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Course: Complex Analysis

Assignment - 2
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Problem 1 Ahlfors Page 47: Problem 1

For real y show that every remainder in the series for $\cos y$ and $\sin y$ has the same sign as the leading term

Solution: The series for both cosine and sine are

$$\cos(y) = \sum_{k=0}^{\infty}(-1)^k \frac{y^{2k}}{(2k)!} = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots$$

$$\sin(y) = \sum_{k=0}^{\infty}(-1)^k \frac{y^{2k+1}}{(2k+1)!} = y - \frac{y^3}{3!} + \frac{y^6}{6!} - \cdots$$

We can write Taylor's formula as $f(y) = T_n(y) + R_n(y)$ where

$$f(y) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} y^k + \frac{1}{k!}\int_0^y (y-t)^k f^{(k+1)}(t)dt.$$

Now, we can write cosine and sine of y as

$$\cos(y) = \sum_{k=0}^{n} \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{n!}\int_0^y (y-t)^n \cos^{n+1}(t)dt$$

$$\sin(y) = \sum_{k=0}^{n-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{n!}\int_0^y (y-t)^n \sin^{n}(t)dt$$

Now we have

$$\sin^{(2m)}(t) = (-1)^m \sin t \qquad \cos^{(2m+1)}(t) = (-1)^{m+1}\sin t$$

For cosine and sine, let $n = 2m$ and $n = 2m-1$, respectively. Then

$$\begin{aligned}
\cos(y) &= \sum_{k=0}^{m} \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{(2m)!}\int_0^y (y-t)^{2m} \cos^{2m+1}(t)dt \\
&= \sum_{k=0}^{m} \frac{(-1)^k}{(2k)!} y^{2k} + \frac{(-1)^{m+1}}{(2m)!}\int_0^y (y-t)^{2m} \sin t dt
\end{aligned}$$

$$\begin{aligned}
\sin(y) &= \sum_{k=0}^{m-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{(2m-1)!}\int_0^y (y-t)^{2m-1} \sin^{2m-1}(t)dt \\
&= \sum_{k=0}^{m-1} \frac{(-1)^k}{(2k+1)!} y^{2k+1} + \frac{(-1)^{m}}{(2m-1)!}\int_0^y (y-t)^{2m-1} \sin t dt
\end{aligned}$$

So it remains to see that

$$\int_0^y (y-t)^k \sin t\, dt > 0$$

for all $y > 0$ and $k > 0$. But that follows since $(y-t)^k$ is a strictly decreasing positive function, so while $2p\pi \leqslant y$ where $p \in \mathbb{N}$ we have

$$\int_{2n\pi}^{2(n+1)\pi} (y-t)^k \sin t\, dt = \underbrace{\int_{2n\pi}^{(2n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{\sin t}_{>0}\, dt}_{>0} + \underbrace{\int_{(2n+1)\pi}^{2(n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{\sin t}_{<0}\, dt}_{<0}$$

Now

$$\int_{2n\pi}^{(2n+1)\pi} (y-t)^k \sin t\, dt > (y-(2n+1)\pi)^k \int_{2n\pi}^{2(n+1)\pi} \sin t\, dt$$
$$= 2(y-(2n+1)\pi)^k$$

$$\int_{(2n+1)\pi}^{2(n+1)\pi} (y-t)^k \sin t\, dt = -\int_{(2n+1)\pi}^{2(n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{(-\sin t)}_{>0}\, dt$$
$$> -(y-(2n+1)\pi)^k \int_{(2n+1)\pi}^{2(n+1)\pi} (-\sin t) dt$$
$$= -2(y-(2n+1)\pi)^k$$

