

Problem 1 Ahlfors Page 47: Problem 1

For real y show that every remainder in the series for $\cos y$ and $\sin y$ has the same sign as the leading term

Solution: The series for both cosine and sine are

$$\begin{aligned}\cos(y) &= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \\ \sin(y) &= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\end{aligned}$$

We can write Taylor's formula as $f(y) = T_n(y) + R_n(y)$ where

$$f(y) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} y^k + \frac{1}{k!} \int_0^y (y-t)^k f^{(k+1)}(t) dt.$$

Now, we can write cosine and sine of y as

$$\begin{aligned}\cos(y) &= \sum_{k=0}^n \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{n!} \int_0^y (y-t)^n \cos^{n+1}(t) dt \\ \sin(y) &= \sum_{k=0}^{n-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{n!} \int_0^y (y-t)^n \sin^n(t) dt\end{aligned}$$

Now we have

$$\sin^{(2m)}(t) = (-1)^m \sin t \quad \cos^{(2m+1)}(t) = (-1)^{m+1} \sin t$$

For cosine and sine, let $n = 2m$ and $n = 2m - 1$, respectively. Then

$$\begin{aligned}\cos(y) &= \sum_{k=0}^m \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{(2m)!} \int_0^y (y-t)^{2m} \cos^{2m+1}(t) dt \\ &= \sum_{k=0}^m \frac{(-1)^k}{(2k)!} y^{2k} + \frac{(-1)^{m+1}}{(2m)!} \int_0^y (y-t)^{2m} \sin t dt \\ \sin(y) &= \sum_{k=0}^{m-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{(2m-1)!} \int_0^y (y-t)^{2m-1} \sin^{2m-1}(t) dt \\ &= \sum_{k=0}^{m-1} \frac{(-1)^k}{(2k+1)!} y^{2k+1} + \frac{(-1)^m}{(2m-1)!} \int_0^y (y-t)^{2m-1} \sin t dt\end{aligned}$$

So it remains to see that

$$\int_0^y (y-t)^k \sin t dt > 0$$

for all $y > 0$ and $k > 0$. But that follows since $(y-t)^k$ is a strictly decreasing positive function, so while $2p\pi \leq y$ where $p \in \mathbb{N}$ we have

$$\int_{2n\pi}^{2(n+1)\pi} (y-t)^k \sin t dt = \underbrace{\int_{2n\pi}^{(2n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{\sin t}_{>0} dt}_{>0} + \underbrace{\int_{(2n+1)\pi}^{2(n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{\sin t}_{<0} dt}_{<0}$$

Now

$$\begin{aligned}
\int_{2n\pi}^{(2n+1)\pi} (y-t)^k \sin t \, dt &> (y - (2n+1)\pi)^k \int_{2n\pi}^{(2n+1)\pi} \sin t \, dt \\
&= 2(y - (2n+1)\pi)^k \\
\int_{(2n+1)\pi}^{2(n+1)\pi} (y-t)^k \sin t \, dt &= - \int_{(2n+1)\pi}^{2(n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{(-\sin t)}_{>0} \, dt \\
&> -(y - (2n+1)\pi)^k \int_{(2n+1)\pi}^{2(n+1)\pi} (-\sin t) \, dt \\
&= -2(y - (2n+1)\pi)^k
\end{aligned}$$

Hence

$$\int_{2n\pi}^{2(n+1)\pi} (y-t)^k \sin t \, dt > 0$$

This is true for all $n \in \{0, 1, \dots, p\}$. Therefore

$$\int_0^{2p\pi} (y-t)^k \sin t \, dt > 0$$

Now if $2p\pi \leq y \leq (2p+1)\pi$ then

$$\int_{2p\pi}^y (y-t)^k \sin t \, dt \geq 0$$

If $(2p+1)\pi \leq y \leq 2(p+1)\pi$

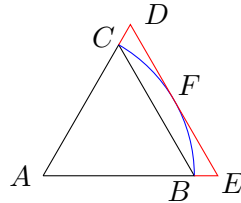
$$\begin{aligned}
\int_{2p\pi}^y (y-t)^k \sin t \, dt &= \int_{2p\pi}^{(2p+1)\pi} (y-t)^k \sin t \, dt + \int_{(2p+1)\pi}^y (y-t)^k \sin t \, dt \\
&= \int_{2p\pi}^{(2p+1)\pi} (y-t)^k \sin t \, dt + \int_{(2p+1)\pi}^{2(p+1)\pi} (y-t)^k \sin t \, dt \\
&> (y - (2p+1)\pi)^k \int_0^\pi \sin t \, dt - (y - (2p+1)\pi)^k \int_{(2p+1)\pi}^{2(p+1)\pi} \sin t \, dt \\
&> 0
\end{aligned}$$

