

Project I: Solving Equations of Motion

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1 The Harmonic Oscillator

1.1 Introduction

Used to model oscillatory motion, the harmonic oscillator differential equation is a standard problem of classical mechanics and, provided there is no friction or driving force applied, has a simple analytical solution. The differential equation is given by,

$$\ddot{x}(t) = -\Omega^2 x(t) \quad (1)$$

where $\Omega = \sqrt{k/m}$ for a spring constant k and mass m . As said before, the above differential equation has a known analytical solution, which is,

$$x(t) = x_0 \sin(\Omega t + \phi) \quad (2)$$

where x_0 is the initial starting position, or amplitude, of the oscillator, and ϕ is the phase angle.

The energy of the harmonic oscillator system is an equally simple expression. Because there is no friction or driving force applied, the harmonic oscillator is a conservative system. This means the potential energy of the harmonic oscillator is the negative integral of the force due to the spring, $U = -\int(-kx)dx$. Therefore the potential energy is given by $U = \frac{1}{2}kx^2$. The total energy of the system is U plus the kinetic energy, or $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. In the following computations, the mass of the system, m , is "set to one." With $m = 1$, the angular frequency is given by $\Omega = \sqrt{k}$. Thus, the total energy for the system is expressed by,

$$E = \frac{1}{2}v^2 + \frac{1}{2}\Omega^2 x^2 \quad (3)$$

Because of the simple solution, the harmonic oscillator is an ideal problem to compare the known analytical solution, equation (2), with those of different numerical methods. Likewise, we can compare the energy of the system, equation (3), with the energy given by the numerical methods to check if energy is conserved when equation (1) is integrated numerically.

The following section will introduce how to numerically integrate the harmonic oscillator ordinary differential equation (ODE), and the three different methods that are used. These methods are the Euler method, the Second-Order Runge-Kutta (referred to as RK2) method, and the Stormer-Verlet method.

1.2 Numerical Methods

Before we describe the numerical methods used to solve the harmonic oscillator problem, we note that equation (1) is an explicit second-order ODE. The harmonic oscillator is an initial value problem with two initial values because it is a second-order ODE. The initial values needed are the initial

velocity, $\dot{x}(0) = v_0$, and the initial position, $x(0) = x_0$. Therefore, the initial value problem to be solved is actually

$$\begin{aligned}\ddot{x}(t) &= -\Omega^2 x(t) \\ \dot{x}(0) &= v_0 \\ x(0) &= x_0\end{aligned}\tag{4}$$

However, the following numerical methods require an initial value problem of the form [1]

$$\begin{aligned}\dot{\mathbf{y}}(t) &= \mathbf{f}(\mathbf{y}, t) \\ y(0) &= y_0\end{aligned}\tag{5}$$

where \mathbf{y} is a n -dimensional vector of independent variables, y_0 is the initial value of y for its corresponding element in the vector \mathbf{y} , and \mathbf{f} is an n -dimensional vector containing the first derivative of each corresponding element in \mathbf{y} . Equation (5) is a *first-order* ODE whereas the harmonic oscillator, equation (4), is a *second-order* ODE. Fortunately, any differential equation of order n and the form $y^{(n)} = f(t; y, \dot{y}, \dots, y^{(n-1)})$ can be transformed into a set of explicit first-order differential equations of the form, [1]

$$\begin{aligned}\dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ &\dots \\ &\dots \\ \dot{y}_{n-1} &= y_n, \\ \dot{y}_n &= f(t; y, \dot{y}, \dots, y^{(n-1)})\end{aligned}\tag{6}$$

thus creating the aforementioned n -dimensional vectors \mathbf{y} and \mathbf{f} in equation (5). In the case of the harmonic oscillator, we note that the first derivative of position is velocity, $\dot{x} = v$, and the first derivative of velocity is acceleration, $\dot{v} = a$. Recall that the acceleration of the harmonic oscillator is $a = -\Omega^2 x$, which means $f(\mathbf{y}, t) = -\Omega^2 x$ and the transformed second-order ODE is thus,

$$\begin{aligned}\dot{x}(t) &= v(t) \\ \dot{v}(t) &= -\Omega^2 x(t)\end{aligned}\tag{7}$$

Our vectors of independent variables and their first derivatives are thus $\mathbf{y} = (x, v)$ and $\mathbf{f}(\mathbf{y}, t) = (v, -\Omega^2 x)$, respectively. Equation (7) is the desired set of first-order ODEs that describe the harmonic oscillator and can be numerically integrated by the following methods.

1.2.1 Euler Method

All numerical methods begin by discretizing the time coordinate t with a small discrete time step h , explicitly $t_n = t_0 + nh$, where t_0 is the initial time and $n \geq 1$. Our discretized set of first-order ODEs is now

$$\dot{\mathbf{y}}_n = \mathbf{f}(\mathbf{y}_n, t_n)\tag{8}$$

We wish to predict the \mathbf{y}_{n+1} terms using the known \mathbf{y}_n terms and their derivatives \mathbf{f}_n . It seems reasonable, then, that the \mathbf{y}_{n+1} terms will equal the current n th term plus the integral of the first derivative over one time step, from t_n to t_{n+1} . Explicitly, this is $\mathbf{y}_{n+1} = \mathbf{y}_n + \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{y}(t), t) dt$. Taylor expanding the integrand in the previous equation about t_n gives us the integral [1]

$$\int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{y}(t), t) dt = \int_{t_n}^{t_{n+1}} [\mathbf{f}_n + (t - t_n)\mathbf{f}'_n + (t - t_n)^2\mathbf{f}''_n] dt\tag{9}$$

We note that $(t_{n+1} - t_n) = h$, our defined time step. After integrating (9), we assume that h^2 is sufficiently small and drop all terms of power two or greater, leaving us with

$$\int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{y}(t), t) dt = h\mathbf{f}_n + \mathcal{O}(h^2) \quad (10)$$

The Euler Method of solving first-order ODEs is therefore

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}_n(\mathbf{y}_n, t_n) + \mathcal{O}(h^2) \quad (11)$$

Note that the error made by approximating h^2 as small goes like h^2 . Because h^2 is one power greater than the highest power term in the Euler Method, $h\mathbf{f}_n(\mathbf{y}_n, t_n)$, we say that the Euler Method is 1st Order.

Substituting our vectors of independent variables and their derivatives for the harmonic oscillator problem into the Euler Method gives us the following equations,

$$\begin{aligned} x_{n+1} &= x_n + hv_n \\ v_{n+1} &= v_n - h\Omega^2 x_n \end{aligned} \quad (12)$$

which, given initial values x_0 , v_0 , and t_0 for $n = 0$, will generate a set of data points that represents the solution the harmonic oscillator ODE. A plot of x_n v. t_n is the numerical counterpart of equation (2) and a plot of v_n v. x_n is the phase space of the harmonic oscillator. Finally, we can discretize the energy of the system as

$$E_{n+1} = \frac{1}{2}v_{n+1}^2 + \frac{1}{2}\Omega^2 x_{n+1}^2 \quad (13)$$

A plot of E_n v. t_n will indicate whether or not the energy of system is conserved using the Euler Method. If E_n v. t_n is constant, or at least constrained within a constant envelope, then we say the numerical method conserves energy, more or less. Equation (13) is generic and applies to all three numerical methods for solving the harmonic oscillator.

1.2.2 Second-Order Runge-Kutta Method (RK2)

The next numerical method used to solve the harmonic oscillator problem is the Second-Order Runge-Kutta Method, known as RK2. RK2 attempts to improve accuracy over the Euler Method by calculating intermediate grid-points within the time step h , from t_n to t_{n+1} . That is to say, it calculates the point $y_{n+\frac{1}{2}}$ at $t_{n+\frac{1}{2}}$, where $t_{n+\frac{1}{2}} = t_n + \frac{h}{2}$. As mentioned in [1], this is approximation is made by the central rectangular rule,

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) + \mathcal{O}(h^3) \quad (14)$$

The central rectangular rule is derived in a similar manner to the Euler Method in that we Taylor expand the integrand of $\int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{y}(t), t) dt$ and then integrate. However, as mentioned above we wish to consider intermediate grid-points at $t_{n+\frac{1}{2}}$. So, instead of Taylor expanding $\mathbf{f}(\mathbf{y}(t), t)$ about t_n we expand the integrand about $t_{n+\frac{1}{2}}$ which, after integrating and dropping terms of power three or greater, gives us equation (14). We then approximate $\mathbf{y}_{n+\frac{1}{2}}$ by using the previously defined Euler Method,

$$\mathbf{y}_{n+\frac{1}{2}} = \mathbf{y}_n + \frac{h}{2}\mathbf{f}(\mathbf{y}_n, t_n) \quad (15)$$

Plugging equation (15) and $t_{n+\frac{1}{2}} = t_n + \frac{h}{2}$ into equation (14) gives us

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}[\mathbf{y}_n + \frac{h}{2}\mathbf{f}(\mathbf{y}_n, t_n), t_n + \frac{h}{2}] + \mathcal{O}(h^3) \quad (16)$$

which is our desired result. However the RK2 method is typically expressed by the following three equations that illustrate the three approximations made,

$$\begin{aligned}\mathbf{k}_1 &= h\mathbf{f}(\mathbf{y}_n, t_n), \\ \mathbf{k}_2 &= h\mathbf{f}\left(\mathbf{y}_n + \frac{1}{2}\mathbf{k}_1, t_n + \frac{h}{2}\right), \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \mathbf{k}_2 + \mathcal{O}(h^3).\end{aligned}\tag{17}$$

Notice that the error in the third equation, a result of the approximation made by the central rectangular rule, goes like h^3 while \mathbf{k}_2 is of power one in h . This means that RK2 is 2nd Order, as implied by its name. Because the error goes like h^3 , we expect RK2 to better approximate the solution to the harmonic oscillator ODE than the Euler Method. Higher order Runge-Kutta approximations exist that improve the accuracy even further.

