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Course: Complex Analysis

Assignment - 4
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Problem 1 Ahlfors Page 130: Problem 5

Prove that an isolated singularity of $f(z)$ is removable as soon as either $\Re(f(z))$ or $\Im(f(z))$ is bounded above or below. *Hint:* Apply a fractional linear transformation.

Solution:

$$\Re > c \qquad \Im > c$$
$$\Re < c \qquad \Im < c$$

Note that the fractional linear transformation $T(z) = \frac{z-1}{z+1}$ maps right half plane ($\Re > 0$) to the unit disc. Consequently, the map $T_c(z) = \frac{z-1-c}{z+1-c}$ maps the region $\Re > c$ onto the unit disc.

If $\Im > c, \Re < c, \Im < c$, we can always rotate these domains onto $\Re > c$. Hence we can map these regions to the unit disc via

$$T_{c,n}(z) = \frac{i^n z - 1 - c}{i^n z + 1 - c}$$

for $n = 0, 1, 2, 3$ (corresponding to the four cases, in order).

Hence, without loss of generality we may assume $\Re > 0$. Consider now the composite map

$$g(z) = \frac{f(z) - 1}{f(z) + 1}$$

If g has a removable singularity at zero, then

$$\lim_{z \to 0} z g(z) = 0$$

and consequently,

$$\lim_{z \to 0} z[f(z) - 1] = \lim_{z \to 0} z f(z) = 0$$

□

Problem 2 Ahlfors Page 130: Problem 6

Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$. *Hint:* f and e^f cannot have a common pole (why?). Now apply Theorem 9.

Solution: Let z_0 is an isolated singularity of f. Now if z_0 is a pole of f then $h(z) = f^{-1}(z)$ can be made analytic in some open disk D around z_0 with $h(z_0) = 0$.

Now $h(z)$ maps the open desk D to some open neighborhood U of 0. Hence there is an open disk D_1 containing 0 inside U. Now let $G_1 = h^{-1}(D_1)$. G_1 is open and contains z_0. So take a open disk D' inside G_1 and again continue like we did for D and repeat again and again to get G_n and D_n. Thus we get a chain of open sets each contained in the previous one

$$G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots \supseteq G_n \supseteq \cdots$$

where $h(G_k) = D_k$. Let r_{d-k} denote the radius of D_k. Hence $f(G_k) = \{z \mid |z| \geq r_{D_k}\}$. Therefore $f(G_k) \subseteq f(G_{k+1})$.

Now if we take D_k's in such way that $r_{D_k} > 2R_{D_{k+1}}$ and take the sequence $x_k \in G_k \setminus G_{k+1}$. Since each D_k contains 0 we take x_i such that $\Re(f(x_i)) = 0$ then

$$\lim_{n \to \infty} x_n = z_0$$

Therefore

$$\lim_{n\to\infty} |e^{f(x_n)}| = \lim_{n\to\infty} |e^{\Re(f(x_n))}| = \lim_{n\to\infty} |e^0| = 1$$

Hence z_0 is not a pole of $e^{f(z)}$

Now if z_0 is an removable singularity it is also a removable singularity for $e^{f(z)}$

If z_0 is an essential singularity then consider any non-zero $c \in \mathbb{C}$. By the Theorem 9, there is a sequence $z_n \to z_0$ such that $f(z_n) \to \log(c)$. So $\exp(f(z_n)) \to c$. Since this is true for all non-zero c, $\exp(f(z))$ must have an essential singularity at z_0.

Hence z_0 is not a pole of $e^{f(z)}$.

□

Problem 3 Ahlfors Page 133: Problem 3

Apply the representation $f(z) = w_o + \zeta(z)^n$ to $\cos z$ with $z_o = 0$. Determine $\zeta(z)$ explicitly.

Solution: Take $g(z) = \cos z - 1$. Now $g'(z) = -\sin z$, $g''(z) = -\cos z$ and $g(0) = 0 = g'(0)$, $g''(0) = -1$. Therefore $n = 2$. Now

$$\cos z - 1 = -2\sin^2\left(\frac{z}{2}\right) \iff \cos z = 1 - 2\sin^2\left(\frac{z}{2}\right) \quad \forall\ z \in \mathbb{C}$$

Hence

$$\cos z - 1 = \left(\pm\sqrt{2}i \sin\frac{z}{2}\right)^2$$

Hence $\zeta(z) = \sqrt{2}i \sin\frac{z}{2}$ (we are taking the positive branch)

□

Problem 4 Ahlfors Page 133: Problem 4

If $f(z)$ is analytic at the origin and $f'(0) \neq 0$, prove the existence of an analytic $g(z)$ such that $f(z^n) = f(0) + g(z)^n$ in a neighborhood of 0.

