Dictionary of Algebraic Geometry

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In this document, the word "ring" means a commutative ring with 1.
Some good reference:

- Stacks Project
- Wikipedia: Glossary of algebraic geometry

1 Topology And Algebra

1.1 Topology

Lemma 1.1 (Gluing). *Let U be an open subset of a topological space X, $\{U_i\}_i$ be an open covering of U, and for each i, let f_i be a continuous map $U_i \to \mathbb{R}$. If $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there exists a unique continuous map $f : X \to \mathbb{R}$ such that $f|_{U_i} = f_i$ for each i.*

Definition 1.2 (Product). Let X and Y be topological spaces. The product $X \times Y$ of X and Y is the product of them in the category of topological spaces.

Proposition 1.3. *The product of topological spacecs X and Y always exists.*

Definition 1.4 (Diagonal Map). The diagonal map of a topological space X is the natural morphism $X \to X \times X$ in the universal property defining the product.

Remark 1.5. The underlying set of $X \times Y$ can be chosen as the Cartesian product of them and the diagonal map $X \to X \times X$ is just $x \mapsto (x, x)$ as you may expect.

Definition 1.6 (Irreducible). A topological space is irreducible if it is nonempty and cannot be the union of two proper closed subsets.

Definition 1.7 (Generic Point). Let Z be an irreducible closed subset of a topological space X. Then a generic point of Z is a point in Z whose closure is Z.

Definition 1.8 (Dimension). The dimension of a topological space is the supreme of the length n of the chains $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ where all Z_i are irreducible closed subspaces.

Definition 1.9 (Quasi-Compact Space). A topological space is quasi-compact if every open cover admits a finite subcover.

Remark 1.10. Algebra geometers like to use the strange word "quasi-compact" which just means "compact" elsewhere. Well, that's probably because they use the word "compact" to mean "compact" + "Hausdorff".

Definition 1.11 (Quasi-Compact Map). A continuous map between topological spaces is quasi-compact if the preimage of a quasi-compact open set is quasi-compact.

Definition 1.12 (Open, Closed). A map $f : X \to Y$ between topological spaces X, Y is open (closed resp.) if the image of an open (closed resp.) subset of X is open (closed resp.) in Y.

Definition 1.13 (Hausdorff). A topological space is Hausdorff if its diagonal map is closed.

Remark 1.14. This definition of Hausdorff spaces is equivalent to the ordinary definition in terms of the separation of open sets.

1.2 Algebra

Definition 1.15 (Spectrum). Let R be a ring. The spectrum $\mathrm{Spec}(R)$ of R is a topological space, where its underlying set is $\{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal of } R\}$, and the closed sets are the sets of the form $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$ for some ideal I of R.

Proposition 1.16 (Distinguished Set). *Let R be a ring. The distinguished sets $\{D(f)|f \in R\}$ form an open basis of* $\mathrm{Spec}(R)$*, where $D(f) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\}$.*

Definition 1.17 (Local Ring). A local ring is a ring R with a unique maximal ideal $\mathfrak{m}$, normally written as $(R, \mathfrak{m})$.

Definition 1.18 (Local Ring Map). A local ring map φ from a local ring $(R, \mathfrak{m})$ to a local ring $(R', \mathfrak{m}')$ is a ring homomorphism such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{m}'$.

Definition 1.19 (Residue Field of a Local Ring). The residue field $\kappa(R)$ of a local ring $(R, \mathfrak{m})$ is the field $R/\mathfrak{m}$.

Definition 1.20 (Residue Field of a Prime Ideal). The residue field $\kappa(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of a ring R is the residue field of the localization $\kappa(R_\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

Definition 1.21 (Algebra over a Ring). Let R be a ring. An algebra over R is a ring A together with a ring homomorphism $R \to A$ (called the structure homomorphism).

Remark 1.22. This definition of algebras is equivalent to the ordinary definition which says that an algebra over R is a ring and also an R-module that satisfies some compability.

Remark 1.23. Let $\mathfrak{p}$ be a prime ideal of a ring R. Note that $\kappa(\mathfrak{p})$ is naturally an algebra over R via the natural ring homomorphism $R \to R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = \kappa(\mathfrak{p})$.

Definition 1.24 (Morphism of Algebras). Let R be a ring. The morphism between R-algebras (or R-algebra homomorphism) is a ring homomorphism and also an R-linear map.

