Algebraic Databases

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Introduction

- This talk describes a new "algebraic" way to formalize databases based on category theory.
- Category theory was designed to migrate **theorems** from one **area of mathematics** to another, but researchers at MIT developed a way to use it to migrate **data** from one **computer system** to another.
- Research has culminated in an open-source prototype ETL and data integration tool, AQL (Algebraic Query Language), available at catinf.com.
- Outline:
  - Part 0: Review of basic category theory.
  - Part 1: Introduction to AQL.
  - AQL demo.
  - Optional interlude: additional AQL constructions.
  - Part 2: How AQL instances model the simply-typed λ-calculus.
  - AQL demo.

Category Theory

- A category $\mathcal{C}$ consists of
  - a set of *objects*, $\mathsf{Ob}(\mathcal{C})$
  - forall $X, Y \in \mathsf{Ob}(\mathcal{C})$, a set $\mathcal{C}(X, Y)$ of *morphisms* a.k.a *arrows*
  - forall $X \in \mathsf{Ob}(\mathcal{C})$, a morphism $id \in \mathcal{C}(X, X)$
  - forall $X, Y, Z \in \mathsf{Ob}(\mathcal{C})$, a function $\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ s.t.

$$f \circ id = f \qquad id \circ f = f \qquad\qquad (f \circ g) \circ h = f \circ (g \circ h)$$

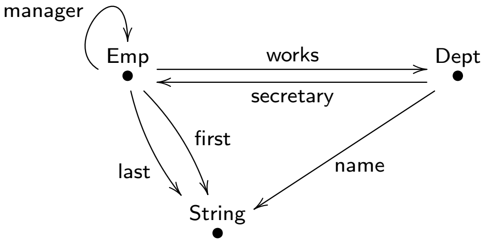
- The category **Set** has sets as objects and functions as arrows, and the "category" **Haskell** has types as objects and expressions as arrows.

- A functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}, \mathcal{D}$ consists of
  - a function $\mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D})$
  - forall $X, Y \in Ob(\mathcal{C})$, a function $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ s.t.

$$F(id) = id \qquad F(f \circ g) = F(f) \circ F(g)$$

- The functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ takes each set to its power set, and the functor $\mathsf{List} : \mathbf{Haskell} \to \mathbf{Haskell}$ takes each type t to the type $\mathsf{List}\ t$.

Schemas and Instances



works $\circ$ manager $=$ works works $\circ$ secretary $= id$

Emp				
ID	**mgr**	**works**	**first**	**last**
101	103	q10	Al	Akin
102	102	x02	Bob	Bo
103	103	q10	Carl	Cork

Dept		
ID	**sec**	**name**
q10	102	CS
x02	101	Math

String
ID
Al
Bob
...

Categorical Semantics of Schemas and Instances

- The meaning of a schema S is a category $[\![S]\!]$.
  - Ob($[\![S]\!]$) is the nodes of S.
  - Forall nodes X, Y, $[\![S]\!](X, Y)$ is the set of finite paths $X \to Y$, modulo the path equivalences in S.
  - Path equivalence in S may not be decidable! ("the word problem")
- A morphism of schemas (a "schema mapping") $S \to T$ is a functor $[\![S]\!] \to [\![T]\!]$.
  - It can be defined as an equation-preserving function:

$$nodes(S) \to nodes(T) \qquad edges(S) \to paths(T).$$

- An S-instance is a functor $[\![S]\!] \to \mathbf{Set}$.
  - It can be defined as a set of tables, one per node in S and one column per edge in S, satisfying the path equivalences in S.
- A morphism of S-instances $I \to J$ (a "data mapping") is a natural transformation $I \to J$.
  - Instances on S and their mappings form a category, written S-inst.

Why "Algebraic"?

- A schema can be identified with an algebraic (equational) theory.

$$\mathsf{Emp}\ \mathsf{Dept}\ \mathsf{String} : \mathsf{Type} \qquad \mathsf{first}\ \mathsf{last} : \mathsf{Emp} \to \mathsf{String} \qquad \mathsf{name} : \mathsf{Dept} \to \mathsf{String}$$

$$\mathsf{works} : \mathsf{Emp} \to \mathsf{Dept} \qquad \mathsf{mgr} : \mathsf{Emp} \to \mathsf{Emp} \qquad \mathsf{secr} : \mathsf{Dept} \to \mathsf{Emp}$$

$$\forall e : \mathsf{Emp}.\ \mathsf{works}(\mathsf{manager}(e)) = \mathsf{works}(e) \qquad \forall d : \mathsf{Dept}.\ \mathsf{works}(\mathsf{secretary}(d)) = d$$

- This perspective makes it easy to add functions such as $+ : \mathsf{Int}, \mathsf{Int} \to \mathsf{Int}$ to a schema. See *Algebraic Databases*.

