

**Problem 1** Ahlfors Page 47: Problem 1

For real  $y$  show that every remainder in the series for  $\cos y$  and  $\sin y$  has the same sign as the leading term

**Solution:** The series for both cosine and sine are

$$\begin{aligned}\cos(y) &= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \\ \sin(y) &= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\end{aligned}$$

We can write Taylor's formula as  $f(y) = T_n(y) + R_n(y)$  where

$$f(y) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} y^k + \frac{1}{k!} \int_0^y (y-t)^k f^{(k+1)}(t) dt.$$

Now, we can write cosine and sine of  $y$  as

$$\begin{aligned}\cos(y) &= \sum_{k=0}^n \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{n!} \int_0^y (y-t)^n \cos^{n+1}(t) dt \\ \sin(y) &= \sum_{k=0}^{n-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{n!} \int_0^y (y-t)^n \sin^n(t) dt\end{aligned}$$

Now we have

$$\sin^{(2m)}(t) = (-1)^m \sin t \quad \cos^{(2m+1)}(t) = (-1)^{m+1} \sin t$$

For cosine and sine, let  $n = 2m$  and  $n = 2m - 1$ , respectively. Then

$$\begin{aligned}\cos(y) &= \sum_{k=0}^m \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{(2m)!} \int_0^y (y-t)^{2m} \cos^{2m+1}(t) dt \\ &= \sum_{k=0}^m \frac{(-1)^k}{(2k)!} y^{2k} + \frac{(-1)^{m+1}}{(2m)!} \int_0^y (y-t)^{2m} \sin t dt \\ \sin(y) &= \sum_{k=0}^{m-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{(2m-1)!} \int_0^y (y-t)^{2m-1} \sin^{2m-1}(t) dt \\ &= \sum_{k=0}^{m-1} \frac{(-1)^k}{(2k+1)!} y^{2k+1} + \frac{(-1)^m}{(2m-1)!} \int_0^y (y-t)^{2m-1} \sin t dt\end{aligned}$$

So it remains to see that

$$\int_0^y (y-t)^k \sin t dt > 0$$

for all  $y > 0$  and  $k > 0$ . But that follows since  $(y-t)^k$  is a strictly decreasing positive function, so while  $2p\pi \leq y$  where  $p \in \mathbb{N}$  we have

$$\int_{2n\pi}^{2(n+1)\pi} (y-t)^k \sin t dt = \underbrace{\int_{2n\pi}^{(2n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{\sin t}_{>0} dt}_{>0} + \underbrace{\int_{(2n+1)\pi}^{2(n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{\sin t}_{<0} dt}_{<0}$$

Now

$$\begin{aligned}
\int_{2n\pi}^{(2n+1)\pi} (y-t)^k \sin t \, dt &> (y - (2n+1)\pi)^k \int_{2n\pi}^{(2n+1)\pi} \sin t \, dt \\
&= 2(y - (2n+1)\pi)^k \\
\int_{(2n+1)\pi}^{2(n+1)\pi} (y-t)^k \sin t \, dt &= - \int_{(2n+1)\pi}^{2(n+1)\pi} \underbrace{(y-t)^k}_{>0} \underbrace{(-\sin t)}_{>0} \, dt \\
&> -(y - (2n+1)\pi)^k \int_{(2n+1)\pi}^{2(n+1)\pi} (-\sin t) \, dt \\
&= -2(y - (2n+1)\pi)^k
\end{aligned}$$

