Analysis 2 Lecture Notes – Upendra Kulkarni

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Abstract

This is the lecture notes scribed by me. If you find any mistakes in the notes please email me at sohamc@cmi.ac.in.

The whole course is taken by Prof. Upendra Kulkarni, online. If you want the lectures then you can find them in this link. Sir mainly followed Prof. Pramath Sastry's Notes (https://www.cmi.ac.in/~pramath/teaching.html#ANA2). You can find all the assignments problems in the following drive link. Through out the course the books we followed is Principles of Mathematical Analysis by Walter Rudin.

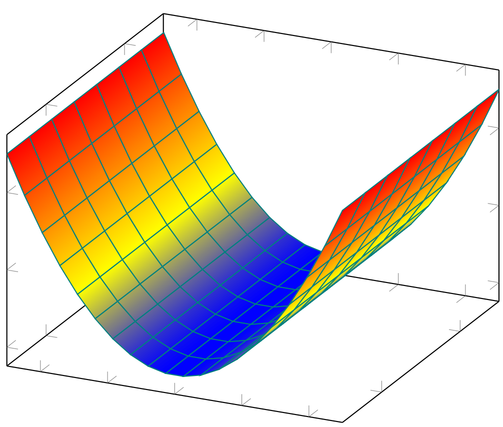
Chapter 1

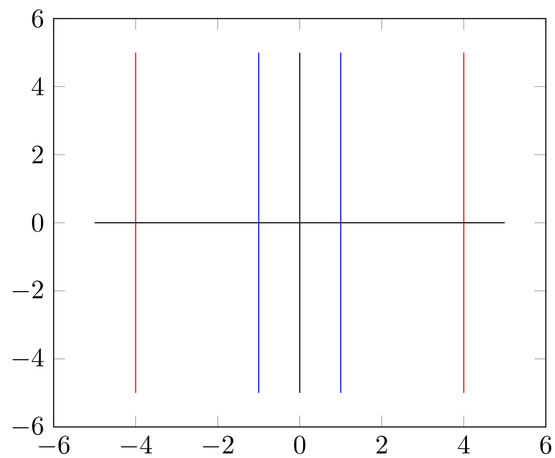
Examples of Functions and Analyze Critical Points

Graph of $\Phi(x) = \Phi(x_1, \ldots, x_n)$ is in $\mathbb{R}^{n+1}$. We can visualize it in $\mathbb{R}^n$ by drawing level sets, namely plot $\Phi(x_1, \ldots, x_n) = c$ for various values of constant c in $\mathbb{R}$

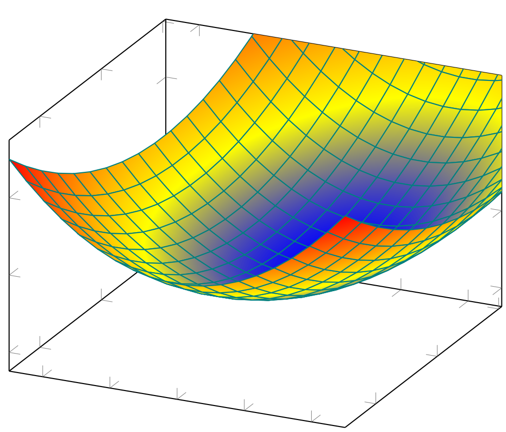
Examples

① $f(x, y) = x^2$

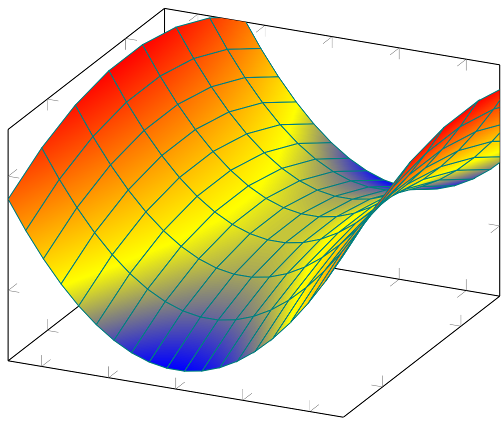


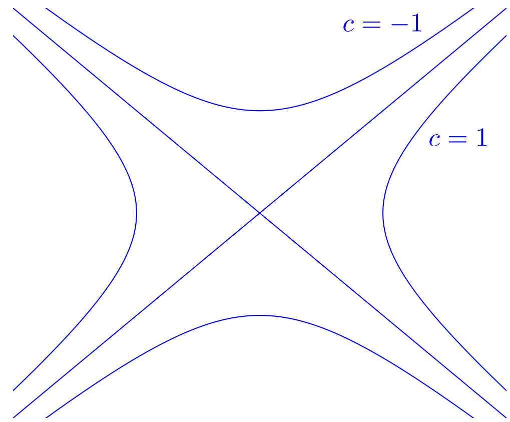


② $f(x, y) = x^2 + y^2$. Level Sets = Circles centered at $(0, 0)$



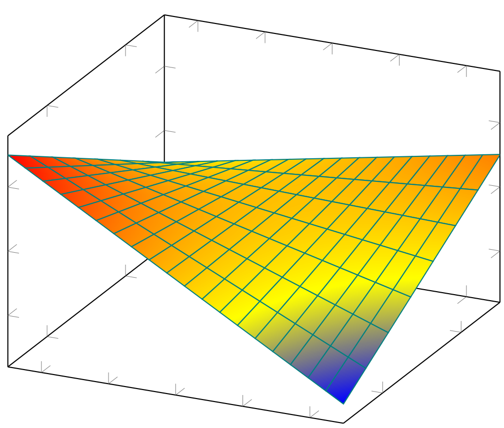
③ $f(x,y) = x^2 - y^2$. Level Sets $c = 0 \implies x = \pm y$, $c = 1 \implies x^2 - y^2 = 1$, $c = -1 \implies x^2 - y^2 = -1$

