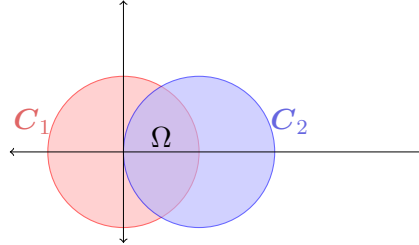


**Problem 1** Ahlfors Page 96: Problem 1

Map the common part of the disks  $|z| < 1$  and  $|z - 1| < 1$  on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

**Solution:** Let  $C_1 : |z| = 1$  and  $C_2 : |z - 1| = 1$ . Let the common region between them is  $\Omega$



The circles intersect when

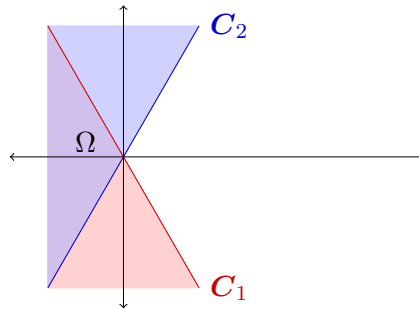
$$|z| = |z - 1| \iff z\bar{z} = (z - 1)(\bar{z} - 1) \iff 1 = z + \bar{z}$$

Hence  $\Re(z) = \frac{1}{2}$ . Therefore  $\Im(z) = \pm \frac{\sqrt{3}}{2}$  since  $|z| = 1$ . Therefore  $C_1$  and  $C_2$  intersect at  $-\omega$  and  $-\omega^2$ .

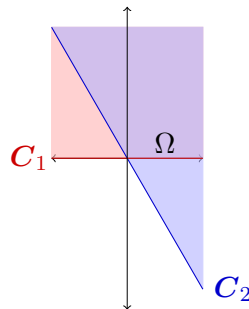
Now we send  $-\omega^2 \rightarrow \infty$  and  $-\omega \rightarrow 0$  by the conformal map  $f_1(z) = \frac{z+\omega}{z+\omega^2}$ . Then

$$f_1(1) = \frac{1+\omega}{1+\omega^2} = \frac{-\omega^2}{-\omega} = \omega \quad f_1(0) = \frac{\omega}{\omega^2} = \frac{1}{\omega} = \bar{\omega} = \omega^2$$

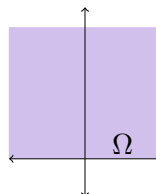
Hence  $f_1(C_1)$  = line joining 0 and  $\omega$  and  $f_1(C_2)$  = line joining 0 and  $\omega^2$



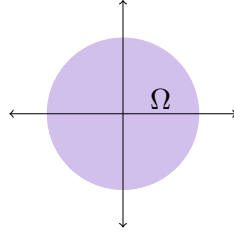
Now we rotate the region  $\Omega$  by  $\frac{2\pi}{3}$  clockwise by the conformal map  $f_2(z) = e^{-i\frac{2\pi}{3}} z = \omega z$



Now we map the common region  $\Omega$  to the upper half of the plane by the conformal map  $f_3(z) = z^{\frac{3}{2}}$



Now we want to map the upper half plane to inside of the unit circle. We do it with the conformal map  $f_4(z) = \frac{z-\omega}{z-\omega^2}$



Hence the final conformal map which maps the region  $\Omega$  to the inside of unit disk is

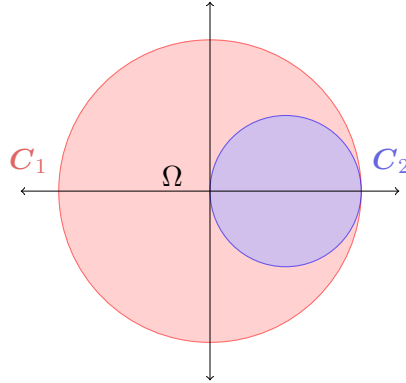
$$\begin{aligned} f_4 \circ f_3 \circ f_2 \circ f_1(z) &= f_4 \circ f_3 \circ f_2 \left( \frac{z+\omega}{z+\omega^2} \right) = f_4 \circ f_3 \left( \omega \frac{z+\omega}{z+\omega^2} \right) = f_4 \circ f_3 \left( \frac{\omega^2 z + 1}{\omega z + 1} \right) \\ &= f_4 \left( \left[ \frac{\omega^2 z + 1}{\omega z + 1} \right]^{\frac{3}{2}} \right) = \frac{\left[ \frac{\omega^2 z + 1}{\omega z + 1} \right]^{\frac{3}{2}} - \omega}{\left[ \frac{\omega^2 z + 1}{\omega z + 1} \right]^{\frac{3}{2}} - \omega^2} \end{aligned}$$

□

### Problem 2 Ahlfors Page 96: Problem 2

Map the region between  $|z| = 1$  and  $|z - \frac{1}{2}| = \frac{1}{2}$  on a half plane.

