Chabauty-Kim

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1 Apply to $\mathbb{G}_m(\mathbb{Z})$

For any prime l consider the diagram

$$\begin{tikzcd}
\mathbb{G}_m(\mathbb{Z}) \arrow[r] \arrow[d, "j"] & \mathbb{G}_m(\mathbb{Z}_l) \arrow[d, "j_l"] \\
H^1(G_{\mathbb{Q}}, \mathbb{Q}_p(1)) \arrow[r, "\mathrm{loc}_l"] & H^1(G_l, \mathbb{Q}_p(1))
\end{tikzcd} \tag{1}$$

Let K be a field of characteristic 0. We have short exact sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \xrightarrow{\times p^n} \mathbb{G}_m \longrightarrow 0$$

which induces long exact sequence of Galois cohomology on $\overline{K}$ points

$$\mu_{p^n} \longrightarrow \overline{K}^\times \xrightarrow{\times p^n} \overline{K}^\times \longrightarrow H^1(G_K, \mu_{p^n}(\overline{K})) \longrightarrow H^1(G_K, \overline{K}^\times)$$

By Hilbert 90, $H^1(G_K, \overline{K}^\times) = 0$. So the above long exact sequence gives us the following short exact sequence

$$0 \longrightarrow K^\times/(K^\times)^{p^n} \longrightarrow H^1(G_K, \mu_{p^n}(\overline{K})) \longrightarrow 0$$

So

$$K^\times/(K^\times)^{p^n} \simeq H^1(G_K, \mu_{p^n}(\overline{K})).$$

By taking the limit $n \to \infty$, we get

$$\varprojlim_n K^\times/(K^\times)^{p^n} \simeq H^1(G_K, \mathbb{Z}_p(1)).$$

Thus,

$$K^\times \otimes \mathbb{Q}_p := \left(\varprojlim_n K^\times/(K^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^1(G_K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^1(G_K, \mathbb{Q}_p(1))$$

where the kummer map $\mathbb{G}_m(K) = K^\times \to H^1(G_K, \mathbb{Q}_p(1))$ is given by the natural map

$$K^\times \to K^\times \otimes \mathbb{Q}_p := \left(\varprojlim_n K^\times/(K^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In particular,

$$\mathbb{Q}^\times \otimes \mathbb{Q}_p := \left(\varprojlim_n \mathbb{Q}^\times/(\mathbb{Q}^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^1(G_{\mathbb{Q}}, \mathbb{Q}_p(1)) \tag{2}$$

$$\mathbb{Q}_l^\times \otimes \mathbb{Q}_p := \left(\varprojlim_n \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^1(G_l, \mathbb{Q}_p(1)) \tag{3}$$

The l-valuation map $v_l : \mathbb{Q}_l^\times \to \mathbb{Z}$ gives us the commutative diagram below with rows exact

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_l^\times \arrow[r] \arrow[d, "\times p^n"] & \mathbb{Q}_l^\times \arrow[r, "v_l"] \arrow[d, "\times p^n"] & \mathbb{Z} \arrow[r] \arrow[d, "\times p^n"] & 0 \\
0 \arrow[r] & \mathbb{Z}_l^\times \arrow[r] & \mathbb{Q}_l^\times \arrow[r, "v_l"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

By snake lemma, we have the following short exact sequence

$$0 \longrightarrow \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n} \longrightarrow \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n} \xrightarrow{v_l} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0$$

By taking the limit $n \to \infty$, we have

$$0 \longrightarrow \varprojlim_n \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n} \longrightarrow \varprojlim_n \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n} \xrightarrow{v_l} \mathbb{Z}_p \longrightarrow 0$$

Note that $v_l : \mathbb{Q}_l^\times \to \mathbb{Z}$ has a section $s_l : \mathbb{Z} \to \mathbb{Q}_l^\times, m \mapsto l^m$ such that $v_l \circ s_l = \mathrm{id}_{\mathbb{Z}}$. So the above short exact sequences split, i.e.,

$$\varprojlim_n \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n} \xrightarrow{\simeq} \left(\varprojlim_n \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n}\right) \oplus \mathbb{Z}_p \tag{4}$$

where the map from $\varprojlim_n \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n} \to \varprojlim_n \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n}$ is induced from a section $s : \mathbb{Q}_l^\times \to \mathbb{Z}_l^\times$ of the inclusion $i : \mathbb{Z}_l^\times \to \mathbb{Q}_l^\times$ with $s \circ i = \mathrm{id}_{\mathbb{Z}_l^\times}$, and the map $\varprojlim_n \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n} \to \mathbb{Z}_p$ is induced from the l-valuation map $v_l : \mathbb{Q}_l^\times \to \mathbb{Z}$.

