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Course: Complex Analysis

Assignment - 3
Roll: BMC202175
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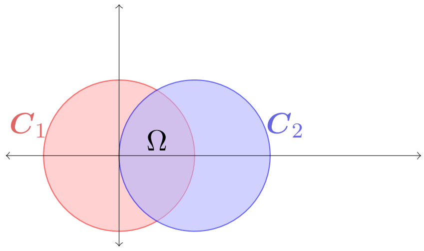
Problem 1 Ahlfors Page 96: Problem 1

Map the common part of the disks $|z| < 1$ and $|z-1| < 1$ on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

Solution: Let $C_1 : |z| = 1$ and $C_2 : |z-1| = 1$. Let the common region between them is Ω



The circles intersect when

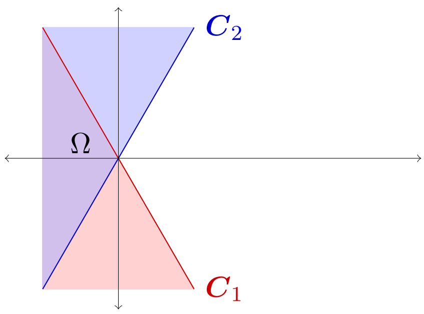
$$|z| = |z-1| \iff z\overline{z} = (z-1)(\overline{z}-1) \iff 1 = z + \overline{z}$$

Hence $\Re(z) = \frac{1}{2}$. Therefore $\Im(z) = \pm\frac{\sqrt{3}}{2}$ since $|z| = 1$. Therefore C_1 and C_2 intersects at $-\omega$ and $-\omega^2$.

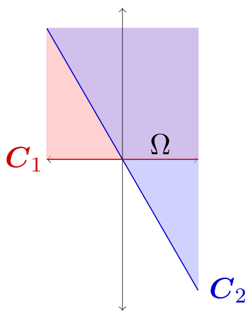
Now we send $-\omega^2 \to \infty$ and $-\omega \to 0$ by the conformal map $f_1(z) = \frac{z+\omega}{z+\omega^2}$. Then

$$f_1(1) = \frac{1+\omega}{1+\omega^2} = \frac{-\omega^2}{-\omega} = \omega \qquad f_1(0) = \frac{\omega}{\omega^2} = \frac{1}{\omega} = \bar{\omega} = \omega^2$$

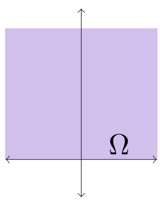
Hence $f_1(C_1) =$ line joining 0 and ω and $f_1(C_2) =$ line joining 0 and ω^2



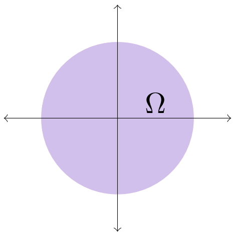
Now we rotate the region Ω by $\frac{2\pi}{3}$ clockwise by the conformal map $f_2(z) = e^{-i\frac{2\pi}{3}}z = \omega z$



Now we map the common region Ω to the upper half of the plane by the conformal map $f_3(z) = z^{\frac{3}{2}}$



Now we want to map the upper half plane to inside of the unit circle. We do it with the conformal map $f_4(z) = \frac{z-\omega}{z-\omega^2}$



Hence the final conformal map which maps the region Ω to the inside of unit disk is

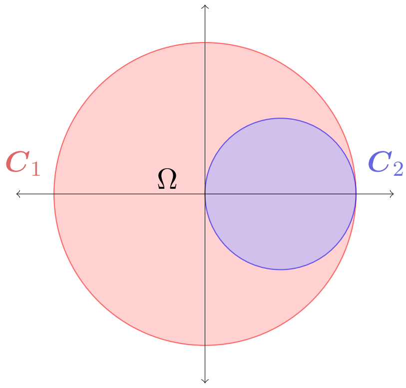
$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = f_4 \circ f_3 \circ f_2 \left(\frac{z+\omega}{z+\omega^2}\right) = f_4 \circ f_3 \left(\omega \frac{z+\omega}{z+\omega^2}\right) = f_4 \circ f_3 \left(\frac{\omega^2 z+1}{\omega z+1}\right)$$
$$= f_4\left(\left[\frac{\omega^2 z+1}{\omega z+1}\right]^{\frac{3}{2}}\right) = \frac{\left[\frac{\omega^2 z+1}{\omega z+1}\right]^{\frac{3}{2}} - \omega}{\left[\frac{\omega^2 z+1}{\omega z+1}\right]^{\frac{3}{2}} - \omega^2}$$

□

Problem 2 Ahlfors Page 96: Problem 2

Map the region between $|z| = 1$ and $\left|z - \frac{1}{2}\right| = \frac{1}{2}$ on a half plane.

