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Course: Complex Analysis

Assignment - 3
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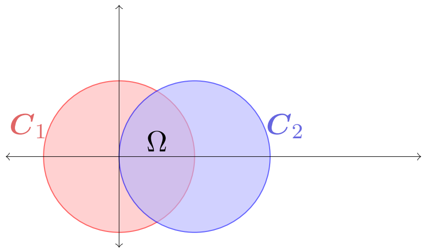
Date: March 8, 2023

Problem 1 Ahlfors Page 96: Problem 1

Map the common part of the disks $|z| < 1$ and $|z-1| < 1$ on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

Solution: Let $C_1 : |z| = 1$ and $C_2 : |z-1| = 1$. Let the common region between them is Ω



The circles intersect when

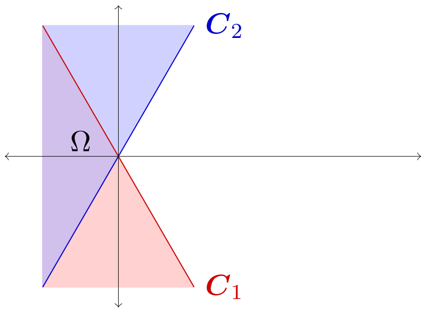
$$|z| = |z-1| \iff z\overline{z} = (z-1)(\overline{z}-1) \iff 1 = z + \overline{z}$$

Hence $\Re(z) = \frac{1}{2}$. Therefore $\Im(z) = \pm\frac{\sqrt{3}}{2}$ since $|z| = 1$. Therefore C_1 and C_2 intersects at $-\omega$ and $-\omega^2$.

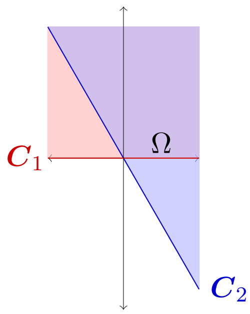
Now we send $-\omega^2 \to \infty$ and $-\omega \to 0$ by the conformal map $f_1(z) = \frac{z+\omega}{z+\omega^2}$. Then

$$f_1(1) = \frac{1+\omega}{1+\omega^2} = \frac{-\omega^2}{-\omega} = \omega \qquad f_1(0) = \frac{\omega}{\omega^2} = \frac{1}{\omega} = \bar{\omega} = \omega^2$$

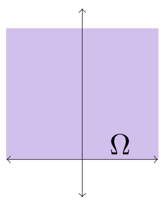
Hence $f_1(C_1) =$ line joining 0 and ω and $f_1(C_2) =$ line joining 0 and ω^2



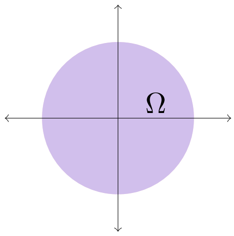
Now we rotate the region Ω by $\frac{2\pi}{3}$ clockwise by the conformal map $f_2(z) = e^{-i\frac{2\pi}{3}}z = \omega z$



Now we map the common region Ω to the upper half of the plane by the conformal map $f_3(z) = z^{\frac{3}{2}}$



Now we want to map the upper half plane to inside of the unit circle. We do it with the conformal map $f_4(z) = \frac{z-\omega}{z-\omega^2}$



Hence the final conformal map which maps the region Ω to the inside of unit disk is

$$
\begin{aligned}
f_4 \circ f_3 \circ f_2 \circ f_1(z) = f_4 \circ f_3 \circ f_2 \left(\frac{z+\omega}{z+\omega^2} \right) = f_4 \circ f_3 \left(\omega \frac{z+\omega}{z+\omega^2} \right) = f_4 \circ f_3 \left(\frac{\omega^2 z + 1}{\omega z + 1} \right) \\
= f_4 \left(\left[\frac{\omega^2 z + 1}{\omega z + 1} \right]^{\frac{3}{2}} \right) = \frac{\left[\frac{\omega^2 z+1}{\omega z+1} \right]^{\frac{3}{2}} - \omega}{\left[\frac{\omega^2 z+1}{\omega z+1} \right]^{\frac{3}{2}} - \omega^2}
\end{aligned}
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Problem 2

Solution:

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