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Course: Complex Analysis

Assignment - 5
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Problem 1 Ahlfors Page 154: Problem 1

How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$? Hint: Look for the biggest term when $|z| = 1$ and apply Rouche's theorem.

Solution: Take $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$ and $f(z) = 6z^3$. Then on $|z| = 1$

$$|g(z) - f(z)| = |z^7 - 2z^5 - z + 1| \leq |z|^7 + 2|z|^5 + |z| + 1 \leq 1 + 2 + 1 + 1 = 5 < 6 = |f(z)|$$

Hence by Rouche's Theorem $f(z)$ and $g(z)$ has same number of zeros inside $|z| < 1$. Now f has only three zero (one zero with order three). Hence $g(z)$ has three zeros inside $|z| < 1$.

□

Problem 2 Ahlfors Page 154: Problem 2

How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2 ?

Solution: Let $g(z) = z^4 - 6z + 3$. In this case we find how many zeros g has inside $|z| < 1$ and $|z| < 2$. then we subtract the number of zeros inside the smaller disk from the number of zeros inside the bigger disk.

Take $f(z) = -6z$. Then on $|z| = 1$ we have

$$|g(z) - f(z)| = |z^4 + 3| \leq |z|^4 + 1 = 4 < 6 = |f(z)|$$

Hence by Rouche's theorem f and g has same number of zeros inside $|z| < 1$. Now f has only one zero in $|z| < 1$. Hence g has one zero in $|z| < 1$.

Now take $f(z) = z^4$. Then $|z| = 2$ we have

$$|g(z) - f(z)| = 6z + 3| \leq 6|z| + 3 = 6 \times 2 + 3 = 15 < 16 = |f(z)|$$

Hence by Rouche's Theorem f and g has same number of zeros inside $|z| < 2$. Now f has only four zeros in $|z| < 2$ (one zero with order four). Therefore g has four zeros in $|z| < 2$.

Hence g has $4 - 1 = 3$ zeros in the region between $|z| < 1$ and $|z| < 2$.

□

Problem 3 Ahlfors Page 161: Problem 3

Evaluate the following integrals by the method of residues:

(a) $\int\limits_0^{\pi/2} \frac{dx}{a + \sin^2 x}, |a| > 1,$

(b) $\int\limits_0^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6},$

(c) $\int\limits_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx,$

(d) $\int\limits_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)^3}, a \text{ real},$

(e) $\int\limits_0^{\infty} \frac{\cos x}{x^2 + a^2} dx, a \text{ real},$

(f) $\int\limits_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx, a \text{ real},$

(g) $\int\limits_0^{\infty} \frac{x^{1/3}}{1 + x^2} dx,$

(h) $\int\limits_0^{\infty} (1 + x^2)^{-1} \log x dx,$

(i) $\int\limits_0^{\infty} \log(1 + x^2) \frac{dx}{x^{1+\alpha}} (0 < \alpha < 2)$. (Try integration by parts.)

Solution:

(a) We know $\sin^2 x = \frac{1-\cos 2x}{2}$ Hence

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{dx}{a+\frac{1-\cos 2x}{2}} = \int_0^{\frac{\pi}{2}} \frac{2dx}{2a+1-\cos 2x} = \int_0^{\pi} \frac{dt}{2a+1-\cos t} \text{ [Substituting } t = 2x]$$

Now take $-2a-1 = b$. Threrefore $|b| > 1$. Hence we need to evaluate $\displaystyle\int_0^{\pi} \frac{dx}{b-\cos x}$

Case 1: $b > 1$

Then this is like Ahlfors Case 1 Example. Then

$$\int_0^{\pi} \frac{dx}{b-\cos x} = \frac{\pi}{\sqrt{b^2-1}} = \frac{\pi}{\sqrt{(2a+1)^2-1}} = \frac{\pi}{2\sqrt{a}\sqrt{a+1}}$$

