Affine Space

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1 Definition of Affine Spaces

Definition 1.1. Let V be a vector space over a field k, and A be a nonempty set. Let $\beta : A \times A \to V$ and $\eta : V \times A \to A$ be two maps. If the followings conditions hold:

- **(A1)** $\forall a \in A$, $\beta(a, a) = 0$;
- **(A2)** $\forall a \in A$, $\eta(0, a) = a$;
- **(A3)** $\forall a, a' \in A, v \in V$, $\beta(\eta(v, a'), a) = v + \beta(a', a)$;
- **(A4)** $\forall a, a', a'' \in A$, $\eta(\beta(a', a), a'') = \eta(\beta(a'', a), a')$,

then we say that (A, V, β, η) is an **affine space (for V over k)**.

(Note: 0 denotes the zero element of the vector space V.)

Example 1.2. Let V be a vector space. Let $\beta : V \times V \to V$, $(v, v') \mapsto v - v'$, $\eta : V \times V \to V$, $(v, v') \mapsto v + v'$. Then (V, V, β, η) is an affine space, called the **trivial affine space for V**, and often denoted by $(V, V, -, +)$.

Remark 1.3. As Example 1.2 suggests, informally speaking, β is the "subtraction" of vectors, and η is the "addition" of vectors. No one remembers the conditions (A1)(A2)(A3)(A4) as what it exactly states above in Definition 1.1. Instead, what keeps in your mind should look like this:

- **(A1)** $\forall a \in A$, $a - a = 0$;
- **(A2)** $\forall a \in A$, $0 + a = a$;
- **(A3)** $\forall a, a' \in A, v \in V$, $(v + a') - a = v + (a' - a)$;
- **(A4)** $\forall a, a', a'' \in A$, $(a' - a) + a'' = (a'' - a) + a'$.

After seeing this, you may question that "What's the difference between a vector space and an affine space?" After all, these conditions just look similar to the axioms of vector spaces, and Example 1.2 just shows that a vector space can be an affine space. However, Example 1.2 is actually poorly chosen, because it does not reveal the essentials of affine spaces. The only reason that I put Example 1.2 here is that it is perhaps the easiest one. The following Example 1.4 makes much more sense.

Example 1.4. Let V be a vector space over a field k. Denote the vector space $\tilde{V} := V \oplus k$, the canonical projection $\mathrm{pr} : \tilde{V} \to V$ by forgetting the component of k, and the canonical inclusion $i : V \to \tilde{V}$ by adding $0 \in k$. Let A be the subset of $\tilde{V}$ given by $\{v + 1 | v \in V\}$. Let $\beta : A \times A \to V$, $(a, a') \mapsto \mathrm{pr}(a - a')$ and $\eta : V \times A \to A$, $(v, a) \mapsto i(v) + a$. Then (A, V, β, η) is an affine space, called the **model of affine space for V**, and often denoted by $(\mathrm{Mod}_V, V, -, +)$. In particular, if $V = \mathbb{R}^n$, then the model $(\mathrm{Mod}_V, V, -, +)$ is often denoted by $(\mathbb{A}^n, \mathbb{R}^n, -, +)$.

Remark 1.5. In a plain language, $A := \mathbb{A}^n$ is a *subset* of $\mathbb{R}^{n+1}$ that consists of points with the last entry equal to 1. This captures an important feature of the affine space. That is, A is *not* a subspace of $\mathbb{R}^{n+1}$ (i.e., the addition of two points of A lies out of A), whereas, though the subtraction $a - a'$ of two points of A also lies out of A, it lies in a *subspace* V of $\mathbb{R}^{n+1}$ (where V consists of those points in $\mathbb{R}^{n+1}$ whose last entry is zero. It is easy to see that $V \simeq \mathbb{R}^n$). This gives the real meaning of $\beta : A \times A \to V$. And similarly, when we have a subtracted vector $v := a - a' \in V$ that has been already lived in the *subspace* V, we can then add v back with any element a'' in A to get a new element $(a - a') + a'' \in A$. This makes sense since the last entries add: $0+1=1$, which gives the real meaning of $\eta : V \times A \to A$. In other words, the affine space only cares about the parallel structure and ignores the "base point". We will discuss it further later.

