Extremes of Stationary Sequences

Exercises

February 26, 2023

Theory exercises

Exercise 1: Equivalent independent processes

Suppose that the maximum of the stationary sequence $X_1, \dots, X_n$ has a limiting extreme value distribution $G(x)$ and extremal index θ. If the sequence $X_1^\star, \dots, X_n^\star$ is i.i.d. with the same marginal distribution as the X_i, and the limiting distribution for $\max\{X_1^\star, \dots, X_n^\star\}$ is $\mathrm{GEV}(\mu, \sigma, \xi)$, find the parameters of the GEV distribution $G(x)$ in terms of θ, μ, σ, and ξ.

Exercise 2: Moving scaled maxima

Let $X_1, \dots, X_n$ be a stationary time series with $X_{i+1} = \max(\alpha X_i, \varepsilon_{i+1})$ and $X_1 = \varepsilon_1$ where the ε_i's are a sequence of *i.i.d.* variables with distribution

$$F_{\varepsilon_i}(x) = \exp\{-(1-\alpha)/x\}$$

for $x > 0$, and $0 \leq \alpha < 1$. Show that the marginal distribution of the X_i sequence is

$$F_X(x) = \exp(-1/x)$$

and hence show that the extremal index is $\theta = 1 - \alpha$.
[HINT: What constraint must the ε_i satisfy for $\max\{X_1, \dots, X_n\} \leq x$?]

Exercise 3: Derivation of the intervals estimator

Let $\{X_t\}_{t=1}^n$ be a stationary time series with marginal distribution F and $1 \leq S_1 < \cdots < S_N \leq n$ denote exceedance times of u by $\{X_t\}_{t=1}^n$. Recall that for the interexceedance time $T_i := T_i(u) = S_{i+1} - S_i$, we have that $\{1 - F(u)\}T_i \xrightarrow{d} T_\theta$ as $u \to \sup\{x : F(x) < 1\}$ where

$$T_\theta = \begin{cases} 0, & \text{with probability } (1-\theta), \\ E_\theta, & \text{with probability } \theta, \end{cases}$$

and E_θ follows an exponential distribution with mean θ^{-1}. Assuming that the rescaled interexceedance times $\{1-F(u)\}T_i$ follow exactly the limiting mixture distribution, construct the moment-based estimator $\widehat{\theta}^\delta$ for the extremal index θ by considering the first two moments of T_θ and equating them to their empirical counterparts. Refine your estimate using the fact that θ is related to the coefficient of variation, ν, of the interexceedance times by $1 + \nu^2 = \mathbb{E}(T_\theta^2)/\mathbb{E}(T_\theta)^2 = 2\theta^{-1}$.

R exercises

Exercise 4:

First-order autoregressive sequence with discrete uniform noise
Consider the first-order autoregressive time series given by

$$X_n = \frac{1}{\lambda} X_{n-1} + \varepsilon_n,$$

where $\lambda \geq 2$ is integer, and ε_n (independent of X_{n-1}) are *i.i.d.* and uniformly distributed on $\{0, 1/\lambda, \ldots, (\lambda-1)/\lambda\}$. For a range of λ values, generate 1000 observations from the Chernick first-order process, estimate the extremal index and build a block bootstrap confidence interval. Compare with the theoretical value for the extremal index of this AR process, $\theta = (\lambda - 1)/\lambda$.

Exercise 5:

Montreal summer daily temperature maxima We shall use the runs estimator to estimate the extremal index and find clusters for the Montreal Montreal summer daily temperature maxima.

1. For a suitable range of thresholds and run lengths r, plot the estimated extremal index against threshold.

2. What difference does changing the run length make? What seems a sensible run length to use?

3. For a chosen run length and threshold, decluster the data. Fit a GPD to the cluster maxima and consider the goodness of fit.