THEORY AND MODELS OF LAMBDA CALCULUS: UNTYPED AND TYPED

Session 2: *Modeling by Enumeration Operators*

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Basics of Topology

Definition. A ***topological space*** is a set (space) $\mathscr{S}$ of ***points*** together with a notion of ***"closeness"*** or ***"convergence"***.

NOTE: There at two popular ways of presenting topological structures: (1) specifying the collection of open subsets, or (2) providing a neighborhood base.

Axioms for Opens: Opens for $\mathscr{S}$ consist of a given collection $\mathcal{O}$ of subsets of the space closed under ***arbitrary unions*** and ***finite intersections***.

Axioms for Neighborhoods: A ***base*** for $\mathscr{S}$ consists of a collection $\mathcal{N}$ of subsets (neighborhoods) such that every finite intersection from $\mathcal{N}$ ***is*** the union of the neighborhoods it contains.

Arbitrary opens of $\mathscr{S}$ are then the unions of sets of neighborhoods.

Convergence and Continuity

Convergence: A sequence of points of a space $\mathcal{S}$ ***converges*** to a ***limit*** iff every neighborhood of the limit contains all but a ***finite number*** of points of the sequence.

Continuity: A function $\boldsymbol{F} : \mathcal{S} \longrightarrow \mathcal{T}$ between topological spaces is ***continuous*** iff for every point `x` of $\mathcal{S}$ and every $\mathcal{T}$-neighborhood `U` of ***F***`(x)`, there is an $\mathcal{S}$-neighborhood `V` of `x` such that ***F***`(y)` is in `U` for all `y` in `V`.

Inverse Images: Note that continuous functions are ***exactly*** those where the inverse image of every open set is open.

Euclidean Spaces: In 3-D for example, use ***open balls*** as forming a neighborhood base. Connect this idea with the ***distance metric***. ***How many*** balls do we actually need?

Those εs and δs: Recall from ***Calculus*** the ε–δ definition of continuity and check that it agrees with our definitions here.

Separation Axioms in Topology

Note: ”T" stands for ***Trennung***, a German word for "separation".

(T0)	Kolmogorov space
(T1)	Fréchet space
(T2)	Hausdorff space
(T2½)	Urysohn space
(T3)	Regular Hausdorff space
(T3½)	Tychonoff space
(T4)	Normal Hausdorff space
(T5)	Completely normal Hausdorff space
(T6)	Perfectly normal Hausdorff space

A ***T0-space*** is one where each point is ***uniquely determined*** by its neighborhoods.

A ***T1-space*** is one where, for every pair of distinct points, each has a neighborhood ***not containing*** the other.

A ***T2-space*** is one where two distinct points always have ***disjoint*** neighborhoods.

Note: The points of a ***T0-space*** are ***partially ordered*** by by a ***specialization order*** , where x ≦ y iff every neighborhood of x is a neighborhood of y.

T0-spaces that are not T1-spaces are exactly those spaces where the specialization order is a ***nontrivial*** partial ordering.

(Such spaces naturally occur in Computer Science, specifically in Denotational Semantics.)

The Powerset of the Integers

The powerset $\mathcal{P}(\mathbb{N}) = \{\, X \mid X \subseteq \mathbb{N} \,\}$ becomes a ***T_0-topological space*** with the sets of the form $\{\, X \subseteq \mathbb{N} \mid E \subseteq X \,\}$ as a ***neighborhood base***, where E is taken as a ***finite*** set.

Exercise 1: What are the ***open*** subsets of $\mathcal{P}(\mathbb{N})$ in this topology?

Exercise 2: Show that the ***continuous*** functions $F\colon \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ are those where for all $X \in \mathcal{P}(\mathbb{N})$ and all finite $E \in \mathcal{P}(\mathbb{N})$ we have $E \subseteq F(X)$ iff there is a finite $D \subseteq X$ with $E \subseteq F(D)$.

Exercise 3: Prove that all continuous functions are ***monotone*** for the relationship of ***set inclusion***.

Exercise 4: Find a monotone function that is ***not*** continuous.

Exercise 5: Which of the usual ***Boolean operations*** are continuous?

Enumeration Operators

Definitions. (1) ***Pairing:*** $(n,m) = 2^n(2m+1)-1$.

(2) ***Sequence numbers:*** $\langle\rangle = 0$ and

$\langle n_0, n_1, \ldots, n_{k-1}, n_k\rangle = (\langle n_0, n_1, \ldots, n_{k-1}\rangle, n_k)+1$.

(3) ***Sets:*** $\mathbf{set}(0) = \varnothing$ and $\mathbf{set}((n,m)+1) = \mathbf{set}(n) \cup \{m\}$.

(4) ***Kleene star:*** $X^* = \{n \mid \mathbf{set}(n) \subseteq X\}$, for sets $X \subseteq \mathbb{N}$.