Hence

$$\int_{2n\pi}^{2(n+1)\pi} (y-t)^k \sin t \, dt > 0$$

This is true for all  $n \in \{0, 1, \dots, p\}$ . Therefore

$$\int_0^{2p\pi} (y-t)^k \sin t \, dt > 0$$

Now if  $2p\pi \leq y \leq (2p+1)\pi$  then

$$\int_{2p\pi}^y (y-t)^k \sin t \, dt \geq 0$$

If  $(2p+1)\pi \leq y \leq 2(p+1)\pi$

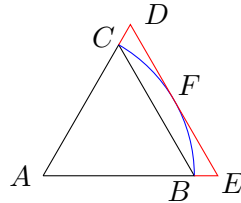
$$\begin{aligned}
\int_{2p\pi}^y (y-t)^k \sin t \, dt &= \int_{2p\pi}^{(2p+1)\pi} (y-t)^k \sin t \, dt + \int_{(2p+1)\pi}^y (y-t)^k |\sin t| \, dt \\
&= \int_{2p\pi}^{(2p+1)\pi} (y-t)^k \sin t \, dt + \int_{(2p+1)\pi}^{2(p+1)\pi} (y-t)^k |\sin t| \, dt \\
&> (y - (2p+1)\pi)^k \int_0^\pi \sin t \, dt - (y - (2p+1)\pi)^k \int_{(2p+1)\pi}^{2(p+1)\pi} \sin t \, dt \\
&> 0
\end{aligned}$$

□

### Problem 2 Ahlfors Page 47: Problem 2

Prove, for instance that  $3 < \pi < 2\sqrt{3}$

**Solution:**



The triangles  $\triangle ABC$  and  $\triangle DAE$  are equilateral triangles. Side length of  $\triangle ABC$  is 1. The radius of the circular sector  $ACFBA$  is 1. Hence the height of the triangle  $\triangle ADE$  is 1. Therefore the side length of

the  $\triangle ADE$  is  $\frac{2}{\sqrt{3}}$ . Now the arc length of  $BFC$  is greater than the side length of the triangle  $\triangle ABC$ . Therefore

$$\frac{2 \times \pi \times 1}{6} > 1 \implies \pi > 3$$

So the we have

$$\text{Area}(ABFCA) < \text{Area}(\triangle ADE)$$

Now,  $\text{Area}(ABFCA) = \frac{1}{6} \pi 1^2 = \frac{\pi}{6}$  and  $\text{Area}(\triangle ADE) = \frac{\sqrt{3}}{4} \left( \frac{2}{\sqrt{3}} \right)^2 = \frac{1}{\sqrt{3}}$ . Therefore

$$\frac{\pi}{6} < \frac{1}{\sqrt{3}} \implies \pi < 2\sqrt{3}$$

Therefore

$$3 < \pi < 2\sqrt{3}$$

□

### Problem 3 Ahlfors Page 47: Problem 4

For what values of  $z$  is  $e^z$  equal to  $2, -1, i, -i/2, -1, -i, 1 + 2i$ ?

**Solution:**

- Let for  $z = a + ib$   $e^z = 2$ . Then

$$e^{a+ib} = 2 \implies e^a e^{ib} = 2$$

Now  $|e^{ib}| = 1$ . Hence  $e^a = 2$  then  $a = \ln 2$ . Hence  $e^{ib} = 1 = \cos b + i \sin b$ . Then  $\cos b = 1$  and  $\sin b = 0$ . Then  $b = 2n\pi \forall n \in \mathbb{Z}$ . Then for  $z = \ln 2 + i2n\pi \forall n \in \mathbb{Z}$   $e^z = 2$

- Let for  $z = a + ib$   $e^z = -1$ . Then

$$e^{a+ib} = -1 \implies e^a e^{ib} = -1$$

Now  $|e^{ib}| = 1$ . Hence  $e^a = 1$  then  $a = 0$ . Hence  $e^{ib} = -1 = \cos b + i \sin b$ . Then  $\cos b = -1$  and  $\sin b = 0$ . Then  $b = (2n+1)\pi \forall n \in \mathbb{Z}$ . Then for  $z = i(2n+1)\pi \forall n \in \mathbb{Z}$   $e^z = -1$

- Let for  $z = a + ib$   $e^z = i$ . Then

$$e^{a+ib} = i \implies e^a e^{ib} = i$$

Now  $|e^{ib}| = 1$ . Hence  $e^a = 1$  then  $a = 0$ . Hence  $e^{ib} = i = \cos b + i \sin b$ . Then  $\cos b = 0$  and  $\sin b = 1$ . Then  $b = (4n+1)\frac{\pi}{2} \forall n \in \mathbb{Z}$ . Then for  $z = i(4n+1)\frac{\pi}{2} \forall n \in \mathbb{Z}$   $e^z = i$

