Rigid Geometry

Xiang Li

In the following, $\mathcal{V}$ is a valuation ring with fractional field K and residue field k.

1 Rigid Analytic Variety

2 Formal Scheme

3 Rigid Cohomology

Definition 3.1 (**Generic Fibre, Special Fibre**)**.** Let $\mathcal{X}$ be a formal scheme over $\mathcal{V}$. Then

- the generic fibre of $\mathcal{X}$ is $\mathcal{X}_K$ which is a rigid analytic variety over K;
- the special fibre of $\mathcal{X}$ is $\mathcal{X}_k$ which is an algebraic variety over k.

Lemma 3.2. *Any formal scheme P over $\mathcal{V}$ has a special fibre P_k with an embedding $P_k \hookrightarrow P$ which is actually a homeomorphism $P_k \simeq P$.*

Definition 3.3 (**Specialization Map**)**.** Let P_K be the generic fibre of the formal scheme P over $\mathcal{V}$. There is a specialization map $\mathrm{sp} : P_K \to P$ which is basically the reduction $P_K \to P_k$.

Definition 3.4 (**Tube**)**.** Let $\iota : X \hookrightarrow P$ be a locally closed immersion of the algebraic variety X over k into the formal scheme P over $\mathcal{V}$. Then the tube of X in P is the rigid analytic variety $]X[_P := \mathrm{sp}^{-1}(\iota(X))$ (i.e., consists of points in P_K which reduce to points in X).

Lemma 3.5. $]\bigcup_i X_i[_P = \bigcup_i]X_i[_P, \quad]\bigcap_i X_i[_P = \bigcap_i]X_i[_P, \quad]Y \backslash X[_P =]Y[_P \backslash]X[_P.$

4 Colemann Integration

Definition 4.1 (**Rigid Triple/Frame**)**.** The rigid triple/frame is (X, Y, P) where

- $X \hookrightarrow Y$ is an open immersion of the algebraic variety X over k to the algebraic variety Y over k;
- $Y \hookrightarrow P$ is a closed immersion to the formal scheme P over $\mathcal{V}$.

Definition 4.2 (**Strict Neighborhood**)**.** Let V be a rigid analytic variety. An admissible open $U \subseteq V$ is called a strict neighborhood of W if $(U, V \backslash W)$ is an admissible covering of V.

Definition 4.3 (**Dagger Functor**)**.** Let (X, Y, P) be a rigid triple and V be a strict neighborhood of $]X[_P$ in $]Y[_P$. Define the dagger functor $j^\dagger$ from the category of sheaves on V to itself by $j^\dagger(F) = \varinjlim_U j_{U*}(F|_U)$ where the limit is over all U which are strict neighborhood of $]X[_P$ in $]Y[_P$ contained in V and $j_U : U \hookrightarrow V$ is the canonical embedding.

Definition 4.4 (**Isocrystal**)**.** An isocrystal on the rigid triple (X, Y, P) is a $j^\dagger\mathcal{O}_{]Y[_P}$-module together with an integrable connection.

Definition 4.5 (**Overconvergent Isocrystal**)**.** An isocrystal is called overconvergent if the Taylor expansion map gives an isomorphism on a strict neighborhood of the diagonal.

Remark 4.6. The category of overconvergent isocrystals on (X, Y, P) is independent from Y and P up to equivalence, and thus is called the category of overconvergent isocrystals on X.

Definition 4.7 (F-isocrystal over K). An (overconvergent) F-isocrystal over K is an (overconvergent) isocrystal on $(\mathrm{Spec}(k), \mathrm{Spec}(k), \mathrm{Spf}(\mathcal{V}))$. Essentially:

- An F-isocrystal over K is a vector space over K together with a K-linear automorphism;
- An overconvergent F-isocrystal over K is just a vector space over K.

Definition 4.8 (Abstract Coleman Functions). Let (X, Y, P) be a rigid triple, and $\mathcal{F}$ be a locally free $j^\dagger \mathcal{O}_{]Y[_P}$-module. An abstract Coleman function is a triple (M, s, y) where

- $M = (M, \nabla)$ is a unipotent isocrystal on (X, Y, P);
- $s \in \mathrm{Hom}(M, \mathcal{F})$;
- y is a collection of flat sections $\{y_x \in M(]x[_P]) \mid x \in X\}$.

Definition 4.9. 1