

**Problem 1 Ahlfors Page 154: Problem 1**

How many roots does the equation  $z^7 - 2z^5 + 6z^3 - z + 1 = 0$  have in the disk  $|z| < 1$ ? Hint: Look for the biggest term when  $|z| = 1$  and apply Rouché's theorem.

**Solution:** Take  $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$  and  $f(z) = 6z^3$ . Then on  $|z| = 1$

$$|g(z) - f(z)| = |z^7 - 2z^5 - z + 1| \leq |z|^7 + 2|z|^5 + |z| + 1 \leq 1 + 2 + 1 + 1 = 5 < 6 = |f(z)|$$

Hence by Rouché's Theorem  $f(z)$  and  $g(z)$  has same number of zeros inside  $|z| < 1$ . Now  $f$  has only three zero (one zero with order three). Hence  $g(z)$  has three zeros inside  $|z| < 1$ . □

**Problem 2 Ahlfors Page 154: Problem 2**

How many roots of the equation  $z^4 - 6z + 3 = 0$  have their modulus between 1 and 2?

**Solution:** Let  $g(z) = z^4 - 6z + 3$ . In this case we find how many zeros  $g$  has inside  $|z| < 1$  and  $|z| < 2$ . then we subtract the number of zeros inside the smaller disk from the number of zeros inside the bigger disk.

Take  $f(z) = -6z$ . Then on  $|z| = 1$  we have

$$|g(z) - f(z)| = |z^4 + 3| \leq |z|^4 + 1 = 4 < 6 = |f(z)|$$

Hence by Rouché's theorem  $f$  and  $g$  has same number of zeros inside  $|z| < 1$ . Now  $f$  has only one zero in  $|z| < 1$ . Hence  $g$  has one zero in  $|z| < 1$ .

Now take  $f(z) = z^4$ . Then  $|z| = 2$  we have

$$|g(z) - f(z)| = |6z + 3| \leq 6|z| + 3 = 6 \times 2 + 3 = 15 < 16 = |f(z)|$$

Hence by Rouché's Theorem  $f$  and  $g$  has same number of zeros inside  $|z| < 2$ . Now  $f$  has only four zeros in  $|z| < 2$  (one zero with order four). Therefore  $g$  has four zeros in  $|z| < 2$ .

Hence  $g$  has  $4 - 1 = 3$  zeros in the region between  $|z| < 1$  and  $|z| < 2$ . □

**Problem 3 Ahlfors Page 161: Problem 3**

Evaluate the following integrals by the method of residues:

(a)  $\int_0^{\pi/2} \frac{dx}{a + \sin^2 x}, |a| > 1,$

(f)  $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx, a \text{ real},$

(b)  $\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6},$

(g)  $\int_0^\infty \frac{x^{1/3}}{1 + x^2} dx,$

(c)  $\int_{-\infty}^\infty \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx,$

(h)  $\int_0^\infty (1 + x^2)^{-1} \log x dx,$

(d)  $\int_0^\infty \frac{x^2 dx}{(x^2 + a^2)^3}, a \text{ real},$

(i)  $\int_0^\infty \log(1 + x^2) \frac{dx}{x^{1+\alpha}} (0 < \alpha < 2).$  (Try integration by parts.)

(e)  $\int_0^\infty \frac{\cos x}{x^2 + a^2} dx, a \text{ real},$

**Solution:**

(a) We know  $\sin^2 x = \frac{1-\cos 2x}{2}$  Hence

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{dx}{a + \frac{1-\cos 2x}{2}} = \int_0^{\frac{\pi}{2}} \frac{2dx}{2a + 1 - \cos 2x} = \int_0^{\pi} \frac{dt}{2a + 1 - \cos t} \text{ [Substituting } t = 2x]$$

Now take  $-2a - 1 = b$ . Therefore  $|b| > 1$ . Hence we need to evaluate  $\int_0^{\pi} \frac{dx}{b - \cos x}$

