Dictionary of Algebraic Geometry

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In this document, the word "ring" means a commutative ring with 1.
Some good reference:

- The Stacks Project
- Wikipedia: Glossary of algebraic geometry

1 Topology And Algebra

1.1 Topology

Lemma 1.1 (Gluing). *Let U be an open subset of a topological space X, $\{U_i\}_i$ be an open covering of U, and for each i, let f_i be a continuous map $U_i \to \mathbb{R}$. If $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there exists a unique continuous map $f : X \to \mathbb{R}$ such that $f|_{U_i} = f_i$ for each i.*

Definition 1.2 (Product). Let X and Y be topological spaces. The product $X \times Y$ of X and Y is the product of them in the category of topological spaces.

Lemma 1.3. *The product of topological spacecs X and Y always exists.*

Definition 1.4 (Diagonal Map). The diagonal map of a topological space X is the natural morphism $X \to X \times X$ in the universal property defining the product.

Remark 1.5. The underlying set of $X \times Y$ can be chosen as the Cartesian product of them and the diagonal map $X \to X \times X$ is just $x \mapsto (x, x)$ as you may expect.

Definition 1.6 (Irreducible). A topological space is irreducible if it is nonempty and cannot be the union of two proper closed subsets.

Definition 1.7 (Generic Point, Special Point). Let Z be an irreducible closed subset of a topological space X. Then a generic point of Z is a point in Z whose closure is Z. A special point of Z is a closed point in Z.

Definition 1.8 (Dimension). The dimension of a topological space is the supreme of the length n of the chains $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ where all Z_i are irreducible closed subspaces.

Definition 1.9 (Noetherian). A topological space is Noetherian if every ascending chain of open subsets terminates.

Definition 1.10 (Quasi-Compact Space). A topological space is quasi-compact if every open cover admits a finite subcover.

Remark 1.11. Algebra geometers like to use the strange word "quasi-compact" which just means "compact" elsewhere. Well, that's probably because they use the word "compact" to mean "compact" + "Hausdorff".

Definition 1.12 (Quasi-Compact Map). A continuous map between topological spaces is quasi-compact if the preimage of a quasi-compact open set is quasi-compact.

Definition 1.13 (Open, Closed). A map $f : X \to Y$ between topological spaces X, Y is open (closed resp.) if the image of an open (closed resp.) subset of X is open (closed resp.) in Y.

Definition 1.14 (Hausdorff). A topological space is Hausdorff if its diagonal map is closed.

Remark 1.15. This definition of Hausdorff spaces is equivalent to the ordinary definition in terms of the separation of open sets.

1.2 Algebra

Definition 1.16 (**Spectrum**)**.** Let R be a ring. The spectrum $\mathrm{Spec}(R)$ of R is a topological space, where its underlying set is $\{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal of } R\}$, and the closed sets are the sets of the form $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$ for some ideal I of R.

Proposition 1.17 (**Distinguished Set**)**.** *Let R be a ring. The distinguished sets $\{D(f) | f \in R\}$ form an open basis of $\mathrm{Spec}(R)$, where $D(f) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\}$.*

Proposition 1.18. *Let $\{R_i\}_i$ be a set of rings. Then $\mathrm{Spec}(\prod_i R_i)$ is homeomorphic to $\bigsqcup \mathrm{Spec} R_i$.*

Definition 1.19 (**Local Ring**)**.** A local ring is a ring R with a unique maximal ideal $\mathfrak{m}$, normally written as $(R, \mathfrak{m})$.

Definition 1.20 (**Local Ring Map**)**.** A local ring map φ from a local ring $(R, \mathfrak{m})$ to a local ring $(R', \mathfrak{m}')$ is a ring homomorphism such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{m}'$.

Definition 1.21 (**Residue Field of a Local Ring**)**.** The residue field $\kappa(R)$ of a local ring $(R, \mathfrak{m})$ is the field $R/\mathfrak{m}$.

Definition 1.22 (**Residue Field of a Prime Ideal**)**.** The residue field $\kappa(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of a ring R is the residue field of the localization $\kappa(R_\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

Definition 1.23 (**Algebra over a Ring**)**.** Let R be a ring. An algebra over R is a ring A together with a ring homomorphism $R \to A$ (called the structure homomorphism).

Remark 1.24. This definition of algebras is equivalent to the ordinary definition which says that an algebra over R is a ring and also an R-module that satisfies some compability.

Remark 1.25. Let $\mathfrak{p}$ be a prime ideal of a ring R. Note that $\kappa(\mathfrak{p})$ is naturally an algebra over R via the natural ring homomorphism $R \to R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = \kappa(\mathfrak{p})$.

Definition 1.26 (**Morphism of Algebras**)**.** Let R be a ring. The morphism between R-algebras (or R-algebra homomorphism) is a ring homomorphism and also an R-linear map.

Definition 1.27 (**Tensor Product**)**.** Let R be a ring and A, B be R-algebras. The tensor product $A \otimes_R B$ of A and B is the coproduct of them in the category of algebras over R.

Lemma 1.28. *The tensor product of two algebras A and B over a ring R always exists.*

Definition 1.29 (**Base Change**)**.** Let R be a ring and A, B be algebras over R. The base change of A to B is the algebra $A \otimes_R B$ over B with the structure homomorphism being the inclusion homomorphism $B \to A \otimes_R B$.

Definition 1.30 (**Fibre**)**.** Let R be a ring, A be an R-algebra with the structure homomorphism $\varphi : R \to A$, and $\mathfrak{p} \in \mathrm{Spec} R$. The fibre of φ over $\mathfrak{p}$ is the base change of A to $\kappa(\mathfrak{p})$, i.e., the algebra $A \otimes_R \kappa(\mathfrak{p})$ over $\kappa(\mathfrak{p})$.

Definition 1.31 (**Krull Dimension**)**.** The Krull dimension of a ring R is the dimension of the topological space $\mathrm{Spec}(R)$.

Definition 1.32 (**Reduced**)**.** A reduced ring is a ring which has no nonzero nilpotent elements.

Definition 1.33 (**Integral**)**.** An integral domain is a ring with $0 \neq 1$ which has no nonzero zero divisor.

Definition 1.34 (**Noetherian**)**.** A Noetherian ring is a ring such that every ascending chain of ideals terminates.

