Chabauty-Kim Methods and S-unit Equations

Xiang Li

First-Year Report
Graduate School of Mathematics
University of Edinburgh
August 9, 2024

Abstract

The Chabauty-Kim method is a p-adic method to find rational or integral points on algebraic curves effectively. The report discusses the Chabauty-Kim methods on S-unit equations. We begin the report by some background knowledge on de Rham fundamental groups and Hodge filtrations. Then we introduce the methodology of refined Chabauty-Kim methods and weight filtration methods. Finally we focus on the case $|S| = 3$ over $\mathbb{Q}$ to give a new upper bound of the number of solutions, and also focus on the case over number fields.

Contents

Chapter 1

Introduction

1.1 Introduction of Chabauty-Kim Methods

Settings:

- X is a smooth proper curve with Jacobian J of genus g.
- p is a prime of good reduction.
- For a field K, $G_K := \mathrm{Gal}(\overline{K}/K)$ is the absoulte Galois group of K.
- U is (a quotient of) the $\mathbb{Q}_p$-prounipotent étale fundamental group with the basepoint $b \in X$.
- $U_n := U/U^{n+1}$ is the n-th quotient of the lower central series of U.

We have a commutative diagram.

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d, "j"'] & X(\mathbb{Q}_p) \arrow[d, "j_p"] \\ H^1(G_{\mathbb{Q}}, U_n) \arrow[r, "\mathrm{loc}_p"'] & H^1(G_{\mathbb{Q}_p}, U_n) \end{tikzcd}$$

In the diagram, j (resp. j_p) is the *gobal* (resp. *local*) *Kummer map*, which maps x to the equivalent class of its path torsor where X is over $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}_p}$).

Define H^1_f as the crystalline cohomology, which is a subscheme of H^1. More concretely, the *local Selmer scheme* $H^1_f(G_{\mathbb{Q}_p}, U_n)$ is the Zariski closure of the image of j_p, and the *glolbal Selmer scheme* $\mathrm{Sel}_n := H^1(G_{\mathbb{Q}}, U_n)$ consists of classes α such that $\mathrm{loc}_l(\alpha) \in H^1_f(G_{\mathbb{Q}_l}, U_n)$ for all prime l. We have H^1_f is the moduli space of U-torsors with Galois action.

The above diagram induces the following diagram, which is the fundamental diagram of Chabauty-Kim methods (“Kims cutter”):[12]

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d, "j"'] & X(\mathbb{Q}_p) \arrow[d, "j_p"] \arrow[dr, "j^{\mathrm{dR}}"] & \\ \mathrm{Sel}_n \arrow[r, "\mathrm{loc}_p"'] & H^1_f(G_{\mathbb{Q}_p}, U_n) \arrow[r, "\mathrm{D}", "\simeq"'] & F^0U^{\mathrm{dR}} \backslash U^{\mathrm{dR}} \end{tikzcd}$$

In the diagram, U^{dR} is the de Rham fundamental group and $F^\bullet$ is the Hodge filtration (which will be defined and discussed in details in Chapter 2).

In particular, for depth $n = 1$, we have $U_1 = \pi_1^{\mathbb{Q}_p}(X;b)^{\mathrm{ab}} = V_pJ := T_pJ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which recovers the ordinary Chabuaty-Coleman method.

The Kim's cutter has a version of S-integral points: (Given a finite set of primes $S \subseteq \mathbb{Z}$, denote $\mathbb{Z}_S := \mathbb{Z}[1/S]$, i.e., the ring of those rational numbers whose denominators does not contain the prime factors that are not in S)

$$\begin{tikzcd}
X(\mathbb{Z}_S) \arrow[r] \arrow[d, "j"'] & X(\mathbb{Z}_p) \arrow[d, "j_p"] \arrow[dr, "j^{\mathrm{dR}}"] & \\
\mathrm{Sel}_{S,n} \arrow[r, "\mathrm{loc}_p"'] & H^1_f(G_{\mathbb{Q}_p}, U_n) \arrow[r, "\mathrm{D}", "\simeq"'] & F^0U^{\mathrm{dR}}\backslash U^{\mathrm{dR}}
\end{tikzcd}$$

where $\mathrm{Sel}_{S,n} := H^1_f(G_{S,\mathbb{Q}}, U_n)$ is the *S-global Selmer scheme* where $G_{S,\mathbb{Q}}$ is the Galois group of the maximal extension that is unramified outside S over $\mathbb{Q}$.

Definition 1.1.1. [12] The *Chabauty-Kim locus in depth n* is defined by

$$X(\mathbb{Z}_p)_{S,n} := (j_p)^{-1}(\mathrm{im}(\mathrm{loc}_p)).$$

We have a nested sequence:

$$X(\mathbb{Z}_p) \supseteq X(\mathbb{Z}_p)_{S,1} \supseteq X(\mathbb{Z}_p)_{S,2} \supseteq \cdots \supseteq X(\mathbb{Z}_S)$$

Theorem 1.1.2. *[12] If loc_p is non-dominant ($\dim H^1_f(G_{\mathbb{Q}}, U_n) < \dim H^1_f(G_{\mathbb{Q}_p}, U_n)$), then $X(\mathbb{Q})$ (resp. $X(\mathbb{Z}_S)$) is finite.*

Proof. There is a nonzero $\alpha : F^0U^{\mathrm{dR}}\backslash U^{\mathrm{dR}} \to \mathbb{Q}_p$ vanishing on $\mathrm{im}(D \circ \mathrm{loc}_p)$. So $(\alpha \circ j^{\mathrm{dR}})|_{X(\mathbb{Q})} = 0$. But can show the finiteness of the zero set. $\square$

In particular, in ordinary Chabauty, the condition $\mathrm{rk}(J) < g$ implies that $X(\mathbb{Q})$ is finite. It is conjectured that loc_p is non-dominant for sufficiently large n whenever $g > 2$, which implies Mordell conjecture.

Conjecture 1.1.3 (Kim's conjecture[12]). *$X(\mathbb{Z}_S) = X(\mathbb{Z}_p)_{S,n}$ for large enough n.*

So the Kim's cutter and Kim's conjecture actually provide an efficient algorithm to compute the S-integral points of the curve X.

The above Chabauty-Kim method is over $\mathbb{Q}$. It can be generalized to be over a number field, which will be discusseed a bit in Chapter 4.

1.2 Introduction of S-unit Equations

Denote the group of units of $\mathbb{Z}_S$ by $\mathbb{Z}_S^\times$. The *S-unit equation problem* is to find all $x \in \mathbb{Z}_S^\times$ such that $x + y = 1$ for some $y \in \mathbb{Z}_S^\times$. Using the language of algebraic geometry, the S-unit equation problem can be restated as finding $X(\mathbb{Z}_S)$, the set of $\mathbb{Z}_S$-points of the variety $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$.

By a quick parity analysis, if $2 \notin S$, then $X(\mathbb{Z}_S) = \varnothing$. So we may assume $2 \in S$. The group $\left\{z, 1-z, \frac{1}{z}, \frac{z-1}{z}, \frac{1}{1-z}, \frac{z}{z-1}\right\} \simeq S_3$ acts on $X(\mathbb{Z}_S)$. If $2 \in S$, then there is always one orbit of three elements, namely $\left\{2, -1, \frac{1}{2}\right\}$, and any other orbit (if exists) has six elements.

We first introduce some basic notions:

Definition 1.2.1. A prime number is *Fermat prime* if it is of the form $2^n + 1$ for some $n \in \mathbb{Z}^+$, and is *Mersenne prime* if it is of the form $2^n - 1$ for some $n \in \mathbb{Z}^+$.

The following theorem completely solves the S-unit equation problem for S of size one and two:

Theorem 1.2.2. *[3]*

- *Given $S = \{2\}$, we have $X(\mathbb{Z}_S) = \left\{2, -1, \frac{1}{2}\right\}$.*
- *Given $S = \{2, p\}$ where p is an odd prime,*
  - *if p is neither Fermat or Mersenne, then $X(\mathbb{Z}_S) = \left\{2, -1, \frac{1}{2}\right\}$;*
  - *if $p \neq 3$ is Fermat, then $X(\mathbb{Z}_S)$ has 9 elements, which contains an orbit of 2 and an orbit of p;*
  - *if $p \neq 3$ is Mersenne, then $X(\mathbb{Z}_S)$ has 9 elements, which contains an orbit of 2 and an orbit of $-p$;*
  - *if $p = 3$, then $X(\mathbb{Z}_S)$ has 21 elements, which contains an orbit of 2, an orbit of 3, an orbit of -3, and an orbit of 9.*

However, we don't know much about $X(\mathbb{Z}_S)$ for S of size at least 3. Siegel first showed the following ground-breaking theorem:

Theorem 1.2.3 (Siegel 1926[15]). *Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$. For any set S of finitely many prime numbers, $X(\mathbb{Z}_S)$ is a finite set.*

It is worth mentioning that Minhyong Kim gave an alternative proof[10] of the theorem of Siegel in 2005 using Chabauty-Kim methods, which is effective.

There is also a result on the bound of the number of the solutions.

Theorem 1.2.4 (Evertse 1984[9]). *Let $s = \#S$. Then $\#X(\mathbb{Z}_S) \leq 3 \cdot 7^{2s+3}$.*

The bound is not so good, especially when s is small. Let's focus on the case for S of size 3 now. The above bound gives us $X(\mathbb{Z}_S) \leq 1.211 \times 10^8$, whereas the numerical experiment suggests $X(\mathbb{Z}_S) \leq 99$.

