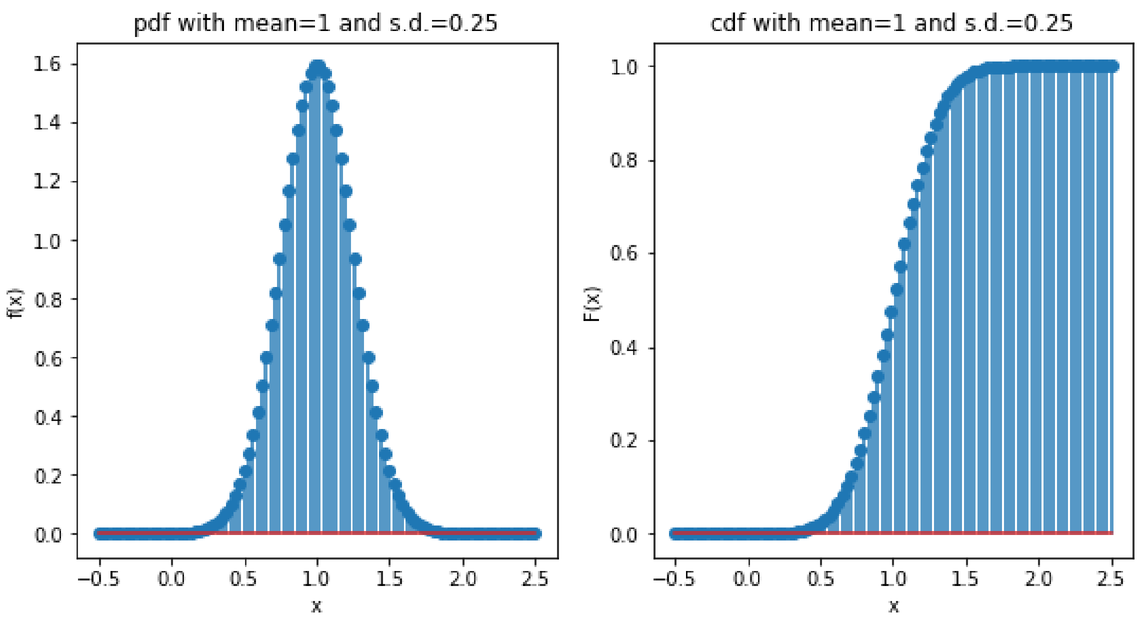
In [1]:

```
import numpy as np
import matplotlib.pyplot as pl
%matplotlib inline

from scipy import stats as st
sd为标准差
mean为均值
概率密度pdf
累计分布cdf
```

In [2]:

```
import numpy as np
import matplotlib.pyplot as pl
%matplotlib inline

from scipy import stats as st
mean, sd = 1, 0.25
x = np.linspace(mean - 6*sd, mean + 6*sd, 100)
f = st.norm.pdf(x=x, loc=mean, scale=sd)
F = st.norm.cdf(x=x, loc=mean, scale=sd)

pl.figure(figsize=(10,5))
pl.subplot(121)
pl.stem(x, f); pl.xlabel('x'); pl.ylabel('f(x)'); pl.title('pdf with mean={} and
 s.d.={}'.format(mean, sd))
pl.subplot(122)
pl.stem(x, F); pl.xlabel('x'); pl.ylabel('F(x)'); pl.title('cdf with mean={} and
 s.d.={}'.format(mean, sd))
pl.show()
```



