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Course: Quantum Algorithmic Thinking

Assignment - 1
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Problem 1

Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices.

Solution: Pauli matrices are

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For I for all vectors v $Iv = v$. S0 every vector is an eigenvector and its eigenvalue is 1. Since I is already in its diagonal representation I's diagonal representation is I itself.

Since $\sigma_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\sigma_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have

$$\sigma_x \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \sigma_x \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = - \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

So the for the eignevalue 1 the corresponding eignevector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and for the eigenvalue -1 the corresponding eigenvalue is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Since $\sigma_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -i \end{bmatrix}$ and $\sigma_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 0 \end{bmatrix}$ we have

$$\sigma_y \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -i \end{bmatrix} + i \begin{bmatrix} i \\ 0 \end{bmatrix} = -1 \left(i \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \quad \sigma_y \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -i \end{bmatrix} - i \begin{bmatrix} i \\ 0 \end{bmatrix} = -i \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So the for the eignevalue 1 the corresponding eignevector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and for the eigenvalue -1 the corresponding eigenvalue is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Since $\sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\sigma_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the for the eignevalue 1 the corresponding eignevector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and for the eigenvalue -1 the corresponding eigenvalue is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now $\sigma_x, \sigma_y, \sigma_z$ has eigenvalues 1 and -1. So if we write in their corresponding eigenbasis then we will obtain the same diagonalized matrices where all the eigenvalues are in the diagonal positions i.e. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

□

Problem 2

Show that a normal matrix is Hermitian if and only if it has real eigenvalues. Show that a positive operator is necessarily Hermitian.

Solution: Let A is normal and it is hermitian. Then $A = A^\dagger$. Let v be an eigenvector of A with eigenvalue λ. Then $v^\dagger A v = v^\dagger \lambda v = \lambda |v|^2$. Also $v^\dagger A v = v^\dagger A^\dagger v = (Av)^\dagger v = \lambda^\dagger v^\dagger v = \lambda^\dagger |v|^2$. So we have $\lambda = \lambda^\dagger$. Which implies λ is real. Hence all eigenvalues of A are real.

For the opposite direction we need some lemmas.

Lemma 1: The product of two unitary matrices is unitary

Proof: Let U, V are two unitary matrices then $(UV)^\dagger = V^\dagger U^\dagger$. Now $(UV)(UV)^\dagger = U(VV^\dagger U^\dagger) = UIU^\dagger = I$.

Lemma 2: If A is any square complex matrix then there is an upper triangular complex matrix T and a unitary matrix U so that $A = UTU^\dagger$

Proof: Let A is a $n \times n$ matrix. Let v_1 be a eigenvector of A with the corresponding eigenvalue λ_1. We can take x_1 to be of unit length. Now by Gram-Schmidt process we can extend x_1 to an orthonormal basis $\{x_1, v_2, \dots, v_n\}$; Let $S_0 = \begin{bmatrix} x_1 & v_2 & \cdots & v_n \end{bmatrix}$ then S_0 is unitary and

$$S_0^\dagger A S_0 = \begin{bmatrix} \lambda_1 & * \\ 0 & A_1 \end{bmatrix}$$

where A_1 is an $(n-1) \times (n-1)$ matrix. Again suppose x_2 is an eigenvector of A_1 and the corresponding eigenvalue is λ_2. Then again for A_1 we extend x_2 to an orthonormal basis $\{x_2, \tilde{v}_2, \dots, \tilde{v}_{n-1}\}$ and take $\hat{S}_1 = \begin{bmatrix} x_2, \tilde{v}_2, \cdots, \tilde{v}_{n-1} \end{bmatrix}$ then S_1 is also unitary and we have $\hat{S}_1^\dagger A_1 \hat{S}_1 = \begin{bmatrix} \lambda_2 & * \\ 0 & A_2 \end{bmatrix}$ where A_2 is a $(n-2) \times (n-2)$ matrix. So we take $S_1 = S_0 \begin{bmatrix} 1 & 0 \\ 0 & \hat{S}_1 \end{bmatrix}$. Then

$$S_1^\dagger A S_1 = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & A_2 \end{bmatrix}$$

We continue like this letting $S_k = S_{k-1} \begin{bmatrix} I_k & 0 \\ 0 & \hat{S}_k \end{bmatrix}$ thus at the end we obtain $U := S_n$ such that $U^\dagger A U = T$ which is an upper triangular matrix. Hence we have $A = UTU^\dagger$