Definition 1.35 (**Normal/Integrally Closed Domain**)**.** A normal ring (integrally closed domain) is an integral domain R which is integrally closed in $K := \mathrm{Frac}(R)$ (i.e., all of the roots in K of a monic polynomial over R are in R).

Definition 1.36 (**Regular**)**.** A regular local ring is a Noetherian local ring $(R, \mathfrak{m})$ such that $\dim_{\kappa(R)} \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Definition 1.37 (Dedekind). A Dedekind domain is a normal Noetherian ring of Krull dimension 0 or 1 (i.e., every nonzero prime ideal is maximal).

Definition 1.38 (Discrete Valuation Ring/DVR). A discrete valuation ring (DVR) is a local Dedekind domain of Krull dimension 1 (i.e., not a field).

Definition 1.39 (Finite). A ring homomorphism $\varphi : R \to A$ is finite if the R-algebra A with the structure homomorphism φ is finitely generated as an R-module.

Definition 1.40 (of Finite Type). A ring homomorphism $\varphi : R \to A$ is of finite type if the R-algebra A with the structure homomorphism φ is finitely generated as an R-algebra.

Proposition 1.41. *A ring homomorphism $\varphi : R \to A$ is of finite type if and only if the R-algebra A with the structure homomorphism φ is isomorphic as R-algebras to a quotient of a polynomial ring in finitely many variables.*

Definition 1.42 (of Finite Presentation). A ring homomorphism $\varphi : R \to A$ is of finite presentation if the R-algebra A with the structure homomorphism φ is isomorphic as R-algebras to a quotient of a polynomial ring in finitely many variables by a finitely generated ideal.

Definition 1.43 (Quasi-Finite). A ring homomorphism $\varphi : R \to A$ is quasi-finite if it is of finite type and for any $\mathfrak{p} \in \operatorname{Spec}(R)$, the structure homomorphism of the fibre of φ over $\mathfrak{p}$ (i.e., the inclusion $\kappa(\mathfrak{p}) \to A \otimes_R \kappa(\mathfrak{p})$) is finite.

Definition 1.44 (Flat). A ring homomorphism $\varphi : R \to A$ is flat if the R-algebra A with the structure homomorphism φ is flat as an R-module, i.e., $- \otimes_R A : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is an exact functor.

2 Sheaf

Definition 2.1 (Open Category). Let X be a topological space. The open category $\operatorname{Open}(X)$ of X is a category whose objects are open sets of X, and whose morphisms are the inclusion maps between open sets.

Definition 2.2 (Presheaf). Let X be a topological space and C be a cocomplete category. A presheaf of C on X is a contravariant functor from $\operatorname{Open}(X)$ to C.

Remark 2.3. We often consider $C =$**Set**/**Ab**/**Ring** (the category of sets/abelian groups/rings).

Definition 2.4 (Morphism of Presheaves). A morphism between presheaves is a natural transformation between those functors.

Definition 2.5 (Section). Let $\mathcal{F}$ be a presheaf on a topological X, and $U \subseteq X$ open. Then a section of $\mathcal{F}$ over U is an element of $\mathcal{F}(U)$. A global section of $\mathcal{F}$ is an element of $\mathcal{F}(X)$.

Definition 2.6 (Restriction Map). Let $\mathcal{F}$ be a presheaf on a topological space X, open subsets $V \subseteq U \subseteq X$ with inclusion $i : V \to U$, and $s \in \mathcal{F}(U)$. Define the restriction $s|_V$ of s on V is $\mathcal{F}(i)(s) \in \mathcal{F}(V)$.

Remark 2.7. For an open $U \subseteq X$, also denote $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$. For an inclusion $i : V \to U$ of open sets, also denote $\operatorname{res}^U_V := \mathcal{F}(i) : \mathcal{F}(U) \to \mathcal{F}(V)$.

Definition 2.8 (Constant Presheaf). A constant presheaf is a presheaf which is a constant functor.

Definition 2.9 (Sheaf). A sheaf on a topological space X is a presheaf $\mathcal{F}$ on X such that the following is an equalizer for any open subset U of X and any open cover $\{U_i\}_i$ of U.

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

Remark 2.10. Concretely, a sheaf on X is essentially a presheaf on X with the gluing condition. Let U be an open subset of X, $\{U_i\}_i$ be an open cover of U, and for each i let $s_i \in \mathcal{F}(U_i)$. If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Definition 2.11 (**Morphism of Sheaves).** A morphism between sheaves is a morphism between presheaves.

Notation 2.12. Let X be a topological space and C be a cocomplete category. The category of Sheaves of C on X is denoted by $\mathbf{Sh}^C(X)$.

Definition 2.13 (**Stalk).** Let $\mathcal{F}$ be a presheaf on a topological space X, $x \in X$. Then the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is the colimit of $\mathcal{F}|_I$ (the functor $\mathcal{F}$ restricting on I) where I is the full subcategory of $\mathrm{Open}(X)$ whose objects are open subsets of X containing x (i.e., $\mathcal{F}_x = \varinjlim_{x \in U \text{ open}} \mathcal{F}(U)$).

Lemma 2.14. *The stalk of a presheaf on a topological space X at $x \in X$ always exists.*

Definition 2.15 (**Germ).** Let $\mathcal{F}$ be a presheaf on a topological space X, $x \in X$, $U \subseteq X$ an open subset containing x, and $s \in \mathcal{F}(U)$. The universal property of the colimit gives us the canonical map $\mathcal{F}_x^U : \mathcal{F}(U) \to \mathcal{F}_x$. The germ s_x of s at x is $\mathcal{F}_x^U(s) \in \mathcal{F}_x$.

Definition 2.16 (**Induced Map on Stalks).** Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X, $x \in X$. By the universal property of colimit, there is a unique map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ such that the following diagram commutes for any open $U \subseteq X$ containing x. The map f_x is called the induced map of f on stalks.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, "\mathcal{F}_x^U"'] & \mathcal{G}(U) \arrow[d, "\mathcal{G}_x^U"] \\
\mathcal{F}_x \arrow[r, "f_x"'] & \mathcal{G}_x
\end{tikzcd}$$

Theorem 2.17. *A morphism of sheaves that induces isomorphisms on all stalks is an isomorphism.*

Definition 2.18 (**Sheafification).** The sheafification is a functor that is left adjoint to the inclusion functor from the category of sheaves to the category of presheaves (So the sheafification of a presheaf is a sheaf).