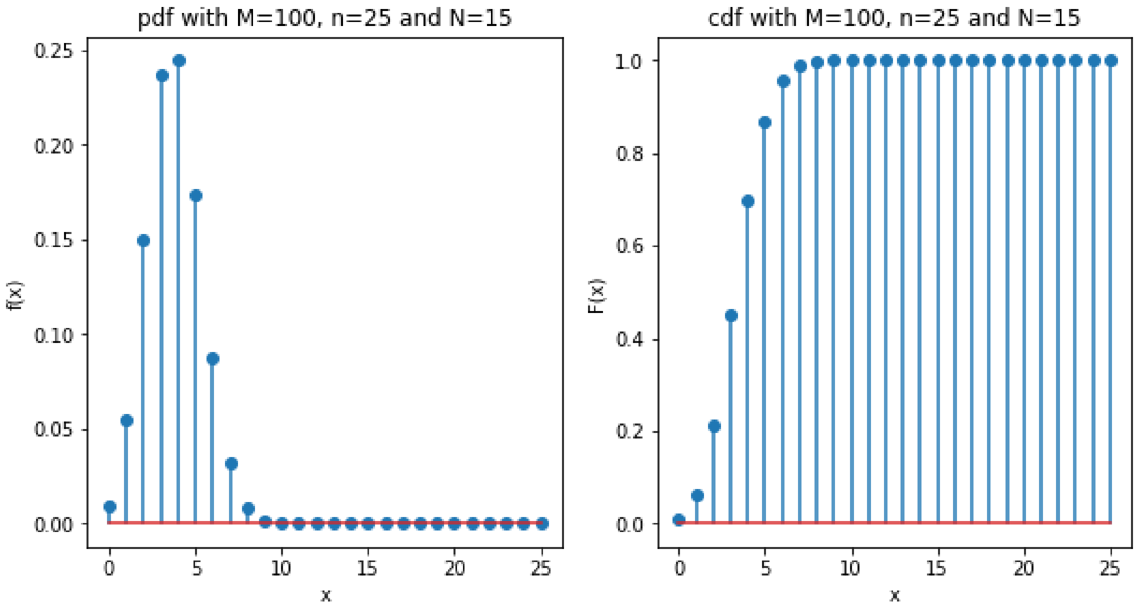
Exercise 1.1.1 Change location and scale parameters (e.g. loc=1, scale=0.25) and observe results.

1.2 Discrete random variables

Exercise 1.2.1 In an urn which contains M=100 different currency bills, n=25 bills are Australian dollars. Plot the pmf and cdf if N=15 bills are randomly drawn *without replacement*.

In [3]:

```
#Answer
M, n, N = 100, 25, 15 #Population, Successes, No of draws
x = np.arange(0, n+1)
f = st.hypergeom.pmf(x, M, n, N)
F = st.hypergeom.cdf(x, M, n, N)

pl.figure(figsize=(10,5))
pl.subplot(121)
pl.stem(x, f); pl.xlabel('x'); pl.ylabel('f(x)'); pl.title('pdf with M={}, n={}
 and N={}'.format(M, n, N))
pl.subplot(122)
pl.stem(x, F); pl.xlabel('x'); pl.ylabel('F(x)'); pl.title('cdf with M={}, n={}
 and N={}'.format(M, n, N))
pl.show()
```



2. Sampling

2.1 Pseudo-random number generators (PRNG)

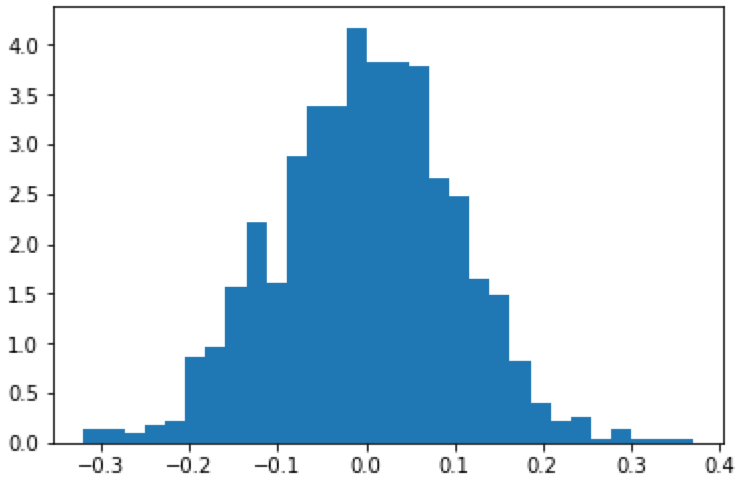
Exercise 2.1.1 Discuss how random numbers are generated in a computer (Turing machine).

