Root System

Associated Lie Algebras

Correspondence: Reduced root systems correspond to finite dimensional semisimple Lie algebras; Irreducible reduced root systems correspond to finite dimensional simple Lie algebras.

A_n corresponds to $\mathfrak{sl}_{n+1}$.

B_n corresponds to $\mathfrak{so}_{2n+1}$.

C_n corresponds to $\mathfrak{sp}_{2n}$.

D_n corresponds to $\mathfrak{so}_{2n}$.

E_6, E_7, E_8, F_4, G_2 correspond to exceptional simple Lie algebras.

Remark:

$\mathfrak{so}_n = \left\{A \in \mathfrak{gl}_n | JA + A^T J = 0\right\}$ where J 的反对角线上都是 1 其余都是 0.

$\mathfrak{so}_n$的特征：关于反对角线对称的两个entries之和为0. 反对角线上的每个entry都等于0.

$\mathfrak{sp}_{2n} = \left\{A \in \mathfrak{gl}_{2n} | JA + A^T J = 0\right\}$ where J 的反对角线的右上部分都是 -1，左下部分都是 1，其余都是 0.

$\mathfrak{sp}_{2n}$的特征：对于 $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}_{2n}$，在 A 和 D 中，关于反对角线对称的两个entires之和为0；在 B 和 C 中，关于反对角线对称的两个entires相等。此外反对角线上的entries可任意取值。

Classification of Irreducible Reduced Root Systems

Irreducible reduced simply laced (ADE classification): A_n, D_n $(n \geq 4)$, E_6, E_7, E_8.

Irreducible reduced not simply laced: $B_2 \simeq C_2$, B_n $(n \geq 3)$, C_n $(n \geq 3)$, F_4, G_2.

Remark: Some isomorphisms in low dimension:

- $A_1 \simeq B_1 \simeq C_1$, $\mathfrak{sl}_2 \simeq \mathfrak{so}_3 \simeq \mathfrak{sp}_2$.
- $B_2 \simeq C_2$, $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$.
- $D_2 \simeq A_1 \times A_1$, $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.
- $D_3 \simeq A_3$, $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$.

Roots

$R_{A_n} = \{e_i - e_j | i \neq j\}$ $(1 \leq i, j \leq n+1)$.

$R_{B_n} = \{\pm e_i\} \cup \{\pm e_i \pm e_j | i \neq j\}$ $(1 \leq i, j \leq n)$.

$R_{C_n} = \{\pm 2e_i\} \cup \{\pm e_i \pm e_j | i \neq j\}$ $(1 \leq i, j \leq n)$.

$R_{D_n} = \{\pm e_i \pm e_j | i \neq j\}$ $(1 \leq i, j \leq n)$.

$R_{E_6} = \{e_i - e_j | 1 \leq i \neq j \leq 6\} \cup \{\pm(e_7 - e_8)\} \cup \left\{\frac{1}{2}\sum_{i=1}^{8} k_i e_i | k_i = \pm 1, \sum_{i=1}^{8} k_i = 0, k_7 + k_8 = 0\right\}$.

$R_{E_7} = \{e_i - e_j | i \neq j\} \cup \left\{\frac{1}{2}(\pm e_1 \pm \cdots \pm e_8) | \text{four - signs}\right\}$ $(1 \leq i, j \leq 8)$.

$R_{E_8} = \{\pm e_i \pm e_j | i \neq j\} \cup \left\{\frac{1}{2}(\pm e_1 \pm \cdots \pm e_8) | \text{even number of - signs}\right\}$ $(1 \leq i, j \leq 8)$.

$R_{F_4} = \{\pm e_i\} \cup \{\pm e_i \pm e_j | i \neq j\} \cup \left\{\frac{1}{2}(\pm e_1 \pm \cdots \pm e_4)\right\}$ $(1 \leq i, j \leq 4)$.

$R_{G_2} = \{e_i - e_j | i \neq j\} \cup \{\pm(2e_i - e_j - e_k) | i, j, k \text{ distinct}\}$ $(1 \leq i, j, k \leq 3)$.

Remark: $R^\vee_{C_n} = R_{B_n}$.