**Case 1:**  $b > 1$

Then this is like Ahlfors Case 1 Example. Then

$$\int_0^{\pi} \frac{dx}{b - \cos x} = \frac{\pi}{\sqrt{b^2 - 1}} = \frac{\pi}{\sqrt{(2a + 1)^2 - 1}} = \frac{\pi}{2\sqrt{a}\sqrt{a + 1}}$$

**Case 2:**  $b < -1$

Now

$$\int_0^{\pi} \frac{dx}{b + \cos x} = -i \int_{|z|=1} \frac{dz}{z^2 + 2bz + 1}$$

Hence  $z^2 + 2bz + 1 = (z - \alpha)(z - \beta)$  where

$$\alpha = -b + \sqrt{b^2 - 1} \quad \beta = -b - \sqrt{b^2 - 1}$$

Then just like Ahlfors Case 1 Example instead of taking the root  $\alpha = -b + \sqrt{b^2 - 1}$  we will choose the root  $\beta = -b - \sqrt{b^2 - 1}$ , cause  $|\alpha| > 1$  and  $|\beta| < 1$ . Therefore residue at  $\beta$  is  $\frac{1}{\beta - \alpha} = -\frac{1}{2\sqrt{b^2 - 1}}$ . Then we get

$$\int_0^{\pi} \frac{dx}{b + \cos x} = -\frac{\pi}{\sqrt{b^2 - 1}} = -\frac{\pi}{2\sqrt{a}\sqrt{a + 1}}$$

Therefore

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x} = \begin{cases} \frac{\pi}{2\sqrt{a}\sqrt{a + 1}} & \text{when } a > 1 \\ -\frac{\pi}{2\sqrt{a}\sqrt{a + 1}} & \text{when } a < -1 \end{cases}$$

□

(b)

$$\int_0^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6}$$

Now this is like Ahlfors Case 2. Using that we get

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6} = 2\pi i \sum_{y>0} \text{Res} \frac{z^2}{z^4 + 5z^2 + 6}$$

Now

$$z^4 + 5z^2 + 6 = (z^2 + 3)(z^2 + 2) = (z + \sqrt{2}i)(z - \sqrt{2}i)(z + \sqrt{3}i)(z - \sqrt{3}i)$$

Now

$$\begin{aligned} \text{Res}_{z=\sqrt{3}i} \frac{z^2}{z^4 + 5z^2 + 6} &= \lim_{z \rightarrow \sqrt{3}i} \frac{z^2(z - \sqrt{3}i)}{z^4 + 5z^2 + 6} = \frac{-3}{2\sqrt{3}i(-3 + 2)} = \frac{\sqrt{3}}{2i} \\ \text{Res}_{z=\sqrt{2}i} \frac{z^2}{z^4 + 5z^2 + 6} &= \lim_{z \rightarrow \sqrt{2}i} \frac{z^2(z - \sqrt{2}i)}{z^4 + 5z^2 + 6} = \frac{-2}{2\sqrt{2}i(-2 + 3)} = -\frac{\sqrt{2}}{2i} \end{aligned}$$

Therefore

$$\begin{aligned} 2\pi i \sum_{y>0} \text{Res} \frac{z^2}{z^4 + 5z^2 + 6} &= 2\pi i \left( \text{Res}_{z=\sqrt{3}i} \frac{z^2}{z^4 + 5z^2 + 6} + \text{Res}_{z=\sqrt{2}i} \frac{z^2}{z^4 + 5z^2 + 6} \right) \\ &= 2\pi i \left( \frac{\sqrt{3}}{2i} - \frac{\sqrt{2}}{2i} \right) = \pi(\sqrt{3} - \sqrt{2}) \end{aligned}$$

Hence

$$\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6} = \frac{\pi}{2}(\sqrt{3} - \sqrt{2})$$

□

(c) Now this is like Ahlfors Case 2. Using that we get

$$\int_{-\infty}^\infty \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} = 2\pi i \sum_{y>0} \text{Res} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$$

