Soham Chatterjee
Email: sohamc@cmi.ac.in
Course: Complex Analysis

Assignment - 1
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Problem 1

Find the general analytic function $f = u + iv$, such that $u = x^2 - y^2$.

Solution: Given that $u = x^2 - y^2$. Then $\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = -2y$. Since the function is analytic u, v follows the Cauchy Riemann Equations. Hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $\frac{\partial v}{\partial x} = 2x$ and $\frac{\partial v}{\partial y} = 2y$. Now since $\frac{\partial v}{\partial x} = 2y$ we can assume $v = 2xy + g(y)$ where g is some real valued function. But then $\frac{\partial v}{\partial y} = 2x$ implies that $g'(y) = 0$ hence g is some constant function. Hence $v = 2xy + c$ where $c \in \mathbb{R}$ some constant. Hence

$$f(x, y) = x^2 - y^2 + i(2xy + c) == x^2 - y^2 + 2ixy + ic = (x + iy)^2 + ic \iff f(z) = z^2 + ic$$

□

Problem 2

Solution:

□

Problem 3

Prove Cauchy's inequality: Let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ be two complex vectors, then

$$(a \cdot b)^2 \leq \|a\|^2 \|b\|^2$$

where $(a \cdot b)$ is the scalar product of vectors.

Solution: Consider the vector $a + \lambda b$. Now since for any vector v, $\sqrt{(v \cdot v)} = \|v\| \geq 0$ we have $\sqrt{((a + \lambda b) \cdot (a + \lambda b))} \geq 0$. Now

$$\begin{aligned}((a + \lambda b) \cdot (a + \lambda b)) &= (a \cdot (a + \lambda b)) + \lambda(b \cdot (a + \lambda b)) \\ &= (a \cdot a) + \lambda(a \cdot b) + \lambda(b \cdot a) + \lambda^2 (b \cdot b) \\ &\|a\|^2 + 2\lambda(a \cdot b) + \lambda^2 \|b\|^2\end{aligned}$$

Since $((a + \lambda b) \cdot (a + \lambda b)) \geq 0$ the discriminant of the polynomial, $p(\lambda) = \|a\|^2 + 2\lambda(a \cdot b) + \lambda^2 \|b\|^2$ is non-positive. Hence

$$4(a \cdot b)^2 \leq 4\|a\|^2 \|b\|^2 \iff (a \cdot b)^2 \leq \|a\|^2 \|b\|^2$$

□

Problem 4 Ahlfors Exercise 2.1 Problem 1

If $g(w)$ and $f(z)$ are analytic functions, show that $g(f(z))$ is also analytic.

Solution: Let $g(w) = u_g(x, y) + iv_g(x, y)$ and $f(z) = u_f(x, y) + iv_f(x, y)$. Then u_g, u_f, v_g, v_f have continuous partial derivatives and satisfy Cauchy Riemann Equations.

$$\frac{\partial u_g}{\partial x} = \frac{\partial v_g}{\partial y} \qquad \frac{\partial u_g}{\partial y} = -\frac{\partial v_g}{\partial x}$$
$$\frac{\partial u_f}{\partial x} = \frac{\partial v_f}{\partial y} \qquad \frac{\partial u_f}{\partial y} = -\frac{\partial v_f}{\partial x}$$

Then

$$h(z) = u_h(x, y) + iv_h(x, y) = u_g(u_f(x, y), v_f(x, y)) + iv_g(u_f(x, y), v_f(x, y))$$

Therefore

$$\frac{\partial u_h}{\partial x} = \frac{\partial u_g}{\partial u_f}\frac{\partial u_f}{\partial x} + \frac{\partial u_g}{\partial v_f}\frac{\partial v_f}{\partial x} \qquad \frac{\partial u_h}{\partial y} = \frac{\partial u_g}{\partial u_f}\frac{\partial u_f}{\partial y} + \frac{\partial u_g}{\partial v_f}\frac{\partial v_f}{\partial y}$$
$$\frac{\partial v_h}{\partial x} = \frac{\partial v_g}{\partial u_f}\frac{\partial u_f}{\partial x} + \frac{\partial v_g}{\partial v_f}\frac{\partial v_f}{\partial x} \qquad \frac{\partial v_h}{\partial y} = \frac{\partial v_g}{\partial u_f}\frac{\partial v_f}{\partial y} + \frac{\partial v_g}{\partial v_f}\frac{\partial v_f}{\partial y}$$

□

Problem 5 Ahlfors Exercise 2.1 Problem 2

Verify Cauchy-Riemann's equations for the functions z^2 and z^3

Solution:

$$z^2 = (x + iy)^2 = \underbrace{x^2 - y^2}_{u_1} + \underbrace{2xy}_{v_1}\, i$$

$$z^3 = (x + iy)^3 = \left(x^2 - y^2 + 2xyi\right)(x + yi) = \left[\underbrace{x^3 - 3xy^2}_{u_2}\right] + \left[\underbrace{3x^2y - y^3}_{v_2}\right] i$$

For z^2

$$\frac{\partial u_1}{\partial x} = 2x = \frac{\partial v_1}{\partial y} \qquad \frac{\partial u_1}{\partial y} = -2y = -\frac{\partial v_1}{\partial y}$$

and for z^3

$$\frac{\partial u_2}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v_2}{\partial Y} \qquad \frac{\partial u_2}{\partial y} = -6xy = -\frac{\partial v_2}{\partial x}$$

Hence both the functions follow the Cauchy-Riemann's Equations

□

Problem 6 Ahlfors Exercise 2.1 Problem 3

Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

Solution:

□

Problem 7 Ahlfors Exercise 2.1 Problem 4

Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Solution: If $|f(z)| = 0 \ \forall\ z \in \mathbb{C}$ then we have $f(z) = 0$. Now let $|f(z)| = c > 0 \ \forall\ z \in \mathbb{C}$. Let $f = u + iv$ then $|f(z)| = \sqrt{u^2(x, y) + v^2(x, y)} = c$. Therefore

$$\frac{\partial}{\partial x}(u^2 + v^2) = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$
$$\frac{\partial}{\partial y}(u^2 + v^2) = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$$

We can write this in matrix form that

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now if $\begin{bmatrix} u & -v \\ v & u \end{bmatrix}$ is not invertible then determinant$=u^2+v^2=0$ which is not possible then $\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Hence all the partial derivatives of f is 0. So $f'(z) = 0$ or $f(z) = c$

$\square$

Problem 8 Ahlfors Exercise 2.1 Problem 5

Prove rigorously that the functions $f(z)$ and $\overline{f(z)}$ are simultaneously analytic.

Solution:

$\square$

Problem 9 Ahlfors Exercise 2.1 Problem 6

Prove that the functions $u(z)$ and $u(\overline{z})$ are simultaneously harmonic.

Solution:

$\square$