Applied Numerical Methods - Math 151B

Homework 1

Erik Lamb

April 4, 2016

1 Parts a and b

For this problem, we were to solve the following initial value problem (IVP) numerically using Euler's method.

$$\begin{aligned} y'(t) &= r(1 - \frac{y}{K})y, \qquad 0 \le t \le 50 \\ y(0) &= y_0 \end{aligned} \tag{1}$$

The above IVP models population growth, and the it was solved using the values $r = 0.2$, $K = 4000$, and $y_0 = 1000$. The exact solution to the IVP above is given by

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \tag{2}$$

Euler's method is a simple numerical method that was used to solve the IVP given by (1). An approximation is evaluated at discrete times separated by a constant time-step h. The time-step is given by $h = (b - a)/N$ where $b = 50$, $a = 0$, and N the number of iterations. We approximate the solution to (1) at these discrete time-steps using

$$\begin{aligned} w_0 &= y_0 \\ w_{i+1} &= w_i + hf(t_i, w_i) \end{aligned} \tag{3}$$

where $f(t, y) = r(1 - \frac{y}{K})y$ for this problem. Approximations to the solution of the IVP (1) were found for the different time-steps $h = 10$, $h = 1$, and $h = 0.1$, which correspond to approximations with $N = 5$, $N = 50$, and $N = 500$ iterations or time-steps, respectively. These approximations and the actual value of the solution, given by (2), are shown in Figure 1 below.

In Figure 1, the actual solution for $h = 0.1$ represents the actual solution for all time from $t = 0$ to $t = 50$. However, the other two actual solutions using the same discrete time values t_i are also given. All solutions have the same, very general behavior, in that they appear to not approximate the actual solution for small time values but more closely approximate the actual solution for larger time values. All solutions also rapidly increase and begin to level off for large time, representing what occurs when a population approaches its carrying capacity. The three approximations however, differ significantly in just how closely they approximate the corresponding actual solutions. Larger h values lead to jagged, very inaccurate approximations to the actual solution while smaller h values decrease the absolute error to zero.

Figure 1 on the next page shows how dramatic of an effect the step size has on the accuracy of the solution. For $h = 10$, only five approximations are made, creating a wildly inaccurate solution plot to the IVP. However, as the time-step is decreased and more data points are used, the approximation plots more closely approximate the actual solution at the same times, given by (2). Although the approximation for $h = 1$ is still somewhat inaccurate for small time values, by about $t = 20$ the approximation is virtually indistinguishable from the actual solution. By $h = 0.1$, where

500 approximations are made, the Euler approximation appears to match the actual solution over the entire time interval. It appears that the more data points or approximations that are made, the more accurate the numerical solutions to an IVP. This makes sense because Euler's method as given by (3), so closely depends on previous approximations w_i. If a previous approximation w_i is separated by a large time step h from the next approximation, we should not expect it to give us a good description of future approximations w_{i+1}, leading to larger error. Hence, if we use smaller time-steps, the approximations are closer together and the information given by previous approximations w_i is relevant to current approximations w_{i+1}.

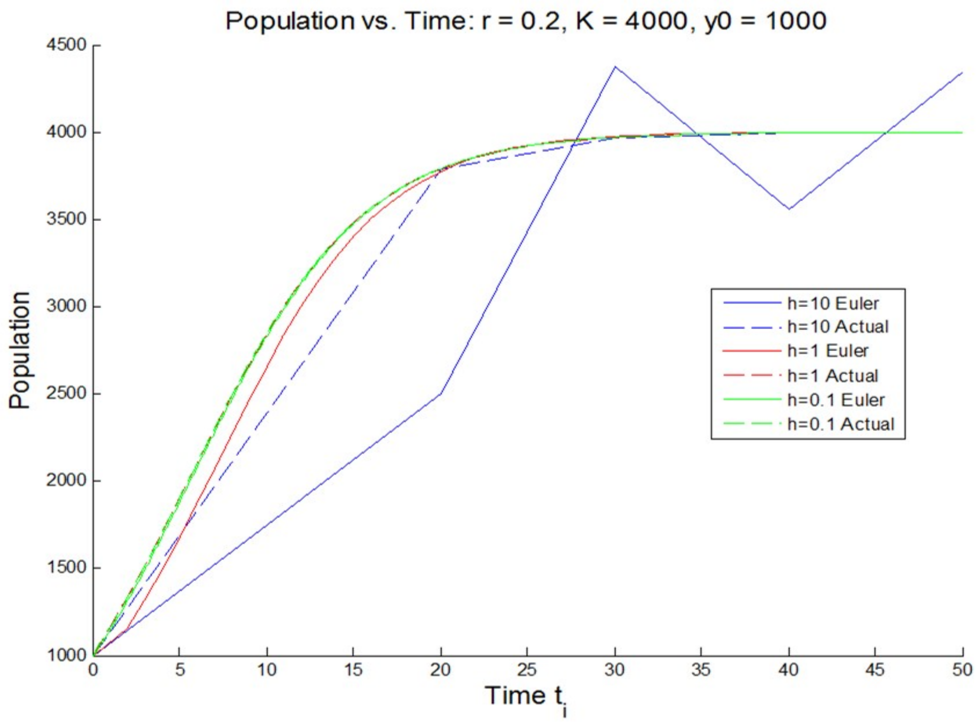

Figure 1: Euler approximations to the IVP given by (1) and the corresponding actual values at the same discrete times given by (2). Solid lines are the Euler approximations for a given time-step h and dashed lines are the actual values at the same times. Notice how as h is decreased the Euler approximations approach the actual solution.