Proposition 1.6. Let (A, V, β, η) be an affine space and $a \in A$ be a fixed element. Let $\beta_a = \beta(-, a) : A \to V$, $a' \mapsto \beta(a', a)$ and $\eta_a = \eta(-, a) : V \to A, v \mapsto \eta(v, a)$. Then $\beta_a \circ \eta_a = \mathrm{id}_V$ and $\eta_a \circ \beta_a = \mathrm{id}_A$.

Proof. For one direction, (A3)(A1) gives $\beta_a(\eta_a(v)) = \beta(\eta(v, a), a) = v + \beta(a, a) = v + 0 = v$.

Conversely, (A4)(A2) gives $\eta_a(\beta_a(a')) = \eta(\beta(a', a), a) = \eta(\beta(a, a), a') = \eta(0, a') = a'$. □

Remark 1.7. From the above proposition, we know that $\beta_a : A \to V$ is a bijection between A and V, and hence we can endow A a vector space structure through β_a. However, there is no canonical bijection between A and V, since different β_a give rise to different bijections (and hence different vector space structures), and there is no canonical choice of the fixed element $a \in A$. Unlike the affine space A, the vector space V has a special element $0 \in V$. Therefore, we always say that an affine space is basically a vector space that forgets the zero element.

Digression 1.8. You may notice that in Definition 1.1 there is an odd requirement that A is a nonempty set. Actually, if A is empty, then the conditions (A1)(A2)(A3)(A4) still hold (indeed, hold trivially). But at this time, $A = \emptyset$ is no longer bijective to V (since a vector space V has at least one element $0 \in V$)! Notice that Proposition 1.6 still holds if A is empty, though the statement sounds more like a vacuous truth. That means, it is *true* that $\beta_a : A \to V$ is a bijection *for all* $a \in A$ even when $A = \emptyset$. But since $A = \emptyset$, we cannot pick any $a \in A$, which makes such bijection $\beta_a : A \to V$ fails to *exist*. Due to this fact, it is a reasonable to require that A is nonempty in Definition 1.1 to keep the property of bijection $A \simeq V$.

Proposition 1.9. For a vector space V, a nonempty set A, two maps $\beta : A \times A \to V$ and $\eta : V \times A \to A$, the followings are equivalent:

- (A, V, β, η) is an affine space, i.e., (A1)(A2)(A3)(A4) (say **conditions A**) hold.
- The following conditions (say **conditions E**) hold:
  - **(E1)** $\forall a \in A$, $\eta(0, a) = a$;
  - **(E2)** $\forall a \in A, v, v' \in V$, $\eta(v, \eta(v', a)) = \eta(v + v', a)$;
  - **(E3)** $\forall a, a' \in A$, $\eta(\beta(a', a), a) = a'$;
  - **(E4)** $\forall a \in A$, $\eta_a = \eta(-, a) : V \to A$ is injective.
- The following conditions (say **conditions B**) hold:
  - **(B1)** $\forall a \in A$, $\beta(a, a) = 0$;
  - **(B2)** $\forall a, a', a'' \in A$, $\beta(a, a') + \beta(a', a'') = \beta(a, a'')$;
  - **(B3)** $\forall a \in A, v \in V$, $\beta(\eta(v, a), a) = v$;
  - **(B4)** $\forall a \in A$, $\beta_a = \beta(-, a) : A \to V$ is injective.

Proof. (The following proof is a tedious verification and you can skip it if you wish.)

conditions A $\Rightarrow$ conditions E:

Observe that (E1) is just (A2), and (E4) follows from Proposition 1.6.

For (E2), we have $\beta_a(\eta(v, \eta(v', a))) = \beta(\eta(v, \eta(v', a)), a) = v + \beta(\eta(v', a), a) = v + v' + \beta(a, a) = v + v' = \beta_a(\eta_a(v + v')) = \beta_a(\eta(v + v', a))$. However β_a is a bijection by Proposition 1.6. So we can cancel β_a from both sides to get (E2).

For (E3), $\eta(\beta(a', a), a) = \eta(\beta(a, a), a') = \eta(0, a') = a'$.

conditions E $\Rightarrow$ conditions A:

Observe that (A2) is just (E1).