**Solution:** Let  $C_1 : |z| = 1$  and  $C_2 : |z - \frac{1}{2}| = \frac{1}{2}$ . Let the common region between them is  $\Omega$

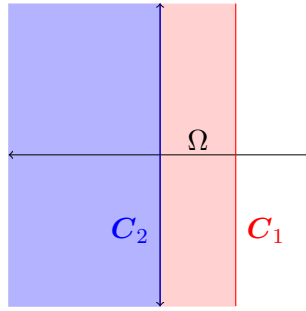


The two circles touch each other at 1.

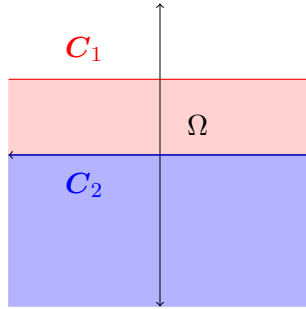
Now we send  $1 \rightarrow \infty$  and  $0 \rightarrow 0$  with the conformal map  $f_1(z) = \frac{z}{z-1}$ . Hence

$$\begin{aligned} f_1(-1) &= \frac{-1}{-1-1} = \frac{1}{2} & f_1\left(\frac{1}{2}\right) &= \frac{\frac{1}{2}}{\frac{1}{2}-1} = -1 \\ f_1(i) &= \frac{i}{i-1} = \frac{i(-i-1)}{2} = \frac{1}{2} - \frac{i}{2} & f_1(-i) &= \frac{-i}{-i-1} = \frac{i(1-i)}{2} = +\frac{1}{2} + \frac{i}{2} \end{aligned}$$

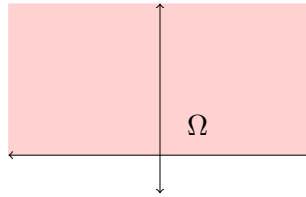
Hence  $f(C_1)$  is the line parallel to imaginary axis passing through  $\frac{1}{2}$ .  $f(C_2)$  is imaginary axis. Hence the region  $\Omega$  is mapped to the  $\frac{1}{2}$  width strip parallel to imaginary axis enclosed between imaginary axis and  $\Re(z) = \frac{1}{2}$ .



Now we rotate the region  $\Omega$  by  $90^\circ$  counter clockwise with the conformal map  $f_2(z) = iz$



Since the strip width is  $\frac{1}{2}$  we take the conformal map  $f_3(z) = (e^z)^{\frac{\pi}{1/2}} = (e^z)^{2\pi} = e^{2\pi z}$  which maps the strip of width  $\frac{1}{2}$  to the upper half plane.



Hence the final conformal map which maps the region between the two circles  $C_1$  and  $C_2$  to the upper half plane is

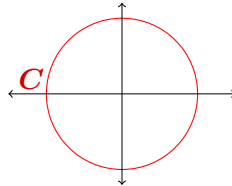
$$f_3 \circ f_2 \circ f_1(z) = f_3 \circ f_2 \left( \frac{z}{z-1} \right) = f_3 \left( i \frac{z}{z-1} \right) = e^{\frac{2\pi iz}{z-1}}$$

□

### Problem 3 Ahlfors Page 97: Problem 3

Map the complement of the arc  $|z| = 1, y \geq 0$  on the outside of the unit circle so that the points at  $\infty$  correspond to each other

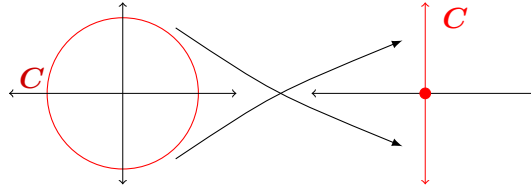
**Solution:** Let  $C : |z| = 1$ .



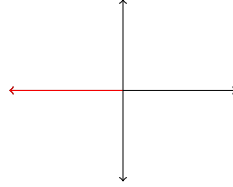
We map the semi arc containing  $i$  to the negative real axis then ultimately to the right half plane. We first map  $1 \rightarrow \infty$  and  $-1 \rightarrow 0$  with the conformal map  $f_1(z) = \frac{z+1}{z-1}$ . Then

$$f(\infty) = 1 \quad f(i) = \frac{i+1}{i-1} = \frac{(i+1)(-1-i)}{2} = -i \quad f(-i) = \frac{-i+1}{-i-1} = \frac{(-i+1)(-1+i)}{2} = i$$

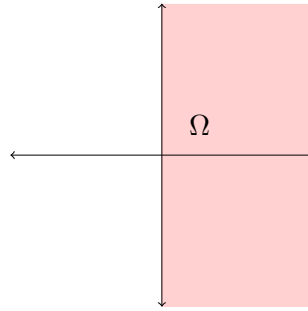
Hence  $f_1(C) = \text{Imaginary axis}$



Now we will only concentrate on the lower ray i.e.  $\{z \mid \Re(z) = 0, \Im(z) \leq 0\}$ . Now we want to rotate this by  $90^\circ$  clockwise to map it to the real axis by the conformal map  $f_2(z) = -iz$ . Hence  $f_2(1) = -i$ .



