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Course: Parallel Algorithms and Complexity

Assignment - 9
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Problem 1

For a real $n \times n$ matrix M define $\|M\| = \sup\limits_{0 \neq x \in \mathbb{R}^n} \frac{\|xM\|}{\|x\|}$. Then

1. $\|M + N\| \leq \|M\| + \|N\|$
2. $\|MN\| \leq \|M\|\|N\|$
3. $\|M\| = \max\limits_i \left(\sum\limits_j |M[i,j]| \right)$
4. If M is a transition probability matrix i.e. all entries are non negative and the sum of entries in each row is 1, then $\|M\| = 1$
5. If all entries of M are bounded in absolute value by ϵ then $\|M\| \leq n\epsilon$

Solution:

1. For any $x \in \mathbb{R}^n$ we have $\|x(M+N)\| = \|xM + xN\| \leq \|xM\| + \|xN\|$. Hence after taking supremum over all nonzero $x \in \mathbb{R}^n$ we have $\|M+N\| \leq \|M\| + \|N\|$.

2. We have $\|M\| = \sup\limits_{0 \neq x \in \mathbb{R}^n} \frac{\|xM\|}{\|x\|}$. Hence we have for any $x \in \mathbb{R}^n$ $\|xM\| \leq \|M\||x|$. So now

$$\|MN\| = \sup_{0 \neq x \in \mathbb{R}^n} \frac{\|x(MN)\|}{\|x\|} = \sup_{0 \neq x \in \mathbb{R}^n} \frac{\|(xM)N\|}{\|x\|} \leq \sup_{0 \neq x \in \mathbb{R}^n} \frac{\|xM\|\|N\|}{\|x\|} = \left(\sup_{0 \neq x \in \mathbb{R}^n} \frac{\|xM\|}{\|x\|} \right) \|N\| = \|M\|\|N\|$$

3. Let's denote the rows of M by $M_1, \ldots, M_n$. Then for every $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
\|xM\| &= \left\| \sum_{j=1}^{n} x_j \cdot M_j \right\| \\
&\leqslant \sum_{j=1}^{n} \|x_j \cdot M_j\| \\
&= \sum_{j=1}^{n} |x_j| \cdot \|M_j\| \\
&\leqslant \max\left\{ \|M_j\| : 1 \leqslant j \leqslant n \right\} \left(\sum_{j=1}^{n} |x_j| \right) \\
&= \max\left\{ \|M_j\| : 1 \leqslant j \leqslant n \right\} \cdot \|x\|
\end{aligned}$$

That shows that

$$\|M\|_1 \leqslant \max\left\{ \|M_j\| : 1 \leqslant \nu \leqslant n \right\},$$

Let k is the index for which the column sum of M has maximum. Then choosing $x = e_k$ shows the opposite inequality. Hence we have the equality

4. By the previous part $\|M\| = \max_i \left(\sum_j |M[i,j]| \right)$. Since for each row of M the row sum is 1 we have the $\|M\| = \max_i \left(\sum_j |M[i,j]| \right) = \max_i 1 = 1$.

5. By the part (3) we have $\|M\| = \max_i \left(\sum_j |M[i,j]| \right)$. Now all entries of M is bounded by ϵ in absolute value. So each rowsum is bounded by $n\epsilon$ in absolutevalue. Hence $\|M\| \leq n\epsilon$.

$\square$