For (A1), $\eta_a(\beta(a, a)) = \eta(\beta(a, a), a) = a = \eta(0, a) = \eta_a(0)$. Since η_a is injective, we get (A1).

For (A3), $\eta_a(\beta(\eta(v, a'), a)) = \eta(\beta(\eta(v, a'), a), a) = \eta(v, a') = \eta(v, \eta(\beta(a', a), a)) = \eta(v + \beta(a', a), a) = \eta_a(v + \beta(a', a))$. Since η_a is injective, we get (A3).

For (A4), $\eta(\beta(a', a), a'') = \eta(\beta(a', a), \eta(\beta(a'', a), a)) = \eta(\beta(a', a) + \beta(a'', a), a) = \eta(\beta(a'', a) + \beta(a', a), a) = \eta(\beta(a'', a), \eta(\beta(a', a), a)) = \eta(\beta(a'', a), a')$.

conditions A $\Rightarrow$ conditions B:

Observe that (B1) is just (A1), and (B4) follows from Proposition 1.6.

For (B2), $\beta(a,a'') = \beta(\eta_{a'}(\beta_{a'}(a)),a'') = \beta(\eta(\beta_{a'}(a),a'),a'') = \beta_{a'}(a)+\beta(a',a'') = \beta(a,a')+\beta(a',a'')$.

For (B3), $\beta(\eta(v,a),a) = v+\beta(a,a) = v+0 = v$.

conditions B $\Rightarrow$ conditions A:

Observe that (A1) is just (B1).

For (A2), $\beta_a(\eta(0,a)) = \beta(\eta(0,a),a) = 0 = \beta(a,a) = \beta_a(a)$. Since β_a is injective, we get (A2).

For (A3), $\beta(\eta(v,a'),a) = \beta(\eta(v,a'),a')+\beta(a',a) = v+\beta(a',a)$.

For (A4), $\beta_{a'}(\eta(\beta(a',a),a'')) = \beta(\eta(\beta(a',a),a''),a') = \beta(\eta(\beta(a',a),a''),a'')+\beta(a'',a') = \beta(a',a)+\beta(a'',a') = \beta(a'',a) = \beta(\eta(\beta(a'',a),a'),a') = \beta_{a'}(\eta(\beta(a'',a),a'))$. Since $\beta_{a'}$ is injective, we get (A4). □

Corollary 1.10. Let (A,V,β,η) be an affine space. Then we have

- **(B5)** $\forall a,a' \in A, \beta(a,a') = -\beta(a',a)$.

Proof. By (B2) and (B1), $\beta(a,a')+\beta(a',a) = \beta(a,a) = 0$. □

Digression 1.11. There is an obvious duality between the conditions E and B, and the original conditions A in Definition 1.1 is kind of "mixture" of the conditions G and B. Basically, (A1)(A2) talking about the zero case are "inverses" to each other (one is about β and the other one is about η), and (A3)(A4) talking about the general case are also "inverses" to each other (one is approximately about $\beta(\eta(\cdots))$ and the other one is about $\eta(\beta(\cdots))$).

Let's discuss about the last two conditions in E and B now. First notice that (E3) and (E4) actually tell us that η_a is a bijection (observe that (E3) just says that η_a is a surjection where the preimage is given by β). And it's similar for (B3) and (B4). Since η_a (and β_a resp.) is a bijection, the expression of (E3) (and (B3) resp.) uniquely determines β (and η resp.), explicitly by $\beta(a',a) = \eta_a^{-1}(a')$ (and $\eta(v,a) = \beta_a^{-1}(v)$ resp.). Thus, we can recover one of β,η from the other one, which implies that one of β,η is actually *redundant*! That means, we can define an affine space as a *triad* (A,V,η) (or (A,V,β)) rather than a *quartet* (A,V,β,η). For instance, using conditions E, we can redefine the affine space as a pair (A,V,η) that satisfies:

- **(E1)** $\forall a \in A$, $\eta(0,a) = a$;
- **(E2)** $\forall a \in A, v,v' \in V$, $\eta(v,\eta(v',a)) = \eta(v+v',a)$;
- **(E3+E4)** $\forall a \in A$, $\eta_a = \eta(-,a) : V \to A$ is a *bijection*.

