Tensor Category in Arithmetic, Geometry and Physics: or how I learned to stop worrying and love fusion categories

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1 Introduction

The significance of tensor categories in mathematics physics emerged from Moore-Seiberg's paper on conformal field theories. In that paper, they first define what is a modular tensor category. Then Reshetikhin and Turaev's combinatorial constructions of invariants of links and 3-manifolds relate tensor categories with topological field theory. In this sense, tensor categories encode the symmetries of both conformal field theories and topological field theories. So, as groups encode classical symmetries, tensor categories provide a unifying language to encode the symmetries that arise in many quantum contexts, including those mentioned above. The mathematical study of tensor categories itself, such as fusion categories and modular tensor categories, has since become a hugely important field in its own right with connections to many mathematical field, for example the Hopf algebra (and quantum groups), representation theory, homotopy theory, even Arithmetic.

In our notes, we will first introduce the definition of modular tensor category in section 2. It is almost the "smallest" tensor category. It's definition will involove the definition of monoidal categories, tensor categories, fusion categories. Note that not all parts will use the full definition of modular tensor categories. Usually, we just need part of this definition.

Then in section 3, we will introduce the notion of factorization algebra to show the mathematical structure of pointed-like observables or pointed-like defects in a quantum field theory (QFT). After given physical intuition of factorization algebra, we will temporarily focus on topological case. For non-topological case will show in section 5. Of course, we can not just consider pointed-like defect, we can consider higher dimensional defects, see 3.3. For topological case, this will lead a notion of "extended" topological field theories. If we consider all dimensional defect, this will lead a notion of "fully extended" topological field theories. For (framed) full extended topological field theory, there is a pretty nice classification theorem, called Cobordism Hypothesis, see 3.4. In the same time, these higher dimensional operators themself will form a higher fusion categories structure. In particular, in low dimenisonal example, they will form fusion categories structure or modular tensor categories structure, see section 4.

After that, in section 4, we will see how symmetries in physics can be defined as *defects*, and an intuition on where it physically come from. We will then follow the construction in [5] giving the defects a structure so that they form a fusion category, then we will see an example of a theory with those defects. We will also go through how we can mathematically construct how those symmetries act on a topological field theory in the Freed Teleman Moore picture [13].

In section 5, we will consider non-topological field theory, that is 2D conformal field theory. The Lie algebra of conformal transformations in two dimensions is infinite-dimensional, which has remarkable consequences for 2D conformal field theories (Belavin, Polyakov, Zamolodchikov). Such theories are, in many cases, almost entirely determined by how the quantum fields transform under conformal transformations, that is with respect to the action of Virasoro algebra. More generally, the algebraic structure underlying conformal field theory is known as a vertex operator algebra. Well known examples include those corresponding to the Virasoro algebra, affine Kac–Moody algebras and W-algebras.

Finally, in section 6, we will show an application of tensor categories in Arthmetic. Minhyong Kim has developed the non-abelian Chabauty method for finding rational points on a curve effectively. The section starts with the notion of neutral Tannakian category, which is a kind of symmetric tensor category. It can be shown that such category is equivalent to $\mathrm{Rep}_{\Bbbk} G$ for some pro-algebraic group G. For a variety, we can use differential geometry as an analogy to define the algebraic unipotent vector bundle with flat connection. The category of such can be shown to be neutral Tannakian, and hence is equivalent to $\mathrm{Rep}_{\Bbbk} G$ for some G. This G is the de Rham fundamental group of the variety, which makes sense of algebraic iterated integrals, and whose Hodge filtration detects some information of the rational points on the variety. Finally, we take the glimpse of the Kim's cutter.

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2 Modular Tensor Category

Let first see some example of categories:

Example 2.1. The category of finite-dimensional vector spaces over $\mathbb{C}$, denoted by $\mathrm{Vec}_{\mathbb{C}}$ or simply Vec, is defined by the following data:

- Objects: finite-dimensional vector spaces over $\mathbb{C}$.
- Morphisms: $\mathbb{C}$-linear maps.
- Usual composition of maps;
- Usual identity map. ♡

Example 2.2. Let G be a group. The category of finite-dimensional G-representations over $\mathbb{C}$, denoted by $\mathrm{Rep}(G)$, is defined by the following data:

- Objects: $(V, \rho : G \to \mathrm{GL}(V))$.
- morphisms: G-representation maps. ♡

Example 2.3. Let G be a group. The category of finite-dimensional G-graded vector spaces (over $\mathbb{C}$), denote by Vec_G, is defined by the following data:

- Objects: finite-dimensional G-graded vector space, i.e. a collection of finite-dimensional vector spaces $\{V_g \in \mathrm{Vec}\}_{g\in G}$. The direct sum $V := \bigoplus_{g\in G} V_g$ is called total space.
- A morphism from V to W is a collection of $\mathbb{C}$-linear maps $\{f_g : V_g \to W_g\}_{g\in G}$. Note that $f := \bigoplus_{g\in G} f_g : V \to W$ is a linear map between two total spaces. ♡

The categories $\mathrm{Vec}, \mathrm{Rep}(G)$ and Vec_G have more structures than an ordinary category. Recall for every vector spaces V and W, $\mathrm{Hom}_{\mathrm{Vec}}(V,W)$ is naturally a vector space. Moreover the composition map $\mathrm{Hom}(V,W) \times \mathrm{Hom}(U,V) \to \mathrm{Hom}(U,W)$ is $\mathbb{C}$-bilinear.

Definition 2.4. A $\mathbb{C}$-*linear category* is a category in which each hom set is equipped with a structure of a finite-dimensional vector space, such that the composition of morphisms is $\mathbb{C}$-bilinear. ■

Example 2.5. All $\mathrm{Vec}, \mathrm{Rep}(G), \mathrm{Vec}_G$ are $\mathbb{C}$-linear categories. ♡

Definition 2.6. Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. An object $x \in \mathcal{C}$ is called *simple* if $\mathrm{Hom}(x,x) = \mathbb{C}$. We say $\mathcal{C}$ is *finite semisimple* if it satisfies the following conditions:

1. There are finite isomorphism classes of simple objects.
2. every object in $\mathcal{C}$ is a finite direct sum of simple objects. ■

Example 2.7. Let G be a finite group. By Maschke's theorem, $\mathrm{Rep}(G)$ is a finite semisimple category. ♡

Example 2.8. $\mathrm{Rep}(\mathbb{Z})$ is not finite semisimple. ♡

Definition 2.9. A *monoidal category* consists of the following data:

- a category $\mathcal{C}$;
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, where $\otimes(x,y)$ is also denoted by $x \otimes y$;
- a distinguished object $\mathbb{1} \in \mathcal{C}$, call the *tensor unit*.
- a natural isomorphism $\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, called the *associator*;
- a natural isomorphism $\lambda_x : \mathbb{1} \otimes x \to x$, called the *left unitor*;

- a natural isomorphism $\rho_x : x \otimes \mathbb{1} \to x$, called the *right unitor*;

  and these data satisfy pentagon equation and triangle equation. ■

Example 2.10. Let G be a finite group. For $(V,\rho),(W,\sigma) \in \mathrm{Rep}(G)$, their tensor product is the G-representation defined by the vector space $V \otimes_{\mathbb{C}} W$ equipped with the G-action

$$g \mapsto \rho(g) \otimes_{\mathbb{C}} \sigma(g), \forall g \in G.$$

This tensor product induces a $\mathbb{C}$-linear monoidal structure on $\mathrm{Rep}(G)$. The tensor unit is the trivial G-representation.♡

Example 2.11. Let G be a finite group. For $V,W \in \mathrm{Vec}_G$, their tensor product is the G-graded vector space defined by the total space $V \otimes_{\mathbb{C}} W$ equipped with the G-grading

$$(V \otimes_{\mathbb{C}} W)_g := \bigoplus_{h \in G} V_h \otimes_{\mathbb{C}} W_{h^{-1}g}, \quad g \in G$$

However, there are different $\mathbb{C}$-linear monoidal structures on Vec_G with this tensor product. For each 3-cocycle $\omega \in Z^3(G;\mathbb{C}^\times)$, there is a monoidal category, denoted by Vec_G^ω, whose underlying category is Vec_G and associator is determined by ω. Its tensor unit is $\mathbb{C}_e$. ♡

Definition 2.12. Let $\mathcal{C}$ be a monoidal category and $x \in \mathcal{C}$. A *left dual* of x is an object $x^L \in \mathcal{C}$ equipped with two morphisms $\mathrm{ev}_x : x^L \otimes x \to \mathbb{1}$ and $\mathrm{coev}_x : \mathbb{1} \to x \otimes x^L$ satisfying two zig-zag equations. Similarly one can define *right dual*. If every object admits both left duals and right duals, we say that $\mathcal{C}$ is rigid monoidal category. ■

Example 2.13. $\mathrm{Vec}, \mathrm{Rep}(G), \mathrm{Vec}_G^\omega$ are all rigid monoidal categories. ♡

Definition 2.14. A *fusion category* is a $\mathbb{C}$-linear finite semisimple rigid monoidal (abelian) category such that $\mathbb{1}$ is simple. ■

Let $\mathcal{C}$ be a fusion category. Since $\mathcal{C}$ is semisimple, the tensor product of two simple objects $x,y \in \mathcal{C}$ is the direct sum of simple objects. Thus we have

$$x \otimes y \simeq \bigoplus_{z \in \mathrm{Irr}(\mathcal{C})} N_{xy}^z \cdot z$$

for some $N_{xy}^z \in \mathbb{N}$. These numbers $\{N_{xy}^z\}_{x,y,z \in \mathrm{Irr}(\mathcal{C})}$ are called the *fusion rules* of $\mathcal{C}$.

In a rigid monoidal category, given an object x, possibly there is no isomorphism between x^L and x^R, or equivalently, no isomorphism between x and x^{LL}.

Definition 2.15. A *pivotal structure* on a rigid monoidal category $\mathcal{C}$ is a natural isomorphism $a_x : x \to x^{LL}$ satisfy $a_x \otimes a_y = a_{x \otimes y}$ (this condition means that a is a monoidal natural isomorphism). ■

In a pivotal fusion category $(\mathcal{C}, a)$, we can draw a "circle" for an morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(x,x)$. However, in this case there are two different ways to draw and they may not be equal, and these two results are called *left* and *right trace* of x:

$$\mathrm{tr}_a^L(f) := \mathrm{ev}_{x^L} \circ ((a_x \circ f) \otimes \mathrm{id}_{x^L}) \circ \mathrm{coev}_x, \qquad \mathrm{tr}_a^R(f) := \mathrm{ev}_x \circ (\mathrm{id}_{x^L} \otimes (f \circ a_x^{-1})) \circ \mathrm{coev}_{x^L}.$$

Definition 2.16. A pivotal structure on a fusion category is called *spherical* if the left and right traces are equal. ■

In particular, we define the *left* and *right quantum dimensions* of x by $\dim_a^L(x) = \mathrm{tr}_a^L(\mathrm{id}_x)$ and $\dim_a^R(x) := \mathrm{tr}_a^R(\mathrm{id}_x)$. Since $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbb{C}$, a trace can be regarded as an element of $\mathbb{C}$.

Definition 2.17. A *braiding* structure on a monoidal category $\mathcal{C}$ is a natural isomorphism $c_{x,y} : x \otimes y \to y \otimes x$ satisfy two hexagon equations.

Moreover, we say $\mathcal{C}$ is a *symmetric monoidal category* if $c_{y,x} \circ c_{x,y} = \mathrm{id}_{x \otimes y}$ for all $x,y \in \mathcal{C}$. ■

Example 2.18. There is a natural braiding on Vec or $\mathrm{Rep}(G)$ defined by

$$\begin{aligned} c_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w \otimes v \end{aligned}$$

Moreover this braiding is symmetric. ♡

Let $\mathcal{C}$ be a spherical braided fusion category. For two simple objects $x, y \in \mathcal{C}$, define

$$S_{xy} := \mathrm{tr}(c_{y,x^*} \circ c_{x^*,y}) = \mathrm{tr}(c_{y^*,x} \circ c_{x,y^*})$$

The matrix $S := (S_{xy})_{x,y \in \mathrm{Irr}(\mathcal{C})}$ is called the S matrix of $\mathcal{C}$.

Definition 2.19. Let $\mathcal{C}$ be a braided fusion category. A *ribbon* structure on $\mathcal{C}$ is a natural isomorphism $\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ such that $\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y}$ and $\theta_x^L = \theta_{x^L}$ for all $x, y \in \mathcal{C}$. ■

For each simple object $x \in \mathcal{C}$, since $\mathrm{Hom}_{\mathcal{C}}(x, x) \simeq \mathbb{C}$ we know that $\theta_x = T_x \cdot \mathrm{id}_x$ for some scalar $T_x \in \mathbb{C}$. The diagonal matrix T is called the *T-matrix* of the ribbon fusion category.

Definition 2.20. Let $\mathcal{C}$ be a braided fusion category. The *muger center* $\mathfrak{Z}_2(\mathcal{C})$ of $\mathcal{C}$ is the full subcategory consisting of the objects which have trivial double braiding with all objects in $\mathcal{C}$, i.e.

$$\mathfrak{Z}_2(\mathcal{C}) := \{x \in \mathcal{C} | c_{y,x} \circ c_{x,y} = \mathrm{id}, \forall y \in \mathcal{C}\}$$

$\mathcal{C}$ is *nondegenerate* if $\mathfrak{Z}_2(\mathcal{C}) \simeq \mathrm{Vec}$. ■

Definition 2.21. A *modular tensor category* is just a nondegenerate ribbon fusion category. ■

Finally, let's give a simple example of (unitary) modular tensor categories: $\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))$:

- There are only four simple objects $\mathbb{1}, e, m, f$.
- The fusion rules are given by $e \otimes m = f = m \otimes e$ and $e \otimes e = m \otimes m = f \otimes f = \mathbb{1}$. This implies that each simple objects is self-dual and has quantum dimension 1. The associator and left/right unitor are identities.
- The braiding structure is not 'gauge invariant'. It depends on many artificial choices. Different choices will lead equivalent braided fusion categories. Only double braiding $c_{y,x} \circ c_{x,y}$ is gauge invariant. So up to equivalence, we only need to consider double braiding or S-matrix:

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

• The ribbon structure is implied by T-matrix:

$$T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

3 Factorization algebras

Unless noted explicitly, every manifold is smooth and compact.

3.1 Physical intuition of factorization algebra

3.1.1 Classical field theory

Let's first consider a classical field theory. Let X be a manifold. For time interval $I \in \mathbb{R}$, the space of maps

$$\mathrm{Hom}(I, X)$$

is all the paths a particle could take. However, in the real world, the paths that a particle can take will be subject to constraints. These constraints are called *equations of motion* or *Euler-Lagrangian equations*. These equations are determined by a map

$$S : \mathrm{Hom}(I, X) \to \mathbb{R}$$

called *action*. Then the allowed paths $\mathrm{EL} = \{\mathrm{f} : \mathrm{I} \to \mathrm{X} | (\mathrm{dS})(\mathrm{f}) = 0\}$ is a subset of $\mathrm{Hom}(I, X)$. This is called *critical locus* of S.

More generally, we can consider more general spacetime n-manifold M not just I. Now we can give the following definition:

Definition 3.1. A *classical field theory* has the following data:

- a space of fields $\mathrm{Hom}(M, X)$; The dimension of M is called the *dimension* of the classical field theory.
- action: $S : \mathrm{Hom}(M, X) \to \mathbb{R}$.

such that S must be local: it must arise as the integral over M of some polynomial in a field $\phi \in \mathrm{Hom}(M, X)$ and its derivatives. ■

Example 3.2. (Classical massless free theory). Let $I = [a, b]$. Takes fields $\mathrm{Hom}(I, \mathbb{R}^n)$. The action is given by

$$\begin{aligned} S : \mathrm{Hom}(I, \mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \int_a^b \langle \frac{df(t)}{dt}, \frac{df(t)}{dt} \rangle dt \end{aligned}$$

Here we are taking the inner product on $\mathbb{R}^n$. In this case, the critical locus is straight lines. ♡

3.1.2 Observables and factorization algebra

Given a classical field theory

$$\mathrm{EL} \subset \mathrm{Hom}(\mathrm{I}, \mathrm{X}),$$

the classical observables $\mathrm{Obs}^{\mathrm{cl}} := \mathrm{Hom}(\mathrm{EL}, \mathbb{R})$ are the real functions on the critical locus. Thus an observable takes in an allowed path and spits out a real number: the measurement on that path.

Remark 3.3. The kind of functions are depending on the context. ♢

The vector space of functions $\mathrm{Obs}^{\mathrm{cl}}$ has a commutative algebra structure:

$$(fg)(u) = f(u)g(u). \quad \forall f, g : \mathrm{EL} \to \mathbb{R}$$

Physically, this is due to that in a classical theory, we can perform two observables (e.g position and momentum) at the same time to get a new measurement fg.

However, this is exactly what fails in the quantum world!

Theorem 3.4. (Heisenberg Uncertainly Principle). In quantum field theory, one cannot precisely know both the position and the momentum of a particle at the same time.

What can we do? Let $I = [a, b]$ be our spacetime. Let f, g be quantum observables. We can form a new observables on I that on (t_1, t_2) does f, on (t_3, t_4) does g and does nothing in between. Nothing means the observable that sending every path to zero. Note that (t_1, t_2) and (t_3, t_4) should be disjoint because of Heisenberg Uncertainly Principle. So we have a way to combine two quantum observables on disjoint open intervals.

More generally, we can consider n-manifold M instead of I. We can perform different observables on disjoint open n-disks in M.

Conclusion 3.1. Whatever quantum field theory is, the set of observables has a algebraic structure controlled by disjoint open n-disks in M.

Definition 3.5. A *colored operad* $\mathcal{C}$ consists of the following data:

- A set of objects $\{x, y, z, \cdots\}$;
- For a finite collection of objects $\{x_1, \cdots, x_n\}$ and $y \in \mathcal{C}$, there corresponds a set $\mathrm{Mul}_{\mathcal{C}}(\{x_1, \cdots, x_n\}, y)$;
- Composition maps;
- For each $x \in \mathcal{C}$, an identity $\mathrm{id}_x \in \mathrm{Mul}_{\mathcal{C}}(\{x\}, x)$.

  such that composition maps are associative and the identity maps are unital. ■

Example 3.6. Let M be a manifold. The colored operad $\mathrm{Open}(M)$ has the following data:

- objects are all connected open subsets in M;
- For every finite collections of opens $\{U_i\}_{i\in I}$ and open subset V, one defines

$$\mathrm{Mul}_{\mathrm{Open}(M)}(\{U_i\}, V) = \begin{cases} \{\bullet\} & \text{If any two } U_i \text{ are disjoint, and all } \{U_i\} \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

♡

Given a colored operad $\mathcal{C}$, there is a symmetric monoidal category $\overline{\mathcal{C}}$ of $\mathcal{C}$, called *monoidal envelope of* $\mathcal{C}$, constructed as following:

- Objects in $\overline{C}$ are the formal expressions $x_1 \otimes \cdots \otimes x_n$.
- Hom spaces are defined by $\mathrm{Hom}_{\overline{C}}(x_1 \otimes \cdots \otimes x_n, y) = \mathrm{Mul}_{\mathcal{C}}(\{x_1, \cdots, x_n\}, y)$.
- The tensor product of $x_1 \otimes \cdots \otimes x_n$ and $y_1 \otimes \cdots \otimes y_m$ is $x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m$.

Example 3.7. Let $\overline{\mathrm{Open}(M)}$ denote the following symmetric monoidal category:

- The objects of $\overline{\mathrm{Open}(M)}$ are pairs $(U, i_U : U \hookrightarrow M)$ where U is a topological space, and i_U is an open embedding on each connected component of U.
- A map from (U, i_U) to (V, i_V) is a map $f : U \to V$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i_U"'] & V \arrow[dl, "i_V"] \\ M & \end{tikzcd}$$

- symmetric monoidal structure is given by disjoint union;
- tensor unit is given by $\emptyset$. ♡

Definition 3.8. A *prefactorization algebra* on M with values in a symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal functor

$$\mathcal{F} : \overline{\mathrm{Open}(M)} \to \mathcal{C}$$

A *factorization algebra* is a prefactorization algebra $\mathcal{F}$ satisfied Weiss cosheaf condition, which I will not give the details here. See [8] [2] ■

Remark 3.9. In general, $\mathcal{C}$ is Vect or Chain. ♢

Whatever a QFT is, we can also consider the observables on it. Note that only knowing observables can not determine the full QFT.

Theorem 3.10. (Costello-Gwilliam) The observables $\mathrm{Obs}^{\mathrm{q}}$ of a perturbative QFT on a Riemannian manifold M has the structure of a factorization algebra on M.

Remark 3.11. In nonperturbative setting, the observables form a prefactorization algebra. ♢

3.2 Topological field theory and locally constant factorization algebra

We have talked about that classical observables form a commutative algebra, and quantum observables form a factorization algebra. This section I want to talk about how this structure behaves when we have a *topological* field theory.

Informally, a field theory is topological if it does not depend on a metric. For example, consider the value $\mathrm{Obs}^{\mathrm{q}}(\mathrm{B_r}(0))$ on a ball of radius r, centered at the origin. Since $\mathrm{Obs}^{\mathrm{q}}$ does not know about size,

$$\mathrm{Obs}^{\mathrm{q}}(\mathrm{B_r}(0)) \xrightarrow{\sim} \mathrm{Obs}^{\mathrm{q}}(\mathrm{B_{r'}}(0))$$

for $r < r'$. Moreover, it does not know distance from the origin. So if we move $B_r(0)$ around inside $B_{r'}(0)$, that will not effect the answer.

Definition 3.12. A factorization algebra $\mathcal{F}$ on M is *locally constant* if for any inclusion of disks

$$D_1 \subset D_2$$

in M, the induced map

$$\mathcal{F}(D_1) \to \mathcal{F}(D_2)$$

is an equivalence. ■

Definition 3.13. A field theory is *topological* if $\mathrm{Obs}^{\mathrm{q}}$ is locally constant. ■

Example 3.14.
1. A locally constant factorization algebra on $\mathbb{R}$ value in $\mathrm{Vec}_{\mathbb{C}}$ is an associative $\mathbb{C}$-algebra.
2. A locally constant factorization algebra on $\mathbb{R}^{\infty}$ value in $\mathrm{Vec}_{\mathbb{C}}$ is a commutative $\mathbb{C}$-algebra. ♡

Theorem 3.15. Let M be any topological n-manifold. The ∞-category of locally constant factorization algebras on M is equivalent to the ∞-category of locally constant factorization algebras on $\mathbb{R}^n$.

Lemma 3.16. A commutative $\mathbb{C}$-algebra A is a $\mathbb{C}$-vector space equipped with two multiplications $\times_1, \times_2$ and a single unit 1 such that $\times_2$ is a $\times_1$-homomorphism, i.e $(a \times_1 b) \times_2 (c \times_1 d) = (a \times_2 c) \times_1 (b \times_2 d)$ for all $a, b, c, d \in A$.

Proof. If $(A, \times, 1)$ is a commutative algebra, then we can define $\times_1 = \times_2 = \times$.

Conversely, if we have a vector space with two multiplications:

First let's check that unit is the unit of both $\times_1$ and $\times_2$. Assume 1_1 is the unit of $\times_1$ and 1_2 is the unit of $\times_2$. Then $1_1 = 1_1 \times_1 1_1 = (1_2 \times_2 1_1) \times_1 (1_1 \times_2 1_2) = (1_2 \times_1 1_1) \times_2 (1_1 \times_1 1_2) = 1_2 \times_2 1_2 = 1_2$.

Then we check $a \times_1 b = a \times_2 b$. Since $a \times_2 b = (a \times_1 1) \times_2 (1 \times_1 b) = (a \times_2 1) \times_1 (1 \times_2 b) = a \times_1 b$.

Finally we check $a \times_1 b = b \times_1 a$. Since $a \times_1 b = (1 \times_2 a) \times_1 (b \times_2 1) = (1 \times_1 b) \times_2 (a \times_1 1) = b \times_2 a = b \times_1 a$. □

This implies a notion called *E_n-algebra*, which intuitatively means there are n-kinds mulitiplication. With more multiplications, an algebra will be more "commutative".

Example 3.17.
1. E_1-algebras in $\mathrm{Vec}_{\mathbb{C}}$ are associtative $\mathbb{C}$-algebras.
2. E_2-algebras (or E_{∞}-algebras) in $\mathrm{Vec}_{\mathbb{C}}$ are commutative algebras.
3. E_1-algebras in $\mathcal{C}$at are monoidal categories.
4. E_2-algebras in $\mathcal{C}$at are braided monoidal categories.

5. E_∞-algebras in $\mathcal{C}at$ are symmetric monoidal categories. ♡

Theorem 3.18. (Lurie) Locally constant factorization algebras on $\mathbb{R}^n$ are the same as E_n-algebras.

Remark 3.19. We can also talk about non-topological field theory and its factorization algebra. An example of non-topological field theory is called *Holomorphic translation invariant field theory*. The observables Obs^q of a holomorphic translation-invariant field theory is a holomorphically translation-invariant factorization algebra. Say we are working over $\mathbb{C}$. Assume that our field theory additionally is S^1-invariant. In this situation, we will get a nice algebraic description of the observables, like we did for locally constant factorization algebras as E_n-algebras. This nice algebra is called *vertex algebra*. More Details see section 5 ◇

3.3 Higher dimensional operators

To record what we learned, the pointed observables has a factorization algebra structure in general, and in the topological case we get a E_n-algebra.