2.2 Sampling from a distribution

Normal distribution

In [4]:

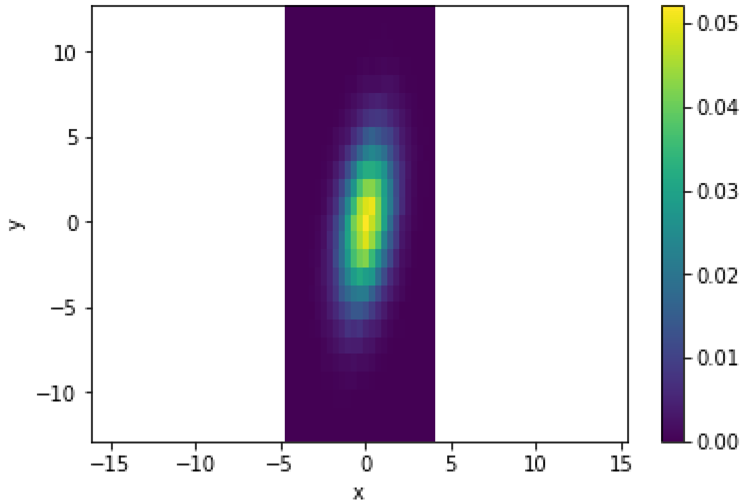
```
mu, sigma = 0, 0.1 # mean and standard deviation
samples = np.random.normal(mu, sigma, 1000) # or use scipy.stats.rvs(size=1000)

count, bins, ignored = pl.hist(samples, bins=30, normed=True)
pl.show()
```



Multivaraiate normal

In [5]:

```
mean = (0, 0)
cov = [[1, 1], [1, 10]]
x, y = np.random.multivariate_normal(mean, cov, 100000).T

#pl.scatter(x, y)
pl.hist2d(x, y, 25, normed=True) #hexbin
pl.xlabel('x'); pl.ylabel('y')
pl.colorbar()
pl.axis('equal')
pl.show()
del x, y
```

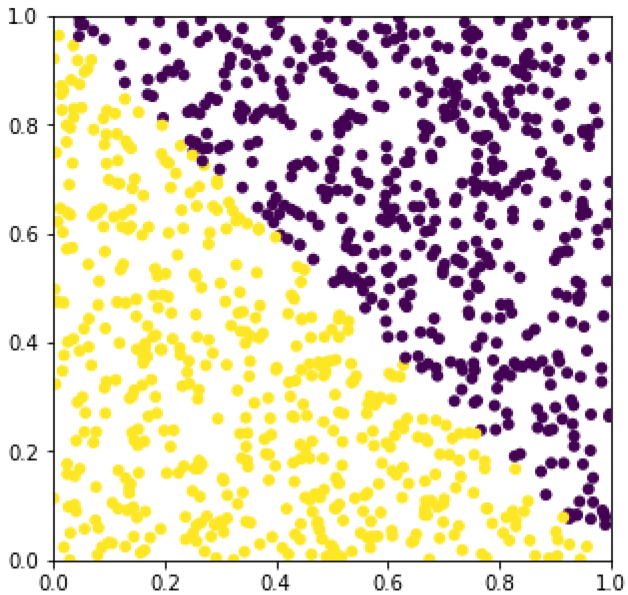


Exercise 2.2.1 Observe the probability distribution by varying mean and covariance.