Solution: Let $C_1 : |z| = 1$ and $C_2 : \left|z - \frac{1}{2}\right| = \frac{1}{2}$. Let the common region between them is Ω



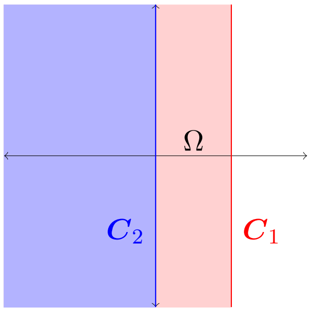
The two circles touch each other at 1.

Now we send $1 \to \infty$ and $0 \to 0$ with the conformal map $f_1(z) = \frac{z}{z-1}$. Hence

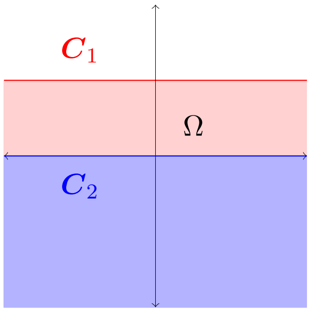
$$f_1(-1) = \frac{-1}{-1-1} = \frac{1}{2} \qquad f_1\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{2}-1} = -1$$

$$f(i) = \frac{i}{i-1} = \frac{i(-i-1)}{2} = \frac{1}{2} - \frac{i}{2} \qquad f(-i) = \frac{-i}{-i-1} = \frac{i(1-i)}{2} = +\frac{1}{2} + \frac{i}{2}$$

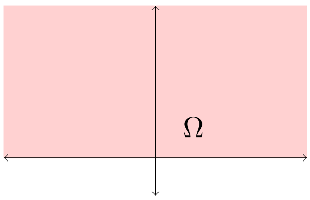
Hence $f(C_1) =$ is the line parallel to imaginary axis passing through $\frac{1}{2}$. $f(C_2) =$ imaginary axis. Hence the region Ω is mapped to the $\frac{1}{2}$ width strip parallel to imaginary axis enclosed between imaginary axis and $\Re(z) = \frac{1}{2}$.



Now we rotate the region Ω by $90°$ counter clockwise with the conformal map $f_2(z) = iz$



Since the strip width is $\frac{1}{2}$ we take the conformal map $f_3(z) = (e^z)^{\frac{\pi}{\frac{1}{2}}} = (e^z)^{2\pi} = e^{2\pi z}$ which maps the strip of width $\frac{1}{2}$ to the upper half plane.



Hence the final conformal map which maps the region between the two circles C_1 and C_2 to the upper half plane is

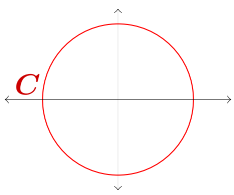
$$f_3 \circ f_2 \circ f_1(z) = f_3 \circ f_2 \left(\frac{z}{z-1}\right) = f_3 \left(i\frac{z}{z-1}\right) = e^{\frac{2\pi iz}{z-1}}$$

$\square$

Problem 3 Ahlfors Page 97: Problem 3

Map the complement of the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle so that the points at ∞ correspond to each other

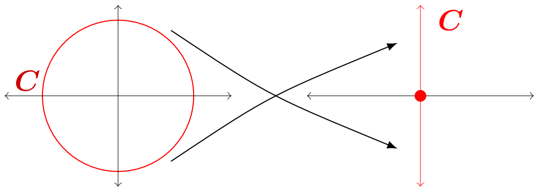
Solution: Let $C : |z| = 1$.



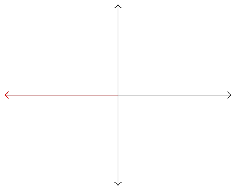
We map the semi arc containg i to the negative real axis then ultimately to the right half plane. We first map $1 \to \infty$ and $-1 \to 0$ with the conformal map $f_1(z) = \frac{z+1}{z-1}$. Then

$$f(\infty) = 1 \quad f(i) = \frac{i+1}{i-1} = \frac{(i+1)(-1-i)}{2} = -i \quad f(-i) = \frac{-i+1}{-i-1} = \frac{(-i+1)(-1+i)}{2} = i$$

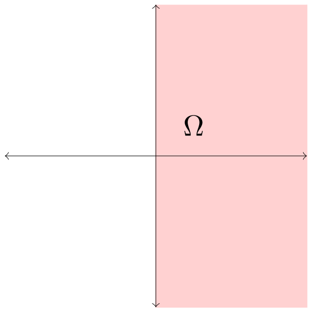
Hence $f_1(C) =$ Imaginary axis



Now we will only concentrate on the lower ray i.e. $\{z \mid \Re(z) = 0, \Im(z) \leq 0\}$. Now we want to rotate this by $90°$ clockwise to map it to the real axis by the conformal map $f_2(z) = -iz$. Hence $f_2(1) = -i$.