In this report, I will use the refined Chabauty-Kim methods and weight filtration methods (discussed in Chapter 3) to show in Chapter 4 the following proposition as my current result:

Proposition 1.2.5. *Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and S is a set of 3 distinct primes. Then*

$$\#X(\mathbb{Z}_S) \leq \begin{cases} 9399 & 3 \notin S \\ 18297 & 5 \notin S \\ 26985 & S = \{2, 3, 5\} \end{cases}.$$

Chapter 2

Background Knowledge

2.1 de Rham Fundamental Group

2.1.1 Tannakian Formalism And Unipotent Completion

Let K be a field of characteristic 0, Vec_K be the category of finite dimensional K-vector spaces.

Definition 2.1.1. A *Tannakian category* over K is a symmetric tensor category $\mathcal{C}$ over K (i.e., symmetric k-linear rigid monoidal abelian category with 1 simple) that admits a braided tensor functor $F : \mathcal{C} \to \mathrm{Vec}_K$ (called the fiber functor).

An example of Tannakian category is $\mathrm{Rep}_K G$, the category of representations of a finite group G on finite dimensional K-vector spaces, with the fiber functor being the forgetful functor $F : \mathrm{Rep}_K G \to \mathrm{Vec}_K$.

Definition 2.1.2. [6] A *neutral Tannakian category* over K is a Tannakian category over K whose fiber functor is exact and faithful.

Theorem 2.1.3 (Tannakian Duality)**.** *[6] If $\mathcal{C}$ is a neutral Tannakian category over K with an exact and faithful fiber functor F, then there exists a unique affine group scheme G over K (up to affine group scheme isomorphism) such that the functor $\mathcal{C} \to \mathrm{Rep}_K G$ defined by F is an equivalence of braided tensor categories.*

Definition 2.1.4. [5] Let $\mathcal{C}$ be an abelian monoidal category. An object $F \in \mathrm{Obj}(\mathcal{C})$ is *unipotent* if there is a finite filtration $F = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_{n-1} \supseteq \mathcal{F}_n = 0$ such that each quotient $\mathcal{F}_i/\mathcal{F}_{i+1} \simeq 1$ where 1 is the unit.

Lemma 2.1.5. *[5] Let G be a group. The full subcategory $\mathrm{Rep}_K^{un} G$ of $\mathrm{Rep}_K G$ that consists of unipotent objects is a neutral Tannkian category, with the fiber functor being the forgetful functor.*

Definition 2.1.6. [5] Let G be a group. By Lemma 2.1.5 and Theorem 2.1.3, there exists a unique affine group scheme G_K over K up to isomorphism such that there is an equivalence $\mathrm{Rep}_K^{un} G \simeq \mathrm{Rep}_K G_K$ defined by the forgetful functor. Such G_K called the *K-unipotent completion* (or *K-Malcev completion*) of G.

2.1.2 de Rham Fundamental Group

Let X be a variety over K.

Definition 2.1.7. The full subcategory of $\mathrm{Flat}(X)$ (the category of algebraic vector bundles with flat connections on X) that consists of unipotent objects is denoted by $\mathrm{Un}(X)$, whose elements are called *unipotent algebraic vector bundle with flat connection* on X.

Choose a base point $b \in X$. Define the fiber functor $F_b : \mathrm{Un}(X) \to \mathrm{Vec}_K(X)$ that maps $(\mathcal{E}, \nabla) \in \mathrm{Un}(X)$ to the stalk $\mathcal{E}_b$ and maps a morphism $(\mathcal{F}, \nabla_1) \to (\mathcal{G}, \nabla_2)$ to the indcued map $\mathcal{F}_b \to \mathcal{G}_b$ on the stalk level.

Lemma 2.1.8. *[5]* $\mathrm{Un}(X)$ *with the fiber functor* $F_b : \mathrm{Un}(X) \to \mathrm{Vec}_\neg(X)$ *is a neutral Tannakian category.*

Definition 2.1.9. [5] By Lemma 2.1.8 and Theorem 2.1.3, there is a unique affine group scheme $\pi_1^{\mathrm{dR}}(X, b)$ over K up to isomorphism such that there exists an equivalence $\mathrm{Un}(X) \simeq \mathrm{Rep}_K \pi_1^{\mathrm{dR}}(X, b)$ defined by the fiber functor F_b. Such $\pi_1^{\mathrm{dR}}(X, b)$ is called the *de-Rham fundamental group* of X at b.

Definition 2.1.10. [12] For $x \in X$, the *de Rham path torsor* $P^{\mathrm{dR}}(X; b, x) := \mathrm{isom}^{\otimes}(F_b, F_x)$.

In particular, $P^{\mathrm{dR}}(X; b, b) = \mathrm{Aut}^{\otimes}(F_b) \simeq \pi_1^{\mathrm{dR}}(X, b)$ by Tannakian formalism.
Now suppose that X is a variety over $\mathbb{C}$. Then it makes sense to consider its underlying manifold, denoted by X^{an}. We have the comparison theorem.

Theorem 2.1.11 (Comparison Theorem)**.** *[5] For a smooth variety X over $\mathbb{C}$ and $b \in X$, $\pi_1^{\mathrm{dR}}(X, b) \simeq \pi_1(X^{\mathrm{an}}, b)_{\mathbb{C}}$, the $\mathbb{C}$-unipotent completion of $\pi_1(X^{\mathrm{an}}, b)$.*

Remark 2.1.12. If the variety X is over a field k, the comparison theorem becomes $\pi_1^{\mathrm{dR}}(X, b) \otimes_k \mathbb{C} \simeq \pi_1(X^{\mathrm{an}}, b)_{\mathbb{C}}$.

Example 2.1.13. Consider $X = \mathbb{G}_{a/\mathbb{C}} = \mathrm{Spec}\mathbb{C}[t]$ over $\mathbb{C}$. Its underlying manifold is $X^{\mathrm{an}} = \mathbb{C}$. By comparison theorem, $\pi_1^{\mathrm{dR}}(X) = \pi_1(X^{\mathrm{an}})_{\mathbb{C}} = \{1\}_{\mathbb{C}} = \mathrm{Spec}\mathbb{C}$.

Example 2.1.14. Consider $X = \mathbb{G}_{m/\mathbb{C}} = \mathrm{Spec}\mathbb{C}[t, t^{-1}]$ over $\mathbb{C}$. Its underlying manifold is $X^{\mathrm{an}} = \mathbb{C} \setminus \{0\}$. By comparison theorem, $\pi_1^{\mathrm{dR}}(X) = \pi_1(X^{\mathrm{an}})_{\mathbb{C}} = \mathbb{Z}_{\mathbb{C}} = \mathbb{G}_{a/\mathbb{C}}$.

Remark 2.1.15. There is another notion of fundamental group in arithmetic geometry called "étale fundamental group $\pi_1^{\text{ét}}$". For a variety over $\mathbb{C}$, unlike the fact that de Rham fundamental group is the unipotent completion of the usual fundamental group, the étale fundamental group is the pro-finite completion of that. So for example, $\pi_1^{\text{ét}}(\mathbb{G}_{a/\mathbb{C}}) = 1$, $\pi_1^{\text{ét}}(\mathbb{G}_{m/\mathbb{C}}) = \hat{\mathbb{Z}}(1) := \varprojlim_n \mu_n$ where $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ is the group of n-th unity roots.

2.2 Hodge Filtration

2.2.1 Basic Notions

Definition 2.2.1. [14] A mixed Hodge structure is a $\mathbb{Z}$-module $H_{\mathbb{Z}}$ together with an increasing filtration (called weight filtration) $H_{\mathbb{Q}} \subseteq \cdots \subseteq W_0 \subseteq W_1 \subseteq \cdots$ of $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and a decreasing filtration (called Hodge filtration) $H_{\mathbb{C}} \supseteq \cdots \supseteq F^0 \supseteq F^1 \supseteq \cdots$ of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ such that F induces a pure Hodge structure of weight k on the graded piece

$$\mathrm{Gr}_k^W H_{\mathbb{Q}} = W_k / W_{k-1}.$$

Example 2.2.2. The filtrations on the tensor product and dual is given by

$$W_m(A \otimes B) = \sum_{i+j=m} W_i A \otimes W_j B, \quad F^p(A \otimes B) = \sum_{i+j=p} F^i A \otimes F^j B;$$

$$W_m A^\vee = \{f \in A^\vee : \forall n, f(W_n A) \subseteq W_{n+m}\mathbb{Q}\}, \quad F^p A^\vee = \left\{f \in A^\vee : \forall n, f(F^n A) \subseteq F^{n+p}\mathbb{C}\right\}.$$

Example 2.2.3. For an integer n, the Tate-Hodge structure $\mathbb{Z}(n)$ is $H_{\mathbb{Z}} = \mathbb{Z}$ together with the following filtrations:

$$W_m = \begin{cases} 0 & m < -2n \\ \mathbb{Q} & m \geq -2n \end{cases}, \quad F^p = \begin{cases} \mathbb{C} & p \leq -n \\ 0 & p > -n \end{cases}.$$

Let U be a smooth complex variety. Let $X \supseteq U$ be a compactification. Let $D = X - U$ be a normal crossing divisor (basically it means that D locally looks like the crossing of coordinate hyperplanes). A differential form ω on U is said to be have logarithm poles along D if ω and $d\omega$ have at most a pole of order one along D. They constitute a complex $\Omega_X^\bullet(\log D)$ called the logarithm de Rham complex, where $\Omega_X^r(\log D)$ is generated by differential forms of the shape $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_j}{z_j} \wedge \alpha$ where z_i is a local equation of a component of D, $j \leq r$, and $\alpha \in \Omega_X^{r-j}$.