And wikipedia just did in this way! The reason that I made the redundant choice (A,β,η) in Definition 1.1 is that it looks more symmetric.

Now let's focus on the first two conditions in E and B. For (E1)(E2), it just says that $\eta : V \times A \to A$ is an abelian group $(V,+)$ action on A.

2 Affine Homomorphisms And Isomorphisms

Lemma 2.1. For two affine spaces (A_1,V_1,β_1,η_1) and (A_2,V_2,β_2,η_2), a map $\varphi : A_1 \to A_2$, and a map $\Lambda : V_1 \to V_2$, the followings are equivalent:

- **(H1)** $\forall a,a' \in A_1$, $\beta_2(\varphi(a'),\varphi(a)) = \Lambda(\beta_1(a',a))$;
- **(H2)** $\forall a \in A_1, v \in V_1$, $\eta_2(\Lambda(v),\varphi(a)) = \varphi(\eta_1(v,a))$.

Proof. **(H1) $\Rightarrow$ (H2):** Suppose that (H1) holds. For any $a \in A_1, v \in V_1$, let $a' = \eta_{1a}(v) = \eta_1(v,a) \in A_1$. Then $v = \beta_{1a}(\eta_{1a}(v)) = \beta_{1a}(a') = \beta_1(a',a)$. We use (H1)(E3) to get

$$\eta_2(\Lambda(v),\varphi(a)) = \eta_2(\Lambda(\beta_1(a',a)),\varphi(a)) = \eta_2(\beta_2(\varphi(a'),\varphi(a)),\varphi(a)) = \varphi(a') = \varphi(\eta_1(v,a)).$$

(H2) $\Rightarrow$ (H1): Conversely, suppose that (H2) holds. For any $a,a' \in A_1$, let $v = \beta_{1a}(a') = \beta_1(a',a) \in V_1$. Then $a' = \eta_{1a}(\beta_{1a}(a')) = \eta_{1a}(v) = \eta_1(v,a)$. We use (H2)(B3) to get

$$\beta_2(\varphi(a'),\varphi(a)) = \beta_2(\varphi(\eta_1(v,a)),\varphi(a)) = \beta_2(\eta_2(\Lambda(v),\varphi(a)),\varphi(a)) = \Lambda(v) = \Lambda(\beta_1(a',a)).$$

□

Lemma 2.2. Let $(A_1, V_1, \beta_1, \eta_1)$, $(A_2, V_2, \beta_2, \eta_2)$ be affine spaces. Let $\varphi : A_1 \to A_2$ be a map and $a \in A_1$ be a fixed point. If a map $\Lambda : V_1 \to V_2$ satisfies (H1), then $(\beta_2)_{\varphi(a)} \circ \varphi = \Lambda \circ \beta_{1a}$.

Proof. For any $a' \in A$, $(\beta_2)_{\varphi(a)}(\varphi(a')) = \beta_2(\varphi(a'), \varphi(a)) = \Lambda(\beta_1(a', a)) = \Lambda(\beta_{1a}(a'))$. □

Lemma 2.3. Let $(A_1, V_1, \beta_1, \eta_1)$, $(A_2, V_2, \beta_2, \eta_2)$ be affine spaces. Let $\varphi : A_1 \to A_2$ be a map. Then there is at most one map $\Lambda : V_1 \to V_2$ that satisfies (H1).

Proof. It follows from Lemma 2.2 and the fact that β_{1a} is a bijection. □

Definition 2.4. Let $(A_1, V_1, \beta_1, \eta_1)$, $(A_2, V_2, \beta_2, \eta_2)$ be affine spaces over a field k. Let $\varphi : A_1 \to A_2$ be a map. If one of (H1)(H2) holds (and hence both hold by Lemma 2.1) for some (and hence unique by Lemma 2.3) *linear* map $\Lambda : V_1 \to V_2$, then we say that φ is an **affine homomorphism** from $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$. We also say the unique linear map Λ induced by φ as the **push-forward** of φ, which is denoted by φ_*.

