges (This follows from Real analysis). Since $\mathcal{F}$ is uniformly bounded, there exists subssequence $\{f_{2n}\} = f_{2,1}, f_{2,2}, \dots$ of $\{f_{1,n}\}$ such that $f_{2,n}(w_2)$ converges. Note that $f_{2,n}(w_1)$ also converges. For each $k \geq 1$, we can construct a subsequence $\{f_{k,n}\}$ of $\{f_n\}$ such that $f_{k,n}(w_j)$ converges for $j = 1, \dots, k$.

Now we want a subsequence that converges for every point. Denote by $g_n = f_{n,n}$. $(g_n(w_j))_{n\in\mathbb{N}}$ converges for all j and is a subsequence of $\{f_{1,n}\}$. This converges for all points dense in Ω.

We want to prove that $\{g_n\}$ converges uniformly on K. Now we'll use that $\mathcal{F}$ is equicontinuous. Let $\varepsilon > 0$. Since $\mathcal{F}$ is equicontinuous, $\exists \delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \qquad \forall f \in \mathcal{F}.$$

Take $\varepsilon > 0$ and fix $\delta > 0$ from this definition. It is clear that $K \bigcup_{w\in K} \cup D_\delta(w)$. This is an open cover: $\{D_\delta(w) \mid w \in K\}$. Since K is compact, $\exists$ finite subcover $D_\delta(w_1), D_\delta(w_M)$.

Since $g_n(w_1)$ converges, $\exists N_1 > 0$ such that $\forall n, m > N_1$:

$$|g_n(w_1) - g_m(w_1)| < \varepsilon.$$

Since $g_n(w_2)$ converges, $\exists N_2 > 0$, ... Take the maximum. Then $\forall n > N$:

$$|g_n(w_j) - g_m(w_j)| < \varepsilon, \quad \forall j.$$

Now we prove the theorem.

Let $z \in K$. Then $z \in D_\delta(w_j)$ for some j. Then for all $n, m > N$, we have

$$\begin{aligned} |g_n(z) - g_N(z)| &= |g_n(z) - g_n(w_j) + g_n(w_j) - g_m(w_j) + g_m(w_j) - g_m(z)| \\ &\leq |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| \quad \leq 3\varepsilon. \end{aligned}$$

- The first is less than ε because of the equicontinuitity
- The last term is also less than ε because of equicontinuitity
- The middel term follows from above.

Therefore g_n converges uniformly on K.

Why? Chauchy criteria for uniform convergence.

Done? Nope. We proved that for a sequence f_n of functions from $\mathcal{F}$ and a compact subset K, there exists a subsequence of functions that converges uniformly on K.

We need to find a function that converges on *every* compact subset, and not only on K. Now, we proved last lecture that there always exist an exhaustion by compact subsets. Therefore take a subsubsubsubsubsequence. Let $K_1 \subset K_2 \subset K_3 \subset \dots$ be an exhaustion of Ω. Take $g_n = f_{n,n} \Rightarrow g_n$ converges uniformly on every K_j. Subsequence that converges uniformly on K_1 ... Definition of exhaustion: $K_n \subset K_{n+1}$ and $\forall K \subset \Omega \exists n : K \subset K_n$ $\square$

Theorem 41 (About sequences of injective holomorphic function)**.** Let $\Omega \subset \mathbb{C}$ be open and connected. If $\{f_n\}$ is a sequence of injective functions on Ω such that $(f_n)_n \to f$ uniformly on compact subsets. Then[a] f is either injective or constant.

[a]we already know that f is holomorphic (long time ago)

Proof. By contrary. Suppose f is neither injective, nor constant. Since f is not injective, $\exists z_1, z_2$ such that $f(z_1) = f(z_2)$. Denote by $g_n(z) = f_n(z) - f_n(z_1)$. Therefore, $g_n(z_1) = 0$, and for all $z \in \Omega \setminus \{z_1\}$, $g_n(z) \neq 0$ (since g_n is injective) g_n converges uniformly on compact subsets of Ω to $g(z) = f(z) - f(z_1)$. $g(z_1) = g(z_2) = 0$. Since f is not constant, $g(z)$ is also not constant. Therefore $\exists \varepsilon > 0 : \forall z \in D_\varepsilon(z_2) \setminus \{z_2\} : g(z) \neq 0$ ('z_2 its important') as zeros are isolated. (This is where we use connected (How?)) Denote by $\gamma = \partial D_\varepsilon(z_2)$. Therefore by argument principle,

$$\frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz = 1 - 0 = 1,$$

the number of zeros minus the number of poles. From the other size:

$$\frac{1}{2\pi i} \int_\gamma \frac{g_n'(z)}{g_n(z)} = 0 - 0.$$

Since $f_n(z)$ converges uniformly on compact subsets of Ω to f and γ is a circle (which is compact) we have that

$$0 = \int_\gamma \frac{g_n'(z)}{g_n(z)} dz \to \int_\gamma \frac{g'(z)}{g(z)} dz = 2\pi i.$$

□

Recall the following main question:

- (Which domains are conformally equivalent?)
- Which domains are conformally equivalent to the unit disk?

