Net of Conics and Cubic Curves

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Abstract

In this report we study smooth three dimensional hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 2)$. We show, via a detailed study of cubic curves, that, under generality conditions, these hypersurfaces (and therefore nets of conics) can be put into one of three standard forms by linear change of coordinates.

Contents

1 Introduction

Generally speaking, one may be interested in projective geometry as it simplifies the language and process of Euclidean geometry, particularly when we consider projective space over an algebraically closed field such as $\mathbb{C}$, as we do in this report. You will see this in Section 2, where we look at definitions that we need for this report. For example, in projective geometry, we know that we can express all conics as

$$\begin{pmatrix} x & y & z \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

By utilising classical results from linear algebra such as the Gramm-Schmidtt algorithm, we are able to completely classify all conics into one of three types, up to projective transformations, based on the rank of M; as you will see in Section 3. Comparatively, in $\mathbb{R}^2$ we have an extensive list of conics up to affine transformations. This demonstrates our motivation to use projective geometry in order to simplify our understanding of such objects.

In Section 4, we apply the same reasoning to families of conics, called pencils of conics which are exactly the one-dimensional vector subspaces of the defining polynomial. Naturally, our next consideration is one degree higher. In Section 5 we consider projective transformations of cubic curves in $\mathbb{P}^2_{\mathbb{C}}$ and the standard forms of these including the Weierstrass and Legendre forms and also their classifications. As an alternative way, we can put the cubic into the form

$$x^3 + y^3 + z^3 + \lambda xyz = 0$$

for some constant λ. This is called the Hesse form and in Section 6 we show that every smooth cubic curve on $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to this form.

Then main result of this report follows in Section 7 and concerns the classification of nets of conics which is a more difficult problem. The paper 'Nets of Conics and associated Artinian algebras of length 7' [1] dives into it in more detail, and was only published in 2021 after being found by Ivan Cheltsov, who encouraged the authors to translate and publish it. There are several ways to approach this, we may view these objects as a vector space or as a hypersurface of degree $(1, 2)$ in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$. We restrict our view, as before, to only smooth cases. This object is exactly a Fano threefold in family 2-24, in particular it is a three dimensional generalisation of Del Pezzo surfaces, and they play an important role in modern day algebraic geometry. Our primary aim is to apply a similar classification method as has been done in the case of conics and cubics.

Main Theorem. [2] Let X be a smooth surface with bi-degree $(1, 2)$ in

$\mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz}$, i.e. X is represented by

$$F(u, v, w; x, y, z) = uf_2(x, y, z) + vg_2(x, y, z) + wh_2(x, y, z) = 0.$$

Then we can choose coordinate such that X is given by one of the following equations:

- $(\mu yz + x^2)u + (\mu xz + y^2)v + (\mu xy + z^2)w = 0$ for some $\mu \in \mathbb{C}$ such that $\mu^3 \neq -1$
- $(yz + x^2)u + (xz + y^2)v + z^2 w = 0$
- $(yz + x^2)u + y^2 v + z^2 w = 0$

Our technique of proving the main theorem is considering the cubic discriminant curve $C_3 \subseteq \mathbb{P}^2_{uvw}$ represented by $G(u, v, w) := \det \mathrm{Hess}(F)$ as a homogeneous polynomial of u, v, w with degree 3, where

$$\mathrm{Hess}(F) = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}.$$

If X is smooth, then it can be shown that C_3 is one of the following:

- a smooth cubic curve,
- an irreducible nodal cubic curve,
- a union of three lines that do not intersect at one point,
- a union of an irreducible conic and a line that does not intersect the conic tangentially.

Let us give a brief overview of next steps of our proof. We have an equation for a Fano 3-fold in $\mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz}$. We employ the same method as we did when classifying conics and view the 3-fold as given by

$$\begin{pmatrix} u & v & w \end{pmatrix} M \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0.$$

However, M is now a 3×3 matrix with all entries being polynomials of degree 1 in x, y, z. So, we know that $\det(M)$ is a polynomial of degree three in x, y, z which we simplify by changing coordinates at first, then putting the resulting cubic into the Hesse form.

We also explore simplifying three dimensional threefolds in the Fano threefold 3-13 in Section 8, which is a subset X in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ with coordinates that satisfy the system of equations

$$\begin{cases} \mathbf{x}^T M_1 \mathbf{y} = 0 \\ \mathbf{y}^T M_2 \mathbf{z} = 0 \\ \mathbf{z}^T M_3 \mathbf{x} = 0 \end{cases},$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{P}^2_{\mathbb{C}}$ and M_1, M_2, M_3 are 3×3 matrices. One can simplify it either by reducing to the case $M_1 = M_2 = \mathbb{1}$, or by choosing appropriate M_1, M_2, M_3 such that the system has S_3 symmetry.

2 Preliminary: Projective Geometry

Many of the following definitions can be found in the book 'Undergraduate Algebraic Geometry' by Miles Ried [9].

2.1 Projective Plane

The protective plane, $\mathbb{P}^2 = (k^3 \setminus (0,0,0))/\sim$, is the set of equivalence classes of nonzero points in k^3, where k is an algebraically closed field and the equivalence relation is given by

$$[x : y : z] \sim [\lambda x : \lambda y : \lambda z]$$

for nonzero $\lambda \in k$. Let $U \subset \mathbb{P}^2 := \{[x : y : z] : z \neq 0\}$, there is a bijective map from $U \mapsto k^2$ given by:

$$[x : y : z] \sim \left[\frac{x}{z} : \frac{y}{z} : 1\right] \longleftrightarrow \left(\frac{x}{z}, \frac{y}{z}\right)$$

And for the complement $\mathbb{P}^2 \setminus U \subset \mathbb{P}^2 := \{[x : y : z] : z = 0, (x, y) \neq (0, 0)\}$, there is a 1:1 correspondence between points in $\mathbb{P}^2 \setminus U$ and points in $\mathbb{P}^1$ given by:

$$[x : y : 0] \longleftrightarrow [x : y].$$

So, $\mathbb{P}^2$ is the union of U and a *line at infinity.*

Throughout this report, we only consider the case $k = \mathbb{C}$.

2.2 Lines in the Projective Plane

In order to discuss problems in projective spaces, we first consider the simplest geometric object in $\mathbb{P}^2_{\mathbb{C}}$. We know a line from the definition of projective plane $L : z = 0$, which is the line at infinity. Consequently, we can define a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ to be the set of equivalence classes which solve the equation:

$$ax + by + cz = 0, \ [x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$$

where $(a, b, c) \neq (0, 0, 0) \in \mathbb{C}^3$. Then there is a natural bijection,

$$[a : b : c] \longleftrightarrow \{[x : y : z] : ax + by + cz = 0\}.$$

The line in projective space is determined uniquely by the point $[a : b : c]$. Notice that given two lines $L_1, L_2 \in \mathbb{P}^2_{\mathbb{C}}$, either $L_1 = L_2$ or $|L_1 \cap L_2| = 1$, since we can find the solution of equations:

$$\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{aligned}$$

2.3 Conics in the Projective Plane

A conic in $\mathbb{P}^2_{\mathbb{C}}$ is the subset $\mathcal{C} \in \mathbb{P}^2_{\mathbb{C}}$ given by solutions of the equation

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0,\ [x, y, z] \in \mathbb{P}^2_{\mathbb{C}}$$

where $(a, b, c, d, e, f) \neq (0, 0, 0, 0, 0, 0) \in \mathbb{C}^6$. We can find a bijecitve map if we encode the equation in the symmetric matrix,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 \longleftrightarrow \mathbf{x}B\mathbf{x}^T$$

where $\mathbf{x} = [x : y : x]$ and B is the symmetric matrix

$$B = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

A conic $\mathcal{C} \in \mathbb{P}^2_{\mathbb{C}}$ is irreducible if and only if $\det(B) \neq 0$. If a conic is not irreducible, we call it *degenerate*.
An important geometric property of curves is smoothness. Suppose $\mathcal{C} \in \mathbb{P}^2_{\mathbb{C}}$ is an irreducible curve. A point $P = [x : y : z]$ on $\mathcal{C}$ is smooth if the gradient

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\bigg|_p \neq (0, 0, 0)$$

A curve is smooth if every point is smooth [10].

2.4 Projective Transformations

Working with curves in projective plane, projective transformations help us to simplify equations of curves. A projective transformation is a map $\phi : \mathbb{P}^2_{\mathbb{C}} \mapsto \mathbb{P}^2_{\mathbb{C}}$ that maps

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where A is an invertible 3×3 matrix. The composition of two projective transformations is also a projective transformation.

If there exist a projective transformation ϕ such that $\phi(\mathcal{C}_1) = \mathcal{C}_2$, then $\mathcal{C}_1, \mathcal{C}_2$ are *projectively equivalent*.

3 Classification of Conics

It is useful and interesting to note that in projective geometry when considering objects, in this case, conics, we can often simplify and classify them in a way which helps us study and gives us a better understanding.

The following theorem demonstrates this exactly, as we note that any conics fall into one of the three categories.

An arbitrary conic $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$ has the defining equation of the form

$$f_1(x,y,z) = c_1x^2 + c_2xy + c_3y^2 + c_4xz + c_5yz + c_6z^2 = 0$$

for $(c_1, c_2, c_3, c_4, c_5, c_6) \in \mathbb{C}^6$.
Recall that the *Hessian* of f is the 3×3 symmetric matrix

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z \partial z} \end{pmatrix}.$$

Note here that the polynomial $f_1(x,y,z) = 0$ is as above, and that we have:

$$h(f_1)(x,y,z) = \mathbf{x}H(f_1)\mathbf{x}^T,$$

where $\mathbf{x} = [x : y : z]$=0.

Theorem 1. (Classification of Conics) Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then, up to projective transformation, $\mathcal{C}$ is of the form

1. $x^2 + y^2 + z^2 = 0$ (an irreducible conic);
2. $xy = 0$ (a reducible conic of two distinct lines); or
3. $x^2 = 0$ (a reducible conic consisting of a double line).

Proof. Let us work systematically through this list.

1. Suppose $\mathcal{C}$ is given by the irreducible polynomial $f_1(x,y,z) = 0$. We know that

$$h(f_1)(x,y,z) = \mathbf{x}H\mathbf{x}^T = 0$$

also characterises $\mathcal{C}$, where $\mathbf{x} = [x : y : z]$.
By results from linear algebra, there must exist a basis of $\mathbb{C}^3$ so that we can take a coordinate system (u, v, w), and with respect to this basis we have

$$\mathbf{x}H(f_1)\mathbf{x}^T = \mathbf{u}\mathbb{I}_3\mathbf{u}^T = u^2 + v^2 + w^2 = 0$$

where $\mathbf{u} = [u : v : w]$, and $\mathbb{I}_3$ is the 3×3 identity matrix. It is clearly irreducible. Given that a change of basis (in $\mathbb{C}^3$) is a linear transformation (of $\mathbb{C}^3$), we automatically get an induced projective transformation (in $\mathbb{P}^2_{\mathbb{C}}$). So we are done in this case.

Another convenient form of this is $xy = z^2$, where we change the coordinate of x, y, z to be $x = \frac{x'+y'}{2}$, $y = \frac{i(x'-y')}{2}$, and $z = iz'$.

2. Suppose now that the polynomial representation of $\mathcal{C}$ is reducible, then it can be written as;

$$f_1(x, y, z) = g_1(x, y, z)h_1(x, y, z) = 0$$

where h_1 and g_1 are non-zero, degree-one polynomials such that $h_1 \neq \lambda g_1$ for $\lambda \in \mathbb{C}$. Then we have two lines: $L_1 : h_1 = 0$ and $L_2 : g_1 = 0$ where $\mathcal{C} = L_1 \cup L_2$. We know that there exists a unique intersection point $P \in \mathbb{P}^2_{\mathbb{C}}$, such that $P = L_1 \cap L_2$.

Applying a proper projective transformation, we may assume that $L_1 : x = 0$, $P = [0 : 0 : 1]$, $L_2 : ax + by + cz = 0$ and $L_2 \neq L_1$. Since $P \in L_2$, we have $c = 0$ and the equation of $\mathcal{C} = L_1 \cup L_2$ is now reduced to $x(ax + by) = 0$. By the linear independent condition $h_1 \neq \lambda g_1$, we have $b \neq 0$. By another projective transformation $x = u, y = -(a/b)u + (1/b)v, z = w$, the equation of $L_1 \cup L_2$ is reduced to $uv = 0$ and so we are done.

3. Finally, suppose $\mathcal{C}$ can be written as:

$$f_1(x, y, z) = (f_2(x, y, z))^2$$

where f_2 is a non-zero, degree-one polynomial. Then our conic is uniquely defined by a double line L given by $f_2 = 0$ (counted with multiplicity).

Consider distinct points $P_1, P_2 \in L$. Now we can find a projective transformation $\xi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $P_1 = [0 : 1 : -1]$ and $P_2 = [0 : 1 : 1]$. From here it is clear that L is given by $x = 0$ and consequently that $\mathcal{C}$ is given by $x^2 = 0$. So, we are done.

□

4 Pencil of Conics

Throughout this report, we work over the algebraically closed field $\mathbb{C}$. Let $\mathcal{Q}$ be the 6-dimensional vector space of the symmetric bilinear form over $\mathbb{C}^3$ and $\mathbb{P}_{\mathcal{Q}}$ the associated 5-dimensional projective space (up to scaling)[5]. An element of $\mathbb{P}_{\mathcal{Q}}$ is a conic section over $\mathbb{P}^2_{\mathbb{C}}$.

We define a pencil of conics in $\mathbb{P}_{\mathcal{Q}}$ to be a 1-dimensional linear system. Since we know that any conic is uniquely determined by five points (no three collinear) in a plane, we can say that the set of conics which pass through exactly four of these five is exactly a pencil of conics. So, a pencil is uniquely described by any pair of distinct conics as any two conics intersect in four points (counted with multiplicity) and these four points specify a pencil.

Algebraically, we may represent a pencil of conics as a linear combination of two distinct conics in a projective plane over an algebraically closed field. Precisely, for every $\lambda, \mu \in \mathbb{C}$ not both zero we have that the polynomial

$$\lambda f_1 + \mu f_2$$

represents a conic in the pencil which is determined by $\mathcal{C}_1 : f_1 = 0$ and $\mathcal{C}_2 : f_2 = 0$.