Now

$$z^4 + 10z^2 + 9 = (z^2 + 9)(z^2 + 1) = (z + 3i)(z - 3i)(z + i)(z - i)$$

Now

$$\begin{aligned} \text{Res}_{z=3i} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} &= \lim_{z \rightarrow 3i} \frac{(z^2 - z + 2)(z - 3i)}{z^4 + 10z^2 + 9} = \frac{-9 - 3i + 2}{6i(-9 + 1)} = \frac{7 + 3i}{48i} \\ \text{Res}_{z=i} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} &= \lim_{z \rightarrow i} \frac{(z^2 - z + 2)(z - i)}{z^4 + 10z^2 + 9} = \frac{-1 - i + 2}{(-1 + 9)2i} = \frac{1 - i}{16i} = \frac{3 - 3i}{48i} \end{aligned}$$

Therefore

$$\begin{aligned} 2\pi i \sum_{y>0} \text{Res} \frac{z^2}{z^4 + 5z^2 + 6} &= 2\pi i \left( \text{Res}_{z=3i} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} + \text{Res}_{z=i} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} \right) \\ &= 2\pi i \left( \frac{7 + 3i}{48i} + \frac{3 - 3i}{48i} \right) = \pi \frac{10}{24} = \frac{5\pi}{12} \end{aligned}$$

Hence

$$\int_{-\infty}^\infty \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} = \frac{5\pi}{12}$$

□

(d) If  $a = 0$  then the integral is infinite. Hence let's assume  $a \neq 0$ .

$$\int_0^\infty \frac{x^2 dx}{(x^2 + a^2)^3} = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2 dx}{(x^2 + a^2)^3}$$

Now this is like Ahlfors Case 2. Using that we get

$$\int_{-\infty}^\infty \frac{x^2 dx}{(x^2 + a^2)^3} = 2\pi i \sum_{y>0} \text{Res} \frac{z^2}{(z^2 + a^2)^3}$$

Now

$$(z^2 + a^2)^3 = (z + ai)^3(z - ai)^3$$

Now WLOG assume  $a > 0$ . Now

$$\frac{d^2}{dz^2} \frac{z^2(z - ai)^3}{(z^2 + a^2)^3} = \frac{d}{dz} \left[ \frac{2z}{(z + ai)^3} - \frac{3z^2}{(z + ai)^4} \right] = \frac{d^2}{dz^2} \frac{z^2}{(z + ai)^3} = \frac{2}{(z + ai)^3} - \frac{12z^2}{(z + ai)^4} + \frac{12z^2}{(z + ai)^5}$$

$$\begin{aligned}\text{Res}_{z=ai} \frac{z^2}{(z^2+a^2)^3} &= \lim_{z \rightarrow ai} \frac{1}{2!} \frac{d^2}{dz^2} \frac{z^2(z-ai)^3}{(z^2+a^2)^3} = \frac{1}{2} \lim_{z \rightarrow ai} \left( \frac{2}{(z+ai)^3} - \frac{12z}{(z+ai)^4} + \frac{12z^2}{(z+ai)^5} \right) = \frac{1}{16a^3i} \\ \text{Res}_{z=-ai} \frac{z^2}{(z^2+a^2)^3} &= \lim_{z \rightarrow -ai} \frac{1}{2!} \frac{d^2}{dz^2} \frac{z^2(z+ai)^3}{(z^2+a^2)^3} = \frac{1}{2} \lim_{z \rightarrow -ai} \left( \frac{2}{(z-ai)^3} - \frac{12z}{(z-ai)^4} + \frac{12z^2}{(z-ai)^5} \right) = -\frac{1}{16a^3i}\end{aligned}$$

Therefore

$$2\pi i \sum_{y>0} \text{Res} \frac{z^2}{(z^2+a^2)^3} = \begin{cases} 2\pi i \text{Res}_{z=ai} \frac{z^2}{(z^2+a^2)^3} = 2\pi i \frac{1}{16a^3i} = \frac{2\pi}{16a^3} & \text{when } a > 0 \\ 2\pi i \text{Res}_{z=-ai} \frac{z^2}{(z^2+a^2)^3} = 2\pi i \frac{1}{16(-a)^3i} = \frac{2\pi}{16(-a)^3} & \text{when } a < 0 \end{cases}$$