Lemma 2.19. *The sheafification always exists.*

Definition 2.20 (**Constant Sheaf).** A constant sheaf is the sheafification of a constant presheaf.

Definition 2.21 (**Direct Image/Pushforward).** Let X, Y be topological spaces, $\mathcal{F}$ be a sheaf on X, and $f : X \to Y$ a continuous map. The direct image (pushforward) $f_*\mathcal{F}$ of $\mathcal{F}$ by f is a presheaf on Y that sends an open $U \subseteq Y$ to $\mathcal{F}(f^{-1}(U))$, and sends an inclusion $V \to U$ to $\mathcal{F}(j)$ where j is the inclusion $f^{-1}(V) \to f^{-1}(U)$.

Proposition 2.22. *The direct image of a sheaf is also a sheaf, which makes the direct image f_* actually a functor from the category of sheaves on X to the category of sheaves on Y.*

Definition 2.23 (**Inverse Image/Pullback).** Let $f : X \to Y$ be a continuous map between topological spaces X, Y. The inverse image (pullback) f^{-1} is the left adjoint functor to the direct image f_* (So the inverse image $f^{-1}(\mathcal{G})$ of a sheaf $\mathcal{G}$ on Y is a sheaf on X).

Lemma 2.24. *The inverse image of a sheaf always exists.*

Definition 2.25 (**Restriction of Sheaf).** Let $\mathcal{F}$ be a presheaf on a topological space X and $U \subseteq X$ an open subset of X. Let $\mathrm{Open}(U)$ be the open category of U (subspace topology), which is naturally the subcategory of $\mathrm{Open}(X)$. Then the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ on U is $\mathcal{F}|_{\mathrm{Open}(U)}$.

Theorem 2.26. *Let X be a topological space. The category $\mathbf{Sh}^{\mathbf{Ab}}(X)$ of sheaves of abelian groups on X can be defined naturally as a pre-additive, and furthermore an abelian category.*

3 Scheme

3.1 Definition of a Scheme

Definition 3.1 (Ringed Space). A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space (called the underlying topological space) and $\mathcal{O}_X$ is a sheaf of rings (means a sheaf of the category of rings) on X (called the structure sheaf).

Definition 3.2 (Morphism of Ringed Spaces). A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f^{\text{top}}, f^{\#})$ where $f^{\text{top}} : X \to Y$ is a continuous map and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves on Y.

Remark 3.3. Since the inverse image f^{-1} is the left adjoint functor to the direct image $f^{\#}$, a morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y naturally corresponds to a morphism $f_\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X, and vice versa.

Definition 3.4 (Locally Ringed Space). A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x is always a local ring.

Definition 3.5 (Morphism of Locally Ringed Spaces). A morphism of locally ringed space $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed space such that for all $x \in X$, the indcued map on stalks $f^{\#}_x := g_x \circ (f^{\#})_{f(x)} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map, where $g_x : (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ is the canonical map given by the universal property of the colimit.

Remark 3.6. g_x above is given by (in the following $V \subseteq Y$ open and $U \subseteq X$ open)

$$g_x : (f_*\mathcal{O}_X)_{f(x)} = \varinjlim_{f(x)\in V} f_*\mathcal{O}_X(V) = \varinjlim_{x\in f^{-1}(V)} \mathcal{O}_X(f^{-1}(V)) \to \varinjlim_{x\in U} \mathcal{O}_X(U) = \mathcal{O}_{X,x}.$$

Proposition 3.7 (Structure Sheaf of Spec). *Let R be a ring. Then there is a unique (up to sheaf isomorphism) sheaf $\mathcal{O}_{\mathrm{Spec}R}$ of rings on* $\mathrm{Spec}R$ *(called the structure sheaf of* $\mathrm{Spec}R$*) such that $\mathcal{O}_{\mathrm{Spec}R}(D(f))$ is isomorphic as rings to the localization R_f for every $f \in R$. Furtheremore,* $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ *is a locally ringed space.*

Definition 3.8 (Affine Schemes). An affine scheme is a locally ringed space which is isomorphic as locally ringed spaces to $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ for some ring R.

Definition 3.9 (Morphism of Affine Schemes). A morphism of affine schemes is a morphism of locally ringed spaces.

Theorem 3.10. *The functor* $\mathrm{Spec} : R \mapsto (\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ *defines an equivalence from the opposite category of rings to the category of affine schemes.*

Remark 3.11. As a result of the theorem above, we can identify an affine scheme X as $\mathrm{Spec}R$ for some ring R, and a ring homomorphism $R \to A$ is in $1-1$ correspondence to a morphism of affine schemes $\mathrm{Spec}A \to \mathrm{Spec}R$.

Definition 3.12 (Scheme). A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $x \in X$, there is an open subset $U \subseteq X$ containing x such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Definition 3.13 (Morphism of Schemes). A morphism of schemes is a morphism of locally ringed spaces.

Definition 3.14 (Open Subscheme). Let $(X, \mathcal{O}_X)$ be a scheme. Then an open subscheme of $(X, \mathcal{O}_X)$ is a scheme $(U, \mathcal{O}_X|_U)$ for some open subspace U of X.

Notation 3.15. The category of schemes is denoted by **Sch**. From now on, we will write a scheme as X rather than $(X, \mathcal{O}_X)$ for simplicity.

3.2 Base Change

Definition 3.16 (Scheme over a Scheme). Let S be a scheme. A scheme over S is a scheme X together with a morphism of schemes $X \to S$ (called the structure morphism).

Definition 3.17 (**Morphism of Schemes over a Scheme**)**.** Let S be a scheme. A morphism from a scheme X over S with the strcture morphsim $f_X : X \to S$ to a scheme Y over S with $f_Y : Y \to S$ is a morphism of schemes $f : X \to Y$ such that $f_Y \circ f = f_X$.

Remark 3.18 (**Scheme over a Ring**)**.** A scheme over a ring R is a scheme over the scheme $\mathrm{Spec}(R)$. For schemes X, Y and a ring R, $X \times_R Y$ means $X \times_{\mathrm{Spec}(R)} Y$.

Notation 3.19. Let S be a scheme and R be a ring. The category of schemes over S is denoted by $\mathbf{Sch}/S$, and the category of schemes over R is denoted by $\mathbf{Sch}/R$.