Solution: Take $h(z) = f(z^n)$. Then

$$\begin{aligned}
h'(z) &= nz^{n-1}f'(z^n) \\
h''(z) &= n(n-1)z^{n-2}f'(z^n) + n^2z^{2(n-1)}f''(z^n) = n(n-1)z^{n-2}f'(z^n) + \tilde{g}_2(z) \\
h^{(3)}(z) &= n(n-1)n(2)z^{n-3}f'(z^n) + \tilde{g}_3(z) \\
\vdots &\qquad\qquad \vdots \\
h^{(n)}(z) &= n!f'(z^n) + \tilde{g}_n(z)
\end{aligned}$$

Now forall $k \in \{1, 2, \ldots, n\}$ $\tilde{g}_k(0) = 0$ and forall $k \in \{1, 2, \ldots, n-1\}$, $h^{(k)}(0) = 0$ and $g^{(n)}(0) = n!f'(0)$. Since given that $f'(0) \neq 0$ we have $g^{(n)}(0) \neq 0$. Hence there exists a function $h(z)$ such that $h(0) \neq 0$ and

$$g(z) - f(0) = (z-0)^n h(z)$$

Now since $H()z)$ is continuous for $\epsilon = |h(z_0)|$ there exists δ such that

$$\forall\ |z - z_0| < \delta \implies |h(z) - h(z_0)| < |h(z_0)$$

Hence we have a single valued branch of $\sqrt[n]{h(z)}$, $\zeta(z) = (z - z_0)\sqrt[n]{h(a)}$. Then we have

$$h(z) - f(0) = \zeta^n(z) \iff f(z^n) = f(0) + \zeta^n(z)$$

□

Problem 5 Ahlfors Page 136: Problem 1

Show by use of (36), or directly, that $|f(z)| \le 1$ for $|z| \le 1$ implies

$$\frac{|f'(z)|}{(1-|f(z)|^2)} \le \frac{1}{1-|z|^2}$$

Solution: Here the given function f, $|f(z)| \le 1$ for $|z|| \le 1$. Then choose any $z_0 \in \overline{B(0,1)}$ and let $w_0 = f(z_0)$. Hence we have

$$\begin{aligned}
&\left|\frac{f(z)-w_0}{1-\overline{w_0}f(z)}\right| \le \left|\frac{z-z_0}{1-\overline{z_0}z}\right| \\
\implies &\frac{\left|\frac{f(z)-f(z_0)}{z-z_0}\right|}{|1-\overline{w_0}f(z)|} \le \left|\frac{1}{1-\overline{z_0}z}\right| \\
\implies &\frac{|f'(z_0)|}{\left|1-\overline{f(z_0)}f(z_0)\right|} \le \frac{1}{|1-\overline{z_0}z_0|} && [\text{As } z \to z_0] \\
\implies &\frac{|f'(z_0)|}{1-|f(z_0)|^2} \le \frac{1}{|1-|z_0|^2}
\end{aligned}$$

Since z_0 is arbitrary we have

$$\frac{|f'(z)|}{(1-|f(z)|^2)} \le \frac{1}{1-|z|^2}$$

□

Problem 6 Ahlfors Page 136: Problem 2

If $f(z)$ is analytic and $\Im f(z) \ge 0$ for $\Im z > 0$, show that

$$\frac{|f(z)-f(z_0)|}{\left|f(z)-\overline{f(z_0)}\right|} \le \frac{|z-z_0|}{|z-\bar{z}_0|}$$

and

$$\frac{|f'(z)|}{\Im f(z)} \le \frac{1}{y} \quad (z = x+iy)$$

Solution: Let z_0 be any complex number and $w_0 = f(z_0)$. Take $F(\zeta) = Sf(T^{-1}\zeta)$ like in the proof of the inequality in (36) in Ahlfors for some linear transformation T, S. Now define

$$Tz = \frac{z-z_0}{z-\overline{z_0}} \quad \text{and} \quad Sz = \frac{z-w_0}{z-\overline{w_0}}$$

Now if we can show that $F(\zeta)$ satisfies the conditions of Schwarz Lemma then we have $|F(\zeta)| \le |\zeta|$ by setting $Tz = \zeta$.