Positive Roots

Definition: $\alpha \in R$ is positive if $f(\alpha) > 0$ by choosing a linear $f : V \to \mathbb{R}$ with $0 \notin f(R)$.

$R^+_{A_n} = \{e_i - e_j | i < j\}$ $(1 \le i, j \le n+1)$ by choosing $f(e_1) = n+1$, $f(e_2) = n$, $\cdots$, $f(e_{n+1}) = 1$.

$R^+_{B_n} = \{e_i\} \cup \{e_i \pm e_j | i < j\}$ $(1 \le i, j \le n)$ by choosing $f(e_1) = n$, $\cdots$, $f(e_n) = 1$.

$R^+_{C_n} = \{2e_i\} \cup \{e_i \pm e_j | i < j\}$ $(1 \le i, j \le n)$ by choosing $f(e_1) = n$, $\cdots$, $f(e_n) = 1$.

$R^+_{D_n} = \{e_i \pm e_j | i < j\}$ $(1 \le i, j \le n)$ by choosing $f(e_1) = n$, $\cdots$, $f(e_n) = 1$.

$R^+_{E_6} = \{e_i - e_j | 1 \le i < j \le 6\} \cup \{e_7 - e_8\} \cup \left\{\frac{1}{2}\sum_{i=1}^8 k_i e_i | k_1 = 1, k_i = \pm 1, \sum_{i=1}^8 k_i = 0, k_7 + k_8 = 0\right\}$ by choosing $f(e_1) = 28$, $f(e_i) = 9 - i$ for $2 \le i \le 8$.

$R^+_{E_7} = \{e_i - e_j | i < j\} \cup \left\{\frac{1}{2}(e_1 \pm \cdots \pm e_8) | \text{four - signs}\right\}$ $(1 \le i, j \le 8)$ by choosing $f(e_1) = 28$, $f(e_i) = 9 - i$ for $2 \le i \le 8$.

$R^+_{E_8} = \{e_i \pm e_j | i < j\} \cup \left\{\frac{1}{2}(e_1 \pm \cdots \pm e_8) | \text{even number of - signs}\right\}$ $(1 \le i, j \le 8)$ by choosing $f(e_1) = 28$, $f(e_i) = 9 - i$ for $2 \le i \le 8$.

$R^+_{F_4} = \{e_i\} \cup \{e_i \pm e_j | i < j\} \cup \left\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\}$ $(1 \le i, j \le 4)$ by choosing $f(e_1) = 7$, $f(e_i) = 5 - i$ for $2 \le i \le 4$.

$R^+_{G_2} = \{e_i - e_j | i < j\} \cup \{2e_1 - e_2 - e_3, e_1 - 2e_2 + e_3, e_1 + e_2 - 2e_3\}$ $(1 \le i, j \le 3)$ by choosing $f(e_1) = 8$, $f(e_2) = 3$, $f(e_3) = 1$.

Simple Roots

Definition: $\alpha \in R^+$ is simple if it is not the sum of two positive roots.

For the followings, we write $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ to label the order.

$\Pi_{A_n} = \{e_1 - e_2, \cdots, e_{n-1} - e_n, e_n - e_{n+1}\}$.

$\Pi_{B_n} = \{e_1 - e_2, \cdots, e_{n-1} - e_n, e_n\}$.

$\Pi_{C_n} = \{e_1 - e_2, \cdots, e_{n-1} - e_n, 2e_n\}$.

$\Pi_{D_n} = \{e_1 - e_2, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$.

$\Pi_{E_6} = \left\{e_2 - e_3, e_3 - e_4, e_4 - e_5, \frac{1}{2}(e_1 + e_5 + e_6 + e_8 - e_2 - e_3 - e_4 - e_7), e_7 - e_8, e_5 - e_6\right\}$.

$\Pi_{E_7} = \left\{e_2 - e_3, \cdots, e_7 - e_8, \frac{1}{2}(e_1 + e_6 + e_7 + e_8 - e_2 - \cdots - e_5)\right\}$.

$\Pi_{E_8} = \left\{e_2 - e_3, \cdots, e_7 - e_8, \frac{1}{2}\left(e_1 + e_8 - e_2 - \cdots - e_7\right), e_7 + e_8\right\}$.

$\Pi_{F_4} = \left\{e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\right\}$.