Hence

$$\int_{2n\pi}^{2(n+1)\pi} (y-t)^k \sin t\, dt > 0$$

This is true for all $n \in \{0, 1, \ldots, p\}$. Therefore

$$\int_0^{2p\pi} (y-t)^k \sin t\, dt > 0$$

Now if $2p\pi \le y \le (2p+1)\pi$ then

$$\int_{2p\pi}^{y} (y-t)^k \sin t\, dt \ge 0$$

If $(2p+1)\pi \le y \le 2(p+1)\pi$

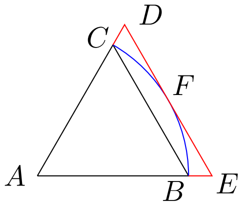
$$\int_{2p\pi}^{y} (y-t)^k \sin t\, dt = \int_{2p\pi}^{(2p+1)\pi} (y-t)^k \sin t\, dt + \int_{(2p+1)\pi}^{y} (y-t)^k |\sin t|\, dt$$
$$= \int_{2p\pi}^{(2p+1)\pi} (y-t)^k \sin t\, dt + \int_{(2p+1)\pi}^{2(p+1)\pi} (y-t)^k |\sin t|\, dt$$
$$> (y-(2p+1)\pi)^k \int_0^{\pi} \sin t\, dt - (y-(2p+1)\pi)^k \int_{(2p+1)\pi}^{2(p+1)\pi} \sin t\, dt$$
$$> 0$$

$\square$

Problem 2 Ahlfors Page 47: Problem 2

Prove, for instance that $3 < \pi < 2\sqrt{3}$

Solution:



The triangles $\triangle ABC$ and $\triangle DAE$ are equilateral triangles. Side length of $\triangle ABC$ is 1. The radius of the circular sector $ACFBA$ is 1. Hence the height of the triangle $\triangle ADE$ is 1. Therefore the side length of the $\triangle ADE$ is $\frac{2}{\sqrt{3}}$. Now the arc length of BFC is greater than the side length of the triangle $\triangle ABC$. Therefore

$$\frac{2 \times \pi \times 1}{6} > 1 \implies \pi > 3$$

So the we have

$$Area(ABFCA) < Area(\triangle ADE)$$

Now, $Area(ABFCA) = \frac{1}{6}\pi 1^2 = \frac{\pi}{6}$ and $Area(\triangle ADE) = \frac{\sqrt{3}}{4}\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{1}{\sqrt{3}}$. Therefore

$$\frac{\pi}{6} < \frac{1}{\sqrt{3}} \implies \pi < 2\sqrt{3}$$

Therefore

$$3 < \pi < 2\sqrt{3}$$

$\square$

Problem 3 Ahlfors Page 47: Problem 4

For what values of z is e^z equal to $2, -1, i, -i/2, -1, -i, 1+2i$?

Solution:

- Let for $z = a + ib$ $e^z = 2$. Then

  $$e^{a+ib} = 2 \implies e^a e^{ib} = 2$$

  Now $|e^{ib}| = 1$. Hence $e^a = 2$ then $a = \ln 2$. Hence $e^{ib} = 1 = \cos b + i \sin b$. Then $\cos b = 1$ and $\sin b = 0$. Then $b = 2n\pi \ \forall\ n \in \mathbb{Z}$. Then for $z = \ln 2 + i2n\pi \ \forall\ n \in \mathbb{Z}$ $e^z = 2$

- Let for $z = a + ib$ $e^z = -1$. Then

  $$e^{a+ib} = -1 \implies e^a e^{ib} = -1$$

  Now $|e^{ib}| = 1$. Hence $e^a = 1$ then $a = 0$. Hence $e^{ib} = -1 = \cos b + i \sin b$. Then $\cos b = -1$ and $\sin b = 0$. Then $b = (2n+1)\pi \ \forall\ n \in \mathbb{Z}$. Then for $z = i(2n+1)\pi \ \forall\ n \in \mathbb{Z}$ $e^z = -1$