As we have seen, factorization algebras, E_n-algebras, and n-disk algebras are all related by the fundamental transformation of thinking of embedding disks, or points at their centers. Therefore, observables we consider before are sometimes called *pointed observables* or *pointed operators*. More generally, We described observable of codimensional l in a k-dimensional manifold M, where $k \geq 0$ and $1 \leq l \leq k$. Let $Z \subset M$ be a submanifold of codimension l, and let $v \in M$ be an open tubular neighborhoods of $Z \subset M$. Let $\overline{v}$ be the closure of v. It turns out that $\overline{v}$ can always be identified with a disk bundle over Z with fibers the closed l-dimensional disk $\mathbb{R}^l$ [Topology and geometry, Bredon]. The fiber over $p \in Z$ is denoted $\overline{v}_p$. Its boundary $\partial \overline{v}_p$ is just $l-1$-dimensional sphere S^{l-1}. This sphere is so-called *link* of $Z \subset M$ at p.

The factorization algebra of pointed observables in TFT only depends on the data of $\mathcal{F}(\mathbb{R}^n)$. I will also denote $\mathcal{F}(\mathbb{R}^n) = \mathcal{Z}(S^{n-1})$ for its dependence on the linking of a point.

Pointed observables gave us some interesting algebras to think about, but there is some structure of the field theory that it misses.

Example 3.20. Consider G-gauge theory on M. Given a loop γ in M and a representation ρ of G, we can define a map

$$W_\rho[\gamma] : \mathrm{Bun}_G^\nabla(\mathrm{M}) \to \mathbb{R}$$

sending a connection A on principal G-bundle $P \to M$ to $\mathrm{tr}_\rho(\mathrm{Hol}_A(\gamma))$, where $\mathrm{Hol}_A(\gamma)$ is the holonomy of A. This map is called *closed wilson loop operator*. Every connection A will give a holonomy. The set of all holonomies for a given principal G-bundle form a group which must be a subgroup of G. So in particular the holonomy $\mathrm{Hol}_A(\gamma)$ of A can be view as an element in G, then $\mathrm{tr}_\rho(\mathrm{Hol}_A(\gamma))$ make sense. Note that the set of all wilson lines is in one-to-one correspondence with the representations of the gauge group G. ♡

In general, this type of construction giving observables depending on loops or lines in spacetime are called *line operators*.

Similarly like pointed observables, For a topological theory on $\mathbb{R}^n$, line operators form an E_{n-1}-monoidal category $\mathcal{Z}(S^{n-1})$.

Remark 3.21. If we consider all $< n$-dimensional defects (of some nice topological field theory) in a n-manifold, we will get a fusion $(n-1)$-category. For details on low dimensional manifold case, see section 4. ◇

3.4 Cobordism Hypothesis

The problem when we try to classify Atiyah-Segal TFTs in higher dimensions is that the objects become too complicated. Up to reversing orientation and taking disjoint unions, the categories $\mathrm{Cob}_1(1)$ and $\mathrm{Cob}_1(2)$ have a unique object, $\bullet$ and S^1, respectively. For $n = 3$, there are infinitely many oriented 2-manifold, one for each genus g. A question is can we reduced n-dimensional TFT down to information about 1-dimensional TFTs. So we give another model for TFT, which is called fully extended TFT:

Definition 3.22. Define an symmetric monoidal (∞, n)-category Bord_n with

- Objects are 0-manifolds;
- 1-morphisms are cobordisms between 0-manifolds;

- 2-morphisms are cobordisms between cobordisms
- $\cdots$
- n-morphisms are n-manifolds with corners
- (n+1)-morphisms are diffeomorphisms
- (n+2)-morphisms are isotopies between diffeomorphisms.
- $\cdots$ ■

Definition 3.23. Let $\mathcal{C}$ be a symmetric monoidal (∞,n)-category. An **fully extended** $\mathcal{C}$-valued n-dimensional TFT is a symmetric monoidal functor

$$\mathcal{Z}^{fe} : \mathrm{Bord}_n \to \mathcal{C}$$ ■

Example 3.24. Take $\mathcal{C}$ so that $\Omega^n\mathcal{C} = \mathbb{C}, \Omega^{n-1}\mathcal{C} = \mathrm{Vec}_{\mathbb{C}}$ and $\Omega^{n-2}\mathcal{C} = \mathrm{LinCat}_{\mathbb{C}}$. Then $\mathcal{Z}^{fe}(S^{n-2}) \in \mathrm{LinCat}_{\mathbb{C}}$ is an E_{n-1}-monoidal linear 1-category of line operators. Similarly, $\mathcal{Z}^{fe}(S^{n-k})$ is an E_{n-k+1}-monoidal $(k-1)$-category of $(k-1)$-dimensional defects. Compared with Remark 3.21. ♡

The purpose of this definition is to allow us to reduce n-dimensional TFTs down to information about 1-dimensional TFTs. As we all know, a 1-dimensional TFT is determined by its value on a point. Thus we may have the following guass: An fully extended field theory is determined by its value on a single point.

There are two problems with this guess.

1. Even in 1-dimension, not every vector space determined a TFT. We need to restrict to finite-dimensional ones. The analogue in higher dimensions will be something called "fully dualizable objects".
2. Orientation in dimension 1 is the same as a framing. This is not true in higher dimensions. We actually wanted a framing not just orientation. Note that a framing determines an orientation. However orientability does not imply the existence of framing. For example, any even-dimensional sphere doesn't admits framing but all spheres are orientable.

Given a symmetric monoidal (∞,n)-category we would like to pick out the largest fully dualizable subcategory $\mathcal{C}^{fd}$. In general, we can obtain $\mathcal{C}^{fd}$ from $\mathcal{C}$ by repeatedly discarding morphisms that don't admit adjoints and objects that don't admit duals.

Definition 3.25. Let $\mathcal{C}$ be a symmetric monoidal (∞,n)-category. An object $x \in \mathcal{C}$ is called *fully dualizable* in $\mathcal{C}$ if $x \in \mathcal{C}^{fd}$. ■

Remark 3.26. This means in general, a fully dualizable object need its evaluation map and coevaluation map has ajoints, and the higher evaluation map of evaluation map and its adjoint also need ajoints. Same with coevaluation map. However, in dimension 2, there are some simple criteria for testing whether or not an object is fully dualizable: Let $\mathcal{C}$ be a symmetric monoidal $(\infty,2)$-category. Then $x \in \mathcal{C}$ is fully dualizable iff it admits a dual x^* and the evaluation map $\mathrm{ev} : x \otimes x^* \to \mathbb{1}$ has both a left and a right adjoint. ♢

Theorem 3.27. (Cobordism Hypothesis: Framed Version [24]). Let $\mathcal{C}$ be a symmetric monoidal (∞,n)-category. Let $\pi_{<\infty}(\mathcal{C})$ be the ∞-groupoid of $\mathcal{C}$. Then the evaluation functor $Z \mapsto Z(\bullet)$ induces an equivalence of ∞-groupoid

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \xrightarrow{\sim} \pi_{<\infty}(\mathcal{C}^{fd})$$

So a fully-extended framed TFT $\mathcal{Z}$ is completely determined by $\mathcal{Z}(\bullet)$. So how does one obtain $\mathcal{Z}(N)$? It is expected that there is a version of factorization homology for (∞,n)-categories so that

$$\int_N \mathcal{Z}(\bullet) = \mathcal{Z}(N)$$

for all manifold N of dimension $0,\cdots,n$. This kind of factorization homology is called "β-factorization homology" [3].

Remark 3.28. This give us relation of Atiyah-Segal TFT and fully extended TFT. For example consider a fixed physical sense 2D TFT. For Atiyah-Segal TFT model $\mathcal{Z}^{AS}$, it is completely determined by $\mathcal{Z}^{AS}(S^1)$ which is a commutative Frobenius algebra. However, for 2D fully extended framed TFT model $\mathcal{Z}$, i.e. $(2,1,0)$-TFT, it is completely determined by $\mathcal{Z}(\bullet) = A$ which is a Frobenius algebra. We can compute $\mathcal{Z}(S^1) = \int_{S^1}(\mathcal{Z}(\bullet)) = A/[A,A]$ as vector space. Since A is a Frobenius algebra, $A/[A,A] = Z(A)^*$. The pants-like cobordisms will equip $Z(A)^*$ a commutative Frobenius algebra structure, and $\mathcal{Z}^{AS}(S^1) = Z(A)^*$. ♢

Take $\mathcal{C}$ to be a suitable choice of an (∞, n)-category Mor of algebras up to Morita equivalences. Given a fully extended framed nD TFT $\mathcal{Z}$, $\mathcal{Z}(\bullet)$ is a certain type of E_{n-1}-algebras. Usually, we are thinking of an n-dimensional TQFT as corresponding to its E_n-algebra of pointed observables, so what is up with the E_{n-1}-algebra? How do we get from $\mathcal{Z}(\bullet)$ to $\mathcal{Z}(S^{n-1})$? Of course, we can compute $\mathcal{Z}(S^{n-1})$ by $\int_{S^{n-1}} \mathcal{Z}(\bullet)$. However, there is a more explicit result: $\mathcal{Z}(S^{n-1}) = \mathfrak{Z}_{n-1}(\mathcal{Z}(\bullet))$, where $\mathfrak{Z}_{n-1}$ is the E_{n-1}-center, Example see 3.28 and 4.3.1.

4 Tensor categories as encoding higher symmetries in Physics

4.1 Defects in a Topological Quantum Field Theory

A Topological Quantum Field theory is defined as a functor from a category of bordisms to a category of vector fields. We want to define *defects* on the space time manifold of the field theory, that will encode a symmetries in Physics. In [5], defects are defined in the following way.

Definition 4.1. Let $\mathcal{T}$ be the domain category of an n-dimensional topological field theory (i.e., the space-time manifold M_n dimension is n). A p-dimensional *defect* are conditions D_p applied on a p-dimensional submanifold Σ_p of M_n. The conditions D_p make the vector fields on M_n change "uncontinuously" along Σ_p. ■

In order to understand what we mean, we should define some objects of differential geometry. Let M_n be an n-dimensional manifold.

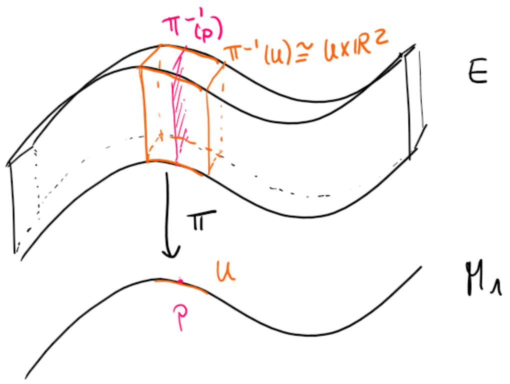

Figure 1: Example of a vector bundle on a manifold of dimension 1

Definition 4.2. A *vector bundle* on the *base space* M_n consists of a topological space E, called *total space*, and a surjection $\pi : E \to M_n$ such that

1. for all p in M_n, the *fiber* $\pi^{-1}(p)$ is a vector space of finite dimension ;
2. for all p in M_n, there exists an open $U \subset M_n$, a natural number k, and a homeomorphism

$$\phi : U \times \mathbb{R}^k \cong \pi^{-1}(U),$$

   such that for all x in U,

$$(\pi \circ \phi)(x, v) = x \quad \forall v \in \mathbb{R}^k,$$

   and the map $v \mapsto \phi(x, v)$ is a linear isomorphism between $\mathbb{R}^k$ and $\pi^{-1}(\{x\})$ ■

You can see a picture of an example of vector bundle on M_1 in Figure 1.

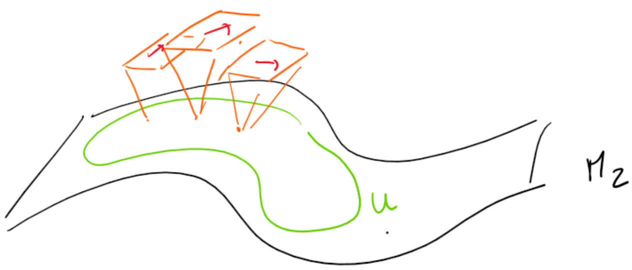

Figure 2: Example of a vector field on a manifold of dimension 2

Definition 4.3. Let $U \subset M_n$ an open subset, and $\pi : E \to M_n$ a vector bundle. A *section/vector field* of π on U is a continuous function $s : U \to E$ such that for all $u \in U$,

$$\pi \circ s(u) = u. \qquad \blacksquare$$

You can see an example of a vector field on M_2 in Figure 16.

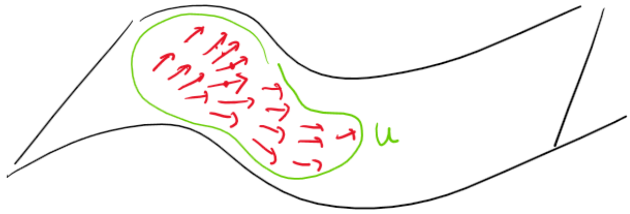

Figure 3: Representation of a vector field on a manifold of dimension 2

Now imagine you have a vector field on M_2. By definition we know that it should be continuous. So we can represent it in an intuitive way as in Figure 3. Let us see what is the action of a defect on this.

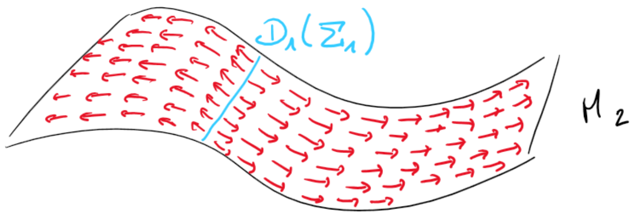

Figure 4: Example of a defect on a vector field on a manifold of dimension 2

In Figure 4, the vector field is acting on $U = M_2$ and the insertion of the line defect $D_1(\Sigma_1)$ breaks the continuity along the line Σ_1.

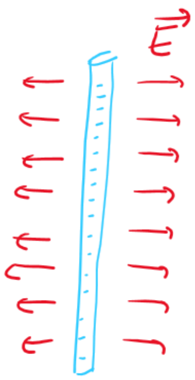

Figure 5: Electric field induced by a line of electrons

This defect in particular, can be understood as a line of electrons in vacuum, where the vector field is depicting the electric field, as we can see in Figure 5.

Definition 4.4. We say that a defect D_p is *topological* if changing its locus Σ_p to another Σ'_p will not change the global theory. $\blacksquare$

In our example, the defect is topological. And we could add another line defect $D'_1(\Sigma'_1)$ to get a capacitor (i.e., we add a line of protons).

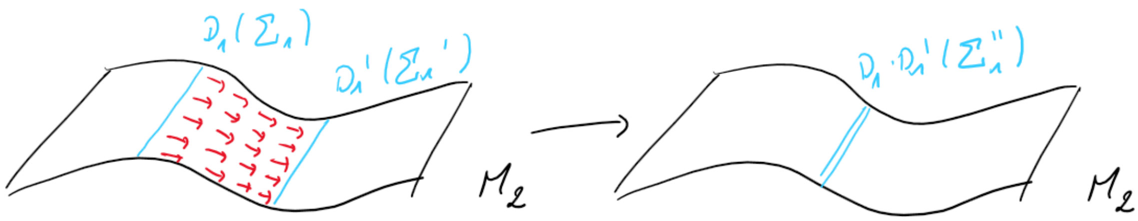

Figure 6: Capacitor merging to identity defect

Then as we can see in Figure 6, if we make the capacitor bounds closer and closer, the resulting electric field is then none on the all space. This means that merging those two defects together result as having "no defect" or "identity defect". This inspires us to construct a multiplication operation by getting two topological defects closer and closer. If we have two defects, corresponding of a change of angle α and β in the vector field, as in Figure 7, if

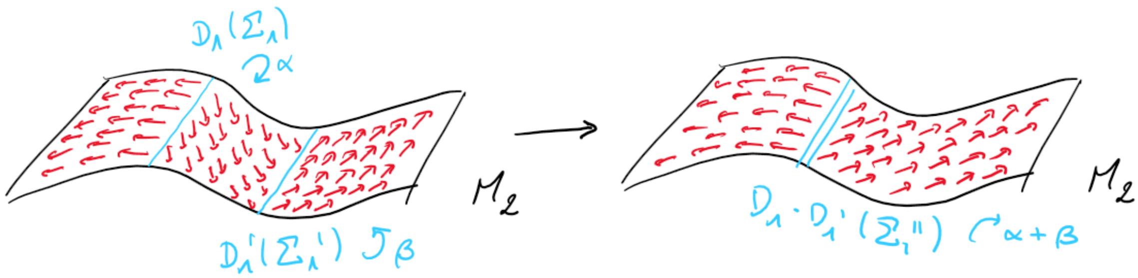

Figure 7: Multiplication of two defects

we get them closer and closer, the resulting defect is a change of angle $\alpha + \beta$ in the vector field.

We can also construct morphisms between defects. In our example, if we add a point charge in the middle of

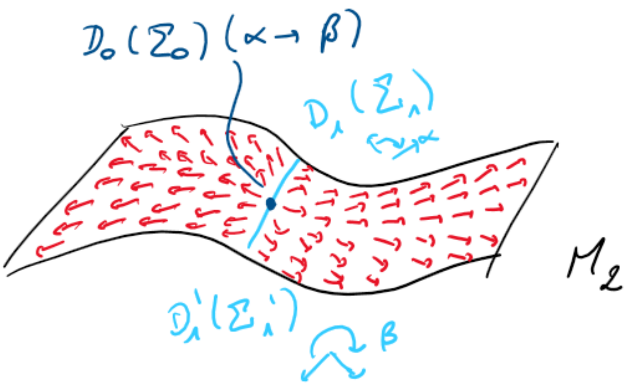

Figure 8: Adding a point charge leads to a morphism between two line defects

our line of electrons, we get the vector fields depicted in Figure 8.

So now we have some intuition of how to construct a structure on our defects. In our example, they form a group, but in general it is not sure you can find inverses, but we'll always be able to construct morphisms and to "tensor" defects of the same dimension, and we will get *fusion categories*. In the next paragraph, we'll see how to formalize the construction of those categories.

4.2 Symmetry category

In their paper *Non-invertible Higher-Categorical Symmetries* [5], Lakshya Bhardwaj, Lea E. Bottini, Sakura Schäfer-Nameki and Apoorv Tiwari construct higher categories out of defects.

Let $\mathcal{T}$ be the domain category of an n-dimensional topological field theory (i.e., the space-time manifold M_n dimension is n). We can construct an $(n-1)$-category $\mathcal{C}_{\mathcal{T}}$, called the *symmetry category* from the defects in $\mathcal{T}$.

- *Objects*. The objects in $\mathcal{C}_\mathcal{T}$ are the topological codimension-1 topological defects D_{n-1}. There exists an additive structure on topological defects that implies an additive structure on objects. So if $D_{n-1} = \oplus_i n_i D_{n-1}^{(i)}$ with $n_i > 0$, and the $D_{n-1}^{(i)}$ beeing distinct codimension-1 defects, the number n_i represent the number of vacua where D_{n-1} acts as $D_{n-1}^{(i)}$. We can also tensor two objects $D_{n-1}^{(1)}$ and $D_{n-1}^{(2)}$, and get

$$D_{n-1}^{(12)} := D_{n-1}^{(1)} \otimes D_{n-1}^{(2)} \tag{4.1}$$

by taking the *fusion* of those defects. We will not explain it here but you can see a detailed construction of this fusion in [4]. The picture of this construction is depicted in Figure 9 from [5].

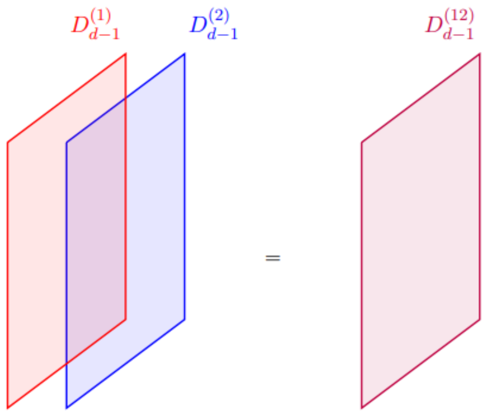

Figure 9: Fusion of defects.

- 1-*morphisms*. The 1-morphisms in $\mathcal{C}_\mathcal{T}$ are topological codimension-2 topological defects D_{n-2}. More precisely,

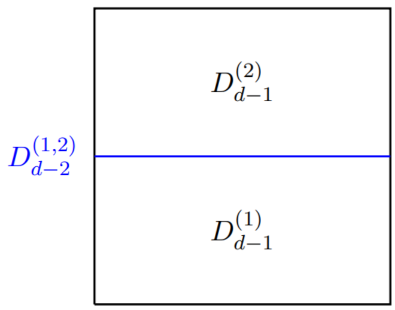

Figure 10: Morphism of defects.

a morphism between defects $D_{n-1}^{(1)}$ and $D_{n-1}^{(2)}$ is a defect $D_{n-2}^{(1,2)}$ located in the junction of the loci of $D_{n-1}^{(1)}$ and $D_{n-1}^{(2)}$. See Figure 10 from [5] for q picture of those morphisms. Similarly as for objects, there is an additive structure on $n-2$ topological defects implying additive structure on $1-morphisms$. So if $D_{n-2} = \oplus_i n_i D_{n-2}^{(i)}$ with $n_i > 0$, and the $D_{n-2}^{(i)}$ beeing distinct codimension-1 defects, the number n_i represent the number of vacua where D_{n-2} acts as $D_{n-2}^{(i)}$. Morphisms can be composed in the following way. Let $D_{n-2}^{(1,2)}$ be a morphism from $D_{n-1}^{(1)}$ to $D_{n-1}^{(2)}$ and $D_{n-2}^{(2,3)}$ be a morphism from $D_{n-1}^{(2)}$ to $D_{n-1}^{(3)}$. Let us define $D_{n-2}^{(2,3)} \circ D_{n-2}^{(1,2)} := D_{n-2}^{(1,2)} \otimes_{D_{n-1}^{(2)}} D_{n-2}^{(2,3)}$ by the fusion of $D_{n-2}^{(1,2)}$ and $D_{n-2}^{(2,3)}$ along $D_{n-1}^{(2)}$, as depicted in Figure 11 from [5]. Now let $D_{n-2}^{(1,2)}$ be a 1-morphism from $D_{n-1}^{(1)}$ to $D_{n-1}^{(2)}$ and $D_{n-2}^{(3,4)}$ be a 1-morphism from $D_{n-1}^{(3)}$ to $D_{n-1}^{(4)}$. Then we can construct their fusion

$$D_{n-2}^{(1,2)} \otimes D_{n-2}^{(3,4)},$$

which is a 1-morphism from $D_{n-1}^{(12)}$ to $D_{n-1}^{(34)}$ as defined in (4.1).

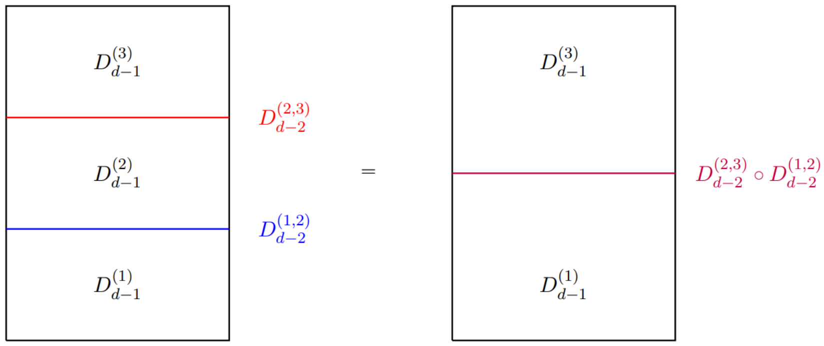

Figure 11: Composition of morphism of defects.

- *Higher morphisms.* Higher morphisms and their structure can be constructed similarly.

4.3 The 0-form gauging and the Ising category

Two defects of the same dimension can be linked by *condensation*. If they are, then it is possible to construct one defect from the other by the operation of *gauging*, which is detailed in [5]. In particular, if a theory $\mathcal{T}$ has a 0-form symmetry group G, then we can gauge it and construct the theory $\mathcal{T}/G$. We will explain how to construct this gauging going through the example 5.2 from [5].

4.3.1 The $\mathbb{Z}_2$ topological Gauge theories

First we will describe what are $\mathbb{Z}_2$ topological theories in general.

Definition 4.5. [27] Let $n \geq 0$. The *standard n-simplex* is the space $\Delta^n = \{(t_0,...t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1$ and $t_i \geq 0$ for all $i\}$. For $0 \leq i \leq n$ the *i-th face map* $d^i : \Delta^{n-1} \to \Delta^n$ is defined by $d_i(t_0,...,t_{n-1}) = (t_0,...,t_{n-1},0,t_i,...,t_{n-1})$■

Definition 4.6. [27] Let X be a space and let $n \geq 0$. A *singular n-simplex in X* is a continuous map $\sigma : \Delta^n \to X$. We will write $\sigma := (v_0, v_1, ..., v_n)$. Define $C_n(X)$ to be the free abelian group with basis the set of singular n-simplices of X. Thus an element of $C_n(X)$, called an *n—chain in X*, is a finite formal sum $c_1\sigma_1 + ... + c_r\sigma_r$ where the c_i are integers and the σ_i are singular n-simplices in X. ■

Definition 4.7. We can now define more generally the *n-chain* $C_n(X,A)$ of X on A, where A is an abelian group by

$$C_n(X,A) = C_n(X) \otimes_{\mathbb{Z}} A,$$

thus an element of $C_n(X,A)$ is a finite formal sum $c_1\sigma_1 + ... + c_r\sigma_r$ where the c_i are elements of A and the σ_i are singular n-simplices in X. Note that $C_n(X,\mathbb{Z}) = C_n(X)$. ■

Definition 4.8. [27] Let X be a space, A be an abelian group and $n \geq 1$. The *boundary map* $\delta_n : C_n(X,A) \to C_{n-1}(X,A)$ is the homomorphism which is defined on basis elements by sending a singular n—simplex σ to the sum

$$\delta_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d^i \qquad ■$$

Lemma 4.9. [27] The composite $\delta_{n-1} \circ \delta_n$ is zero.

