Applied Numerical Methods - Math 151B

Homework 2

Erik Lamb

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1 Part a: 2nd Order Taylor Method

For this section we used the 2nd Order Taylor method to approximate the given IVP. Figure 1 shows three approximations of the IVP using the 2nd Order Taylor Method (henceforth called taylor2) for three time-steps h that decrease in size. For an initial step size of $h = 0.2$, taylor2 does not approximate the exact solution very well, except for at the beginning of the interval where all approximations should have the same value. In fact, taylor2 best approximates the actual solution for lower values of time t for all time steps h. This approximation worsens as time progresses. Deceasing the time step h, however, does improve the approximation of taylor2 as decreasing the values of h gradually increases the values of the approximation. The exact solution for the IVP is best approximated by taylor2 using $h = 0.05$, in this case.

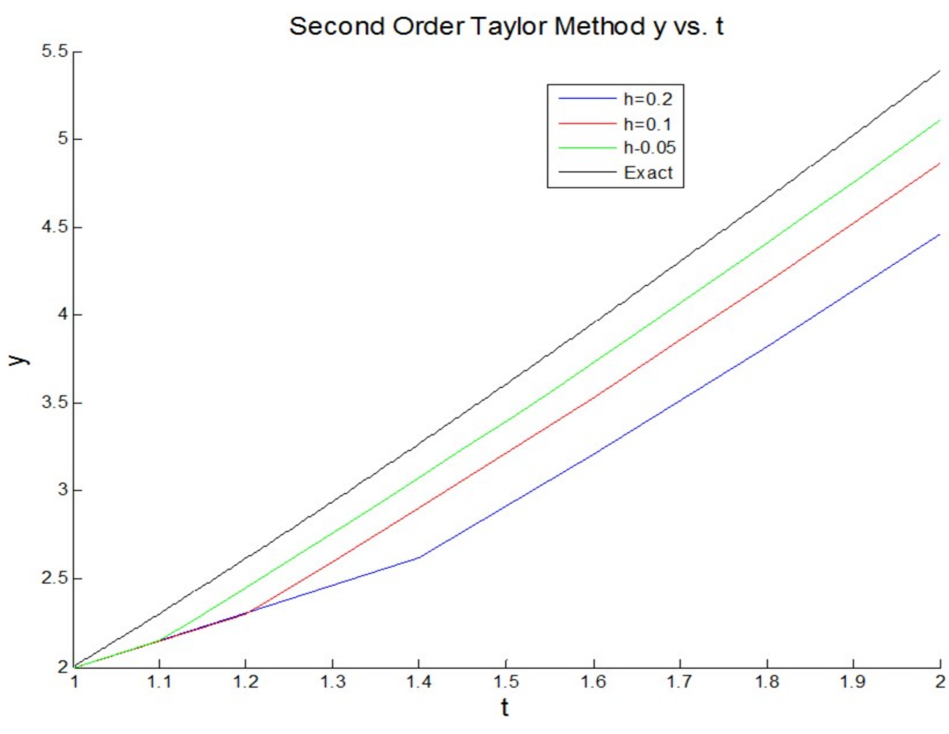

Figure 1: 2nd Order Taylor method approximations for the given IVP. Notice how the 2nd Order Taylor method under approximates the exact solution of the IVP. As the step size h is decreased, the 2nd Order Taylor method better approximates the exact solution.

Next, we consider the error of the taylor2 method. As expected, as we decrease the step size h, the absolute error decreases and is the least for the step size $h = 0.05$. By comparing the largest values of the error for each step size h, we can determine the relationship between the absolute error $e(h)$ and the time step h. For $h = 0.2$, the maximum error is about $e(0.2) \approx 0.9$. When we cut the time step in half to $h = 0.1$, the error decreases by about half, having a maximum error of $e(0.1) \approx 0.45$. As we cut the time step in half again to $h = 0.05$, the maximum error is once again about cut in half to around $e(0.05) \approx 0.225$. The fact that the absolute error is cut in half when the time step is cut in half suggests the error has a linear relationship in the time step h. Therefore, the 2nd Order Taylor method is of order one.