Definition 1.25 (Tensor Product). Let R be a ring and A, B be R-algebras. The tensor product $A \otimes_R B$ of A and B is the coproduct of them in the category of algebras over R.

Proposition 1.26. *The tensor product of two algebras A and B over a ring R always exists.*

Definition 1.27 (Base Change). Let R be a ring and A, B be algebras over R. The base change of A to B is the algebra $A \otimes_R B$ over B with the structure homomorphism being the inclusion homomorphism $B \to A \otimes_R B$.

Definition 1.28 (Fibre). Let R be a ring, A be an R-algebra with the structure homomorphism $\varphi : R \to A$, and $\mathfrak{p} \in \mathrm{Spec}R$. The fibre of φ over $\mathfrak{p}$ is the base change of A to $\kappa(\mathfrak{p})$, i.e., the algebra $A \otimes_R \kappa(\mathfrak{p})$ over $\kappa(\mathfrak{p})$.

Definition 1.29 (Krull Dimension). The Krull dimension of a ring R is the dimension of the topological space $\mathrm{Spec}(R)$.

Definition 1.30 (Reduced). A reduced ring is a ring which has no nonzero nilpotent elements.

Definition 1.31 (Integral). An integral domain is a ring with $0 \neq 1$ which has no nonzero zero divisor.

Definition 1.32 (Noetherian). A Noetherian ring is a ring such that every ascending chain of ideals terminates.

Definition 1.33 (Normal/Integrally Closed Domain). A normal ring (integrally closed domain) is an integral domain R which is integrally closed in $K := \mathrm{Frac}(R)$ (i.e., all of the roots in K of a monic polynomial over R are in R).

Definition 1.34 (Regular). A regular local ring is a Noetherian local ring $(R, \mathfrak{m})$ such that $\dim_{\kappa(R)} \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Definition 1.35 (Dedekind). A Dedekind domain is a normal Noetherian ring of Krull dimension 0 or 1 (i.e., every nonzero prime ideal is maximal).

Definition 1.36 (Discrete Valuation Ring/DVR). A discrete valuation ring (DVR) is a local Dedekind domain of Krull dimension 1 (i.e., not a field).

Definition 1.37 (Finite). A ring homomorphism $\varphi : R \to A$ is finite if the R-algebra A with the structure homomorphism φ is finitely generated as an R-module.

Definition 1.38 (of Finite Type). A ring homomorphism $\varphi : R \to A$ is of finite type if the R-algebra A with the structure homomorphism φ is finitely generated as an R-algebra.

Definition 1.39 (Quasi-Finite). A ring homomorphism $\varphi : R \to A$ is quasi-finite if it is of finite type and for any $\mathfrak{p} \in \mathrm{Spec}(R)$, the structure homomorphism of the fibre of φ over $\mathfrak{p}$ (i.e., the inclusion $\kappa(\mathfrak{p}) \to A \otimes_R \kappa(\mathfrak{p})$) is finite.

Definition 1.40 (Flat). A ring homomorphism $\varphi : R \to A$ is flat if the R-algebra A with the structure homomorphism φ is flat as an R-module, i.e., $- \otimes_R A : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is an exact functor.

Definition 1.41 (Finite Étale). A ring homomorphism $\varphi : R \to A$ is finite étale if φ is finite, flat, and for any $\mathfrak{p} \in \mathrm{Spec}(R)$, the fibre of φ over $\mathfrak{p}$ (i.e., $A \otimes_R \kappa(\mathfrak{p})$) is isomorphic as $\kappa(\mathfrak{p})$-algebras to a finite product of finite separable extensions of $\kappa(\mathfrak{p})$.

Definition 1.42 (Finite Étale Algebra). Let R be a normal ring. A finite étale algebra over R is an R-algebra which is isomorphic as R-algebras to a finite direct product $\prod B_i$ of R-algebras B_i where for each i, B_i is the integral closure of R in some finite separable extension of $\mathrm{Frac}(R)$ and the structure ring homomorphism $R \to B_i$ is finite étale.

2 Sheaf And Scheme

2.1 Sheaf

Definition 2.1 (Open Category). Let X be a topological space. The open category $\mathrm{Open}(X)$ of X is a category whose objects are open sets of X, and whose morphisms are the inclusion maps between open sets.