Now we know $f_3(z) = \sqrt{z}$ maps the $\mathbb{C}\backslash$negative real axis to right half plane.

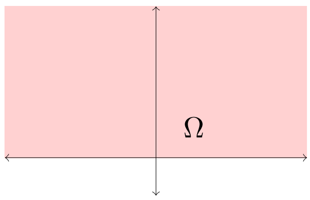


Hence $f_3(-i) = e^{-i\frac{\pi}{4}}$. Hence at this stage ∞ is mapped to $e^{-i\frac{\pi}{4}}$. Now we have to map $e^{-i\frac{\pi}{4}}$ to 1. This is achieved by conformal map of the form $f_4(z) = az + b$ where $a \in \mathbb{R}, b \in \mathbb{C}$, $b = b_1 + ib_2$. Therefore

$$f_4\left(e^{-i\frac{\pi}{4}}\right) = 1 \iff ae^{-i\frac{\pi}{4}} + b = 1 \iff \frac{a}{\sqrt{2}}(1-i) + b = \iff b_2 = \frac{a}{\sqrt{2}},\ b_1 = \frac{a}{\sqrt{2}} - 1$$

We can take $a = \sqrt{2}$, $b_1 = 0$ and $b_2 = 1$. then we have $f_4(z) = \sqrt{2}z + i$. Hence now ∞ is mapped to 1

Now we will rotate the right half plane by $\frac{\pi}{2}$ counter clockwise to map it to upper half plane by the conformal map $f_5(z) = iz$



Finally we use the map $f_6(z) = \frac{z-i}{z+i}$ which maps the upper half plane to the inside of unit disk. And also we have $f_6(i) = \infty$. Hence the final map

$$f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(z)$$

maps $\infty \to \infty$. and this map, maps the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle so that the points at ∞ correspond to each other

$\square$

Problem 4 Ahlfors Page 108: Problem 1

Compute

$$\int_\gamma x dz$$

where γ is the directed line segment from 0 to $1+i$.

Solution:

γ is the directed line segment from 0 to $1+i$. Hence $z = t(1+i)$ where $t \in [0,1]$. Then we have

$$dz = (1+i)dt$$

then

$$\begin{aligned}\int_\gamma x dz &= \int_0^1 \Re(t(1+i))(1+i)dt \\ &= \int_0^1 t(1+i)dt \\ &= (1+i)\int_0^1 t dt \\ &= (1+i)\left[\frac{t^2}{2}\right]_0^1 \\ &= \frac{1+i}{2}\end{aligned}$$

□

Problem 5 Ahlfors Page 108: Problem 2

Compute

$$\int_{|z|=r} x dz$$

for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that $x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.

Solution:

- Given that $|z| = r$. Therefore $z = re^{i\theta}$ where $\theta \in [0, 2\pi]$. Hence

$$dz = ire^{i\theta}d\theta$$

$$
\begin{aligned}
\int_{|z|=r} x dz &= \int_0^{2\pi} \Re(re^{i\theta}) \left(ire^{i\theta} d\theta\right) \\
&= \int_0^{2\pi} \Re(r\left(\cos\theta + i\sin\theta\right)) \left(ir\left(\cos\theta + i\sin\theta\right)\right) d\theta \\
&= ir^2 \int_0^{2\pi} \cos\theta \left(\cos\theta + i\sin\theta\right) d\theta \\
&= ir^2 \left[\int_0^{2\pi} \cos^2\theta d\theta + i\int_0^{2\pi} \cos\theta\sin\theta d\theta\right] \\
&= ir^2 \left[\frac{1}{2}\int_0^{2\pi} (\cos 2\theta + 1) d\theta + \frac{i}{2}\int_0^{2\pi} \sin 2\theta d\theta\right] \\
&= ir^2 \left[\frac{1}{2}\int_0^{2\pi} d\theta\right] \\
&= ir^2 \frac{1}{2}(2\pi - 0) \\
&= i\pi r^2
\end{aligned}
$$

-

$$
\begin{aligned}
\int_{|z|=r} x dz = \int_{|z|=r} \frac{1}{2}(z+\bar{z}) dz &= \int_{|z|=r} \frac{1}{2}\left(z + \frac{r^2}{z}\right) dz \\
&= \underbrace{\frac{1}{2}\int_{|z|=r} z dz}_{\substack{=0 \\ \text{As } f \text{ is analytic}}} + \frac{r^2}{2}\int_{|z|=r} \frac{1}{z} dz \\
&= \frac{r^2}{2} 2\pi i = i\pi r^2
\end{aligned}
$$

□

Problem 6 Ahlfors Page 108: Problem 3

Compute

$$
\int_{|z|=2} \frac{dz}{z^2 - 1}
$$

for the positive sense of the circle.