□

Problem 2 Ahlfors Page 47: Problem 2

Prove, for instance that $3 < \pi < 2\sqrt{3}$

Solution:



The triangles $\triangle ABC$ and $\triangle DAE$ are equilateral triangles. Side length of $\triangle ABC$ is 1. The radius of the circular sector $ACFBA$ is 1. Hence the height of the triangle $\triangle ADE$ is 1. Therefore the side length of

the $\triangle ADE$ is $\frac{2}{\sqrt{3}}$. Now the arc length of BFC is greater than the side length of the triangle $\triangle ABC$. Therefore

$$\frac{2 \times \pi \times 1}{6} > 1 \implies \pi > 3$$

So the we have

$$\text{Area}(\triangle ABFCA) < \text{Area}(\triangle ADE)$$

Now, $\text{Area}(\triangle ABFCA) = \frac{1}{6} \pi 1^2 = \frac{\pi}{6}$ and $\text{Area}(\triangle ADE) = \frac{\sqrt{3}}{4} \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{1}{\sqrt{3}}$. Therefore

$$\frac{\pi}{6} < \frac{1}{\sqrt{3}} \implies \pi < 2\sqrt{3}$$

Therefore

$$3 < \pi < 2\sqrt{3}$$

□

Problem 3 Ahlfors Page 47: Problem 4

For what values of z is e^z equal to $2, -1, i, -i/2, -1, -i, 1 + 2i$?

Solution:

- Let for $z = a + ib$ $e^z = 2$. Then

$$e^{a+ib} = 2 \implies e^a e^{ib} = 2$$

Now $|e^{ib}| = 1$. Hence $e^a = 2$ then $a = \ln 2$. Hence $e^{ib} = 1 = \cos b + i \sin b$. Then $\cos b = 1$ and $\sin b = 0$. Then $b = 2n\pi \forall n \in \mathbb{Z}$. Then for $z = \ln 2 + i2n\pi \forall n \in \mathbb{Z}$ $e^z = 2$

- Let for $z = a + ib$ $e^z = -1$. Then

$$e^{a+ib} = -1 \implies e^a e^{ib} = -1$$

Now $|e^{ib}| = 1$. Hence $e^a = 1$ then $a = 0$. Hence $e^{ib} = -1 = \cos b + i \sin b$. Then $\cos b = -1$ and $\sin b = 0$. Then $b = (2n+1)\pi \forall n \in \mathbb{Z}$. Then for $z = i(2n+1)\pi \forall n \in \mathbb{Z}$ $e^z = -1$

- Let for $z = a + ib$ $e^z = i$. Then

$$e^{a+ib} = i \implies e^a e^{ib} = i$$

Now $|e^{ib}| = 1$. Hence $e^a = 1$ then $a = 0$. Hence $e^{ib} = i = \cos b + i \sin b$. Then $\cos b = 0$ and $\sin b = 1$. Then $b = (4n+1)\frac{\pi}{2} \forall n \in \mathbb{Z}$. Then for $z = i(4n+1)\frac{\pi}{2} \forall n \in \mathbb{Z}$ $e^z = i$

- Let for $z = a + ib$ $e^z = -\frac{i}{2}$. Then

$$e^{a+ib} = -\frac{i}{2} \implies e^a e^{ib} = -\frac{i}{2}$$

Now $|e^{ib}| = 1$. Hence $e^a = \frac{1}{2}$ then $a = \ln \frac{1}{2}$. Hence $e^{ib} = -i = \cos b + i \sin b$. Then $\cos b = 0$ and $\sin b = -1$. Then $b = (4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$. Then for $z = \ln \frac{1}{2} + i(4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$ $e^z = -\frac{i}{2}$

