

Fractional Taylor Series of \exp , \sin , \cos

We demonstrate the fractional Taylor series approximation of three fundamental functions and demonstrate that our series expansion need not be initiated at an integer number for k .

Exp

Manipulate[Expand[{D[E^x, {x, 0}] /. {x → 0}} +

$$\sum_{k=1}^n \left(\frac{1}{\Gamma[k]} * \int_0^y \{D[E^x, \{x, k\}] /. \{x \rightarrow 0\}\} * (y - t)^{k-1} dt \right)], \{n, 0, 10, 1\}]$$

n

$$\left\{ 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} \right\}$$

Sin

Manipulate[Expand[{D[Sin[x], {x, 0}] /. {x → 0}} +

$$\sum_{k=1}^n \left(\frac{1}{\Gamma[k]} * \int_0^y \{D[\sin[x], \{x, k\}] /. \{x \rightarrow 0\}\} * (y - t)^{k-1} dt \right)], \{n, 0, 20, 1\}]$$

n

$$\left\{ y - \frac{y^3}{6} + \frac{y^5}{120} - \frac{y^7}{5040} + \frac{y^9}{362880} \right\}$$

COS

Manipulate[Expand[{D[Cos[x], {x, 0}] /. {x → 0}} +

$$\sum_{k=1}^n \left(\frac{1}{\Gamma[k]} * \int_0^y \{D[\cos[x], \{x, k\}] /. \{x \rightarrow 0\}\} * (y - t)^{k-1} dt \right)], \{n, 0, 20, 1\}]$$

n

$$\left\{ 1 - \frac{y^2}{2} + \frac{y^4}{24} - \frac{y^6}{720} + \frac{y^8}{40320} - \frac{y^{10}}{3628800} + \frac{y^{12}}{479001600} \right\}$$

Non-integer Summation

Up to now, k was an integer number and we expanded the terms in integer steps. We demonstrate the semi-series expansion of $\cos(x)$

This means that the series expansion of $\cos(x)$ will be initiated at $k = \frac{1}{2}$ and each new term will be obtained by counting in $\frac{1}{2}$ steps

Manipulate[**Expand**[**D**[**Cos**[**x**], {**x**, **0**}] /. {**x** → **0**} +

$$\sum_{k=\frac{1}{2}}^n \left(\frac{1}{\text{Gamma}[k]} * \int_0^x \left(\text{Cos}\left[x + \frac{\text{Pi } k}{2}\right] \right) /. \{x \rightarrow 0\} * (x - t)^{k-1} dt \right), \{n, 0, 20, \frac{1}{2}\}]$$

n

$$1 + \sqrt{\frac{2}{\pi}} \sqrt{x} - \frac{2}{3} \sqrt{\frac{2}{\pi}} x^{3/2} - \frac{4}{15} \sqrt{\frac{2}{\pi}} x^{5/2} +$$

$$\frac{8}{105} \sqrt{\frac{2}{\pi}} x^{7/2} + \frac{16}{945} \sqrt{\frac{2}{\pi}} x^{9/2} - \frac{32}{10395} \sqrt{\frac{2}{\pi}} x^{11/2} - \frac{64}{135135} \sqrt{\frac{2}{\pi}} x^{13/2} +$$

$$\frac{128}{2027025} \sqrt{\frac{2}{\pi}} x^{15/2} + \frac{256}{34459425} \sqrt{\frac{2}{\pi}} x^{17/2} - \frac{512}{654729075} \sqrt{\frac{2}{\pi}} x^{19/2} - \frac{1024}{13749310575} \sqrt{\frac{2}{\pi}} x^{21/2}$$

Manipulate[Expand[(D[Sin[x], {x, 0}] /. {x → 0}) +

$$\sum_{k=\frac{1}{2}}^n \left(\frac{1}{\Gamma[k]} * \int_0^x \left(\sin\left[x + \frac{\pi k}{2}\right] \right) /. \{x \rightarrow 0\} * (x - t)^{k-1} dt \right)], \{n, 0, 20, \frac{1}{2}\}]$$