- An S-instance can be identified with the initial algebra of an algebraic theory extending S.

$$\mathsf{101}\ \mathsf{102}\ \mathsf{103} : \mathsf{Emp} \qquad \mathsf{q10}\ \mathsf{x02} : \mathsf{Dept}$$

$$\mathsf{mgr}(\mathsf{101}) = \mathsf{103} \qquad \mathsf{works}(\mathsf{101}) = \mathsf{q10} \qquad \ldots$$

- Treating instances as theories allows instances that are infinite or inconsistent (e.g., Alice=Bob).

Functorial Data Migration

A schema mapping $F\colon S \to T$ induces three data migration functors:

- $\Delta_F\colon T$-inst $\to S$-inst (like project)

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[rr, bend right, "\Delta_F(I) \;:=\; I \circ F"'] & T \arrow[r, "I"] & \mathbf{Set} \end{tikzcd}$$

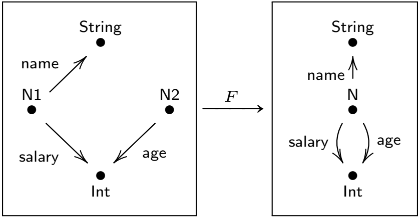
- $\Pi_F\colon S$-inst $\to T$-inst (right adjoint to Δ_F; like join)

$$\forall I, J. \quad S\text{-inst}(\Delta_F(I), J) \cong T\text{-inst}(I, \Pi_F(J))$$

- $\Sigma_F\colon S$-inst $\to T$-inst (left adjoint to Δ_F; like outer union then merge)

$$\forall I, J. \quad S\text{-inst}(J, \Delta_F(I)) \cong T\text{-inst}(\Sigma_F(J), I)$$

Δ (Project)



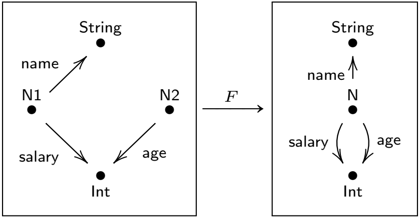
N1		
ID	**name**	**salary**
1	Alice	\$100
2	Bob	\$250
3	Sue	\$300

N2	
ID	**age**
4	20
5	20
6	30

$\xleftarrow{\Delta_F}$

N			
ID	**name**	**salary**	**age**
a	Alice	\$100	20
b	Bob	\$250	20
c	Sue	\$300	30

Π (Product)



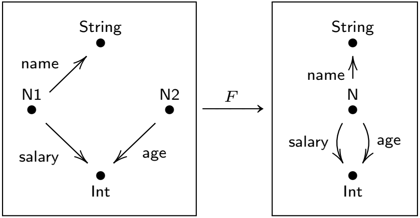
N1		
ID	**name**	**salary**
1	Alice	\$100
2	Bob	\$250
3	Sue	\$300

N2	
ID	**age**
4	20
5	20
6	30

$\xrightarrow{\Pi_F}$

N			
ID	**name**	**salary**	**age**
a	Alice	\$100	20
b	Alice	\$100	20
c	Alice	\$100	30
d	Bob	\$250	20
e	Bob	\$250	20
f	Bob	\$250	30
g	Sue	\$300	20
h	Sue	\$300	20
i	Sue	\$300	30

Σ (Outer Union)



N1		
ID	**Name**	**Salary**
1	Alice	\$100
2	Bob	\$250
3	Sue	\$300

N2	
ID	**Age**
4	20
5	20
6	30

$\xrightarrow{\Sigma_F}$

N			
ID	**Name**	**Salary**	**Age**
a	Alice	\$100	$null_1$
b	Bob	\$250	$null_2$
c	Sue	\$300	$null_3$
d	$null_4$	$null_5$	20
e	$null_6$	$null_7$	20
f	$null_8$	$null_9$	30

