Fractional Calculus

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$\boxed{\text{Introduction}}$

- Operators
- Semi-Group
- Interpolation Formulae

1. Interpolation Formulae...

2. Semi-Group. . .

Interpolation Formulae

Apply Successively

Recursively Defined

Semi-Group

Satisfies Scheme:

1. Closure
2. Associativity
3. Identity
4. Inverse (conditionally)

D and $\int$ not necessarily (left or right) inverses

1. Study consequences of interchanging D & $\int$

2. Study non-integer application of D & $\int$

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1. Gives rise to different formalisms

   - Riemann-Liouville

   - Caputo and Others

   - Restrict our attention to Riemann-Liouville

2. Gives rise to Interpolation Formulae

Formalisms

Riemann-Liouville	**Caputo**
Simpler Formulae	Formulae "expand"
Analytic Methods	Numerical Methods

Note:

1. Riemann-Liouville takes derivative on the left

2. Caputo takes derivative on the right.

$\boxed{\text{Definitions}}$

Definition 1. *Let $n \in \mathbb{N}$. Then we define the factorial function as*

$$\begin{aligned} n! &= \prod_{k=1}^{n} k \\ n! &= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot (n) \end{aligned}$$

Definition 2. *The gamma function, $\Gamma(n)$ is an extension of the factorial function to complex arguments related by*

$$\Gamma(n) = (n-1)!$$

It is defined as an improper definite integral for complex arguments $z \in \mathbb{C}$ having positive real part, $\Re(z) > 0$, as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Given $t \in \mathbb{R}^+$ and $z \in \mathbb{C}$, the power t^z is defined by

$$t^z = e^{z \ln(b)}$$

Definition 3. *The Beta function is defined as the definite integral*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt$$

where $\Re(z) > 0$ *and* $\Re(w) > 0$

Definition 4.

Let $\Omega = [a,b]$ *where* $-\infty < a < b < \infty$ *be a finite closed interval on the Real axis* $\mathbb{R}$. *The left-sided and right-sided Riemann-Liouville fractional integrals* ${}_{a+}J^{\alpha}f$ *and* ${}_{b-}J^{\alpha}f$ *of order* $\alpha \in \mathbb{R}$ *based at* a *and* b *respectively is defined*

$${}_{a+}J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \ ,(x>a)$$

and

$${}_{b-}J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt \ ,(x<b)$$

Definition 5. *The left-sided and right-sided Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{R}$ based at a and b, respectively, is defined as*

$$D_{a+}^{\alpha} f(x) = \left(\frac{d}{dx}\right)^m \left[J_{a+}^{(m-\alpha)} f(x)\right], x > a$$

$$D_{b-}^{\alpha} f(x) = \left(-\frac{d}{dx}\right)^m \left[J_{b-}^{(m-\alpha)} f(x)\right], x < b$$

where $m = \lceil \alpha \rceil$

Definition 6. *The left-sided and right-sided Caputo fractional derivatives of order* $\alpha \in \mathbb{R}$ *based at a and b, respectively, is defined as*

$$D^{\alpha}_{a+}f(x) = J^{(m-\alpha)}_{a+}\left[\left(\frac{d}{dx}\right)^{m} f(x)\right], x > a$$

$$D^{\alpha}_{b-}f(x) = J^{(m-\alpha)}_{b-}\left[\left(-\frac{d}{dx}\right)^{m} f(x)\right], x < b$$

where $m = \lceil \alpha \rceil$

integration second

$\Longrightarrow$ gives rise to transient terms

$\Longrightarrow$ provides clue to define Taylor series

Examples

Notation

Riemann-Liouville definition forces us to write

1. $D^{\frac{1}{3}}f(x) = D^1 D^{-\frac{2}{3}}f(x)$

2. $D^{\frac{4}{3}}f(x) = D^2 D^{-\frac{2}{3}}f(x)$

- Avoids computation of $\Gamma(p)$

- Fractional Derivative well defined

$\boxed{\text{La Croix' semi-derivative}}$

In 1819 La Croix used Euler's (1729) Gamma function and

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

to derive

$$\boxed{\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} t = 2\sqrt{\frac{t}{\pi}}} \text{ and } \boxed{\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} t^0 = \frac{1}{\sqrt{\pi t}}, \text{ not } 0}$$

