# Hodge

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## 1 Pure Hodge Structure

### 1.1 General Theory

**Definition 1.1.** A pure Hodge structure of weight $n$ is a $\mathbb{Z}$-module $H_{\mathbb{Z}}$ and a decomposition of $H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ into $H^{p,q}$ where $p+q=n$, i.e.,

$$H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

$W_n =$ everything
$W_{n-1} = 0$

such that $H^{q,p} = \overline{H^{p,q}}$.

**Remark 1.2.** The Hodge decompostion $H^{p,q}$ corresponds to Hodge Filtration $F^pH$ in the following ways.

$$F^pH = \bigoplus_{i \geq p} H^{i,n-i},$$
$$H^{p,q} = F^pH \cap \overline{F^qH}.$$

The Hodge Filtration satisfies the following properties:

$$\forall p, q \text{ with } p+q = n+1, F^pH \cap \overline{F^qH} = 0 \text{ and } F^pH \oplus \overline{F^qH} = H.$$

**Definition 1.3.** Hodge number

$$h^{p,q}(H) := \dim_{\mathbb{C}} H^{p,q}.$$

Since $H^{q,p} = \overline{H^{p,q}}$, we have for Hodge number, $h^{q,p} = h^{p,q}$.

**Definition 1.4.** A polarization of a pure Hodge structure $H_{\mathbb{Z}}$ of weight $n$ is a bilinear form $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ which extends $\mathbb{C}$-linearly to a bilinear form $Q : H_{\mathbb{C}} \times H_{\mathbb{C}} \to \mathbb{C}$ that satisfies:

- $Q(u,v) = (-1)^n Q(v,u)$;
- $Q(H^{p,q}, H^{p',q'}) = 0$ for $p \neq q'$ (or equivalently, $Q(F^p, F^{n-p+1}) = 0$);
- $i^{p-q}Q(u, \overline{u}) > 0$ for $u \in H^{p,q}$ and $u \neq 0$.

A pure hodge structure that admits a polarization is said to be polarizable.

### 1.2 Pure Hodge Structure on Cohomology of Kähler Manifolds

An important example is the pure Hodge structure on the cohomology of the Kähler manifold (an integrable almost complex manifold with a Hermitian metric whose associated closed 2-form of type $(1,1)$ is closed. See the details of the definition in Appendix A). For instance, the smooth projective complex variety is a Kähler manifold. We have the following theorems:

**Theorem 1.5.** *Let $X$ be a Kähler manifold. Then*

$$H^n(X;\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$$

*$(0 \leq p, q \leq n)$ where*

$$H^{p,q} = H^q(X, \Omega^p_X) = \mathcal{H}^{p,q}(X).$$

*Here, $\Omega^\bullet_X$ is the de Rham complex of $X$, and $\mathcal{H}^{p,q}(X)$ is the harmonic forms of type $(p,q)$.*

In short, the $n$-th cohomology of a Kähler manifold has a pure Hodge structure of weight $n$.

For the sheaf cohomology $H^q(X, \Omega_X^p)$, we have the Serre duality.

**Theorem 1.6** (Serre Duality)**.** *Let $X$ be a smooth complex manifold of complex dimension $d$. Then $H^q(X, \Omega_X^p) \simeq H^{d-q}(X, \Omega_X^{d-p})^\vee$.*

In particular, for Hodge number, we have $h^{p,q} = h^{d-p,d-q}$.

Let $X$ be a Kähler manifold of complex dimension $d$ with the associated 2-form $\omega$ of type $(1,1)$. Then $H_\mathbb{Z} := H^n(X;\mathbb{Z})$ has the polarization $Q : H_\mathbb{Z} \times H_\mathbb{Z} \to \mathbb{Z}$:

$$Q(\xi, \eta) = \int_X \xi \wedge \eta \wedge \omega^{d-n}.$$

**Example 1.7.** The cohomology of $X = \mathbb{P}^n$ over $\mathbb{C}$ is

$\Omega^1 \simeq \mathcal{O}(-2)$

$$H^0\mathcal{O}(n) = \begin{cases} 0 & n<0 \\ \mathbb{C} & n=0 \\ \text{homo dgn} & n>0 \end{cases}$$

$$H^i(\mathbb{P}^n;\mathbb{C}) = \begin{cases} \mathbb{C} & i \text{ is even and } 0 \le i \le 2n \\ 0 & \text{otherwise} \end{cases}$$

It can be shown that

$$H^{p,q} = H^q(X, \Omega_X^p) = \begin{cases} \mathbb{C} & 0 \le p = q \le n \\ 0 & \text{otherwise} \end{cases}.$$

So Hodge number

$$h^{p,q} = \begin{cases} 1 & 0 \le p = q \le n \\ 0 & \text{otherwise} \end{cases}.$$

**Example 1.8.** Let $X$ be a complex torus of genus $g$. Then the cohomology

$$H^n(X;\mathbb{C}) = \begin{cases} \mathbb{C}^{2g} & n = 1 \\ \mathbb{C} & n = 0 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}.$$

We have

$$H^{p,q} = H^q(X, \Omega_X^p) = \begin{cases} \mathbb{C}^g & (p,q) = (0,1) \text{ or } (1,0) \\ \mathbb{C} & (p,q) = (0,0) \text{ or } (1,1) \\ 0 & \text{otherwise} \end{cases}.$$