We put two filtrations on $\Omega_X^\bullet(\log D)$ making it a bifiltered complex:[14]

$$W_m = \begin{cases} \Omega_X^i(\log D) & m \geq i \\ \Omega_X^{i-m} \wedge \Omega_X^m(\log D) & 0 \leq m \leq i \\ 0 & m < 0 \end{cases}, \quad F^p = \begin{cases} 0 & p > i \\ \Omega_X^i(\log D) & p \leq i \end{cases}. \tag{2.1}$$

Thus, we have the filtrations on the hypercohomology $\mathbb{H}^n(X, \Omega_X^\bullet(\log D)) \simeq H^n(U; \mathbb{C})$. It can be further shown that the filtration W can be defined over $\mathbb{Q}$. This gives us a mixed Hodge structure on $H^n(U)$.

Example 2.2.4. Let $X = \mathbb{P}^1$. If $D = \varnothing$, then $H^{2d}(U) \simeq \mathbb{Z}(-d)$ (isomorphic as mixed Hodge structure). If D consists of r points, then $H^1(U) \simeq \mathbb{Z}(-1)^{r-1}$.

2.2.2 Hodge Filtrations of π_1^{dR} via Universal Connections

Let C be a smooth projective curve of genus g over a field k of characteristic 0. Let D be a nonempty divisor of size r and let $X := C - D$. Let $\alpha_0, \cdots, \alpha_{2g+r-2}$

form a k-basis of $H^1_{\mathrm{dR}}(X;k)$ so that $\alpha_0, \cdots, \alpha_{g-1}$ form a k-basis of $H^0(C, \Omega^1_{X/k})$. Let $V_{\mathrm{dR}} := H^1_{\mathrm{dR}}(X;k)^\vee$ with basis A_i dual to α_i. Let R be the tensor algebra of V_{dR}, i.e.,[11]

$$R := \bigoplus_{i=0}^{\infty} V_{\mathrm{dR}}^{\otimes i}.$$

Let I be the ideal generated by $A_0, \cdots, A_{2g+r-2}$. Let

$$R_n := R/I^{n+1} \simeq \bigoplus_{i=0}^{n} V_{\mathrm{dR}}^{\otimes i}.$$

Let $\mathcal{E}_n := R_n \otimes \mathcal{O}_X$ and let $\mathcal{E}$ be the limit of $\mathcal{U}_n$. So $\mathcal{E} = R \otimes \mathcal{O}_X$. Then $\mathcal{E}$ is the pro-universal object in the category of unipotent vector bundles on X of flat connections. Let $\mathcal{A} := \mathcal{E}^\vee$ be the dual bundle of $\mathcal{E}$.

By Example 2.2.4 and the filtration on the dual, we have the We have the Filtration on V_{dR}: $F^p V_{\mathrm{dR}} = V_{\mathrm{dR}}$ when $p < 0$, and $F^p V_{\mathrm{dR}} = 0$ when $p \geq 0$. Then by filtration on the tensor product and on the dual, we can compute the filtration on the stalk $\mathcal{E}_x$ and $\mathcal{A}_x$ for $x \in X$:

$$F^p(\mathcal{E}_x) = \begin{cases} 0 & p > 0 \\ R_{-p} & p \leq 0 \end{cases}, \quad \mathcal{F}^p(\mathcal{A}_x) = \bigoplus_{i=p}^{\infty} H^i_{\mathrm{dR}}(X;k)^{\otimes i}$$

To compute the filtrations on $\pi_1^{\mathrm{dR}}(X,x)$, we need the following proposition.

Proposition 2.2.5.
- *The Lie algebra* $\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)$ *is isomorphic to primitive elements of the fibre* $\mathcal{E}_x$.
- *Thus we deduce*

$$\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) = \left\{ \sum_{j\geq 1} c_{i_1 \cdots i_j} [A_{i_1}, [A_{i_2}, \cdots, [A_{i_j - 1}, A_{i_j}]]] \in \mathcal{E}_x \quad \right\}$$

As a result, one can first use the filtrations on $\mathcal{E}_x$ to get the filtrations on $\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)$. Then one can get the filtrations on $\pi_1^{\mathrm{dR}}(X,x)$ by exponential map.

Example 2.2.6. Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and choose a point $x \in X$. Let A_0, A_1 be a basis of V_{dR}. We can compute the Filtration on $\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)$:

$$\begin{aligned} F^0 \mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) &= \{0\} \\ F^{-1} \mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) &= \{c_0 A_0 + c_1 A_1\} \\ F^{-2} \mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x) &= \{c_0 A_0 + c_1 A_1 + c_{01}[A_0, A_1]\} \\ &\cdots \end{aligned}$$

In general, $F^{-n}\mathrm{Lie}\pi_1^{\mathrm{dR}}(X,x)$ is generated by $\leq n$ brackets of A_i. In particular, via the exponential map, we find that $F^0 \pi_1^{\mathrm{dR}}(X,x)$ is trivial.

Chapter 3

Methodology

3.1 Refined Chabauty-Kim Methods

This section is mainly from [3]. Let S be a finite set of prime numbers, $\mathcal{X} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathbb{Z}_S$, X the generic fibre of $\mathcal{X}$.

3.1.1 Key Diagram and $a_{l,q}$

For the depth $n = 2$, we have the short exact sequence:

$$1 \longrightarrow U^2/U^3 \simeq \mathbb{Q}_p(2) \longrightarrow U_2 \longrightarrow U_1 \simeq \mathbb{Q}_p(1) \times \mathbb{Q}_p(1) \longrightarrow 1$$

So $U_2 \simeq \mathbb{A}^3$. And the short exact sequnece induces that of cohomology.

$$1 \longrightarrow H^1(G_{\mathbb{Q}}, \mathbb{Q}_p(2)) \longrightarrow H^1(G_{\mathbb{Q}}, U_2) \longrightarrow H^1(G_{\mathbb{Q}}, U_1) \longrightarrow 1$$

By Soule's vanishing, $H^1(G_{\mathbb{Q}}, \mathbb{Q}_p(2)) = 0$. So $H^1(G_{\mathbb{Q}}, U_2) \simeq H^1(G_{\mathbb{Q}}, U_1)$. As a result, $\mathrm{Sel}_{S,2} = \mathrm{Sel}_{S,1} \simeq \mathbb{A}^S \times \mathbb{A}^S$. And $H^1_f(G_{\mathbb{Q}_p}, U_2) \simeq U_2^{\mathrm{dR}} \simeq \mathbb{A}^3$. Hence, the fundamental diagram for Chabauty-Kim methods in this case is shown below:[3]

$$\begin{tikzcd}
 & \mathcal{X}(\mathbb{Z}_S) \arrow[r] \arrow[dl] \arrow[d, "j"'] & \mathcal{X}(\mathbb{Z}_p) \arrow[d, "j_p"] \arrow[dr, "j^{\mathrm{dR}}"] & \\
\mathbb{A}^S \times \mathbb{A}^S \arrow[r, equal] & \mathrm{Sel}_{S,2} \arrow[r, "\mathrm{loc}_p"'] \arrow[d, equal, "\pi_*"] & H^1_f(G_{\mathbb{Q}_p}, U_2) \arrow[r, equal] \arrow[d, "\pi_*"] & \mathbb{A}^3 \arrow[d, "p_{1,2}"] \\
 & \mathrm{Sel}_{S,1} \arrow[r, "\mathrm{loc}_p"'] & H^1_f(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1)^2) \arrow[r, equal] & \mathbb{A}^2
\end{tikzcd}$$

where

- $j : \mathcal{X}(\mathbb{Z}_S) \to \mathrm{Sel}_{S,2} = \mathbb{A}^S \times \mathbb{A}^S, \quad z \mapsto ((v_l(z))_{l \in S}, (v_l(1-z))_{l \in S});$
- $j^{\mathrm{dR}} : \mathcal{X}(\mathbb{Z}_p) \to \mathbb{A}^3, \quad z \mapsto (\log(z), \log(1-z), -\mathrm{Li}_2(z));$
- $\mathrm{loc}_p : \mathrm{Sel}_{S,2} \to H^1_f(G_{\mathbb{Q}_p, U_2}),$
  $((x_l)_{l \in S}, (y_l)_{l \in S}) \mapsto (\sum_{l \in S} \log(l) x_l, \sum_{l \in S} \log(l) y_l, h_3(x, y))$ where $h_3(x, y) =$

$\sum_{l,q\in S} a_{l,q}x_l y_q$ is a bilinear form, and the coefficients $a_{l,q}$ has twisted anti-symmetry $a_{l,q} + a_{q,l} = \log(l)\log(q)$ (In particular, $a_{l,l} = \frac{1}{2}\log(l)^2$).

There is an algorithm to compute $a_{l,q}$ by a decomposition into a sum of Steinberg elements. One can show that for Fermat or Mersenne prime $q = 2^n \pm 1$,[3]

$$a_{2,q} = -\frac{1}{n}\mathrm{Li}_2(1 \mp q), \quad a_{q,2} = -\frac{1}{n}\mathrm{Li}_2(\pm q) \tag{3.1}$$

3.1.2 Why We Need Refined Selmer Schemes

Kim's method holds if $\dim \mathrm{Sel}_{S,n} < \dim H^1_f(G_p, U_n^{\text{ét}})$. For $n = 2$, $\dim \mathrm{Sel}_{S,2} = 2|S|$ whereas $\dim H^1_f(G_p, U_2^{\text{ét}}) = 3$. So it won't work if $|S| \geq 2$.