General case, $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}$

$$\begin{aligned}
\frac{d^\alpha}{dt^\alpha} t^\lambda &= D^m \left[J^{m-\alpha} t^\lambda \right] \\
&= D^m \left[\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + m - \alpha + 1)} t^{\lambda + m - \alpha} \right] \\
&= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda - \alpha}
\end{aligned}$$

Result independent of m

Substitute $\lambda = 1$ and $\alpha = \frac{1}{2}$ to obtain

$$D^{\frac{1}{2}}t = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}t^{1-\frac{1}{2}} = 2\sqrt{\frac{t}{\pi}}$$

Interestingly, applying semi-derivative again yields expected result

$$\begin{aligned} D^{\frac{1}{2}}D^{\frac{1}{2}}t &= \frac{2}{\sqrt{\pi}}D^{\frac{1}{2}}\sqrt{t} \\ &= \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}+1)}t^{\frac{1}{2}-\frac{1}{2}} \\ &= \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} \\ &= 1 \end{aligned}$$

To compute $D^{\frac{1}{2}}$ of constant function $f(t) = c$, we use interpolation formula

$$\frac{d^\alpha}{dt^\alpha}\ c\ t^\lambda = c\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$$

and apply the limit as $\lambda \to 0$ to obtain

$$\begin{aligned} D^\alpha\ c &= \lim_{\lambda\to 0}\ D^\alpha\ c\ t^\lambda \\ &= c\ \lim_{\lambda\to 0}\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha} \\ &= \frac{c}{\Gamma(1-\alpha)} t^{-\alpha} \end{aligned}$$

Letting $\alpha = \frac{1}{2}$, we obtain $D^{\frac{1}{2}}\ c = \frac{c}{\sqrt{\pi t}}$

Properties

Previous examples relied on

- Linearity of D and $\int$
- Composition of D and $\int$

Linearity

Let λ, μ be scalars and fix $\alpha \in \mathbb{R}$. Given real valued functions f and g, we have

L1. $J^{\alpha}(\lambda\ f + \mu\ g)(x) = \lambda\ J^{\alpha} f(x) + \mu\ J^{\alpha} g(x)$

L2. $D^{\alpha}(\lambda\ f + \mu\ g)(x) = \lambda\ D^{\alpha} f(x) + \mu\ D^{\alpha} g(x)$

Composition

Fix $\alpha > \beta > 0$. For J and D we have the following composition rules

C1. $J^{\alpha}J^{\beta}f(x) = J^{\alpha+\beta}f(x) = J^{\beta}J^{\alpha}f(x)$

C2. $D^{\alpha}D^{\beta}f(x) = D^{\alpha+\beta}f(x) = D^{\beta}D^{\alpha}f(x)$

C3. $D^{\alpha}J^{\beta}f(x) = D^{\alpha}D^{-\beta}f(x) = D^{\alpha-\beta}f(x)$

C4. $J^{\alpha}D^{\beta}f(x) = D^{-\alpha}D^{\beta}f(x) = D^{-\alpha+\beta}f(x)$

if and only if $0 = f(a) = f'(a) = \ldots$

Proof Strategies and Remarks

L1. follows easily from the definition of the differ-integral operator and the properties of integer-order integration

L2. follows from L1 and the properties of the integer-order differentiation operator

C3. motivates the Riemann-Liouville formalism

C4. motivates the Caputo formalism

C1. We discuss a proof-outline of C1 from which C2 follows as corollary

$\boxed{\text{Proof Outline of C1}}$

Write the expression for $\left(J^{\alpha}J^{\beta}f\right)(x)$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_a^t(x-t)^{\alpha-1}(t-s)^{\beta-1}\ f(s)\ ds\ dt$$

