Honours Research Project
on Fractional Calculus

University of Johannesburg

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In partial fulfilment for the degree Baccalaureus Honores Mathematica

under supervision of Dr. A. Swartz

To a Mathematician, Epsilon and Delta is the Alpha and Omega

Abstract

Differentiation and integration are usually regarded as discrete operations. In other words, our interpretation of $D^n f(x)$ and $I^n f(x)$ takes $n \in \mathbb{N}$ and differentiates or integrates the function f, n times. The aim of section 5 is to unify and develop the notion of fractional order differentiation and integration, such that $D^n f(x)$ or $I^n f(x)$ assumes $n \in \mathbb{R}$ to produce a useful result. This generalized differentiation and integration operator, later called *Differintegral* and denoted by $J^n f(x)$, obeys certain properties which are discussed in subsection 5.3. Fractional Taylor series and related examples are discussed in section 6. In section 8 we discuss future directions and open questions followed by a brief motivation and applications for this projet.

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A Letter of Gratitude

We would like to mention a kind word of appreciation and gratitude towards the help of the project managers. Those who have lead us through our learning experiences and guided us to enjoy the subject of interest. The journey to higher mathematics is rather humbling and priceless. To Dr. A Swartz, a dearest worth of thank you. From the authors of the books we've studied, to the students in class with whom we co-study, a thank you for your motivation. A sincere blessing to the departments' help and friendliness over correspondence and a whole two years of fulfilment in a beautiful setting of learning experiences.

1 Introduction

Mathematical ideas are ever-changing. Often, mathematicians explore older intentions using contemporary tools, with the hope of striking deeper connotation. After all, recursion is intrinsic to mathematics.

Although some discoveries are stumbled upon, really nothing in mathematics is a coincidence. Patterns occur abundantly in nature; it is the duty of a scholar to formalise these notions. One such a fundamental tool is that of calculus. All over the world, independent cultures exploited this idea and it survived criticism of mathematical flux. This article will pursue the objective of introducing the fractional calculus. A stepping stone for understanding this renewed approach to the analysis of the fractional Taylor series is built on the key concepts of fractional calculus. The author realises the difficulties encountered in processing new concepts without the illustrative capacity of examples. Accordingly, one may reasonably refer directly to section 5 onwards of this article, conveniently omitting abundant theory covered in all prior sections. The formalism of this text does not attempt to present a new mathematical theory, but is rather based on, and supplements well known results such as those used in calculus, and other relevant branches and invokes the pleasing simplicity of treating polynomials.

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A Brief History

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Leibniz philosophied in 1695 that $d^{\frac{1}{2}}x = x\sqrt{dx}$, which gave rise to the idea of finding formulae for such expressions (Oldham & Spanier, 1974, p. 3). It was Lacroix who, almost 200 years later, first derived explicitly (Oldham & Spanier, 1974, p. 4)

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}$$

Lagrange showed in 1772 that for integers p and q the law of indices hold (Oldham & Spanier, 1974, p. 3)

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n} = \frac{d^{m+n}}{dx^{m+n}}$$

The Cauchy-Goursat theorem in complex analysis was first investigated by Carl Friedrich Gauss in 1811, who wrote in a letter to Friedrich Wilhelm Bessel "This is a very beautiful theorem, for which I will give a not difficult proof at a suitable opportunity" (Bottazzini, 1987, p. 156). However, Cauchy was the first to publish it's proof and a particular consequence is the generalized Cauchy formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(\xi)\ d\xi}{(\xi - z)^{n+1}}$$

where f is an analytic function inside and on the simple closed curve γ and $z \in \mathbb{C}$ is any point inside γ with n natural. Notice the close connection with Cauchy's repeated integral formula for n natural and f continuous

$$f^{(-n)}(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$$

Next, Niels Hendrik Abel applied fractional calculus in 1832 to solve the tautchrone (isochrone) problem, as discussed in section 8.1 (Oldham & Spanier, 1974, p. 4). In 1847 Riemann sought a generalization of Taylor's series expansion and defined the Riemann-Liouville fractional integral operator in his undergraduate notes (Oldham & Spanier, 1974, p. 6).

$$\frac{d^{-r}}{dx^{-r}}u(x) = \frac{1}{\Gamma(r)}\int_c^x\ (x-k)^{r-1}u(k)\ dk$$

In 1859 Greer developed formulae for semi-derivatives of $\sin x$ and $\cos x$ using Liouville's results (Oldham & Spanier, 1974, p. 7).

$$D^{\frac{1}{2}}e^{mx} = m^{\frac{1}{2}}e^{mx}$$

Nine years later, Letnikov (Oldham & Spanier, 1974, p. 7) proved for arbitrary order p and q that

$$[D^q D^p f(x)]_{x_0}^x = [D^{q+p} f(x)]_{x_0}^x$$

Laurent generalized Cauchy's integral formula in 1884 and made advances on Leibniz' generalized product rule (not discussed in this paper) (Oldham & Spanier, 1974, p. 8).

Notation was improved by H. Davis (Oldham & Spanier, 1974, p. 9) around 1928 who proposed the use of ${}_cD_x^{-v}f(x)$ to mean

$$\frac{1}{\Gamma(v)}\int_c^x\ (x-t)^{v-1}f(t)\ dk$$

and utilized this to solve equations such as

$${}_cD_x^{\frac{1}{2}}u + \lambda u = f(x)$$

By mention of these historical contributions, we see that the future in fractional calculus is promising and endless. We discuss some connections to other areas of mathematics in section 8.

2 Preliminaries

Definition 2.0.1 (Exponential Order). *(Edwards & Penney, 1989, p. 273) A function f is said to be of exponential order as $t \to \infty$ if there exist nonnegative constants M, c and T such that*

$$|f(t)| \leq Me^{ct}, \ \forall t \geq T$$

Definition 2.0.2 (Linear Operator). *(Kreyszig, 1978, p. 82) A linear operator is a mapping $T : \mathcal{D}(T) \to \mathcal{R}(T)$ such that:*

(i) the domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ of T lies in a vector space over the same field

(ii) for all $x, y \in \mathcal{D}(T)$ and scalars α,

$$T(x + y) = Tx + Ty$$

$$T(\alpha x) = \alpha Tx$$

Theorem 2.0.1 (Fundamental Theorem of Calculus - Part 1). *(Stewart, 2012, p. 388) Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all $x \in [a, b]$, by*

$$F(x) = \int_a^x f(t)dt$$

Then F is continuous on $[a, b]$, and differentiable on the open interval (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$.

Theorem 2.0.2 (Fundamental Theorem of Calculus - Part 2). *(Stewart, 2012, p. 391) Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F denote the antiderivative of f on $[a, b]$, i.e. $F'(x) = f(x)$. If f is integrable on on $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Theorem 2.0.3 (Leibniz' Integral Rule). *(Collins, 2006, p. 3) Given an integral of the form*

$$\int_{a(x)}^{b(x)} f(x, t) \ dt$$

where $-\infty < a(x), b(x) < \infty$, then the derivative of this integral can be expressed as

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x, t)dt\right) = f(x, b(x))\frac{d}{dx}b(x) - f(x, a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x, t) \ dt$$

Theorem 2.0.4 (Inverse Operator)**.** *(Kreyszig, 1978, p. 88) Let X, Y be vector spaces, both real or both complex. Let $T : \mathcal{D}(T) \to Y$ be a linear operator with domain $\mathcal{D}(T) \subset X$ and range $\mathcal{R}(T) \subset Y$. Then:*

(i) The inverse $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ exists if and only if

$$Tx = 0 \implies x = 0$$

(ii) If T^{-1} exists, then it is a linear operator

(iii) If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{D}(T) = \dim \mathcal{R}(T)$

2.1 The Factorial Function

According to (Biggs, 1979, p. 109 to 136), the factorial of a number n, as we impose meaning on it today, comes a long way. Its origin dates back at least as early as the 12th century to Indian scholars. Its roots are found in combinatorics and the notation $n!$ was introduced in 1808 by French Mathematician Christian Kramp in his *Élémens d'Arithmétique Universelle* (Higgins, 2008, p. 12).

Definition 2.1.1 (Factorial Function)**.** *(Dutka, 1991, p. 225) Let $n \in \mathbb{N}$. Then we define the factorial function as*

$$\begin{aligned} n! &= \prod_{k=1}^{n} k \\ n! &= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot (n) \end{aligned}$$

Remark. *In order for this recurrence relation to be extended to $n = 0$, it is necessary to define $0! = 1$ for all application purposes. The factorial and double factorial function stand in close connection with infinite products, such as English mathematician, John Wallis' product for π*

$$\prod_{k=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots = \frac{\pi}{2}$$

2.2 Gamma Function

For some an interest in mathematics arise from the usefulness of its tools to formulate methods that produce exact results. It is often thought of as an interpolation problem, where one might want to determine a compact formula that can simplify rote calculation.

By definition 2.1.1, the factorial function takes $n \in \mathbb{N}$ and produces again a number $n! \in \mathbb{N}$. Euler and Goldbach's correspondence pondered whether or not it is possible to determine a function that could take $n \notin \mathbb{N}$ to produce a useful result, hence extending its domain. The desire to determine such a formula for the factorial function, lead the expedition in formulating Euler's gamma function.

The derivation of the gamma function demanded a great deal of pre-developed work. It was via experimenting with the infinite product $\prod_{k=1}^{\infty} \frac{(n+1)^2}{n^s} \cdot \frac{n}{n+s}$, that Euler derived the gamma function.

Adrien Marie Legendre further extended the domain of the gamma function, by a simple transformation (Davis, 1959, p. 855).