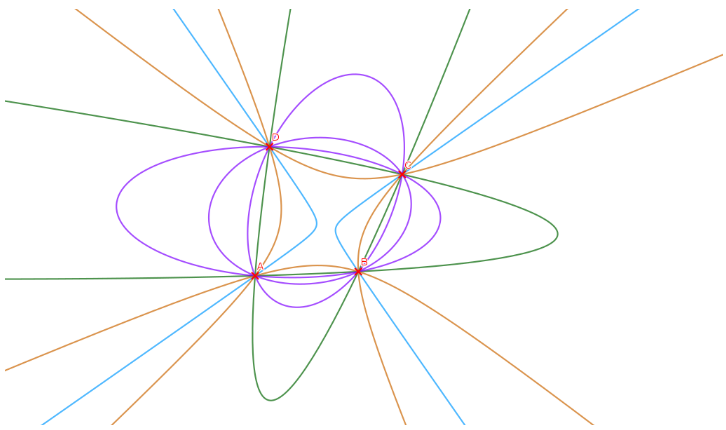

Figure 1: The set of conics pass through four points in the figure. This is in the real plane as it is difficult to visualize the complex pencils. [8]

4.1 Features of Pencils of Conics

We know that symmetric bilinear forms can be represented at 3×3 matrices and thus the expression above can also be represented as such. Since we know that a conic is irreducible if and only if the determinant of the corresponding matrix is non-zero, it is clear that a conic in a specified pencil is irreducible if and only if

$$\det(\lambda M_1 + \mu M_2) \neq 0.$$

where $M_1 = \mathrm{Hess}(f_1), M_2 = \mathrm{Hess}(f_2)$ are 3×3 matrices representing the conics $\mathcal{C}_1, \mathcal{C}_2$ respectively.

Proposition 2 ([5]). *Suppose we have a pencil of conics $\mathcal{P}$ in $\mathbb{P}^2_{\mathbb{C}}$ with at least one non-degenerate conic. Then the pencil has at most 3 degenerate conics.*

Proof. This follows directly from the fact that a cubic has at least three roots. □

4.2 Classification of Pencils of Conics

We have known that four intersection points (called base locus) of two conics specify a pencil, given that f_1, f_2 do not have common factors. Thus, the classification of pencils depends on the positions of base locus. There are five cases for base locus:

1. $\{P, Q, R, S\}$;
2. $\{2P, Q, R\}$;
3. $\{3P, Q\}$;
4. $\{2P, 2Q\}$;
5. $\{4P\}$.

We do a case-by-case discussion:

- **Case 1** $\{P, Q, R, S\}$**:** Since f_1, f_2 do not share common factors, no three points of P, Q, R, S lie on a line. So we assume $P = [1 : 0 : 0]$, $Q = [0 : 1 : 0]$, $R = [0 : 0 : 1]$, $S = [1 : 1 : 1]$ by a projective transformation.
  It is easy to check that the four points impose an independent condition on conics (by writing out the coefficient matrix), so the dimension of the solution space is 2. We find that $\mathcal{C}_1 : (x - z)y = 0$ and $\mathcal{C}_2 : (x - y)z = 0$ are two linearly independent conics that pass through the four points. Thus, in this case, the pencil is generated by $\mathcal{C}_1, \mathcal{C}_2$.
  Finally, $g(\lambda) := \det(\lambda M_1 + M_2) = -2\lambda(\lambda + 1)$ has two roots. Plus that $\mathcal{C}_1$ itself is degenerate, we have that in this pencil, there are precisely three degenerate conics.

- **Case 2** $\{2P, Q, R\}$**:** In this case, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at Q, R transversely and at P with tangent line L. Choose a point $S \neq P$ on the line L. Then no three points of P, Q, R, S lie on a line. We assume $P = [0 : 0 : 1]$, $Q = [1 : 0 : 0]$, $R = [1 : 1 : 1]$, $S = [0 : 1 : 0]$ by a projective transformation.
  Then the tangent line L is given by $x = 0$. By writing out $f(x, y, z) = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$ and considering the system $\nabla f|_P = [1 : 0 : 0]$, $f(P) = f(R) = f(S) = 0$, we have $A = E = F = 0$ and $D = -B - C$. Thus, the two linearly independent solutions are given by $\mathcal{C}_1 : x(y - z) = 0$ and $\mathcal{C}_2 : y^2 - xz = 0$, and thus the pencil is generated by $\mathcal{C}_1, \mathcal{C}_2$.
  Finally, $g(\lambda) := \det(\lambda M_1 + M_2) = -2(\lambda + 1)^2$ has one root. Plus

that $\mathcal{C}_1$ is degenerate, there are precisely two degenerate conics in this pencil.

- **Case 3** $\{3P, Q\}$**:** In this case, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at Q transversely and at P with tangent line L, where the intersection multiplicity $I_P(\mathcal{C}_1, \mathcal{C}_2)$ at P is 3. Choose a point $R \neq P$ on the line L. Then P, Q, R don't lie on a line. We assume $P = [0 : 0 : 1]$, $Q = [1 : 0 : 0]$, $R = [0 : 1 : 0]$ and hence L is given by $x = 0$. By writing out $f(x, y, z) = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$ and considering the system $\nabla f|_P = [1 : 0 : 0]$, $f(P) = f(R) = 0$, we have $A = E = F = 0$.
  Furthermore, not all $\mathcal{C}_1, \mathcal{C}_2$ with $A = E = F = 0$ satisfy $I_P(\mathcal{C}_1, \mathcal{C}_2) = 3$ (For instance, if $\mathcal{C}_1 : y^2 - xz = 0$ and $\mathcal{C}_2 : xz = 0$, then $I_P(\mathcal{C}_1, \mathcal{C}_2) = 2$). So the dimension of the solution space is no more than 2. Note that $\mathcal{C}_1 : xy = 0$ and $\mathcal{C}_2 : y^2 - xz = 0$ satisfy $A = E = F = 0$ and $I_P(\mathcal{C}_1, \mathcal{C}_2) = 3$. So the pencil is generated by $\mathcal{C}_1, \mathcal{C}_2$.
  Finally, $g(\lambda) := \det(\lambda M_1 + M_2) = -2 \neq 0$, so the only degenerate conic in this pencil is $\mathcal{C}_1$.

- **Case 4** $\{2P, 2Q\}$**:** In this case, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at P, Q with tangent lines L_1, L_2 respectively. Choose $R \neq P$ on L_1 and $S \neq Q$ on L_2. Then no three of P, Q, R, S lie on a line. We assume $P = [0 : 0 : 1]$, $Q = [1 : 0 : 0]$, $R = [0 : 1 : 0]$, $S = [1 : 1 : 1]$. So $L_1 : x = 0$ and $L_2 : y - z = 0$.
  By writing out $f(x, y, z) = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$ and considering the system $\nabla f|_P = [1 : 0 : 0]$, $\nabla f|_Q = [0 : 1 : -1]$, $f(P) = f(Q) = 0$, we have $A = E = F = 0$ and $B = -D$. Thus, the two linearly independent solutions are given by $\mathcal{C}_1 : x(y - z) = 0$ and $\mathcal{C}_2 : y^2 = 0$.
  Finally, $g(\lambda) := \det(\lambda M_1 + M_2) = -2\lambda^2$ has one root. Plus that $\mathcal{C}_1$ is degenerate, there are 2 degenerate conics in this pencil.

- **Case 5** $\{4P\}$**:** In this case, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at P with the tangent line L. Assume $P = [0 : 0 : 1]$ and $L : x = 0$. By writing out $f(x, y, z) = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$ and considering the system $\nabla f|_P = [1 : 0 : 0]$, $f(P) = 0$, we have $E = F = 0$. We split it into two sub-cases.
  - **Case 5.1** If the pencil contains at least one irreducible conic, suppose $\mathcal{C}_1 : y^2 - xz = 0$. Since the intersection multiplicity

    $$\begin{aligned} 4 = I_P(\mathcal{C}_1, \mathcal{C}_2) &= \dim \frac{k[[x, y]]}{(y^2 - x, Ax^2 + Bxy + Cy^2 + Dx)} \\ &= \dim \frac{k[[y]]}{(Ay^4 + By^3 + (C + D)y^2)}, \end{aligned}$$

    we have $B = C + D = 0$. So two linearly independent solutions are $\mathcal{C}_1$ itself and $\mathcal{C}_2 : x^2 = 0$. So the pencil is generated by $\mathcal{C}_1$ and

$\mathcal{C}_2$. Finally, $g(\lambda) := \det(\lambda M_1 + M_2) = -2\lambda^3$ has one root. Since $\mathcal{C}_1$ is irreducible, the only degenerate conic in the pencil is $\mathcal{C}_2$.

– **Case 5.2** If all conics in the pencil are degenerate, we assume $\mathcal{C}_1$ has a component $x = 0$ and $\mathcal{C}_2$ has a component $y = 0$, and thus $\mathcal{C}_1 : x(ax + by) = 0$ and $\mathcal{C}_2 : y(cx + dy) = 0$ by a projective transformation. So the pencil is given by

$$\begin{aligned}&\lambda x(ax + by) + \mu y(cx + dy)\\ =&a\lambda \left(x + \frac{b\lambda + c\mu}{2a\lambda} y\right)^2 + \left(d\mu - \frac{(b\lambda + c\mu)^2}{4a\lambda}\right) y^2 = 0.\end{aligned}$$

By changing coordinates, it becomes $\lambda x^2 + \mu y^2 = 0$. So we pick $\mathcal{C}_1 : x^2 = 0$, $\mathcal{C}_2 : y^2 = 0$, and the pencil is generated by $\mathcal{C}_1$, $\mathcal{C}_2$.

5 Classification of Cubics

Before we begin, we define the notion of inflection points.

Definition 3. Let $\mathcal{C} \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a curve. We say that a point $P \in \mathcal{C}$ is an inflection point, if it is smooth and the intersection multiplicity $I_P(L, C) \geq 3$, where L is the tangent line at P of C.

By Bezout's theorem, we note that the inequality above is actually an equality $I_P(L, C) = 3$ if $\mathcal{C}$ is a cubic curve.

5.1 Smooth Cubic Curves

For the very first step of classifying cubic curves, we consider the smooth ones and put them into the 'standard forms'.

Theorem 4. Any smooth cubic in $\mathbb{P}^2_{\mathbb{C}}$ can be put in the *Weierstrass form*,

$$y^2 z = x^3 + axz^2 + bz^3 \tag{1}$$

by a projective transformation.

Proof. Consider an arbitrary cubic $\mathcal{C}_3$ in $\mathbb{P}^2_{\mathbb{C}}$, then it has the equation

$$\mathcal{C}_3 : Ax^3 + Bx^2z + Cxz^2 + Dz^3 + Eyz^2 + Fxyz + Gx^2y + Hxy^2 + Iy^2z + Jy^3 = 0$$

Since $\mathcal{C}_3$ is a smooth cubic with $\deg(\mathcal{C}_3) = 3$ and $\mathbb{C}$ is an algebraically closed field, it follows that $\mathcal{C}_3$ has 9 distinct inflection points (see Lemma 11). Let us call one of the inflection points P and we may assume (for simplicity) that $P = [0 : 1 : 0]$ (after a projective transformation) and the tangent line at P is $z = 0$. Now, our general equation for $\mathcal{C}_3$ is simplified in the following way

- $P \in \mathcal{C}_3$ forces $J = 0$
- The gradient at P is given by $(H, 0, I)$, Clearly, $H = 0$ in order to arrive at the specified tangent line. Also, since P is smooth we need $I \neq 0$, so up to projective transformation, we may assume $I = -1$
- Hess($\mathcal{C}_3$) at P is

$$\begin{vmatrix} 2G & 0 & F \\ 0 & 0 & 2I \\ F & 2I & 2E \end{vmatrix} = -8GI^2$$

  Since P is an inflection point, det(Hess($\mathcal{C}_3$)) at $P = 0$, which gives that $G = 0$.

Putting this information back into our equation for $\mathcal{C}_3$, we get :

$$y^2z = Ax^3 + Bx^2z + Cxz^2 + Dz^3 + Eyz^2 + Fxyz$$

Recall the we assume $\mathcal{C}_3$ to be smooth so A is non-zero. The following steps simplify the equation to the desired form [12]:

1. Change of variables $x \mapsto x, y \mapsto y + \alpha x + \beta z, z \mapsto z$ fixes E = F = 0;
2. Change of variables $x \mapsto x + \gamma z, y \mapsto y, z \mapsto z$ fixes $B = 0$;
3. Change of variables $x \mapsto x, y \mapsto y, z \mapsto \delta z$ fixes $A = 1$.

This is exactly the *Weierstrass form* (1). □

We can arrive at the *Legendre form* using the *Weierstrass form* of a cubic, another useful form for smooth cubic curves.

Lemma 5. Let $\mathcal{C}_3 \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth cubic. There exists a projective transformation which takes it to the *Legendre form*:

$$y^2z = x(x - z)(x - \lambda z) \tag{2}$$

where $\lambda \notin \{0, 1\}$.

Proof. Take the Weierstrass form of $\mathcal{C}_3$, a change of variables $x \mapsto x + \alpha z$ puts one of the roots of the right hand side of the *Weierstrass form* (1) at $x = 0$. Rescaling once more, we arrive at a second root at $x = z$. □

So far we have classified smooth cubics. We will now consider possible classifications of singular irreducible cubics and reducible cubics.

5.2 Singular Irreducible Cubic Curves

In Section 5.1, we looked at reducible cubic curves. Following on from this, we find out that we can classify singular irreducible cubic curves into two forms.