The double integral taken over the region
$R = \{(s,t)|\ a \le s \le t$ and $a \le t \le x\}$

Change the order of integration

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_s^t\int_a^x(x-t)^{\alpha-1}(t-s)^{\beta-1}\ f(s)\ dt\ ds$$

Region of integration expressible as
$R' = \{(s,t)|\ s \le t \le x$ and $a \le s \le t\}$

- change of variables $t = s + (x - s)r$
- manipulation with beta function

$$
\begin{aligned}
& \left(J^{\alpha} J^{\beta} f\right)(x) \\
= \; & \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-s)^{\alpha+\beta-1} \; f(s) \; ds \\
= \; & \left(J^{\alpha+\beta} f\right)(x)
\end{aligned}
$$

$$
\left(J^{\alpha} J^{\beta} f\right)(x) = \left(J^{\alpha+\beta} f\right)(x) = \left(J^{\beta} J^{\alpha} \; f\right)(x)
$$

Equally suitable alternative formalisms

Variations

Grünwald-Letnikov

- Riemann-Liouville and Caputo derives differ-integral from repeated $\int$

- Grünwald-Letnikov derives differ-integral from repeated D

Definition 7. *Let f be a real valued function defined in an open neighbourhood of $a \in \mathbb{R}$*

If the limit exists, the classical definition of the derivative is defined as the difference quotient

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Applying the definition to $f'(x)$ we obtain

$f''(x)$

$\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$

$\lim_{h\to 0} \frac{\lim_{h\to 0} \frac{f(x+h+h)-f(x+h)}{h} - \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}{h}$

$\lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$

Recursively, after n differentiations, we obtain

$$\frac{d^n}{dx^n} f(x) = \lim_{h\to 0} \frac{1}{h^q} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x-mh)$$

- replace binomial coefficients with gamma coefficients

- $\frac{t-a}{h} \to \infty$ as $h \to 0$

$$\frac{d^q}{dx^q} f(x) = \lim_{h \to 0} \frac{1}{h^q} \sum_{m=0}^{\frac{t-a}{h}} \frac{(-1)^m \, \Gamma(q+1)}{m! \; \Gamma(q-m+1)} \, f(x-mh)$$

Grünwald-Letnikov non-integer order integral defined by replacing order of differentiation q by $-q$ expressible as

$$\frac{d^{-q}}{dx^{-q}} f(x) = \lim_{h \to 0} h^q \sum_{m=0}^{\frac{t-a}{h}} \frac{(-1)^m \, \Gamma(q+m)}{m! \; \Gamma(q)} \, f(x-mh)$$

Grünwald-Letnikov formalism is equivalent to the Riemann-Liouville formalism

(proof is lengthy)

Cauchy Integral Formula

Suppose that the Riemann-Liouville fractional derivative is defined from the Riemann-Liouville fractional integral by replacing α, the order of integration, by $-\alpha$, to denote differentiation

$$D^{\alpha}f(x) = J^{-\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_a^x (x-t)^{-\alpha-1}f(t)dt$$

Reconcile this formula with Cauchy's Integral formula from complex analysis

$$\frac{d^n}{dz^n}f(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

where C is a closed contour surrounding point z and enclosing a region of analyticity of f

1. Replace positive integer n by non-integer q

2. $(\zeta - z)^{-q-1}$ has a branch cut rather than a pole at $\zeta = z$

define in the quadrant $\Re(\zeta) \leq 0$ and $\Im(\zeta) \leq 0$

$$\frac{d^q}{dz^q} f(z) = \frac{\Gamma(q+1)}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{q+1}} d\zeta$$