Definition 4.10. [27] The *singular chain complex of X on A*, denoted $C(X,A)$, is the following chain complex:

$$\dots \xrightarrow{\delta_4} C_3(X,A) \xrightarrow{\delta_3} C_2(X,A) \xrightarrow{\delta_2} C_1(X,A) \xrightarrow{\delta_1} C_0(X,A) \longrightarrow 0 \qquad ■$$

Definition 4.11. Let X be a space and A an abelian group. The *n-cochains* $C^n(X,A)$ are the duals of the chains, i.e.,

$$C^n(X,A) := \mathrm{Hom(C_n(X,A),A)}. \qquad \blacksquare$$

Definition 4.12. Let X be a space, A be an abelian group and $n \geq 1$. The *coboundary map* or *differential* $\tilde{\delta}_n : C^n(X,A) \to C^{n+1}(X,A)$ is defined by

$$\tilde{\delta}_n(f) = f \circ \delta_{n+1},$$

for all $f : C_n(X,A) \to A$. $\blacksquare$

Definition 4.13. Let X be a space, A be an abelian group and $p,q \in \mathbb{N}$. The *cup product* $\smile: C^p(X,A) \times C^q(X,A) \to C^{p+q}(X,A)$ is defined as follows. For all $\alpha \in C^p(X,A)$, $\beta \in C^q(X,A)$, and $\sigma = (v_0, v_1, ..., v_{p+q}) : \Delta^{p+q} \to X$,

$$\alpha \smile \beta(\sigma) := \alpha(v_0, ..., v_p) \cdot \beta(v_p, ..., v_{p+q}) \qquad \blacksquare$$

Definition 4.14. Let $\mathcal{C}$ be a monoidal category. Its *Drinfeld center* is a monoidal category $\mathfrak{Z}_1(\mathcal{C})$ whose

- objects are pairs (X, ϕ) of an object $X \in \mathcal{C}$ and a natural isomorphism (braiding morphism)

$$\phi : X \otimes (-) \to (-) \otimes X$$

  such that for all $Y, Z \in \mathcal{C}$ we have

$$\phi_{Y \otimes Z} = (id_Y \otimes \phi_Z) \circ (\phi_Y \otimes id_Z),$$

  i.e., the following diagram commutes:

$$\begin{tikzcd}
 & (Y \otimes X) \otimes Z \arrow[r, "\cong"] & Y \otimes (X \otimes Z) \arrow[dr, "id_Y \otimes \phi_Z"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\phi_Y \otimes id_Z"] \arrow[dr, "\cong"'] & & & Y \otimes (Z \otimes X) \arrow[dl, "\cong"] \\
 & X \otimes (Y \otimes Z) \arrow[r, "\phi_{Y \otimes Z}"'] & (Y \otimes Z) \otimes X, &
\end{tikzcd}$$

- morphisms are given by

$$\mathrm{Hom((X,\phi),(Y,\psi)) = \{f \in Hom_{\mathcal{C}}(X,Y) \mid (id_Z \otimes f) \circ \phi_Z = \psi_Z \circ (f \otimes id_Z), \forall Z \in \mathcal{C}\}},$$

  i.e., $f \in \mathrm{Hom((X,\phi),(Y,\psi))}$ if and only if the following diagram commutes:

$$\begin{tikzcd}
X \otimes Z \arrow[r, "\phi_Z"] \arrow[d, "f \otimes id_Z"'] & Z \otimes X \arrow[d, "id_Z \otimes f"] \\
Y \otimes Z \arrow[r, "\psi_Z"'] & Z \otimes Y.
\end{tikzcd}$$

- the tensor product is given by

$$(X, \phi) \otimes (Y, \psi) = (X \otimes Y, (\phi \otimes id_Y) \circ (id_X \otimes \psi)),$$

  by which we mean that for all $Z \in \mathcal{C}$, we take the composition

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "id_X \otimes \psi_Z"] & X \otimes (Z \otimes Y) \arrow[r, "\cong"] & (X \otimes Z) \otimes Y \arrow[r, "\phi_Z \otimes id_Y"] & (Z \otimes X) \otimes Y \arrow[d, "\cong"] \\
(X \otimes Y) \otimes Z \arrow[u, "\cong"] \arrow[rrr, "((\phi \otimes id_Y) \circ (id_X \otimes \psi))_Z"'] & & & Z \otimes (X \otimes Y).
\end{tikzcd} \qquad \blacksquare$$

For G a group, let us define the category RepG to be the category of finite dimensinoal representations of G.

A $\mathbb{Z}_2$-Gauge field theory in $3d$ is a functor $\mathcal{Z} : \mathrm{Bord}_3(\mathcal{F}) \to \mathcal{C}$ as defined in definitions 3.22 and 3.23 with background field $\mathcal{F} = C^1(M,A) \times C^1(M,A)$ with M the space time manifold and A an abelian group, such that $\mathcal{Z}$ acts as follows on closed manifolds

- $\mathcal{Z} : M^3 \mapsto x$, i.e., $\mathcal{Z}$ maps closed 3-manifolds M^3 to a number x ;
- $\mathcal{Z} : \Sigma \mapsto V$, i.e., $\mathcal{Z}$ maps closed 2-manifolds Σ to a vector space V ;
- $\mathcal{Z} : S^1 \mapsto \mathfrak{Z}_1(\mathrm{Rep}\mathbb{Z}_2)$, i.e., $\mathcal{Z}$ maps closed 1-manifolds to the Drinfeld center of $\mathrm{Rep}\mathbb{Z}_2$;
- $\mathcal{Z} : \bullet \mapsto \mathrm{Rep}\mathbb{Z}_2$, i.e., $\mathcal{Z}$ maps closed 0-manifolds to the tensor category $\mathrm{Rep}\mathbb{Z}_2$.

The Drinfeld center of $\mathrm{Rep}\mathbb{Z}_2$ is braided equivalent to $\mathrm{Rep}\mathbb{Z}_2 \boxtimes \mathrm{Vec}\mathbb{Z}_2$, where $\boxtimes$ is Deligne tensor product.

Definition 4.15. Let G be a topological group. The *trivial G-principal bundle* over a topological space X is a topological space P equipped with

- the projection map $p_1 : X \times G \to X$;
- the action of G on $X \times G$ by right multiplication of G on itself. ■

Definition 4.16. A *G-principal bundle* over a topological space X is a topological space P equipped with

- a continuous function $p : P \to X$;
- an *action* $\rho : P \times G \to P$ of G on P over X, hence fitting into a coequalizing diagram

$$\begin{tikzcd} P \times G \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\rho"] \\ P \arrow[d, "p"] \\ X, \end{tikzcd}$$

  with $p_1 : P \times G \to P$ the proection map on the first component.

- such that this is is *locally trivial* in the sense that there exists a cover $U \to X$ and a continuous map $P_{|U} \to U \times G$ from the pullback of P to the cover to the trivial G-principal bundle on the cover, which is an isomorphism of G-actions over U

$$\begin{tikzcd} U \times G \arrow[dr, "p_1"'] & P_{|U} \arrow[l, "\simeq_G"'] \arrow[r] & P \arrow[d, "p"] \\ & U \arrow[r] & X \end{tikzcd}$$

■

For a closed 3-manifold M^3, the number associated to M^3 by $\mathcal{Z}$ is

$$\mathcal{Z}(M^3) = \sum_{C_i} \frac{1}{|C_i|},$$

with the sum beeing on all class of isomorphism of $\mathbb{Z}_2$-bundles on M^3 C_i. and $|C_i|$ beeing the number of automorphisms in the class. The fields in $\mathbb{Z}_2$- Gauge theory on a space time manifold M are of the kind $C^1(M,A) \times C^1(M,A)$, with A an abelian group. In the example we are following from [5], we look at the action on a 3d manifold M,

$$S = i\pi \int_M b_1 \smile \tilde{\delta} a_1,$$

where $a_1, b_1 \in C^1(M, \mathbb{Z}_2)$. It has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry, depicted by the line defects corresponding to the electric and magnetic fields $D_1^{(e)}(\gamma) = exp\left(i\pi \oint_\gamma a_1\right)$ and $D_1^{(m)}(\gamma) = exp\left(i\pi \oint_\gamma b_1\right)$. Let us just stop here to understand those integrals. We know that $a_1, b_1 \in C^1(M, \mathbb{Z}_2)$.

$$C_1(M, \mathbb{Z}_2) = \left\{ \sum_i c_i \sigma_i \mid \sigma_i : \Delta^1 \to M, c_i \in \{0, 1\} \right\},$$

but since $\Delta^1 \cong I = [0,1]$, the singular 1-simplex σ_i are paths in M. Now since $C^1(M,\mathbb{Z}_2) = \mathrm{Hom}(\mathrm{C_1(M},\mathbb{Z}_2),\mathbb{Z}_2)$, we have that every map $f \in C^1(M,\mathbb{Z}_2)$ is determined by what value between 0 and 1 they assign to every path σ in M. We take the convention that

$$f(\sigma_1)+f(\sigma_2) = \begin{cases} 0 & \text{if } f(\sigma_1) = f(\sigma_2) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

This means that $\oint_\gamma a_1 = a_1(\gamma) \in \{0,1\}$ and $\oint_\gamma b_1 = b_1(\gamma) \in \{0,1\}$. Which means now that $D_1^{(e)}(\gamma) = exp\left(i\pi \oint_\gamma a_1\right) \in \{\pm 1\} = \mathbb{Z}_2^{(e)}$ and $D_1^{(m)}(\gamma) = exp\left(i\pi \oint_\gamma b_1\right) \in \{\pm 1\} = \mathbb{Z}_2^{(m)}$. The 1-form symmetry is then the group

$$\Gamma^{(1)} = \mathbb{Z}_2^e \times \mathbb{Z}_2^m.$$

The diagonal subgroup of $\Gamma^{(1)}$ is generated by the fermionic line $D_1^{(f)}(\gamma) = D_1^{(e)}(\gamma) D_1^{(m)}(\gamma)$.

4.3.2 0-form gauging

In the example model we are looking at, there is also a 0-form symmetry

$$\Gamma^{(0)} = \mathbb{Z}_2^{em},$$

which acts on the line defects by exchanging $D_1^{(e)}$ with $D_1^{(e)}$, and leaving the rest unchanged. In this configuration, we will be able to gauge the model. So let us go back to theory.

Suppose we have a theory $\mathcal{T}$ on a n-dimensional space-time manifold. Let us define the category $\mathcal{C}_{\mathrm{id},\mathcal{T}}$ to be the subcategory of $\mathcal{C}_\mathcal{T}$ obtained by forgetting about all the non-trivial codimension-1 defects. Suppose moreover that there is a 0-form symmetry corresponding to a group G. Then we can be gauge to obtain the theory $\mathcal{T}/G$. So $\mathcal{C}_\mathcal{T}$ contains codimension-1 defects $D_{n-1}^{(g)}$ parametrized by $g \in G$. We also assume that there is no 1-morphism between $D_{n-1}^{(g)}$ and $D_{n-1}^{(g')}$ for $g \neq g'$. The tensor product of these objects follows the group operation on G:

$$D_{d-1}^{(g)} \otimes D_{d-1}^{(g')} = D_{d-1}^{(gg')}.$$

The action of G on defects can be depicted as in Figure 12 from [5]. So if we go back to our example, the defects

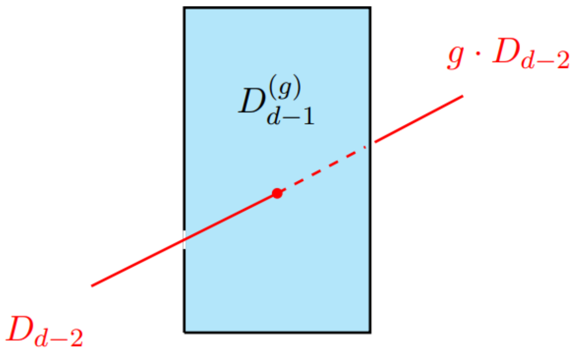

Figure 12: Fusion of defects.

$D_2^{(-1)}$ could be represented as a kind of electromagnetic converter since this "wall" should convert electric field to magnetic field and vice versa.

Let us go back to theory, but suppose from now that we are in $3d$. We can then construct the category $\mathcal{C}_{\mathrm{id},\mathcal{T}/G}$ of the $3d$ theory $\mathcal{T}/G$ obtained after gauging the 0-form symmetry G of the $3d$ theory $\mathcal{T}$. The objects of $\mathcal{C}_{\mathrm{id},\mathcal{T}/G}$ are the genuine line defects in the theory $\mathcal{T}/G$. They form a sub category RepG. The irreducible representation of G will be the *simple* objects of $\mathcal{C}_{\mathrm{id},\mathcal{T}/G}$. However, not every objects in $\mathcal{C}_{\mathrm{id},\mathcal{T}}$ will be objects in $\mathcal{C}_{\mathrm{id},\mathcal{T}/G}$ since we gauge by G, so only the objects that are left invariant by G will become an object in $\mathcal{C}_{\mathrm{id},\mathcal{T}/G}$, the others will form one simple object with all the other objects of their orbit $\mathcal{O}$. A simple object $D_1^{(\mathcal{O})}$ arising this way is described as

$$D_1^{(\mathcal{O})} \equiv \bigoplus_{i \in \mathcal{O}} D_1^{(i)},$$

in $\mathcal{C}_{\mathrm{id},\mathcal{T}}$, where $D_1^{(i)}$ are distinct simple objects of $\mathcal{C}_{\mathrm{id},\mathcal{T}}$ lying in an orbit $\mathcal{O}$ of the G action. Now there is a last kind of simple objects. For all orbit $\mathcal{O}$, there is a subgroup $G_{\mathcal{O}} \subset G$, being the stabilizer of every object $D_1^{(i)}$ for $i \in \mathcal{O}$. This means that the object $D_1^{(\mathcal{O})}$ can actually be represented by all the representations of $G_{\mathcal{O}}$, so the simple objects corresponding to the orbit $\mathcal{O}$ will be represented by

$$D_1^{(\mathcal{O},R_{\mathcal{O}})},$$

where $R_{\mathcal{O}}$ is an irreducible representation of $G_{\mathcal{O}}$. $D_1^{(\mathcal{O})} = \bigoplus_{i\in\mathcal{O}} D_1^{(i)}$ is obtained by choosing $R_{\mathcal{O}}$ to be the trivial representation. The simple objects in the subcategory RepG are actully special case of those last defined simple object, taking $\mathcal{O}$ to be the orbit of the identity object of $\mathcal{C}_{id,\mathcal{T}}$, for which the stabilizer is the whole group G. We represent those simple objects as

$$D_1^{(R)},$$

where R is an irreducible representation of G. The identity object of $\mathcal{C}_{id,\mathcal{T}/G}$ is denoted as

$$D_1^{(id)},$$

obtained by taking the trivial representation of G.

In our example, the simple objects are

- the simple objects left invariant by $\Gamma^(0)$, i.e., $D_1^{(id)}$ and $D_1^{(f)}$ split into two objects since $\mathbb{Z}_2$ has two irreducible representations, the trivial one and $\rho^- : \mathbb{Z}_2 \to \mathbb{C}$

$$\rho(1) = 1,$$
$$\rho(-1) = -1.$$

  So we have
$$\left(D_1^{(id)}, D_1^{(f)}\right)_{\mathcal{C}_{\mathbb{Z}_2}} \mapsto \left(D_1^{(id)}, D_1^{(-)}, D_1^{(f)}, D_1^{(f-)}\right)_{\mathcal{C}_{\mathbb{Z}_2/\Gamma^{(0)}}}.$$

- The objects $D_1^{(e)}$ and $D_1^{(f)}$ combine to one simple object

$$D_1^{(e,m)} := (D_1^{(e)} \oplus D_1^{(m)})_{\mathcal{C}_{\mathbb{Z}_2}},$$

  since their stabilizer is the identity.

Then using the fusion rules defined in [5, 4.2], we get that $\mathcal{C}_{\mathbb{Z}_2/\Gamma^{(0)}}$ can be identified to a subcategory of the fusion category Ising $\times$ $\overline{\text{Ising}}$. The objects of which are

$$\mathcal{C}^{\mathrm{ob}}_{\mathrm{Ising}} = \left\{D_1^{(1)}, D_1^{(\psi)}, D_1^{(\sigma)}\right\}$$
$$\mathcal{C}^{\mathrm{ob}}_{\overline{\mathrm{Ising}}} = \left\{D_1^{(1)}, D_1^{(\overline{\psi})}, D_1^{(\overline{\sigma})}\right\},$$

identifying the labels $(\mathrm{id}) \sim (1), (\mathrm{f}) \sim (\psi), (\mathrm{f}-) \sim (\overline{\psi}), (-) \sim (\psi\overline{\psi})$ and $(e,m) \sim (\sigma\overline{\sigma})$.

nNow that we have discussed how to construct a category of defects, we will see in the next paragraph how we can make them act on a quantum field theory.

4.4 Action of defects on Field Theories

We now discuss how D.S. Freed, G.W. Moore and C. Teleman describe the action of defects on quantum field theory in their joint work [13]. The idea is that if you have an n-dimensional theory ρ with defect, you want to extend it to an $n+1$-theory with one boundary, and this boundary beeing ρ. This build what is called a *quiche*, since quiche in french means a pie with crust under but not above, so the crust will be ρ, and the filling will be σ, as depicted in Figure 13. Then if we want to apply those defect on another n-dimensional theory F, we will have to find an n-dimensional theory $\tilde{F}$ such that when we close our quiche to make a *sandwich*, then the sandwich is isomorphic to F. We will study topological field theories $\sigma : Bord_m(\mathcal{F}) \to \mathcal{C}$ with $\mathcal{F}$ its background field.

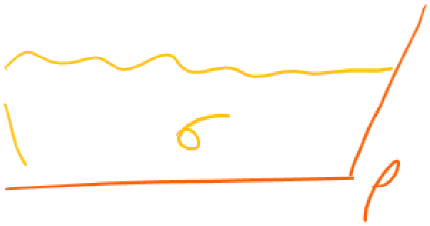

Figure 13: Quiche.

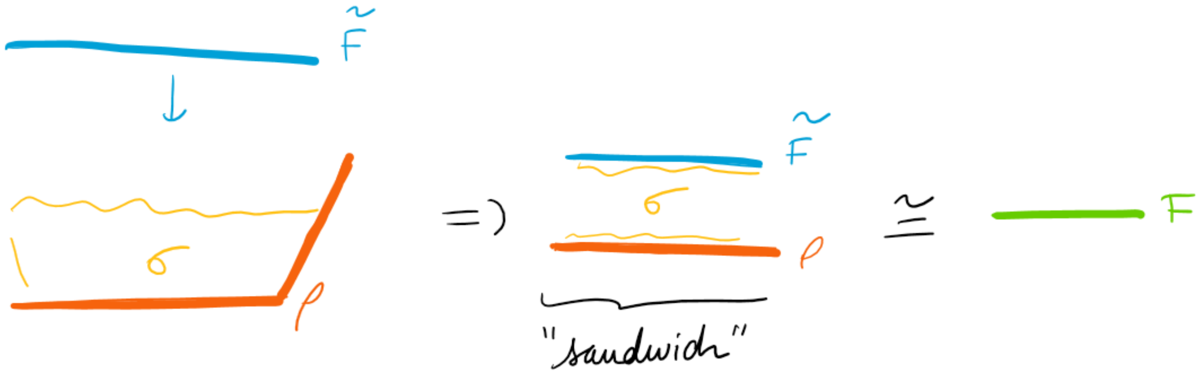

Figure 14: Action of defects on a theory F being a quiche with a boundary, i.e., a sandwich.

Definition 4.17. Let σ_1, σ_2 be $(n+1)$-dimensional theories on background fields $\mathcal{F}_1$, $\mathcal{F}_2$ with common codomain $\mathcal{C}$. A *domain wall* $(\sigma_1, \sigma_2, \delta)$ assigns a codimension-1 defect $D_n^{(12)} = \delta(M^1_{n+1}, M^2_{n+1})$ from $D^{(1)}_{n+1} = \sigma_1(M^1_{n+1})$ to $D^{(2)}_{n+1}\sigma_2(M^2_{n+1})$ to every M^1_{n+1} and M^2_{n+1} being $n+1$-manifolds with corners in Bord_{n+1}. ■

Definition 4.18. Let σ, be $(n+1)$-dimensional theory, then a *right boundary* or of σ or *right toplogical σ-module* is a domain wall $\sigma \to \mathbb{1}$; a *left boundary* of σ *right toplogical σ-module* is a domain wall $\mathbb{1} \to \sigma$. ■

Remark 4.19. There is a more mathematical definition of defects that you can find in [13] using the notion of link from section 3.3. ◇

Definition 4.20. [13] Fix $n \in \mathbb{Z}$. An *n-dimensional quiche* is a pair (σ, ρ), where $\sigma : \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n+1)$-dimensional topological field theory and ρ is a right topological σ-module. ■

The idea is that we will now see how this defect ρ acts on a quantum field theory of dimension n.

Definition 4.21. [6] A higher category $\mathcal{C}$ is called *fully dualizable* if its objects have duals and its morphisms have adjoints at all levels. More generally, $\mathcal{C}$ is *k-dualizable* if all objects have duals and all morphisms of degree less than k have adjoints. ■

Definition 4.22. [13] Suppose $\mathcal{C}'$ is a symmetric monoidal n-category and σ is an $(n+1)$-dimensional topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $A = \sigma(\mathrm{pt})$. Then A is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n+1)$-dualizable. Assume that the *right regular module* A_A (i.e., the vector space A furnished with the right action of A by multiplication) is n-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory ρ determined by A_A is the *right regular boundary theory* of σ, or the *right regular σ-module* ■

Remark 4.23. The regular boundary theory is often calleda *Dirichlet boundary theory*. ◇

Definition 4.24. [13] Let (σ, ρ) be an n-dimensional quiche. Let F be an n-dimensional field theory. A (σ, ρ)-*module structure* on F is a pair $(\tilde{F}, \theta)$ in which $\tilde{F}$ is a left σ-module and θ is an isomorphism

$$\theta : \rho \otimes_\sigma \tilde{F} \xrightarrow{\cong} F,$$

of absolute n-dimensional theories. ■

Here $\rho \otimes_\sigma \tilde{F}$ notates the dimensional reduction of σ along the closed interval with boundaries colored with ρ and $\tilde{F}$. The bulk theory σ with its right and left boundary theories ρ and $\tilde{F}$ is sometimes called a *sandwich*. And now we have an idea of how to make a symmetry act on a quantum field theory.

5 Constructing Correlation Functions for Rational Conformal Field Theories

5.1 Outline

In our study of correlators of rational conformal field theories as in other parts of this report symmetries shall play a crucial role. Hence, we shall begin by studying vertex algebras which encode the symmetries, in the form of the Ward identities, of chiral CFTs. Then in section 5.3 we shall explore the structure of the category of modules over a vertex algebra. Inspired by the OPE of primary fields we shall introduce the fusion rules which give this category the structure of a braided tensor category. This will lead us to the notion of a rational VOA were it is known such a fusion[1] product always exists assuming a technical condition.

In section 5.4 we begin our study of correlation functions by introducing conformal blocks. We begin by defining conformal blocks for the affine Kac-Moody VOA and to connect this abstract definition to physics we shall show how this definition implies the global Ward-identities. We then discuss how conformal blocks vary as we vary the marked points and the sewing procedure. Finally in section 5.5 we introduce the Fuchs-Runkel-Schweigert construction of RCFT correlators. We then provide physical motivation for the role of the special symmetric Frobenius algebra in the construction and look at a simple example.

5.2 Vertex Algebras

5.2.1 What is a Vertex Algebra?

As we have outlined our goal is to construct correlators of rational conformal field theories. However, as full CFTs are quite complicated to study mathematically we shall start in a setting where we have a solid mathematical understanding which is the symmetry algebra of the chiral part of the theory, which is described by the theory of vertex algebras. Vertex algebras include the symmetries of many well studied examples in physics such as the Wess-Zumino-Witten and Liouville theories which will serve as motivating examples in this section. Now before we can define a vertex algebra, which is a vector space V of states to which we associate a field $Y(A,z)$ for every $A \in V$ by a state-field correspondence we must first introduce the notion of a field.

Definition 5.1. Let V be a vector space over $\mathbb{C}$. We say a formal power series $A(z) \in \mathrm{End}V[[z^{\pm 1}]]$, which may write as

$$A(z) = \sum_{j \in \mathbb{Z}} A_j z^{-j} \in \mathrm{End}V[[z^{\pm 1}]], \tag{5.1}$$

is a *field* on V if for every $v \in V$ we have that $A_j v = 0$ for large enough j. ∎

If V is a $\mathbb{Z}$-graded vector space we can introduce the notion of conformal dimension.

Definition 5.2. Suppose V is a $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

Then we say a linear map $\phi : V \longrightarrow V$ is *homogeneous of degree m* if $\phi(V_n) \subset V_{n+m}$ for all n. ∎

Definition 5.3. We say a field $A(z)$ is of *conformal dimension* $\Delta \in \mathbb{Z}$ if it is an $\mathrm{End}V$ valued formal power series of the form as in equation (5.1) and such that each A_j is a homogeneous linear operator on V of degree $-j + \Delta$. ∎

We now have all the preliminary definitions we need to define a vertex algebra following the statement in [14].

Definition 5.4. A *vertex algebra* consists of the following collection of data:

- **Space of States**: A vector space V;
- **Vacuum Vector**: A vector $|0\rangle \in V$;

[1] In this section we shall use physics convention and often replace the use of the word tensor in section 2 with the word fusion. This is to avoid confusion between the fusion product and the ordinary tensor product of representations both of which shall play a role in this section.