Now we know  $f_3(z) = \sqrt{z}$  maps the  $\mathbb{C} \setminus \text{negative real axis}$  to right half plane.

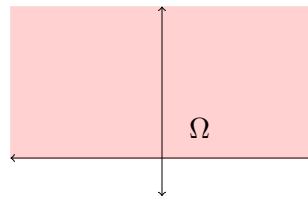


Hence  $f_3(-i) = e^{-i\frac{\pi}{4}}$ . Hence at this stage  $\infty$  is mapped to  $e^{-i\frac{\pi}{4}}$ . Now we have to map  $e^{-i\frac{\pi}{4}}$  to 1. This is achieved by conformal map of the form  $f_4(z) = az + b$  where  $a \in \mathbb{R}, b \in \mathbb{C}, b = b_1 + ib_2$ . Therefore

$$f_4\left(e^{-i\frac{\pi}{4}}\right) = 1 \iff ae^{-i\frac{\pi}{4}} + b = 1 \iff \frac{a}{\sqrt{2}}(1 - i) + b = 1 \iff b_2 = \frac{a}{\sqrt{2}}, b_1 = \frac{a}{\sqrt{2}} - 1$$

We can take  $a = \sqrt{2}$ ,  $b_1 = 0$  and  $b_2 = 1$ . then we have  $f_4(z) = \sqrt{2}z + i$ . Hence now  $\infty$  is mapped to 1

Now we will rotate the right half plane by  $\frac{\pi}{2}$  counter clockwise to map it to upper half plane by the conformal map  $f_5(z) = iz$



Finally we use the map  $f_6(z) = \frac{z-i}{z+i}$  which maps the upper half plane to the inside of unit disk. And also we have  $f_6(i) = \infty$ . Hence the final map

$$f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(z)$$

maps  $\infty \rightarrow \infty$ . and this map, maps the arc  $|z| = 1, y \geq 0$  on the outside of the unit circle so that the points at  $\infty$  correspond to each other

□

**Problem 4** Ahlfors Page 108: Problem 1

Compute

$$\int_{\gamma} x dz$$

where  $\gamma$  is the directed line segment from 0 to  $1 + i$ .

**Solution:**

$\gamma$  is the directed line segment from 0 to  $1 + i$ . Hence  $z = t(1 + i)$  where  $t \in [0, 1]$ . Then we have

$$dz = (1 + i)dt$$

then

$$\begin{aligned} \int_{\gamma} x dz &= \int_0^1 \Re(t(1 + i))(1 + i) dt \\ &= \int_0^1 t(1 + i) dt \\ &= (1 + i) \int_0^1 t dt \\ &= (1 + i) \left[ \frac{t^2}{2} \right]_0^1 \\ &= \frac{1 + i}{2} \end{aligned}$$

□

**Problem 5** Ahlfors Page 108: Problem 2

Compute

$$\int_{|z|=r} x dz$$

for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that  $x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2} \left( z + \frac{r^2}{z} \right)$  on the circle.

**Solution:**

- Given that  $|z| = r$ . Therefore  $z = re^{i\theta}$  where  $\theta \in [0, 2\pi]$ . Hence

$$dz = ire^{i\theta} d\theta$$

$$\begin{aligned}
\int_{|z|=r} x dz &= \int_0^{2\pi} \Re(re^{i\theta}) (ire^{i\theta} d\theta) \\
&= \int_0^{2\pi} \Re(r(\cos \theta + i \sin \theta)) (ir(\cos \theta + i \sin \theta)) d\theta \\
&= ir^2 \int_0^{2\pi} \cos \theta (\cos \theta + i \sin \theta) d\theta \\
&= ir^2 \left[ \int_0^{2\pi} \cos^2 \theta d\theta + i \int_0^{2\pi} \cos \theta \sin \theta d\theta \right] \\
&= ir^2 \left[ \frac{1}{2} \int_0^{2\pi} (\cos 2\theta + 1) d\theta + \frac{i}{2} \int_0^{2\pi} \sin 2\theta d\theta \right] \\
&= ir^2 \left[ \frac{1}{2} \int_0^{2\pi} d\theta \right] \\
&= ir^2 \frac{1}{2} (2\pi - 0) \\
&= i\pi r^2
\end{aligned}$$

•

$$\begin{aligned}
\int_{|z|=r} x dz &= \int_{|z|=r} \frac{1}{2} (z + \bar{z}) dz = \int_{|z|=r} \frac{1}{2} \left( z + \frac{r^2}{z} \right) dz \\
&= \underbrace{\frac{1}{2} \int_{|z|=r} z dz}_{=0} + \frac{r^2}{2} \int_{|z|=r} \frac{1}{z} dz \\
&\quad \text{As } f \text{ is analytic} \\
&= \frac{r^2}{2} 2\pi i = i\pi r^2
\end{aligned}$$

□

### Problem 6 Ahlfors Page 108: Problem 3

Compute

$$\int_{|z|=2} \frac{dz}{z^2 - 1}$$

for the positive sense of the circle.