- Let for $z = a + ib$ $e^z = -i$. Then

$$e^{a+ib} = -i \implies e^a e^{ib} = -i$$

Now $|e^{ib}| = 1$. Hence $e^a = 1$ then $a = 0$. Hence $e^{ib} = -i = \cos b + i \sin b$. Then $\cos b = 0$ and $\sin b = -1$. Then $b = (4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$. Then for $z = i(4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$ $e^z = -i$

- Let for $z = a + ib$ $e^z = 1 + 2i$. Then

$$e^{a+ib} = 1 + 2i \implies e^a e^{ib} = 1 + 2i$$

Now $|e^{ib}| = 1$. Hence $e^a = \sqrt{5}$ then $a = \ln \sqrt{5}$. Hence $e^{ib} = \frac{1}{\sqrt{5}}(1+2i) = \cos b + i \sin b$. Then $\cos b = \frac{1}{\sqrt{5}}$ and $\sin b = \frac{2}{\sqrt{5}}$. Then $b = 2n\pi + \sin^{-1} \frac{2}{\sqrt{5}} \forall n \in \mathbb{Z}$. Then for $z = \ln \sqrt{5} + i \left(2n\pi + \sin^{-1} \frac{2}{\sqrt{5}} \right) \forall n \in \mathbb{Z}$ $e^z = 1 + 2i$

□

Problem 4 Ahlfors Page 47: Problem 6

Determine all values of 2^i , i^i , $(-1)^{2i}$

Solution:

- $2^i = \exp(i \ln 2) = \cos \ln 2 + i \sin \ln 2$
- $i^i = \exp(i \log i) = \exp \left(i \ln \left(e^{i(4n+1)\frac{\pi}{2}} \right) \right) = \exp \left(i \left(i(4n+1)\frac{\pi}{2} \right) \right) = \exp \left(-(4n+1)\frac{\pi}{2} \right)$
- $(-1)^{2i} = i^{4i} = (i^i)^4 = \left(\exp \left(-(4n+1)\frac{\pi}{2} \right) \right)^4 = \exp \left(-2(4n+1)\pi \right)$

□

Problem 5 Ahlfors Page 47: Problem 7

Determine the real and imaginary parts of z^z

Solution: Let $z = x + iy$. Then

$$z^z = (x + iy)^{x+iy} = \exp((x + iy) \log(x + iy))$$

Now let $e^{a+ib} = x + iy$. Since $|e^{ib}| = 1$. We have $e^a = \sqrt{x^2 + y^2}$. Therefore $a = \ln \sqrt{x^2 + y^2}$. Now

$$e^{ib} = \cos b + i \sin b = \frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}}$$

Hence $b = 2n\pi + \tan^{-1} \frac{y}{x} \forall n \in \mathbb{Z}$. Therefore

$$a + ib = \ln \sqrt{x^2 + y^2} + i \left(2n\pi + \tan^{-1} \frac{y}{x} \right)$$

Hence

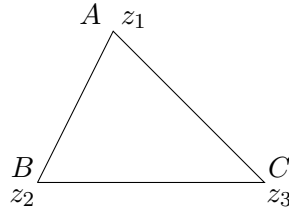
$$\begin{aligned} \exp((x + iy) \ln(x + iy)) &= \exp \left((x + iy) \ln \left(e^{\ln \sqrt{x^2 + y^2} + i(2n\pi + \tan^{-1} \frac{y}{x})} \right) \right) \\ &= \exp \left((x + iy) \left(\ln \sqrt{x^2 + y^2} + i \left(2n\pi + \tan^{-1} \frac{y}{x} \right) \right) \right) \\ &= \exp \left(x \ln \sqrt{x^2 + y^2} - y \left(2n\pi + \tan^{-1} \frac{y}{x} \right) \right. \\ &\quad \left. + i \left(y \ln \sqrt{x^2 + y^2} + x \left(2n\pi + \tan^{-1} \frac{y}{x} \right) \right) \right) \\ &= e^{x \ln \sqrt{x^2 + y^2} - y(2n\pi + \tan^{-1} \frac{y}{x})} \left[\cos \left(\left(y \ln \sqrt{x^2 + y^2} + x \left(2n\pi + \tan^{-1} \frac{y}{x} \right) \right) \right) \right. \\ &\quad \left. + i \sin \left(\left(y \ln \sqrt{x^2 + y^2} + x \left(2n\pi + \tan^{-1} \frac{y}{x} \right) \right) \right) \right] \end{aligned}$$

