Complexity Theory I Jan – May, 2023

Lecture 1: Introduction

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1 Reference

Books:-

- Introduction to the Theory of Computation by Michael Sipser [**?**]
- Computational Complexity: A Modern Approach by Sanjeev Arora and Boaz Barak [**?**]
- Computational Complexity: A Conceptual Perspective by Oded Goldreich [**?**]
- Mathematics and Computation: A Theory Revolutionizing Technology and Science by Avi Wigderson [**?**]

Lecture Notes:-

- Madhu Sudan: 2018 and 2021
- Venkat Guruswamy: 2011 and 2009
- Luca Trevisan: 2015, 2014, 2012, 2010
- Salil Vedhan: Notes
- Prahlad Harsha: 2021, 2020, 2018, 2014, 2013, 2012, 2011
- Jay Kumar Radhakrishnan: 2004

2 Basic Classes

Note:-

All the classes in this course are subsets of decidable problems

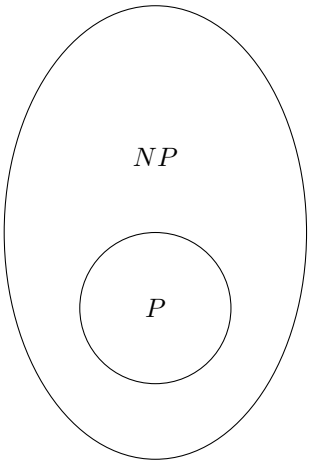
We know for any problem P a language L_P is associated.
$\boldsymbol{P} \coloneqq$ Class of problems that can be decided in deterministic polynomial time
$\boldsymbol{NP} \coloneqq$ Class of problems for which witness can be verified in deterministic polynomial time. Now it is obvious that P is contained in NP.

But we don't know if $P = NP$ or not.

Open Question 1

$$P \subsetneq NP$$

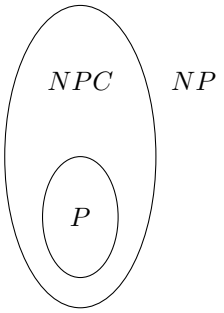
But it is believed that $P \neq NP$

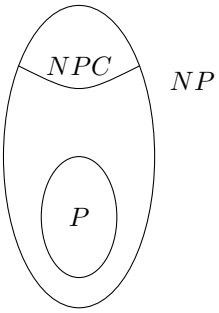


This is called the Complexity Zoo. As the course will further progress more and more things will be added

Example 1 ($NP-complete$ (NPC))
$SAT, 3COL, VC, HAM, \ldots$
$3COL$:=Given a graph G, check whether the graph is 3-colorable
[**?**] a has a lot of examples of $NP-complete$ problems

Now there was question that what is the set $NP \setminus P$ if $NP \neq P$. Which one of the following true



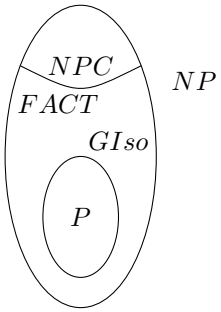


Ladner proved in [**?**] that there exists an infinite hierarchy of problems sitting from P to $NP-complete$, $L \in NP \setminus (P \cup NPC)$.

But we know very less natural problems in $NP \setminus (P \cup NPC)$. Eg: Factoring (it is believed), Graph Isomorphism or $GIso$ (It was believed that $GIso$ is in $NP \setminus (P \cup NPC)$ but Laci Babai in [**?**] brought it to very closed to P).

Primality was used to be believed in $NP \setminus (P \cup NPC)$ but later it was discovered that it is in P in [**?**]
Graph Isomorphism (***GIso***) := $G_1 = (V_1, E_1)$, $G_2 = (v-2, E_2)$. The question is if there exists $\phi : V_1 \to V_2$ such that $(u, v) \in E_1 \iff (\phi(u), \phi(v)) \in E_2$

Now the picture is



3 Space/Memory

Now comes another question. Are the problems in P "memory" efficient?.

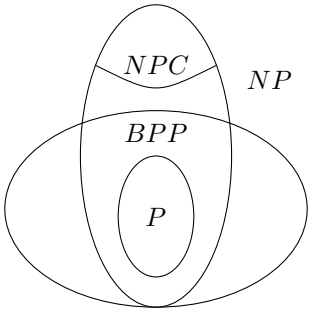
(G, E) is an undirected graph. If $u, v \in V$ then is $u \to v$ reachable? Any search algorithm like DFS, BFS works$\approx \xrightarrow{2004-Reingold[?]} O(\log n)-$memory

4 Randomness

Now we allow the Turing Machine to toss a coin to take decisions and then walk accordingly. It contains P as we can say stop tossing. Hence $P \subseteq BPP$

Here randomized algorithms are used to solve problems. Eg. Miller Robin Ptimality Test.

Now the picture is



BPP $\coloneqq$ Class of problems which can be solved in polynomial time with the power of randomness

Open Question 2

$$P = BPP$$

But it is believed that $P = BPP$. This gives a philosophical question that can every randomized polynomial time algorithm be converted to a polynomial time deterministic algorithm with comparable time?

Note:-

BPP has no such comlete problems like $NP-complete$ problems in NP class. We have to use pseudorandom generator. So the task is to build a powerfull pseudoranddom generator

5 Lower Bounds

One such question on lower bound to show $P \neq NP$ is

Question 1

Let ψ be a boolean formula and $|\psi| = n$. Prove that if $\psi \in SAT$ it requires at least super linear time $(n^{1.1})$

We don't even know that

We can put some constraints or cut down some power of turing machine to show some lower bounds.

6 Diagonalization

Theorem 1

$$DTIME(n^2) \subsetneq DTIME(n^3)$$

People used to believe that these kind of ideas can be used to prove $P \neq NP$. But Natural Proofs [?] crushed this idea.