

The Iterated Mod Problem

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Introduction

- This paper is Iterated Mod Problem by Karloff and Ruzzo [KR89]
- Sequential algorithm for computing \gcd is based on Euclidean Algorithm $r_0 = a, r_1 = b$. Then

$$r_2 = r_0 \bmod r_1, \quad r_3 = r_1 \bmod r_2, \quad \dots$$

\gcd is the last nonzero r_i .

- But parallel complexity of \gcd is poorly understood. Fastest parallel algorithm takes $O\left(\frac{n}{\log n}\right)$ time [CG90]
- The problem we will study related to the \gcd problem. It is given integers or polynomials $x, m_n, m_{n-1}, \dots, m_1$ find if

$$((x \bmod m_n) \bmod m_{n-1}) \cdots \bmod m_1 = 0$$

Iterated Integer Mod Problem

Introduction

Problem:

Given positive integers $x, m_n, m_{n-1}, \dots, m_1$ find if

$$((x \bmod m_n) \bmod m_{n-1}) \cdots \bmod m_1 = 0$$

Theorem

Iterated Integer Mod $\in P$

For any 2 numbers a and b , $a \bmod b$ is in P . Here we are doing n iterated mods. So it still takes polynomial time. So $IIM \in P$.

Circuit Value Problem

Theorem ([Lad75])

Circuit Value Problem is P-complete.

- Enough to take *CVP* for circuits with only *NAND* gates, *NANDCVP*

$\text{Gates} \in [G]$

Input Variables: $y_i, i \in [r]$, Input Bits: $Y_i, i \in [r]$

$NANDCVP \leq_I IIM$

Log-Space Reduction

Let $n = 2G$.

- x is $n + 1$ -bit integer whose i th bit is Y_j if the i th edge is incident from the input y_j . Otherwise it is 1.
- $1 \leq g \leq G$

$$m_{2g} = 2^{2g} + 2^{2g-1} + \sum_{\substack{j \text{th edge} \\ \text{out-edge from } g}} 2^j \text{ and } m_{2g-1} = 2^{2g-1}$$

Remark: Here m_{2g} and m_{2g-1} simulate the gate g

$NANDCVP \leq_I IIM$

Correctness

Theorem

Let $x_{G+1} = x$. And for all $1 \leq g \leq G$

$$x_g = ((\cdots ((x \bmod m_{2G}) \bmod m_{2g-1}) \cdots \bmod m_{2g}) \bmod m_{2g-1}) = 0.$$

Then:

- ① For all $1 \leq g \leq G + 1$, $x_g \leq 2^{2g-1}$
- ② For all $1 \leq g \leq G + 1$, $0 \leq j \leq 2g - 1$ if the j th edge is an outgoing edge from an input node or from a gate h such that $h \geq g$ then x_g 's j th bit is the value carried by j th edge otherwise 1

$NANDCVP \leq_I IIM$

Correctness

Prove by downward induction:

Base Case ($g = G + 1$): We have $x < 2^{2(G+1)-1} = 2^{2G+1} = 2^n$. True as x is n -bit number. And second condition follows by constuction. Let the theorem holds for all $g > k$.

$NANDCVP \leq_I IIM$ III

Correctness

Part (a):

$x_k = (x_{k+1} \bmod m_{2k}) \bmod m_{2g-1}$. $m_{2k-1} = 2^{2k-1}$. So x_k has $2k-1$ bits so $x_k < 2^{2k-1}$. So Part (a) is proved.