So Hodge number

$$h^{p,q} = \begin{cases} g & (p,q) = (0,1) \text{ or } (1,0) \\ 1 & (p,q) = (0,0) \text{ or } (1,1) \\ 0 & \text{otherwise} \end{cases}.$$

# 2 Mixed Hodge Structure

## 2.1 General Theory

**Definition 2.1.** A mixed Hodge structure is a $\mathbb{Z}$-module $H_\mathbb{Z}$ together with an increasing filtration (called weight filtration) $H_\mathbb{Q} \subseteq \cdots \subseteq W_0 \subseteq W_1 \subseteq \cdots$ of $H_\mathbb{Q} := H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ and a decreasing filtration (called Hodge filtration) $H_\mathbb{C} \supseteq \cdots \supseteq F^0 \supseteq F^1 \supseteq \cdots$ of $H_\mathbb{C} := H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ such that $F$ induces a pure Hodge structure of weight $k$ on the graded piece

$$\mathrm{Gr}_k^W H_\mathbb{Q} = W_k / W_{k-1}.$$

**Remark 2.2.** How does it induce?

$$F^p(W_k/W_{k-1}) := (W_k \cap F^p + W_{k-1} \otimes \mathbb{C})/(W_{k-1} \otimes \mathbb{C}).$$

**Definition 2.3.** Hodge number

$$h^{p,q}(H) := \dim_{\mathbb{C}} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W (H_{\mathbb{C}}).$$

**Definition 2.4.** We say that the mixed Hodge structure is graded-polarizable if $\mathrm{Gr}_k^W H_{\mathbb{Q}}$ are pure polarizable $\mathbb{Q}$-Hodge structures.

The category of mixed Hodge structure is a rigid abelian monoidal category, which is defined as the followings:

**Zero:**

$$\begin{aligned} W_m 0 &= 0 \\ F^p 0 &= 0 \end{aligned}$$

**Direct Sum:**

$$\begin{aligned} W_m(A \oplus B) &= W_m(A) \oplus W_m(B) \\ F^p(A \oplus B) &= F^p(A) \oplus F^p(B) \end{aligned}$$

**Tensor Unit:**

$$W_m \mathbb{Q} = \begin{cases} \mathbb{Q} & p \geq 0 \\ 0 & p < 0 \end{cases} \tag{1}$$

$$F^p \mathbb{C} = \begin{cases} \mathbb{C} & p \leq 0 \\ 0 & p > 0 \end{cases} \tag{2}$$

**Tensor Product:**

$$\begin{aligned} W_m(A \otimes B) &= \sum_{i+j=m} W_i A \otimes W_j B \\ F^p(A \otimes B) &= \sum_{i+j=p} F^i A \otimes F^j B \end{aligned}$$

**Dual:**

$$W_m A^\vee = \{f \in A^\vee : \forall n, f(W_n A) \subseteq W_{n+m}\mathbb{Q}\} \tag{3}$$

$$F^p A^\vee = \left\{f \in A^\vee : \forall n, f(F^n A) \subseteq F^{n+p}\mathbb{C}\right\} \tag{4}$$

**Quotient:**

W

$$F^p(A/B) = (B + F^p A)/B \simeq F^p A/(B \cap F^p A)$$

It can be further shown that the category of mixed Hodge structure is a neutral Tannakian category, which is equivalent to the category of the representations of an affine group scheme.

For a mixed Hodge structure $H$, define $H^{\otimes 0}$ as the tensor unit in the category of mixed Hodge structures, $H^{\otimes(n+1)} = H^{\otimes n} \otimes H$ for $n \geq 0$, and $H^{\otimes(-n)} = (H^{\otimes n})^\vee$ for $n \geq 0$.

**Example 2.5.** The Tate-Hodge structure $\mathbb{Z}(1)$ is $H_{\mathbb{Z}} = \mathbb{Z}$ together with the following filtrations:

$$W_m = \begin{cases} 0 & m < -2 \\ \mathbb{Q} & m \geq -2 \end{cases}$$

$$F^p = \begin{cases} \mathbb{C} & p \leq -1 \\ 0 & p > -1 \end{cases}$$

Define $\mathbb{Z}(n)$ as $\mathbb{Z}(1)^{\otimes n}$, whose filtrations can be shown as the followings:

$$W_m = \begin{cases} 0 & m < -2n \\ \mathbb{Q} & m \geq -2n \end{cases} \tag{5}$$

$$F^p = \begin{cases} \mathbb{C} & p \leq -n \\ 0 & p > -n \end{cases} \tag{6}$$

It can be seen that $\mathbb{Z}(n)$ is pure of type $(-n, -n)$ (means it is actually a pure Hodge structure with $H^{-n,-n} = \mathbb{C}$ and $H^{p,q} = 0$ for other $(p, q)$).

**Definition 2.6.** Given a mixed Hodge structure $H$, its $n$-th Tate twist is $H(n) := H \otimes \mathbb{Z}(n)$.

**Example 2.7.** The $n$-th twist of $\mathbb{Z}(m)$ is $\mathbb{Z}(m + n)$.

## 2.2 Mixed Hodge Structure on Cohomology of Smooth Varieties

Now we turn to an important example, which is the mixed Hodge structure on cohomology of smooth varieties. Before that, we introduce the notion of hypercohomology.