□

Problem 6 Ahlfors Page 47: Problem 9

Show how to define the “angles: in a triangle, bearing in mind they should lie between 0 and π . With this definition, prove that the sum of the angles is π .

Solution:



The angle between the sides AB and AC defined to be the angle between the sides in the anti clockwise direction. Hence we can define the angle

$$\angle A = \Im \log \left(\frac{z_3 - z_1}{z_2 - z_1} \right) = \Im (\log(z_3 - z_1) - \log(z_2 - z_1)) = \Im \log(z_3 - z_1) - \Im \log(z_2 - z_1)$$

Therefore for other angles we have

$$\angle B = \Im \log \left(\frac{z_1 - z_2}{z_3 - z_2} \right) = \Im \log(z_1 - z_2) - \Im \log(z_3 - z_2)$$

$$\angle C = \Im \log \left(\frac{z_2 - z_3}{z_1 - z_3} \right) = \Im \log(z_2 - z_3) - \Im \log(z_1 - z_3)$$

Therefore

$$\begin{aligned} \angle A + \angle B + \angle C &= \Im \log \left(\frac{z_3 - z_1}{z_2 - z_1} \right) + \Im \log \left(\frac{z_1 - z_2}{z_3 - z_2} \right) + \Im \log \left(\frac{z_2 - z_3}{z_1 - z_3} \right) \\ &= \Im \log \left(\frac{z_3 - z_1}{z_2 - z_1} \frac{z_1 - z_2}{z_3 - z_2} \frac{z_2 - z_3}{z_1 - z_3} \right) \\ &= \Im \log \left(\frac{z_1 - z_2}{z_2 - z_1} \frac{z_2 - z_3}{z_3 - z_2} \frac{z_3 - z_1}{z_1 - z_3} \right) \\ &= \Im \log ((-1)^3) = \Im \log(-1) = \pi \end{aligned}$$

□

Problem 7 Ahlfors Page 72: Problem 1

Give a precise definition of a single-valued branch of $\sqrt{a+z} + \sqrt{1-z}$ in a suitable region, and prove that it is analytic.

Solution: We recall that for \sqrt{z} we choose the region Ω which is the complement of the negative real axis $x \leq 0, y = 0$. Hence, to define $\sqrt{1+z}$ we should choose Ω_1 as the complement of $x \leq -1, y = 0$ and for $\sqrt{1-z}$ we should choose Ω_2 as the complement of $1 \leq x, y = 0$. In total, to define $\sqrt{1+z} + \sqrt{1-z}$ we choose the region $\Omega = \Omega_1 \cap \Omega_2$. This is actually the same region used in defining $\arccos z$. The branch chosen is that which has positive real part. As simple transformations of \sqrt{z} in an analytic region, it follows that $\sqrt{1+z} + \sqrt{1-z}$ is analytic.

□

Problem 8 Ahlfors Page 72: Problem 3

Suppose that $f(z)$ is analytic and satisfies the condition $|f^2(z) - 1| < 1$ in a region Ω . Show that either $\Re f(z) > 0$ or $\Re f(z) < 0$ throughout Ω

Solution: Suppose $\Re f(z) = 0$ at a point $z \in \Omega$. Then $f(z) = iy$ for some $y \in \mathbb{R}$, and thus $f(z)^2 = -y^2$. By the condition $|f(z)^2 - 1| < 1$ we have $|-y^2 - 1| < 1$ and thus $|y^2 + 1| < 1$. This is clearly impossible, so that $\Re f(z) \neq 0$ throughout Ω . But, $\Re f(z)$ is continuous and Ω is connected, so either $\Re f(z) > 0$ or $\Re f(z) < 0$.