As before, we substitute our vectors of independent variables and their derivatives for the harmonic oscillator problem into RK2 to give us,

$$\begin{aligned}kx_1 &= hv_n, & kv_1 &= -h\Omega^2 x_n, \\ kx_2 &= h(v_n - \frac{h}{2}\Omega^2 x_n), & kv_2 &= -h\Omega^2(x_n + \frac{h}{2}v_n), \\ x_{n+1} &= x_n + kx_2 & v_{n+1} &= v_n + kv_2\end{aligned}\tag{18}$$

where kx_i and kv_i are the RK2 approximations for position and velocity, respectively. Equation (18) will numerically solve the harmonic oscillator ODE, with the two equations in the third line generating our future position and velocity data given x_0 and v_0 at t_0 . We also check energy conservation using the discretized energy equation (13).

1.2.3 Stormer-Verlet Method

The third and last method we used to numerically solve the harmonic oscillator ODE is the Stormer-Verlet Method. The Stormer-Verlet Method is a symplectic integrator which are a family of integrators that are specifically for solving Hamiltonian systems numerically [2]. Stormer-Verlet is also time-centered and avoids asymmetries in the discretization. We derive the Stormer-Verlet method by first Taylor expanding $\mathbf{y}_{n\pm 1}$:

$$\mathbf{y}_{n\pm 1} = \mathbf{y}_n \pm \dot{\mathbf{y}}_n h + \frac{1}{2}\ddot{\mathbf{y}}_n h^2 \pm \frac{1}{6}\dddot{\mathbf{y}}_n h^3 + \mathcal{O}(h^4)\tag{19}$$

Adding \mathbf{y}_{n+1} with \mathbf{y}_{n-1} gives us the Stormer-Verlet method:

$$\mathbf{y}_{n+1} = 2\mathbf{y}_n - \mathbf{y}_{n-1} + \ddot{\mathbf{y}}_n h^2 + \mathcal{O}(h^4)\tag{20}$$

If instead of adding \mathbf{y}_{n+1} and \mathbf{y}_{n-1} from equation (19) we subtract them, then $\dot{\mathbf{y}}_n$ is retained and we can explicitly solve for, finding

$$\mathbf{v}_n = \frac{\mathbf{x}_{n+1} - \mathbf{x}_{n-1}}{2h} + \mathcal{O}(h^2)\tag{21}$$

where the error goes like $\mathcal{O}(h^2)$ since $\frac{\mathcal{O}(h^3)}{h} = \mathcal{O}(h^2)$. We can then use (21) to find an expression for \mathbf{v}_{n+1} and use (19) to find \mathbf{x}_{n+1} , truncating all terms with power three or greater. This gives us the following two equations that are known as the Velocity Verlet Method.

$$\begin{aligned}\mathbf{x}_{n+1} &= \mathbf{x}_n + h(\mathbf{v}_n + \frac{h}{2}\mathbf{a}(\mathbf{x}_n)) + \mathcal{O}(h^3) \\ \mathbf{v}_{n+1} &= \mathbf{v}_n + \frac{h}{2}\mathbf{a}(\mathbf{x}_n) + \frac{h}{2}\mathbf{a}(\mathbf{x}_{n+1}) + \mathcal{O}(h^3)\end{aligned}\tag{22}$$

Note that acceleration terms $\mathbf{a}(\mathbf{x}_n)$ depend on the position \mathbf{x}_n and are assumed to depend on position in the Velocity Verlet Method [3]. This means that the Velocity Verlet Method is only suited to solve ODEs describing equations of motion, thus explaining its categorization as a symplectic integrator. Also note that the error of the Verlet Method goes like h^3 , meaning the Velocity Verlet Method is 2nd Order, the same as RK2, and is expected to have less error and conserve energy better than the Euler Method. Recall that the harmonic oscillator problem is one dimensional, therefore we will have only one position equation and one velocity equation using the Verlet Method. These are,

$$\begin{aligned} x_{n+1} &= x_n + h(v_n - \frac{h}{2}\Omega^2 x_n) \\ v_{n+1} &= v_n - \frac{h}{2}\Omega^2 x_n - \frac{h}{2}\Omega^2 x_{n+1} \end{aligned} \quad (23)$$

The above equations will generate the solution to the harmonic oscillator problem using the Velocity Verlet Method. Energy conservation, once again, is checked with equation (13).

1.3 Results

All three methods described above were implemented by placing their equations for x_{n+1} , v_{n+1} , and E_{n+1} in a loop using Python. Given initial values x_0 , v_0 , and E_0 , each successive n th term can be calculated and stored in a vector until the maximum number of steps, N , is reached.

What follows are the results to the harmonic oscillator ODE, generated by the numerical methods described, in the form of plots. The results include solution plots (position versus time), phase space plots, and energy plots for different time step h values using different numerical methods. All computations have $m = 1$, $x_0 = 1$, $v_0 = 0$, and $\Omega = 0.5$. Lastly, the numerical results are compared against known analytical results.

1.3.1 Solution Plots

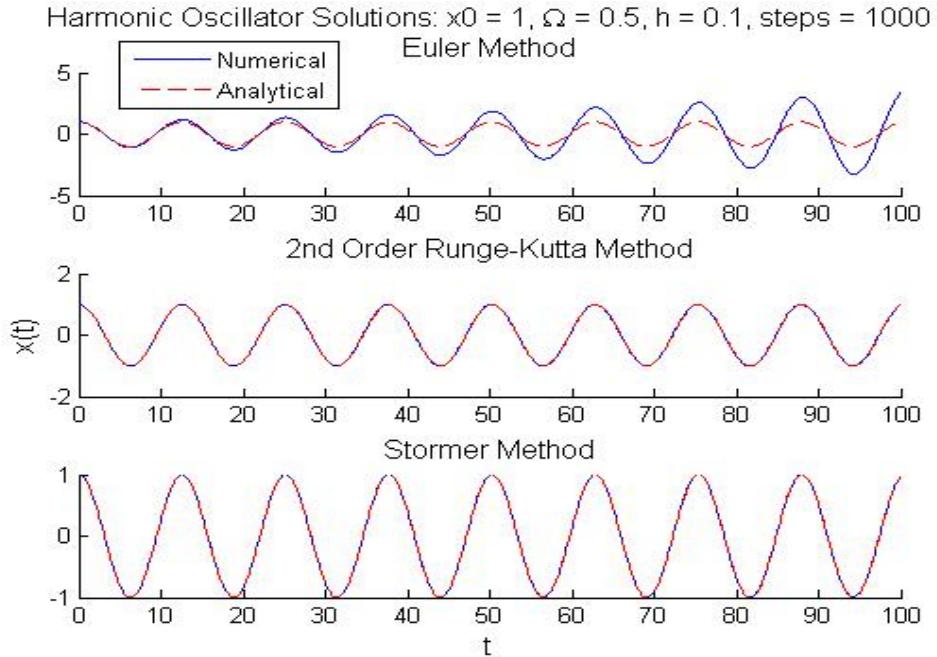


Figure 1: The above plots are the numerical and analytical solutions for the harmonic oscillator ODE, $x(t)$ v. t . The numerical solutions have a solid blue line while the analytical solutions have a red dashed line. The analytical solution is given by $x(t) = \sin(0.5t + \frac{\pi}{2})$.