Definition 2.2.1. *(Baleanu, 2012, p. 5) The (complete) gamma function, $\Gamma(n)$ is defined to be an extension of the factorial function to complex and real number arguments and is related to it by $\Gamma(n) = (n-1)!$. It is also defined as an improper definite integral for complex arguments $z \in \mathbb{C}$ having positive real part, $\Re(z) > 0$, as*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

Claim. *The gamma function satisfies $\Gamma(x+1) = x\Gamma(x)$.*

Proof.

$$\begin{aligned}
\frac{d}{dt} t^x e^{-t} &= x t^{x-1} e^{-t} - t^x e^{-t} \\
\int_0^{\infty} d(t^x e^{-t}) &= \int_0^{\infty} t^{x-1} e^{-t} dt - \int_0^{\infty} t^x e^{-t} \\
\lim_{b \to \infty} \left[\frac{b^x}{e^b} \right] &= x\Gamma(x) - \Gamma(x+1) \\
0 &= x\Gamma(x) - \Gamma(x+1) \\
\Gamma(x+1) &= x\Gamma(x)
\end{aligned}$$

QED

The gamma function is useful in many application instances, such as in computing factorials of non-integer values as the example below illustrates:

Example 2.2.1. *In this example we compute $\Gamma(\frac{1}{2})$.*

Solution. *Letting $x = \frac{1}{2}$ and substituting into the gamma function, we get*

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-t} t^{-\frac{1}{2}} dt$$

We make the change of variables $u = t^{\frac{1}{2}}$ so that $\frac{d}{dt}u = \frac{1}{2}t^{-\frac{1}{2}}$, noting that $u(\infty) = \infty$; $u(0) = 0$, *We now get that*

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du$$

and by squaring, we obtain

$$\Gamma^2\left(\frac{1}{2}\right) = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\ du\ dv$$

We transform this integral into polar coordinates over the region

$$R = \{(x, y) \mid 0 \leq x < \infty\ ;\ 0 \leq y < \infty\}$$

by the substitutions $u = r\cos(\theta)$ *and* $v = r\sin(\theta)$ *so that*

$$R = \{(\theta, r) \mid 0 \leq r < \infty\ ;\ 0 \leq \theta \leq \frac{\pi}{2}\}$$

and the Jacobian

$$J(\theta, r) = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r$$

The problem now reduces to

$$\Gamma^2\left(\frac{1}{2}\right) = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\ r\ dr\ d\theta$$

which is easily solved to give

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

We now state two useful identities relating to the gamma (Oldham & Spanier, 1974, p. 17 to 18). It turns out that $\Gamma(\frac{1}{2}+n)$ and $\Gamma(\frac{1}{2}-n)$ are multiples of $\sqrt{\pi}$

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n n!\sqrt{\pi}}{(2n)!}$$

2.3 Beta Function

The beta function was studied extensively by Legendre and Euler, due to its close connection with the gamma function.

Definition 2.3.1. *(Baleanu, 2012, p. 7) The Beta function is defined as the definite integral*

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt$$

where $\Re(z) > 0$ *and* $\Re(w) > 0$

Theorem 2.3.1. *The relationship between the gamma and beta function is given by*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Proof.

$$\Gamma(u)\Gamma(v) = \int_0^\infty \int_0^\infty t^{u-1} s^{v-1} e^{-(t+s)}\ dt\ ds$$

Applying the change of variables $t + s = x$ we obtain the Jacobian

$$J(t,s) = \begin{vmatrix} y & x \\ 1-y & -x \end{vmatrix} = -x$$

with

$$dt\ ds = \left| \frac{\partial(t,s)}{\partial(x,y)} \right| dx\ dy = x\ dx\ dy$$

$$\begin{aligned}
\implies \quad \Gamma(u)\Gamma(v) &= \int_0^1 \int_0^\infty e^{-x} x^{u+v-1} y^{u-1} (1-y)^{v-1}\ dx\ dy \\
\implies \quad \Gamma(u)\Gamma(v) &= \int_0^\infty e^{-x} x^{u+v-1} dx \int_0^1 y^{u-1}(1-y)^{v-1}\ dy \\
\implies \quad \Gamma(u)\Gamma(v) &= \Gamma(u+v) B(u,v) \\
\implies \quad B(u,v) &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}
\end{aligned}$$

QED

Remark. *Theorem 2.3.1 can also be proved using the Laplace transform techniques discussed in section 3.*

3 Laplace Transform

The differentiation operator D, can be viewed as a transformation, which, when applied to a function $f(t)$, yields a new function $D\{f(t)\} = f'(t)$.

Definition 3.0.1 (Laplace Transform)**.** *(Collins, 2006, p. 298) Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is defined as a function for all values of s for which the improper integral converges*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Theorem 3.0.1 (Existence of Laplace Transform)**.** *(Edwards & Penney, 1989, p. 273) If the function f is piecewise continuous for $t \geq 0$ and is of exponential order (definition 2.0.1) as $t \to \infty$, then the Laplace transform $\mathcal{L}\{f(t)\}$ exists.*

Proof. We can take $T = 0$, for by piecewise continuity $|f|$ is bounded on $[0, T]$. We can therefore assume that $|f(t)| \leq M$, increasing M as necessary, hence it follows that $|f(t)| \leq Me^{ct}$ for all $t \geq 0$.

Now, since absolute convergence implies convergence, it suffices to prove that the integral

$$\lim_{b \to \infty} \int_0^b |e^{-st} f(t)| dt$$

exists for $s \geq c$. We show that the integral remains bounded as $b \to \infty$

$$\begin{aligned}
\int_0^b |e^{-st} f(t)| dt &\leq \int_0^b |e^{-st} M e^{ct}| dt \\
&\leq M \int_0^b e^{-(s-c)t} dt \\
&\leq \frac{M}{s - c}
\end{aligned}$$

QED

Example 3.0.1. *In this example, we compute $\mathcal{L}\{t^a\}$ for $a > -1$.*

Solution.

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a dt$$

We then make the substitution $u = st$ so that

$$\mathcal{L}\{t^a\} = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du = \frac{\Gamma(a+1)}{s^{a+1}}$$

for all $s > 0$.

Example 3.0.2. *Laplace transforms of functions such as $\cos(kt)$ and $\sinh(kt)$ are also easily computed if they are expressed in exponential form.*

The Laplace transforms of certain known functions are given in the table below:

$f(t)$	$\mathcal{L}\{f(t)\}$	
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s > 0$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$s > \|k\|$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$s > \|k\|$

3.1 Convolution Product

Definition 3.1.1 (Convolution Product)**.** *(Edwards & Penney, 1989, p. 296) The convolution product $(f * g)(t)$ of the piecewise continuous functions f and g, is defined for $t \geq 0$ by*

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Theorem 3.1.1 (Convolution Theorem)**.** *(Edwards & Penney, 1989, p. 297) The convolution product is commutative:*

$$(f * g)(t) = (g * f)(t)$$

Proof. To prove that $(f * g)(t) = (g * f)(t)$, we use make the substitution $u = t - \tau$, then $d\tau = -du$

$$\begin{aligned} (f * g)(t) &= \int_0^t f(\tau)\ g(t-\tau)\ d\tau \\ &= \int_t^0 f(t-u)\ g(u)\ (-du) \\ &= \int_0^t f(t-u)\ g(u)\ du \\ &= \int_0^t g(u)\ f(t-u)\ du \\ &= (g * f)(t) \end{aligned}$$

QED

3.2 Linearity of Laplace Transform

The Laplace transform obeys the linearity property which may be useful when computing transforms of functions such as $\cosh(kt)$ or if combined with Theorem 3.3.1, allows us to determine transforms of functions such as te^{at}.

Theorem 3.2.1 (Linearity Property). *(Collins, 2006, p. 299) If a and b are constants, then*

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

for all s such that both Laplace transforms of functions f and g exist.

Proof.

$$\begin{aligned}
\mathcal{L}\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}\left[af(t)+bg(t)\right]dt \\
&= \lim_{c\to\infty}\int_0^c \left[ae^{-st}f(t)+be^{-st}g(t)\right]dt \\
&= \lim_{c\to\infty}\int_0^c ae^{-st}f(t)dt + \lim_{c\to\infty}\int_0^c be^{-st}g(t)dt \\
&= a\lim_{c\to\infty}\int_0^c e^{-st}f(t)dt + b\lim_{c\to\infty}\int_0^c e^{-st}g(t)dt \\
&= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}
\end{aligned}$$

QED

3.3 Laplace transform of Derivatives and Integrals

Theorem 3.3.1 (Transform of Derivatives). *(Edwards & Penney, 1989, p. 277) Suppose that the function $f(t)$ is continuous and piecewise smooth for $t \geq 0$, and is of exponential order as $t \to \infty$, so that there exists nonnegative constants M, c and T such that $|f| \leq Me^{ct}$ for all $t \geq T$, then*

$$\mathcal{L}\{f'(t)\}$$

exists for $s > c$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof. The case where $f(t)$ is merely continuous is simple, leaving the case where $f(t)$ is piecewise smooth.

