Extremes of Stationary Sequences

Exercises 2

February 26, 2023

Theory exercises

Exercise 1:

Conditioned limit laws: bivariate logistic max-stable distribution

Suppose (X_0, X_1) follows a bivariate logistic max-stable distribution F with standard Laplace margins and dependence parameter $\kappa \in (0,1)$, that is,

$$\mathbb{P}(X_0 \le x_0, X_1 \le x_1) = \exp\{-V\left(T(x_0), T(x_1)\right)\},$$

where $T(x) = -1/\log\left(F_L(x)\right)$,

$$V(x,y) = \left(x^{-1/\kappa} + y^{-1/\kappa}\right)^{\kappa} \qquad \text{and} \qquad V_1(x,y) = \frac{\partial}{\partial x} V(x,y).$$

The conditional distribution for this example is

$$\mathbb{P}(X_1 \le x_1 \mid X_0 = x_0) = -T(x_0)^2\, e^{1/T(x_0)}\, V_1\left\{T(x_0), T(x_1)\right\} \exp\left[-V\left\{T(x_0), T(x_1)\right\}\right].$$

1. Show that the conditioned limit law is

   $$\mathbb{P}(X_1 - X_0 \le z \mid X_0 = u) \to [1 + \exp(-z/\kappa)]^{\kappa - 1}, \qquad u \to \infty.$$

   for $z \in \mathbb{R}$.

2. Consider a Markov process $\{X_t : t = 0, 1, \ldots\}$ that satisfies $(X_t, X_{t+1}) \sim F$ for all $t \ge 0$. Explain in simple terms why such a Markov process exists. Derive a recurrence relation for the tail chain $\{Z_t\}_{t \ge 0}$ associated with this process.

Exercise 2:

Conditioned limit laws: bivariate inverted logistic max-stable distribution
Suppose (X_0, X_1) follows a bivariate inverted logistic max-stable distribution F with unit exponential margins and dependence parameter $\kappa \in (0,1)$, that is,

$$\mathbb{P}(X_0 > x_0, X_1 > x_1) = \exp\{-V(1/x_0, 1/x_1\},$$

where

$$V(x,y) = \left(x^{-1/\kappa} + y^{-1/\kappa}\right)^{\kappa}.$$

The conditional distribution this example is

$$\mathbb{P}(X_1 \leq x_1 \mid X_0 = x_0) = 1 - V_1(1, x_0/x_1)\exp\{x_0 - x_0 V(1, x_0/x_1)\}.$$

1. Show that the conditioned limit law for is

$$\mathbb{P}\left(\frac{X_1}{X_0^{1-\kappa}} \leq z \mid X_0 = u\right) \to 1 - \exp(-\kappa z^{1/\kappa}) \qquad u \to \infty.$$

   for $z \in \mathbb{R}_+$.

2. Consider a Markov process $\{X_t : t = 0, 1, \ldots\}$ that satisfies $(X_t, X_{t+1}) \sim F$ for all $t \geq 0$. Explain in simple terms why such a Markov process exists. Derive a recurrence relation for the tail chain $\{Z_t\}_{t\geq 0}$ associated with this process.

Exercise 3:

Fitting time-series conditional extreme-value models

Using the starter code provided on GitHub, fit Markov conditional extreme-value models of order 1 to the Orléans data. Obtain realisations from the forward simulation procedure, given that an extreme above v is observed in time $t = 1$, where v is the 0.95–quantile of the data. Based on these forward simulations, obtain Monte Carlo estimates of the following quantities:

1. The expected maximum of daily maximum temperatures over the next d lags:

$$e_1(v, d) = \mathbb{E}(\max \boldsymbol{X}_{1\,:\,d} \mid X_1 > v).$$

2. The expected mean of the daily maximum temperatures at the next d lags:

$$e_1(v, d) = \mathbb{E}\left(\frac{1}{d}\sum_{i=1}^{d} X_i \mid X_1 > v\right).$$

3. The expected number of daily maximum temperatures above X_1 over the next d lags:

$$e_1(v, d) = \mathbb{E}\left(\sum_{i=1}^{d} \mathbb{1}[X_i > v] \mid X_1 > v\right).$$

Compare the estimated quantities under different orders (2, and 3) of fitted Markov models.