Necessary condition. Suppose $f : \Omega \to \mathbb{D}$ is a conformal mapping What can we about Ω?

- Ω is not $\mathbb{C}$ (as f's image would be bounded and therefore be constant: Liouville's theorem).
- Ω is connected. (easy)
- Ω is simply connected. (we did it in an exercise session)

These conditions are also sufficient.

Opmerking (Examen)**.** Residue theorem en volgende theorem meest belangrijk.

> **Theorem 42** (Riemann mapping theorem)**.** Let $\Omega \subset \mathbb{C}$ be open, connected, simply connected and $\Omega \neq \mathbb{C}$. Let $z_0 \in \Omega$. Then there exists an unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$ (which implies that $F'(z_0)$ is real)

Proos (Uniqueness). By contrary, suppose there are 2 maps: $F : \Omega \to \mathbb{D}$ and $G : \Omega \to \mathbb{D}$ such that $F(z_0) = G(z_0) = 0$ and $F'(z_0) > 0, G'(z_0) > 0$. Then $G \circ F^{-1} : \mathbb{D} \to \mathbb{D}$ is an autormophism of $\mathbb{D}$. Moreover, $G \circ F^{-1}(0) = G(F^{-1}(0)) = G(z_0) = 0$. Now we can apply the corollary of theorem about automorphisms of the disk. We get that $G \circ F^{-1}$ is a rotation. $\forall z \in \mathbb{D} : G \circ F^{-1}(z) = e^{i\theta} z$. Looking to the derivatives at 0, $(G \circ F^{-1})'(0) =$ exercise, we conclude that

$$e^{i\theta} = 1.$$

Therefore $G \circ F^{-1}(z) = z$, which means that $G = F$.

$\square$

Opmerking. Examen. 6 oefeningen (3 op 2 punten, 3 op 3 punten), niet per se makkelijk naar moeilijk en dan nog 5 punten op de theorie. 3 exercises where you have to think and 3 where you don't have to think: e.g. find integral.

30 min op theorie, in total 4u30. Time is more than enough.

First exercise is difficult.

Les 13: Riemann mapping theorem

di 21 mei 16:04

Continuing on the Riemann mapping theorem.

Note that we can assume $0 \notin \Omega, z_0 = 1$ Indeed, consider the linear map

$$\begin{aligned} f_1 : \Omega &\longrightarrow \Omega_1 \\ z &\longmapsto \frac{z - \alpha}{z_0 - \alpha} \quad \text{where } \alpha \notin \Omega. \end{aligned}$$

f_1 is conformal, $f_1(\alpha) = 0 \notin \Omega_1$ and $f_1(z_0) = 1 \in \Omega_1$.

Proof.

Step 1. Claim: There exists an injective holomorphic $g : \Omega \to \mathbb{D}$, s.t. $g(1) = 0$ and $g'(1) \neq 0$.

(This doesn't prove it, we want it conformal) Since $0 \notin \Omega$ and $1 \in \Omega$, we can define $f_1 = \log_\Omega$ on Ω. Since $e^{f_1(z)} = z$, we have that f_1 is injective. Note that $\forall z \in \Omega : f_1(z) \neq 2\pi i$.

Proof by contrary. Suppose $f_1(z_0) = 2\pi i$. Therefore $z_0 = e^{f_1(z_0)} = e^{2\pi i} = 1$. Therefore $z_0 = 1$, but $f_1(1) = 0$. ↯ $\square$

Moreover, $\exists R > 0$ such that $\forall z \in \Omega : |f_1(z) - 2\pi i| > R$

Proof by contrary. Suppose $\exists z_n \in \Omega$ such that $f_1(z_n) \xrightarrow{n\to\infty} 2\pi i$, which implies that $e^{f_1(z_n)} \to e^{2\pi i}$, so $z_n \xrightarrow{n\to\infty} 1$. Therefore $f_1(z_n) \to 0$. ↯ $\square$

Consider the function $f_2(z) = \frac{R}{f_1(z)-2\pi i}$. Since $f_1(z) \neq 2\pi i$, $f_2(z)$ is holomorphic. It is also injective. Note that

$$|f_2(z)| < \left|\frac{R}{f_1(z) - 2\pi i}\right| < 1.$$

So $f_2 : \Omega \to \mathbb{D}$, injective and holomorphic.

Now, $f_2(1) \neq 0\ldots$ To fix this, consider $g = \psi_{f_2(1)} \circ f_2$, where $\psi = \ldots$. Therefore $g(1) = 0$.

Using direct calculations, we can see that $g'(1) \neq 0$.

Step 2. Claim: $\exists$ (not unique) injective holomorphic function $g : \Omega \to \mathbb{D}$ with $|g'(0)|$ maximum possible (of all functions $f : \Omega \to \mathbb{D}$ with $f(1) = 0$).