Let $X$ be a topological space and $A^\bullet$ be a complex in the category of sheaves on $X$ (denoted by $\mathcal{C}$ here). Then there exists a complex $I^\bullet$ of injective elements in $\mathcal{C}$ such that $A^\bullet$ and $I^\bullet$ are quasi-isomorphic (means the induced maps on cohomology of sheaves are isomorphisms). Then the hypercohomology of $A$ is defined by

$$\mathbb{H}^i(X, A^\bullet) := H^i(\Gamma(X, I^\bullet)).$$

In practice, we can replace the injective resolution by the acyclic resolution (A sheaf is acyclic iff cohomology of all positive degrees vanishes). Appendix B (which is taken from another article) shows the details.

If we have an increasing filration $W$ on the complex $A^\bullet$, then it induces an increasing filratiohn on its hypercohomology $\mathbb{H}^n(X, A^\bullet)$ via

$$W_m\mathbb{H}^n(X, A^\bullet) := \mathrm{im}(\mathbb{H}^n(X, W_{m-n}A^\bullet) \to \mathbb{H}^n(X, A^\bullet)). \tag{7}$$

Similarly, if we have a decreasing filtration $F$ on the complex $A^\bullet$, then it induces a decreasing filtration on its hypercohomology $\mathbb{H}^n(X, A^\bullet)$ via

$$F^p\mathbb{H}^n(X, A^\bullet) := \mathrm{im}(\mathbb{H}^n(X, F^pA^\bullet) \to \mathbb{H}^n(X, A^\bullet)). \tag{8}$$

Now we can state how to give the mixed Hodge structure on the cohomology of smooth varieties.

Let $U$ be a smooth complex variety. Let $X \supseteq U$ be a compactification. Let $D = X - U$ be a normal crossing divisor (basically it means that $D$ locally looks like the crossing of coordinate hyperplanes). A differential form $\omega$ on $U$ is said to be have logarithm poles along $D$ if $\omega$ and $d\omega$ have at most a pole of order one along $D$. They constitute a complex $\Omega_X^\bullet(\log D)$ called the logarithm de Rham complex, where $\Omega_X^r(\log D)$ is generated by differential forms of the shape $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_j}{z_j} \wedge \alpha$ where $z_i$ is a local equation of a component of $D$, $j \leq r$, and $\alpha \in \Omega_X^{r-j}$.

**Remark 2.8.** Note: $\Omega_X^\bullet(\log \varnothing) = \Omega_X^\bullet$, and $\Omega_X^0(\log D) = \Omega_X^0$.

We can show that the ordinary cohomology coincides with the hypercohomology of the logarithm de Rham complex as in the following theorem. So in order to put a mixed Hodge structure on the former, we only need to put one on the latter.

**Theorem 2.9.** $H^n(U; \mathbb{C}) = \mathbb{H}^n(X, \Omega_X^\bullet(\log D))$.

We first put two filtrations on $\Omega_X^\bullet(\log D)$ making it a bifiltered complex:

$$W_m\Omega_X^i(\log D) = \begin{cases} \Omega_X^i(\log D) & m \geq i \\ \Omega_X^{i-m} \wedge \Omega_X^m(\log D) & 0 \leq m \leq i \\ 0 & m < 0 \end{cases} \tag{9}$$

$$F^p\Omega_X^i(\log D) = \begin{cases} 0 & p > i \\ \Omega_X^i(\log D) & p \leq i \end{cases}. \tag{10}$$

Thus, by (7)(8), we have the increasing filtration $W$ and the decreasing filtration $F$ on the hypercohomology $\mathbb{H}^n(X, \Omega^\bullet_X(\log D)) \simeq H^n(U;\mathbb{C})$. It can be further shown that the filtration $W$ can be defined over $\mathbb{Q}$. This gives us a mixed Hodge structure on $H^n(U)$.

There are some nice descriptions of the filtrations $W$ and $F$ on $H^n(U)$.

**Proposition 2.10.**

$$W_m H^n(U) = \begin{cases} 0 & m < n \\ \mathrm{im}(H^n(X) \to H^n(U)) & m = n \\ H^n(U) & m > n \end{cases} \tag{11}$$

**Proposition 2.11.** *Take an acyclic complex $I^\bullet$ which is quasi-isomorphic to $\Omega^\bullet_X(\log D)$. Then*

$$F^p H^n(U;\mathbb{C}) = \begin{cases} H^n(U;\mathbb{C}) & p < n \\ \mathrm{im}(\ker(\Gamma(X, I^n) \to \Gamma(X, I^{n+1})) \to H^n(U;\mathbb{C})) & p = n \\ 0 & p > n \end{cases} \tag{12}$$

**Example 2.12.** Let $X = U = \mathbb{P}^1$ (i.e., $D = \varnothing$). We want to put a mixed Hodge structure on $H^2(\mathbb{P}^1)$. Recall that for $k = \mathbb{Q}$ or $\mathbb{C}$, we have $H^2(\mathbb{P}^1; k) = k$. Then by (11), we have

complex dim

$$W_m H^2(\mathbb{P}^1;\mathbb{Q}) = \begin{cases} \mathbb{Q} & m \geq 2 \\ 0 & m < 2 \end{cases} \quad H^{2d} \simeq \mathbb{Q}(-d)$$

By Example (1.7), $H^2(\mathbb{P}^1;\mathbb{C}) = \bigoplus_{p+q=2} H^{p,q}$ where $H^{1,1} = \mathbb{C}$ and $H^{p,q} = 0$ for other $(p, q)$ with $p + q = 2$. So

$$F^p H^2(\mathbb{P}^1;\mathbb{C}) = \bigoplus_{i \geq p} H^{i,2-i} = \begin{cases} \mathbb{C} & p \geq 1 \\ 0 & p < 1 \end{cases}$$

By comparing with (5)(6), we find that this mixed Hodge structure is $\mathbb{Z}(-1)$.