Unit of $\Sigma_F \dashv \Delta_F$

N1		
ID	**Name**	**Salary**
1	Alice	$100
2	Bob	$250
3	Sue	$300

N2	
ID	**Age**
4	20
5	20
6	30

$\xrightarrow{\Sigma_F}$

N			
ID	**Name**	**Salary**	**Age**
a	Alice	$100	$null_1$
b	Bob	$250	$null_2$
c	Sue	$300	$null_3$
d	$null_4$	$null_5$	20
e	$null_6$	$null_7$	20
f	$null_8$	$null_9$	30

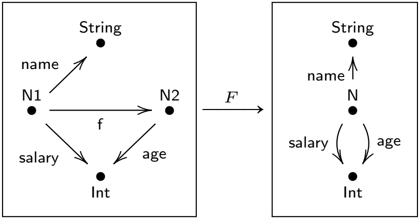
Δ_F

$\downarrow \eta$

N1		
ID	**Name**	**Salary**
a	Alice	$100
b	Bob	$250
c	Sue	$300
d	$null_4$	$null_5$
e	$null_6$	$null_7$
f	$null_8$	$null_9$

N2	
ID	**Age**
a	$null_1$
b	$null_2$
c	$null_3$
d	20
e	20
f	30

A Foreign Key



N1			
ID	**name**	**salary**	**f**
1	Alice	$100	4
2	Bob	$250	5
3	Sue	$300	6

N2	
ID	**age**
4	20
5	20
6	30

$\xleftarrow{\Delta_F}$

$\xrightarrow{\Pi_F, \Sigma_F}$

N			
ID	**name**	**salary**	**age**
a	Alice	$100	20
b	Bob	$250	20
c	Sue	$300	30

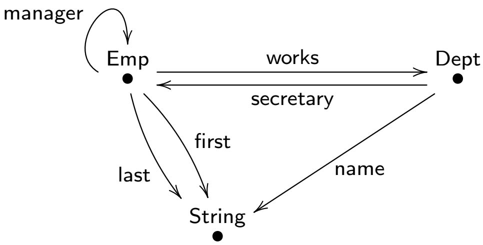
AQL vs SQL

- Data migration triplets of the form

$$\Sigma_F \circ \Pi_G \circ \Delta_H$$

  can be expressed using relational algebra and keygen, provided:
    - F is a discrete op-fibration (ensures union compatibility).
- The difference-free fragment of relational algebra can be expressed using such triplets. See *Relational Foundations*.
- Such triplets can be written in “foreign-key aware” SQL-ish syntax.

Select-From-Where Syntax



Find the name of every manager's department:

```
AQL
select e.manager.works.name
from Emp as e
```

```
SQL
select d.name
from Emp as e1, Emp as e2, Dept as d
where e1.manager = e2.ID and
      e2.works = d.ID
```

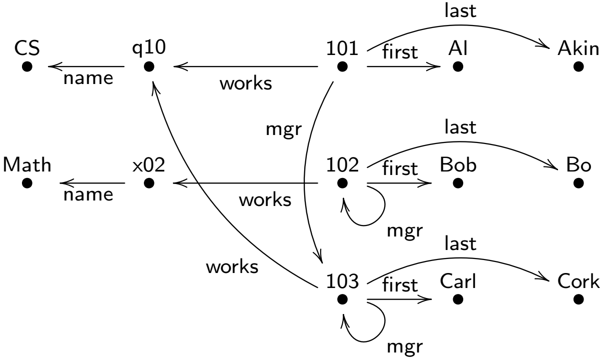
AQL Demo

- The AQL IDE implements Δ, Σ, Π in software.
  - catinf.com
- The AQL IDE "execution engine" is an automated theorem prover.
  - Value proposition: AQL catches mistakes at compile time that existing ETL / data integration tools catch at runtime – if at all.
  - Data import and export by JDBC-SQL and CSV.

Interlude - Additional Constructions

- Pivot.
- Pushouts.
- Non-equational data integrity constraints.
- AQL vs comprehension calculi.