$\Pi_{G_2} = \{e_2 - e_3, e_1 - 2e_2 + e_3\}$.

Rank, Cardinality And Dimension

Formulae: Let R be a reduced root system, and $\mathfrak{g}$ be the corresponding finite dimensional semisimple Lie algebra. We have $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ and

- $\dim \mathfrak{t} = \operatorname{rank}(R) = \#\Pi$.
- $\dim \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha = \#R$.
- Thus, $\dim \mathfrak{g} = \operatorname{rank}(R) + \#R$.

The following lists the triple $(\operatorname{rank}(R), \#R, \dim \mathfrak{g})$ for each reduced root system R.

$A_n : (n, n^2 + n, n^2 + 2n)$

$B_n : (n, 2n^2, 2n^2 + n)$

$C_n : (n, 2n^2, 2n^2 + n)$

$D_n : (n, 2n^2 - 2n, 2n^2 - n)$

$E_6 : (6, 72, 78)$

$E_7 : (7, 126, 133)$

$E_8 : (8, 240, 248)$

$F_4 : (4, 48, 52)$

$G_2 : (2, 12, 14)$

Cartan Matrix

Correspondence: Cartan matrices correspond to finite dimensional semisimple Lie algebras (existence and uniqueness theorem).

Definition: For $\Pi = \{\alpha_1, \cdots, \alpha_n\}$, $A = (a_{ij})_{1 \le i,j \le n}$ where $a_{ij} = (\alpha_i^\vee, \alpha_j)$.

$$A_n\text{:} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \in \mathbb{Z}^{n \times n}$$

$$B_n\text{:} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix} \in \mathbb{Z}^{n \times n}$$

$$C_n\text{:} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix} \in \mathbb{Z}^{n \times n}$$

$$D_n\text{:} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix} \in \mathbb{Z}^{n \times n}$$

$$E_6\text{:} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & -1 \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & \\ & & -1 & & & 2 \end{pmatrix}$$

$$E_7\text{:} \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & -1 \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ & & & -1 & & & 2 \end{pmatrix}$$

$$E_8\text{:} \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

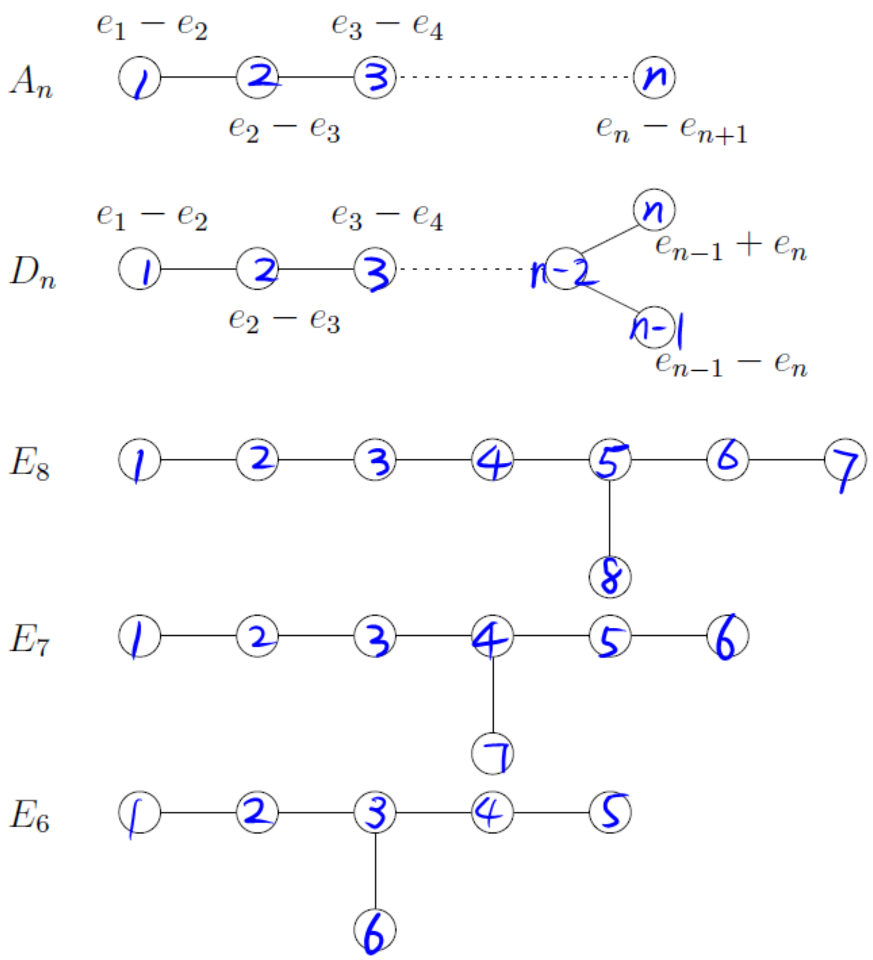
F_4: $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -2 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$

