Extremes of Stationary Sequences

Solutions 2

February 26, 2023

Theory exercises

Exercise 1:

1)
We are interested in computing

$$\lim_{u\to\infty} \mathbb{P}(X_1 - X_0 \le z \mid X_0 = u) = \lim_{u\to\infty} \mathbb{P}(X_1 \le z + u \mid X_0 = u).$$

From the question statement, we have that

$$\mathbb{P}(X_1 \le z + u \mid X_0 = u) = -T(u)^2\, e^{1/T(u)}\, V_1\left\{T(u), T(z+u)\right\} \exp\left[-V\left\{T(u), T(z+u)\right\}\right].$$

We seek an approximation of $T(u)$, hence of $\log F_L(u)$, as $u \to \infty$. Recall the Mercator series for $\log(1+x) = x - x^2/2 + O(x^3)$. For positive u, this yields that

$$-\log F_L(u) = -\log\{1 - \exp(-u)/2\} = \frac{\exp(-u)}{2} + O(\exp(-2u)) \;\sim \frac{\exp(-u)}{2},\text{ as } u \to \infty.$$

It follows that $T(u) \sim 2\exp(u)$, $u \to \infty$. Hence, as $u \to \infty$, we have that $\mathbb{P}(X_1 \le z + u \mid X_0 = u)$ is asymptotically equivalent to

$$-4\exp(2u)\,\exp\left\{\frac{2}{\exp(u)}\right\} V_1\left\{2\exp(u), 2\exp(u+z)\right\} \exp\left[-V\left\{2\exp(u), 2\exp(u+z)\right\}\right]$$

$$\sim -4\exp(2u)\, V_1\left\{2\exp(u), 2\exp(u+z)\right\} \exp\left[-V\left\{2\exp(u), 2\exp(u+z)\right\}\right]$$

$$= 4\exp(2u)\,\{2\exp(u)\}^{-1-1/\kappa}\left[\{2\exp(u)\}^{-1/\kappa} + \{2\exp(u+z)\}^{-1/\kappa}\right]^{\kappa-1} \times \exp\left[-\left[\{2\exp(u)\}^{-1/\kappa} + \{2\exp(u+z)\}^{-1/\kappa}\right]^{\kappa}\right]$$

$$= \{2\exp(u)\}^{1-1/\kappa}\left[\{2\exp(u)\}^{-1/\kappa} + \{2\exp(u+z)\}^{-1/\kappa}\right]^{\kappa-1} \exp\left[-\left[\{2\exp(u)\}^{-1/\kappa} + \{2\exp(u+z)\}^{-1/\kappa}\right]^{\kappa}\right].$$

Now, as $u \to \infty$,

$$\exp\left[-\left[\{2\exp(u)\}^{-1/\kappa} + \{2\exp(u+z)\}^{-1/\kappa}\right]^{\kappa}\right] \to 1.$$

Hence,

$$\begin{aligned}\mathbb{P}(X_1 \le z + u \mid X_0 = u) &\sim \{2\exp(u)\}^{1-1/\kappa} \left[\{2\exp(u)\}^{-1/\kappa} + \{2\exp(u+z)\}^{-1/\kappa}\right]^{\kappa-1} \\ &= \{2\exp(u)\}^{1-1/\kappa} \{2\exp(u)\}^{-(\kappa-1)/\kappa} \left[1 + \{\exp(z)\}^{-1/\kappa}\right]^{\kappa-1} \\ &\to \left[1 + \exp(-z/\kappa)\right]^{\kappa-1}.\end{aligned}$$

2)
By definition of the process, we have

$$X_1 - X_0 \mid \{X_0 = u\} \xrightarrow{d} \varepsilon_1, \qquad u \to \infty.$$

Let $X_0 = u$. For sufficiently large u, we have (approximately) that

1. $X_1 = X_0 + \varepsilon_1$
2. $X_2 = X_1 + \varepsilon_2 = X_0 + \varepsilon_1 + \varepsilon_2$

   $\vdots$

t. $X_t = X_{t-1} + \varepsilon_t = X_0 + \sum_{i=1}^{t} \varepsilon_i$.

where steps 2 to t are justified because $X_1, \ldots, X_{t-1}$ can be made arbitrarily large by X_0 being large. Hence, for any $t \in \{1, 2, \ldots\}$,

$$X_t - X_0 \mid \{X_0 = u\} \xrightarrow{d} \sum_{i=1}^{t} \varepsilon_i =: Z_t,$$

which gives $Z_t = Z_{t-1} + \varepsilon_t$ where $Z_0 = 0$.