The reduction modulo l map $\mathrm{mod}_l : \mathbb{Z}_l^\times \to \mathbb{F}_l^\times$ gives us the commutative diagram below with rows exact:

$$\begin{tikzcd}
0 \arrow[r] & 1 + l\mathbb{Z}_l \simeq \mathbb{Z}_l \arrow[r, "\exp"] \arrow[d, "\times p^n"] & \mathbb{Z}_l^\times \arrow[r, "\mathrm{mod}_l"] \arrow[d, "\times p^n"] & \mathbb{F}_l^\times \arrow[r] \arrow[d, "\times p^n"] & 0 \\
0 \arrow[r] & 1 + l\mathbb{Z}_l \simeq \mathbb{Z}_l \arrow[r, "\exp"] & \mathbb{Z}_l^\times \arrow[r, "\mathrm{mod}_l"] & \mathbb{F}_l^\times \arrow[r] & 0
\end{tikzcd}$$

(In the above, the inclusion map $\iota : 1 + l\mathbb{Z}_l \to \mathbb{Z}_l^\times$ is identified with the the exponential map $\exp : \mathbb{Z}_l \to \mathbb{Z}_l^\times$)

By snake lemma, we have the following short exact sequence

$$0 \longrightarrow \mathbb{Z}_l/p^n\mathbb{Z}_l \xrightarrow{\exp} \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n} \xrightarrow{\mathrm{mod}_l} \mathbb{F}_l^\times/(\mathbb{F}_l^\times)^{p^n} \longrightarrow 0$$

By taking the limit $n \to \infty$,

$$0 \longrightarrow \varprojlim_n \mathbb{Z}_l/p^n\mathbb{Z}_l \xrightarrow{\exp} \varprojlim_n \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n} \xrightarrow{\mathrm{mod}_l} \varprojlim_n \mathbb{F}_l^\times/(\mathbb{F}_l^\times)^{p^n} \longrightarrow 0$$

But note that $\varprojlim_n \mathbb{F}_l^\times/(\mathbb{F}_l^\times)^{p^n} = 0$. So we have

$$\varprojlim_n \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n} \underset{\simeq}{\xrightarrow{\log}} \varprojlim_n \mathbb{Z}_l/p^n\mathbb{Z}_l. \tag{5}$$

Now we claim that

Lemma 1.1.

$$H^1(G_l, \mathbb{Q}_p(1)) \simeq \begin{cases} 0 \oplus \mathbb{Q}_p & l \neq p \\ \mathbb{Q}_p \oplus \mathbb{Q}_p & l = p \end{cases}$$

where the map to the first component comes from the logarithm, and the map to the second component comes from the l-valuation.

Proof. When $l \neq p$, we have $\varprojlim_n \mathbb{Z}_l/p^n\mathbb{Z}_l = 0$. By (5), $\varprojlim_n \mathbb{Z}_l^\times/(\mathbb{Z}_l^\times)^{p^n} = 0$. By (4), $\varprojlim_n \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n} \simeq \mathbb{Z}_p$ induced by l-valuation. By (3), we have

$$H^1(G_l, \mathbb{Q}_p(1)) \simeq \left(\varprojlim_n \mathbb{Q}_l^\times/(\mathbb{Q}_l^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathbb{Q}_p.$$

When $l = p$, we have $\varprojlim_n \mathbb{Z}_p/p^n\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$. By (5), $\varprojlim_n \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^{p^n} \xrightarrow[\simeq]{\log} \mathbb{Z}_p$. By (4), $\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n} \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$ where the map to the first $\mathbb{Z}_p$ is via log and the map to the second $\mathbb{Z}_p$ is via l-valuation. By (3), we have

$$H^1(G_p, \mathbb{Q}_p(1)) \simeq \left(\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (\mathbb{Z}_p \oplus \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p.$$

$\square$

Now consider the local Kummer map $j_l : \mathbb{G}_m(\mathbb{Z}_l) \to H^1(G_l, \mathbb{Q}_p(1))$ in our diagram (1).