G_2: $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Dynkin Diagram

Correspondence: Reduced root systems correspond to Dynkin diagrams; Irreducible reduced root systems correspond to connected Dynkin diagrams.

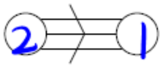
Definition: Let $A = (a_{ij})$ be Cartan matrix. Vertices correspond to simple roots. Between vertices α_i and α_j, there are $a_{ij}a_{ji}$ lines. If there are more than 1 lines, we put an arrow towards the short root.



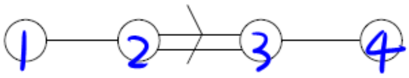
B_n

C_n

G_2



F_4



Weight Lattice / Root Lattice P/Q

Definition: $Q = \mathbb{Z}R = \bigoplus_{\alpha_i \in \Pi} \mathbb{Z}\alpha_i$.
$P = \{\gamma \in \mathbb{Q}R | (\gamma, \alpha^\vee) \in \mathbb{Z}, \forall \alpha \in R\}$.

How to Compute: The Smith normal form of the Cartan matrix A gives the f.g. abelian group structure of P/Q. In particular, $\#(P/Q) = \det A$.

$A_n : P/Q \simeq \mathbb{Z}/(n+1)\mathbb{Z}$.

$B_n : P/Q \simeq \mathbb{Z}/2\mathbb{Z}$.

$C_n : P/Q \simeq \mathbb{Z}/2\mathbb{Z}$.

$D_n : P/Q \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & n \text{ odd} \end{cases}$.

$E_6 : P/Q \simeq \mathbb{Z}/3\mathbb{Z}$.

$E_7 : P/Q \simeq \mathbb{Z}/2\mathbb{Z}$.

$E_8 : P/Q$ is trivial.

$F_4 : P/Q$ is trivial.

$G_2 : P/Q$ is trivial.

Fundamental Weights

Definition: dual basis to simple roots, $(\omega_i, \alpha_j^\vee) = \delta_{ij}$.

How to compute: Let A be the Cartan matrix and S_e be the matrix of simple roots w.r.t. the standard basis $\{e_i\}$ (coefficients in column). Then the matrix of fundamental weights w.r.t. simple roots is $W_s = A^{-1}$, and hence the matrix of fundamental weights w.r.t. standard basis $\{e_i\}$ is $W_e = S_e W_s = S_e A^{-1}$.

Now we state the result.

- A_n:
  $\omega_1 = \frac{n}{n+1} e_1 - \frac{1}{n+1}(e_2 + \cdots + e_{n+1})$,
  $\omega_2 = \frac{n-1}{n+1}(e_1 + e_2) - \frac{2}{n+1}(e_3 + \cdots + e_{n+1})$,
  $\cdots$,
  $\omega_n = \frac{1}{n+1}(e_1 + \cdots + e_n) - \frac{n}{n+1} e_{n+1}$.
- B_n:
  $\omega_1 = e_1$,
  $\omega_2 = e_1 + e_2$,

$\cdots,$
$\omega_{n-1} = e_1 + \cdots + e_{n-1},$
$\omega_n = \frac{1}{2}(e_1 + \cdots + e_n).$