Pivot (Instance ⇔ Schema)



Emp				
ID	**mgr**	**works**	**first**	**last**
101	103	q10	Al	Akin
102	102	x02	Bob	Bo
103	103	q10	Carl	Cork

Dept	
ID	**name**
q10	CS
x02	Math

Richer Constraints

- Not all data integrity constraints are equational (e.g., keys).
- A data mapping $\varphi : A \to E$ defines a constraint: instance I satisfies φ if for every $\alpha : A \to I$ there exists an $\epsilon : E \to I$ s.t $\alpha = \epsilon \circ \varphi$.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & I \\ E \arrow[ur, dashed, "\epsilon"'] & \end{tikzcd}$$

- Most constraints used in practice can be captured the above way.
- See *Database Queries and Constraints via Lifting Problems* and *Algebraic Model Management*.

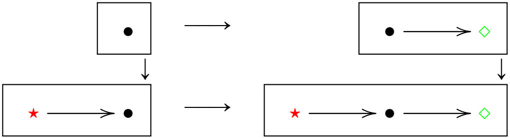
Pushouts

- A pushout of p, q is f, q s.t. for every f', g' there is a unique m s.t.:

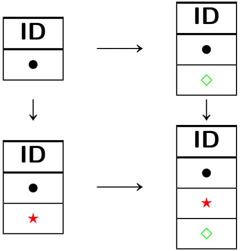
$$\begin{tikzcd}
& \bullet \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\bullet \arrow[dr, "f"] \arrow[ddr, bend right, "f'"'] & & \bullet \arrow[dl, "g"'] \arrow[ddl, bend left, "g'"] \\
& \bullet \arrow[d, dashed, "m"] & \\
& \bullet &
\end{tikzcd}$$

- The category of schemas has all pushouts.
- For every S, the category S-inst has all pushouts.
- See *Algebraic Data Integration*.

Pushouts of Schemas



Pushouts of Instances



AQL vs LINQ

- Treating entity sets as types rather than terms makes AQL a conceptual dual to comprehension calculi (e.g., LINQ). See *QINL: Query-Integrated Languages, or co-LINQ*.
- LINQ enriches programs with (schemas, queries and instances).
  - Collections are *terms*

$$\mathrm{Employee} \colon \mathrm{Set}\ \mathrm{Int} \quad \mathrm{manager} \colon \mathrm{Set}\ (\mathrm{Int} \times \mathrm{Int})$$

  - $\mathrm{e} \colon \mathrm{Employee}$ *is not* a judgment.
  - There *is* a term $\in \colon \mathrm{Int} \times \mathrm{Set}\ \mathrm{Int} \to \mathrm{Bool}$.
- AQL enriches (schemas, queries and instances) with programs.
  - Collections are *types*

$$\mathrm{Employee} \colon \mathrm{Type} \quad \mathrm{manager} \colon \mathrm{Employee} \to \mathrm{Employee}$$

  - $\mathrm{e} \colon \mathrm{Employee}$ *is* a judgment.
  - There *is not* a term $\in \colon \mathrm{Employee} \times \mathrm{Type} \to \mathrm{Bool}$.
- LINQ is more popular, but AQL's style is common in Coq, Agda, etc.

AQL is "one level up" from LINQ

- LINQ
  - Schemas are collection types over a base type theory

    $$\mathsf{Set}\ (\mathsf{Int} \times \mathsf{String})$$

  - Instances are terms

    $$\{(1, \mathsf{CS})\} \cup \{(2, \mathsf{Math})\}$$

  - Data migrations are functions

    $$\pi_1 \colon \mathsf{Set}\ (\mathsf{Int} \times \mathsf{String}) \to \mathsf{Set\ Int}$$

- AQL
  - Schemas are type theories over a base type theory

    $$\mathsf{Dept},\ \mathsf{name} \colon \mathsf{Dept} \to \mathsf{String}$$

  - Instances are term models (initial algebras) of theories

    $$\mathsf{d}_1, \mathsf{d}_2 \colon \mathsf{Dept},\ \mathsf{name}(\mathsf{d}_1) = \mathsf{CS},\ \mathsf{name}(\mathsf{d}_2) = \mathsf{Math}$$

  - Data migrations are functors

    $$\Delta_{\mathsf{Dept}} \colon (\mathsf{Dept}, \mathsf{name} \colon \mathsf{Dept} \to \mathsf{String})\text{-}\mathsf{inst} \ \to\ (\mathsf{Dept})\text{-}\mathsf{inst}$$

Part 2

- For every schema S, S-inst models simply-typed λ-calculus (STLC).
- The STLC is the core of typed functional languages ML, Haskell, etc.
- We will use the internal language of a cartesian closed categor, which is equivalent to the STLC.
- Lots of "point-free" functional programming ahead.