Definition 2.2 (Presheaf). Let X be a topological space and C be a cocomplete category. A presheaf of C on X is a contravariant functor from $\mathrm{Open}(X)$ to C.

Remark 2.3. We often consider $C =$**Set**/**Ab**/**Ring** (the category of sets/abelian groups/rings).

Definition 2.4 (Morphism of Presheaves). A morphism between presheaves is a natural transformation between those functors.

Definition 2.5 (Section). Let $\mathcal{F}$ be a presheaf on a topological X, and $U \subseteq X$ open. Then a section of $\mathcal{F}$ over U is an element of $\mathcal{F}(U)$. A global section of $\mathcal{F}$ is an element of $\mathcal{F}(X)$.

Definition 2.6 (Restriction Map). Let $\mathcal{F}$ be a presheaf on a topological space X, open subsets $V \subseteq U \subseteq X$ with inclusion $i : V \to U$, and $s \in \mathcal{F}(U)$. Define the restriction $s|_V$ of s on V is $\mathcal{F}(i)(s) \in \mathcal{F}(V)$.

Remark 2.7. For an open $U \subseteq X$, also denote $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$. For an inclusion $i : V \to U$ of open sets, also denote $\mathrm{res}^U_V := \mathcal{F}(i) : \mathcal{F}(U) \to \mathcal{F}(V)$.

Definition 2.8 (Sheaf). A sheaf on a topological space X is a presheaf $\mathcal{F}$ on X that satisfies gluing condition: Let U be an open subset of X, $\{U_i\}_i$ be an open covering of U, and for each i let $s_i \in \mathcal{F}(U_i)$. If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.

Definition 2.9 (Morphism of Sheaves). A morphism between sheaves is a morphism between presheaves.

Definition 2.10 (Stalk). Let $\mathcal{F}$ be a presheaf on a topological space X, $x \in X$. Then the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is the colimit of $\mathcal{F}|_I$ (the functor $\mathcal{F}$ restricting on I) where I is the subcategory of $\mathrm{Open}(X)$ whose objects are open subsets of X containing x (i.e., $\mathcal{F}_x = \varinjlim_{x \in U \text{ open}} \mathcal{F}(U)$).

Proposition 2.11. *The stalk of a presheaf on a topological space X at $x \in X$ always exists.*

Definition 2.12 (Germ). Let $\mathcal{F}$ be a presheaf on a topological space X, $x \in X$, $U \subseteq X$ an open subset containing x, and $s \in \mathcal{F}(U)$. The universal property of the colimit gives us the canonical map $\mathcal{F}_x^U : \mathcal{F}(U) \to \mathcal{F}_x$. The germ s_x of s at x is $\mathcal{F}_x^U(s) \in \mathcal{F}_x$.

Proposition 2.13 (Induced Map on Stalks). *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X, $x \in X$. Then there is a unique map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ (called the induced map of f on stalks) such that the following diagram commutes for any open $U \subseteq X$ containing x:*

$$\begin{array}{ccc} \mathcal{F}(U) & \overset{f(U)}{\longrightarrow} & \mathcal{G}(U) \\ {\scriptstyle \mathcal{F}_x^U}\downarrow & & \downarrow{\scriptstyle \mathcal{G}_x^U} \\ \mathcal{F}_x & \underset{f_x}{\longrightarrow} & \mathcal{G}_x \end{array}$$

Definition 2.14 (Direct Image/Pushforward). Let X, Y be topological spaces, $\mathcal{F}$ be a sheaf on X, and $f : X \to Y$ a continuous map. The direct image (pushforward) $f_*\mathcal{F}$ of $\mathcal{F}$ by f is a presheaf on Y that sends an open $U \subseteq Y$ to $\mathcal{F}(f^{-1}(U))$, and sends an inclusion $V \to U$ to $\mathcal{F}(j)$ where j is the inclusion $f^{-1}(V) \to f^{-1}(U)$.

Proposition 2.15. *The direct image of a sheaf is also a sheaf, which makes the direct image f_* actually a functor from the category of sheaves on X to the category of sheaves on Y.*

Definition 2.16 (Inverse Image/Pullback). Let $f : X \to Y$ be a continuous map between topological spaces X, Y. The inverse image (pullback) f^{-1} is the left adjoint functor to the direct image f_* (So the inverse image $f^{-1}(\mathcal{G})$ of a sheaf $\mathcal{G}$ on Y is a sheaf on X).