- Let for $z = a + ib$ $e^z = i$. Then

  $$e^{a+ib} = i \implies e^a e^{ib} = i$$

  Now $|e^{ib}| = 1$. Hence $e^a = 1$ then $a = 0$. Hence $e^{ib} = i = \cos b + i \sin b$. Then $\cos b = 0$ and $\sin b = 1$. Then $b = (4n+1)\frac{\pi}{2} \ \forall\ n \in \mathbb{Z}$. Then for $z = i(4n+1)\frac{\pi}{2} \ \forall\ n \in \mathbb{Z}$ $e^z = i$

- Let for $z = a + ib$ $e^z = -\frac{i}{2}$. Then

  $$e^{a+ib} = -\frac{i}{2} \implies e^a e^{ib} = -\frac{i}{2}$$

  Now $|e^{ib}| = 1$. Hence $e^a = \frac{1}{2}$ then $a = \ln \frac{1}{2}$. Hence $e^{ib} = -i = \cos b + i \sin b$. Then $\cos b = 0$ and $\sin b = -1$. Then $b = (4n+3)\frac{\pi}{2} \ \forall\ n \in \mathbb{Z}$. Then for $z = \ln \frac{1}{2} + i(4n+3)\frac{\pi}{2} \ \forall\ n \in \mathbb{Z}$ $e^z = -\frac{i}{2}$

- Let for $z = a + ib$ $e^z = -i$. Then

  $$e^{a+ib} = -i \implies e^a e^{ib} = -i$$

  Now $|e^{ib}| = 1$. Hence $e^a = 1$ then $a = 0$. Hence $e^{ib} = -i = \cos b + i \sin b$. Then $\cos b = 0$ and $\sin b = -1$. Then $b = (4n+3)\frac{\pi}{2} \ \forall\ n \in \mathbb{Z}$. Then for $z = i(4n+3)\frac{\pi}{2} \ \forall\ n \in \mathbb{Z}$ $e^z = -i$

- Let for $z = a + ib$ $e^z = 1 + 2i$. Then

$$e^{a+ib} = 1 + 2i \implies e^a e^{ib} = 1 + 2i$$

Now $|e^{ib}| = 1$. Hence $e^a = \sqrt{5}$ then $a = \ln\sqrt{5}$. Hence $e^{ib} = \frac{1}{\sqrt{5}}(1+2i) = \cos b + i\sin b$. Then $\cos b = \frac{1}{\sqrt{5}}$ and $\sin b = \frac{2}{\sqrt{5}}$. Then $b = 2n\pi + \sin^{-1}\frac{2}{\sqrt{5}}$ $\forall\, n \in \mathbb{Z}$. Then for $z = \ln\sqrt{5} + i\left(2n\pi + \sin^{-1}\frac{2}{\sqrt{5}}\right)$ $\forall\, n \in \mathbb{Z}$ $e^z = 1 + 2i$

$\square$

Problem 4 Ahlfors Page 47: Problem 6

Determine all values of 2^i, i^i, $(-1)^{2i}$

Solution:

- $2^i = \exp(i\ln 2) = \cos\ln 2 + i\sin\ln 2$
- $i^i = \exp(i\log i) = \exp\left(i\ln\left(e^{i(4n+1)\frac{\pi}{2}}\right)\right) = \exp\left(i\left(i(4n+1)\frac{\pi}{2}\right)\right) = \exp\left(-(4n+1)\frac{\pi}{2}\right)$
- $(-1)^{2i} = i^(4i) = (i^i)^4 = \left(\exp\left(-(4n+1)\frac{\pi}{2}\right)\right)^4 = \exp\left(-2(4n+1)\pi\right)$

$\square$

Problem 5 Ahlfors Page 47: Problem 7

Determine the real and imaginary parts of z^z

Solution: Let $z = x + iy$. Then

$$z^z = (x+iy)^{x+iy} = \exp((x+iy)\log(x+iy))$$

Now let $e^{a+ib} = x + iy$. Since $|e^{ib}| = 1$. We have $e^a = \sqrt{x^2+y^2}$. Therefore $a = \ln\sqrt{x^2+y^2}$. Now

$$e^{ib} = \cos b + i\sin b = \frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}$$