Figure 1 above represents the solutions, which are $x(t)$ v. t plots, to the harmonic oscillator ODE using the Euler, RK2, and Stormer-Verlet numerical methods. Each method used the same initial conditions $x_0 = 1$, $v_0 = 0$, $\Omega = 0.5$ and a time step of $h = 0.1$ over 1000 steps. The analytical solution had the same initial conditions and is represented by $x(t) = \sin(0.5t + \frac{\pi}{2})$. From the plots above it is quite clear that the RK2 and Stormer-Verlet methods give very accurate solutions, as the curves generated by those methods are virtually indistinguishable from the actual, analytical solution. The Euler Method, on the other hand, does not give a very accurate solution as it is clear that the curve generated is beginning to diverge from the actual solution. Since the amplitude of the solution is growing, it appears that the Euler Method is not conserving energy and is, in fact, causing the system to gain energy.

Recall the error of the RK2 and Stormer-Verlet methods both go like $\mathcal{O}(h^3)$ and that they are second order methods while the Euler Method is first order with an error that goes like $\mathcal{O}(h^2)$. This error is the source of the well behaved solutions given by RK2 and Stormer-Verlet and the poor solution given by Euler. Because the time step $h < 1$, then for any power n , $h^n < h^{n-1} < \dots < h < 1$. Therefore, the higher the power the truncation error of the numerical method, the smaller the error. Evidently a time step of $h = 0.1$ works well for RK2 and Stormer-Verlet but not for Euler because they are high order methods than the Euler method. In order to achieve an equally accurate solution using the Euler Method, we must decrease the time step and increase the number steps accordingly. After trying different steps, I found that a time step of $h = 0.001$ with $N = 100000$ steps gives an Euler Method solution in close agreement with the analytical solution. This is shown in Figure 2 below.

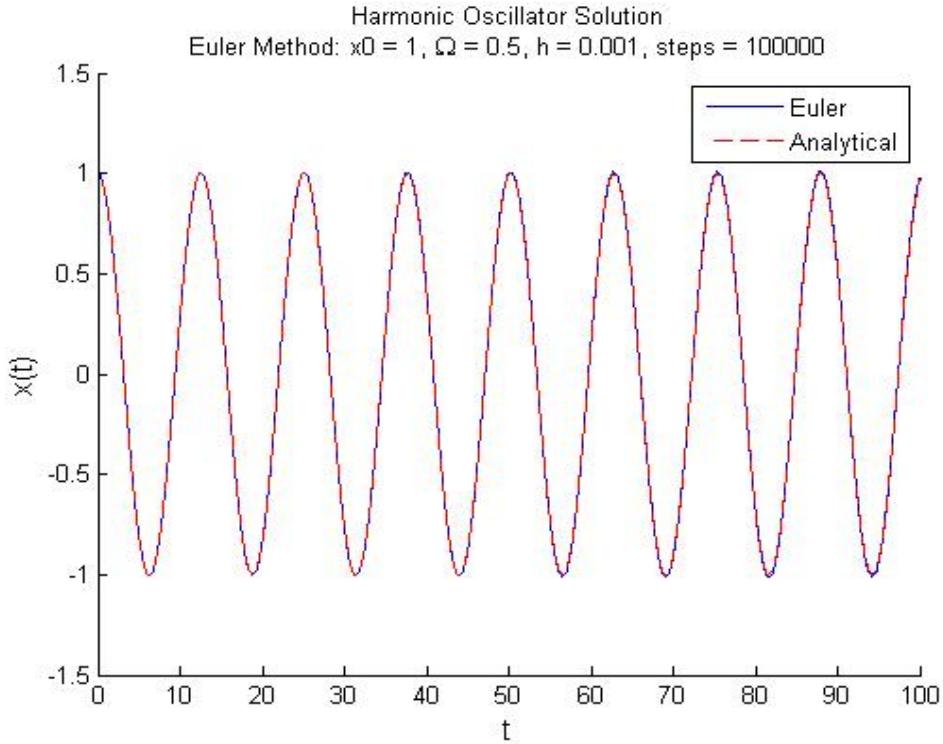


Figure 2: The above plot is the Euler Method solution with a time step of $h = 0.001$ iterated over $N = 100000$ steps. Note that this result much more closely approximates the actual solution than the Euler Method did with a time step of $h = 0.1$.

Because the solutions given by RK2 and Stormer-Verlet were so exact with a time step of $h = 0.1$, as shown in Figure 1, re-plotting their solutions with a time step of $h = 0.001$ would result in a negligible difference. It is important to note however, that decreasing h absolutely decreases the

error for all methods, however this comes at the cost of computation time. The smaller the time step, the more steps that are needed, and the longer it takes to solve a problem numerically. While low order methods like the Euler Method can produce an accurate solution given a small enough time step, the high order methods such as RK2 and Stormer-Verlet are more desirable as they can produce an equally, if not more, accurate solution with a larger time step, less steps, and ultimately less computational time.

1.3.2 Phase Space Plots

Phase Space is a plot of momentum or velocity versus position. Below are phase space plots generated by each method for a time step of $h = 0.1$.

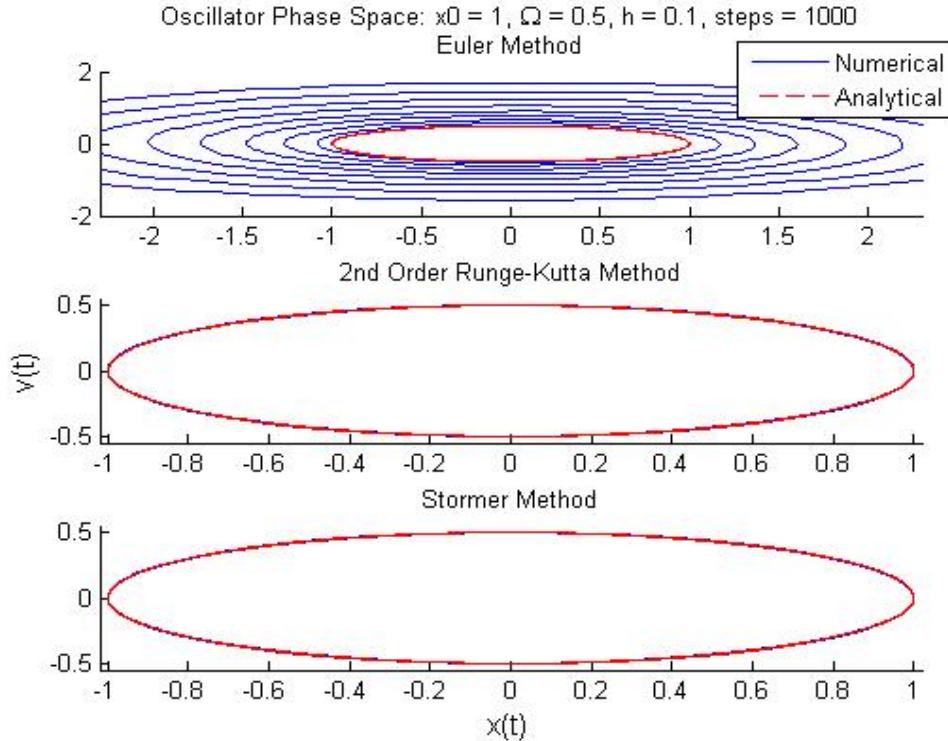


Figure 3: Oscillator phase space plots. The actual phase space plot is represented by $(\sin(0.5t + \frac{\pi}{2}), 0.5 \cos(0.5t + \frac{\pi}{2}))$, which is the dashed red curve in the plots above.

The phase space plot above is simply another means of illustrating what was shown in the solution plots. Once again, it is clear that the RK2 and Stormer-Verlet methods do a very good job of approximating the actual phase space of the harmonic oscillator since the numerical solutions and the actual solutions are virtually indistinguishable. The Euler Method clearly generates an unstable solution, as the phase space plot is spiraling away from the actual solution at the center. Decreasing the time step from $h = 0.1$ to $h = 0.001$ generates a closer approximation to the phase space when using the Euler method, as shown in the next two plots.

Although from Figure 4 the approximation appears closer when using the smaller time step h , the Euler Method's phase space still does not perfectly overlap the actual phase space as it did with RK2 and Stormer-Verlet in Figure 3 for $h = 0.1$. Closer inspection leads us to Figure 5, which shows that the Euler phase space is still spiraling away from the actual phase space. A time step even smaller than $h = 0.001$, therefore, is required for the Euler Method to better approximate the actual harmonic oscillator solution. With a time step of $h = 0.001$, the Euler Method solution is still unstable and in gaining energy since the phase space is spiraling away from the actual solution.