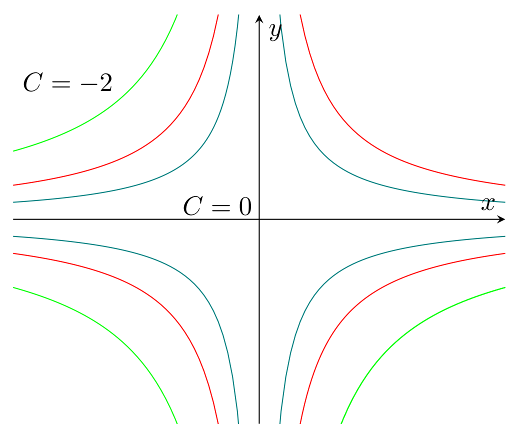




④ $f(x,y) = xy$

$u = \frac{x+y}{\sqrt{2}}, v = \frac{x-y}{\sqrt{2}}$. Then $x = \frac{u+v}{\sqrt{2}}, y = \frac{u-v}{\sqrt{2}}$ and $f(x,y) = \frac{u^2-v^2}{2}$. Here $A = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$





We should understand graphs of 'Quadratic Hypersurfaces' $\Phi(x) = 0$, where $\Phi(x)$ is a quadratic polynomial in n variables.

'Standard Form' is $\lambda_1 x_2^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 +$ Constant. We will see that by a shift of origin and orthogonal change of coordinates, we can express any general quadratic Φ to the Standard Form

① Getting Rid of Linear Part

$$\begin{aligned} &\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 + p_1 x_1 + \cdots + p_n x_n + \text{ constant} \\ =&\lambda_1 (x_1 - a_1)^2 + \cdots + \lambda_n (x_n - a_n)^2 + \text{ another constant} \quad [-2\lambda_i a_i = p_i \implies a_i = -\frac{p_i}{2\lambda_i}, \text{ assuming } \lambda_i \neq 0] \end{aligned}$$

② In general we express x in terms of new basis consisting of orthonormal eigenvectors of A.

Nationalizing a matrix A, $\Gamma^{-1} A \Gamma = D$-diagonal matrix where columns of $\Gamma =$ eigen basis corresponding to matrix A. Here Γ is orthogonal matrix $\Gamma\Gamma^T = \Gamma^T\Gamma = I$ and we have $\Gamma^T A \Gamma = D \implies A = \Gamma D \Gamma^T$. Now

$$\Phi(x) = x^T A x + pX + r$$

Let x^* = coordinate vector of x in terms of new basis consisting of columns of Γ

$$\begin{aligned} x^* &= \Gamma^{-1}x = \Gamma^T x \text{ we use this to formulate } \Phi \\ &= (x^T\Gamma)D(\Gamma^T x) + p\Gamma(\Gamma^T x) + r = \Phi(x) \\ &= \underset{\substack{\downarrow \\ \text{standard} \\ \text{form}}}{x^{*T}Dx^*} + \underset{\substack{\downarrow \\ \text{linear} \\ \text{form}}}{p\Gamma x^*} + r = \Psi(x^*) \end{aligned}$$

Use step 1 to eliminate the linear term

Now we will look into some more examples.

① $f(x,y) = x^2 - xy + y^2$

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

H is positive definite because diagonal entries are positive and determinant $= 3 > 0$. So the unique critical point $(0,0)$ is a local minima

Note:-

2×2 symmetric matrix $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$ is positive definite $\iff \begin{cases} a, b > 0 \\ ab - c^2 > 0 \end{cases}$

② $\Phi(x) = 2x^2 + 3y^2 - 4xy - 12x - 14y + 21 = \begin{bmatrix} x \\ y \end{bmatrix}^T A \begin{bmatrix} x \\ y \end{bmatrix} + p \begin{bmatrix} x \\ y \end{bmatrix} + r$

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \text{ and } H = \begin{bmatrix} 4 & -4 \\ -4 & 6 \end{bmatrix} \text{ and } p = \begin{bmatrix} -12 & 14 \end{bmatrix}$$

H is positive definite as diagonal entries are positive and determinant $= 8 > 0$. The critical point is the solution of the equation

$$H \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} -12 \\ 13 \end{bmatrix} \iff \begin{bmatrix} 4 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} -12 \\ 14 \end{bmatrix}$$

Hence $x = 2$, $y = -1$. Therefore minimum value $\Phi(2,-1) = 2$

Note:-

Another way: Complete the squares

$$\begin{aligned} \Phi(x) &= 2(x-2)^2 + 4(y+1)^2 - 4(x-2)(y+1) + 2 \\ &= 2u^2 + 3v^2 - 4uv + 2 \end{aligned}$$

③ $f(x,y) = x^3 + y^3 - 3x - 3y$

$$f'(x,y) = \begin{bmatrix} 3x^2 - 3 & 3y^2 - 3 \end{bmatrix}, \qquad \nabla f = \begin{bmatrix} 3x^2 - 3 \\ 3y^2 - 3 \end{bmatrix}$$

Critical points are (x,y) such that $f'(x,y) = 0$ i.e. $\begin{cases} 3x^2 - 3 = 0 \\ 3y^2 - 3 = 0 \end{cases}$. There are 4 critical points $= (\pm 1, \pm 1)$

$$\text{Hessian } H = \begin{bmatrix} 6x & 0 \\ 0 & 6x \end{bmatrix}$$

$$(1,1,) \to \text{local min}, (-1,-1) \to \text{local max}, (\pm 1, \mp 1) \to \text{saddle points}$$

Note:-

For $x^3 - y^2 + 3x - 3y$ there are no critical points