In addition to Fig. 1, to see how closely the Euler approximations approach the actual solutions for $h = 1$ and $h = 0.1$, a zoomed in plot is shown below.

Another plot given in Figure 3 is the plot of the absolute error, $|y(t_i) - w_i|$, versus time for each approximation. Fig. 3 shows even more clearly how the absolute error of each Euler approximation decreases with time and approaches zero as the time-step is decreased. Both decreases in error can be attributed to the fact that more data points are taken into consideration. From Fig. 3 we see that each approximation has its own maximum error $max|y(t_i) - w_i|$. For $h = 10$ the maximum error is $max|y(t_i) - w_i| = 1291.7$, for $h = 1$ it is $max|y(t_i) - w_i| = 229.16$, and for $h = 0.1$ it is $max|y(t_i) - w_i| = 22.889$. Recall that the error bound for Euler's method as given by Theorem 1 from Lecture 2 is given by

$$|y(t_i) - w_i| \leq \frac{hM}{2L}(e^{L(t_i - a)} - 1) \tag{4}$$

where M is upper bound of the second derivative of the solution $y(t)$ and L is the Lipschitz constant. A figure of the second derivative is shown in Figure 4 and from it we see that $M \approx 15$. The Lipschitz constant L, unfortunately, cannot be determined from the given IVP, but holding L constant, we can compare (4) to the actual absolute values given earlier. We see that, like the values given above, the error bound of the Euler method decreases with h, and appears to approach zero as h approaches zero, as our given maximum values and Fig. 3 suggests. We know, however, that due to round-off error we cannot decreases h indefinitely to eliminate error, as the error due to round-off will eventually become significant. Nonetheless, our values of h are well above the minimum h value where round-off error becomes significant, thus decreasing h results in smaller error. Unlike equation (4), our absolute error appears to decrease with time as more data points are used. This is not a concern, however, since (4) merely gives an upper bound on the error for Euler's method.

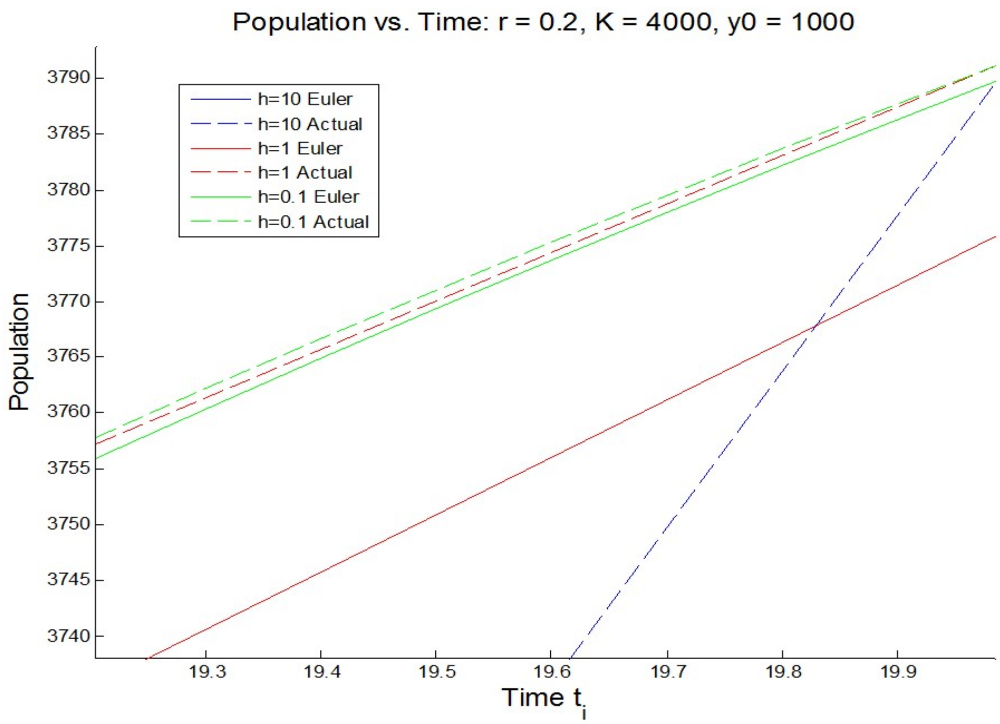

Figure 2: Euler approximations to the IVP given by (1) and the corresponding actual values at the same discrete times given by (2). This is a zoomed in image of Fig. 1 to show how smaller h values lead to better approximations.