Because taylor2 is of order one, if we wanted to an accuracy of 10^{-4} for $y(2)$, we would need a step size of about $h = 2 \times 10^{-5}$. This is because the error for $h = 0.2$ is about one, thus if we want to decrease the error from about one to 10^{-4}, and because the error is related linearly to h, we simply decrease h by factor of 10^{-4}.

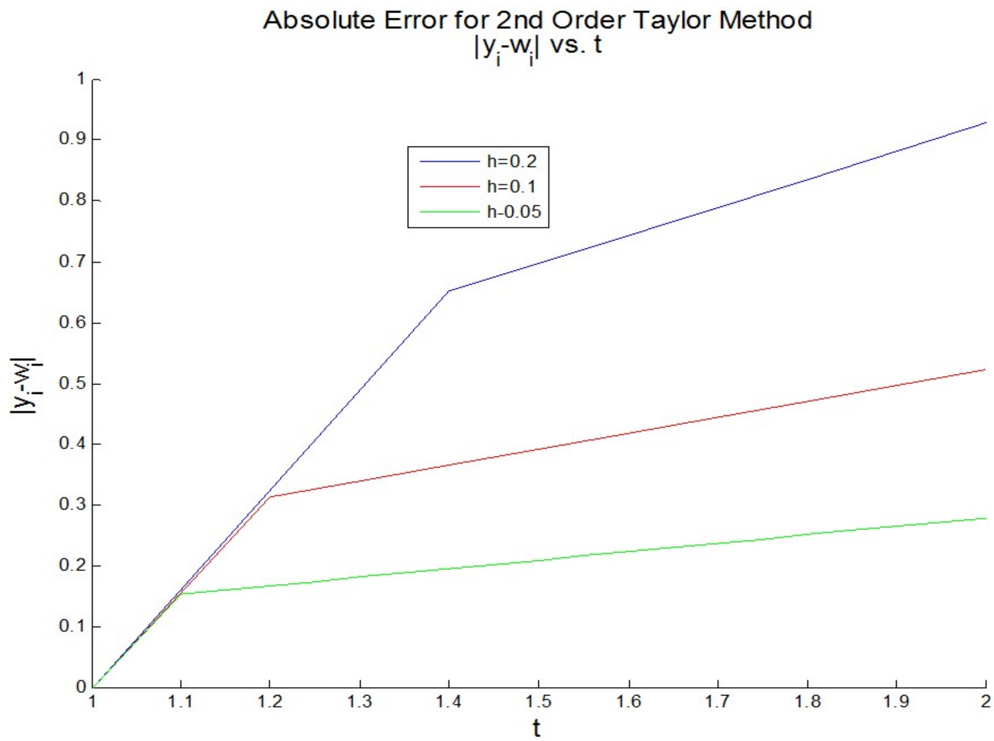

Figure 2: Absolute error $|y(t_i) - w_i|$ for each approximation with different time-step h. Notice how the error gets smaller as the time step is decreased. In fact, the error appears to have a linear relationship with h, being halved when h is halved. This would suggest taylor2 is of order 1.

2 Part b: Midpoint Method

For this section we consider the Midpoint method. In figure 3 below is the approximation due to the midpoint method for the same different values of h as before. By looking at figure 3, we notice it is identical to the different approximations using taylor2. Therefore the same discussion applies to the midpoint method as it did for taylor2; the method best approximates the exact solution for low values of time t and for small values of h.