**Solution:** Given that  $|z| = 2$ .

$$\begin{aligned}
\int_{|z|=2} \frac{dz}{z^2 - 1} &= \frac{1}{2} \int_{|z|=2} \frac{(z+1) - (z-1)}{z^2 - 1} dz \\
&= \frac{1}{2} \int_{|z|=2} \left[ \frac{1}{z-1} - \frac{1}{z+1} \right] dz \\
&= \frac{1}{2} \left[ \int_{|z|=2} \frac{dz}{z-1} - \int_{|z|=2} \frac{dz}{z+1} \right] \\
&= \frac{1}{2} [2\pi i - 2\pi i] = 0
\end{aligned}$$

□

### Problem 7 Ahlfors Page 118: Problem 3

The *Jordan curve theorem* asserts that every Jordan curve in the plane determines exactly two

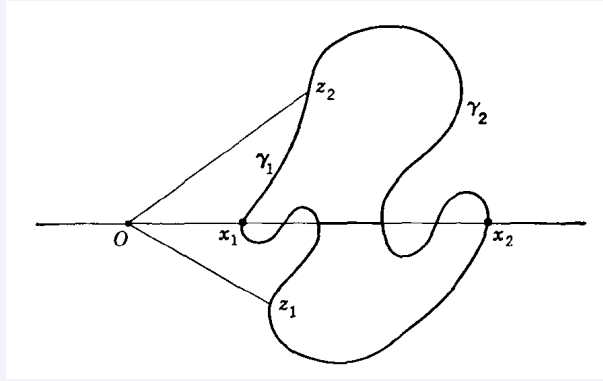
regions. The notion of winding number leads to a quick proof of one part of the theorem, namely that the complement of a Jordan curve  $\gamma$  has at least two components. This will be so if there exists a point  $a$  with  $n(\gamma, a) \neq 0$ .

We may assume that  $\Re(z) > 0$  on  $\gamma$ , and that there are points  $z_1, z_2 \in \gamma$  with  $\Im(z_1) < 0, \Im(z_2) > 0$ . These points may be chosen so that there are no other points of  $\gamma$  on the line segments from 0 to  $z_1$  and from 0 to  $z_2$ . Let  $\gamma_1$  and  $\gamma_2$  be the arcs of  $\gamma$  from  $z_1$  to  $z_2$  (excluding the end points).

Let  $\sigma_1$  be the closed curve that consists of the line segment from 0 to  $z_1$  followed by  $\gamma_1$  and the segment from  $z_2$  to 0, and let  $\sigma_2$  be constructed in the same way with  $\gamma_2$  in the place of  $\gamma_1$ . Then  $\sigma_1 - \sigma_2 = \gamma$  or  $-\gamma$ .

The positive real axis intersects both  $\gamma_1$  and  $\gamma_2$  (why?). Choose the notation so that the intersection  $x_2$  farthest to the right is with  $\gamma_2$  (Figure). Prove the following:

- (a)  $n(\sigma_1, x_2) = 0$ , hence  $n(\sigma_1, z) = 0$  for  $z \in \gamma_2$ ;
- (b)  $n(\sigma_1, x) = n(\sigma_2, x) = 1$  for small  $x > 0$  (Lemma 2);
- (c) the first intersection  $x_1$  of the positive real axis with  $\gamma$  lies on  $\gamma_1$ ;
- (d)  $n(\sigma_2, x_1) = 1$ , hence  $n(\sigma_2, z) = 1$  for  $z \in \gamma_1$ ;
- (e) there exists a segment of the positive real axis with one end point on  $\gamma_1$ , the other on  $\gamma_2$ , and no other points on  $\gamma$ . The points  $x$  between the end points satisfy  $n(\gamma, x) = 1$  or  $-1$ .



### **Solution:**

- (a) We claim that  $x_2$  lies outside the curve  $\sigma_1$ . Indeed, consider the ray joining 0 and  $x_2$ . This ray can not intersect  $\gamma$  beyond the point  $x_2$  because  $x_2$  is the farthest point. Thus the whole ray  $(x_2, \infty)$  is contained in one region defined by the curve, but  $\gamma$  must be bounded. Thus  $x_2$  must lie in the unbounded region defined by  $\sigma_1$ .

Now pick  $z \in \gamma_2$ . Then  $x_2, z$  are path connected, hence they must lie in the same connected region of  $\sigma_1$ . Hence  $n(\sigma_1, z) = n(\sigma_1, x_2) = 0$