Beginning with the definition of $\mathcal{L}\{f'(t)\}$ and applying integration by parts, the result follows easily

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt = \left[e^{-st}f(t)\right]_{t=0}^\infty + s\int_0^\infty e^{-st}f(t)dt = s\mathcal{L}\{f(t)\} - f(0)$$

QED

Corollary 3.3.2 (Transform of Higher Derivatives). *(Edwards & Penney, 1989, p. 278) Suppose that the functions $f, f', f'', \ldots, f^{(n-1)}$ are continuous and piecewise smooth for $t \geq 0$, and that each of these functions is of exponential order as $t \to \infty$, so that there exists nonnegative constants M, c and T such that $|f| \leq Me^{ct}, |f'| \leq Me^{ct}, \ldots, |f^{(n-1)}| \leq Me^{ct}$, for all $t \geq T$, then*

$$\mathcal{L}\{f^{(n)}(t)\}$$

exists for $s > c$ and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Proof. Corollary 3.3.2 follows inductively from the previous theorem QED

Theorem 3.3.3 (Transform of Integrals)**.** *(Edwards & Penney, 1989, p. 284) If $f(t)$ is a piecewise continuous function for $t \geq 0$, and satisfies the condition of exponential order, namely $|f(t)| \leq Me^{ct}$ for $t \geq T$, then*

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\left\{f(t)\right\}$$

for $s > c$

Proof. Since f is piecewise continuous, the Fundamental Theorem of Calculus (Part 1 and Part 2) implies that

$$g(t) = \int_0^t f(\tau)\ d\tau$$

is continuous and that $g'(t) = f(t)$ where f is continuous; thus g is continuous and piecewise smooth for $t \geq 0$. Furthermore,

$$|g(t)| \leq \int_0^t |f(t)|\ d\tau \leq M \int_0^t e^{c\tau}\ d\tau = \frac{M}{c}\ (e^{ct} - 1)\ < \frac{M}{c}\ e^{ct}$$

so g is of Exponential Order as $t \to \infty$, hence we may apply Theorem 3.3.1 to g with $g(0) = 0$;

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$$

$$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\}$$

and division by s yields

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\left\{f(t)\right\}$$

QED

3.4 Inverse Laplace Transform

Definition 3.4.1 (Inverse Laplace Transform)**.** *(Collins, 2006, p. 315) If $F(s) = \mathcal{L}\{f(t)\}$ then $f(t)$ is the inverse Laplace transform of $F(s)$ formally defined as the principal value contour integral*

$$\mathcal{L}^{-1}\left\{F(s)\right\} = f(t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\sigma-R}^{\sigma+R} F(s)\ e^{st}\ ds$$

where σ is large enough so that $F(s)$ is defined for $\Re(s) \geq \sigma$

Claim. *The inverse Laplace transform $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$ is a linear operator.*

Proof. We utilize Theorem 2.0.4 and show that $\mathcal{L}^{-1}$ satisfies the criteria in the theorem statement.

Notice that the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ is the set of all real valued functions, $f(t)$ such that $f(t)e^{-st}$ is integrable over $[0, \infty)$ and the range $\mathcal{R}(\mathcal{L})$ is the set of all complex valued functions $F(s)$ which are analytic in the region $\Re(s) \geq \sigma$ where $s \in \mathbb{C}$.

To show that $\mathcal{L}^{-1}$ exists, we need to show that $\mathcal{L}(f(t)) = 0 \implies f(t) = 0$.

Suppose that

$$0 = \mathcal{L}(f(t)) = \int_0^\infty e^{-st}\ f(t)\ dt$$

Then

$$\frac{d}{dt}0 = \frac{d}{dt}\int_0^\infty e^{-st}\ f(t)\ dt$$

by the Fundamental Theorem of Calculus we obtain

$$0 = f(t)e^{-st}$$

which implies that $f(t) = 0$ since $e^{-st} \neq 0$ for all t. Hence we see that $\mathcal{L}^{-1}$ exists.

To show that $\mathcal{L}^{-1}$ is linear, we argue as follows

Since $\mathcal{L}$ is linear, for suitable functions f and g,

$$\mathcal{L}\left(\alpha f(t) + \beta g(t)\right) = \alpha\mathcal{L}(f(t)) + \beta\mathcal{L}(g(t))$$

$$\mathcal{L}\left(\alpha f(t) + \beta g(t)\right) = \alpha F(s) + \beta G(s)$$

where F and G are the Laplace transforms of f and g respectively. We now apply the inverse Laplace transform over the equation, such that

$$\mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$$

and obtain the following

$$\mathcal{L}^{-1}(\mathcal{L}(\alpha f(t) + \beta g(t))) = \mathcal{L}^{-1}(\alpha F(s) + \beta G(s))$$

$$\alpha f(t) + \beta g(t) = \mathcal{L}^{-1}(\alpha F(s) + \beta G(s))$$

$$\alpha\mathcal{L}^{-1}(F(s)) + \beta\mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}(\alpha F(s) + \beta G(s))$$

which shows that $\mathcal{L}^{-1}$ is indeed a linear operator

Naturally we would want $\mathcal{D}(\mathcal{L}^{-1}) = \mathcal{R}(\mathcal{L})$ and $\mathcal{R}(\mathcal{L}^{-1}) = \mathcal{D}(\mathcal{L})$. This result is further guaranteed by Theorem 2.0.4. QED

Example 3.4.1. *The inverse Laplace transforms of certain known functions are shown:*

$\mathcal{L}^{-1}\{F(s)\}$	$f(t)$
$\frac{\Gamma(n+1)}{s^{n+1}}$	t^n
$\frac{n!}{(s-a)^{n+1}}$	$t^n e^{at}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

Where $s > a$

4 Cauchy's Repeated Integral Formula

The Cauchy formula for repeated integration, named after Augustin-Louis Cauchy, allows one to compress n antidifferentiations of a function f into a single integral. It is the central concept on which fractional calculus is built.

Theorem 4.0.1 (Cauchy's Formula for Repeat Integration)**.** *(Oldham & Spanier, 1974, p. 37) let f be a continuous function on the real line. Then the n^{th} repeated integral of f based at a,*

$$f^{(-n)}(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n)\ d\sigma_n \cdots d\sigma_2\ d\sigma_1$$

is given by single integration as

$$f^{(-n)}(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$$

Proof. The proof is given by induction

Since f is continuous, the base case follows from the Fundamental Theorem of Calculus:

$$\frac{d}{dx} f^{(-1)}(x) = \frac{d}{dx}\int_a^x f(t)\ dt$$

where

$$f^{(-1)}(a) = \int_a^a f(t)\ dt = 0$$

Now, suppose the theorem holds for $n = k$, i.e.

$$f^{(-k)}(x) = \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1} f(t)dt$$

Consider the case when $n = k+1$;

First, applying Leibniz' rule for integrals, we get

$$\frac{d}{dx}\left[\frac{1}{(k)!}\int_a^x (x-t)^k f(t)dt\right] = \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1} f(t)dt$$

Applying the inductive hypothesis

$$\begin{aligned} f^{-(k+1)}(x) &= \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_k} f(\sigma_{k+1})\ d\sigma_{k+1}\ d\sigma_k \cdots d\sigma_2\ d\sigma_1 \\ &= \int_a^x \frac{1}{(k-1)!} \int_a^{\sigma_1} (\sigma_1 - t)^{k-1}\ f(t)\ dt\ d\sigma_1 \\ &= \int_a^x \frac{d}{d\sigma_1}\left[\frac{1}{k!}\int_a^{\sigma_1} (\sigma_1 - t)^k\ f(t)\ dt\right] d\sigma_1 \\ &= \frac{1}{k!}\int_a^x (x-t)^k\ f(t)\ dt \end{aligned}$$

Hence, Theorem 4.0.1 holds for all $n \in \mathbb{N}$ QED

5 Differintegral Operator

Definition 5.0.1 (Differintegral Operator)**.** *The Differintegral operator applied α times to a function f, denoted $J^\alpha f$, is a combination of the differentiation operator D and integral operator I into a single symbol operator J, where negative values for α correspond to differentiation and positive values for α correspond to antidifferentiation. In the case where $\alpha = 0$, we obtain the function f.*

5.1 Riemann-Liouville Integral

The Riemann-Liouville integral is in essence an attempt to extend the domain of the Cauchy formula for repeated integration. There are several other formulations of the fractional integral operator which will be briefly discussed.

Definition 5.1.1 (Riemann-Liouville Integral)**.** *(Baleanu, 2012, p. 10 to 11) Let $\Omega = [a, b]$ where $-\infty < a < b < \infty$ be a finite closed interval on the Real axis $\mathbb{R}$. The left-sided and right-sided Riemann-Liouville fractional integrals ${}_{a+}J^\alpha f$ and ${}_{b-}J^\alpha f$ of order α based at a and b respectively is defined by*

$${}_{a+}J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt\ , (x > a; \Re(\alpha) > 0)$$

and

$${}_{b-}J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt\ , (x < b; \Re(\alpha) > 0)$$

where α is a complex number in the Half Plane $\Re(z) > 0$.

Theorem 5.1.1 (Semi-Group Property of J)**.** *(Baleanu, 2012, p. 13) The integral operator, J satisfies*

$$({}_aJ^\alpha)({}_aJ^\beta f)(x) = ({}_aJ^{\alpha+\beta} f)(x)$$

Proof. Without loss of generality, taking $a = 0$

$$\begin{aligned}(J^{\alpha})(J^{\beta}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}(J^{\beta}f)(t)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\ \frac{1}{\Gamma(\beta)}\int_0^x (t-s)^{\beta-1}\ f(s)\ ds\ dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\ \int_0^x\ \int_0^t\ (x-t)^{\alpha-1}\ (t-s)^{\beta-1}\ f(s)\ ds\ dt\end{aligned}$$

For simplicity, we now let $f(s,t) = (x-t)^{\alpha-1}(t-s)^{\beta-1}\ f(s)$

$$(J^{\alpha})(J^{\beta}f)(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\ \int_0^x\ \int_0^t\ f(s,t)\ ds\ dt$$

which can be expressed an area integral

$$(J^{\alpha})(J^{\beta}f)(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\ \int\int_{D_1}\ f(s,t)\ dA$$

where

$$D_1 = \{(s,t)\ |\ 0 \leq s \leq t, 0 \leq t \leq x\}$$

D_1 is a type I Region (See Figure 1)

To change the order of integration, we must express D_1 as a type II Region;

$$D_2 = \{(s,t)\ |\ s \leq t \leq x, 0 \leq s \leq x\}$$

D_2 is a type II Region (See Figure 2)