Hence

$$\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6} = \frac{\pi}{16|a|^3} \quad \text{when } a \neq 0$$

□

(e) If  $a = 0$  then

$$\int_0^\infty \frac{\cos x}{x^2} dx \geq - \int_0^\infty \frac{dx}{x^2}$$

which is divergent hence the integral is infinite. Hence let's assume  $a \neq 0$ . Since  $\frac{\sin x}{x^2+a^2}$  is an odd function we have  $\int_{-\infty}^\infty \frac{\sin x}{x^2+a^2} dx = 0$ . Hence we have

$$\int_0^\infty \frac{\cos x}{x^2+a^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos x}{x^2+a^2} dx = \frac{1}{2} \left[ \int_{-\infty}^\infty \frac{\cos x}{x^2+a^2} dx + i \int_{-\infty}^\infty \frac{\sin x}{x^2+a^2} dx \right] = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix}}{x^2+a^2} dx$$

Now this is like Ahlfors Case 3

$$\int_{-\infty}^\infty \frac{e^{ix}}{x^2+a^2} dx = 2\pi i \sum_{y>0} \text{Res} \frac{e^{iz}}{z^2+a^2}$$

now  $z^2 + a^2 = (z+ai)(z-ai)$ . WLOG assume  $a > 0$

$$\text{Res}_{z=ai} \frac{e^{iz}}{z^2+a^2} = \lim_{z \rightarrow ai} \frac{e^{iz}(z-ai)}{z^2+a^2} = \frac{e^{-a}}{2ai}$$

Hence

$$2\pi i \sum_{y>0} \text{Res} \frac{1}{z^2+a^2} = 2\pi i \text{Res}_{z=ai} \frac{e^{iz}}{z^2+a^2} = 2\pi i \frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}$$

Hence

$$\int_0^\infty \frac{\cos x}{x^2+a^2} dx = \frac{\pi e^{-a}}{2a}$$

□

(f) If  $a = 0$  then we have  $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$  by Ahlfors Case 3 Example (Page 158)

Since  $\frac{x \cos x}{x^2+a^2}$  is an odd function we have  $\int_{-\infty}^\infty \frac{x \cos x}{x^2+a^2} dx = 0$ . Hence we have

$$i \int_0^\infty \frac{x \sin x}{x^2+a^2} dx = i \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2+a^2} dx = \frac{1}{2} \left[ \int_{-\infty}^\infty \frac{x \sin x}{x^2+a^2} dx + i \int_{-\infty}^\infty \frac{x \cos x}{x^2+a^2} dx \right] = \frac{1}{2} \int_{-\infty}^\infty \frac{x e^{ix}}{x^2+a^2} dx$$

Now this is like Ahlfors Case 3

$$\int_{-\infty}^{\infty} \frac{x e^{ix} dx}{x^2 + a^2} = 2\pi i \sum_{y>0} \text{Res} \frac{z e^{iz}}{z^2 + a^2}$$

now  $z^2 + a^2 = (z + ai)(z - ai)$ . WLOG assume  $a > 0$

$$\text{Res}_{z=ai} \frac{z e^{iz}}{z^2 + a^2} = \lim_{z \rightarrow ai} \frac{z e^{iz}(z - ai)}{z^2 + a^2} = \frac{a i e^{-a}}{2ai} = \frac{e^{-a}}{2}$$

Hence

$$2\pi i \sum_{y>0} \text{Res} \frac{1}{z^2 + a^2} = 2\pi i \text{Res}_{z=ai} \frac{z e^{iz}}{z^2 + a^2} = 2\pi i \frac{e^{-a}}{2} = \pi e^{-a} i$$

Hence

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \Im \left( \frac{1}{2} \pi e^{-a} i \right) = \frac{\pi e^{-a}}{2}$$

