

ALL ABOUT A FOLD*

GClaramunt

* Not all, actually

YOU COULD'VE
INVENTED FOLD...

HOW TO SUM ALL ELEMENTS OF A LIST?

[1, 7, 4, 11, 3, 9]

sum :: [Int] -> Int

HOW TO SUM ALL ELEMENTS OF A LIST?

[1, 7, 4, 11, 3, 9]

sum :: [Int] -> Int

sum [] = ?

sum (x:xs) = ?

HOW TO SUM ALL ELEMENTS OF A LIST?

[1, 7, 4, 11, 3, 9]

sum :: [Int] -> Int

sum [] = 0

sum (x:xs) = x + sum xs

HOW TO SUM ALL ELEMENTS OF A LIST? (IN SCALA)

```
List(1, 7, 4, 11, 3, 9)
```

```
def sum(nums: List[Int]): Int = nums match {  
  case Nil => 0  
  
  case x::xs => x + sum(xs)  
  
}
```

CONVERT TO STRING ALL ELEMENTS OF A LIST?

[1, 7, 4, 11, 3, 9]

toString :: [Int] -> String

toString [] = ?

toString (x:xs) = ?

CONVERT TO STRING ALL ELEMENTS OF A LIST?

```
[1, 7, 4, 11, 3, 9]
```

```
toString :: [Int] -> String
```

```
toString [] = ""
```

```
toString (x:xs) = show x ++ toString xs
```


ALL ELEMENTS OF A LIST SATISFY A PROPERTY?

[1, 7, 4, 11, 3, 9]

`all :: (a->Bool) -> [a] -> Bool`

`all _ [] = ?`

`all p (x:xs) = ?`

ALL ELEMENTS OF A LIST SATISFY A PROPERTY?

```
[1, 7, 4, 11, 3, 9]
```

```
all :: ( a->Bool ) -> [a] -> Bool
```

```
all _ [] = True
```

```
all p (x:xs) = p x && all p xs
```

HOW WE DID RECURSION?

We have:

- One definition for the empty case
- One definition for the head/tail case

HOW WE DID RECURSION?

We have:

- One definition for the empty case
- One definition for the head/tail case

We are doing recursion in the
structure of the list!

LET'S GENERALIZE!

LET'S GENERALIZE!

```
sum [] = 0
```

```
sum (x:xs) = x + sum xs
```

```
toString [] = ""
```

```
toString (x:xs) =
```

```
    show x ++ toString xs
```

```
all _ [] = True
```

```
all p (x:xs) = p x && all p xs
```

LET'S GENERALIZE!

```
sum [] = 0
```

```
sum (x:xs) = x + sum xs
```

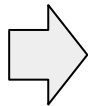
```
toString [] = ""
```

```
toString (x:xs) =
```

```
    show x ++ toString xs
```

```
all _ [] = True
```

```
all p (x:xs) = p x && all p xs
```



```
sum [] = 0
```

```
sum (x:xs) = (+) x (sum xs)
```

```
toString [] = ""
```

```
toString (x:xs) =
```

```
    ((++).show) x (toString xs)
```

```
all _ [] = True
```

```
all p (x:xs) = ((&&).p) x (all p xs)
```

LET'S GENERALIZE!

```
sum [] = 0
```

```
sum (x:xs) = (+) x (sum xs)
```

```
toString [] = ""
```

```
toString (x:xs) =
```

```
    ((++).show) x (toString xs)
```

```
all _ [] = True
```

```
all p (x:xs) = ((&&).p) x (all p xs)
```



A value **z** for the empty case

A function **f** for the head/tail case that combines the head with the result of the **recursive call** on the tail

LET'S GENERALIZE!

A value `z` for the empty case

A function `f` for the head/tail case that combines the head with the result of the `recursive call` on the tail

```
rec_list f z [] = z
```

```
rec_list f z (x:xs) = f x (rec_list f z xs)
```

LET'S GENERALIZE!

A value `z` for the empty case

A function `f` for the head/tail case that combines the head with the result of the `recursive call` on the tail

```
rec_list f z [] = z
```

```
rec_list f z (x:xs) = f x (rec_list f z xs)
```

Is “foldr” !

FOLD!

`foldr :: (a -> b -> b) -> b -> [a] -> b`

Usually “given a function f that combines an element with the accumulation and an initial value z , starting with z traverses the list (backwards) applying f producing a single result”

The result is $f(a_1 \ f(a_2 \ \dots (f \ a_n \ z)) \dots))$

(sadly, not tail recursive)

WHAT ABOUT OTHER
DATATYPES?

WHAT ABOUT OTHER DATATYPES?

What happens with other datatypes ?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

(and what about Either or Maybe?)

HOW TO SUM ALL ELEMENTS OF A TREE?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

```
sum :: BTree Int -> Int
```

HOW TO SUM ALL ELEMENTS OF A TREE?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

```
sum :: BTree Int -> Int
```

```
sum (Leaf a) = ?
```

```
sum (Branch t1 t2) = ?
```

HOW TO SUM ALL ELEMENTS OF A TREE?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

```
sum :: BTree Int -> Int
```

```
sum (Leaf a) = a
```

```
sum (Branch t1 t2) = sum t1 + sum t2
```


CONVERT TO STRING ALL ELEMENTS OF A TREE?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

```
toString :: BTree Int -> String
```

```
toString (Leaf a) = ?
```

```
toString (Branch t1 t2) = ?
```

CONVERT TO STRING ALL ELEMENTS OF A TREE?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

```
toString :: BTree Int -> String
```

```
toString (Leaf a) = show a
```

```
toString (Branch t1 t2) = toString t1 ++ toString t2
```

HOW TO FOLD A TREE?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

```
rec_tree :: (b -> b -> b) -> (a -> b) -> Tree a -> b
```

```
rec_tree _ g (Leaf a) = ?
```

```
rec_tree f g (Branch t1 t2) = ?
```

HOW TO FOLD A TREE?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a

rec_tree :: (b -> b -> b) -> (a -> b) -> Tree a -> b

rec_tree _ g (Leaf a) = g a

rec_tree f g (Branch t1 t2) =
    f (rec_tree f g t1) (rec_tree f g t2)
```

WHAT IS A FOLD?

WHAT IS A FOLD?