Case 2: $b < -1$

Now

$$\int_0^{\pi} \frac{dx}{b+\cos x} = -i\int_{|z|=1} \frac{dz}{z^2+2bz+1}$$

Hence $z^2+2bz+1 = (z-\alpha)(z-\beta)$ where

$$\alpha = -b+\sqrt{b^2-1} \quad \beta = -b-\sqrt{b^2-1}$$

Then just like Ahlfors Case 1 Example instead of taking the root $\alpha = -b+\sqrt{b^2-1}$ we will choose the root $\beta = -b-\sqrt{b^2-1}$, cause $|\alpha| > 1$ and $|\beta| < 1$. Therefore residue at β is $\frac{1}{\beta-\alpha} = -\frac{1}{2\sqrt{b^2-1}}$. Then we get

$$\int_0^{\pi} \frac{dx}{b+\cos x} = -\frac{\pi}{\sqrt{b^-1}} = -\frac{\pi}{2\sqrt{a}\sqrt{a+1}}$$

Therefore

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} = \begin{cases} \dfrac{\pi}{2\sqrt{a}\sqrt{a+1}} & \text{when } a > 1 \\ -\dfrac{\pi}{2\sqrt{a}\sqrt{a+1}} & \text{when } a < -1 \end{cases}$$

□

(b)

$$\int_0^{\infty} \frac{x^2dx}{x^4+5x^2+6} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2dx}{x^4+5x^2+6}$$

Now this is like Ahlfors Case 2. Using that we get

$$\int_{-\infty}^{\infty} \frac{x^2dx}{x^4+5x^2+6} = 2\pi i \sum_{y>0} \operatorname{Res} \frac{z^2}{z^4+5z^2+6}$$

Now

$$z^4+5z^2+6 = (z^2+3)(z^2+2) = (z+\sqrt{2}i)(z-\sqrt{2}i)(z+\sqrt{3}i)(z-\sqrt{3}i)$$

Now

$$\operatorname{Res}_{z=\sqrt{3}i} \frac{z^2}{z^4+5z^2+6} = \lim_{z\to\sqrt{3}i} \frac{z^2(z-\sqrt{3}i)}{z^4+5z^2+6} = \frac{-3}{2\sqrt{3}i(-3+2)} = \frac{\sqrt{3}}{2i}$$

$$\operatorname{Res}_{z=\sqrt{2}i} \frac{z^2}{z^4+5z^2+6} = \lim_{\to\sqrt{2}i} \frac{z^2(z-\sqrt{2}i)}{z^4+5z^2+6} = \frac{-2}{2\sqrt{2}i(-2+3)} = -\frac{\sqrt{2}}{2i}$$

Therefore

$$2\pi i \sum_{y>0} \text{Res}\, \frac{z^2}{z^4+5z^2+6} = 2\pi i \left(\text{Res}_{z=\sqrt{3}i}\, \frac{z^2}{z^4+5z^2+6} + \text{Res}_{z=\sqrt{2}i}\, \frac{z^2}{z^4+5z^2+6} \right)$$
$$= 2\pi i \left(\frac{\sqrt{3}}{2i} - \frac{\sqrt{2}}{2i} \right) = \pi(\sqrt{3}-\sqrt{2})$$

Hence

$$\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6} = \frac{\pi}{2}(\sqrt{3}-\sqrt{2})$$

□

(c) Now this is like Ahlfors Case 2. Using that we get

$$\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9} = 2\pi i \sum_{y>0} \text{Res}\, \frac{z^2-z+2}{z^4+10z^2+9}$$

Now

$$z^4+10z^2+9 = (z^2+9)(z^2+1) = (z+3i)(z-3i)(z+i)(z-i)$$

Now

$$\text{Res}_{z=3i}\, \frac{z^2-z+2}{z^4+10z^2+9} = \lim_{z\to 3i} \frac{(z^2-z+2)(z-3i)}{z^4+10z^2+9} = \frac{-9-3i+2}{6i(-9+1)} = \frac{7+3i}{48i}$$
$$\text{Res}_{z=i}\, \frac{z^2-z+2}{z^4+10z^2+9} = \lim_{z\to i} \frac{(z^2-z+2)(z-i)}{z^4+10z^2+9} = \frac{-1-i+2}{(-1+9)2i} = \frac{1-i}{16i} = \frac{3-3i}{48i}$$