Definition 3.1.1. [3] Let $\mathcal{Y}/\mathbb{Z}_S$ be a model of hyperbolic curve $Y/\mathbb{Q}$. A cohomology class $\alpha \in H^1(G_\mathbb{Q}, U_n^{\text{ét}})$ is locally geometric if for all primes l, $\mathrm{loc}_l(\alpha)$ is contained in

$$\begin{cases} j_l(\mathcal{Y}(\mathbb{Z}_l))^{\mathrm{Zar}}, & \text{if } l \notin S \\ j_l(Y(\mathbb{Q}_l))^{\mathrm{Zar}}, & \text{if } l \in S \end{cases}.$$

The refined Selmer scheme $\mathrm{Sel}^{\min}_{S,n}$ represents the subfunctor of $R \mapsto H^1(G_\mathbb{Q}, U_n^{\text{ét}}(R))$ given by locally geometric cohomology classes. We can also define the refined locus $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,n}$. The refined Kim's conjecture claims that $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,n} = \mathcal{Y}(\mathbb{Z}_S)$ holds for large enough n.

Lemma 3.1.2. *[3] $j_l(\mathcal{X}(\mathbb{Q}_l))^{\mathrm{Zar}}$ in $\mathbb{A}^2$ is the union of three lines $x = 0$, $y = 0$, and $x = y$.*

Let $p_{\diagup}(x,y) = x - y$, $p_{|}(x,y) = x$, $p_{-}(x,y) = y$.

Lemma 3.1.3. *[3] Let $n = 2$. For $\mathcal{X}$,*

- *If $2 \notin S$, then $\mathrm{Sel}^{\min}_{S,n} = \varnothing$;*
- *Otherwise, $\mathrm{Sel}^{\min}_{S,n}$ is the union of $\{((x_l)_{l\in S}, (y_l)_{l\in S}) : p_{i_l}(x_l, y_l) = 0\ \forall l \in S\} \subseteq \mathbb{A}^S \times \mathbb{A}^S$ for the $3^{|S|}$ choices of tuples of conditions $\Sigma = (i_l)_{l\in S} \in \{\diagup, |, -\}^S$.*

By lemma 3.1.3, for $n = 2$, $\dim \mathrm{Sel}^{\min}_{S,n} = |S|$. So Kim's method works for $|S| \leq 2$ now.

Let $\mathcal{X}(\mathbb{Z}_p)^{\Sigma}_{S,n} = \{z \in \mathcal{X}(\mathbb{Z}_p)_{S,n} : j_p(z) \in \mathrm{im}(\mathrm{loc}_p)\}$. Then we have $\mathcal{X}(\mathbb{Z}_p)^{\min}_{S,n} = \bigcup_\Sigma \mathcal{X}(\mathbb{Z}_p)^{\Sigma}_{S,n}$.

3.1.3 Computation for the case $S = \{2\}$

Since S_3 acts transitively on $\{\diagup, |, -\}$, we consider $|$ only.

- Depth $n = 1$: $\mathrm{loc}_p : \mathrm{Sel}_{\{2\},1} \to \mathbb{A}^2$, $(x_2, y_2) \mapsto (\log(2)x_2, \log(2)y_2)$. $\mathrm{Sel}^{|}_{\{2\},1}$ is cut out by $x_2 = 0 \Rightarrow \mathrm{loc}_p(0, y_2) = (0, \log(2)y_2)$, i.e., the image is cut out by $u = 0$, which becomes $\log(z) = 0$. So $\mathcal{X}(\mathbb{Z}_p)^{\min}_{\{2\},1}$ consists of nontrivial $(p-1)$-st roots of unity (plus orbits). So refined Kim's conjecture holds for $p = 3$ in this case.

- Depth $n=2$: $\mathrm{loc}_p : \mathrm{Sel}_{\{2\},2} \to \mathbb{A}^3$, $(x_2, y_2) \mapsto (\log(2)x_2, \log(2)y_2, \frac{1}{2}\log(2)^2 x_2 y_2)$. On $\mathrm{Sel}^{|}_{\{2\},2}$, $\mathrm{loc}_p(0, y_2) = (0, \log(2)y_2, 0)$. So $\mathcal{X}(\mathbb{Z}_p)^{|}_{\{2\},2}$ is cut out by $\log(z) = 0$, $\mathrm{Li}_2(z) = 0$. So $\mathcal{X}(\mathbb{Z}_p)^{\min}_{\{2\},2}$ consists of nontrivial $(p-1)$-th roots of unity which is also a zero of Li_2 (plus orbits). So refined Kim's conjecture holds for $3 \leq p \leq 10^5$ in this case.

3.1.4 Computation for the case $S = \{2, q\}$

The action of S_3 on $\{\diagup, |, -\}$ is 2-transitive $\Rightarrow$ two orbits in $\{\diagup, |, -\}^2$: $|, |$ and $|, -$. Let $n = 2$.

$$\mathrm{loc}_p : \mathrm{Sel}_{S,2} = \mathbb{A}^2 \times \mathbb{A}^2 \to \mathbb{A}^3, (x, y) \mapsto \begin{pmatrix} \log(l)x_2 + \log(q)x_q \\ \log(l)y_2 + \log(q)y_q \\ \frac{1}{2}\log(2)^2 x_2 y_2 + a_{2,q}x_2 y_q + a_{q,2}x_q y_2 + \frac{1}{2}\log(q)^2 x_q y_q \end{pmatrix}$$

- On $\mathrm{Sel}^{|,|}_{S,2}$ (i.e., $x_2 = x_q = 0$), $\mathrm{loc}_p(0, 0, y_2, y_q) = (0, \log(2)y_2 + \log(q)y_q, 0)$. So $\mathrm{Sel}^{|,|}_{S,2}$ is cut out by $u = 0$, $w = 0$. So $\mathcal{X}(\mathbb{Z}_p)^{|,|}_{S,2}$ is cut out by $\log(z) = 0$, $\mathrm{Li}_2(z) = 0$.

- On $\mathrm{Sel}^{|,-}_{S,2}$ (i.e., $x_2 = 0$, $y_q = 0$), $\mathrm{loc}_p(0, x_q, y_2, 0) = (\log(q)x_q, \log(2)y_2, a_{q,2}x_q y_2)$. So $\mathrm{Sel}^{|,-}_{S,2}$ is cut out by $a_{q,2}uv - \log(2)\log(q)w = 0$. So $\mathcal{X}(\mathbb{Z}_p)^{|,-}_{S,2}$ is cut out by $a_{q,2}\log(z)\log(1-z) + \log(2)\log(q)\mathrm{Li}_2(z) = 0$, which can be equivalently written as $a_{2,q}\mathrm{Li}_2(z) = a_{q,2}\mathrm{Li}_2(1-z)$.

Note that $\mathcal{X}(\mathbb{Z}_p)^{|,|}_{S,2} \subseteq \mathcal{X}(\mathbb{Z}_p)^{|,-}_{S,2}$. In summary, $\mathcal{X}(\mathbb{Z}_p)^{\min}_{S,2}$ is cut out (up to orbits) by

$$a_{2,q}\mathrm{Li}_2(z) = a_{q,2}\mathrm{Li}_2(1-z). \tag{3.2}$$

By analysis of Newton polygon, for $q > 3$ in $S = \{2, q\}$ and $p = 3$, $\mathcal{X}(\mathbb{Z}_3)^{\min}_{S,2}$ contains $\left\{2, -1, \frac{1}{2}\right\}$ and at most one more orbit (occurs iff $\min\{v_3(a_{2,q}), v_3(a_{q,2})\} = 1 + v_3(\log(q))$ $\star$). Note that for Fermat or Mersenne prime $q = 2^n \pm 1$, $z = \pm q$ satisfies (3.2) by (3.1) which provides one more orbit, so refined Kim's conjecture holds. For other primes in then range $q \leq 1000$, there are 31 primes do not satisfy $\star$, so refined Kim's conjecture holds for those values as well.[3]

Finally, for $S = \{2, 3\}$ and $p = 5$, there is one extra solution of (3.2) which does not correspond to the actual solution and appears transcendental. Another paper[13] shows that by going to depth $n = 4$, refined Kim's conjecture holds for $S = \{2, 3\}$, $p = 5$.

3.2 Weight Filtrations on Selmer Schemes

This section is mainly from [4]. Let K_v be a finite extension of $\mathbb{Q}_p$ and $X = \overline{X} \backslash D$ be a smooth hyperbolic curve of genus g over K_v with the divisor D of size r.

3.2.1 Weight Filtrations and Hilbert Series

Definition 3.2.1 (**Abstract Coleman Algebraic Functions**[4])**.** An *abstract Coleman algebraic function* on X with the basepoint b is a triple $(\mathcal{E}, \sigma_b, \tau)$ consisting of:

- a unipotent vector bundle $\mathcal{E}$ with log-connection;
- an $\mathcal{O}_X$-linear map $\tau : \mathcal{E} \to \mathcal{O}_X$; and
- a point $\sigma_b \in \mathcal{E}_b$ in the fibre of $\mathcal{E}$ at the basepoint b.