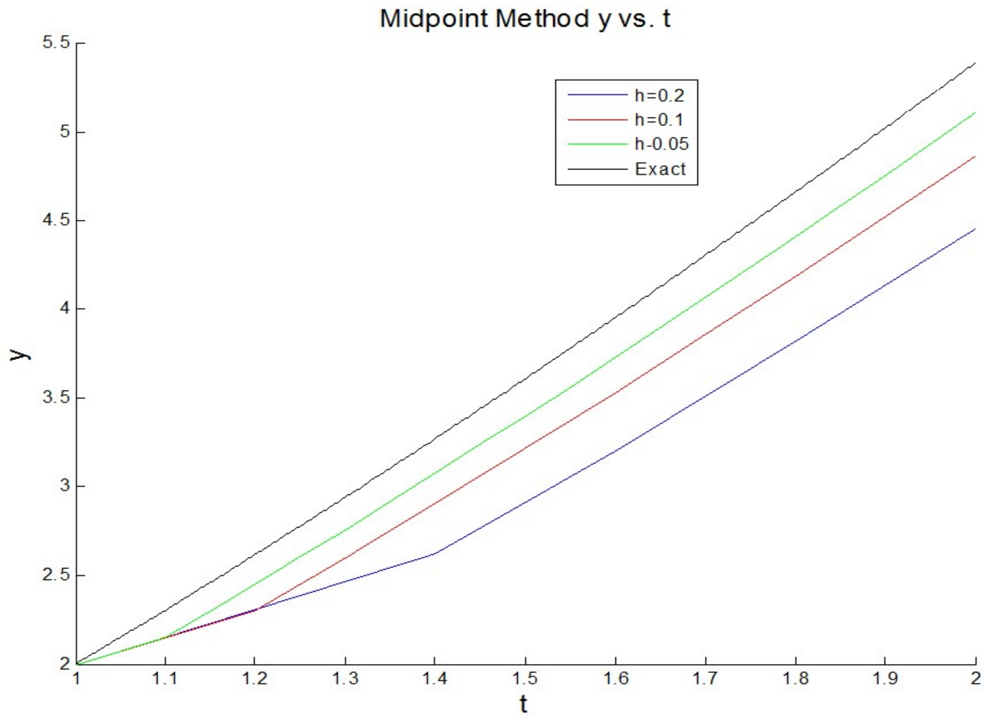

Figure 3: Midpoint method approximations for the given IVP. Notice how the Midpoint method under approximates the exact solution of the IVP. As the step size h is decreased, the Midpoint method better approximates the exact solution. Also notice that this figure is identical to Fig. 1, the approximations using taylor2.

As before we next consider the absolute error of the midpoint method as a function of h. The results are shown in figure 4, and once again they appear identical to their taylor2 counterpart. By the exact same analysis, we determine that the midpoint method is of order one and would require a time step of $h = 2 \times 10^{-5}$ in order to approximate the exact solution to 10^{-4}.