Theorem 6. Suppose $\mathcal{C}_3 \subset \mathbb{P}^2_{\mathbb{C}}$ is a singular irreducible conic. Then it is projectively equivalent to one of the following forms:

1. $zy^2 = x^2(x+z)$
2. $zy^2 = x^3$

Proof. Take the singular point to be $P = [0 : 0 : 1]$ (up to projective transformation). Then, we may simplify the general form of a cubic curve to arrive at

$$ax^3 + by^3 + xc^2y + dxy^2 = z(ex^2 + fy^2 + gxy).$$

Now, suppose $(ex^2 + fy^2 + gxy)$ is a perfect square (i.e. the quadratic has a repeated root). We can choose this root to be $y = 0$ by projective transformation. This means we can choose $f = 1$, $e = g = 0$. Apply transformations $z \mapsto z + \alpha x + \beta y$ and $x \mapsto x + \gamma y$ we have $zy^2 = x^3$.

Now consider the case that $(ex^2 + fy^2 + gxy)$ is not a square, therefore, we have two distinct roots. Up to projective transformation, we may choose these roots to be $x = 0$, $y = 0$, so we can assume $e = f = 0$ and $g = 1$. A transformation on z gives the curve $xyz = ax^3 + by^3$. If either a or b is zero, we have a reducible cubic (contradicting our assumption that $\mathcal{C}_3$ is a singular irreducible conic). Therefore, both of a, b are non-zero, and we get $xyz = x^3 + y^3$. We arrive at $zy^2 = x^2(x+z)$ by substituting:

- $x \mapsto \frac{(x-y)}{2}$
- $y \mapsto \frac{(x+y)}{2}$
- $z \mapsto -4z - 3x$.

$\square$

Due to the shape of the resultant graph (see Figure 2), we call cubics of the form in Therorem 6.1 ($zy^2 = x^2(x+z)$), *nodal cubics* and cubics of the form in Therorem 6.2 ($zy^2 = x^3$) *cuspidal cubics.*

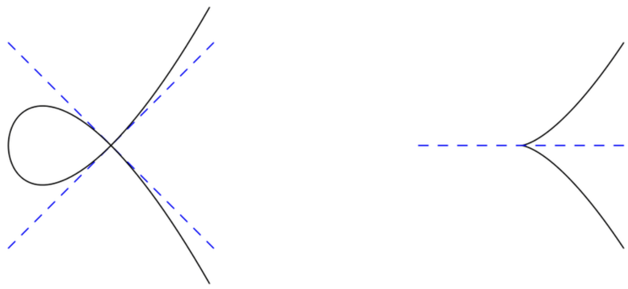

Figure 2: Real part of Nodal and Cuspidal Cubic (left to right) and their tangent lines [6].

Example 7. Two cubic curves are projectively equivalent if, and only if we can find a projective transformation between them. As an explicit example of Theorem 6,

$$zy^2 + \lambda x^2 y + \mu x^3 = 0 \quad \text{and} \quad zy^2 + x^3 = 0 \qquad (\mu \neq 0)$$

are projectively equivalent. Actually, for the cubic $zy^2 + \lambda x^2 y + \mu x^3 = 0$, if we substitute

$$\begin{aligned} x &= \frac{1}{\sqrt[3]{\mu}} x' - \frac{\lambda}{3\mu} y', \\ y &= y', \\ z &= \frac{\lambda^2}{3\mu\sqrt[3]{\mu}} x' - \frac{2\lambda^3}{27\mu^2} y' + z', \end{aligned}$$

then it becomes $z'y'^2 + x'^3 = 0$.

5.3 Reducible Cubic Curves

Finally, we can classify reducible cubic curves.

Theorem 8. Suppose $\mathcal{C}_3$ is an arbitrary reducible cubic in $\mathbb{P}^2_{\mathbb{C}}$. Then there exists a projective transformation which takes it to one of the following forms:

1. $x(zy + x^2) = 0$;
2. $x(zx + y^2) = 0$;
3. $xyz = 0$;
4. $xy(x + y) = 0$;
5. $x^2 y = 0$;

6. $x^3 = 0$.

Proof. We know that a reducible conic is exactly the union of a line and a conic. Therefore, this result follows exactly from the classification of conics in the previous section and Theorem 1. $\square$

Not discussed in this report is the possible applications of our understanding of the cubic curves, their classification and their geometry. One such example is discussed in Wenping Wan's paper: 'Computing quadric surface intersections based on an analysis of plane cubic curves' [11], which provides an algebraic method for computing and parameterising the intersection of quadric surfaces. It turns out that this is a central problem is solid modelling and computer graphics.

6 Hesse Pencil

In section 5.1, we see that a smooth cubic curve on the projective plane can be transformed to the Weierstrass form or the Legendre form:

- (Weierstrass form) $y^2z = x^3 + axz^2 + bz^3$;
- (Legendre form) $y^2z = x(x-z)(x-\lambda z)$.

However, neither of them are 'symmetric' enough. Now we introduce the notion of Hesse pencil, which gives us a nicer form for cubic curves.

Definition 9. A *Hesse pencil* is a family of cubic curves on $\mathbb{P}^2_{\mathbb{C}}$ of the form

$$\mu(x^3 + y^3 + z^3) + \lambda xyz = 0 \qquad (\mu, \lambda \in \mathbb{C}).$$

If a smooth curve is of the Hesse form, then $\mu \neq 0$ (since $xyz = 0$ is not an irreducible curve). By scaling, we assume $\mu = 1$ and we expect that every smooth cubic curve has the form $x^3 + y^3 + z^3 + \lambda xyz = 0$. Notice that when $\lambda = -3, -3\omega, -3\omega^2, \infty$ ($\omega = (-1+\sqrt{3}i)/2$), the curve is the union of three lines, which is not smooth, and the equations could be expressed as follows,

- $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$,
- $x^3 + y^3 + z^3 - 3\omega xyz = (\omega x+y+z)(\omega x+\omega y+\omega^2 z)(\omega x+\omega^2 y+\omega z)$,
- $x^3 + y^3 + z^3 - 3\omega^2 xyz = (\omega^2 x+y+z)(\omega^2 x+\omega y+\omega^2 z)(\omega^2 x+\omega^2 y+\omega z)$,
- $x^3 + y^3 + z^3 - 3\infty xyz = 0 \implies \frac{xyz}{x^3+y^3+z^3} = 0 \implies xyz = 0$.

The following theorem summarises the above discussion, which is well-known but hard to find in literature.

Theorem 10. Every smooth cubic curve on $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to $x^3 + y^3 + z^3 + \lambda xyz = 0$ for some $\lambda \in \mathbb{C}$ ($\lambda \notin \{-3, -3\omega, -3\omega^2, \infty\}$).

The following procedure was taken in order to prove this theorem.

1. A smooth cubic curve has 9 distinct inflection points, which form a Hesse configuration;

2. A Hesse configuration can be projectively transformed to the standard position;

3. A smooth cubic curve that passes through all 9 points of the standard position must have the form $x^3 + y^3 + z^3 + \lambda xyz = 0$.

Here, the "standard position" is a set of 9 distinct points $\{Q_1, \cdots, Q_9\}$ with homogeneous coordinates

$$\begin{aligned} &Q_1 = [0:1:-1], \qquad &&Q_2 = [1:0:-1], \qquad &&Q_3 = [1:-1:0], \\ &Q_4 = [0:1:-\omega], \qquad &&Q_5 = [1:0:-\omega], \qquad &&Q_6 = [1:-\omega:0], \\ &Q_7 = [0:1:-\omega^2], \qquad &&Q_8 = [1:0:-\omega^2], \qquad &&Q_9 = [1:-\omega^2:0]. \end{aligned} \tag{3}$$

In the remaining part of the section, we will prove the statement in each step shown above, which completes the proof of Theorem 10.

6.1 Step 1

We will now introduce the Hesse configuration of Plane Cubic Curves which consists of 9 points and 12 lines, which we will be considering in $\mathbb{P}^2_{\mathbb{C}}$. Every point within the configuration lies on 4 different lines and each line contains exactly three points (see Figure 4).

Take any non-singular cubic curve in $\mathbb{P}^2_{\mathbb{C}}$, call it $\mathcal{C}$ such that $\mathcal{C}$ is given by the homogeneous equation $F(x, y, z) = 0$. We define the Hessian curve, $\text{Hess}(\mathcal{C})$, of $\mathcal{C}$ by the equation $\text{Hess}(F) = 0$ where $\text{Hess}(F)$ denotes the determinant of the matrix of second partial derivatives of F. We find 9 points of intersection between C and $\text{Hess}(\mathcal{C})$ [4], these we call the inflection points of $\mathcal{C}$.

The Hesse configuration is precisely the 9 inflection points of a non-singular cubic curve [3] and the following lemma gives that the Hesse configuration is the unique configuration of the form $(12_3, 9_4)$.

Lemma 11. Let $\Sigma \subseteq \mathbb{P}^2$, $|\Sigma| = 9$ points. Suppose that the points in Σ are not collinear. Then there exists a line $L \subseteq \mathbb{P}^2$ such that $L \cap \Sigma = 2$ points except when Σ forms a Hesse configuration.

Proof. The Sylvester-Gallai Theorem states that for any finite collection of points $\Sigma \subset \mathbb{R}^2$, either all points are collinear or there exists a line that passes through exactly two points in Σ.

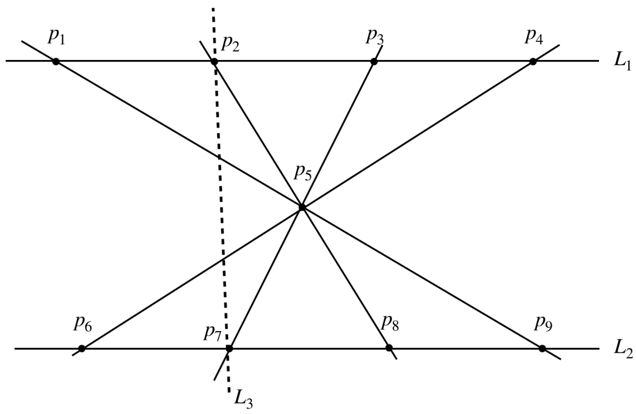

Figure 3: 9 distinct points.

This does not hold in $\mathbb{P}^2_{\mathbb{C}}$ as we have a counter example, which is exactly the standard position (3).

However, if we require that Σ is not projectively equivalent to the points above, then (3) is true. There does not exist a projective transformation transformation ϕ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\Sigma)$ forms the points in the standard position (3).

Suppose that Σ does not form a Hesse configuration and there does not exist a line L such that $|L \cap \Sigma| = 2$. Then there exists a line L_1 that passes through 4 points, say $p_1, p_2, p_3, p_4 \in \Sigma$. Since the points in Σ are assumed non-collinear, there must exist a $p_5 \in \Sigma$ such that p_5 does not lie on L_1. By the assumption, there exists a p_6 on the line determined by p_1, p_5. We construct p_7, p_8, p_9 in the same way so $p_6, p_7, p_8, p_9 \in \Sigma$ lie on the line L_2. We can see this in Figure 3.

Consider the line L_3 passing through p_2 and p_7. It obviously doesn't pass through p_1, p_3 or p_4 as p_7 doesn't lie on L_1. It also doesn't pass through p_6, p_8 or p_9 as our points are distinct and are on the same line L_2. Since p_7 and p_8 are collinear, if L_3 passes through p_5, $p_7 = p_8$ which is impossible as our points are distinct.

So, there exists a line L_3 that only passes through p_2 and p_7 and $|L \cap \Sigma| = 2$. This is a contradiction and so there exists a line $L \subseteq \mathbb{P}^2$ such that $L \cap \Sigma = 2$ points except when Σ forms a Hesse configuration. [4] $\square$

As a corollary of this lemma, we can show the statement in Step 1.

Corollary 12. The 9 inflection points on a smooth cubic curve C in $\mathbb{P}^2_{\mathbb{C}}$ form a Hesse configuration.

Proof. Denote Σ as the set of those 9 inflection points. We observe that points in Σ are not colinear, since $|\mathcal{C} \cap L| \leq 3$ for any line L by Bezout's theorem.

If Σ does not form a Hesse configuration, then $|L \cap \Sigma| = 2$ for some line L by Lemma 11. Say $L \cap \Sigma = \{P_1, P_2\}$. Note that L cannot be the tangent to $\mathcal{C}$ at P_i $(i = 1, 2)$ (Otherwise, suppose that L is tangent to $\mathcal{C}$ at P_1. Then the intersection multiplicity $I_{P_1}(L, \mathcal{C}) = 3$ by the definition of inflection points, and hence $I_{P_1}(L, \mathcal{C}) + I_{P_2}(L, \mathcal{C}) \geq 4 > 3$, which contradicts the Bezout's theorem).

Thus, L must intersect with $\mathcal{C}$ at another point P_3 that is different from P_1 and P_2. Equip $\mathcal{C}$ with addition such that P_1 is the identity element $\mathcal{O}$. Then $P_2 + P_3 = P_1 + P_2 + P_3 = \mathcal{O}$. Since P_2 is an inflection point, $3P_2 = \mathcal{O}$, and hence $3P_3 = -3P_2 = \mathcal{O}$. So P_3 is an inflection point, which means $P_3 \in \Sigma$. But now $\{P_1, P_2, P_3\} \subseteq L \cap \Sigma$, which contradicts $|L \cap \Sigma| = 2$. $\square$

6.2 Converse of Step 1

Now we want to show that The Hesse Configuration is precisely the 9 inflection points of a irreducible cubic curve.

Lemma 13. Let $\mathcal{C}_3$ be an smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$, and let a Hesse Configuration Σ be a subset of $\mathcal{C}_3$ in $\mathbb{P}^2_{\mathbb{C}}$ such that $|\Sigma| = 9$. And

$$\Sigma \subset \mathcal{C}_3, \text{ and } \Sigma := \{P_i, i \in 1, ..., 9\}$$

Then Σ is the set of inflection points in $\mathcal{C}_3$.

Proof. First, we equip $\mathcal{C}_3$ with the group operation $+$, such that $\mathcal{O} :=$ zero, and if Q_1, Q_2 and Q_3 counted with multiplicity are in $\mathcal{L} \cap \mathcal{C}_3$, where $\mathcal{L}$ a line in $\mathbb{P}^2_{\mathbb{C}}$, then,

$$Q_1 + Q_2 + Q_3 = \mathcal{O}.$$

If $Q = \mathcal{L} \cap \mathcal{C}_3$, then Q is an inflection points of $\mathcal{C}_3$ with multiplicity 3, and $Q = \mathcal{O}$.