**Example 2.13.** Let $X = \mathbb{P}^1$ over $\mathbb{C}$, $D = \{p_1, \cdots, p_r\}$ be $r$ points, and $U = X - D$. The logarithm de Rham complex $\Omega^\bullet_X(\log D)$ is given by the followings:

$$\Omega^0_{\mathbb{P}^1} \xrightarrow{d} \Omega^1_{\mathbb{P}^1}(\log\{p_1, \cdots, p_r\}) \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

Since each term in the above complex is acyclic, we can just take the total complex $I^\bullet$ of the resolution to coincide with the above complex when computing hypercohomology.

Taking global sections,

$$\boxed{\mathbb{C} \simeq \Gamma(\mathbb{P}^1, \Omega^0_{\mathbb{P}^1}) \xrightarrow{d=0} \mathbb{C}^{r-1} \simeq \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log\{p_1, \cdots, p_r\})) \xrightarrow{0} 0 \xrightarrow{0} \cdots}$$

From the above complex of global sections, we directly see that

$n$ $\quad W_n = \text{every}$

$W_{n-1} = 0$

$$\mathbb{Q}(-1)^{r-1} \quad H^i(U;\mathbb{C}) = \mathbb{H}^i(\mathbb{P}^1, \Omega^\bullet_{\mathbb{P}^1}(\log\{p_1, \cdots, p_r\})) = \begin{cases} 0 & i > 2 \\ \mathbb{C}^{r-1} & i = 1 \\ \mathbb{C} & i = 0 \end{cases}$$

$W_n/W_{n-1} = \{ \mathbb{Q}^{r-1} \quad \otimes \quad m=2$

By (11)(12), we have

$$W_m H^1(U;\mathbb{Q}) = \begin{cases} 0 & m < 1 \\ \mathrm{im}(H^1(\mathbb{P}^1;\mathbb{Q}) = 0 \to H^1(U;\mathbb{Q})) = 0 & m = 1 \\ H^1(U;\mathbb{Q}) = \mathbb{Q}^{r-1} & m > 1 \end{cases} \tag{13}$$

$$F^p H^1(U;\mathbb{C}) = \begin{cases} H^1(U;\mathbb{C}) = \mathbb{C}^{r-1} & p < 1 \\ \mathrm{im}(\ker(\mathbb{C}^{r-1} \to 0) \to H^1(U;\mathbb{C})) = \mathbb{C}^{r-1} & p = 1 \\ 0 & p > 1 \end{cases} \tag{14}$$

(In particular, when $r = 2$, i.e., $U = \mathbb{G}_m$, the mixed Hodge structure on $H^1(U)$ given above coincides with the dual of Tate-Hodge structure $\mathbb{Z}(1)^\vee = \mathbb{Z}(-1)$ by comparing with (5)(6).)

And

$$W_m H^0(U;\mathbb{Q}) = \begin{cases} 0 & m<0 \\ \mathrm{im}(H^0(\mathbb{P}^1;\mathbb{Q}) = \mathbb{Q} \to H^0(U;\mathbb{Q}) = \mathbb{Q}) = \mathbb{Q} & m=0 \\ H^0(U;\mathbb{Q}) = \mathbb{Q} & m>0 \end{cases}$$

$$F^p H^0(U;\mathbb{C}) = \begin{cases} H^0(U;\mathbb{C}) = \mathbb{C} & p<0 \\ \mathrm{im}(\ker(d=0:\mathbb{C}\to\mathbb{C}^{r-1}) \to H^0(U;\mathbb{C})) = \mathbb{C} & p=0 \\ 0 & p>0 \end{cases}$$

Then (in the following $\Omega^\bullet$ represents $\Omega^\bullet_{\mathbb{P}^1}(\log\{p_1,\cdots,p_r\})$.)

$$h^{p,q}H^1(U;\mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} \mathbb{H}^1(\mathbb{P}^1,\Omega^\bullet) = \begin{cases} r-1 & p=q=1 \\ 0 & \text{otherwise} \end{cases}$$

$$h^{p,q}H^0(U;\mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} \mathbb{H}^0(\mathbb{P}^1,\Omega^\bullet) = \begin{cases} 1 & p=q=0 \\ 0 & \text{otherwise} \end{cases}.$$

# 3 Hodge Filtrations of $\pi_1^{\mathrm{dR}}$ via Universal Connections

Let $C$ be a smooth projective curve of genus $g$ over a field $k$ of characteristic 0. Let $D$ be a nonempty divisor of size $r$ and let $X := C - D$. Let $\alpha_0, \cdots, \alpha_{2g+r-2}$ form a $k$-basis of $H^1_{\mathrm{dR}}(X;k)$ so that $\alpha_0,\cdots,\alpha_{g-1}$ form a $k$-basis of $H^0(C,\Omega^1_{X/k})$. Let $V_{\mathrm{dR}} := H^1_{\mathrm{dR}}(X;k)^\vee$ with basis $A_i$ dual to $\alpha_i$. Let $R$ be the tensor algebra of $V_{\mathrm{dR}}$, i.e.,

$$R := \bigoplus_{i=0}^{\infty} V_{\mathrm{dR}}^{\otimes i}.$$

Let $I$ be the ideal generated by $A_0, \cdots, A_{2g+r-2}$. Let

$$R_n := R/I^{n+1} \simeq \bigoplus_{i=0}^{n} V_{\mathrm{dR}}^{\otimes i}.$$

Let $\mathcal{E}_n := R_n \otimes \mathcal{O}_X$ and let $\mathcal{E}$ be the limit of $\mathcal{U}_n$. So $\mathcal{E} = R\otimes\mathcal{O}_X$. Then $\mathcal{E}$ is the pro-universal object in the category of unipotent vector bundles on $X$ of flat connections.