- (b) Let  $S := \{z \mid z \in \gamma, z \in \mathbb{R}\}$ . Since  $S$  is compact we can pick the smallest element of  $S$ , considered as a subset of  $\mathbb{R}$ , let's call it  $x_1$ . We claim that  $n(\sigma_1, x) = n(\sigma_2, x) = 0$  for  $x \in (0, x_1)$ . By Lemma 2, it suffices to prove that  $\gamma_1, \gamma_2$  doesn't pass through any point to the left of  $x_1$ , which is obvious by the construction of  $x$ . Also, the path  $z_1 \rightarrow 0 \rightarrow z_2$  intersects the X-axis only at 0.
- (c) Suppose the first intersection  $x_1$  of the positive real axis with  $\gamma$  does not lie on  $\gamma_1$ ; then it must lie on  $\gamma_2$ . In part (b) we proved that  $n(\sigma_1, x) = 0$  for  $x \in (0, x_1)$ . But now we can consider the path from  $x$  to  $x_1$  along the positive X-axis and then use the path along  $\gamma_2$  from  $x_1$  to  $x_2$ . Thus  $x$  and  $x_2$  are path-connected and both are in the same region of  $\sigma_1$ . But then  $1 = n(\sigma_1, x) = n(\sigma_1, x_2) = 0$ , where the first equality is from part (a) and the last equality is from part (b).
- (d) Because  $x_1$  lies on  $\gamma_1$ , again by applying Lemma 2, we get  $n(\sigma_2, x_1) = 1$ . Again, because any  $z \in \gamma_2$  is path-connected along  $\gamma_2$ , we must have  $n(\sigma_2, z) = 1$

- (e) Consider the sets  $S_1 := \{z \in \mathbb{R} \mid z \in \gamma_1\}$  and  $S_2 := \{z \in \mathbb{R} \mid z \in \gamma_2\}$ . Since  $S_1$  and  $S_2$  are compact we can find two points  $a \in S_1, b \in S_2$  such that there is no point of  $\gamma$  between  $a$  and  $b$ , otherwise we can construct a sequence of points in  $S_1$  or  $S_2$  such that it converges to a point in another set, contradicting the fact that  $S_1, S_2$  are closed and disjoint. Now pick point  $x$  between  $a$  and  $b$ . Let us assume  $\gamma = \sigma_1 - \sigma_2$ . Then  $n(\gamma, x) = n(\sigma_1, x) - n(\sigma_2, x)$ . But  $n(\sigma_1, x) = n(\sigma_1, b) = 0$  by part (a) and  $n(\sigma_2, x) = n(\sigma_2, a) = 1$  by part (d).

□

**Problem 8** Ahlfors Page 120: Problem 1

Compute

$$\int_{|z|=1} \frac{e^z}{z} dz$$

**Solution:**  $f(z) = e^z$  is analytic on  $\mathbb{C}$ . By Cauchy's Integral Formula we have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta d\zeta}{\zeta - z}$$

Hence

$$1 = e^0 = f(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta}{\zeta} d\zeta \iff \int_{|z|=1} \frac{e^z}{z} dz = 2\pi i$$

□

**Problem 9** Ahlfors Page 120: Problem 2

Compute

$$\int_{|z|=2} \frac{dz}{z^2 + 1}$$

by decomposition of the integrand in partial fractions.

**Solution:**

$$\begin{aligned} \int_{|z|=2} \frac{dz}{z^2 + 1} &= \frac{1}{2i} \int_{|z|=2} \frac{(z+i) - (z-i)}{(z+i)(z-i)} dz \\ &= \frac{1}{2i} \int_{|z|=2} \left[ \frac{1}{z-i} - \frac{1}{z+i} \right] dz \\ &= \frac{1}{2i} \left[ \int_{|z|=2} \frac{1}{z-i} dz - \int_{|z|=2} \frac{1}{z+i} dz \right] \\ &= \frac{1}{2i} [2\pi i - 2\pi i] = 0 \end{aligned}$$

□



**Problem 10** Ahlfors Page 120: Problem 3

Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition  $|a| \neq \rho$ . Hint: make use of the equations  $z\bar{z} = \rho^2$  and

$$|dz| = -i\rho \frac{dz}{z}.$$

**Solution:** We have  $|dz| = -i\rho \frac{dz}{z}$ . Therefore

$$\begin{aligned} \int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= -i\rho \int_{|z|=\rho} \frac{dz}{z|z-a|^2} = -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\bar{z}-\bar{a})} \\ &= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(z\bar{z}-\bar{a}z)} \\ &= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)\left(\frac{\rho^2}{z}z-\bar{a}z\right)} \\ &= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2-\bar{a}z)} \end{aligned}$$

Now if  $\rho < |a|$ , then  $|z-a|^2 > 0$ . Hence the function  $\frac{1}{(z-a)(\rho^2-\bar{a}z)}$  is analytic and its integral along  $|z| = \rho$  is 0

If  $\rho > |a|$  then if  $\rho^2 \neq \bar{a}z$  because if it is then

$$\rho^2 \neq \bar{a}z \iff |z| = \frac{\rho^2}{|a|} \iff \rho = \frac{\rho^2}{|a|} \iff |a| = \rho$$

which is not possible. Hence  $f(z) = \frac{1}{\rho^2-\bar{a}z}$  is analytic in the  $\rho$ -disk. Hence

$$\int_{|z|=\rho} \frac{dz}{\rho^2-\bar{a}z} = 0$$

. Then by Cauchy's Integral Formula we have

$$f(a) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)dz}{z-a} = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2-\bar{a}z)} \iff$$