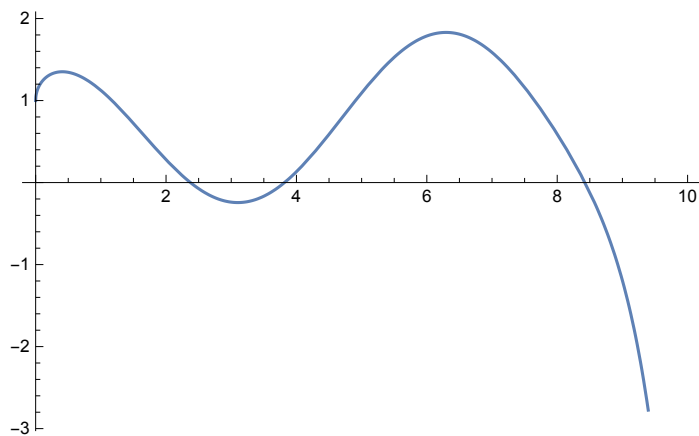
n

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \sqrt{x} + \frac{2}{3} \sqrt{\frac{2}{\pi}} x^{3/2} - \frac{4}{15} \sqrt{\frac{2}{\pi}} x^{5/2} - \frac{8}{105} \sqrt{\frac{2}{\pi}} x^{7/2} + \frac{16}{945} \sqrt{\frac{2}{\pi}} x^{9/2} + \frac{32 \sqrt{\frac{2}{\pi}} x^{11/2}}{10395} - \\ & \frac{64 \sqrt{\frac{2}{\pi}} x^{13/2}}{135135} - \frac{128 \sqrt{\frac{2}{\pi}} x^{15/2}}{2027025} + \frac{256 \sqrt{\frac{2}{\pi}} x^{17/2}}{34459425} + \frac{512 \sqrt{\frac{2}{\pi}} x^{19/2}}{654729075} - \frac{1024 \sqrt{\frac{2}{\pi}} x^{21/2}}{13749310575} - \\ & \frac{2048 \sqrt{\frac{2}{\pi}} x^{23/2}}{316234143225} + \frac{4096 \sqrt{\frac{2}{\pi}} x^{25/2}}{7905853580625} + \frac{8192 \sqrt{\frac{2}{\pi}} x^{27/2}}{213458046676875} - \frac{16384 \sqrt{\frac{2}{\pi}} x^{29/2}}{6190283353629375} - \\ & \frac{32768 \sqrt{\frac{2}{\pi}} x^{31/2}}{191898783962510625} + \frac{65536 \sqrt{\frac{2}{\pi}} x^{33/2}}{6332659870762850625} + \frac{131072 \sqrt{\frac{2}{\pi}} x^{35/2}}{221643095476699771875} - \\ & \frac{262144 \sqrt{\frac{2}{\pi}} x^{37/2}}{8200794532637891559375} - \frac{524288 \sqrt{\frac{2}{\pi}} x^{39/2}}{319830986772877770815625} \end{aligned}$$

Graphs

cos

$$\text{Plot}\left[1 + \sqrt{\frac{2}{\pi}} \sqrt{x} - \frac{2}{3} \sqrt{\frac{2}{\pi}} x^{3/2} - \frac{4}{15} \sqrt{\frac{2}{\pi}} x^{5/2} + \frac{8}{105} \sqrt{\frac{2}{\pi}} x^{7/2} + \frac{16}{945} \sqrt{\frac{2}{\pi}} x^{9/2} - \frac{32}{10395} \sqrt{\frac{2}{\pi}} x^{11/2} - \frac{64}{135135} \sqrt{\frac{2}{\pi}} x^{13/2} + \frac{128}{2027025} \sqrt{\frac{2}{\pi}} x^{15/2} + \frac{256}{34459425} \sqrt{\frac{2}{\pi}} x^{17/2} - \frac{512}{654729075} \sqrt{\frac{2}{\pi}} x^{19/2} - \frac{1024}{13749310575} \sqrt{\frac{2}{\pi}} x^{21/2} + \frac{2048}{316234143225} \sqrt{\frac{2}{\pi}} x^{23/2} + \frac{4096}{7905853580625} \sqrt{\frac{2}{\pi}} x^{25/2} - \frac{8192}{213458046676875} \sqrt{\frac{2}{\pi}} x^{27/2} - \frac{16384}{6190283353629375} \sqrt{\frac{2}{\pi}} x^{29/2} + \frac{32768}{191898783962510625} \sqrt{\frac{2}{\pi}} x^{31/2} + \frac{65536}{6332659870762850625} \sqrt{\frac{2}{\pi}} x^{33/2} - \frac{131072}{221643095476699771875} \sqrt{\frac{2}{\pi}} x^{35/2} - \frac{262144}{8200794532637891559375} \sqrt{\frac{2}{\pi}} x^{37/2} + \frac{524288}{319830986772877770815625} \sqrt{\frac{2}{\pi}} x^{39/2}, \{x, 0, 10\}\right]$$



sin

$$\begin{aligned}
 \text{Plot} \left[\sqrt{\frac{2}{\pi}} \sqrt{x} + \frac{2}{3} \sqrt{\frac{2}{\pi}} x^{3/2} - \frac{4}{15} \sqrt{\frac{2}{\pi}} x^{5/2} - \frac{8}{105} \sqrt{\frac{2}{\pi}} x^{7/2} + \frac{16}{945} \sqrt{\frac{2}{\pi}} x^{9/2} + \frac{32 \sqrt{\frac{2}{\pi}} x^{11/2}}{10395} - \right. \\
 \frac{64 \sqrt{\frac{2}{\pi}} x^{13/2}}{135135} - \frac{128 \sqrt{\frac{2}{\pi}} x^{15/2}}{2027025} + \frac{256 \sqrt{\frac{2}{\pi}} x^{17/2}}{34459425} + \frac{512 \sqrt{\frac{2}{\pi}} x^{19/2}}{654729075} - \frac{1024 \sqrt{\frac{2}{\pi}} x^{21/2}}{13749310575} - \\
 \frac{2048 \sqrt{\frac{2}{\pi}} x^{23/2}}{316234143225} + \frac{4096 \sqrt{\frac{2}{\pi}} x^{25/2}}{7905853580625} + \frac{8192 \sqrt{\frac{2}{\pi}} x^{27/2}}{213458046676875} - \frac{16384 \sqrt{\frac{2}{\pi}} x^{29/2}}{6190283353629375} - \\
 \frac{32768 \sqrt{\frac{2}{\pi}} x^{31/2}}{191898783962510625} + \frac{65536 \sqrt{\frac{2}{\pi}} x^{33/2}}{6332659870762850625} + \frac{131072 \sqrt{\frac{2}{\pi}} x^{35/2}}{221643095476699771875} - \\
 \left. \frac{262144 \sqrt{\frac{2}{\pi}} x^{37/2}}{8200794532637891559375} - \frac{524288 \sqrt{\frac{2}{\pi}} x^{39/2}}{319830986772877770815625}, \{x, 0, 10\} \right]
 \end{aligned}$$