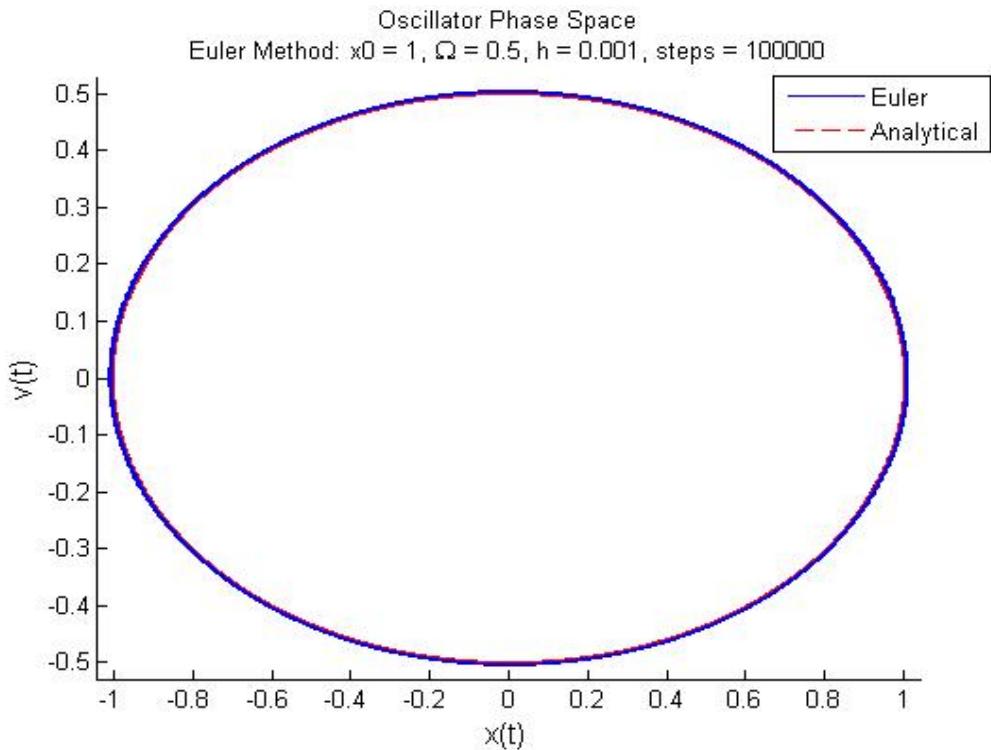


Figure 4: Euler phase space plot with $h = 0.001$. The actual phase space plot is represented by $(\sin(0.5t + \frac{\pi}{2}), 0.5 \cos(0.5t + \frac{\pi}{2}))$, which is the dashed red curve in the plots above. Notice that even with a smaller time step, the Euler phase space does not perfectly overlap with the actual phase space.

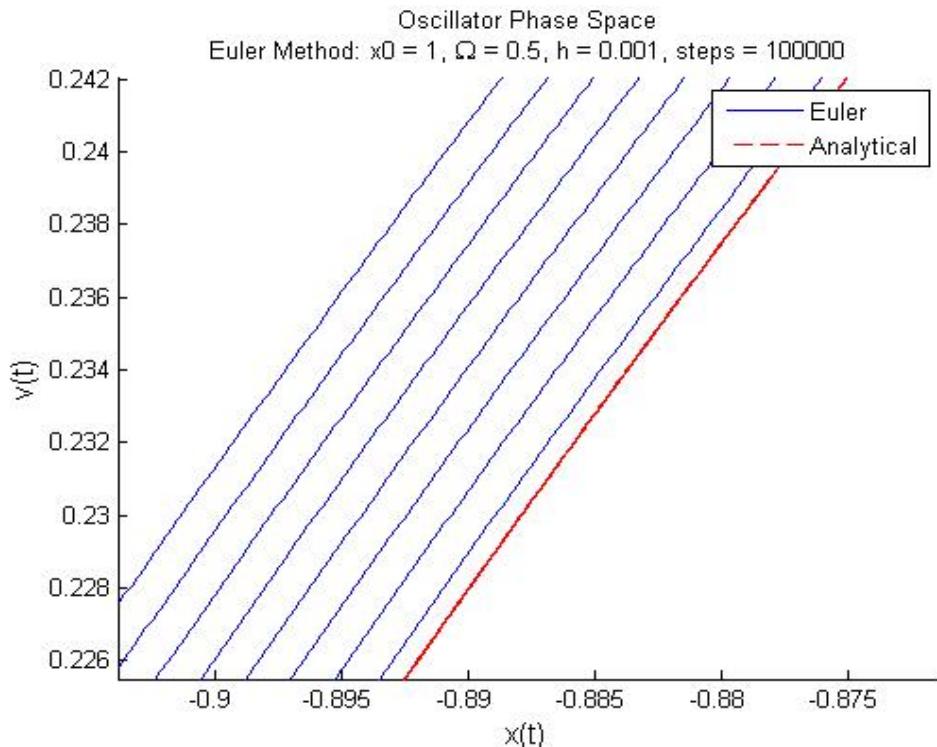


Figure 5: Euler phase space plot with $h = 0.001$ and zoomed in. The actual phase space plot is shown as the dashed red line. Note that even with $h = 0.001$, the Euler Method's phase space still spirals away from the actual phase space, indicating an inaccurate solution and unstable system that's gaining energy.

1.3.3 Energy Plots

Though we have discussed energy conservation of the numerical methods indirectly, we now look at the energy versus time plots directly to see if energy is in fact conserved by our numerical methods. The total energy of the harmonic oscillator is constant, equal to the potential energy evaluated at the initial position x_0 and it is $E_{TOT} = \frac{1}{2}\Omega^2x_0^2 = 0.5 * 0.5^2 * 1 = 0.125$. Figure 6 below shows the energy of the system for each numerical method with the large time step of $h = 0.1$. As expected from our previous results, the energy of the Euler Method is growing exponentially, and is not bounded by any constant value. A somewhat surprising result is that the RK2 method does not appear to conserve energy very well either. Though the energy of the RK2 solution grows by only 0.002 over 1000 steps, it is linear in its growth and given enough steps the error in the solution and the conservation of its energy will become unacceptable and noticeable. Only the Stormer-Verlet Method appears to conserve energy very well. Although its energy versus time plot is oscillating, the energy of the Stormer-Verlet method never exceeds the actual energy of the harmonic oscillator, $E = 0.125$, and the minimum of energy is no lower than 0.1249, at most a difference of 0.0001 from the actual energy value. The energy of the Stormer-Verlet method is bounded, meaning the Stormer-Verlet method conserves energy.

Figure 7 shows the energy of the system after decreasing the time step to $h = 0.001$. At this small time step, the Euler Method now conserves energy almost as well as RK2 with $h = 0.1$ in that its growth appears linear. The RK2 method still has linear growth in energy, however its curve is very close to the actual energy value of 0.125 after 100000 steps. Even more steps would be required, then, for the RK2 method to show a substantial growth in energy and an error in the solution. Finally, the energy conservation of the Sotrmer-Verlet method remains unchanged, a testament to how well the method conserves energy even with large time steps.

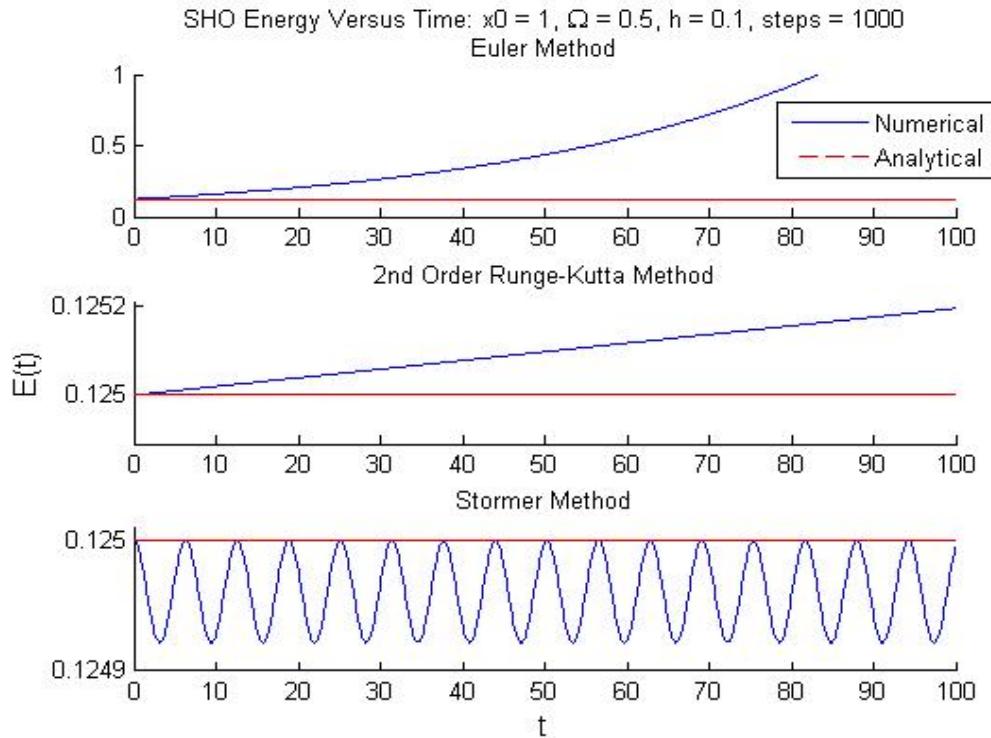


Figure 6: Energy versus time plots for the three different numerical solution with $h = 0.1$. The actual energy of the system is constant, given by the potential energy at x_0 , that is $E_{TOT} = \frac{1}{2}\Omega^2x_0^2 = 0.5 * 0.5^2 * 1 = 0.125$.