Proposition 3.20. $\mathrm{Spec}\mathbb{Z}$ *is the terminal object in the category of schemes. Or equivalently, the category* $\mathbf{Sch}$ *is isomorphic to* $\mathbf{Sch}/\mathbb{Z}$.

Proposition 3.21. *Let X be a scheme over a ring R. Then the structure morphism* $X \to \mathrm{Spec}R$ *makes the structure sheaf $\mathcal{O}_X$ actually a sheaf of R-algebras on X.*

Theorem 3.22. *The functor of taking the global sections* $\Gamma : \mathbf{Sch} \to \mathbf{Alg}_R^{\mathrm{op}}, X \mapsto \Gamma(X, \mathcal{O}_X)$ *is the left adjoint of the functor* $\mathrm{Spec} : \mathbf{Alg}_R^{\mathrm{op}} \to \mathbf{Sch}, A \mapsto \mathrm{Spec}(A)$. *In particular, the global sections* $\Gamma(X, \mathcal{O}_X)$ *is naturally bijective to* $\mathrm{Hom}_{\mathbf{Sch}/R}(X, \mathrm{Spec}(R[x]))$.

Definition 3.23 (**Functor of Points**)**.** Let X be a scheme over a scheme S. Its functor of points is the contravariant Hom functor $\mathrm{Hom}_{\mathbf{Sch}/S}(\bullet, X)$ from $\mathbf{Sch}/S$ to $\mathbf{Set}$, which is representable by the scheme X over S.

Theorem 3.24. *Let S be a scheme. Sending a scheme over S to its functor of points yields an equivalence from the category* $\mathbf{Sch}/S$ *to the category of representable contravariant functors from* $\mathbf{Sch}/S$ *to* $\mathbf{Set}$ *(where the morphism is the natural transformation).*

Definition 3.25 (**Fibre Product**)**.** Let S be a scheme, and X, Y be schemes over S. The fibre product $X \times_S Y$ of X and Y is the product of them in the category of schemes over S.

Lemma 3.26. *The fibre product of schemes X and Y over a scheme S always exists.*

Theorem 3.27. *Let R be a ring. The functor* $\mathrm{Spec} : A \mapsto \mathrm{Spec}(A)$ *defines an equivalence from the opposite category of R-algebras to the category of affine schemes over R, taking tensor product to fibre product.*

Definition 3.28 (**Base Change**)**.** Let X and S' be schemes over a scheme S. The base change of X to S' is the scheme $X \times_S S'$ over S', where the structure morphism $X \times_S S' \to S'$ is the natural one.

Theorem 3.29. *Let X and S' be schemes over a scheme S, and let* $F_X : \mathbf{Sch}/S \to \mathbf{Set}$ *be the functors of points of X. Let F be the natural functor* $\mathbf{Sch}/S' \to \mathbf{Sch}/S$. *Then the functor of points of the base change of X to S' is* $F_X \circ F : \mathbf{Sch}/S' \to \mathbf{Set}$.

Definition 3.30 (**Diagonal Morphism**)**.** Let X be a scheme over a scheme S. The diagonal morphism $\Delta_{X/S}$ of X over S is the natural morphism $X \to X \times_S X$ in the universal property defining the product.

Definition 3.31 (**Section**)**.** A section of a scheme X over a scheme S with the structure morphism $f : X \to S$ is a morphism $g : S \to X$ over S (where S is viewed as the scheme over itself with the identity morphism), i.e., a morphism $g : S \to X$ with $f \circ g = \mathrm{id}_S$.

Definition 3.32 (**Residue Field**)**.** Let X be a scheme and $x \in X$. Take an affine open $U = \mathrm{Spec}(R)$ containing x. Then x corresponds to a prime ideal $\mathfrak{p} \subseteq R$. The residue field $\kappa(x)$ of the point x on the scheme X is the residue field $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Remark 3.33. The residue field of a point on a scheme does not depend on the choice of U.

Definition 3.34 (**Fibre**)**.** Let X be a scheme over a scheme S with the structure morphism $f : X \to S$, $p \in S$, and view $\mathrm{Spec}\kappa(p)$ as a scheme over S with the inclusion morphism $\mathrm{Spec}\kappa(p) \to S$. The fibre X_p of f over p is the base change of X to $\kappa(p)$, i.e. $X \times_S \mathrm{Spec}\kappa(p)$ over $\mathrm{Spec}\kappa(p)$.

Proposition 3.35. *Let X be a scheme over a scheme S with the structure morphism $f : X \to S$, and $p \in S$. Then the underlying topological space of the fibre* $X \times_S \mathrm{Spec}\kappa(p)$ *of f over p is homeomorphic to the set-theoretic fibre $f^{-1}(p) \subseteq X$ endowed with the subspace topology.*

Definition 3.36 (**Generic Fibre, Special Fibre**)**.** A generic fibre (special fibre resp.) of a morphism of schemes f is a fibre of f over a generic point (special point resp.).

4 Properties of Schemes

4.1 Topological Properties

Definition 4.1 (Irreducible). A scheme is irreducible if its underlying topological space is irreducible.

Definition 4.2 (Dimension). The dimension of a scheme is the dimension of its underlying topological space.

Definition 4.3 (Quasi-Compact). A scheme is quasi-compact if its underlying topological space is quasi-compact.

4.2 Algebraic Properties

The Stacks Project: Types of schemes defined by properties of rings

Definition 4.4 (Reduced). A scheme X is reduced if for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is a reduced ring.

Definition 4.5 (Integral). A scheme X is integral if it is nonempty and for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral domain.

Proposition 4.6. *A scheme is integral if and only if it is reduced and irreducible.*

Lemma 4.7. *Let X be an integral scheme. Then the fractional fields of $\mathcal{O}_X(U)$ for affine open U are all the same, and furtheremore isomorphic to $\mathcal{O}_{X,x}$ for the generic point $x \in X$.*

Definition 4.8 (Function Field). Let X be an integral scheme. The function field of X is the fractional field of $\mathcal{O}_X(U)$ for an affine open U.

Remark 4.9. By the above lemma, the function field does not depend on the choice of U.

Definition 4.10 (Locally Noetherian). A scheme is locally Noetherian if it can be covered by affine open subsets $\mathrm{Spec} A_i$ where A_i are all Noetherian rings.