Let $\mathcal{A} := \mathcal{E}^\vee$ be the dual bundle of $\mathcal{E}$.

## 3.1 Hodge Filtrations on $\mathcal{O}_X$ and $V_{dR}$

The filtrations on $\mathcal{O}_X$ is given by

- $F^p\mathcal{O}_X = \mathcal{O}_X$ when $p \le 0$, and $F^p\mathcal{O}_X = 0$ when $p>0$.
- $W_m\mathcal{O}_X = \mathcal{O}_X$ when $m \ge 0$, and $W_m\mathcal{O}_X = 0$ when $m<0$.

The filtrations on $V_{\mathrm{dR}} := H^1_{\mathrm{dR}}(X)^\vee$ are the dual filtrations on $H^1(X)$, which has the following explicit expressions based on (11) (12).

$$W_m H^1(X) = \begin{cases} 0 & m<1 \\ \mathrm{im}(H^1(C)\to H^1(X)) & m=1 \\ H^1(X) & m>1 \end{cases}$$

$$F^p H^1(X;\mathbb{C}) = \begin{cases} H^1(X;\mathbb{C}) & p<1 \\ \mathrm{im}(\ker(\Gamma(C,I^1)\to\Gamma(C,I^2)) \to H^1(U;\mathbb{C})) & p=1 \\ 0 & p>1 \end{cases}$$

where $I^\bullet$ is an acyclic complex which is quasi-isomorphic to $\Omega^\bullet_C(\log D)$.

By the dual filtration, we have

$$W_m V_{\mathrm{dR}} = \begin{cases} V_{\mathrm{dR}} & m \geq -1 \\ 0 & m \leq -3 \end{cases}$$

$$F^p V_{\mathrm{dR}} = \begin{cases} 0 & p \geq 1 \\ V_{\mathrm{dR}} & p \leq -1 \end{cases}$$

whereas $W_{-2}V_{\mathrm{dR}}$ and $F^0 V_{\mathrm{dR}}$ depend on cases.

**Example 3.1.** When $D = \varnothing$ (i.e., $X = C$), then $W_1 H^1(X) = H^1(X)$. So $W_{-2}V_{\mathrm{dR}} = 0$.

**Example 3.2.** Let $C = \mathbb{P}^1$ over $k$, $D = \{p_1, \cdots, p_r\}$ be $r$ points, and $X = C - D$. By (13)(14), we have $W_1 H^1(X) = 0$ and $F^1 H^1(X) = H^1(X)$. So $W_{-2}V_{\mathrm{dR}} = V_{\mathrm{dR}}$ and $F^0 V_{\mathrm{dR}} = 0$.

## 3.2 Hodge Filtrations on $\mathcal{E}$ and $\mathcal{A}$

Note that $\mathcal{E}$ and $\mathcal{A}$ are bundles only in terms of compositions of direct sums, tensor products and duals in terms of $\mathcal{O}_X$ and $V_{\mathrm{dR}}$. Therefore, once we get the filtrations on $\mathcal{O}_X$ and $V_{\mathrm{dR}}$ (as in previous subsection), we immediately get the filtrations on $\mathcal{E}$ and $\mathcal{A}$ since the category of mixed Hodge structure is rigid abelian monoidal. Let me show you an example.

**Example 3.3.** Let $C = \mathbb{P}^1$ over $k$, $D = \{p_1, \cdots, p_r\}$ be $r$ points, and $X = C - D$. In this case, the genus $g = 0$. We have:

**Filtration on $\mathcal{O}_X$:** $F^p\mathcal{O}_X = \mathcal{O}_X$ when $p \leq 0$, and $F^p\mathcal{O}_X = 0$ when $p > 0$.

**Filtration on $V_{\mathrm{dR}}$:** $F^p V_{\mathrm{dR}} = V_{\mathrm{dR}}$ when $p < 0$, and $F^p V_{\mathrm{dR}} = 0$ when $p \geq 0$.

**Filtration on $V_{\mathrm{dR}}^{\otimes i}$:** Use the tensor product filtration $F^p(A \otimes B) := \bigoplus_{i+j=p} F^i A \otimes F^j B$. In this case,

$$F^p V_{\mathrm{dR}}^{\otimes i} = \begin{cases} 0 & p > -i \\ V_{\mathrm{dR}}^{\otimes i} & p \leq -i \end{cases}.$$

**Filtration on $R_n$:** Use the direct sum filration $F^p(A \oplus B) = F^p A \oplus F^p B$. In this case,

$$F^p R_n = F^p \left( \bigoplus_{i=0}^{n} V_{\mathrm{dR}}^{\otimes i} \right) = \begin{cases} 0 & p > 0 \\ R_{-p} & -n \leq p \leq 0 \\ R_n & p < -n. \end{cases}$$

**Filtration on $\mathcal{E}_n$:** Use the tensor product filtration on $\mathcal{E}_n = R_n \otimes \mathcal{O}_X$.

$$F^p \mathcal{E}_n = F^p(R_n \otimes \mathcal{O}_X) = \begin{cases} 0 & p > 0 \\ \mathcal{E}_{-p} & -n \leq p \leq 0 \\ \mathcal{E}_n & p < -n. \end{cases}$$

(not the fund.

p-adic iterated.