- C_n:
  $\omega_1 = e_1,$
  $\omega_2 = e_1 + e_2,$
  $\cdots,$
  $\omega_n = e_1 + \cdots + e_n.$
- D_n:
  $\omega_1 = e_1,$
  $\omega_2 = e_1 + e_2,$
  $\cdots,$
  $\omega_{n-2} = e_1 + \cdots + e_{n-2},$
  $\omega_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1}) - \frac{1}{2}e_n,$
  $\omega_n = \frac{1}{2}(e_1 + \cdots + e_{n-1} + e_n).$
- E_6:
  $\omega_1 = \frac{2}{3}(e_1 + e_2) - \frac{1}{3}(e_3 + \cdots + e_6),$
  $\omega_2 = \frac{4}{3}e_1 + \frac{1}{3}(e_2 + e_3) - \frac{2}{3}(e_4 + e_5 + e_6),$
  $\omega_3 = 2e_1 - e_5 - e_6,$
  $\omega_4 = \frac{5}{3}e_1 - \frac{1}{3}(e_2 + \cdots + e_6),$
  $\omega_5 = \frac{5}{6}e_1 - \frac{1}{6}(e_2 + \cdots + e_6) + \frac{1}{2}(e_7 - e_8),$
  $\omega_6 = e_1 - e_6.$
- E_7:
  $\omega_1 = \frac{3}{4}(e_1 + e_2) - \frac{1}{4}(e_3 + \cdots + e_8),$
  $\omega_2 = \frac{3}{2}e_1 + \frac{1}{2}(e_2 + e_3) - \frac{1}{2}(e_4 + \cdots + e_8),$
  $\omega_3 = \frac{9}{4}e_1 + \frac{1}{4}(e_2 + e_3 + e_4) - \frac{3}{4}(e_5 + \cdots + e_8),$
  $\omega_4 = 3e_1 - e_6 - e_7 - e_8,$
  $\omega_5 = 2e_1 - e_7 - e_8,$
  $\omega_6 = e_1 - e_8,$
  $\omega_7 = \frac{7}{4}e_1 - \frac{1}{4}(e_2 + \cdots + e_8).$
- E_8:
  $\omega_1 = e_1 + e_2,$
  $\omega_2 = 2e_1 + e_2 + e_3,$
  $\omega_3 = 3e_1 + e_2 + e_3 + e_4,$
  $\omega_4 = 4e_1 + e_2 + e_3 + e_4 + e_5,$
  $\omega_5 = 5e_1 + e_2 + e_3 + e_4 + e_5 + e_6,$
  $\omega_6 = \frac{7}{2}e_1 + \frac{1}{2}(e_2 + \cdots + e_7) - \frac{1}{2}e_8,$
  $\omega_7 = 2e_7,$
  $\omega_8 = \frac{5}{2}e_1 + \frac{1}{2}(e_2 + \cdots + e_7 + e_8).$
- F_4:
  $\omega_1 = e_1 + e_2,$
  $\omega_2 = 2e_1 + e_2 + e_3,$
  $\omega_3 = \frac{3}{2}e_1 + \frac{1}{2}(e_2 + e_3 + e_4),$
  $\omega_4 = e_1.$
- G_2:
  $\omega_1 = 2\alpha_1 + \alpha_2,$
  $\omega_2 = 3\alpha_1 + 2\alpha_2.$

Remark: For A_n, if we regard $e_i \in \mathfrak{t}^* \subseteq (\mathfrak{sl}_{n+1})^*$, then we have a relation $\sum_{i=1}^{n+1} e_i = 0$. Under this relation,
$\omega_1 = e_1,$
$\omega_2 = e_1 + e_2,$
$\cdots,$
$\omega_n = e_1 + \cdots + e_n.$
However, this version does not preserve the inner product. For example, in A_1, we should have $(\omega_1, \omega_1) = (\frac{1}{2}(e_1 - e_2), \frac{1}{2}(e_1 - e_2)) = \frac{1}{2}$. But if we write $\omega_1 = e_1$, we find $(\omega_1, \omega_1) = (e_1, e_1) \neq 1$.

Rho ρ

Definition: $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, the half of the sum of all positive roots.

How to Compute: $\rho = \sum \omega_i$, the sum of all fundamental weights.

$\rho_{A_n} = \sum_{j=1}^{n+1} (\frac{n}{2} - j + 1) e_j$.

$\rho_{B_n} = \sum_{j=1}^{n} (n - j + \frac{1}{2}) e_j$.

$\rho_{C_n} = \sum_{j=1}^{n} (n - j + 1) e_j$.