- **Translation Operator**: A linear operator $T : V \longrightarrow V$;
- **Vertex Operator**: A linear operation

$$Y(\cdot, z) : V \longrightarrow \mathrm{End}V[[z^{\pm 1}]],$$

taking each $A \in V$ to a field acting on V we notate by

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}.$$

This collection of data is subject to the following axioms

- **Vacuum axiom**: $Y(|0\rangle, z) = id_V$. Furthermore, for any $A \in V$ we have that

$$Y(A, z)|0\rangle \in V[[z]],$$

and hence $Y(A, z)|0\rangle$ has a well defined value at $z = 0$. We require that

$$Y(A, z)|0\rangle|_{z=0} = A,$$

which means that $A_{(n)}|0\rangle = 0$ for $n \geq 0$ and $A_{(-1)}|0\rangle = A$.

- **Translation Axiom**: For any $A \in V$,

$$[T, Y(A, z)] = \partial_z Y(A, z),$$

and we require that $T|0\rangle = 0$.

- **Locality Axiom**: All fields $Y(A, z)$ are local with respect to each other, meaning for every $A, B \in V$ there exists a non negative integer N such that

$$(z - w)^N [Y(A, z), Y(B, w)] = 0 \tag{5.2}$$

.

If V is a $\mathbb{Z}$-graded vector space we require that $|0\rangle$ is a vector of degree 0, T is a linear operator of degree 1, and for $A \in V_m$ the field $Y(A, z)$ has conformal dimension m, meaning $\deg A_{(n)} = -n + m - 1$. ■

At first this long list of axioms may be intimidating but it captures many of the aspects of CFT known in examples. For instance, one of the most important properties of a CFT is the state field correspondence, which states that every field $\phi(z)$ is in one to one correspondence with a state $|\phi\rangle$, where the correspondence is given by

$$|\phi\rangle = \lim_{z \to 0} \phi(z)|0\rangle \tag{5.3}$$

This principle motivates the introduction of what in the mathematics literature is called a vertex operator[2]. The vertex operator $Y(., z)$ gives a way to associate a field to each element of the vertex algebra giving one direction of the correspondence and the vacuum axiom conversely ensure that when we apply $Y(A, z)$ to $|0\rangle$ and take the limit as z goes to 0 that I get back A. A precise statement of the state-field correspondence for vertex algebras is given by Goodard's uniqueness theorem following the statement in [14].

Theorem 5.5. Let V be a vertex algebra and let $A(z)$ be a field on V. Suppose that there exists a vector $a \in V$ such that

$$A(z)|0\rangle = Y(a, z)|0\rangle,$$

and $A(z)$ is local with respect to $Y(b, z)$ for all $b \in V$. Then $A(z) = Y(a, z)$.

[2]This terminology can be confusing as they are not the same as what a physicists would call a vertex operator. For instance, what a physicist would call a vertex operator for the world sheet CFT for bosonic string theory a mathematician would call a intertwining operators of Herisenberg algebra modules.

The proof of this theorem is a nice example to illustrate how one can use these axioms. We have to show $A(z)b = Y(a,z)b$ for every $b \in B$, but the vacuum axiom allows us to replace $b|0\rangle$ with $Y(B,w)|_{w=0}|0\rangle$, and the vertex operators are then easier to manipulate as we have the locality axiom.

Proof. By locality we know there exists N large enough we get the following equalities in $V[[z^{\pm 1}, w^{\pm 1}]]$ for every $b \in V$

$$\begin{aligned}(z-w)^N A(z)Y(b,w)|0\rangle =&(z-w)^N Y(b,w)A(z)|0\rangle \\ =&(z-w)^N Y(b,w)Y(a,z)|0\rangle \\ =&(z-w)^N Y(a,z)Y(b,w)|0\rangle\end{aligned}$$

where the first and second equality use our assumptions and the last uses the locality axiom. Then applying the vacuum axiom we may take the limit as w goes to 0 to get $z^N A(z)b = z^N Y(a,z)b$ and since we are in $V[[z^{\pm 1}, w^{\pm 1}]]$ we may divide out by z^N to get $A(z)b = Y(a,z)b$ for every $b \in V$ and thus $Y(a,z) = A(z)$. □

To motivate the locailty axiom, one should think of a vertex algebra as being inserted at a point x on a Riemann surface with local coordinate z centred at x. Then one may think of the translation axiom as telling us how our vertex operators changes under infinitesimal translation, which is captured in a lemma one can prove using the translation axiom that

$$Y(TA,z) = \partial_z Y(A,z)$$

for all $A \in V$. We shall come back to this idea of varying the insertion point of our vertex algebra, when we introduce vertex operator algebras.

Finally we have already seen how the locality axiom is useful in proving Goddard's uniqueness theorem. At first one may wonder why we need to multiply by $(z-w)^N$, however, this is necessary as otherwise we would get a commutative vertex algebra, and most of the physically interesting examples such as those associated to WZW and Lioville theories are not commutative.

We shall finish this subsection by introducing the concept of a vertex operator algebra (also called a conformal vertex algebra). In order to do this we must introduce the Virasoro algebra.

Definition 5.6. Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Then the *Virasoro algebra* is the central extension of $Der\mathcal{K} = \mathbb{C}((t))\partial_t$ such that

$$0 \longrightarrow \mathbb{C}C \longrightarrow Vir \longrightarrow Der\mathcal{K} \longrightarrow 0$$

defined via the commutation relations

$$[f(t)\partial_t, g(t)\partial_t] = (fg' - f'g)\partial_t - \frac{1}{12}(Res_{t=0} f g''' dt)C \tag{5.4}$$

■

The Virasoro algebra has topological generators C and

$$L_n = -t^{n+1}\partial_t$$

satisfying

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{n^3-n}{12}\delta_{n,-m}C. \tag{5.5}$$

Now we are ready to define a VOA

Definition 5.7. A $\mathbb{Z}$-graded vertex algebra V is called a *vertex operator algebra (VOA)*, with central charge $c \in \mathbb{C}$, if there exists a non-zero conformal vector $\omega \in V_2$ such that the Fourier coefficients L_n^V of the corresponding vertex operator

$$T(z) := Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^V z^{-n-2}, \tag{5.6}$$

satisfy the defining relations of the Virasoro algebra with central charge c. ■

In physics one usually thinks of L_0 as the energy operator, as our vertex algebra are usually given a grading by eigenspaces of L_0, meaning that $V = \oplus V_n$, such that $L_0 V_n = nV_n$. Furthermore, one can show that L_{-1} is the translation operator as in [14]. One should think of this additional structure as telling us how our vertex algebra varies under change of coordinates. This allows one to define vertex algebras on a Riemann surface locally using the above definition and then using the conformal structure to define changes of coordinates. This perspective is spelled out in detail in [14] chapters 6 to 8.

5.2.2 Example: Affine Kac-Moody Vertex Algebra

First we shall look at the Affine Kac-Moody algebra, which builds a vertex algebra from a finite dimensional, simple Lie algebra which captures the symmetries of the chiral part of the WZW theory.

Central Extensions and the Universal Enveloping Algebra

We begin with $\mathfrak{g}$ a finite dimensional, simple Lie algebra and then we define.

Definition 5.8. A *1-dimensional central extension* $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ is a Lie algebra defined via the short exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0,$$

such that $\mathfrak{a} = \mathbb{C} \cdot 1$ is central, meaning $[a, g] = 0$ for every $a \in \mathfrak{a}$ and $g \in \widehat{\mathfrak{g}}$. The Lie algebra structure on $\widehat{\mathfrak{g}}$ which is central with respect to $\mathfrak{a}$ is defined by

$$[x, y]_{\widehat{\mathfrak{g}}} = [x, y]_{\mathfrak{g}} + c(x, y)1, \tag{5.7}$$

for $x, y \in \mathfrak{g}$ and $c : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$, a 2-cocyle, meaning $c(x, y) = -c(y, x)$ and $c(x, [y, z]) + c(y, [z, x]) + c(z, [x, y]) = 0$. ■

Definition 5.9. Let $\mathfrak{g}$ be a Lie algebra, then we define

$$U(\mathfrak{g}) = \left(\bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}\right)/I, \tag{5.8}$$

where I is the two sided ideal generated by $x \otimes y - y \otimes x - [x, y]$ for every $x, y \in \mathfrak{g}$. We call $U(\mathfrak{g})$ the *universal enveloping algebra of* $\mathfrak{g}$ and it is an associative algebra with a unit. ■

This definition identifies the Lie bracket with the commutator meaning that any Lie algebra representation may be thought of as a module over its universal enveloping algebra.

Remark 5.10. By the Poincaré–Birkhoff–Witt (PBW) theorem if $\{x_i\}_{i\in I}$ is an ordered basis of $\mathfrak{g}$ then the lexicographically ordered monomials $x_{i_i} \ldots . x_{i_n}$ for $i_1 \preccurlyeq i_2 \preccurlyeq \ldots \preccurlyeq i_n$ form a basis of $U(\mathfrak{g})$ called the PBW basis.

The Affine Kac-Moody Algebra

Before we can define a Kac-Moody algebra we will need to discuss invariant bilinear forms for a simple Lie algebra $\mathfrak{g}$.

Definition 5.11. A bilinear form on a simple Lie algebra $\mathfrak{g}$ is called *invariant* if

$$([x, y], z) + (y, [x, z]) = 0,$$

for every $x, y, z \in \mathfrak{g}$. ■

This condition is the infinitesimal version of requiring the form to be invariant under the adjoint action.

Definition 5.12. An example of such a form is given by starting with a faithful finite-dimensional representation $\rho_V : \mathfrak{g} \longrightarrow \mathrm{End}V$ of $\mathfrak{g}$ and then defining

$$(x, y) = Tr_V(\rho_V(x)\rho_V(y)).$$

In the case where ρ_V is the adjoint representation this is called the *Killing form* and is denoted $(.,.)_K$. Then for $h^\vee$ the dual Coxeter number we define

$$(x,y) = \frac{1}{2h^\vee}(x,y)_K.$$

Now we can define a affine Kac-Moody algebra as a central extension of the loop algebra of a Lie algebra. We begin with $\mathfrak{g}$ a finite-dimensional simple Lie algebra over $\mathbb{C}$. We define the formal *loop algebra* of $\mathfrak{g}$ as the Lie algebra

$$L\mathfrak{g} = \mathfrak{g}((t)) = \mathfrak{g} \otimes \mathbb{C}((t)),$$

with Lie bracket

$$[A \otimes f(t), B \otimes g(t)] = [A,B] \otimes f(t)g(t).$$

■

Definition 5.13. We define an *affine Kac-Moody algebra* $\widehat{\mathfrak{g}}$ as a central extension

$$0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{\mathfrak{g}} \longrightarrow L\widehat{\mathfrak{g}} \longrightarrow 0,$$

with bracket $[K,\cdot] = 0$ and

$$[A \otimes f(t), B \otimes g(t)] = [A,B] \otimes f(t)g(t) - (Res_{t=0} f\, dg)(A,B)K. \tag{5.9}$$

■

Example 5.14. An example of an affine Kac-Moody algebra is the Heisenberg algebra in the case $\mathfrak{g} = \mathbb{C}$ which we denote $\mathcal{H}$. ♡

Before we can define a vertex algebra structure on $\widehat{\mathfrak{g}}$ we must first discuss its representation theory. We shall induct from a trivial representation of $\mathbb{C}[[t]] \oplus \mathbb{C}K$. For an affine Kac-Moody algebra we begin by noting the bracket is trivial on $\mathfrak{g}[[t]]$ and hence the central extension is trivial on this subspace. Therefore, $\mathfrak{g}[[t]] \oplus \mathbb{C}K$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$.

Definition 5.15. Let $\mathbb{C}_k$ be the one-dimensional representation on $\mathbb{C}$ where $\mathfrak{g}[[t]]$ acts trivially and K acts by multiplication by $k \in \mathbb{C}$. Then we define the *vacuum representation of level k of* $\widehat{\mathfrak{g}}$ as

$$V_k(\mathfrak{g}) = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K} \mathbb{C}_k = U(\widehat{\mathfrak{g}}) \otimes_{\mathfrak{g}[[t]] \oplus \mathbb{C}K} \mathbb{C}_k. \tag{5.10}$$

■

Notation 5.16. We shall denote the vertex algebra associated to the Heisenberg algebra by π, and shall call it the Heisenberg vertex algebra.

Vertex Algebra Structure

We shall first define some notation. Let $\{J^a\}_{a=1,\dots,\dim \mathfrak{g}}$ be an ordered basis of $\mathfrak{g}$. For any $A \in \mathfrak{g}$ and $n \in \mathbb{Z}$ we denote

$$A_n := A \otimes t^n \in L\mathfrak{g}.$$

Therefore, J^a_n for $n \in \mathbb{Z}$ and K form a topological basis for $\widehat{\mathfrak{g}}$. The commutator is then given by

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n(J^a, J^b)\delta_{n,-m}K. \tag{5.11}$$

Then let v_k be the image of $1 \otimes 1 \in U\widehat{\mathfrak{g}} \otimes \mathbb{C}_k$ in V_k, which we call the vacuum vector. By the PBW theorem we have a PBW basis for $V_k(\mathfrak{g})$ given by

$$J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_k,$$

for $n_1 \leq \dots \leq n_m < 0$, and if $n_i = n_{i+1}$ then $a_i \leq a_{i+1}$. We are now ready to define the vertex algebra structure.

Definition 5.17. We define a $\mathbb{Z}_+$-graded vertex algebra structure on $V_k(\mathfrak{g})$ as follows,

1. A vacuum vector $|0\rangle = v_k$.
2. A translation operator T defined by $Tv_k = 0$ and $[T, J^a_n] = -nJ^a_{n-1}$.
3. Vertex operators defined by $Y(v_k, z) = Id$,

$$Y(J^a_{-1}v_k, z) = J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1},$$

and in general

$$Y(J^{a_1}_{n_1}....J^{a_m}_{n_m}v_k, z) = \frac{1}{(-n_1-1)!...(-n_m-1)!} : \partial_z^{-n_1-1}J^{a_1}(z)....\partial_z^{-n_m-1}J^{a_m}(z) : . \tag{5.12}$$

4. A $\mathbb{Z}_+$-gradation given by

$$\deg J^{a_1}_{n_1}....J^{a_m}_{n_m}v_k = -\sum_{i=1}^{m} n_i.$$

■

For those who have not seen normal ordering before the : .... : denotes the normal ordering of fields defined by

$$: A(z)B(w) := \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} A_{(m)}B_{(n)}z^{-m-1} + \sum_{m\geq 0} B_{(n)}A_{(m)}z^{-m-1} \right) w^{-n-1}.$$

Then one can prove as [14] chapter 2 that this data defines a vertex algebra.

Finally we can define a conformal structure using the Segal-Sugawara vector. To construct this vector we pick a basis $\{J^a\}_{a=1,..,d}$ for $\mathfrak{g}$ and we denote its dual basis by $\{J_a\}_{a=1,..,d}$. Then we write $J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$ and $J_a(z) = \sum_{n\in\mathbb{Z}} J_{a,n}z^{-n-1}$, and we set

$$S = \frac{1}{2}\sum_{a=1}^{d} J_{a,-1}J^a_{-1}v_k. \tag{5.13}$$

For $k \neq h^\vee$, the vector $\omega = S/(k+h^\vee)$ is a conformal vector with central charge

$$c(k) = \frac{k \dim \mathfrak{g}}{k+h^\vee}.$$

A proof may be found in [14] using the OPE which we shall later introduce.

5.2.3 Example: Virasoro Vertex Algebra

In order to define our Virasoro VOA, the first bit of data we need is a vector space which we shall construct as a representation of the Virasoro algebra. We begin by noticing that $Der\mathcal{O} = \mathbb{C}[[t]]\partial_t$ is a Lie subalgebra and therefore we can induct the 1 -dimensional representation $\mathbb{C}_c$, where C acts by multiplication by $c \in \mathbb{C}$ and $Der\mathcal{O}$ acts trivially, to get

$$Vir_c = Ind^{Vir}_{Der\mathcal{O}\oplus\mathbb{C}C}\mathbb{C}_c = U(Vir)\otimes_{U(Der\mathcal{O}\oplus\mathbb{C}C)}\mathbb{C}_c \tag{5.14}$$

By the PBW theorem Vir_c has a basis consisting of monomials of the form

$$L_{j_1}......L_{j_m}v_c, \qquad j_1 \leq j_2 \leq ..... \leq j_m \leq -2. \tag{5.15}$$

We can then put a $\mathbb{Z}$-gradation on Vir_c by defining $degL_n = -n$ and $degv_c = 0$.

Now that we have defined a vector space we must provide the rest of the data of a VOA. We shall take $v_c \in Vir_c$ as the vacuum vector and L_{-1} as the translation operator. First we shall define the field corresponding to the state $L_2 v_c$ by

$$Y(L_{-2}v_c, z) = T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{5.16}$$

which we can use to define our fields in general by

$$Y(L_{j_1}......L_{j_m}v_c, z) = \frac{1}{(-j_1-2)!}....\frac{1}{(-j_m-2)!} :\partial_z^{-j_1-2}T(z).....\partial_z^{-j_m-2}T(z): \tag{5.17}$$

We shall conclude by noting that $L_{-2}v_c$ is the conformal vector as by definition $Y(L_{-2}v_c, z) = T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

5.3 Rep$V$ as a Modular Fusion Category

In [26], Moore and Seiberg show that the category of representations of a vertex algebra is given the structure of a modular fusion category by the fusion product. Our goal in this section will be to give an intuition of how we can get such a structure from a vertex algebra primarily following [29] and [15].

5.3.1 The Associativity Property and Operator Product Expansions

In the same spirit as how one can think of the locality axiom

$$(z-w)^N[Y(A,z), Y(B,w)] = 0,$$

for all A, B in a vertex algebra V, as a generalisation of commutativity, one may think of the equality

$$Y(A,z)Y(B,w)C = Y(Y(A,z-w)B,w)C \tag{5.18}$$

for $A, B, C \in V$ as a generalisation of associativity. This is called the associativity formula and is proven in chapter 3 of [14].

Remark 5.18. Experts should recall that the above equation is an abuse of notation as the left hand side lives in the ring $V((z))((w))$ and the right hand side lives in $V((w))((z-w))$ and we mean that both are expansions in their respective rings of a common element of $V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$. ◊

Remark 5.19. This property can be proven from the axioms of a vertex algebra by using some computational lemmas to rewrite both the left and right hand side in a form that allows one to apply the locality axiom[3] ◊

Now we can write (5.18), equivalently as

$$Y(A,z)Y(B,w)C = \sum_{n\in\mathbb{Z}} \frac{Y(A_{(n)}\cdot B, w)}{(z-w)^{n+1}}C, \tag{5.19}$$

which in physics is called the OPE formula. One can also show that

$$Y(A,z)Y(B,w) = \sum_{n\geq 0} \frac{Y(A_{(n)}\cdot B, w)}{(z-w)^{n+1}} + :Y(A,z)Y(B,w):, \tag{5.20}$$

In physics one often ignores the regular part so it is common to write

$$Y(A,z)Y(B,w) \sim \sum_{n\geq 0} \frac{Y(A_{(n)}\cdot B, w)}{(z-w)^{n+1}}. \tag{5.21}$$

[3] For the experts we wish to note the most useful statement of locality in this case is the one in terms of expansions of both matrix elements $\langle\varphi, Y(A,z)Y(B,w)v\rangle$ and $\langle\varphi, Y(B,w)Y(A,z)v\rangle$ in their respective rings of a common element $f_{\varphi,v} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.

Example 5.20. We shall give an example of an OPE by computing the OPE of two current fields in the affine Kac-Moody VOA

$$
\begin{aligned}
J^a(z)J^b(y) \sim & \sum_{n\geq 0} \frac{Y(J_n^a \cdot J_{-1}^b|0\rangle, w)}{(z-w)^{n+1}} \\
= & \frac{Y(J_1^a J_{-1}^b|0\rangle, w)}{(z-w)^2} + \frac{Y(J_0^a J_{-1}^b|0\rangle, w)}{(z-w)^1},
\end{aligned}
$$

where all other higher terms vanish since $[J_n^a, J_{-1}^b] = 0$ and $J_n^a|0\rangle = 0$ for $n \geq 2$. Then using the commutation relations of the Kac-Moody algebra we can compute,

$$
\begin{aligned}
= & \frac{Y(k(J^a, J^b)|0\rangle, w)}{(z-w)^2} + \frac{Y([J^a, J^b]|0\rangle, w)}{(z-w)^1} \\
= & \frac{k(J^a, J^b)}{(z-w)^2} + \frac{[J^a, J^b](w)}{(z-w)}.
\end{aligned}
$$

♡

5.3.2 The Fusion Product

Before we can define the fusion product we must first introduce modules over vertex algebras. Some physical example of elements of modules over vertex algebras are the states corresponding to primary fields.

Definition 5.21. [14] Let $(V, |0\rangle, T, Y)$ be a vertex algebra. A complex vector space M is called a *V-module* if it is equipped with an operation $Y_M : V \longrightarrow \mathrm{End} M[[z^{\pm 1}]]$ which assigns to each $A \in V$ a field

$$Y_M(A, z) = \sum_{n\in\mathbb{Z}} A_{(n)}^M z^{-n-1}, \tag{5.22}$$

on M, such that the following axioms hold

1. $Y_M(|0\rangle, z) = Id_M$;
2. for all $A, B \in V$ and $C \in M$

$$Y_M(A, z)Y(B, w)C = Y_M(Y(A, z-w)B, w)C.$$

If V is a $\mathbb{Z}$ graded vertex algebra, then a V-module M is called graded if M is a $\mathbb{Z}$-graded vector space and if for $A \in V_m$ the field $Y_M(A, z)$ has conformal dimension m, meaning $A_{(n)}^M$ is homogeneous of degree $-n+m-1$. ■

Example 5.22. Let $\lambda \in \mathbb{C}$, then we can define a module, which we shall denote π_λ, over the Weyl algebra $\tilde{\mathcal{H}} = U(\mathcal{H})/(1-\mathbf{1})$. We can define this module, which will be generated by the vector $|\lambda\rangle$, by requiring that

$$b_n|\lambda\rangle = 0, \qquad n > 0, \qquad b_0|\lambda\rangle = \lambda|\lambda\rangle. \tag{5.23}$$

As a vector space $\pi_\lambda \cong \mathbb{C}[b_n]_{n<0}|\lambda\rangle$, where the generators b_n for $n < 0$ act by multiplication and b_n for $n > 0$ act as $n\frac{\partial}{\partial b_{-n}}$ and b_0 acts as scalar multiplication by λ. Hence, this action induces an action by the fields of π, which are derivatives and normal ordered products of the field

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$$

and hence we can define

$$Y_{\pi_\lambda}(b_{n_1} \ldots . b_{n_m}|0\rangle) = \frac{1}{(-n_1-1)! \ldots (-n_m-1)!} : \partial_z^{-n_1-1} b(z) \ldots . \partial_z^{-n_m-1} b(z) :\in End\pi_\lambda[[z^{\pm 1}]] \tag{5.24}$$

♡

Notation 5.23. We may also think of the map $Y_M : V \longrightarrow EndM[[z^{\pm 1}]]$ as a linear map from the tensor product $V \otimes M$ to $M((z))$ which we denote by

$$Y^M : V \otimes M \longrightarrow M((z))$$

Definition 5.24. For any pair of modules M_1 and M_2 over a vertex algebra V, a *fusion product* is a module $M_1 \otimes_V M_2$ over V together with an intertwining operator

$$\mathcal{Y}^{M_1,M_2} : M_1 \otimes M_2 \longrightarrow M_1 \otimes_V M_2\{z\}[\log(z)] \tag{5.25}$$

such that for any module X over V and intertwining operator $\mathcal{I} : M_1 \otimes M_2 \longrightarrow X\{z\}[\log(z)]$ there exists a unique morphism $\phi : M_1 \otimes_V M_2 \longrightarrow X$ such that the following diagram

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "\mathcal{Y}^{M_1,M_2}"] \arrow[dr, "\mathcal{I}"'] & M_1 \otimes_V M_2\{z\}[\log(z)] \arrow[d, dotted, "\exists! \phi"] \\ & X\{z\}[\log(z)] \end{tikzcd}$$

commutes ■

If a fusion product exists then it equips $\mathcal{C} = Rep(V)$ with the structure of a braided tensor category.

Notation 5.25. We denote the vector space of intertwiners from $M_1 \otimes M_2$ to M_3 by $\binom{M_3}{M_1,M_2}$.