Exercise 2:

1)
We are interested in computing

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{X_1}{X_0^{1-\kappa}} \le z \mid X_0 = u\right) = \lim_{u\to\infty} \mathbb{P}(X_1 \le z u^{1-\kappa} \mid X_0 = u).$$

From the question statement, we have that, for $z > 0$,

$$\mathbb{P}(X_1 \le z x^{1-\kappa} \mid X_0 = u) = 1 - V_1\left(1, \frac{u^\kappa}{z}\right) \exp\left\{u - uV\left(1, \frac{u^\kappa}{z}\right)\right\}.$$

Since

$$V_1(x,y) = -x^{-1-1/\kappa}\left(x^{-1/\kappa} + y^{-1/\kappa}\right)^{\kappa-1},$$

we have that

$$V_1(1, u^\kappa/z) = \left(1 + \frac{u}{z^\kappa}\right)^{\kappa-1}.$$

Hence, for $z > 0$,

$$\mathbb{P}(X_1 \le z x^{1-\kappa} \mid X_0 = u) = 1 - \left(1 + \frac{z^{1/\kappa}}{u}\right)^{\kappa-1} \exp\left\{u - u\left(1 + \frac{z^{1/\kappa}}{u}\right)^{\kappa}\right\},$$

and by the binomial series $(1+x)^\alpha \sim 1 + \alpha x + O(x^2)$ as $x \to 0$, one obtains that as $u \to \infty$,

$$\left(1 + \frac{z^{1/\kappa}}{u}\right)^{\kappa-1} \sim \left\{1 + (\kappa - 1)\frac{z^{1/\kappa}}{u}\right\} \to 1$$

and

$$u - u\left(1 + \frac{z^{1/\kappa}}{u}\right)^{\kappa} \sim u - \left(u + \kappa z^{1/\kappa}\right) = -\kappa z^{1/\kappa},$$

This in turn yields

$$\mathbb{P}(X_1 \le z x^{1-\kappa} \mid X_0 = u) \to 1 - \exp\left(-\kappa z^{1/\kappa}\right), \quad u \to \infty.$$

2)
Due to stationarity, and from subpart 1, we have that for all $t \in \{0, 1, \ldots\}$

$$\frac{X_1}{X_0^{1-\kappa}} \mid \{X_0 = u\} \xrightarrow{d} \varepsilon_1, \qquad u \to \infty,$$

Now, let $X_0 = u$. Then for sufficiently large u, we have (approximately) that

1. $X_1 = \varepsilon_1 X_0^{1-\kappa}$
2. $X_2 = \varepsilon_2 X_1^{1-\kappa} = \varepsilon_2 \left(\varepsilon_1 X_0^{1-\kappa}\right)^{1-\kappa} = \varepsilon_2 \varepsilon_1^{1-\kappa} X_0^{(1-\kappa)^2}$

3. $X_3 = \varepsilon_3 X_2^{1-\kappa} = \varepsilon_3 \left(\varepsilon_2 \varepsilon_1^{1-\kappa} X_0^{(1-\kappa)^2}\right)^{1-\kappa} = \varepsilon_3 \varepsilon_2^{1-\kappa} \varepsilon_1^{(1-\kappa)^2} X_0^{(1-\kappa)^3}$

$\vdots$

t. $X_t = \varepsilon_t X_{t-1}^{1-\kappa} = X_0^{(1-\kappa)^t} \prod_{i=0}^{t-1} \varepsilon_{t-i}^{(1-\kappa)^i}$,

where steps 2 to t are justified because $\kappa < 1$, so that $X_1, \ldots, X_{t-1}$ can be made arbitrarily large (recall that ε_t is a positive random variable). Hence, for any $t \in \{1, 2, \ldots\}$,

$$\frac{X_t}{X_0^{(1-\kappa)^t}} \mid \{X_0 = u\} \xrightarrow{d} \prod_{i=0}^{t-1} \varepsilon_{t-1-i}^{(1-\kappa)^i} = \varepsilon_t \prod_{i=1}^{t-1} \varepsilon_{t-i}^{(1-\kappa)^i} =: Z_t,$$

and

$$\frac{X_{t-1}}{X_0^{(1-\kappa)^{(t-1)}}} \mid \{X_0 = u\} \xrightarrow{d} \prod_{i=0}^{t-2} \varepsilon_{t-1-i}^{(1-\kappa)^i} = \prod_{i=1}^{t-1} \varepsilon_{t-i}^{(1-\kappa)^{i-1}} =: Z_{t-1},$$

which gives, $Z_t = \varepsilon_t Z_{t-1}^{1-\kappa}$ where $Z_0 = 1$.

Exercise 3:

Fitting time-series conditional extreme-value model
See solutions on GitHub.