$\rho_{D_n} = \sum_{j=1}^{n} (n - j) e_j$.

$\rho_{E_6} = \frac{15}{2} e_1 + \left(\sum_{j=2}^{6} (\frac{5}{2} - j) e_j \right) + \frac{1}{2}(e_7 - e_8)$.

$\rho_{E_7} = \frac{49}{4} e_1 + \sum_{j=2}^{8} (\frac{13}{4} - j) e_j$.

$\rho_{E_8} = 23 e_1 + \sum_{j=2}^{8} (8 - j) e_j$.

$\rho_{F_4} = \frac{11}{2} e_1 + \frac{5}{2} e_2 + \frac{3}{2} e_3 + \frac{1}{2} e_4$.

$\rho_{G_2} = 5\alpha_1 + 3\alpha_2$.

Highest Roots

Definition: The highest root $\theta \in R^+$ is the unique positive root such that $\forall \alpha_i \in \Pi, \theta + \alpha_i \notin R^+$.

$\theta_{A_n} = e_1 - e_{n+1} = \alpha_1 + \cdots + \alpha_n = \omega_1 + \omega_n$.

$\theta_{B_n} = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n = \omega_2$.

$\theta_{C_n} = 2e_1 = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n = 2\omega_1$.

$\theta_{D_n} = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n = \omega_2$.

$\theta_{E_6} = e_1 - e_6 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = \omega_6$.

$\theta_{E_7} = e_1 - e_8 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 = \omega_6$.

$\theta_{E_8} = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = \omega_1$.

$\theta_{F_4} = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \omega_1$.

$\theta_{G_2} = 2e_1 - e_2 - e_3 = 3\alpha_1 + 2\alpha_2 = \omega_2$.

Extended Cartan matrix

Definition: Extend the Cartan matrix by setting $\alpha_0 = -\theta$. Then $\tilde{A} = (a_{ij})_{0 \le i,j \le n}$ where $a_{ij} = (\alpha_i^\vee, \alpha_j)$.

$$A_n\text{:} \begin{pmatrix} \color{red}{2} & \color{red}{-1} & \color{red}{0} & \color{red}{\cdots} & \color{red}{0} & \color{red}{-1} \\ \color{red}{-1} & 2 & -1 & & & \\ \color{red}{0} & -1 & 2 & -1 & & \\ \color{red}{\vdots} & & \ddots & \ddots & \ddots & \\ \color{red}{0} & & & -1 & 2 & -1 \\ \color{red}{-1} & & & & -1 & 2 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times(n+1)}$$

$$B_n\text{:} \begin{pmatrix} \color{red}{2} & \color{red}{0} & \color{red}{-1} & \color{red}{0} & \color{red}{\cdots} & \color{red}{0} \\ \color{red}{0} & 2 & -1 & & & \\ \color{red}{-1} & -1 & 2 & -1 & & \\ \color{red}{0} & & \ddots & \ddots & \ddots & \\ \color{red}{\vdots} & & & -1 & 2 & -1 \\ \color{red}{0} & & & & -2 & 2 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times(n+1)}$$

$$C_n\colon \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -2 & 2 & -1 & & & & \\ 0 & -1 & 2 & -1 & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ \vdots & & & -1 & 2 & -1 & \\ \vdots & & & & -1 & 2 & -2 \\ 0 & & & & & -1 & 2 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times(n+1)}$$

$$D_n\colon \begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & -1 & & & & \\ -1 & -1 & 2 & -1 & & & \\ 0 & & \ddots & \ddots & \ddots & & \\ \vdots & & & -1 & 2 & -1 & -1 \\ \vdots & & & & -1 & 2 & \\ 0 & & & & -1 & & 2 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times(n+1)}$$

$$E_6\colon \begin{pmatrix} 2 & & & & & & -1 \\ & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & -1 \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ -1 & & & -1 & & & 2 \end{pmatrix}$$

$$E_7\colon \begin{pmatrix} 2 & & & & & & -1 & \\ & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ -1 & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