Example 5.26. [14] Note that we can equip π with a conformal structure by taking the conformal vector to be $\omega = \frac{1}{2} b_{-1}^2$. Then for N a positive integer we have a conformal π-module structure on

$$V_{\sqrt{N}\mathbb{Z}} := \bigoplus_{m \in \mathbb{Z}} \pi_{m\sqrt{\mathbb{Z}}}, \tag{5.26}$$

given by using the structure on each of the summands. Then for $\lambda, \mu \in \sqrt{N}\mathbb{Z}$ we can define a fusion product

$$\pi_\lambda \otimes_\pi \pi_\mu := \pi_{\lambda+\mu}. \tag{5.27}$$

To complete the definition of the fusion product we must define intertwining operators. To do so it is sufficient to define a field

$$V_\lambda(z) : \pi_\mu \longrightarrow \pi_{\mu+\lambda}[[z^{\pm 1}]] \tag{5.28}$$

given explicitly by the formula [4]

$$V_\lambda(z) = S_\lambda z^{\lambda b_0} \exp\left(-\lambda \sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(-\lambda \sum_{n>0} \frac{b_n}{n} z^{-n}\right) \tag{5.29}$$

where $S_\lambda : \pi_\mu \longrightarrow \pi_{\lambda+\mu}$ is the shift operator defined by

$$S_\lambda|\mu = |\mu + \lambda\rangle, \qquad [S_\lambda, b_n] = 0, \qquad n \neq 0 \tag{5.30}$$

♡

Remark 5.27. [29]

Recall for a tensor category $(\mathcal{C}, \otimes)$ we required natural isomorphisms $r : id \otimes X \longrightarrow X$ and $l : X \otimes id \longrightarrow X$ for every object $X \in \mathcal{C}$. For the category of modules over a vertex algebra which we shall denote $Rep(V)$ with a fusion product this requirement means we need

$$l_M(\mathcal{Y}^{V,M}(A,z)m) = Y^M(A,z)m \tag{5.31}$$

for every $A \in V$ and $m \in M$. Hence we should identify the intertwiner $\mathcal{Y}^{V,M}$ with the action of V on M by Y^M

[4] These are the vertex operators in bosonic string theory

2. The associativity constraint corresponds to identifying the intertwining operators via

$$A_{M_1,M_2,M_3}(\mathcal{Y}^{M_1,M_2\otimes_V M_3}(m_1,z_1)\mathcal{Y}^{M_2,M_3}(m_2,z_2)m_3) \tag{5.32}$$
$$=\mathcal{Y}^{M_1\otimes_V M_2,M_3}(\mathcal{Y}^{M_1,M_2}(m_1,z_1-z_2)m_2,z_2)m_3, \tag{5.33}$$

for every $m_i \in M_i$. As one should think of the modules M_i as being inserted at the point z_i, then geometrically one should think of the left hand side as expanding in a domain where z_2 is close to z_3 to allow us to compute the intertwiner M_2 and M_3 and then we compute the intertwinder with M_1. Whilst the right hand side we are expanding in the case z_1 is close to z_2 so we fist take the intertwiner of M_1 and M_2 and then take the intertwiner of their fusion product with z_3.

3. One should regard the bariding isomorphism c as identifying the intertwining operator $\mathcal{Y}^{M_1,M_2}$ at z with the intertwining operator $\mathcal{Y}^{M_2,M_1}$ transported to $-z$, meaning that

$$c_{M_2,M_1}(\mathcal{Y}^{M_2,M_1}(m_2,z)m_1 = e^{zL_{-1}}\mathcal{Y}^{M_1,M_2}(m_1,-z)m_2, \tag{5.34}$$

for every $m_i \in M_i$. First one should think of M_1 as inserted at z and M_2 as inserted at 0. Then multiplication by -1 rotates z to $-z$ and thus M_1 is now located at $-z$. Then acting by $e^{zL_{-1}}$ translates both points by $-z$ so now M_2 is located at z and M_1 is located at 0 and hence one should think of the braiding as wrapping the points our modules are inserted at around each other which looks like a braiding.

4. Finally their is a twist isomorphism θ defined by $\theta_M = \exp(2\pi i L_0)|_M$ which is balanced with respect to the braiding and hence

$$\theta_{M_1\otimes_V M_2} = c_{M_2,M_1} \circ c_{M_1,M_2} \circ (\theta_{M_1} \otimes_V \theta_{M_2}). \tag{5.35}$$

Furthermore, if the category is rigid then θ defines a ribbon structure. ◇

Example 5.28. The simplest example of such a categorical structure comes from the minimal model[5] describing the 2nd order phase transition of the Ising model. In this case the category associated to this theory has three simple objects $1, \epsilon, \sigma$. Here 1 is the unit for the fusion product and the non trivial multiplications are given by

$$\epsilon \otimes_{Vir} \sigma = \sigma \tag{5.36}$$
$$\epsilon \otimes_{Vir} \epsilon = 1 \tag{5.37}$$
$$\sigma \otimes_{Vir} \sigma = 1 + \epsilon \tag{5.38}$$

♡

In general constructing a fusion product structure on the category of modules of a vertex algebra is a very difficult problem and leads us to study rational vertex operator algebra, where such a product has been constructed by Huang and Lepowsky.

In particular they show if V satisfies a technical condition called C_2-cofinitemness, there there is a natural choice of category, called the admissible modules, that has finitly many simple objects, every object has finite length and all Hom spaces are finite-dimensional. Then they show

Theorem 5.29. [21] If V is C_2-cofinite and if the category of admissible modules is semisimple, then it is a modular fusion category.

This leads us to make the following definition

Definition 5.30. If V is a VOA that is C_2-cofinite and with a semisimple category of admissible modules then we say V is a *rational VOA*. ■

Before finishing this section we wish to make one final remark about a remarkable result called the Verlinde formula. To begin we shall define the character of a rational VOA module

[5] The state spaces in minimal models are constructed by taking quotients of the Virasoro vertex algebra.

Definition 5.31. Let V be a rational VOA and let M be a module over V. Then for $q = \exp(2\pi i\tau)$ and $\tau \in \mathbb{H}$ we define

$$\chi_M(\tau) = tr_M(q^{L_0 - c/24}) \tag{5.39}$$

■

Then we can define modular transformations called the S and T transformations by

$$S(\chi_M(\tau)) = \chi_M\left(\frac{-1}{\tau}\right), \qquad T(\chi_M(\tau)) = \chi_M(\tau+1). \tag{5.40}$$

Then as V is rational we can fix a finite set I such that $\{M_i\}_{i\in I}$ of representatives of isomorphism classes of simple modules with $M_0 = V$. Then the S and T transformations can be realized by matrices whose entries are defined by

$$S(\chi_{M_i}) = \sum_{j\in I} S_{i,j}\chi_{M_j}, \qquad T(\chi_{M_i}) = \sum_{j\in I} T_{i,j}\chi_{M_j}, \tag{5.41}$$

for all $i \in I$. Then using our same fixed basis we can define structure constants for the fusion product by

$$[M_i] \otimes_V [M_j] = \sum_k N_{ij}^k [M_k], \tag{5.42}$$

and then the Verlinde formula described these coefficients using the S transformation by

$$N_{ij}^k = \sum_l \frac{S_{jl}S_{il}(S^{-1})_{lk}}{S_{0l}}. \tag{5.43}$$

Firstly, we note that this result is computationally useful it is often much easier to compute how the characters transform and then use the Verlinde formula than to compute fusion product directly. This for instance is how one could compute the fusion products in the Ising model example. Secondly it is also remarkable as it relates the coefficients N_{ij}^k which contain local data about OPE coefficients to global data about how characters transform under the action of the generators $SL_2(\mathbb{Z})$.

5.4 Conformal Blocks

5.4.1 Defining Conformal Blocks for Affine Kac-Moody VOAs

For affine Kac-Moody algebras we will be able to make a simple definition of conformal blocks following [14] and [29]. This is because the current fields transform covariantly under changes which makes it easier to define a global version of a vertex algebra, where one should recall that one should think of a vertex algebra as living at some point on a Riemann surface such that z is a local coordinate and this point corresponds to $z = 0$. For the case of a general vertex algebra the construction of a global vertex algebra is more involved and we refer any interested reader to chapter 9 and 10 of [14] for the construction of conformal blocks for general VOAs.

In order to make our definition we begin with X a Riemann surface with n marked points $p_1, ..., p_n$ (which we shall collectively denote by $\vec{p}$) and $\mathfrak{g}$ a finite dimensional complex simple Lie algebra. Then we define $\mathcal{F}_{X,\vec{p}}$ to be the space of holomorphic functions on $X \setminus \vec{p}$ that at each point have at most a finite order pole. Then we can consider the Lie algebra $\mathfrak{g} \otimes \mathcal{F}_{X,\vec{p}}$ of $\mathfrak{g}$-valued rational functions, which captures the global properties of our vertex algebra.

Let $\tilde{M}$ be a $\mathfrak{g}$-module. Then we can construct a representation M of $\hat{\mathfrak{g}}$ of level k by induction. In particular, we can define an action of the subalgebra $\mathfrak{g}[[t]] = \mathfrak{g} \otimes \mathbb{C}[[t]]$ on $\tilde{M}$ by using the existing $\mathfrak{g} = \mathfrak{g} \otimes 1$ action and defining that $\mathfrak{g} \otimes t\mathbb{C}[[t]]$ acts trivially. Then we can have the central element K act on M by multiplication by k. Hence, we can define a $\hat{\mathfrak{g}}$-module by

$$M = Ind_{\mathfrak{g}[[t]]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} \tilde{M}. \tag{5.44}$$

Now to define conformal blocks let $\tilde{M}_1, ...\tilde{M}_n$ be $\mathfrak{g}$-modules and let $M_1, ..., M_n$ be the induced $\hat{\mathfrak{g}}$-modules defined via the above procedure. Then we can define an action of $\mathfrak{g} \otimes \mathcal{F}_{X,\vec{p}}$ on $M_1 \otimes .... \otimes M_n$ as follows. Let ξ_i be a local holomorphic coordinate centred at p_i for each i. Then we can expand any function $f \in \mathcal{F}_{X,\vec{p}}$ in a Laurent series as

$$f^{(i)}(\xi_i) = \sum_{n>>-\infty} a_n^{(i)} \xi_i^n \tag{5.45}$$

Then we define that action of $J_0^a \otimes f$ on $M_1 \otimes .... \otimes M_n$ as the sum

$$\sum_{i=1}^{n} \left(id \otimes .... \otimes \left(\sum_{n>>-\infty} a_n^{(i)} J_n^{a,(i)} \right) \otimes ... \otimes id \right) \tag{5.46}$$

where we are using that the Lie algebra[6] $\hat{\mathfrak{g}}^{(i)}$ acts on M_i as a $\hat{\mathfrak{g}}$-module.

Remark 5.32. Another equivalent way to define this action by saying that $J^a \otimes f \in \mathfrak{g} \otimes \mathcal{F}_{X,\vec{p}}$ acts on $M_1 \otimes ... \otimes M_n$ via

$$\sum_i Res_{t=p_i} f(t) J^a(t) dt \tag{5.47}$$

◇

Now we can define the space of coinvariants as the quotient of $M_1 \otimes ... \otimes M_n$ by this action and the space of conformal blocks as the space of invariants of the induced action on the dual space. In particular we denote the space of coinvariants by

$$H(X, \vec{p}, \vec{M}) := (M_1 \otimes ... \otimes M_n)/(\mathfrak{g} \otimes \mathcal{F}_{X,\vec{p}}(M_1 \otimes ... \otimes M_n)) \tag{5.48}$$

and the space of conformal blocks by

$$C(X, \vec{p}, \vec{M}) := \hom_{\mathfrak{g} \otimes \mathcal{F}_{X,\vec{p}}}(M_1 \otimes ... \otimes M_n, \mathbb{C}) \tag{5.49}$$

$$= \{ f : M_1 \otimes ... \otimes M_n \longrightarrow \mathbb{C} : f((g \otimes f) \cdot (v_1, ..., v_n) = 0, \text{ for } g \otimes f \in \mathfrak{g} \otimes \mathcal{F}_{X,\vec{p}} \text{ and } v_i \in M_i \} \tag{5.50}$$

One may naively think an element in the space of conformal blocks as the chiral part of a correlation function, however, this is naive as one may pick up mondromy as you move the insertion points.

The definition of conformal blocks now allows us to make precise our statement early that a vertex algebra encodes the symmetries of the chiral part of a CFT. This is since the vertex algebra defines the action of $g \otimes f$ on $M_1 \otimes ..... \otimes M_n$. These constraints on the correlation functions are refered to in the physics literature as the chiral Ward identities.

Remark 5.33. An analogous definition of conformal blocks to that given above can be made for the Virasoro vertex algebra on the Riemann sphere. This is since on the Riemann sphere the stress energy tensor transforms tensorially, however, this definition cannot be used on other Riemann surfaces as the stress energy tensor only transforms tensorially if the Schwarzian derivative vanishes and the schwarzian derivative does not vanish for the change of coordinate functions on Riemann surfaces with non zero genus. ◇

5.4.2 The Global Chiral Ward Identities

To begin with we must first introduce the notion of a primary field.

Definition 5.34. Let A be an element of a module over a VOA V. Then we say that A is a *primary vector with conformal dimension* Δ if

$$L_0 A = \Delta A, \qquad L_n A = 0 \text{ for } n > 0 \tag{5.51}$$

■

In physics the fields corresponding to primary vectors are called primary fields and we shall denote them by $V_\Delta(z)$. Those corresponding to vectors generated by A, such as $J_{-n_k}^{a_k} ....J_{-n_1}^{a_1} A$ in the affine Kac-Moody case, are called descendants.

In order to derive the global Ward identities it will be useful to compute the OPE of the stress energy tensor with a primary field of conformal dimension Δ which can by computed using the OPE formula which gives

$$\begin{aligned} T(z) V_\Delta(z) &= \sum_{n \in \mathbb{Z}} \frac{(L_n V_\Delta)(z)}{(z-w)^{n+2}} \\ &= \sum_{n \leq 0} \frac{(L_n V_\Delta)(z)}{(z-w)^{n+2}} \\ &= \frac{\Delta V_\Delta(z)}{(z-w)^2} + \frac{\partial_z V_\Delta(z)}{z-w} + regular, \end{aligned}$$

[6]The notation (i) means that in the construction of the affine Lie algebra we are using the local coordinate ξ_i

where we denote the field corresponding to the descendent vector $L_n A$ by $(L_n V_\Delta)(z)$ and have used that $(L_{-1}V_\Delta)(z) = \partial_z V_\Delta(z)$ as L_{-1} is the translation operator.

We now shall use the defining property of conformal blocks to derive[7] the Global Ward identities on $X = \mathbb{P}^1$ for primary fields based on the calculation in [28]. In order to do this we recall that conformal blocks are invariant under the action of $\mathfrak{g}$ valued functions. Furthermore, as $V_k(\mathfrak{g})$ has a conformal structure given by the Segal-Sugawara vector then this induces an action of the Virasoro algebra which the conformal blocks will also have to be invariant under. Therefore, using the invariance of conformal blocks under the action by $L_{-1} = \oint_\infty T(z)$ gives

$$\begin{aligned}
0 &= \oint_\infty T(z) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle \\
&= \sum_{i=1}^n \oint_{z_i} \left\langle V_{\Delta_1}(z_1).....T(z)V_{\Delta_i}(z_i)...V_{\Delta_n}(z_n) \right\rangle dz \\
&= \sum_{i=1}^n \oint_{z_i} \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i} \right) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle \\
&= \sum_{i=1}^n \partial_{z_i} \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle,
\end{aligned}$$

where in the second equality we used the residue theorem, in the third equality we used the OPE formula and in the final equality we computed the residue formula. We can then perform similar calculations to get remaining Ward identities. Now using invariance for $L_0 = \oint_\infty zT(z)$ gives

$$\begin{aligned}
0 &= \oint_\infty zT(z) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle \\
&= \sum_{i=1}^n \oint_{z_i} z \left\langle V_{\Delta_1}(z_1).....T(z)V_{\Delta_i}(z_i)...V_{\Delta_n}(z_n) \right\rangle dz \\
&= \sum_{i=1}^n \oint_{z_i} \left(\frac{z\Delta_i}{(z-z_i)^2} + \frac{z\partial_{z_i}}{z-z_i} \right) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle \\
&= \sum_{i=1}^n (\Delta_i + z_i \partial_{z_i}) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle,
\end{aligned}$$

and finally using invariance for $L_1 = \oint_\infty z^2 T(z)$ gives

$$\begin{aligned}
0 &= \oint_\infty z^2 T(z) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle \\
&= \sum_{i=1}^n \oint_{z_i} z^2 \left\langle V_{\Delta_1}(z_1).....T(z)V_{\Delta_i}(z_i)...V_{\Delta_n}(z_n) \right\rangle dz \\
&= \sum_{i=1}^n \oint_{z_i} \left(\frac{z^2\Delta_i}{(z-z_i)^2} + \frac{z^2\partial_{z_i}}{z-z_i} \right) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle \\
&= \sum_{i=1}^n (2z_i\Delta_i + z_i^2 \partial_{z_i}) \left\langle \prod_{i=1}^n V_{\Delta_i}(z_i) \right\rangle.
\end{aligned}$$

[7] In this calculation we shall label the marked points by z_i and use physics notation where we replace elements $v_i \in V_i$ by the corresponding fields $V_{\Delta_i}(z)$.

Hence, we get the global Ward identities for chiral conformal blocks

$$\sum_{i=1}^{n}\partial_{z_i}\left\langle\prod_{i=1}^{n}V_{\Delta_i}(z_i)\right\rangle=0, \tag{5.52}$$

$$\sum_{i=1}^{n}(\Delta_i+z_i\partial_{z_i})\left\langle\prod_{i=1}^{n}V_{\Delta_i}(z_i)\right\rangle=0, \tag{5.53}$$

$$\sum_{i=1}^{n}(2z_i\Delta_i+z_i^2\partial_{z_i})\left\langle\prod_{i=1}^{n}V_{\Delta_i}(z_i)\right\rangle=0. \tag{5.54}$$

Using a more computationally involved version of the above calculation one can show that invariance under each element of $\mathfrak{g}\otimes\mathcal{F}_{X,\vec{z}}$ corresponds to a Ward identity. The above three Ward identities are called the global Ward identities in the physics literature as they encode how the conformal blocks transform under Möbius transformations and hence one can show

$$V_{\Delta}\left(\frac{az+b}{cz+d}\right)=(cz+d)^{2\Delta}V_{\Delta}(z). \tag{5.55}$$

5.4.3 Bundles of Conformal Blocks and the Knizhnik–Zamolodchikov Equations

Up untill this point we have only considered conformal blocks for a fixed collection of points $p_1,...,p_n$ on X. However, we will also want to consider how the space of conformal blocks varies as we vary the marked points. Therefore we should first note that a collection of n marked points on a Riemann surface of genus g is a point in the moduli space $\mathcal{M}_{g,n}$. However, for the construction of conformal blocks we in fact need not just the data of the surface and the marked points but also a choice of local coordinates around each marked points. We therefore need to consider the moduli space $\widetilde{\mathcal{M}_{g,n}}$ whose points are exactly Riemann surfaces of genus g with n marked points and a choice of local coordinates around each marked point. Now we can state the following proposition

Proposition 5.35. [9] The vector spaces $C(X,\vec{p},\vec{M})$ define a quasi-coherent sheaf on $\widetilde{\mathcal{M}_{g,n}}$. Furthermore if V is rational and C_2-cofinite, then the sheaf is coherent and descends to a sheaf on the moduli space $\mathcal{M}_{g,n}$.

This proposition is useful as it allows us to talk about projectivly flat connections on this bundle. In particular this bundle is equipped with a projectivly flat connection $\Delta = d + L_{-1}dz$, called the Knizhnik–Zamolodchikov connection. One may then think of a conformal block as a section of this bundle and when taking this approach the sections being horizontal implies that the conformal blocks satisfy a first order differential equation called the Knizhnik–Zamolodchikov equation. For $V = V_k(\mathfrak{g})$, $M_1,....,M_n$ highest weight modules[8] as constructed in section 5.4.1, $X = \mathbb{P}^1$ with marked points $z_1,...,z_n$ and Φ a horizontal section of the KZ connection, the KZ equations are given by

$$(k+h^{\vee})\frac{\partial}{\partial z_i}\Phi=\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}\Phi, \tag{5.56}$$

for $i = 1,...,n$ and $\Omega=\sum_{a=1}^{\dim\mathfrak{g}}J_a\otimes J^a$. The derivation uses a similar style of calculation to the derivation of the global Ward identities. Furthermore, a key trick is that using the invariance one can rewrite any element in a tenor product of highest weight modules (i.e. one could have a tensor product of descendant fields) as a sum of highest weight vectors modulo coinvariants. Thus, we only need to consider the action of conformal blocks on tensor products of highest weight vectors. Hence, one only has to compute how $L_{-1}^{(i)}$ acts on a tensor product of highest weight states which is much more tractable than its action on descendants. Then one simplifies the resulting expression using the invariance properties of conformal blocks to get the KZ equations.

5.4.4 Factorisation/The Sewing Procedure

Up until now we have been dealing with conformal blocks with a fixed number of puncture n. However, we know in physics it is possible to 'sew' conformal blocks together allowing one to compute for instance 4 point functions on

[8]It is also interesting to consider non-highest weight modules, such as those coming from spectrally flowed $\mathfrak{sl}_2$ WZW models which play an important role in the AdS_3/CFT_2 correspondence. In this case the calculation is similar but more complicated as more terms survive each step of the calculation which previously were killed due to the highest weight condition. For details see [19].

the sphere as a product of 3 point functions. In order to formalise this we first note that many of our definitions and arguments also apply not only if X is smooth, but also if it has a mild form of singularity called a ordinary double point. By an ordinary double point we mean a point p on X which can be blown up, meaning their is a smooth curve X' and a projection to X such that p has two preimages $p_{\pm}$. This allows us to state the following theorem.

Theorem 5.36. [9] If X is a curve with a ordinary double point p, then there exists a canonical isomorphism

$$g_{X,X'} : C(X, \vec{p}, \vec{M}) \longrightarrow \bigoplus_{M' \in I} C(X', \vec{p} \cup \{p_+, p_-\}, \vec{M} \otimes M' \otimes M'^{\vee}), \tag{5.57}$$

where the direct sum is over all the isomorphism classes of simple objects of $Rep(V)$.

In physics this isomorphism is referred to as the sewing procedure. In what follows one should take our vertex algebra to be either the Virasoro[9] or affine Kac-Moody vertex algebra and the only thing that will change between these two cases is what primary fields and descendants one will have to sum over.

Let $\mathcal{O}_1, ..., \mathcal{O}_n$ be local fields near the points $z_1., ..., z_n$ on a Riemann surface N and let $\mathcal{O}_{n+1}, ..., \mathcal{O}_{n+m}$ be local fields near the points $z_{n+1}., , , .z_{n+m}$ on a Riemann surface M, then the conformal block on M sew N, denoted by $M \infty N$, is defined by

$$\langle \mathcal{O}_1 ... \mathcal{O}_{m+n} \rangle_{N \infty M} = \sum_{i, \vec{n}, \vec{m}} \langle \mathcal{O}_1, ..., \mathcal{O}_n \mathcal{L}_{-\vec{n}} \phi_i \rangle_N \langle \mathcal{L}_{-\vec{n}} \phi_i \mathcal{L}_{-\vec{m}} \phi_i \rangle^{-1} \langle \mathcal{L}_{-\vec{m}} \phi_i \mathcal{O}_{n+1}, ..., \mathcal{O}_{n+m} \rangle_M \tag{5.58}$$

Where the i runs over all the primary fields and the indices $\vec{n}$, $\vec{m}$ run over all descendants, where we define

$$\mathcal{L}_{-\vec{m}} \phi = L_{-m_1} .... L_{-m_K} \phi_i \tag{5.59}$$

in the Virasoro VOA case, and

$$\mathcal{L}_{-\vec{m}} \phi = J^{a_1}_{-m_1} .... J^{a_k}_{-m_k} \phi_i \tag{5.60}$$

in the affine Kac-Moody case.

An example by taking $n = m = 2$ one can use this procedure to compute a 4 point function on the sphere by summing over products of 3-point functions. A picture of how one sews the two spheres with 3 punctures to get a sphere with 4 punctures is shown below.

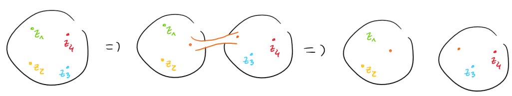

Figure 15: Sewing Procedure for the 4-point function

5.5 The Fuchs-Runkel-Schweigert Construction of RCFT Correlators from 3d TFTs

5.5.1 What is a Full CFT Correlator?

In the previous subsections we studied vertex algebras, modules over vertex algebras and their conformal blocks. However, until this point we have only considered the chiral part of the CFT. Our goal in this section will instead be to outline a construction that gives a consistent system of correlators on the whole CFT.

Firstly we recall that the symmetries of the chiral part of a CFT are encoded in a vertex operator algebra V and that we realise these symmetries as Ward identities of chiral conformal blocks. However, to study the full CFT we will need to combine left movers and right movers. Hence, we will need two vertex operator algebras V_L and V_R, and the appropriate category to encode this symmetry is Rep($V_L \otimes_{\mathbb{C}} V_R$), which under suitable finitness conditions is equivalent to $Rep(V_L) \boxtimes Rep(V_R)$, where $\boxtimes$ is the Deligne product. One may think of the Deligne product as combining left movers and right movers in the physics literature. In what follows we shall simplify our situation by restricting to rational CFTs where the left and right movers are governed by the same VOA, meaning $\mathcal{C}_L = \mathcal{C}$ and $\mathcal{C}_R = \mathcal{C}^{rev}$, where the superscript means we have reversed the braiding on $\mathcal{C}$ a modular fusion category. We are now ready to define what a consistent system of correlators is

[9]For this discussion whose goal is to provide physical intuition for the above theorem we shall ignore that our definition of conformal blocks will not work for the Virasoro VOA when X is not a genus 0 Riemann surface.

Definition 5.37. We define a *worldsheet* to be a topological 2 dimensional manifold, possibly with boundary and punctures. Then we say a *consistent system of correlators* is an assignment

$$S \longrightarrow Cor_{\mathcal{C}}(S) \in Bl_{\mathcal{C}}(S) \tag{5.61}$$

that specifies for every world sheet S an element, we call a correlator $Cor_{\mathcal{C}}(S)$, in the vector space $Bl_{\mathcal{C}}(S)$ of conformal blocks such that

1. $Cor_{\mathcal{C}}(S)$ is invariant under the action of $Map(S)$ the mapping class group of S,
2. and the assignment is compatible with the sewing of worldsheets. ■

Remark 5.38. The first condition is to ensure that our correlation functions are actually single valued rather than multivalued functions. This is since if one defined conformal blocks as horizontal sections of the KZ connections, then these sections may have monodromy and hence would not be single valued. ◇

5.5.2 Outline of FRS

The key to the FRS construction to construct a consistent system of correlators is to use a 3d topological field theory called the Reshetikhin-Turaev TFT. For every modular fusion category $\mathcal{C}$, which in our case one may think of as $Rep(V)$, one can construct a RT type TFT we shall label $RT_{\mathcal{C}}$. For the RT TFT our 2-dimensional manifolds are extended surfaces which are surfaces equipped with arcs which one should think of as insertion points plus the topological remnant of the information of local coordinates at the marked points. The RT TFT associates to each extended surface S a complex vector space $RT_{\mathcal{C}}(S)$, which we will interpret as the space of conformal blocks on S. Furthermore, to a three manifold M with ribbon graphs, where the ribbons are labelled by objects in the modular fusion category $\mathcal{C}$, we get a linear map $RT_{\mathcal{C}}(M) : RT_{\mathcal{C}}(\partial M_+) \longrightarrow RT_{\mathcal{C}}(\partial M_-)$, where $\partial M = \partial M_+ \coprod \partial M_-$.