Definition 4.11 (Noetherian). A scheme is Noetherian if it is locally Noetherian and quasi-compact (hence we can choose finite number of such affine open subsets in the above definition).

Definition 4.12 (Normal). A scheme X is normal if every stalk $\mathcal{O}_{X,x}$ is a normal ring.

Definition 4.13 (Regular). A scheme X is regular if every stalk $\mathcal{O}_{X,x}$ is a regular local ring.

Definition 4.14 (Dedekind). A Dedekind scheme is a normal and locally Noetherian scheme of dimension 0 or 1.

5 Properties of Morphisms of Schemes

5.1 Topological Properties

Definition 5.1 (Quasi-Compact). A morphism of schemes is quasi-compact if its underlying map of topological spaces is quasi-compact.

Definition 5.2 (Closed). A morphism of schemes is closed if its underlying map of topological spaces is closed.

Definition 5.3 (Universally Closed). A morphism of schemes $f : X \to Y$ is universally closed if the natural morphism $X \times_Y Z \to Z$ is closed for all morphisms of schemes $Z \to Y$.

Definition 5.4 (Open Immersion). A morphism of schemes $f : X \to Y$ is an open immersion if f is a homeomorphism onto an open subset of Y, and $f_\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

Proposition 5.5 (Open Immersion). *A morphism of schemes $f : X \to Y$ is an open immersion if and only if there exists an open subscheme U of X such that $f = i \circ g$ where $g : X \to U$ is an isomorphism of schemes and $i : U \to Y$ is the inclusion morphism.*

Definition 5.6 (**Closed Immersion**). A morphism of schemes $f : X \to Y$ is a closed immersion if f is a homeomorphism onto a closed subset of Y, and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Definition 5.7 (**Immersion**). A morphism of schemes $f : X \to Y$ is an immersion if $f = g \circ h$ for some open immersion g and closed immersion h.

Definition 5.8 (**Quasi-Separated**). A morphism of schemes $f : X \to Y$ is quasi-separated if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is quasi-compact.

Definition 5.9 (**Separated**). A morphism of schemes $f : X \to Y$ is separated if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion.

5.2 Algebraic Properties

The Stacks Project: Types of morphisms defined by properties of ring maps

Definition 5.10 (**Affine**). A morphism of schemes is affine if the preimage of an affine open set is affine open.

Definition 5.11 (**Finite**). A morphism of schemes $f : X \to Y$ is finite if it is affine and for every affine open $\mathrm{Spec}(R) = V \subseteq Y$ with preimage $\mathrm{Spec}(A) = f^{-1}(V) \subseteq X$, the corresponding ring homomorphism $R \to A$ is finite.

Definition 5.12 (**Locally of Finite Type**). A morphism of schemes $f : X \to Y$ is locally of finite type if there is an affine open covering of Y such that for each $V = \mathrm{Spec} R$ in the covering, $f^{-1}(V)$ can be covered by affine open subsets $U_i = \mathrm{Spec} A_i$ where every corresponding ring homomorphism $R \to A_i$ is of finite type.

Definition 5.13 (**of Finite Type**). A morphism of schemes is of finite type if it is locally of finite type and quasi-compact (hence we can choose finite number of U_j in the above definition).

Definition 5.14 (**Locally of Finite Presentation**). A morphism of schemes is locally of finite presentation if the ring homomorphism $R \to A_i$ in Definition 5.12 is of finite presentation.

Definition 5.15 (**of Finite Presentation**). A morphism of schemes is of finite presentation if it is locally of finite presentation, quasi-compact and quasi-separated.

Definition 5.16 (**Locally of Quasi-Finite**). A morphism of schemes is locally of quasi-finite if it is locally of finite type and the underlying topological space of every fibre is discrete.

Remark 5.17. By Proposition 3.35, we don't need to distinguish whether the "fibre" in the above definition means the scheme-theoretic fibre X_y or means the set-theoretic fibre $f^{-1}(y)$.

Definition 5.18 (**Quasi-Finite**). A morphism of schemes is quasi-finite if it is locally of quasi-finite and quasi-compact (equivalently, of finite type and the underlying topological space of every fibre has finite elements).

Theorem 5.19 (**Zariski Main Theorem**). *Let $f : X \to Y$ be a quasi-finite and separated morphism of schemes where Y is quasi-compact and the unique morphism $Y \to \mathrm{Spec}\mathbb{Z}$ is quasi-separated. Then $f = g \circ h$ for some open immersion h and finite morphism g.*

Definition 5.20 (**Flat**). A morphism of schemes $f : X \to Y$ is flat if every induced ring homomorphism between stalks is flat (i.e., $\forall x \in X, \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat).

Definition 5.21 (**Proper**). A morphism of schemes $f : X \to Y$ is proper if it is separated, of finite type and universally closed.

Proposition 5.22 (**Valuative Criterion**). *Let X, Y be locally Noetherian schemes, $f : X \to Y$ be a morphism of finite type. Then*

- *f is separated if and only if there is at most one dotted arrow such that the new diagram commutes for any DVR R and any commutative diagram with solid arrows;*
- *f is universally closed if and only if there is at least one dotted arrow such that $\cdots$;*
- *f is proper if and only if there is a unique dotted arrow such that $\cdots$.*

The diagram mentioned above refers to the following diagram:

$$\begin{tikzcd}
\mathrm{Spec}(\mathrm{Frac}(R)) \arrow[r] \arrow[d, "i_*"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(R) \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

(where i_* is the morphism induced by the inclusion homomorphism $i : R \to \mathrm{Frac}(R)$.)

6 Sheaf of Modules

6.1 Definitions

Definition 6.1 (**Mod Categroy**)**.** The category **Mod** consists of objects which are triples (R, M, m) where R is a ring, M is an abelian group, $m : R \times M \to M$ is a map making M an R-module. The morphism from (R_1, M_1, m_1) to (R_2, M_2, m_2) in **Mod** are pairs (φ, f) where $\varphi : R_1 \to R_2$ is a ring homomorphism, $f : M_1 \to M_2$ is an abelian group homomorphism such that the following diagram commutes:

$$\begin{tikzcd}
R_1 \times M_1 \arrow[r, "m_1"] \arrow[d, "\varphi\times f"'] & M_1 \arrow[d, "f"] \\
R_2 \times M_2 \arrow[r, "m_2"] & M_2
\end{tikzcd}$$

Definition 6.2 (**(Sheaf of)$\mathcal{O}_X$-modules**)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. A (sheaf of) $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of the category **Mod** on X such that for each open $U \subseteq X$, $\mathcal{F}(U) = (\mathcal{O}_X(U), \cdots, \cdots)$, and for every inclusion $i : V \hookrightarrow U$ of open sets in X, $\mathcal{F}(i) = (\mathcal{O}_X(i), \cdots)$.