Therefore we have

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho f(a)2\pi i = -i\rho \frac{2\pi i}{\rho^2 - a\bar{a}} = \frac{2\pi\rho}{\rho^2 - a\bar{a}}$$

□

**Problem 11** Ahlfors Page 123: Problem 1

Compute

$$\int_{|z|=1} e^z z^{-n} dz, \quad \int_{|z|=2} z^n (1-z)^m dz, \quad \int_{|z|=\rho} |z-a|^{-4} |dz| (|a| \neq \rho).$$

**Solution:**

- Let  $f(z)e^z$  Then we have

$$e^z = f^{((n-1))}(z) = \frac{(n-1)!}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)d\zeta}{(\zeta-z)^n} = \frac{(n-1)!}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta d\zeta}{(\zeta-z)^n}$$

Therefore

$$f(0) = e^0 = 1 = \frac{(n-1)!}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta}{\zeta^n} dz \iff \int_{|\zeta|=1} \frac{e^\zeta}{\zeta^n} dz = \frac{2\pi i}{(n-1)!}$$

- We have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)d\zeta}{\zeta - z}$$

4 cases possible

**Case 1:**  $m \geq 0, n \geq 0$

Then  $z^n(1-z)^m$  is analytic on  $\mathbb{C}$ . Hence

$$\int_{|z|=2} z^n(1-z)^m dz = 0$$

**Case 2:**  $m < 0, n \geq 0$

$$\begin{aligned} \int_{|z|=2} z^n(1-z)^m dz &= \int_{|z|=2} \frac{z^n}{(1-z)^{|m|}} dz = (-1)^m \int_{|z|=2} \frac{z^n}{(z-1)^{|m|}} dz \\ &= \frac{2\pi i (-1)^m}{(|m|-1)!} \frac{d^{|m|-1}}{dz^{|m|-1}} z^n \Big|_{z=1} = \frac{2\pi i n! (-1)^m}{(|m|-1)!(n-(|m|-1))!} = 2\pi i (-1)^m \binom{n}{|m|-1} \end{aligned}$$

If  $|m|-1 > n$  then the above is zero

**Case 3:**  $m \geq 0, n < 0$

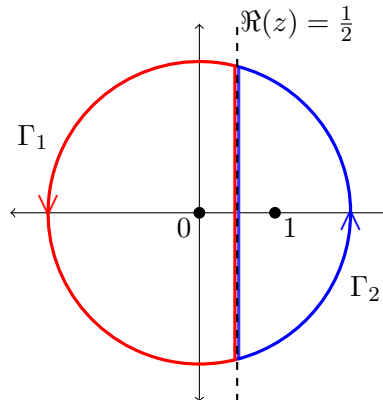
$$\begin{aligned} \int_{|z|=2} z^n(1-z)^m dz &= \int_{|z|=2} \frac{(1-z)^m}{z^{|n|}} dz = \frac{2\pi i}{(|n|-1)!} \frac{d^{|n|-1}}{dz^{|n|-1}} (1-z)^m \Big|_{z=0} \\ &= \frac{2\pi i m! (-1)^{|n|-1}}{(|n|-1)!(m-(|n|-1))!} = 2\pi i (-1)^{|n|-1} \binom{m}{|n|-1} \end{aligned}$$

If  $|n|-1 > m$ , we interpret the above to be zero.

**Case 4:**  $m < 0, n < 0$

$$\int_{|z|=2} z^n(1-z)^m dz = \int_{|z|=2} \frac{1}{(1-z)^{|m|} z^{|n|}} dz$$

In this case we divide the closed curve  $|z| = 2$  into addition of two paths  $\Gamma_1$  and  $\Gamma_2$  where when  $\Re(z) = \frac{1}{2}$   $\Gamma_1$  goes from lower half plane to upper half plane along the line  $\Re(z) = \frac{1}{2}$  and  $\Gamma_2$  does the opposite after that both paths follow the circle perimeter like in the figure



Now  $(1 - z)^{-|m|}$  is analytic on  $\Gamma_1$ . Hence

$$\frac{2\pi i}{(|n| - 1)!} \frac{d^{|n|-1}}{dz^{|n|-1}} \frac{1}{(1 - z)^{|m|}} \Big|_{z=0} = \int_{\Gamma_1} \frac{1}{(1 - z)^{|m|} |z|^{|n|}} dz = \frac{(|m| + |n| - 2)!}{(m| - 1)! (|n| - 1)!} 2\pi i$$

$z^{-|n|}$  is analytic on  $\Gamma_2$ . Hence

$$\frac{2\pi i}{(|m| - 1)!} (-1)^{|m|} \frac{d^{|m|-1}}{dz^{|m|-1}} \frac{1}{z^{|m|}} \Big|_{z=1} = \int_{\Gamma_2} \frac{1}{(1 - z)^{|m|} |z|^{|n|}} dz = -\frac{(|m| + |n| - 2)!}{(m| - 1)! (|n| - 1)!} 2\pi i$$

Hence

$$\int_{|z|=2} z^n (1 - z)^m dz = \int_{|z|=2} \frac{1}{(1 - z)^{|m|} |z|^{|n|}} dz = 0$$