**Filtration on $\mathcal{E}$:** Just take the limit of $\mathcal{E}_n$.

$R_{-p} \otimes \mathcal{O}_X$

$$F^p \mathcal{E} = \begin{cases} 0 & p > 0 \\ \mathcal{E}_{-p} & p \leq 0 \end{cases}.$$

$\bigoplus_{i=0}^{-p} V_{dr}^{\otimes i}$

$\int A_0 . A_1 \cdots$

**Filtration on $\mathcal{A}$:** Use the filtrations on the dual.

$$F^p \mathcal{A} = \bigoplus_{i=p}^{\infty} (V_{\mathrm{dR}}^{\otimes i})^{\vee} \otimes \mathcal{O}_X = \bigoplus_{i=p}^{\infty} H^i_{\mathrm{dR}}(X;k)^{\otimes i} \otimes \mathcal{O}_X.$$

$\underbrace{A_1 - A_d}_{\#p \text{ length.}}$ )

## 3.3 Hodge Filtrations on $\mathcal{E}_x$ and $\mathcal{A}_x$

The Hodge filtrations of $\mathcal{E}_x$ and $\mathcal{A}_x$ are given by

$$\begin{aligned}\mathcal{F}^p(\mathcal{E}_x) &= (\mathcal{F}^p\mathcal{E})_x \\ \mathcal{F}^p(\mathcal{A}_x) &= (\mathcal{F}^p\mathcal{A})_x.\end{aligned}$$

**Example 3.4.** Let $C = \mathbb{P}^1$ over $k$, $D = \{p_1, \cdots, p_r\}$ be $r$ points, and $X = C - D$.

For any $x \in X$, the stalk of $\mathcal{O}_X$ at $x$ in this case is isomorphic to $k[t]_{(t)}$, and by quotient out the maximal ideal, the fibre of $\mathcal{O}_X$ at $x$ is just $k$. So for any $k$-vector space $E$, the fibre of $E \otimes \mathcal{O}_X$ at $x$ is $E \otimes_k k = E$. For instance,

$$\mathcal{E}_x = R \otimes_k k = R = \bigoplus_{i=0}^{\infty} V_{\mathrm{dR}}^{\otimes i}$$

In this case, we have

**Filtration on $\mathcal{E}_x$ :**

$$\boxed{F^p(\mathcal{E}_x) = (F^p\mathcal{E})_x = \begin{cases} 0 & p > 0 \\ R_{-p} = \bigoplus_{i=0}^{-p} V_{\mathrm{dR}}^{\otimes i} & p \leq 0 \end{cases}}$$

**Filtration on $\mathcal{A}_x$ :**

$$\mathcal{F}^p(\mathcal{A}_x) = (F^p\mathcal{A})_x = \bigoplus_{i=p}^{\infty} H^i_{\mathrm{dR}}(X;k)^{\otimes i}$$

## 3.4 Calculation of $\pi_1^{\mathrm{dR}}$ and $\mathrm{Lie}\pi_1^{\mathrm{dR}}$

Since $\mathcal{E}$ is the pro-universal object, this gives us a canonical map $\Delta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ which makes the fibre $\mathcal{E}_x$ for every $x \in X$ a coalgebra over $k$.

**Definition 3.5.** Let $A$ be a coalgebra over $k$, where the comultiplication is $\Delta : A \to A \otimes A$ and the counit is $e^* : A \to k$.

- The group-like elements of $A$ is those $g \in A$ with $\Delta(g) = g \otimes g$ and $e^*(g) = 1$.
- The primitive elements of $A$ is those $x \in A$ with $\Delta(x) = 1 \otimes x + x \otimes 1$.

If $A$ is a Hopf algebra over $k$, then the group-like elements of $A$ form a group under the multiplication in $A$ as the group operation, and the primitive elements of $A$ form a Lie algebra under $[x, y] := xy - yx$ as the Lie bracket.

**Theorem 3.6.** *Let $x \in X$. The fibre $\mathcal{E}_x$ is a Hopf algebra over $k$, and we have:*

- *The group $\pi_1^{\mathrm{dR}}(X, x)$ is isomorphic to group-like elements of the fibre $\mathcal{E}_x$.*
- *The Lie algebra $\mathrm{Lie}\pi_1^{\mathrm{dR}}(X, x)$ is isomorphic to primitive elements of the fibre $\mathcal{E}_x$.*

**Example 3.7.** Let $C = \mathbb{P}^1$ over $k$, $D = \{p_1, \cdots, p_r\}$ be $r$ points, and $X = C - D$. Recall the the fibre

$$\mathcal{E}_x = R \otimes_k k = R = \bigoplus_{i=0}^{\infty} V_{\mathrm{dR}}^{\otimes i}$$

The coalgebra structure of $\mathcal{E}_x$ is determined $k$-linearly by:

$$\begin{aligned}\Delta(A_i) &= 1 \otimes A_i + A_i \otimes 1 \\ e^*(A_i) &= 0\end{aligned}$$