Solution: Given that $|z| = 2$.

$$
\begin{aligned}
\int_{|z|=2} \frac{dz}{z^2 - 1} &= \frac{1}{2}\int_{|z|=2} \frac{(z+1) - (z-1)}{z^2 - 1} dz \\
&= \frac{1}{2}\int_{|z|=2} \left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz \\
&= \frac{1}{2}\left[\int_{|z|=2} \frac{dz}{z-1} - \int_{|z|=2} \frac{dz}{z+1}\right] \\
&= \frac{1}{2}[2\pi i - 2\pi i] = 0
\end{aligned}
$$

□

Problem 7 Ahlfors Page 118: Problem 3

The *Jordan curve theorem* asserts that every Jordan curve in the plane determines exactly two regions. The notion of winding number leads to a quick proof of one part of the theorem, namely that the complement of a Jordan curve γ has at least two components. This will be so if there exists a point a with $n(\gamma, a) \neq 0$.

We may assume that $\Re(z) > 0$ on γ, and that there are points $z_1,\ z_2 \in \gamma$ with $\Im(z_1) < 0, \Im(z)_2 > 0$. These points may be chosen so that there are no other points of γ on the line segments from 0 to z_1 and from 0 to z_2. Let γ_1 and γ_2 be the ares of γ from z_1 to z_2 (excluding the end points).

Let σ_1 be the closed curve that consists of the line segment from 0 to z_1 followed by γ_1 and the segment from z_2 to 0 , and let σ_2 be constructed in the same way with γ_2 in the place of γ_1. Then $\sigma_1 - \sigma_2 = \gamma$ or $-\gamma$.

The positive real axis intersects both γ_1 and γ_2 (why?). Choose the notation so that the intersection x_2 farthest to the right is with γ_2 (Figure). Prove the following:

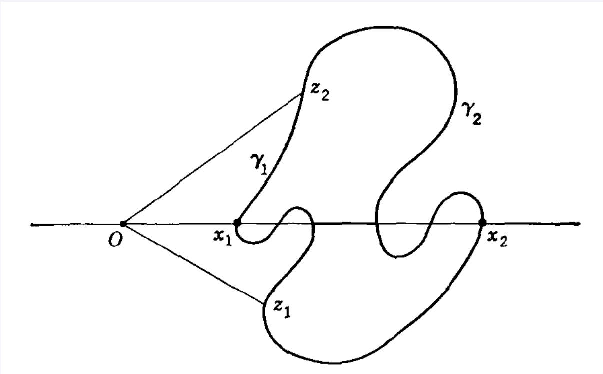
(a) $n(\sigma_1, x_2) = 0$, hence $n(\sigma_1, z) = 0$ for $z \epsilon \gamma_2$;

(b) $n(\sigma_1, x) = n(\sigma_2, x) = 1$ for small $x > 0$ (Lemma 2);

(c) the first intersection x_1 of the positive real axis with γ lies on γ_1;

(d) $n(\sigma_2, x_1) = 1$, hence $n(\sigma_2, z) = 1$ for $z \epsilon \gamma_1$;

(e) there exists a segment of the positive real axis with one end point on γ_1, the other on γ_2, and no other points on γ. The points x between the end points satisfy $n(\gamma, x) = 1$ or -1 .



Solution:

(a) We claim that x_2 lies outside the curve σ_1. Indeed, consider the ray joining 0 and x_2. This ray can not intersect γ beyond the point x_2 because x_2 is the farthest point. Thus the whole ray (x_2, ∞) is contained in one region defined by the curve, but γ must be bounded. Thus x_2 must lie in the unbounded region defined by σ_1.

Now pick $z \in \gamma_2$. Then x_2, z are path connected, hence they must lie in the same connected region of σ_1. Hence $n(\sigma_1, z) = n(\sigma_1, x_2) = 0$

(b) Let $S := \{z \mid z \in \gamma, z \in \mathbb{R}\}$. Since S is compact we can pick the smallest element of S, considered as a subset of $\mathbb{R}$, let's call it x_1. We claim that $n(\sigma_1, x) = n(\sigma_2, x) = 0$ for $x \in (0, x_1)$. By Lemma 2 , it suffices to prove that γ_1, γ_2 doesn't pass through any point to the left of x_1, which is obvious by the construction of x. Also, the path $z_1 \to 0 \to z_2$ intersects the X-axis only at 0 .

(c) Suppose the first intersection x_1 of the positive real axis with γ does not lie on γ_1; then it must lie on γ_2. In part (b) we proved that $n(\sigma_1, x) = 0$ for $x \in (0, x_1)$. But now we can consider the path from x to x_1 along the positive X-axis and then use the path along γ_2 from x_1 to x_2. Thus x and x_2 are path-connected and both are in the same region of σ_1. But then $1 = n(\sigma_1, x) = n(\sigma_1, x_2) = 0$, where the first equality is from part (a) and the last equality is from part (b).