Note: X^* **consists of all the sequence numbers representing all the finite subsets of the set** X**.**

Definition. An ***enumeration operator*** $\boldsymbol{F}:\mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ is a mapping ***determined by*** a given subset $F \subseteq \mathbb{N}$ by the formula:

$$\boldsymbol{F}(X) = \{m \mid \exists n \in X^*. (n,m) \in F\}$$

Exercise: Show that the enumeration operators on $\mathcal{P}(\mathbb{N})$ are ***exactly*** the continuous functions.

The λ-Calculus Model

Application: $F(X) = \{\, m \mid \exists n \in X^{*}.\,(n,m) \in F \,\}$

Abstraction: $\lambda X.[..X..] = \{\, (n,m) \mid m \in [..\mathbf{set}(n)..] \,\}$,
where $X \mapsto [..X..]$ is ***continuous***.

Exercise 1: Application is a continuous function of ***two*** variables.

Exercise 2: If $\boldsymbol{F}(X)$ is continuous, then $\lambda X.\boldsymbol{F}(X)$ is the ***largest*** *set* F where for all sets T, we have $F(T) = \boldsymbol{F}(T)$.

Note: Generally we only have $F \subseteq \lambda X.F(X)$.

Exercise 3: If the function $\boldsymbol{F}(X,Y)$ is continuous, then the abstraction term $\lambda X.\boldsymbol{F}(X,Y)$ is continuous in the ***other variable***.

Some Historical Background

This model clearly satisfies all the axioms but the last, and it could easily have been defined in 1957!!

John R. Myhill: **Born:** 11 August 1923, Birmingham, UK
Died: 15 February 1987, Buffalo, NY

John Shepherdson: **Born:** 7 June 1926, Huddersfield, UK
Died: 8 January 2015, Bristol, UK

Hartley Rogers, Jr.: **Born:** 6 July, 1926, Buffalo, NY
Died: 17 July, 2015, Waltham, MA

- John Myhill and John C. Shepherdson, *Effective operations on partial recursive functions*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik**, vol. 1 (1955), pp. 310-317.

- Richard M. Friedberg and Hartley Rogers Jr., *Reducibility and completeness for sets of integers*, **Mathematical Logic Quarterly**, vol. 5 (1959), pp. 117-125. Some earlier results are presented in an abstract in **The Journal of Symbolic Logic**, vol. 22 (1957), p. 107.

- Hartley Rogers, Jr., **Theory of Recursive Functions and Effective Computability**, McGraw-Hill, 1967, xix + 482 pp.

Some λ-Properties & Computability

Theorem. For all sets of integers F and G we have:

$$\lambda X.F(X) \subseteq \lambda X.G(X) \text{ iff } \forall X.F(X) \subseteq G(X),$$

$$\lambda X.(F(X) \cap G(X)) = \lambda X.F(X) \cap \lambda X.G(X),$$

and

$$\lambda X.(F(X) \cup G(X)) = \lambda X.F(X) \cup \lambda X.G(X).$$

Definition. An enumeration operator $\boldsymbol{F}(X_0, X_1, \ldots, X_{n-1})$ is ***computable*** iff in the model this set is **RE**:

$$F = \lambda X_0 \lambda X_1 \ldots \lambda X_{n-1}.\boldsymbol{F}(X_0, X_1, \ldots, X_{n-1}).$$

Fixed Points and Recursion

Three Basic Theorems.

- All pure λ-terms define ***computable*** operators.
- **Y**(`F`) is the ***least fixed point*** of the enumeration operator `F`.
- The least fixed point of a ***computable*** operator is computable.

A Principal Theorem. These computable operators:

$$\mathbf{Succ}(X) = \{n+1 \mid n \in X\},$$

$$\mathbf{Pred}(X) = \{n \mid n+1 \in X\}, \text{ and}$$

$$\mathbf{Test}(Z)(X)(Y) = \{n \in X \mid 0 \in Z\} \cup \{m \in Y \mid \exists k.\, k+1 \in Z\},$$

together with λ-calculus, suffice for defining ***all RE sets***.

Hint of Proof: **Show first that all primitive recursive functions have the form** $P(\{n\})$ **with** P **λ-definable.**

Gödel Numbering RE Sets

Theorem. There is a computable $\mathbf{V} = \lambda \mathtt{x}.\ \mathbf{V}(\mathtt{x})$ where

(i) $\mathbf{V}(\{0\}) = \lambda \mathtt{y}.\lambda \mathtt{x}.\ \mathtt{y} = \mathbf{K}$,

(ii) $\mathbf{V}(\{1\}) = \lambda \mathtt{z}.\lambda \mathtt{y}.\lambda \mathtt{x}.\ \mathtt{z}(\mathtt{x})(\mathtt{y}(\mathtt{x})) = \mathbf{S}$,

(iii) $\mathbf{V}(\{2\}) = \mathbf{Test}$,

(iv) $\mathbf{V}(\{3\}) = \mathbf{Succ}$,

(v) $\mathbf{V}(\{4\}) = \mathbf{Pred}$, and

(vi) $\mathbf{V}(\{5 + (\mathtt{n},\mathtt{m})\}) = \mathbf{V}(\{\mathtt{n}\})(\mathbf{V}(\{\mathtt{m}\}))$.

