Applied Numerical Methods - Math 151B

Homework 6

Erik Lamb

May 9, 2016

In this assignment, we were to solve the Van der Pol's equation for current in a vacuum tube with three internal elements, given by,

$$y'' = \mu(y^2 - 1) - y, \quad \mu > 0; \quad y(0) = 0, \quad y(2) = 1. \tag{1}$$

Equation (1) is in fact a Boundary Value Problem (BVP), a non-linear BVP, which requires special numerical methods to solve. Namely, solving a non-linear BVP requires turning it into a regular Initial Value Problem (IVP) and then using a numerical method to find the final solution y. For this assignment, we were to implement the Nonlinear Shooting method and the Nonlinear Finite-Difference method to solve the Van der Pol BVP given by (1). The Nonlinear Shooting method was implemented using algorithm 11.2 from *Numerical Analysis* by Burden, Faires and Burden by writing a function in MATLAB, named shoot.m. The Nonlinear Finite-Difference method was partly implemented using algorithm 11.4 from *Numerical Methods*, but was mostly derived from example 1 from lecture 16, and was also written in MATLAB with the function name fd.m. Both methods utilize the Newton method in their approximations, however shoot.m uses the Newton method in converting the BVP to an IVP and then uses RK4 to solve for y and y' whereas fd.m uses the Newton method to solve for y. Both methods can be expressed as tridiagonal matrices, but only the function fd.m utilizes the MATLAB "left division" operator for matrices which calculates $S(\overrightarrow{w}^{(k)}) = J(\overrightarrow{w}^{(k)})^{-1}F(\overrightarrow{w}^{(k)})$, where F is the matrix of functions and J is the Jacobean matrix of all F_i. This allows us to simply implement the final step using the Nonlinear Finite-Difference method which is simply the Newton approximation $\overrightarrow{w}^{k+1} = \overrightarrow{w}^{k} + S(\overrightarrow{w}^{(k)})$.

Having briefly discussed the origin of our numerical functions shoot.m and fd.m, let us now discuss their approximations for the solution of the BVP given by (1). As mentioned, all approximations used a spatial step of $h = 0.1$, and the Van der Pol equation was solved by each method for $\mu = 0.1, 1,$ and 2. In Figure 1 on the next page, we give our initial attempt for our approximations of the Van der Pol equation. Solution curves using the function fd.m are solid blue and solution curves using shoot.m are dashed red. Notice that there are three curves per figure, with each curve representing a solution for a different value of μ. The bottom curve is for $\mu = 0.1$, the middle is for $\mu = 1$, and the top is for $\mu = 2$, as shown by the labels. This, of course, describes the relation of the solutions with μ. Although all solutions must have the same boundary conditions and all start at $(0, 0)$ and end at $(2, 1)$, the paths traveled differ from μ to μ. The large the μ the larger the amplitude of the solution. Physically, this would mean the current moving through the vacuum tube is larger for a larger constant μ, which represents some physical quantity of the tube. Also of note is that the slope increases with larger μ, which makes sense because if a solution is to have a large amplitude in within the same domain of $x = [0, 2]$ then the solution must increase more rapidly.