- Let for  $z = a + ib$   $e^z = -\frac{i}{2}$ . Then

$$e^{a+ib} = -\frac{i}{2} \implies e^a e^{ib} = -\frac{i}{2}$$

Now  $|e^{ib}| = 1$ . Hence  $e^a = \frac{1}{2}$  then  $a = \ln \frac{1}{2}$ . Hence  $e^{ib} = -i = \cos b + i \sin b$ . Then  $\cos b = 0$  and  $\sin b = -1$ . Then  $b = (4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$ . Then for  $z = \ln \frac{1}{2} + i(4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$   $e^z = -\frac{i}{2}$

- Let for  $z = a + ib$   $e^z = -i$ . Then

$$e^{a+ib} = -i \implies e^a e^{ib} = -i$$

Now  $|e^{ib}| = 1$ . Hence  $e^a = 1$  then  $a = 0$ . Hence  $e^{ib} = -i = \cos b + i \sin b$ . Then  $\cos b = 0$  and  $\sin b = -1$ . Then  $b = (4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$ . Then for  $z = i(4n+3)\frac{\pi}{2} \forall n \in \mathbb{Z}$   $e^z = -i$

- Let for  $z = a + ib$   $e^z = 1 + 2i$ . Then

$$e^{a+ib} = 1 + 2i \implies e^a e^{ib} = 1 + 2i$$

Now  $|e^{ib}| = 1$ . Hence  $e^a = \sqrt{5}$  then  $a = \ln \sqrt{5}$ . Hence  $e^{ib} = \frac{1}{\sqrt{5}}(1+2i) = \cos b + i \sin b$ . Then  $\cos b = \frac{1}{\sqrt{5}}$  and  $\sin b = \frac{2}{\sqrt{5}}$ . Then  $b = 2n\pi + \sin^{-1} \frac{2}{\sqrt{5}} \forall n \in \mathbb{Z}$ . Then for  $z = \ln \sqrt{5} + i \left(2n\pi + \sin^{-1} \frac{2}{\sqrt{5}}\right) \forall n \in \mathbb{Z}$   $e^z = 1 + 2i$

□

#### Problem 4 Ahlfors Page 47: Problem 6

Determine all values of  $2^i$ ,  $i^i$ ,  $(-1)^{2i}$

**Solution:**

- $2^i = \exp(i \ln 2) = \cos \ln 2 + i \sin \ln 2$
- $i^i = \exp(i \log i) = \exp\left(i \ln \left(e^{i(4n+1)\frac{\pi}{2}}\right)\right) = \exp\left(i \left(i(4n+1)\frac{\pi}{2}\right)\right) = \exp\left(-(4n+1)\frac{\pi}{2}\right)$
- $(-1)^{2i} = i^{4i} = (i^i)^4 = \left(\exp\left(-(4n+1)\frac{\pi}{2}\right)\right)^4 = \exp(-2(4n+1)\pi)$

□

#### Problem 5 Ahlfors Page 47: Problem 7

Determine the real and imaginary parts of  $z^z$

**Solution:** Let  $z = x + iy$ . Then

$$z^z = (x + iy)^{x+iy} = \exp((x + iy) \log(x + iy))$$

Now let  $e^{a+ib} = x + iy$ . Since  $|e^{ib}| = 1$ . We have  $e^a = \sqrt{x^2 + y^2}$ . Therefore  $a = \ln \sqrt{x^2 + y^2}$ . Now

$$e^{ib} = \cos b + i \sin b = \frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}}$$