In order to construct correlation functions we begin with our topological worldsheet S. We can then take the double $\hat{S}$ defined by

$$\hat{S} := S \times \{\pm 1\} / \sim \tag{5.62}$$

where $(p,+) \sim (p,-)$ for $p \in \partial S$. Then we shall construct a space of conformal blocks associated to S by

$$Bl_{\mathcal{C}}(S) = RT_{\mathcal{C}}(\hat{S}). \tag{5.63}$$

Now to construct a correlation function we will use a bordism from the empty set to $\hat{S}$. We start by defining

$$\tilde{M}_s = M \times [-1,1] / \sim, \tag{5.64}$$

with the identification $(x,t) \sim (x,-t)$ for all $x \in \partial S$ and $t \in [-1,1]$. Then one should note that $\partial \tilde{M}_S = \hat{S}$. Up until now we have only used the data of the modular fusion category $\mathcal{C}$, however, in the FRS construction one also needs to specify the data of a special symmetric Frobenius algebra A in $\mathcal{C}$. This A should be thought of as providing data about possible boundary conditions, in particular to each component of ∂S a module over A will specify the boundary conditions. We shall explore the physical motivation of this construction further in the next section. Now to finish outlining the construction we can use A to specify a ribbon in $X \times \{0\} \subset \tilde{M}_S$ and we call this 3 manifold with ribbon M_S. Details of the construction of this ribbon graph are given in [15], this construction involves a number of choices and the Frobenius algebra must be special and symmetric to ensure the construction is independent of these choices.

As $\partial M_S = \hat{S}$ then

$$RT_{\mathcal{C}}(M_S) : \ RT_{\mathcal{C}}(\emptyset) \longrightarrow RT_{\mathcal{C}}(\hat{S}) \tag{5.65}$$
$$\mathbb{C} \longrightarrow Bl_{\mathcal{C}}(S), \tag{5.66}$$

and we define

$$Cor_{\mathcal{C}}(S) = RT_{\mathcal{C}}(M_S)(1), \tag{5.67}$$

and thus we can state the main result of the FRS papers

Theorem 5.39. [15][17][16][18][12] The assignment $S \mapsto Cor_{\mathcal{C}}(S)$ for each topological worldsheet gives a consistent system of correlators.

Remark 5.40. One advantage of this construction is that it allows one to construct a consistent system of correlators from purely categorical data, i.e. a modular fusion category and a special symmetric Frobenius algebra in that category. In particular this means we can study CFT correlation functions without the hard work of studying the representation theory of vertex algebras which is a very difficult subject.

Furthermore, as a special case of this theorem one can recover the holographic relation between WZW theory and Chern-Simons theory. In this context one should think of Chern-Simons theory as a 3-d TQFT where one interprets the vector spaces associated to 2-d manifolds by the Chern-Simons TQFT as the spaces of conformal blocks for the WZW model.

5.5.3 Where does the Special Symmetric Frobenius Algebra come from?

In order to understand where the algebra A comes from we will need to consider CFTs with boundary as in [15]. We shall argue that each boundary condition on a RCFT with chiral symmetries given by a vertex algebra V, determines an algebra object in the modular category $RepV$.

For concreteness we shall take our worldsheet to the the upper half plane, including the real line as the boundary. Then let M denote a boundary condition on the real line and let $\mathcal{H}$ denote the space of states corresponding to boundary fields living on a boundary segment with boundary condition M. Unlike in the bulk where our spaces of state were modules over $V \otimes V^{rev}$, as we combined left movers and right movers, on the boundary our states are modules over just V. Hence, we may write

$$\mathcal{H} = \bigoplus_{a \in I} n^a U_a, \tag{5.68}$$

where I denotes a set of representatives of isomorphism classes of irreducible highest weight modules U_a over V, and $n^a \in \mathbb{Z}_{\geq 0}$ gives their multiplicities.

Using the correspondence of states in $\mathcal{H}$ and fields that live in the boundary, we shall denote by $\Psi_{a,\alpha}(x)$ the primary boundary field, labelled by a, denoting the representation U^a of V and with α denoting the multiplicity index. Furthermore, we should note as this is a boundary field then $x \in \mathbb{R}$. In simple cases such as minimal models the OPEs of boundary fields take the form

$$\Psi_{a\alpha}(x)\Psi_{b\beta}(y) = \sum_{c,\gamma} C^{c\gamma}_{a\alpha,b\beta}(x-y)^{\Delta_c - \Delta_a - \Delta_b}[\Psi_{c\gamma}(y) + \text{terms with descendants}]. \tag{5.69}$$

In general however we may have more complicated fusion rules than in the case of minimal models. In particular, we could have more than one independent way for representations a and b to fuse into c, or for a given pair a and b, the primary field corresponding to c may not appear in the OPE. Recall that the dimension of the space of coulpings from a and b to c is the fusion rule N^c_{ab}

In order to describe the OPEs in general we will need a vertex operator associating to every vector v in a highest weight module U_a, a collection of N^c_{ab} fields $V^{c,\delta}_{ab} : U_a \longrightarrow \hom(U_a, U_b)[[z, z^{-1}]]$ for $\delta = 1, 2, ..., N^c_{ab}$. Then the OPE is given by

$$\Psi_{a\alpha}(x)\Psi_{b\beta}(y) = \sum_{c \in I}\sum_{\gamma=1}^{n^c}\sum_{\delta=1}^{N^c_{ab}} C^{c\gamma;\delta}_{a\alpha,b\beta}(x-y)^{\Delta_c - \Delta_a - \Delta_b}\sum_{C}\langle v^C_c | V^{c,\delta}_{ab}(v_a, z=1)|v_b\rangle (x-y)^{\Delta(v^C_c) - \Delta_c}\Psi^C_{c\gamma}(y), \tag{5.70}$$

where $\{v^D_d\}$ denotes an orthonormal basis of L_0 eigenvectors in U_d, for $v^0_d = v_d$ a highest weight vector, and $\psi^D_{d\alpha}$ the corresponding descendent field.

Now we can use the data of the boundary OPEs to construct an algebra A in Rep$V$. We begin by defining

$$A \cong \bigoplus_{a \in I} n^a U_a \tag{5.71}$$

as in the definition of $\mathcal{H}$. Then fixing a basis of $\hom(U_a, A)$ one can define a multiplication on A in terms of this bases by taking the coefficients $m_{a\alpha,b\beta}^{c\gamma;\delta}$ as the OPE coefficients

$$m_{a\alpha,b\beta}^{c\gamma;\delta} := C_{a\alpha,b\beta}^{c\gamma;\delta} \tag{5.72}$$

The unit of the algebra is given by the identity field on M and the associativity of the multiplication can be seen as a result of the sewing constraint that arises when factorising the correlators of four boundary fields on a disk with boundary condition M. Then in [15] theorem 3.6 they show that the nondegeneracy of two point functions and properties of the 1-point functions on annuli imply that this algebra is a special symmetric Frobenius algebra.

5.5.4 FRS Construction for the Trivial Frobenius Algebra

As a sanity check we shall consider the case that $\mathcal{C} = \mathrm{Rep}V$ for V a rational vertex operator algebra, $A = 1$ the trivial Frobenius algebra and S has no boundary. Then in this case $M_S = \tilde{M}_S$, meaning that their is no ribbon. Furthermore, in this case using the functorial properties of the RT TFT we have that

$$\begin{aligned} RT_{\mathrm{Rep}V}(\hat{S}) =& RT_{\mathrm{Rep}V}(S \coprod -S) \\ =& RT_{\mathrm{Rep}V}(S) \otimes RT_{\mathrm{Rep}V}(-S) \\ =& Bl_{\mathrm{Rep}V}(S) \otimes \overline{Bl_{\mathrm{Rep}V}(S)} \end{aligned}$$

where $\overline{Bl_{\mathrm{Rep}V}(S)}$ denotes the space of conformal blocks for the antiholomorphic theory. Then we have that

$$Cor_{\mathcal{C}}(S) = RT_{\mathcal{C}}(M_S)(1) \in Bl_{\mathrm{Rep}V}(S) \otimes \overline{Bl_{\mathrm{Rep}V}(S)} \tag{5.73}$$

and hence in this simple case where we have no boundary or defects a correlator is simply a combination of holomorphic and anti-holomorphic conformal blocks as expected.

6 Tensor Category in Arithmetic

Throughout this whole section, $\Bbbk$ is a field and we always assume that the characteristic of $\Bbbk$ is 0. A ring means a commutative ring with the multiplicative identity and a $\Bbbk$-algebra means a commutative algebra over $\Bbbk$.

6.1 Tannakian Formalism

6.1.1 The category $\mathrm{Rep}_{\Bbbk}G$

Let G be a finite group, $\mathrm{Vec}_{\Bbbk}$ be the category of finite dimensional $\Bbbk$-vector spaces, $\mathrm{Rep}_{\Bbbk}G$ be the category of representations of G on finite dimensional $\Bbbk$-vector spaces, $F : \mathrm{Rep}_{\Bbbk}G \to \mathrm{Vec}_{\Bbbk}G$ be the forgetful functor. Let's explore what properties the category $\mathrm{Rep}_{\Bbbk}G$ with the forgetful functor F has.

From the field $\Bbbk$ and the group G, we can form a group algebra $\Bbbk[G]$ whose elements are formal $\Bbbk$-linear combinations of finitely many elements in G, and the formal multiplication makes it a commutative $\Bbbk$-algebra structure. And even better, $\Bbbk[G]$ is a commutative Hopf algebra over $\Bbbk$, where

- the diagonal map $G \to G \times G, g \mapsto (g, g)$ induces the comultiplication $\Delta : \Bbbk[G] \to \Bbbk[G \times G] \simeq \Bbbk[G] \otimes \Bbbk[G]$;
- the counit is the augmentation map $e^* : \Bbbk[G] \to \Bbbk$, which sends $g \in G$ to 1 and extends linearly;
- the antipode is given by the inverse map $G \to G$ and extends linearly.

There is a natural representation of G on $\Bbbk[G]$, where G acts by left multiplication on $G \subseteq \Bbbk[G]$ and extends linearly, which makes $\Bbbk[G] \in \mathrm{Obj}(\mathrm{Rep}_{\Bbbk}G)$ (the notation Obj means "object"). Actually, $\Bbbk[G]$ is "universal" in some sense. Let $\mathrm{Rep}^p_{\Bbbk}G$ be the category of pointed representations (i.e., the objects are representations (V, v) with a chosen vector $v \in V$, and the morphisms $(V, v) \to (V', v')$ are required to send v to v'). The precise statement of the universality of $\Bbbk[G]$ is shown below.

Proposition 6.1. [23] $(\Bbbk[G], 1)$ is the universal object in $\mathrm{Rep}^p_{\Bbbk}G$, i.e., for any $(V, v) \in \mathrm{Obj}(\mathrm{Rep}^p_{\Bbbk}G)$, there exists a unique morphism $(\Bbbk[G], 1) \to (V, v)$ in $\mathrm{Rep}^p_{\Bbbk}G$.

A question can be asked. How do we recover the group G and the "universal" object $\Bbbk[G]$ from the category $\mathrm{Rep}_{\Bbbk}G$ with the forgetful functor F? Here is the answer.

Proposition 6.2. [23] Let G be a finite group and $F : \mathrm{Rep}_{\Bbbk}G \to \mathrm{Vect}_{\Bbbk}$ be the forgetful functor.

1. (Recover G from F) The group G is isomorphic to $\mathrm{Aut}^{\otimes}(F)$, the group of the tensor-compatible automorphism of the forgetful functor F (tensor-compatible means $\alpha_{V\otimes W} = \alpha_V \otimes \alpha_W$ and $\alpha_k = \mathrm{id}_k$ for $\alpha \in \mathrm{Aut}(F)$);
2. (Recover $\Bbbk[G]$ from F) $\mathrm{End}(F) \simeq \Bbbk[G]$;
3. (Recover G from $\Bbbk[G]$) G can be recovered as the group-like elements of $\Bbbk[G]$, i.e., $a \in \Bbbk[G]$ such that $e^*(a) = 1$ and $\Delta(a) = a \otimes a$.

Remark 6.3. In the language of tensor category, the tensor-compatibility condition just says the automorphism should be a monoidal natural isomorphism. $\diamondsuit$

Proof. (Sketch)

1. The isomorphism $G \to \mathrm{Aut}^{\otimes}(F)$ is given by $g \mapsto (\rho \mapsto \rho(g))$ where $\rho : G \to \mathrm{GL}(V)$ is in $\mathrm{Obj}(\mathrm{Rep}_{\Bbbk}G)$.
2. The isomorphism is given by $a \mapsto a_{\Bbbk[G]}(1)$.
3. For any $\sum_i k_i g_i \in \Bbbk[G]$ where $k_i \in \Bbbk$ and $g_i \in G$, the condition $e^*(a) = 1$ means $\sum_i k_i = 1$ and the condition $\Delta(a) = a \otimes a$ means $k_i = 0$ or 1, and $k_i k_j = 0$ for $i \neq j$. These are equivalent to saying $a \in G$. $\square$

6.1.2 Tannakian Category and Neutral Tannakian Category

Now we generalize the notion of $\mathrm{Rep}_{\Bbbk}G$ with the forgetful functor to the Tannakian category.

Definition 6.4. A *Tannakian category* over $\Bbbk$ is a symmetric tensor category $\mathcal{C}$ over k (i.e., symmetric $\Bbbk$-linear rigid monoidal abelian category with 1 simple) that admits a braided tensor functor $F : \mathcal{C} \to \mathrm{Vec}_{\Bbbk}$ (called the fiber functor). ■

Example 6.5. Let G be a finite group. Then $\mathrm{Rep}_{\Bbbk}G$ is a Tannakian category with the forgetful functor $\mathrm{Rep}_{\Bbbk}G \to \mathrm{Vec}_{\Bbbk}$ as the fiber functor. ♡

Theorem 6.6. [11] Let $\mathcal{C}$ be a symmetric fusion category over $\Bbbk$ which is algebraically closed. Then $\mathcal{C}$ is a Tannakian category if and only if there exists a finite group G such that $\mathcal{C}$ is equivalent to $\mathrm{Rep}_{\Bbbk}G$ as a braided fusion category.

Remark 6.7. The “fusion" means finite semisimple conditions. In particular, there are finitely many isomorphism classes of simple objects. That’s why we can require the existence of a "finite" group in the above theorem. ♢

As you may expect, for a tannakian category $\mathcal{C}$ with the fiber functor $F : \mathcal{C} \to \mathrm{Vec}_{\Bbbk}$, the group G in $\mathcal{C} \simeq \mathrm{Rep}_{\Bbbk}G$ can be obtianed by $\mathrm{Aut}^{\otimes}(F)$.

Definition 6.8. [10] A *neutral Tannakian category* over $\Bbbk$ is a Tannakian category over $\Bbbk$ whose fiber functor is exact and faithful. ■

Before giving the main theorem about the neutral Tannakian category, let’s first introduce the notion of group scheme briefly.

Definition 6.9. A *group scheme* over a scheme S is a group object in the category of schemes over S, i.e., a scheme G over S together with three morphisms $m : G \times_S G \to G$, $e : S \to G$, $\iota : G \to G$ satisfying some commutative diagrams as the axiom of groups. ■

Our main focus is the case of an affine group scheme G over a field $\Bbbk$. Write $G = \mathrm{Spec}A$ for some $\Bbbk$-algebra A. For any $\Bbbk$-algebra R, the set $G(R)$ of morphisms $\mathrm{Spec}R \to G$ over $\Bbbk$ (or equivalently, the set of $\Bbbk$-alegbra homomorphisms $A \to R$) is actually a group under some operations defined in terms of the fiber product of $\Bbbk$-schemes (or equivalently, tensor product of $\Bbbk$-algebras). Thus, G defines a functor from $\mathrm{Alg}_{\Bbbk}$ (the category of $\Bbbk$-algebras) to Grp (the category of groups), which is representable by A.

Definition 6.10. The *morphism of affine group schemes* $f : G_1 \to G_2$ over $\Bbbk$ is the morphism of schemes over $\Bbbk$ such that for every k-algebra $\mathbb{R}$, the induced map $f(R) : G_1(R) \to G_2(R)$ is a group homomorphism. ■

Also, for an affine group scheme $G = \mathrm{Spec}A$ over $\Bbbk$, A is a commutative Hopf algebra over $\Bbbk$ where the comultiplication, counit and the antipode of A come from m, e, ι respectively (but in a reverse direction). Conversely, given a commutative Hopf $\Bbbk$-algebra A, $\mathrm{Spec}A$ is an affine group scheme over $\Bbbk$ where m, e, ι comes from the comultiplication, counit and the antipode of A.

To summarize, there are three ways viewing affine group schemes over $\Bbbk$:

1. as group objects in the category of affine group schemes over $\Bbbk$;
2. as representable functors from $\mathrm{Alg}_{\Bbbk}$ to Grp;
3. as (the spectrum of) commutative Hopf algebras over $\Bbbk$.

Example 6.11. Over the field $\Bbbk$, $\mathbb{G}_{a/\Bbbk} := \mathrm{Spec}\,\Bbbk[t] = \mathbb{A}^1_k$, $\mathbb{G}_{m/\Bbbk} := \mathrm{Spec}\,\Bbbk[t, t^{-1}] = \mathbb{A}^1_k \setminus \{0\}$ are two examples of affine group scheme, with $\mathbb{G}_{a/\Bbbk}(\Bbbk) \simeq \Bbbk$ and $\mathbb{G}_{m/\Bbbk}(\Bbbk) \simeq \Bbbk^{\times}$. Actually, both of them are *algebraic groups*, which are group schemes and algebraic varieties simultaneously. In general, an affine group scheme is not necessarily an algebraic group, but a *pro-alegbraic group*, i.e., the limit of algebraic groups in the category of affine group scheme.♡

Theorem 6.12 (Main Theorem). [10] If $\mathcal{C}$ is a neutral Tannakian category over $\Bbbk$ with an exact and faithful fiber functor F, then there exists a unique affine group scheme G over $\Bbbk$ (up to affine group scheme isomorphism) such that the functor $\mathcal{C} \to \mathrm{Rep}_{\Bbbk}G$ defined by F is an equivalence of braided tensor categories.

We omit the details[10] about how to define such functor by F. The spirit is that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\simeq"] \arrow[d, "F"'] & \mathrm{Rep}_{\Bbbk} G \arrow[dl, "\text{forgetful functor}"] \\ \mathrm{Vec}_{\Bbbk} & \end{tikzcd}$$

Remark 6.13. Here $\mathrm{Rep}_{\Bbbk} G$ actually means $\mathrm{Rep}_{\Bbbk} G(\Bbbk)$. We adopt this abuse of notation which denotes the group $G(\Bbbk)$ by G as well for an affine group scheme G over $\Bbbk$ from now on. For example, $\Bbbk[G]$ should be interpreted as $\Bbbk[G(\Bbbk)]$ for an affine group scheme G over $\Bbbk$. $\diamondsuit$

Remark 6.14. If $\mathcal{C}$ is furthermore a fusion category over $\Bbbk$ which is algebraically closed, then G in the Main theorem can be required to be a finite group scheme, i.e., the structure morphism $G \to \mathrm{Spec}\,\Bbbk$ is finite. And $G(\Bbbk)$ is a finite group in this case. $\diamondsuit$

6.1.3 Recovering the Affine Group Scheme Structure

Given a group G not necessarily finite, Proposition 6.2 still holds and which shows that we can use $\mathrm{Aut}^{\otimes}(F)$ to recover the group G from the category $\mathrm{Rep}_{\Bbbk} G$ with the forgetful functor $F : \mathrm{Rep}_{\Bbbk} G \to \mathrm{Vec}_{\Bbbk}$. However, Proposition 6.1 does not quite hold, because $\Bbbk[G]$ in this case is not necessarily finite dimensional, hence is not necessarily in $\mathrm{Obj}(\mathrm{Rep}_{\Bbbk} G)$. Actually, $(\Bbbk[G], 1)$ is the pro-universal (rather than just universal in general) object in $\mathrm{Rep}^{p}_{\Bbbk} G$.

Now given an affine group scheme G over $\Bbbk$, how can we recover the affine group scheme structure of G from $\mathrm{Rep}_{\Bbbk} G$ and the forgetful functor $F : \mathrm{Rep}_{\Bbbk} G \to \mathrm{Vec}_{\Bbbk}$?

By the second way of viewing affine group schemes over $\Bbbk$, we want to make $\mathrm{Aut}^{\otimes}(F)$ actually a functor from $\mathrm{Alg}_{\Bbbk}$ to Grp, not just a group. Actually we can, and the definition of the functor $\mathrm{Aut}^{\otimes}(F)$ is in a similar manner. For any $\Bbbk$-algebra R, $\mathrm{Aut}^{\otimes}(F)(R)$ is a group that consists of families of linear automorphisms $\alpha_X : X \otimes R \to X \otimes R$ ranging over $X \in \mathrm{Obj}(\mathrm{Rep}_{\Bbbk} G)$ with tensor-compatibility $\alpha_{X\otimes Y} = \alpha_X \otimes \alpha_Y$, $\alpha_k = \mathrm{id}_R$ and $(f \otimes \mathrm{id}_R) \circ \alpha_X = \alpha_Y \circ (f \otimes \mathrm{id}_R)$ for any morphism $f : X \to Y$ in $\mathrm{Rep}_{\Bbbk} G$.

Remark 6.15. The above construction of the functor $\mathrm{Aut}^{\otimes}(F)$ can be generalized to any fiber functor $F : \mathcal{C} \to \mathrm{Vec}_{\Bbbk}$ of an arbitrary neutral Tannakian category $\mathcal{C}$. $\diamondsuit$

Proposition 6.16. [10] An affine group scheme G over $\Bbbk$ as a functor from $\mathrm{Alg}_{\Bbbk}$ to Grp is naturally isomorphic to the functor $\mathrm{Aut}^{\otimes}(F)$ defined above.

Corollary 6.17. [10] Let $G = \mathrm{Spec}\,A$ be an affine group scheme over $\Bbbk$. Then the functor $\mathrm{Aut}^{\otimes}(F)$ defined above is represented by A.

Corollary 6.18. The uniqueness part in Main Theorem 6.12 is correct.

Proof. It suffices to show, for affine group schems G, G' over $\Bbbk$, $\mathrm{Rep}_{\Bbbk} G \simeq \mathrm{Rep}_{\Bbbk} G'$ with the following commutative diagram implies $G \simeq G'$ as group schemes, where F and F' are corresponding forgetful functors.

$$\begin{tikzcd} \mathrm{Rep}_{\Bbbk} G \arrow[r, "\simeq"] \arrow[d, "F"'] & \mathrm{Rep}_{\Bbbk} G' \arrow[dl, "F'"] \\ \mathrm{Vec}_{\Bbbk} & \end{tikzcd}$$

Indeed, both G, G' as functors $\mathrm{Alg}_{\Bbbk} \to \mathrm{Grp}$ must be isomorphic to the functor $\mathrm{Aut}^{\otimes}(F) \simeq \mathrm{Aut}^{\otimes}(F')$ by Proposition 6.16. And if $G \simeq G'$ as functors, then $G \simeq G'$ as group schemes. $\square$

Now we managed to recover G as a functor via the the functor $\mathrm{Aut}^{\otimes}(F)$. But how do we know that this functor is arised from a group scheme? So we have to find a commutative Hopf algebra A over $\Bbbk$ such that $G \simeq \mathrm{Spec}\,A$ from the third way of viewing an affine group scheme over $\Bbbk$ (as said previously, the comultiplication, counit and antipode correspond to m, e, ι, which makes $\mathrm{Spec}\,A$ really a group scheme).

First we use Proposition 6.2 to recover the "pro-universal" object $\Bbbk[G]$. Then we construct the vector space $A = \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk)$, i.e., the dual space of the vector space $\Bbbk[G]$. A is actually a commutative Hopf k-alegrba where

- the algebra structure of A comes from the coalgebra structure of $\Bbbk[G]$ via the dual map;
- the coalgebra structure of A comes from the algebra structure of $\Bbbk[G]$ via the dual map;
- the antipode of A comes from the antipode of $\Bbbk[G]$ via the dual map.

For example, wirting out the comultiplication of $\Bbbk[G]$, i.e., $\Delta : \Bbbk[G] \to \Bbbk[G \times G] \simeq \Bbbk[G] \otimes \Bbbk[G]$. Then the the multiplication of A is given by its dual map $\Delta^* : A \otimes A \simeq \mathrm{Hom}_{\Bbbk}(\Bbbk[G] \otimes \Bbbk[G], \Bbbk) \to A$.

Proposition 6.19. [23] An affine group scheme G over $\Bbbk$ is isomorphic as affine group schemes to $\mathrm{Spec}(A)$ where $A = \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk)$ with the commutative Hopf $\Bbbk$-algebra structure defined above.

In particular, $\mathrm{Hom}_{\mathrm{Alg}_{\Bbbk}}(A, \Bbbk) = G(\Bbbk) = \mathrm{Spec}(A)(\Bbbk)$.

