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Course: Complex Analysis

Assignment - 3
Roll: BMC202175
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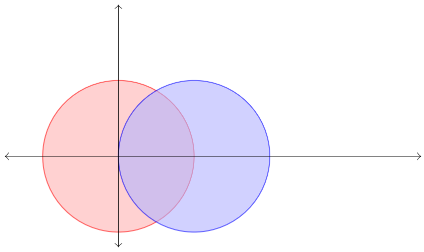
Problem 1 Ahlfors Page 96: Problem 1

Map the common part of the disks $|z| < 1$ and $|z - 1| < 1$ on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

Solution:



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Problem 2 Ahlfors Page 96: Problem 2

Map the region between $|z| = 1$ and $\left|z - \frac{1}{2}\right| = \frac{1}{2}$ on a half plane.

Solution:

□

Problem 3 Ahlfors Page 97: Problem 3

Map the complement of the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle so that the points at ∞ correspond to each other

Solution:

□

Problem 4 Ahlfors Page 108: Problem 1

Compute

$$\int_\gamma x dz$$

where γ is the directed line segment from 0 to $1 + i$.

Solution:

γ is the directed line segment from 0 to $1 + i$. Hence $z = t(1 + i)$ where $t \in [0, 1]$. Then we have

$$dz = (1 + i)dt$$

then

$$
\begin{aligned}
\int_\gamma x dz &= \int_0^1 \Re(t(1+i))(1+i)dt \\
&= \int_0^1 t(1+i)dt \\
&= (1+i)\int_0^1 t dt \\
&= (1+i)\left[\frac{t^2}{2}\right]_0^1 \\
&= \frac{1+i}{2}
\end{aligned}
$$

□

Problem 5 Ahlfors Page 108: Problem 2

Compute

$$\int_{|z|=r} x dz$$

for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that $x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.

Solution:

- Given that $|z| = r$. Therefore $z = re^{i\theta}$ where $\theta \in [0, 2\pi]$. Hence

$$dz = ire^{i\theta} d\theta$$

$$
\begin{aligned}
\int_{|z|=r} x dz &= \int_0^{2\pi} \Re(re^{i\theta})\left(ire^{i\theta} d\theta\right) \\
&= \int_0^{2\pi} \Re(r(\cos\theta + i\sin\theta))\left(ir(\cos\theta + i\sin\theta)\right) d\theta \\
&= ir^2 \int_0^{2\pi} \cos\theta\left(\cos\theta + i\sin\theta\right) d\theta \\
&= ir^2\left[\int_0^{2\pi} \cos^2\theta d\theta + i\int_0^{2\pi} \cos\theta\sin\theta d\theta\right] \\
&= ir^2\left[\frac{1}{2}\int_0^{2\pi} (\cos 2\theta + 1) d\theta + \frac{i}{2}\int_0^{2\pi} \sin 2\theta d\theta\right] \\
&= ir^2\left[\frac{1}{2}\int_0^{2\pi} d\theta\right] \\
&= ir^2 \frac{1}{2}(2\pi - 0) \\
&= i\pi r^2
\end{aligned}
$$

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$$\int_{|z|=r} x dz = \int_{|z|=r} \frac{1}{2}(z+\bar{z}) dz = \int_{|z|=r} \frac{1}{2}\left(z+\frac{r^2}{z}\right) dz$$
$$= \underbrace{\frac{1}{2}\int_{|z|=r} z dz}_{\substack{=0 \\ \text{As } f \text{ is analytic}}} + \frac{r^2}{2}\int_{|z|=r} \frac{1}{z} dz$$
$$= \frac{r^2}{2} 2\pi i = i\pi r^2$$

□

Problem 6 Ahlfors Page 108: Problem 3

Compute

$$\int_{|z|=2} \frac{dz}{z^2-1}$$

for the positive sense of the circle.

Solution: Given that $|z| = 2$.

$$\int_{|z|=2} \frac{dz}{z^2-1} = \frac{1}{2}\int_{|z|=2} \frac{(z+1)-(z-1)}{z^2-1} dz$$
$$= \frac{1}{2}\int_{|z|=2} \left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz$$
$$= \frac{1}{2}\left[\int_{|z|=2} \frac{dz}{z-1} - \int_{|z|=2} \frac{dz}{z+1}\right]$$
$$= \frac{1}{2}[2\pi i - 2\pi i] = 0$$

□

Problem 7 Ahlfors Page 118: Problem 3

The *Jordan curve theorem* asserts that every Jordan curve in the plane determines exactly two regions. The notion of winding number leads to a quick proof of one part of the theorem, namely that the complement of a Jordan curve γ has at least two components. This will be so if there exists a point a with $n(\gamma, a) \neq 0$.

We may assume that $\Re(z) > 0$ on γ, and that there are points z_1, $z_2 \in \gamma$ with $\Im(z_1) < 0, \Im(z)_2 > 0$. These points may be chosen so that there are no other points of γ on the line segments from 0 to z_1 and from 0 to z_2. Let γ_1 and γ_2 be the ares of γ from z_1 to z_2 (excluding the end points).