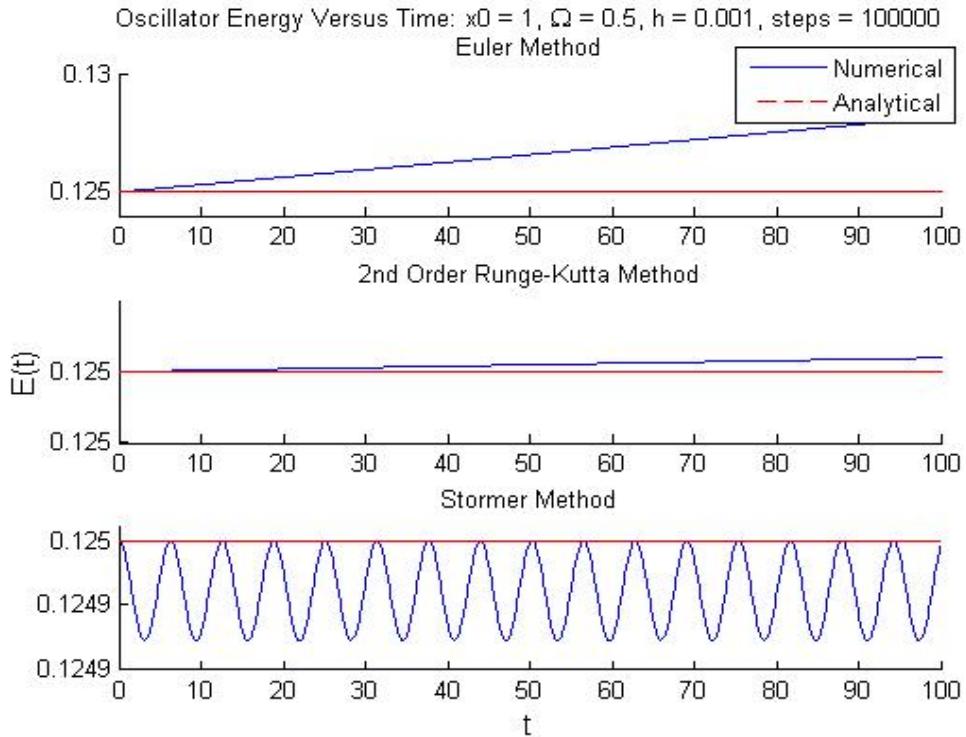


Figure 7: Energy versus time plots for the three different numerical solution with $h = 0.001$. The actual energy of the system is given by the potential energy at x_0 , that is $E_{TOT} = \frac{1}{2}\Omega^2x_0^2 = 0.5 * 0.5^2 * 1 = 0.125$.

1.4 Conclusion

In conclusion, the Stormer-Verlet Method generates the best solutions and conserves energy the best among the three numerical methods considered in solving the harmonic oscillator problem. The RK2 method does an equally good job of generating correct solutions, however it does not conserve energy as well as the Stormer-Verlet method for a particular time step. Finally, the Euler Method does a horrible job of generating a solutions and conserving energy due to the fact that it is a 1st Order method. The Euler Method is heavily dependent of the time step h in order to generate an acceptable solution, with a time step of $h = 0.001$ or smaller necessary to create solutions without noticeable error. Even with this small time step, the Euler Method's results are not as good as the Stormer-Verlet or even the RK2 method's with the large time step $h = 0.1$.

Also considered but not shown in any plot is what happens when we consider larger values of x_0 and Ω . For larger values of these quantities the error of the three numerical methods used, and in fact any computation made using a machine that only represents numbers with finite digits, increases with each iteration. This is because when x_0 and Ω are large, the calculated values for x_{n+1} and v_{n+1} become large, but they can only be represented by a finite amount of digits. This leads to round off or chopping of the actual value, known as round-off error [1], that accumulates with each iteration and leads to unacceptable error in our results. Round-off error can also be caused by subtracting two number that are close in value, multiplying by a large number, and dividing by a small number. As we will see in the Kepler problem, round-off error can be mitigated by using smaller time steps.

2 The Kepler Problem

2.1 Introduction

We now move on to a more complicated and important problem of astrodynamics; the Kepler Problem. The Kepler Problem explains planetary motion due to the gravitational attraction between astronomical bodies. Though as many bodies can be considered as possible, here we only consider a two body system, which still has a nontrivial analytical solution. The two body problem can further be simplified as a one body problem by considering the motion of planets in a plane and writing the equations of motion in the center of mass frame, which are

$$\begin{aligned}\dot{\mathbf{r}} &= \mathbf{v}, \\ \dot{\mathbf{v}} &= -\frac{G(M+m)\mathbf{r}}{r^3}\end{aligned}\tag{24}$$

Where \mathbf{r} is the distance between the planet and its sun. This set of first order ODEs does have a simple enough analytical solution against which we can compare our numerical result, although finding it is nontrivial and motivates the use of numerical analysis for more complicated problems. The solution is, in polar coordinates,

$$r(\theta(t)) = \frac{C}{1 - \epsilon \cos(\theta(t) - \theta_0)}\tag{25}$$

where for an initial distance x_0 , $C = x_0^2$ and $\epsilon = |1 - |x_0||$, the latter of which is known as the eccentricity. However, of more importance than the analytical solution of the the Kepler Problem are Kepler's Laws of Planetary Motion which are:

1. Planets move in elliptical orbits with Sun at one focus of the ellipse.

2. The amount of area per unit time swept out by a line joining a planet to the Sun is constant.

3. The period of the orbit T is proportional to $a^{3/2}$, where a is the semi-major axis of the ellipse.

We wish to confirm these laws by solving equation (24) numerically and checking if the orbit of our result is an ellipse, checking the second law by seeing if angular momentum is conserved, and checking the third law by plotting the orbital period T versus the semi-major axis a in a LogLog scale plot and finding that the slope is $\frac{3}{2}$. Also of note is the fact that the eccentricity determines that shape of the orbit, where $\epsilon = 0$ is a circle, $\epsilon < 1$ is an ellipse, $\epsilon = 1$ is a parabola, and $\epsilon > 1$ is a hyperbola. Because we want to confirm the first law, we will only consider closed orbits, meaning our initial position x_0 may only have the range $0 < x_0 < 2$. We now briefly explain the numerical methods used to solve the Kepler problem.

2.2 Numerical Methods

As explained in the harmonic oscillator section, our numerical methods require a set of first-order ODEs. Fortunately, equation (24) is already in the necessary form, however we wish to make them simpler by normalizing the units of the equations. The distance between planets \mathbf{r} will be normalized and measured in units of a_0 , suggested as being the mean distance from the Earth to the Sun. Time will be normalized by $\frac{1}{\Omega}$, where Ω is the angular frequency of the orbit is is equal to $\Omega = \sqrt{G(M+m)/a_0^3}$. Finally, mass will be normalized by the reduced mass μ . This normalization gives us the following simplified set of ODEs,

$$\begin{aligned}\dot{\mathbf{r}} &= \mathbf{v}, \\ \dot{\mathbf{v}} &= -\frac{\mathbf{r}}{r^3}\end{aligned}\tag{26}$$

Equation (26) can be further broken up into xy components, giving us the following four equations,

$$\begin{aligned}\dot{x} &= v_x, \\ \dot{y} &= v_y, \\ \dot{v}_x &= -\frac{x}{r^3}, \\ \dot{v}_y &= -\frac{y}{r^3},\end{aligned}\tag{27}$$

where $r = \sqrt{x^2 + y^2}$. Our vectors of independent variables and their first derivatives are therefore $\mathbf{y} = (x, y, v_x, v_y)$ and $\mathbf{f}(\mathbf{y}, t) = (v_x, v_y, -\frac{x}{r^3}, -\frac{y}{r^3})$.