- from Problem 10 we have  $|dz| = -i\rho \frac{dz}{z}$ . Therefore

$$\begin{aligned} \int_{|z|=\rho} \frac{|dz|}{|z - a|^4} &= -i\rho \int_{|z|=\rho} \frac{dz}{z|z - a|^4} = -i\rho \int_{|z|=\rho} \frac{dz}{z(z - a)^2(\bar{z} - \bar{a})^2} \\ &= -i\rho \int_{|z|=\rho} \frac{zdz}{(z - a)^2(z\bar{z} - \bar{a}z)^2} \\ &= -i\rho \int_{|z|=\rho} \frac{zdz}{(z - a)^2 \left( \frac{\rho^2}{z} z - \bar{a}z \right)^2} \\ &= -i\rho \int_{|z|=\rho} \frac{zdz}{(z - a)^2 (\rho^2 - \bar{a}z)^2} \end{aligned}$$

Now if  $\rho < |a|$ , then  $|z - a|^4 > 0$ . Hence the function  $\frac{1}{(z - a)^2(\rho^2 - \bar{a}z)^2}$  is analytic and its integral along  $|z| = \rho$  is 0

If  $\rho > |a|$  then if  $\rho^2 \neq \bar{a}z$  because if it is then

$$\rho^2 \neq \bar{a}z \iff |z| = \frac{\rho^2}{|a|} \iff \rho = \frac{\rho^2}{|a|} \iff |a| = \rho$$

which is not possible. Hence  $f(z) = \frac{z}{(\rho^2 - \bar{a}z)^2}$  is analytic in the  $\rho$ -disk. Then by Cauchy's Integral Formula we have

$$\begin{aligned} \int_{|z|=\rho} \frac{zdz}{(z - a)^2 (\rho^2 - \bar{a}z)^2} &= \frac{d}{dz} \frac{z}{(\rho^2 - \bar{a}z)^2} \Big|_{z=a} \frac{2\pi i}{1!} \\ &= \left( \frac{1}{(\rho^2 - \bar{a}a)^2} + \frac{2\bar{a}a}{(\rho^2 - \bar{a}a)^3} \right) 2\pi i \\ &= 2\pi i \frac{\rho^2 - |a|^2 + 2|a|^2}{(\rho^2 - |a|^2)^3} = 2\pi i \frac{\rho^2 + |a|^2}{(\rho^2 - |a|^2)^3} \end{aligned}$$

Hence

$$\int_{|z|=\rho} \frac{|dz|}{|z - a|^4} = -i\rho \int_{|z|=\rho} \frac{zdz}{(z - a)^2 (\rho^2 - \bar{a}z)^2} = -i\rho \left[ 2\pi i \frac{\rho^2 + |a|^2}{(\rho^2 - |a|^2)^3} \right] = 2\pi\rho \frac{\rho^2 + |a|^2}{(\rho^2 - |a|^2)^3}$$

□

**Problem 12** Ahlfors Page 123: Problem 2

Prove that a function which is analytic in the whole plane and satisfies an inequality  $|f(z)| < |z|^n$  for some  $n$  and all sufficiently large  $|z|$  reduces to a polynomial.

**Solution:** Given  $|f(z)| < |z|^n$ . To show  $f$  is a polynomial it is enough to show  $\exists n \in \mathbb{N}$  such that  $f^{(n)}(z) = 0$ . We have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|s|=\rho} \frac{f(s)ds}{(s-z)^{n+1}}$$

Now if  $|z| > \rho$  then

$$\begin{aligned} |f^{(n+1)}(z)| &\leq \frac{(n+1)!}{2\pi} \left| \int_{|t|=\rho} \frac{f(t)}{(t-z)^{n+2}} dt \right| \\ &\leq \frac{(n+1)!}{2\pi} \int_{|t|=\rho} \frac{|f(t)|}{|t-z|^{n+2}} |dt| \\ &\leq \frac{(n+1)!}{2\pi} \int_{|t|=\rho} \frac{|t|^n}{(|t|-|z|)^{n+2}} |dt| \\ &\leq \frac{(n+1)!}{2\pi} \frac{\rho^n}{(\rho-|z|)^{n+2}} 2\pi\rho = \frac{\rho^{n+1}(n+1)!}{(\rho-|z|)^{n+2}} \end{aligned}$$

Hence as  $|z| \rightarrow \infty$  and  $\rho \rightarrow \infty$  we have  $\frac{\rho^{n+1}(n+1)!}{(\rho-|z|)^{n+2}} \rightarrow 0$ . Hence

$$|f^{(n+1)}(z)| \leq 0 \iff f^{(n+1)}(z) = 0$$

Now if  $|z| \leq \rho$  then  $f$  has a maximum  $M$  in that closed  $\rho$ -disk. Hence

$$|f^{(n+1)}(z)| \leq \frac{M(n+1)!}{\rho^{n+1}}$$

Now as  $\rho \rightarrow \infty$ ,  $\frac{(n+1)!}{\rho^{n+1}} \rightarrow 0$ . Hence

$$|f^{(n+1)}(z)| \leq 0 \iff f^{(n+1)}(z) = 0$$

2

□

**Problem 13** Ahlfors Page 123: Problem 3

If  $f(z)$  is analytic and  $|f(z)| \leq M$  for  $|z| \leq R$ , find an upper bound for  $|f^{(n)}(z)|$  in  $|z| \leq \rho < R$ .