$NANDCVP \leq_I IIM IV$

Correctness

Part (b):

- The only bits differ between x_{k+1} and x_k are the bits corresponding to edges incident on k th vertex (in and out). In x_{k+1} the j th bits are 1 if j th edge going out from gate k .
- The $2k$ and $2k - 1$ th edges are in edges of gate k . So in x_{k+1} the $(2k)$ th and $(2k - 1)$ th bits are the value carried by the $(2k)$ and $(2k - 1)$ th edges. Two cases to consider:

$NANDCVP \leq_I IIM \vee$

Correctness

Both $(2k)$ and $(2k + 1)$ th bits are 1:

$m_{2k} \leq x_{k+1} < 2m_{2k}$. So

$$(x_{k+1} \bmod m_{m_{2k}}) \bmod m_{2k-1} = x_{k+1} - m_{2k}$$

So in x_{2k} at output bits position of m_{2k} the 1 is replaced by 0

At least one of the bits is 0:

$$x_{k+1} < m_{2k} \implies x_{k+1} \bmod m_{2k} = x_{k+1}$$

So in x_{2k} at output bits position of m_{2k} has 1.

IIM is P -complete

$x_1 < 2^1$ is the value carried by the 0th edge, value of the CVP instance.

Theorem

$NANDCVP \leq_I \text{Iterated Integer Mod}$

Theorem

Integer Iterated Mod Problem is P -complete

Super Increasing Knapsack Problem (SIK)

Introduction

Definition (0-1 Knapsack Problem)

Given an integer w and a sequence of integers w_1, w_2, \dots, w_n is there a sequence of 0 – 1 valued variables x_1, \dots, x_n such that $w = \sum_{i=1}^n x_i w_i$.

- 0-1 Knapsack Problem is known to be NP -complete. [GJ90]
- A knapsack instance is called super increasing (SIK) if each weight w_i is larger than the sum of the previous weights i.e. for all $2 \leq i \leq n$ we have $w_i > \sum_{j=1}^{i-1} w_j$

Super Increasing Knapsack Problem (SIK)

Introduction

Theorem

Super Increasing Knapsack Problem $\in P$

Greedy strategy considering the w_i' in decreasing order gives a linear time algorithm for solving super increasing knapsack problem.

SIK is P -complete I

Theorem

If w_1, \dots, w_n are such that $\forall i \in [n-1] \sum_{k=1}^i w_k < w_{i+1}$ then there is a 0-1 sequence of variables x_1, \dots, x_n such that $\sum_{i=1}^n x_i w_i = w$ iff

$$((\dots ((w \bmod w_n) \bmod w_{n-1}) \dots) \bmod w_2) \bmod w_1 = 1$$

SIK is P -complete II

Observe: The previous reduction the modulo numbers doesn't satisfy super increasing knapsack condition.

- Need to find another reduction of $NANDCVP$ to IIM where modulo numbers are super increasing to work with above theorem !!

SIK is P -complete III

- Let x is $n + 1$ -length base 4 number whose i th digit is Y_j if the i th edge is incident from the input y_j . Otherwise it is 1.
- $1 \leq g \leq G$

$$m_{2g} = 4^{2g} + 4^{2g-1} + \sum_{\substack{j\text{th edge} \\ \text{out-edge from } g}} 4^j$$

$$m_{2g-0.5} = 4^{2g} - 4^{2g-1}, \quad m_{2g-1} = 4^{2g-1}$$

SIK is P -complete IV

Define for all $1 \leq g \leq G$,

$$x_g = (((\cdots (((x \bmod m_{2G}) \bmod m_{2G-0.5}) \bmod m_{2G-1}) \cdots) \bmod m_{2g}) \bmod m_{2g-0.5}) \bmod m_{2g-1} = 0 \text{ and } x_{G+1} = x.$$

- $x_g \leq 4^{2g-1}$ for all $1 \leq g \leq G+1$

SIK is P -complete V

Theorem

For all $1 \leq g \leq G + 1, 0 \leq j \leq 2g - 1$ if the j th edge is an outgoing edge from an input node or from a gate h such that $h \geq g$ then x_g 's j th bit is the value carried by j th edge otherwise 1