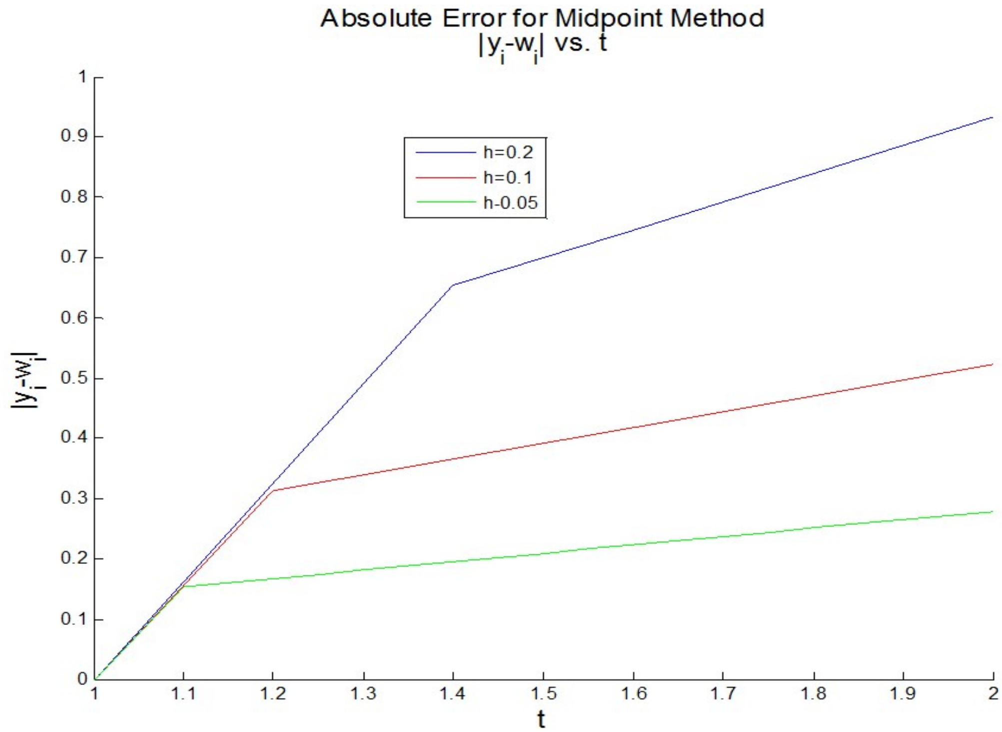

Figure 4: Absolute error $|y(t_i) - w_i|$ for each approximation with different time-step h. Notice how the error gets smaller as the time step is decreased. In fact, the error appears to have a linear relationship with h, being halved when h is halved. This would suggest the midpoint method is of order 1, just as we found for the taylor method.

3 Part c

Finally, we compare the run times T of taylor2 with the midpoint method. We plotted the run times as a function of the time step h for both methods in figure 5. Both curves for both methods have essentially the same shape, however the midpoint method clearly requires less time to complete running across all time steps h. The fact that the midpoint method is faster than the 2nd order Taylor method, but is otherwise indistinguishable in the approximations and the error, suggests that the midpoint method is the more superior and more efficient method of the two.

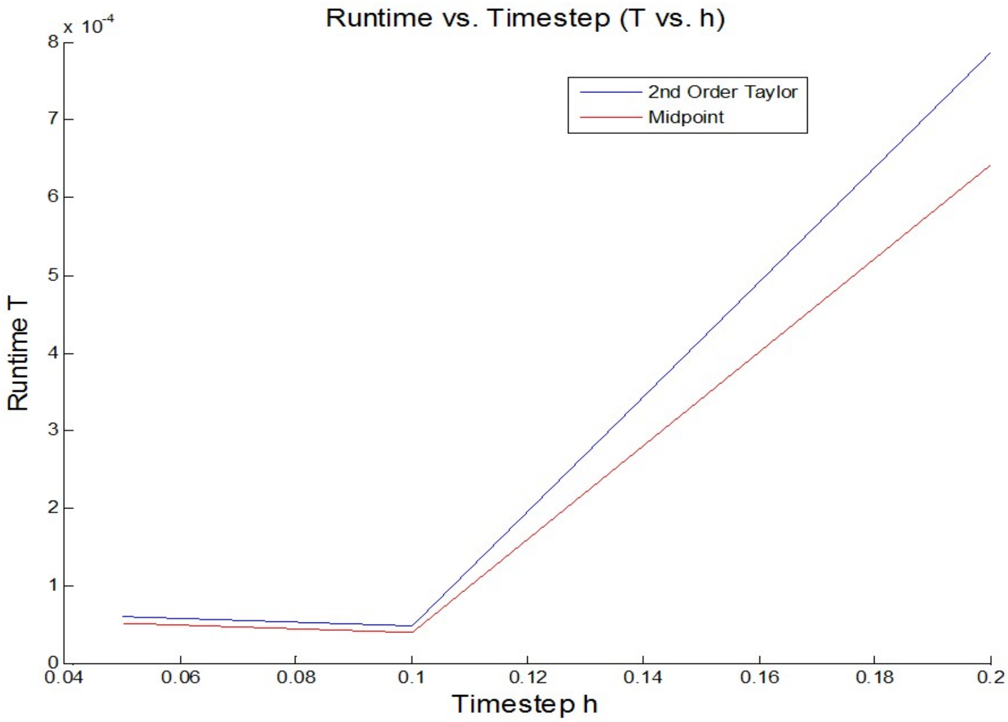

Figure 5: Plot of the runtime T as a function of h for both the 2nd order Taylor method and the midpoint method. Notice that the midpoint method has a smaller runtime for all values of h, indicating it is the more efficient method.