- C initiated and terminated at $\zeta = 0$

- Deform C to obtain (Hankel) contour C' about z

- Introduce change of variables

$$(\zeta - z) \rightarrow (\zeta - z) e^{2\pi z i}$$

$$\frac{d^q}{dz^q}f(z) = \frac{\Gamma(q+1)}{2\pi i}\left(1 - e^{-2\pi i(q+1)}\right)\int_0^z \frac{f(\zeta)}{(\zeta-z)^{q+1}}d\zeta$$

$$= \Gamma(q+1)\frac{e^{-\pi i(q+1)}}{\pi}\frac{e^{\pi i(q+1)}-e^{-\pi i(q+1)}}{2i}\int_0^z \frac{f(\zeta)d\zeta}{(\zeta-z)^{q+1}}$$

$$= \Gamma(q+1)\frac{(-1)^{-(q+1)}}{\pi}\sin((q+1)\pi)\int_0^z \frac{f(\zeta)d\zeta}{(\zeta-z)^{q+1}}$$

$$= \frac{\pi}{\sin((q+1)\pi)\Gamma(-q)}\frac{(-1)^{-(q+1)}}{\pi}\sin((q+1)\pi)\int_0^z \frac{f(\zeta)d\zeta}{(\zeta-z)^{q+1}}$$

$$= \frac{1}{\Gamma(-q)}\int_0^z \frac{f(\zeta)d\zeta}{(\zeta-z)^{q+1}}$$

Substituted $z = q + 1$ in symmetry formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)} \text{ where } 0 \leq \Re(z) \leq 1$$

Laplace Transform

Riemann-Liouville's integral can be defined in terms of the Laplace transform

Definition 8. *The Laplace transform of a function $f(t)$ defined for $t \in \mathbb{R}^+$ is the function $\mathcal{L}\{f(t)\} = F(s)$ defined for complex $s = \sigma + i\omega$ for which the improper integral converges*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\ e^{-st}\ dt$$

The non-integer order integral operator could also have been defined in terms of the Laplace transform as

$$J^\alpha f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^\alpha}\mathcal{L}\{f(t)\}\right)$$

- Derived using the convolution product

- Used as an alternative for computation, demonstrated next

using $\mathcal{L}\{t^\lambda\} = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}$ we compute $J^\alpha t^\lambda$

$$\begin{aligned} J^\alpha t^\lambda &= \mathcal{L}^{-1}\left(\frac{1}{s^\alpha}\mathcal{L}\{t^\lambda\}\right) \\ &= \mathcal{L}^{-1}\left(\frac{\Gamma(\lambda+1)}{s^{\alpha+\lambda+1}}\right) \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}t^{\alpha+\lambda} \end{aligned}$$

corresponds with previous result

$\boxed{\text{Taylor Series}}$

Non-integer order Caputo derivative of a function $f(x)$ is given by

$$D^{\alpha} f(x) = J^{(m-\alpha)} \left[\left(\frac{d}{dx} \right)^{m} f(x) \right]$$

where $m = \lceil \alpha \rceil$

integration second

$\Longrightarrow$ gives rise to transient terms

$\Longrightarrow$ provides clue to define Taylor series

Definition 9. *The fractional Taylor series of a function $f(x)$ centred at a point a is given by*

$$f(x) = f(a) + \sum_{k>0} {}_{a}J_{x}^{k} \left({}_{a}D_{x}^{k} f(x) \right)$$

The formula in definition 9 is expressible as

$$f(x) = f(a) + \sum_{k>0} \left(\frac{d^k}{dx^k} f(x) \right)_{x=a} \frac{1}{\Gamma(k)} \int_a^x (x-t)^{k-1} \, dt$$

since $\left(\frac{d^k}{dx^k} f(x) \right)_{x=a}$ is constant for all k

- $f(a)$ cannot be included in the summation since integral for $k = 0$ involves computation of $\Gamma(0)$ which is undefined

- Demonstrate the power of the fractional Taylor series by expanding the function $f(x) = (x + y)^n$ about the point $a = 0$ and reconcile it with the classical binomial expansion