Remark 3.2.2. An abstract Coleman algebraic function $(\mathcal{E}, \sigma_b, \tau)$ is related to a Coleman function f from a subset of $X(\mathbb{Q}_p)$ to $\mathbb{Q}_p$ via parallel transport. To be more precise, suppose $b \in U$ for some open $U \subseteq X$ and choose $\sigma \in \mathcal{E}(U)$ whose stalk at b is σ_b. Then $\tau(\sigma) \in \mathcal{O}_X(U)$ is just a morphism $U \to \mathbb{A}^1_{\mathbb{Q}_p}$, which passes to the map f of $\mathbb{Q}_p$ points.

A unipotent vector bundle $\mathcal{E}$ with log-connection on (X, D) is said to have weight at most m when it admits a ∇-stable filtration[4]

$$0 = W_{-1}\mathcal{E} \leq W_0\mathcal{E} \leq \cdots \leq W_m\mathcal{E} = \mathcal{E}$$

such that $W_i\mathcal{E}/W_{i-1}\mathcal{E}$ is a trivial vector bundle with connection (direct sum of copies of $(\mathcal{O}_X, \mathrm{d})$) and the connection on $W_i\mathcal{E}/W_{i-2}\mathcal{E}$ is regular on X (i.e., takes values in $\Omega^1_X \otimes_{\mathcal{O}_X} (W_i\mathcal{E}/W_{i-2}\mathcal{E})$). And an abstract Coleman algebraic function $(\mathcal{E}, \sigma_b, \tau)$ is said to have weight at most m if $\mathcal{E}$ has weight at most m.

Let $\{V_n\}_{n\geq 1}$ be the graded pieces of the descending central series and let W_m be the space of abstract Coleman algebraic functions on X of weight $\leq m$.

Example 3.2.3. For $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, W_{2m} is spanned by

$$\left\{ \log^{k_0}(z)\mathrm{Li}_1^{k_1}(z)\mathrm{Li}_2^{k_2}(z) \cdots \mathrm{Li}_m^{k_m}(z) : k_0 + k_1 + 2k_2 + \cdots + mk_m = m \right\}.$$

Theorem 3.2.4. *[4] Let c_i^{glob} and c_i^{loc} be the i-th coefficient of the power series*

$$\mathrm{HS}_{\mathrm{glob}}(t) := \prod_{n\geq 1}^{\infty} (1 - t^n)^{-\dim H^1_{f,T}(G_{\mathbb{Q}}, V_n)}, \quad \mathrm{HS}_{\mathrm{loc}}(t) := \prod_{n\geq 1}^{\infty} (1 - t^n)^{-\dim H^1_f(G_p, V_n)}.$$

Let $m \in \mathbb{N}$. If $\sum_{i=0}^m c_i^{\mathrm{glob}} < \sum_{i=0}^m c_i^{\mathrm{loc}}$, then $X(\mathbb{Z}_S)_\Sigma \subseteq W_m$ for every reduction type Σ.

3.2.2 Differential Operators and the Bound of the Number

Given 1-form ω and a differential operator $\mathcal{D} = \sum_{i=0}^N g_i(t) \frac{\mathrm{d}^i}{\mathrm{d}t^i}$, define

- $\mathrm{div}(\omega)^+ := \mathrm{div}(\omega) + (\mathrm{supp}(\mathrm{div}(\omega)) \cap X)$;
- $\mathrm{div}(\mathcal{D}) := \min((\mathrm{div}(g_i) - i\mathrm{div}(\omega)^+)_{0\leq i\leq N}, 0)$.

Definition 3.2.5. [4] Let $\mathcal{O}_v[\![t]\!]^{\mathrm{PD}}$ be the algebra of power series $f = \sum_{i=0}^{\infty} a_i t^i / i!$ with $a_i \in \mathcal{O}_v$. A differential operator $\mathcal{D} = \sum_{i=0}^{N} g_i(t) \frac{\mathrm{d}^i}{\mathrm{d}t^i}$ is called *PD-nice* if each $g_i \in \mathcal{O}_v[\![t]\!]^{\mathrm{PD}}$ and $g_N \in \mathcal{O}_v[\![t]\!]^{\mathrm{PD},\times}$.

The number N in the definition is called the *order* of the operator $\mathcal{D}$. It is easy to show that $\mathrm{ord}(\mathcal{D}_1 \circ \mathcal{D}_2) \leq \mathrm{ord}(\mathcal{D}_1) + \mathrm{ord}(\mathcal{D}_2)$.

Lemma 3.2.6. *[4] Let $f \in K_v[\![t]\!]$ be nonzero and $\mathcal{D}$ be a PD-nice differential operator of order N with $\mathcal{D}(f) = 0$. Then for every $\lambda > 1/(p-1)$, the number of zeros of f in the closed disc of radius $p^{-\lambda}$ is at most*

$$\left(1 + \frac{1}{(\lambda - \frac{1}{p-1}) \log(p)}\right) \cdot (N-1).$$

Define $\theta_v := \lceil \frac{e_v+1}{p-1} \rceil$ (e_v ramification index, f_v residue degree) and

$$\kappa_v := p^{(\theta_v - 1) f_v} \left(1 + \frac{e_v}{(\theta_v - \frac{e_v}{p-1}) \log(p)}\right).$$

Theorem 3.2.7. *[4] Suppose $f \in W_m$. Then*

$$\#X(\mathcal{O}_v) \leq \kappa_v \cdot \#X_0(k_v) \cdot (4g + 2r - 2)^m \prod_{i=1}^{m-1} (c_i + 1)$$

where $X_0(k_v)$ is the set of points of special fibre of X which are rational over the residue of K_v.

Lemma 3.2.8. *[4] Let f be a non-zero Coleman algebraic function on $\mathbb{P}^1_{\mathbb{Z}_p} \setminus \{0, 1, \infty\}$ of weight at most $2m$. Then the number of $\mathbb{Z}_p$-integral zeros of f is at most $\kappa_p \cdot (p-2) \cdot (2^{m+1} - 2)$.*

The key idea to prove the lemma is to construct the following operator:

$$\mathcal{D}_m := (z - z^2)^{2^m} \frac{\mathrm{d}^{2^m}}{\mathrm{d}z^{2^m}} \cdot (z - z^2)^{2^{m-1}} \frac{\mathrm{d}^{2^{m-1}}}{\mathrm{d}z^{2^{m-1}}} \cdot \ldots \cdot (z - z^2) \frac{\mathrm{d}}{\mathrm{d}z}.$$

Theorem 3.2.9. *[4] Let S be a finite set of primes of size s. Then*

$$\#(\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathbb{Z}_S) \leq 8 \cdot 6^s \cdot 2^{4^s}$$

Proof.
- $(\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathbb{Z}_S)$ is the union of 3^s subsets.
- $\kappa_p \cdot (p-2) \leq 2^{s+2}$ and m can be chosen $\leq 4^s$.

□

Chapter 4

Current Work And Future Plans

4.1 The Case for $|S| = 3$ over $\mathbb{Q}$

4.1.1 Write out Coleman Functions

Let $\mathrm{Sel}_{S,\mathrm{PL},n} = H^1_{f,S}(G_{\mathbb{Q}}, U_{\mathrm{PL},n})$, where $U_{\mathrm{PL},n}$ is the polylograithm quotient[1] of the $\mathbb{Q}_p$-unipotent fundamental group. Let $\mathrm{Sel}^{\min}_{S,\mathrm{PL},n}$ be the refined Selmer scheme.

- $\dim H^1_f(G_p, U_{\mathrm{PL},n}) = n+1$, generated by $\log, \mathrm{Li}_1, \cdots, \mathrm{Li}_n$.
- $\dim \mathrm{Sel}_{S,\mathrm{PL},n} = 2|S| + \lfloor \frac{n-1}{2} \rfloor$, with coordinates $(x_l)_{l\in S}, (y_l)_{l\in S}, (z_i)_{\text{odd } i\in[3,n]}$.
- $\dim \mathrm{Sel}^{\min}_{S,\mathrm{PL},n} = |S| + \lfloor \frac{n-1}{2} \rfloor$, with equations $x_l y_l (x_l - y_l) = 0$, $\forall l \in S$.

Refined Condition: $\dim \mathrm{Sel}^{\min}_{S,\mathrm{PL},n} < \dim H^1_f(G_p, U_{\mathrm{PL},n}) \Leftrightarrow |S| + \lfloor \frac{n-1}{2} \rfloor < n+1$. For example, when $|S| = 3$ and $n = 4$, the refined condition holds.

The localization map in this case is:[1]

$$\begin{aligned}
\mathrm{loc}^{\#}(\log) &= \sum_{l\in S} \log(l) x_l \\
\mathrm{loc}^{\#}(\mathrm{Li}_1) &= \sum_{l\in S} \log(l) y_l \\
\mathrm{loc}^{\#}(\mathrm{Li}_2) &= \sum_{l,q\in S} a_{\tau_l \tau_q} x_l y_q \\
\mathrm{loc}^{\#}(\mathrm{Li}_3) &= \sum_{l_1,l_2,q\in S} a_{\tau_{l_1}\tau_{l_2}} x_{l_1} x_{l_2} y_q + \zeta(3) z_3 \\
\mathrm{loc}^{\#}(\mathrm{Li}_4) &= \sum_{l_1,l_2,l_3,q\in S} a_{\tau_{l_1}\tau_{l_2}\tau_{l_3}\tau_q} x_{l_1} x_{l_2} x_{l_3} y_q + \sum_{l\in S} a_{\sigma_3 \tau_l} x_l z_3
\end{aligned}$$

Let $S = \{2, q, r\}$, $\mathcal{X} = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ over $\mathbb{Z}_S$, $\mathbf{x} = (x_2, x_q, x_r)$, $\mathbf{y} = (y_2, y_q, y_r)$.