Therefore

$$2\pi i \sum_{y>0} \text{Res}\, \frac{z^2}{z^4+5z^2+6} = 2\pi i \left(\text{Res}_{z=3i}\, \frac{z^2-z+2}{z^4+10z^2+9} + \text{Res}_{z=i}\, \frac{z^2-z+2}{z^4+10z^2+9} \right)$$
$$= 2\pi i \left(\frac{7+3i}{48i} + \frac{3-3i}{48i} \right) = \pi \frac{10}{24} = \frac{5\pi}{12}$$

Hence

$$\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9} = \frac{5\pi}{12}$$

□

(d) If $a=0$ then the integral is infinite. Hence lets assume $a \neq 0$.

$$\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^3} = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2 dx}{(x^2+a^2)^3}$$

Now this is like Ahlfors Case 2. Using that we get

$$\int_{-\infty}^\infty \frac{x^2 dx}{(x^2+a^2)^3} = 2\pi i \sum_{y>0} \text{Res}\, \frac{z^2}{(z^2+a^2)^3}$$

Now

$$(z^2+a^2)^3 = (z+ai)^3(z-ai)^3$$

Now WLOG assume $a>0$. Now

$$\frac{d^2}{dz^2} \frac{z^2(z-ai)^3}{(z^2+a^2)^3} = \frac{d}{dz}\left[\frac{2z}{(z+ai)^3} - \frac{3z^2}{(z+ai)^4} \right] = \frac{d^2}{dz^2} \frac{z^2}{(z+ai)^3} = \frac{2}{(z+ai)^3} - \frac{12z^2}{(z+ai)^4} + \frac{12z^2}{(z+ai)^5}$$

$$\mathrm{Res}_{z=ai}\,\frac{z^2}{(z^2+a^2)^3} = \lim_{z\to ai}\frac{1}{2!}\frac{d^2}{dz^2}\frac{z^2(z-ai)^3}{(z^2+a^2)^3} = \frac{1}{2}\lim_{z\to ai}\left(\frac{2}{(z+ai)^3}-\frac{12z}{(z+ai)^4}+\frac{12z^2}{(z+ai)^5}\right) = \frac{1}{16a^3i}$$

$$\mathrm{Res}_{z=-ai}\,\frac{z^2}{(z^2+a^2)^3} = \lim_{z\to -ai}\frac{1}{2!}\frac{d^2}{dz^2}\frac{z^2(z+ai)^3}{(z^2+a^2)^3} = \frac{1}{2}\lim_{z\to -ai}\left(\frac{2}{(z-ai)^3}-\frac{12z}{(z-ai)^4}+\frac{12z^2}{(z-ai)^5}\right) = -\frac{1}{16a^3i}$$

Therefore

$$2\pi i\sum_{y>0}\mathrm{Res}\,\frac{z^2}{(z^2+a^2)^3} = \begin{cases} 2\pi i\,\mathrm{Res}_{z=ai}\,\dfrac{z^2}{(z^2+a^2)^3} = 2\pi i\dfrac{1}{16a^3i} = \dfrac{2\pi}{16a^3} & \text{when } a>0 \\ 2\pi i\,\mathrm{Res}_{z=-ai}\,\dfrac{z^2}{(z^2+a^2)^3} = 2\pi i\dfrac{1}{16(-a)^3i} = \dfrac{2\pi}{16(-a)^3} & \text{when } a<0 \end{cases}$$

Hence

$$\int_0^\infty \frac{x^2dx}{x^4+5x^2+6} = \frac{\pi}{16|a|^3} \quad \text{when } a\neq 0$$

□

(e) If $a=0$ then

$$\int_0^\infty \frac{\cos x}{x^2}dx \geq -\int_0^\infty \frac{dx}{x^2}$$

which is divergent hence the integral is infinite. Hence lets assume $a\neq 0$. Since $\frac{\sin x}{x^2+a^2}$ is an odd function we have $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x}{x^2+a^2}dx = 0$. Hence we have

$$\int_0^\infty \frac{\cos x}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}dx = \frac{1}{2}\left[\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}dx + i\int_{-\infty}^\infty \frac{\sin x}{x^2+a^2}\right] = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{ix}dx}{x^2+a^2}$$