(d) Because x_1 lies on γ_1, again by applying Lemma 2, we get $n(\sigma_2, x_1) = 1$. Again, because any $z \in \gamma_2$ is path-connected along γ_2, we must have $n(\sigma_2, z) = 1$

(e) Consider the sets $S_1 := \{z \in \mathbb{R} \mid z \in \gamma_1\}$ and $S_2 := \{z \in \mathbb{R} \mid z \in \gamma_2\}$. Since S_1 and S_2 are compact we can find two points $a \in S_1, b \in S_2$ such that there is no point of γ between a and b, otherwise we can construct a sequence of points in S_1 or S_2 such that it converges to a point in another set. contradicting the fact that S_1, S_2 are closed and disjoint. Now pick point x between a and b. Let us assume $\gamma = \sigma_1 - \sigma_2$. Then $n(\gamma, x) = n(\sigma_1, x) - n(\sigma_2, x)$. But $n(\sigma_1, x) = n(\sigma_1, b) = 0$ by part (a) and $n(\sigma_2, x) = n(\sigma_1, a) = 1$ by part (d).

$\square$

Problem 8 Ahlfors Page 120: Problem 1

Compute

$$\int_{|z|=1} \frac{e^z}{z} dz$$

Solution: $f(z) = e^z$ is analytic on $\mathbb{C}$. By Cauchy's Integral Formula we have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta d\zeta}{\zeta - z}$$

Hence

$$1 = e^0 = f(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta}{\zeta} d\zeta \iff \int_{|z|=1} \frac{e^z}{z} dz = 2\pi i$$

$\square$

Problem 9 Ahlfors Page 120: Problem 2

Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

by decomposition of the integrand in partial fractions.

Solution:

$$\begin{aligned}
\int_{|z|=2} \frac{dz}{z^2+1} &= \frac{1}{2i} \int_{|z|=2} \frac{(z+i)-(z-i)}{(z+i)(z-i)} dz \\
&= \frac{1}{2i} \int_{|z|=2} \left[\frac{1}{z-i} - \frac{1}{z+i} \right] dz \\
&= \frac{1}{2i} \left[\int_{|z|=2} \frac{1}{z-i} dz - \int_{|z|=2} \frac{1}{z+i} dz \right] \\
&= \frac{1}{2i} [2\pi i - 2\pi i] = 0
\end{aligned}$$

$\square$

Problem 10 Ahlfors Page 120: Problem 3

Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition $|a| \neq \rho$. Hint: make use of the equations $z\bar{z} = \rho^2$ and

$$|dz| = -i\rho \frac{dz}{z}.$$

Solution: We have $|dz| = -i\rho\frac{dz}{z}$. Therefore

$$\begin{aligned}\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= -i\rho \int_{|z|=\rho} \frac{dz}{z|z-a|^2} = -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\overline{z}-\overline{a})} \\ &= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)\left(z\overline{z}-\overline{a}z\right)} \\ &= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)\left(\frac{\rho^2}{z}z-\overline{a}z\right)} \\ &= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)\left(\rho^2-\overline{a}z\right)}\end{aligned}$$

Now if $\rho < |a|$, then $|z-a|^2 > 0$. Hence the function $\frac{1}{(z-a)(\rho^2-\overline{a}z)}$ is analytic and its integral along $|z| = \rho$ is 0

If $\rho > |a|$ then if $\rho^2 \neq \overline{a}z$ because if it is then

$$\rho^2 \neq \overline{a}z \iff |z| = \frac{\rho^2}{|a|} \iff \rho = \frac{\rho^2}{|a|} \iff |a| = \rho$$

which is not possible. Hence $f(z) = \frac{1}{\rho^2-\overline{a}z}$ is analytic in the ρ-disk. Hence

$$\int_{|z|=\rho} \frac{dz}{\rho^2-\overline{a}z} = 0$$

. Then by Cauchy's Integral Formula we have

$$f(a) = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(z)dz}{z-a} = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{dz}{(z-a)\left(\rho^2-\overline{a}z\right)} \iff$$

Therefore we have

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho f(a) 2\pi i = -i\rho\frac{2\pi i}{\rho^2 - a\overline{a}} = \frac{2\pi\rho}{\rho^2 - a\overline{a}}$$

□

Problem 11 Ahlfors Page 123: Problem 1

Compute

$$\int_{|z|=1} e^z z^{-n} dz, \quad \int_{|z|=2} z^n(1-z)^m dz, \quad \int_{|z|=\rho} |z-a|^{-4}|dz| (|a| \neq \rho).$$

Solution:

- Let $f(z)e^z$ Then we have

$$e^z = f^{((n-1))}(z) = \frac{(n-1)!}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)d\zeta}{(\zeta-z)^n} = \frac{(n-1)!}{2\pi i}\int_{|\zeta|=1} \frac{e^\zeta d\zeta}{(\zeta-z)^n}$$

  Therefore

$$f(0) = e^0 = 1 = \frac{(n-1)!}{2\pi i}\int_{|\zeta|=1} \frac{e^z}{z^n}dz \iff \int_{|\zeta|=1} \frac{e^z}{z^n}dz = \frac{2\pi i}{(n-1)!}$$

- We have
$$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)d\zeta}{\zeta-z}$$
4 cases possible

Case 1: $m \geq 0$, $n \geq 0$

Then $z^n(1-z)^m$ is analytic on $\mathbb{C}$. Hence
$$\int_{|z|=2} z^n(1-z)^m dz = 0$$

Case 2: $m < 0$, $n \geq 0$

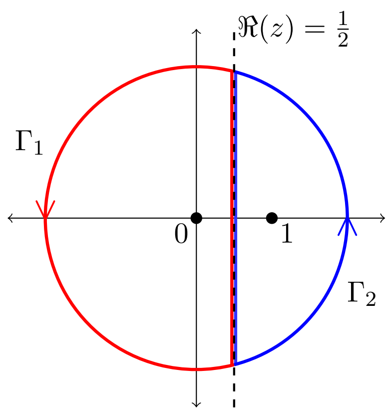
$$\begin{aligned}\int_{|z|=2} z^n(1-z)^m dz &= \int_{|z|=2}\frac{z^n}{(1-z)^{|m|}}dz = (-1)^m\int_{|z|=2}\frac{z^n}{(z-1)^{|m|}}dz\\ &= \frac{2\pi i(-1)^m}{(|m|-1)!}\frac{d^{|m|-1}}{dz^{|m|-1}}z^n\bigg|_{z=1} = \frac{2\pi i n!(-1)^m}{(|m|-1)!(n-(|m|-1))!} = 2\pi i(-1)^m\binom{n}{|m|-1}\end{aligned}$$
If $|m|-1>n$ then the above is zero

Case 3: $m \geq 0$, $n < 0$

$$\begin{aligned}\int_{|z|=2} z^n(1-z)^m dz &= \int_{|z|=2}\frac{(1-z)^m}{z^{|n|}}dz = \frac{2\pi i}{(|n|-1)!}\frac{d^{|n|-1}}{dz^{|n|-1}}(1-z)^m\bigg|_{z=0}\\ &= \frac{2\pi i m!(-1)^{|n|-1}}{(|n|-1)!(m-(|n|-1))!} = 2\pi i(-1)^{|n|-1}\binom{m}{|n|-1}\end{aligned}$$
If $|n|-1>m$, we interpret the above to be zero.

Case 4: $m < 0$, $n < 0$

$$\int_{|z|=2} z^n(1-z)^m dz = \int_{|z|=2}\frac{1}{(1-z)^{|m|}z^{|n|}}dz$$
In this case we divide the closed curve $|z|=2$ into addition of two paths Γ_1 and Γ_2 where when $\Re(z)=\frac{1}{2}$ Γ_1 goes from lower half plane to upper half plane along the line $\Re(z)=\frac{1}{2}$ and Γ_2 does the opposite after that both paths follows the circle perimeter like in the figure



Now $(1-z)^{-|m|}$ is analytic on Γ_1. Hence

$$\frac{2\pi i}{(|n|-1)!}\frac{d^{|n|-1}}{dz^{|n|-1}}\left.\frac{1}{(1-z)^{|m|}}\right|_{z=0} = \int_{\Gamma_1}\frac{1}{(1-z)^{|m|}z^{|n|}}dz = \frac{(|m|+|n|-2)!}{(m|-1)!(|n|-1)!}2\pi i$$

$z^{-|n|}$ is analytic on Γ_2. Hence

$$\frac{2\pi i}{(|m|-1)!}(-1)^{|m|}\frac{d^{|m|-1}}{dz^{|m|-1}}\left.\frac{1}{z^{|m|}}\right|_{z=1} = \int_{\Gamma_2}\frac{1}{(1-z)^{|m|}z^{|n|}}dz = -\frac{(|m|+|n|-2)!}{(m|-1)!(|n|-1)!}2\pi i$$

