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Course: Complex Analysis

Assignment - 1
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Problem 1

Find the general analytic function $f = u + iv$, such that $u = x^2 - y^2$.

Solution: Given that $u = x^2 - y^2$. Then $\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = -2y$. Since the function is analytic u, v follows the Cauchy Riemann Equations. Hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $\frac{\partial v}{\partial x} = 2x$ and $\frac{\partial v}{\partial y} = 2y$. Now since $\frac{\partial v}{\partial x} = 2y$ we can assume $v = 2xy + g(y)$ where g is some real valued function. But then $\frac{\partial v}{\partial y} = 2x$ implies that $g'(y) = 0$ hence g is some constant function. Hence $v = 2xy + c$ where $c \in \mathbb{R}$ some constant. Hence

$$f(x, y) = x^2 - y^2 + i(2xy + c) == x^2 - y^2 + 2ixy + ic = (x + iy)^2 + ic \iff f(z) = z^2 + ic$$

$\square$

Problem 2

Let $f(x, y) = (u(x, y), v(x, y))$be a function defined on an open set U of the plane, and taking values in the plane. We may think of f as a complex valued function defined on the open subset U of the complex numbers in the obvious way. Show that f is differentiable as a complex valued function at a point p of U if and only if the total derivative of f commutes with multiplication by complex numbers i. $T(ab) = bT(a)$

Solution: Total derivative of f at p will be the matrix

$$T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Let $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Since $M^2 = T_2$ one can treat this matrix M as the $\sqrt{-1} = i$. Now we define a ring homomorphism $\varphi : \mathbb{C} \to M_2(\mathbb{R})$ where

$$\varphi(a) = aI_2 \ \forall \ a \in \mathbb{R} \qquad \varphi(i) = M$$

Therefore for any $a + ib \in \mathbb{C}$

$$\varphi(a + ib) = \varphi(a) + \varphi(ib) = aI_2 + \varphi(i)\varphi(b) = aI_2 + MbI_2 = aI_2 + bM = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Now if $a + ib \in \ker(\varphi)$ then

$$\varphi(a + ib) = 0 \implies \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 0 \implies a = b = 0$$

Hence $\ker \varphi = \{0\}$. By First Isomorphism Theorem $\mathbb{C} \cong \Im(\varphi)$

Let f is differentiable at p. Hence f follows the Cauchy Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

. Hence Hence total derivative of f commutes with complex multiplication is equivalent to showing T commutes with M.

$$TM = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \end{bmatrix}$$

and

$$MT = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial v}{\partial x} & -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$$

Therefore $TM = MT$.

Let total derivative of f commutes with complex multiplication. Hence $TM = MT$. Therefore

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence f is differentiable at p and it follows the Cauchy Riemann Equations. Hence f is complex differentiable at p. mat

$\square$

Problem 3

Prove Cauchy's inequality: Let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ be two complex vectors, then

$$(a \cdot b)^2 \leq \|a\|^2 \|b\|^2$$

where $(a \cdot b)$ is the scalar product of vectors.

Solution: Consider the vector $a + \lambda b$. Now since for any vector v, $\sqrt{(v \cdot v)} = \|v\| \geq 0$ we have $\sqrt{((a + \lambda b) \cdot (a + \lambda b))} \geq 0$. Now

$$\begin{aligned} ((a + \lambda b) \cdot (a + \lambda b)) &= (a \cdot (a + \lambda b)) + \lambda(b \cdot (a + \lambda b)) \\ &= (a \cdot a) + \lambda(a \cdot b) + \lambda(b \cdot a) + \lambda^2 (b \cdot b) \\ \|a\|^2 + 2\lambda(a \cdot b) + \lambda^2 \|b\|^2 \end{aligned}$$

Since $((a + \lambda b) \cdot (a + \lambda b)) \geq 0$ the discriminant of the polynomial, $p(\lambda) = \|a\|^2 + 2\lambda(a \cdot b) + \lambda^2 \|b\|^2$ is non-positive. Hence

$$4(a \cdot b)^2 \leq 4\|a\|^2 \|b\|^2 \iff (a \cdot b)^2 \leq \|a\|^2 \|b\|^2$$

$\square$

Problem 4 Ahlfors Exercise 2.1 Problem 1

If $g(w)$ and $f(z)$ are analytic functions, show that $g(f(z))$ is also analytic.