□

Problem 9 Ahlfors Page 78: Problem 1

Prove that the reflection $z \rightarrow \bar{z}$ is not a linear transformation

Solution: Suppose $\varphi(z) : z \mapsto \bar{z}$ is a linear fractional transformation, and thus it must be of the form

$$\bar{z} = \varphi(z) = \frac{az + b}{cz + d}, \quad \forall z \in \mathbb{C}$$

Note that if $\Im z = 0$ then $\bar{z} = z$. In particular,

$$0 \mapsto 0 \implies \varphi(0) = \frac{b}{d} = 0 \implies b = 0$$

Plugging in different values yields

$$\begin{aligned} 1 \mapsto \varphi(1) &= \frac{a}{c+d} = 1 \\ -1 \mapsto \varphi(-1) &= \frac{-a}{-c+d} = -1 \end{aligned}$$

Or,

$$\begin{aligned} c + d &= a \\ d - c &= a \end{aligned}$$

Thus, $a = d$ and hence $c = 0$. But then we have

$$\bar{z} = \frac{az}{d} = z$$

Hence contradiction. φ is not linear transformation

□

Problem 10 Ahlfors Page 78: Problem 2

If

$$T_1 z = \frac{z+2}{z+3} \quad T_2 z = \frac{z}{z+1}$$

Find $T_1 T_2 z$, $T_2 T_1 z$, $T_1^{-1} T_2 z$

Solution: Here we compute several compositions:

$$\begin{aligned} T_1 T_2 z &= \frac{\frac{z}{z+1} + 2}{\frac{z}{z+1} + 3} = \frac{\frac{3z+2}{z+1}}{\frac{4z+3}{z+1}} = \frac{3z+2}{4z+3} \\ T_2 T_2 z &= \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3} + 1} = \frac{\frac{z+2}{z+3}}{\frac{2z+5}{z+3}} = \frac{z+2}{2z+5} \end{aligned}$$

Now note that

$$T_1^{-1}(w) = \frac{3w-2}{1-w}$$

Thus,

$$T_1^{-1}T_2z = \frac{3\frac{z}{z+1}-2}{1-\frac{z}{z+1}} = \frac{z-2}{1} = z-2$$

□

Problem 11 Ahlfors Page 78: Problem 3

Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by reflection in the real axis.

Solution: Let φ be a general transformation as given. Then, we have that $\varphi(0) = 0$ and for any pair $(z, w) \in \mathbb{C} \times \mathbb{C}$:

$$|z - w| = |\varphi(z) - \varphi(w)|$$

Thus

$$|\varphi(1)| = |\varphi(1) - \varphi(0)| = 1 = |\varphi(i)| = |\varphi(i) - \varphi(0)|$$

and

$$|\varphi(1) - \varphi(i)| = \sqrt{2}$$

Thus $\varphi(0)$, $\varphi(1)$, $\varphi(i)$, must form a right-angled triangle where the angle $\varphi(1)$ and $\varphi(i)$ makes at origin is 90° . Let $\varphi(1) = e^{i\theta}$. Hence $\varphi(i) = ie^{i\theta}$ or $-ie^{i\theta}$. For any $z \in \mathbb{C}$ $|z| = |\varphi(z)|$. Hence

$$|z| = |z - 0| = |\varphi(z)| \quad |z - 1| = |\varphi(z) - \varphi(1)| \quad |z - i| = |\varphi(z) - \varphi(i)| \quad (1)$$

Since $0, 1, i$ are not co-linear then the circles can intersect at only one point at-most. Thus $\varphi(z)$ lies on that intersection

Case 1: $\varphi(i) = ie^{i\theta}$

Then we claim that $\varphi(z) = ze^{i\theta}$. We just need to check the equalities in (1)

$$\begin{aligned} |\varphi(z) - \varphi(0)| &= |ze^{i\theta}| = |z| \\ |\varphi(z) - \varphi(1)| &= |ze^{i\theta} - e^{i\theta}| = |(z-1)e^{i\theta}| = |z-1| \\ |\varphi(z) - \varphi(i)| &= |ze^{i\theta} - ie^{i\theta}| = |(z-i)e^{i\theta}| = |z-i| \end{aligned}$$