First we have to show $F(0) = 0$. Now $Tz_0 = 0$ hence $T^{-1}0 = z_0$. Similarly $Sw_0 = 0$. Since $f(z_0) = w_0$ we have $F(0) = 0$

Now we have to show $|F(\zeta)| \le 1$. Now Tz maps the upper half plane to inside of the unit disk. Hence im $\Im z \ge 0$ then $|\zeta| < 1$. Similarly S also maps the upper half plane to inside of the unit disk. Therefore $T^{-1}(\zeta)$ maps the inside of the unit disk to upper half plane. And given that if $\Im z \ge 0$ then $\Im f(z) \ge 0$. Hence $|F(\zeta)| \le 1$ for $|\zeta| < 1$.

Since F satisfies the conditions of Schwarz Lemma we have

$$|F(\zeta)| \le |\zeta| \implies |Sf(T^{-1}\zeta)| \le |\zeta| \implies |Sf(z)| \le |Tz| \implies$$

$$\left|\frac{f(z)-w_0}{f(z)-\overline{w_0}}\right| \le \left|\frac{z-z_0}{z-\overline{z_0}}\right| \implies \left|\frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}}\right| \le \left|\frac{z-z_0}{z-\overline{z_0}}\right|$$

Now

$$\left|\frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}}\right| \le \left|\frac{z-z_0}{z-\overline{z_0}}\right| \implies \left|\frac{f(z)-f(z_0)}{z-z_0}\right| \le \left|\frac{f(z)-\overline{f(z_0)}}{z-\overline{z_0}}\right|$$

By taking $z \downarrow z_0$ we have

$$\lim_{z\downarrow z_0}\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \le \lim_{z\downarrow z_0}\left|\frac{f(z)-\overline{f(z_0)}}{z-\overline{z_0}}\right| \implies |f'(z_0)| \le \frac{2(\Im f(z_0))}{2(\Im z_0)} \implies \frac{|f'(z_0)|}{\Im f(z_0)} \le \frac{1}{\Im z_0} \text{ for } \Im z_0 \ge 0$$

Since z_0 is arbitrary in the upper half plane, hence for any $z = x + iy$, with $y \ge 0$ we have

$$\frac{|f'(z)|}{\Im f(z)} \le \frac{1}{y}$$

□

Problem 7 Ahlfors Page 136: Problem 3

In Problem 5 and Problem 6, prove that equality implies that $f(z)$ is a linear transformation.

Solution: Take $F(\zeta) = Sf(T^{-1}\zeta)$ like in the proof of the inequality in (36) in Ahlfors for some linear transformation T, S. Now $|F(\zeta)| \le 1$ and $F(0) = 0$. Hence if $|F(\zeta)| = |\zeta|$ then $F(z) = cz$ for some constant $c \in \mathbb{C}$ by Schwarz Lemma. Hence

$$Sf(T^{-1}\zeta) = cs \iff f(z) = S^{-1}(cT(z)) \quad [\text{take } z = T^{-1}(\zeta)]$$

Therefore $f(z)$ is a linear combination

□

Problem 8 Ahlfors Page 148: Problem 2

Prove that the region obtained from a simply connected region by removing m points has the connectivity $m+1$, and find a homology basis.

Solution:

- Let the region Ω is obtained from the simply connected region S by removing m points. Let the points are $x_1, \ldots, x_m$. Hence $\mathbb{C} - \Omega = (\mathbb{C} - S) \cup \{x_1, \ldots, x_m\}$.

  $\mathbb{C} - S$ is connected in the extended plane by definition. Hence $\mathbb{C} - S$ is counted as one connected component. Since the m points are chosen from the region S, the singleton sets $\{x_i\}$ form connected components for all $i \in \{1, 2, \ldots, m\}$. Hence there are $m+1$ connected components in $\mathbb{C} - \Omega$ which are $\mathbb{C} - S$ and $\{x_i\}$ $\forall\, i \in \{1, 2, \ldots, m\}$

- Since the points are distinct and finite we can take the circles as the curves around each point choose their radius in such a way that no two curves intersect each other and no curv pass through any of those m points. These m many circles will be the homology basis.

□

Problem 9 Ahlfors Page 48: Problem 5

Show that a single-valued analytic branch of $\sqrt{1-z^2}$ can be defined in any region such that the points ± 1 are in the same component of the complement. What are the possible values of

$$\int \frac{dz}{\sqrt{1-z^2}}$$

over a closed curve in the region?

