Extremes of Stationary Sequences

Solutions

February 26, 2023

Theory exercises

Exercise 1:

Let the GEV distributions of maxima from the independent process $X_1^\star, \ldots, X_n^\star$ and the stationary process $X_1, \ldots, X_n$ be denoted by $G^\star(x)$ and $G(x)$ respectively. Then we have that

$$G(x) = G^\star(x)^\theta.$$

Now,

$$G(x) = G^\star(x)^\theta = \exp\left\{ - \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]_+^{-1/\xi} \right\}^\theta = \exp\left\{ -\theta \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]_+^{-1/\xi} \right\}$$

$$= \exp\left\{ - \left[\theta^{-\xi} + \theta^{-\xi}\xi\left(\frac{x-\mu}{\sigma}\right)\right]_+^{-1/\xi} \right\} = \exp\left\{ - \left[1 + (\theta^{-\xi}-1) + \theta^{-\xi}\xi\left(\frac{x-\mu}{\sigma}\right)\right]_+^{-1/\xi} \right\}$$

$$= \exp\left\{ - \left[1 - \frac{\theta^\xi - 1}{\theta^\xi} + \xi\left(\frac{x-\mu}{\theta^\xi\sigma}\right)\right]_+^{-1/\xi} \right\} = \exp\left\{ - \left[1 - \xi\frac{\sigma(\theta^\xi - 1)/\xi}{\theta^\xi\sigma} + \xi\left(\frac{x-\mu}{\theta^\xi\sigma}\right)\right]_+^{-1/\xi} \right\}$$

$$= \exp\left\{ - \left[+\xi\left(\frac{x-(\mu+\sigma(\theta^\xi-1)/\xi)}{\theta^\xi\sigma}\right)\right]_+^{-1/\xi} \right\}.$$

Hence, we have that $G(x)$ is GEV($\tilde{\mu}$,$\tilde{\sigma}$,ξ), with

$$\tilde{\mu} = \mu + \sigma(\theta^\xi - 1)/\xi)$$

and

$$\tilde{\sigma} = \theta^\xi\sigma.$$

Exercise 2:

Moving scaled maxima

Since $X_i = \max(\alpha X_{i-1}, \varepsilon_i)$, and since X_{i-1} is independent of ε_i by assumption, we have that

$$\begin{aligned}
\mathbb{P}(X_i \le x) &= \mathbb{P}(\alpha X_{i-1} \le x, \varepsilon_i \le x) = \mathbb{P}(\alpha X_{i-1} \le x)\mathbb{P}(\varepsilon_i \le x) = \mathbb{P}(\alpha X_{i-1} \le x)F_\varepsilon(x) \\
&= \mathbb{P}(\alpha \max(\alpha X_{i-2}, \varepsilon_{i-1}) \le x)F_\varepsilon(x) = \mathbb{P}(\alpha^2 X_{i-2} \le x, \alpha\varepsilon_{i-1} \le x)F_\varepsilon(x) \\
&= \mathbb{P}(\alpha^2 X_{i-2} \le x)F_\varepsilon(x/\alpha)F_\varepsilon(x) = \mathbb{P}(\alpha^2 X_{i-2} \le x)\prod_{j=0}^{1} F_\varepsilon(x/\alpha^j) = \cdots \\
&= \mathbb{P}(\alpha^{i-1} X_1 \le x)\prod_{j=0}^{i-2} F_\varepsilon(x/\alpha^j) \\
&= \mathbb{P}(\alpha^{i} X_0 \le x)F_\varepsilon(x/\alpha^{i-1})\prod_{j=0}^{i-2} F_\varepsilon(x/\alpha^j) \\
&= \mathbb{P}(\alpha^{i} \varepsilon_1 \le x)\prod_{j=0}^{i-1} F_\varepsilon(x/\alpha^j) = \prod_{j=0}^{i} F_\varepsilon(x/\alpha^j)
\end{aligned}$$

Now, since $F_{\varepsilon_i}(x) = \exp\{-(1-\alpha)/x\}$, we have

$$\mathbb{P}(X_i \le x) = \prod_{j=0}^{i} F_\varepsilon(x/\alpha^j) = \prod_{j=0}^{i} \exp\left\{-\frac{1-\alpha}{x/\alpha^j}\right\} = \exp\left\{-\frac{(1-\alpha)\sum_{j=0}^{i}\alpha^j}{x}\right\}.$$