SIK is P -complete VI

- Prove by downward induction. Base case $g = G + 1$ is true.
- x_{k+1} and x_k differs at the positions corresponding to the edges incident on k th vertex.
- $2k$ and $2k - 1$ th edges are in-edges of vertex k so they are the values carried by $2k$ and $2k - 1$ th edges

SIK is P -complete VII

If both of them 1:

$$4m_{2k} > x_{k+1} \geq m_{2k} \implies x_{k+1} \bmod m_{2k} = x_{k+1} - m_{2k} < 4^{2k-1}$$

$$(x_{k+1} - m_{2k} \bmod m_{2k-0.5}) \bmod m_{2k-1} = x_{k+1} - m_{2k}$$

In x_k the positions where m_{2k} has 1 will have 0.

SIK is P -complete VIII

If at least one of them 0:

$x_{k+1} \bmod m_{2k} = x_{k+1}$. In x_k positions where m_{2k} has 1 will have 1.

$$x_{k+1} = a \times 4^{2k} + b \times 4^{2k-1} + c \text{ where } a, b \in \{0, 1\}$$

- $a = 1, b = 0$:

$$(x_{k+1} \bmod m_{2k-0.5}) \bmod m_{2k-1} = 1 \times 4^{2k-1} + c \bmod m_{2k-1} = c$$

- $b = 0, 1$:

$$(x_{k+1} \bmod m_{2k-0.5}) \bmod m_{2k-1} = b \times 4^{2k-1} + c \bmod m_{2k-1} = c$$

SIK is P -complete IX

After m_1 , $x_1 < 2^1$ is the value carried by the 0th edge, the value of the CVP.

- **Notice:** The modulus satisfies the super increasing knapsack problem.

Since

$$\sum_{g=1}^k m_{2g} + m_{2g-0.5} + m_{2g-1} = \sum_{g=1}^k m_{2g} + 4^{2g} < 4^{2k+1} = m_{2(k+1)-1}$$

SIK is P -complete X

- ① Sum of weights till m_{2k} is strictly $< m_{2(k+1)-1}$
- ② Sum of weights till $m_{2(k+1)-1}$
 $=$ (sum of weights till m_{2k}) $+ m_{2(k+1)-1}$
 $< 2 \times 4^{2(k+1)-1} < 3 \times 4^{2(k+1)-1} = m_{2(k+1)-0.5}$
- ③ Sum of weights till $m_{2(k+1)-0.5}$
 $=$ (sum of weights till m_{2k}) $+ m_{2(k+1)-1} + m_{2(k+1)-0.5}$
 $< 2 \times 4^{2(k+1)-1} + 3 \times 4^{2(k+1)+1}$
 $= 4^{2(k+1)} + 4^{2(k+1)-1} < m_{2(k+1)}$

SIK is P -complete XI

Theorem

$NANDCVP \leq_1 \text{Super Increasing Knapsack}$

Theorem

Super Increasing Knapsack Problem is P -complete.

Polynomial Iterated Mod Problem

Introduction

Definition (Polynomial Iterated Mod Problem)

Given univariate polynomials $a(x), b_1(x), \dots, b_n(x)$ over a field \mathbb{F} compute the residue

$$((\dots ((a(x) \bmod b_1(x)) \bmod b_2(x)) \dots) \bmod b_{n-1}(x)) \bmod b_n(x)$$

- A polynomial mod can't test for two bits

$$(10)_2 \bmod (11)_2 = (10)_2 \text{ but } (x^2 + 0x) \bmod (x^2 + x) = 0x^2 - x$$

Theorem

Polynomial Iterated Mod Problem is in P

Lower Triangular Matrix Inversion

Theorem ([Hel74],[Hel78])

For any field \mathbb{F} , lower triangular matrix inversion is in Arithmetic – NC

Theorem ([BvzGH82],[BCP84])

Lower triangular matrix inversion is in NC over finite fields and \mathbb{Q}

Reduction I

Given $a(x), b_1(x), \dots, b_n(x)$ over \mathbb{F} .