Corollary 2.5. Let φ be an affine homomorphism from $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$, Then

- $(\beta_2)_{\varphi(a)} \circ \varphi = \varphi_* \circ \beta_{1a}$ for any $a \in A_1$.
- φ is a bijection if and only if φ_* is a linear isomorphism.

Proof. The first statement is basically rewriting Lemma 2.2, and the second statement holds because both $(\beta_2)_{\varphi(a)}$ and β_{1a} are bijections, and φ_* is a linear map. □

Proposition 2.6. If φ_1 and φ_2 are affine homomorphisms from $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$, then $\varphi_1 = \varphi_2$ if and only if $\varphi_{1*} = \varphi_{2*}$ and $\varphi_1(a) = \varphi_2(a)$ for some $a \in A_1$.

Proof. $\Rightarrow$: Obvious.
$\Leftarrow$: By Corollary 2.5, $\varphi_1 = (\beta_2)^{-1}_{\varphi_1(a)} \circ \varphi_{1*} \circ \beta_{1a} = (\beta_2)^{-1}_{\varphi_2(a)} \circ \varphi_{2*} \circ \beta_{1a} = \varphi_2$. □

Proposition 2.7. If φ_1 is an affine homomorphism from $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$ and φ_2 is an affine homomorphism from $(A_2, V_2, \beta_2, \eta_2)$ to (A_3, V_3, f_3, g_3), then $\varphi_2 \circ \varphi_1$ is an affine homomorphism from $(A_1, V_1, \beta_1, \eta_1)$ to (A_3, V_3, f_3, g_3), and $(\varphi_2 \circ \varphi_1)_* = \varphi_{2*} \circ \varphi_{1*}$.

Notation 2.8. Let k be a field. Write $\mathbf{Aff}_k$ as the category whose objects are affine spaces over k and morphisms are affine homomorphisms. Write $\mathbf{Vect}_k$ as the category whose objects are vector spaces over k and morphisms are linear maps.

Proposition 2.9. Let k be a field. Then there is a unique functor F from the category $\mathbf{Aff}_k$ to $\mathbf{Vect}_k$ that sends the object (A, V, β, η) to V, and the morphism ϕ to ϕ_*.

Proof. The key point is that Proposition 2.7 gives us the functoriality of the push-forward. □

Definition 2.10. Let φ be an affine homomorphism from the affine space over k $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$. Then φ is said to be an **affine isomorphism**, if it is an isomorphism in the category $\mathbf{Aff}_k$, i.e., there exists an affine homomorphism $\tilde{\varphi}$ from $(A_2, V_2, \beta_2, \eta_2)$ to $(A_1, V_1, \beta_1, \eta_1)$ such that $\varphi \circ \tilde{\varphi} = \mathrm{id}_{A_2}$ and $\tilde{\varphi} \circ \varphi = \mathrm{id}_{A_1}$.

Proposition 2.11. A map φ is an affine isomorphism if and only if φ is a bijective affine homomorphism. In this case, $(\varphi^{-1})_* = (\varphi_*)^{-1}$.

Proof. $\Rightarrow$: Obvious.
$\Leftarrow$: Let's say φ is a bijective affine homomorphism from $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$. To show that φ is an affine isomorphism, it suffices to show that φ^{-1} is an affine *homomorphism* from $(A_2, V_2, \beta_2, \eta_2)$ to $(A_1, V_1, \beta_1, \eta_1)$ by Definition 2.10. By Corollary 2.5, $\varphi_* : V_1 \to V_2$ is a linear isomorphism, and hence $(\varphi_*)^{-1} : V_2 \to V_1$ is a linear map. Thus, it suffices to show that φ^{-1} and the linear map $(\varphi_*)^{-1}$ satisfy (H1) (but in a reverse direction), i.e., it suffices to show that for any $a, a' \in A_2$, $\beta_1(\varphi^{-1}(a'), \varphi^{-1}(a)) = (\varphi_*)^{-1}(\beta_2(a', a))$. Actually, it can be proven in the following direct computation (by using the fact that φ and the linear map φ_* satisfy (H1)):

$$\begin{aligned} &\beta_1(\varphi^{-1}(a'), \varphi^{-1}(a)) = (\varphi_*)^{-1}(\varphi_*(\beta_1(\varphi^{-1}(a'), \varphi^{-1}(a)))) \\ &= (\varphi_*)^{-1}(\beta_2(\varphi(\varphi^{-1}(a')), \varphi(\varphi^{-1}(a)))) = (\varphi_*)^{-1}(\beta_2(a', a)). \end{aligned}$$