Hence $b = 2n\pi + \tan^{-1}\frac{y}{x}$ $\forall\, n \in \mathbb{Z}$. Therefore

$$a + ib = \ln\sqrt{x^2+y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

Hence

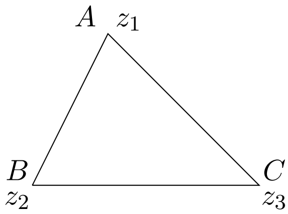
$$\begin{aligned}
\exp((x+iy)\ln(x+iy)) &= \exp\left((x+iy)\ln\left(e^{\ln\sqrt{x^2+y^2}+i\left(2n\pi+\tan^{-1}\frac{y}{x}\right)}\right)\right)\\
&= \exp\left((x+iy)\left(\ln\sqrt{x^2+y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)\right)\right)\\
&= \exp\left(x\ln\sqrt{x^2+y^2} - y\left(2n\pi + \tan^{-1}\frac{y}{x}\right)\right.\\
&\qquad\qquad \left. + i\left(y\ln\sqrt{x^2+y^2} + x\left(2n\pi + \tan^{-1}\frac{y}{x}\right)\right)\right)\\
&= e^{x\ln\sqrt{x^2+y^2}-y\left(2n\pi+\tan^{-1}\frac{y}{x}\right)}\left[\cos\left(\left(y\ln\sqrt{x^2+y^2} + x\left(2n\pi + \tan^{-1}\frac{y}{x}\right)\right)\right)\right.\\
&\qquad\qquad \left. + i\sin\left(\left(y\ln\sqrt{x^2+y^2} + x\left(2n\pi + \tan^{-1}\frac{y}{x}\right)\right)\right)\right]
\end{aligned}$$

$\square$

Problem 6 Ahlfors Page 47: Problem 9

Show how to define the "angles: in a triangle, bearing in mind they should lie between 0 and π. With this definition, prove that the sum of the angles is π.

Solution:



The angle between the sides AB and AC defined to be the angle between the sides in the anti clockwise direction. Hence we can define the angle

$$\angle A = \Im \log\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \Im\left(\log(z_3 - z_1) - \log(z_2 - z_1)\right) = \Im \log(z_3 - z_1) - \Im \log(z_2 - z_1)$$

Therefore for other angles we have

$$\angle B = \Im \log\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \Im \log(z_1 - z_2) - \Im \log(z_3 - z_2)$$

$$\angle C = \Im \log\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \Im \log(z_2 - z_3) - \Im \log(z_1 - z_3)$$

Therefore

$$\begin{aligned}
\angle A + \angle B + \angle C &= \Im \log\left(\frac{z_3 - z_1}{z_2 - z_1}\right) + \Im \log\left(\frac{z_1 - z_2}{z_3 - z_2}\right) + \Im \log\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \\
&= \Im \log\left(\frac{z_3 - z_1}{z_2 - z_1}\frac{z_1 - z_2}{z_3 - z_2}\frac{z_2 - z_3}{z_1 - z_3}\right) \\
&= \Im \log\left(\frac{z_1 - z_2}{z_2 - z_1}\frac{z_2 - z_3}{z_3 - z_2}\frac{z_3 - z_1}{z_1 - z_3}\right) \\
&= \Im \log\left((-1)^3\right) = \Im \log(-1) = \pi
\end{aligned}$$

□

Problem 7 Ahlfors Page 72: Problem 1

Give a precise definition of a single-valued branch of $\sqrt{a+z} + \sqrt{1-z}$ in a suitable region, and prove that it is analytic.