In this case, by an elementary computation using Theorem 3.6, we have

$X = \mathbb{P}^1 \setminus 3$ points

**Proposition 3.8.**

$$\pi_1^{\mathrm{dR}}(X,x) = \left\{ 1 + \sum_{j \geq 1} c_{i_1 \cdots i_j} A_{i_1} \cdots A_{i_j} \in \mathcal{E}_x \quad \middle| \quad \textit{some relations of } c_{i_1 \cdots i_j} \right\}$$

*where the relations are given by, for all $s, t \geq 1$ and indexes $i_1, \cdots, i_s, i'_1, \cdots, i'_t$,*

$$\sum_{\sigma \in \mathrm{Sym}(s+t)\ \textit{shuffle of type}(s,t)} c_{\sigma(i_1 \cdots i_s i'_1 \cdots i'_t)} = c_{i_1 \cdots i_s} \cdot c_{i'_1 \cdots i'_t}.$$

**Remark 3.9.** The relation is pretty much like product formula for iterated integrals:

$$\sum_\sigma \int_\gamma \omega_{\sigma(1)} \cdots \omega_{\sigma(s+t)} = \int_\gamma \omega_1 \cdots \omega_s \int_\gamma \omega_{s+1} \cdots \omega_{s+t}$$

where the sum ranges over $\sigma \in \mathrm{Sym}(s+t)$ where $\sigma$ is a shuffle of type $(s,t)$.

**Proposition 3.10.** $V_{dR} = H^1(X)^\vee = \mathrm{span}\{A_0, A_1\}$

$$\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) = \left\{ \sum_{j \geq 1} c_{i_1 \cdots i_j} A_{i_1} \cdots A_{i_j} \in \mathcal{E}_x \quad \middle| \quad \textit{some relations of } c_{i_1 \cdots i_j} \right\}$$

$[A_0, [A_1, \cdots]]$ $\qquad [X,Y] = XY - YX$

*where the relations are given by, for all $s, t \geq 1$ and indexes $i_1, \cdots, i_s, i'_1, \cdots, i'_t$,*

$$\sum_{\sigma \in \mathrm{Sym}(s+t)\ \textit{shuffle of type }(s,t)} c_{\sigma(i_1 \cdots i_s i'_1 \cdots i'_t)} = 0.$$

### 3.5 Hodge Filtrations on $\mathrm{Lie}\pi_1^{\mathrm{dR}}$ and $\pi_1^{\mathrm{dR}}$

There is the exp map from $\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)$ to $\pi_1^{\mathrm{dR}}(X,x)$:

$$\exp : \mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) \to \pi_1^{\mathrm{dR}}(X,x), \quad A \mapsto \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{15}$$

The Hodge filtration on $\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)$ is given by

$$F^p\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) = \mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) \cap F^p(\mathcal{E}_x). \tag{16}$$

$(F^p \mathcal{E})_x$

And the Hodge filtration on $\pi_1^{\mathrm{dR}}(X,x)$ is given by

$$F^p\pi_1^{\mathrm{dR}}(X,x) = \exp\left(F^p\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)\right). \tag{17}$$

**Example 3.11.** Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and choose a point $x \in X$. Let $A_0, A_1$ be a basis of $V_{\mathrm{dR}}$.

**Filtration on $\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)$:** By (16) and Proposition 3.10, we have $[A_0, A_1]$

$$F^0\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) = \{0\}$$
$$F^{-1}\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) = \{c_0A_0 + c_1A_1\}$$
$$F^{-2}\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) = \{c_0A_0 + c_1A_1 + c_{01}A_0A_1 - c_{01}A_1A_0\}$$
$$\cdots$$

$[A_0, A_1]$ $\quad F^{-1}$ $\quad F^{-2}$ $\quad F^{-n}$ $\quad F^{-1}/F^{-3}$

$c_{01}[A_0, A_1]$

**Filtration on $\pi_1^{\mathrm{dR}}(X,x)$:** By (17) and (15), we have

$[A_0, A_1]$ $\qquad A_0 A_1$ $\qquad V_{dr} = H^1(X)^\vee$

$$F^0\pi_1^{\mathrm{dR}}(X,x) = \{1\}$$
$$F^{-1}\pi_1^{\mathrm{dR}}(X,x) = \left\{ 1 + c_0A_0 + c_1A_1 + \frac{1}{2}\left(c_0^2A_0A_0 + c_0c_1(A_0A_1 + A_1A_0) + c_1^2A_1A_1\right) + \cdots \right\}$$
$$\cdots$$

$\mathcal{E}$ $\qquad V_{dr} \subseteq L$ $\qquad U = \mathrm{Lie}(L) = L$ $\qquad L^2$

schema 130 $\qquad x \cdot y =$ $\qquad U/U^2 \cong \mathbb{Q}(-1)^2$

$U = \pi_1^{dR}$

$U^n = [U^{(n-1)}, U]$

$U^n/U^{n+1} = L^n/L^{n+1} = \mathbb{Q}(-n)^{r_n}$ as Hodge.

**Filtration on $U^n/U^{n+1}$:** Let $U = \pi_1^{\mathrm{dR}}(X, x)$.

$L^n = [L^{n-1}, L]$

# Appendix A Kähler Manifold

**Definition A.1.** An almost complex structure on a real vector space $V$ is a linear endomorphism $I : V \to V$ such that $I^2 = -\mathrm{id}$.