Remark 6.3. For a ringed space $(X, \mathcal{O}_X)$, note that $\mathcal{O}_X$ itself is an $\mathcal{O}_X$-module because for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is trivially an $\mathcal{O}_X(U)$-module.

Definition 6.4 (**Morphism of $\mathcal{O}_X$-modules**)**.** A morphism of $\mathcal{O}_X$-modules is a morphism of sheaves of **Mod** on X.

Proposition 6.5 (**Sheaf Associated to Modules**)**.** *Let R be a ring, M be an R-module, and $X = \mathrm{Spec}R$ be an affine scheme. Then there is a unique (up to $\mathcal{O}_X$-module isomorphism) $\mathcal{O}_X$-module $\tilde{M}$ (called the sheaf associated to the module M) such that $\tilde{M}(D(f))$ is isomorphic in the category* **Mod** *to $M \otimes_R R_f$ for every $f \in R$.*

Definition 6.6 (**Tensor Product**)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of $\mathcal{F}, \mathcal{G}$ is an $\mathcal{O}_X$-module, which is the sheafification of the presheaf sending an open $U \subseteq X$ to $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Definition 6.7 (**Direct Sum**)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\{\mathcal{F}_i\}_i$ be a set of $\mathcal{O}_X$-modules. The direct sum of $\{\mathcal{F}_i\}_i$ is the coproduct of those $\mathcal{F}_i$ in the category of $\mathcal{O}_X$-modules.

Lemma 6.8. *The direct sum of $\mathcal{O}_X$-modules always exists.*

Theorem 6.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. The category of $\mathcal{O}_X$-modules is a symmetric monoidal abelian category, where the tensor product is in the sense of Definition 6.6, the addition of $\varphi, \psi \in$* **Hom**$(\mathcal{F}, \mathcal{G})$ *is defined pointwisely, and the direct sum is in the sense of Definition 6.7.*

Definition 6.10 (**Ideal Sheaf/Sheaf of Ideals**)**.** An ideal sheaf (sheaf of ideals) J on a ringed space $(X, \mathcal{O}_X)$ is a subobject of $\mathcal{O}_X$ in the abelian category of $\mathcal{O}_X$-modules.

6.2 Properties

Definition 6.11 (**Quasi-Coherent**)**.** A quasi-coherent sheaf on a scheme X is an $\mathcal{O}_X$-module $\mathcal{F}$ such that there is an affine open cover $\{U_i\}_i$ with $U_i = \mathrm{Spec}R_i$ such that for each i, $\mathcal{F}|_{U_i}$ is isomorphic as $\mathcal{O}_X|_{U_i}$-modules to the sheaf $\tilde{M}_i$ associated to some R_i-module M_i.

Definition 6.12 (**Coherent).** A coherent sheaf is a quasi-coherent sheaf such that each M_i is finitely generated as R_i-module in the above definition.

Proposition 6.13 (**Quasi-Coherent).** *Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is a quasi-coherent sheaf if and only if for every affine open $U = \mathrm{Spec}R \subseteq X$, $\mathcal{F}|_U$ is isomorphic as $\mathcal{O}_X|_U$-modules to the sheaf $\tilde{M}$ associated to some R-module M.*

Theorem 6.14. *Let X be a scheme. Then the category of quasi-coherent sheaves on X as a subcategory of $\mathcal{O}_X$-modules is an abelian category.*

Theorem 6.15. *Let R be a ring. Then the category of quasi-coherent sheaves on* $\mathrm{Spec}R$ *is equivalent to the category of R-modules.*

Definition 6.16 (**Locally Free).** Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is locally free if for every point $x \in X$, there exists an open $U \subseteq X$ such that $\mathcal{F}|_U$ is isomorphic as $\mathcal{O}_X|_U$-modules to a direct sum of some copies of $\mathcal{O}_X|_U$.

Definition 6.17 (**Locally Free of Finite Rank (or Rank n)).** Let X be a scheme. An $\mathcal{O}_X$-module is locally free of finite rank (or rank n resp.) if the direct sum is of finite (or n resp.) copies of $\mathcal{O}_X|_U$ in the above definition.

Remark 6.18. The category of locally free $\mathcal{O}_X$-modules is not an abelian catgeory.

Proposition 6.19. *A locally free $\mathcal{O}_X$-module is quasi-coherent.*

Definition 6.20 (**Invertible Sheaf/Line Bundle).** An invertible sheaf (line bundle) on a scheme X is an $\mathcal{O}_X$-module that is locally free of rank 1.

Definition 6.21 (**Support).** Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-module. The support of $\mathcal{F}$ is the subspace of X where the stalk of $\mathcal{F}$ is nonzero.

Proposition 6.22 (**Defining Ideal Sheaf).** *Let Y be a scheme. The closed subschemes of Y correspond to the quasi-coherent sheaves of ideals on Y, which is given by the followings:*

- *A sheaf of ideals $\mathcal{I}$ on Y corresponds to a closed subscheme $(X, \mathcal{O}_Y/\mathcal{I})$ where X is the support of $\mathcal{O}_Y/\mathcal{I}$, which is called the closed subscheme defined by $\mathcal{I}$.*
- *A closed immersion $i : X \to Y$ corresponds to the quasi-coherent sheaf of ideal $\mathcal{I}_{X/Y} := \ker i^\#$ on Y where $i^\# : \mathcal{O}_Y \to i_*\mathcal{O}_X$, which is called the ideal sheaf of Y defining $i(X)$.*

7 Sheaf Cohomology

Proposition 7.1. *Let X be a topological space. The category $\mathbf{Sh}^{\mathbf{Ab}}(X)$ of sheaves of abelian groups as an abelian category has enough injectives.*

Proposition 7.2. *Let X be a topological space. The functor of taking global sections of sheaves of abelian groups $\Gamma(X, \bullet) : \mathbf{Sh}^{\mathbf{Ab}}(X) \to \mathbf{Ab}$, $\mathcal{F} \to \Gamma(X, \mathcal{F})$ is left exact.*

Definition 7.3 (**Sheaf Cohomology).** Let X be a topological space. The sheaf cohomology on X is the right derived functor of the functor of taking global sections of sheaves of abelian groups $\Gamma(X, \bullet) : \mathbf{Sh}^{\mathbf{Ab}}(X) \to \mathbf{Ab}$. That is, $H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$.

