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Course: Parallel Algorithms and Complexity

Assignment - 4
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Problem 1

Prove that the $6 - Color - Rooted - Tree$ Algorithm produces a valid 6-coloring of a tree.

Solution: Let L_k denote the number of bits used to represent vertices at k-th iteration. Now for $i = 1$ we have

$$L_1 = \lceil \log n \rceil + 1 \leq 2\lceil \log n \rceil$$

Now let for $i = k - 1$ we have $L_{k-1} \leq 2\lceil \log^{(k-1)} n \rceil$ and $\lceil \log^{(k)} n \rceil \geq 2$. Now

$$L_k = \lceil \log L_{k-1} \rceil + 1 \leq \left\lceil \log 2 \left\lceil \log^{(k-1)} n \right\rceil \right\rceil + 1 \leq 2 \left\lceil \log^{(k)} n \right\rceil$$

Hence if $\lceil \log^{(k)} n \rceil \geq 2$ we have $L_k \leq 2\lceil \log^{(k)} n \rceil$ Therefore the number of bits to represent the vertices decreases with each iteration and after $O(\log^* n)$ many iteration L_k reaches the value of 3 (The limit L of $\lim\limits_{k\to\infty} L_k = \lim\limits_{k\to\infty} \lceil \log L_{k-1} \rceil + 1$ is the solution of $L = \lceil \log L \rceil + 1$). In those 3 bits the i_v takes 3 possible values and the b_v takes 2 possible values for each vertex v. Hence total number of colors is $3 \times 2 = 6$.

□

Problem 2

- Prove that every weakly connected component of a pseudoforest contains at most one cycle
- Find a 3 Coloring pseudoforest algorithm in $O(\log^* n)$ time

Solution:

- Suppose there are two cycles C_1, C_2 in a weakly connected component. Now C_1 and C_2 has to be disjoint cause other wise there will be a vertex in $CC_! \cap C_2$ from which two edges have gone out side one for the next vertex in C_1 and the other one for the next vertex in C_2. This is not possible since in a pseudoforest each vertex has out-degree exactly 1. So C_1 and C_2 are disjoint. Since they remain in same weakly connented component for $u \in C_1$ and $v \in C_2$ there exists a path $u \rightsquigarrow v$ or $v \rightsquigarrow u$. WLOG let the path $u \rightsquigarrow v$ exists. Let the path is P. Now there exists an edge $(x, y) = e \in P$ such that $e \notin C_1$ but the tail x of the edge is in C_1. Now since $x \in C_1$ there is an edge going outward from x towards the next vertex in C_1. And also the edge (x, y) is going outwards along P. Hence out-degree of x is at least 2. Which is not possible. Hence every weakly connected component of a pseudoforest has at most one cycle.
- After applying $6 - Color - Rooted - Tree$ if there are more than 3 colors we shift down every color that is replace every non-root node's color with the color of the parent and color the root nodes with something different from the color before shifting and then for a color c we replace c with the smallest color other than c, the color of its child nodes and the color of the parent nodes (we can do this since there are more than 3 colors). In this shifting process total number of color is reduced by 1. Hence after 3 iteration of this shifting process number of colors is reduced to 3. In a shifting process the new coloring can be done for each vertex parallely and removing the color c for each vertex can also be done in parallel. So each shifting process takes constant time. Since we do thisshifting process 3 times it takes constant time to reduce the number of colors to 3. Hence to 3 color a tree it takes $O(\log^* n)$ time.

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Problem 3

Is Maximum Independent Set for bounded degree graph $NP - hard$?

Solution:

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