□

(g) Now this is like Ahlfors Case 4. Hence we have

$$\int_0^{\infty} \frac{x^{1/3} dx}{1 + x^2} = \frac{2}{1 - e^{2\pi i/3}} \sum \text{Res} \frac{z^{1/3}}{1 + z^2}$$

Now

$$1 + z^2 = (z + i)(z - i)$$

Hence

$$\begin{aligned} \text{Res}_{z=i} &= \lim_{z \rightarrow i} \frac{z^{1/3}(z - i)}{1 + z^2} = \frac{i^{1/3}}{2i} = \frac{e^{i\pi/6}}{2} \\ \text{Res}_{z=-i} &= \lim_{z \rightarrow -i} \frac{z^{1/3}(z + i)}{1 + z^2} = \frac{(-i)^{1/3}}{-2i} = -\frac{e^{i\pi/2}}{2} \end{aligned}$$

Hence

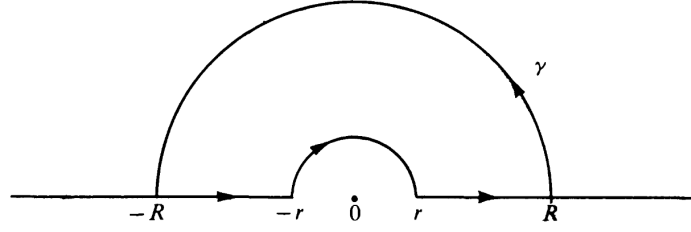
$$\begin{aligned} \frac{2}{1 - e^{2\pi i/3}} \sum \text{Res} \frac{z^{1/3}}{1 + z^2} &= \frac{2}{1 - e^{2\pi i/3}} \left[ \frac{e^{i\pi/6}}{2} - \frac{e^{i\pi/2}}{2} \right] \\ &= \frac{1}{(1 - e^{\pi i/3})(1 + e^{\pi i/3})} e^{i\pi/6} (1 - e^{i\pi/3}) \\ &= \frac{\pi e^{i\pi/6}}{1 + e^{i\pi/3}} = \frac{\pi}{e^{-i\pi/6} + e^{i\pi/6}} = \frac{\pi}{2 \cos \frac{\pi}{6}} = \frac{\pi}{\sqrt{3}} \end{aligned}$$

Hence

$$\int_0^{\infty} \frac{x^{1/3} dx}{1 + x^2} = \frac{\pi}{\sqrt{3}}$$

□

(h) Let  $\gamma$  be the following curve for  $0 < r < R$



$$\int_{\gamma} \frac{\log z}{1+z^2} dz = \int_r^R \frac{\log x}{1+x^2} dx + iR \int_0^{\pi} \frac{\log R + i\theta}{1+R^2 e^{2i\theta}} e^{i\theta} d\theta + \int_{-R}^{-r} \frac{\log |x| + \pi i}{1+x^2} dx + ir \int_{\pi}^0 \frac{[\log r + i\theta]}{1+r^2 e^{2i\theta}} e^{i\theta} d\theta$$

Now

$$\int_r^R \frac{\log x}{1+x^2} dx + \int_{-R}^{-r} \frac{\log |x| + \pi i}{1+x^2} dx = 2 \int_r^R \frac{\log x}{1+x^2} dx + \pi i \int_r^R \frac{dx}{1+x^2}$$

Now as  $r \rightarrow 0$  and  $R \rightarrow \infty$  we have

$$\int_r^R \frac{dx}{1+x^2} = \frac{\pi}{2}$$

Now

$$\left| R \int_0^{\pi} \frac{[\log R + i\theta]}{1+R^2 e^{2i\theta}} e^{i\theta} d\theta \right| \leq \frac{R|\log R|}{|1-R^2|} \int_0^{\pi} d\theta + \frac{R}{|1-R^2|} \int_0^{\pi} \theta d\theta = \frac{\pi R|\log R|}{|1-R^2|} + \frac{R\pi^2}{2|1-R^2|}$$