Now this is like Ahlfors Case 3

$$\int_{-\infty}^\infty \frac{e^{ix}dx}{x^2+a^2} = 2\pi i\sum_{y>0}\mathrm{Res}\,\frac{e^{iz}}{z^2+a^2}$$

now $z^2+a^2=(z+ai)(a-ai)$. WLOG assume $a>0$

$$\mathrm{Res}_{z=ai}\,\frac{e^{iz}}{z^2+a^2} = \lim_{z\to ai}\frac{e^{iz}(z-ai)}{z^2+a^2} = \frac{e^{-a}}{2ai}$$

Hence

$$2\pi i\sum_{y>0}\mathrm{Res}\,\frac{1}{z^2+a^2} = 2\pi i\,\mathrm{Res}_{z=ai}\,\frac{e^{iz}}{z^2+a^2} = 2\pi i\frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}$$

Hence

$$\int_0^\infty \frac{\cos x}{x^2+a^2}dx = \frac{\pi e^{-a}}{2a}$$

□

(f) If $a=0$ then we have $\displaystyle\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}]$ by Ahlfors Case 3 Example (Page 158)

Since $\frac{x\cos x}{x^2+a^2}$ is an odd function we have $\displaystyle\int_{-\infty}^{\infty}\frac{x\cos x}{x^2+a^2}dx = 0$. Hence we have

$$i\int_0^\infty \frac{x\sin x}{x^2+a^2}dx = i\frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\left[\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx + i\int_{-\infty}^\infty \frac{x\cos x}{x^2+a^2}\right] = \frac{1}{2}\int_{-\infty}^\infty \frac{xe^{ix}dx}{x^2+a^2}$$

Now this is like Ahlfors Case 3

$$\int_{-\infty}^{\infty} \frac{xe^{ix}dx}{x^2+a^2} = 2\pi i \sum_{y>0} \text{Res}\, \frac{ze^{iz}}{z^2+a^2}$$

now $z^2+a^2 = (z+ai)(a-ai)$. WLOG assume $a > 0$

$$\text{Res}_{z=ai} \frac{ze^{iz}}{z^2+a^2} = \lim_{z\to ai} \frac{ze^{iz}(z-ai)}{z^2+a^2} = \frac{aie^{-a}}{2ai} = \frac{e^{-a}}{2}$$

Hence

$$2\pi i \sum_{y>0} \text{Res}\, \frac{1}{z^2+a^2} = 2\pi i\, \text{Res}_{z=ai} \frac{ze^{iz}}{z^2+a^2} = 2\pi i \frac{e^{-a}}{2} = \pi e^{-a} i$$

Hence

$$\int_0^{\infty} \frac{x\sin x}{x^2+a^2} dx = \Im\left(\frac{1}{2}\pi e^{-a} i\right) = \frac{\pi e^{-a}}{2}$$

$\square$

(g) Now this is like Ahlfors Case 4. Hence we have

$$\int_0^{\infty} \frac{x^{1/3}dx}{1+x^2} = \frac{2}{1-e^{2\pi i/3}} \sum \text{Res}\, \frac{z^{\frac{1}{3}}}{1+z^2}$$

Now

$$1+z^2 = (z+i)(z-i)$$

Hence

$$\text{Res}_{z=i} = \lim_{z\to i} \frac{z^{\frac{1}{3}}(z-i)}{1+z^2} = \frac{i^{\frac{1}{3}}}{2i} = \frac{e^{\frac{i\pi}{6}}}{2}$$
$$\text{Res}_{z=-i} = \lim_{z\to -i} \frac{z^{\frac{1}{3}}(z+i)}{1+z^2} = \frac{(-i)^{\frac{1}{3}}}{-2i} = -\frac{e^{\frac{i\pi}{2}}}{2}$$