Lemma 3: A matrix A is diagonalizable with a unitary matrix if and only if A is normal

Proof: Let A is normal. Then by Lemma 2 there is a unitary matrix U and a upper traingular matrix T such that $A = UTU^\dagger$. Then

$$\begin{aligned} TT^\dagger = U^\dagger AU(U^\dagger AU)^\dagger = U^\dagger AUU^\dagger A^\dagger U = U^\dagger AA^\dagger U \\ = U^\dagger A^\dagger AU = U^\dagger A^\dagger UU^\dagger AU = (U^\dagger AU)^\dagger U^\dagger AU = T^\dagger T \end{aligned}$$

Now let $T + (t_{i,j})_{1 \le i,j \le n}$. Then the first diagonal entry of $TT^\dagger$ is

$$\sum_{i=1}^{n} t_{1,i} \overline{t_{1,i}} = \sum_{i=1}^{n} |t_{1,i}|^2$$

Now the first diagonal entry of $T^\dagger T$ is $t_{1,1}\overline{t_{1,1}} = |t_{1,1}|^2$. These two are equal. Hence for all $2 \le i \le n$ we have $t_{1,i} = 0$. Similarly comparing the second diagonal entry of $TT^\dagger$ and $T^\dagger T$ we have that all the nondiagonal entries of second row of T is 0. Continuing like this we have that T is diagonal.

Now suppose that A is any matrix such that there exists an unitary matrix U such that $U^\dagger AU = D$ where D is diagonal. Then

$$\begin{aligned} AA^\dagger = UDU^\dagger(UDU^\dagger)^\dagger = UDU^\dagger UD^\dagger U^\dagger = UDD^\dagger U^\dagger \\ = UD^\dagger DU^\dagger = UD^\dagger U^\dagger UDU^\dagger = (UDU^\dagger)^\dagger UDU^\dagger = A^\dagger A \end{aligned}$$

So A is normal.

Now coming back to the original question we have that the eigenvalues of A are real. A is normal. Then there exists an unitary matrix U such that $U^\dagger AU = D$ where D is diagonal. Since all eigenvalues of A are real $D^\dagger = D$. Then we have

$$A^\dagger = (U^\dagger DU)^\dagger = U^\dagger D^\dagger U = U^\dagger DU = A$$

So A is hermitian

Now suppose A is positive operator. Then for all $v \in V$ we have

$$v^\dagger Av \ge 0 \implies v^\dagger Av = (v^\dagger Av)^\dagger = v^\dagger A^\dagger v \ge 0 \implies v^\dagger(A - A^\dagger)v = 0$$

Now also we have

$$(A - A^\dagger)(A - A^\dagger)^\dagger = (A - A^\dagger)(A^\dagger - A) = AA^\dagger - A^\dagger A^\dagger - AA + A^\dagger A$$
$$= (A^\dagger - A)(A - A^\dagger) = (A - A^\dagger)^\dagger (A - A^\dagger)$$

So $A - A^\dagger$ is a normal operator. Hence by Lemma 3 there exists an unitary matrix U such that $U^\dagger(A - A^\dagger)U = D$ where D is a diagonal matrix. Now for standard basis for any e_i

$$e_i^\dagger De_i = e^\dagger U^\dagger (A - A^\dagger)Ue_i = (Ue_i)^\dagger (A - A^\dagger)(Ue_i) = 0$$

Now $e_i^\dagger De_i$ is the i-th diagonal element of D which we got is 0. Since this is true for all $i \in [n]$ we have D is a null matrix. So

$$U^\dagger (A - A^\dagger)U = 0 \iff A - A^\dagger = U0U^\dagger = 0 \iff A = A^\dagger$$

Hence A is hermitian.

□

Problem 3

Suppose that A and B are Hermitian operators. Then show that the commutator $[A, B] = 0$ if and only if there exists an orthonormal basis such that both A and B are diagonal with respect to that basis.

Solution: If there exists an orthonormal basis such that both A and B are diagonal with respect to that basis then let we have $P^\dagger AP = D_A$ and $P^\dagger P - D_B$. Then

$$AB - BA = PD_A P^\dagger PD_B P^\dagger - PD_B P^\dagger PD_A P^\dagger = PD_A D_B P^\dagger - PD_B D_A P^\dagger = P(D_A D_B - S_B D_A)P^\dagger = 0$$

The last equality comes because D_A and D_B are diagonal matrices so $D_A D_B = D_B D_A$.