Proposition 2.17. *The inverse image of a sheaf always exists.*

Definition 2.18 (Restriction of Sheaf). Let $\mathcal{F}$ be a presheaf on a topological space X and $U \subseteq X$ an open subset of X. Let $\mathrm{Open}(U)$ be the open category of U (subspace topology), which is naturally the subcategory of $\mathrm{Open}(X)$. Then the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ on U is $\mathcal{F}|_{\mathrm{Open}(U)}$.

2.2 Scheme

Definition 2.19 (Ringed Space). A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space (called the underlying topological space) and $\mathcal{O}_X$ is a sheaf of rings (means a sheaf of the category of rings) on X (called the structure sheaf).

Definition 2.20 (Morphism of Ringed Spaces). A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f^{\mathrm{top}}, f^{\#})$ where $f^{\mathrm{top}} : X \to Y$ is a continuous map and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves on Y.

Remark 2.21. Since the inverse image f^{-1} is the left adjoint functor to the direct image $f^{\#}$, a morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y naturally corresponds to a morphism $f_\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X, and vice versa.

Definition 2.22 (Locally Ringed Space). A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x is always a local ring.

Definition 2.23 (Morphism of Locally Ringed Spaces). A morphism of locally ringed space $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed space such that for all $x \in X$, the indcued map on stalks $f_x^{\#} := g_x \circ (f^{\#})_{f(x)} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map, where $g_x : (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ is the canonical map given by the universal property of the colimit.

Remark 2.24. g_x above is given by (in the following $V \subseteq Y$ open and $U \subseteq X$ open)

$$g_x : (f_*\mathcal{O}_X)_{f(x)} = \varinjlim_{f(x) \in V} f_*\mathcal{O}_X(V) = \varinjlim_{x \in f^{-1}(V)} \mathcal{O}_X(f^{-1}(V)) \to \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}.$$

Proposition 2.25 (Structure Sheaf of Spec). *Let R be a ring. Then there is a unique (up to sheaf isomorphism) sheaf $\mathcal{O}_{\mathrm{Spec}R}$ of rings on $\mathrm{Spec}R$ (called the structure sheaf of $\mathrm{Spec}R$) such that $\mathcal{O}_{\mathrm{Spec}R}(D(f))$ is isomorphic as rings to the localization R_f for every $f \in R$. Furtheremore, $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ is a locally ringed space.*

Definition 2.26 (**Affine Schemes**)**.** An affine scheme is a locally ringed space which is isomorphic as locally ringed spaces to $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ for some ring R.

Definition 2.27 (**Morphism of Affine Schemes**)**.** A morphism of affine schemes is a morphism of locally ringed spaces.

Theorem 2.28. *The category of affine schemes is equivalent to the opposite category of rings.*

Definition 2.29 (**Scheme**)**.** A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $x \in X$, there is an open subset $U \subseteq X$ containing x such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Definition 2.30 (**Morphism of Schemes**)**.** A morphism of schemes is a morphism of locally ringed spaces.

Definition 2.31 (**Open Subscheme**)**.** Let $(X, \mathcal{O}_X)$ be a scheme. Then an open subscheme of $(X, \mathcal{O}_X)$ is a scheme $(U, \mathcal{O}_X|_U)$ for some open subspace U of X.

Notation 2.32. From now on, we will write a scheme as X rather than $(X, \mathcal{O}_X)$ for simplicity.

2.3 Base Change

Definition 2.33 (**Scheme over a Scheme**)**.** Let S be a scheme. A scheme over S is a scheme X together with a morphism of schemes $X \to S$ (called the structure morphism).

Definition 2.34 (**Morphism of Schemes over a Scheme**)**.** Let S be a scheme. A morphism from a scheme X over S with the strcture morphsim $f_X : X \to S$ to a scheme Y over S with $f_Y : Y \to S$ is a morphism of schemes $f : X \to Y$ such that $f_Y \circ f = f_X$.

Proposition 2.35. $\mathrm{Spec}\mathbb{Z}$ *is the terminal object in the category of schemes. Or equivalently, the category of schemes is isomorphic to the category of schemes over* $\mathrm{Spec}\mathbb{Z}$*.*

Definition 2.36 (**Fibre Product**)**.** Let S be a scheme, and X, Y be schemes over S. The fibre product $X \times_S Y$ of X and Y is the product of them in the category of schemes over S.

