Don't Bet on an Expected Value

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0.1 To play or not to play?

Imagine a game where you toss a fair coin. If it comes up heads your monetary wealth increases by 50%; if it comes up tails, it is reduced by 40%. You're not only doing this once, but many times; for example, once per week for the rest of your life. Would you accept the rules of our game? Would you play this game if given the opportunity?

0.2 Solution

Every run of the game is independent and success equally likely. Thus X_i, a random variable returning 1 on success and 0 on failure, is nothing else than a $Bernoulli(1/2)$. X, the random variable that counts the number of successful outcomes in n runs of the game, would be defined as:

$$X = \sum_{k=1}^{n} X_i$$

Then $X \sim Bin(n, 1/2)$.

After n coin tosses, our random variable final wealth W_n can be modeled as:

$$W_n = g(X)$$
$$W_n = w_0 * 1.5^X * 0.6^{n-X}$$

In order to decide if we would accept to play this game for the rest of our lives we would have to check what happens to W_n when $n \to \infty$.

According to the Strong Law of Large Numbers, which we already know holds for the Bernoulli distribution, we will prove that X is equal to its expected value $np = n/2$ with probability equal to 1 when $n \to \infty$

$$\mathbb{P}(\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_i}{n} = \mu_{X_i}) = 1$$
$$\mathbb{P}(\lim_{n\to\infty} \frac{\cancel{n} * \sum_{k=1}^{n} X_i}{\cancel{n}} = np) = 1$$
$$\mathbb{P}(\lim_{n\to\infty} X = \tfrac{n}{2}) = 1$$

As n grows, the possible outcomes of W_n start to concentrate around an only value with probability equal to 1. When $n \to \infty$, W_n will become a degenerate random variable with no randomness. We can see that by applying $g()$ at both sides

$$\mathbb{P}(\lim_{n\to\infty} g(X) = g(\tfrac{n}{2})) = 1$$
$$\mathbb{P}(\lim_{n\to\infty} W_n = w_0 * 1.5^{n/2} * 0.6^{n-n/2}) = 1$$

Now, what is that only value? Lets solve the limit

$$\begin{aligned}
\lim_{n\to\infty} w_0 * 1.5^{n/2} * 0.6^{n-n/2} &= \lim_{n\to\infty} w_0 * 1.5^{n/2} * 0.6^{n/2} \\
&= \lim_{n\to\infty} w_0 * (\sqrt{1.5 * 0.6})^n \\
&= \lim_{n\to\infty} w_0 * (\sqrt{0.9})^n \\
&= 0
\end{aligned}$$

So:

$$\mathbb{P}(\lim_{n\to\infty} W_n = 0) = 1$$

With probability equal to 1, our wealth will decrease to 0 when $n \to \infty$ independently of our starting wealth. The answer to our initial question should be: no, I do not want to play since I'm certain to go bust.

0.3 The expected value

A common erroneous way of approaching the problem is to calculate the expected value of the return that modifies your wealth, a random variable that we call V_n:

$$\mathbb{E}[V_{n+1}] = V_n * \mathbb{E}[R]^n$$

With R being a random variable defined as:

$$R = \begin{cases} 1.5 & \text{with probability } 1/2 \\ 0.6 & \text{with probability } 1/2 \end{cases}$$

Let's calculate $\mathbb{E}[R]$:

$$\mathbb{E}[R] = 1.5 * 0.5 + 0.6 * 0.5 = 1.05$$

The expected value of R is 1.05. This might lead us to conclude that the gamble is worth taking since $\mathbb{E}[R] > 1$ and we *expect* our wealth to increase.

Expected value doesn't tell us if a gamble is worth taking. It tells us what would happen on average if a group of people were to take the bet on parallel.

That is because it is not the same to calculate the average returns of a hundred people who go the casino one night and place just one bet than to calculate what happens to the wealth of an individual who visits the casino a hundred days in a row. The average return of a hundred people going once tells us nothing about what would happen to a person going over and over again.