Since $0 \le \alpha < 1$, $\lim_{i\to\infty}\sum_{j=0}^{i}\alpha^j = 1/(1-\alpha)$, and thus $\lim_{i\to\infty}\mathbb{P}(X_i \le x) = \exp(-1/x)$. By construction of the process, we have

$$\max_{1\le i\le n} X_i = \max_{1\le i\le n} \varepsilon_i.$$

Hence, $\mathbb{P}(\max_{1\le i\le n} X_i) = \mathbb{P}(\max_{1\le i\le n} \varepsilon_i)$, and

$$\begin{aligned}
\mathbb{P}(\max_{1\le i\le n} X_i/n \le x) &= \mathbb{P}(\max_{1\le i\le n} \varepsilon_i/n \le x) = \prod_{i=0}^{n}\mathbb{P}(\varepsilon_i \le nx) \\
&= \exp\left\{-\frac{1-\alpha}{nx}\right\}^n = \exp\left\{-\frac{1-\alpha}{x}\right\} = \exp\left\{-\frac{1}{x}\right\}^{1-\alpha} = [G_X(x)]^{1-\alpha},
\end{aligned}$$

where $G_X(x)$ is the distribution of the maxima of independent Fréchet random variables. It follows that $\theta = 1-\alpha$.

Exercise 3:

Derivation of the intervals estimator

We note that T_θ is equal in distribution to a mixture of a point mass at 0 (D_0) and of an exponential ($E_\theta \sim$Exp(θ)) random variable so that

$$T_\theta = \begin{cases} 0, & \text{with probability } (1-\theta), \\ E_\theta, & \text{with probability } \theta, \end{cases}$$

We consider the first two moments of T_θ. The first moment, $\mathbb{E}[T_\theta]$, is obtained through the law of total expectation:

$$\begin{aligned} \mathbb{E}(T_\theta) &= \mathbb{E}(T_\theta \mid T_\theta = 0)\mathbb{P}(T_\theta = 0) + \mathbb{E}(T_\theta \mid T_\theta > 0)\mathbb{P}(T_\theta > 0) \\ &= 0(1-\theta) + \theta\mathbb{E}(E_\theta) = \theta/\theta = 1. \end{aligned}$$

Similarly, the second moment of T_θ is obtained through

$$\begin{aligned} \mathbb{E}(T_\theta^2) &= \mathbb{E}(T_\theta^2 \mid T_\theta = 0)\mathbb{P}(T_\theta = 0) + \mathbb{E}(T_\theta^2 \mid T_\theta > 0)\mathbb{P}(T_\theta > 0) \\ &= 0(1-\theta) + \theta\{\mathrm{Var}(E_\theta) + \mathbb{E}(E_\theta)^2\} \\ &= \theta\left(\frac{1}{\theta^2} + \frac{1}{\theta^2}\right) = \frac{2}{\theta}. \end{aligned}$$

Equating the second moment with its empirical estimate, we obtain

$$\frac{2}{\theta} = \mathbb{E}(T_\theta^2) \approx \mathbb{E}\left[\{1-F(u)\}^2 T_i^2\right] = \frac{1}{N-1}\sum_{i=1}^{N-1}\{1-F(u)\}^2 T_i^2$$

An empirical estimator for $1-F(u)$ consisting of the number of exceedances of u, N, and the total number of observations n from $\{X_t\}_{t=1}^n$ can be constructed as

$$\widehat{\bar{F}(u)} = 1 - \widehat{F(u)} = \frac{N}{n},$$

so that

$$\widehat{\theta} = \frac{2(N-1)}{\widehat{\bar{F}(u)}^2}\left\{\sum_{i=1}^{N-1} T_i^2\right\}^{-1} = \frac{2n^2(N-1)}{N^2}\left\{\sum_{i=1}^{N-1} T_i^2\right\}^{-1}.$$

We now refine the estimator by using the second hint. Since

$$1 + \nu^2 = \frac{\mathbb{E}(T_\theta^2)}{\mathbb{E}(T_\theta)^2} = \frac{2}{\theta},$$

we can obtain $\widehat{\theta}^\delta$ through the refinement

$$\widehat{\theta}^\delta = \frac{2\widehat{\mathbb{E}}(T_\theta)^2}{\widehat{\mathbb{E}}(T_\theta^2)} = \widehat{\mathbb{E}}(T_\theta)^2 \frac{2n^2(N-1)}{N^2} \left\{ \sum_{i=1}^{N-1} T_i^2 \right\}^{-1}.$$

Using the moment based estimator

$$\widehat{\mathbb{E}}(T_\theta)^2 = \widehat{\mathbb{E}}\{\bar{F}(u)T_i\}^2 = \left(\frac{N}{n} \frac{1}{N-1} \sum_{i=1}^{N-1} T_i \right)^2,$$

one obtains that

$$\widehat{\theta}^\delta = \left(\frac{N}{n} \frac{1}{N-1} \sum_{i=1}^{N-1} T_i \right)^2 \frac{2n^2(N-1)}{N^2} \left\{ \sum_{i=1}^{N-1} T_i^2 \right\}^{-1} = \frac{2\left(\sum_{i=1}^{N-1} T_i\right)^2}{(N-1)\sum_{i=1}^{N-1} T_i^2}.$$

R exercises

See GitHub repository for solutions.