Proposition 2.37. *The fibre product of schemes X and Y over a scheme S always exists.*

Definition 2.38 (**Diagonal Morphism**)**.** Let X be a scheme over a scheme S. The diagonal morphism $\Delta_{X/S}$ of X over S is the natural morphism $X \to X \times_S X$ in the universal property defining the product.

Definition 2.39 (**Base Change**)**.** Let X and S' be schemes over a scheme S. The base change of X to S' is the scheme $X \times_S S'$ over S', where the structure morphism $X \times_S S' \to S'$ is the natural one.

Remark 2.40 (**Scheme over a Ring**)**.** A scheme over a ring R is a scheme over the scheme $\mathrm{Spec}(R)$. For schemes X, Y and a ring R, $X \times_R Y$ means $X \times_{\mathrm{Spec}(R)} Y$.

Definition 2.41 (**Residue Field**)**.** Let X be a scheme and $x \in X$. Take an affine open $U = \mathrm{Spec}(R)$ containing x. Then x corresponds to a prime ideal $\mathfrak{p} \subseteq R$. The residue field $\kappa(x)$ of the point x on the scheme X is the residue field $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Remark 2.42. The residue field of a point on a scheme does not depend on the choice of U.

Definition 2.43 (**Fibre**)**.** Let X be a scheme over a scheme S with the structure morphism $f : X \to S$, $p \in S$, and view $\mathrm{Spec}\kappa(p)$ as a scheme over S with the inclusion morphism $\mathrm{Spec}\kappa(p) \to S$. The fibre of f over p is the base change of X to $\kappa(p)$, i.e. $X \times_S \mathrm{Spec}\kappa(p)$ over $\mathrm{Spec}\kappa(p)$.

Proposition 2.44. *Let X be a scheme over a scheme S with the structure morphism $f : X \to S$, and $p \in S$. Then the underlying topological space of the fibre $X \times_S \mathrm{Spec}\kappa(p)$ of f over p is homeomorphic to the set-theoretic fibre $f^{-1}(p) \subseteq X$ endowed with the subspace topology.*

3 Properties of Schemes

3.1 Topological Properties

Definition 3.1 (Irreducible). A scheme is irreducible if its underlying topological space is irreducible.

Definition 3.2 (Dimension). The dimension of a scheme is the dimension of its underlying topological space.

Definition 3.3 (Quasi-Compact). A scheme is quasi-compact if its underlying topological space is quasi-compact.

3.2 Algebraic Properties

Definition 3.4 (Reduced). A scheme X is reduced if for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is a reduced ring.

Definition 3.5 (Integral). A scheme X is integral if it is nonempty and for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral domain.

Proposition 3.6. *A scheme is integral if and only if it is reduced and irreducible.*

Lemma 3.7. *Let X be an integral scheme. Then the fractional fields of $\mathcal{O}_X(U)$ for affine open U are all the same, and furtheremore isomorphic to $\mathcal{O}_{X,x}$ for the generic point $x \in X$.*

Definition 3.8 (Function Field). Let X be an integral scheme. The function field of X is the fractional field of $\mathcal{O}_X(U)$ for an affine open U.

Remark 3.9. By the above lemma, the function field does not depend on the choice of U.

Definition 3.10 (Locally Noetherian). A scheme is locally Noetherian if it can be covered by affine open subsets $\mathrm{Spec} A_i$ where A_i are all Noetherian rings.

Definition 3.11 (Noetherian). A scheme is Noetherian if it is locally Noetherian and quasi-compact (hence we can choose finite number of such affine open subsets in the above definition).

Definition 3.12 (Normal). A scheme X is normal if every stalk $\mathcal{O}_{X,x}$ is a normal ring.

Definition 3.13 (Regular). A scheme X is regular if every stalk $\mathcal{O}_{X,x}$ is a regular local ring.

Definition 3.14 (Dedekind). A Dedekind scheme is a normal and locally Noetherian scheme of dimension 0 or 1.

4 Properties of Morphisms of Schemes

4.1 Topological Properties

Definition 4.1 (Quasi-Compact). A morphism of schemes is quasi-compact if its underlying map of topological spaces is quasi-compact.