Let real vector space $V$ be endowed with an almost complex structure $I : V \to V$ and $W_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Then $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$. The $I : V \to V$ naturally induces an almost structure $I : W_{\mathbb{C}} \to W_{\mathbb{C}}$. It has eigenvalues $i$ and $-i$, and thus has the eigenspace decomposition corresponding to $i$ and $-i$

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}.$$

Let

$$\begin{aligned} W^{1,1} &:= W^{1,0} \otimes_{\mathbb{C}} W^{0,1} \subseteq \wedge^2 W_{\mathbb{C}} \\ W^{1,1}_{\mathbb{R}} &:= W^{1,1} \cap \wedge^2 W_{\mathbb{R}}. \end{aligned}$$

Note that an element of $\wedge^2 W_{\mathbb{R}}$ corresponding to an anti-symmetric bilinear form $V \times V \to \mathbb{R}$.

**Definition A.2.** Let $V$ be a real vector space. An anti-symmetric bilinear form $V \times V \to \mathbb{R}$ is called of type $(1,1)$ if it corresponds to an element in $W^{1,1}_{\mathbb{R}}$.

For a real vector space $V$ with an almost complex structure $I : V \to V$, it naturally makes $V$ also a complex vector space, where the scalar product is defined by $(a + bi) \cdot v = a \cdot v + b \cdot I(v)$.

**Definition A.3.** Let $V$ be a complex vector space. A Hermitian form on $V$ is a conjugate-symmetric bilinear form $V \times V \to \mathbb{C}$.

**Lemma A.4.** *Let $V$ be a real vector space with an almost complex structure (which makes it also a complex vector space). Then there is a natural one-to-one correspondence between the Hermitian forms on the complex vector space $V$ and the elements of $W^{1,1}_{\mathbb{R}}$ (i.e., forms of type $(1,1)$) given by $h \mapsto -\mathrm{Im}(h)$, where $\mathrm{Im}$ means the imaginary part.*

*Furthermore, the correspondence preserves the non-degenerateness.*

**Definition A.5.** A complex manifold of dimension $n$ is a real manifold of dimension $2n$ with complex structure, i.e., there is a chart such that each $U_i$ in the chart is diffeomorphic to $\mathbb{C}^n$ and the transition maps are holomorphic.

**Definition A.6.** An almost complex structure on a real manifold $M$ is an endomorphism $I : TM \to TM$ such that $I^2 = -\mathrm{id}$.

A complex manifold has a natural induced almost complex structure $I$.

**Definition A.7.** For a real manifold $M$ with the almost complex structure $I$, if it is induced by some complex manifold, then $I$ is called integrable, and $M$ is called an integrable almost complex manifold.

Let $I$ be an almost complex structure on a real manifold $M$. Then it induces on each point $x \in M$ an almost complex structure $I_x : TM_x \to TM_x$ of $TM_x$, making $TM_x$ also a complex vector space.

**Definition A.8.** A Hermitian metric $h$ on a real manifold $M$ with an almost complex structure $I$ is a smooth varying non-degenerate Hermitian form on each tangent space $TM_x$.

Let's denote $h_x : TM_x \times TM_x \to \mathbb{C}$ be the Hermitian form on $TM_x$ as above. By lemma A.4, it corresponds a non-degenerate anti-symmetric bilinear form $TM_x \times TM_x \to \mathbb{R}$ of type $(1,1)$. Varying points smoothly, it gives rise to a non-degenerate 2-form $\omega \in \Omega^2_M$ of type $(1,1)$, called the associated 2-form of type $(1,1)$ to the Hermitian metric $h$.

**Definition A.9.** A Kähler manifold is an integrable almost complex manifold with a Hermitian metric whose associated 2-form of type $(1,1)$ is closed.

# Appendix B Hypercohomology via Acyclic Resolution

We begin with some background on hypercohomology and spectral sequences; a more detailed discussion may be found in [35, Ch. 8]. Let

$$\cdots \xrightarrow{d} A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

be a complex of sheaves on a space $X$. Recall that the *hypercohomology* of the complex $(A^\bullet, d)$ is defined by choosing an acyclic resolution of $A^\bullet$ by a double complex $(I^{\bullet,\bullet}, d, d')$, i.e. a diagram with exact rows and columns

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & I^{0,1} \arrow[r, "d"'] \arrow[u, "d'"'] & I^{1,1} \arrow[r, "d"'] \arrow[u, "d'"'] & I^{2,1} \arrow[r, "d"'] \arrow[u, "d'"'] & \cdots \\
\cdots \arrow[r] & I^{0,0} \arrow[r, "d"'] \arrow[u, "d'"'] & I^{1,0} \arrow[r, "d"'] \arrow[u, "d'"'] & I^{2,0} \arrow[r, "d"'] \arrow[u, "d'"'] & \cdots \\
\cdots \arrow[r] & A^0 \arrow[r, "d"'] \arrow[u, "d'"'] & A^1 \arrow[r, "d"'] \arrow[u, "d'"'] & A^2 \arrow[r, "d"'] \arrow[u, "d'"'] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and $H^n(X, I^{i,j}) = 0$ for $n \geq 0$. Such a resolution always exists. Let $I^n := \bigoplus_{i+j=n} I^{i,j}$ denote the total complex with differential $\delta = d + (-1)^i d'$, then the $n$th hypercohomology of $(A^\bullet, d)$ is defined to be the $n$th cohomology of the complex of global sections of $I^\bullet$,

$$\mathbb{H}^n(X, A^\bullet) := H^n_\delta \Gamma(X, I^\bullet),$$

and this definition is independent of the choice of $I^{\bullet,\bullet}$.