Finally, the above argument gives us $(\varphi^{-1})_* = (\varphi_*)^{-1}$. □

Lemma 2.12. Let (A, V, β, η) be an affine space. Let $a \in A$ be a fixed point. Then $\beta_a : A \to V$ is an affine isomorphism from (A, V, β, η) to $(V, V, -, +)$ with $(\beta_a)_* = \mathrm{id}_V$.

Proof. β_a is a bijection by Proposition 1.6. So by Proposition 2.11, it suffices to show that β_a is an affine homomorphism with $(\beta_a)_* = \mathrm{id}_V$, i.e., $\beta_a : A \to V$ and the linear map $\mathrm{id}_V : V \to V$ satisfy (H1). Actually, for any $a', a'' \in A$, we use (B2)(B5) to deduce

$$\beta_a(a') - \beta_a(a'') = \beta(a', a) - \beta(a'', a) = \beta(a', a) + \beta(a, a'') = \beta(a', a'') = \mathrm{id}_V(\beta(a', a'')),$$

which means that (H1) holds. □

Lemma 2.13. Let V_1, V_2 be vector spaces over a field k, and $\varphi : V_1 \to V_2$ is a linear map. Then

- φ is an affine homomorphism from $(V_1, V_1, -, +)$ to $(V_2, V_2, -, +)$ with $\varphi_* = \varphi$.
- If in addition φ is a linear isomorphism, then φ is an affine isomorphism from $(V_1, V_1, -, +)$ to $(V_2, V_2, -, +)$.

Proof. For the first statement, the *map* φ (which plays the role of φ in (H1)) and the *linear map* φ itself (which plays the role of Λ in (H1)) obviously satisfy (H1), since at this time (H1) is just saying $\varphi(a') - \varphi(a) = \varphi(a' - a)$ for any $a, a' \in V_1$. So φ is an affine homomorphism from $(V_1, V_1, -, +)$ to $(V_2, V_2, -, +)$ with $\varphi_* = \varphi$.

For the second statement, note that now φ is a bijective affine homomorphism. By Proposition 2.11, φ is an affine isomorphism. □

Theorem 2.14. *Let $(A_1, V_1, \beta_1, \eta_1)$ and $(A_2, V_2, \beta_2, \eta_2)$ be affine spaces over a field k. Then*

- *If φ is an affine isomorphism from $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$, then $\varphi_* : V_1 \to V_2$ is a linear isomorphism.*
- *If $\Lambda : V_1 \to V_2$ is a linear map, then for any $a \in A_1$ and $a' \in A_2$, the map $\varphi = \eta_{2a'} \circ \Lambda \circ \beta_{1a}$ is the unique affine homomorphism from $(A_1, V_1, \beta_1, \eta_1)$ to $(A_2, V_2, \beta_2, \eta_2)$ such that $\varphi_* = \Lambda$ and $\varphi(a) = a'$. Furthermore, if Λ is a linear isomorphism, then φ is an affine isomorphism.*

Proof. For the first statement, note that $\varphi : A_1 \to A_2$ is an affine homomorphism and a bijection. Then by Corollary 2.5, φ_* is a linear isomorphism.