6.2 Unipotent Completion

6.2.1 Definition of Unipotent Completion

Definition 6.20. [7] Let $\mathcal{C}$ be an abelian monoidal category. An object $F \in \mathrm{Obj}(\mathcal{C})$ is *unipotent* if there is a finite filtration $F = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_{n-1} \supseteq \mathcal{F}_n = 0$ such that each quotient $\mathcal{F}_i/\mathcal{F}_{i+1} \simeq 1$ where 1 is the unit.

When the unit 1 is simple, this filtration is the Jordan-Hölder series of F, and its length n is uniquely determined. We call n the index of unipotency of F. ■

Remark 6.21. Note that 0 is not the unit 1 in general. 0 is part of the structure of the abelian category, whereas 1 is that of the monoidal category. ♢

Example 6.22. In the category $\mathrm{Vec}_{\Bbbk}$, 0 is the zero vector space, 1 is the vector space $\Bbbk$, and every object $V \in \mathrm{Obj}(\mathrm{Vec}_{\Bbbk})$ is unipotent. ♡

Example 6.23. In the category $\mathrm{Rep}_{\Bbbk} G$ where G is a finite group, 0 is the unique representation on the zero vector space, and 1 is the trivial representation on $\Bbbk$ (More concretely, $G \to \mathrm{GL}(\Bbbk), g \mapsto \mathrm{id}_{\Bbbk}$). It is easy to see that every finite direct sum of trivial representations is an unipotent object in $\mathrm{Rep}_{\Bbbk} G$.

Actually, the converse is also true, and we argue as the followings.

For a unipotent object $V \in \mathrm{Obj}(\mathrm{Rep}_{\Bbbk} G)$ of index of unipotency 1, we have $V = \mathcal{F}_0 \supseteq \mathcal{F}_1 = 0$ with $V = \mathcal{F}_0 \simeq \mathcal{F}_0/\mathcal{F}_1 \simeq 1$.

For a unipotent object $V \in \mathrm{Obj}(\mathrm{Rep}_{\Bbbk} G)$ of index of unipotency 2, we have $V = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 = 0$. So we have a short exact sequence:

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow V = \mathcal{F}_0 \longrightarrow \mathcal{F}_0/\mathcal{F}_1 \longrightarrow 0,$$

By the definition of unipotent object, $\mathcal{F}_0/\mathcal{F}_1 \simeq 1$. And we just argued that $\mathcal{F}_1$ as a unipotent object of index of unipotency 1 is isomorphic to T. Note that $\mathrm{Rep}_{\Bbbk} G$ is a semisimple category by Maschke's theorem, and every short exact sequence splits in a semisimple category. Thus, $V \simeq \mathcal{F}_1 \oplus (\mathcal{F}_0/\mathcal{F}_1) \simeq 1 \oplus 1$.

Now for a unipotent object $V \in \mathrm{Obj}(\mathrm{Rep}_{\Bbbk} G)$ of index of unipotency n, by induction, we can similarly argue that $V \simeq 1 \oplus \cdots \oplus 1$, the n copies of direct sums of 1. ♡

Example 6.24. In the category $\mathrm{Rep}_{\Bbbk} \mathbb{Z}$, the representation $V : n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ is unipotent of index of unipotency 2. Indeed, $V = \mathcal{F}_2 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_0 = 0$ with $\mathcal{F}_1 = 1$. However, V is not a direct sum of trivial representations. ♡

Lemma 6.25. [7] Let G be a group. The full subcategory $\mathrm{Rep}^{un}_{\Bbbk} G$ of $\mathrm{Rep}_{\Bbbk} G$ that consists of unipotent objects is a neutral Tannkian category, with the fiber functor being the forgetful functor.

Definition 6.26. [7] Let G be a group. By Lemma 6.25 and Theorem 6.12, there exists a unique affine group scheme $G_{\Bbbk}$ over $\Bbbk$ up to isomorphism such that there is an equivalence $\mathrm{Rep}^{un}_{\Bbbk} G \simeq \mathrm{Rep}_{\Bbbk} G_{\Bbbk}$ defined by the forgetful functor. Such $G_{\Bbbk}$ called the *$\Bbbk$-unipotent completion* (or *$\Bbbk$-Malcev completion*) of G. ■

Example 6.27. Let G be a finite group. We previously have seen that $\mathrm{Rep}^{un}_{\Bbbk} G$ precisely consists of finite direct sums of trivial representations from Example 6.23. But also note that all of the representations of the trivial group is exactly the finite direct sums of trivial representations. Therefore, the $\Bbbk$-unipotent completion of a finite group G is an affine group scheme $G_{\Bbbk}$ over $\Bbbk$ such that $G_{\Bbbk}(\Bbbk)$ is a trivial group. That means, $G_{\Bbbk} = \mathrm{Spec}\,\Bbbk$.

More generally, if $\mathrm{Rep}_{\Bbbk} G$ is a semisimple category, then the argument in Example 6.23 still holds, which means $G_{\Bbbk}$ in this general case is still $\mathrm{Spec}\,\Bbbk$. ♡

Example 6.28. What about the $\mathbb{C}$-unipotent completion of $\mathbb{Z}$? We spot the following two facts:

1. $\mathbb{Z}$ is dense in the affine line $\mathbb{C}$ with respect to Zariski topology over $\mathbb{C}$;
2. Over $\mathbb{C}$, every unipotent representation of $\mathbb{Z}$ on V (means $V \in \mathrm{Rep}^{un}_{\mathbb{C}}\mathbb{Z}$) extends to a representation of $\mathbb{C}$ on V.

These two facts can be summarised in the following commutative diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\rho_{\mathbb{Z}}"] \arrow[dr, "dense"'] & \mathrm{GL}(V) \\ & \mathbb{C} \arrow[u, dotted, "\exists \rho_{\mathbb{C}}"'] \end{tikzcd}$$

The first fact is based on the definition of Zariski topology. And we sketch the proof of the second fact. For every $V \in \mathrm{Rep}^{un}_{\mathbb{C}}\mathbb{Z}$, all the elements of the image of $\rho_{\mathbb{Z}} : \mathbb{Z} \to \mathrm{GL}(V)$ are unipotent matrices (means matrix g such that $(I-g)^n = 0$ for some $n \in \mathbb{N}$). Now for a complex number $z \in \mathbb{C}$, define $\rho_{\mathbb{C}}(z) = \exp(z \log(\rho_{\mathbb{Z}}(1)))$ where $\exp(g) := \sum_{i=0}^{\infty} g^i/i!$ and $\log(g) := -\sum_{i=1}^{\infty}(I-g)^i/i$ for a matrix g. Note that exp always absolutely converges, and log in this case has only finitely many terms because of the condition $(I-g)^n = 0$ for some n.

A theorem tells us that if the above two facts hold, then $\mathbb{Z}_{\mathbb{C}}$, the $\mathbb{C}$-unipotent completion of $\mathbb{Z}$, must satisfy $\mathbb{Z}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}$, which means $\mathbb{Z}_{\mathbb{C}} = \mathbb{G}_{a/\mathbb{C}}$. ♡

6.2.2 Properties of Unipotent Completion

The unipotent completion has an explicit form. Let G be a group. Let J be the augmentation ideal of $\Bbbk[G]$ (i.e., the kernel of the augmentation map $e^* : \Bbbk[G] \to \Bbbk$). Then the commutative Hopf $\Bbbk$-algebra structure of $\Bbbk[G]$ induces that of $B_m := \mathrm{Hom}_{\Bbbk}(\Bbbk[G]/J^m, \Bbbk) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[G]/J^m, \Bbbk)$ for every $m \in \mathbb{N}$, and hence induces that of $B_{\Bbbk} := \varinjlim_m B_m$. By the third way of viewing affine group scheme over $\Bbbk$, $\mathrm{Spec}\, B_{\Bbbk}$ is such.

Proposition 6.29. [7] Let G be a group. Then $G_{\Bbbk} \simeq \mathrm{Spec}\, B_{\Bbbk}$ as affine group schemes.

Furthermore, the unipotent completion satisfies the universal property.

Proposition 6.30. [7] Let G be a group. There is a canonical group homomorphism $u : G \to G_{\Bbbk}(\Bbbk)$ such that for any unipotent algebraic group U over $\Bbbk$ and group homomorphism $f : G \to U(\Bbbk)$, there is a unique morphism of affine group schemes $f_u : G_{\Bbbk} \to U$ such that after passing $\Bbbk$-points $f_u(\Bbbk) : G_{\Bbbk}(\Bbbk) \to U(\Bbbk)$, we have $f = f_u \circ u$.

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[dr, "f"'] & G_{\Bbbk}(\Bbbk) \arrow[d, dotted, "f_u(\Bbbk)"] \\ & U(\Bbbk) \end{tikzcd}$$

6.3 Algebraic Analogy of Notions in Differential Geometry

6.3.1 Algebraic Differential Forms

Here we introduce basic knowledge of algebraic differential forms with reference [25]. Let R be a ring, and X be a scheme over a scheme S with the structure morephism $f : X \to S$.

Definition 6.31. Let A be an R-algebra and M be an A-module. An *R-derivation* of A into M is an R-linear map $d : A \to M$ satisfying the Leibniz rule: $d(a_1a_2) = a_1da_2 + a_2da_1$ for any $a_1, a_2 \in A$. ■

Definition 6.32. Let A be an R-algebra. The *module of relative differentials* of A over R is an A-module $\Omega^1_{A/R}$ together with an R-derivation $d_{A/R} : A \to \Omega^1_{A/R}$ of A (called the *universal derivation*) such that for any A-module M and R-derivation $d' : A \to M$ of A, there exists a unique A-linear map $\phi : \Omega^1_{A/R} \to M$ such that $d' = \phi \circ d_{A/R}$.

$$\begin{tikzcd} A \arrow[r, "d_{A/R}"] \arrow[dr, "d'"'] & \Omega^1_{A/R} \arrow[d, dotted, "\exists ! \phi"] \\ & M \end{tikzcd}$$

■

Definition 6.33. Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on a topological space X, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-module. An $\mathcal{O}_1$-*derivation* of $\mathcal{O}_2$ into $\mathcal{F}$ is a morphism $d : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the Leibniz rule $d(U)(ab) = ad(U)(b) + d(U)(a)b$ for all open $U \subseteq X$ and $a, b \in \mathcal{O}_2(U)$. ■

Definition 6.34. Let $\Delta : X \to X \times_S X$ be the diagonal morphism, $\mathcal{I} = \ker \Delta^\#$ be the sheaf of $\mathcal{O}_X$-ideals defining $\Delta(X)$. The *sheaf of relative differentials* of X over S is the sheaf of $\mathcal{O}_X$-modules $\Omega^1_{X/S} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$. ■

Definition 6.35. The *universal derivation* $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ is an $f^{-1}\mathcal{O}_S$-derivation of $\mathcal{O}_X$ into $\Omega^1_{X/S}$ such that it has basically the same universal property as Definition 6.32. ■

Definition 6.36. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes \mathcal{G}$ is a sheaf of $\mathcal{O}_X$-module, which sends an open $U \subseteq X$ to the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ and takes the sheafification. ■

Definition 6.37. Suppose $S = \operatorname{Spec} R$ for some ring R. For an R-module E and a sheaf of $\mathcal{O}_X$-module $\mathcal{F}$, $E \otimes \mathcal{F}$ is the sheaf of $\mathcal{O}_X$-modules that sends an affine open $U \subseteq X$ to $E \otimes_R \mathcal{F}(U)$ and then glues and takes the sheafification.

Alternatively, $E \otimes \mathcal{F}$ can be also defined as $f^*\tilde{E} \otimes \mathcal{F}$ where $\tilde{E}$ is the sheaf of $\mathcal{O}_S$-modules associated to E. ■

Definition 6.38. [1] Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-module. The *tensor algebra* $T(\mathcal{F})$ of $\mathcal{F}$ is the sheaf of noncommutative $\mathcal{O}_X$-algebras $T(\mathcal{F}) = \bigoplus_{n=0}^\infty T^n(\mathcal{F})$ where $T^n(\mathcal{F}) = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$ (n times). The *exterior algebra* $\Lambda(\mathcal{F})$ of $\mathcal{F}$ is the quotient of $T(\mathcal{F})$ by the two sided ideals generated by $s \otimes s \in T^2(\mathcal{F})$ ranging over $s \in \mathcal{F}(U)$. The sheaf $\Lambda^n\mathcal{F}$ is the sheafification of the presheaf that sedns an open $U \subseteq X$ to $\Lambda^n_{\mathcal{O}_X(U)}\mathcal{F}(U)$. ■

Definition 6.39. The *sheaf of differential forms of degree n* on X over S is the sheaf of $\mathcal{O}_X$-module $\Omega^n_{X/S} := \Lambda^n\Omega^1_{X/S}$, whose elements are called *algebraic differential forms of degree n* on X over S. ■

Definition 6.40. The universal derivation $d := d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ extends naturally to a cochain complex (i.e., $d \circ d = 0$ in the following sequence), which is called the *algebraic de Rham complex*.

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \xrightarrow{d} \cdots$$

■

6.3.2 Algebraic Vector bundle, Connection and Curvature

From now on, let X be a variety over $\Bbbk$, and $d := d_{X/\Bbbk} : \mathcal{O}_X \to \Omega^1_{X/\Bbbk}$ be the universal derivation.

Definition 4.2 tells us what the vector bundle in differential geometry means. In alegbraic geometry, we also have an analogus notion called "algebraic vector bundle", which is defined as the following.

Definition 6.41. An *algebraic vector bundle* of rank r on X is a sheaf of $\mathcal{O}_X$-module $\mathcal{E}$ which is locally free of rank r, i.e., for each $x \in X$, there exists an open $U \subseteq X$ such that $\mathcal{E}|_U$ is isomorphic to a direct sum of r-copies of $\mathcal{O}_X|_U$.■

As in differential geometry, we can similarly define the notion of connection.

Definition 6.42. [7] Let $\mathcal{E}$ be an algebraic vector bundle on X. The *connection* on $\mathcal{E}$ is a morphism of sheaf of $\mathcal{O}_X$-modules $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X/\Bbbk}$ that satisfies the Leibniz rule, i.e., for any open $U \subseteq X$, $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{E}(U)$,

$$\nabla(U)(fs) = f\nabla(U)(s) + \nabla(U)(s) \otimes df.$$

For an algebraic vector bundle $\mathcal{E}$ of order r on X, locally on open $U \subseteq X$, we have $\mathcal{E}|_U \simeq \mathcal{O}_X^r|_U$. So we can choose a basis $s_1, \cdots, s_r \in \mathcal{E}(U)$ such that for every $s \in \mathcal{E}(U)$, it's a linear combination of those s_i with coefficients $f_i \in \mathcal{O}_X(U)$. We thus identify $s \in \mathcal{E}(U)$ with an r-dimensional vector $(f_1, \cdots, f_r)$.

For $1 \le i, j \le r$, we define the *Christoffel symbols* $\Gamma_{ij} \in \Omega^1_{X/k}(U)$ by

$$\nabla(U)(s_j) = \sum_{i=1}^r s_i \otimes \Gamma_{ij}.$$

So for a general $s = (f_1, \cdots, f_r) \in \mathcal{E}(U)$, by Lebniz rule,

$$\nabla(U)(s) = \sum_{i=1}^r s_i \otimes \left(df_i + \sum_{j=1}^r f_j\Gamma_{ij} \right).$$

Let Γ denote the matrix of $\{\Gamma_{ij}\}$. Then we may write the previous equation in the simple form

$$\nabla = d + \Gamma.$$

Now we can define the curvature.

Definition 6.43. [7] Let $(\mathcal{E}, \nabla)$ be an algebraic vector bundle with connection. The *curvature* of ∇ is $\nabla^2 := \nabla \circ \nabla \in \operatorname{End}(\mathcal{E}) \otimes \Omega^2_{X/\Bbbk}$. ■

Proposition 6.44. Let $(\mathcal{E}, \nabla)$ be an algebraic vector bundle with connection. Locally on open U, we have

$$\nabla^2|_U := d\Gamma + \Gamma \wedge \Gamma$$

where $d\Gamma$ is componentwise exterior differentiation and $\Gamma \wedge \Gamma$ is the matrix multiplication using the wedge product.

Definition 6.45. [7] As in differential geometry, a connection ∇ is *flat* (or *integrable*) if its curvature $\nabla^2 = 0$. A section $s \in \mathcal{E}(U)$ is *horizontal* if $\nabla(U)(s) = 0$. ■

Denote the category of algebraic vector bundles of finite rank on X with connection by $\operatorname{Conn}(X)$, where the morphisms in $\operatorname{Conn}(X)$ is the morphisms of algebraic vector bundles (i.e., morphisms of sheaves of $\mathcal{O}_X$-modules) that preserve the connection. Denote the full subcategory of those with flat connection by $\operatorname{Flat}(X)$.

Definition 6.46. [7] For $(\mathcal{E}_1, \nabla_1), (\mathcal{E}_2, \nabla_2) \in \operatorname{Obj}(\operatorname{Flat}(X))$, define the *tensor product bundle* of them to be $(\mathcal{E}_1 \otimes \mathcal{E}_2, \nabla_1 \otimes \nabla_2) \in \operatorname{Obj}(\operatorname{Flat}(X))$, where $\mathcal{E}_1 \otimes \mathcal{E}_2$ is in the sense of Definition 6.36 and $\nabla := \nabla_1 \otimes \nabla_2$ satisfies for each open $U \subseteq X$, $s_1 \in \mathcal{E}_1(U)$ and $s_2 \in \mathcal{E}_2(U)$,

$$\nabla(U)(s_1 \otimes s_2) = \nabla_1(U)(s_1) \otimes s_2 + s_1 \otimes \nabla_2(U)(s_2).$$

By this means, $\operatorname{Flat}(X)$ is a monoidal category, where the unit object is $(\mathcal{O}_X, d)$ where $d : \mathcal{O}_x \to \Omega^1_{X/\Bbbk}$ is the universal derivation. Furthermore, it can be shown that it is also an abelian category.

6.3.3 Algebraic Parallel Transport

Let M be a manifold. We also denote the category of usual geometric vector bundle (in the sense of 4.2) with connection on M by $\operatorname{Conn}(M)$, and the full subcategroy of those with flat connections by $\operatorname{Flat}(M)$.

We first introduce the notion of parallel transport in differential geometry.

Proposition 6.47. Let $(E, \nabla) \in \operatorname{Obj}(\operatorname{Conn}(M))$, $\gamma : I \to M$ be a path in M, and $v_0 \in E_{\gamma(0)}$. Then there exists a unique section $\mathscr{X} : I = \gamma^* M \to E$ along γ (called the *parallel transport* of v_0 along γ) such that $\nabla \mathscr{X} = 0$ and $\mathscr{X}_0 = v_0$.

Proof. The horizontal condition $\nabla \mathscr{X} = 0$ defines an ordinary differential equation and the condition $\mathscr{X}_0 = v_0$ provides the initial condition. By the existence and uniqueness theorem, such $\mathscr{X}$ exists uniquely along γ. □

The following picture shows a visual example of the parallel transport when the vector bundle E is the tangent bundle TM. In this case, the horizontal condition makes the vectors in $\mathscr{X}$ move "parallelly" along γ, and hence the name of the word “parallel transport".

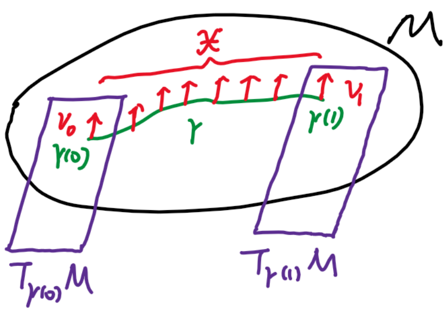

Figure 16: Picture of the parallel transport when the vector bundle is the tangent bundle

Let $a := \gamma(0) \in M$, $b := \gamma(1) \in M$. Then the parallel transport along γ defines a linear map $E_a \to E_b$ that sends $v_0 = \mathscr{X}_0 \in E_a$ to $v_1 := \mathscr{X}_1 \in E_b$. Furthermore, it is a linear isomorphism, where the inverse is given by the parallel transport along $\overline{\gamma}$, the inverse path of γ. Thus, a path γ from a to b defines a linear isomorphism $E_a \to E_b$, which we denote $p_{(E,\nabla)}(\gamma) : E_a \to E_b$ here[7]. Such linear isomorphism can be described as an $r \times r$ invertible matrix valued in $\mathbb{C}$ where r is the dimension of M, called the *associated parallel transport matrix* along γ.

Now let's consider the case when γ is a loop, i.e., $a = b$ and when the connection ∇ is flat. It can be shown that the parallel transport along γ only depends on the homotopy class of γ. Thus, we get a well-defined action of $\pi_1(M, b)$:

Definition 6.48. Let $(E, \nabla) \in \mathrm{Flat}(M)$ and $b \in M$. For a loop γ, the map $\gamma \mapsto p_{(E,\nabla)}(\gamma)$ descends to the map $\pi_1(M, b) \to \mathrm{GL}(E_b), [\gamma] \mapsto p_{(E,\nabla)}(\gamma)$, which is called *the natural representation* of $\pi_1(M, b)$ on E_b. ■

It can be shown that if we have a morphism $f : (E, \nabla) \to (E', \nabla')$ in the caetgory $\mathrm{Conn}(M)$, then the following diagram commutes:

$$\begin{tikzcd} E_a \arrow[r, "p_{(E,\nabla)}(\gamma)"] \arrow[d, "(f_*)_a"'] & E_b \arrow[d, "(f_*)_b"] \\ E'_a \arrow[r, "p_{(E,\nabla)}(\gamma)"'] & E'_b \end{tikzcd}$$

In other words, for $m \in M$, define the fiber functor F_m from the category of $\mathrm{Conn}(M)$ to $\mathrm{Vec}_{\mathbb{C}}$ that sends (E, ∇) to E_m. Then given a path γ from a to b, the parallel transport determines a natural isomorphism from F_a to F_b.

We can use this as an analogy to define the algebraic version of parallel transport. Let X be a variety over $\mathbb{k}$.

Definition 6.49. For any point $x \in X$, define the fiber functor from the category $\mathrm{Conn}(X)$ to $\mathrm{Vect}_{\mathbb{k}}$ that sends $(\mathcal{E}, \nabla) \in \mathrm{Obj}(\mathrm{Flat}(X))$ to the stalk $\mathcal{E}_x$. Given points $a, b \in X$, define an *algebraic parallel transport* from a to b as a natural isomorphism from the fiber functor F_a to F_b. ■

Let $a, b \in X$, and $\mathcal{E}$ be an algebraic vector bunlde of rank r with connection ∇. An algebraic parallel transport $\alpha : F_a \to F_b$ defines a linear isomorphism $\alpha_{(\mathcal{E},\nabla)} : \mathcal{E}_a \simeq \mathbb{k}^r \to \mathcal{E}_b \simeq \mathbb{k}^r$. So an algebraic parallel transport from a to b on $(\mathcal{E}, \nabla)$ can be seen as an $r \times r$ invertible matrix valued in $\mathbb{k}$ where r is the rank of $\mathcal{E}$.

6.4 Iterated Integrals on Manifolds

6.4.1 Iterated Integrals and Fundamental Group

Let M be a manifold, PM be the set of piecewise smooth paths in M.

Definition 6.50. [20] Let $\gamma \in PM$, $\omega_1, \cdots, \omega_r$ be $\mathbb{C}$-valued smooth 1-forms on M, $\gamma^*\omega_j = f_j(t)dt$. Define the *iterated integrals*

$$\int_\gamma \omega_1 \cdots \omega_r := \int_{0 \le t_1 \le \cdots \le t_r \le 1} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r.$$

Remark 6.51. When $r = 0$, the integral is intepreted to be 1. ◇

There are some basic properties of the iterated integrals.

Proposition 6.52. [20] Let $\omega_1, \omega_2, \cdots$ be $\mathbb{C}$-valued smooth 1-forms on M.

- The value of the iterated integral is independent of the parameterisation of the path $\gamma \in PM$.
- (Product Formula) .

  $$\int_\gamma \omega_1 \cdots \omega_r \int_\gamma \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_\gamma \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)}$$

  where the sum ranges over all shuffle permutations $\sigma \in \mathrm{Sym}(r+s)$ of type (r, s).
- (Coproduct Formula)

  $$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r$$

  where $\alpha, \beta \in PM$ such that $\alpha(1) = \beta(0)$, and $\alpha\beta \in PM$ is the concatenated path of α and β.

- (Inverse Formula)
$$\int_{\overline{\gamma}} \omega_1 \cdots \omega_r = (-1)^r \int_{\gamma} \omega_r \cdots \omega_1$$
where $\overline{\gamma} \in PM$ is the inverse path of $\gamma \in PM$.

The product formula, coproduct formula and inverse formula actually corresponds to multiplication, comultiplication and antipode of some Hopf algebra, which is actually $\pi_1(M,b)_{\mathbb{C}}$. We will see how the iterated integrals and the $\mathbb{C}$-unipotent completion of the fundamental group are related through the following steps[7].