2.2 Monte Carlo (MC) methods - optional

In [6]:

```
N = 1000
x_samples, y_samples = np.random.random(N), np.random.random(N)

condition = x_samples + y_samples < 1

pl.figure(figsize=(5,5))
pl.scatter(x_samples, y_samples, c=condition, edgecolor='')
pl.axis([0, 1, 0, 1])
pl.show()

print('Number of samples = {}'.format(N))
print('Number of samples that satisfies the condition = {}'.format(np.count_nonzero(condition)))
print('Proportion of samples that satisfies the condition = {}'.format(np.count_nonzero(condition)/N))
```




```
Number of samples = 1000
Number of samples that satisfies the condition = 482
Proportion of samples that satisfies the condition = 0.482
```

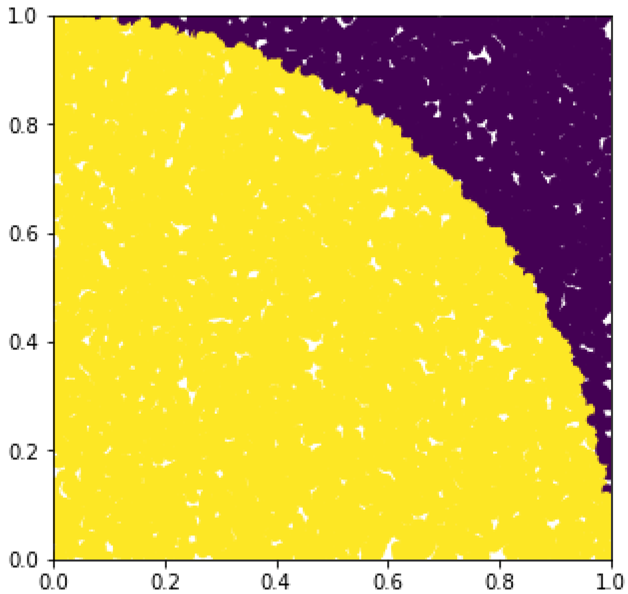
Exercise 2.2.1 Vary the number of samples N and observe results.

Exercise 2.2.2 Use Monte Carlo simulation to estimate the value of π.

Hint: The equation of a circle is $x^2 + y^2 = r^2$ and area is πr^2.

In [7]:

```
#Answer
N = 10000
x_samples, y_samples = np.random.random(N), np.random.random(N)

condition = x_samples**2 + y_samples**2 < 1

pl.figure(figsize=(5,5))
pl.scatter(x_samples, y_samples, c=condition, edgecolor='')
pl.axis([0, 1, 0, 1])
pl.show()

print('Number of samples inside the square = {}'.format(N))
print('Number of samples inside the quarter-circle = {}'.format(np.count_nonzero
(condition)))
print('Value of pi = {}'.format(4*np.count_nonzero(condition)/N ))
```




```
Number of samples inside the square = 10000
Number of samples inside the quarter-circle = 7865
Value of pi = 3.146
```

2.4 Central Limit Theorem (CLT) - optional

Crude idea: The distibution of averaging of N random varaibles, each having a mean μ and variance σ^2, follows a normal distibution with mean μ and variance σ^2/N as N increases.

Theorem - Let $X_1, X_2, \ldots, X_N$ be a sequence of *iid* (independent and identically distributed) random variables (discrete or continuous). Let $E[X_i] = \mu$ and $Var[X_i] = \sigma^2 > 0$ and both are finite (*moment-generating-function* exists). As $N \to \infty$ the sample average $\bar{X}_n = (1/n)\Sigma_{i=1}^N \xrightarrow{a.s.} \mu$.

Laws of Large Numbers:

- Definition - Convergence in probability: A sequence of random variables $X_1, X_2, \ldots, X_N$ *converges in probability* to a random variable X if,
$$\lim_{N\to\infty} P(|X_N - X| < \epsilon) = 1, \forall\epsilon > 0$$
- Theorem - Weak Law of Large Numbers (WLLN): Let $X_1, X_2, \ldots, X_N$ be a sequence of *iid* random variables. Let $\mathrm{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 > 0$ (both are finite). Define $\bar{X}_n = (1/n)\Sigma_{i=1}^N$.
$$\lim_{N\to\infty} P(|\bar{X}_N - \mu| < \epsilon) = 1, \forall\epsilon > 0$$
i.e. $\bar{X}_N$ converges in probability to μ.
- Theorem - Strong Law of Large Numbers (SLLN): Let $X_1, X_2, \ldots, X_N$ be a sequence of *iid* random variables. Let $\mathrm{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 > 0$ (both are finite). Define $\bar{X}_n = (1/n)\Sigma_{i=1}^N$.
$$P(\lim_{N\to\infty} |\bar{X}_N - \mu| < \epsilon) = 1, \forall\epsilon > 0$$
i.e. $\bar{X}_N$ converges *almost surely (a.s.)* (meaning, with probability = 1) to μ.