Hence

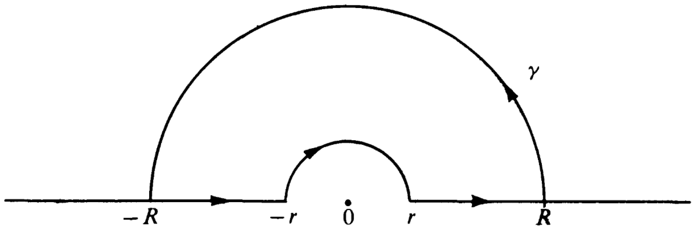
$$\begin{aligned}
\frac{2}{1-e^{2\pi i/3}} \sum \text{Res}\, \frac{z^{\frac{1}{3}}}{1+z^2} &= \frac{2}{1-e^{2\pi i/3}} \left[\frac{e^{\frac{i\pi}{6}}}{2} - \frac{e^{\frac{i\pi}{2}}}{2}\right] \\
&= \frac{1}{(1-e^{\pi i/3})(1+e^{\pi i/3})} e^{\frac{i\pi}{6}} \left(1-e^{\frac{i\pi}{3}}\right) \\
&= \frac{\pi e^{\frac{i\pi}{6}}}{1+e^{\frac{i\pi}{3}}} = \frac{\pi}{e^{-\frac{i\pi}{6}} + e^{\frac{i\pi}{6}}} = \frac{\pi}{2\cos\frac{\pi}{6}} = \frac{\pi}{\sqrt{3}}
\end{aligned}$$

Hence

$$\int_0^{\infty} \frac{x^{1/3}dx}{1+x^2} = \frac{\pi}{\sqrt{3}}$$

$\square$

(h) Let γ be the following curve for $0 < r < R$



$$\int_\gamma \frac{\log z}{1+z^2}dz = \int_r^R \frac{\log x}{1+x^2}dx + iR\int_0^\pi \frac{\log R + i\theta}{1+R^2e^{2i\theta}}e^{i\theta}d\theta + \int_{-R}^{-r}\frac{\log|x|+\pi i}{1+x^2}dx + ir\int_\pi^0 \frac{[\log r + i\theta]}{1+r^2e^{2i\theta}}e^{i\theta}d\theta$$

Now

$$\int_r^R \frac{\log x}{1+x^2}dx + \int_{-R}^{-r}\frac{\log|x|+\pi i}{1+x^2}dx = 2\int_r^R \frac{\log x}{1+x^2}dx + \pi i\int_r^R \frac{dx}{1+x^2}$$

Now as $r \to 0$ and $R \to \infty$ we have

$$\int_r^R \frac{dx}{1+x^2} = \frac{\pi}{2}$$

Now

$$\left|R\int_0^\pi \frac{[\log R + i\theta]}{1+R^2e^{i\theta}}e^{i\theta}d\theta\right| \le \frac{R|\log R|}{|1-R^2|}\int_0^\pi d\theta + \frac{R}{|1-R^2|}\int_0^\pi \theta d\theta = \frac{\pi R|\log R|}{|1-R^2|} + \frac{R\pi^2}{2\,|1-R^2|}$$

Hence as $R \to \infty$ we have $\frac{\pi R|\log R|}{|1-R^2|} + \frac{R\pi^2}{2|1-R^2|} \to 0$. Similarly

$$\left|r\int_0^\pi \frac{[\log r + i\theta]}{1+r^2e^{i\theta}}e^{i\theta}d\theta\right| \le \frac{r|\log r|}{|1-r^2|}\int_0^\pi d\theta + \frac{r}{|1-r^2|}\int_0^\pi \theta d\theta = \frac{\pi r|\log r|}{|1-r^2|} + \frac{r\pi^2}{2\,|1-r^2|}$$

Hence as $r \to 0$ we have $\frac{\pi r|\log r|}{|1-r^2|} + \frac{r\pi^2}{2|1-r^2|} \to 0$. Now by Residue Theorem we have

$$\int_\gamma \frac{\log z}{1+z^2}dz = \sum_{y>0} \text{Res}\,\frac{\log z}{1+z^2}$$

Now $1+z^2 = (z+i)(z-i)$. Therefore

$$\text{Res}_{z=i}\,\frac{\log z}{1+z^2} = \lim_{z\to i}\frac{(z-i)\log z}{1+z^2} = \frac{\log i}{2i} = \frac{\frac{i\pi}{2}}{2i} = \frac{\pi}{4}$$