For the opposite direction suppose v be an eigenvector with corresponding eigenvector λ of A then $Av = \lambda v$. Now

$$A(Bv) = BAv = B\lambda v = \lambda Bv$$

Hence for any eigenvector v of A Bv is also an eigenvector and if Bv is zero then still it is an eigenvector of A for same eigenvalue.

Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of A. Then the corresponding eigenspaces of A are V_{λ_i} for $i \in [k]$. Then we have $B(V_{\lambda_i}) \subseteq V_{\lambda_i}$ for all $i \in [k]$. Now let β be an eigenvalue of B with corresponding eigenvector is y. Then for any $i \in [k]$ we can think $y = y_1 + y_2$ where $y_1 \in V_{\lambda_i}$ and and $y_2 \in \bigoplus_{j\neq i} V_{\lambda_j}$. Then $By = \beta y = \beta y_1 + \beta y_2$. also we have $By = By_2 + By_2$. Since $B(V_{\lambda_i}) \subseteq V_{\lambda_i}$ and $B\left(\bigoplus_{j\neq i} V_{\lambda_j}\right) \subseteq \bigoplus_{j\neq i} V_{\lambda_j}$ we can say $By_1 = \beta y_1$ and $By_2 = \beta y_2$. Now if the V_β is the corresponding eigenspace fo the eigenvalue β then

$$V_\beta = [V_\beta \cap V_{\lambda_i}] \oplus \left[V_\beta \cap \bigoplus_{j\neq i} V_{\lambda_j}\right] = \bigoplus_{i=1}^{k} V_{\lambda_i} \cap V_\beta$$

Now if $\beta_1, \ldots, \beta_l$ are the eigenvalues of B then we have

$$\bigoplus_{i=1}^{l} V_{\beta_i} = \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{k} V_{\lambda_j} \cap V_{\beta_i}\right) = \bigoplus_{\substack{1\leq i\leq l \\ 1\leq j\leq k}} V_{\beta_i} \cap V_{\lambda_j}$$

Let us denote $V_{i,j} = V_{\beta_i} \cap V_{\lambda_j}$ then for each $V_{i,j}$ we take an orthogonal basis for all i, j. Then taking union of all of them we have an orthogoanl basis for both A and B such that both A and B are diagonal. Now for each vector in the basis after normalizing we get an orthonormal basis such that both A and B are diagonal with respect to that basis.

□

Problem 4

Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that $|\psi\rangle$ is a product state if and only if the reduced density matrices ρ_A and ρ_B are pure states.

Solution:

- Let the $|\psi\rangle$ is a product state. Then $\exists\ |\psi_1\rangle \in A, |\psi_2\rangle \in B$ such that $|\psi\rangle = |\psi_1\rangle |\psi_2\rangle$. Now by Schmidt Decomposition there exists an orthonormal basis $\{|i_A\rangle\}$ for system A and orthonormal basis $\{|i_B\rangle\}$ for system B such that
$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |i_A\rangle |i_B\rangle$$
where $\lambda_i \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i^2 = 1$. We have there exists at least one $\lambda_i \neq 0$. WLOG $\lambda_1 \neq 0$ Now we also have
$$|\psi_1\rangle = \sum_{i=1}^{n} \lambda_{i,A} |i_A\rangle \qquad |\psi_2\rangle = \sum_{i=1}^{n} \lambda_{i,B} |i_B\rangle$$
then we have
$$\sum_{i=1}^{n} \lambda_i |i_A\rangle |i_B\rangle = |\psi\rangle = \left(\sum_{i=1}^{n} \lambda_{i,A} |i_A\rangle\right)\left(\sum_{i=1}^{n} \lambda_{i,B} |i_B\rangle\right) = \sum_{1\leq i,j\leq n} \lambda_{i,A}\lambda_{j,B} |i_A\rangle |j_B\rangle$$
Comparing the coefficients we have $\lambda_i = \lambda_{i,A}\lambda_{i,B}$ and for all $\lambda_{i,A}\lambda_{j,B} = 0$ where $i \neq j$. Since $\lambda_1 \neq 0$ we have $\lambda_{1,A}, \lambda_{1,B} \neq 0$. Since for all $j \neq 1$, $\lambda_{1,A}\lambda_{j,B} = 0$ we have $\lambda_{j,B} = 0$ for all $2 \leq j \leq n$. Similarly since for all $i \neq 1$, $\lambda_{i,A}\lambda_{1,B} = 0$ we have $\lambda_{i,A} = 0$ for all $2 \leq i \leq n$. So we have $\lambda_i = 0$ for all $2 \leq i \leq n$. So $|\psi\rangle = \lambda_1 |i_A\rangle |i_B\rangle$. Hence $|\psi\rangle$ has Schmidt Number 1.