Let σ_1 be the closed curve that consists of the line segment from 0 to z_1 followed by γ_1 and the segment from z_2 to 0 , and let σ_2 be constructed in the same way with γ_2 in the place of γ_1. Then $\sigma_1 - \sigma_2 = \gamma$ or $-\gamma$.

The positive real axis intersects both γ_1 and γ_2 (why?). Choose the notation so that the intersection x_2 farthest to the right is with γ_2 (Figure). Prove the following:

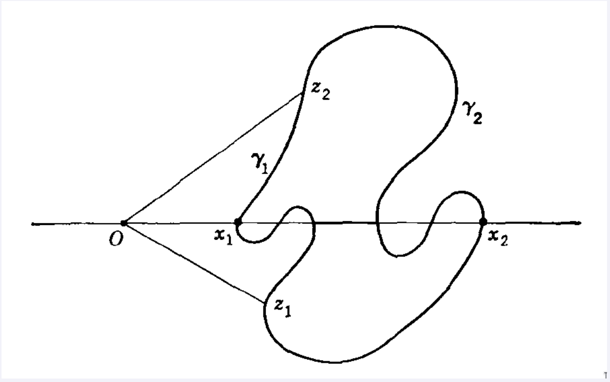
(a) $n(\sigma_1, x_2) = 0$, hence $n(\sigma_1, z) = 0$ for $z \epsilon \gamma_2$;

(b) $n(\sigma_1, x) = n(\sigma_2, x) = 1$ for small $x > 0$ (Lemma 2);

(c) the first intersection x_1 of the positive real axis with γ lies on γ_1;

(d) $n(\sigma_2, x_1) = 1$, hence $n(\sigma_2, z) = 1$ for $z \epsilon \gamma_1$;

(e) there exists a segment of the positive real axis with one end point on γ_1, the other on γ_2, and no other points on γ. The points x between the end points satisfy $n(\gamma, x) = 1$ or -1 .



Solution:

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Problem 8 Ahlfors Page 120: Problem 1

Compute

$$\int_{|z|=1} \frac{e^z}{z} dz$$

Solution: $f(z) = e^z$ is analytic on $\mathbb{C}$. By Cauchy's Integral Formula we have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta d\zeta}{\zeta - z}$$

Hence

$$1 = e^0 = f(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{e^\zeta}{\zeta} d\zeta \iff \int_{|z|=1} \frac{e^z}{z} dz = 2\pi i$$

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Problem 9 Ahlfors Page 120: Problem 2

Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

by decomposition of the integrand in partial fractions.

Solution:

$$\begin{aligned}
\int_{|z|=2} \frac{dz}{z^2+1} &= \frac{1}{2i} \int_{|z|=2} \frac{(z+i)-(z-i)}{(z+i)(z-i)} dz \\
&= \frac{1}{2i} \int_{|z|=2} \left[\frac{1}{z-i} - \frac{1}{z+i} \right] dz \\
&= \frac{1}{2i} \left[\int_{|z|=2} \frac{1}{z-i} dz - \int_{|z|=2} \frac{1}{z+i} dz \right] \\
&= \frac{1}{2i} [2\pi i - 2\pi i] = 0
\end{aligned}$$

□

Problem 10 Ahlfors Page 120: Problem 3

Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition $|a| \neq \rho$. Hint: make use of the equations $z\bar{z} = \rho^2$ and

$$|dz| = -i\rho\frac{dz}{z}.$$

Solution: We have $|dz| = -i\rho\frac{dz}{z}$. Therefore

$$\begin{aligned}
\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= -i\rho \int_{|z|=\rho} \frac{dz}{z|z-a|^2} = -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\overline{z}-\overline{a})} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(z\overline{z}-\overline{a}z)} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)\left(\frac{\rho^2}{z}z-\overline{a}z\right)} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2-\overline{a}z)}
\end{aligned}$$

Now if $\rho < |a|$, then $|z-a|^2 > 0$. Hence the function $\frac{1}{(z-a)(\rho^2-\overline{a}z)}$ is analytic and its integral along $|z| = \rho$ is 0

If $\rho > |a|$ then if $\rho^2 \neq \overline{a}z$ because if it is then

$$\rho^2 \neq \overline{a}z \iff |z| = \frac{\rho^2}{|a|} \iff \rho = \frac{\rho^2}{|a|} \iff |a| = \rho$$

which is not possible. Hence $f(z) = \frac{1}{\rho^2-\overline{a}z}$ is analytic in the ρ-disk. Hence

$$\int_{|z|=\rho} \frac{dz}{\rho^2-\overline{a}z} = 0$$

. Then by Cauchy's Integral Formula we have

$$f(a) = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(z)dz}{z-a} = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2-\overline{a}z)} \iff$$

Therefore we have

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -i\rho f(a)2\pi i = -i\rho\frac{2\pi i}{\rho^2-a\overline{a}} = \frac{2\pi\rho}{\rho^2-a\overline{a}}$$

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Problem 11 Ahlfors Page 123: Problem 1