Notice that $f(0) = (0 + y)^n = y^n$ and

$\left(\frac{d^k}{dx^k}(x+y)^n\right)_{a=0} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)} y^{n-k}$

so that we get

$$\begin{aligned} & f(x) \\ = \ & y^n + \sum_{k=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} y^{n-k} \left(\frac{1}{\Gamma(k)} \int_0^x (x-t)^{k-1} dt \right) \\ = \ & y^n + \sum_{k=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} y^{n-k} \frac{x^k}{\Gamma(k+1)} \\ = \ & y^n + \sum_{k=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} x^k y^{n-k} \\ = \ & y^n + \sum_{k=1}^{\infty} \binom{n}{k} x^k y^{n-k} \\ = \ & \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k} \end{aligned}$$

which is the concise expression for the classical binomial expansion as we know it

We illustrate the fractional Taylor series expansion for the three functions $\sin x$, $\cos x$ and $\exp x$

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Refer to Mathematica Notebook

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The value of k in the summation need not be an integer number, but may also assume non-integer values as demonstrated

$\boxed{\text{Tautochrone Problem}}$

In words, the tautochrone problem states:

Determine the curve in the (x, y)-plane such that the time required for the particle with mass m, to slide down the curve to its lowest point (ignoring friction), is a minimum, independent of its initial placement on the curve

Parameters and Variables:

(X, Y) initial placement of the particle
$(0, 0)$ initial placement of the particle
(x, y) intermediate placement of the particle
σ arc length measured from the origin $(0, 0)$
m mass of the particle
g gravitational acceleration constant
t independent variable, time

Conservation of energy law from physics:

$\boxed{\textit{Gain in kinetic energy}}$

$\boxed{\textit{is equal to}}$

$\boxed{\textit{loss of potential energy}}$

In an equation, this becomes

$$\begin{aligned}
&\Longrightarrow \quad \frac{1}{2}m\left(\frac{d\sigma}{dt}\right)^2 = mg(Y-y) \\
&\Longrightarrow \quad \left(\frac{d\sigma}{dt}\right)^2 = 2g(Y-y) \\
&\Longrightarrow \quad \frac{d\sigma}{dt} = \pm\sqrt{2g(Y-y)} \\
&\Longrightarrow \quad d\sigma = -\sqrt{2g(Y-y)}dt
\end{aligned}$$

Since $\frac{d\sigma}{dt} < 0$ because the arc length decreases as the particle slides down its path.

σ is a function of height y which is in turn dependent on time t so that $\sigma = \sigma(y(t))$. By the chain rule we have

$$d\sigma = \frac{d\sigma}{dy} dy$$

Setting $\sigma^{(1)}(y) = \frac{d\sigma}{dy}$ we get the separable ordinary differential equation

$$\frac{d\sigma}{\sqrt{Y-y}} = -\sqrt{2g}dt \implies \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = -\sqrt{2g}dt$$

- Integrating from $y = Y$ to $y = 0$

- Corresponds to $t_Y = 0$ and $t_0 = T$

$$\int_Y^0 \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = -\int_0^T \sqrt{2g}dt$$

$$\Longrightarrow \int_0^Y \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = \sqrt{2g}\ T$$

The left hand side is expressible using the convolution product

$$\int_0^Y \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = y^{-\frac{1}{2}} * \sigma^{(1)} \Longrightarrow \sqrt{2g}\, T = y^{-\frac{1}{2}} * \sigma^{(1)}$$

- Apply the Laplace transform to both sides

- Applying the convolution theorem

$$\frac{1}{s}\sqrt{2g}\ T = \sqrt{\frac{\pi}{s}}\mathcal{L}\left(\frac{d\sigma}{dy}\right)$$

- Applying the Inverse Laplace Transform

- Solving for $\frac{d\sigma}{dy}$

$$\frac{d\sigma}{dy} = \frac{\sqrt{2g}\ T}{\pi}\ y^{-\frac{1}{2}}$$

Let us also write $f(Y) = \sqrt{2g}\ T$, then we finally obtain Abel's solution to the Tautochrone problem

$$\sigma(y) = \frac{1}{\pi}\ \int_0^y \frac{f(Y)}{\sqrt{y-Y}}\ dY$$

expressed in terms of the height y along the curve.