  For the opposite direction $|\psi\rangle$ has Schmidt Number 1. So $|\psi\rangle = |i_A\rangle |i_B\rangle$ Here are $|i_A\rangle$ is a state of system A and $|i_B\rangle$ is a state of system B. Hence $|\psi\rangle$ is already in a product state. Hence $|\psi\rangle$ is a product state of the composite system AB.

- $|\psi\rangle$ is a product state. Hence it has Schmidt Number 1. So there exists an orthonormal basis $\{|i_A\rangle\}$ for system A and orthonormal basis $\{|i_B\rangle\}$ for system B such that $|\psi\rangle = |i_A\rangle |i_B\rangle$. Then
$$\rho_{AB} = |\psi\rangle \langle\psi| = (|i_A\rangle |i_B\rangle)(\langle i_A| \langle i_B|) = |i_A\rangle \langle i_A| \otimes |i_B\rangle \langle i_B|$$
Now
$$\rho_A = tr_B(\rho_{AB}) = tr_B(|i_A\rangle \langle i_A| \otimes |i_B\rangle \langle i_B|) = |i_A\rangle \langle i_A| \, tr(|i_B\rangle \langle i_B|) = |i_A\rangle \langle i_A|$$
and similarly
$$\rho_B = tr_A(\rho_{AB}) = tr_A(|i_A\rangle \langle i_A| \otimes |i_B\rangle \langle i_B|) = tr(|i_A\rangle \langle i_A|)\, |i_A\rangle \langle i_B| = |i_B\rangle \langle i_B|$$
So ρ_A and ρ_B are pure states.

  Let ρ_A and ρ_B are pure states. Let $|\psi\rangle = |\psi_1\rangle |\psi_2\rangle$ Then
$$|\psi\rangle \langle\psi| = \left(\sum_{i=1}^{n} \lambda_i |i_A\rangle |i_B\rangle\right)\left(\sum_{j=1}^{n} \lambda_j \langle j_A| \langle j_B|\right) = \sum_{i=1}^{n} \lambda_i^2 |i_A\rangle \langle i_A| \otimes |i_B\rangle \langle i_B|$$
There exists at least one $\lambda_i \neq 0$. WLOG $\lambda_1 =\neq 0$. Now
$$\rho_A = \mathrm{tr}_B\left(\sum_{i=1}^{n} \lambda_i^2 |i_A\rangle \langle i_A| \otimes |i_B\rangle \langle i_B|\right) = \sum_{i=1}^{n} \lambda_i^2 |i_A\rangle \langle i_A| \, tr(|i_B\rangle \langle i_B|) = \sum_{i=1}^{n} \lambda_i^2 |i_A\rangle \langle i_A|$$
and
$$\rho_B = \mathrm{tr}_A\left(\sum_{i=1}^{n} \lambda_i^2 |i_A\rangle \langle i_A| \otimes |i_B\rangle \langle i_B|\right) = \sum_{i=1}^{n} \lambda_i^2 tr(|i_A\rangle \langle i_A|)\, |i_B\rangle \langle i_B| = \sum_{i=1}^{n} \lambda_i^2 |i_B\rangle \langle i_B|$$

Since ρ_A and ρ_B are pure states there exists $k, l \in [n]$ such that $\rho_A = \lambda_k |k_A\rangle \langle k_A|$ and $\rho_B = \lambda_l |l_B\rangle \langle l_B|$ since we already know that $\lambda_1 \neq 0$ we have $k = l = 1$ for all $2 \leq i \leq n$ $\lambda_i = 0$. So $\rho_A = |1_A\rangle \langle 1_A|$ and $\rho_B = |1_A\rangle \langle 1_B|$. Hence $|\psi\rangle = \lambda_1 |1_A\rangle |1_B\rangle$. So $|\psi\rangle$ has Schmidt Number 1. So $|\psi\rangle$ is a product state of the composite system AB.

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Problem 5

Write a self-contained proof that single qubit gates and $CNOT$ gates are universal.

Solution:

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Problem 6

Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices.

Solution:

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