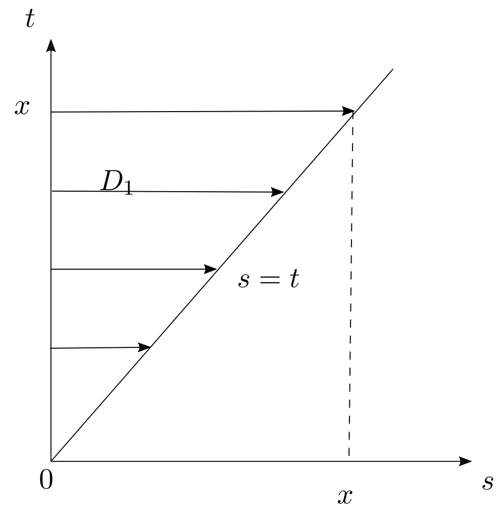


Figure 1: D_1 is a Type 1 Region

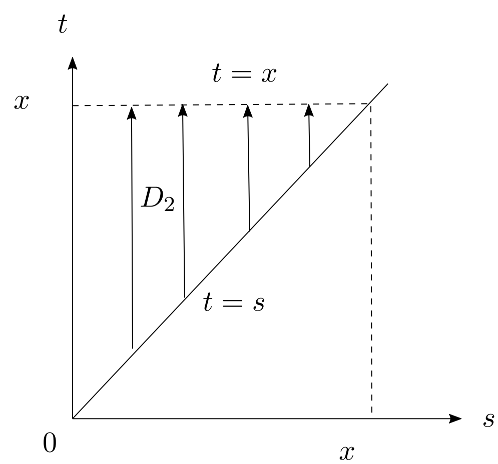


Figure 2: D_2 is a Type 2 Region

We now get the integral

$$(J^\alpha)(J^\beta f)(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int\int_{D_2} f(s,t)\ dA$$

$$\begin{aligned}
(J^\alpha)(J^\beta f)(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_s^x (x-t)^{\alpha-1}\ (t-s)^{\beta-1}\ f(s)\ dt\ ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s) \int_s^x (x-t)^{\alpha-1}\ (t-s)^{\beta-1}\ dt\ ds
\end{aligned}$$

The change of variables $t = s + (x-s)r$ gives the following

$$\frac{d}{dr}t = (x-s) \implies dt = (x-s)dr$$

and the upper and lower limits of integration

$$t = s \implies r = 0$$

$$t = x \implies r = 1$$

which results in the following integral

$$\begin{aligned}
& (J^\alpha)(J^\beta f)(x) \\
= & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s) \int_0^1 (x-(s+(x-s)r))^{\alpha-1}\ (s+(x-s)r-s)^{\beta-1}(x-s)\ dr\ ds \\
= & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s) \int_0^1 (x(1-r)-s(1-r))^{\alpha-1}\ ((x-s)r)^{\beta-1}(x-s)\ dr\ ds \\
= & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s) \int_0^1 ((x-s)(1-r))^{\alpha-1}\ ((x-s)r)^{\beta-1}(x-s)\ dr\ ds \\
= & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s) \int_0^1 (x-s)^{\alpha+\beta-1}\ (1-r)^{\alpha-1}r^{\beta-1}\ dr\ ds \\
= & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s)\ (x-s)^{\alpha+\beta-1} \int_0^1 (1-r)^{\alpha-1}r^{\beta-1}\ dr\ ds \\
= & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s)\ (x-s)^{\alpha+\beta-1}\ B(\alpha,\beta)\ ds \\
= & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x f(s)\ (x-s)^{\alpha+\beta-1}\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ ds \\
= & \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\ \Gamma(\alpha)\ \Gamma(\beta)} \int_0^x f(s)\ (x-s)^{\alpha+\beta-1}\ ds \\
= & \frac{1}{\Gamma(\alpha+\beta)} \int_0^x f(s)\ (x-s)^{\alpha+\beta-1}\ ds \\
= & (J^{\alpha+\beta} f)(x)
\end{aligned}$$

QED

Example 5.1.1. *In this example, we find the fractional integral for $f(x) = x^n$ based at $a = 0$*

Solution. *We apply the Riemann-Liouville integral to obtain*

$$J^\alpha x^k = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\ t^k\ dt$$

Now we apply the change of variables

$$u = \frac{t}{x} \implies ux = t \implies xdu = dt$$

with the upper and lower limits of integration

$$u(0) = 0 \text{ and } u(x) = 1$$

to obtain the following integral

$$\begin{aligned}
J^\alpha x^k &= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-ux)^{\alpha-1}\ (ux)^k\ x\ du \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1}\ u^k\ x^{\alpha+k}\ du \\
&= \frac{x^{\alpha+k}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1}\ u^k\ du \\
&= \frac{x^{\alpha+k}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1}\ u^{(k+1)-1}\ du \\
&= \frac{x^{\alpha+k}}{\Gamma(\alpha)}\ B(k+1,\alpha) \\
&= \frac{x^{\alpha+k}}{\Gamma(\alpha)} \frac{\Gamma(k+1)\Gamma(\alpha)}{\Gamma(\alpha+k+1)} \\
&= \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\ x^{\alpha+k}
\end{aligned}$$

We conclude that

$$J^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\ x^{\alpha+k}$$

5.2 Fractional Calculus and the Laplace Transform

The integral operator J explored in subsection 5.1 can be derived from an alternative approach, using that of the Laplace transform method discussed in section 3.

The next corollary follows inductively from Theorem 3.3.3 of subsection 3.3

Corollary 5.2.1. *If $f(t)$ is a piecewise continuous function for $t \geq 0$, and satisfies the condition of exponential order $|f(t)| \leq Me^{ct}$, for $t \geq T$, then*

$$\mathcal{L}\left\{J^{\alpha} f(t)\right\} = \frac{1}{s^{\alpha}} \mathcal{L}\left\{f(t)\right\}$$

for $s > 0$. Equivalently

$$J^{\alpha} f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}} \mathcal{L}\left\{f(t)\right\}\right\}$$

Proof. Let $g(x) = x^{\alpha-1}$ and using the convolution product

$$\mathcal{L}\left\{f(t) * g(t)\right\} = \mathcal{L}\left\{f(t)\right\} \mathcal{L}\left\{g(t)\right\}$$

Starting with the Riemann-Liouville integral, we get

$$\begin{aligned}
J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (t-\tau)^{\alpha-1} f(\tau) d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^x g(t-\tau) f(\tau) d\tau \\
&= \frac{1}{\Gamma(\alpha)} (g * f)(t) \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}^{-1}\left\{\mathcal{L}\left\{g(t)\right\} \mathcal{L}\left\{f(t)\right\}\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}^{-1}\left\{\mathcal{L}\left\{t^{\alpha-1}\right\} \mathcal{L}\left\{f(t)\right\}\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}^{-1}\left\{s^{-\alpha}\, \Gamma(\alpha) \mathcal{L}\left\{f(t)\right\}\right\} \\
&= \mathcal{L}^{-1}\left\{s^{-\alpha}\, \mathcal{L}\left\{f(t)\right\}\right\}
\end{aligned}$$

QED

Remark. *Recall that $\mathcal{L}\left\{t^k\right\} = \frac{\Gamma(k+1)}{s^{k+1}}$, so that this gives*

$$J^{\alpha} t^k = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}} \mathcal{L}\left\{t^k\right\}\right\} = \mathcal{L}^{-1}\left\{\frac{\Gamma(k+1)}{s^{\alpha+k+1}}\right\} = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} t^{\alpha+k}$$

which corresponds with our result in example 5.1.1

5.3 Properties of the Integral Operator

Theorem 5.3.1 (Semi-Group Property)**.** *(Baleanu, 2012, p. 13) The Riemann-Liouville integral satisfies, for $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ the semi-group property*

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$$

Proof.

$$J^{\alpha} J^{\beta} f(x) = \frac{1}{\Gamma(\alpha)\ \Gamma(\beta)} \int_0^x dt \int_0^t (x-t)^{\alpha-1}\ (t-u)^{\beta-1}\ f(u)\ dt\ du$$

We now make the change of variables $t = u + s(x-u)$ to obtain

$$J^{\alpha} J^{\beta} f(x) = \frac{B(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x (x-u)^{\alpha+\beta-1} f(u)\ du \ = \ J^{\alpha+\beta} f(x)$$

QED

Theorem 5.3.2 (Linearity of Integral Operator)**.** *The Riemann-Liouville integral satisfies, for $\alpha > 0$, the linearity property*

$$J^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda J^{\alpha} f(x) + \mu J^{\alpha} g(x)$$

Proof. The proof of Theorem 5.3.2 is trivial, and is left to the reader QED

5.4 Properties of the Fractional Differential Operator

Definition 5.4.1. *(Baleanu, 2012, p. 11) The left-sided and right-sided fractional derivatives ${}_{a+}D_x^m$ and ${}_{b-}D_x^m$ of order α based at a and b respectively is defined in terms of the fractional integral as*

$${}_{a+}D_x^{\alpha} f(t) = \left(\frac{d}{dx}\right)^m \left[{}_{a+}D_x^{-(m-\alpha)} f(t)\right] = \left(\frac{d}{dx}\right)^m \left[{}_{a+}I_x^{(m-\alpha)} f(t)\right] \ ; (x > a)$$

and

$${}_{b-}D_x^{\alpha} f(t) = \left(-\frac{d}{dx}\right)^m \left[{}_{b-}D_x^{-(m-\alpha)} f(t)\right] = \left(-\frac{d}{dx}\right)^m \left[{}_{b-}I_x^{(m-\alpha)} f(t)\right] \ ; (x < b)$$

where m is the nearest integer greater than the real part α, i.e. $m = \lceil \Re(\alpha) \rceil$. This typeset will conveniently be omitted whenever the context is clear from the text and we seize denoting the many subscripts unless otherwise necessary.

If the order of the fractional derivative is $\alpha = \frac{1}{2}$, then it is said to be a semi-derivative.

An analogous result to that of the fractional integral for a power function $f(x) = x^n$ for fractional derivatives exists and is stated below.