$b_0(x) = r_0(x) = a(x)$ and $d_i = \deg b_i(x)$ for all $0 \leq i \leq n$.

Assume $d_0 \geq d_1 > \dots > d_n$

$$\begin{aligned} a(x) &= q_1(x)b_1(x) + r_1(x) \\ &= q_1(x)b_1(x) + q_2(x)b_2(x) + r_2(x) \\ &\vdots \\ &= q_1(x)b_1(x) + \dots + q_n(x)b_n(x) + r_n(x) \end{aligned}$$

$r_{i-1}(x) = q_i(x) \cdot b_i(x) + r_i(x)$ with $\deg r_i < \deg b_i = d_i$ or $r_i = 0$

Reduction II

The coefficient of x^j in $a(x), b_i(x), q_i(x), r_i(x)$ are $a_j, b_{i,j}, q_{i,j}, r_{i,j}$.

- $\deg q_1 = d_0 - d_1, \deg q_i \leq d_{i-1} - d_i - 1$
- Compare the coefficients of x^j in both direction.
- $(d_0 + 1) \times (d_0 + 1)$ matrix M . Denote the variable matrix for coefficients of q_i and r_n as X

Reduction III

$d_0 - i$ -th entry of MX is coefficient of degree i . $d_k \leq i < d_{k-1}$.

$r_n(x) + \sum_{i=K+1}^n q_i(x)b_i(x)$ doesn't take part in coefficient of x^i .

$$i = d_k + (d_{k-1} - d_k - 1 - (d_{k-1} - 1 - i)) = d_k + (i - d_k)$$

Can't go lower $(d_{k-1} - d_k - 1 - (d_{k-1} - 1 - i))$ for coefficient of q_k

$$d_0 - i = (d_0 - d_1 + 1) + (d_1 - d_2) + \cdots (d_{k-2} - d_{k-1}) + (d_{k-1} - 1 - i)$$

So M has at $(d_0 - i, d_0 - i)$ th entry b_{k,d_k} and after that all entries are 0 in that row. Hence M is lower triangular.

Matrix is non-singular since the diagonal entries are the leading coefficients of $b_i(x)$

Reduction IV

We need to inverse M which is in *Arithmetic* – NC for general fields and for finite field and \mathbb{Q} in NC. So we have

Theorem

Iterated Polynomial Mod Problem is in NC for finite field and \mathbb{Q} and in Arithmetic – NC for general field.

Thank You!

References I

- [BCP84] A Borodin, S Cook, and N Pippenger.
Parallel computation for well-endowed rings and
space-bounded probabilistic machines.
Inf. Control, 58(1–3):113–136, jul 1984.
- [BvzGH82] Allan Borodin, Joachim von zur Gathen, and John
Hopcroft.
Fast parallel matrix and gcd computations.
In *Proceedings of the 23rd Annual Symposium on
Foundations of Computer Science*, SFCS '82, page 65–71,
USA, 1982. IEEE Computer Society.
- [CG90] Benny Chor and Oded Goldreich.
An improved parallel algorithm for integer GCD.
Algorithmica, 5(1):1–10, 1990.

References II

- [GJ90] Michael R. Garey and David S. Johnson.
Computers and Intractability; A Guide to the Theory of NP-Completeness.
W. H. Freeman & Co., USA, 1990.
- [Hel74] Don Heller.
A determinant theorem with applications to parallel algorithms.
SIAM J. Numer. Anal., 11(3):559–568, jun 1974.
- [Hel78] Don Heller.
A survey of parallel algorithms in numerical linear algebra.
SIAM Rev., 20(4):740–777, oct 1978.
- [KR89] Howard J. Karloff and Walter L. Ruzzo.
The iterated mod problem.
Information and Computation, 80(3):193–204, 1989.

References III

- [Lad75] Richard E. Ladner.
The circuit value problem is log space complete for p.
SIGACT News, 7(1):18–20, jan 1975.