Definition 4.2 (Closed). A morphism of schemes is closed if its underlying map of topological spaces is closed.

Definition 4.3 (Universally Closed). A morphism of schemes $f : X \to Y$ is universally closed if the natural morphism $X \times_Y Z \to Z$ is closed for all morphisms of schemes $Z \to Y$.

Definition 4.4 (Open Immersion). A morphism of schemes $f : X \to Y$ is an open immersion if f is a homeomorphism onto an open subset of Y, and $f_\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

Proposition 4.5 (Open Immersion). *A morphism of schemes $f : X \to Y$ is an open immersion if and only if there exists an open subscheme U of X such that $f = i \circ g$ where $g : X \to U$ is an isomorphism of schemes and $i : U \to Y$ is the inclusion morphism.*

Definition 4.6 (**Closed Immersion**). A morphism of schemes $f : X \to Y$ is a closed immersion if f is a homeomorphism onto a closed subset of Y, and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Definition 4.7 (**Immersion**). A morphism of schemes $f : X \to Y$ is an immersion if $f = g \circ h$ for some open immersion g and closed immersion h.

Definition 4.8 (**Quasi-Separated**). A morphism of schemes $f : X \to Y$ is quasi-separated if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is quasi-compact.

Definition 4.9 (**Separated**). A morphism of schemes $f : X \to Y$ is separated if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion.

4.2 Algebraic Properties

Definition 4.10 (**Affine**). A morphism of schemes is affine if the preimage of an affine open set is affine open.

Definition 4.11 (**Finite**). A morphism of schemes $f : X \to Y$ is finite if it is affine and for every affine open $\mathrm{Spec}(R) = V \subseteq Y$ with preimage $\mathrm{Spec}(A) = f^{-1}(V) \subseteq X$, the corresponding ring homomorphism $R \to A$ is finite.

Definition 4.12 (**Locally of Finite Type**). A morphism of schemes $f : X \to Y$ is locally of finite type if there is an affine open covering of Y such that for each $V = \mathrm{Spec} R$ in the covering, $f^{-1}(V)$ can be covered by affine open subsets $U_i = \mathrm{Spec} A_i$ where every corresponding ring homomorphism $R \to A_i$ is of finite type.

Definition 4.13 (**of Finite Type**). A morphism of schemes is of finite type if it is locally of finite type and quasi-compact (hence we can choose finite number of U_j in the above definition).

Definition 4.14 (**Quasi-Finite**). A morphism of schemes $f : X \to Y$ is quasi-finite if it is of finite type and for each $x \in X$, the structure morphism of the fibre of f over $f(x)$ (i.e., $X \times_Y \mathrm{Spec}(\kappa(f(x))) \to \mathrm{Spec}(\kappa(f(x)))$) is finite.

Theorem 4.15 (**Zariski Main Theorem**). *Let $f : X \to Y$ be a quasi-finite and separated morphism of schemes where Y is quasi-compact and the unique morphism $Y \to \mathrm{Spec}\mathbb{Z}$ is quasi-separated. Then $f = g \circ h$ for some open immersion h and finite morphism g.*

Definition 4.16 (**Flat**). A morphism of schemes $f : X \to Y$ is flat if every induced ring homomorphism between stalks is flat (i.e., $\forall x \in X, \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat).

Definition 4.17 (**Proper**). A morphism of schemes $f : X \to Y$ is proper if it is separated, of finite type and universally closed.

Proposition 4.18 (**Valuative Criterion**). *Let X, Y be locally Noetherian schemes, $f : X \to Y$ be a morphism of finite type. Then*

- *f is separated if and only if there is at most one dotted arrow such that the new diagram commutes for any DVR R and any commutative diagram with solid arrows;*
- *f is universally closed if and only if there is at least one dotted arrow such that $\cdots$;*
- *f is proper if and only if there is a unique dotted arrow such that $\cdots$.*

The diagram mentioned above refers to the following diagram:

$$\begin{tikzcd} \mathrm{Spec}(\mathrm{Frac}(R)) \arrow[r] \arrow[d, "i_*"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

(where i_* is the morphism induced by the inclusion homomorphism $i : R \to \mathrm{Frac}(R)$.)