For the second statement, we first prove the homomorphism part. Note $\beta_{1a} : A_1 \to V_1$, $\beta_{2a'} : A_2 \to V_2$ are both affine isomorphisms with $(\beta_{1a})_* = \mathrm{id}_{V_1}$, $\beta_{2a'} = \mathrm{id}_{V_2}$ by Lemma 2.12, and $\Lambda : V_1 \to V_2$ is an affine homomorphism with $\Lambda_* = \Lambda$ by Lemma 2.13. Then $\eta_{2a'} = \beta_{2a'}^{-1} : V_2 \to A_2$ is also an affine isomorphism and $(\eta_{2a'})_* = ((\beta_{2a'})_*)^{-1} = \mathrm{id}_{V_2}$ by Proposition 2.11. Thus by Proposition 2.7, φ is an affine homomorphism and $\varphi_* = (\eta_{2a'})_* \circ \Lambda_* \circ (\beta_{1a})_* = \Lambda$. Also $\varphi(a) = \eta_{2a'}(\Lambda(\beta_{1a}(a))) = \eta_{2a'}(\Lambda(0)) = \eta_{2a'}(0) = a'$. Such φ is unique, since if another affine homomorphism $\varphi' : A_1 \to A_2$ satisfies $\varphi'_* = \Lambda$ and $\varphi'(a) = a'$, then $\varphi_* = \varphi'_*$ and $\varphi(a) = \varphi'(a)$, which implies $\varphi = \varphi'$ by Proposition 2.6. We next prove the isomorphism part. When Λ is a linear isomorphism, Λ is also an affine isomorphism by Lemma 2.13, and hence φ is an affine isomorphism as the composite of affine isomorphisms. □

Corollary 2.15. Affine spaces $(A_1, V_1, \beta_1, \eta_1)$ and $(A_2, V_2, \beta_2, \eta_2)$ over a field k are affine isomorphic if and only if V_1 and V_2 are linearly isomorphic.

Proof. It directly follows from Theorem 2.14 (and the fact that A_1 and A_2 are nonempty by Definition 1.1, see Digression 1.8). □

Notation 2.16. In an arbitrary category $\mathbf{C}$, we denote $\mathrm{Hom}_{\mathbf{C}}(A, B)$ as the set of morphisms from the object A to the object B, and $\mathrm{Isom}_{\mathbf{C}}(A, B)$ as that of isomorphisms. We also denote $\mathrm{End}_{\mathbf{C}}(A) := \mathrm{Hom}_{\mathbf{C}}(A, A)$ as the set of **endomorphisms** of the object A, and $\mathrm{Aut}_{\mathbf{C}}(A) := \mathrm{Isom}_{\mathbf{C}}(A, A)$ as the set of **automorphisms** of the object A. It is easy to see that $\mathrm{End}_{\mathbf{C}}(A)$ is indeed a monoid and $\mathrm{Aut}_{\mathbf{C}}(A)$ is indeed a group under the composition of morphisms.

Corollary 2.17. Let $(A_1, V_1, \beta_1, \eta_1)$ and $(A_2, V_2, \beta_2, \eta_2)$ be affine spaces over a field k. Then

- $\mathrm{Hom}_{\mathbf{Aff}_k}(A_1, A_2) = \{\eta_{2a'} \circ \Lambda \circ \beta_{1a} : A_1 \to A_2 \mid a \in A_1, a' \in A_2, \Lambda \in \mathrm{Hom}_{\mathbf{Vect}_k}(V_1, V_2)\}$;
- $\mathrm{Isom}_{\mathbf{Aff}_k}(A_1, A_2) = \{\eta_{2a'} \circ \Lambda \circ \beta_{1a} : A_1 \to A_2 \mid a \in A_1, a' \in A_2, \Lambda \in \mathrm{Isom}_{\mathbf{Vect}_k}(V_1, V_2)\}$.

Definition 2.18. Let (A, V, β, η) be an affine space, and $v \in V$ be a fixed vector. Define a map $T_v = \eta(v, -) : A \to A, a \mapsto \eta(v, a)$, which is called the **translation by** v.

Remark 2.19. For a vector space V and $v \in V$, $T_v : V \to V$ is just a map $v' \mapsto v + v'$ by considering the trivial affine space $(V, V, -, +)$. Note that T_v is not a linear map if $v \neq 0$.

Proposition 2.20. Let (A, V, β, η) be an affine space, and $v \in V$ be a fixed vector. Then the translation $T_v : A \to A$ is an affine automorphism of (A, V, β, η).

Proposition 2.21. Let φ be an affine homomorphism from $(V_1, V_1, -, +)$ to $(V_2, V_2, -, +)$. Then there exists a unique linear map $\Lambda : V_1 \to V_2$ and a unique $v \in V_2$ such that $\varphi = T_v \circ \Lambda$. Furthermore, the unique Λ is exactly φ_*.