Theorem. Every ***recursively enumerable set*** is of the form $\mathbf{V}(\{\mathtt{n}\})$.

NOTE: The operator V is the analogue of the Universal Turing Machine.

The Category of Closure Operators

Definition. A set $\mathtt{C} = \lambda \mathtt{X}.\mathtt{C}(\mathtt{X})$ represents a ***closure operator*** iff for all $\mathtt{X} \subseteq \mathbb{N}$ we have $\mathtt{X} \subseteq \mathtt{C}(\mathtt{X}) = \mathtt{C}(\mathtt{C}(\mathtt{X}))$.

Note. The set of ***fixed points*** of a closure operator form a ***lattice*** and uniquely determine the operator. They give examples (up to isomorphism) of all ***countably based algebraic lattices***.

Theorem. We have ***function spaces*** and thus a ***category*** for closure operators via these definitions:

$$\mathtt{F}\colon\ \mathtt{C} \to \mathtt{D} \text{ iff } \mathtt{F} = \mathtt{D} \circ \mathtt{F} \circ \mathtt{C}$$

and

$$(\mathtt{C} \to \mathtt{D}) = \lambda \mathtt{F}.\ \mathtt{D} \circ \mathtt{F} \circ \mathtt{C}, \text{ where } \mathtt{F} \circ \mathtt{G} = \lambda \mathtt{X}.\ \mathtt{F}(\mathtt{G}(\mathtt{X})).$$

Products of Closures

Definition. ***Pairing functions*** for sets in $\mathcal{P}(\mathbb{N})$

can be defined by these enumeration operators:

$$\textbf{Pair}(X)(Y)=\{2n \mid n \in X\} \cup \{2m+1 \mid m \in Y\}$$

$$\textbf{Fst}(Z)=\{n \mid 2n \in Z\} \quad \text{and} \quad \textbf{Snd}(Z)=\{m \mid 2m+1 \in Z\}.$$

Theorem. In the category of closure operators, ***products*** of closures can be defined as:

$$(C \times D) = \boldsymbol{\lambda} Z.\textbf{Pair}(C(\textbf{Fst}(Z)))(D(\textbf{Snd}(Z))).$$

Exercise: Show that $\mathbf{I} = (\mathbf{I} \times \mathbf{I})$. Are there ***other*** such closures?

Note. The closure operators as a ***cartesian closed category*** is equivalent to the category of ***countably based algebraic lattices***.

A Universal Closure Operator

Theorem. Every enumeration operator $\mathtt{F}$ ***generates*** a closure operator $\mathtt{C}$ with range $\{\mathtt{X} \subseteq \mathbb{N} \mid \mathtt{F}(\mathtt{X}) \subseteq \mathtt{X}\}$, where we can define

$$\mathtt{C}(\mathtt{X}) = \mathbf{Y}(\lambda \mathtt{Y}.(\mathtt{X} \cup \mathtt{F}(\mathtt{Y}))).$$

Hint. The set $\{\mathtt{X} \subseteq \mathbb{N} \mid \mathtt{F}(\mathtt{X}) \subseteq \mathtt{X}\}$ is closed under ***arbitrary intersections***, and we want $\mathtt{C}(\mathtt{X})$ to be defined as ***the least*** $\mathtt{Y}$ in that set with $\mathtt{X} \subseteq \mathtt{Y}$. Thus we can make the theorem into an ***operator*** by the definition:

$$\mathbf{Clos} = \lambda \mathtt{F}.\lambda \mathtt{X}.\mathbf{Y}(\lambda \mathtt{Y}.(\mathtt{X} \cup \mathtt{F}(\mathtt{Y}))).$$

Theorem. $\mathbf{Clos}(\mathbf{Clos}) = \mathbf{Clos}.$

Using Fixed Points of Closures

Definition. In the category of closure operators, define:

$$\mathbf{D} = \mathbf{Y}(\lambda \mathtt{D}.\,\mathbf{Clos}(\mathtt{D} \to \mathbf{I})).$$

Theorem. Because $\mathbf{D} = \mathbf{D} \to \mathbf{I}$ and $\mathbf{I} = \mathbf{I} \times \mathbf{I}$, we have

$$\mathbf{D} \cong \mathbf{D} \times \mathbf{D} \quad \text{and} \quad \mathbf{D} \cong \mathbf{D} \to \mathbf{D},$$

where we invoke the idea of ***isomorphism***
in the cartesian closed category of closure operators.

Conclusion: The range of D can be made into a model of *all* the three λ-calculus axioms along with a *surjective pairing*.

An Easier Model?

Definition. In the category of closure operators, define:

$$\mathbf{D} \;=\; \mathbf{Y}(\lambda \mathtt{D}.\mathbf{Clos}(\mathtt{D} \to \mathtt{D})).$$

Exercise: Show that $\mathbf{D}(\varnothing) = \varnothing$ and $\mathbf{D}(\mathbb{N}) = \mathbb{N}$.

Conclusion: Because we have D = (D → D),

and because the range of D is *non-trivial*,

the range of D is very directly (without isomorphisms)

a model of *all* the three λ-calculus axioms.