Hence  $b = 2n\pi + \tan^{-1} \frac{y}{x} \forall n \in \mathbb{Z}$ . Therefore

$$a + ib = \ln \sqrt{x^2 + y^2} + i \left(2n\pi + \tan^{-1} \frac{y}{x}\right)$$

Hence

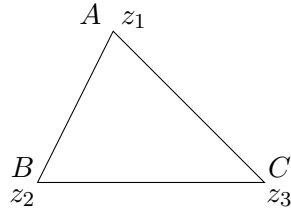
$$\begin{aligned} \exp((x + iy) \ln(x + iy)) &= \exp\left((x + iy) \ln \left(e^{\ln \sqrt{x^2 + y^2} + i(2n\pi + \tan^{-1} \frac{y}{x})}\right)\right) \\ &= \exp\left((x + iy) \left(\ln \sqrt{x^2 + y^2} + i \left(2n\pi + \tan^{-1} \frac{y}{x}\right)\right)\right) \\ &= \exp\left(x \ln \sqrt{x^2 + y^2} - y \left(2n\pi + \tan^{-1} \frac{y}{x}\right) \right. \\ &\quad \left. + i \left(y \ln \sqrt{x^2 + y^2} + x \left(2n\pi + \tan^{-1} \frac{y}{x}\right)\right)\right) \\ &= e^{x \ln \sqrt{x^2 + y^2} - y(2n\pi + \tan^{-1} \frac{y}{x})} \left[ \cos \left( \left(y \ln \sqrt{x^2 + y^2} + x \left(2n\pi + \tan^{-1} \frac{y}{x}\right)\right) \right) \right. \\ &\quad \left. + i \sin \left( \left(y \ln \sqrt{x^2 + y^2} + x \left(2n\pi + \tan^{-1} \frac{y}{x}\right)\right) \right) \right] \end{aligned}$$

□

**Problem 6** Ahlfors Page 47: Problem 9

Show how to define the “angles: in a triangle, bearing in mind they should lie between 0 and  $\pi$ . With this definition, prove that the sum of the angles is  $\pi$ .

**Solution:**



The angle between the sides  $AB$  and  $AC$  defined to be the angle between the sides in the anti clockwise direction. Hence we can define the angle

$$\angle A = \Im \log \left( \frac{z_3 - z_1}{z_2 - z_1} \right) = \Im (\log(z_3 - z_1) - \log(z_2 - z_1)) = \Im \log(z_3 - z_1) - \Im \log(z_2 - z_1)$$

Therefore for other angles we have

$$\angle B = \Im \log \left( \frac{z_1 - z_2}{z_3 - z_2} \right) = \Im \log(z_1 - z_2) - \Im \log(z_3 - z_2)$$

$$\angle C = \Im \log \left( \frac{z_2 - z_3}{z_1 - z_3} \right) = \Im \log(z_2 - z_3) - \Im \log(z_1 - z_3)$$

Therefore

$$\begin{aligned} \angle A + \angle B + \angle C &= \Im \log \left( \frac{z_3 - z_1}{z_2 - z_1} \right) + \Im \log \left( \frac{z_1 - z_2}{z_3 - z_2} \right) + \Im \log \left( \frac{z_2 - z_3}{z_1 - z_3} \right) \\ &= \Im \log \left( \frac{z_3 - z_1}{z_2 - z_1} \frac{z_1 - z_2}{z_3 - z_2} \frac{z_2 - z_3}{z_1 - z_3} \right) \\ &= \Im \log \left( \frac{z_1 - z_2}{z_2 - z_1} \frac{z_2 - z_3}{z_3 - z_2} \frac{z_3 - z_1}{z_1 - z_3} \right) \\ &= \Im \log ((-1)^3) = \Im \log(-1) = \pi \end{aligned}$$

□

**Problem 7** Ahlfors Page 72: Problem 1

Give a precise definition of a single-valued branch of  $\sqrt{a+z} + \sqrt{1-z}$  in a suitable region, and prove that it is analytic.

**Solution:** We recall that for  $\sqrt{z}$  we choose the region  $\Omega$  which is the complement of the negative real axis  $x \leq 0, y = 0$ . Hence, to define  $\sqrt{1+z}$  we should choose  $\Omega_1$  as the complement of  $x \leq -1, y = 0$  and for  $\sqrt{1-z}$  we should choose  $\Omega_2$  as the complement of  $1 \leq x, y = 0$ . In total, to define  $\sqrt{1+z} + \sqrt{1-z}$  we choose the region  $\Omega = \Omega_1 \cap \Omega_2$ . This is actually the same region used in defining  $\arccos z$ . The branch chosen is that which has positive real part. As simple transformations of  $\sqrt{z}$  in an analytic region, it follows that  $\sqrt{1+z} + \sqrt{1-z}$  is analytic.