For any $P_i \in \Sigma$ given, we choose collinear $P_l, P_m, P_n \in \Sigma$ such that none of them is P_i. We know that there exist $P'_l, P'_m, P'_n \in \Sigma$ are collinear to P and P_l, P_m, P_n respectively. By the group law we introduced on $\mathcal{C}_3$, we get

$$P_i + P_m + P'_m = P_i + P_n + P'_n = P_i + P_l + P'_l = \mathcal{O}.$$

We rearrange the equation and use the collinearity of P_l, P_m, P_n so that

$$3P_i + P_m + P'_m + P_n + P'_n + P_l + P'_l + = 3P_i + P'_l + P'_m + P'_n = \mathcal{O}.$$

To show that P'_l, P'_m, P'_n are collinear, we recall that Σ is a Hesse Configuration. Therefore $P_i = \mathcal{O}$.

We conclude that each P_i is a inflection point of $\mathcal{C}_3$ $\square$

6.3 Step 2

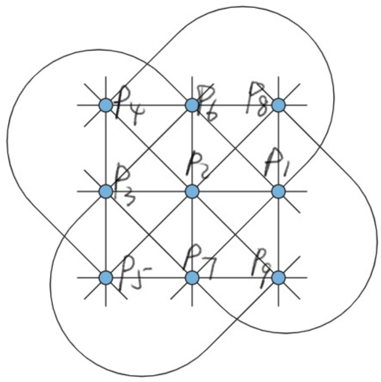

Figure 4: Hesse Configuration

We can also show that the Hesse configuration is unique up to projective transformation, which proves the statement in Step 2.

Lemma 14. If $\Sigma = \{P_1, \cdots, P_9\}$ is a Hesse configuration in $\mathbb{P}^2_{\mathbb{C}}$, then there exists a projective transformation $T \in \mathrm{PGL}(3, \mathbb{C})$ such that $T(\Sigma)$ is the standard position (3).

Proof. Choose a proper $T \in \mathrm{PGL}(3, \mathbb{C})$ such that $T(P_1) = Q_1, T(P_2) = Q_2$, $T(P_4) = Q_4$, $T(P_5) = Q_5$. Since $T \in \mathrm{PGL}(3, \mathbb{C})$, $T(\Sigma)$ is still a Hesse configuration. Denote $T(P_i) = [x_i : y_i : z_i]$ for $i = 1, \cdots, 9$.

Since T preserves the relation between lines and points, we must have

- $T(P_1), T(P_2), T(P_3)$ are colinear;
- $T(P_4), T(P_3), T(P_5)$ are colinear.

By the conditions above, we have the following equations

$$\det \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ x_3 & y_3 & z_3 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & -\omega \\ x_3 & y_3 & z_3 \\ 1 & 0 & -\omega \end{pmatrix} = 0.$$

That is,

$$-x_3 - y_3 - z_3 = 0, \tag{4}$$
$$\omega x_3 + \omega y_3 + z_3 = 0. \tag{5}$$

The solutions of $(x_3, y_3, z_3)^T$ are spanned by $(1, -1, 0)^T$. So,

$$T(P_3) = [1 : -1 : 0] = Q_3.$$

Hence, $T(P_i) = Q_i$ for $i = 1, \cdots, 5$.

Now we want to determine $T(P_8)$ and $T(P_9)$. We have relations:

- $T(P_5), T(P_2), T(P_8)$ are colinear;
- $T(P_4), T(P_2), T(P_9)$ are colinear;
- $T(P_8), T(P_1), T(P_9)$ are colinear.

We have the equations

$$\det\begin{pmatrix} 1 & 0 & -\omega \\ 1 & 0 & -1 \\ x_8 & y_8 & z_8 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & -\omega \\ 1 & 0 & -1 \\ x_9 & y_9 & z_9 \end{pmatrix} = \det\begin{pmatrix} x_8 & y_8 & z_8 \\ 0 & 1 & -1 \\ x_9 & y_9 & z_9 \end{pmatrix} = 0$$

That is,

$$(1-\omega)y_8 = 0 \tag{6}$$

$$-x_9 - \omega y_9 - z_9 = 0 \tag{7}$$

$$-x_9 y_8 + x_8 y_9 - x_9 z_8 + x_8 z_9 = 0 \tag{8}$$

So $y_8 = 0$ from (6). Now $T(P_8) = [x_8 : 0 : z_8]$.

If $x_8 = 0$, then $z_8 \neq 0$. We assume $z_8 = 1$. Then (8) becomes $x_9 = 0$. Thus, (7) becomes $-\omega y_9 - z_9 = 0$. We can let $y_9 = 1, z_9 = -\omega$. So $T(P_9) = [0 : 1 : -\omega] = Q_4 = T(P_4)$. Since T is invertible, $P_9 = P_4$, which is a contradiction. Therefore, $x_8 \neq 0$.We assume $x_8 = 1$. Now $T(P_8) = [1 : 0 : z_8]$.

If $x_9 = 0$, then (7) becomes $-\omega y_9 - z_9 = 0$. Let $y_9 = 1, z_9 = -\omega$, so (8) becomes $1 - \omega = 0$, which is a contradiction. Therefore, $x_9 \neq 0$, and we assume $x_9 = 1$. Now $T(P_9) = [1 : y_9 : z_9]$, and the equations become:

$$-1 - \omega y_9 - z_9 = 0 \tag{9}$$

$$y_9 - z_8 + z_9 = 0 \tag{10}$$

Similarly, we have the following relations:

- $T(P_5), T(P_6), T(P_1)$ are colinear.
- $T(P_4), T(P_1), T(P_7)$ are colinear.
- $T(P_6), T(P_2), T(P_7)$ are colinear.

We have the equations:

$$\det\begin{pmatrix} 1 & 0 & -\omega \\ x_6 & y_6 & z_6 \\ 0 & 1 & -1 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & -\omega \\ 0 & 1 & -1 \\ x_7 & y_7 & z_7 \end{pmatrix} = \det\begin{pmatrix} x_6 & y_6 & z_6 \\ 1 & 0 & -1 \\ x_7 & y_7 & z_7 \end{pmatrix} = 0$$

That is,

$$-\omega x_6 - y_6 - z_6 = 0 \tag{11}$$
$$(-1+\omega)x_7 = 0 \tag{12}$$
$$-x_7 y_6 + x_6 y_7 + y_7 z_6 - y_6 z_7 = 0 \tag{13}$$

Analogous to how we deal with (6)(7)(8), we deduce that $T(P_7) = [0 : 1 : z_7]$ and $T(P_6) = [x_6 : -\omega : z_6]$ from (11)(12)(13), and the equations become:

$$-\omega x_6 + \omega - z_6 = 0 \tag{14}$$
$$x_6 + z_6 + \omega z_7 = 0 \tag{15}$$

Next, we will determine $T(P_6) = [x_6 : -\omega : z_6]$, $T(P_7) = [0 : 1 : z_7]$, $T(P_8) = [1 : 0 : z_8]$, $T(P_9) = [1 : y_9 : z_9]$ by the following relations:

- $T(P_4), T(P_6), T(P_8)$ are colinear;
- $T(P_6), T(P_3), T(P_9)$ are colinear;
- $T(P_5), T(P_7), T(P_9)$ are colinear;
- $T(P_3), T(P_7), T(P_8)$ are colinear.

We have the equations:

$$\det\begin{pmatrix} 0 & 1 & \omega \\ x_6 & -\omega & z_6 \\ 1 & 0 & z_8 \end{pmatrix} = \det\begin{pmatrix} x_6 & -\omega & z_6 \\ 1 & -1 & 0 \\ 1 & y_9 & z_9 \end{pmatrix}$$
$$= \det\begin{pmatrix} 1 & 0 & -\omega \\ 0 & 1 & z_7 \\ 1 & y_9 & z_9 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & z_7 \\ 1 & 0 & z_8 \end{pmatrix} = 0.$$

That is,

$$(1+\omega) + z_6 - x_6 z_8 = 0 \tag{16}$$
$$z_6 + y_9 z_6 + \omega z_9 - x_6 z_9 = 0 \tag{17}$$
$$\omega - y_9 z_7 + z_9 = 0 \tag{18}$$
$$-z_7 + z_8 = 0 \tag{19}$$

By (19), $z_8 = z_7$. Thus, we substitute z_8 with z_7 in the equations.

By (10), $z_7 = y_9 + z_9$. Substituting (18) with $z_7 = y_9 + z_9$, and combining it with (9) gives

$$-1 - \omega y_9 - z_9 = 0,$$
$$\omega - y_9(y_9 + z_9) + z_9 = 0.$$

The solutions of y_9, z_9 (including $z_7 = y_9 + z_9$) are

$$\begin{cases} y_9 = -\omega^2 \\ z_9 = 0 \\ z_7 = -\omega^2 \end{cases} \quad \text{or} \quad \begin{cases} y_9 = -\omega \\ z_9 = -2-\omega \\ z_7 = -2-2\omega \end{cases} \tag{20}$$

By (15), $z_7 = -\omega^2 x_6 - \omega^2 z_6$. Substituting (16) with $z_7 = -\omega^2 x_6 - \omega^2 z_6$, and combining it with (14) gives

$$\begin{aligned} -\omega x_6 + \omega - z_6 &= 0, \\ 1 + \omega + z_6 - x_6(-\omega^2 x_6 - \omega^2 z_6) &= 0. \end{aligned}$$

The solutions of x_6, z_6 (including $z_7 = -\omega^2 x_6 - \omega^2 z_6$) are

$$\begin{cases} x_6 = 1 \\ z_6 = 0 \\ z_7 = -\omega^2 \end{cases} \quad \text{or} \quad \begin{cases} x_6 = \omega^2 \\ z_6 = -1+\omega \\ z_7 = -2-2\omega \end{cases}. \tag{21}$$

By combining (20) with (21), we get

$$\begin{cases} T(P_6) = [1 : -\omega : 0] = Q_6 \\ T(P_7) = [0 : 1 : -\omega^2] = Q_7 \\ T(P_8) = [1 : 0 : -\omega^2] = Q_8 \\ T(P_9) = [1 : -\omega^2 : 0] = Q_9 \end{cases} \quad \text{or} \quad \begin{cases} T(P_6) = [\omega^2 : -\omega : -1+\omega] \\ T(P_7) = [0 : 1 : -2-2\omega] \\ T(P_8) = [1 : 0 : -2-2\omega] \\ T(P_9) = [1 : -\omega : -2-\omega] \end{cases}.$$

For the former case, $T(P_i) = Q_i$ for $i = 1, \cdots, 9$ and the proof is completed. For the latter case, denote

$$T' = \begin{pmatrix} -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ -\frac{1}{2} + \frac{i}{2\sqrt{3}} & -1 - \frac{i}{\sqrt{3}} & -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ -1 - \frac{i}{\sqrt{3}} & -\frac{1}{2} + \frac{i}{2\sqrt{3}} & -\frac{1}{2} + \frac{i}{2\sqrt{3}} \end{pmatrix}.$$

Let $\tilde{T} = T' \circ T$. By performing a computation, we get

$$\begin{array}{lll} \tilde{T}(P_1) = Q_6, & \tilde{T}(P_2) = Q_5, & \tilde{T}(P_3) = Q_1, \\ \tilde{T}(P_4) = Q_4, & \tilde{T}(P_5) = Q_7, & \tilde{T}(P_6) = Q_2, \\ \tilde{T}(P_7) = Q_8, & \tilde{T}(P_8) = Q_9, & \tilde{T}(P_9) = Q_3. \end{array}$$

Thus, $\tilde{T}(\Sigma) = \{Q_1, \cdots, Q_9\}$ and the proof is completed. $\square$

6.4 Step 3

Now we show the statement in Step 3, which is the final step to prove Theorem 10.

Lemma 15. If a smooth cubic curve $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$ passes through all the points $Q_1, \cdots, Q_9$ of the standard position (3), then C has the form $x^3 + y^3 + z^3 + \lambda xyz = 0$ for some $\lambda \in \mathbb{C}$ such that $\lambda \notin \{-3, -3\omega, -3\omega^2, \infty\}$.

Proof. Putting $Q_1, \cdots, Q_9$ into $a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_5x^2z + a_6xyz + a_7y^2z + a_8xz^2 + a_9yz^2 + a_{10}z^3 = 0$, we get a system of equations $\mathbf{A}\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (a_1, \cdots, a_{10})^T$, and

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\omega & 0 & \omega^2 & -1 \\ 1 & 0 & 0 & 0 & -\omega & 0 & 0 & \omega^2 & 0 & -1 \\ 1 & -\omega & \omega^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\omega^2 & 0 & \omega & -1 \\ 1 & 0 & 0 & 0 & -\omega^2 & 0 & 0 & \omega & 0 & -1 \\ 1 & -\omega^2 & \omega & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\operatorname{rank}(\mathbf{A}) = 8$ and hence the dimension of the solutions $\mathbf{x}$ is $10 - 8 = 2$. Note that $\mathbf{x}_1 = (1, 0, 0, 1, 0, 0, 0, 0, 0, 1)^T$ (corresponding to $x^3 + y^3 + z^3$) and $\mathbf{x}_2 = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0)^T$ (corresponding to xyz) are linearly independent solutions.

So for any solution $\mathbf{x}$, $\mathbf{x} = \mu\mathbf{x}_1 + \lambda\mathbf{x}_2$ for some $\lambda, \mu \in \mathbb{C}$, which corresponds to the curve $\mu(x^3 + y^3 + z^3) + \lambda xyz = 0$. Finally, $\mu \neq 0$ since the curve is smooth, which can be assumed to be 1 up to scaling, and $\lambda \neq -3, -3\omega, -3\omega^2, \infty$ by what we have discussed at the beginning of the section. This gives us the desired form $x^3 + y^3 + z^3 + \lambda xyz = 0$. □

7 Net of Conics: Fano Threefolds 2-24

Gino Fano started studying what we now know as the Fano variety in 1932. They were classified in the 70s by Vasilii Iskovskikh, Shigefumi Mori and Shigeru Mukai, with their results being found the book 'Fano Varieties' [7].