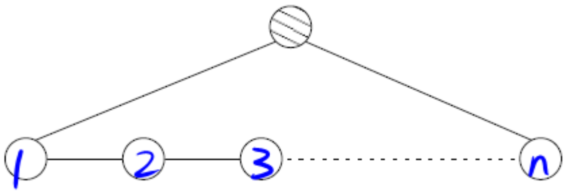
$$E_8\colon \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & -1 & & -1 \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & \\ & & & & & -1 & & & 2 \end{pmatrix}$$

$$F_4\colon \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -2 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

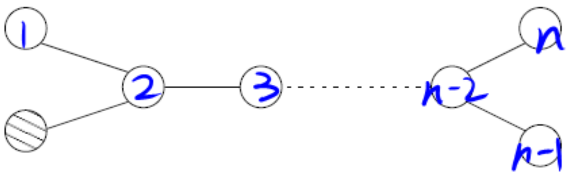
$$G_2\colon \begin{pmatrix} 2 & & -1 \\ & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}$$

Extended Dynkin Diagram

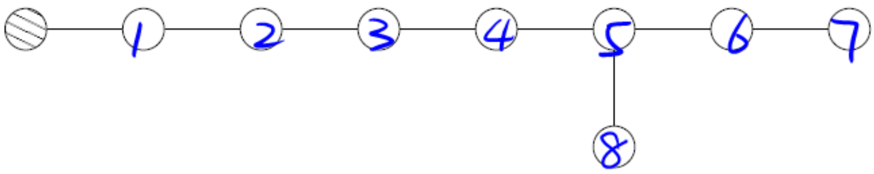
$\tilde{A}_n$



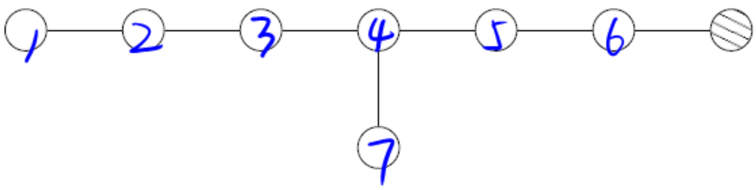
$\tilde{D}_n$



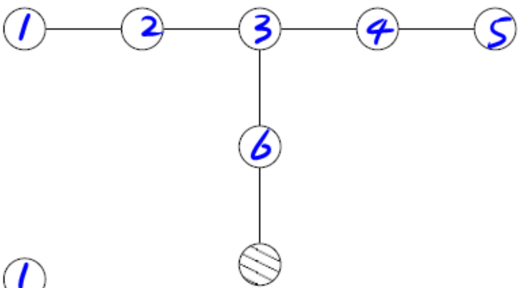
$\tilde{E}_8$



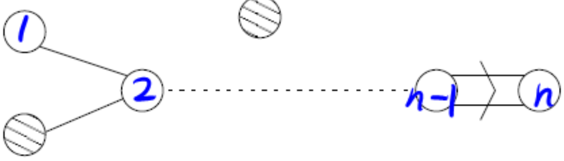
$\tilde{E}_7$



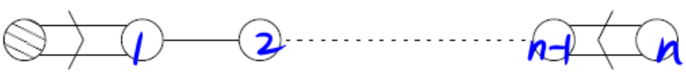
$\tilde{E}_6$



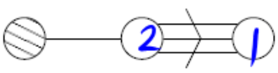
$\tilde{B}_n$



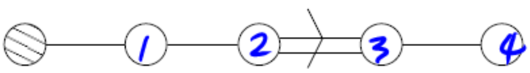
$\tilde{C}_n$



$\tilde{G}_2$



$\tilde{F}_4$



Weyl Group

Definition: the subgroup of $GL(\mathbb{R}R)$ generated by $\{s_\alpha|\alpha \in R\}$ where $s_\alpha(v) = v - (v, \alpha^\vee)\alpha$.

$W_{A_n} \simeq S_{n+1}$.

$W_{B_n} \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.

$W_{C_n} \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.

$W_{D_n} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$.

W_{E_6} is a group of order $2^7 \cdot 3^4 \cdot 5$.

W_{E_7} is a group of order $2^{10} \cdot 3^4 \cdot 5 \cdot 7$.

W_{E_8} is a group of order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

W_{F_4} is a group of order $2^7 \cdot 3^2$.

$W_{G_2} \simeq D_6$ of order 12.