Since the S_3-action on $\{\diagup, |, -\}$ is i-transitive for $i \in \{1, 2, 3\}$, we only consider

$$(|, |, |), (|, |, -), (|, -, |), (-, |, |), (|, -, \diagup).$$

The case $(|,|,|)$: For $\mathbf{x}=0$, we have

$$\mathrm{loc}_p(\mathbf{x},\mathbf{y},z_3) = \begin{pmatrix} 0 \\ \log(2)y_2+\log(q)y_q+\log(r)y_r \\ 0 \\ \zeta(3)z_3 \\ 0 \end{pmatrix}$$

So $\mathcal{X}(\mathbb{Z}_p)^{(|,|,|)}_{S,\mathrm{PL},4}$ is cut out by $\log(z)=0$, $\mathrm{Li}_2(z)=0$, $\mathrm{Li}_4(z)=0$.

The case $(|,|,-)$: For $x_2=x_q=0$ and $y_r=0$, we have

$$\mathrm{loc}_p(\mathbf{x},\mathbf{y},z_3) = \begin{pmatrix} \log(r)x_r \\ \log(2)y_2+\log(q)y_q \\ a_{\tau_r\tau_2}x_ry_2+a_{\tau_r\tau_q}x_ry_q \\ a_{\tau_r\tau_r\tau_2}x_r^2y_2+a_{\tau_r\tau_r\tau_q}x_r^2y_q+\zeta(3)z_3 \\ a_{\tau_r\tau_r\tau_r\tau_2}x_r^3y_2+a_{\tau_r\tau_r\tau_r\tau_q}x_r^3y_q+a_{\sigma_3\tau_r}x_rz_3 \end{pmatrix}$$

So $\mathcal{X}(\mathbb{Z}_p)^{(|,|,-)}_{S,\mathrm{PL},4}$ is cut out by $f_1(z)=0$ where the function $f_1(z)$ is given by

$$\begin{aligned}
&\big(a_{\sigma_3\tau_r}(a_{\tau_r\tau_q}a_{\tau_r\tau_r\tau_2}-a_{\tau_r\tau_2}a_{\tau_r\tau_r\tau_q})+\zeta(3)(a_{\tau_r\tau_q}a_{\tau_r\tau_r\tau_2}+a_{\tau_r\tau_2}a_{\tau_r\tau_r\tau_q})\big)\,(\log(z))^3\mathrm{Li}_1(z)\\
&+\log(r)\big(a_{\sigma_3\tau_r}(\log(q)a_{\tau_r\tau_r\tau_2}+\log(2)a_{\tau_r\tau_r\tau_q})\\
&\qquad\qquad+\zeta(3)(\log(q)a_{\tau_r\tau_r\tau_2}-\log(2)a_{\tau_r\tau_r\tau_q})\big)(\log(z))^2\mathrm{Li}_2(z)\\
&+(\log(r))^2a_{\sigma_3\tau_r}\left(\log(q)a_{\tau_r\tau_2}-\log(2)a_{\tau_r\tau_q}\right)\log(z)\,\mathrm{Li}_3(z)\\
&+(\log(r))^3\zeta(3)\left(\log(q)a_{\tau_r\tau_2}+\log(2)a_{\tau_r\tau_q}\right)\mathrm{Li}_4(z)
\end{aligned}$$

In particular, $-1\in\mathcal{X}(\mathbb{Z}_p)^{(|,|,|)}_{S,\mathrm{PL},4}\subseteq\mathcal{X}(\mathbb{Z}_p)^{(|,|,-)}_{S,\mathrm{PL},4}$.

The case $(|,-,|)$: In this case, $x_2=x_r=0$ and $y_q=0$. So $\mathcal{X}(\mathbb{Z}_p)^{(|,-,|)}_{S,\mathrm{PL},4}$ is cut out by $f_2(z)=0$, where $f_2(z)$ is given by $f_1(z)$ by switching $q\leftrightarrow r$. In particular, $-1\in\mathcal{X}(\mathbb{Z}_p)^{(|,|,|)}_{S,\mathrm{PL},4}\subseteq\mathcal{X}(\mathbb{Z}_p)^{(|,-,|)}_{S,\mathrm{PL},4}$.

The case $(-,|,|)$: In this case, $x_q=x_r=0$ and $y_2=0$. So $\mathcal{X}(\mathbb{Z}_p)^{(-,|,|)}_{S,\mathrm{PL},4}$ is cut out by $f_3(z)=0$, where $f_3(z)$ is given by $f_2(z)$ by switching $2\leftrightarrow q$. In particular, $-1\in\mathcal{X}(\mathbb{Z}_p)^{(|,|,|)}_{S,\mathrm{PL},4}\subseteq\mathcal{X}(\mathbb{Z}_p)^{(-,|,|)}_{S,\mathrm{PL},4}$.

The case $(|, -, \diagup)$: For $x_2 = 0$, $y_q = 0$ and $y_r = x_r$, we have $\mathrm{loc}_p(\mathbf{x}, \mathbf{y}, z_3) =$

$$\left(\begin{array}{c} \log(q)x_q + \log(r)x_r, \\ \log(2)y_2 + \log(r)x_r, \\ a_{\tau_q\tau_2}x_qy_2 + a_{\tau_q\tau_r}x_qx_r + a_{\tau_r\tau_2}x_ry_2 + \frac{1}{2}(\log(r))^2x_r^2, \\ a_{\tau_q\tau_q\tau_2}x_q^2y_2 + (a_{\tau_q\tau_r\tau_2} + a_{\tau_r\tau_q\tau_2})x_qx_ry_2 + a_{\tau_r\tau_r\tau_2}x_r^2y_2 + a_{\tau_q\tau_q\tau_r}x_q^2x_r \\ +(a_{\tau_q\tau_r\tau_r} + a_{\tau_r\tau_q\tau_r})x_qx_r^2 + a_{\tau_r\tau_r\tau_r}x_r^3 + \zeta(3)z_3, \\ a_{\tau_q\tau_q\tau_q\tau_2}x_q^3y_2 + a_{\tau_r\tau_r\tau_r\tau_2}x_r^3y_2 + a_{\tau_q\tau_q\tau_q\tau_r}x_q^3x_r + (a_{\tau_q\tau_q\tau_r\tau_2} + a_{\tau_q\tau_r\tau_q\tau_2} + a_{\tau_r\tau_q\tau_q\tau_2})x_q^2x_ry_2 \\ +(a_{\tau_r\tau_r\tau_q\tau_2} + a_{\tau_r\tau_q\tau_r\tau_2} + a_{\tau_q\tau_r\tau_r\tau_2})x_qx_r^2y_2 + (a_{\tau_q\tau_q\tau_r\tau_r} + a_{\tau_q\tau_r\tau_q\tau_r} + a_{\tau_r\tau_q\tau_q\tau_r})x_q^2x_r^2 \\ +(a_{\tau_r\tau_r\tau_q\tau_r} + a_{\tau_r\tau_q\tau_r\tau_r} + a_{\tau_q\tau_r\tau_r\tau_r})x_qx_r^3 + a_{\tau_r\tau_r\tau_r\tau_r}x_r^4 + a_{\sigma_3\tau_q}x_qz_3 + a_{\sigma_3\tau_r}x_rz_3 \end{array}\right)$$

So $\mathcal{X}(\mathbb{Z}_p)_{S,\mathrm{PL},4}^{(|,-,\diagup)}$ is cut out by $f_4(z) = 0$ where the function $f_4(z)$ is in the span of

$$(\log(z))^{k_0}(\mathrm{Li}_1(z))^{k_1} \cdots (\mathrm{Li}_4(z))^{k_4}$$

where

$$k_i \in \mathbb{Z}, \quad k_0 + k_1 + 2k_2 + 3k_3 + 4k_4 = 8, \quad k_0 + k_2 + k_4 > 0.$$

In particular, $-1 \in \mathcal{X}(\mathbb{Z}_p)_{S,\mathrm{PL},4}^{(|,|,|)} \subseteq \mathcal{X}(\mathbb{Z}_p)_{S,\mathrm{PL},4}^{(|,-,\diagup)}$.

In summary, $\mathcal{X}(\mathbb{Z}_p)_{S,\mathrm{PL},4}^{\min}$ is the union of zeros of f_i $(i = 1, 2, 3, 4)$ up to orbits.