Now, we consider smooth hypersurfaces with bi-degree $(1, 2)$ in $\mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz}$. As mentioned in the introduction, this object is exactly a Fano threefold in the family 2-24. It is a three dimensional generalisation of Del Pezzo surface. Iskovskikh and Prokhorov describe 2-24 as a a divisor on $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ with bidegree $(1, 2)$. [7, Page 219]

In the following sections we will be looking at surfaces in this Fano variety.

7.1 Properties of the Discriminant Curve

We now arrive at one of the main results in this report which combines our understanding of conics and cubics in $\mathbb{P}^2_{\mathbb{C}}$ and it allows us to classify the cubics further.

Theorem 16. Let X be a smooth surface with bi-degree $(1,2)$ in $\mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz}$, i.e. X is represented by

$$F(u,v,w;x,y,z) = uf_2(x,y,z) + vg_2(x,y,z) + wh_2(x,y,z) = 0.$$

Denote $G(u,v,w) := \det \operatorname{Hess}(F)$ as a homogeneous polynomial of u,v,w with degree 3, where

$$\operatorname{Hess}(F) = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}, F_{xx} = \frac{\partial^2 F}{\partial x \partial x}$$

If X is smooth, then the cubic curve $\mathcal{C}_3 \subseteq \mathbb{P}^2_{uvw}$ represented by $G(u,v,w)$ is one of the following:

- a smooth cubic curve,
- an irreducible nodal cubic curve,
- a union of three lines that do not intersect at one point,
- a union of an irreducible conic and a line that does not intersect the conic tangentially.

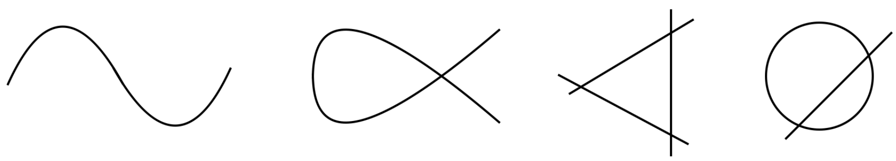

Figure 5: Examples of the real parts of a smooth cubic curve, an irreducible nodal cubic curve, a union of three lines that do not intersect at one point and a union of an irreducible conic and a line that does not intersect the conic tangentially.

Remark 17. The smoothness of X in the theorem is equivalent to the condition that three conics $f_2(x,y,z) = 0$, $g_2(x,y,z) = 0$ and $h_2(x,y,z) = 0$ do not intersect, i.e., f_2, g_2, h_2 cannot vanish simultaneously. Actually, if f_2, g_2, h_2 do not vanish simultaneously, then (F_u, F_v, F_w) is nonzero and hence $\nabla F = (F_u, F_v, F_w, F_x, F_y, F_z) \neq 0$, which implies that X is smooth. Conversely, if X is smooth, we will show that the three conics do not intersect by contradiction. Assume that they intersect at $[0:0:1]$ up to projective transformation. Then we can write

$$\begin{aligned} f_2(x,y,z) &= ax + by + (\text{high order terms}), \\ g_2(x,y,z) &= a'x + b'y + (\text{high order terms}), \\ h_2(x,y,z) &= a''x + b''y + (\text{high order terms}). \end{aligned}$$

By linear algebra, (a,b), (a',b') and (a'',b'') are linearly dependent. Assume that (a'',b'') is a linear combination of (a,b) and (a',b') without loss of generality. Then there exists $\alpha, \beta \in \mathbb{C}$ such that

$$h_2(x,y,z) = \alpha f_2(x,y,z) + \beta g_2(x,y,z) + (\text{high order terms}).$$

By a proper projective transformation, we send h_2 to $h_2 - \alpha f_2 - \beta g_2$, which makes h_2 vanish at first order, i.e.,

$$h_2(x,y,z) = (\text{high order terms}).$$

Now the smooth curve F is given by

$$F(u,v,w;x,y,z) = u(ax+by) + v(a'x+b'y) + w(\text{high order terms}).$$

Look at $P := ([0:0:1];[0:0:1]) \in \mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz}$. We find

$$\begin{aligned} F_u(P) = f_2(P) = 0,\ F_v(P) = g_2(P) = 0,\ F_w(P) = h_2(P) = 0, \\ F_x(P) = \big(ua + va'\big)\,|_P = 0,\ F_y(P) = \big(ub + vb'\big)\,|_P = 0,\ F_z(P) = 0. \end{aligned}$$

So $\nabla F|_P = 0$, which contradicts the smoothness of F.

Proof of Theorem 16. Denote $\mathrm{pr}_1, \mathrm{pr}_2$ as projections $\mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz} \to \mathbb{P}^2_{uvw}, \mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz} \to \mathbb{P}^2_{xyz}$ respectively. We eliminate the following possibilities of $\mathcal{C}_3$ to derive the result.

The possibilities of $\mathcal{C}_3$ that we wish to eliminate are:

1. A cuspidal cubic curve $G(u,v,w) = wu^2 - v^3$,

2. A union of an irreducible conic and a line that tangents it $G(u,v,w) = v(u^2 + vw)$,

3. A union of three lines that all pass through one point $G(u,v,w) = vu(v-u)$,

4. A union of two lines such that one is taken with multiplicity two $G(u,v,w) = u^2 v$,

5. A triple line $G(u,v,w) = u^3$, and;

6. Zero polynomial $G(u, v, w) = 0$.

Note $F(0, 0, 1; x, y, z) = h_2(x, y, z) = 0$, which is a conic. There are four possibilities for h_2: a zero polynomial, a double line, a union of two lines that intersect and a union of two lines that intersect.

When h_2 is a zero polynomial, $h_2(x, y, z) = 0$ we have that $F(x, y, z; u, v, w) = uf_2(x, y, z) + vg_2(x, y, z)$. Since f_2 and g_2 are two conics, the equations

$$\begin{cases} f_2(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{cases}$$

must have at least one solution in $\mathbb{P}^2_{xyz}$, namely $[\alpha : \beta : \gamma]$. By computation, we find $\nabla F\Big|_{(0,0,1;\alpha,\beta,\gamma)} = 0$, which contradicts the smoothness of X.

So, for each of our cases, h_2 is not the zero polynomial.

Consider the case where h_2 represents a smooth conic $h_2(x, y, z) = xy - z^2$. Then, $F(x, y, z; u, v, w) = uf_2(x, y, z) + vg_2(x, y, z) + w(xy - z^2)$. Let $M_f := \mathrm{Hess}(f_2) = (a_{ij})$, $M_g := \mathrm{Hess}(g_2) = (b_{ij})$, and

$$M_h := \mathrm{Hess}(xy - z^2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathrm{Hess}(F) &= uM_f + vM_g + wM_h \\ &= \begin{pmatrix} ua_{11} + vb_{11} & ua_{12} + vb_{12} + w & ua_{13} + vb_{13} \\ ua_{12} + vb_{12} + w & ua_{22} + vb_{22} & ua_{23} + vb_{23} \\ ua_{13} + vb_{13} & ua_{23} + vb_{23} & ua_{33} + vb_{33} - 2w \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \det(\mathrm{Hess}(F)) =& u^3(a_{11}a_{22}a_{33} - a_{11}a_{23}^2 - a_{12}^2a_{33} + 2a_{12}a_{13}a_{23} - a_{13}^2a_{22}) \\ &+ u^2v(a_{11}a_{22}b_{33} - 2a_{11}a_{23}b_{23} + a_{11}a_{33}b_{22} - a_{12}^2b_{33} \\ &+ 2a_{12}a_{13}b_{23} + 2a_{12}a_{23}b_{13} - 2a_{12}a_{33}b_{12} - a_{13}^2b_{22} \\ &- 2a_{13}a_{22}b_{13} + 2a_{13}a_{23}b_{12} + a_{22}a_{33}b_{11} - a_{23}^2b_{11}) \\ &+ u^2w(-2a_{11}a_{22} + 2a_{12}^2 - 2a_{12}a_{33} + 2a_{13}a_{23}) \\ &+ uv^2(a_{11}b_{22}b_{33} - a_{11}b_{23}^2 - 2a_{12}b_{12}b_{33} + 2a_{12}b_{13}b_{23} \\ &+ 2a_{13}b_{12}b_{23} - 2a_{13}b_{13}b_{22} + a_{22}b_{11}b_{33} - a_{22}b_{13}^2 \\ &- 2a_{23}b_{11}b_{23} + 2a_{23}b_{12}b_{13} + a_{33}b_{11}b_{22} - a_{33}b_{12}^2) \\ &+ uvw(-2a_{11}b_{22} + 4a_{12}b_{12} - 2a_{12}b_{33} + 2a_{13}b_{23} - 2a_{22}b_{11} \\ &+ 2a_{23}b_{13} - 2a_{33}b_{12}) + uw^2(4a_{12} - a_{33}) + v^3(b_{11}b_{22}b_{33} \\ &- b_{11}b_{23}^2 - b_{12}^2b_{33} + 2b_{12}b_{13}b_{23} - b_{13}^2b_{22}) + v^2w(-2b_{11}b_{22} \\ &+ 2b_{12}^2 - 2b_{12}b_{33} + 2b_{13}b_{23}) + vw^2(4b_{12} - b_{33}) + 2w^3. \end{aligned}$$

So $G(u, v, w) = \det \operatorname{Hess}(F)$ has the term $2w^3$. However, there is no w^3 term in any of our cases listed above and so h_2 cannot be a smooth conic in any of our cases.

Now there are only two possibilities for h_2: a double line and a union of two lines that intersect.

The following goes through each possible G and h_2 to arrive at the result by contradiction.

1. **A cuspidal cubic curve $G(u, v, w) = wu^2 - v^3$:**

   (a) *h_2 represents a double line $h_2(x, y, z) = x^2$:*
   In this case, $F(x, y, z; u, v, w) = uf_2(x, y, z) + vg_2(x, y, z) + wx^2$. Let $M_f := \operatorname{Hess}(f_2) = (a_{ij})$, $M_g := \operatorname{Hess}(g_2) = (b_{ij})$, $M_h := \operatorname{Hess}(xy) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

   $$\begin{aligned} \operatorname{Hess}(F) &= uM_f + vM_g + wM_h \\ &= \begin{pmatrix} ua_{11} + vb_{11} + 2w & ua_{12} + vb_{12} & ua_{13} + vb_{13} \\ ua_{12} + vb_{12} & ua_{22} + vb_{22} & ua_{23} + vb_{23} \\ ua_{13} + vb_{13} & ua_{23} + vb_{23} & ua_{33} + vb_{33} \end{pmatrix} \end{aligned}$$

   See python codes in appendix A.1.
   (runs for 30 minutes without results)

   (b) *h_2 represents a union of two lines that intersect $h_2(x, y, z) = xy$:*
   In this case, $F(x, y, z; u, v, w) = uf_2(x, y, z) + vg_2(x, y, z) + wxy$. Let $M_f := \operatorname{Hess}(f_2) = (a_{ij})$, $M_g := \operatorname{Hess}(g_2) = (b_{ij})$, $M_h := \operatorname{Hess}(xy) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

   $$\begin{aligned} \operatorname{Hess}(F) &= uM_f + vM_g + wM_h \\ &= \begin{pmatrix} ua_{11} + vb_{11} & ua_{12} + vb_{12} + w & ua_{13} + vb_{13} \\ ua_{12} + vb_{12} + w & ua_{22} + vb_{22} & ua_{23} + vb_{23} \\ ua_{13} + vb_{13} & ua_{23} + vb_{23} & ua_{33} + vb_{33} \end{pmatrix} \end{aligned}$$

   The coefficients of the term uw^2 and vw^2 are $-a_{33}$ and $-b_{33}$ respectively in $\det \operatorname{Hess}(F)$, which do not appear in $G(u, v, w) = wu^2 - v^3$. So $a_{33} = b_{33} = 0$. Now the coefficient of v^2w becomes $b_{13}b_{23}$ in $\det \operatorname{Hess}(F)$, which again must be zero. It means either $b_{13} = 0$ or $b_{23} = 0$. If $b_{13} = 0$, then the coefficient of uvw is $a_{13}b_{23} = 0$. But now the coefficient of wu^2 has the factor a_{13} and that of v^3 has the factor b_{23}, so one of them must be zero, which contradicts $G(u,v,w) = wu^2 - v^3$. So $b_{13} \neq 0$. Similarly, we can argue that $b_{23} \neq 0$. Thus, we get a contradiction.

2. **A union of an irreducible conic and a line that tangents it** $G(u,v,w) = v(u^2+vw)$**:**

   (a) *h_2 represents a double line $h_2(x,y,z) = x^2$:*
   In this case, $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wx^2$. Let $M_f := \mathrm{Hess}(f_2) = (a_{ij})$, $M_g := \mathrm{Hess}(g_2) = (b_{ij})$, $M_h := \mathrm{Hess}(x^2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
   The below code solves this case: We first substitute simple cases to reduce the number of equations Sympy needs to solve. The output shows that the system of equations has no solutions so such a case is not possible given our assumptions.
   See python codes in appendix A.2.

   (b) *h_2 represents a union of two lines that intersect $h_2(x,y,z) = xy$:*
   In this case, $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wxy$. Let $M_f := \mathrm{Hess}(f_2) = (a_{ij})$, $M_g := \mathrm{Hess}(g_2) = (b_{ij})$, $M_h := \mathrm{Hess}(xy) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} \mathrm{Hess}(F) &= uM_f + vM_g + wM_h \\ &= \begin{pmatrix} ua_{11}+vb_{11} & ua_{12}+vb_{12}+w & ua_{13}+vb_{13} \\ ua_{12}+vb_{12}+w & ua_{22}+vb_{22} & ua_{23}+vb_{23} \\ ua_{13}+vb_{13} & ua_{23}+vb_{23} & ua_{33}+vb_{33} \end{pmatrix} \end{aligned}$$

   We used the following Python code (see appendix A.3, first part) in order to find an expression for the determinant of Hess(F) in order to compare coefficients to $G(u,v,w) = v(u^2+vw)$.
   We require that the coefficients of vu^2, v^2w both be 1 and all other coefficients be zero. By the output in Python, we can generate a system of simultaneous equations (see appendix A.3, second part).
   This verifies that there are no solutions to the above which satisfy our condition that $G(u,v,w) = v(u^2+vw)$

3. **A union of three lines that all pass through one point** $G(u,v,w) = vu(v-u)$**:**

(a) *h_2 represents a double line $h_2(x,y,z) = x^2$:*
In this case, $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wx^2$. Let $M_f := \text{Hess}(f_2) = (a_{ij})$, $M_g := \text{Hess}(g_2) = (b_{ij})$, $M_h := \text{Hess}(x^2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
Again, we wish to take $\det(M_f + M_g + M_h)$ and compare coefficients with our expression for $G(u,v,w) = vu(v-u)$.
The Python code in appendix A.4 sets up our expression, substitutes clear values and then shows the remaining conditions have no solutions.