Note: assumption of finite variance can be relaxed.

Note: In order for SLLN to hold P should converge pointwise (limit is inside P). SLLN $\implies$ WLLN.

In [17]:

```
N = 10
x = np.random.random((N, 1E5)) #np.random.beta(10, 0.5,(N, 1E5)) change N to 100

pl.figure(figsize=(15,5))

Na = 1
pl.subplot(131)
pl.hist(x[:Na, :].mean(0), 100) #let the bin size be 100
#pl.axis([0, 1, 0, None])
pl.title('N = {}'.format(Na))

pl.subplot(132)
Nb = 2
pl.hist(x[:Nb, :].mean(0), 100)
pl.axis([0, 1, 0, None])
pl.title('N = {}'.format(Nb))

pl.subplot(133)
Nc = N
pl.hist(x[:Nc, :].mean(0), 100)
pl.axis([0, 1, 0, None])
pl.title('N = {}'.format(Nc))

pl.show()
del x
```

```
---------------------------------------------------------------------------
TypeError                                 Traceback (most recent call last)
<ipython-input-17-e6df3c9a458b> in <module>()
      1 N = 10
----> 2 x = np.random.random((N, 1E5)) #np.random.beta(10, 0.5,(N, 1E5)) change N to 100
      3 
      4 pl.figure(figsize=(15,5))
      5 

mtrand.pyx in mtrand.RandomState.random_sample()

mtrand.pyx in mtrand.cont0_array()

TypeError: 'float' object cannot be interpreted as an integer
```

3. Information Theory

3.1 Entropy

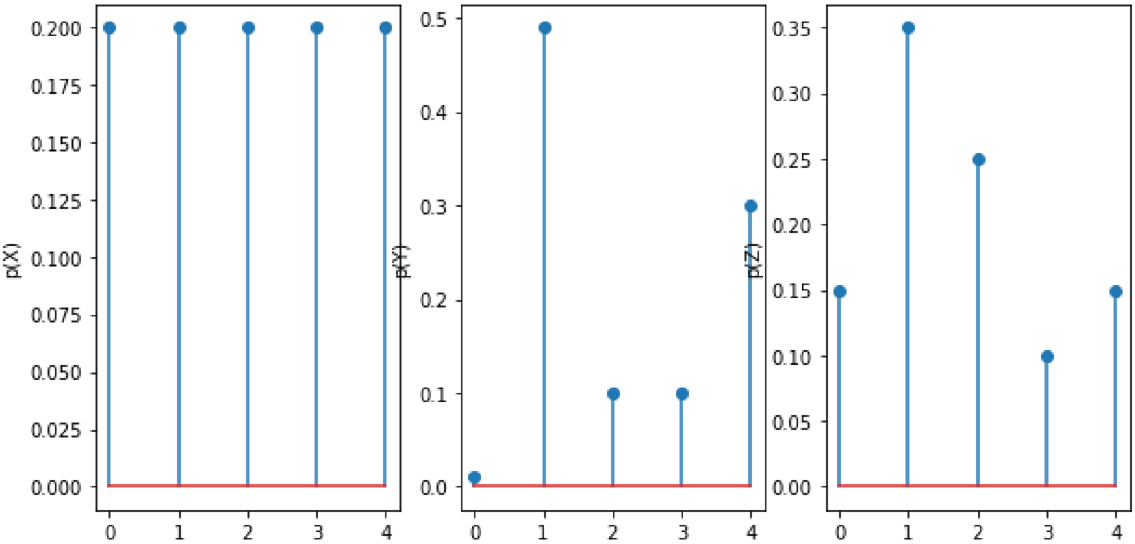
Entropy is a measure of uncertainty.

$$\mathrm{H}[x] := \sum_x p(x) log_n \frac{1}{p(x)} = -\sum_x p(x) log_n \left(p(x)\right)$$

If n=2, the unit of measurement is in bits.