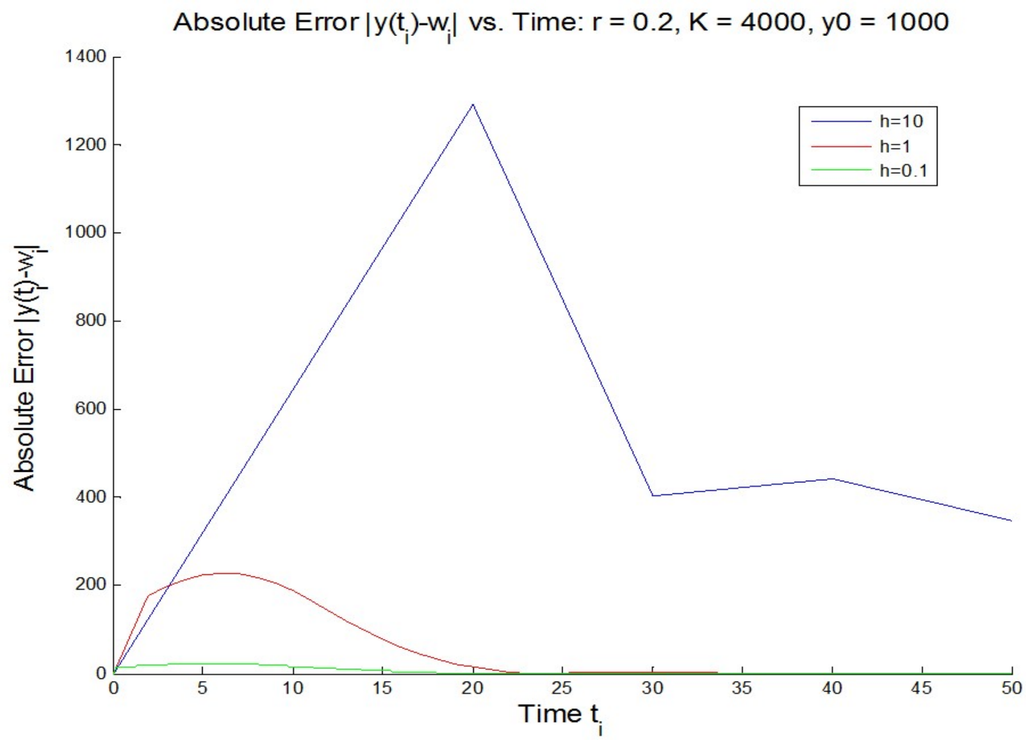

Figure 3: Absolute error $|y(t_i) - w_i|$ for each approximation with different time-step h. Notice how the error gets smaller as time increase and approaches zero at h is decreased.

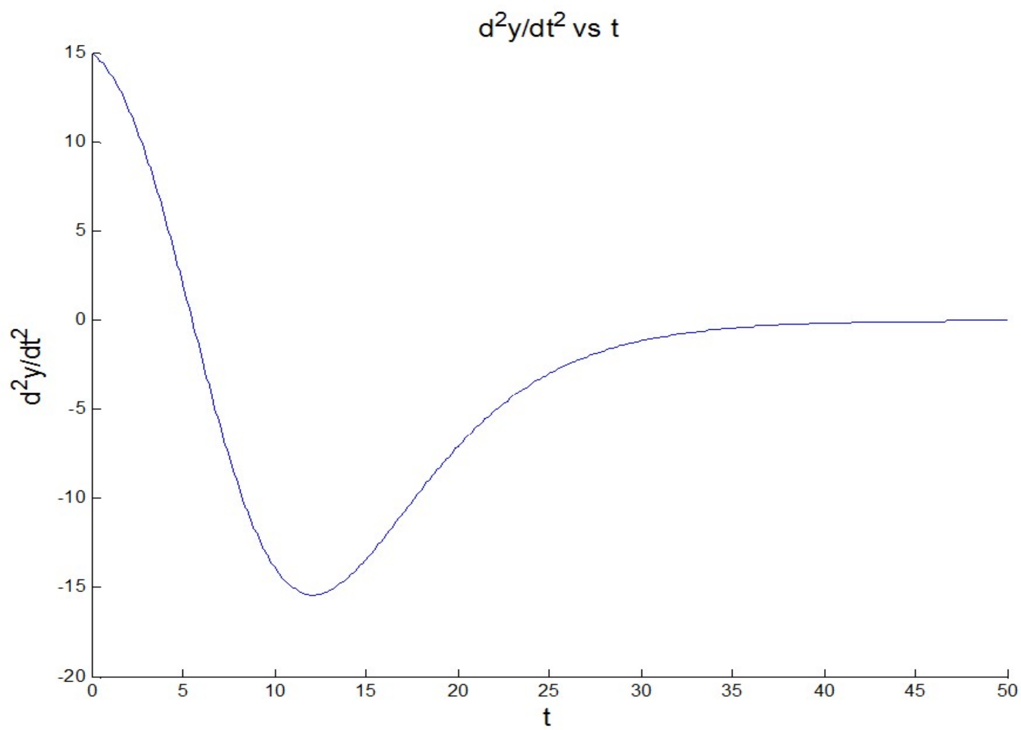

Figure 4: Figure of $y''(t)$. From this figure we can see that $|y''(t)| \leq 15$.