# The untyped lambda calculus

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## 1 Untyped lambda calculus

This document describes the untyped lambda-calculus, with the following grammar. References are to definitions in Barendregt, "The Lambda Calculus: Its Syntax and Semantics".

$$
\begin{array}{lll}
tm,\ t,\ u \quad ::= & & \text{terms} \\
\mid & x & \text{variables} \\
\mid & \lambda x.t & \text{abstractions} \\
\mid & t\ u & \text{function application}
\end{array}
$$

In $\beta$-reduction, the argument of an application is substituted for the bound variable in an abstraction. We use the $t[x \rightsquigarrow u]$ notation for substituting the term $u$ for the variable $x$ in the term $t$.

$\boxed{t\ \beta\text{-reduces to}\ u}$            *(Definition 3.1.3)*

$$
\text{BETA-REDUCT} \\
\frac{}{(\lambda x.t)\ u\ \beta\text{-reduces to}\ t[x \rightsquigarrow u]}
$$

We can define a deterministic, small-step evaluation relation by reducing in the heads of applications. This is a *call-by-name* semantics and performs head-reduction.

$\boxed{t \rightsquigarrow t'}$            *(Small-step evaluation)*

$$
\text{S-APP} \quad\quad\quad\quad\quad \text{S-BETA} \\
\frac{t \rightsquigarrow t'}{t\ u \rightsquigarrow t'\ t} \quad\quad\quad \frac{}{(\lambda x.t)\ u \rightsquigarrow t[x \rightsquigarrow u]}
$$

We can also define a nondeterministic single-step full-reduction by performing $\beta$-reduction in any subterm. Iterating this reduction will convert a term into its $\beta$-normal form.

$\boxed{t \longrightarrow_\beta u}$                                                 *(Full $\beta$-reduction)*

$$\frac{}{(\lambda x.t)\ u \longrightarrow_\beta t[x \rightsquigarrow u]} \text{ F-\textsc{beta}} \qquad \frac{t \longrightarrow_\beta t'}{\lambda x.t \longrightarrow_\beta \lambda x.t'} \text{ F-\textsc{abs}} \qquad \frac{t \longrightarrow_\beta t'}{t\ u \longrightarrow_\beta t'\ u} \text{ F-\textsc{app}1}$$

$$\frac{u \longrightarrow_\beta u}{t\ u \longrightarrow_\beta t\ u'} \text{ F-\textsc{app}2}$$

We can define when two terms are equivalent up to $\beta$.

$\boxed{t \equiv_\beta u}$                          *($\beta$-conversion or $\beta$-equality (Definition 2.1.4))*

$$\frac{}{(\lambda x.t)\ u \equiv_\beta t[x \rightsquigarrow u]} \text{ EQ-\textsc{beta}} \quad \frac{}{t \equiv_\beta t} \text{ EQ-\textsc{refl}} \quad \frac{u \equiv_\beta t}{t \equiv_\beta u} \text{ EQ-\textsc{sym}} \quad \frac{t_1 \equiv_\beta t_2 \qquad t_2 \equiv_\beta t_3}{t_1 \equiv_\beta t_3} \text{ EQ-\textsc{trans}}$$

$$\frac{t \equiv_\beta t'}{t\ u \equiv_\beta t'\ u} \text{ EQ-\textsc{app}1} \qquad \frac{u \equiv_\beta u'}{t\ u \equiv_\beta t\ u'} \text{ EQ-\textsc{app}2} \qquad \frac{t \equiv_\beta t'}{\lambda x.t \equiv_\beta \lambda x.t'} \text{ EQ-\textsc{abs}}$$

Finally, the Church-Rosser Theorem relies on the definition of parallel reduction. This is a version of reduction that is confluent.

