Soham Chatterjee
Email: sohamc@cmi.ac.in
Course: Parallel Algorithms and Complexity

Assignment - 3
Roll: BMC202175
Date: August 24, 2023

Problem 1

Show that $FracPowering \leq_{cd} Powering$.

Solution: Let $0 < x < 2^n$, $0 < y < 2^n$ and k are the inputs. We have to calculate $\left\lfloor \frac{x^k}{y^k} \right\rfloor$. We first calculate x^k and y^k in parallel. Let $2^{j-1} \leq y^k < 2^j$. Let $u = 1 - 2^{-j}y^k \implies y^{-k} = \dfrac{2^{-j}}{1-u}$. Therefore we also get $|u| \leq \frac{1}{2}$. Now

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k$$

Since we cannot do infinite sum we will do our best approximation. Take $\widetilde{y^{-1}} = 2^{-j}\sum\limits_{k=0}^{n-1} u^k$. Now

$$\left| y^{-k} - \widetilde{y^{-k}} \right| \leq 2^{-j}\sum_{k=n}^{\infty} |u|^k \leq 2^{-j}\sum_{k=n}^{\infty}\left(\frac{1}{2}\right)^k = 2^{-j}\frac{1}{2^n}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2^{-j}\frac{2}{2^n} \leq 2^n$$

Hence

$$\left| x^k y^{-k} - x^k\widetilde{y^{-k}} \right| \leq |x|2^{-n} \leq 1$$

Hence $x^k y^{-k}$ and $x^k\widetilde{y^{-k}}$ differs at most at the last bit. So we can multiply y^k with $x^k\widetilde{y^{-k}}$ and check for which one it becomes x^k and output accordingly. Hence $FracPowering \leq Powering$

$\square$

Problem 2

In Mahajan Vinay's Paper prove that

$$\sum_{p:s\rightsquigarrow t_+} w(p) - \sum_{p:s\rightsquigarrow t_-} w(p) = \sum_{\mathcal{W}:\text{ Clow Sequence}} sgn(\mathcal{W})w(\mathcal{W})$$

Solution: We will find a bijection between $s \rightsquigarrow t_+$ paths and the clow sequences of positive sign. And similarly bijection between $s \rightsquigarrow t_-$ paths and the clow sequences of negative sign.

The edges of the last level as weight 1 and it takes care of the parity of n. Hence it is enough to show that the clow sequences with an even number of components correspond to $s \rightsquigarrow q_+$ paths and similalry clow sequences with an odd number of components correspond to $s \rightsquigarrow q_-$ paths. Let $\mathcal{W} = (C_1, C_2, \dots, C_{2k})$ be a clow sequence. Let h_i is the head of C_i and n_i is the number of edges in clows $C_1, \dots, C_{i-1}$. We will show a path from s to q_+ in H_A. The path will go through $[p, h_i, h_i, n_i,]$ where if i is odd $p = 0$ and otherwise $p = 1$.

Now from s we can go to $[0, h_1, h_1, 0]$. Now let the path has reached $[p, h_i, h_i, n_i]$. Suppose $C_i = (h_i, v_1, \dots, v_{k-1})$ a closed walk of length k. From $[p, h_1, h_i, n_i]$ H_A has a path through the verteices $[p, h_i, v_1, n_i + 1], \dots, [p, h_i, v_{k-1}, n_i + (k-1)]$ and then $[\overline{p}, h_{i+1}, h_{i+1}, n_i + k] = [\overline{p}, h_{i+1}, h_{i+1}, n_{i+1}]$.

At the last clow, starting from $[1, h_{2k}, h_{2k}, n_{2k}]$, H_A will have a path tracing out the vertices of clow C_{2k} and in the end a transition to q_+. The weight of the path is identical to the weight of the clow sequences.

Now p be an $s \rightsquigarrow q_+$ path in H_A. In the sequences of vertices visited in the path the second component of the vertex labels is monotonically non-decreasing. Suppose it takes m distinct values $h_1, \dots, h_m$. Also the first component changes exactly when the second component does. It is 0 at h_1 and 1 at h_m (to allow an edge to q_+). So m must be even. Consider the maximal segment of the path with second component h_i. The third components along this segment constitute a clow with leader h_i in G_A. When this clow is completely traversed a new clow with a larger head must be started and the parity of number of components must change. But this is precisely modelled by the edges of H_A. Therefore p corresponds to a clow sequence with an even number od components in G_A.

Similarly we get a bijection between the paths from $s \rightsquigarrow q_-$ and clow sequences with an odd number of componentsm, preserving weights. Hence we get

$$\sum_{p:s \rightsquigarrow t_+} w(p) - \sum_{p:s \rightsquigarrow t_-} w(p) = \sum_{\mathcal{W}\text{: Clow Sequence}} sgn(\mathcal{W})w(\mathcal{W})$$

$\square$