Thus φ rotates every complex number by a fixed number

Case 2: $\varphi(i) = -ie^{i\theta}$

We claim that $\varphi(z) = \bar{z}e^{i\theta} = \overline{ze^{i\theta}}$. We just need to check the equalities in (1)

$$\begin{aligned} |\varphi(z) - \varphi(0)| &= |\bar{z}e^{i\theta}| = |\bar{z}| = |z| \\ |\varphi(z) - \varphi(1)| &= |\bar{z}e^{i\theta} - e^{i\theta}| = |(\bar{z}-1)e^{i\theta}| = |\bar{z}-1| = |z-1| \\ |\varphi(z) - \varphi(i)| &= |\bar{z}e^{i\theta} + ie^{i\theta}| = |(\bar{z}+i)e^{i\theta}| = |\bar{z}+i| = |\bar{z}-\bar{i}| = |z-i| \end{aligned}$$

Thus φ first rotates every complex number by a fixed number then takes the conjugate of the number i.e. φ is a rotation followed by a reflection in the real axis

□

Problem 12 Ahlfors Page 78: Problem 4

Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

Solution: Let $\varphi(z)$ be a linear transformation with the additional restriction that $\varphi(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. Hence

$$\varphi(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{C}$. We will now find $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ so that $\varphi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$. We need to distinguish separate cases:

If $ad - bc = 0$:

$$\varphi(z) = \frac{\alpha z + \beta}{\gamma z + \delta} = \frac{bc - ad}{c^2 \left(z + \frac{d}{c}\right)} + \frac{a}{c} = \frac{a}{c}$$

Now $\varphi(0) = \frac{a}{c} \in \mathbb{R} \implies \frac{a}{c} \in \mathbb{R}$

If $ad - bc \neq 0$:

If $a \neq 0$:

$$\varphi(z) = \frac{az + b}{cz + d} \cdot \frac{\bar{a}}{a} = \frac{a'z + b'}{c'z + d'}, \quad a' \in \mathbb{R}$$

The important thing to observe is that we will then have $\Im a' = 0$, i.e $a' \in \mathbb{R}$. We now have a representation

$$\varphi(z) = \frac{a'z + b'}{c'z + d'}, \quad a' \in \mathbb{R}$$

In the above we see that $z \rightarrow \infty \implies \phi(\infty) = \frac{a'}{c'}$ implying that $c' \in \mathbb{R}$ as well as a' . The transformation has an inverse

$$\varphi^{-1}(z) = \frac{d'z - b'}{a' - c'z}$$

If $d' = 0$ then taking $z = 0$ we see $\varphi(0) = b'$ and hence achieve $b' \in \mathbb{R}$. We now need to show that the same holds whenever $d' \neq 0$. In the above we see that $z \rightarrow \infty \implies \varphi^{-1}(\infty) = -\frac{d'}{c'}$ implying that $d' \in \mathbb{R}$ as well as c' .

$$\varphi(0) = \frac{b'}{d'} \in \mathbb{R} \implies b' \in \mathbb{R}$$

If $a = 0$:

If $a = 0$ then we have $\varphi(z) = \frac{b}{cz + d}$. Here if $b = 0$ we have $\varphi(z)$ is non-invertible which is impossible. So

$$\varphi(z) = \frac{b}{cz + d} \cdot \frac{\bar{b}}{b}$$

thus obtaining

$$\varphi(z) = \frac{b'}{c'z + d'}, \quad b' \in \mathbb{R}$$

Taking $z = 0$ we see that $\varphi(0) = \frac{b'}{d'}$. We then have $d' \in \mathbb{R}$. If we take $z = 1$ then

$$\varphi(1) = \frac{b'}{c' + d'}$$

Since $b', d, \in \mathbb{R}$ if $c' \notin \mathbb{R}$ then $c' + d' \notin \mathbb{R}$ but $c' + d' \in \mathbb{C}$ hence $\frac{b'}{c'+d'} \notin \mathbb{R}$ but $\frac{b'}{c'+d'} \in \mathbb{C}$. But $\varphi(1) \in \mathbb{R}$. So $c' \in \mathbb{R}$.

Hence φ can be written with real coefficients

□