Normalizing our set of ODEs also normalizes the expressions for angular momentum and energy. They are now

$$L = xv_y - yv_x\tag{28}$$

and

$$E = \frac{1}{2}v^2 - \frac{1}{r}\tag{29}$$

Finally, the initial conditions for the Kepler problem will always take the form $\mathbf{r}(0) = (x_0, 0)$ and $\mathbf{v}(0) = (0, 1)$. We now discuss how the above equations are implemented using the Stormer-Verlet method of numerical integration.

2.2.1 Stormer-Verlet Method

Recall the Stormer-Verlet Method, in particular the Velocity Verlet Method, described in the harmonic oscillator section. Inserting our vectors of independent variables and their first derivatives for the Kepler problem into the Velocity Verlet Method, equation (22), gives us the following four equations with which we can numerically integrate equation (26),

$$\begin{aligned}x_{n+1} &= x_n + h(v_n^x - \frac{h}{2} \frac{x_n}{(x_n^2 + y_n^2)^{3/2}}) \\ y_{n+1} &= y_n + h(v_n^y - \frac{h}{2} \frac{y_n}{(x_n^2 + y_n^2)^{3/2}}) \\ v_{n+1}^x &= v_n^x - \frac{h}{2} \frac{x_n}{(x_n^2 + y_n^2)^{3/2}} - \frac{h}{2} \frac{x_{n+1}}{(x_{n+1}^2 + y_{n+1}^2)^{3/2}} \\ v_{n+1}^y &= v_n^y - \frac{h}{2} \frac{y_n}{(x_n^2 + y_n^2)^{3/2}} - \frac{h}{2} \frac{y_{n+1}}{(x_{n+1}^2 + y_{n+1}^2)^{3/2}}\end{aligned}\tag{30}$$

where I used the explicit value for $r_n = (x_n^2 + y_n^2)^{3/2}$. Thus for initial conditions $\mathbf{r}(0) = (x_0, 0)$ and $\mathbf{v}(0) = (0, 1)$ we can generate a set of data representing the solution to the Kepler problem. Notice that the Kepler problem is a set of ODEs describing motion and that the acceleration terms all depend on position, in accordance with the assumptions made by the Velocity Verlet Method. Thus, the Kepler problem is an appropriate problem to solve using the symplectic Velocity Verlet Method. As before, when discretize the angular momentum and the energy of the system, giving us

$$L_{n+1} = x_{n+1}v_{n+1}^y - y_{n+1}v_{n+1}^x\tag{31}$$

and

$$E_{n+1} = \frac{1}{2}((v_{n+1}^x)^2 + (v_{n+1}^y)^2) - \frac{1}{\sqrt{x_{n+1}^2 + y_{n+1}^2}}\tag{32}$$

since $v_{n+1} = \sqrt{(v_{n+1}^x)^2 + (v_{n+1}^y)^2}$. Therefore, starting with our initial conditions we can find the angular momentum and energy and any future time. Using equations (30)-(32) we can now completely solve the Kepler problem numerically and confirm Kepler's Laws.

2.3 Results

As with the harmonic oscillator, we used a program written in Python to loop through equations (30)-(32) until a maximum number of steps, N , was reached. Recall that the maximum number of steps depends on the time step, which in turn depends on the method and, especially for the Kepler problem, the initial conditions.

What follows are solution plots the Kepler problem, equation (26), and confirmations of the Kepler's Law through numerical integration. Results are given for various initial conditions $\mathbf{r}(0) = (x_0, 0)$ and various time steps h , whereas the initial velocity is always $\mathbf{v}(0) = (0, 1)$. We begin by confirming Kepler's First Law of elliptical motion of planets.

2.3.1 Confirmation of Kepler's First Law

We first confirm Kepler's First Law by plotting the solutions of Kepler's ODEs, equation (26), both numerical and analytical, the latter of which is given by equation (25). For the Kepler problem, the solution plot is a plot of the plane of motion of the planets; a plot of the planet's y -position versus its x -position. Our initial x_0 is always less than or equal to 1 since, with the definition of the eccentricity as $\epsilon = |1 - |x_0||$, all possible values of eccentricity can be achieved using $0 < x_0 \leq 1$. In addition, interesting effects occur due to using a small value for x_0 , which will be discussed shortly. Our first solution plot contains plots for $x_0 = 0.9$ and $x_0 = 0.6$ at the time step values of $h = 0.1$ and $h = 0.001$,

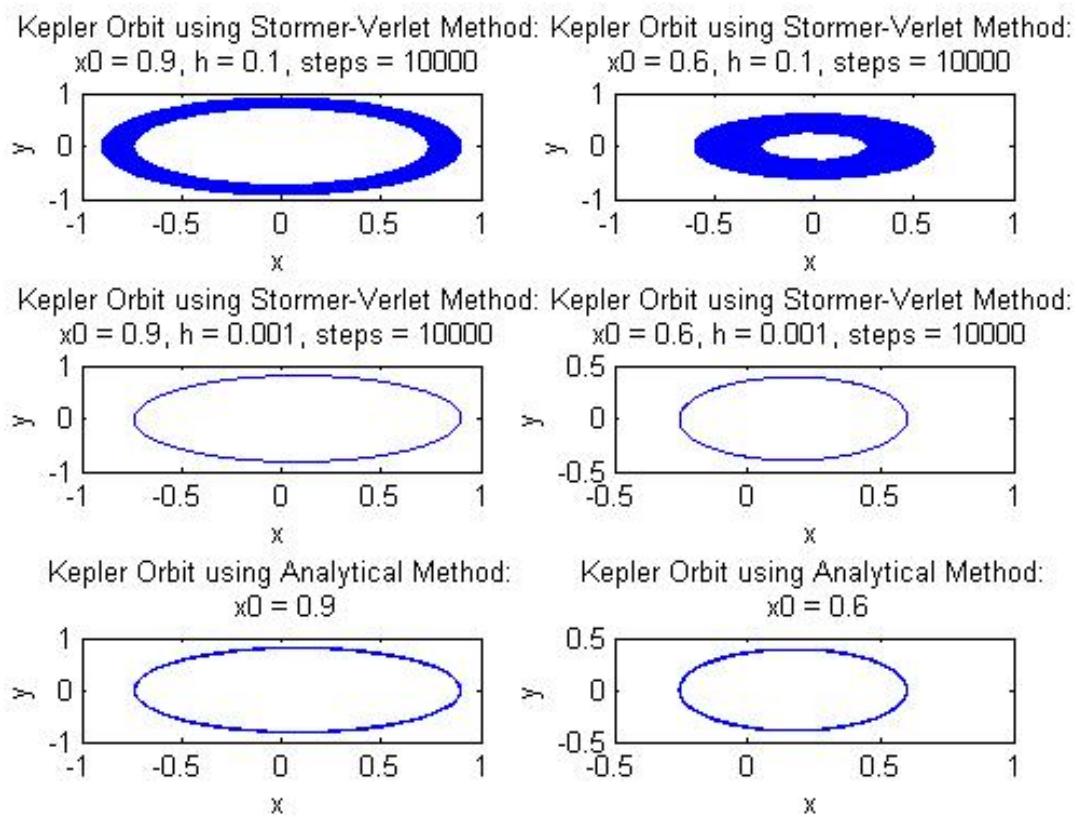


Figure 8: Kepler solution plots. The left column is for $x_0 = 0.9$ while the right column is for $x_0 = 0.6$. The top row is for $h = 0.1$, middle is $h = 0.001$, and the last row is the analytical solution, equation (26). Note the top two plots appear to have precession.

Figure 8 illustrates the interesting results foreshadowed earlier in regards to using a small value for x_0 . In the top plots we have what looks like a closed elliptical orbit, however precession seems to have occurred. This is odd because precession should only occur for at least a three body problem, e.g. the Sun, the Earth, and the Moon. For a two body problem such as this one, the solution given by our numerical computations should be a perfect ellipse, as shown in the third row. Notice, however, that middle row exhibits perfect elliptical orbits that agree with the analytical solutions, for the same x_0 as the top row. The difference between the rows is the time step h , and the reason this effects the orbital path is because of the round-off error discussed at the end of the harmonic oscillator section. Recall that the gravitational potential is proportional to $1/r^3$, therefore when we use an $x_0 < 1$, then $r^3 \ll 1$, which means we are dividing by a small number when numerically integrating using equation (30). As mentioned before, dividing by a small number causes a significant round-off error due to the machine's representation of numbers using finite-digits. This round-off error occurs at every iteration, becoming more significant as the planet approaches the Sun and r goes to zero. However, because the Stormer-Verlet Method does an excellent job of conserving energy, as shown in the harmonic oscillator section, it actually constrains the shape of the orbit to an ellipse, while the round-off error rotates the axis of the ellipse, creating the illusion of precession. What is shown in Figure 8 is that the round-off error is more significant for smaller x_0 , see $x_0 = 0.9$ versus $x_0 = 0.6$, and that the way to counter round-off error due to dividing by a small number is to multiply by a correspondingly time step. Using $h = 0.001$ instead of $h = 0.1$ in Figure 8 completely eliminates the precession due to round-off error, giving us perfectly elliptical orbits thereby confirming Kepler's First Law.