Hence

$$\int_\gamma \frac{\log z}{1+z^2}dz = 2\pi i\frac{\pi}{4} = \frac{\pi^2 i}{2} = 2\int_0^\infty \frac{\log x}{1+x^2}dx + \pi i\frac{\pi}{2} \implies \int_0^\infty \frac{\log x}{1+x^2}dx = 0$$

(i)

$$\int_0^\infty \frac{\log\left(1+x^2\right)}{x^{1+\alpha}}dx = -\left.\frac{\log\left(1+x^2\right)}{\alpha x^\alpha}\right|_0^\infty + \frac{1}{\alpha}\int_0^\infty \frac{d/dx\log\left(1+x^2\right)}{x^\alpha}dx = \frac{2}{\alpha}\int_0^\infty \frac{x}{x^\alpha\left(1+x^2\right)}dx$$

Now there are three cases

Case 1: $\alpha = 1$

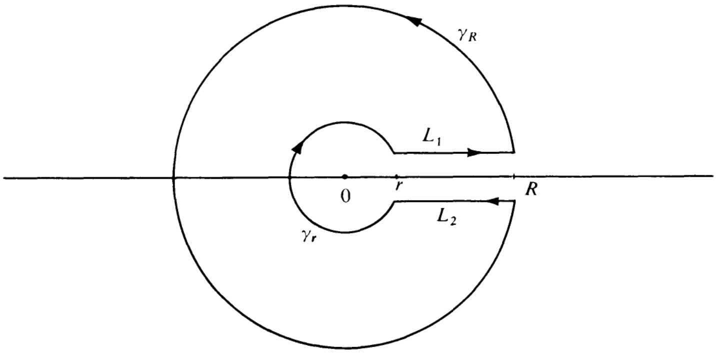
Then
$$\frac{2}{\alpha}\int_0^\infty \frac{x}{x^\alpha\left(1+x^2\right)}dx = 2\int_0^\infty \frac{x}{x\left(1+x^2\right)}dx = 2\int_0^\infty \frac{1}{\left(1+x^2\right)}dx = 2\frac{\pi}{2} = \pi$$

Case 2: $0 < \alpha < 1$

Then take $a = 1-\alpha$. Then $0 < a < 1$. Hence
$$\int_0^\infty \frac{x}{x^\alpha\left(1+x^2\right)}dx = \int_0^\infty \frac{x^{1-\alpha}}{1+x^2}dx = \int_0^\infty \frac{x^a}{1+x^2}dx$$
This is like Ahlfors Case 4. Now $1+z^2 = (z+i)(z-i)$. Hence
$$\mathrm{Res}_{z=i}\frac{z^a}{1+z^2} = \lim_{z\to i}\frac{z^a(z-i)}{1+z^2} = \frac{i^a}{2i} = \frac{e^{\frac{ia\pi}{2}}}{2i}$$
$$\mathrm{Res}_{z=-i}\frac{z^a}{1+z^2} = \lim_{z\to -i}\frac{z^a(z+i)}{1+z^2} = \frac{(-i)^a}{-2i} = -\frac{e^{\frac{i3a\pi}{2}}}{2i}$$
Hence
$$\begin{aligned}\int_0^\infty \frac{x}{x^\alpha\left(1+x^2\right)}dx &= \frac{2\pi i}{1-e^{2\pi i a}}\left[\frac{e^{\frac{ia\pi}{2}}}{2i} - \frac{e^{\frac{i3a\pi}{2}}}{2i}\right] \\ &= \frac{\pi e^{\frac{ia\pi}{2}}}{(1-e^{\pi i a})(1+e^{\pi i a})}\left(1-e^{\pi i a}\right) \\ &= \frac{\pi e^{\frac{ia\pi}{2}}}{1+e^{\pi i a}} \\ &= \frac{\pi}{e^{-\frac{ia\pi}{2}}+e^{\frac{ia\pi}{2}}} = \frac{\pi}{2\cos\frac{a\pi}{2}} = \frac{\pi}{2\sin\frac{\alpha\pi}{2}}\end{aligned}$$
Hence
$$\int_0^\infty \frac{\log\left(1+x^2\right)}{x^{1+\alpha}}dx = \frac{2}{\alpha}\frac{\pi}{2\sin\frac{\alpha\pi}{2}} = \frac{\pi}{\alpha\sin\frac{\alpha\pi}{2}}$$