Solution: Since f is analytic

$$\lim_{h \to 0} \frac{|f(z+h) - f(z) - f'(z)h|}{|h|} = 0$$

and Let $f(z) = b$ then $g'(f(z)) = g'(b)$ then

$$\lim_{h \to 0} \frac{|g(b+k) - g(b) - g'(b)k|}{|k|} = 0$$

Let

$$\alpha(h) = f(z+h) - f(a) - f'(z)h \qquad\qquad \epsilon(h) = \frac{|\alpha(h)|}{|h|} \to 0 \text{ as } h \to 0$$

$$\beta(k) = g(b+k) - g(b) - g'(b)k \qquad\qquad \eta(k) = \begin{cases} \frac{|\beta(k)|}{|k|} \to 0 \text{ as } k \to 0 \\ 0 \text{ when } k = 0 \end{cases}$$

η is continuous at $k = 0$. We want to show that

$$\lim_{h\to 0} \frac{|g(f(z+h)) - g(f(z)) - g'(b)f'(z)h|}{|h|} = 0 \iff \lim_{k\to 0} \frac{|g(b+k) - g(b) - g'(b)f'(z)h|}{|h|} = 0$$

where $f(z+h) = b + k \iff k = f(z+h) - f(z)$. We have taken a specific value of k depending on h. g'(b)o now k is a function of h. Hence $f'(z)h = f(z+h) - f(z) - \alpha(h) = k - \alpha(h)$

$$\begin{aligned} & g(b+k) - g(b) - g'(b)f'(z)h \\ =& g(b+k) - g(b) - g'(b)(k - \alpha(h)) \\ =& g(b+k) - g(b) - g'(b)k + g'(b)\alpha(h) \end{aligned}$$

Therefore

$$\frac{|g(b+k) - g(b) - g'(b)f'(z)h|}{|h|} \leq \overbrace{\frac{|g(b+k) - g(b) - g'(b)k|}{|h|}}^{\substack{\beta(k) \\ \|}} + \frac{|g'(b)\alpha(h)|}{|h|}$$

want to bound each of these separately

$$\frac{|g'(b)\alpha(h)|}{|h|} = |g'(b)|\frac{|\alpha(h)|}{|h|} \to 0$$

Now how to bound the first term. In the first term $\frac{|\beta(k)|}{|h|} = |\eta(k)|\frac{|k|}{|h|}$. Now

$$\begin{aligned} & k = f'(z)h + \alpha(h) \\ \implies & |k| = |f'(z)h| + |\alpha(h)| \\ \implies & \frac{|k|}{|h|} \leq \frac{|f'(z)h|}{|h|} + \frac{|\alpha(h)|}{|h|} = \frac{|f'(z)||h|}{|h|} + \frac{|\alpha(h)|}{|h|} = |f'(z)| + \frac{|\alpha(h)|}{|h|} \end{aligned}$$

Hence

$$\frac{|\beta(k)|}{|k|} = |\eta(k)|\frac{|k|}{|h|} \leq \eta(k)\left[|f'(z)| + \frac{|\alpha(h)|}{|h|}\right]$$

As $h \to 0$ $|f'(z)| + \frac{|\alpha(h)|}{|h|} \to |f'(z)| + 0$ which is finite. And as $h \to 0$, $k \to 0 \implies |\eta(k)| \to 0$ because η is continuous at 0. Hence $g \circ f$ is differentiable at z $\forall$ $z \in \mathbb{C}$. Therefore $g \circ f$ is analytic.

□

Problem 5 Ahlfors Exercise 2.1 Problem 2

Verify Cauchy-Riemann's equations for the functions z^2 and z^3

Solution:

$$z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u_1} + \underbrace{2xy}_{v_1} i$$

$$z^3 = (x+iy)^3 = \left(x^2 - y^2 + 2xyi\right)(x+yi) = \left[\underbrace{x^3 - 3xy^2}_{u_2}\right] + \left[\underbrace{3x^2y - y^3}_{v_2}\right] i$$

For z^2

$$\frac{\partial u_1}{\partial x} = 2x = \frac{\partial v_1}{\partial y} \qquad \frac{\partial u_1}{\partial y} = -2y = -\frac{\partial v_1}{\partial y}$$

and for z^3

$$\frac{\partial u_2}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v_2}{\partial Y} \qquad \frac{\partial u_2}{\partial y} = -6xy = -\frac{\partial v_2}{\partial x}$$

Hence both the functions follow the Cauchy-Riemann's Equations

□

Problem 6 Ahlfors Exercise 2.1 Problem 3

Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

Solution: Solution: Let $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$. Then,

$$\frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2 \qquad \frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2$$
$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by \qquad \frac{\partial^2 u}{\partial y^2} = 2cx + 6dy$$

So, for u to be harmonic, it must be that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2(3a + c)x + 2(b + 3d)y = 0$$

evidently, this occurs if and only if $c = -3a$ and $b = -3d$. Hence

$$u(x, y) = ax^3 - 3dx^y - 3axy^2 + dy^3$$

Now, to find the conjugate harmonic polynomial v it must be that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$. Using the first relation, we find that

$$v = -\int bx^2 + 2cxy + 3dy^2 dx = \int 3dx^2 + 6axy - 3dy^2 dx = dx^3 + 3ax^2y - 3dxy^2 + \varphi(y)$$

where we have substituted the values of c and b found, and $\varphi(y)$ is some function of y. Differentiating this with respect to y gives

$$3ax^2 - 6dxy + \varphi'(y) = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2 = 3ax^2 - 6dxy - 3ay^2.$$