Solution: We recall that for $\sqrt{z}$ we choose the region Ω which is the complement of the negative real axis $x \leq 0$, $y = 0$. Hence, to define $\sqrt{1+z}$ we should choose Ω_1 as the complement of $x \leq -1$, $y = 0$ and for $\sqrt{1-z}$ we should choose Ω_2 as the complement of $1 \leq x$, $y = 0$. In total, to define $\sqrt{1+z} + \sqrt{1-z}$ we choose the region $\Omega = \Omega_1 \cap \Omega_2$. This is actually the same region used in defining $\arccos z$. The branch chosen is that which has positive real part. As simple transformations of $\sqrt{z}$ in an analytic region, it follows that $\sqrt{1+z} + \sqrt{1-z}$ is analytic.

□

Problem 8 Ahlfors Page 72: Problem 3

Suppose that $f(z)$ is analytic and satisfies the condition $|f^2(z) - 1| < 1$ in a region Ω. Show that either $\Re f(z) > 0$ or $\Re f(z) < 0$ through out Ω

Solution: Suppose $\Re f(z) = 0$ at a point $z \in \Omega$. Then $f(z) = iy^2$ for some $y \in \mathbb{R}$, and thus $f(z)^2 = -y^2$. By the condition $|f(z)^2 - 1| < 1$ we have $|-y^2 - 1| < 1$ and thus $|y^2 + 1| < 1$. This is clearly impossible, so that $\Re f(z) \neq 0$ throughout Ω. But, $\Re f(z)$ is continuous and Ω is connected, so either $\Re f(z) > 0$ or $\Re f(z) < 0$.

□

Problem 9 Ahlfors Page 78: Problem 1

Prove that the reflection $z \to \overline{z}$ is not a linear transformation

Solution: Suppose $\varphi(z) : z \mapsto \overline{z}$ is a linear fractional transformation, and thus it must be of the form

$$\overline{z} = \varphi(z) = \frac{az+b}{cz+d}, \quad \forall\, z \in \mathbb{C}$$

Note that if $\Im z = 0$ then $\overline{z} = z$. In particular,

$$0 \mapsto 0 \implies \varphi(0) = \frac{b}{d} = 0 \implies b = 0$$

Plugging in different values yields

$$\begin{aligned} 1 \mapsto \varphi(1) &= \frac{a}{c+d} = 1 \\ -1 \mapsto \varphi(-1) &= \frac{-a}{-c+d} = -1 \end{aligned}$$

Or,

$$\begin{aligned} c + d &= a \\ d - c &= a \end{aligned}$$

Thus, $a = d$ and hence $c = 0$. But then we have

$$\overline{z} = \frac{az}{d} = z$$

Hence contradiction. φ is not linear transformation

□

Problem 10 Ahlfors Page 78: Problem 2

If

$$T_1 z = \frac{z+2}{z+3} \qquad T_2 z = \frac{z}{z+1}$$

Find T_1T_2z, T_2T_1z, $T_1^{-1}T_2z$

Solution: Here we compute several compositions:

$$\begin{aligned} T_1T_2z &= \frac{\frac{z}{z+1} + 2}{\frac{z}{z+1} + 3} = \frac{\frac{3z+2}{z+1}}{\frac{4z+3}{z+1}} = \frac{3z+2}{4z+3} \\ T_2T_2z &= \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3} + 1} = \frac{\frac{z+2}{z+3}}{\frac{2z+5}{z+3}} = \frac{z+2}{2z+5} \end{aligned}$$

Now note that

$$T_1^{-1}(w) = \frac{3w-2}{1-w}$$

Thus,

$$T_1^{-1}T_2 z = \frac{3\frac{z}{z+1} - 2}{1 - \frac{z}{z+1}} = \frac{z-2}{1} = z - 2$$

□

Problem 11 Ahlfors Page 78: Problem 3

Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by reflexion in the real axis.