**Solution:** We have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|s|=\rho} \frac{f(s)ds}{(s-z)^{n+1}}$$

Let

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi} \left| \int_{|t|=R} \frac{f(t)}{(t-z)^{n+1}} dt \right| \\ &\leq \frac{n!}{2\pi} \int_{|t|=R} \frac{|f(t)|}{|t-z|^{n+1}} |dt| \\ &\leq \frac{n!}{2\pi} \int_{|t|=R} \frac{M}{(|t|-|z|)^{n+1}} |dt| \\ &\leq \frac{n!}{2\pi} \frac{M}{(R-|z|)^{n+1}} 2\pi R \\ &= \frac{Mn!R}{(R-|z|)^{n+1}} \leq \frac{Mn!R}{(R-\rho)^{n+1}} \end{aligned}$$

□

**Problem 14** Ahlfors Page 123: Problem 4

If  $f(z)$  is analytic for  $|z| < 1$  and  $|f(z)| \leq 1/(1-|z|)$ , find the best estimate of  $|f^{(n)}(0)|$  that Cauchy's inequality will yield.

**Solution:** We have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|s|=\rho} \frac{f(s)ds}{(s-z)^{n+1}}$$

**Claim:**  $|f^{(n)}(z)| \leq (n+1)!e$

**Proof:** Let for any  $k \in \mathbb{N}$ ,  $|z| = 1 - \frac{1}{k} = r_k$ . Then

$$|f(z)| \leq \frac{1}{1-|z|} = \frac{1}{1-\frac{1}{k}} = k$$

Then we have

$$|f^{(n)}(0)| = \left| \frac{n!}{2\pi} \int_{|z|=r_k} \frac{f(z)dz}{z^{n+1}} \right| \leq \frac{n!}{2\pi} \int_{|z|=r_k} \frac{|f(z)|}{|z|^{n+1}} |dz| \leq \frac{n!}{2\pi} \frac{k}{r_k^{n+1}} 2\pi r_k = \frac{kn!}{(1-\frac{1}{k})^{n+1}} = \frac{n!k^{n+2}}{(k-1)^{n+1}}$$

Now taking  $k = n+1$  we have

$$|f^{(n)}(0)| \leq \frac{n!(n+1)^{n+2}}{n^{n+1}} = (n+1)! \frac{(n+1)^{n+1}}{n^{n+1}} = (n+1)! \left(1 + \frac{1}{n}\right)^{n+1} \leq (n+1)!e$$

Hence we have the best estimate of  $|f^{(n)}(0)|$  which is  $|f^{(n)}(0)| \leq (n+1)!e$

□

**Problem 15** Ahlfors Page 123: Problem 5

Show that the successive derivatives of an analytic function at a point can never satisfy  $|f^{(n)}(z)| > n!n^n$ . Formulate a sharper theorem of the same kind.

**Solution:** We have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|s|=\rho} \frac{f(s)ds}{(s-z)^{n+1}}$$

Hence

$$|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_{|s|=\rho} \frac{f(s)ds}{(s-z)^{n+1}} \right| \leq \frac{n!}{2\pi} \int_{|s|=\rho} \left| \frac{f(s)ds}{(s-z)^{n+1}} \right| = \frac{n!}{2\pi} \int_{|s|=\rho} \frac{|f(s)|}{|s-z|^{n+1}} |ds|$$

Since  $f$  is continuous in the  $\rho$ -disk it is bounded by some value  $M$ . Therefore

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{|s|=\rho} \frac{|f(s)|}{|s-z|^{n+1}} |ds| \leq \frac{n!}{2\pi} \int_{|s|=\rho} \frac{M}{|s-z|^{n+1}} |ds| \leq \frac{Mn!}{2\pi} \frac{1}{|\rho-z|^{n+1}} 2\pi\rho = Mn! \frac{\rho}{|\rho-z|^{n+1}}$$

Since  $\rho > |z|$  we have

$$|f^{(n)}(z)| \leq Mn! \frac{\rho}{|\rho-z|^{n+1}} \leq Mn! \frac{\rho}{(|\rho|-|z|)^{n+1}} \leq Mn! \frac{\rho}{(|\rho|)^{n+1}} = \frac{Mn!}{\rho^n}$$

Using the given inequality we have

$$n!n^n < |f^{(n)}(z)| \leq \frac{Mn!}{\rho^n} \iff n!n^n < \frac{Mn!}{\rho^n} \iff (n\rho)^n < M$$

which is not possible as  $n \rightarrow \infty$ . Hence  $f$  doesn't satisfy  $|f^{(n)}(z)| > n!n^n$

□