```
data List a = Cons a (List a) | Nil
```

```
(or data [] a = a : [a] | [] )
```

```
rec_list :: (a -> b -> b) -> b -> List a -> b
```

```
Cons 1 (Cons 2 (Cons 3 (Nil))) ~>
```

```
  f 1 (f 2 (f 3 z))
```

WHAT IS A FOLD?

```
data BTree a = Branch (BTree a) (BTree a) | Leaf a
```

```
rec_tree :: (b -> b -> b) -> (a -> b) -> BTree a -> b
```

```
Branch (Branch (Leaf 1) (Leaf 2)) (Leaf 3) ~>
```

```
  f (f (g 1) (g 2)) (g 3)
```

A FOLD REPLACES THE
DATATYPE CONSTRUCTORS
WITH FUNCTIONS

WHAT ABOUT OTHER DATATYPES?

Maybe a = Nothing | Just a

Either a b = Left a | Right b

WHAT ABOUT OTHER DATATYPES?

Maybe a = Nothing | Just a

`fold_m :: b -> (a -> b) -> Maybe a -> b`

(“maybe” in Haskell)

Either a b = Left a | Right b

`fold_e :: (a->c) -> (b->c) -> Either a b -> c`

(“either” in Haskell)

WHAT IS A FOLD?

Transforms the input into something else, following the structure of the datatype

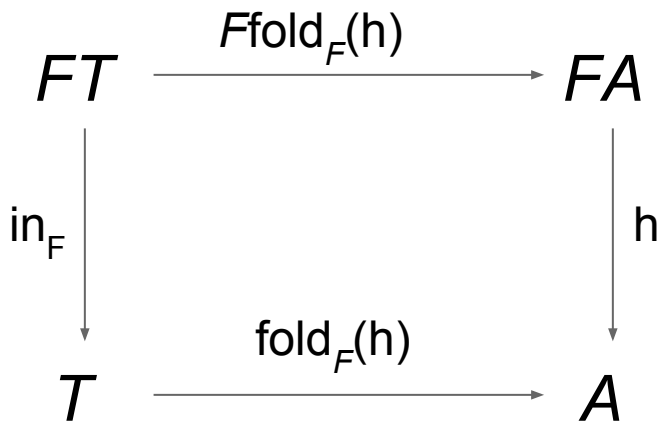
Catamorphism

Greek 'κατα-' meaning "downward or according to"

“There’s a truly marvellous category theory explanation for this which this slide is too narrow to contain”

CATAMORPHISMS!

“Catamorphisms are generalizations of the concept of a fold in functional programming. A catamorphism deconstructs a data structure with an F-algebra for its underlying functor”



Given an F-algebra $h : F A \rightarrow A$,

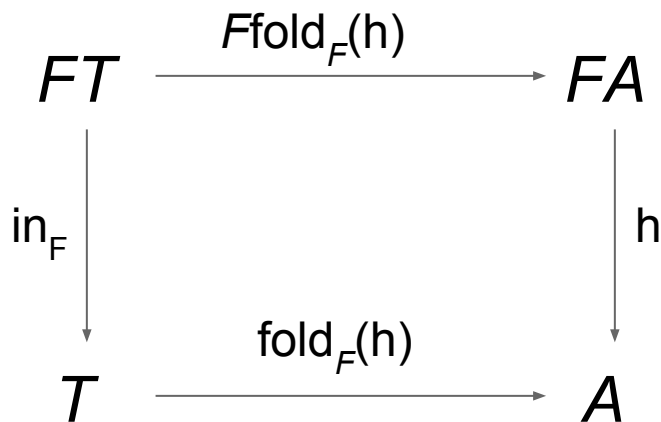
$\text{fold}_F(h)$ is the unique homomorphism between in_F and h

Fold replaces each constructor for the corresponding function in the algebra.

Fold corresponds to definitions by structural recursion over the type

CATAMORPHISMS!

“Catamorphisms are generalizations of the concept of a fold in functional programming. A catamorphism deconstructs a data structure with an F-algebra for its underlying functor”



Given an F-algebra $h : F A \rightarrow A$,

$\text{fold}_F(h)$ is the unique homomorphism between in_F and h

Fold replaces each constructor for the corresponding function in the algebra.

Fold corresponds to definitions by structural recursion over the type

(An algebra of functors $1, K, I, +, *$ can describe regular datatypes and be an initial algebra for all of them)

“AFTER ALL, A FOLD IS ORIGINATED BY THE
UNIQUE HOMOMORPHISM THAT EXISTS
BETWEEN THE INITIAL ALGEBRA AND ANY
OTHER ALGEBRA, WHAT’S THE PROBLEM?”

THANK YOU!

@GCLARAMUNT

BONUS TRACK

`foldr (:) [] [1,2,3] == [1,2,3]`

What about:

`foldr ((:).f) []` or `fold ((Leaf).f)(Branch)`

What's the result of `foldr ((:).(+2)) [] [1,2,3]` ?

BONUS TRACK

`foldr (:) [] [1,2,3] == [1,2,3]`

What about:

`foldr ((:).f) []` or `fold ((Leaf).f)(Branch)`

What's the result of `foldr ((:).(+2)) [] [1,2,3]` ?

That's the map function!