$\boxed{t \Longrightarrow_\beta u}$                                    *(Parallel reduction (3.2.3))*

$$\frac{t \Longrightarrow_\beta \lambda x.t' \qquad u \Longrightarrow_\beta u'}{t\ u \Longrightarrow_\beta t'[x \rightsquigarrow u']} \text{ P-\textsc{beta}} \quad \frac{}{x \Longrightarrow_\beta x} \text{ P-\textsc{var}} \quad \frac{t \Longrightarrow_\beta t'}{\lambda x.t \Longrightarrow_\beta \lambda x.t'} \text{ P-\textsc{abs}} \quad \frac{t \Longrightarrow_\beta t' \qquad u \Longrightarrow_\beta u'}{t\ u \Longrightarrow_\beta t'\ u'} \text{ P-\textsc{app}}$$

# 2   Relation operations

Many of the operations above can be generated by applying the following closure operations to the $\beta$ reduction relation. These operations are parameterized by an arbitrary relation $R$.

Note that $t \to_R u$ with $R$ equal to $\beta$ is the same relation as $t \longrightarrow_\beta u$. And, the compatible, reflexive, symmetic and transitive closure of $\beta$ is the same relation as $t \equiv_\beta u$.

$\boxed{t \to_R u}$                            *(Compatible closure of R (aka one-step reduction (3.1.5))*

$$\frac{R\ t\ u}{t \to_R u} \text{ CC-\textsc{rel}} \qquad \frac{t \to_R u}{\lambda x.t \to_R \lambda x.u} \text{ CC-\textsc{abs}} \qquad \frac{t \to_R t'}{t\ u \to_R t'\ u} \text{ CC-\textsc{app}1} \qquad \frac{u \to_R u'}{t\ u \to_R t\ u'} \text{ CC-\textsc{app}2}$$

$\boxed{t \to_{\overline{\overline{R}}} u}$  *(reflexive closure of R (3.1.4))*

$$\frac{\text{R-REL}}{\quad R\ t\ u \quad}{t \to_{\overline{\overline{R}}} u}$$

$$\frac{\text{R-REFL}}{t \to_{\overline{\overline{R}}} t}$$

$\boxed{t \to_R^+ u}$  *(transitive closure of R (3.1.4))*

$$\frac{\text{T-REL}}{\quad R\ t\ u \quad}{t \to_R^+ u}$$

$$\frac{\text{T-TRANS}}{t_1 \to_R^+ t_2 \qquad t_2 \to_R^+ t_3}{t_1 \to_R^+ t_3}$$

$\boxed{t \to_R^* u}$  *(reflexive-transitive closure of R)*

$$\frac{\text{RT-REL}}{\quad R\ t\ u \quad}{t \to_R^* u}$$

$$\frac{\text{RT-REFL}}{t \to_R^* t}$$

$$\frac{\text{RT-TRANS}}{t_1 \to_R^* t_2 \qquad t_2 \to_R^* t_3}{t_1 \to_R^* t_3}$$

$\boxed{t \leftrightarrow_R u}$  *(symmetric-transitive closure of R)*

$$\frac{\text{ST-REL}}{\quad R\ t\ u \quad}{t \leftrightarrow_R u}$$

$$\frac{\text{ST-SYM}}{\quad u \leftrightarrow_R t \quad}{t \leftrightarrow_R u}$$

$$\frac{\text{ST-TRANS}}{t_1 \leftrightarrow_R t_2 \qquad t_2 \leftrightarrow_R t_3}{t_1 \leftrightarrow_R t_3}$$

# 3 Eta-reduction

By changing the definition of the underlying primitive reduction, we can also reason about $\eta$-reduction and $\beta\eta$-equivalence. Note that the rule for $\eta$-reduction has been stated in a way that generates the appropriate output for the locally-nameless representation in Coq. In this output, the fact that $x \notin \mathsf{fv}\,t$ is implicit and does not need to be added as a precondition to the rule.

$\boxed{t\ \eta\text{-reduces to } u}$  *(Definition 3.1.1 (i))*

$$\frac{\text{ETA-REDUCT}}{\quad t' = t\ x \quad}{\lambda x.t'\ \eta\text{-reduces to } t}$$

$\boxed{t\ \beta\eta\text{-reduces to } u}$  *(Either $\beta$ or $\eta$ reduction (Definition 3.1.1 (ii)))*

$$\frac{\text{ETA-BETA}}{\quad t\ \beta\text{-reduces to } u \quad}{t\ \beta\eta\text{-reduces to } u}$$

$$\frac{\text{ETA-ETA}}{\quad t\ \eta\text{-reduces to } u \quad}{t\ \beta\eta\text{-reduces to } u}$$