Connection between Abel's solution and semi-integral of $\sigma^{(1)}$:

$$J^{\frac{1}{2}}\sigma^{(1)}(y) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^Y (Y-y)^{\frac{1}{2}-1}\sigma^{(1)}(y)\ dy$$

$$\Longrightarrow \quad J^{\frac{1}{2}}\sigma^{(1)}(y) = \frac{1}{\sqrt{\pi}}\int_0^Y \frac{\sigma^{(1)}(y)}{\sqrt{Y-y}}\ dy$$

$$\Longrightarrow \quad \sqrt{\pi}\ J^{\frac{1}{2}}\sigma^{(1)}(y) = \int_0^Y \frac{\sigma^{(1)}(y)}{\sqrt{Y-y}}\ dy$$

$$\implies \quad f(y) = \sqrt{\pi}\ J^{\frac{1}{2}}\sigma^{(1)}(y)$$

$$\implies \quad J^{\frac{1}{2}}f(y) = \sqrt{\pi}\ J^{\frac{1}{2}}J^{\frac{1}{2}}\sigma^{(1)}(y)$$

$$\implies \quad \sigma(y) = \frac{1}{\sqrt{\pi}}J^{\frac{1}{2}}f(y)$$

$$\implies \quad \sigma(y) = \frac{1}{\sqrt{\pi}}\ \frac{1}{\sqrt{\pi}}\int_0^y \frac{f(Y)}{\sqrt{Y-y}}\ dy$$

$$\implies \quad \sigma(y) = \frac{1}{\pi}\ \int_0^y \frac{f(Y)}{\sqrt{Y-y}}\ dy$$

$$\implies \quad \sigma(y) = \frac{1}{\pi}\ \int_0^y \frac{\sqrt{2g}\ T}{\sqrt{Y-y}}\ dy$$

Corresponds to Abel's solution of the Tautochrone problem

- Avoids use of Laplace transforms

- Avoids evaluation of complicated integrals

Future Directions

Connection to Functional Analysis

Behaviour of non-integer order operators through scope of functional analysis, considering algebraic behaviour

1. Elegance of algebraic properties

2. Solvability of operator equations

Discuss Fredholm's linear integral equations generalized to normed spaces

Let X be a normed space over the complex field and $0 \neq \lambda \in \mathbb{C}$.

Consider the compact linear operator $T : X \to X$ and the adjoint operator $T^{\times} : X' \to X'$. Let $y \in X$ and $g \in X'$

The four equations below are of interest:

A1: $Tx - \lambda x = y$

A2: $Tx - \lambda x = 0$

B1: $T^{\times}x - \lambda x = g$

B2: $T^{\times}x - \lambda x = 0$

The solvability of these equations are summarised as follows:

1. A1 has a solution x iff $f(y) = 0$ for all solutions f of B2.
   Hence if $f = 0$ is the only solution of B2, then the equation A1 is solvable for all y

2. B1 has a solution f iff $g(x) = 0$ for all solutions x of A2.
   Hence if $x = 0$ is the only solution of A2, then equation B1 is solvable for all g

3. A1 has a solution x for every $y \in X$ iff $x = 0$ is the only solution of A2

4. B1 has a solution f for every $g \in X'$ iff $f = 0$ is the only solution of B2.

5. A2 and B2 have the same number of linearly independent solutions.

We invoke the question of how one can generalize the concept of solvability to suit equations involving non-integer order operators and what would be analogous criteria as stated above for such operators?

A reasonable approach to answer these type of questions could be achieved by generalizing the sequence spaces l^p and l^q in which the classical conjugate exponents equation

$$\frac{1}{p} + \frac{1}{q} = 1$$

no longer holds, but rather consider the relation between p and q to be such that in $\mathbb{R}^n$

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

and $1 \leq p < \frac{n}{\alpha}$ where α is the order of a linear operator from $l^p(\mu)$ and μ a measure.