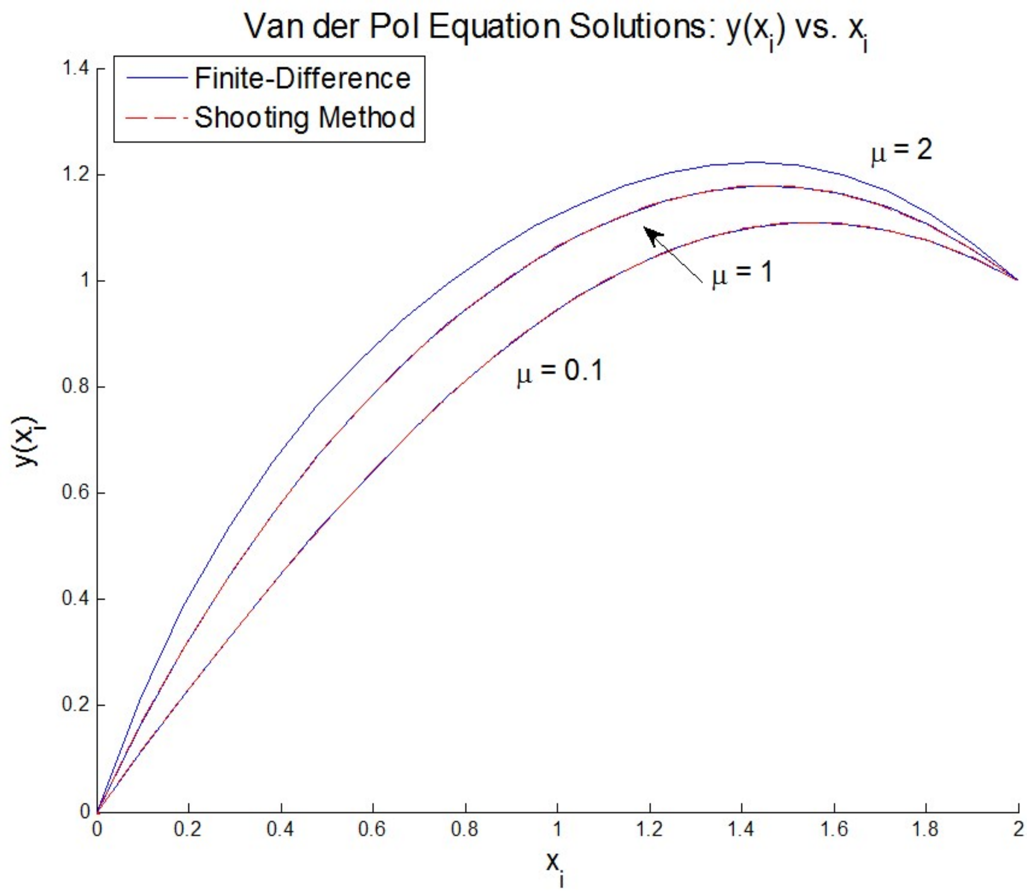

Figure 1: This is a plot of solutions to the Van der Pol's BVP given by (1) using both the Nonlinear Finite-Difference method and the Nonlinear Shooting method. Each curve represents a solution for different μ. Notice that solutions for the finite-difference method are solid blue and solutions for the shooting method are dashed red. Also notice that the shooting method does not have a unique solution for (1) for $\mu = 2$. This is because the solution to (1) rapidly increases for $\mu = 2$ and the shooting method is sensitive to round-off error and thus needs a good initial approximation of the slope t_0, because of its utilization of Newton's method, otherwise the shooting method will not converge. The initial slope can no longer be approximated by (2) because the shooting method does not have a unique solution for $\mu = 2$ with initial slope (2). The finite-difference method does not have this issue because the Newton's method is not used to find roots as it is in the shooting method.

Solutions using both numerical methods appear identical, and are difficult to distinguish in figure 1. However, upon closer inspection we notice that there *is no* solution for $\mu = 2$ using the nonlinear shooting method. Indeed only the nonlinear finite-difference method provides a solution for $\mu = 2$. This is due to our initial approximation for the slope y' in the algorithm implemented in the function shoot.m. As given by algorithm 11.2, our initial approximation for the slope of the solution of (1) is given by

$$t_0 = \frac{\beta - \alpha}{b - a}, \tag{2}$$

for boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ on the domain $x = [a, b]$. According the remark in lecture 14, a nonlinear BVP of the form $y'' = f(x, y, y')$ will converge for any initial approximation of the slope t_0 only if $f(x, y, y')$ satisfies the conditions required for existence and uniqueness of solutions of the BVP. Also, recall that the nonlinear shooting method uses the Newton method in its root-finding approximation, as seen in lecture 14. While this gives fast convergence, for rapidly increasing functions it is very sensitive to round-off error, requiring a good initial approximation for t_0. Therefore, it appears that while the nonlinear finite-difference method is convergent as has an unique solution for all μ, the nonlinear shooting method does not and is not convergent for $\mu = 2$. Evidently, the BVP with $\mu = 2$ has a solution that rapidly increases, which leads to the shooting method being unstable for bad first approximations given by (2). Thus, the shooting method does not have a unique solution of the Van der Pol's BVP for $\mu = 2$ and initial approximation given by (2)