In [9]:

```
#Observe how Entropy changes with the spread of data
X = np.arange(5)
Z = Y = X

pX = np.array([ 0.2, 0.2, 0.2, 0.2, 0.2])
pY = np.array([ 0.01, 0.49, 0.1, 0.10, 0.30])
pZ = np.array([ 0.15, 0.35, 0.25, 0.1, 0.15])

pl.figure(figsize=(10,5))
pl.subplot(131)
pl.stem(X, pX)
pl.ylabel('p(X)')

pl.subplot(132)
pl.stem(Y, pY)
pl.ylabel('p(Y)')

pl.subplot(133)
pl.stem(Z, pZ)
pl.ylabel('p(Z)')

pl.show()

def calc_entropy(p):
    return -np.sum(p*np.log2(p)) #from scipy import stats as st; st.entropy(pX,
 base=2)

print(calc_entropy(pX), calc_entropy(pY), calc_entropy(pZ))
```

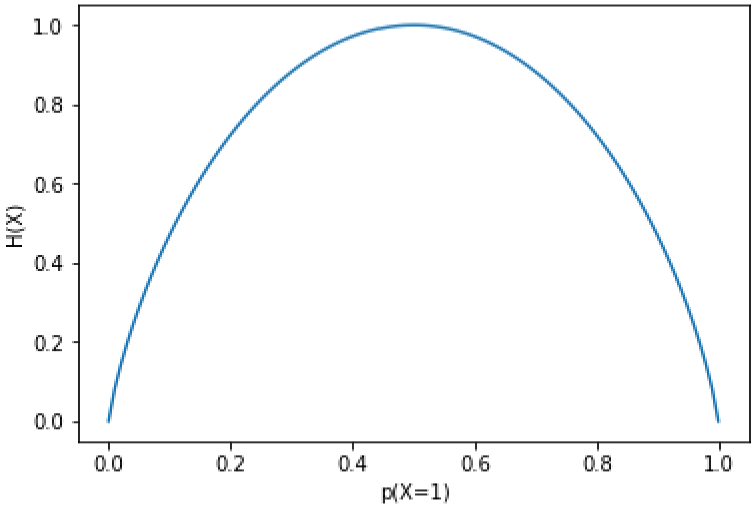



```
2.321928094887362 1.7561955684982449 2.1833830982290134
```

Exercise 3.1.1 Consider the Bernoulli trial of tossing a coin (unfair coin) with event space X. Let $p(X=1)$ be the probability of obtaining a head. Calculate the entopy for $p(X=1) \in [0,1]$ and show that entopy decreases as the uncertainty decreases.

In [10]:

```
eps = np.finfo(float).eps
p = np.linspace(0+eps, 1-eps, 100)
H = np.empty(p.shape)
for i, theta in enumerate(p):
    H[i] = calc_entropy(np.array([theta, 1-theta]))

pl.plot(p, H)
pl.xlabel('p(X=1)'); pl.ylabel('H(X)')
pl.show()
```



3.2 Kullback-Leibler (KL) divergence or Relative Entropy

KL divergence measures how dissimilar two probability distributions are.

$$\mathrm{KL}(p||q) := \sum_x p(x) log_n \frac{p(x)}{q(x)}$$

In [11]:

```
def calc_KL(p, q):
    return np.sum(p*np.log2(p/q)) #from scipy import stats as st; st.entropy(p,
 q, base=2)

print('KL(p||p)', calc_KL(pX, pX)) #Note min(KL) is zero when iff p=q
print('KL(p||q)', calc_KL(pX, pY))
print('KL(q||p)', calc_KL(pY, pX)) # Note KL is not symmetric
```

```
KL(p||p) 0.0
KL(p||q) 0.888836768987672
KL(q||p) 0.5657325263891176
```