Claim. *The fractional derivative of the power function $f(t) = t^{\lambda}$ is given by*

$$\frac{d^{\alpha}}{dt^{\alpha}} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$$

Proof. The proof follows easily from the definition 5.4.1 and use of the Riemann-Liouville integral 5.1.1;

$$\begin{aligned} D^{\alpha}t^{\lambda} &= D^{m}\left[I^{(m-\alpha)}t^{\lambda}\right] \\ &= D^{m}\left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m-\alpha+1)}t^{\lambda+m-\alpha}\right] \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha} \end{aligned}$$

QED

Example 5.4.1. *This allows us to compute the semi-derivative of $f(t) = t$, $D^{\frac{1}{2}}t$, when $\alpha = \frac{1}{2}$*

Solution. *Let $\alpha = \frac{1}{2}$, then using our claim with $\lambda = 1$, we get*

$$D^{\frac{1}{2}}t = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}t^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})}\sqrt{t} = 2\sqrt{\frac{t}{\pi}}$$

Example 5.4.2. *In this example we compute the semi-derivative for a constant function, i.e. $D^{\frac{1}{2}}c$, where c is any constant.*

Solution. *We use the above claim again on a function $f(t) = ct^{\lambda}$ and take the limit as $\lambda \to 0$.*

$$\lim_{\lambda\to 0} D^{\alpha}\ c\ t^{\lambda} = c\ \lim_{\lambda\to 0}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha} = \frac{c}{\Gamma(1-\alpha)\ t^{\alpha}}$$

Observation.

$$\begin{aligned} D^{\frac{1}{2}}\ \left[D^{\frac{1}{2}}\ t\right] &= D^{\frac{1}{2}}\ \left[2\ \sqrt{\frac{t}{\pi}}\right] \\ &= \frac{2}{\sqrt{\pi}}\ D^{\frac{1}{2}}\ \left[\sqrt{t}\right] \\ &= \frac{2}{\sqrt{\pi}}\ \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}+1\right)}\ t^{\frac{1}{2}-\frac{1}{2}} \\ &= \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}\ x^{0} \\ &= \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\ =\ 1 \end{aligned}$$

Remark. *It should be noted that in general $D^{\alpha}D^{\beta}f(t) \neq D^{\alpha+\beta}$.It is necessary to apply the fractional derivative after the integer derivative has been applied to a function, for example*

$$D^{\frac{3}{2}}f(x) = D^{\frac{1}{2}}D^{1}f(x)$$

6 Power Series

Power series serve to generalize polynomials and polynomial-like functions. Two main types of power series are usually considered, namely Taylor Series and Laurentz Series. However, we will restrict our attention to Taylor series and Maclaurin Series.

Definition 6.0.1 (Power Series). *(Stewart, 2012, p. 741) A power series centred at point a is a series of the form*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

We are interested to know when certain classes of functions have power series representations and how such representations can be found. From elementary calculus, we have the following theorem without proof.

Theorem 6.0.1 (Convergence of Power Series). *(Stewart, 2012, p. 743) For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:*

(i) The series converges only when $x = a$

(ii) The series converges for all x

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

Remark. *The number R in case (iii) is called the radius of convergence of the power series. By convention, we define $R = 0$ in case (i) and $R = \infty$ in case (ii).*

6.1 Classical Taylor Series

In this section, we start off with Taylor's theorem, which is intended to determine the coefficients c_n of a power series expansion.

Theorem 6.1.1 (Coefficients of Power Series). *(Stewart, 2012, p. 754) If f has a power series representation at point a, such that*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

where $|x-a| < R$, then its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

and $f(x)$ can be written as

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

Remark. *Whenever $a=0$, the above series is called the Maclaurin series of $f(x)$*

6.2 Fractional Taylor Series

The fractional Taylor series expansion is an extension of the classical Taylor series expansion, with exponents taken from $\mathbb{R}$.

Definition 6.2.1 (Fractional Series Expansion)**.** *(Li, Ren, & Zhu, 2009, p. 6 to 7) The fractional Taylor series of $f(x)$ centred at point a is given by*

$$f(x)=f(a)+\sum_{k>0}{}_aJ_x^k({}_aD_x^kf(x))$$

which is equivalent to

$$f(x)=f(a)+\sum_{k>0}\left(\frac{d^k}{dx^k}f(x)\right)_{x=a}\left(\frac{1}{\Gamma(k)}\int_a^x(x-t)^{k-1}dt\right)$$

Before moving on further, let us discuss an example for a better understanding of this formula.

Example 6.2.1. *Expand $f(x)=(x+y)^n$ about the point $a=0$ starting with $k=1$*

Solution. *Notice that $f(0)=y^n$ and $\frac{d^k}{dx^k}(x+y)^n|_{x=0}=\frac{\Gamma(n+1)}{\Gamma(n-k+1)}(x+y)^{n-k}|_{x=0}=\frac{\Gamma(n+1)}{\Gamma(n-k+1)}y^{n-k}$ so that we have*

$$\begin{aligned}
f(x) &= y^n+\sum_{k=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-k+1)}y^{n-k}\cdot\frac{1}{\Gamma(k)}\int_a^x(x-t)^{k-1}dt\\
f(x) &= y^n+\sum_{k=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-k+1)}y^{n-k}\cdot\frac{x^k}{\Gamma(k+1)}\\
f(x) &= y^n+\sum_{k=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}\cdot x^ky^{n-k}\\
f(x) &= y^n+\sum_{k=1}^{\infty}\binom{n}{k}x^ky^{n-k}\\
f(x) &= \sum_{k=0}^{\infty}\binom{n}{k}x^ky^{n-k}
\end{aligned}$$

which is the concise expression for the classical binomial expansion as we know it.

Example 6.2.2. *Compute the fractional order derivative of* $\sin(x)$

Solution.

$$D^1 \sin(x) = sin(x+\frac{\pi}{2}), D^2 \sin(x) = sin(x+2\cdot\frac{\pi}{2}), D^3 \sin(x) = sin(x+3\cdot\frac{\pi}{2}) \ldots D^\alpha \sin(x) = sin(x+\alpha\frac{\pi}{2})$$

Example 6.2.3. *Using* $D^\alpha \sin(x) = sin(x+\alpha\frac{\pi}{2})$ *from example 6.2.2, the following Wolfram Alpha code is used and executed using Mathematica software to illustrate the fractional Taylor series approximations of the sinus function up to twenty terms initiated at various k-values for* $\alpha \in \mathbb{R}$ *ranging between the limits of summation in unit intervals. Colour coded graphics are included on pages 25 to 27*

```
f[x_] = Sin[x];

g[x_] = Expand[(D[Sin[x], {x, 0}] /. {x → 0}) +
    Sum[(1/Gamma[k] * Integrate[(Sin[x + Pi k/2] /. {x → 0}) * (x - t)^(k-1), {t, 0, x}]), {k, 1, 20}]];

h[x_] = Expand[(D[Sin[x], {x, 0}] /. {x → 0}) +
    Sum[(1/Gamma[k] * Integrate[(Sin[x + Pi k/2] /. {x → 0}) * (x - t)^(k-1), {t, 0, x}]), {k, 1/2, 20}]];

i[x_] = Expand[(D[Sin[x], {x, 0}] /. {x → 0}) +
    Sum[(1/Gamma[k] * Integrate[(Sin[x + Pi k/2] /. {x → 0}) * (x - t)^(k-1), {t, 0, x}]), {k, 1/4, 20}]];

j[x_] = Expand[(D[Sin[x], {x, 0}] /. {x → 0}) +
    Sum[(1/Gamma[k] * Integrate[(Sin[x + Pi k/2] /. {x → 0}) * (x - t)^(k-1), {t, 0, x}]), {k, 1/6, 20}]];

k[x_] = Expand[(D[Sin[x], {x, 0}] /. {x → 0}) +
    Sum[(1/Gamma[k] * Integrate[(Sin[x + Pi k/2] /. {x → 0}) * (x - t)^(k-1), {t, 0, x}]), {k, 1/8, 20}]];

l[x_] = Expand[(D[Sin[x], {x, 0}] /. {x → 0}) +
    Sum[(1/Gamma[k] * Integrate[(Sin[x + Pi k/2] /. {x → 0}) * (x - t)^(k-1), {t, 0, x}]), {k, 1/10, 20}]];

p1 = Plot[f[x], {x, 0, 12}, PlotStyle → Purple];

p2 = Plot[g[x], {x, 0, 12}, PlotStyle → Blue];

p3 = Plot[h[x], {x, 0, 12}, PlotStyle → Green];

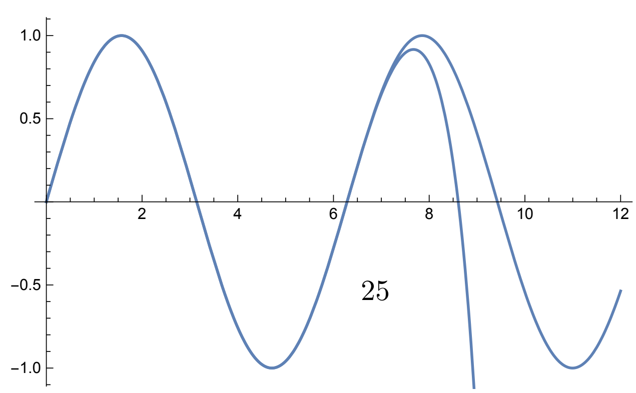
p4 = Plot[i[x], {x, 0, 12}, PlotStyle → Yellow];

p5 = Plot[j[x], {x, 0, 12}, PlotStyle → Orange];

p6 = Plot[k[x], {x, 0, 12}, PlotStyle → Pink];

p7 = Plot[l[x], {x, 0, 12}, PlotStyle → Red];

Show[p1, p2]
```



```
Show[p1, p3]
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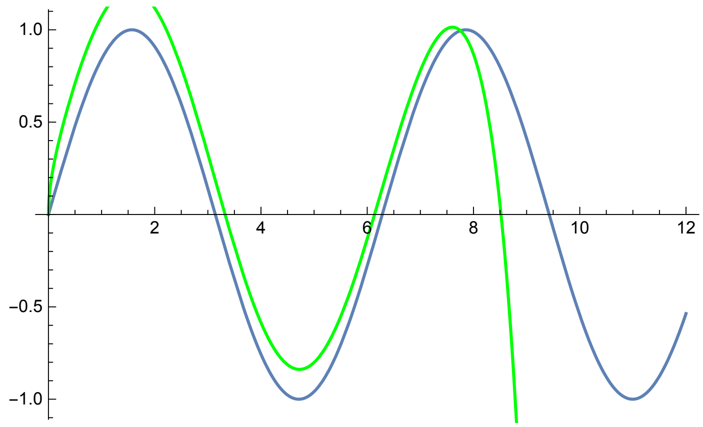
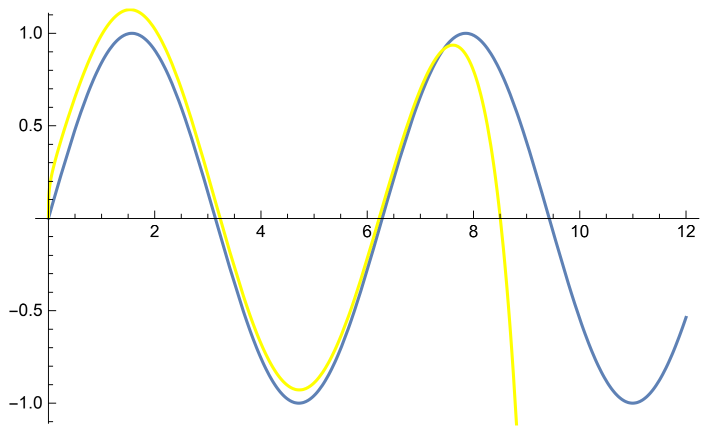
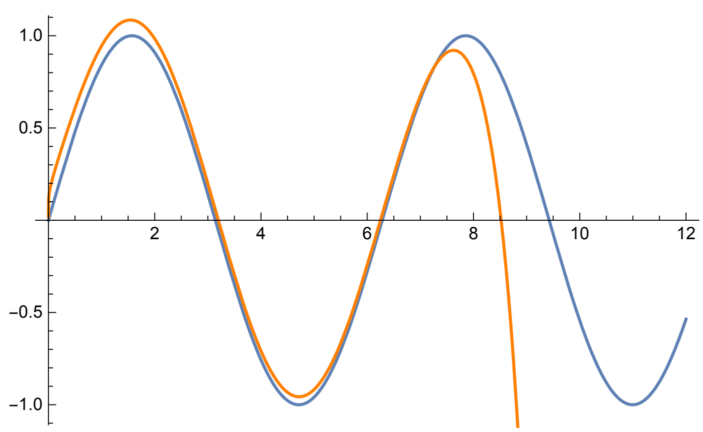


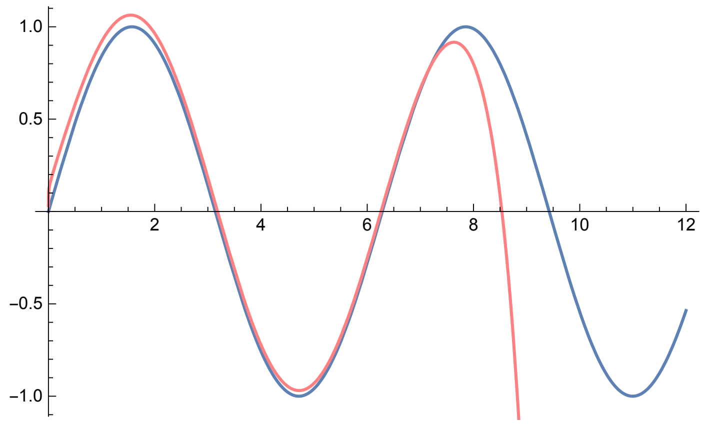