When $l \neq p$, then the map j_l is identified with $j_l : \mathbb{Z}_l^\times \to \mathbb{Q}_p$ which comes from the l-valuation $v_l : \mathbb{Q}_l^\times \to \mathbb{Z}$. However, $v_l(\mathbb{Z}_l^\times) = 0$, which means that the local Kummer map $j_l : \mathbb{Z}_l^\times \to \mathbb{Q}_p$ in this case is just the zero map. This tells us that the Selmer scheme $\mathrm{Sel}_\infty(\mathbb{G}_m) = 0$.

When $l = p$, then the map j_p is identified with $j_p : \mathbb{Z}_p^\times \to \mathbb{Q}_p \oplus \mathbb{Q}_p$ where the map to the first component $\mathbb{Z}_p^\times \to \mathbb{Q}_p$ comes from the logarithm, and the map to the second component $\mathbb{Z}_p^\times \to \mathbb{Q}_p$ comes from the valuation, which is the zero map by what we discussed previously. So we can view j_p as the logarithm map $\log : \mathbb{Z}_p^\times \to j_p(\mathbb{G}_m(\mathbb{Z}_p)) \subseteq \mathbb{Q}_p$. Note that $j_p(\mathbb{G}_m(\mathbb{Z}_l))$ has infinitely many points in $\mathbb{Q}_p$. So $H^1_f(G_p, \mathbb{Q}_p(1)) := j_p(\mathbb{G}_m(\mathbb{Z}_p))^{\mathrm{Zar}}$, the Zariski closure of $j_p(\mathbb{G}_m(\mathbb{Z}_p))$, is the whole $\mathbb{Q}_p$ (recall that the Zariski topology on $\mathbb{Q}_p$ is the cofinite topology).

In summary, the Chabauty-Kim diagram

$$\begin{tikzcd}
\mathbb{G}_m(\mathbb{Z}) \arrow[r] \arrow[d, "j"] & \mathbb{G}_m(\mathbb{Z}_p) \arrow[d, "j_p"] \\
\mathrm{Sel}_\infty(\mathbb{G}_m) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, \mathbb{Q}_p(1))
\end{tikzcd}$$

can be identified with

$$\begin{tikzcd}
\mathbb{G}_m(\mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{G}_m(\mathbb{Z}_p) \arrow[d, "\log"] \\
0 \arrow[r, "\mathrm{loc}_p"] & \mathbb{Q}_p
\end{tikzcd}$$

Hence,

$$\begin{aligned}
\mathbb{G}_m(\mathbb{Z}_p)_\infty &:= \{x \in \mathbb{G}_m(\mathbb{Z}_p) \mid j_p(x) \in \mathrm{im}(\mathrm{loc}_p)\} \\
&= \left\{x \in \mathbb{Z}_p^\times \mid \log(x) = 0\right\} \\
&= \left\{x \in \mathbb{Z}_p^\times \mid x \in \mu_{p^n}(\mathbb{Q}_p) \text{ for some } n\right\}
\end{aligned}$$

Recall that we have the short exact sequence

$$0 \longrightarrow 1 + p\mathbb{Z}_p \simeq \mathbb{Z}_p \xrightarrow{\exp} \mathbb{Z}_p^\times \xrightarrow{\mathrm{mod}_p} \mathbb{F}_p^\times \longrightarrow 0$$

and it splits for odd p because exp has a section log. So $\mathbb{Z}_p^\times \simeq \mathbb{Z}_p \times \mathbb{F}_p^\times$. Its p^n-torsion points for some n lie in $\mathbb{F}_p^\times$. So

$$\mathbb{G}_m(\mathbb{Z}_p)_\infty \simeq \mathbb{F}_p^\times.$$

By choosing $p = 3$, we find

$$\mathbb{G}_m(\mathbb{Z}) \subseteq \mathbb{G}_m(\mathbb{Z}_3)_\infty = \{1, -1\}.$$