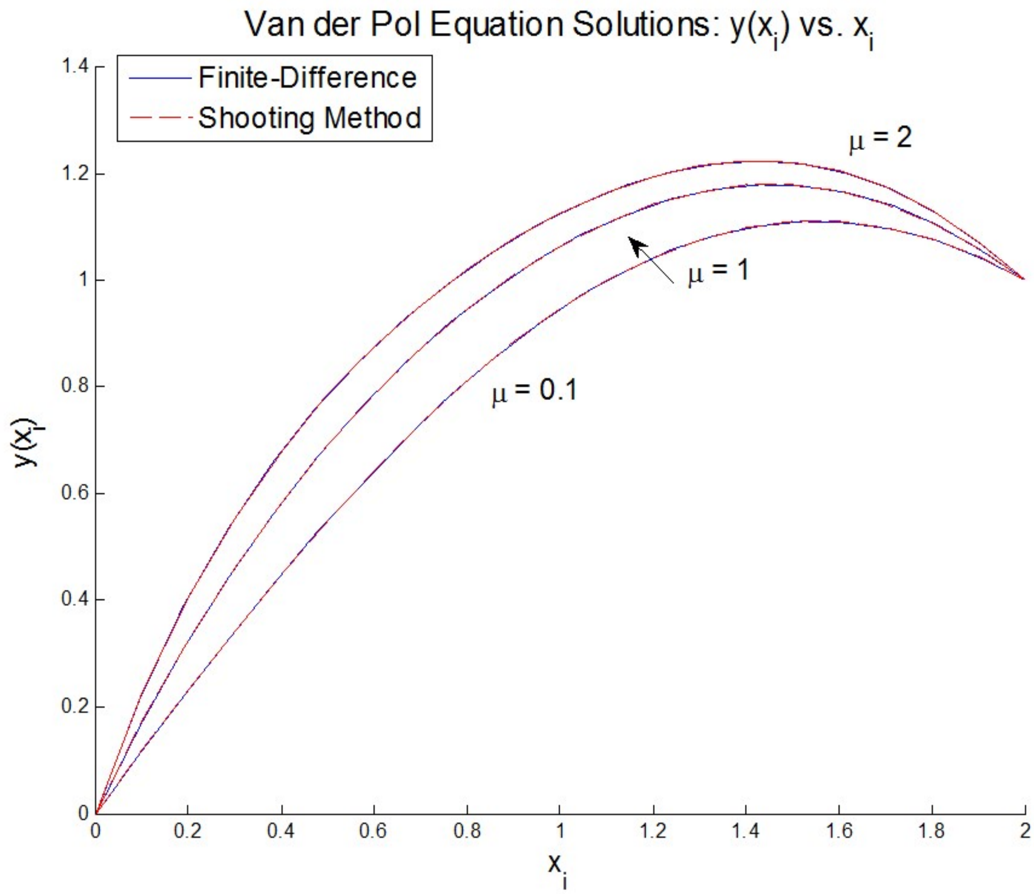

Figure 2: This plot is identical to figure 1 with the exception that the shooting method now provides a solution for $\mu = 2$. This is because we provided a better first approximation of the slope of the solution of the Van der Pol's equation, y'. This first approximation is given by (3), and was determined by calculating the slope of the first two data points approximated by the finite-difference method, finding $t_0 = 2.12$. Because the shooting method was convergent and had unique solutions for $\mu = 0.1$, and 1, changing the initial approximation t_0 has no effect on their curves. It does, however, make the approximation for (1) with μ using the shooting method convergent, meaning the shooting method has a unique solution for the Van der Pol's BVP for all $\mu = 0.1, 1,$ and 2, with initial slope approximation $t_0 = 2.12$.

From the boundary conditions in (1), we see that the initial approximation of the slope at $x = 0$ is $t_0 = 0.5$. By inspection of the solution for (1) using the finite-difference method in figure 1, we see that $t_0 = 0.5$ is far too small of an approximation for the initial slope using the shooting method. In reality, the initial slope is closer to

$$t_0 = 2.12, \tag{3}$$

found by calculating the slope of tangent line of the first two data points using the finite-difference method. Using the initial approximation $t_0 = 2.12$ for the shooting method produces figure 2. From figure 2 we see that the shooting method is now convergent and has a unique solution for all values of μ with initial slope approximation $t_0 = 2.12$. The solution using the shooting method is now identical to that using the finite-difference method. Though we changed the initial approximation t_0, note that the shooting method approximations for smaller μ are the same, since these methods were already stable.