```
Show[p1, p4]
```



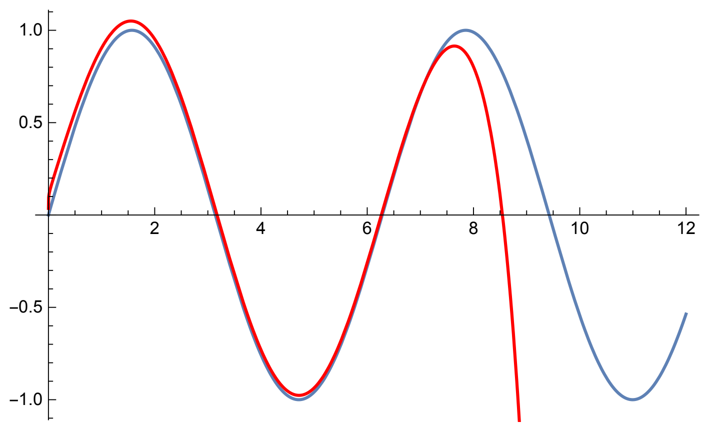

```
Show[p1, p5]
```



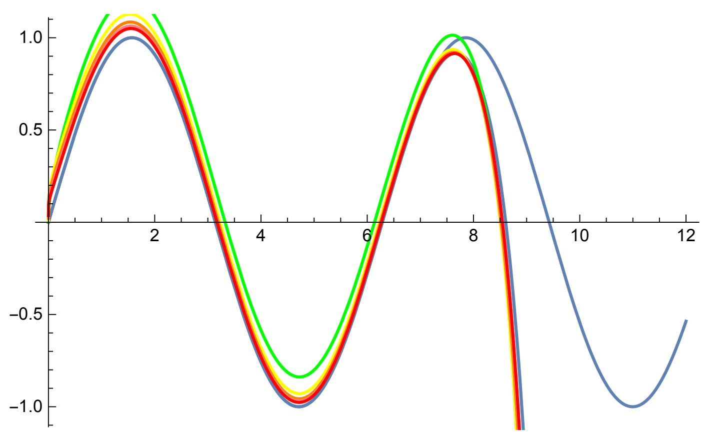
```
Show[p1, p6]
```




```
Show[p1, p7]
```




```
Show[p1, p2, p3, p4, p5, p6, p7]
```



$$\mathtt{Manipulate}\Big[\mathtt{Expand}\Big[\big(\mathtt{D[Sin[x],\{x,0\}]\ /.\ \{x\to 0\}}\big)+\sum_{k=p}^{n}\left(\frac{1}{\mathtt{Gamma[k]}}*\int_0^x\left(\mathtt{Sin}\left[x+\frac{\mathtt{Pi}\,k}{2}\right]\mathtt{\ /.\ \{x\to 0\}}\right)*(x-t)^{k-1}\,dt\right)\Big],\ \{n,0,10,p\},\ \left\{p,\frac{1}{10},1,\frac{1}{10}\right\}\Big]$$

n

p

$$\frac{10\,x^{7/10}\,\mathrm{Cos}\left[\frac{3\pi}{20}\right]}{7\,\mathrm{Gamma}\left[\frac{7}{10}\right]}-\frac{10\,x^{27/10}\,\mathrm{Cos}\left[\frac{3\pi}{20}\right]}{27\,\mathrm{Gamma}\left[\frac{27}{10}\right]}+\frac{10\,x^{47/10}\,\mathrm{Cos}\left[\frac{3\pi}{20}\right]}{47\,\mathrm{Gamma}\left[\frac{47}{10}\right]}-\frac{10\,x^{67/10}\,\mathrm{Cos}\left[\frac{3\pi}{20}\right]}{67\,\mathrm{Gamma}\left[\frac{67}{10}\right]}+\frac{10\,x^{87/10}\,\mathrm{Cos}\left[\frac{3\pi}{20}\right]}{87\,\mathrm{Gamma}\left[\frac{87}{10}\right]}+\frac{10\,x^{17/10}\,\mathrm{Sin}\left[\frac{3\pi}{20}\right]}{17\,\mathrm{Gamma}\left[\frac{17}{10}\right]}-\frac{10\,x^{37/10}\,\mathrm{Sin}\left[\frac{3\pi}{20}\right]}{37\,\mathrm{Gamma}\left[\frac{37}{10}\right]}+\frac{10\,x^{57/10}\,\mathrm{Sin}\left[\frac{3\pi}{20}\right]}{57\,\mathrm{Gamma}\left[\frac{57}{10}\right]}-\frac{10\,x^{77/10}\,\mathrm{Sin}\left[\frac{3\pi}{20}\right]}{77\,\mathrm{Gamma}\left[\frac{77}{10}\right]}$$

The following code shows the interactive comparison of the classical and fractional taylor series expansions of $\cos(x)$ and e^x

Taylor Series Expansion of cos(x) about x=0

```
Manipulate[Series[Cos[x], {x, 0, n}], {n, 0, 20, 1}]
```

n

$$1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\frac{x^{10}}{3628800}+\frac{x^{12}}{479001600}-\frac{x^{14}}{87178291200}+\frac{x^{16}}{20922789888000}-\frac{x^{18}}{6402373705728000}+\frac{x^{20}}{2432902008176640000}+O[x]^{21}$$

$$\text{Manipulate}\Big[\text{Expand}\Big[\{\text{D}[\text{Cos}[x],\{x,0\}]\ /.\ \{x\to 0\}\}+\sum_{k=1}^{n}\Big(\frac{1}{\text{Gamma}[k]}*\int_0^y \{\text{D}[\text{Cos}[x],\{x,k\}]\ /.\ \{x\to 0\}\}*(y-t)^{k-1}\,dt\Big)\Big],\{n,0,20,1\}\Big]$$

n

$$\Big\{1-\frac{y^2}{2}+\frac{y^4}{24}-\frac{y^6}{720}+\frac{y^8}{40320}-\frac{y^{10}}{3628800}+\frac{y^{12}}{479001600}-\frac{y^{14}}{87178291200}+\frac{y^{16}}{20922789888000}-\frac{y^{18}}{6402373705728000}+\frac{y^{20}}{2432902008176640000}\Big\}$$

Taylor Series Expansion of e^x about x=0