Solution:

- Let Ω be any region. Now given that in $\mathbb{C}-\Omega$, 1 and -1 lie in same component. Now we have to define a single valued analytic branch of $\sqrt{1-z^2}$.

  First we will define an single valued analytic branch of $\log\left(\frac{1-z}{1+z}\right)$ in Ω. Let γ be any closed curve in Ω. Then
$$\int_\gamma \left(\frac{1}{z-1} - \frac{1}{z+1}\right) dz = n(\gamma;1) - n(\gamma;-1) = 0$$
  since 1, -1 lie in same component. Hence it has an anti-derivative function $f(z)$ which is analytic defined in the region Ω. Now we have
$$\frac{d}{dz}\left(\frac{z+1}{z-1}\right)e^f(z) = e^z\left(\frac{(z-1)-(z+1)}{(z-1)^2} + \frac{z+1}{z-1}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\right) = 0$$
  Hence $\frac{z+1}{z-1}e^f(z)$ is a constant function. Hence $\frac{z+1}{z-1}e^{f(z)} = cz$ for some $c \in \mathbb{C}$. Let $e^a = c$ then let $g(z) = f(z) - a$ then
$$\frac{z+1}{z-1}e^{f(z)} = cz \implies e^{-a}e^{f(z)} = \frac{z-1}{z+1} \implies e^{f(z)-a} = \frac{z-1}{z+1} \implies e^{g(z)} = \frac{z-1}{z+1}$$
  Hence we can define $\log\left(\frac{z-1}{z+1}\right) = g(z) = f(z) - a$. Hence now we have an analytic function $g(z)$ defined in Ω.

  Now
$$e^{g(z)} = \frac{z-1}{z+1} = -\frac{1-z^2}{(1+z)^2} \implies -(z+1)^2e^{g(z)} = 1-z^2 \implies \left(i(z+1)e^{\frac{g(z)}{2}}\right) = 1-z^2$$
  Let $h(z) = i(z+1)e^{\frac{g(z)}{2}}$ then $h(z)$ defines an analytic branch or $\sqrt{1-z^2}$.

  $\sqrt{1-z^2}$ vanishes at both 1 and -1. If γ doesn't pass through $[-1,1]$ interval, removing the branch cut, we can pick an analytic branch of $\sqrt{1-z^2}$ in $\mathbb{C}-[-1,1]$. For our region we can extend the branch of $\sqrt{1-z^2}$ to $\mathbb{C}-[-1,1]$ in this case. So we can replace the integrand function with the extended branch and assume that it is analytic in $\mathbb{C}-[-1,1]$ and $\Omega = \mathbb{C}-[-1,1]$. And WLOG we can also assume that the component in which $1,-1$ lies, is the line segment $[-1,1]$. Now we can take any circle of radius r where $r > 1$ to be the homology basis for the component of the line segment $[-1,1]$.

  Now suppose γ passes through $[-1,1]$. Now there exists another path p which is not cut by γ. Now we extend the branch to $\mathbb{C}-p$ and take circle of radius r such that path p lies in side the disk of radius r. This circle will form a homology basis corresponding to p.

- Since $\lim\limits_{z\to\infty} \frac{1-z^2}{z^2} = -1$ we have $\lim\limits_{z\to\infty} \frac{\sqrt{1-z^2}}{z} = i$ or $-i$ Hence for the first case
$$\frac{1}{\sqrt{1-z^2}} - \frac{1}{iz} = \frac{iz - \sqrt{1-z^2}}{iz\sqrt{1-z^2}} = O(z^{-3})$$

Similarly in the second case

$$\frac{1}{\sqrt{1-z^2}} + \frac{1}{iz} = \frac{iz + \sqrt{1-z^2}}{iz\sqrt{1-z^2}} = O(z^{-3})$$

Hence we have $\frac{1}{\sqrt{1-z^2}} = \pm\frac{1}{iz} + O(z^{-3})$. Hence

$$\int_\gamma \frac{dz}{\sqrt{1-z^2}} = n(\gamma;0)\int_{|z|=R}\frac{dz}{\sqrt{1-z^2}} = n(\gamma;0)\lim_{R\to\infty}\int_{|z|=R}\left(\pm\frac{1}{iz} + O(z^3)\right)dz = n(\gamma;0)\left(\underbrace{\pm\int_{|z|=R}\frac{dz}{iz}}_{\pm 2\pi}\right)$$

Hence $\int\limits_\gamma \frac{dz}{\sqrt{1-z^2}}$ is integral multiple of 2π

$\square$