Fractional Taylor Series of exp, sin, cos

We demonstrate the fractional Taylor series approximation of three fundamental functions and demonstrate that our series expansion need not be initiated at an integer number for *k*.

Exp

$$\text{Manipulate}\Big[\text{Expand}\Big[\{\text{D[E^x, \{x, 0\}] /. \{x}\to 0\}\} + \sum_{k=1}^{n}\left(\frac{1}{\text{Gamma}[k]} * \int_{0}^{y}\{\text{D[E^x, \{x, k\}] /. \{x}\to 0\}\} * (y-t)^{k-1}\,dt\right)\Big], \{n, 0, 10, 1\}\Big]$$

n

$$\left\{1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}\right\}$$

Sin

$$\text{Manipulate}\Big[\text{Expand}\Big[\{\text{D[Sin[x], \{x, 0\}] /. \{x}\to 0\}\} + \sum_{k=1}^{n}\left(\frac{1}{\text{Gamma}[k]} * \int_{0}^{y}\{\text{D[Sin[x], \{x, k\}] /. \{x}\to 0\}\} * (y-t)^{k-1}\,dt\right)\Big], \{n, 0, 20, 1\}\Big]$$

n

$$\left\{y - \frac{y^3}{6} + \frac{y^5}{120} - \frac{y^7}{5040} + \frac{y^9}{362\,880}\right\}$$

cos

$$\text{Manipulate}\Big[\text{Expand}\Big[\{\text{D[Cos[x], \{x, 0\}] /. \{x}\to 0\}\} + \sum_{k=1}^{n}\left(\frac{1}{\text{Gamma}[k]} * \int_{0}^{y}\{\text{D[Cos[x], \{x, k\}] /. \{x}\to 0\}\} * (y-t)^{k-1}\,dt\right)\Big], \{n, 0, 20, 1\}\Big]$$

n

$$\left\{1 - \frac{y^2}{2} + \frac{y^4}{24} - \frac{y^6}{720} + \frac{y^8}{40\,320} - \frac{y^{10}}{3\,628\,800} + \frac{y^{12}}{479\,001\,600}\right\}$$

Non-integer Summation

Up to now, k was an integer number and we expanded the terms in integer steps. We demonstrate the semi-series expansion of cos (x)

This means that the series expansion of cos (x) will be initiated at $k=\frac{1}{2}$ and each new term will be obtained by counting in $\frac{1}{2}$ steps

$$\texttt{Manipulate}\Big[\texttt{Expand}\Big[\big(\texttt{D[Cos[x], \{x, 0\}] /. \{x} \to \texttt{0\}}\big) + \sum_{k=\frac{1}{2}}^{n}\left(\frac{1}{\texttt{Gamma[k]}} * \int_0^x \left(\texttt{Cos}\left[x+\frac{\texttt{Pi k}}{2}\right] \texttt{ /. \{x} \to \texttt{0\}}\right) * (x-t)^{k-1}\,dt\right)\Big], \left\{n, 0, 20, \frac{1}{2}\right\}\Big]$$

n

$$1+\sqrt{\frac{2}{\pi}}\sqrt{x}-\frac{2}{3}\sqrt{\frac{2}{\pi}}\,x^{3/2}-\frac{4}{15}\sqrt{\frac{2}{\pi}}\,x^{5/2}+\frac{8}{105}\sqrt{\frac{2}{\pi}}\,x^{7/2}+\frac{16}{945}\sqrt{\frac{2}{\pi}}\,x^{9/2}-\frac{32\sqrt{\frac{2}{\pi}}\,x^{11/2}}{10\,395}-\frac{64\sqrt{\frac{2}{\pi}}\,x^{13/2}}{135\,135}+\frac{128\sqrt{\frac{2}{\pi}}\,x^{15/2}}{2\,027\,025}+\frac{256\sqrt{\frac{2}{\pi}}\,x^{17/2}}{34\,459\,425}-\frac{512\sqrt{\frac{2}{\pi}}\,x^{19/2}}{654\,729\,075}-\frac{1024\sqrt{\frac{2}{\pi}}\,x^{21/2}}{13\,749\,310\,575}$$

$$\text{Manipulate}\Big[\text{Expand}\Big[\big(\text{D}[\text{Sin}[x],\{x,0\}]\ /.\ \{x\to 0\}\big)+\sum_{k=\frac{1}{2}}^{n}\left(\frac{1}{\text{Gamma}[k]}*\int_0^x\left(\text{Sin}\left[x+\frac{\text{Pi}\,k}{2}\right]\ /.\ \{x\to 0\}\right)*(x-t)^{k-1}\,dt\right)\Big],\ \left\{n,0,20,\frac{1}{2}\right\}\Big]$$