```
Manipulate[Series[E^x, {x, 0, n}], {n, 0, 10, 1}]
```

n

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+\frac{x^8}{40320}+\frac{x^9}{362880}+\frac{x^{10}}{3628800}+O[x]^{11}$$

$$\text{Manipulate}\Big[\text{Expand}\Big[\{\text{D}[\text{E}\verb|^|x,\{x,0\}]\ /.\ \{x\to 0\}\}+\sum_{k=1}^{n}\Big(\frac{1}{\text{Gamma}[k]}*\int_0^y \{\text{D}[\text{E}\verb|^|x,\{x,k\}]\ /.\ \{x\to 0\}\}*(y-t)^{k-1}\,dt\Big)\Big],\{n,0,10,1\}\Big]$$

n

$$\Big\{1+y+\frac{y^2}{2}+\frac{y^3}{6}+\frac{y^4}{24}+\frac{y^5}{120}+\frac{y^6}{720}+\frac{y^7}{5040}+\frac{y^8}{40320}+\frac{y^9}{362880}+\frac{y^{10}}{3628800}\Big\}$$

7 Integral Equations

There are four main types of integral equations, though each stated initially, will also be discussed in their respective subsections.

Definition 7.0.1 (Integral Equation Classes). *(Collins, 2006, p. 5) Suppose $f : [a, b] \to \mathbb{R}$ and $K : [a, b]^2 \to \mathbb{R}$ are continuous and λ, a and b are constants, then four main types of integral equations are defined by*

A1: Volterra Homogeneous Integral Equattion

$$y(x) = \int_a^x K(x,t)y(t)dt$$

A2: Volterra Non-Homogeneous Integral Equation

$$y(x) = f(x) + \int_a^x K(x,t)y(t)dt$$

B1: Fredholm Homogeneous Integral Equation

$$y(x) = \lambda \int_a^b K(x,t)y(t)dt$$

B2: Fredholm Non-Homogeneous Integral Equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$$

where $x \in [a, b]$ and $K(x, t)$ is called the kernel.

Remark. *If $K(x, t) = K(t, x)$ then the kernel is said to be symmetric*

To fully comprehend the relationship between differential and integral equations, we need the following Lemma:

Lemma 7.0.1 (Replacement Lemma). *(Collins, 2006, p. 6) Suppose $f : [a, b] \to \mathbb{R}$ is continuous for all $x \in [a, b]$, then*

$$\int_a^x \int_a^{x_1} f(t)\ dt\ dx_1 = \int_a^x (x - t)\ f(t)\ dt$$

Proof. To prove the above lemma, we need to use Leibniz' integral rule (Theorem 2.0.3)

Define $F : [a, b] \to \mathbb{R}$, for $x \in [a, b]$ by

$$F(x) = \int_a^x (x - t)f(t)\ dt$$

Since $(x-t)$ and $\frac{\partial}{\partial\, x}[(x-t)f(t)]$ are continuous for all $x, t \in [a,b]$, by Theorem 2.0.3,

$$F'(x) = [(x-t)f(t)]|_{t=x} \ \frac{d}{dx}x \ + \int_a^x \frac{\partial}{\partial x}[(x-t)f(t)]\ dt = \int_a^x f(t)\ dt$$

Since $\int_a^x f(t)\ dt$ and $\frac{\partial}{\partial x}F$ are continuous on $[a,b]$, we can use the Fundamental Theorem of Calculus Theorem 2.0.2 to deduce that

$$F'(x_1) = F(x_1) - F(a) = \int_a^{x_1} F'(x)\ dx = \int_a^{x_1}\int_a^x f(t)\ dt\ dx = \int_a^x \int_a^{x_1} f(t)\ dt\ dx_1$$

QED

Remark. *Repeated application of the replacement lemma is equivalent to the Riemann-Liouville integral operator. The replacement lemma is definition 5.1.1 with* $\alpha = 2$

Before we discuss the various types of integral equations, let us set up the preliminarie:

Consider the differential equation defined for all $x \in [0, L]$

$$y'' + \lambda y = g(x) \tag{1}$$

where λ is a positive constant and g is continuous on $[0, L]$. Integrating from 0 to x yields

$$y'(x) + y'(0) + \lambda \int_0^x y(t)\ dt = \int_0^x g(t)\ dt$$

By integrating a second time from 0 to x, we obtain

$$y(x) - y(0) - xy'(0) + \lambda \int_0^x \int_0^{x_1} y(t)\ dt\ dx_1 = \int_0^x \int_0^{x_1} g(t)\ dt\ dx_1$$

and by using the replacement lemma, we can rewrite

$$y(x) - y(0) - xy'(0) + \lambda \int_0^x (x-t)y(t)\ dt = \int_0^x (x-t)g(t)\ dt$$

then solving for $y(x)$ we obtain

$$y(x) = y(0) + xy'(0) + \int_0^x (x-t)g(t)\ dt - \lambda \int_0^x (x-t)y(t)\ dt \tag{*}$$

Remark. *To obtain the original differential equation from a given integral equation, we may use the Leibniz rule Theorem 2.0.3*

Remark. *The replacement lemma is also closely related to the convolution Product by*

$$\int_0^x \int_0^{x_1} f(t)\ dt\ dx_1 = \int_0^x (x-t)f(t)\ dt = x * f(x)$$

7.1 Volterra Integral Equations

If we impose initial conditions $y(0) = 0$ and $y'(0) = A$ on * where A is a real constant, then

$$y(x) = Ax + \int_0^x (x-t)g(t)\ dt - \lambda \int_0^x (x-t)y(t)\ dt$$

and define the kernel $K(x,t)$ and function $f(x)$ as

$$\begin{aligned} K(x,t) &= \lambda(t-x) \\ f(x) &= Ax + \int_0^x (x-t)g(t)\ dt \end{aligned}$$

then we obtain the Volterra non-homogeneous integral equation

$$y(x) = f(x) + \int_0^x K(x,t)y(t)\ dt$$

Remark. *Under the assumption that $f(x) = 0$, we obtain the Volterra homogeneous integral equation*

7.2 Fredholm integral Equations

If we impose the boundary value conditions where y is given at the end-points of an interval, $y(0) = 0$ and $y(L) = B$, and B is a constant, by setting $y''(x) = u(x)$ and two integrations, we obtain

$$y(x) = y'(0)x + \int_0^x (x-t)u(t)\ dt \tag{**}$$

By substituting $x = L$ we obtain

$$y'(0) = \frac{B}{L} - \frac{1}{L}\int_0^L (L-t)u\ dt \tag{***}$$

Now, by substituting *** back into ** we get

$$y(x) = \frac{Bx}{L} - \frac{x}{L}\int_0^L (L-t)u(t)\ dt + \int_0^x (x-t)u(t)\ dt \tag{****}$$

Substituting $u(x)$ and equation **** back into 1 and using the rule

$$\int_0^L h(t)\ dt = \int_0^x h(t)\ dt + \int_x^L h(t)\ dt$$

we obtain after manipulating

$$y(x) = g(x) - \frac{\lambda Bx}{L} + \lambda\int_0^x \frac{t}{L}(L-x)u(t)\ dt + \lambda\int_x^L \frac{x}{L}(L-t)u(t)\ dt$$

If we let

$$
\begin{aligned}
f(x) &= g(x) - \frac{\lambda B x}{L} \\
K(x,t) &= \begin{cases} \frac{t}{L}(L-x) & \text{if } 0 \le t \le x \le L \\ \frac{x}{L}(L-t) & \text{if } 0 \le x \le t \le L \end{cases}
\end{aligned}
$$

we obtain the Fredholm non-homogeneous integral equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt$$

Remark. *Under the assumption that $f(x) = 0$, we obtain the Fredholm homogeneous integral equation*

$$y(x) = \lambda \int_a^b K(x,t) y(t) dt$$

8 Application, Motivation and Future Directions

8.1 Application of Fractional Calculus

Fractional calculus has most of its application in fractional differential equations (sometimes referred to as extraordinary differential equations) which provides the necessary tools to solve complicated initial value problems by various methods, such as the Laplace and Fourier transform method, the power series method and transformation to ordinary differential equations method.

In this section we will discuss the application of fractional calculus to the tautochrone problem.

In words, the tautochrone problem states:

Determine the curve in the (x, y)-plane such that the time required for a particle with mass m, to slide down the curve to its lowest point (ignoring friction), is a minimum, independent of its initial placement on the curve

Figure 3 below provides a visual illustration of the problem where

(X, Y) is the initial placement of the particle
$(0, 0)$ is the final placement of the particle
(x, y) is the intermediate placement of the particle
σ is the arc length measured from the origin $(0, 0)$
m is the mass of the particle
g is the gravitational acceleration constant
t is the independent variable time

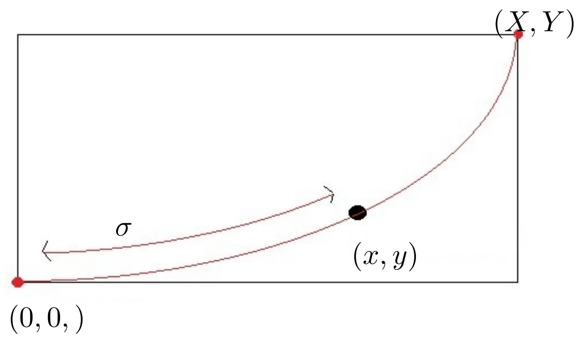


Figure 3: The curve of a point particle

We set up an equation by implementing the conservation of energy law from classical physics which states that

The gain in kinetic energy is equal to the loss of the potential energy

In an equation, this becomes

$$\begin{aligned}
&\implies \quad \frac{1}{2}m\left(\frac{d\sigma}{dt}\right)^2 = mg(Y-y) \\
&\implies \quad \left(\frac{d\sigma}{dt}\right)^2 = 2g(Y-y) \\
&\implies \quad \frac{d\sigma}{dt} = \pm\sqrt{2g(Y-y)} \\
&\implies \quad d\sigma = -\sqrt{2g(Y-y)}dt
\end{aligned}$$

Since $\frac{d\sigma}{dt} < 0$ because the arc length decreases as the particle slides down its path. Take note that σ is a function of height y which is in turn dependent on time t so that $\sigma = \sigma(y(t))$. Then by the chain rule we have

$$d\sigma = \frac{d\sigma}{dy}dy$$

By letting $\sigma^{(1)}(y) = \frac{d\sigma}{dy}$ we get the following separable ordinary differential equation

$$\frac{d\sigma}{\sqrt{Y-y}} = -\sqrt{2g}dt \implies \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = -\sqrt{2g}dt$$

Now, integrating from $y = Y$ to $y = 0$ and keeping in mind that $t_Y = 0$ and $t_0 = T$ (T is the time taken for the particle to slide from its initial to final position), we get

$$\int_Y^0 \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = -\int_0^T \sqrt{2g}dt$$

$$\implies \quad \int_0^Y \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = \sqrt{2g}\ T$$

Notice that the left hand side can be expressed using the convolution product