(b) *h_2 represents a union of two lines that intersect $h_2(x,y,z) = xy$:*
In this case, $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wxy$. Let $M_f := \text{Hess}(f_2) = (a_{ij})$, $M_g := \text{Hess}(g_2) = (b_{ij})$, $M_h := \text{Hess}(xy) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} \text{Hess}(F) &= uM_f + vM_g + wM_h \\ &= \begin{pmatrix} ua_{11} + vb_{11} & ua_{12} + vb_{12} + w & ua_{13} + vb_{13} \\ ua_{12} + vb_{12} + w & ua_{22} + vb_{22} & ua_{23} + vb_{23} \\ ua_{13} + vb_{13} & ua_{23} + vb_{23} & ua_{33} + vb_{33} \end{pmatrix} \end{aligned}$$

We used the Python code in appendix A.5 in order to find an expression for the determinant of Hess(F) in order to compare coefficients to $G(u,v,w) = vu(v-u)$.

4. **A union of two lines such that one is taken with multiplicity two $G(u,v,w) = u^2v$:**

(a) *h_2 represents a double line $h_2(x,y,z) = x^2$:*
In this case, $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wx^2$. Let $M_f := \text{Hess}(f_2) = (a_{ij})$, $M_g := \text{Hess}(g_2) = (b_{ij})$, $M_h := \text{Hess}(x^2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
Again, we will take $\det(M_f + M_g + M_h)$ and compare coefficients with our expression of $G(u,v,w) = u^2v$.
We use the following Python code to find det(Hessain) in order to compare with u^2v, the coefficient of $G(u,v,w)$

It shows that there is no solution in this case.

See python codes in appendix A.6

(b) *h_2 represents a union of two lines that intersect $h_2(x,y,z) = xy$:* In this case, $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wxy$. Let $M_f := \text{Hess}(f_2) = (a_{ij})$, $M_g := \text{Hess}(g_2) = (b_{ij})$, $M_h := \text{Hess}(xy) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We use the Python code in appendix A.7 to eliminate this case. And this verifies that there is no solution to $h = xy$ that satisfy $G = u^2v$

5. **A triple line $G(u,v,w) = u^3$:**

(a) *h_2 represents a double line $h_2(x,y,z) = x^2$:* In this case $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wx^2$. Let $M_f := \text{Hess}(f_2) = (a_{ij})$, $M_g := \text{Hess}(g_2) = (b_{ij})$ and $M_h := \text{Hess}(x^2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as we have seen before.

We take $\det(uM_f + vM_g + wM_h)$ and compare with coefficients with $G(u,v,w) = u^3$. We expand to get

$$\begin{aligned}
\det(M_f + M_g + M_h) =& u^3(a_{11}a_{22}a_{33} - a_{11}a_{23}^2 - a_{12}^2a_{33} \\
&+ 2a_{12}a_{13}a_{23} - a_{13}^2a_{22}) + u^2v(a_{11}a_{22}b_{33} \\
&- 2a_{11}a_{23}b_{23} + a_{11}a_{33}b_{22} - a_{12}^2b_{33} \\
&+ 2a_{12}a_{13}b_{23} + 2a_{12}a_{23}b_{13} - 2a_{12}a_{33}b_{12} \\
&- a_{13}^2b_{22} - 2a_{13}a_{22}b_{13} + 2a_{13}a_{23}b_{12} \\
&+ a_{22}a_{33}b_{11} - a_{23}^2b_{11}) + u^2w(2a_{22}a_{33} \\
&- 2a_{23}^2) + uv^2(a_{11}b_{22}b_{33} - a_{11}b_{23}^2 \\
&- 2a_{12}b_{12}b_{33} + 2a_{12}b_{13}b_{23} + 2a_{13}b_{12}b_{23} \\
&- 2a_{13}b_{13}b_{22} + a_{22}b_{11}b_{33} - a_{22}b_{13}^2 \\
&- 2a_{23}b_{11}b_{23} + 2a_{23}b_{12}b_{13} + a_{33}b_{11}b_{22} \\
&- a_{33}b_{12}^2) + uvw(2a_{22}b_{33} - 4a_{23}b_{23} \\
&+ 2a_{33}b_{22}) + v^3(b_{11}b_{22}b_{33} - b_{11}b_{23}^2 \\
&- b_{12}^2b_{33} + 2b_{12}b_{13}b_{23} - b_{13}^2b_{22}) \\
&+ v^2w(2b_{22}b_{33} - 2b_{23}^2)
\end{aligned}$$

We use the code in Appendix A.8 to show that it is not possible for $\det(uM_f + vM_g + wM_h) = u^3$.

(b) *h_2 represents a union of two lines that intersect $h_2(x,y,z) = xy$:* In this case, $F(x,y,z;u,v,w) = uf_2(x,y,z) + vg_2(x,y,z) + wxy$.

Let $M_f := \mathrm{Hess}(f_2) = (a_{ij})$, $M_g := \mathrm{Hess}(g_2) = (b_{ij})$, $M_h := \mathrm{Hess}(xy) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we have

$$\begin{aligned}\det(uM_f + vM_g + wM_h) =& 2a_{13}a_{23}u^2w + 2b_{13}b_{23}v^2w \\ & + u^3(a_{11}a_{23}^2 + 2a_{12}a_{13}a_{23}a_{13}^2a_{22}) \\ & + u^2v(2a_{11}a_{23}b_{23} + 2a_{12}a_{13}b_{23} \\ & + 2a_{12}a_{23}b_{13}a_{13}^2b_{22}2a_{13}a_{22}b_{13} \\ & + 2a_{13}a_{23}b_{12}a_{23}^2b_{11}) \\ & + uv^2(a_{11}b_{23}^2 + 2a_{12}b_{13}b_{23} \\ & + 2a_{13}b_{12}b_{23}2a_{13}b_{13}b_{22} \\ & a_{22}b_{13}^2 2a_{23}b_{11}b_{23} + 2a_{23}b_{12}b_{13}) \\ & + uvw(2a_{13}b_{23} + 2a_{23}b_{13}) \\ & + v^3(-b_{11}b_{23}^2 + 2b_{12}b_{13}b_{23} - b_{13}^2b_{22}).\end{aligned}$$

This can never be equal to $G(u, v, w) = u^3$, see the code in appendix A.9 which is a contradiction.

6. **Zero polynomial** $G(u, v, w) = 0$**:** This case is the most straight forward since, if G is a zero polynomial we know that in the polynomial expansion of X we must have a_33 and b_33 non-zero (and hence we can choose it to be one up to scaling). By the code in A.10 we can see that either of these substitutions results in the terms $-uw^2$ and $-vw^2$ in the expansion of the determinant which contradicts $G(u, v, w) = 0$

So, F must be a smooth cubic curve, an irreducible nodal cubic curve, a union of three lines that do not intersect at one point or a union of an irreducible conic and a line that does not intersect the conic tangentially. □

7.2 Classifying Hypersurfaces

We recall from Theorem 10 we get that every smooth cubic curve is projectively equivalent to the Hessse form:

$$x^3 + y^3 + z^3 + \lambda xyz = 0 \tag{22}$$

for some $\lambda \in \mathbb{C}$ $(\lambda \notin \{-3, -3\omega, -3\omega^2, \infty\})$

We can link the results in sections 7.1 and this theorem to aid the proof of the following theorem.

Theorem 18 (**Main Theorem** [2])**.** Let X be a smooth surface with bidegree $(1, 2)$ in $\mathbb{P}^2_{uvw} \times \mathbb{P}^2_{xyz}$. Then we can choose coordinate such that X is given by one of the following equations:

1. $(\mu yz + x^2)u + (\mu xz + y^2)v + (\mu xy + z^2)w = 0$ for some $\mu \in \mathbb{C}$ such that $\mu^3 \neq -1$
2. $(yz + x^2)u + (xz + y^2)v + z^2w = 0$
3. $(yz + x^2)u + y^2v + z^2w = 0$

Remark 19. The following proof is found in the book 'The Calabi problem for Fano threefolds' [2] which is a working paper.

Proof. First we show that each of our cases are equivalent to

$$(a_1yz + a_2x^2)u + (b_1\mu xz + b_2y^2)v + (c_1xy + c_2z^2)w = 0 \tag{23}$$

for some numbers $a_1, a_2, b_1, b_2, c_1, c_2$. Suppose that X is given by (23). Since X is smooth, we can appropriately scale x, y and z so that $a_2 = b_2 = c_2 = 1$. Choose a, b and c so that $a^3 = a_1$, $b^3 = b_1$ and $c^3 = c_1$. If $abc \neq 0$ and we scale our coordinates by

$$\begin{aligned} u &\mapsto u, & v &\mapsto v\alpha^2, & w &\mapsto w\beta^2 \\ x &\mapsto x, & y &\mapsto \frac{y}{\beta}, & z &\mapsto \frac{z}{\alpha} \end{aligned}$$

for $\alpha = \frac{a}{b}, \beta = \frac{a}{c}$, we are in case (1). We have $\mu = abc$, with X being singular only when $\mu^3 = -1$, and it follows from Theorem 10, where we proved that every smooth cubic curve can be put into the form $u^3 + v^3 + w^3 + \lambda uvw = 0$ and it is singular when $\lambda = -3, -3\omega, -3\omega^2$ ($\omega = \frac{-1+\sqrt{3}i}{2}$). In a similar way, if $abc = 0$, we can show that (23) is equivalent to either cases (2) or (3).

Now, we prove that X is given by (23) if we scale x, y, z, u, v, w appropriately.

By definition, X is represented by

$$F(u, v, w; x, y, z) = uf_2(x, y, z) + vg_2(x, y, z) + wh_2(x, y, z) = 0.$$

and we let $G(u, v, w) := \det \mathrm{Hess}(F)$.

We know from Theorem 16 that since X is smooth the discriminant curve, $G(u, v, w)$, is either smooth or nodal. If G is reducible, the result is well-known (see [2, § 10]). Therefore, we may assume that G is irreducible and so it follows from Theorem 10 that G is given by (22).

Let $[1 : 0 : 0] \in f_2$, $[0 : 1 : 0] \in g_2$ and $[0 : 0 : 1] \in h_2$. Since $[1 : 0 : 0]$ and $[0 : 1 : 0]$ are not in G, we know that f_2 and g_2 are smooth. So we can choose x, y and z such that f_2 is given by $xz + y^2 = u = w = 0$. So, X is given by

$$u(xz + y^2) + vg_2(x, y, z) + wh_2(x, y, z) = 0$$

where g_2 and h_2 are quadratic polynomials such that $g_2(x, y, z) = u = w = 0$ and $h_2(x, y, z) = u = v = 0$.

If G is singular, we have that $[0:0:1]$ is the singularity of G so that h_2 is a double line.

We can choose x, y, z so that $[0:0:1] \in g_2$ and $[0:1:0] \in h_2$ by the smoothness of X (see [2, Proof of Lemma A.7.10]). Then

$$g_2(x,y,z) = a_1 y^2 + a_2 x^2 + a_3 xy + a_4 yz + a_5 xz,$$
$$h_2(x,y,z) = b_1 z^2 + b_2 x^2 + b_3 xy + b_4 yz + b_5 xz,$$

for some numbers a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5.

Remark 20. The following changes of coordinates and substitutions are found in more detail in the proof of Lemma A.7.10 in [2].

Then, by some change of coordinates we arrive at G being given by

$$\begin{gathered}u^3 - \big(a_3a_4a_5 - a_2a_4^2 - a_5^2\big)v^3 - \big(b_3b_4b_5 - b_2b_4^2 - b_3^2\big)w^3 - \\ - \big(4 - 2a_2b_4 + a_3b_5 - 2a_4b_2 - 2a_4b_4 + a_5b_3\big)uvw + \\ + \big(a_2+2a_4\big)u^2v - \big(b_2+2b_4\big)u^2w - \big(a_3a_5 - 2a_2a_4 - a_4^2\big)uv^2 - \big(b_3b_5 - 2b_2b_4 - b_4^2\big)uw^2 - \\ - \big(4a_2 - 2a_2a_4b_4 + a_3a_4b_5 + a_3a_5b_4 - a_4^2b_2 + a_4a_5b_3 - a_3^2 - 2a_5b_5\big)v^2w - \\ - \big(4b_2 - a_2b_4^2 + a_3b_4b_5 - 2a_4b_2b_4 + a_4b_3b_5 + a_5b_3b_4 - 2a_3b_3 - b_5^2\big)vw^2 = 0.\end{gathered}$$

But, we know that G is given by (22) and so we arrive at the equations

$$\begin{gathered}a_2 + 2a_4 = 0, b_2 + 2b_4 = 0, a_3a_5 - 2a_2a_4 - a_4^2 = 0, b_3b_5 - 2b_2b_4 - b_4^2 = 0, \\ 4a_2 - 2a_2a_4b_4 + a_3a_4b_5 + a_3a_5b_4 - a_4^2b_2 + a_4a_5b_3 - a_3^2 - 2a_5b_5 = 0, \\ 4b_2 - a_2b_4^2 + a_3b_4b_5 - 2a_4b_2b_4 + a_4b_3b_5 + a_5b_3b_4 - 2a_3b_3 - b_5^2 = 0.\end{gathered}$$

After some subsitutions, these simplify to

$$\begin{gathered}a_2 = -2a_4, b_2 = -2b_4, 3a_4^2 + a_3 = 0, b_3b_5 + 3b_4^2 = 0, \\ a_3a_4b_5 + 6a_4^2b_4 - a_3^2 + a_3b_4 + a_4b_3 - 8a_4 - 2b_5 = 0, \\ a_3b_4b_5 + a_4b_3b_5 + 6a_4b_4^2 - 2a_3b_3 + b_3b_4 - b_5^2 - 8b_4 = 0,\end{gathered}$$

and X is given by

$$\big(x^2+yz\big)u + \big(v^2+xz-3a_4^2xy-2a_4x^2+a_4yz\big)v + \big(b_3xy-2b_4x^2+b_4yz+b_5xz+z^2\big)w = 0.$$

After some more change of corrdinates, we get that X is given by

$$\big(x^2+yz\big)u + \big(xz+ay^2\big)v + \big(x^2+c_1z^2+c_2y^2+c_3xy+c_4yz+c_5xz+x^2\big)w = 0,$$

where $a = a_4^3 + 1$, $c_1 = 4a_4^2 + 2a_4b_5 - 2b_4$, $c_2 = a_4^4 + a_4^3b_5 - 3a_4^2b_4 - a_4b_3$, $c_3 = -2a_4^2 - a_4b_5 + b_4$, $c_4 = -4a_4^3 - 3a_4^2b_5 + 6a_4b_4 + b_3$, $c_5 = 4a_4 + b_5$.