4.1.2 Construct Differential Operators

Construct

$$\mathfrak{D}_0 = \mathrm{id}$$
$$\mathfrak{D}_m = z^{2^m - 1}(z-1)^{2^m}\frac{\mathrm{d}^{2^{m-1}}}{\mathrm{d}z^{2^{m-1}}} \circ z\frac{\mathrm{d}^{2^{m-1}}}{\mathrm{d}z^{2^{m-1}}} \circ \mathfrak{D}_{m-1}$$

We have $\mathfrak{D}_m$ vanishes on

$$V_m := \mathrm{span}\langle \log^{k_0}\mathrm{Li}_1^{k_1}\mathrm{Li}_2^{k_2}\cdots\mathrm{Li}_m^{k_m} : k_0 + k_1 + 2k_2 + \cdots + mk_m = m, \text{ and } k_1 < m\rangle,$$

and

$$\mathrm{order}(\mathfrak{D}_m) \leq 2^1 + 2^2 + \cdots + 2^m = 2^{m+1} - 2$$

So for a non-zero Coleman algebraic function f on $\mathcal{X}/\mathbb{Z}_p$ vanishing on V_m, we have

$$\#\mathrm{zeros}(f) \leq \kappa_p \cdot (p-2) \cdot (2^{m+1} - 3) \tag{4.1}$$

where

$$\kappa_p = \begin{cases} 1 + \frac{p-1}{(p-2)\log(p)} & p \text{ odd prime} \\ 2 + \frac{2}{\log(2)} & p = 2 \end{cases}.$$

Also construct

$$\tilde{\mathfrak{D}} = \frac{\mathrm{d}^5}{\mathrm{d}z^5}\circ(z-1)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\circ z^5(z-1)\frac{\mathrm{d}^3}{\mathrm{d}z^3}\circ(z-1)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\circ z^3(z-1)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\circ z\frac{\mathrm{d}}{\mathrm{d}z}\circ(z-1)^2\frac{\mathrm{d}}{\mathrm{d}z}\circ z\frac{\mathrm{d}}{\mathrm{d}z}$$

We have $\tilde{\mathfrak{D}}$ vanishes on $\tilde{V} := \mathrm{span}\langle \log^3 \mathrm{Li}_1, \log^2 \mathrm{Li}_2, \log \mathrm{Li}_3, \mathrm{Li}_4\rangle$, and

$$\mathrm{order}(\tilde{\mathfrak{D}}) \leq 1+1+1+2+2+3+2+5 = 17$$

So for a non-zero Coleman algebraic function f on $\mathcal{X}/\mathbb{Z}_p$ vanishing on $\tilde{V}$, we have

$$\#\mathrm{zeros}(f) \leq \kappa_p \cdot (p-2) \cdot 16 \tag{4.2}$$

4.1.3 Bound the Number of Solutions

Note:

- $\mathcal{X}(\mathbb{Z}_S) \subseteq \mathcal{X}(\mathbb{Z}_p)^{\min}_{S,\mathrm{PL},4}$;
- Each zero of f_i (if it's a solution) contributes an orbit of $\mathcal{X}(\mathbb{Z}_S)$;
- -1 is always a zero of f_i;
- Each orbit of $\mathcal{X}(\mathbb{Z}_S)$ contains exactly 6 elements except the orbit $\left\{2, -1, \frac{1}{2}\right\}$.

So for $S = \{2, q, r\}$,

$$\#\mathcal{X}(\mathbb{Z}_S) \leq 3 + 6\sum_{i=1}^{4} \left(\#\mathrm{zeros}(f_i) - 1\right). \tag{4.3}$$

Note that f_i $(i = 1, 2, 3)$ are all in $\tilde{V}$, and f_4 is in V_8. Thus, by (4.1)(4.2)(4.3), we have

$$\#\mathcal{X}(\mathbb{Z}_S) \leq \begin{cases} 9399 & 3 \notin S \\ 18297 & 5 \notin S \\ 26985 & S = \{2,3,5\} \end{cases}. \tag{4.4}$$

4.1.4 Future Plans

Further study the S-unit equation on $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ for the case $|S| = 3$ over $\mathbb{Q}$.

One of the future goals is to get a sharper bound to approach the bound suggested by numerical experiments in Appendix A. A possible way to accomplish the goal is to use the information of the coefficients (such as $a_{\tau_q\tau_r\tau_r\tau_r\tau_2}$) rather than just the information of weight. The main challenge is to generalize the following theorem by replacing W_m by an appropriate smaller subsapce:

Theorem 4.1.1. *[4] Suppose the operator $\mathcal{D}_m$ vanishes on W_m. Then for every $f \in W_{m+1}$, $\mathcal{D}_m(f)$ is a rational function lying in $H^0(X, \mathcal{O}_X(-\mathrm{div}(\mathcal{D}_m)))$.*

The other future goal is to show some of the conjectures in Appendix A.

For both goals, I have to learn more knowledge in the area of refined Chabauty-Kim methods, weight filtration methods and any other methods (for example, some Diophantine approximation) that might be helpful.

4.2 The Case over Number Fields

4.2.1 Kim's Cutter over Number Fields

Let X be a hyperbolic curve over a number field K. Let S be a finite set of primes in K, p be a rational prime of good recution for X which splits compeletly in K, and $T = S \cup \{\text{primes ramifying in } K\} \cup \{\text{primes of bad reduction for } X\} \cup \{v \mid p\}$. We have the following commutative diagram[8]

$$\begin{CD} X(\mathcal{O}_{K,S}) @>>> \prod_{v\in S} X(K_v) \times \prod_{v\in T\setminus S} X(\mathcal{O}_{K_v}) \\ @VjVV @VV\prod_{v\in T} j_vV \\ H^1(G_T, U_n) @>\mathrm{loc}>> \prod_{v\in T} H^1(G_{K_v}, U_n) \end{CD} \tag{4.5}$$

which induces the Kim's cutter[8]

$$\begin{CD} X(\mathcal{O}_{K,S}) @>>> \prod_{v\mid p} X(\mathcal{O}_{K_v}) \\ @VjVV @VV\prod_{v\mid p} j_vV \\ H^1_f(G_T, U_n) @>\mathrm{loc}_p>> \prod_{v\mid p} H^1_f(G_{K_v}, U_n) \end{CD} \tag{4.6}$$

4.2.2 Example over $\mathbb{Q}(i)$

Now let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $K = \mathbb{Q}(i)$, $S = \{1 + i, \lambda_1, \lambda_2\}$ be the primes in $\mathcal{O}_K$ lying over 2, $l = 2^m + 1$ for some $m \in \mathbb{N}$, and $T = S \cup \{v_1, v_2\}$ where $p = v_1 v_2$ splits in K. We denote by W Deligne's quotient[7] of the $\mathbb{Q}_p$-unipotent fundamental group U of X with tangential basepoint $b = \vec{01}$.

For the depth $n = 2$, we are interested in $W_2 = W/W^3$, which is generated by words $A, B, C := [A, B]$ with $\mathbb{Q}_p A \simeq \mathbb{Q}_p B \simeq \mathbb{Q}_p(1)$ and $\mathbb{Q}_p C \simeq \mathbb{Q}_p(2)$. There is a Galois-equivariant action of $\mathbb{A}^1 \times \mathbb{A}^1$ on W_2 such that (t, s) acts as $A \mapsto tA$, $B \mapsto sB$, $C \mapsto tsC$.

Let $\alpha_1 = 1 - i2^m$ and $\alpha_2 = 1 + i2^m$ so that $l = \alpha_1\alpha_2$, which are clearly S-units. Denote by (a_i, b, c_i) the cocycle associated to α_i (The b-components are the same because $1 - \alpha_1$ and $1 - \alpha_2$ differs by a unit). Also we can compute $dc_i = (1/2)(a_i \cup b - b \cup a_i)$. From this and $\mathbb{A}^1 \times \mathbb{A}^1$ action, we deduce that $(t_1a_1 + t_2a_2, sb, t_1sc_1 + t_2sc_2)$ is a cocycle valued in W_2 for any $t_1, t_2, s \in \mathbb{Q}_p$.

Lemma 4.2.1. *The dimension of the refined Selmer scheme* $\dim \mathrm{Sel}^{\min}_{S,2} = 4$ *and a general element of* $\mathrm{Sel}^{\min}_{S,2}$ *can be written as* $(t_1a_1 + t_2a_2, sb, t_1sc_1 + t_1sc_2 + r\xi)$ *for a nonzero class* $\xi \in H^1(G_T, \mathbb{Q}_p(2))$.

Proof. It can be shown $\dim H^1_f(G_T, \mathbb{Q}_p(1)) = |S| = 3$ using Kummer theory. So $\dim H^1_f(G_T, W/W^2) = 6$. But if we consider the refined conditions, we should substract $|S| = 3$ dimensions, which means that $\mathrm{Sel}^{\min}_{S,2}$ on W/W^2 (i.e., two $\mathbb{Q}_p(1)$ copies) has dimension 3. But the $\mathbb{Q}_p(2)$ copy provides an additional $\dim H^1(G_T, \mathbb{Q}_p(2)) = 1$ dimension, which implies $\dim \mathrm{Sel}^{\min}_{S,2} = 4$. It can be shown $(t_1a_1 + t_2a_2, sb, t_1sc_1 + t_2sc_2 + r\xi)$ are cocycles, and they have already spann 4 dimensional spaces, so those are all the elements. □

Denote $\Phi = D \circ \mathrm{loc}_p$ where $D : H^1_f(G_{K_{v_1}}, W_2) \times H^1_f(G_{K_{v_1}}, W_2) \to W_1^{\mathrm{dR}} \times W_2^{\mathrm{dR}} \simeq \mathbb{A}^3 \times \mathbb{A}^3$ is the Blach-Kato isomorphism. We can always choose ξ so that $\Phi(\xi) = (1, -1)$ by considering the $\sigma \in G_{\mathbb{Q}}/G_T$ action where σ is the complex conjugation.

Lemma 4.2.2.

$$\begin{aligned}
&\Phi(t_1a_1 + t_2a_2, sb, t_1sc_1 + t_2sc_2 + r\xi) \\
=&((t_1 \log(\alpha_1) + t_2 \log(\alpha_2), sm \log(2), -t_1s\mathrm{Li}_2(\alpha_1) - t_2s\mathrm{Li}_2(\alpha_2)), \\
&(t_1 \log(\alpha_2) + t_2 \log(\alpha_1), sm \log(2), -t_1s\mathrm{Li}_2(\alpha_2) - t_2s\mathrm{Li}_2(\alpha_1)))
\end{aligned}$$

Proof. Note Φ map (a_1, b, c_1) and (a_2, b, c_2) to the followings respectively:

- $((\log(\alpha_1), m \log(2), -\mathrm{Li}_2(\alpha_1)), (-\log(\alpha_2), m \log(2), -\mathrm{Li}_2(\alpha_2)))$;
- $((\log(\alpha_2), m \log(2), -\mathrm{Li}_2(\alpha_2)), (-\log(\alpha_1), m \log(2), -\mathrm{Li}_2(\alpha_1)))$.