3.3 Mutual Information (MI) - optional

MI measures how much knowing one random variable tells about the other (dependence).

$$\mathrm{I}(X;Y) := \sum_{x} \sum_{y} p(x,y) log_n \frac{p(x,y)}{p(x)p(y)}$$

Note: If two random variables are independent, then $p(X,Y) = p(X)p(Y)$ and hence $\mathrm{I}(X;Y) = 0$.

For two events (not random variables), pointwise mutual information (PMI) is defined as,

$$\mathrm{PMI}(x,y) := log_n \frac{p(x,y)}{p(x)p(y)}$$

Exercise 3.3.1 Associated Press Newswire Corpus (1988) contains 44 million words. Individual words "set", "up", "off", "out", "on", "in" and "about" have been found 13046,64601, 20693, 47956, 258170, 739932 and 82319 times respectively. Similarly, hypothetical phrasal verbs "set up", "set off", "set out", "set on", "set in" and "set about" have been found 2713, 463, 301, 162, 795 and 16 respectively. Use PMI to determine which of the hypothetical phrasal verbs are more likely to be commonly used phrasal verbs.

Reference: W.C. Kenneth and H. Patrick. "Word association norms, mutual information, and lexicography". Computational Linguistics. vol.16(1) 1990.

In [12]:

```
#Answer
import numpy as np

verb = np.array([13046])[:,np.newaxis]
preps_list = ['up', 'off', 'out', 'on', 'in', 'about']
preps = np.array([64601, 20693, 47956, 258170, 739932, 82319])[:,np.newaxis]
co_oc = np.array([2713, 463, 301, 162, 795, 16])[:,np.newaxis]

#corpus of 1988 Associated Press newswire

def PMI(p_x, p_y, p_xy, total):
    return np.log2(total*p_xy/(p_x*p_y)) #/ (-np.log2(p_xy/total)) normalization

for i in range(preps.shape[0]):
    PMI_score = PMI(verb[0,:], preps[i, :], co_oc[i, :], 44E6)
    print('PMI of set ' + preps_list[i], PMI_score)

#Note: This corpus is old and most of the articles are written in American Engli
sh by journalists.
```

```
PMI of set up [7.14608473]
PMI of set off [6.23769346]
PMI of set out [4.40387624]
PMI of set on [1.08156844]
PMI of set in [1.85745816]
PMI of set about [-0.60925756]
```

4. Bayes' Rule

$$p(Y=y|X=x) = \frac{p(X=x|Y=y)p(Y=y)}{\sum_{y'} p(X=x|Y=y')p(Y=y')}$$

Exercise 4.1.1

0.4% of a population is having a particular genetic disorder. In order to test the disorder, a person has undergone a medical test which has a **sensitivity** of 80% (if a person has the disorder, the test result will be positive with a probability of 0.8) and a **false alarm** of 10%. If the test is positive, what is the probability of person the having the particular genetic disorder?

Answer

Let $x = 1$ be positive test results and $x = 0$ be negative test results.

Let $y = 1$ be the person has a cancer and $y = 0$ be the person does not have a cancer.

- prior: $p(Y = 1) = 0.004$ and $p(Y = 0) = 1 - 0.004 = 0.996$
- sensitivity: $p(X = 1|Y = 1) = 0.8$
- false positive/alarm: $p(X = 1|Y = 0) = 0.1$

$$p(Y = 1|X = 1) = \frac{p(X = 1|Y = 1)p(Y = 1)}{p(X = 1|Y = 1)p(Y = 1) + p(X = 1|Y = 0)p(Y = 0)} = \frac{0.8 \times 0.004}{0.8 \times 0.004 + 0.1 \times 0.996} \approx 3\%$$