Hence

$$\int_{|z|=2} z^n(1-z)^m dz = \int_{|z|=2}\frac{1}{(1-z)^{|m|}z^{|n|}}dz = 0$$

- from Problem 10 we have $|dz| = -i\rho\frac{dz}{z}$. Therefore

$$\begin{aligned}
\int_{|z|=\rho}\frac{|dz|}{|z-a|^4} &= -i\rho\int_{|z|=\rho}\frac{dz}{z|z-a|^4} = -i\rho\int_{|z|=\rho}\frac{dz}{z(z-a)^2(\overline{z}-\overline{a})^2}\\
&= -i\rho\int_{|z|=\rho}\frac{zdz}{(z-a)^2\left(z\overline{z}-\overline{a}z\right)^2}\\
&= -i\rho\int_{|z|=\rho}\frac{zdz}{(z-a)^2\left(\frac{\rho^2}{z}z-\overline{a}z\right)^2}\\
&= -i\rho\int_{|z|=\rho}\frac{zdz}{(z-a)^2\left(\rho^2-\overline{a}z\right)^2}
\end{aligned}$$

Now if $\rho < |a|$, then $|z-a|^4 > 0$. Hence the function $\frac{1}{(z-a)^2(\rho^2-\overline{a}z)^2}$ is analytic and its integral along $|z|=\rho$ is 0

If $\rho > |a|$ then if $\rho^2 \neq \overline{a}z$ because if it is then

$$\rho^2 \neq \overline{a}z \iff |z| = \frac{\rho^2}{|a|} \iff \rho = \frac{\rho^2}{|a|} \iff |a| = \rho$$

which is not possible. Hence $f(z) = \frac{z}{(\rho^2-\overline{a}z)^2}$ is analytic in the ρ-disk. Then by Cauchy's Integral Formula we have

$$\begin{aligned}
\int_{|z|=\rho}\frac{zdz}{(z-a)^2\left(\rho^2-\overline{a}z\right)^2} &= \frac{d}{dz}\left.\frac{z}{(\rho^2-\overline{a}z)^2}\right|_{z=a}\frac{2\pi i}{1!}\\
&= \left(\frac{1}{\left(\rho^2-\bar{a}a\right)^2}+\frac{2\bar{a}a}{\left(\rho^2-\bar{a}a\right)^3}\right)2\pi i\\
&= 2\pi i\,\frac{\rho^2-|a|^2+2|a|^2}{(\rho^2-|a|^2)^3} = 2\pi i\,\frac{\rho^2+|a|^2}{(\rho^2-|a|^2)^3}
\end{aligned}$$

Hence

$$\int_{|z|=\rho}\frac{|dz|}{|z-a|^4} = -i\rho\int_{|z|=\rho}\frac{zdz}{(z-a)^2\left(\rho^2-\overline{a}z\right)^2} = -i\rho\left[2\pi i\,\frac{\rho^2+|a|^2}{(\rho^2-|a|^2)^3}\right] = 2\pi\rho\,\frac{\rho^2+|a|^2}{(\rho^2-|a|^2)^3}$$

□

Problem 12 Ahlfors Page 123: Problem 2

Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Solution: Given $|f(z)| < |z|^n$. To show f is a polynomial it is enough to show $\exists\ n \in \mathbb{N}$ such that $f^{(n)}(z) = 0$. We have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|s|=\rho}\frac{f(s)ds}{(s-z)^{n+1}}$$

Now if $|z| > \rho$ then

$$\begin{aligned}\left|f^{(n+1)}(z)\right| &\leq \frac{(n+1)!}{2\pi}\left|\int_{|t|=\rho}\frac{f(t)}{(t-z)^{n+2}}dt\right| \\ &\leq \frac{(n+1)!}{2\pi}\int_{|t|=\rho}\frac{|f(t)|}{|t-z|^{n+2}}|dt| \\ &\leq \frac{(n+1)!}{2\pi}\int_{|t|=\rho}\frac{|t|^n}{(|t|-|z|)^{n+2}}|dt| \\ &\leq \frac{(n+1)!}{2\pi}\frac{\rho^n}{(\rho-|z|)^{n+2}}2\pi\rho = \frac{\rho^{n+1}(n+1)!}{(\rho-|z|)^{n+2}}\end{aligned}$$

Hence as $|z| \to \infty$ and $\rho \to \infty$ we have $\frac{\rho^{n+1}(n+1)!}{(\rho-|z|)^{n+2}} \to 0$. Hence

$$\left|f^{(n+1)}(z)\right| \leq 0 \iff f^{(n+1)}(z) = 0$$

Now if $|z| \leq \rho$ then f has a maximum M in that closed $\rho-$disk. Hence

$$\left|f^{(n+1)}(z)\right| \leq \frac{M(n+1)!}{\rho^{n+1}}$$

Now as $\rho \to \infty$, $\frac{(n+1)!}{\rho^{n+1}} \to 0$. Hence

$$\left|f^{(n+1)}(z)\right| \leq 0 \iff f^{(n+1)}(z) = 0$$

2

□

Problem 13 Ahlfors Page 123: Problem 3

If $f(z)$ is analytic and $|f(z)| \leqq M$ for $|z| \leqq R$, find an upper bound for $\left|f^{(n)}(z)\right|$ in $|z| \leqq \rho < R$.