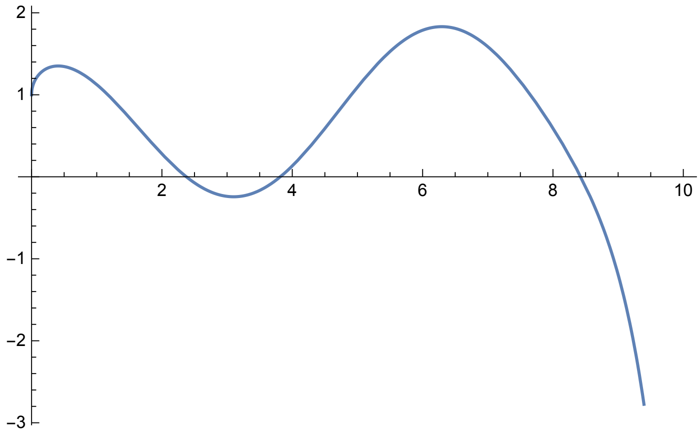
n

$$\sqrt{\frac{2}{\pi}}\sqrt{x}+\frac{2}{3}\sqrt{\frac{2}{\pi}}\,x^{3/2}-\frac{4}{15}\sqrt{\frac{2}{\pi}}\,x^{5/2}-\frac{8}{105}\sqrt{\frac{2}{\pi}}\,x^{7/2}+\frac{16}{945}\sqrt{\frac{2}{\pi}}\,x^{9/2}+\frac{32\sqrt{\frac{2}{\pi}}\,x^{11/2}}{10\,395}-$$
$$\frac{64\sqrt{\frac{2}{\pi}}\,x^{13/2}}{135\,135}-\frac{128\sqrt{\frac{2}{\pi}}\,x^{15/2}}{2\,027\,025}+\frac{256\sqrt{\frac{2}{\pi}}\,x^{17/2}}{34\,459\,425}+\frac{512\sqrt{\frac{2}{\pi}}\,x^{19/2}}{654\,729\,075}-\frac{1024\sqrt{\frac{2}{\pi}}\,x^{21/2}}{13\,749\,310\,575}-$$
$$\frac{2048\sqrt{\frac{2}{\pi}}\,x^{23/2}}{316\,234\,143\,225}+\frac{4096\sqrt{\frac{2}{\pi}}\,x^{25/2}}{7\,905\,853\,580\,625}+\frac{8192\sqrt{\frac{2}{\pi}}\,x^{27/2}}{213\,458\,046\,676\,875}-\frac{16\,384\sqrt{\frac{2}{\pi}}\,x^{29/2}}{6\,190\,283\,353\,629\,375}-$$
$$\frac{32\,768\sqrt{\frac{2}{\pi}}\,x^{31/2}}{191\,898\,783\,962\,510\,625}+\frac{65\,536\sqrt{\frac{2}{\pi}}\,x^{33/2}}{6\,332\,659\,870\,762\,850\,625}+\frac{131\,072\sqrt{\frac{2}{\pi}}\,x^{35/2}}{221\,643\,095\,476\,699\,771\,875}-$$
$$\frac{262\,144\sqrt{\frac{2}{\pi}}\,x^{37/2}}{8\,200\,794\,532\,637\,891\,559\,375}-\frac{524\,288\sqrt{\frac{2}{\pi}}\,x^{39/2}}{319\,830\,986\,772\,877\,770\,815\,625}$$