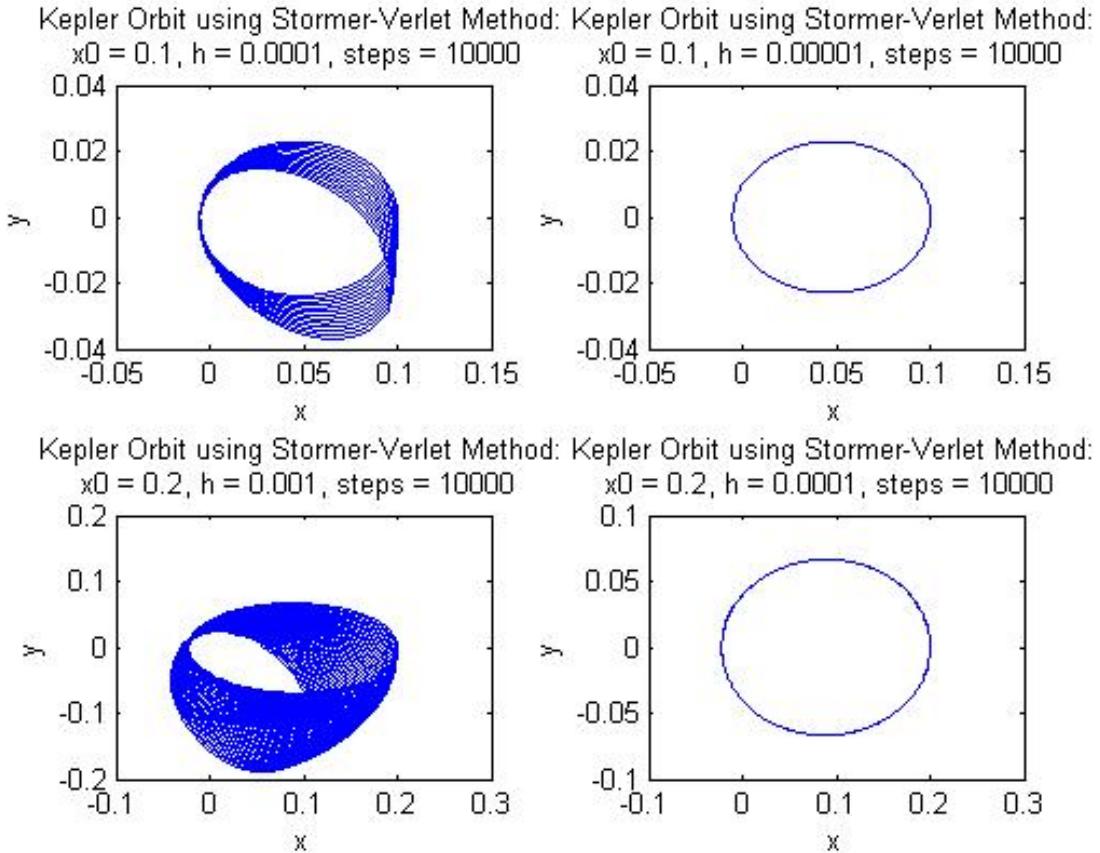


Figure 9: Kepler solution plots. The top row is for $x_0 = 0.1$ while the bottom row is for $x_0 = 0.2$. Note the different time steps needed to eliminate round-off error for the different values for x_0 .

A more extreme example of round-off error and its effects on the elliptical orbit for very small x_0 are shown in Figure 9 above. In general, the smaller the value of x_0 , the smaller the time step h needs to be to counter round-off error. Even x_0 that differ by 0.1, such is the case in Figure 9, require values of h that differ by a factor of ten.

Despite round-off error and its effects on the solution to the Kepler problem, Kepler's First Law is nonetheless confirmed since for a given x_0 , a proper choice for h will result in a perfectly elliptical orbit. To further confirm Kepler's First Law, note that the eccentricity for all of the orbits in Figure 8 and 9 is $\epsilon < 1$, which is what it should be for an ellipse. Figure 10 below is a Kepler solution where $\epsilon = 0$, meaning the orbit should be and be a circle. The orbit of the analytical and numerical solutions are indistinguishable.

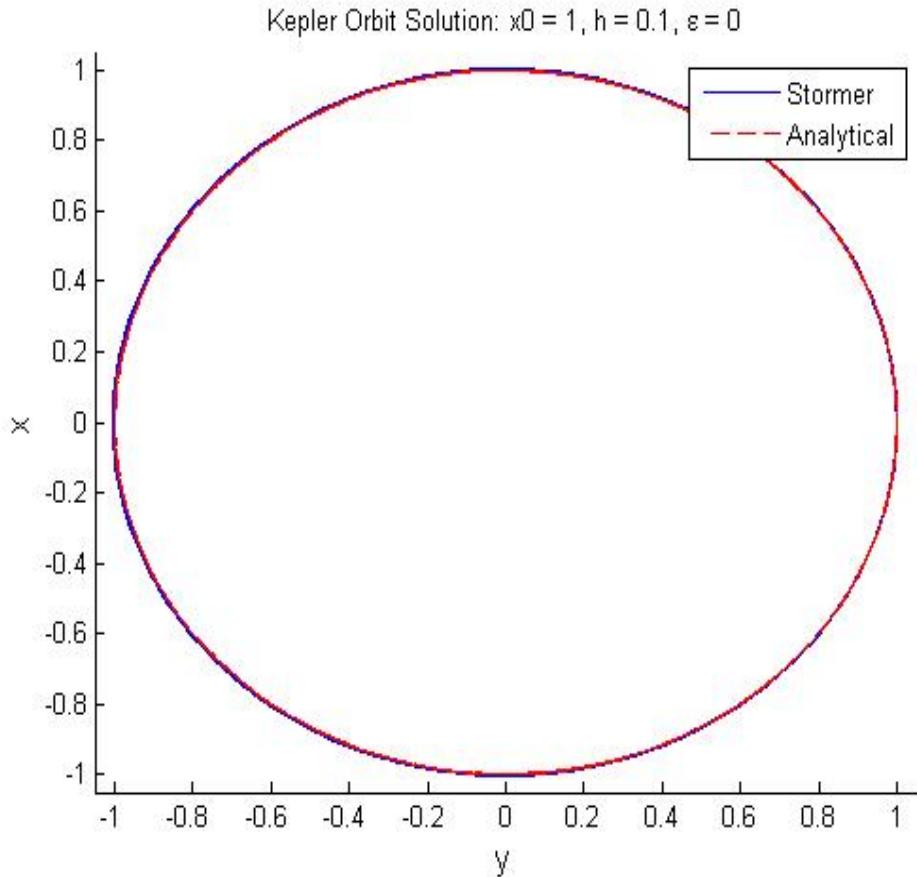


Figure 10: Kepler solution plot for $x_0 = 1$ and an eccentricity of $\epsilon = 0$. Notice that the orbit is a perfect circle, as it should be for $\epsilon = 0$, further confirming Kepler's First Law. Also note that a large time step of $h = 0.1$ does not result in round-off error because x_0 is not less than one.

2.4 Confirmation of Kepler's Second Law

Kepler's Second Law states that the amount of area per unit time swept out by a line joining a planet to the Sun is constant. Though rather daunting at first, this law is nothing more than a statement that angular momentum is conserved. Confirming Kepler's Second Law, then, is rather straight forward in that we use the discretized and normalized formulation of angular momentum, equation (31), and plot it against time. Because our initial conditions are always $\mathbf{r}(0) = (x_0, 0)$ and $\mathbf{v}(0) = (0, 1)$, our expression for the total angular momentum is just $L_{TOT} = L_0 = x_0$. Therefore our total angular momentum should be a constant value, always equal to the initial position x_0 . A

plot of the angular momentum as a function of time for different values of x_0 and time steps h is shown below.