Categorical Abstract Machine Language (CAML)

- Types t:

$$t ::= 1 \mid t \times t \mid t^t$$

- Terms f, g:

$$id_t : t \to t \qquad ()_t : t \to 1 \qquad \pi^1_{s,t} : s \times t \to s \qquad \pi^2_{s,t} : s \times t \to t$$

$$eval_{s,t} : t^s \times s \to t \qquad \frac{f : s \to u \qquad g : u \to t}{g \circ f : s \to t} \qquad \frac{f : s \to t \qquad g : s \to u}{(f, g) : s \to t \times u}$$

$$\frac{f : s \times u \to t}{\lambda f : s \to t^u}$$

- Equations:

$$id \circ f = f \qquad f \circ id = f \qquad f \circ (g \circ h) = (f \circ g) \circ h \qquad () \circ f = ()$$

$$\pi^1 \circ (f, g) = f \qquad \pi^2 \circ (f, g) = g \qquad (\pi^1 \circ f, \pi^2 \circ f) = f$$

$$eval \circ (\lambda f \circ \pi^1, \pi^2) = f \qquad \lambda(eval \circ (f \circ \pi^1, \pi^2)) = f$$

Programming AQL in CAML

- For every schema S, the category S-inst is cartesian closed.
  - Given a type t, you get an S-instance $[t]$.
  - Given a term $f : t \to t'$, you get a data mapping $[f] : [t] \to [t']$.
- AQL implements the translation $[-]$ when $[\![S]\!]$ is finite.
- S-inst is further a topos (model of higher-order logic / set theory).
- We consider the following schema in the examples that follow:

$$\boxed{\overset{a}{\bullet} \xrightarrow{f} \overset{b}{\bullet}}$$

Programming AQL in CAML: Unit

- The unit instance 1 has one row per table:

a	
ID	**f**
x	x

b
ID
x

- The data mapping $()_t : t \to 1$ sends every row in t to the only row in 1. For example,

$t :=$

a	
ID	**f**
p	q
r	t

b
ID
q
t

$\xrightarrow{()_t}$

a	
ID	**f**
x	x

b
ID
x

$:= 1$

$$\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{t} \xrightarrow{()_t} \mathsf{x}$$

Programming AQL in CAML: Products

- Products $s \times t$ are computed row-by-row, with evident projections $\pi^1 : s \times t \to s$ and $\pi^2 : s \times t \to t$. For example:

a	
ID	**f**
1	3
2	3

b
ID
3
4

$\times$

a	
ID	**f**
a	c
b	c

b
ID
c
d

$=$

a	
ID	**f**
(1,a)	(3,c)
(1,b)	(3,c)
(2,a)	(3,c)
(2,b)	(3,c)

b
ID
(3,c)
(3,d)
(4,c)
(4,d)

- Given data mappings $f : s \to t$ and $g : s \to u$, how to define $(f, g) : s \to t \times u$ is left to the reader.
  - hint 1: try it on π^1 and π^2 and verify that $(\pi^1, \pi^2) = id$.
  - hint 2: try it in the AQL tool (requires "FQL" mode).

Programming AQL in CAML: Exponentials

- Exponentials t^s are given by finding all data mappings $s \to t$:

a	
ID	**f**
1	3
2	3

b
ID
3
4

$\to$

a	
ID	**f**
a	c
b	c

b
ID
c
d

$=$

a	
ID	**f**
$1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto b, 2 \mapsto a, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto a, 2 \mapsto a, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto b, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto a, 2 \mapsto b, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$
$1 \mapsto b, 2 \mapsto a, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$
$1 \mapsto a, 2 \mapsto a, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$
$1 \mapsto b, 2 \mapsto b, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$

b
ID
$3 \mapsto c, 4 \mapsto c$
$3 \mapsto c, 4 \mapsto d$
$3 \mapsto d, 4 \mapsto c$
$3 \mapsto d, 4 \mapsto d$

- Defining $eval$ and λ are left to the reader.

AQL Demo II

Concusion

- We described a new "algebraic" approach to databases based on category theory.
  - Schemas are categories, instances are set-valued functors.
  - Three adjoint data migration functors, Σ, Δ, Π.
  - Instances on a schema model the simply-typed λ-calculus.
- Our approach is implemented in AQL, an open-source project, available at catinf.com.
- Collaborators welcome!