Graphs

cos

$$\text{Plot}\Big[1+\sqrt{\tfrac{2}{\pi}}\sqrt{x}-\frac{2}{3}\sqrt{\tfrac{2}{\pi}}x^{3/2}-\frac{4}{15}\sqrt{\tfrac{2}{\pi}}x^{5/2}+\frac{8}{105}\sqrt{\tfrac{2}{\pi}}x^{7/2}+\frac{16}{945}\sqrt{\tfrac{2}{\pi}}x^{9/2}-\frac{32\sqrt{\frac{2}{\pi}}x^{11/2}}{10395}-\frac{64\sqrt{\frac{2}{\pi}}x^{13/2}}{135135}+\frac{128\sqrt{\frac{2}{\pi}}x^{15/2}}{2027025}+\frac{256\sqrt{\frac{2}{\pi}}x^{17/2}}{34459425}-\frac{512\sqrt{\frac{2}{\pi}}x^{19/2}}{654729075}-\frac{1024\sqrt{\frac{2}{\pi}}x^{21/2}}{13749310575}+\frac{2048\sqrt{\frac{2}{\pi}}x^{23/2}}{316234143225}+\frac{4096\sqrt{\frac{2}{\pi}}x^{25/2}}{7905853580625}-\frac{8192\sqrt{\frac{2}{\pi}}x^{27/2}}{213458046676875}-\frac{16384\sqrt{\frac{2}{\pi}}x^{29/2}}{6190283353629375}+\frac{32768\sqrt{\frac{2}{\pi}}x^{31/2}}{191898783962510625}+\frac{65536\sqrt{\frac{2}{\pi}}x^{33/2}}{6332659870762850625}-\frac{131072\sqrt{\frac{2}{\pi}}x^{35/2}}{221643095476699771875}-\frac{262144\sqrt{\frac{2}{\pi}}x^{37/2}}{8200794532637891559375}+\frac{524288\sqrt{\frac{2}{\pi}}x^{39/2}}{319830986772877770815625},\ \{x,0,10\}\Big]$$



sin

$$\text{Plot}\Big[\sqrt{\frac{2}{\pi}}\sqrt{x}+\frac{2}{3}\sqrt{\frac{2}{\pi}}x^{3/2}-\frac{4}{15}\sqrt{\frac{2}{\pi}}x^{5/2}-\frac{8}{105}\sqrt{\frac{2}{\pi}}x^{7/2}+\frac{16}{945}\sqrt{\frac{2}{\pi}}x^{9/2}+\frac{32\sqrt{\frac{2}{\pi}}x^{11/2}}{10\,395}-\frac{64\sqrt{\frac{2}{\pi}}x^{13/2}}{135\,135}-\frac{128\sqrt{\frac{2}{\pi}}x^{15/2}}{2\,027\,025}+\frac{256\sqrt{\frac{2}{\pi}}x^{17/2}}{34\,459\,425}+\frac{512\sqrt{\frac{2}{\pi}}x^{19/2}}{654\,729\,075}-\frac{1024\sqrt{\frac{2}{\pi}}x^{21/2}}{13\,749\,310\,575}-\frac{2048\sqrt{\frac{2}{\pi}}x^{23/2}}{316\,234\,143\,225}+\frac{4096\sqrt{\frac{2}{\pi}}x^{25/2}}{7\,905\,853\,580\,625}+\frac{8192\sqrt{\frac{2}{\pi}}x^{27/2}}{213\,458\,046\,676\,875}-\frac{16\,384\sqrt{\frac{2}{\pi}}x^{29/2}}{6\,190\,283\,353\,629\,375}-\frac{32\,768\sqrt{\frac{2}{\pi}}x^{31/2}}{191\,898\,783\,962\,510\,625}+\frac{65\,536\sqrt{\frac{2}{\pi}}x^{33/2}}{6\,332\,659\,870\,762\,850\,625}+\frac{131\,072\sqrt{\frac{2}{\pi}}x^{35/2}}{221\,643\,095\,476\,699\,771\,875}-\frac{262\,144\sqrt{\frac{2}{\pi}}x^{37/2}}{8\,200\,794\,532\,637\,891\,559\,375}-\frac{524\,288\sqrt{\frac{2}{\pi}}x^{39/2}}{319\,830\,986\,772\,877\,770\,815\,625},\ \{x,0,10\}\Big]$$