Proceeding in a similar fashion, we get that X has the representation

$$\big(x^2c_5^2 + c_5^2yz\big)u + \left(c_5xz - \frac{y^2c_5}{8}\right)v + \left(\big(2c_5^2 + c_1\big)z^2 + \frac{(c_5^2 - 4c_1)xy}{4}\right)w.$$

This is a special case of (23) and so we are done. $\square$

8 Fano Threefolds in the Family 3-13

Can we simplify more complex geometric objects? Consider a subset X in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ with coordinates that satisfy the system of equations

$$\begin{cases} \mathbf{x}^T M_1 \mathbf{y} = 0 \\ \mathbf{y}^T M_2 \mathbf{z} = 0 \\ \mathbf{z}^T M_3 \mathbf{x} = 0 \end{cases}, \tag{24}$$

where M_1, M_2, M_3 are 3×3 matrices. In other words, X is a complete intersection of hypersurfaces.

Suppose also that X is smooth. Taking the chart of a point in X gives you three quadratic equations in $\mathbb{C}^6$. The partial diffentials of these with respect to x, y and z will give you a linear polynomial. If these are linearly independent, X is smooth at this point.

We can characterise it as a Fano threefold in the family 3-13.

Remark 21. In many books the Fano variety 3-13 is defined as a blow up of the divisor $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ in a curve C of bidegree $(2, 2)$ such that the composition $C \hookrightarrow W \to \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{p_i} \mathbb{P}^2$ is an embedding for each projection p_1 and p_2 [7, Page 221].

However, we will not look into this definition as the concept of 'blowing up' is beyond the scope of this report.

In this section we will be looking at such surfaces.

8.1 Simplifying Three-folds

Consider the transformations

$$\mathbf{x} \to A\mathbf{x}, \quad \mathbf{y} \to B\mathbf{y}, \quad \mathbf{z} \to C\mathbf{z} \tag{25}$$

So that (24) becomes

$$\begin{cases} \mathbf{x}^T A^T M_1 B \mathbf{y} = 0 \\ \mathbf{y}^T B^T M_2 C \mathbf{z} = 0 \\ \mathbf{z}^T C^T M_3 A \mathbf{x} = 0 \end{cases}. \tag{26}$$

To make this as simple as possible we can choose A, B such that,

$$A^T M_1 B = \mathbb{1}, \quad B^T M_2 C = \mathbb{1}. \tag{27}$$

and (24) is now

$$\begin{cases} \mathbf{x}^T \mathbb{1} \mathbf{y} = 0 \\ \mathbf{y}^T \mathbb{1} \mathbf{z} = 0 \\ \mathbf{z}^T M \mathbf{x} = 0 \end{cases}. \tag{28}$$

Now, consider the (different) transformations

$$\mathbf{x} \to A'\mathbf{x}, \quad \mathbf{y} \to B'\mathbf{y}, \quad \mathbf{z} \to C'\mathbf{z} \tag{29}$$

So that (28) becomes

$$\begin{cases} \mathbf{x}^T A'^T B'\mathbf{y} = 0 \\ \mathbf{y}^T B'^T C'\mathbf{z} = 0 \\ \mathbf{z}^T C'^T M A'\mathbf{x} = 0 \end{cases} . \tag{30}$$

To make this as simple as possible we can choose

$$A'^T B' = \mathbb{1}, \quad B'^T C' = \mathbb{1}. \tag{31}$$

This means that $C'^T = B'^{-1}$ and $A' = (B'^{-1})^T$. So, $C'^T M A' = B'^{-1} M (B'^{-1})^T$.

Say that $B'^{-1} = N^T$. To simplify (24) further we want $N^T M N$ to be as nice as possible.

When M is symmetric, we know that we can diagonalize it with our matrix N. However, this is not possible when M is not symmetric.

In this case we can write

$$M = \frac{M + M^T}{2} + \frac{M - M^T}{2} \tag{32}$$

where we can see that $M + M^T/2 = M_S$ is a symmetric matrix and $M - M^T/2 = M_{SS}$ is skew symmetric.

Option 1 Looking at the conic $x^2 + y^2 + z^2 = 0$:

We can choose $N^T M_S N = \mathbb{1}$ as M_S is symmetric and see that $N^T M_{SS} N$ is skew symmetric. So, we have

$$N^T M N = N^T M_S N + N^T M_{SS} N = \mathbb{1} + \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}. \tag{33}$$

Option 2 Looking at the conic $2xy = 2z^2$:
We encode the symmetric matrix as $M_S = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{smallmatrix}\right)$. So, we have

$$\begin{aligned} N^T M N &= N^T M_S N + N^T M_{SS} N \\ &= N^T \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} N + N^T \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} N. \end{aligned} \tag{34}$$

We want to make the matrix M_{SS} as simply as possible.
For example, $a = b = 0, c = 1$ or $b = c = 0$.

8.2 Simplifying Three-folds Further

In last section, we simplified X as

$$\begin{cases} \mathbf{x}^T \mathbb{1} \mathbf{y} = 0 \\ \mathbf{y}^T \mathbb{1} \mathbf{z} = 0 \\ \mathbf{z}^T M \mathbf{x} = 0 \end{cases} \tag{35}$$

for some certain form of M. One choice is

$$M = \begin{pmatrix} 0 & 1+s & 0 \\ 1-s & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{36}$$

Alternatively, we can try another model. To see this, let's find how we permute $\mathbf{x}, \mathbf{y}, \mathbf{z}$ such that the form of (35) is fixed. One way is $([x_0, x_1, x_2], [y_0, y_1, y_2], [z_0, z_1, z_2]) \mapsto ([y_1, y_0, y_2], [x_1, x_0, x_2], [z_1, z_0, z_2])$, which permutes $x \leftrightarrow y$ and subscripts $0 \leftrightarrow 1$ while leaves z and 2 fixed. From this, we see that the system has $\mathbb{Z}_2$ symmetry.

However, we want

$$\begin{cases} \mathbf{x}^T M_1 \mathbf{y} = 0 \\ \mathbf{y}^T M_2 \mathbf{z} = 0 \\ \mathbf{z}^T M_3 \mathbf{x} = 0 \end{cases} \tag{37}$$

for some M_1, M_2, M_3 such that the system has S_3 symmetry, which permutes $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $0, 1, 2$ in the way as the following diagram shows:

$$\begin{tikzcd} & x \arrow[dl] & \\ y \arrow[rr] & & z \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl] & \\ 1 \arrow[rr] & & 2 \arrow[ul] \end{tikzcd}$$

(i.e., $([x_0, x_1, x_2], [y_0, y_1, y_2], [z_0, z_1, z_2]) \mapsto ([y_1, y_2, y_0], [z_1, z_2, z_0], [x_1, x_2, x_0])$)

Take whatever M_1. For instance,

$$M_1 = \begin{pmatrix} 0 & 1+t & 0 \\ 1-t & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{38}$$

We can determine M_2, M_3 such that the form of (37) is unchanged after S_3 acting on the system (37).

$$M_2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 1+t \\ 0 & 1-t & 0 \end{pmatrix}; \tag{39}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 1-t \\ 0 & -2 & 0 \\ 1+t & 0 & 0 \end{pmatrix}. \tag{40}$$

Now we want to find how the (35) is transformed into (37).

$$\begin{cases} \mathbf{x}^T \mathbb{1} \mathbf{y} = 0 \\ \mathbf{y}^T \mathbb{1} \mathbf{z} = 0 \\ \mathbf{z}^T M \mathbf{x} = 0 \end{cases} \rightarrow \begin{cases} \mathbf{x}^T M_1 \mathbf{y} = 0 \\ \mathbf{y}^T M_2 \mathbf{z} = 0 \\ \mathbf{z}^T M_3 \mathbf{x} = 0 \end{cases} \tag{41}$$

$$\mathbf{x} \rightarrow A\mathbf{x}, \quad \mathbf{y} \rightarrow B\mathbf{y}, \quad \mathbf{z} \rightarrow C\mathbf{z} \tag{42}$$

$$\begin{cases} A^T B = M_1 \\ B^T C = M_2 \\ C^T M A = M_3 \end{cases} . \tag{43}$$

Let $D := (B^{-1})^T$. Then $A = DM_1^T$, $C = DM_2$, and

$$D^T M D = (M_2^T)^{-1} M_3 (M_1^T)^{-1}. \tag{44}$$

The left-hand side is a matrix in terms of s (according to (36)) and the right-hand side is a matrix in terms of t (according to (38)). Using python codes in Appendix B, we solve

$$D = \begin{pmatrix} -\frac{(t+1)(t^2-4t+7)}{8c(t-1)^2} & 0 & 0 \\ 0 & 0 & c \\ 0 & \sqrt{\frac{-1}{2(t+1)}} & 0 \end{pmatrix}, \tag{45}$$

$$s = \frac{(t-3)(t^2+3)}{(t+1)(t^2-4t+7)}, \tag{46}$$

where $c \neq 0$ is an arbitrary constant. Normally, we set $c = 1$.

Example 22. Consider a concrete example of the following 3-fold given by

$$\begin{cases} \mathbf{x}^T \mathbb{1} \mathbf{y} = 0 \\ \mathbf{y}^T \mathbb{1} \mathbf{z} = 0 \\ \mathbf{z}^T M \mathbf{x} = 0 \end{cases}$$

where

$$M = \begin{pmatrix} 0 & 6 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

It corresponds to the case $s = 5$ in (36). The task is, finding a transformation $\mathbf{x} \rightarrow A\mathbf{x}, \mathbf{y} \rightarrow B\mathbf{y}, \mathbf{z} \rightarrow C\mathbf{z}$ that transforms the above 3-fold into the form of (37) with S_3 symmetry, i.e., finding all of M_1, M_2, M_3, A, B, C.

By (46), $\frac{(t-3)(t^2+3)}{(t+1)(t^2-4t+7)} = 5$. We get a solution $t = 1 - \sqrt[3]{12}$ (There are other two solutions; but we only need one). The 3-fold is now turned into

$$\begin{cases} \mathbf{x}^T M_1 \mathbf{y} = 0 \\ \mathbf{y}^T M_2 \mathbf{z} = 0 \\ \mathbf{z}^T M_3 \mathbf{x} = 0 \end{cases}$$

where M_1, M_2, M_3 are determined by (38)(39)(40) as the followings,

$$M_1 = \begin{pmatrix} 0 & 2-\sqrt[3]{12} & 0 \\ \sqrt[3]{12} & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

$$M_2 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 2-\sqrt[3]{12} \\ 0 & \sqrt[3]{12} & 0 \end{pmatrix};$$

$$M_3 = \begin{pmatrix} 0 & 0 & \sqrt[3]{12} \\ 0 & -2 & 0 \\ 2-\sqrt[3]{12} & 0 & 0 \end{pmatrix}.$$

By (45) (setting $c = 1$), we have

$$D = \begin{pmatrix} \frac{\sqrt[3]{12}}{24} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2(-2+\sqrt[3]{12})}} & 0 \end{pmatrix}.$$

So the transformation is given by $\mathbf{x} \to A\mathbf{x}, \mathbf{y} \to B\mathbf{y}, \mathbf{z} \to C\mathbf{z}$ where

$$A = DM_1^T = \begin{pmatrix} 0 & \frac{\sqrt[3]{18}}{12} & 0 \\ 0 & 0 & -2 \\ -\frac{\sqrt{-4+2\sqrt[3]{12}}}{2} & 0 & 0 \end{pmatrix};$$

$$B = (D^T)^{-1} = \begin{pmatrix} 4\sqrt[3]{18} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \sqrt{2(-2+\sqrt[3]{12})} & 0 \end{pmatrix};$$

$$C = DM_2 = \begin{pmatrix} -\frac{\sqrt[3]{12}}{12} & 0 & 0 \\ 0 & \sqrt[3]{12} & 0 \\ 0 & 0 & -\frac{\sqrt{-4+2\sqrt[3]{12}}}{2} \end{pmatrix}.$$

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Appendix A Codes for Proof of Theorem 16

A.1 Codes for 1(a)

```
import sympy as sym
a11,a12,a13,a22,a23,a33=sym.symbols('a11,a12,a13,a22,a23,a33')
b11,b12,b13,b22,b23,b33=sym.symbols('b11,b12,b13,b22,b23,b33')
u,v,w=sym.symbols('u,v,w')
Hess_f = Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hess_g = Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hess_h = Matrix([[2,0,0],[0,0,0],[0,0,0]])
Mat = u*Hess_f + v*Hess_g + w*Hess_h
A = factor(det(Mat),u,v,w)
A1 = A.subs(b22*b33,b23**2)
f = sym.Eq(a11*a22*a33 - a11*a23**2 - a12**2*a33 + 2*a12*a13*a23
- a13**2*a22,0)
g = sym.Eq(a11*a22*b33 - 2*a11*a23*b23 + a11*a33*b22
- a12**2*b33 + 2*a12*a13*b23 + 2*a12*a23*b13 - 2*a12*a33*b12
- a13**2*b22 - 2*a13*a22*b13 + 2*a13*a23*b12 + a22*a33*b11
- a23**2*b11,0)
h = sym.Eq(2*a22*a33 - 2*a23**2, 1)
k = sym.Eq(-2*a12*b12*b33 + 2*a12*b13*b23 + 2*a13*b12*b23
- 2*a13*b13*b22 + a22*b11*b33 - a22*b13**2 - 2*a23*b11*b23
+ 2*a23*b12*b13 + a33*b11*b22 - a33*b12**2,0)
l = sym.Eq(2*a22*b33 - 4*a23*b23 + 2*a33*b22,0)
m = sym.Eq(-b12**2*b33 + 2*b12*b13*b23 - b13**2*b22,-1)
sym.solve([f,g,h,k,l,m])
```