Hence as  $R \rightarrow \infty$  we have  $\frac{\pi R|\log R|}{|1-R^2|} + \frac{R\pi^2}{2|1-R^2|} \rightarrow 0$ . Similarly

$$\left| r \int_0^{\pi} \frac{[\log r + i\theta]}{1+r^2 e^{2i\theta}} e^{i\theta} d\theta \right| \leq \frac{r|\log r|}{|1-r^2|} \int_0^{\pi} d\theta + \frac{r}{|1-r^2|} \int_0^{\pi} \theta d\theta = \frac{\pi r|\log r|}{|1-r^2|} + \frac{r\pi^2}{2|1-r^2|}$$

Hence as  $r \rightarrow 0$  we have  $\frac{\pi r|\log r|}{|1-r^2|} + \frac{r\pi^2}{2|1-r^2|} \rightarrow 0$ . Now by Residue Theorem we have

$$\int_{\gamma} \frac{\log z}{1+z^2} dz = \sum_{y>0} \text{Res} \frac{\log z}{1+z^2}$$

Now  $1+z^2 = (z+i)(z-i)$ . Therefore

$$\text{Res}_{z=i} \frac{\log z}{1+z^2} = \lim_{z \rightarrow i} \frac{(z-i) \log z}{1+z^2} = \frac{\log i}{2i} = \frac{\frac{i\pi}{2}}{2i} = \frac{\pi}{4}$$

Hence

$$\int_{\gamma} \frac{\log z}{1+z^2} dz = 2\pi i \frac{\pi}{4} = \frac{\pi^2 i}{2} = 2 \int_0^{\infty} \frac{\log x}{1+x^2} dx + \pi i \frac{\pi}{2} \implies \int_0^{\infty} \frac{\log x}{1+x^2} dx = 0$$

(i)

$$\int_0^{\infty} \frac{\log(1+x^2)}{x^{1+\alpha}} dx = - \left. \frac{\log(1+x^2)}{\alpha x^{\alpha}} \right|_0^{\infty} + \frac{1}{\alpha} \int_0^{\infty} \frac{d/dx \log(1+x^2)}{x^{\alpha}} dx = \frac{2}{\alpha} \int_0^{\infty} \frac{x}{x^{\alpha}(1+x^2)} dx$$

Now there are three cases

**Case 1:**  $\alpha = 1$

Then

$$\frac{2}{\alpha} \int_0^\infty \frac{x}{x^\alpha (1+x^2)} dx = 2 \int_0^\infty \frac{x}{x(1+x^2)} dx = 2 \int_0^\infty \frac{1}{(1+x^2)} dx = 2 \frac{\pi}{2} = \pi$$

**Case 2:**  $0 < \alpha < 1$

Then take  $a = 1 - \alpha$ . Then  $0 < a < 1$ . Hence

$$\int_0^\infty \frac{x}{x^\alpha (1+x^2)} dx = \int_0^\infty \frac{x^{1-\alpha}}{1+x^2} dx = \int_0^\infty \frac{x^a}{1+x^2} dx$$

This is like Ahlfors Case 4. Now  $1+z^2 = (z+i)(z-i)$ . Hence

$$\text{Res}_{z=i} \frac{z^a}{1+z^2} = \lim_{z \rightarrow i} \frac{z^a(z-i)}{1+z^2} = \frac{i^a}{2i} = \frac{e^{\frac{ia\pi}{2}}}{2i}$$

$$\text{Res}_{z=-i} \frac{z^a}{1+z^2} = \lim_{z \rightarrow -i} \frac{z^a(z+i)}{1+z^2} = \frac{(-i)^a}{-2i} = -\frac{e^{\frac{i3a\pi}{2}}}{2i}$$

Hence

$$\begin{aligned} \int_0^\infty \frac{x}{x^\alpha (1+x^2)} dx &= \frac{2\pi i}{1 - e^{2\pi ia}} \left[ \frac{e^{\frac{ia\pi}{2}}}{2i} - \frac{e^{\frac{i3a\pi}{2}}}{2i} \right] \\ &= \frac{\pi e^{\frac{ia\pi}{2}}}{(1 - e^{\pi ia})(1 + e^{\pi ia})} (1 - e^{\pi ia}) \\ &= \frac{\pi e^{\frac{ia\pi}{2}}}{1 + e^{\pi ia}} \\ &= \frac{\pi}{e^{-\frac{ia\pi}{2}} + e^{\frac{ia\pi}{2}}} = \frac{\pi}{2 \cos \frac{a\pi}{2}} = \frac{\pi}{2 \sin \frac{\alpha\pi}{2}} \end{aligned}$$

