HONOURS PROJECT PRESENTATION

UNDER SUPERVISION OF DR. A SWARTZ

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INTRODUCTION

The two final subjects that I had in my part time honours course was functional analysis from the pure and partial differential equations from the applied mathematics department. It was fun to study fractional calculus as a project because it seemed like it shared many overlapping traits with both the aforementioned subjects.

It dealt mostly with differentials and integrals as operators. In this brochure I basically just want to discuss the motivation which led me to want to do the project and elaborate on the theory in writing. At first it might seem kind of like a magic trick, or as mathematicians would put it, "hand-wavy", but the backbone of this phenomenon has been and is currently being widely studied and documented.

MOTIVATION

Consider the binomial

$$(x+y)^n$$

We are all well familiar with its expansion methods. So let's go ahead and do this:

$$\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x^2+2xy+y^2\\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\
(x+y)^4 &= x^4+4x^3y+\cdots\\
&\vdots
\end{aligned}$$

This is where I would like to stop and ask a question: Without actually knowing what pascal's triangle looks like or performing combinatorial calculations, is there any way to determine the third term and the terms that follow it in the last row?

Well, here's an informal description of a method:

We would start with the the first term x^4, and then differentiate it to get $\frac{d}{dx}x^4 = 4x^3$, after which we will multiply it by the definite integral $\int_0^y ds_1 = y$, so that the second term is given by $(4x^3)(y) = 4x^3y$.

To obtain the third term in this row, we would perform the above operation twice; i.e. differentiate x^4 twice to get $\frac{d^2}{dx^2}x^4 = 12x^2$ and multiply it by the double integral $\int_0^y\int_0^y dsds = \frac{y^2}{2}$ to get $(12x^2)\left(\frac{y^2}{2}\right) = 6x^2y^2$

And so the fourth and fifth terms are given by:

$$\left(\frac{d^3}{dx^3}x^4\right)\left(\int_0^y\int_0^y\int_0^y ds\ ds\ ds\right) = (24x)\left(\frac{y^3}{6}\right) = 4xy^3$$

and

$$\left(\frac{d^4}{dx^4}x^4\right)\left(\int_0^y\int_0^y\int_0^y\int_0^y ds\ ds\ ds\ ds\right) = (24)\left(\frac{y^4}{24}\right) = y^4$$

by putting all these terms in their respective places, we obtain

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

Notice that the coefficients work out along the way.

If we generalize this by use of a formula, we get the following:

$$(x+y)^n = x^n+\sum_{k=1}^{n}\frac{d^k}{dx^k}x^n\cdot\frac{1}{\Gamma(k)}\int_0^y (y-t)^{k-1}\,dt$$

This is not surprising, since in the fractional calculus, the Taylor series expansion for a function f about a point a is given by

$$f(x) = f(a)+\sum_{k>0}\left(\frac{d^k}{dx^k}f(x)\right)_{x=a}\left(\frac{1}{\Gamma(k)}\int_a^x (x-t)^{k-1}dt\right)$$