Complexity Theory I	Jan – May, 2023

Lecture 1: Introduction

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1 References

Books:

- Introduction to the Theory of Computation by Michael Sipser [Sip13]
- Computational Complexity: A Modern Approach by Sanjeev Arora and Boaz Barak [AB09]
- Computational Complexity: A Conceptual Perspective by Oded Goldreich [Gol08]
- Mathematics and Computation: A Theory Revolutionizing Technology and Science by Avi Wigderson [Wig19]

Lecture-Notes:

- Madhu Sudan: 2018 and 2021
- Venkat Guruswamy: 2011 and 2009
- Luca Trevisan: 2015, 2014, 2012, 2010, 2009, 2008, 2007, 2005
- Salil Vedhan: Notes
- Prahlad Harsha: 2021, 2020, 2018, 2014, 2013, 2012, 2011
- Jay Kumar Radhakrishnan: 2004

2 Basic Classes

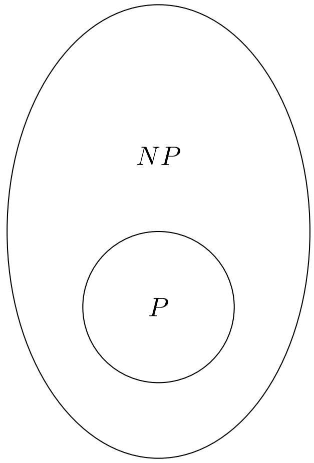
Note:-
All the classes in this course are subsets of decidable problems

We know for any problem P a language L_P is associated.
$\boldsymbol{P}$:= Class of problems that can be decided in deterministic polynomial time
$\boldsymbol{NP}$:= Class of problems for which witness can be verified in deterministic polynomial time
Now it is obvious that P is contained in NP. But we don't know if $P = NP$ or not.

Open Question 1

$$P \subsetneq NP$$

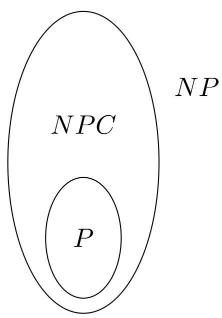
But it is believed that $P \neq NP$

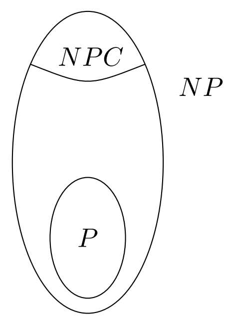


This is called the Complexity Zoo. As the course will further progress more and more things will be added

Example 1 *($NP - complete(NPC)$)*
SAT, $3COL$, VC, $HAM\ldots.$
$3COL :=$ Given a graph G, check whether the graph is $3-$colorable
[MRG79] has a lot of examples of $NP - complete$ problems

Now there was question that what is the set $NP \setminus P$ if $NP \neq P$. Which one of the following true



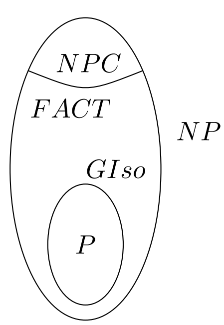


Ladner proved in [Lad75] that there exists an infinite hierarchy of problems sitting from P to $NP-complete$, $L \in NP \setminus (P \cup NPC)$.

But we know very less natural problems in $NP \setminus (P \cup NPC)$. Eg: Factoring (it is believed), Graph Isomorphism or $GIso$ (It was believed that $GIso$ is in $L \in NP \setminus (P \cup NPC)$ but Laci Babai in [Bab16] brought it to very closed to P) .

Primality was used to be believed in $NP \setminus (P \cup NPC)$ but later it was discovered that it is in P in [MA04]
Graph Isomorphism ($GIso$):= $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$. The question is if there exists $\phi : V_1 \to V_2$ such that $(u, v(\in E_1 \iff (\phi(u_1), \phi(u_2)) \in E_2)$.

Now the picture is



3 Space/Memory

Now comes another question. Are the problems in P "memory" efficient?.

(G, E) is an undirected graph. If $u, v \in V$ then is $u \rightsquigarrow v$ reachable? Any search algorithm like DFS, BFS works $\approx O(n)-$memory $\xrightarrow{\text{2004}-\text{Reingold [Rei08]}} O(\log n)-$memory.

4 Randomness

Now we allow the Turing Machine to toss a coin to take decisions and then walk accordingly. It contains P as we can say stop tossing. Hence $P \subseteq BPP$

Here randomized algorithms are used to solve problems. Eg. Miller Robin Ptimality Test.

$\boldsymbol{BPP}$ **:=** Class of problems which can be solved in polynomial time with the power of randomness

Open Question 2

$$P = BPP$$

But it is believed that $P = BPP$. This gives a philosophical question that can every randomized polynomial time algorithm be converted to a polynomial time deterministic algorithm with comparable time?

Note:-

BPP has no such *comlete* problems like $NP-complete$ problems in NP class. We have to use pseudorandom generator. So the task is to build a powerfull pseudoranddom generator

5 Lower Bounds

One such question on lower bound to show $P \neq NP$ is

Question 1

Let ψ be a boolean formula and $|\psi| = n$. Prove that if $\psi \in SAT$ it requires at least super linear time $(n^{1.1})$

We don't even know that

We can put some constraints or cut down some power of turing machine to show some lower bounds.

6 Diagonalization

Theorem 1

$DTIME(n^2) \subsetneq DTIME(n^3)$

People used to believe that these kind of ideas can be used to prove $P \neq NP$. But Natural Proofs [RR97] crushed this idea.

References

[AB09] Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 1st edition, 2009.

[Bab16] László Babai. Graph isomorphism in quasipolynomial time. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pages 684–697, 2016.

[Gol08] Oded Goldreich. *Computational Complexity: A Conceptual Perspective*. Cambridge University Press, 2008.

[Lad75] Richard E. Ladner. On the structure of polynomial time reducibility. *J. ACM*, 22(1):155–171, jan 1975.

[MA04] Nitin Saxena Manindra Agrawal, Neeraj Kayal. Primes is in p. *Annals of Mathematics*, 160 (2):781–793, 2004.

[MRG79] David S. Johnson Michael R. Garey. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Series of Books in the Mathematical Sciences. W. H. Freeman, first edition edition, 1979.

[Rei08] Omer Reingold. Undirected connectivity in log-space. *J. ACM*, 55(4), sep 2008.

[RR97] Alexander A Razborov and Steven Rudich. Natural proofs. *Journal of Computer and System Sciences*, 55(1):24–35, 1997.

[Sip13] Michael Sipser. *Introduction to the Theory of Computation*. Cengage India Private Limited, third edition, 2013.

[Wig19] Avi Wigderson. *Mathematics and Computation: A Theory Revolutionizing Technology and Science*. Princeton University Press, 2019.