Case 3: $1 < \alpha < 2$

Then take $b = \alpha - 1$. Then $0 < b < 1$. Hence
$$\int_0^\infty \frac{x}{x^\alpha\left(1+x^2\right)}dx = \int_0^\infty \frac{1}{x^{\alpha-1}(1+x^2)}dx = \int_0^\infty \frac{1}{x^b(1+x^2)}dx$$
Let γ be the following curve for $0 < r < R$.
Hence
$$\left|\int_{\gamma_R}\frac{1}{z^b\left(1+z^2\right)}dz\right| \leq \frac{\pi R}{R^{\alpha-1}\left|R^2-1\right|} = \frac{\pi R^{2-\alpha}}{\left|R^2-1\right|}$$
Since $1 < \alpha < 2$ we have $0 < 2-\alpha < 1$. Hence as $R\to\infty$, $\frac{\pi R^{2-\alpha}}{|R^2-1|}\to 0$. Similarly
$$\left|\int_{\gamma_r}\frac{1}{z^b\left(1+z^2\right)}dz\right| \leq \frac{\pi r}{r^{\alpha-1}\left|r^2-1\right|} = \frac{\pi r^{2-\alpha}}{\left|r^2-1\right|}$$
Hence as $r\to 0$ we have $\frac{\pi r^{2-\alpha}}{|r^2-1|}\to 0$ Now $1+z^2 = (z+i)(z-i)$. Hence
$$\mathrm{Res}_{z=i}\frac{1}{z^b(1+z^2)} = \lim_{z\to i}\frac{z-i}{z^b(1+z^2)} = \frac{i^{-b}}{2i} = \frac{e^{-\frac{ib\pi}{2}}}{2i}$$



$$\operatorname{Res}_{z=-i} \frac{1}{z^b(1+z^2)} = \lim_{z \to -i} \frac{z+i}{z^b(1+z^2)} = \frac{(-i)^{-b}}{-2i} = -\frac{e^{-\frac{i3b\pi}{2}}}{2i}$$

Hence

$$\begin{aligned}
\int_0^\infty \frac{1}{x^b(1+x^2)} dx &= \frac{2\pi i}{1-e^{-2\pi i b}} \left[\frac{e^{-\frac{ib\pi}{2}}}{2i} - \frac{e^{-\frac{i3b\pi}{2}}}{2i} \right] \\
&= \frac{\pi e^{-\frac{ib\pi}{2}}}{(1-e^{-\pi i b})(1+e^{-\pi i b})} \left(1 - e^{-\pi i b}\right) \\
&= \frac{\pi e^{-\frac{ib\pi}{2}}}{1+e^{-\pi i b}} \\
&= \frac{\pi}{e^{\frac{ib\pi}{2}} + e^{-\frac{ib\pi}{2}}} = \frac{\pi}{2\cos\frac{b\pi}{2}} = \frac{\pi}{2\sin\frac{\alpha\pi}{2}}
\end{aligned}$$

Hence

$$\int_0^\infty \frac{\log\left(1+x^2\right)}{x^{1+\alpha}} dx = \frac{2}{\alpha} \frac{\pi}{2\sin\frac{\alpha\pi}{2}} = \frac{\pi}{\alpha\sin\frac{\alpha\pi}{2}}$$

Therefore we get $\forall\ 0 < \alpha < 2$ we have

$$\int_0^\infty \frac{\log\left(1+x^2\right)}{x^{1+\alpha}} dx = \frac{2}{\alpha} \frac{\pi}{2\sin\frac{\alpha\pi}{2}} = \frac{\pi}{\alpha\sin\frac{\alpha\pi}{2}}$$

□

Problem 4 Ahlfors Page 186: Problem 2

Let Ω be a doubly connected region whose complement consists of the components E_1, E_2. Prove that every analytic function $f(z)$ in Ω can be written in the form $f_1(z)+f_2(z)$ where $f_1(z)$ is analytic outside of E_1 and $f_2(z)$ is analytic outside of E_2. (The precise proof requires a construction like the one in Chap. 4 , Sec. 4.5.)

Solution:

□