A.2 Codes for 2(a)

```
Hess_f = Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hess_g = Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hess_h = Matrix([[0,0,0],[2,0,0],[0,0,0]])
Mat = u*Hess_f + v*Hess_g + w*Hess_h
A = factor(det(Mat),u,v,w)
A1 = A.subs(b12*b33,b13*b23)
A2 = A1.subs(a12*a33,a13*a23)
f = sym.Eq(a11*a22*a33-a11*a23**2+a12*a13*a23-a13**2*a22,0)
g = sym.Eq(a11*a22*b33-2*a11*a23*b23+a11*a33*b22-a12**2*b33
+2*a12*a13*b23+2*a12*a23*b13-a13**2*b22-2*a13*a22*b13+a22*a33*b11
-a23**2*b11,1)
h = sym.Eq(a11*b22*b33-a11*b23**2+2*a13*b12*b23-2*a13*b13*b22
+a22*b11*b33-a22*b13**2-2*a23*b11*b23+a23*b12*b13+a33*b11*b22
```

```
-a33*b12**2,0)
k = sym.Eq(-2*a12*b33+2*a13*b23+2*a23*b13-2*a33*b12,0)
l = sym.Eq(b11*b22*b33-b11*b23**2+b12*b13*b23-b13**2*b22,0)
sym.solve([f,g,h,k,l])
```

A.3 Codes for 2(b)

First part:

```
a11,a12,a13,b11,b12,b13,a22,a23,b22,b23,a33,b33
= symbols('a11,a12,a13,b11,b12,b13,a22,a23,b22,b23,a33,b33')
u,v,w = symbols('u,v,w')
Hess_f = Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hess_g = Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hess_h = Matrix([[0,1,0],[1,0,0],[0,0,0]])
Mat = u*Hess_f + v*Hess_g + w*Hess_h
factor(Mat.det(),u,v,w)
```

Second part:

```
L = factor(det(Mat),u,v,w)
L1 = L.subs(a33,0)
L2 = L1.subs(b33,0)
L3 = L2.subs(2*b13*b23,1)
L4 = L3.subs(a13*b23,-a23*b13)
L5 = L4.subs(a13*a23,0)
F = sym.Eq(-a11*a23**2-a13**2*a22,0)
G = sym.Eq(-2*a11*a23*b23-a13**2*b22-2*a13*a22*b13-a23**2*b11,1)
K = sym.Eq(-a11*b23**2+a12-2*a13*b13*b22-a22*b13**2-2*a23*b11*b23,0)
H = sym.Eq(-b11*b23**2+b12-b13**2*b22,0)
sym.solve([F,G,K,H]).
```

A.4 Codes for 3(a)

```
Hess_f = Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hess_g = Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hess_h = Matrix([[0,0,0],[2,0,0],[0,0,0]])
Mat = u*Hess_f + v*Hess_g + w*Hess_h
c = factor(det(Mat),u,v,w)
# WANT v^2u, -u^2v
c1 = c.subs(a12*a33,a13*a23)#term in u^2w
c2 = c1.subs(b12*b33, b13*b23)#term in v^2w
# now I'll solve with Sympy
a1 = sym.Eq(a11*a22*a33-a11*a23**2+a12*a13*a23-a13**2*a22,0)#coefficient
a2 = sym.Eq(a11*a22*b33-2*a11*a23*b23+a11*a33*b22-a12**2*b33
```

```
+2*a12*a13*b23+2*a12*a23*b13-a13**2*b22-2*a13*a22*b13+a22*a33*b11
-a23**2*b11,-1)
a3 = sym.Eq(a11*b22*b33-a11*b23**2+2*a13*b12*b23-2*a13*b13*b22
+a22*b11*b33-a22*b13**2-2*a23*b11*b23+2*a23*b12*b13+a33*b11*b22
-a33*b12**2,1)
a4 = sym.Eq(-2*a12*b33+2*a13*b23+2*a23*b13-2*a33*b12,0)#term in uvw
a5 = sym.Eq(b11*b22*b33-b11*b23**2+b12*b13*b23-b13**2*b22,0)#term in v^3
sym.solve([a1,a2,a3,a4,a5])
```

A.5 Codes for 3(b)

```
Hess_f = Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hess_g = Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hess_h = Matrix([[0,1,0],[1,0,0],[0,0,0]])
Mat = u*Hess_f + v*Hess_g + w*Hess_h
c = factor(det(Mat),u,v,w)
\text{we want our expression to reduce to v^2u, -u^2v}
c1 = c.subs(a12*a33,a13*a23)#term in u^2w
c2 = c1.subs(b12*b33, b13*b23)#term in v^2w
# Now I'll solve with Sympy
a1 = sym.Eq(a11*a22*a33-a11*a23**2+a12*a13*a23-a13**2*a22,0)
# coefficient of u^3 = 0
a2 = sym.Eq(a11*a22*b33-2*a11*a23*b23+a11*a33*b22-a12**2*b33+2*a12*a13*b2
a3 = sym.Eq(a11*b22*b33-a11*b23**2+2*a13*b12*b23-2*a13*b13*b22+a22*b11*b3
a4 = sym.Eq(-2*a12*b33+2*a13*b23+2*a23*b13-2*a33*b12,0)\text{coefficient
a5 = sym.Eq(b11*b22*b33-b11*b23**2+b12*b13*b23-b13**2*b22,0)
sym.solve([a1,a2,a3,a4,a5])
```

A.6 Codes for 4(a)

```
from sympy import *
#x^2 case#
a11,a12,a13,a22,a23,a33=symbols('a11 a12 a13 a22 a23 a33')
b11,b12,b13,b22,b23,b33=symbols('b11 b12 b13 b22 b23 b33')
u,v,w=symbols('u,v,w')
Hessf=Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hessg=Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hessh=Matrix([[2,0,0],[0,0,0],[0,0,0]])
Mat=u*Hessf+v*Hessg+w*Hessh
F=factor(det(Mat),u,v,w)
C1=F.subs(2*a22*a33 ,2*a23**2)
C2=C1.subs(2*b22*b33,2*b23**2)
```

```
A1=Eq(a11*a22*a33 - a11*a23**2 - a12**2*a33 + 2*a12*a13*a23
- a13**2*a22,0)
A2=Eq(a11*a22*b33 - 2*a11*a23*b23 + a11*a33*b22 - a12**2*b33
+ 2*a12*a13*b23 +
2*a12*a23*b13 - 2*a12*a33*b12 - a13**2*b22 - 2*a13*a22*b13 +
2*a13*a23*b12 + a22*a33*b11 - a23**2*b11,1)
A3=Eq(a11*b22*b33 - a11*b23**2 - 2*a12*b12*b33 + 2*a12*b13*b23
+ 2*a13*b12*b23 - 2*a13*b13*b22 + a22*b11*b33 - a22*b13**2
- 2*a23*b11*b23 +
2*a23*b12*b13 + a33*b11*b22 - a33*b12**2,0)
A4=Eq(2*a22*b33 - 4*a23*b23 + 2*a33*b22,0)
A5=Eq(b11*b22*b33 - b11*b23**2 - b12**2*b33 + 2*b12*b13*b23
- b13**2*b22,0)
solve([A1,A2,A3,A4,A5])
```

A.7 Codes for 4(b)

```
from sympy import *
#xy case#

Hessf=Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hessg=Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hessh=Matrix([[1,0,0],[0,1,0],[0,0,0]])
Mat=u*Hessf+v*Hessg+w*Hessh
F=factor(det(Mat),u,v,w)

A1=F.subs(a33,0)
A2=A1.subs(b33,0)
A3=A2.subs(a13*b13,-a23*b23)
A4=A3.subs(-a13**2,a23**2)
A5=A4.subs(-b13**2,b23**2)
print(A5)

E1=Eq(-a11*a23**2 + 2*a12*a13*a23 + a22*a23**2,0)
E2=Eq(-2*a11*a23*b23 + 2*a12*a13*b23 + 2*a12*a23*b13
+ 2*a13*a23*b12 + 2*a22*a23*b23 - a23**2*b11 + a23**2*b22,1)
E3=Eq(-a11*b23**2 + 2*a12*b13*b23 + 2*a13*b12*b23 +
        a22*b23**2 - 2*a23*b11*b23 + 2*a23*b12*b13 + 2*a23*b22*b23,0)
E4=Eq(-b11*b23**2 + 2*b12*b13*b23 + b22*b23**2,0)
solve([E1,E2,E3,E4])
```

A.8 Codes for 5(a)

```
import sympy as sym
from sympy import *
a11, a12, a13, b11, b12, b13, a22, a23, b22, b23, a33, b33 = symbols( 'a1
u, v, w = symbols('u, v, w')
Hessf = Matrix( [ [ a11 , a12 , a13 ] , [ a12 , a22 , a23 ] , [ a13 , a23
Hessg = Matrix( [ [ b11 , b12 , b13 ] , [ b12 , b22 , b23 ] , [ b13 , b23
Hessh = Matrix( [ [ 2 , 0 , 0 ] , [ 0 , 0 , 0 ] , [ 0 , 0 , 0 ] ] )
Mat = u*Hessf + v*Hessg+ w*Hessh
F=factor(Mat.det() ,u ,v ,w)
print(F)
A1 = Eq(a11*a22*a33 - a11*a23**2 - a12**2*a33 + 2*a12*a13*a23 - a13**2*a2
A2 = Eq(a11*a22*b33 - 2*a11*a23*b23 + a11*a33*b22 - a12**2*b33 + 2*a12*a1
A3 = Eq(2*a22*a33 - 2*a23**2,0)
A4 = Eq(a11*b22*b33 - a11*b23**2 - 2*a12*b12*b33 + 2*a12*b13*b23 + 2*a13*
A5 = Eq(2*a22*b33 - 4*a23*b23 + 2*a33*b22,0)
A6 = Eq(b11*b22*b33 - b11*b23**2 - b12**2*b33 + 2*b12*b13*b23 - b13**2*b2
A7 = Eq(2*b22*b33 - 2*b23**2,0)
solve([A1,A2,A3,A4,A5,A6,A7])
```

A.9 Codes for 5(b)

```
import sympy as sym
from sympy import *
a11, a12, a13, b11, b12, b13, a22, a23, b22, b23, a33, b33 = symbols( 'a1
u, v, w = symbols('u, v, w')
Hessf = Matrix( [ [ a11 , a12 , a13 ] , [ a12 , a22 , a23 ] , [ a13 , a23
Hessg = Matrix( [ [ b11 , b12 , b13 ] , [ b12 , b22 , b23 ] , [ b13 , b23
Hessh = Matrix( [ [ 0 , 1 , 0 ] , [ 1 , 0 , 0 ] , [ 0 , 0 , 0 ] ] )
Mat = u*Hessf + v*Hessg+ w*Hessh
F=factor(Mat.det() ,u ,v ,w)
F1= F.subs(a33,0)
F2=F1.subs(b33,0)
print(F2)
1 = Eq(2*a13*a23,0)
A2 = Eq(2*b13*b23,0)
A3 = Eq(-a11*a23**2 + 2*a12*a13*a23 - a13**2*a22,1)
A4 = Eq(-2*a11*a23*b23 + 2*a12*a13*b23 + 2*a12*a23*b13 - a13**2*b22 - 2*a
A5 = Eq(-a11*b23**2 + 2*a12*b13*b23 + 2*a13*b12*b23 - 2*a13*b13*b22 - a22
A6 = Eq(2*a13*b23 + 2*a23*b13,0)
A7 = Eq(-b11*b23**2 + 2*b12*b13*b23 - b13**2*b22,0)
solve([A1, A2, A3, A4, A5, A6, A7])
```

A.10 Codes for 6

```
a11,a12,a13,b11,b12,b13,a22,a23,b22,b23,a33,b33 = symbols('a11,a12,a13,b1
u,v,w = symbols('u,v,w')
Hess_f = Matrix([[a11,a12,a13],[a12,a22,a23],[a13,a23,a33]])
Hess_g = Matrix([[b11,b12,b13],[b12,b22,b23],[b13,b23,b33]])
Hess_h1 = Matrix([[0,1,0],[1,0,0],[0,0,0]])
Mat1 = u*Hess_f + v*Hess_g + w*Hess_h1

L = factor(det(Mat1),u,v,w)
f = L.subs(a33,1)
ff = f.subs(b33,1)
ff
```

Appendix B Codes for Section 8.2

```
import sympy as sym
s,t = sym.symbols('s,t')
d11,d12,d13,d21,d22,d23,d31,d32,d33 = sym.symbols('d11,d12,d13,
d21,d22,d23,d31,d32,d33')
M1 = sym.Matrix ([[0,1+t,0],[1-t,0,0],[0,0,-2]])
M2 = sym.Matrix ([[-2,0,0],[0,0,1+t],[0,1-t,0]])
M3 = sym.Matrix ([[0,0,1-t],[0,-2,0],[1+t,0,0]])
M = sym.Matrix ([[0,1+s,0],[1-s,0,0],[0,0,-2]])
D = sym.Matrix ([[d11,d12,d13],[d21,d22,d23],[d31,d32,d33]])
LHS = D.transpose()*M*D
RHS = ((M2.transpose())**(-1))*M3*((M1.transpose())**(-1))
LR = LHS-RHS
A=[]
for i in range(3):
    for j in range(3):
        A.append(sym.Eq(LR[i,j],0))
sym.solve(A,d11,d12,d13,d21,d22,d23,d31,d32,d33,s)
```