$$\int_0^Y \frac{\sigma^{(1)}(y)dy}{\sqrt{Y-y}} = y^{-\frac{1}{2}} * \sigma^{(1)}$$

so that we can write

$$\sqrt{2g}\ T = y^{-\frac{1}{2}} * \sigma^{(1)}$$

We now apply the Laplace transform to both sides and applying the convolution theorem, we obtain

$$\frac{1}{s}\sqrt{2g}\ T = \sqrt{\frac{\pi}{s}}\mathcal{L}\left(\frac{d\sigma}{dy}\right)$$

So by applying the inverse Laplace Transform and solving for $\frac{d\sigma}{dy}$ we get

$$\frac{d\sigma}{dy} = \frac{\sqrt{2g}\ T}{\pi}\ y^{-\frac{1}{2}}$$

Let us also write $f(Y) = \sqrt{2g}\ T$, then we finally obtain Abel's solution to the tautochrone problem

$$\sigma(y) = \frac{1}{\pi}\int_0^y \frac{f(Y)}{\sqrt{y-Y}}\ dY$$

which is expressed in terms of the height y along the curve.

What we would like to point out in the current situation is the connection of Abel's solution to the semi-integral of $\sigma^{(1)}$:

$$J^{\frac{1}{2}}\sigma^{(1)}(y) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^Y (Y-y)^{\frac{1}{2}-1}\sigma^{(1)}(y)\ dy$$

$$\implies \quad J^{\frac{1}{2}}\sigma^{(1)}(y) = \frac{1}{\sqrt{\pi}}\int_0^Y \frac{\sigma^{(1)}(y)}{\sqrt{Y-y}}\ dy$$

$$\implies \quad \sqrt{\pi}\ J^{\frac{1}{2}}\sigma^{(1)}(y) = \int_0^Y \frac{\sigma^{(1)}(y)}{\sqrt{Y-y}}\ dy$$

$$\implies \quad f(y) = \sqrt{\pi}\ J^{\frac{1}{2}}\sigma^{(1)}(y)$$

$$\implies \quad J^{\frac{1}{2}}f(y) = \sqrt{\pi}\ J^{\frac{1}{2}}J^{\frac{1}{2}}\sigma^{(1)}(y) \ \ \text{(applying} J^{\frac{1}{2}}\text{as operator on both sides)}$$

$$\implies \quad \sigma(y) = \frac{1}{\sqrt{\pi}}J^{\frac{1}{2}}f(y)$$

$$\implies \quad \sigma(y) = \frac{1}{\sqrt{\pi}}\ \frac{1}{\sqrt{\pi}}\int_0^y \frac{f(Y)}{\sqrt{Y-y}}\ dy$$

$$\implies \quad \sigma(y) = \frac{1}{\pi}\ \int_0^y \frac{f(Y)}{\sqrt{Y-y}}\ dy$$

$$\implies \quad \sigma(y) = \frac{1}{\pi}\ \int_0^y \frac{\sqrt{2g}\ T}{\sqrt{Y-y}}\ dy$$

which corresponds to Abel's solution of the tautochrone problem. As we can observe, the fractional calculus avoided the use of Laplace transforms and evaluation of complicated integrals, which is one of the advantages of using this as a tool in mathematics.

8.2 Motivation

In this section we motivate briefly that fractional calculus is an interesting topic which arises naturally and we provide an intuitive explanation.

Suppose we wanted to find a quicker method for computing the expansion below

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n$$

Observe the patterns of the powers on the right hand side. The coefficients will be dealt with during development of the discussion. The exponent of x reduces by 1 with each successive term - This suggests differentiation. The exponent of y increases by 1 with each successive term - This suggests integration. We might as well have done it as follows, starting with x^n lone term:

(1) Our first term is x^n

(2) compute the product of $\frac{d}{dx}x^n$ and $\int_0^y 1d\sigma_1$ to find the second term

(3) compute the product of $\frac{d^2}{dx^2}x^n$ and $\int_0^y \int_0^{\sigma_1} 1d\sigma_2 d\sigma_1$ to find the third term

$\vdots$

(k) compute the product of $\frac{d^k}{dx^k}x^n$ and $\int_0^y \int_0^{\sigma_1} \cdots \int_0^{\sigma_{k-1}} 1d\sigma_k d\sigma_{k-1} \cdots d\sigma_1$ to find the kth term

$\vdots$

Inherent to the computations, the coefficients will work out as we will see after rewriting using Cauchy's repeated integral formula and the power rule for derivatives.

(1) Our first term is x^n

(2) compute the product of nx^{n-1} and $\frac{1}{\Gamma(1)}\int_0^y (y-t)^0 dt$ to find the second term

(3) compute the product of $n(n-1)x^{n-2}$ and $\frac{1}{\Gamma(2)}\int_0^y (y-t)^1 dt$ to find the third term

$\vdots$

(k) compute the product of $\frac{\Gamma(n+1)}{\Gamma(n-k+1)}x^{n-k}$ and $\frac{1}{\Gamma(k)}\int_0^y (y-t)^{k-1} dt$ to find the kth term

$\vdots$

As can be seen, the binomial coefficients naturally arise as a result of these operations described above. Our final task is to sum these terms together which can be expressed as

$$(x+y)^n = x^n + \sum_{k=1}^{n} \left(\frac{d^k}{dx^k}x^n\right)\left(\frac{1}{\Gamma(k)}\int_0^y (y-t)^{k-1}\ dt\right)$$

This expression resembles that of the Taylor series expansion of $(x+y)^n$ about the point 0 discussed in example 6.2.1. This natural result partially stimulated the investigation for the project on fractional calculus.

8.3 Future Directions

The Leibniz rule generalises the product rule to arbitrary order $n \in \mathbb{N}$ and is stated in the following theorem

Theorem 8.3.1 (Generalized Leibniz Rule)**.** *If f and g are n times differentiable functions, then the product (fg) is also n times differentiable and its n^{th} derivative is given by*

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\ g^{(k)}(x)$$

It is a relief that the n^{th} derivative of a product of functions obey an analogously simple algebraic property to that of the binomials. It seems far to obvious to ignore that there is some intricate connection, perhaps on the indices of various mathematical expressions, yet there is no clearly cut relationship defined between the binomial expansion and Theorem 8.3.1. Perhaps a deeper study of the fractional calculus could reveal such information.

==
Connection to Functional Analysis
==

It is a natural question to ask about the behaviour of these fractional order operators through the scope of functional analysis, considering their algebraic behaviour. We would like to know whether it is possible to treat fractional differential equations as elegantly as we do ordinary differential equations. For example, the wave equation from physics

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\frac{\partial^2}{\partial x^2}u(x,t) = 0$$

can be "factorized" in a sense

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u(x,t) = 0$$

since the underlying theory for this has already been well developed by the works of F. Riesz, J Schauder, Volterra and Fredholm himself, just to mention a few (Kreyszig, 1978, p436)

In section 7 we briefly discussed Fredholm's linear integral equations where we now generalize the theory to normed spaces.

Let X be a normed space over the complex field and $0 \neq \lambda \in \mathbb{C}$.

Consider the compact linear operator $T : X \to X$ and the adjoint operator $T^\times : X' \to X'$.

The four equations below are of interest:

A1: $Tx - \lambda x = y$

A2: $Tx - \lambda x = 0$

B1: $T^\times x - \lambda x = g$

B2: $T^\times x - \lambda x = 0$

where $y \in X$ and $g \in X'$

The solvability of these equations are summarised as follows

1. A1 has a solution x iff $f(y) = 0$ for all solutions f of B2.
   Hence if $f = 0$ is the only solution of B2, then the equation A1 is solvable for all y
2. B1 has a solution f iff $g(x) = 0$ for all solutions x of A2.
   Hence if $x = 0$ is the only solution of A2, then equation B1 is solvable for all g

3. A1 has a solution x for every $y \in X$ iff $x = 0$ is the only solution of A2
4. B1 has a solution f for every $g \in X'$ iff $f = 0$ is the only solution of B2.
5. A2 and B2 have the same number of linearly independent solutions.

We invoke the question of how one can generalize the concept of solvability to suit equations involving non-integer order operators and what would be analogous criteria as stated above for such operators?

A reasonable approach to answer these type of questions could be achieved by generalizing the sequence spaces l^p and l^q in which the classical conjugate exponents equation

$$\frac{1}{p} + \frac{1}{q} = 1$$

no longer holds, but rather consider the relation between p and q to be such that

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

and $1 \leq p < \frac{n}{\alpha}$ where α is the order of a linear operator from $l^p(\mu)$ and μ a measure.

Many new interesting results are achieved by refuting the classical $\frac{1}{p} + \frac{1}{q} = 1$. For example, recently Furuta (I Gil, 2009, p. 57 to 58 - abstract) showed the operator inequality

$$(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$$

where A and B are bounded linear operators on a Hilbert space satisfying $0 \leq B \leq A$ and positive real numbers p and r satisfy for $q \geq 1$

$$p + 2r \leq (1 + 2r)q$$

As a last note to the reader, we want to mention that the final comments will be addressed in a potential follow-up article. We wish to thank the reader for their interest up to this point and welcome any suggestions regarding improvement and progress towards a better write-up.

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