Compute

$$\int_{|z|=1} e^z z^{-n} dz, \quad \int_{|z|=2} z^n(1-z)^m dz, \quad \int_{|z|=\rho} |z-a|^{-4}|dz|(|a| \neq \rho).$$

Solution:

- Let $f(z)e^z$ Then we have

$$e^z = f^{((n-1))}(z) = \frac{(n-1)!}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)d\zeta}{(\zeta-z)^n} = \frac{(n-1)!}{2\pi i}\int_{|\zeta|=1} \frac{e^\zeta d\zeta}{(\zeta-z)^n}$$

  Therefore

$$f(0) = e^0 = 1 = \frac{(n-1)!}{2\pi i}\int_{|\zeta|=1} \frac{e^z}{z^n}dz \iff \int_{|\zeta|=1} \frac{e^z}{z^n}dz = \frac{2\pi i}{(n-1)!}$$

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Problem 12 Ahlfors Page 123: Problem 2

Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Solution:

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Problem 13 Ahlfors Page 123: Problem 3

If $f(z)$ is analytic and $|f(z)| \leqq M$ for $|z| \leqq R$, find an upper bound for $\left|f^{(n)}(z)\right|$ in $|z| \leqq \rho < R$.

Solution:

□

Problem 14 Ahlfors Page 123: Problem 4

If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leqq 1/(1-|z|)$, find the best estimate of $\left|f^{(n)}(0)\right|$ that Cauchy's inequality will yield.

Solution: We have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|s|=\rho} \frac{f(s)ds}{(s-z)^{n+1}}$$

Claim: $\left|f^{(n)}(z)\right| \leq (n+1)!e$
Proof: Let for any $k \in \mathbb{N}$, $|z| = 1 - \frac{1}{k} = r_k$. Then

$$|f(z)| \leq \frac{1}{1-|z|} = \frac{1}{1-\frac{1}{k}} = k$$

Then we have

$$\left|f^{(n)}(0)\right| = \left|\frac{n!}{2\pi}\int_{|z|=r_k} \frac{f(z)dz}{z^{n+1}}\right| \leq \frac{n!}{2\pi}\int_{|z|=r_k} \frac{|f(z)|}{|z|^{n+1}}|dz| \leq \frac{n!}{2\pi}\frac{k}{r_k^{n+1}}2\pi r_k = \frac{kn!}{(1-\frac{1}{k})^{n+1}} = \frac{n!k^{n+2}}{(k-1)^{n+1}}$$

Now taking $k = n+1$ we have

$$\left|f^{(n)}(0)\right| \leq \frac{n!(n+1)^{n+2}}{n^{n+1}} = (n+1)!\frac{(n+1)^{n+1}}{n^{n+1}} = (n+1)!\left(1+\frac{1}{n}\right)^{n+1} \leq (n+1)!e$$

Hence we have the best estimate of $\left|f^{(n)}(0)\right|$ which is $\left|f^{(n)}(0)\right| \leq (n+1)!e$

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Problem 15 Ahlfors Page 123: Problem 5

Show that the successive derivatives of an analytic function at a point can never satisfy $\left|f^{(n)}(z)\right| > n!n^n$. Formulate a sharper theorem of the same kind.

Solution: We have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|s|=\rho}\frac{f(s)ds}{(s-z)^{n+1}}$$

Hence

$$\left|f^{(n)}(z)\right| = \left|\frac{n!}{2\pi i}\int_{|s|=\rho}\frac{f(s)ds}{(s-z)^{n+1}}\right| \leq \frac{n!}{2\pi}\int_{|s|=\rho}\left|\frac{f(s)ds}{(s-z)^{n+1}}\right| = \frac{n!}{2\pi}\int_{|s|=\rho}\frac{|f(s)|}{|s-z|^{n+1}}|ds|$$

Since f is continuous in the ρ-disk it is bounded by some value M. Therefore

$$\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi}\int_{|s|=\rho}\frac{|f(s)|}{|s-z|^{n+1}}|ds| \leq \frac{n!}{2\pi}\int_{|s|=\rho}\frac{M}{|s-z|^{n+1}}|ds| \leq \frac{Mn!}{2\pi}\frac{1}{|\rho-z|^{n+1}}2\pi\rho = Mn!\frac{\rho}{|\rho-z|^{n+1}}$$

Since $\rho > |z|$ we have

$$\left|f^{(n)}(z)\right| \leq Mn!\frac{\rho}{|\rho-z|^{n+1}} \leq Mn!\frac{\rho}{(|\rho|-|z|)^{n+1}} \leq Mn!\frac{\rho}{(|\rho|)^{n+1}} = \frac{Mn!}{\rho^n}$$

Using the given inequality we have

$$n!n^n < \left|f^{(n)}(z)\right| \leq \frac{Mn!}{\rho^n} \iff n!n^n < \frac{Mn!}{\rho^n} \iff (n\rho)^n < M$$

which is not possible as $n \to \infty$. Hence f doesn't satisfy $\left|f^{(n)}(z)\right| > n!n^n$

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