Denote by $\mathcal{F} = \{f \mid f : \Omega \to \mathbb{D}, \text{injective, holomorphic}, f(1) = 0\}$ We showed that $\mathcal{F} \neq \emptyset$. Note that $\mathcal{F}$ is uniformly bounded ($|.| < 1$ as $\to \mathbb{D}$) on compact subsets of Ω. By Montel's theorem, this family is normal.

Denote by $s = \sup_{f\in\mathcal{F}} |f'(1)|$. By step 1, $s > 0$. As $1 \in \Omega$, $\exists D_\varepsilon(1) \subset \Omega$. Denote by $\gamma = \partial D_\varepsilon(1)$. By Cauchy's integral formula for derivatives, we have

$$|f'(1)| \leq \frac{2\pi}{\varepsilon} \sup_{z\in\gamma} |f(z)| \quad \forall f \in \mathcal{F}.$$

Since $\mathcal{F}$ is uniformly bounded, $s < \infty$, i.e. $s \in \mathbb{R}_{>0}$. Let f_n be a sequence of functions from $\mathcal{F}$ such that $|f_n'(1)| \xrightarrow{n\to\infty} s$. Since $\mathcal{F}$ is normal, $\exists$ a subsequence f_{n_k} such that f_{n_k} converges to g uniformly on compact subsets of Ω. It's clear that $|g'(1)| = s$.

We need to prove that $g \in \mathcal{F}$.

- $g : \Omega \to \mathbb{D}$ since $\forall f_{n_k}$, $f_{n_k} : \Omega \to \mathbb{D}$
- $g(1) = 0$, since $\forall f_{n_k}(1) = 0$
- Since $|g'(1)| = s > 0$, g is not constant. As we proved earlier, this implies that it is injective!

Therefore $g \in \mathcal{F}$, which means that g is a injective, holomorphic function $\Omega \to \mathbb{D}$ such that $g(1) = 0$.

Step 3. The function $g : \Omega \to \mathbb{D}$ from step 2 is surjective!

By contrary. Suppose that g is not surjective. $\exists \alpha \in \mathbb{D}$ such that $g(z) \neq \alpha$ for all $z \in \Omega$.

Denote by $U = (\psi_\alpha \circ g)(\Omega) \subset \mathbb{D}$.

Since $g(z) \neq \alpha$ for all $z \in \Omega$, we have that $0 \notin U$. Since $0 \neq U$, we can define a holomorphic function h such that $h : U \to \ldots : h(w) = e^{\frac{1}{2}\log w} =$

$\sqrt{w}$ [1] Note that h is injective on U. Denote by

$$F = \psi_{h(\alpha)} \circ h \circ \psi_\alpha \circ g.$$

Note that $F : \Omega \to \mathbb{D}$. Note that F is injective and $F(1) = 0$.

Denote by $\phi : \mathbb{D} \to \mathbb{D}$ defined by $\phi(z) = z^2$.

$$g = \psi_\alpha^{-1} \circ \phi \circ \psi_{h(\alpha)}^{-1} \circ F =: \Phi \circ F.$$

Then $\Phi : \mathbb{D} \to \mathbb{D}$, not injective (because of z^2) Therefore because of Schwarz lemma, $|\Phi'(0)| < 1$. (Because if it is 1, it would be a rotation, it would be injective. But is not.) Finally, not that $g'(1) = (\Phi \circ F)'(1) = \Phi'(F(1))F'(1) = \Phi(0)F'(1)$. Which means that $|g'(1)| \leq |\Phi'(0)||F'(1)| < |F'(1)|$. This contradicts the choice of g.

This means that $\exists F : \Omega \to \mathbb{D}$ conformal map such that $F(1) = 0$. Denote by $\theta = \arg F'(1)$. The map $z \mapsto e^{-i\theta} F(z)$ is a conformal map such that $1 \mapsto 0$ and $F'(1) > 0$. $\square$

Note. F is injective on U, but Φ is not not injective in the whole unit disk. Therefore $\Phi' < 1$.

Where have we used $0 \notin U$? To define $\sqrt{}$.

Opmerking (Examen)**.** Bad news. Correction: 4 hours in total! Easier excersises. (0u30 + 3u30)

Oefeningen gelijkaardig zoals DONDERDAG???

Ook: schrap 2.

Hint: When you think about power series: cauchy-hadamar: boundary!

13: Function must be continuous. (Or find the answer in the book)

15: we: closed simple curve, book: closed curve. (Any of these two will be marked remark)

20: Uniqueness (but write it explicitly, e.g. add it is on a connected set?)

?? :classification three conditions:

- Bounded 0
- Limit ∞
- Other: essential

[1] Maar niet per se $1 \in U$, maar we kunnen wel nog altijd iets log-achtig definieren, zolang maar 0 er niet in zit.

Les 13: Riemann mapping theorem

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di 29 jul 16:00

Les 14: Title of the lecture

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