□

**Problem 8** Ahlfors Page 72: Problem 3

Suppose that  $f(z)$  is analytic and satisfies the condition  $|f^2(z) - 1| < 1$  in a region  $\Omega$ . Show that either  $\Re f(z) > 0$  or  $\Re f(z) < 0$  throughout  $\Omega$

**Solution:** Suppose  $\Re f(z) = 0$  at a point  $z \in \Omega$ . Then  $f(z) = iy^2$  for some  $y \in \mathbb{R}$ , and thus  $f(z)^2 = -y^2$ . By the condition  $|f(z)^2 - 1| < 1$  we have  $|-y^2 - 1| < 1$  and thus  $|y^2 + 1| < 1$ . This is clearly impossible, so that  $\Re f(z) \neq 0$  throughout  $\Omega$ . But,  $\Re f(z)$  is continuous and  $\Omega$  is connected, so either  $\Re f(z) > 0$  or  $\Re f(z) < 0$ .

□

**Problem 9** Ahlfors Page 78: Problem 1

Prove that the reflection  $z \rightarrow \bar{z}$  is not a linear transformation

**Solution:** Suppose  $\varphi(z) : z \mapsto \bar{z}$  is a linear fractional transformation, and thus it must be of the form

$$\bar{z} = \varphi(z) = \frac{az + b}{cz + d}, \quad \forall z \in \mathbb{C}$$

Note that if  $\Im z = 0$  then  $\bar{z} = z$ . In particular,

$$0 \mapsto 0 \implies \varphi(0) = \frac{b}{d} = 0 \implies b = 0$$

Plugging in different values yields

$$\begin{aligned} 1 \mapsto \varphi(1) &= \frac{a}{c+d} = 1 \\ -1 \mapsto \varphi(-1) &= \frac{-a}{-c+d} = -1 \end{aligned}$$

Or,

$$\begin{aligned} c + d &= a \\ d - c &= a \end{aligned}$$

Thus,  $a = d$  and hence  $c = 0$ . But then we have

$$\bar{z} = \frac{az}{d} = z$$

Hence contradiction.  $\varphi$  is not linear transformation

□

**Problem 10** Ahlfors Page 78: Problem 2

If

$$T_1 z = \frac{z+2}{z+3} \quad T_2 z = \frac{z}{z+1}$$

Find  $T_1 T_2 z$ ,  $T_2 T_1 z$ ,  $T_1^{-1} T_2 z$

**Solution:** Here we compute several compositions:

$$\begin{aligned} T_1 T_2 z &= \frac{\frac{z}{z+1} + 2}{\frac{z}{z+1} + 3} = \frac{\frac{3z+2}{z+1}}{\frac{4z+3}{z+1}} = \frac{3z+2}{4z+3} \\ T_2 T_2 z &= \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3} + 1} = \frac{\frac{z+2}{z+3}}{\frac{2z+5}{z+3}} = \frac{z+2}{2z+5} \end{aligned}$$

Now note that

$$T_1^{-1}(w) = \frac{3w - 2}{1 - w}$$

Thus,

$$T_1^{-1}T_2z = \frac{3\frac{z}{z+1} - 2}{1 - \frac{z}{z+1}} = \frac{z - 2}{1} = z - 2$$

□

### Problem 11 Ahlfors Page 78: Problem 3

Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by reflexion in the real axis.

**Solution:** Let  $\varphi$  be a fractional linear transformation as given. Then, we have that  $\varphi(0) = 0$  and for any pair  $(z, w) \in \mathbb{C} \times \mathbb{C}$ :

$$|z - w| = |\varphi(z) - \varphi(w)|$$

since we assume that  $\varphi$  preserved distance under transformation. Since  $\varphi$  is fractional linear transformation we have

$$\varphi(z) = \frac{az + b}{cz + d}$$

where  $a, b, c, d \in \mathbb{C}$ . The assumption that  $0 \mapsto 0$  yields immediately

$$0 = \varphi(0) = \frac{b}{d}$$

and thus we have  $b = 0$ . Now, as  $\varphi(0) = 0$  we see

$$|z| = |\varphi(z) - \varphi(0)| = |\varphi(z) - 0| = \left| \frac{az}{cz + d} \right| = |a| \cdot \frac{|z|}{|cz + d|}, \quad \forall z \in \mathbb{C}$$