Definition 7.4 (**Acyclic).** A sheaf of abelian groups is acyclic if its sheaf cohomology vanishes for every positive degree.

Remark 7.5. Concretely, the sheaf cohomology of a sheaf of abelian groups is the cohomology of its injective resolutions. It can be shown that the injective resolutions can be replaced by the acyclic resolutions in the above, and every injective sheaf is indeed acyclic.

Theorem 7.6 (**Grothendieck Vanishing).** *Let X be a topological space. If X is Noetherian with dimension $\leq d$, then $H^i(X, \mathcal{F}) = 0$ for all $i > d$ and any sheaf $\mathcal{F}$ of abelian groups.*

8 Sheaf of Differential Forms

This section is imcomplete. Please see section 6.3 of my report Tensor Categroy in Arithmetic Geometry and Physics for the complete version.

8.1 Sheaf of Relative Differentials of Degree 1

Definition 8.1 (**Derivation into a Module).** Let A be an R-algebra and M be an A-module. An R-derivation of A into M is an R-linear map $d : A \to M$ such that $d(a_1a_2) = a_1da_2 + a_2da_1$ for any $a_1, a_2 \in A$.

Definition 8.2 (**Module of Relative Differentials).** Let A be an R-algebra. The module of relative differentials of A over R is an A-module $\Omega^1_{A/R}$ together with an R-derivation $d_{A/R}$: $A \to \Omega^1_{A/R}$ of A such that for any A-module M and R-derivation $d' : A \to M$ of A, there exists a unique A-linear map $\phi : \Omega^1_{A/R} \to M$ such that $d' = \phi \circ d_{A/R}$.

Lemma 8.3. *Given an R-algebra A, $\Omega^1_{A/R}$ always exists.*

Definition 8.4 (**Derivation into a Sheaf of Modules).** https://stacks.math.columbia.edu/tag/01UN

Proposition 8.5 (**Sheaf of Relative Differentials).** *Let X be a scheme over a scheme S with the structure morphism $f : X \to S$. Then there exists a unique (up to $\mathcal{O}_X$-module isomorphism) quasi-coherent sheaf $\Omega^1_{X/S}$ on X (called the sheaf of relative differentials of X over S) together with a unique $\mathcal{O}_X$-module morphism $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ (called the universal S-derivation of X) such that for any affine open $V = \mathrm{Spec}A \subseteq Y$, any affine open $U = \mathrm{Spec}R \subseteq f^{-1}(V) \subseteq X$, and any $x \in X$, we have $\Omega^1_{X/S}|_U$ corresponds to (see Theorem 6.15) $\Omega^1_{A/R}$, $(\Omega^1_{X/S})_x \simeq \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}$ and .*

Definition 8.6. https://stacks.math.columbia.edu/tag/01CF
https://stacks.math.columbia.edu/tag/0FKL

9 Variety

Definition 9.1 (**Geometric Point).** A geometric point of a scheme X is a morphism of schemes $\mathrm{Spec}(k) \to X$ for some separably closed field k.

Definition 9.2 (**Geometrically Integral).** A scheme X over a field k is geometrically integral if its base change to $\overline{k}$ is integral (i.e., $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is integral).

Definition 9.3 (**Variety).** A variety over a field k is a scheme X over k which is integral, and the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

Definition 9.4 (**Algebraic Group).** An algebraic group over a field k is a variety over k and also a group such that the multiplication and the inverse are morphisms of schemes.

Definition 9.5 (**Complete).** A variety X over a field k is complete if the structure morphism $X \to \mathrm{Spec}(k)$ is proper.

Definition 9.6 (**Curve).** A curve is a variety of dimension 1.

Definition 9.7 (**Group Scheme).** A group scheme over a scheme S is a group object in the category of schemes over S, i.e., a scheme G over S together with three morphisms $m : G \times_S G \to G$, $e : S \to G$, $\iota : G \to G$ satisfying some commutative diagrams as the axiom of groups.

Lemma 9.8. *Let (G, m, e, ι) be a group scheme over a scheme S. For any scheme T over S, $G(T) := \mathbf{Hom}_{\mathbf{Sch}/S}(T, G)$ is a group whose operation is defined as the following. For $f_1, f_2 \in G(T)$, by the universal property of fibre product, there exists a unique morphism $f : T \to G \times_S G$. Then we define the multiplication of f_1 and f_2 in $G(T)$ is $m \circ f \in G(T)$.*

Definition 9.9 (**Morphism of Group Schemes).** A morphism from the group scheme (G_1, m_1, e_1, ι_1) to (G_2, m_2, e_2, ι_2) over a scheme S is a morphism $f : G_1 \to G_2$ over S such that for any scheme T over S, the map $G_1(T) \to G_2(T)$, $g \mapsto f \circ g$ is a group homomorphism.

Theorem 9.10. *Let G be a group scheme over a scheme S. Then the rule $(T \in \mathbf{Obj}(\mathbf{Sch}/S)) \mapsto (G(T) \in \mathbf{Obj}(\mathbf{Grp}))$, $(f \in \mathbf{Hom}_{\mathbf{Sch}/S}(T_1, T_2)) \mapsto (g \in G(T_2) \mapsto g \circ f \in G(T_1))$ is a contravariant functor from the category of schemes over S to the category of groups, whose composition with the forgetful functor to the category of sets is representable by the set $G(S)$.*

Remark 9.11. There are three ways viewing affine group schemes over a field k:

- as group objects in the category of affine group schemes over k (by definition);
- as representable functors from the category of k-algebras to the category of groups (by Theorem 9.10 and Theorem 3.27);
- as (the spectrum of) commutative Hopf algebras over k (where the comultiplication, counit, and the antipode are given by the reverse of m, e, ι respectively. For example, say $G = \mathrm{Spec}A$, then the morphism of schemes $m : G \times_k G \to G$ corresponds to a ring homomorphism $A \to A \otimes_k A$, which is the comultiplication in the Hopf algebra).