5 $\mathcal{O}_X$-modules

5.1 Definitions

Definition 5.1 (**$\mathcal{O}_X$-modules**). Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian groups on X together with a morphism of sheaves of sets $m : (F_2 \circ \mathcal{O}_X) \times (F_1 \circ \mathcal{F}) \to F_1 \circ \mathcal{F}$ (where $F_1 : \mathbf{Ab} \to \mathbf{Set}$ and $F_2 : \mathbf{Ring} \to \mathbf{Set}$ are forgetful functors) such that the map of sets $m(U)$ makes $\mathcal{F}(U)$ a module over the ring $\mathcal{O}_X(U)$ for each open $U \subseteq X$.

Remark 5.2. Note that m is a morphism of sheaves, i.e., a natural transformation. So there is a commutative diagram hidden in the above definition.

Definition 5.3 (**Morphism of $\mathcal{O}_X$-modules**). Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{O}_X$-modules. A morphism from an $\mathcal{F}$ to $\mathcal{G}$ is a morphism of sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes

$$\begin{array}{ccc} (F_2 \circ \mathcal{O}_X) \times (F_1 \circ \mathcal{F}) & \xrightarrow{m_{\mathcal{F}}} & F_1 \circ \mathcal{F} \\ {\scriptstyle \mathrm{id}\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (F_2 \circ \mathcal{O}_X) \times (F_1 \circ \mathcal{G}) & \xrightarrow{m_{\mathcal{G}}} & F_1 \circ \mathcal{G} \end{array}$$

Proposition 5.4 (**Sheaf Associated to Modules**). *Let R be a ring, M be an R-module, and $X = \mathrm{Spec}R$ be an affine scheme. Then there is a unique (up to $\mathcal{O}_X$-module isomorphism) $\mathcal{O}_X$-module $\tilde{M}$ (called the sheaf associated to the module M) such that $\tilde{M}(D(f))$ is isomorphic as abelian groups to $M \otimes_A A_f$ for every $f \in R$.*

5.2 Properties

Definition 5.5 (**Quasi-Coherent Sheaf**). A quasi-coherent sheaf on a scheme X is an $\mathcal{O}_X$-module $\mathcal{F}$ such that there is an affine open cover $\{U_i\}_i$ with $U_i = \mathrm{Spec}R_i$ such that for each i, $\mathcal{F}|_{U_i}$ is isomorphic as $\mathcal{O}_X|_{U_i}$-modules to the sheaf $\tilde{M}_i$ associated to some R_i-module M_i.

Definition 5.6 (**Coherent Sheaf**). A coherent sheaf is a quasi-coherent sheaf such that each M_i is finitely generated as R_i-module in the above definition.

Proposition 5.7. *Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is a quasi-coherent sheaf if and only if for every affine open $U = \mathrm{Spec}R \subseteq X$, $\mathcal{F}|_U$ is isomorphic as $\mathcal{O}_X|_U$-modules to the sheaf $\tilde{M}$ associated to some R-module M.*

Theorem 5.8. *Let R be a ring. Then the category of quasi-coherent sheaves on* $\mathrm{Spec}R$ *is equivalent to the category of R-modules.*

6 Variety

Definition 6.1 (**Geometric Point**). A geometric point of a scheme X over a field k is a morphism of schemes $\mathrm{Spec}(\overline{k}) \to X$.

Definition 6.2 (**Geometrically Integral**). A scheme X over a field k is geometrically integral if its base change to $\overline{k}$ is integral (i.e., $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is integral).

Definition 6.3 (**Variety**). A variety over a field k is a scheme X over k which is integral, and the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

Definition 6.4 (**Algebraic Group**). An algebraic group over a field k is a variety over k and also a group such that the multiplication and the inverse are morphisms of schemes.

Definition 6.5 (**Complete**). A variety X over a field k is complete if the structure morphism $X \to \mathrm{Spec}(k)$ is proper.

Definition 6.6 (**Curve**). A curve is a variety of dimension 1.

Definition 6.7 (**Group Scheme**). A group scheme over a scheme S is a group object in the category of schemes over S.

Definition 6.8 ($\mathbb{G}_m$)**.** The group scheme $\mathbb{G}_m$ over a field k is $\mathrm{Spec}(k[x, x^{-1}])$.

Definition 6.9 (**Abelian Variety**)**.** An abelian variety over a field k is a group scheme over k and also a geometrically integral complete variety over k.