Optimisation Methods: Assignment 6

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The purpose of this assignment is to familiarise yourself with the terminology and optimality conditions in constrained optimisation.

It is due for **Wednesday 14th May at 2 pm**. Only the third part (Python implementation) will be graded but you can also submit your answers for the first part and the latter will also be corrected (though not graded). For the first part, you need to submit a Python notebook on iCorsi and follow these instructions:

- Put the answers for each part of the question into separate cells.
- Before each cell, put a markdown header that says which part of the question comes in the following cell.
- Good coding style is part of the grade: add clear comments throughout the code when it is necessary.
- Before you submit your notebook, make sure it runs.

Part 0: reading

Read the Chapter 12 of the book "Numerical Optimisation" (Nocedal & Wright).

Part 1: theory

Exercise 1 (Convex problems)
Consider the problem

$$\min_{x \in \Omega} f(x).$$

1. *Show that if f is convex and the feasible set $\Omega \subset \mathbb{R}^n$ is convex, then any local minimiser is a global minimiser.*
2. *Prove that the set of global minimisers is convex.*

Exercise 2 (First-order necessary optimality condition with convex feasible set)
Consider the problem

$$\min_{x \in \Omega} f(x),$$

with $\Omega \subset \mathbb{R}^n$ a convex set and f a continuously differentiable function. Prove that if x^* is a local minimiser, then

$$\nabla f(x^*)^T(x - x^*) \geq 0, \qquad \forall x \in \Omega.$$

Part 2: exam-type exercises

Exercise 3
Consider the halfspace defined by

$$\Omega = \{x \in \mathbb{R}^n : a^T x + b \geq 0\}$$

with $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$. Write and solve the optimisation problem for finding the point $x \in \Omega$ that has the smallest Euclidean norm. Hint: the $x \in \Omega$ with the smallest Euclidean norm is also the $x \in \Omega$ with the smallest squared Euclidean norm.

Exercise 4
Consider the unit disk in $\mathbb{R}^2$ defined as

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}.$$

1. *Find the maximiser of $f(x) = x_1 x_2$ over Ω.*
2. *Find the minimiser of f over the unit circle defined as*

$$C = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}.$$

Exercise 5
Consider the polyhedron $P \subset \mathbb{R}^2$ defined by the following set of constraints:

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 &\leq 2 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

1. *Draw P.*
2. *Write P in standard form.*
3. *Enumerate all basic solutions and basic feasible solutions of P.*

Part 3: programming problems

Problem 1 (Active constraints)
Consider the following constrained and linear optimisation problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \; & 2x_1 + x_2 \\ \textit{subject to} \quad & x_1 + x_2 \geq 3 \\ & x_1 + 2x_2 \leq 6 \\ & x_1 \geq 1 \\ & x_2 \geq 1. \end{aligned}$$

1. *For each constraint, write a function* `is_ci_verified` *that checks for any point x if the constraint is verified at x (this function should return a boolean).*

2. *Using the previous functions, write a function* `is_feasible` *that checks if a point x is feasible (this function should return a boolean).*

3. *Test it on $x^0 = (1, 2)$ and $x^1 = (3, 1.2)$. Are x^0 and x^1 feasible?*

4. *For each constraint, write a function* `is_ci_active` *that checks for any point x if the constraint is active at x (this function should return a boolean).*

5. *Test these functions on x^0 and x^1. Which constraints are active at x^0? Same question for x^1.*

6. *We now want to plot the feasible region Ω of this problem. Since the constraints are linear, each of them defines a halfspace and the intersection of the four halfspaces is the feasible region. To find the halfspace defined by each constraint, we can first find the equation of the line dividing the space into 2 halfspaces. For instance, for the first constraint $x_1 + x_2 \geq 3$, the corresponding line is $x_1 + x_2 = 3$ which is the same as $x_2 = 3 - x_1$. We can thus plot the latter line and the corresponding halfspace is $x_2 \geq 3 - x_1$, i.e., the region "above" the line. Repeat this strategy for all four constraints and annotate in your plot the location of Ω.*

7. *Add on the previous plot the location of the points x^0 and x^1. Observe that x^0 belongs to the lines corresponding to the active constraints at x^0.*

8. *We now want to determine the solution of this constrained problem. Since the objective function is linear, for any scalar $b \in \mathbb{R}$, $f(x) = b$ is the equation of a line. Observe that this defines a family of parallel lines, indexed by different values of b. Thus, to find the optimum, we need to find b such that at least one point of the line $f(x) = b$ falls in the feasible region. Check that this happens for $b = 4$ and that x^0 is the global minimiser by plotting the corresponding line.*

Problem 2 (KKT conditions)
Consider the following constrained problem:

$$\min_{x \in \mathbb{R}} x^2$$
$$\text{subject to} \quad 1 \leq x \leq 4.$$

1. *Re-write this problem in the general form given in the lectures by finding the constraint functions c_i and for each of them, define a function that compute $c_i(x)$ for any point x.*

2. *Write a function* `is_feasible` *that checks if a point x is feasible (this function should return a boolean).*

3. *Test this functions on $x^0 = 0$, $x^1 = 1$ and $x^2 = 3$.*

4. *Define a function that computes the Lagragian for any point x and vector of multipliers λ.*

5. *Define a function that that computes the gradient of the Lagragian function w.r.t. x and for any point x and vector of multipliers λ.*

6. *Check that the KKT conditions are verified at* $(x^* = 1, \lambda^* = \begin{pmatrix} 0 \\ 2 \end{pmatrix})$.

Problem 3 (Feasible directions)
Consider a feasible set defined by

$$\Omega = \{x \in \mathbb{R}^2 : -x_0 + x_1 \leq 2, x_0 + x_1 \leq 4, x_0 \geq 0, x_1 \geq 1\}.$$

1. *Draw the lines delimitating the feasible set and annotate the area corresponding to the feasible set.*

2. *Write a function* `is_feasible` *that checks is a point $x \in \mathbb{R}^2$ is feasible (this function should return a boolean).*

3. *Write a function* `active_set` *that returns the indices of active constraints at a point $x \in \mathbb{R}^2$.*

4. *Test the functions* `is_feasible` *and* `active_set` *on the points:*

   - $x = (0, 0)$
   - $x = (0, 4)$
   - $x = (1, 3)$

5. *Write a function* `is_feasible_direction` *that checks if a direction $d \in \mathbb{R}^2$ is in the set of linearised feasible directions at a feasible point x (this function should return a boolean). Recall that*

$$\mathcal{F}(x) = \{x \in \mathbb{R}^2 : \nabla c_i(x)^T d \geq 0, i \in \mathcal{A}(x) \cap \mathcal{I}, \nabla c_i(x)^T d = 0, i \in \mathcal{E}\}$$

6. *Test the function* `is_feasible_direction` *on the points $x = (0, 0)$ and $\bar{x} = (1, 3)$ and the directions:*

   - $d = (0, -1)$
   - $d = (1, 0)$
   - $d = (0, 1)$