Definition 9.12 (Algebraic Group). An algebraic group over a field k is a group scheme over k and also a variety over k.

Lemma 9.13. *An affine group scheme $G = \mathrm{Spec}A$ over a field k is an algebraic group over k if and only if the k-algebra A is finitely generated as a k-algebra.*

Definition 9.14 (Additive Group). The additive group $\mathbb{G}_a$ over a field k is the algebraic group $\mathrm{Spec}(k[x])$ over k.

Definition 9.15 (Multiplicative Group). The multiplicative group $\mathbb{G}_m$ over a field k is the algebraic group $\mathrm{Spec}(k[x, x^{-1}])$ over k.

Definition 9.16 (General Linear Group). The general linear group $\mathbb{GL}_n$ over a field k is the algebraic group $\mathrm{Spec}\left(\frac{k[t_{11}, \cdots, t_{nn}, y]}{(\det A)y - 1}\right)$ over k where A is the matrix $\begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix}$.

Definition 9.17 (Linear Algebraic Group). A linear algebraic group over a field k is an algebraic group over k which is a closed immersion into the general linear group $\mathbb{GL}_n$ over k.

Proposition 9.18. *An algebraic group over a field k is a linear algebraic group over k if and only if it is affine.*

Proposition 9.19. *An affine group scheme over a field k is a limit of some affine algebraic groups over k in the category of affine group schemes over k.*

Definition 9.20 (Abelian Variety). An abelian variety over a field k is a group scheme over k and also a geometrically integral complete variety over k.

10 Étale Fundamental Group

10.1 Étale Cover

Definition 10.1 (Étale Morphism). A morphism of schemes $f : X \to Y$ is étale if it is flat, locally of finite presentation, and for every $y \in Y$, the fibre X_y of f over y is an affine scheme over $\mathrm{Spec}\kappa(y)$ corresponding to (see Theorem 3.27) a $\kappa(y)$-algebra which is a product of finite seprable extensions of $\kappa(y)$.

Remark 10.2. The fibre condition in the above definition can be replaced by the geometric fibre condition: for every geometric point $\overline{y} : \mathrm{Spec}k \to Y$, the geometric fibre $X_{\overline{y}}$ of f over $\overline{y}$ is an affine scheme over $\mathrm{Spec}k$ corresponding to a k-algebra which is a direct product of copies of k.

Lemma 10.3. *A finite morphism of schemes is quasi-finite (and hence the underlying topological space of every fibre has finite elements).*

Proposition 10.4. *An étale morphism is locally of quasi-finite (so each fibre is discrete), and a finite étale morphism is quasi-finite (so each fibre has finite elements).*

Definition 10.5 (Finite Locally Free). A morphism of schemes $f : X \to Y$ is finite locally free if f is affine and $f_*\mathcal{O}_X$ is an $\mathcal{O}_Y$-module that is locally free of finite rank.

Proposition 10.6. *A morphism of schemes is finite locally free if and only if it is finite, flat, and locally of finite presentation.*

Remark 10.7. By Proposition 1.18, the spectrum of a finite product is the finite disjoint union of the spectrum. Then by Proposition 10.6 and Lemma 10.3, a morphism $f : X \to Y$ is finite étale if and only if it is finite locally free and every fibre X_y is a disjoint union of finite points, each of which is the spectrum of finite separable extension of $\kappa(y)$.

Definition 10.8 (Étale Ring Homomorphism). A ring homomorphism $\varphi : R \to A$ is étale if the corresponding morphism of schemes $\mathrm{Spec}A \to \mathrm{Spec}R$ is étale.

Corollary 10.9. *A ring homomorphism $\varphi : R \to A$ is finite étale if and only if it is finite, flat, of finite presentation, and for any $\mathfrak{p} \in \mathrm{Spec}R$, the fibre $A \otimes_R \kappa(\mathfrak{p})$ of φ over $\mathfrak{p}$ is isomorphic as $\kappa(\mathfrak{p})$-algebras to a finite product of finite separable extensions of $\kappa(\mathfrak{p})$ (or equivalently, every geometric fibre $A \otimes_R \overline{\kappa(\mathfrak{p})}$ is isomorphic as $\overline{\kappa(\mathfrak{p})}$-algebras to a direct product of copies of $\overline{\kappa(\mathfrak{p})}$).*

Remark 10.10. In the case that $R = k$ is a field, $\mathfrak{p} \in \mathrm{Spec}k$ can only be the zero ideal, and hence necessarily $\kappa(\mathfrak{p}) = k$, $A \otimes_R \kappa(\mathfrak{p}) = A$ in the above corollary.

Lemma 10.11. *Suppose that X is an nonempty scheme. Then a finite locally free morphism of schemes $f : X \to Y$ is surjective if and only if Y is connected.*

Definition 10.12 (Finte Étale Cover). An finite étale cover over a scheme S is a scheme X over S with the structure morphism being a (surjective) finite étale morphism $f : X \to S$.

Remark 10.13. Some literature require the surjectivity condition in the above definition whereas others do not. By Lemma 10.11, the surjectivity is satisfied most of the time. So we don't need to worry about it too much.

11 Site and Stack

Definition 11.1 (Site/Grothendieck Topology). A site consists of the data of

- a category C;
- for every object X in C, a set $\mathrm{Cov}(X)$ of families of morphisms in C,

that satisfies the following properties:

- If $\phi : X' \to X$ is an isomorphism in C, then $\{\phi : X' \to X\} \in \mathrm{Cov}(X)$;
- If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism in C, then the fibre product $X_i \times_X Y$ exists, and $\{X_i \times_X Y \to X\}_{i\in I} \in \mathrm{Cov}(X)$;
- If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j\in J}$ for each $i \in I$, then $\{X_{ij} \to X\}_{(i,j)\in I\times J} \in \mathrm{Cov}(X)$.

Definition 11.2 (Sheaf on a Site). A sheaf $\mathcal{F}$ on a site $(C, \{\mathrm{Cov}(X)\}_{X\in\mathrm{Obj}(C)})$ is a contravariant functor from C to **Set** such that for any $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$, the following is an equalizer.

$$\mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j)$$