1. The iterated integral defines a map $\int \omega_1 \cdots \omega_r : PM \to \mathbb{C}, \gamma \mapsto \int_\gamma \omega_1 \cdots \omega_r$.
2. For each $b \in M$, by the following lemma,

   Lemma 6.53. The iterated integral $\int_\gamma \omega_1 \cdots \omega_r$ depends only on the homotopy class of γ relative to endpoints iff each ω_i is closed and $\omega_i \wedge \omega_{i+1} = 0$ for $i = 1, \cdots, r-1$ (it is called homotopy invariant condition).

   under the homotopy invariant condition, it defines a map $\int \omega_1 \cdots \omega_r : \pi_1(M,b) \to \mathbb{C}$, $[\gamma] \mapsto \int_\gamma \omega_1 \cdots \omega_r$.
3. We extend the above map linearly to a $\mathbb{Q}$-linear map $\int \omega_1 \cdots \omega_r : \mathbb{Q}[\pi_1(M,b)] \to \mathbb{C}$.
4. By the following lemma,

   Lemma 6.54. The map $\int \omega_1 \cdots \omega_r : \mathbb{Q}[\pi_1(M,b)] \to \mathbb{C}$ vanishes on elements of J^m for any $m > r$ where J is the augmentation ideal of $\mathbb{Q}[\pi_1(M,b)]$.

   we get a $\mathbb{Q}$-linear map $\int \omega_1 \cdots \omega_r : \mathbb{Q}[\pi_1(M,b)]/J^m \to \mathbb{C}$, i.e., $\int \omega_1 \cdots \omega_r \in B_m$ for any $m > r$ where $B_m := \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1(M,b)]/J^m, \mathbb{C})$.
5. Since it holds for any $m > r$, $\int \omega_1 \cdots \omega_r \in B_{\mathbb{C}}$ where $B_{\mathbb{C}} := \varinjlim_m B_m$. By Propositon 6.29, $\mathrm{Spec} B_{\mathbb{C}} \simeq \pi_1(M,b)_{\mathbb{C}}$, or equivalently, $B_{\mathbb{C}} \simeq \mathcal{O}(\pi_1(M,b)_{\mathbb{C}})$ where $\mathcal{O}$ is the global section. So $\int \omega_1 \cdots \omega_r \in \mathcal{O}(\pi_1(M,b)_{\mathbb{C}})$.

6.4.2 Iterated Integrals and Parallel Transport

Let M be a manifold, $\omega_1, \cdots, \omega_r$ be smooth 1-forms on M, $a, b \in M$ and γ is a path from a to b. Then evaluating $\int_\gamma \omega_1 \cdots \omega_r$ is the same[7] as solving the differential equations $df_i = f_{i-1}\omega_i$ with the initial conditions $f_i(a) = 0$ for $1 \le i \le r$ so that $f_r(b) = \int_\gamma \omega_1 \cdots \omega_r$ (Here f_0 should be interpreted as the constant function 1).

Let $\mathbf{f}$ be the column vector of $(f_0, f_1, \cdots, f_r)$. Then the differential equations become
$$d\mathbf{f} = \mathbf{A}\mathbf{f}$$
with the initial condition $\mathbf{f}(a) = \mathbf{v} := (1, 0, 0, \cdots, 0)$ where
$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega_r & 0 \end{pmatrix}. \tag{6.1}$$

In other words, the solutions to the differential equations are horizontal sections of the connection ∇ on $E := \mathcal{O}_M^{r+1}$ (the vector bundle given by $(r+1)$-copies of sheaf of C^∞ functions) given by $\Gamma := -\mathbf{A}$.

The curvature
$$\nabla^2 = d\Gamma + \Gamma \wedge \Gamma = -\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ d\omega_1 & 0 & 0 & 0 & \cdots & 0 \\ \omega_1 \wedge \omega_2 & d\omega_2 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 \wedge \omega_3 & d\omega_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \omega_{r-1} \wedge \omega_r & d\omega_r & 0 \end{pmatrix}$$

So ∇ is flat if and only if each ω_i is closed and $\omega_i \wedge \omega_{i+1} = 0$, which is just the homotopy invariant condition in Lemma 6.53.

Note that the associated parallel transport matrix $\mathbf{P}(\gamma)$ along γ is[7]

$$\mathbf{P}(\gamma) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \int_\gamma \omega_1 & 1 & 0 & \cdots & 0 \\ \int_\gamma \omega_1\omega_2 & \int_\gamma \omega_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \int_\gamma \omega_1\cdots\omega_r & \int_\gamma \omega_2\cdots\omega_r & \cdots & \int_\gamma \omega_r & 1 \end{pmatrix}.$$

The fact that the matrix $\mathbf{P}(\gamma)$ is a unipotent matrix corresponds to the fact that (E,∇) is an unipotent object in $\mathrm{Flat}(M)$. And we see that the iterated integrals are actually the coefficients in the associated parallel transport matrix! Or more preciesely, if we denote $\mathrm{pr}_{r+1} : \mathbb{C}^{r+1} \to \mathbb{C}$ to be the projection on the last entry, then

$$\int_\gamma \omega_1\cdots\omega_r = \mathrm{pr}_{r+1}(\mathbf{P}(\gamma)\mathbf{v}) \in \mathbb{C}.$$

6.5 de Rham Fundamental Group

6.5.1 Definition of de Rham Fundamental Group

Let X be a variety over $\mathbb{k}$.

Definition 6.55. The full subcategory of $\mathrm{Flat}(X)$ that consists of unipotent objects is denoted by $\mathrm{Un}(X)$, whose elements are called *unipotent algebraic vector bundle with flat connection* on X. ■

Choose a base point $b \in X$. Define the fiber functor $F_b : \mathrm{Un}(X) \to \mathrm{Vec}_{\mathbb{k}}(X)$ that maps $(\mathcal{E},\nabla) \in \mathrm{Un}(X)$ to the stalk $\mathcal{E}_b$ and maps a morphism $(\mathcal{F},\nabla_1) \to (\mathcal{G},\nabla_2)$ to the indcued map $\mathcal{F}_b \to \mathcal{G}_b$ on the stalk level.

Lemma 6.56. [7] $\mathrm{Un}(X)$ with the fiber functor $F_b : \mathrm{Un}(X) \to \mathrm{Vec}_{\mathbb{k}}(X)$ is a neutral Tannakian category.

Definition 6.57. [7] By Lemma 6.56 and Theorem 6.12, there is a unique affine group scheme $\pi_1^{\mathrm{dR}}(X,b)$ over $\mathbb{k}$ up to isomorphism such that there exists an equivalence $\mathrm{Un}(X) \simeq \mathrm{Rep}_{\mathbb{k}}\pi_1^{\mathrm{dR}}(X,b)$ defined by the fiber functor F_b. Such $\pi_1^{\mathrm{dR}}(X,b)$ is called the *de-Rham fundamental group* of X at b. ■

Definition 6.58. [23] For $x \in X$, the *de Rham path torsor* $P^{\mathrm{dR}}(X;b,x) := \mathrm{Isom}^{\otimes}(F_b,F_x)$. ■

In particular, $P^{\mathrm{dR}}(X;b,b) = \mathrm{Aut}^{\otimes}(F_b) \simeq \pi_1^{\mathrm{dR}}(X,b)$ by Tannakian formalism.

Let $\mathrm{Un}_b(X)$ be the category of pointed unipotent vector bundles $(\mathcal{V},\nabla_{\mathcal{V}},v)$ with flat connection $\nabla_{\mathcal{V}}$ with a specified $v \in \mathcal{V}_b$ (the fibre at b), and the morphisms are required to send the specified point to the specified point.

Proposition 6.59. [23] There is a pro-universal object $(\mathcal{E}^{\mathrm{dR}},\nabla^{\mathrm{dR}},1)$ in $\mathrm{Un}_b(X)$ (which is a projective system of objects $(\mathcal{E}_n^{\mathrm{dR}},\nabla_n^{\mathrm{dR}},1) \in \mathrm{Un}_b(X)$ satisfying some compatibility). That means, given any $(\mathcal{V},\nabla_{\mathcal{V}},v) \in \mathrm{Obj}(\mathrm{Un}_b(X))$ where $v \in V_b$, there exists a unique morphism $\phi : (\mathcal{E}^{\mathrm{dR}},\nabla^{\mathrm{dR}},1) \to (\mathcal{V},\nabla_{\mathcal{V}},v)$.

In fact, $(\mathcal{E}^{\mathrm{dR}},\nabla^{\mathrm{dR}},1)$ as a pro-universal object in $\mathrm{Un}_b(X)$ corresponds to $(\mathbb{k}[G],1)$ as a pro-universal object in $\mathrm{Rep}_{\mathbb{k}}G$, and $\pi_1^{\mathrm{dR}}(X,b)$ corresponds to G in this context. As in Tannakian formalism, we can recover $\pi_1^{\mathrm{dR}}(X,b)$ and $\mathcal{E}_b^{\mathrm{dR}}$ which is stated below (Compare Proposition 6.2).

Proposition 6.60. [23] With the same notations as above,

1. (Recover $\pi_1^{\mathrm{dR}}(X,b)$ from F_b) $\pi_1^{\mathrm{dR}}(X,b) \simeq \mathrm{Aut}^{\otimes}(F_b)$ as functors $\mathrm{Alg}_{\mathbb{k}} \to \mathrm{Grp}$;
2. (Recover $\mathcal{E}_b^{\mathrm{dR}}$ from F_b) $\mathrm{End}(F_b) \simeq \mathcal{E}_b^{\mathrm{dR}}$;
3. (Recover $\pi_1^{\mathrm{dR}}(X,b)$ from $\mathcal{E}_b^{\mathrm{dR}}$) $\pi_1^{\mathrm{dR}}(X,b)$ is isomorphic to the group-like elements in $\mathcal{E}_b^{\mathrm{dR}}$, while $P^{\mathrm{dR}}(X;b,x)$ is isomorphic to the group-like elements in $\mathcal{E}_x^{\mathrm{dR}}$.

The universal property of $(\mathcal{E}^{\mathrm{dR}}, \nabla^{\mathrm{dR}}, 1)$ gives rise to a unique map of pro-objects in $\mathrm{Un}(X)$

$$\Delta : (\mathcal{E}^{\mathrm{dR}}, \nabla) \to (\mathcal{E}^{\mathrm{dR}}, \nabla)\hat{\otimes}(\mathcal{E}^{\mathrm{dR}}, \nabla)$$

which sends 1 to $1 \otimes 1$. That defines a comultiplication on $\mathcal{E}^{\mathrm{dR}}$ which makes $\mathcal{E}^{\mathrm{dR}}_x$ a $\Bbbk$-coalgebra structure for every $x \in X$. Let $\mathcal{A}^{\mathrm{dR}} = (\mathcal{E}^{\mathrm{dR}})^*$ be the dual bundle. Then the $\Bbbk$-coalgebra structure of $\mathcal{E}^{\mathrm{dR}}_x$ gives $\mathcal{A}^{\mathrm{dR}}_x = \mathrm{Hom}_{\Bbbk}(\mathcal{E}^{\mathrm{dR}}_x, \Bbbk)$ the $\Bbbk$-algebra structure via the dual map of Δ.

Proposition 6.61. [23] $P^{\mathrm{dR}}(X; b, x) = \mathrm{Spec}(\mathcal{A}^{\mathrm{dR}}_x)$. In particular, $\pi_1^{\mathrm{dR}}(X, b) = \mathrm{Spec}(\mathcal{A}^{\mathrm{dR}}_b)$.

6.5.2 Comparison Theorem

Theorem 6.62 (Riemann-Hilbert Correspondence). [7] For a connected smooth manifold M and a point $b \in M$, there is an equivalence of categories from $\mathrm{Flat}(M)$ to $\mathrm{Rep}_{\mathbb{C}}(\pi_1(M, b))$ that sends the vector bundle with flat connection (E, ∇) to the natural representation of $\pi_1(M, b)$ on E_b (see Definition 6.48).

Now suppose that X is a variety over $\mathbb{C}$. Then it makes sense to consider its underlying manifold, denoted by X^{an}. We have the following GAGA principle.

Theorem 6.63 (GAGA Principle). [7] Let X be a smooth variety over $\mathbb{C}$.

- If X is proper, then $\mathrm{Flat}(X) \simeq \mathrm{Flat}(X^{\mathrm{an}})$ as tensor categories.
- $\mathrm{Un}(X) \simeq \mathrm{Un}(X^{\mathrm{an}})$ as tensor categories.

Theorem 6.64 (Comparison Theorem). [7] For a smooth variety X over $\mathbb{C}$ and $b \in X$, $\pi_1^{\mathrm{dR}}(X, b) \simeq \pi_1(X^{\mathrm{an}}, b)_{\mathbb{C}}$.

Proof. By Riemann-Hilbert correspondence 6.62, $\mathrm{Flat}(X^{\mathrm{an}}) \simeq \mathrm{Rep}_{\mathbb{C}}(\pi_1(X^{\mathrm{an}}, b))$. By GAGA 6.63 and the definition 6.26 of unipotent completion, $\mathrm{Un}(X) \simeq \mathrm{Un}(X^{\mathrm{an}}) \simeq \mathrm{Rep}_{\mathbb{C}}(\pi_1(x^{\mathrm{an}}, b)_{\mathbb{C}})$. By the definition 6.57 of de Rham fundamental group, $\mathrm{Un}(X) \simeq \mathrm{Rep}_{\mathbb{C}}(\pi_1^{\mathrm{dR}}(X, b))$. Then it follows from the uniqueness part of Main Theorem 6.12. □

Remark 6.65. If the variety X is over a field k, the comparison theorem becomes $\pi_1^{\mathrm{dR}}(X, b) \otimes_k \mathbb{C} \simeq \pi_1(X^{\mathrm{an}}, b)_{\mathbb{C}}$. ♢

Example 6.66. Consider $X = \mathbb{G}_{a/\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[t]$ over $\mathbb{C}$. Its underlying manifold is $X^{\mathrm{an}} = \mathbb{C}$. By comparison theorem and Example 6.27, $\pi_1^{\mathrm{dR}}(X) = \pi_1(X^{\mathrm{an}})_{\mathbb{C}} = \{1\}_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}$. ♡

Example 6.67. Consider $X = \mathbb{G}_{m/\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[t, t^{-1}]$ over $\mathbb{C}$. Its underlying manifold is $X^{\mathrm{an}} = \mathbb{C} \backslash \{0\}$. By comparison theorem and Example 6.28, $\pi_1^{\mathrm{dR}}(X) = \pi_1(X^{\mathrm{an}})_{\mathbb{C}} = \mathbb{Z}_{\mathbb{C}} = \mathbb{G}_{a/\mathbb{C}}$. ♡

Remark 6.68. There is another notion of fundamental group in arithmetic geometry called "étale fundamental group $\pi_1^{\text{ét}}$". For a variety over $\mathbb{C}$, unlike the fact that de Rham fundamental group is the unipotent completion of the usual fundamental group, the étale fundamental group is the pro-finite completion of that. So for example, $\pi_1^{\text{ét}}(\mathbb{G}_{a/\mathbb{C}}) = 1$, $\pi_1^{\text{ét}}(\mathbb{G}_{m/\mathbb{C}}) = \hat{\mathbb{Z}}(1) := \varprojlim_n \mu_n$ where $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ is the group of n-th unity roots. ♢

6.5.3 de Rham Fundamental Group of Affine Curves

Let X be an affine smooth curve over $\Bbbk$, and choose a base point $b \in X$ and basis $\alpha_1, \cdots, \alpha_m \in H^1_{\mathrm{dR}}(X)$.

Lemma 6.69. [22] Let $(\mathcal{V}, \nabla) \in \mathrm{Obj}(\mathrm{Un}(X))$ of rank r. Then there exists strictly upper-triangular matrices N_i such that

$$(V, \nabla) \simeq \left(\mathcal{O}_X^r, d + \sum_i \alpha_i N_i\right).$$

The free algebra $\Bbbk\langle A_1, \cdots, A_m\rangle$ on symbols $A_1, \cdots, A_m$ is a commutative Hopf $\Bbbk$-algebra, where

- the comultiplication is given by $\Delta(A_i) = A_i \otimes 1 + 1 \otimes A_i$ and extends linearly;
- the antipode is given by $A_i \mapsto -A_i$ and extends linearly.

Let J be the augmentation idedal of $\Bbbk\langle\alpha_1,\cdots,\alpha_m\rangle$. For any n, consider $E_n = \Bbbk\langle\alpha_1,\cdots,\alpha_m\rangle/J^{n+1}$ and $E = \varprojlim_n E_n$. Then E has a natural commutative Hopf $\Bbbk$-algebra structure.

Now let $(\mathcal{E},\nabla_{\mathcal{E}})$ be the vector bundle where $\mathcal{E} = E\otimes\mathcal{O}_X$ (see Definition 6.37) and $\nabla_{\mathcal{E}} f = df - \sum_i A_i f\alpha_i$ for sections $f : X \to E$. There is the distinguished element $1 \in \mathcal{E}_b = E$.

Proposition 6.70. [22] There is a unique isomorphism $(\mathcal{E},\nabla_{\mathcal{E}},1) \simeq (\mathcal{E}^{\mathrm{dR}},\nabla^{\mathrm{dR}},1)$.

Proof. We only need to verify that $(\mathcal{E},\nabla_{\mathcal{E}},1)$ is the pro-universal object. The existence part is easy. Choose $b \in X$. For any $(\mathcal{V},\nabla) \in \mathrm{Obj}(\mathrm{Un}(X))$, by lemma 6.69, we may assume $(\mathcal{V},\nabla) = (\mathcal{O}_X^r, d - \sum_i \alpha_i N_i)$ where N_i are strictly upper triangluar matrices. For any $v \in \mathcal{V}_b$, define the map $\mathcal{E}\to\mathcal{V}$ that sends a section $\sum f_w[w]$ of $\mathcal{E}$ to $\sum_w f_w N_w v$ where N_w is the matrix $N_{i_1}\cdots N_{i_n}$ for $w = A_{i_1}\cdots A_{i_n}$. One can check that this map preserves the connection and sends 1 to v. The uniqueness is a bit tricky, using the iterated integrals, and we omit the proof. □

Let $\mathcal{A} = \mathcal{E}^*$ be the dual bundle.

Theorem 6.71. For an affine smooth curve X over $\Bbbk$, $\pi_1^{\mathrm{dR}}(X,b) \simeq \mathrm{Spec}\,\mathcal{A}_b$ and $P^{\mathrm{dR}}(X;b,x) \simeq \mathrm{Spec}\,\mathcal{A}_x$.

Proof. By Proposition 6.61, $\pi_1^{\mathrm{dR}}(X,b) \simeq \mathrm{Spec}\,\mathcal{A}_b^{\mathrm{dR}}$. By Proposition 6.70, $\mathcal{A}^{\mathrm{dR}} \simeq \mathcal{A}$. Similar for $P^{\mathrm{dR}}(X;b,x)$. □

6.5.4 Algebraic Iterated Integrals

Let X be a smooth variety over $\mathbb{C}$, $\omega_1\cdots\omega_r$ are $\mathbb{C}$-valued smooth 1-forms on X^{an}, and γ be a loop in X^{an} based at point $b \in X$. By comparison theorem 6.64, $\pi_1(X^{\mathrm{an}},b)_{\mathbb{C}} \simeq \pi_1^{\mathrm{dR}}(X,b)$. So $\int \omega_1\cdots\omega_r \in \mathcal{O}(\pi_1(X^{\mathrm{an}},b)_{\mathbb{C}}) = \mathcal{O}(\pi_1^{\mathrm{dR}}(X,b))$. So the iterated integral $\int\omega_1\cdots\omega_r$ can be seen as a map $\pi_1^{\mathrm{dR}}(X,b)\to\mathbb{C}$.

From previous sections, we know that the iterated integral on a manifold can be seen as the coefficients in the associated parallel transport matrix. We apply the same idea to define the *algebraic iterated integrals*, i.e., iterated integrals of algebraic differential forms on varieties.

Let X be a variety over $\Bbbk$, $b\in X$, $\omega_1,\cdots,\omega_r$ be algebraic 1-forms on X satisfying the homotopy invariant condition, and $\gamma\in\pi_1^{\mathrm{dR}}(X,b)(\Bbbk)$. Let $(\mathcal{E},\nabla)\in\mathrm{Obj}(\mathrm{Flat}(X))$ be the vector bundle $\mathcal{E} = \mathcal{O}_X^{r+1}$ with horizontal connection given by $\Gamma = -\mathbf{A}$ where $\mathbf{A}$ is in the same equation of (6.1). It can be shown that $(\mathcal{E},\nabla)\in\mathrm{Un}(X)$. So there is a natural action of $\pi_1^{\mathrm{dR}}(X,b)(\Bbbk)$ on $\mathcal{E}_b\simeq\Bbbk^{r+1}$ by viewing $\pi_1^{\mathrm{dR}}(X,b)(\Bbbk)\simeq\mathrm{Aut}^{\otimes}F_b$ where F_b is the fiber functor. Let $\mathbf{v} := (1,0,\cdots,0)\in\mathcal{E}_b\simeq\Bbbk^{r+1}$ and $\mathrm{pr}_{r+1} : \mathcal{E}_b\simeq\Bbbk^{r+1}\to\Bbbk$ be the projection to the last coordinate.

Definition 6.72. [7] With the notations above, define

$$\int_\gamma \omega_1\cdots\omega_r := \mathrm{pr}_{r+1}(\gamma\cdot\mathbf{v})\in\Bbbk,$$

where $\gamma\cdot\mathbf{v}$ is the natural action of γ on $\mathbf{v}$ mentioned above ■

We can similarly define $\int_\gamma\omega_1\cdots\omega_r$ for $\gamma\in\pi_1^{\mathrm{dR}}(X,b)(R)$ for $\Bbbk$-algebra R by base change, i.e., we should replace $\mathcal{E}_b\simeq\Bbbk^{r+1}$ above by $\mathcal{E}_b\otimes_{\Bbbk}R\simeq R^{r+1}$. Then this gives a map $\int\omega_1\cdots\omega_r : \pi_1^{\mathrm{dR}}(X,b)(R)\to R = \mathbb{G}_{a/\Bbbk}(R)$, and hence gives a morphism $\int\omega_1\cdots\omega_r : \pi_1^{\mathrm{dR}}(X,b)\to\mathbb{G}_{a/\Bbbk}$, i.e., $\int\omega_1\cdots\omega_r\in\mathcal{O}(\pi_1^{\mathrm{dR}}(X,b))$[7].

6.5.5 Hodge Filtration And a Glimpse of Kim's Cutter

Let X be a projective variety over $\Bbbk$. The pro-universal bundle $\mathcal{E}^{\mathrm{dR}}$ carries a Hodge filtration. This is the unique decreasing filtration $\mathcal{F}^i$, $i\le 0$ of $\mathcal{E}^{\mathrm{dR}}$ satisfying the following conditions[23]:

1. Griffiths transversality: $\nabla(\mathcal{F}^i)\subseteq\mathcal{F}^{i-1}\otimes\Omega_X$;
2. The induced filtration on T_n^{dR} coincides with the constant one coming from (co)homology;
3. $1\in\mathcal{F}^0\mathcal{E}_b^{\mathrm{dR}}$.

The Hodge filtration on $\mathcal{E}^{\mathrm{dR}}$ induces Hodge filtration with non-negative degress on teh dual $\mathcal{A}^{\mathrm{dR}}$ and $\mathcal{F}^1\mathcal{A}^{\mathrm{dR}}$ is an ideal. One defines $\mathcal{F}^0P^{\mathrm{dR}}(X;b,x)$ to be the zero set of $\mathcal{F}^1\mathcal{A}_x^{\mathrm{dR}}$. It is a torsor for $\mathcal{F}^0\pi_1^{\mathrm{dR}}(X,b)$, which is a subgroup of $\pi_1^{\mathrm{dR}}(X,b)$. This is an aspect of the fact that the action of $\pi_1^{\mathrm{dR}}(X,b)$ on $P(X;b,x)$ is compatible with Hodge filtration. The action $P(X;b,x)\times\pi_1^{\mathrm{dR}}(X,b)\to P(X;b,x)$ is defined to be the composition of natural transformations

$\mathrm{Isom}^{\otimes}(F_b, F_x) \times \mathrm{Aut}^{\otimes}(F_b) \to \mathrm{Isom}^{\otimes}(F_b, F_x)$. By taking the dual, it corresponds to a co-action map $\mathcal{A}_x^{\mathrm{dR}} \to \mathcal{A}_x^{\mathrm{dR}} \otimes \mathcal{A}_b^{\mathrm{dR}}$, and this is compatible with the Hodge filtration[23].

Let p be a prime. There is a map $\mathcal{O}(\pi_1^{\mathrm{dR}}(X, O)/\mathcal{F}^0\pi_1^{\mathrm{dR}}(X, O)) \to \mathcal{O}(X(\mathbb{Q}_p))$ by sending $\int \omega_1 \cdots \omega_r$ to $(P \mapsto \int_O^P \omega_1 \cdots \omega_r)$ (called *iterated Coleman integral*) where $\omega_1, \cdots, \omega_r$ are algebraic 1-forms on X. Then it induces a map $X(\mathbb{Q}_p) \to \pi_1^{\mathrm{dR}}(X)/\mathcal{F}^0\pi_1^{\mathrm{dR}}(X)$ which we denote $\int$ here.[7]

Now we take the glimpse of the fundamental diagram in non-abelina Chabauty methods, called "Kim's cutter".

Theorem 6.73 (Kim's Cutter). [23] Let X be a smooth proper curve of geneus ≥ 2, p be a prime. The following diagram commutes:

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, "\mathrm{loc}"] \arrow[d, "j"'] & X(\mathbb{Q}_p) \arrow[d, "j_p"'] \arrow[dr, "\int"] & \\
H_f^1(G_{\mathbb{Q}}, \pi_1^{\text{ét}}(X)_{\mathbb{Q}_p,n}) \arrow[r, "\mathrm{loc}"'] & H_f^1(G_{\mathbb{Q}_p}, \pi_1^{\text{ét}}(X)_{\mathbb{Q}_p,n}) \arrow[r, "\simeq"'] & \pi_1^{\mathrm{dR}}(X)/\mathcal{F}^0\pi_1^{\mathrm{dR}}(X)
\end{tikzcd}$$

Above, $\pi_1^{\text{ét}}(X)_{\mathbb{Q}_p,n} := \pi_1^{\text{ét}}(X)_{\mathbb{Q}_p}/\pi_1^{\text{ét}}(X)_{\mathbb{Q}_p}^{n+1}$ is the n-th quotient of the lower central series. loc is the localisation map. We won't make precise of the Selmer group H_f^1 and the maps j, j_p here. The image of the lower loc map is conjectured to be non-dense for large n. Assuming the conjecture, we can deduce that $X(\mathbb{Q})$ is finite, which is Mordell Conjecture (solved by Faltings).

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