If  $|a| = 0$  i.e.  $a = 0$  then  $\varphi(z) = 0$ . Otherwise  $|a| > 0$ . then for all  $z \in \mathbb{C}$  we must have

$$\frac{|a|}{|cz + d|} = 1 \text{ in } \mathbb{C}$$

This can only happen if the denominator is also constant and hence we conclude that we require additionally that  $c = 0$ . We are left with  $\frac{|a|}{|d|} = 1$ , or  $|a| = |d|$ . Hence

$$\varphi(z) = \frac{az}{d}$$

where  $\left| \frac{a}{d} \right| = 1$ . Let  $k = \frac{a}{d}$ . Then

$$\varphi(z) = kz \quad \text{where } |k| = 1$$

Clearly  $\varphi$  is a rotation.

□

### Problem 12 Ahlfors Page 78: Problem 4

Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

**Solution:** Let  $\varphi(z)$  be a linear transformation with the additional restriction that  $\varphi(z) \in \mathbb{R}$  whenever  $z \in \mathbb{R}$ . Hence

$$\varphi(z) = \frac{az + b}{cz + d}$$

for  $a, b, c, d \in \mathbb{C}$ . We will now find  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$  so that  $\varphi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ . We need to distinguish separate cases:

**If  $a \neq 0$  :**

$$\varphi(z) = \frac{az + b}{cz + d} \cdot \frac{\bar{a}}{\bar{a}} = \frac{a'z + b'}{c'z + d'}, \quad a' \in \mathbb{R}$$

The important thing to observe is that we will then have  $\Im a' = 0$ , i.e  $a' \in \mathbb{R}$ . We now have a representation

$$\varphi(z) = \frac{a'z + b'}{c'z + d'}, \quad a' \in \mathbb{R}$$

In the above we see that  $z \rightarrow \infty \implies \phi(\infty) = \frac{a'}{c'}$  implying that  $c' \in \mathbb{R}$  as well as  $a'$ . The transformation has an inverse

$$\varphi^{-1}(z) = \frac{d'z - b'}{a' - c'z}$$

If  $d' = 0$  then taking  $z = 0$  we see  $\varphi(0) = b'$  and hence achieve  $b' \in \mathbb{R}$ . We now need to show that the same holds whenever  $d' \neq 0$ . In the above we see that  $z \rightarrow \infty \implies \varphi^{-1}(\infty) = -\frac{d'}{c'}$  implying that  $d' \in \mathbb{R}$  as well as  $c'$ .

$$\varphi(0) = \frac{b'}{d'} \in \mathbb{R} \implies b' \in \mathbb{R}$$

**If  $a = 0$  :**

If  $a = 0$  then we have  $\varphi(z) = \frac{b}{cz + d}$ . Here if  $b = 0$  we have  $\varphi(z)$  is non-invertible which is impossible. So

$$\varphi(z) = \frac{b}{cz + d} \cdot \frac{\bar{b}}{\bar{b}}$$

thus obtaining

$$\varphi(z) = \frac{b'}{c'z + d'}, \quad b' \in \mathbb{R}$$

Taking  $z = 0$  we see that  $\varphi(0) = \frac{b'}{d'}$ . We then have  $d' \in \mathbb{R}$ . If we take  $z = 1$  then

$$\varphi(1) = \frac{b'}{c' + d'}$$

Since  $b', d' \in \mathbb{R}$  if  $c' \notin \mathbb{R}$  then  $c' + d' \notin \mathbb{R}$  but  $c' + d' \in \mathbb{C}$  hence  $\frac{b'}{c' + d'} \notin \mathbb{R}$  but  $\frac{b'}{c' + d'} \in \mathbb{C}$ . But  $\varphi(1) \in \mathbb{R}$ . So  $c' \in \mathbb{R}$ .

Hence  $\varphi$  can be written with real coefficients

□