Hence

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}} dx = \frac{2}{\alpha} \frac{\pi}{2 \sin \frac{\alpha\pi}{2}} = \frac{\pi}{\alpha \sin \frac{\alpha\pi}{2}}$$

**Case 3:**  $1 < \alpha < 2$

Then take  $b = \alpha - 1$ . Then  $0 < b < 1$ . Hence

$$\int_0^\infty \frac{x}{x^\alpha (1+x^2)} dx = \int_0^\infty \frac{1}{x^{\alpha-1}(1+x^2)} dx = \int_0^\infty \frac{1}{x^b(1+x^2)} dx$$

Let  $\gamma$  be the following curve for  $0 < r < R$ .

Hence

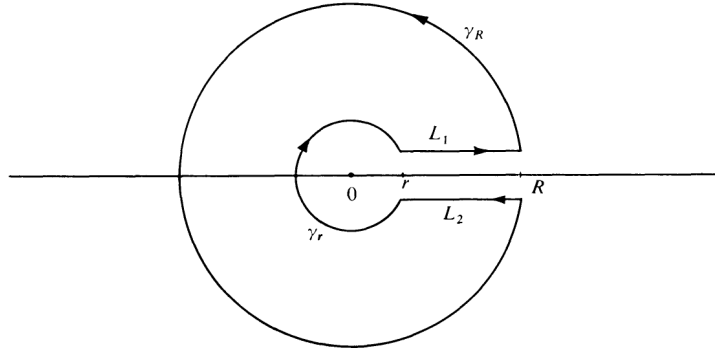
$$\left| \int_{\gamma_R} \frac{1}{z^b(1+z^2)} dz \right| \leq \frac{\pi R}{R^{\alpha-1} |R^2 - 1|} = \frac{\pi R^{2-\alpha}}{|R^2 - 1|}$$

Since  $1 < \alpha < 2$  we have  $0 < 2 - \alpha < 1$ . Hence as  $R \rightarrow \infty$ ,  $\frac{\pi R^{2-\alpha}}{|R^2 - 1|} \rightarrow 0$ . Similarly

$$\left| \int_{\gamma_r} \frac{1}{z^b(1+z^2)} dz \right| \leq \frac{\pi r}{r^{\alpha-1} |r^2 - 1|} = \frac{\pi r^{2-\alpha}}{|r^2 - 1|}$$

Hence as  $r \rightarrow 0$  we have  $\frac{\pi r^{2-\alpha}}{|r^2 - 1|} \rightarrow 0$ . Now  $1+z^2 = (z+i)(z-i)$ . Hence

$$\text{Res}_{z=i} \frac{1}{z^b(1+z^2)} = \lim_{z \rightarrow i} \frac{z-i}{z^b(1+z^2)} = \frac{i^{-b}}{2i} = \frac{e^{-\frac{ib\pi}{2}}}{2i}$$



$$\text{Res}_{z=-i} \frac{1}{z^b(1+z^2)} = \lim_{z \rightarrow -i} \frac{z+i}{z^b(1+z^2)} = \frac{(-i)^{-b}}{-2i} = -\frac{e^{-\frac{i3b\pi}{2}}}{2i}$$