□

From the above, we can compute that in this case, the Coleman function is

$$-\mathrm{Li}_2(z_1) - \mathrm{Li}_2(z_2) + (t_1 + t_2)s(\mathrm{Li}_2(\alpha_1) + \mathrm{Li}_2(\alpha_2))$$

for $z_1 \in X(K_{v_1})$, $z_2 \in X(K_{v_2})$ where

$$\begin{aligned}
t_1 &= \frac{1}{\log^2(\alpha_1) - \log^2(\alpha_2)}(\log(\alpha_1)\log(z_1) - \log(\alpha_2)\log(z_2)) \\
t_2 &= \frac{1}{\log^2(\alpha_1) - \log^2(\alpha_2)}(-\log(\alpha_2)\log(z_1) + \log(\alpha_1)\log(z_2)) \\
s &= \frac{1}{m\log(2)}\log(1 - z_1) = \frac{1}{m\log(2)}\log(1 - z_2).
\end{aligned}$$

4.2.3 Future Plans

Further study the S-unit equation on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over the number field, especially for the cyclotomic field. Plan to find the solution set, prove the result by the Chabauty-Kim methods or other methods, and verify Kim's conjecture for many cases. The main challenge is to deal with cyclotomic fields and all kinds of cohomology. Taking more time to read the book *cohomology of number fields* by Jurgen Neukirch and others is what I plan in the near future.

Appendix A

Numerical Experiment for the Case $|S| = 3$ over $\mathbb{Q}$

Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $S = \{2, q, r\}$ where q and r are distinct odd primes. Our aim is to find $X(\mathbb{Z}_S)$. To do so, we can do some numerical experiments.

Our numerical experiments rely on the explicit abc conjecture to give a bound of the powers $|\alpha_i|, |\beta_i|, |\gamma_i| \leq M$ where $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$ such that

$$2^{\alpha_1} q^{\beta_1} r^{\gamma_1} + 2^{\alpha_2} q^{\beta_2} r^{\gamma_2} = 1. \tag{A.1}$$

Conjecture A.0.1 (**Explicit abc Conjecture, Baker 2004**[2])**.** *If $a + b = c$ where a, b, c are coprime positive integers, then*

$$c < \frac{6}{5} \mathrm{rad}(abc) \frac{(\log(\mathrm{rad}(abc)))^{\omega(abc)}}{(\omega(abc))!} \tag{A.2}$$

where $\mathrm{rad}(n) := \prod_{\mathrm{prime}\ p|n} p$ *and* $\omega(n) := \sum_{\mathrm{prime}\ p|n} 1$.

For example, the explicit abc conjecture allows us to choose $M = 24$ for $q, r < 200$.

Conjecture A.0.2. $\#X(\mathbb{Z}_S) \in \{3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 75, 99\}$*;*

Remark A.0.3. The above conjecture is equivalent to saying

$$\#\mathrm{Orb}_S \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 17\}$$

where Orb_S denote the set of orbits of $X(\mathbb{Z}_S)$ under the S_3-action.

Proposition A.0.4. *In the following ranges (consisting of* 4453 *prime pairs of* (q, r)*)*

- $5 \leq q \leq 7$*, and* $q < r < 7000$*;*
- $11 \leq q \leq 31$*, and* $q < r < 2000$*;*
- $37 \leq q < 200$ *and* $q < r < 200$*,*

the following shows the distribution of $\#X(\mathbb{Z}_{\{2,q,r\}})$ *(assuming explicit abc conjecture):*

- $\#X(\mathbb{Z}_{\{2,q,r\}}) = 33$*:* $(q, r) = (5, 7)$.
- $\#X(\mathbb{Z}_{\{2,q,r\}}) = 27$*:* $(q, r) = (5, 11), (5, 13), (5, 41)$.
- $\#X(\mathbb{Z}_{\{2,q,r\}}) = 21$*:* $(q, r) = (5, 17), (5, 31), (5, 127), (5, 251), (5, 641), (7, 11), (7, 13), (7, 17), (7, 113), (17, 257)$.
- $\#X(\mathbb{Z}_{\{2,q,r\}}) = 15$*: There are* 123 *pairs of such* (q, r).
- $\#X(\mathbb{Z}_{\{2,q,r\}}) = 9$*: There are* 2439 *pairs of such* (q, r).
- $\#X(\mathbb{Z}_{\{2,q,r\}}) = 3$*: There are* 1877 *pairs of such* (q, r).

Proposition A.0.5. *In the range* $3 < r < 7000$ *(consisting of* 898 *prime values of* r*), the following shows the distribution of* $\#X(\mathbb{Z}_{\{2,3,r\}})$ *(assuming explicit abc conjecture):*

- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 99$*:* $r = 5$.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 75$*:* $r = 7$.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 57$*:* $r = 11$.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 51$*:* $r = 13, 17$.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 45$*:* $r = 19$.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 39$*:* $r = 23, 73$.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 33$*:* $r = 29, 31, 37, 41, 43, 47, 61, 97, 431$.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 27$*: There are* 92 *values of such* r.
- $\#X(\mathbb{Z}_{\{2,3,r\}}) = 21$*: There are* 789 *values of such* r.

Conjecture A.0.6. *If* $q = r - 2 \geq 11$*, then* $\#X(\mathbb{Z}_S) \in \{9, 15\}$*. In addition,*

- $\#X(\mathbb{Z}_S) = 15$ *iff one of* q, r *is Fermat or Mersenne. In this case,* $X(\mathbb{Z}_S)$ *consists of an orbit of* 2*, an orbit of* $\frac{q}{r}$*, and an orbit of* q *or* $-q$ *or* r *or* $-r$.
- $\#X(\mathbb{Z}_S) = 9$ *iff none of* q, r *is Fermat or Mersenne. In this case,* $X(\mathbb{Z}_S)$ *consists of an orbit of* 2 *and an orbit of* $\frac{q}{r}$.

Remark A.0.7. The above conjecture has been verified up to $r \leq 1609$ (assuming explicit abc).

Bibliography

[1] Betts L Alexander, Theresa Kumpitsch, and Martin Lüdtke. Chabauty-kim and the section conjecture for locally geometric sections, 2023. URL: `https://arxiv.org/abs/2305.09462v1`.

[2] Alan Baker. Experiments on the *abc*-conjecture. *Publicationes Mathematicae Debrecen*, 65:253–260, 10 2004. `doi:10.5486/pmd.2004.3348`.

[3] Alex Best, L Betts, Theresa Kumpitsch, Martin Lüdtke, Angus McAndrew, Lie Qian, Elie Studnia, and Yujie Xu. Refined selmer equations for the thrice-punctured line in depth two. *Mathematics of Computation*, 93:1497–1527, 10 2023. `doi:10.1090/mcom/3898`.

[4] L. Alexander Betts. Weight filtrations on selmer schemes and the effective chabauty-kim method. *Compositio Mathematica*, 159:1531–1605, 06 2023. `doi:10.1112/s0010437x2300725x`.

[5] David Corwin. From chabauty's method to kim's non-abelian chabauty's method, 08 2021. URL: `https://math.berkeley.edu/~dcorwin/files/ChabautytoKim.pdf`.

[6] P Deligne and J Milne. Tannakian categories, 2018. URL: `https://www.jmilne.org/math/xnotes/tc2018.pdf`.

[7] Par P Deligne. Le groupe fondamental de la droite projective moins trois points. *Mathematical Sciences Research Institute publications*, pages 79–297, 01 1989. `doi:10.1007/978-1-4613-9649-9_3`.

[8] Netan Dogra. Unlikely intersections and the chabauty-kim method over number fields. *Mathematische Annalen*, 05 2023. `doi:10.1007/s00208-023-02638-2`.

[9] J H Evertse. On equations in s-units and the thue-mahler equation. *Inventiones Mathematicae*, 75:561–584, 10 1984. `doi:10.1007/bf01388644`.

[10] Minhyong Kim. The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of siegel. *Invent. math.*, 161:629–656, 04 2005. `doi:10.1007/s00222-004-0433-9`.

[11] Minhyong Kim. The unipotent albanese map and selmer varieties for curves. *Publications of The Research Institute for Mathematical Sciences*, 45:89–133, 01 2009. `doi:10.2977/prims/1234361156`.

[12] Minhyong Kim and Martin Lüdtke. Foundations of the non-abelian method of chabauty. URL: https://drive.google.com/file/d/1c9yeechXKhx529Jkx-7KXysaFE8iT5_z/view?pli=1.

[13] Martin Lüdtke. Refined chabauty-kim computations for the thrice-punctured line over z[1/6]. URL: https://arxiv.org/pdf/2402.03573.

[14] Chris A M Peters, J H M Steenbrink, and Springerlink (Online Service). *Mixed Hodge Structures*. Springer Berlin Heidelberg, 2008.

[15] Carl Ludwig Siegel. The integer solutions of the equation $y^2 = ax^n + bx^{n-1} + \cdots + k$. *J. London Math. Soc.*, 1:66–68, 1926.