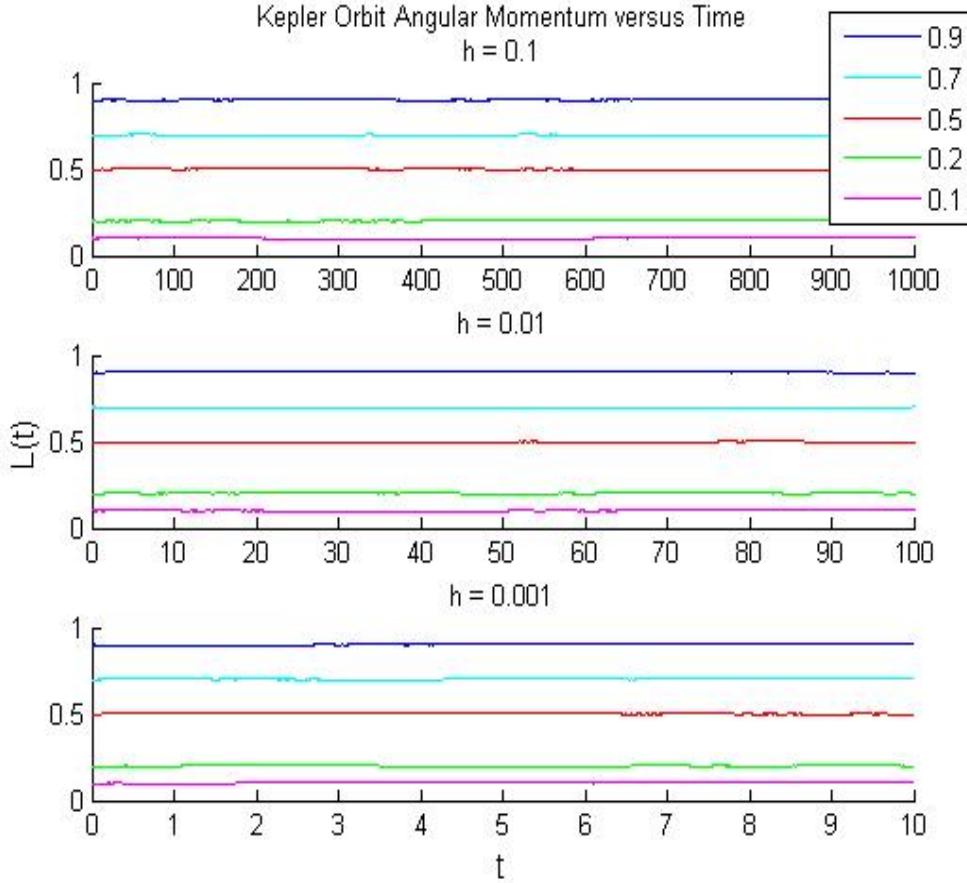


Figure 11: Plot of Kepler orbit angular momentum L versus time for various x_0 and time steps h . The legend at the top right is for different value of the initial position x_0 .

From Figure 11, it is clear that the angular momentum of the Kepler Orbit is well conserved since every plot is virtually a constant line with a value determined by x_0 . Thus, Kepler's Second Law is undoubtedly confirmed for time steps $h = 0.1$ or smaller and for initial positions in the range $0.1 \leq x_0 < 2$.

Though somewhat difficult to tell, it appears that the nearly straight lines for angular momentum are choppier for larger time steps h , and smoother for smaller time steps h . This would make sense as it is explained by both truncation error in the Stormer-Verlet approximation and round-off error just recently discussed. The smaller our time step, the better the approximation, and we can expect large errors to occur for overly large time steps.

2.4.1 Confirmation of Kepler's Third Law

Finally, we must confirm that the period of the orbit T is proportional to $a^{3/2}$, where a is the semi-major axis of the ellipse. The semi-major axis of an ellipse is half of the length of the longer axis of the ellipse. This was determined by finding the maximum and minimum x-coordinate in the Kepler orbit, adding/subtracting them and dividing by two. The maximum x-coordinate was always equal to x_0 , $x_{max} = x_0$. The minimum x-coordinate, x_{min} , was always the most negative coordinate. Therefore the semi-major axis of the elliptical orbit was,

$$a = \frac{x_{max} - x_{min}}{2} \quad (33)$$

where $x_{min} < 0$. The orbital period T was determined by searching through the data generated by the numerical computations. When the most negative value was first found, the period T was simply the time at which it occurred time two. In this manner we were able to generate a set of data points of orbital period T and the semi-major axis a . This data was plotted as a T v. a plot on a LogLog scale. Because $T \propto a^{3/2}$, then on a LogLog scale we have $\log(T) \propto \frac{3}{2} \log(a)$. Thus we should expect a straight line with a slope of about $\frac{3}{2}$ or 1.5. This is exactly what we found, as shown in Figure 12 below.

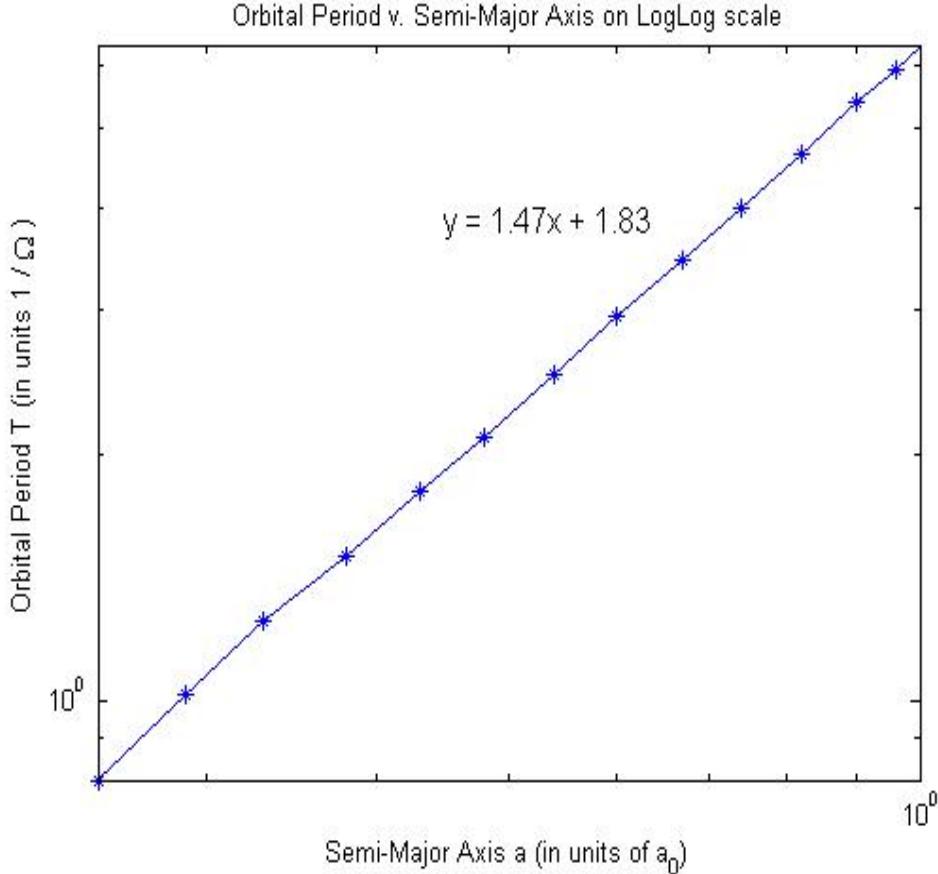


Figure 12: Plot of the orbital period T versus the semi-major axis a on a LogLog scale. Each star indicates a data point. The slope found for the data points used is 1.47, within 2% of the actual value of 1.5.

Clearly, the data points on a LogLog scale generated a fairly straight line. The equation on the figure indicates a slope of 1.47, which is within 2% of 1.5 or $\frac{3}{2}$, the actual value of the slope for T v. a plot. Nonetheless, a slope within 2% of the actual value is close enough to consider Kepler's Third Law confirmed. Possible round-off or truncation error, or human error when determining the orbital period, may be responsible for the slight inaccuracy.

2.5 Conclusion

After integrating the Kepler problem in the form of equation (27) using the Stormer-Verlet method, we were able to confirm all three of Kepler's Laws of Orbital Dynamics, as requested. Confirming the first law of elliptical orbits revealed the full effects of round-off error and the importance of choosing an appropriately small time step h when dividing by a small r value in the gravitational potential. However, the solutions to the Kepler problem also revealed the strength of the Stormer-Verlet

Method in that elliptical orbits were preserved despite precession due to round-off error. The fact that elliptical orbits are preserved by the Stormer-Verlet Method indicates how well it conserves angular momentum. This was point was reinforced when confirming Kepler's Second Law, showing that angular momentum was well conserved even for small x_0 values and large time step h values. Finally, we were able to confirm Kepler's Third Law within to 2% of the actual value, with possible round-off, truncation, or human error responsible for the slight inaccuracy.

In conclusion, the most practical take away from the Kepler problem is being mindful of the physical and computational limitations of the system at hand. As the distance between the Sun and a planet approaches zero, the gravitational potential of Kepler's problem begins to blow up. Likewise, the numerical computations made reflect this limit by introducing large round-off errors which ruin your data. These errors can be counteracted by lowering the time step, literally decreasing the rate at which r approaches zero thus generating a more reasonable solution. When integrating numerically, therefore, it is important to consider all extremes and take the necessary precautions handle them.

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