Hence

$$\begin{aligned} \int_0^\infty \frac{1}{x^b(1+x^2)} dx &= \frac{2\pi i}{1 - e^{-2\pi i b}} \left[ \frac{e^{-\frac{ib\pi}{2}}}{2i} - \frac{e^{-\frac{i3b\pi}{2}}}{2i} \right] \\ &= \frac{\pi e^{-\frac{ib\pi}{2}}}{(1 - e^{-\pi i b})(1 + e^{-\pi i b})} (1 - e^{-\pi i b}) \\ &= \frac{\pi e^{-\frac{ib\pi}{2}}}{1 + e^{-\pi i b}} \\ &= \frac{\pi}{e^{\frac{ib\pi}{2}} + e^{-\frac{ib\pi}{2}}} = \frac{\pi}{2 \cos \frac{b\pi}{2}} = \frac{\pi}{2 \sin \frac{\alpha\pi}{2}} \end{aligned}$$

Hence

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}} dx = \frac{2}{\alpha} \frac{\pi}{2 \sin \frac{\alpha\pi}{2}} = \frac{\pi}{\alpha \sin \frac{\alpha\pi}{2}}$$

Therefore we get  $\forall 0 < \alpha < 2$  we have

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}} dx = \frac{2}{\alpha} \frac{\pi}{2 \sin \frac{\alpha\pi}{2}} = \frac{\pi}{\alpha \sin \frac{\alpha\pi}{2}}$$

□

#### Problem 4 Ahlfors Page 186: Problem 2

Let  $\Omega$  be a doubly connected region whose complement consists of the components  $E_1, E_2$ . Prove that every analytic function  $f(z)$  in  $\Omega$  can be written in the form  $f_1(z) + f_2(z)$  where  $f_1(z)$  is analytic outside of  $E_1$  and  $f_2(z)$  is analytic outside of  $E_2$ . (The precise proof requires a construction like the one in Chap. 4, Sec. 4.5.)

**Solution:** Let  $\Omega$  be the given doubly connected region. At least one of the regions  $E_1$  and  $E_2$  is bounded. WLOG suppose  $E_1$  is bounded. Hence boundary of  $E_1$ ,  $\partial E_1$  is compact. Hence  $\partial E_2$  is closed and disjoint. Let distance between  $\partial E_1$  and  $\partial E_2$  is  $d > 0$

Now we cover the entire plane with the grid of squares of length  $\epsilon$ . We will choose the  $\epsilon > 0$  later. Let  $S$  be the set of squares which intersects  $E_1$ . Since  $E_1$  is bounded  $S$  is a finite set. Let  $S'$  be the set of squares which are not in  $S$  but intersects any square in  $S$ . Take the set  $T = S \cup S'$ . If we take  $\epsilon$  sufficiently small then we can make  $T \subset \Omega$ , take  $\epsilon < \frac{d}{8}$

Now we will define a cycle.

$$\gamma = \sum_{R \in S'} \partial R$$

Now we cancel out the opposite directions of same sides. Now each side of  $\gamma$  have two cases



(1) An edge shared between a square in  $S$  and an square in  $S'$ .

(2) An outer edge of a square in  $S'$ .

Let  $\gamma_1$  be the part of  $\gamma$  which are of type (1). And  $\gamma_2$  be the part of  $\gamma$  which are of type (2). Hence they are disjoint and  $\gamma_1$  is negatively oriented,  $\gamma_2$  is positively oriented.

$$\gamma_1 + \gamma_2 = \gamma$$

Therefore

$$\forall z \in E_1, n(\gamma, z) = n(\gamma_1, z) + n(\gamma_2, z) = -1 + 1 = 0$$

Similarly since  $E_2$  is unbounded we have

$$\forall z \in E_2, n(\gamma, z) = 0$$

since we can make  $\epsilon$  small enough so that  $E_2$  lies outside the region enclosed by  $\gamma_2$ . Hence  $\gamma \sim 0 \pmod{\Omega}$ .

Hence by Generalized Cauchy Theorem

$$f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad k \in \{1, 2\}$$

and we get  $f(z) = f_1(z) + f_2(z)$ .  $f_2$  is analytic outside the union of square in  $S$  and  $f_1$  is analytic outside the union of squares of  $T$

Hence we take  $\epsilon$  small enough so that it approached to  $\partial E_1$  and so we can extend  $f_2$  all over outside  $E_1$ . Similarly we can extend  $f_1$  outside  $E_2$ .

□