It follows that $\varphi'(y) = -3ay^2$ so that $\varphi(y) = -ay^3 + C$. In total,

$$v = dx^3 + 3ax^2y - 3dxy^2 - ay^3 + C$$

for a real constant C. The corresponding analytic function is

$$f(z) = \left[ax^3 - 3dx^2y - 3axy^2 + dy^2\right] + i\left[dx^3 + 3ax^2y - 3dxy^2 - ay^3 + C\right]$$

The formal method states that we can construct $f(z)$ by

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \overline{f}(0, 0) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0, 0) + Ci$$

where C is a real constant. Reducing this gives

$$\begin{aligned} f(z) &= 2\left(a\left(\frac{z}{2}\right)^3 - 3d\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) - 3a\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2 + d\left(\frac{z}{2i}\right)^3\right) + Ci \\ &= \frac{1}{4}\left(az^3 - 3dz^2\left(\frac{z}{i}\right) - 3az\left(\frac{z}{i}\right)^2 + d\left(\frac{z}{i}\right)^3\right) + Ci \\ &= (a + di)z^3 + Ci. \end{aligned}$$

Expanding out the last expression recovers our formula for $f(z)$ deduced from integration

□

Problem 7 Ahlfors Exercise 2.1 Problem 4

Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Solution: If $|f(z)| = 0 \ \forall \ z \in \mathbb{C}$ then we have $f(z) = 0$. Now let $|f(z)| = c > 0 \ \forall \ z \in \mathbb{C}$. Let $f = u + iv$ then $|f(z)| = \sqrt{u^2(x, y) + v^2(x, y)} = c$. Therefore

$$\frac{\partial}{\partial x}(u^2 + v^2) = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$
$$\frac{\partial}{\partial y}(u^2 + v^2) = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$$

We can write this in matrix form that

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now if $\begin{bmatrix} u & -v \\ v & u \end{bmatrix}$ is not invertible then determinant$=u^2+v^2 = 0$ which is not possible then $\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$Hence all the partial derivatives of f is 0. So $f'(z) = 0$ or $f(z) = c$

□

Problem 8 Ahlfors Exercise 2.1 Problem 5

Prove rigorously that the functions $f(z)$ and $\overline{f(z)}$ are simultaneously analytic.

Solution: Suppose first that $f(z)$ is analytic. Let $g(z) = \overline{f(\bar{z})}$. Let u_f, v_f and u_g, v_g denote the real and imaginary parts of f and g respectively. Then, these are related by

$$u_g(x, y) = u_f(x, -y) \quad v_g(x, y) = -v_f(x, -y).$$

Since f is analytic, u_f and v_f have continuous partial derivatives - so does u_g, v_g have. Next we verify the Cauchy-Riemann equations:

$$\frac{\partial u_g}{\partial x} = \frac{\partial u_f}{\partial x} \qquad \frac{\partial v_g}{\partial x} = -\frac{\partial v_f}{\partial x}$$
$$\frac{\partial u_g}{\partial y} = -\frac{\partial u_f}{\partial y} \qquad \frac{\partial v_g}{\partial y} = \frac{\partial v_f}{\partial y}$$

Since f is analytic, u_f and v_f satisfy the Cauchy-Riemann equations. Thus,

$$\frac{\partial u_g}{\partial x} = \frac{\partial u_f}{\partial x} = \frac{\partial v_f}{\partial y} = \frac{\partial v_g}{\partial y}$$
$$\frac{\partial u_g}{\partial y} = -\frac{\partial u_f}{\partial y} = \frac{\partial v_f}{\partial x} = -\frac{\partial v_g}{\partial y}$$

Thus, u_g and v_g satisfy the Cauchy-Riemann equations. It follows that $g(z) = \overline{f(\bar{z})}$ is analytic. Finally, observe that $\overline{g(\bar{z})} = f(z)$ so that the above proof works to show the converse, with the roles of f and g reversed.

□

Problem 9 Ahlfors Exercise 2.1 Problem 6

Prove that the functions $u(z)$ and $u(\overline{z})$ are simultaneously harmonic.

Solution: Since u is the real part of $f(z), u(z) = u(x,y)$ where $z = x + iy$. Suppose $u(z)$ is harmonic. Then $u(z)$ satisfies Laplace equation.

$$\Delta u(z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Now, $u(\bar{z}) = u(x, -y)$ where $\frac{\partial^2}{\partial x^2}u(\bar{z}) = \frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}u(\bar{z}) = \frac{\partial^2 u}{\partial y^2}$ so

$$\Delta u(\overline{z}) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since $u(z)$ is harmonic, $\Delta u(\overline{z}) = 0$ so it follows that $u(\overline{z})$ is harmonic as well. Now observe that $u(z) = u(\overline{\overline{z}})$. Hence above proof works to show the converse.

$\square$