Solution: Let φ be a fractional linear transformation as given. Then, we have that $\varphi(0) = 0$ and for any pair $(z, w) \in \mathbb{C} \times \mathbb{C}$:

$$|z - w| = |\varphi(z) - \varphi(w)|$$

since we assume that φ preserved distance under transformation. Since φ is fractional linear transformation we have

$$\varphi(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$. The assumption that $0 \mapsto 0$ yields immediately

$$0 = \varphi(0) = \frac{b}{d}$$

and thus we have $b = 0$. Now, as $\varphi(0) = 0$ we see

$$|z| = |\varphi(z) - \varphi(0)| = |\varphi(z) - 0| = \left|\frac{az}{cz+d}\right| = |a| \cdot \frac{|z|}{|cz+d|}, \quad \forall z \in \mathbb{C}$$

If $|a| = 0$ i.e. $a = 0$ then $\varphi(z) = 0$. Otherwise $|a| > 0$. then for all $z \in \mathbb{C}$ we must have

$$\frac{|a|}{|cz+d|} = 1 \text{ in } \mathbb{C}$$

This can only happen if the denominator is also constant and hence we conclude that we require additionally that $c = 0$. We are left with $\frac{|a|}{|d|} = 1$, or $|a| = |d|$. Hence

$$\varphi(z) = \frac{az}{d}$$

where $\left|\frac{a}{d}\right| = 1$. Let $k = \frac{a}{d}$. Then

$$\varphi(z) = kz \qquad \text{where } |k| = 1$$

Clearly φ is a rotation.

□

Problem 12 Ahlfors Page 78: Problem 4

Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

Solution: Let $\varphi(z)$ be a linear transformation with the additional restriction that $\varphi(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. Hence

$$\varphi(z) = \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{C}$. We will now find $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ so that $\varphi(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. We need to distinguish separate cases:

If $a \neq 0$:

$$\varphi(z) = \frac{az+b}{cz+d} \cdot \frac{\overline{a}}{\overline{a}} = \frac{a'z+b'}{c'z+d'}, \quad a' \in \mathbb{R}$$

The important thing to observe is that we will then have $\Im a' = 0$, i.e $a \in \mathbb{R}$. We now have a representation

$$\varphi(z) = \frac{a'z+b'}{c'z+d'}, \quad a' \in \mathbb{R}$$

In the above we see that $z \to \infty \implies \phi(\infty) = \frac{a'}{c'}$ implying that $c' \in \mathbb{R}$ as well as a'. The transformation has an inverse

$$\varphi^{-1}(z) = \frac{d'z - b'}{a' - c'z}$$

If $d' = 0$ then taking $z = 0$ we see $\varphi(0) = b$ and hence achieve $b' \in \mathbb{R}$. We now need to show that the same holds whenever $d' \neq 0$. In the above we see that $z \to \infty \implies \varphi^{-1}(\infty) = -\frac{d'}{c'}$ implying that $d' \in \mathbb{R}$ as well as c'.

$$\varphi(0) = \frac{b'}{d'} \in \mathbb{R} \Longrightarrow b' \in \mathbb{R}$$

If $a = 0$:

If $a = 0$ then we have $\varphi(z) = \frac{b}{cz+d}$. Here if $b = 0$ we have $\varphi(z)$ is non-invertible which is impossible. So

$$\varphi(z) = \frac{b}{cz+b} \cdot \frac{\overline{b}}{\overline{b}}$$

thus obtaining

$$\varphi(z) = \frac{b'}{c'z+d'}, \quad b' \in \mathbb{R}$$

Taking $z = 0$ we see that $\varphi(0) = \frac{b'}{d'}$. We then have $d' \in \mathbb{R}$. If we take $z = 1$ then

$$\varphi(1) = \frac{b'}{c'+d'}$$

Since $b', d, \in \mathbb{R}$ if $c' \notin \mathbb{R}$ then $c' + d' \notin \mathbb{R}$ but $c' + d' \in \mathbb{C}$ hence $\frac{b'}{c'+d'} \notin \mathbb{R}$ but $\frac{b'}{c'+d'}' \in \mathbb{C}$. But $\varphi(1) \in \mathbb{R}$. So $c' \in \mathbb{R}$.

Hence φ can be written with real coefficients

$\square$