Solution: We have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|s|=\rho}\frac{f(s)ds}{(s-z)^{n+1}}$$

Let

$$\begin{aligned}\left|f^{(n)}(z)\right| &\leq \frac{n!}{2\pi}\left|\int_{|t|=R}\frac{f(t)}{(t-z)^{n+1}}dt\right| \\ &\leq \frac{n!}{2\pi}\int_{|t|=R}\frac{|f(t)|}{|t-z|^{n+1}}|dt| \\ &\leq \frac{n!}{2\pi}\int_{|t|=R}\frac{M}{(|t|-|z|)^{n+1}}|dt| \\ &\leq \frac{n!}{2\pi}\frac{M}{(R-|z|)^{n+1}}2\pi R \\ &= \frac{Mn!R}{(R-|z|)^{n+1}} \leq \frac{Mn!R}{(R-\rho)^{n+1}}\end{aligned}$$

□

Problem 14 Ahlfors Page 123: Problem 4

If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leqq 1/(1-|z|)$, find the best estimate of $\left|f^{(n)}(0)\right|$ that Cauchy's inequality will yield.

Solution: We have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|s|=\rho}\frac{f(s)ds}{(s-z)^{n+1}}$$

Claim: $\left|f^{(n)}(z)\right| \leq (n+1)!e$
Proof: Let for any $k \in \mathbb{N}$, $|z| = 1 - \frac{1}{k} = r_k$. Then

$$|f(z)| \leq \frac{1}{1-|z|} = \frac{1}{1-\frac{1}{k}} = k$$

Then we have

$$\left|f^{(n)}(0)\right| = \left|\frac{n!}{2\pi}\int_{|z|=r_k}\frac{f(z)dz}{z^{n+1}}\right| \leq \frac{n!}{2\pi}\int_{|z|=r_k}\frac{|f(z)|}{|z|^{n+1}}|dz| \leq \frac{n!}{2\pi}\frac{k}{r_k^{n+1}}2\pi r_k = \frac{kn!}{\left(1-\frac{1}{k}\right)^{n+1}} = \frac{n!k^{n+2}}{(k-1)^{n+1}}$$

Now taking $k = n+1$ we have

$$\left|f^{(n)}(0)\right| \leq \frac{n!(n+1)^{n+2}}{n^{n+1}} = (n+1)!\frac{(n+1)^{n+1}}{n^{n+1}} = (n+1)!\left(1+\frac{1}{n}\right)^{n+1} \leq (n+1)!e$$

Hence we have the best estimate of $\left|f^{(n)}(0)\right|$ which is $\left|f^{(n)}(0)\right| \leq (n+1)!e$

□

Problem 15 Ahlfors Page 123: Problem 5

Show that the successive derivatives of an analytic function at a point can never satisfy $\left|f^{(n)}(z)\right| > n!n^n$. Formulate a sharper theorem of the same kind.

Solution: We have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|s|=\rho}\frac{f(s)ds}{(s-z)^{n+1}}$$

Hence

$$\left|f^{(n)}(z)\right| = \left|\frac{n!}{2\pi i}\int_{|s|=\rho}\frac{f(s)ds}{(s-z)^{n+1}}\right| \leq \frac{n!}{2\pi}\int_{|s|=\rho}\left|\frac{f(s)ds}{(s-z)^{n+1}}\right| = \frac{n!}{2\pi}\int_{|s|=\rho}\frac{|f(s)|}{|s-z|^{n+1}}|ds|$$

Since f is continuous in the ρ-disk it is bounded by some value M. Therefore

$$\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi}\int_{|s|=\rho}\frac{|f(s)|}{|s-z|^{n+1}}|ds| \leq \frac{n!}{2\pi}\int_{|s|=\rho}\frac{M}{|s-z|^{n+1}}|ds| \leq \frac{Mn!}{2\pi}\frac{1}{|\rho-z|^{n+1}}2\pi\rho = Mn!\frac{\rho}{|\rho-z|^{n+1}}$$

Since $\rho > |z|$ we have

$$\left|f^{(n)}(z)\right| \leq Mn!\frac{\rho}{|\rho-z|^{n+1}} \leq Mn!\frac{\rho}{(|\rho|-|z|)^{n+1}} \leq Mn!\frac{\rho}{(|\rho|)^{n+1}} = \frac{Mn!}{\rho^n}$$

Using the given inequality we have

$$n!n^n < \left|f^{(n)}(z)\right| \leq \frac{Mn!}{\rho^n} \iff n!n^n < \frac{Mn!}{\rho^n} \iff (n\rho)^n < M$$

which is not possible as $n \to \infty$. Hence f doesn't satisfy $\left|f^{(n)}(z)\right| > n!n^n$

□